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Transformation - A Fundamental Idea of Mathematics Education

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For Rudolf Sträßer

Introduction

This book intends to open up a discussion on fundamental ideas in didactics of mathematics as a scientific discipline. We want to introduce fundamental ideas as a possible answer to the diversity of theories in the field. Instead of providing a solely theoretical contribution, we suggest entering into this discussion by focusing on the "idea of transformation" that we regard as being fundamental to didactics of mathematics.

Transformation is a matter of interest in many areas of didactics of mathematics conceived of as "the sum of scientific activities to describe, analyze and better understand peoples' joy, tinkering and struggle for/with mathematics" (Sträßer 2009, p. 68): transformations of representations of mathematics and related transformations of mathematics, transformations of artifacts into instruments, transformations of mathematical knowledge, transformation of practice, transformation of solving strategies, and transformation of acquired heuristics to new similar problems, just to name a few. Accordingly, many theoretical approaches aim to conceptualize and grasp transformations: semiotics, the instrumental approach (Rabardel 1995), transposition didactique (Chevallard 1985), and the nested epistemic actions model (Schwarz et al. 2009).

By looking at these theories as being related to the same fundamental idea, we can ask further questions such as: How do we approach transformations research in didactics of mathematics? How is transformation conceptualized in each of these theories? What do we know/ learn about transformations related to the teaching and learning of mathematics?

In the following section, we will elaborate on the theoretical origins of our approach.

Theoretical Background

Our approach is embedded in the debate about the diversity of theories in didactics of mathematics. The diversity of theories has been an issue of discussion ever since the foundation of the discipline. This is documented in the *Theory of Mathematics Education Group* (TME) founded by Steiner and regular study groups at the *International Congress on Mathematics Education* (ICME) and the annual conference

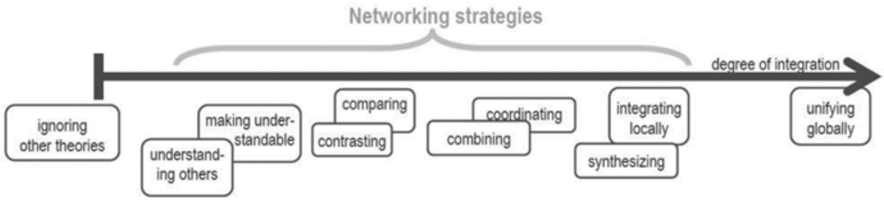

Fig. 1 A landscape of strategies for connecting theoretical approaches (Bikner-Ahsbahs and Prediger 2010, p. 492)

of the *International Group for the Psychology of Mathematics Education* (IGPME). The current significance of this issue as well as the controversy about it can be seen in the comprehensive volume *Theories of Mathematics Education* (Sriraman and English 2010). The tenor of the contributions is that diversity of theories is an inevitable and even welcome hallmark of didactics of mathematics.

The theoretical manifoldness is traced back to the vast variety of goals and research paradigms by many researchers, which are recorded in volumes such as "Didactics of mathematics as a scientific discipline" (Biehler et al. 1994) or the Study of the *International Commission on Mathematical Instruction* (ICMI) "What is research in mathematics education, and what are its results" (cf. Sierpinska and Kilpatrick 1998). Critics such as Steen (1999) argue that a lack of focus and identity pervades the foundations of the discipline:

> there is no agreement among leaders in the field about goals of research, important questions, objects of study, methods of investigation, criteria for evaluation, significant results, major theories, or usefulness of results (Steen 1999, p. 236).

This observation even leads him to question the scientific nature of the field which he describes as

> a field in disarray, a field whose high hopes for a science of education have been overwhelmed by complexity and drowned in a sea of competing theories (Steen 1999, p. 236).

This criticism is often encored by the call for a grand theory of mathematical thinking. Although a growing number of convincing arguments is presented to support the necessity of multiple theories (e.g., Bikner-Ahsbahs and Prediger 2010; Lerman 2006), the related problems of the discipline's missing focus and identity persist. The questions are how we deal with this variety and if there are other ways to promote the development of focus and identity of the discipline than a grand theory of mathematics education.

Bikner-Ahsbahs and Prediger (2010) argue that "the diversity of theories and theoretical approaches should be exploited actively by searching for connecting strategies" in order to "become a fruitful starting point for a further development of the discipline" (p. 490). Based on a meta-analysis of case studies about connecting theories, they suggest different strategies for connecting theories, which they call "networking strategies" (Bikner-Ahsbahs and Prediger 2010, p. 492). These networking strategies are organized according to their degree of integration between the two extremes "ignoring other theories" and "unifying globally" as shown in Fig. 1.

Although this overview of strategies for networking theories in didactics of mathematics provides a fruitful approach to deal with multiple theories, it seems hardly capable of contributing to the discipline's search for focus and identity, because it does not say anything about the phenomena these theories are related to. The networking strategies can be understood as heuristics to connect given theories. However, how to find theories that are worthwhile connecting? Which theories relate to a certain phenomenon?

In order to answer these questions, we suggest reflecting upon fundamental ideas of didactics of mathematics as a scientific discipline. Pointing out fundamental ideas could help to focus on the core issues of the discipline and could provide a means to organize theories in terms of being related to a similar idea.

Fundamental Ideas

In his seminal book "The Process of Education" (1960), Bruner introduced fundamental ideas as a means for curriculum development. For him they provide an answer to the basic problem that learning should serve us in the future which is at the heart of the educational process and therefore a fundamental problem of curriculum development. Students only have limited exposure to exemplary materials they are to learn. How can they learn something that is relevant for the rest of their lives? He argues that this "classic problem of transfer" can be approached by learning about the structure of a subject instead of simply mastering facts and techniques. "To learn structure" for Bruner means "to learn how things are related" (Bruner 1960, p. 7). According to him, transfer is dependent upon the mastery of the structure of a subject matter in the following way:

> in order for a person to be able to recognize the applicability or inapplicability of an idea to a new situation and to broaden his learning thereby, he must have clearly in mind the general nature of the phenomenon with which he is dealing. The more fundamental or basic is the idea he has learned, almost by definition, the greater will be its breadth of applicability to new problems. Indeed, this is almost a tautology, for what is meant by 'fundamental' in this sense is precisely that an idea has wide as well as powerful applicability. (Bruner 1960, p. 18)

Ever since Bruner, fundamental ideas of mathematics have been discussed in mathematics education as a didactical principle to organize curricula, and various catalogues of fundamental ideas of mathematics have been suggested (for an overview see Heymann 2003; Schweiger 2006). We will not discuss these in detail, because it would not support the central claim made here.

In his attempt to characterize mathematics as a cultural phenomenon, Bishop (1991) also arrives at something similar to Bruner's notion of fundamental ideas which he calls "similarities" (Bishop 1991, p. 22). 'Similarities' are similar mathematical activities and ideas that occur in different cultural groups. They are supposed to be a means to overcome the culturo-centrism by focusing on mathematical similarities between different cultural groups rather than on the differences in order to acknowledge that all cultures engage in mathematical activity. Therefore, Bishop's similarities might be understood as a cross-cultural approach to characterize the structure of mathematical activity whereas Bruner's view is limited to a Western/ American perspective. Nevertheless, fundamental ideas or similarities are both means to think about the inner structure of a discipline.

Schweiger, as opposed to Bishop, does not speak of one mathematical culture, but of several mathematical cultures, e.g., "mathematics in everyday life or social practice, mathematics as a toolbox for application, mathematics in school, and mathematics as a science" (Schweiger 2006, p. 63). He claims "it is more fruitful to acknowledge these facts than to try in vain to reconcile these different cultures" (Schweiger 2006, p. 63). Interestingly for him also, fundamental ideas are a way of dealing with this diversity of mathematical cultures by providing an understanding of what mathematics is about (Schweiger 2006, p. 64).

To summarize these reflections on the functions of fundamental ideas, we want to distinguish epistemological functions of fundamental ideas on the one hand from pragmatic functions on the other. From an epistemological point of view, fundamental ideas are a means to elicit the structure of a discipline and build up semantic networks between different areas. Furthermore, they are supposed to elucidate the practice and the essence of a discipline. In doing so, their pragmatic functions are to support the design of curricula and to improve memory.

Although fundamental ideas are discussed in didactics of mathematics to serve these functions with respect to mathematics, it is important to remember that Bruner's introduction of the notion of fundamental ideas was not limited to mathematics, but related to any discipline. Therefore, it seems legitimate to broaden the perspective and to not only discuss fundamental ideas of mathematics in didactics of mathematics, but also contemplate on fundamental ideas of didactics of mathematics itself as a scientific discipline. From the epistemological functions of fundamental ideas, it follows that fundamental ideas could serve as a means to overcome the criticism based on the diversity of theories in the field and to promote the formation of a focus and an identity of the scientific discipline didactics of mathematics.

Although the preceding remarks refer to the functions of fundamental ideas, it remains vague what fundamental ideas are and how they can be identified. Or, as Schweiger puts it, "one has the uneasy feeling there is no agreement about fundamental ideas" (Schweiger 2006, p. 68).

Bruner simply leaves it to specialists in every discipline to identify the fundamental ideas of the discipline:

> It is that the best minds in any particular discipline must be put to work on the task. The decision as to what should be taught in American history to elementary school children or what should be taught in arithmetic is a decision that can best be reached with the aid of those with a high degree of vision and competence in each of these fields. (Bruner 1960, p. 19)

However, even the specialists need to know what they are looking for. Bruner himself does not provide a clear definition of fundamental ideas. Revising the relevant literature on fundamental ideas, Schweiger (2006) offers four descriptive criteria in order to characterize fundamental ideas of mathematics:

Fundamental ideas

- Recur in the historical development of mathematics (time dimension)
- Recur in different areas of mathematics (horizontal dimension)
- Recur at different levels (vertical dimension)
- Are anchored in everyday activities (human dimension; Schweiger 2006, p. 68).

Although these dimensions relate to fundamental ideas of mathematics, they seem to be of a general nature which allows applying them to other disciplines as well. The time dimension and the horizontal dimension can be easily transferred to any other discipline. However, it is not obvious at the first sight what could be conceived of as a vertical dimension and a human dimension in didactics of mathematics. We suggest that different contexts of the disciplines involvement could be regarded as the vertical dimension: Didactics of mathematics is concerned not only with scientific inquiry of issues related to the people's involvement with mathematics, but also issues of teacher education and development. Therefore, ideas recurring as objects of inquiry and as relevant themes for teacher education and development could be conceived of as being fundamental in a vertical sense. Finally, we suggest that important ideas teachers are concerned about in their daily practice could be conceived of as the human dimension of didactics of mathematics.

The question remains how fundamental ideas can be found. It would be easy to just follow Bruner and leave it to "the best minds in any particular discipline". But how will they be able to find fundamental ideas?

Bishop's focus on similarities between different cultural groups in terms of mathematical activities and ideas offers a method to identify such similarities: cross-cultural comparison of mathematical ideas and activities. Accordingly, cross-cultural comparison of ideas informing research in didactics of mathematics could be one way of approaching fundamental ideas of the discipline.

According to Schweiger's characterization, cross-cultural comparison ought to be complemented by historical, horizontal, and vertical analysis of the disciplines areas of study and activity in order to link to the time dimension, the horizontal dimension, and the vertical dimension of fundamental ideas.

Transformation—A Fundamental Idea of Didactics of Mathematics?

In this book, we chose a twofold approach to tackle the issue of fundamental ideas of mathematics education as a scientific discipline. On the one hand, we followed Bruners' advice: "It is a task that cannot be carried out without the active participation of the ablest scholars and scientists" (Bruner 1960, p. 32). The authors that contributed to this book are well-known scholars in mathematics education. They were willing to penetrate the idea of transformation with the expertise of their own field of research. On the other hand, we tried to include researchers from different countries in order to have a multicultural approach to the idea of transformation.

In this book, we restricted the analysis of transformations in didactics of mathematics to three overarching themes:

1. Transformations related to transitions in mathematics education;
2. Transformations related to representations of mathematics;
3. Transformations related to concept and ideas.

The book is structured into three parts and each part is dedicated to one of these themes. Each part begins with an introduction which gives an idea of the overarching theme of each part and thereby sets the scene for the following chapters. At the end of each part, one or two discussions embed the preceding chapters in a broader discourse and relate them to the central idea of the book.

Part I focuses on transformations that occur at transitions in the individuals' course through mathematics education from Kindergarten to University. Two transitions within this course have attracted particular attention lately: the transition from Kindergarten to primary school and the transition from secondary school to university. Most of the chapters in part I focus on the latter transition and related transformations. Chapters 1–5 revolve around the *double discontinuity* in teacher education which was first pointed out by Felix Klein. In Chap. 1, Biermann and Jahnke draw attention to an aspect of the double discontinuity which has faded into obscurity namely the fact that school mathematics and mathematics as a science are “disconnected” and do not develop in mutual relatedness. They illustrate this aspect analyzing the paradigm shift between Eulerian mathematics and the Klein movement at the end of the 19th century using the case of one particular secondary school in Germany. In Chap. 2, Vollstedt, Heinze, Gojdka, and Rach develop a scheme for textbook analysis which covers both, general aspects such as motivation, structure, and visual representation and content specific aspects such as development/understanding of concepts, proof characteristics, and task characteristics. The feasibility study already provides evidence for a transformation of learning that is expected from students at the transition from school to university. This is also substantiated by Deiser and Reiss in Chap. 3, but from a different perspective. They analyze the development of first year university students' mathematical knowledge in terms of fundamental mathematical concepts. In Chap. 4, Pepin carries out a case study with one student which provides a more detailed insight into students' experience of the different contexts, different kinds of feedback and their benefit, and transformations students go through when transiting from school to university. The Chaps. 1–4 aim at a better understanding of the discontinuity between school and university, whereas Kaiser and Buchholtz report on the impact of an innovative teacher training program at the University of Giessen in Germany in Chap. 5. Finally, in Chap. 6, Grevholm widens the perspective by focusing on transformations of an individual's perception of mathematics throughout the course of education in mathematics as a whole.

Part II addresses transformations related to representations of mathematics. Hoffmann (2006) describes the major mathematical activities *mathematization*, *calculation*, *proving*, and *generalization* in terms of related transformations of signs and concludes that the "essence of mathematics consists in working with representations" (p. 279). These activities comprise transformations of several kinds: On the one hand mathematics has to be represented. Therefore, mathematics has to be transformed into the representations. Different representations, e.g., geometrical or algebraic representations, have different affordances and constraints. The mathematical world is seen differently through different representations. Each representation may highlight some aspects of mathematics while others are neglected. On the other hand, humans can only act upon the mathematical reality via representations. Therefore, transformations of mathematical representations are capable of constructing and transforming mathematical reality (Steinbring 2005).

In Chap. 8, Kadunz draws on Pierce's semiotic theory in order to analyze transformations of diagrams that occur while reading a mathematical text. These are related to the transformation of the author's language into the reader's language. Mariotti illuminates the relation between transformations of representations of mathematics in a Dynamic Geometry Environment (DGE) and the mathematical meaning of conditional statement in Chap. 9. In Chap. 10, Hölzl illustrates how a DGE can be implemented to provide opportunities for an investigative style of learning at university level in order to support students in finding their own way of dealing with mathematical content. In Chap. 11, Laborde and Laborde address three different dimensions of transformations that are related to representations of mathematics in a DGE: an epistemological dimension faced by software designers when implementing the features of mathematical objects into the software; a cognitive dimension related to students' learning in dynamic environments; and a didactic dimension concerned with the transformation of tasks in order to exploit the affordances of the dynamic tool. Geiger draws attention to transformational aspects of social setting related to working with technological tools in Chap. 12. In two episodes, he illustrates how students work with technological tools in different social contexts and how this fosters the transformation of students' understanding of mathematics. In Chap. 13, Bessot analyzes how a simulator for reading—marking out activities in vocational education transforms student's encounter with geometry and space. In his discussion of part II, Seeger embeds the theoretical approaches of the chapters in part II in a broader discussion of what has been called an "embodied" perspective on human activity. In a second discussion of part II, Sutherland frames the chapters from a technological point of view and gives ideas on how to promote professional development in using modern tools.

Part III is concerned with transformations related to concepts and ideas. The chapters in this part either introduce genuine didactical concepts that are supposed to grasp transformation processes related to particular mathematical content areas or activities (Chaps. 16–18) or tackle the problem of how to model mathematical competencies in terms of specific instructions for practice or evaluation (Chaps. 19–20). In Chap. 16, Dreyfus and Kidron introduce the notion of *proof image* as an intermediate stage in a learner's production of proof and illuminate the transition from proof image to formal proof. Stanja and Steinbring intend to conceptualize transformations of stochastic thinking at primary level by means of the notion of *elementary stochastic seeing* in Chap. 17. In Chap. 18, Kuzniak introduces the notion *geometric work space* which draws particular attention to figural, instrumental, and discursive geneses and is supposed to conceptualize the transformation of geometric knowledge at school level. Profke tackles the question of how competencies might be achieved through mathematical activities in Chap. 19. He suggests various activities which might contribute to the development of mathematical literacy. In Chap. 20, Klep gives an idea about a future development of didactics of mathematics which he calls "informatical educational science." Using the example of arithmetical competence, he illustrates the algorithm-based implementation of mathematical competences in computer sciences. In her discussion of part III, Prediger comments on the overarching ideas of this book: the notion of fundamental ideas of didactics of mathematics as a scientific discipline in general and the idea of transformation as a candidate for a fundamental idea of didactics of mathematics in particular by referring to the chapters in part III.

Genesis of this Book

The idea of this book was born while preparing for Rudolf Sträßer's 65th birthday celebration. The attempt to find a theme characterizing the work of Rudolf Sträßer led us to an idea, which, at the first sight, seemed not to be related to his work at all. References to one of Rudolf Sträßer's main fields of interest can only be found in Part II. However, to be honest, can you imagine anything more boring than to get a volume with collected works of the field of your own expertise to your 65th birthday? Nevertheless, this book honors the work of Rudolf Sträßer. It relates to his pursuit to develop the field of mathematics education theoretically, it proofs how he encouraged his doctoral students to think independently, to believe in their own ideas and not to hesitate to go off the beaten track. We hope that this book will contribute to the discipline's and to Rudolf Sträßer's personal pursuit to advance the field of mathematics education theoretically.

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Part I
Transformations at Transitions in Mathematics Education

Introduction

Part I deals with transformations related to knowledge which occur throughout the course of education in mathematics. Several perspectives are used to analyze different kinds of transformations in several contexts. Biermann and Jahnke (Chap. 1), Vollstedt et al. (Chap. 2), Deiser and Reiss (Chap. 3) and Pepin (Chap. 4) share the same starting point for their discussion. Even though looking upon different historical periods, they all focus on the gap between mathematics in school and at university. This gap might be characterized by a transformation from an application-oriented school perspective to a formal and deductive paradigm at the university level. It was Klein who first pointed to the so called *double discontinuity,* a term that alludes to the missing link between school and university mathematics, which a teacher student experiences twice: once when he leaves school and enters university to start his studies in mathematics to become a teacher and again when he finishes his studies and goes back to school in order to teach mathematics. In general, this double discontinuity is traced back to the different mathematical paradigms prevalent at school and at university. However, Biermann and Jahnke (Chap. 1) argue that there is another aspect which Klein addressed by his notion of double discontinuity, namely the fact that mathematics in school and mathematics at the university are 'disconnected' and do not develop in mutual relatedness. The transformation of school mathematics is very slow related to the development of mathematics as a science and the transformation of school curricula related to new scientific achievements in mathematics is confronted by several obstacles.

Biermann and Jahnke (Chap. 1) elucidate this second notion of double discontinuity in their chapter. Using the case of a particular secondary school in Germany, they analyse the paradigm shift between Eulerian mathematics and the Klein movement at the end of the nineteenth century. They show that the transformation from the well-established paradigm of algebraic analysis to Klein's concept of *functional thinking* and the introduction of infinitesimal analysis in school mathematics was a tedious process which was underestimated by its proponents. The co-existence of the mentioned paradigms at the beginning of the twentieth century is underlined by the analysis of a school book that fitted to the syllabus of 1901. “Klein thought to solve the problem by a modernization of the school curriculum, but the double discontinuity between school mathematics and university mathematics proved to be a much deeper problem than he could anticipate in his times (Biermann and Jahnke Chap. 1).

Vollstedt et al. (Chap. 2) approach the double discontinuity as it appears at the present from the perspective of textbooks. In order to develop a deeper understanding of the discontinuity from school to university they develop a framework for analyzing textbooks from upper secondary level and university level. The scheme for textbook analysis focuses on general aspects, as well as on content specific criteria. In detail it focuses on ‘motivation’, ‘structure and visual representation’, ‘development/understanding of concepts’, ‘proof characteristics’ and ‘task characteristics’. Feasibility studies based on this scheme already point out important commonalities and differences in school and university textbooks that do not only elucidate the discontinuity from school to university but also point to shortcomings of mathematics textbooks in general. With respect to transformations the analysis shows on the one hand transformations of mathematics and the learning of mathematics at the transition from school to university. On the other hand, the theories backing the categories of the textbook’s analysis scheme themselves point to important transformations that are considered to support learning, e.g. the presentation of contents in different representations.

Deiser and Reiss (Chap. 3) also address the problem of double discontinuity in their chapter. They analyze the development of students’ mathematical knowledge during the first semester by means of an inquiry of students’ understanding of fundamental mathematical notions, as for example *limit of a sequence* or *infimum*. Their analysis mainly focuses on students’ ability to provide formalized definitions of fundamental mathematical notions and thus deals with the shift of paradigms between school and university mathematics. They conclude that “*mathematical knowledge acquired in secondary school does not necessarily constitute a reliable foundation for adopting basic mathematical notions…at the university*” (Deiser and Reiss Chap. 3). Furthermore, they identify typical problems that occur at the transition from school to university concerning *“habits learned in school*” (Deiser and Reiss Chap. 3) and highlight prospects for further research in order to learn more about the transformation of knowledge at the transition from school to university.

Pepin (Chap. 4) addresses the discontinuity between school and university mathematics from the perspective of identifying students’ experiences and kinds of feedback that could be helpful to master independent learning. In a qualitative study she analyzes biographical interviews of students, teachers and professors at university, as well as lectures and curricular documents to identify aspects of feedback and independent learning that students experience. The development of more independent learning practices is stressed as crucial to master the transition from school to university, where transformation processes from lessons into lectures, homework into coursework, textbooks into course materials, tests into examinations and school mathematics into university mathematics take place. Pepin invites higher educational institutions to provide more necessary “tools and instructions to use those" (Pepin Chap. 4) to support learners at the beginning of their studies at the university.

Whereas, the studies by Pepin and Deiser and Reiss take the conditions at school and at university for granted and aim at a better understanding of how students deal with the different expectations at school and at university in terms of social norms or mathematical competence; Kaiser and Buchholtz (Chap. 5) evaluate the effect of changing conditions at the university. In their chapter, they investigate the impact of an innovative teacher education programme at the University of Giessen that was developed to reduce the double discontinuity. In an evaluation study, they investigated how "*the innovative efforts at the University of Giessen actually influenced the development of the local students' competences*" (Kaiser and Buchholtz Chap. 5) compared to that of prospective teacher students from other universities and non-prospective teacher students. In interviews, the students of the University of Giessen who are part of the innovative programme still address the discontinuity between mathematics at school and at university, but predominantly related to courses that have been taught in the traditional way. Altogether, the results of the evaluation study show also that the reorganization of the teacher training programme involving specific courses for prospective mathematics teachers seems to work. Transforming the ways mathematical content is taught at universities in a way that focuses more on understanding than on quantity by means of exemplification, seems to motivate students and have a positive influence on the development of their professional competences (Kaiser and Buchholtz Chap. 5). In this respect the chapter by Kaiser and Buchholtz supports promoting change in teacher education in order to reduce or even overcome the double discontinuity identified by Klein.

Grevholm's approach (Chap. 6) is more open. In her chapter, she addresses the development and transformation of a professional identity over time. Based on a retrospective analysis she identifies influences on the professional identity of one mathematics teacher. Her focus is on transformations "*that take place for example in the professional identity of a student teacher going through teacher education and building up professional experience*" (Grevholm Chap. 6). Grevholm uses narratives to investigate the development of personal conceptions of individuals with the theoretical background of a *concept map model* that combines professional and private identity of individuals to get a comprehensive conceptual understanding of the impact of teacher education. Based on a case study she identifies different kinds of transformations, as for example the "*transformation from passive to active*", "*transformation from child to adult in mathematics*", "*the transformation to experienced user of mathematics*", "*transformation from learner to teacher of mathematics*" or the "*transformation from teacher of mathematics to researcher of mathematics*" (Grevholm Chap. 6). These different transitions can be regarded as processes in the development of an individual's professional identity *"from experience of success in mathematics"* (Grevholm Chap. 6). The identified stages are used in further research to investigate questions, such as: Do weak students of mathematics have a better understanding of special problems regarding the learning process of their pupils? Or is a weakness in mathematics the cause for problems in understanding another person's thinking?

The chapters in this part demonstrate that the analysis of transformations occurring during the course of education in mathematics is approached very differently.

Biermann and Jahnke as well as Vollstedt et al. rely on written, official or historical documents and textbooks in order to analyze transformations of mathematical knowledge. Kaiser and Buchholtz, Deiser and Reiss, Pepin and Grevholm use particular theoretical constructs to grasp transformations: Buchholz and Kaiser, as well as Deiser and Reiss measure specific competences of students in order to find out how students deal with transformations of conditions, whereas Pepin and Grevholm use the notions of didactical contract and identity respectively to grasp individual perceptions of conditions at educational institutions.

So, already the first part of the book indicates how fruitful and broadly conceived the idea of transformation is, although restricted to transition from school to university.

Chapter 1
How Eighteenth-Century Mathematics Was Transformed into Nineteenth-Century School Curricula

Heike Renate Biermann and Hans Niels Jahnke

1.1 The Double Discontinuity

In the introduction to his famous book *Elementary Mathematics from an Advanced Standpoint* (originally published in 1908), Felix Klein coined the frequently cited phrase of a "double discontinuity" between mathematics at schools and universities. The underlying problem, as Klein described it in general terms, is considered as highly relevant still today (Deiser and Reiss, Chap. 3; Pepin, Chap. 4; Buchholtz and Kaiser, Chap. 5). However, at those times there existed an implicit and special connotation of the phrase, which is forgotten today, but should be known to understand Klein adequately.

This connotation becomes obvious in the later parts of the book. In the section on analysis, Klein discussed the exponential and logarithmic functions as paradigmatic cases for some of his views on school mathematics, and it was in this context that he used a concept that might sound mysterious to the modern reader, namely that of "algebraic analysis". Klein designated the procedure generalizing the exponential function step by step from natural numbers to fractions to negative numbers and finally to real numbers as the "Systematic Account of Algebraic Analysis" (Klein n. d., pp. 144–6). The phrase referred to the traditional way in which the exponential functions were handled at schools and which Klein wanted to replace by a more elegant and modern treatment. In a didactical résumé, he stated:

> It is remarkable that this modern development has passed over the schools without having, for the most part, the slightest effect on the instruction, an evil to which I have often alluded. The teacher manages to get along still with the cumbersome algebraic analysis, in spite of its difficulties and imperfections, and avoids the smooth infinitesimal calculus, …

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> The reason for this probably lies in the fact that mathematical instruction in the schools and the onward march of investigation lost all touch with each other after the beginning of the nineteenth century. … I called attention in the preface to this discontinuity, which was of long standing, and which resisted every reform of the school tradition… In a word, Euler remained the standard for the schools. (Klein n. d., p. 155)

This is one of the rare, but by no means singular, statements of Klein's where he alluded to the term "algebraic analysis" as the core component of traditional school mathematics which he wanted to overcome. At another place, he spoke polemically of the "old misery of algebraic analysis" (Klein 1907, p. 105). Thus, Klein's phrase of a "double discontinuity" did not hint only at missing links between school and university mathematics caused by the cognitive distance between research and elementary level or by institutional boundaries. Beyond that, the term designated in Klein's and his contemporaries' view a difference of mathematical conception. As Klein said, the traditional school mathematics of his time was determined by Euler's views, whereas he intended to introduce into school the ideas of contemporary modern mathematics. To use Thomas Kuhn's term, there was a *difference of paradigms* between Klein's conception and that of the school mathematics of his time.

The present chapter briefly sketches the basic conception of eighteenth-century algebraic analysis as conceived by Euler and Lagrange and then analyzes how at the turn of the nineteenth century this conception was transformed by mathematical and cultural forces into a subject of teaching at both universities and the reorganized Prussian gymnasia. It will become clear that the arithmetic algebraic curriculum of Prussian gymnasia was not an arbitrary collection of topics. Rather, there was an inner logic that was convincing to mathematicians and mathematics teachers.

Conceptions on a general level are one thing, teaching concrete students at schools another. Thus, the present chapter accompanies and contrasts the general concepts by information about the teaching of mathematics at a concrete school, the Ratsgymnasium in Bielefeld (see Biermann 2010). The decision to consider just this school is motivated by the fact that in the eyes of the Prussian government it was one of the leading schools in the province of Westfalia. Besides, there exists still extensive archival material concerning the teaching of mathematics, including numerous written tests of students, and the teachers of mathematics were highly committed in the reform movement at the end of the nineteenth century.

> Originally, the Ratsgymnasium in Bielefeld was a church-related Latin school. In 1558 it was taken over by the authorities of the city of Bielefeld. In the course of the Humboldt educational reforms the Ratsgymnasium became a leading gymnasium in Westfalia. During the 19th century the school splitted into two types of secondary schools which remained in the same institution: the classical Gymnasium and the Realgymnasium. The latter had a focus on mathematics and the natural sciences.

In a final part, it will be shown that the old paradigm of algebraic analysis was so strong that F. Klein's efforts to reform the teaching of mathematics did not lead to the replacement of the old paradigm by a new one but to a coexistence of the two paradigms which lasted well until the mid-1920s.

1.2 Euler's Algebraic Analysis

The term "algebraic analysis" suggests how calculus was thought of and practiced in the eighteenth century, especially in the works of Leonhard Euler and Josef-Louis Lagrange, and the continuation of this tradition in the nineteenth century. It was only in the nineteenth century that the term received a technical meaning and appeared in the titles of a certain class of textbooks treating the elementary and preparatory parts of infinitesimal analysis, especially the theory of infinite series. Ironically, this use of the term was presumably established by Cauchy's *Analyse algébrique* (Cauchy 1821) which contributed more than any other book to the final destruction of this tradition, which ended with the article "Algebraische Analysis" by Alfred Pringsheim and Georg Faber in the *Enzyklopädie der Mathematischen Wissenschaften* (Pringsheim and Faber 1909–1921).

The model for these nineteenth-century textbooks was Leonhard Euler's famous *Introductio in analysin infinitorum*, which appeared in two volumes in 1748. The first volume developed algebraic methods and treated functions and their expansions in power series, infinite products, and continued fractions. The second volume contained what today is called analytic geometry. Consequently, the *Introductio* is *not* a book on infinitesimal analysis proper, but provides algebraic and geometric tools to it. Euler explained the motivation of such an introductory book by the observation:

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt to this more subtle art. (English translation according to Euler (1990, v))

The *Introductio* provided algebraic techniques which prepared a student for a deeper understanding of analysis and which were not contained in the ordinary treatises on the elements of algebra. In this way, "the reader gradually and almost imperceptibly becomes acquainted with the idea of the infinite" (l. c.).

Conceptually, there was no clear-cut demarcation between the *Introductio* and infinitesimal analysis proper. The *Introductio* treated infinite symbolic expressions like power series and infinite products, but it did not contain the differential and integral calculus. These were treated in Euler's later textbooks *Institutiones calculi differentialis* (1755) and *Institutiones calculi integralis* (1768–1770). Thus, in a first approximation one could say—and mathematicians at the turn of the nineteenth century did so—that the *Introductio* treated only finite quantities and did not contain infinitely small quantities as symbolized by the differentials dx, dy, and dz. From a modern point of view, this distinction looks rather artificial as both the theory of power series and the differential calculus are based on the concept of limit, but to mathematicians of the time it was plausible.

The somewhat fuzzy demarcation between the two fields becomes apparent again when one considers volume II of the *Introductio*. In this book, among many other things, the tangents to curves are calculated and Euler maintained that this is achieved by purely algebraic methods:

> Thus I have explained a method for defining tangents to curves, their normals, and curvature … Although all of these nowadays are ordinarily accomplished by means of differential calculus, nevertheless, I have here presented them using only ordinary algebra, in order that the transition from finite analysis to analysis of the infinite might be rendered easier. (l. c., vii)

By the end of the eighteenth century, it was an open question what can be afforded by algebraic means and at what point infinitesimal methods were inevitable. In fact, there even emerged various attempts to reduce infinitesimal analysis completely to ordinary algebra.

The most famous approach of this type was Joseph-Louis Lagrange's *Théorie des fonctions analytiques* (1797). In this book, infinitesimal analysis is consistently treated as a calculus of power series. The derivative of a function *f* is defined as the coefficient *p* in the power series expansion

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + \cdots$$

Setting $p = \dfrac{df(x)}{dx} = f'(x)$ and supposing that $f''(x)$ is derived from $f'(x)$ in the same way as $f'(x)$ is derived from $f(x)$, the above series will become the Taylor expansion

$$f(x+i) = f(x) + if'(x) + \frac{i^2}{1\cdot 2} f''(x) + \frac{i^3}{1\cdot 2\cdot 3} f'''(x) + \cdots$$

For details of Lagrange's approach and especially his "proof" that any function can be expanded in a power series, see Fraser (1989).

Given the prominence of its author, Lagrange's attempt of reducing infinitesimal analysis to ordinary algebra was very influential. Nevertheless, at the turn of the nineteenth century it was more common to distinguish between *Analysis of the Finite* and *Infinitesimal Analysis*. The former comprised the contents of Euler's *Introductio*, whereas the latter referred to differential and integral calculus proper. The subject of *Analysis of the Finite* was the analytical treatment of finite quantities including the use of infinite symbolic expressions like power series. The subject of *Infinitesimal Analysis* was the analytical treatment of infinitely small quantities like d*x*, d*y*, and d*z*. This distinction was in line with the global structure Euler had given to the field in his textbooks and was rather different from modern views of analysis.

The contents of *Analysis of the Finite* can be found in a well-known contemporary mathematical encyclopedia as part of the entry *Analysis*.

Analysis of the Finite according to G. S. Klügel's *Mathematisches Wörterbuch* (1803–1808):

I	The theory of functions or of the forms of quantities
II	Introduction to the theory of series
III	Combinatorics
IV	Combinatorial analysis in general
V	Binomial and polynomial theorem
VI	Products of unequal binomial factors
VII	Factorials
VIII	Logarithmic functions
IX	Trigonometric functions
X	Application of trigonometric functions to the decomposition of a function into trinomial real factors
XI	Series, as continuation of Sect. II
XII	Equations in two or more variables
XIII	Analysis of curved lines
XIV	Calculus of finite differences
XV	Connection between the analysis of the finite and the differential calculus:
	1. Through Taylor's theorem and some of its applications to the theory of series
	2. Through the general theorem of Lagrange for the reversion of series
	3. Through the determination of maxima and minima of a function
	4. In the geometry of curved lines, through the determination of tangents, normals and special points; through the formulae for different methods of generating lines by evolution, rotation etc
XVI	Indeterminate or diophantine analysis, which may be viewed as the second main part of algebra

From Klügel's survey, we can conclude that at the turn of the nineteenth century *Taylor's theorem* was situated at the borderline between *Analysis of the Finite* and the *Differential Calculus*. Thus, it could be considered as a topic either in the former or in the latter. A hundred years later, Felix Klein argued that the 1812 syllabus for Prussian gymnasia, the so-called syllabus of Süvern, included differential calculus because Taylor's theorem was mentioned in it. From the above survey it can be seen that this claim was historically not justified. The survey also shows that topics which today are taught as applications of differential calculus (extrema and tangents) were, in Klügel's classification, a part of algebraic analysis (or analysis of the finite) in accordance with Euler's treatment of these topics in Vol. 2 of the *Introductio*.

1.3 The Combinatorial Approach and its Cultural Impact

Klügel's survey contained under numbers III–V the entries "Combinatorics," "Combinatorial analysis in general," and "Binomial and polynomial theorem." These topics refer to a mathematical development which took place in Germany and had a certain importance for the teaching of mathematics. Today the term *combinatorial analysis* is synonymous with combinatorics, but at the end of the eighteenth century it designated an approach to calculus developed by a group of German mathematicians, the so-called *Combinatorial School*. The leading figure of this group was the Leipzig mathematician Carl Friedrich Hindenburg (1741–1808). These mathematicians sought to develop a system of "combinatorial operations" to determine and facilitate calculations with power series, infinite products, and continued fractions. With regard to their general vision, they referred to G. W. Leibniz who in his *Dissertatio de Arte Combinatoria* (1666) and other early writings had developed the idea that the *ars combinatoria* might be considered as a general science of symbolic expressions (combinations of symbols) of which algebra is only a special case (cf. Leibniz 1976, pp. 54–56).

In their approach to infinitesimal analysis, the Combinatorial School followed Lagrange insofar as they considered it as a calculus of power series and infinite products. Their specific idea was to describe calculations with these infinite expressions by means of combinatorics. The reader might consider the so-called "binomial formula"

$$(1+x)^m = \sum_{\nu=1}^{\infty} \binom{m}{\nu} x^\nu$$

and its generalization, the "polynomial formula":

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)^m = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$$

When m is a natural number, these formulae are in fact combinatorial identities. The coefficients A_i on the right side of the latter formula are the so-called "polynomial coefficients" which can be found in any textbook on combinatorics. This is true even if $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdot\cdot$ is an infinite power series, a so-called *infinitinom*, because the calculation of the A_i involves only finite segments of the series, and therefore $A_i = f(a_0, a_1, a_2, \cdots, a_i)$.

It is not so obvious that these formulae can also be interpreted in a combinatorial way when m is a negative or rational number; however, this is in fact possible (Jahnke 1990a, p. 169). Thus, by means of the polynomial formula, arbitrary roots and multiplicative inverses of any power series can be calculated. More than that, in 1793 Heinrich August Rothe, a member of the Combinatorial School, proved a then famous theorem stating that also the reversion of series can be effected by means of the polynomial formula. For the Combinatorial School, algebraic or combinatorial analysis was a *closed system of symbolic expressions.* In this system, the elementary algebraic operations—addition, multiplication, division, exponentiation, extraction of roots of power series, and even the solution of arbitrary equations—are universally performable, and combinatorics, especially the *polynomial theorem,* played an essential role. For Carl Friedrich Hindenburg and his adherents, the polynomial theorem was the "most important theorem of analysis" (Hindenburg 1796).

Of course, this was a purely syntactical approach to power series and infinite products. In terms of modern mathematics, the underlying notion was that of a "formal power series." As is well known, it was not completely clear how the numerical relations which result from such power series expansions could be integrated into this formal approach. Euler and his followers were convinced that divergent series have to be accepted. In addition, at the beginning of the nineteenth century several more or less elaborate approaches for handling divergent series had been developed

(Jahnke, l. c.). None of them was successful. Finally, Cauchy's *Analyse algébrique* of 1821 led to the exclusion of divergent series from mathematics for a long time and thus destroyed the formal approach.

It is a remarkable fact that at the turn of the nineteenth century the combinatorial approach had a considerable impact on intellectual circles outside of mathematics. Philosophers J. F. Herbart (1776–1841), K. Chr. Fr. Krause (1781–1832), J. F. Fries (1783–1843), G. W. F. Hegel (1770–1831), and others as well as the romantic poet Friedrich von Hardenberg (Novalis 1772–1801) referred in their writings on mathematics explicitly to the ideas of the Combinatorial School (Jahnke 1990a, p. 167). Some of these authors had the idea that a view of mathematics as a science of quantity was too narrow. Rather, the two disciplines of arithmetic and combinatorics should be considered as "coordinated" and independent foundations of mathematics (Krause 1807). The same idea was expressed by Justus Günther Grassmann (1779–1852), father of the famous mathematician Hermann Grassmann, who wrote a small booklet on "Geometrische Combinationslehre" in which he developed a three-dimensional vector calculus based on combinatorial considerations (Scholz 1989). In 1819, the famous pedagogue F.A.W. Diesterweg published a small booklet under the title *Geometrische Combinationslehre*.

During the nineteenth century, textbooks written for the teaching of mathematics at gymnasia frequently contained a philosophically oriented introduction on what numbers and quantities are and what mathematics is. For example, in 1853 Carl Friedrich Collmann, teacher of mathematics at the Ratsgymnasium Bielefeld, published a textbook with a lengthy philosophically oriented introduction on the nature of mathematics.

> When contemplating things we are able to ignore and negate special properties of them, and by this we can so to speak pull off or abstract one or several features from things… The science which is concerned with considering relations of form and position and abstract quantities is called mathematics. (Collmann 1853, p. 1 f.)

The authors of these books were aware that students would understand these introductions only "retrospectively" (Tellkampf 1829, p. vi); nevertheless, this seemed to be required by the standards of scientific writing. In some of these introductions, the combinatorial approach with the idea of arithmetic and combinatorics as coordinated and independent foundations of mathematics played an important role (see f.e. Müller 1838).

1.4 The Didactical Conception of Algebraic Analysis

In a complicated process, the mathematical curriculum of Prussian gymnasia and related institutions of higher education emerged and developed in the course of the nineteenth century. The process was framed by interests of the different social groups who sent their children to the institutions of higher education, by political decisions on the structure of the system, by cultural trends, and, last but not least, by mathematical and didactical conceptions and their acceptance by teachers and pupils. Biermann (2010) describes this process in detail discussing the example of the Ratsgymnasium Bielefeld, which in 1866 became a so-called *Doppelanstalt* consisting of a *Gymnasium* and a *Realschule* (later *Realgymnasium*). "Realschulen" were secondary schools which focused on mathematics and the natural sciences. This chapter does not discuss these developments in detail, but rather concentrates on the global structure of the mathematical curriculum and the role of algebraic analysis.

Basically, the mathematical curriculum consisted of three components, namely (1) a course of 2 or 3 years on elementary arithmetical calculations necessary for everyday life and basic commercial applications, (2) synthetic Euclidean geometry, and (3) a scientific course on arithmetic, elementary algebra, and a first introduction to infinite series. Thus, some contents of *Algebraic Analysis* or *Analysis of the Finite* became part of the curriculum.

As F. Klein remarked, it was common use to call the entire arithmetic–algebraic part of the school syllabus "Arithmetic": "The term includes … besides ordinary calculations with letters … the theory of equations and analysis insofar as it is taught" (Klein and Schimmack 1907, p. 101, our translation). Grosso modo the higher parts of this course were a true, though reduced, image of algebraic analysis as exhibited in Klügel's survey. The development started with the ambitious syllabus of Süvern (1812). As was mentioned above, it contained Taylor's theorem but seen as a subject of algebraic analysis. Because of strong resistance against its ambitious requirements in many schools and cities, this syllabus was never made official and finally abandoned. Nevertheless, some calculations with infinite series remained part of the teaching at the upper grades, and, in particular, the binomial theorem for arbitrary exponents as an application of combinatorics was contained in every syllabus until the beginning of the twentieth century. This was obviously a direct result of the combinatorial approach to analysis (see Sect. 3). In the second half of the century, the binomial theorem was confined to natural numbers at the gymnasia and then, of course, it reduced to a finite identity; however, at the Realschulen (Realgymnasien; see above) with more weekly hours for mathematics, the binomial theorem for arbitrary exponents was required until 1901. For the mathematics teachers of the time the binomial theorem was the culmination of the course in arithmetic.

This was also the case at the Ratsgymnasium Bielefeld. From 1830 to 1880, the binomial theorem and infinite series were a constant subject of the upper grades at this school. "Subjects in Prima: over the year in two hours per week combinatorics, binomial theorem, introduction into the theory of infinite series" (Stadtarchiv Bielefeld Ratsgymnasium 111 Lehrmittel 1842–1903). Beginning with the 1880s, there was a shift of subjects to analytic geometry though the binomial theorem was still officially demanded.

In the spirit of the time, teachers of mathematics and textbook authors stressed the systematic unity of the whole course in arithmetic. The regulations for the final examinations at secondary schools (the "Abituredikt") of 1834 required of the students "a clear insight into the coherence of all the theorems in a systematically arranged presentation." At the end of the century, the influential mathematician and teacher trainer Max Simon wrote emotionally:

> The seven arithmetical operations and … the gradually extended concepts of number… form a closed whole of such a firmly established structure and inner necessity that the whole process progresses with elementary energy … if only the right starting point the insight in the process of counting is triggered. The elementary arithmetic from counting until the Binom with arbitrary exponents is the only example of a science which is accessible to schools; only ignorance is able to withhold it from it. (Simon 1908, p. 82, English translation by the authors)

The statement was a clear defense of the nineteenth-century conception of school mathematics and its author Max Simon, a prominent antipode of Felix Klein with regard to the teaching of mathematics (see Klein's remarks on Simon in the introduction of his *Elementary Mathematics from an Advanced Standpoint*). In the above quotation, Simon mentioned the "gradually extended concepts of number," a phrase by which he alluded to the process of adjoining step by step the negative numbers to the natural numbers and the rational to the integers. The mathematical elaboration of the notion of algebraic adjunction as a systematic approach to the concept of number started with M. Ohm's *Versuch eines vollkommen konsequenten Systems der Mathematik [Attempt at a Perfectly Consequential System of Mathematics]* of 1822 (see Bekemeier 1987) and found its most mature expression in H. Hankel's "Principle of Permanence" presented in the *Vorlesungen über die complexen Zahlen und ihre Functionen [Lectures on complex numbers and their functions]* of 1867.

Thus, in the eyes of teachers of mathematics and mathematicians involved in teacher training the arithmetic–algebraic domain as a whole was a systematic theory with the binomial theorem for arbitrary exponents as a culmination. This systematicity constituted the intrinsic meaning of the field.

The extrinsic meaning of the arithmetic–algebraic domain consisted in elementary applications to geometry and physics. In 1834, the government decided to remove analytic geometry from the syllabus. Therefore, the applications were reduced to calculations of geometrical quantities, such as areas and volumes, and to plane and spherical trigonometry. Thus, the whole domain was a structure composed of theory and applications.

Theory	Applications
Arithmetic, algebra, algebraic analysis	Calculations of elementary geometrical and physical quantities and plane and spherical trigonometry

It was common use in school mathematics of the time to call the application of algebra to geometry and physics "algebraic analysis." This referred to the concept of "analysis" in ancient mathematics. The term "algebraic analysis" in this meaning should not be confused with its use as a name for the Eulerian tradition established by the *Introductio* of 1748.

What types of applications students had to work on can be seen from tasks which have been treated at the Ratsgymnasium Bielefeld:

> A prism with base area a [square ft] and height = h [ft] has to be converted into a sphere. (1832)

> The base areas of a layer of a sphere have radii of 12 and 8 inches. the height is 4 inches. The layer is transformed into a regular tetrahedron. What is the distance of the plane which bisects the tetrahedron from its base? (1869)

> An object is thrown down vertically with initial velocity e. How many seconds later has another body with initial velocity c to be thrown down from the same point so that it catches up the first body in t seconds. (1886/1887)

During the nineteenth century, a special didactical education for the teachers of mathematics did not exist, and, consequently, there was no accumulation and systematization of didactical conceptions. Nevertheless, there were some pedagogical and mathematical ideas which, in part, derived from the combinatorial approach to algebra and which are worth considering still today. On a general level, these ideas can be described by the catchword *insight into structural properties of formulae*. In contrast to the standard approach, algebra was not treated as a technique of applying rules to formulae. Rather, the guiding viewpoint was to build up a *network* of formulae connected by structural similarities. The binomial formula might be a good example for illustrating this idea. In the standard treatment

$$(a+b)^2 = a^2 + 2ab + b^2$$

is an isolated entity derived by applying rules such as the distributive, commutative, and associative laws.

The alternative view aims at building up sequences of connected formulae:

$$\begin{aligned}(a+b)^2 &= a^2 + b^2 + 2ab \\ (a+b+c)^2 &= a^2 + b^2 + c^2 + 2(ab + ac + bc) \\ (a+b+c+d)^2 &= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)\end{aligned}$$

The structural similarity of these formulae is obvious, and a teacher could ask a pupil what will happen if a fifth letter occurs, or if one letter is omitted. In other words, the main aim was not to apply rules, but to see the symmetries and regularities of a formula. Today, educators of mathematics would talk of *pattern recognition*. Pattern recognition was also an important feature of Euler's mathematical style, and it was inherent to the combinatorial approach. It is for such reasons that the well-known teacher of mathematics J.H.T. Müller, for example, started the teaching of algebra with "easy combinatorial tasks … in order to accustom the student early to a … law-governed method of arrangement" (Müller 1838, viii). He called tasks involving algebraic pattern recognition as *Symmetrische Aufgaben*. A similar approach can be found in A. Bretschneider's *System der Arithmetik und Analysis* (1856/1857).

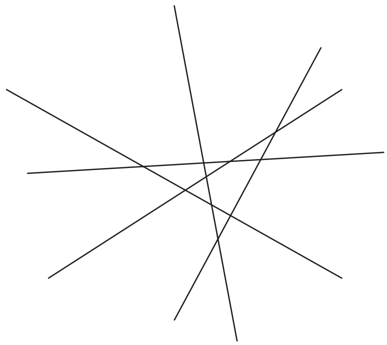

Fig. 1.1 Diesterweg 1819, §11

Algebraic expressions with symmetrical patterns can also be found at the Ratsgymnasium Bielefeld. "The sum of three numbers is 7, the sum of its squares is 45 and their product 14." This leads to the symmetrical equations:

$$a+b+c=7 \qquad a^2+b^2+c^2=45 \qquad a\cdot b\cdot c=14$$

(From the the final examination ("Abitur") in 1873).

The above-mentioned textbook of teacher Collmann (1853, p. 36) contains representations of sums of cubics:

$$1^3=1^2=1$$
$$1^3+2^3=3^2=1+3+5$$
$$1^3+2^3+3^3=6^2=1+3+5+7+9+11$$
$$1^3+2^3+3^3+4^3=10^2=1+3+5+7+9+11+13+15+17+19$$
$$1^3+2^3+3^3+\ldots n^3=(1+2+3+\ldots+n)^2=\left(\frac{(n+1)n}{2}\right)^2$$

Pattern recognition was also the fundamental idea in F.A.W. Diesterweg's *Geometrische Kombinationslehre* (1819). A task in this book is to determine the number of points of intersection of a given number of straight lines (Fig. 1.1).

If the pupil starts with one line (zero points of intersections) and adds stepwise line after line, he will soon observe the pattern that the number of points of intersection of $n+1$ lines is the sum of the points of intersection of n lines plus the number n. Of course, if he has some knowledge of combinatorics he will also realize that the number sought is equal to the number of possibilities of selecting two lines out of n lines that is equal to $\frac{n\cdot(n-1)}{2}$.

Collmann (1853, p. 65) contains, for example, statements about diagonals in an n-gon:

The diagonals from a vertex divide any n-gon in (n−2) triangles. Any n-gon has $\frac{n(n-3)}{2}$ *diagonals.*

1.5 Algebraic Analysis as a "Complete Paradigm"

From these insights into the structure and pedagogical ideas of nineteenth-century school mathematics, it is possible to come to a new evaluation of the question of why differential and integral calculus were not taught in nineteenth-century gymnasia. This question was first posed at the end of the nineteenth century when the reform movement under the influence of Felix Klein demanded the introduction of this subject matter into the school curriculum. Impressed by the heated debates, historiographers of the time answered this question by saying that the absence of infinitesimal calculus was a consequence of the dominance of the classical languages and the suppression of mathematics in neohumanist gymnasia. However, the above analysis leads to a different conclusion. If it is accepted that school mathematics as described above was a complete and closed whole, then the question of why differential and integral calculus were not taught proves largely ahistorical. Infinitesimal calculus was by nature not part of the curriculum, and mathematics teachers themselves defended this position. Of course, there were schools and teachers of mathematics pleading for elements of differential calculus in the upper grades of gymnasia. However, these were singular cases, and the majority of teachers of mathematics adhered to the conception of confining school teaching to analysis of the finite.

To elaborate this argument, it will be shown that algebraic analysis was a complete paradigm for school mathematics. "Complete" means that it was possible to solve problems, usually associated with infinitesimal calculus, for example, the determination of extrema, within algebraic analysis. As noted above, at the beginning of the nineteenth century these problems were counted among the topics of algebraic analysis. Methods adapted to the needs of school teaching were developed during that century. One of them was the so-called "method of Schellbach" (Schellbach 1860).

Karl Heinrich Schellbach (1804–1892) was a well-known mathematics teacher and teacher-trainer in Prussia who exerted a great influence on mathematics teaching. Felix Klein claimed that "Schellbach's method" was "disguised infinitesimal calculus" and that Schellbach did not speak openly of infinitesimal methods because he feared the classical philologists. For the latter assertion, Klein gave no justification, and with regard to the former it is possible to show that Schellbach's method can be interpreted as an integral part of algebraic analysis and that it was didactically well founded.

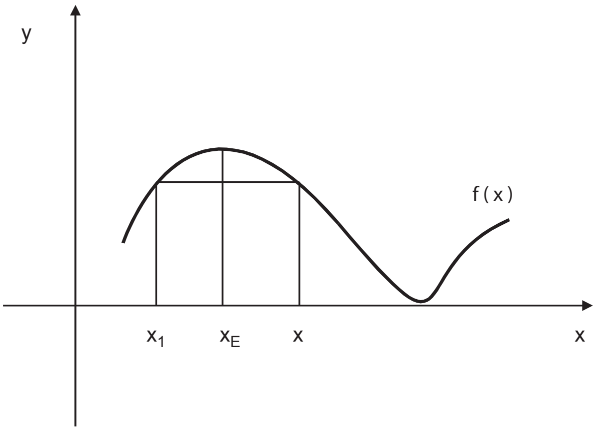

Fig. 1.2 Schellenbach's method

Let f be a given function with an extremal point at x_E (Fig. 1.2).

An arbitrary x is chosen sufficiently near to the extremal point x_E and the parallel to the x-axis is drawn through the point $(x, f(x))$. It intersects the graph of f at $(x_1, f(x_1))$, where x_1 is on the other side of x_E. Thus, to every x there corresponds an x_1. The extremal point is characterized by the condition that x corresponds to itself when the parallel is tangent to the graph of f. By this condition, we can calculate in the following way.

The conditions on x and x_1 imply

$$f(x) - f(x_1) = 0.$$

Schellbach claimed that it is "always possible" to factor $f(x) - f(x_1) = 0$ and to write

$$(x - x_1) \cdot g(x, x_1) = 0.$$

For all x with $x \neq x_1$ it follows that

$$g(x, x_1) = 0.$$

For reasons of continuity, this must also be the case for $x = x_1 = x_E$. Therefore, x_E can be calculated from the equation

$$g(x, x) = 0.$$

This was Schellbach's simple procedure. Felix Klein claimed that this was "disguised infinitesimal calculus" because factoring $f(x) - f(x_1)$ and setting $x = x_1$ may be seen as equivalent to evaluating the limit

$$\lim_{x \to x_1} \frac{f(x_1) - f(x)}{x_1 - x}$$

without mentioning the conceptual difficulties involved.

The whole procedure, however, can be interpreted equally well as a purely algebraic calculation within the framework of analytic geometry. The extremal point is algebraically characterized by the condition that the intersection of the parallel with the graph of the function *f* becomes a single point which must be counted twice. "Schellbach's method" simply imitated, on a more elementary level, Descartes' procedure for calculating tangents and normals in his "La Géométrie" of 1636. Moreover, methods of determining a tangent to a curve algebraically in the style of Euler's *Introductio in analysin infinitorum*, vol. 2, were a common topic in contemporary university textbooks of analytic geometry. Thus, Schellbach's algebraic method agreed completely with Euler's view on tangents and extrema in the *Introductio* (see Sect. 1.2).

The application of this procedure requires deciding whether an extremum exists and where it lies. This investigation uses the concrete conditions of the problem at hand. Clearly, such a concrete investigation is often pedagogically preferable to a blind application of an algorithm requiring mere mechanical calculations of derivatives. Schellbach's method can be used to solve a concrete extremum problem involving a concretely given (algebraic or analytic) function. Questions of existence and uniqueness are treated by recourse to the concrete conditions. For an example of Schellbach's method, the reader should consult Sect. 1.8.

Thus, algebraic analysis provided an effective method for calculating the extrema of concretely given algebraic and analytic functions. It is applicable to all functions that can be factored in the required manner. Infinitesimal calculus becomes necessary only when more general functions have to be treated and when one wishes to give general conditions for the existence and uniqueness of extrema. In addition, the pedagogical superiority of a method which forces the pupils to study a problem concretely and does not use an unnecessary conceptual apparatus is obvious. Schellbach had good mathematical and pedagogical reasons for his method.

Thus, it was only consequential that initiatives to reform mathematics teaching strove to reintroduce into the curiculum analytic geometry, the second part of algebraic analysis as treated in vol. 2 of Euler's *Introductio*. Schellbach's method for determining extrema fitted organically into this field. Apparently, most teachers of mathematics did not feel any need for infinitesimal analysis. In 1860, analytic geometry was introduced into the curriculum of the Realschulen (see above). Thus, mathematics teaching at these schools provided a complete, though of course quite reduced, image of Euler's *Introductio* in both volumes. In 1872, the prominent physiologist and rector of the university of Berlin Émil Du Bois-Reymond delivered a much-discussed speech in which he demanded a reintroduction of analytic geometry into the teaching of mathematics also at Prussian gymnasia (Du Bois Reymond 1877; see Krüger 1999, pp. 118 f.). Du Bois-Reymond justified his claim by a remarkable analysis of the outstanding importance of the mathematical concept of function in science, mathematics, and the general culture. Without using the term “functional thinking,” he spoke of a “graphical method” which comes close to Klein’s later ideas of a qualitative use of graphs of functions.

1.6 The Decline of Algebraic Analysis and the Meran Reform

In the second half of the nineteenth century, algebraic analysis and the Eulerian style of doing analysis got scientifically more and more outmoded. The field lost meaning and status at schools as well. Although for many teachers the binomial formula remained a “venerable center key” of the whole curriculum as Max Nath, a Prussian expert, remarked, the presence of the field in real teaching declined. Yet, in the Prussian syllabus of 1901 it still played a major role.

> At the Ratsgymnasium Bielefeld only a few tasks related to the binomial theorem and infinite series were given in the final examinations of Abitur. One example is from Easter 1837: Expand the fraction $\frac{1}{1-x}$ in an infinite series in x and derive a formula for the sums of all powers of 1/2, 1/3, 1/4 etc.

There were no real teaching environments related to the binomial theorem. Rather, it seems that algebraic analysis and the binomial theorem provided an important *leitmotiv* for the teachers which did not result in a corresponding culture of tasks and teaching activities. During the nineteenth century, the majority of mathematical tasks were in the domains of arithmetic, algebra, equations, calculations of interest, and geometry.

In the beginning of the 1890s, Felix Klein (1849–1925) started to commit himself to a reform of the teaching of mathematics. The key concept for Klein’s ideas became that of *functional thinking*. In its widest sense, the concept was to be understood as *flexible thinking*. Typical applications of the notion of function were the description of law-like relations as, for example, position and velocity of a falling body by a formula and their representation in graphs and tables. It was essential to Klein’s ideas that *functional thinking* had a meaning far beyond the mere application of the concept of function. Rather, *functional thinking* was to pervade all areas of school mathematics and all levels of teaching. In geometry, instead of contemplating static figures the basic account should be to systematically change configurations (see Krüger 1999 for a full analysis of this broader notion and its roots in nineteenth-century mathematics).

Since 1891, Klein and like-minded German professors offered courses of advanced training in Göttingen and Münster and at other places to teachers during their holidays in order to familiarize them with the new ideas. The courses in Göttingen (under Felix Klein) regularly had a mathematical focus. At the Gymnasium and Realgymnasium in Bielefeld, there was a *wide and continuous participation of teachers* in these courses, and they were supported by a municipal grant. Thus, we can conclude that among the mathematics teachers of the Ratsgymnasium Bielefeld there was a positive attitude toward Klein's ideas.

In 1905, at a symposium of the Association of German Natural and Medical Scientists in Meran a syllabus for the teaching of mathematics and science was proposed which came close to Klein's ideas, the Meran syllabus. The proposal was discussed in journals, by principals, teachers, and other interested persons.

One of the main points in this syllabus was the introduction of some elements of infinitesimal analysis into the higher grades of mathematics teaching at gymnasia. The reader can get an idea of the intended new subject matter by considering the textbook *Arithmetische Aufgaben* by Hugo Fenkner. This textbook was used at the Ratsgymnasium Bielefeld. The edition of 1913 contained the forming of the differential quotient and its interpretation as the slope of a tangent, rules and examples for calculating derivatives including the chain rule, representation of a function *f* and its derivative *f'* in *one* coordinate system, curvature and points of inflection, and L'Hopital's rule. The criteria for the existence of extrema were derived by means of the second derivative. Tasks for maxima and minima were considered as "sufficient material for training the application of the differential calculus." Thus, a lot of new theory was added to the existing subject matter. It was nevertheless notable that pupils of the time had high algebraic competencies, thanks to the Eulerian style of teaching.

1.7 Coexistence of Paradigms

All in all, at the beginning of the twentieth century there was among officials and teachers of mathematics some readiness to follow Klein's proposals. This was also the case at the Ratsgymnasium in Bielefeld. On the other hand, there was the old and well-established paradigm of algebraic analysis. Thus, the process of reform did not work in a way that the new (Klein's) conception replaced the old one (Euler's). Rather, there resulted a coexistence of the two paradigms.

This can nicely be seen by studying the above-considered textbook by Hugo Fenkner. It was published from 1890 onward in several editions and, as was said, used at the Ratsgymnasium Bielefeld. Common to all editions was the particular consideration of applications in geometry, physics, and chemistry. In 1892, the theory of minima and maxima had become a new subject at the Realgymnasium but on an elementary level by the means of algebraic analysis. This can be seen in the edition of 1907 of Fenkner's book, which followed the syllabus of 1901.

The determination of minima and maxima is considered in the book as a problem with many interesting applications. The book treated three different methods of determining extrema *without* the use of differential calculus—elementary, arithmetical, and graphical. These were I. The function value appears under a square root, II. Method of Schellbach, and III. Graphical method.

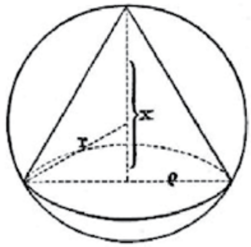

Fig. 1.3 Cone inscribed into a sphere of radius r (Fenkner 1907, p. 15)

I. The Function Value Appears Under a Square Root (Fenkner 1907, p. 11)

The method was applicable to quadratic functions, square root functions and to functions composed of these. It is based on the search for roots under the radicand.

II. Method of Schellbach (Fenkner 1907, p. 15)

The task is to inscribe a cone of maximal volume into a sphere of radius r (Fig. 1.3).

r be the radius of the sphere, x the height of the inscribed cone. Then the volume of the cone is

$$V(x) = \frac{1}{3}x(2r - x)\pi x$$

After setting $\mathrm{V}(x_1)=\mathrm{V}(x_2)$ *which implies* $\mathrm{V}(x_1)-\mathrm{V}(x_2)=0$ *the latter difference is factored into a product* $(x_1 - x_2)\cdot g(x_1, x_2) = 0$ *Then setting* $x_1=x_2=x_0$ *the equation* $g(x_0, x_0) = 0$ *has to be solved to determine the extremal point. The following steps demonstrate an example*:

$$\frac{1}{3}x_1(2r - x_1)\pi x_1 = \frac{1}{3}x_2(2r - x_2)\pi x_2$$
$$2rx_1^2 - x_1^3 = 2rx_2^2 - x_2^3$$
$$(x_1^3 - x_2^3) - 2r(x_1^2 - x_2^2) = 0$$
$$x_1^2 + 2x_1x_2 + x_2^2 - 2r(x_1 + x_2) = 0$$
$$x_1 = x_2 = x_0$$
$$3x_0^2 - 4rx_0 = 0$$

The positive root of the last equation $x_0 = \frac{4}{3}r$ *yields the solution.*

To find out whether x_0 *is a minimum or maximum one has to investigate the behaviour of the function* $V(x_0)$ *near to* x_0.

III. Graphical Method (Fenkner 1907, p. 16)

This method includes an approximate identification of extreme values by drawing a graph. The graph is constructed by calculating some coordinates.

In this way, "maxima and minima" were treated in the 1907 edition of Fenkner's book. The situation changed with the next edition. In general, in the years after the reform document of Meran (1905) had appeared many publishers of schoolbooks reacted by releasing new editions of their textbooks. They integrated some elements of the differential calculus into the volumes for the upper classes—though the official curriculum had not been changed. This practice was tolerated by the Prussian administration. Fenkner's *Arithmetische Aufgaben* was assimilated to the new developments in 1913.

With regard to the topic of "minima and maxima" Fenkner (1913) treated the *three methods* of determining extremal values which were already contained in the former edition without any substantial change. A method using differential calculus was only added as a *fourth method*. Thus, the chapter on "minima and maxima" comprised the subchapters I. "Graphical method," II. "The function value appears under a square root," III. "Method of Schellbach", and IV. "Determination of maxima and minima by means of the differential calculus." There was no unifying idea, no didactical development, and no preference for one of these methods. The teachers could and should decide by themselves what to do. The book reflected a diversity of opinions, not a conception.

However, there was *one* substantial change. The method of Schellbach was reinterpreted by means of the differential calculus. An important step in this method is the factorization, that is, the division by $(x_1 – x_2)$. In Schellbach's view, as also presented in the 1907 edition of Fenkner's book, this was a matter of algebra. Now in the 1913 edition the operation is justified by forming a difference quotient $\frac{y_1 - y_2}{x_1 - x_2}$ giving the slopes of secants. In this way, Schellbach's method was assimilated to the new paradigm.

It is instructive to consider a solution of a minimum problem by a *student* of the time. It is taken from the Abitur examination eastern 1914 at the Ratsgymnasium Bielefeld. The Abitur is a diploma of German secondary schools qualifying for university admission. The task for the students was:

> To inscribe a maximal cylinder with a straight cone on top of it into a hemisphere of radius r. Determine the radius and the height of the cylinder.

The solution below was given by Hermann Schütze, a top ranking student. In 1914, five out of eleven students failed the Abitur examination.

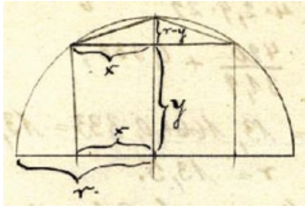

Fig. 1.4 Graph from the solution by Schütze 1914, Stadtarchiv Bielefeld Ratsgymnasium 1067.

First, the student formed an equation of the function which represented the volume of the inscribed solid dependent of the height of the cylinder—the target function.

If the radius of the cylinder is x and the height y, then the volume of the cylinder is given by $x^2\pi y$ (Fig. 1.4). Thus, the whole solid has volume

$$V(y) = \frac{\pi}{3}\left[r\left(2ry - y^2\right) - 2y^3 + r^3\right];$$

For this function the equation of its secant line through the points II and I is formed.

$$\frac{V_2 - V_1}{y_2 - y_1} = \frac{f(y_2) - f(y_1)}{y_2 - y_1}$$

Or

$$\frac{V_2 - V_1}{y_2 - y_1} = \frac{\pi}{3}\left[\frac{r(2ry_2 - y_2^2 - 2ry_1 + y_1^2) + r^3 - r^3 - 2y_2^3 + 2y_1^3}{y_2 - y_1}\right]$$
$$\frac{V_2 - V_1}{y_2 - y_1} = \frac{\pi}{3}\left[\frac{r(2r(y_2 - y_1) - (y_2^2 - y_1^2) - 2(y_2^3 - y_1^3)}{y_2 - y_1}\right];$$
$$\frac{V_2 - V_1}{y_2 - y_1} = \frac{\pi}{3}\left[r(2r - (y_2 + y_1) - 2(y_2^2 + y_2 y_1 + y_1^2)\right]$$

This is the equation of the secant; if point II approaches point I, then the secant changes to the tangent. Its slope is:

$$\left[\frac{V_2 - V_1}{y_2 - y_1}\right]_{y_2 = y_1} = \frac{\pi}{3}\left[2r^2 - 2ry - 6y^2\right]$$

If the slope is 0, there will be a maximum or minimum or a point of inflection. Thus:

$$\frac{\pi}{3}\left(2r^2 - 2ry - 6y^2\right) = 0$$

and

$$y^2 + y \cdot \frac{r}{3} = \frac{r^2}{3};$$

which implies

$$y = \frac{r}{6}(\sqrt{13} - 1)\cdot$$

Because of $x^2 = r^2 - y^2 x$ can be calculated from y. From this the student finally got the solution

$$x = \frac{r}{6}\sqrt{22 + 2\sqrt{13}}\,;\quad y = \frac{r}{6}(\sqrt{13} - 1).$$

The cylinder with height y and radius x has maximal volume and solves the problem.

The student did not prove that the extremum was really a maximum. Following the steps of the solution with regard to the algebraic calculations, the elements of the method of Schellbach can be found. $V_1 - V_2$ is formally divided by $y_1 - y_2$ and the equation is simplified by setting $y_1 = y_2$ and not by determining a limit. Nevertheless, the term in squared brackets is interpreted as the slope of a straight line. The student spoke of secants approaching the tangent, whereas in his calculations he simply set $y_1 = y_2$. Obviously, the student used the method of Schellbach in the new interpretation by Fenkner 1913. We observe that the student demonstrated high competencies of algebraic calculations and made many explaining comments. This shows that he had adequate ideas and was not only applying schematic recipes.

In principle, teachers at the Realgymnasium Bielefeld were open minded toward the introduction of the differential calculus. For example, in the Abitur examination of 1911, the following task was given to the students:

> Investigate the equation of the curve $y = x^3 - 6x^2 + 11x - 6$, and determine maxima, minima and points of inflection. Explore the first and second derivates and draw them.

Nevertheless, it was a long process until the differential calculus became an official subject in the teaching of mathematics at gymnasia. The proposals of Meran were made official in the "Richert" syllabus in 1925, which was the year of Felix Klein's death. In this syllabus, the concept of function and an introduction into differential calculus became the main topic in the teaching of mathematics. At the Gymnasium and Realgymnasium of Bielefeld, the teachers adapted themselves to the innovation with new textbooks and new methods of teaching (Arbeitsunterricht). However, it took a long time for the change to take place and to replace the old paradigm with the new ideas.

1.8 Conclusions

Coming back to the concept of transformation, it can be seen from this study that a process of curricular transformation, like the one which F. Klein inaugurated at the end of the nineteenth century, is much more complicated than its proponents might have anticipated when they started their campaign. In Klein's case, it was a fact that his proposals met an existing tradition of teaching which was coherent and plausible for many mathematics teachers. Thus, the whole reform can be understood as an interaction of two different mathematical paradigms: the elder tradition of algebraic analysis versus the reorganization of teaching due to the needs of the newly introduced infinitesimal calculus. A concrete analysis of the Klein reform movement, which takes into account the interaction of these two paradigms, is still missing.

Klein's concept of a double discontinuity between school mathematics and university mathematics proved to be a much deeper problem than he could anticipate in his times. Klein might have thought that the problem could be solved by a modernization of the school curriculum and its adaptation to the most recent developments in mathematics. Considering the three chapters in this volume referring to this problem (Deiser and Reiss, Chap. 3; Pepin, Chap. 4; Buchholtz and Kaiser, Chap. 5), it is clear that broader and more general issues are behind the double discontinuity which point far beyond a mere curricular renovation.

With regard to the very concept of transformation, it can be seen from the present study that processes of transformations involve, with a certain necessity, partially incompatible components (paradigms in our case) and unintended side effects. As a consequence, projects of reform are involved and escape complete technocratic control.

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Chapter 2
Framework for Examining the Transformation of Mathematics and Mathematics Learning in the Transition from School to University

An Analysis of German Textbooks from Upper Secondary School and the First Semester

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2.1 Introduction

Many first-year students experience the transition from school to university as a challenging enterprise. This is especially true for mathematics programs. Comparatively high dropout rates of freshman students after the first or second semester indicate that this transition is the main obstacle for students to finish their studies in mathematics. For example, in Germany, universities and mathematics departments are faced with dropout rates of up to 50 % of first-year students in mathematics. According to surveys, students report that this is mainly caused by the enormous pressure to perform and a lack of motivation (Heublein et al. 2009). However, most of the surveys do not use instruments detecting the specific situation for the subject mathematics. We assume that the high dropout rate during the transition from school to university is rooted in the necessity of coping with two discontinuities: the discontinuity of the learning subject and the discontinuity of the way of learning. Accordingly, managing the transition from school to university successfully means individually developing two ways of transformation to overcome these discontinuities. First, a transformation from school mathematics to academic mathematics, so that academic mathematics can be recognized as an extension of school mathematics and the individual mathematical knowledge learned in school can serve as a basis for further learning (Deiser and Reiss 2013, Chap. 3). Second, a

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transformation from school learning to academic learning is required, so that learning strategies acquired implicitly in school can be adapted for academic learning processes (Pepin 2013, Chap. 4).

In this contribution, we describe an approach to examine and describe the two discontinuities and the related requirements for the corresponding transformations through textbook comparisons. Mathematics textbooks play a decisive role for students' learning processes at school (Rezat 2009) as well as at university (Alsina 2001). Accordingly, we first assume that the way mathematics is presented in the textbooks can be considered as an indicator of the character of school mathematics and academic mathematics. Second we assume that the didactical structure of the textbooks can be considered as an indicator of the requirements of students' learning strategies. Based on these assumptions, we developed a theory-driven framework to compare textbooks using certain criteria related to the character of mathematics and the requirements for students' learning strategies. Using this framework for textbook comparison, we expect empirical results that help to specify which aspects of mathematics and mathematics learning are constitutive elements of the discontinuities in the transition from school to university. Empirical findings of two feasibility studies using two school textbooks, two university-level textbooks, as well as lecture notes handed out by a mathematics professor indicate that the application of this framework yields reliable results.

2.2 Theoretical Background

During the last decade, the previously mentioned discontinuities and the related challenges for first-year mathematics students were investigated from different perspectives. For example, the investigation took the transformation of mathematical contents, knowledge, learning strategies from school to university into account, as well as the students' motivation and self-regulation (e.g., Deiser and Reiss 2013, Chap. 3; Pepin 2013, Chap. 4; Kaiser and Buchholtz 2013, Chap. 5; De Guzmán et al. 1998; Hoyles et al. 2001; Rach and Heinze 2011). In the following, we discuss theoretical observations and empirical results from studies focusing on the transformation of mathematics from school to academic level as well as the transformation of the corresponding learning processes at school and at university. Moreover, we present some results on textbook research, as our framework for examining transformation processes in the transition phase from school to university is based on a textbook approach.

2.2.1 The Character of Mathematics at School and at University

Mathematics as it is taught in high school is not just academic mathematics in a simplified form; mathematics as "school mathematics" has its own character (e.g., Biermann and Jahnke 2013, Chap. 1; Hoyles et al. 2001; Heinze and Reiss 2007).

Mathematics as a school subject must contribute to the aim of general education. This means, in particular, that the character of school mathematics must enable students to learn mathematics in such a way that they can use their mathematical knowledge for solving everyday problems and as a sound basis for their vocational education (Heymann 2003). Accordingly, mathematical content, which is relevant for the application of mathematics as tool (e.g., percentages and algebraic manipulations) but which is hardly interesting from a scientific mathematical perspective, is comparatively strongly emphasized (e.g., Dörfler and McLone 1986). Mathematics at university has a different character, because it is considered a scientific discipline. Here, the mathematical content is organized and presented in an axiomatic and rigorous manner. In the first semesters, applications of mathematics for solving real-world problems hardly play any role.

Mathematics as a tool and mathematics as a scientific discipline can be considered as two sides of the same domain: "It is a tool in the study of the sciences, and it is an object of study in its own right" (Hoyles et al. 2001, p. 841). The fact that these two sides of mathematics are reflected at school and at university in quite a different way has serious consequences for the role of important characteristics of mathematics like proving, rigor, or formalism. For example, most of the mathematical concepts in school are introduced and used informally (Engelbrecht 2010). Accordingly, students mainly work with a "concept image" of a concept (in the sense of Tall and Vinner 1981), whereas the "concept definition" of most of the concepts does not play a prominent role. In mathematic courses at university, concepts are mostly introduced by a formal definition, i.e., as concept definition (Deiser and Reiss 2013, Chap. 3). This is necessary to meet the standards of rigor. A similar situation can be observed for the role of mathematical proofs. If mathematics is considered as a scientific theory, then there is the need for scientific evidence of statements and for explanations why these statements are true (e.g., Hanna and Jahnke 1996). In contrast, if mathematics is considered as a tool, proofs play a minor role. In this case, proofs are often omitted because it is enough to know that a statement is true (e.g., that the tool works well).

2.2.2 Learning Mathematics at School and at University

Though learners at both school and university learn mathematics, there are remarkable differences in students' learning activities. These differences constitute a discontinuity in the transition phase from school to university which, in consequence, requires a transformation of individual learning strategies. The two most important differences between school and university in this respect are the formal organization of learning opportunities and the individual learning strategies necessary for an effective use of these learning opportunities (Pepin 2013, Chap.4).

In most German universities, teaching and learning mathematics for first-year students is structured in three complementary activities. Each week, there are one or two 90-min lectures given by a mathematics professor, a self-study phase where 3–5 challenging tasks (mainly proof tasks) are solved as obligatory homework, and a 90-minute tutorial where a senior mathematics student discusses the solutions of the homework with a group of 20–30 students. The self-study phase is organized by the students in their private time. Mostly, students work in small groups (2–6 students) on their homework where they are individually and cooperatively involved in problem-solving activities. Moreover, they are supposed to recapitulate their lecture notes and use additional literature. In summary, these learning opportunities are quite different from the learning opportunities in school. In school, German students attend 3–5 mathematics lessons per week (each 45 min) which are prepared and structured by teachers with respect to the cognitive and affective learning prerequisites of the students. Mathematics instruction encompasses phases of teacher talk and student's private work (single, partner, or group work) and class work (in a questioning–answering format). Homework mainly serves as a supplement to the content of the previous lesson and offers opportunities for practicing. In most of the phases, students receive precise instructions about what they should do and what they should achieve. Therefore, it is a kind of guided learning with specific learning tasks which aims at the acquisition of different aspects of competencies (conceptual knowledge, procedural knowledge, etc.; Kaiser 1999; Kawanaka et al. 1999).

The differences between school and university in the formal organization of learning opportunities and in the character of mathematics imply different requirements for students' learning strategies. Because mathematic lectures are both rigorous and formal in university lectures, university students need to apply specific elaboration strategies to understand the mathematical content. New mathematical concepts cannot be grasped through formal concept definitions, so it is necessary that students connect the presented concept definition to an already existing concept image from an intuitive use of this concept in school or that they individually develop a new concept image (cf. Engelbrecht 2010). In addition to learning concepts, students have to acquire problem-solving competencies so that they are able to solve the weekly challenging proof problems as homework. As mathematics is frequently taught using completed theories or as elegant solutions in lectures and tutorials, problem-solving strategies are mainly dealt with implicitly and students are not offered direct accessible models for the trial-and-error process of creating new knowledge (e.g., Dreyfus 1991). Accordingly, they have to elaborate on the proofs and to reflect on proving processes. The use of self-explanations can be considered as an effective learning strategy in this respect (e.g., Chi et al. 1989; Reiss et al. 2006).

In summary, at university, mathematics as a scientific product is presented to the students who, in turn, have to find and apply learning strategies on their own to make this product accessible for their individual learning processes. At school, however, mathematics is presented in the framework of a didactical structure. The teachers prepare mathematics in such a way that it is accessible to the students. Students' learning is guided by sequences of specific chosen tasks which implicitly induces the application of adequate learning strategies (Pepin 2013, Chap. 4).

2.2.3 *Approaching the Discontinuities by Textbook Comparisons*

There are different possibilities to approach the two discontinuities between school and university. For example, taking the student perspective, you can ask students about their perception of mathematics and mathematics learning at school and at university or you can compare mathematical competencies between freshmen before they commence their studies and again after one semester. Similarly, you can take the teacher's perspective and ask teachers at school and university about their view on mathematics and mathematics learning. A third possibility is to take the observer perspective and to observe and analyze mathematics and mathematics learning in both institutions.

In this contribution, we choose the third possibility by taking the observer perspective with a specific focus. We analyzed school and university textbooks and compared our findings to the previously mentioned discontinuities. We are well aware that (1) textbooks obviously only represent a small section of the learning opportunities students are offered at schools and universities. Furthermore, we are also aware that (2) the impact of textbooks strongly depends on the individual use of textbooks which again is influenced by cultural traditions (e.g., Pepin and Haggarty 2001). Nevertheless, research on mathematics textbooks shows that textbooks have a close connection to the curriculum and, therefore, they reflect the differences between the mathematics curriculum at school and at university. Geoffrey Howson (1995) describes textbooks as a mediator between intended and implemented curriculum. They are designed as a means to transfer the intended contents to the lesson or function as a device for self-study phases. However, a textbook cannot be identified with either the intended or the implemented curriculum as publishers as well as teachers choose which contents to include in the book or to impart in the lesson, respectively (cf. Howson 1995). Hence, Schmidt et al. (2001) introduced the notion of a "potentially implemented curriculum" which is represented by a textbook.

There is already sufficient research on textbooks: Some studies take a comparative cultural perspective by investigating the structure and use of textbooks in different countries (Howson 1995; Pepin and Haggarty 2001; Valverde 2002). Other studies look at textbooks from a sociocultural perspective when investigating their structure (Rezat 2006), the students' use of textbooks (Rezat 2009), the difficulty of tasks (Brändström 2005), or the role of textbooks for the establishment of misconceptions (Kajander and Lovric 2009). Most of these studies deal with school textbooks, whereas textbooks at university level are not as well researched.

Regarding our investigation of the two discontinuities during the transition from school to universities, we assume that both the discontinuity in the character of mathematics and the discontinuity in mathematics learning are reflected in the textbooks. According to Pepin and Haggarty (2001), textbooks show mathematical intentions that can be divided into three areas: "What mathematics is represented in textbooks; beliefs about the nature of mathematics that are implicit in textbooks; and the presentation of mathematical knowledge" (Pepin and Haggarty 2001,

p. 160). Therefore, we expect that differences in the character of school mathematics and academic mathematics can be found in mathematics textbooks. In addition, we expect that differences in mathematics learning at school and at university are also reflected in the textbooks. A mathematics textbook in school is a focal point for the interaction between the teacher and mathematics, between the student and the teacher, as well as between the student and mathematics. Rezat (2009, p. 66) therefore suggests enhancing the didactic triangle into a didactic tetrahedron incorporating the textbook as fourth element.

In summary, mathematics textbooks play an important role for students' learning processes at school as well as at university. As textbooks represent potentially implemented curricula, we assume that the way mathematics is presented in the textbooks represents the character of school mathematics and academic mathematics. Moreover, the didactical structure of the textbooks indicates the requirements for students learning strategies because textbooks influence the interaction between teachers, learners, and mathematics.

2.3 Research Objectives

Our research aims at examining and describing the two discontinuities in the transition from school to university regarding the character of mathematics and the way of learning mathematics. To achieve this goal, we use different approaches. In the following, we will present an approach that is based on a comparison of textbooks at school and textbooks at university. Hence, the specific goals of this contribution are as follows:

1. The elaboration of a theory-based framework for analyzing and comparing mathematics textbooks at the upper secondary level and the first semester at universities.
2. The presentation of results of feasibility studies to show that this framework allows a reliable data collection for textbook comparisons.

The feasibility studies were conducted with a small number of textbooks. This means particularly that we cannot yet report clear results concerning differences between textbooks at school and at university. Nevertheless, there are some tendencies that we will address in the discussion section.

2.4 A Framework for Textbook Comparison

For the analysis of mathematics textbooks at school and university levels, we apply a framework that is derived from a psychological and a didactical perspective. It consists of six criteria that can be divided into general and content-specific ones (see Fig. 2.1). General criteria are not bound to mathematical contents but could be

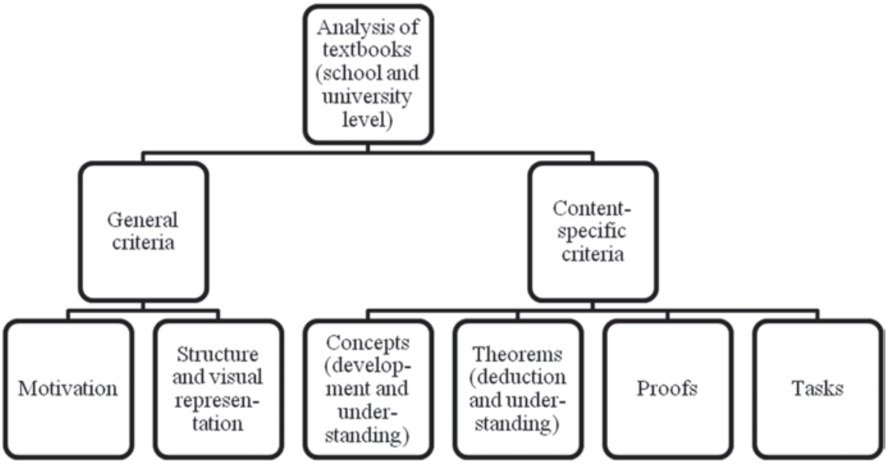

Fig. 2.1 Framework for the analysis of textbooks used at school and university levels

applied for the analysis of textbooks from any other subject. We restrict ourselves to motivation and the structure and visual representation of the contents. Content-specific criteria investigate aspects that are particular for mathematics such as the development and understanding of concepts and the deduction and understanding of theorems, proofs, and tasks.

In the following, we elaborate on these six criteria. In Sect. 2.5, we show examples of the operationalization for some of the criteria that were used as rating schemes for data collection.

2.4.1 General Criteria

The general criteria considered in our framework relate to self-determination theory of motivation and the structure and visual representation.

2.4.1.1 Self-Determination Theory of Motivation

According to self-determination theory (SDT; Deci and Ryan 1985; Ryan and Deci 2002), motivated actions can be distinguished by their degree of self-determination and regulation. Actions can be amotivated, extrinsically motivated, or intrinsically motivated. Forming one end of a self-determination continuum, amotivation is characterized by non-regulation. The other end is marked by intrinsic motivation that is assigned by intrinsic regulation. The different degrees of extrinsic motivation lying in between distinguish four different types of regulations: external, introjected, identified, and integrated regulations (see Ryan and Deci 2002; see also

Pepin 2013, Chap. 4 for the role of self-regulated learning at the transition from school to university).

SDT postulates three basic psychological needs to explain the relation of motivation and goals to health and well-being. The need for competence refers to "feeling effective in one's ongoing interactions with the social environment and experiencing opportunities to exercise and express one's capacities" (Ryan and Deci 2002, p. 7) and the need for social relatedness corresponds to "feeling connected to others, to caring for and being cared for by those others, to having a sense of belongingness both with other individuals and with one's community" (Ryan and Deci 2002, p. 7), while the need for autonomy focuses on "being the perceived origin or source of one's own behavior" (Ryan and Deci 2002, p. 8). These needs are assumed to be innate, culturally universal, and equally relevant for extrinsic and intrinsic motivation (Ryan and Deci 2002).

Different studies have analyzed mathematics lessons with respect to the implementation of the three basic needs (Rakoczy 2008; Kunter 2005; Daniels 2008). In these studies, the following aspects turned out to be important for motivated learning.

Implementation of Perceived Autonomy On the one hand, students should have the possibility to make deliberate choices in their learning process so as to give room for their own demands. In the context of textbooks, they should be able to have the choice as to which explanations, examples, and tasks they want to deal with to organize their own learning process. One way to offer this possibility is to provide different ways of introducing new contents and offering different examples and tasks. It is important that these different ways of approaching a certain concept do not offer different degrees of complexity or contents. On the contrary, they follow the same aim by offering different approaches to the same learning goals and demands disguised in different representations and methods.

On the other hand, the topics dealt with in the lessons should be personally relevant to the students (see also Vollstedt 2011) in order to help them realize the value of their actions. They experience them as leading to their goals concerning their own values. Hence, they have a higher feeling of autonomy which is related to their motivation to learn (see Rakoczy 2008, p. 41). Textbooks therefore implement autonomy when they allow contexts which relate to the students' lives and which are personally relevant to the students. Through this, mathematics can become more important for the students.

Implementation of Perceived Competence This aspect complements the suggestion to give room for own decisions mentioned above. The students perceive themselves as competent when they can come to the right conclusions. Therefore, different levels of difficulty are needed for the tasks and introductory parts of the sections (Rakoczy 2008). Depending on their own level of achievement, students can then choose which task to deal with subsequently. To enable this choice, the task or introduction should be marked, for instance, according to its degree of difficulty.

The second way to foster the students' perception of competence is to give them guidance through the book by following a certain structure. The fragmentation of major contents into subchapters as well as a general guidance through each chapter offer the students a sense of security (Rakoczy 2008; Kunter 2005). Another element of structure is the occurrence of advance organizers at the beginning or the end of the text (Ausubel 1960). Without this structure they might have the feeling of getting lost and not being able to cope with the demands made on them.

Implementation of Perceived Social Relatedness Most of the aspects of the experience of social relatedness in mathematics lessons refer to the relation between the students and the teacher or between the students themselves. These are difficult to transfer to textbooks. It can, however, be evaluated to what extent books stimulate or cultivate cooperative learning so that the students' need for social relatedness is met by the conceptual design of the book.

2.4.1.2 Structure and Visual Representation

This criterion is divided into the following subsections: comprehension of the text, formalism, and visual representation. The variables for the first one result from studies carried out by Langer et al. (1973, 1974, 2006). They were complemented by aspects of formalism (Kettler 1998) as this is one of the characteristic elements of mathematical texts. The last element concerns the role and quality of graphical representations, which is based on the work of Mayer et al. (Mayer and Gallini 1990; Mayer and Moreno 1998; Mayer and Johnson 2008).

Comprehension of the Text and Formalism According to Langer et al., four elements are important to understand texts: simplicity, coherence/organization, conciseness, and motivational additives. Simplicity refers to the diction and the syntax of the text. No matter what level of difficulty characterizes the content, familiar words are combined to short sentences with easy structure and difficult words (foreign words or technical terms) are explained. Coherence describes the inner logical structure of the text in which sentences combine to form a stringent idea, whereas organization refers to the outer structure of the text (sections related to each other are in close distance, sections are divided by headlines, and important aspects are highlighted). The level of conciseness relates to the length of the text in comparison to its informational content, i.e., whether the phrasing is scant or wordy. Motivational additives then embrace elements which the author uses to raise the reader's interest. The complementation of the element formalism adjusts the theory to mathematical texts insofar as it judges the frequency of the occurrence of mathematical elements. According to Kettler (1998), the amount of mathematical symbols can have an impact on the reaction of the reader as the readers' sympathy decreases when the degree of symbolism increases.

Visual Representation Concerning the characteristic visual representation, two major types are distinguished: the role and quality of graphical representations. Mayer and Gallini (1990) differentiate five roles of illustrations:

1. Decoration: The graphical representation has no direct relation to the text but serves as motivational element.

2. Representation: The contents are represented in another way without adding information (e.g., diagrams).
3. Transformation: The graphical representation serves to ease the memory of easily understandable information; additional information may be included.
4. Organization: The graphical representation is supposed to structure the text and to organize its elements.
5. Interpretation: Graphical representations are to help the reader to understand difficult relationships.

The quality of a graphical representation can be judged by the occurrence of the split-attention effect (Mayer and Moreno 1998) and the redundancy effect (Mayer and Johnson 2008). Figures are more difficult to understand when it is necessary to split the reader's attention between more than one source which can only be understood in relation with each other. The effect can be minimized when the sources can be integrated into one main source, for instance, by incorporating the values of angles directly into the figure instead of placing them next to it. The redundancy effect occurs when the same information is given in the text as well as in the figure. The different kinds of representation do not show any relations or help in another way toward a better understanding. The effect can be minimized when only keywords are integrated in the representations, whereas it can be maximized by giving the whole text again in the figure (Sweller 2005). Mayer et al. were able to show that graphical representations can enhance understanding and remembrance of information, whereas improper use hinders them.

2.4.2 Content-Specific Criteria

In contrast to general criteria, which can also be applied to other subjects, content-specific criteria investigate aspects which are specific for mathematics. The following sections give more details about the development and understanding of concepts and theorems, the role of proofs, and tasks.

2.4.2.1 Development and Understanding of Concepts

The way how mathematical concepts are developed influences their fundamental understanding (Vollrath 1984): Can students give a definition of the concept and can they decide whether an example fits the category of the respective concept? Can students give examples or counterexamples and do they know characteristic properties of the concept? Can the concept be applied when solving problems and can the students integrate the concept into a network of subconcepts and generic terms? All but the very first aspect are necessary to develop a deep understanding of concepts.

The background theory we apply to the development and understanding of concepts relates to instructional psychology as well as to the theory of mental models (*Grundvorstellungen*, vom Hofe 1995). Klauer and Leutner (2007) name different possible functions of teaching which are necessary to reach a teaching goal. In our model, we primarily focus on three of them which are important for the development of concepts: First, the *transformation of information* characterized with regard to the way the concept is introduced as well as the precision and formalism of this introduction. The second function concerns the *processing of information* and focuses on the possibilities of understanding, recalling, expanding, and reviewing the concept as well as on the way how concepts are distinguished from others with the help of examples and counterexamples. The third under consideration is *transfer*. This entails looking at the number of adequate mental models and the number of equivalent definitions given in the book.

2.4.2.2 Development and Understanding of Theorems

The development and understanding of theorems is partly analogous to the development and understanding of concepts as well as proofs (see below). The first considered aspect is the way the theorem is introduced with the help of an example or a problem which motivates the theorem. The second aspect then deals with the mathematical development of the theorem. The formulation of the theorem then takes into consideration the precision and formalism of the formulations used to state the theorem. Finally, the last aspect concerns the differentiation of the respective theorem from others with the help of illustrating examples and/or counterexamples for the application of the theorem.

2.4.2.3 Presentation of the Proving Process and Proofs

Proving something is an essential mathematical activity (e.g., Heinze and Reiss 2007). To prove that a mathematical theorem is true, it is crucial to detect connections between mathematical structures and to show that the correctness of these connections can be universally argued. By doing so, learners have the possibility of experiencing mathematics as a process and not as a set science. Boero (1999) distinguishes six phases of a proving process:

1. Production of a conjecture;
2. Formulation of the statement according to shared textual conventions;
3. Exploration of the content (and limits of validity) of the conjecture;
4. Selection and enchaining of coherent, theoretical arguments into a deductive chain;
5. Organisation of the enchained arguments into a proof that is acceptable according to current mathematical standards; and
6. Approaching a formal proof.

From this theoretical basis, we distinguish between elements of the proving process (the role of advanced organizers and the generation of a proof idea) and the formulation of proofs, i.e., preciseness and formalism of the proofs as well as the number of different methods which were presented.

2.4.2.4 Tasks

Tasks are a central element in mathematics textbooks and fundamental for the students' learning process (Rezat 2009). The characteristics taken in our framework to judge the tasks in school and university textbooks are based on the educational standards passed by The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (KMK 2004)[1]. They distinguish between five key content areas, six general mathematical competences (cognitive processes), and three levels of demand. As our study compares textbook sections with similar content only, the different key contents of educational standards can be neglected. The tasks from the different textbooks are therefore analyzed concerning their main mathematical competencies and levels of demand only. Each task has to be judged with respect to the competence:

1. Argue mathematically;
2. Solve problems mathematically;
3. Model mathematically;
4. Use mathematical representations;
5. Deal with symbolic, formal, and technical elements of mathematics; and
6. Communicate.

Moreover, each task has to be judged concerning its level of demand, i.e., whether it is necessary to reproduce, to make connections, or to generalize and reflect.

In addition to the task analysis based on the educational standards, the numbers of different solutions and solution approaches are evaluated. Finally, the tasks are analyzed with respect to their relation to mental models, i.e., whether new mental models are developed or whether known mental models are used.

2.4.3 Summary

In the previous subsections, we present general and content-specific criteria for a textbook analysis. All criteria are based on psychological or didactical theories or models. Their significance for mathematics learning is based on evidence from empirical studies (e.g., in case of learning activities) or on theoretical analyses (in the case of the learning content). Accordingly, we assume that these criteria cover important aspects for a comparison of school mathematics with academic mathematics and for a comparison of the requirements of individual mathematics learn-

[1] The underlying competence model coincides in many respects with the competence model of the PISA 2012 study (see OECD 2010).

ing at school and at university. By identifying differences and commonalities, these aspects help to track down the transformation of mathematical contents and the learning of mathematics at the transition between school and university.

2.5 Feasibility Studies

The model presented above was developed in the context of two feasibility studies. The first one was used to check the validity of the model. Both a school and a university book were rated by two field experts. The focus was on the consistency of the two experts' judgment. Based on the results from this first study, the model was refined and then applied in a second study. The aim of this second study was again to test the model for validity as well as to detect differences concerning the methodological and didactical organization of the textbooks. These results form the basis for statements with respect to the transformation of contents or learning strategies at the transition from school to university level.

The studies reported on in this article are part of an ongoing bigger study that compares textbooks from school and university in different countries. In this article, we restrict ourselves to the first two feasibility studies comparing textbooks at school and university levels which are very frequently used in Germany. For the first study, one book from each level was taken: *Lambacher Schweizer Gesamtband Oberstufe* (Brandt and Reinelt 2009) is one of the most commonly used textbooks at school level. Its section about vector spaces was compared to the respective section in the 'Beutelspacher', a very popular linear algebra textbook at university level using a very explanatory approach (Beutelspacher 2010). For *Lambacher Schweizer Gesamtband Oberstufe*, the experts reach a consensus on 16 out of 32 criteria. For Beutelspacher's textbook, this was the case for 24 out of 34 criteria. The differing number of criteria results from the fact that not all criteria could be applied to both books: *Lambacher Schweizer Gesamtband Oberstufe* does not contain proofs and Beutelspacher's book does not contain pictures. Although the consistency of the rating is higher than the anticipated value, it is obvious that the results could be improved.

A closer look at the results shows that, due to a misunderstanding of the coding scheme, some subitems from proof were accessed for the *Lambacher Schweizer Gesamtband Oberstufe* although this textbook does not contain any proofs. Similarly, the criterion of vividness (one item in structure and visual representation/motivational additives dealing with the way contents are presented) was not judged. Our hypothesis is that it did not become totally clear to what extent the items belong to their main categories. Descriptions were therefore refined to make this clearer.

After refining the framework, the second study was conducted in the field of calculus using the standard school textbooks *Lambacher Schweizer* (Drüke-Noe et al. 2008) and *Elemente der Mathematik* (Griesel and Postel 2001; Griesel et al. 2007, 2008) together with the university-level textbooks *Königsberger* (Königsberger 2004) and *Forster* (Forster 2008), which from experience are often used by undergraduate students. In addition, the lecture notes handed out by one of the professors from the mathematics department at our university were coded. In all cases, the sections dealing with real numbers, continuity, and differentiability were rated.

Attention has to be drawn to some specifics concerning the textbooks used. Forster's book does not contain a chapter dealing with real numbers in detail. They are only treated in a very dense compression on one page at the end of the book. Also, there are no solutions for the tasks posed to students. The lecture notes do not contain any tasks or solutions. Therefore, these sections were not rated for these textbooks in our study. Then, the school textbooks only briefly deals with continuity so that the explanatory power of the comparison in this realm is lowered. We added propositions about typical characteristics of $\mathbb{R}$ like uncountability and the embeddedness of $\mathbb{Q}$ to the topic of real numbers where possible. For *continuity*, we looked at the intermediate value theorem, and for *differentiability*, we observed derivation rules, the calculation of turning points, inflection points, as well as convexity and monotony.

Two master mathematics students were responsible for the rating. The categories were quantified with respect to whether the criterion is a conceptual element of the book, i.e., that it occurs in every chapter considered or whether the criterion just occurs sporadically, i.e., there is at least one chapter in which it does not occur. Criteria were considered as consistent if both raters agreed totally with each other in their judgment. Only those characteristics were interpreted that were rated by both raters.

2.6 Exemplary Results of the Second Feasibility Study

Section 2.4 gave a general introduction to the framework used in the studies described in Sect. 2.5. In the following subsections, exemplary operationalizations are given to illustrate how we transferred the model into ratable characteristic features. Cohen's kappa is reported to indicate the strength of the interrater agreement as a reliability measure. According to the Landis and Koch (1977, p. 165) interpretation scale, the strength of agreement is fair if $0.2<\kappa<0.4$, moderate if $0.4<\kappa<0.6$, substantial if $0.6<\kappa<0.8$, and almost perfect if $0.8<\kappa<1$. In general, a reliability of $\kappa>0.6$ is considered as an acceptable agreement, so that the value of the corresponding criterion can be interpreted.

2.6.1 Motivation

One of the aspects of motivation according to SDT (see above) is the experience of social relatedness. When analyzing textbooks, you therefore have to judge to what extent the book supports group work. The following feature characteristics were developed:

1. The book explicitly invites the students to work on the tasks in groups. This method is part of the book's conceptual design.
2. Students are sporadically invited by the book to work in groups.
3. There are no tasks which are supposed to be worked on in groups.

Interrater agreement on social relatedness in the second feasibility study was substantial ($\kappa = 0.632$).

2.6.2 Structure and Visual Representation

The structure of the textbooks comprises motivational additives. One aspect considered in this realm is vividness:

1. It is part of the book's conceptual design that descriptions are padded with anecdotes and that stories are used to convey facts.
2. Some information is always presented in the same dreary and unvaried way.
3. The text deals with the contents in a very prosaic way, i.e., facts are conveyed by using factual language. There is no supplementary information in terms of anecdotes or stories.

Interrater agreement on vividness was substantial ($\kappa = 0.650$).

The role of graphical representations is operationalized as follows:

1. The graphical representation contains more information than can be found in the text. These mostly comprise tasks that are introduced by a text in which information has to be taken from the corresponding graphical representation.
2. The graphical representation contains the same information as the text but eventually offers another way of access. Graphical representations that present the text in a modified display format belong to this group. They can, for instance, be restructured to be learned or understood more easily.
3. The graphical representation does not have any information content. Pictures with motivating character belong to this group.

Interrater agreement on the role of graphical representation is only moderate ($\kappa = 0.451$).

2.6.3 Development and Understanding of Concepts

To develop a sound concept definition and concept image of a mathematical concept, it has to be linked to inner-mathematical as well as extra-mathematical, i.e., applied contexts and examples. The following characterizations were developed to operationalize the introduction of a new concept:

1. It is part of the book's conceptual design that a new concept is introduced by using an applied or inner-mathematical example or problem.
2. Sporadically, an applied or inner-mathematical example or problem is used to introduce a new concept.
3. There is no introduction.

The characterization of the introduction of a new concept showed substantial interrater agreement ($\kappa = 0.745$).

To understand a mathematical concept properly, the corresponding information has to be processed in different steps. The conceptualization of reviewing is as follows:

1. The book requests the reader (after some time) to be able to actively name and use already known concepts as well as their characteristic properties. Occasionally, contents that have already been learned are referred to, or they are necessary to solve tasks, respectively (active).
2. The book reminds the reader of learned contents and of characteristics of learned concepts (passive).
3. The book proceeds in the contents without testing concepts which have already been learned or including characteristics of learned concepts in the contents. The particular chapters are strictly delimited from each other.

The interrater agreement on reviewing shows moderate strength ($\kappa = 0.548$).

2.6.4 Development and Understanding of Theorems

The operationalization of the development and understanding of theorems is divided into three subsections dealing with the introduction of the theorem, its formulation, and its demarcation from other theorems. The development of the theorem is an example of the first section.

1. It is part of the book's conceptual design that the development of the theorem is described.
2. It is sporadically shown how the theorem can be developed.
3. There is only a formal formulation of the theorem.

Interrater agreement on the development of a theorem showed only moderate strength ($\kappa = 0.417$).

The next operationalization presented is the one of the degree of formalism. It belongs to the formulation of the theorem.

1. The formulation/notation of the theorem equally consists of mathematical symbols and (German) language.
2. The formulation/notation of the theorem consists mainly of mathematical symbols.
3. The formulation/notation of the theorem consists mainly of (German) language.

The interrater agreement on the degree of formalism of this operationalization was substantial ($\kappa = 0.714$).

The demarcation of a mathematical theorem can, for instance, be characterized by using explicating examples and counterexamples as this marks the theorem's applicability. The operationalization of this characteristic shows substantial strength in interrater agreement ($\kappa = 0.696$).

1. There are examples and counterexamples for the theorem given.
2. There are either examples or counterexamples to mark the applicability of the theorem.
3. No examples of applicability of the theorem are used.

2.6.5 Presentation of the Proving Process and Proofs

To learn how to prove a mathematical proposition, it is necessary to understand how to come to the idea of the proof. Therefore, students have to understand how a proof is developed and how to write it down properly. The operationalization of the generation of a proof idea is given below:

1. It is part of the book's conceptual design to show the derivation of the proof ideas.
2. Proof ideas are sporadically derived.
3. It is never shown how a proof idea can develop.

Substantial interrater agreement ($\kappa = 0.632$) could be reached for the generation of a proof idea.

As there are several ways as to how to come to a proof idea, it is necessary to illustrate different approaches or methods on how to reach a proof:

1. It is part of the book's conceptual design that the assertion is proven in different ways or that the proof idea is sketched, respectively.
2. It is sporadically shown how an assertion can be proven in another way.
3. The assertion is proven in at most one way.

The strength of the interrater agreement concerning the number of methods to prove is substantial with $\kappa = 0.632$.

2.6.6 Tasks

To work on mathematical tasks actively is a fundamental part in the process of learning mathematics. Therefore, our model distinguishes between different dimensions of the tasks referring to the national educational standards, contents, and solutions. In the realm of the contents, it was rated to what extent different mental models are part of the book's conceptual design.

1. The tasks only make use of mental models that have been addressed beforehand.
2. New mental models are introduced by means of tasks. They are just briefly presented; there is no sufficient implementation.
3. The readers must develop new mental models on their own while working on the tasks.

The interrater agreement on the use of mental models was perfect ($\kappa = 1$).

One aspect that was rated concerning the solutions of the tasks was the explication of the approach to the solution:

1. The idea of the approach to the solution is described and the approach to the solution is explained.
2. Only the approach to the solution is indicated. It is, however, not stated how it has arisen.
3. Only the solution is given without elaborating on the approach to the solution.

The interrater agreement of this operationalization was moderate ($\kappa = 0.591$).

2.7 Discussion

The goal of the research presented here is to develop a theory-based framework for a mathematics textbook analysis. The aim is to allow a reliable rating of different criteria to compare textbooks for schools and for universities. As presented in Sect. 2.4, the criteria are derived from theories and models concerning mathematical learning activities and the character of mathematics. Two feasibility studies were conducted to evaluate the framework: one to validate and complete the framework and a second to check whether a reliable rating of the criteria is possible. In Sect. 2.6, exemplary results on the reliability values of different rating criteria are presented. The results indicate that the development of operationalizations which allow reliable ratings for a mathematics textbook analysis is possible for many criteria. However, in several cases, the interrater agreement cannot be considered as acceptable, and hence, a further improvement in the feature characteristic descriptions is necessary.

On the basis of the reliable rating criteria presented in Sect. 2.6, some tendencies about commonalities and differences between mathematics textbooks for school and for university can be described. However, as the "sample" of textbooks included in this feasibility studies is quite small (two textbooks for schools, two for universities, and one lecture note for a university course), the results should not be over-interpreted. In our study, we did not find differences between school and university textbooks for:

- The motivation criterion "social relatedness" because there were hardly tasks requiring collaborative activities.
- The criterion relating to the understanding of theorems which addresses the explication of examples and counterexamples. This is because only examples were presented for both types of textbooks.

- The proof criteria “developing a proof idea” and “different proofs for a theorem” because in both types of textbooks the idea was developed for a minimum of proof and, in general, only one proof was presented.
- The task criterion “use of different mental models in tasks,” because, in general, only the mental models introduced before were addressed in the tasks.

In contrast to these commonalities, the textbooks for schools and for universities in our sample also revealed some differences, for example:

- For the structure criterion “motivational additives” the textbooks for schools contain some additional information about the mathematical facts in terms of stories and anecdotes raising the readers’ interests, whereas we did not find such motivational additives in textbooks for universities.
- The introduction of new concepts is in textbooks for schools almost always developed on the basis of inner-mathematical or extra-mathematical examples, whereas in textbooks for universities such an introduction is rarely given.
- The degree of formalism for the formulation of theorems in textbooks for school almost always consists of continuous written language, whereas in textbooks for university a mixture of continuous text and symbols is used.

From analyzing these commonalities and differences, some anticipated findings have become evident which already give indications about the transformation problems in students’ learning during the transition stage from school to universities. For example, mathematical proofs are treated inadequately in both types of textbooks. However, proofs are underemphasized at school so that students do not experience negative consequences. In contrast, at university, proofs are one of the main aspects in mathematics courses; however, university-level textbooks do not give didactical support to learn how to prove a task. Another example is the introduction of new concepts. In school textbooks, there are frequently inner-mathematical or extra-mathematical examples to motivate the new concepts. At university, such motivation is rarely given. This means that students have to elaborate on that question by themselves which requires specific learning strategies.

Already these first ideas from our feasibility studies indicate that the two hypothesized transformations from school to university are not independent but interwoven. The transformation of the character of mathematics with a stronger emphasis on concepts and proofs requires an increasing learning effort. However, the change from school-based to academic learning opportunities requires a transformation of individual learning strategies to grasp the academic mathematics.

To make sound statements about these tendencies revealed from our study, more substantial studies with more textbooks have to be carried out. The findings gained from this study, however, are an initial starting point and can be seen as basis for a future refinement of the model. This refined framework then is supposed to be used in further studies comparing international textbooks.

School and university textbooks should be revised in such a way that their contents and methodologies are better adapted to each other. This would help diminish the transformation challenges experienced during transition from school to university. The transition from school mathematics to university mathematics is supposed to be easier for students if mathematical methods and university standards were applied in school textbooks. The same can be said for improving the didactical quality of university textbooks.

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Chapter 3
Knowledge Transformation Between Secondary School and University Mathematics

Oliver Deiser and Kristina Reiss

Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anders. (Mathematicians are like a sort of Frenchmen: if you talk to them, they translate it into their own language, and then it is immediately something quite different). Johann Wolfgang von Goethe (1749–1832).

3.1 Introduction

Mathematics at school level and mathematics at university level represent one discipline; however, the foundations differ significantly (Freudenthal 1973). Mathematics taught at the university level seeks to describe knowledge within a coherent frame of axioms, definitions, and theorems and their proofs. School mathematics lacks this rigor and makes use of more intuitively accessible knowledge. This difference is crucial and an important cause for difficulties which students encounter when coming to the university. Learning mathematics at this level means in particular mastering the transformation between a phenomenon-oriented view on the subject and a description in terms of formal language. Accordingly, knowledge transformation is more than a translation process but includes the modulation of corresponding components. In the following, we will exemplarily describe how students encounter this transformation in their first year at the university.

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3.2 Theoretical Background

Since Felix Klein wrote his influential books on “Elementarmathematik vom höheren Standpunkte aus,”[1] effective mathematics instruction at the school and the university level has been intensively discussed from the mathematics and the mathematics education points of view. Both points meet when fundamental ideas of mathematics are addressed. Thus, the “reformers” what Felix Klein called the group to which he counted himself could initiate a movement in the mathematics classroom which put an emphasis on the basic notion of function and the “graphical method” (cf. Klein 1925, p. 5 f.). The standards for school mathematics which have been specified in many countries during the last decade adopted some of these ideas and established functional thinking as an important concept in school mathematics (National Council of Teachers of Mathematics 2000; Kultusministerkonferenz 2003; Common Core State Standards Initiative 2010). Graphs of functions are used in depth in order to motivate, introduce, and explore basic analytical notions like continuity, monotony, differentiability, extremal values, and curvature.

However, accepting fundamental ideas does not necessarily mean that school mathematics and university mathematics are regarded as complementing fields by the students. The “double discontinuity” described by Klein concerning the transition of teacher students from the upper secondary level at school to the university and the transition of young teachers from the university back to school still remains (Klein 1925, p. 1; see also Biermann and Jahnke Chap. 1, as well as Buchholtz and Kaiser Chap. 5). In particular, the first transition is regarded a challenging break by many students. They are, on the one hand, used to complex calculation problems and they learn at secondary school, e.g., how to apply the chain rule of derivation, how to compute limits or Newton’s method to find zeros, and how to analyze the curvature of a function by computing and evaluating its second derivative. On the other hand, research suggests that the German mathematics classroom might overemphasize the calculation aspect and disregard the advancement of students’ more profound understanding (Baumert et al. 1997). Mathematics is experienced by students as a subject which is dominated by calculation and guided by recipes. They learn to calculate and to appreciate the correct result but have difficulties with the processes of mathematical work. Identifying mathematical arguments or giving mathematical proofs is a challenging task even for students at the upper secondary school level (Klieme et al. 2003).

At the university level, students experience mathematics as a scientific discipline. In their view, calculation has been the constituting aspect of mathematics; however, it is now losing its dominant role and is replaced by a plethora of definitions, theorems, and proofs, which are presented at a high speed with a hitherto unknown density of information. Living and surviving in this new world will not require a mere extension, but a fundamental transformation of knowledge, and

[1] Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis.

the habits of learning and applying knowledge have to be redefined, tested, and modified. There is a "cognitive distance between research and elementary level" as Biermann and Jahnke (Chap. 1) phrase it, which can be seen as a gap to be bridged. However, school mathematics seems to be far away from the scientific perspective. Accordingly, in particular for preservice teacher students, it is unclear, whether the endeavor of transformation is necessary and meaningful. The requirements of the mathematics classroom, regarded as well known to preservice teacher students because of their 12 years of participation as a student, will cause some of this transformed knowledge to be rediscovered when they find themselves back in the old rooms as mathematics teachers. Therefore, difficulties with the new mathematical knowledge may lead to an argument against its significance for the daily routines at school. Obviously, these arguments cannot be proved true. Teachers need pedagogical knowledge, pedagogical content knowledge, as well as content knowledge (following the classification of Shulman 1986 as well as Shulman 1987). Moreover, it is not only the tradition of university mathematics but also recent research which emphasizes the role of teachers' mathematical content knowledge for the successful classroom work (Baumert et al. 2010; Baumert and Kunter 2006; Darling-Hammond 2000; see also the discussion by Czerwenka and Nölle 2011).

The problems of students of mathematics in their first years (and sometimes beyond) at the university have been discussed in depth (e.g., Cappell et al. 2010); however, there is hardly any research which describes these problems in detail and from an individual point of view. Research concentrates primarily on content knowledge with a clear reference to school mathematics. Mathematical knowledge in these studies (e.g., Baumert et al. 2010; Blömeke et al. 2010) comprises facts and techniques which could have been acquired at school.[2] Moreover, we lack empirical evidence on how learning of mathematics at the university level takes place over time. There are a few introspective reports of problem-solving processes by mathematicians dating back to the first half of the twentieth century (e.g., Wertheimer 1945; Hadamard 1949; for a synopsis see Reiss and Törner 2007) which point out important aspects. These reports as well as recent research in secondary classrooms suggest that looking at mathematics as a subject will not explain the difficulties but point out an important role of intuition and personal views (Heinze and Reiss 2009).

These considerations brought us to conceptualize a study which aimed at identifying preservice teacher students' knowledge and competencies and their development in their first year at the university. We were particularly interested in describing discontinuities in learning.

[2] Baumert et al. (2010) used, e.g., the task "Is $2^{1024}-1$ a prime number?" in their study. Solving this task presupposes the concept of prime number (approx. grade 6 in German curricula) and the formula $x^{2n}-1=(x^n+1)(x^n-1)$ (approx. grade 7 in German curricula).

3.3 Sample and Method

The sample comprised 18 first-year teacher students majoring in mathematics and physics[3] who took part in a test on their mathematical knowledge. All students were enrolled in courses on analysis[4], linear algebra, mathematics education, and pedagogy. By the time of testing, students had participated in all courses for 6 weeks. It should be mentioned that the students attended an innovative teacher education program. This program included exercise phases which were specifically designed for teacher students and addressed the relation of topics to school mathematics (Reiss, Prenzel and Seidel, in prep.). Moreover, most of the students had participated in a precourse on mathematics which aimed at connecting school and university mathematics and had introduced them into the more systematic, abstract, and precise way in which the subject would be taught at the university.

The test encompassed six items related to the analysis course. Two items asked for basic knowledge concerning the geometric series and the handling of complex numbers. Four questions were designed to analyze students' understanding of definitions concerning *infimum and supremum, limit of a sequence, infinite series*, and *subsequence of a sequence.* There were three main reasons for choosing this content. First, the items included basic but nonetheless very important notions of analysis which should be mastered by all students. Second, the notions had, at least from a beginner's point of view, a considerable degree of technical complexity, which was likely to result in answers reflecting specific individual problems. Third, all four notions had a high degree of intuitive meaning, and therefore seemed to be specifically apt to find out details about the transformation processes taking place in the first year in students' minds. In the following, we will concentrate on these mathematical definitions. We will provide the items and give examples of correct answers. Moreover, we will briefly discuss how the concepts had been presented in the analysis class by the time of the test. In order to make the text readable for nonmathematicians, we will also provide some general information about specifics of the concepts before introducing the items.

The first item asked for the definition of the *infimum* of a set of real numbers. The infimum of a set X of real numbers is synonymously called its *greatest lower bound*, and is denoted by $inf(X)$. Some examples are

$$inf(\{1,2\}) = 1, inf\left(\{\frac{1}{n} \mid n \geq 1\} \cup \{0\}\right) = inf\left(\{\frac{1}{n} \mid n \in \mathbb{N}\}\right) = 0, inf([1,2]) = inf([1,2]) = 1.$$

in particular, the infimum of a set might or might not be an element of the set. The examples and the expression *greatest lower bound* give an intuitive meaning, but students were asked to properly define the notion.

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[3] German preservice teacher students who will teach at the upper secondary level are supposed to choose two major subjects during their BA and MA studies as well as elementary courses in pedagogy and psychology.

[4] This means a course at a level somewhat below Walter Rudin's "Principles of Mathematical Analysis" (1976).

Item 1: Greatest Lower Bound (infimum) Let $s \in \mathbb{R}$, and let $X \subseteq \mathbb{R}, X \neq \emptyset$. Define: "*s* is the infimum of *X*, if…"

Example of an expected answer:

"(1) $s \leq X$, i. e. $\forall x \in X s \leq x$, and (2) $\forall t \in \mathbb{R}(t \leq X \Rightarrow t \leq s)$."[5]

In the analysis class, the notions of supremum[6] and infimum were introduced and discussed in its specifics from the very beginning. The corresponding existence principle was part of an axiomatic characterization of the reals. The introduction made use of the established practice of presenting and discussing such a characterization without constructing the real numbers. At length, the irrationality of the square root of 2 was discussed, and $\sqrt{2} = sup(\{q \in \mathbb{Q} \mid q^2 < 2\})$ was pointed out. Following Dedekind, the students learned that square root of 2 marked a gap in the rationals, and that the axiomatic principle of the existence of suprema and infima was one way to precisely state the completeness of the real numbers. Moreover, suprema and infima had already been topics during the precourse. Accordingly, though the students had probably not encountered these notions in their secondary mathematics classrooms, they were not entirely unfamiliar when treated in the analysis class.

The second item addressed the notion of the limit of a sequence. Again, the intuitive meaning of this concept can be illustrated by examples: The limit of $(1, 0.1, 0.01, 0.001, ...)$ is zero, while the limit of $(1, 0, 1, 0, 1, 0, ...)$ does not exist. The limit of the sequence $(1, -0.1, 0.001, -0.0001, ...)$ is zero, too, and this example shows that the notion of a limit cannot be reduced to infima and suprema in a simple way. If a real number *s* is the limit of the sequence $(x_n)_{n \in \mathbb{N}}$, one can say that the sequence *converges to s*.

Item 2: Limit of a Sequence Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$, and let $x \in \mathbb{R}$. Define: "*x* is a limit of $(x_n)_{n \in \mathbb{N}}$, if…"

Examples of expected answers:

$$\text{“} \forall \epsilon > 0 \, \exists n_0 \forall n \geq n_0 \; | x_n - x | < \epsilon \text{.”}$$

"every interval $]x - \epsilon, x + \epsilon[, \epsilon > 0$, contains almost all x_n."

The notion of a limit was advertised to be one of the fundamental concepts of analysis, and it was pointed out that sometimes analysis was even defined to be the study of limits. Moreover, the concept had been introduced in the precourse, too, and had been combined there with a training in the use of quantifiers (e.g., rules for the negation of quantifiers and problem of switching quantifiers). In a tutorial section, the formal definition of the first answer was supplemented with diagrams and then figures of speech like "almost all" appearing in the second answer were introduced.

[5] The notation "$y \leq Y$" was introduced in the course to express that a real number *y* is a lower bound of a set *Y* of reals.

[6] A real number *s* is the *supremum* or *least upper bound* of a set *X* of reals, if it is the smallest real number *s* such that $X \leq s$.

The third item concerned infinite sums of reals. Informally, the infinite sum $x_0 + x_1 + x_2 + \ldots + x_n + \ldots$ is the result of adding up all reals x_n in the order of their appearance, provided this result exists. For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$ (geometric series), while the infinite sums $1 + 2 + 3 + \ldots$ or $1 - 1 + 1 - 1 + \ldots$ do not exist. A proper definition of infinite sums can be given by looking at the limit of the sequence $(s_n)_{n \in \mathbb{N}}$, where the *partial sums* s_n are defined by $s_n = x_0 + x_1 + x_2 + \ldots + x_n$.

Item 3: Infinite Series Explain "$\sum_{n \in \mathbb{N}} x_n$".

Example of an expected answer:

"(1) $\sum_{n \in \mathbb{N}} x_n$ is the sequence $(s_n)_{n \in \mathbb{N}}$, with $s_n = \sum_{k \leq n} x_k$ for all n. In this meaning, it is called an infinite series. (2) $\sum_{n \in \mathbb{N}} x_n$ is the limit of the sequence $(s_n)_{n \in \mathbb{N}}$, if the limit exists. In this meaning, $\sum_{n \in \mathbb{N}} x_n$ is called an infinite sum."

As usual, an infinite series of real numbers was defined in class to be a sequence of partial sums. It was pointed out that the symbolic notation $\sum_{n \in \mathbb{N}} x_n$ was used in the double meaning reflected in the answer provided above. The notions were motivated by looking at the intuitively presented infinite summation $x_0 + x_1 + \ldots + x_n + \ldots$. When one computes this series, a sequence of computations is produced: $x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots$. This sequence of computations serves as the definition of an infinite series. If the sequence converges, its limit is the infinite sum of all x_n. By the time of the test, the students had worked with series like the geometric or harmonic series, and the symbolic notation $\sum_{n \in \mathbb{N}} x_n$ had appeared in both of its meanings several times.

Finally, we asked for the definition of *subsequence of a sequence*. Intuitively, a subsequence of a sequence $(x_0, x_1, x_2, \ldots)$ is produced by selecting infinitely many elements of the sequence, respecting the order of appearance. Thus $(0, 2, 4, 6, \ldots)$ is a subsequence of $(0, 1, 2, 3, \ldots)$, while $(2, 0, 4, 6, \ldots)$ or $(0, 0, 1, 2, 3, \ldots)$ or $(0, 1, 2, 3)$ are not.

Item 4: Subsequence of a Sequence Define the notion of a subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers.

Examples of expected answers:

"A sequence $(y_n)_{n \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$, if there is a strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers such that $y_n = x_{i(n)}$ for all n."

"A sequence is a subsequence of $(x_n)_{n \in \mathbb{N}}$, if it has the form $(x_{i(n)})_{n \in \mathbb{N}}$ for a strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers."

The notion of a subsequence figures prominently in the Bolzano–Weierstraß theorem, which states that every bounded sequence of reals has a convergent subsequence. The theorem had been proven in class, and the recursive construction of the subsequence of the proof had been analyzed in a tutorial section. This led finally to an intuitive summary of the proof.

Table 3.1 Results for 18 students and 4 items.

Student	1	2	3	4	5	6	7	8	9
Item 1:	0	0	0	0	0	1	1	1	2
Item 2:	0	0	0	0	1	0	0	0	0
Item 3:	0	0	0	0	0	0	0	1	0
Item 4:	0	0	0	1	0	0	0	0	0
Student	10	11	12	13	14	15	16	17	18
Item 1:	0	1	0	1	2	2	2	1	2
Item 2:	1	1	2	2	2	2	2	2	2
Item 3:	1	1	0	0	0	0	1	2	2
Item 4:	1	0	1	1	0	1	1	2	2

3.4 Results

As we had expected from former systematic (Kessler 2011) and unsystematic inquiries as well as from teaching experience, the items turned out to be difficult to answer for most of the students. A grading assigned 0, 1, or 2 points to each item (2 points for a fully correct solution, 1 point for solutions with only minor faults, 0 points for all others). The gradings added up to the following results (were the columns represent the 18 participants and are ordered by their scores):

Table 3.1 shows that three students were not able to solve any of the items correctly and only one student got full credit. There were important differences in the individual achievement (i.e., properly defining basic mathematical notions) within the group. Students scored with a mean of m = 2.83 points (max = 8) and a standard deviation of stdev = 2.43 (all items). Moreover, the four items obviously differed in their demand (m = 0.89; stdev = 0.83 for item 1; m = 0.94; stdev = 0.94 for item 2; m = 0.44; stdev = 0.70 for item 3; m = 0.56; stdev = 0.70 for item 4; max = 2 for each item). Though presented as a notion expressing the fundamental difference between the rationals and the reals, the notion of an infimum in its order-theoretic language remained extraneous to many students. Only 5 of 18 answers to item 1 were fully correct. The second item *limit of a sequence* showed the best solution rate, but again around half of the group could not properly define it. Concerning the third item, only two students were able to give a satisfying explanation of the meaning of the frequently used symbolic notation for infinite series and infinite sums. This item turned out to be most difficult for the students. The fourth item was arguably the hardest, as the notion of a subsequence involved the composition of two functions defined on the natural numbers. Again, only two students could properly define this concept.

The rating of items according to a 2-1-0 pattern which reflected full score, partial score, and non credit solutions gives a rough and technical information about students' achievement. More meaningful information can be provided by a detailed analysis of the solutions. Accordingly, correct as well as non-correct solutions were classified in order to understand the variety and nature of students' knowledge, competencies, and problems. We chose the following post-hoc classification, which depended on the specific item and allocated every solution to a single class. We will provide the number of students whose solutions belong to the specific class and will give concrete examples.

Classification of the Answers to Item 1:

- Explanation (translation) as "greatest lower bound" without further comments: 2
- Explanation (translation) using the concept "greatest lower bound" and providing some additional comments: 5
- Erroneous or incomplete explanation of "greatest lower bound": 4
- Answer with small errors: 2
- Correct answer different from the expected one: 1
- Correct answer as expected: 4

The answers to item 1 showed that many students had reasonable ideas about the concept but were struggling with a precise formulation. Several students tried to use formalized language but failed in the end in using it adequately. However, most solutions revealed a rudimentary understanding. There were only two students who translated the concept into another undefined notion, whereas 16 students gave more detailed formulations.

Examples of an explanation (translation) with additional comments are: (1) "There are only finitely many values below this bound and there is no greater lower bound." and (2) "*s* constitutes the least lower limit of *X*. Thus *X* is bounded below and converges against the infimum of *X*." Both answers reveal that students had difficulties in separating this concept from the concept of limit. As an erroneous and incomplete explanation of greatest lower bound may be regarded (3) "*s* is the greatest lower bound of *X*. There are no elements *x* with $x \in X$ below the infimum. $s \leq x$ with $x \in X$." The second sentence of this solution is descriptive, but the third sentence can be read as a definition of "*s* is a lower bound of *X*." It appears to be an explanation of the second sentence, and some kind of translation and transformation process is visible here. The following sentence was rated as a correct answer with small errors: (4) "1 ($\forall x \in X \ x \geq s$ 2) For each s', which is a lower bound too, $s' \leq s$." One could object that "lower bound" was not explicitly defined in the answer, and that it should be "lower bound of X" in part 2). However, it is plausible that the student just needed to write down more carefully what he or she knew. Finally, (5) "$s = \max\{g \in \mathbb{R} \mid g \leq x \forall x \in X\}$" was regarded an unexpected but correct answer. In this case, the infimum *s* of *X* was correctly defined, without using *supremum* to define it.

Classification of the Answers to Item 2:

- Explanation as "the sequence convergences to x" without comments: 4
- "Limit" mistaken with the concepts of "supremum/infimum": 2
- Incorrect use of quantifiers: 4
- Answer with small errors: 1
- Correct answer as expected: 7

This item and the solutions by the students showed two typical phenomena more clearly than the first one: First, even more students presented mere synonymous expressions ("convergent") but did not provide a mathematical definition based on properties of the concept (as shown in the examples). Second, some students knew the need for that type of solution, but failed to use the quantifiers "for all" and "there exists" correctly.

Typical examples for answers showing problems with quantifiers were: (1) "$\exists \epsilon > 0 \forall n_0 \in \mathbb{N}\ \exists n \geq n_0 \mid x_n - x \mid < \epsilon$" and (2) "$\forall \epsilon > 0\ \exists x_0 \forall x > x_0 \mid x - x_0 \mid < \epsilon$." In one instance, the correct answer was encapsulated between two wrong statements: (3) "the sequence $(x_n)_{n\in\mathbb{N}}$ converges strictly increasing against x. $\forall \epsilon > 0 \exists n_0 \forall n \geq n_0 \mid x_n - x \mid < \epsilon$. x is the supremum of $(x_n)_{n\in\mathbb{N}}$." Despite the correct second line, the answer seemed to belong to the second class where *limit* was mistaken with *infimum* and *supremum.*

Classification of the Answers to Item 3:

- Explanation as "series" without further comments: 3
- Explanation using "series" or "partial sums" and some comments added: 4
- Explanation as "$x_0 + x_1 + x_2 + \ldots$" without mentioning "convergence" and "partial sums": 8
- Answer with small errors: 1
- Correct answer as expected: 2

For this item, nearly half of the students gave answers by writing a term like "$x_0 + x_1 + x_2 + \ldots + x_n + \ldots$." They did not recognize that the definition of the symbol $\sum_{n\in\mathbb{N}} x_n$ aimed at transforming an informally presented infinite sum into a precise expression. The problems were multifaceted. Obviously, the definition of a series as the sequence of partial sums has to be regarded as complex and technical, and in the end the symbol is overloaded in denoting always a certain sequence as well as sometimes a real number, too.

One answer of the third category mentioned above was: (1) "It is the infinite sum of all $x_{n.}$ $\sum_{n\in\mathbb{N}} (x_n) = x_1 + x_2 + x_3 + \ldots + x_n$." The missing dots at the end of line indicated that the student knew that partial sums played a role here, but he (or she) failed to write "$\sum_{n\in\mathbb{N}} (x_n) = lim_n (x_1 + x_2 + x_3 + \ldots + x_n)$," which would have been a correct definition of an infinite sum. Finally, there were typical examples that students mixed up concepts: (2) "It indicates the series of the sequence $x_{n.}$ It is the sum of all partial sums of the sequence x_n." Instead of looking at the "limit of all partial sums," this student summed up all partial sums. However, the answer showed that the student was aware that partial sums were needed to define the symbol.

Classification of the Answers to Item 4:

- Incorrect without, in this context, meaningful elements: 2
- Concept mistaken, mentioning of notions like "accumulation point", "Bolzano–Weierstraß": 3
- Explanation as "subset" or "contained in," the concept of "index sequence" was not mentioned: 3

- Descriptive answer listing properties of subsequences: 2
- Concept mistaken with “index sequence”: 4
- Answer with small errors: 2
- Correct answer as expected: 2

Once more, some students simply described the notion without attempting to give a proper definition. Some other students tried to give a definition but failed to properly use the technical notion of an “index sequence.” In contrast to other items, we observed three students who neither described the notion nor attempted to provide a definition, but wrote down something they recalled from the wider context of the notion like the Bolzano–Weierstraß theorem.

This is an example of a descriptive answer: (1) “A subsequence must have the following properties: It must have the sorting of the sequence; all elements of the subsequence have to appear in the sequence; one can jump over arbitrarily many elements of the sequence; the first element of the subsequence may not be the first element of the sequence; the subsequence must not be empty.” A typical example of misunderstanding the concept with index sequences was this one: (2) “A subsequence [crossed out: contains] is an index sequence (i_n) of the sequence $(x_n)_{n\in\mathbb{N}}$, i.e., all elements of the subsequence are elements of the sequence.” This student mixed up the concepts of “index sequence” and “subsequence”; however, the crossed-out word “contains” might indicate some doubts.

3.5 Discussion

Our study provides (more) evidence that the mathematical knowledge acquired in secondary schools does not necessarily constitute a reliable foundation for university mathematics. Adopting basic mathematical notions presented in the formally correct but abstract way at the university is a very difficult task for beginners. However, school mathematics is a starting point not only for calculation or the application of algorithms but also for an intuitive understanding of mathematics. The step from here to rigorous mathematical definitions of concepts like *infimum, limit, infinite sum,* and *subsequence* is still a large one. School mathematics usually lacks a solid foundation; however, this is a characteristic feature and may be regarded the specific difference between mathematics at the school or at the university level (Freudenthal 1973). As a consequence, doing mathematics at the university level requires a different way of using language in class. It is probably not only a change in wording but also the learning of a new language with an unfamiliar structure as well as with a high degree of formality and precision. Appreciating these aspects presupposes a profound understanding of its advantages.

It is not surprising that learners show deficits but also progress while approaching the goal of learning the mathematical language. Our study reveals how different aspects and components come together and add up to a multifaceted picture. They might be regarded as typical for the transition between secondary school and university mathematics.

- Continuation of habits from school: In the context of school mathematics, exact definitions are rarely used. Instead, concepts are often acquired by regarding prototypes and discussing their properties (e.g., "square" and "cube"). Moreover, properties might be supplemented by translations from the Latin origin word to a German origin word which has an intuitive meaning for the students (e.g., "congruent" translated to *deckungsgleich* which means something like *exact covering*). Accordingly, mathematics instruction does primarily target at initiating understanding and the correct identification of examples. This pattern was continued by the students when they explained notions by synonyms or by a description of their properties.
- Discontinuation of habits from school: School mathematics makes use of mathematical definitions, at least at the upper secondary level. However, these definitions do hardly aim at being fundaments, e.g., for further theoretical considerations. It is more important in school instruction to apply concepts and methods in the context of problems: Curve sketching presupposes the concept of a maximum turning point but once it is established students' activities are basically restricted to calculating the first and second derivative of a function.
  Students learned in the few weeks at the university that correct definitions ask for the use of the (new) mathematical language. Accordingly, examples were not used for explaining the concepts involved. On the one hand, this is good news, as students were able to realize the need for formalization in mathematics. On the other hand, this is probably also bad news as examples are often useful in order to prepare a precise formulation and to distinguish between examples and counterexamples.
- There are typical patterns of work which reflect individual intellectual processes. These patterns guided a classification of students' solutions of the items and revealed their errors and problems as well as their creative thoughts.
- Contemporary forms of university instruction include several ways of providing support to the students. In particular, the precourse on mathematics should be regarded such an instrument. Moreover, the degree of abstraction can be lowered, and the pace of work can be reduced. However, it seems that students are confronted with too much new mathematical content in their first weeks at the university. Understanding unfamiliar concepts and ways of working is a challenging task which takes its time. In particular, processes of transition seem to take more time than is usually provided for a smooth change from school to university (see also Kaiser and Buchholtz, Chap. 5).

Looking at the results substantiates that the "double discontinuity" still exists. Is there help for teacher students to master the demands in the beginning of their studies? Certainly this research can only describe students' competencies and discuss their problems. However, the detailed analysis might give information on how university courses should be designed in order to better mediate between the different phases. In our view, the shift from mathematical knowledge based on intuitive concepts with only minor emphasis on an exact wording to mathematical knowledge based on formalization and exactness includes different aspects: Some students would certainly profit from a training in easy formal definitions in order to abandon the old habit of giving mere descriptions, while others would benefit from a "technical" training using quantifiers, sequences, or compositions of functions.

Further research concerning the subject-specific competencies of teachers is needed. Our study will be extended to a larger and more fine-grained analysis of the transformation of knowledge in mathematics between school and university. We will regard the first and second year at the university, because at this time students are confronted with topics bearing the highest relation to topics which are part of the school curriculum. It is still an open question how much understanding of "elementary mathematics from an advanced standpoint" can be achieved in preservice teacher education. However, we know that it is important for the classroom work of a teacher (Baumert and Kunter 2011). As we aim at a better understanding and at more effective learning processes, it is essential to analyze precisely the mathematical difficulties of teacher students and the intellectual processes they pass through. Both components contribute to their content knowledge and their pedagogical content knowledge and are important determinants of their mathematical competencies.

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Chapter 4
Student Transition to University Mathematics Education: Transformations of People, Tools and Practices

Birgit Pepin

4.1 Introduction—Transition from School to Higher Education in Mathematics Education

There has been a widespread concern over the lack of preparedness of students making the transition from upper secondary to university mathematics. It appears that students experience different difficulties at different stages and develop different strategies to make these transitions successful. At the same time institutional practices afford, or hinder, students developing a mathematical disposition and an identity that supports their engagement with mathematically oriented subjects in upper secondary and tertiary education. This links to most European governments' concern about student participation, and success, in mathematics. In the UK the Smith report (2004) stressed the need for more young people to continue to study mathematics, which, as it was suggested, could be achieved by 'wider recognition of the importance of mathematics, improved teacher supply and professional development for teachers, and changes in the curriculum and qualifications pathways, so as to provide appropriate progression for all students' (Brown et al. 2007, p. 18).

A number of studies (e.g. Thomas 2002) have highlighted factors that generally impact on retention rates and performance in higher education, such as 'academic preparedness', or academic experience, to name but a few. Others (e.g. Hager and Hodkinson 2009) have tried to move 'beyond the metaphor of transfer of learning' in higher education. In this paper I want to lean on literature in mathematics education, more particularly mathematics learning, in transition from upper secondary school to higher education mathematics education, that is, transition to mathematically demanding subjects (such as mathematics, engineering, etc.).

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The literature proposes different models for secondary–tertiary transition in mathematics education. In an early paper De Guzman et al. (1998) identified selected difficulties related to this passage; these are broadly categorised as 'epistemological/cognitive'; 'sociological/cultural'; and 'didactical'. Leaning on this work Gueudet (2008) proposes three types of transition in her study: (1) transition 'from one mode of thinking to another', which includes the identification of different 'student thinking modes and knowledge organisation' at transition (e.g. Tall 1991; Sierpinska 2000; Lithner 2000); (2) transition to the 'new world' of proof and different mathematical communication (e.g. laws and language used by mathematicians) (e.g. Nardi 2009) and (3) transition from an institutional perspective (e.g. Bosch et al. 2004). There are also a number of research studies describing and analysing particular, sometimes innovative, practices (e.g. Croft et al. 2009).

In summary, the literature generally shows that transition is often a 'threat' to progress, especially for certain students, and that efforts to align practices on either side of transition can help (e.g. Hoyles et al. 2001). However, mathematics at the secondary/tertiary interface is believed to be particularly problematic for pedagogy, especially for proof and mathematical communication (Hoyles et al. 2001; Nardi 1996), and 'formal' mathematical thinking generally. This is likely to have implications for students' success or failure at this stage of their mathematics learning. Moreover, there are few studies (besides the Manchester projects) that directly address the widening participation agenda with respect to higher education mathematics engagement, learning and identity development. In this paper I address the issue of 'student learning' at the interface between school and university, in particular the different pedagogic practices, types and sources of feedback which are likely to support, or hinder, student transiting to this new 'arena' of mathematics education.

4.2 Theoretical Framework: Feedback and Self-Regulated/Independent Learning

There is a large amount of literature linking student achievement and feedback (e.g. Butler and Winne 1995) where feedback is conceptualised as 'information with which a learner can confirm, add to, overwrite, tune, or restructure information' (p. 275). In their study on the importance of feedback, Hattie and Timperley (2007, p. 81) view feedback as

> information provided by an agent (e.g. teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding. A teacher or parent can provide corrective information, a peer can provide an alternative strategy, a book can provide information to clarify ideas, a parent can provide encouragement, and a learner can look up the answer to evaluate the correctness of a response. Feedback thus is a "consequence" of performance.

There is ample evidence (e.g. Hattie and Jaeger 1998) that the presence of feedback (in whichever form it may be) increases the likelihood that learning will occur. In this study, I conceptualise feedback as feedback not only from teachers/lecturers but also from other sources, such as curriculum materials, or peers, for example. Moreover, and leaning on research by Hattie and Timperley (2007), I distinguish between four levels of feedback: the task level (how well the tasks are understood/performed); the process level (the main process/es needed to understand/perform tasks); the self-monitoring level (directing and regulating actions); and the personal evaluation level (personal evaluation and affect) (p. 87). Butler and Winne (1995) also claim that feedback can have different sources: external (e.g. provided by contexts or other participants) and internal (e.g. self-generated such as monitoring their actions).

Leaning on these conceptualisations, I also explore the role and nature of feedback resulting from 'tools' (e.g. textbook) and their use, or indeed cognitive tools designed to help students develop further understandings of characteristics of mathematical topics. Here a 'tool' can be viewed in different ways. Whilst a tool may have different forms, using a tool in the context of learning mathematics, it is likely to re-frame students' experiences. For example, a new tool is likely to add something to the student's repertoire; equally it may disrupt participants' practice and take something away. This reflects the tool's catalytic quality: it may change participants' perceptions and practice. The individual agency of the student rests with the decisions s/he takes as a result of feedback from the use (of the tool), thus her/his reactions to the feedback.

This provides the link between feedback and self-regulated/independent learning (and I use the latter terms interchangeably). Hattie and Timperley (2007) argue that feedback has a role in teaching self-regulation. Part of this process involves establishing sources to obtain feedback, and they suggest that many students neglect responsibility and 'view feedback as the responsibility of someone else' (p. 101).

Independent learning serves as a comprehensive framework for understanding how students become active and confident agents of their own learning process. Without going into detail, most models define independent/self-regulated learning as 'an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behaviour, guided and constrained by their goals and the contextual features in the environment' (Pintrich 2000, p. 453). Boekaerts (1999) views it as the ability to 'develop knowledge, skills, and attitudes which can be transferred from one learning context to another' (p. 446). It is likely to include 'self-generated thoughts, feelings, and actions that are planned and cyclically adapted to the attainment of personal goals' (Zimmermann 2005, p. 14). The ability to self-regulate one's motivation, cognition, affect, and behaviour seems critical to development and growth (Corno 2009).

According to Pintrich (2000), models of independent/self-regulated learning share a number of common assumptions:

The *active, constructive assumption* where learners are viewed as 'active constructive participants in the learning process' (p. 452).

The *potential for control assumption* where it is assumed that learners can 'potentially monitor, control, and regulate certain aspects of their own cognition, motivation, and behaviour as well as some features of their environment' (p. 454).

The *goal, criterion, or standard assumption* where it is assumed that 'there is some type of criterion or standard (also called goals) against which comparisons are made in order to assess whether the process should continue as is or if some type of change is necessary' (p. 452).

The *mediation assumption* which assumes that 'self-regulatory activities are mediators between personal and contextual characteristics and actual achievement and performance' (p. 453).

Winne (1995) describes independent learners in the following way:

> When they begin to study, self-regulated learners set goals for extending knowledge and sustaining motivation. They are aware of what they know, what they believe, and what the differences between these kinds of information imply for approaching tasks. They have a grasp of their motivation, are aware of their affect, and plan how to manage the interplay between these as they engage with the task. They also deliberate about small-grain tactics and overall strategies, selecting some instead of others based on predictions about how each is able to support progress toward chosen goals. (Winne 1995, p. 173)

In terms of mathematics education and independent learning, mathematics education researchers (e.g. De Corte et al. 2000) adopted the theory of self-regulated learning as an important factor for the learning of mathematics, where students are expected to assume control and take up agency over their own mathematics learning (e.g. problem-solving activities; see Schoenfeld 1992). Pape et al. (2003) argued that in order to develop mathematical thinking and independent learning, several factors are crucial: 'multiple representations and rich mathematical tasks; classroom discourse; environment scaffolding of strategic behaviour; and varying needs for explicitness and support' (p. 179). Overall, it is said that independent/self-regulated learning is 'a major objective of mathematics education…and…a crucial characteristic of effective mathematics learning' (De Corte et al. 2000, p. 721).

Using this larger theoretical frame of 'feedback' I seek to develop deeper understandings, theorise 'feedback' in connection with 'tools', and investigate and relate their connected power to student independent learning.

Thus the research questions are the following:

- What are the contexts, of school and university, in which students work and learn mathematics?
- What are the kinds and sources of feedback students receive in the different contexts, and in which ways do they link to student independent learning?
- What are the 'transformations' students experience and what is their potential for students managing transition into higher education mathematics?

4.3 Research Design

The TransMaths project at the University of Manchester[1] investigated how student experiences of mathematics education practices may interact with various (identified) factors to shape students' development as learners of mathematics, their dispositions and their decision-making at this crucial point. It appears that students experience different difficulties at different stages and develop different strategies to make these transitions successful. At the same time institutional practices afford, or hinder, students developing a mathematical disposition and an identity (Pepin 2009; Boaler and Greeno 2000) that supports their engagement with mathematically oriented subjects in upper secondary and into higher education. The project studied students' identity in relation to their experiences of different mathematics learning-and-teaching practices. Leaning on the work of Cobb et al. (2009) we propose that student developing identities 'can be made tractable for empirical analysis by documenting students' understandings and valuations of their classroom [or institution] obligations'. (p. 223)

The research design (of the whole project) was based on a theoretical framework of mixed methodology design involving longitudinal survey of outcomes, student biographical interviews and case studies of practice. The data selected for the analysis reported in this article were the following:

- Individual biographical interviews (with linked case-study data) of students over a period of two years: in this case Simar and his friends
- Case studies of universities, in this particular case one large traditional university in a large city in the south of England, City University: these include observations of lectures
- Document analysis of policy and curricular documents relevant for the cases: in this case documents related to the school and the university
- Interviews with participants, in this case head teachers and teachers in Simar's secondary school, and lecturers and professors at City University

In order to develop deeper insights into students' experiences at transition from school to university mathematics education, I have analysed the qualitative data (e.g. interviews and case studies of institutional practices) on the basis of my understanding of 'feedback' and 'self-regulated' learning in mathematics education. The theoretical framework for the analysis is provided in the previous section. More practically, a procedure involving the analysis of themes, similar to that described by Woods (1986), was adopted. In addition, and using open-coding with an ongoing formulation and refinement of categories (Strauss and Corbin 1990), I identified the various aspects of the feedback and independent learning that students

[1] http://www.education.manchester.ac.uk/research/centres/lta/LTAResearch/transmaths/

experienced in the different institutions. The results are anchored in particular in the data taken from interviews with Simar and his friends (focus group), and the observations I made in his secondary school and university (City University).

This included, at one level, identification of the different kinds of feedback, such as:

- Generic feedback in lesson/lecture/tutorial (by teacher/lecturer): feedback given on student contribution to the session in interaction with the teacher/lecturer (e.g. model answers, or on common problems)
- Feedback through questioning
- Individual verbal feedback by teacher/lecturer (e.g. on project work)
- Feedback (by teacher/lecturer) to student written work (e.g. home/coursework)
- Class marking/feedback (e.g. of home/coursework)
- Peer feedback: student feedback to each other (e.g. during discussion/group work) in lesson/tutorial
- Electronic feedback (e.g. by computer assisted tool)
- Marking scheme feedback (e.g. of home/course/classwork)
- Summative feedback (e.g. tests)
- Personal development planning/feedback

At another level, I explore the different kinds of feedback provided by the external 'tools' in the classrooms (e.g. persons, objects, tools), and their use and meanings attributed to them, in particular those 'tools' which seem to be perceived as effective, either by the teacher or by the students, and for independent learning.

Subsequently, these different kinds of feedback (characteristic pedagogic practices) are categorised with respect to the four levels of self-regulated learning identified earlier (e.g. task level, process level/task performance, self-monitoring level/ self-regulation and personal evaluation level/self).

4.4 Findings

4.4.1 The Contexts at Simar's School and University

Simar's secondary school (age 11–18) is a high-achieving comprehensive school in a large city in the south of England. One of the head teachers talked about a 'purposeful learning environment' and the focus on learning as their main key for success.

> What the school has been very successful is establishing a very kind of purposeful learning atmosphere around the school. If you walk around the school, the ethos of the place is fantastic…. the learning ethos that we place on the whole way we run the school, I think is absolutely essential, and Ofsted have said that. You know, it's the ethos and the relationships mean that, in the classroom the focus is on learning. (HTeacher, SK, my italics)

He also emphasised the teacher–pupil relationship as a reason for the school's success, in particular an 'atmosphere of trust and mutual respect'.

> I think it's *one i*t's a long-term established ethos, where brothers and sisters have come through the school; it is a feeling of mutual respect. There isn't a 'them' and 'us' feel, you know the teachers, again referring back to Ofsted, the teacher-student relationships are excellent. If you walk around the corridors you don't hear lots of shouting, you know there is a kind of mutual respect, and little things like, we've invested heavily in the fabric of the school. (HTeacher, SK, my italics)

The school's sixth form handbook (for mathematics) provided guidance and help. Amongst others, students were encouraged to

> Get involved in the lessons. You need to be responsible for your understanding, which means that you need to be brave and have a go at offering your ideas and thoughts during class examples and discussions. Don't worry about getting the answer wrong, your teacher will be able to explain to you any mistakes you may make in your thought process.
>
> Discuss your work with other students. If you can explain how to do something, it means you understand it!
>
> Ask for help when you need it. There can be nothing worse than leaving a lesson and knowing that you are going to struggle with the homework because you did not understand the work in class.'
> (extract from the official Student Handbook SK)

Thus, the school environment can be characterised by the development/establishment of an atmosphere of 'trust and mutual respect' and a 'purposeful learning environment', where students were supported in their learning by discussing their work with teachers, peers and explaining to others. They were also encouraged to seek help from the teacher, for example, in terms of pacing their learning and getting ready for examinations.

The stated aims of *City University* were stated in the handbook as:

> …[City University] seeks to teach its students to the very highest academic standards, drawing in creative and innovative ways of its research… to ensure that students, when they leave us, have the mathematical skills most likely to be useful to them and their employers. In particular, these include fluency and accuracy in elementary calculations, ability to reason clearly, critically and with rigour, both orally and in writing… (p. 1-part 3 handbook)

The intake of City University is varied and can be compared to that of Simar's school: many from ethnic backgrounds who have lived in the large city all their lives; there are also many international students. Extracts from interviews showed that students talked about City University as having status, and students generally felt comfortable in a university close to their family or relatives.

The first-year teaching staff (professors), who seemed to be the 'influentials', were mostly experienced professors, who had been at City University for 20 years or longer. Indeed, the programme Director for mainstream mathematics programmes mentioned that it was important to have a first-year team that 'sings from the same hymn sheet', so that students learn 'from day one' that they are not in school but in a university mathematics department.

There was also a clear distance between students and lecturers, which was also mentioned by Simar:

> …in sixth form it was more personalised kind of…. you was closer to the teacher…you was talking to them -…after school you was chatting to them. You saw them around, like here it's so funny cos when we see the lecturers walking around it's like they're like celebrities…we haven't quite got that personalised you know, thing with them so they're from a distance you know. 'That's Professor…, that's Professor…wow!' You're like wow, they're about. So I suppose it's less personal in a way. (DP5, Simar)

It can be argued that this change of 'environment' which included a change of expectations, from school to university appeared to necessitate students becoming more independent learners.

4.4.2 Findings in Terms of Feedback and Self-Regulated Learning

Simar's school One of the aims at SK school was to help students to 'actively learn' mathematics, and teachers chose particular strategies to help students.

> …we get them to teach the class. I've asked students to prepare a starter activity every lesson, but it has to be a starter activity on a topic we have already done…. So that it's not just regurgitating, and they've gotta come up with some ideas, some questions, thoughts on a topic we've completed and may be extended to a little bit further, or just reinforce it, and they are the people that stand at the front-they set the activity, they then go round and they talk about it. They help the people who are getting stuck and go through the answers themselves, which is really good for them…. Like they choose a topic and explain to the rest of the class, they produce their worksheets and they go round and help them in the class, and they do starters, you're gonna be start from year twelve, year eleven, twelve, thirteen, we get them to do starters in lessons…. And we do a lot of group work in A-level and that's how we really try to make them independent. (Teacher 2, SK)

However, it was evident that this was not an easy task, and often not possible, and that students were often very dependent on their mathematics teachers, in particular when students came from other schools with different practices.

> …feeder schools…groom [students] quite a lot and then they get here because they've got wonderful GCSE results, and of course then they struggle a little bit at A-level, because they, you know, they haven't got those same teachers with them, and they haven't got the same sort of structures there. (Teacher 1, SK)

Thus, teachers at this school also felt a responsibility to 'be there for students', to be 'available', in order to support students in their mathematics learning.

> …as teachers we are really always available for them, and after school and lunch time, anytime we are available and they really like that one. And in the class we try to talk to them, why are we doing maths, why are we doing this topic and they find out about things, and they always when we talk about it, explain to them why you know. (SK, T2)

Furthermore, it was made transparent as to what kinds of strategies teachers would use to assess pupil learning, and in interviews both teachers talked about particular strategies to formatively assess pupil learning. The handbook of SK talks about 'Questioning Techniques', 'Sharing information on how learning is assessed' (e.g. focus on feedback—knowing how to improve); 'Feedback'; and 'Peer and

Self-Assessment' as useful and valuable strategies. Whilst not all strategies may have been evident in the classroom, it nevertheless provided students with some techniques to evaluate their learning. In the handbook, students were asked to make comments for each strategy and with respect to the mathematics syllabus.

Simar recalled his experiences at school and emphasises the 'doing of mathematics'.

> I was in the top set [at lower secondary]…we'd have a class of thirty, with a teacher…he'd like teach on the board, kind of throw out examples, a lot of examples…this is how you do this one, now try it with this one…then I went to the sixth form…took maths there…basically more of the same. (DP5, Simar1, p. 10/11)

Thus, it can be argued that Simar's school provided many support mechanisms for students in terms of learning, but these were not always 'visible' in the classroom. It may be that students needed time to adapt to new learning practices (at upper secondary level); it may also be that the most important support at school/college was the individual support of teachers, because they feel responsible for student learning.

City University The main, and most 'esteemed', pedagogic practices at City University were lectures, usually in halls of up to 300 students. Lecturers would typically produce hand-written notes projected onto a screen and talk students through the content. Students would copy those notes; it seemed most of the time with little or no understanding.

Simar talked quite enthusiastically about one of his lecturers, and what he (and his peers) would expect from a *good lecture*, which resembled very much what he was provided with at school in terms of support (e.g. notes).

S: Geometry: the feedback we got from geometry is, basically he's faultless. He's brilliant, he's excellent; the lecture's engaging, the notes are available- clear notes. You can use the notes for the coursework—

Int: The notes are handwritten?

S: Yeah handwritten notes yeah…. you can see the kind of proofs- he doesn't give too much away, but it's just enough to get you thinking in the coursework's, which is excellent…. [students] need something to take away from the lecture and you know, they're gonna ready at home, they're gonna read it, and they understand it…. And they can go to the tutorial, ask whatever questions and do the questions with confidence, knowing that they've done well like because everything's there, available. They don't need to go anywhere else, and if they do, the tutorial's available or the office hours. So really it's probably one of the best.

Int: So do you think they understand because in the lecture he explains well, or do you think they understand because the, it's so well-prepared and written out?

S: I think mainly it's mostly well-prepared, definitely, and then to accompany that, the lectures are brilliant as well. Yeah it's really, really kind of funny. He catches your interest… (DP5, Simar)

However, there were different types of lectures, and some appeared more helpful than others. One criterion was speed, for writing down and for understanding, with no time for questions.

> First of all the speed is amazing…. Doesn't wait for any student response, really rare to see anyone asking questions, I mean truly there's just no time… (DP5, S2, p. 1)

Interestingly, students differentiated between lectures of '*caffeine mode*' type and those of '*sleeping mode*' type. 'Caffeine mode' lectures were characterised by a lively style and lots of movements (by the lecturer)—'you need three or four cameras', and also by the 'ways [the lecturer] speaks about the subject', 'makes connections'- there is a 'story'. 'Sleeping mode' lectures had little movement ('[he] stands where he stands'), they are 'tedious…same voice, same place…uuuh, uuuh, uuuh, blue pen…'. Students perceive that what they need were 'caffeine mode' type lectures (DP5, p. 17).

In terms of lecture notes as support for student learning, students distinguished between three types of notes. Firstly, there were *'understanding notes'* which are wellprepared and developed, and apparently they helped learning and understanding. Secondly, *'comfort notes'* were those where students did not understand but 'you know you've got the notes' and 'you have gone to the lecture' which in their view may have helped for revision and examination purposes. Thirdly, there were *'motivation notes'* which were those that were provided on the web, before the lecture, and which 'makes you want to come to the lecture…because they are different' (DP5, p. 4).

However, often students did not feel provided with lectures at City University, in terms of learning opportunities or strategies how to learn, and they also compared this to what they were expected to do at school.

> Yeah, trying to catch up with what he's saying and he's just talking…he's just writing, and writing and writing, and all your focusing is, on trying to write everything down… Like I know, at first when I started university I thought lectures are like lessons, you learn in the lecture but you don't really. In the lectures you sort of get an understanding but you have to do more, but with his lectures I don't understand anything in the lecture. I have to go away and do it after… (DP5, Focus group interview S5)

Talking about a particular lecture, students realised that they had to identify and seek their own help strategies, if they wanted to survive in this kind of environment.

S3: 'Yeah cos sometimes you know, you're writing so you're trying to catch up with his speed…if you start thinking of, 'let me try and understand', by the time you're understanding, he's already moving on. So you try to write and understand at the same time, so you can't do both of them at the same time which is a bit difficult….

Int: Have you ever asked any questions during the lecture?

S3: No, no,…the only way I understand to do my work is, when I'm doing my coursework and there are help questions to do your coursework…I think the tutorials and the courseworks are more helpful than, the lecture. The lecture you just get the notes.' (DP5, Focus group interview)

As emphasised by tutors in interviews, there were selected support structures for students, and these were commonly related to the pedagogic approaches. In particular, *coursework and tutorials* were expected to help students to give them feedback about their developing learning. All lectures were supported by tutorials where approximately three members of staff (e.g. PhD students) helped students to do the exercises set by the professor. Students had to submit their answers/coursework within a week (of being set by the professor in lectures), and they could get help with these questions in tutorials. However, these tutorials only seemed to be efficient, if students had thought about the exercises beforehand—which relates to students' study skills (or lack of it) and maturity in terms of their learning.

However, the focus group students said that the coursework was so 'time-consuming' and they had to use all resources available to them to get it done. Indeed, it was felt that there was a 'knowledge gap' between the lecture (notes) and the coursework: for them it was not possible to do the coursework simply reviewing the lecture notes. Moreover, there was no time available for 'understanding' the lecture notes.

> I think it's really time consuming yeah…. [I] remember him saying at the beginning of the term, 'oh you should take about an hour attempting the questions and then come to tutorial', and I'm thinking, once I was up till six in the morning and it was on the very first coursework, so it should have been the easiest one, trying to do it using the lecture notes, using three books, using the Internet, and it makes me think that if I need that much, just to do a coursework, what the hell do I need just to understand it? No seriously,…this is learning to do the coursework, so what about just understanding it in general? If it takes me that much time, there's not enough hours in the day to do that. (DP5, focus group interview)
>
> Because I think like when you've, from the lecture notes and then when you start the coursework, the like knowledge gap, there's a bit of space in it where you have to make the links yourself. And that does take time, and because of the time constraints that we have, like it's on a weekly basis that you get the coursework don't you? (DP5, Simar)

It appeared that there were support structures to help students become independent learners, but these would only work if students 'ask for help'; therefore, these were only effective if students practically sought help. It was mentioned by several professors and tutors that students did 'not know how to study on their own' when they arrived at university.

> It's something that they need to learn fairly quickly at university because we don't really have the resources to hold their hands a great deal, even if we wanted to, and I don't think we want to that much. (DP5, FR1)

Thus, it was clear that whilst students were given some formal support, the university and staff members expected them to 'grow up' and assume responsibility for their learning, on their own most likely. Field notes from an induction lecture showed what a professor regarded as 'poor' and 'good' study skills in mathematics. However, in terms of study skills, and besides the short presentation during the 3-day induction, there is no further explicit support of study skills until the second year. For the second year one professor had identified study skills and communicating mathematics as an area of concern. He thus developed a programme module on 'mathematical writing' where he taught students how to write 'precisely' about mathematics 'as a way of helping them to understand'. This was a relatively innovative programme, and regarded with some scepticism by colleagues, but seemed to be welcome nevertheless.

One of the main messages was that students developed an understanding about different *sources of feedback* (e.g. books, internet, and lecture notes), including asking for the help of peers and the tutor.

> Yeah he says it especially… He wants us to go into books, research it…you have to go away, find out for yourself, struggle with it, and then try and get a bit of help and then try it a bit more and he'll come to you. (DP5, Int 2, Simar)

In order to become independent learners, students identified and sought several support mechanisms: e.g. lecture notes, textbooks, coursework and tutorials, and asking peers for help. However, only in exceptional cases did they go to see the lecturers during office hours—quite a contrast to school conditions where the teacher was the first line of support.

To come to terms with such a change was difficult for students. Simar who was regarded as a 'good student' tried to find his own strategies and rhythm of learning:

> I'm the type of person where I have to look at it, look at it, look at it- stare at it, stare at it, try questions, look at it, look at it and then I'll get it. But other people can just, they can just get it. It depends on their kind of, I'm more analytical, I need a full set of notes. I need to go through it all step-by-step, make summaries on it, stare at it more and then do questions, questions, questions and then I'll get it. (DP5, Simar, p. 10)

Thus, Simar's strategy was to listen and take notes during lectures, not necessarily understand. In discussions he talked about the difficulty of 'listening, learning and writing all at the same time', and his tactic to write things down whilst listening, and later going back to his notes either in the library or at home and reading through properly. He claimed that by writing things down he learnt 'subconsciously'. One can argue, and interviews with lecturers supported it, that at the same time students were also 'enculturated' into a 'different kind of thinking' and a different mathematics which included 'rigour, accurate reading, and thinking'.

4.5 Discussions of Findings

At the general level school teachers appeared to provide students with clear and concise instructions what they wanted them to do (see also school handbook), and the expectation was that students learnt according to teacher guidance, and in the ways shown/taught by their teachers. At school this included attending mathematics classes—this is compulsory, and teachers provided particular notes which students either had to copy or they were provided as copies. There were also clear instructions about the course (e.g. the content, expected learning strategies, the modes of instruction, modes of assessment, etc.) in the department handbook. In this school teachers were expected to create opportunities for pupil mathematics learning (e.g. provide particularly useful strategies and worked examples) and recognise when this was not happening (e.g. through assessment), and perhaps offer further possibilities for learning. Equally, students were expected to learn the mathematics skills to solve the exercises provided and the tasks in the tests/examinations. Thus, there was a common obligation and in an atmosphere of 'trust and mutual respect'.

However, at university the responsibility of the lecturers/professors was to provide the lectures and coursework that could be worked upon at tutorials—in short, to 'deliver the content'. Students were expected to manage the learning processes largely by themselves. According to Simar and his friends, students were expected to

- Listen to lectures ('sit quietly and copy')
- 'Take notes' and 'read the lecture notes'
- Go to tutorials and pass tests
- 'Emulate' what/how professors do mathematics

This implied that students were responsible for their own learning, and developed adaptive help seeking, if they did not understand—quite different to school, where the responsibility for learning mathematics was mainlyon the teachers and the school. As one of the lecturers pointed out that

> we do provide quite a lot of support for students, but they have to accept it, they have to go and ask the questions. (DP1, Lecturer1)

Thus, the rhetoric was that they supported their students (and they genuinely wanted to), but practically students did not know 'how to ask the questions'. Students would need to learn how to 'diagnose themselves' to know what their needs/questions were.

On the basis of video footage of selected lectures and pre- and post-video stimulated recall discussions with lecturers, one could identify meanings that were attached to particular practices. Particular lectures reflected the kinds of things that a 'rigorous mathematician' may need to learn:

- 'Reasoning and proof'-based thinking and practices were expected to be developed through geometry and linear algebra
- 'Procedural fluency' (methods) was seen to be developed through calculus
- Practical and context relatedness was regarded to be developed through statistics

Lecturers claimed that whilst students preferred 'recipes', they did not want skill training or recipe-like learning; they wanted mathematical thinking which included 'rigour and proof', at least that was what they claimed. There was an apparent contradiction in terms of what lecturers said and what they did, according to students' understandings (see above, Simar and his friends understandings of expectations): it appeared that, in practice, lecturers wanted students to 'emulate' what they were doing in lecturers, which in turn was interpreted as a recipe for passing the tests.

It can be argued then that there was a clear institutional expectation in terms of mathematics at City University, made explicit in discussion with lecturers and students, and mediated by particular practices. This expectation was about helping students to become a 'rigorous mathematician' and there was an agreement amongst staff members that students should be 'enculturated' into this from day 1. Becoming a 'rigorous mathematician' included a particular approach to mathematical problem solving and reasoning, 'stringing' together a logical argument and writing in a logical manner, setting out the answer in a particular way. It was about 'precision' (e.g. using the right mathematical notation; setting things out properly) and 'clarity' (e.g. what are the assumptions, following an argument), and being able to explain

Table 4.1 Pedagogic practices, feedback and transformations

Pedagogic practices—feedback from	Source of feedback: people/objects/tools involved	Type of feedback with respect to self-regulated learning	Transformation through
Lectures (including planning for lectures, post-lecture work)	Teacher/lecturer Peers Experience	Self-regulation	Confrontation
Tutorial: Class participation Engagement Volunteering Trying things out	Teacher/lecturer Peers Experience	Self Task performance Task	Shared problem space Reflection
Coursework/homework	Teacher/lecturer Peers and parents Experience Self Texts	Task Self-regulation Task performance	Confrontation Shared problem space
Textbook/curriculum materials	Experience Self Texts	Task Self-regulation Task performance	Shared problem space
Lecture notes	Hand written texts	Task Self-regulation Task performance	Reflection
Seeing tutor/teacher	Tutor	Self Task performance Task	Reflection and Coordination
Seeing peers	Peers	Self Task performance Task Self- regulation	Shared problem space Reflection
Tests/examinations	Teacher/lecturer Peers Experience Self Texts	Task performance Task	Objectification

and reason why they did things in a particular way. However, how students were expected to learn and develop these was not clear.

The tests (and preparation for those) were seen (by the university) as getting students on a similar level, but in practice it was more than that—it gave students the message that at City University a different mathematics was taught and learnt than they were used to at school, and with a different rigour—all this necessitated crucial changes from the side of the students.

In terms of mathematics learning and independent learning, I examined what kinds of practices and feedback may have helped students to become independent learners (see Table 4.1). Research (e.g. Schommer 1990) has established that students' beliefs about learning affect self-regulation by influencing the nature of and interpretation of feedback. Alexander et al. (1991) contend that feedback is information with which a learner can confirm, add to, overwrite, tune, or restructure information in memory, whether that information is domain knowledge, metacognitive knowledge, beliefs about self and tasks, or cognitive tactics and strategies. Table 4.1 shows the different practices (at school and university) that are linked to the different types of feedback and people/objects/tools (as sources of feedback) involved.

Different practices and different 'tools' potentially provide different kinds of feedback to students. In principle, there were two sources of feedback: that which students self-generated by monitoring their engagement with learning tasks and that provided externally (e.g. through tutorials). For example, lectures were provided externally, and at the same time students (at City University) worked with lectures in such ways that gave them feedback in terms of regulation of the tasks, the process level. But perhaps most interestingly lectures also gave them feedback in terms of self-regulation and even personal evaluation (Hattie and Timperley 2007): in lectures students were 'confronted' with unknown mathematics, a different pace, for example, and in a language largely unknown to them. Asking questions was often not appropriate, as the lecture pace was too fast to ask suitable questions—thus, adaptive help strategies were not supported by lectures.

However, according to students' comments, potentially and in terms of self-regulation, the lecture became the most potent catalyst for student independent learning. However, in itself it did not provide appropriate information on any of the three self-regulation layers (regulation of self, regulation of learning process, regulation of processing modes), but students who wanted to understand what was going on had to prepare the lectures, and work on them afterwards, perhaps using other tools, such as textbooks. In terms of 'transformation' as a learning mechanism and its associated processes (see Akkerman and Bakker 2011), there is definitely a *confrontation* process, where students were faced with the very different mathematics-teaching practices of 'lecture', and which forced many into self-regulation strategies.

As another example, particular pedagogic practices, such as coursework and tutorials, went handinhand at City University, where tutorials potentially included class participation, engagement, volunteering, or 'trying things out'. Feedback was given by the tutorial leader, by peers, and perhaps students' own experience with the tasks. Asking questions was encouraged, and so was trying things out. This is likely to have provided students with feedback on the mathematical tasks (how well the task is understood), on the process level (how to perform the mathematical task) and perhaps on the self level (personal evaluation—'I can do it'). The timing of coursework was externally enforced, but according to students, comments also helped them to self-regulate and self-monitor. In terms of learning mechanisms and transformation, coursework and tutorials were not a completely new practice for students (there was coursework at school). However, they provided students with 'shared problem spaces' and time for reflection which in turn transformed their learning styles. In fact, it was remarkable how much students valued 'peer' and 'friendship' group work.

Furthermore, the mathematics (and how it was presented) was likely to have had a *confrontation* aspect. Tests and examinations were likely to have provided transformations through 'objectification' and 'reification' (Wenger 1998)—if they succeeded in the tests, they were expected to have learnt the mathematics.

4.6 Conclusions

From our analyses, it appeared that students' experience at upper secondary school did not seem to prepare them well for university learning of mathematics, whether in terms of mathematics ('all new mathematics') or in terms of learning styles, or indeed in terms of autonomously managing the resources. From the interview data, it was clear that most difficult for students were the changes in teaching styles and the associated styles of learning, besides the difference of the mathematics they were taught at school as compared to the'new mathematics' (including more argumentation and proof) at university. This is supported by the literature, and several studies (e.g. Ozga and Sukhnanden 1998) have claimed students' lack of preparedness for learning in higher education. In this study students encountered difficulties because they lacked an understanding of what learning mathematics at university involved, and many tried their 'old' methods (following the teacher), but without success. The tempo was too fast to 'emulate the mathematics', and copy. At the same time lecturers saw it as their responsibility to 'deliver' the content, rather than helping students 'learn to learn'. This is supported by the literature (e.g. Fallow and Steven 2000) which claims that most academics' concerns focus on content, rather than on learning skills, which involved different pedagogic practices, different types of feedback and different tools.

It can be argued that at the point of transition there were distinct transformations with respect to feedback and self-regulated learning from school to university mathematics education, both in terms of sources and types of feedback, which in turn led to different kinds of mathematics learning and responsibilities for such learning. Further, at City University there was a clear distance between students and lecturers (addressed as Professor and Doctor, and not by first name), whereas at school there appeared to be an atmosphere of 'mutual respect and trust'. Moreover, the school mathematics did not fit what was wanted at City University; students were told to 'forget the mathematics you have learnt in school'. At school students needed to acquire the mathematical skills; at university there seemed to be a different kind of learning needed, including reasoning and a deeper mathematical knowledge. It appeared that City University lecturers wanted to 'enculturate' students into a different way of thinking which included rigour and proof, but there was little support of how students could get to that level, except emulation. The university department routinely provided different kinds of support (tutorials, peer group teaching, etc.), but the support students needed most was not provided: 'learning to learn mathematics' (Wingate 2007). The question remains of how this 'transformation'into a 'rigorous mathematician' was expected to happen; neither was it clear how lecturers wanted students to change their learning practices.

Further, I contend that the transformation at transition from school to university mathematics education has implications for students and their mathematics learning strategies. With the change from school to university, teachers 'transform into' lecturers, lessons into lectures, homework into coursework, textbooks into course materials, tests into examinations, and school mathematics into university mathematics. However, students often receive little support how to manage these transformation, and how to steer and direct their learning processes. There are different 'sources' at school and university level in mathematics education that could be regarded as potentially enriching for and supportive of student learning (e.g. texts and textbooks, see Vollstedt et al., Chap. 2). However, these sources are contextualised according to student perceptions and beliefs and prior knowledge, and thus cannot be seen to be uniformly 'useful'. It is perhaps when students look out for these sources, that is when adaptive self-seeking becomes part of student learning, that these are most effective—it is at this moment that they become most relevant and 'problematic' for students.

In terms of theory, the concept of feedback and self-regulated/independent learning has been useful in investigating the transition from school to university mathematics education, and I have argued that 'transformations' of learning practices have serious consequences for students' success (or failure) at these crossroads of their mathematical development, in particular if students are inappropriately supported, or left on their own, to 'bridge the gap' from one to the other. If Higher Education institutions expect students to 'transform' into 'rigorous mathematicians' and self-regulated learners, they have to provide the necessary 'tools' and instruction to use those.

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Chapter 5
Overcoming the Gap Between University and School Mathematics

The Impact of an Innovative Program in Mathematics Teacher Education at the Justus-Liebig-University in Giessen

Gabriele Kaiser and Nils Buchholtz

5.1 Introduction

Prospective teachers, starting to study in order to become a mathematics teacher, contribute in a very special way to transformation processes, which mathematics as a science is actually undergoing in learning and teaching situations. We use the term *transformation* hereby to describe the adequate modification of mathematical content according to situation, intention, and cognition in educational settings. So, transformation is not an oversimplification or trivialization of content but an adequate adaptation of the learning material to the learner's perspective. On the one hand, in university mathematics courses at the beginning of their studies, starting out from their *learner*'s perspective, prospective teachers experience to be taught mathematical content that clearly differs from school mathematics, not only in range but also in formality and stringency. For this reason university teachers, who impart mathematical content, need to be sensitive to transformation processes in order to impart mathematical content in a comprehensible manner. On the other hand, later in their professional life, also the prospective teachers need to be able to *teach* mathematical content and make it accessible to their students in a didactically well-prepared manner. Therefore, they independently need to undertake didactical transformations of mathematical content in the scope of their everyday preparation of teaching as well. In the 1950s, Klafki formed the idea of a didactical reduction of the complexity of the scientific content that has to be learned in school, a transformation of the content via exemplifying, carving out fundamental ideas, and concentrating on elementary aspects. The choice of adequate curricula

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that is known as *didactical analysis* (Klafki 1958) or *didactical transformation* (Aschersleben 1993), its legitimation and the validation of its educational substance as well as the associated reduction of extent and difficulty in order to take into account the learners' cognitive abilities, is considered to be one of the essential requirements for teachers even prior to the choice of adequate methodical approaches. Traditionally prospective teachers are not provided with enough learning opportunities in order to sufficiently acquire the knowledge and abilities that are taught at university for didactically bridging the gap between the academic mathematics and the student-oriented elementary mathematics. High dropout rates among the first-year students indicate that already within the first semester imparting mathematical content in a comprehensible way only partially succeeds. Currently, at the universities there often is a large gap between the specific training on subject-based content and its realization in terms of teaching methodology. It is especially criticized that the link between the separate parts of teacher training, i.e., mathematics, didactics of mathematics, and pedagogy, is insufficient (for an overview cf. Blömeke 2004).

The teacher training program has been criticized to be lacking in practical relevance already for a century. It was at the beginning of the twentieth century that Felix Klein already described the phenomenon known as *double discontinuity*: "The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching." (Klein 1932, p. 1).

Even today the double discontinuity is in the center of discussion concerning the relation between school and university (see Biermann and Jahnke 2013, this volume; Deiser and Reiss 2013, this volume; Pepin 2013, this volume). In the past couple of years there have been several approaches to subtend the discontinuity by altering the conditions of studying; one of them at the University of Giessen was sponsored by the *Deutsche Telekom Stiftung* (German Telekom foundation). This program focuses strongly on the entrance phase of the mathematics teacher training program for those who are becoming mathematics teachers for the higher track of the German tripartite school system (so-called *Gymnasium*). A central assumption of this program is that the discontinuity between school and university can partly be overcome by an adequate teacher-oriented transformation of the mathematical content in the academic lectures. In the following, we report about this program and the achieved changes.

The German Telekom foundation intends to support projects within the teacher training program that helps avoiding breaks in biographic transition periods such as starting the teacher training program in mathematics. In the course of this, the University of Giessen, in cooperation with the University of Siegen, elaborated a research and development program called *Mathematik Neu Denken* (*Thinking Mathematics in a New Way*) that reorientates the teacher training program (Beutelspacher et al. 2011). This project seeks a long-term improvement in the quality of the education of future mathematics teachers for the higher track schools, and, associated with these changes, improvement of mathematics education at schools is intended. The program was realized from 2005 to 2009, respectively, to 2010 with state funding. The students, who become mathematics teachers, were introduced to a new combination of courses that separated them from those who are aiming for a diploma in mathematics during their first two semesters. The subproject of the University of Giessen focuses on the rearrangement of the lecture on linear algebra/analytical geometry to an introductory course that is adapted to school requirements emphasizing the relation to mathematics subject matters and that relies on the vividness and the primacy of geometry. On the one hand, this approach is standing in the tradition of Felix Klein who laid an emphasis on the need of the sensualization of ideal constructs by the use of drawings and models (Klein 1939). On the other hand, it is trying to realize the idea of a didactical transformation of the mathematical content by consciously taking the student teachers' existing preknowledge up and strongly connecting it to extramathematical applications of mathematics with regard to the student teachers' future professional life. Furthermore, laying an emphasis on this kind of transformation offers an opportunity to learn an application-oriented way of teaching.

This chapter is based on data from the evaluation study TEDS-Telekom. The main purpose of this study, which was funded by the German Telekom foundation as well, is the evaluation of the funded project *Thinking Mathematics in a New Way*. In TEDS-Telekom these innovative approaches were evaluated from an external point of view with regard to the impact that was achieved in the area of the development of mathematical, didactical, and pedagogical competences of the students, together with the development of the corresponding beliefs. Among others, it has been drawn on approaches of the international comparative study "Teacher Education and Development Study—Learning to Teach Mathematics" (TEDS-M; Blömeke et al. 2010a, b; Blömeke and Delaney 2012). This IEA study for the efficiency of the education of mathematics teachers presents an external reference framework that allows for specific statements about the innovation potential of the pilot project, funded by the German Telekom foundation, in terms of strengths and weaknesses of the teacher education at the university. Control groups at other universities, which agreed to evaluate their teacher training program too, set another external benchmark. All in all, first-year student cohorts at five universities (Giessen, Siegen, Bielefeld, Essen, and Paderborn) were analyzed. For reasons of confidentiality, the results of the universities that additionally took part in the study will be anonymously communicated throughout the chapter.

5.2 Theoretical Framework and Study Design of the TEDS-Telekom Study

The presented study attempts to answer the question of how far the innovative efforts at the University of Giessen actually influenced the development of the local students' competences. The TEDS-Telekom study is restricted to the examination of influence of institutional educational factors on the individual development of competences with the help of quantitatively oriented written tests. For that reason, the approach of the study, which is mainly able to capture the development of the students' professional competence, is enriched. Qualitatively oriented and problem-focused interviews with prospective teachers of the involved universities were used in order to have an additional perspective. In doing so, the qualitative approach enables us to gain insight into the impact of didactical concepts of the universities and different teaching and learning conditions on students and on their individual internal perception and acceptance of particular components of the university teacher education. The so-called mixed-method design—a qualitative–quantitative mixed study design (cf. Kelle 2008) that has been chosen to be applied—is supposed to compensate "blind spots" in the methods of a single research paradigm and customize a broader range of results.

The term professional competence has been conceptualized in various ways within the scope of empirical studies, such as the international studies MT21—"Mathematics Teaching in the twenty-first Century" (Blömeke et al. 2008) and TEDS-M (Blömeke et al. 2010a, b; Blömeke and Delaney 2012), the German COACTIV study—"Professionswissen von Lehrkräften, kognitiv aktivierender Mathematikunterricht und die Entwicklung mathematischer Kompetenz" (Kunter et al. 2011) or the Michigan LMT-Project "Learning Mathematics for Teaching" (Hill et al. 2008). The evaluation study presented below is based on the conceptualization of professional competence of prospective mathematics teachers as a multidimensional construct, like it has been developed in general by Weinert (1999) and Bromme (1992, 1997) and which forms the theoretical basis of the TEDS-M study too. According to this approach, professional competence includes subject-related and interdisciplinary cognitive dispositions of performance, as well as affective-motivational beliefs as part of a teacher's personality. In addition to that, in his topology of teacher's professional knowledge, Bromme (1992, 1997) underlines the impact of the teachers' personality on the professional competence by describing the knowledge on the philosophy of the school subject and its contained perspective of valuating. Further, elaborately discussed questions in the field of scientific research on the assessment of teachers' competences are about the integration of acting into models of professional competence of teachers and the measurability of competence for action.

The evaluation study TEDS-Telekom is restricted to the analysis of the cognitive components of professional competence (professional knowledge of teachers) and focuses in the area of personality features on *beliefs* concerning the subject and the teaching and learning of the respective subject. The study owes its restriction to the fact that students at the beginning of their studies cannot gain much action

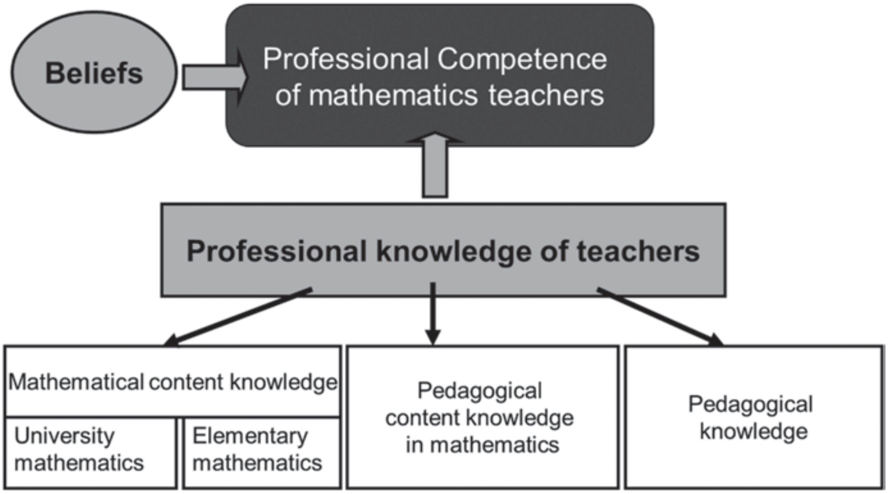


Fig. 5.1 Model about professional competence in the evaluation study TEDS-Telekom

competence, because up to that time most of them did not have the opportunity to get professional experience and, regarding the impact of personality traits on the characteristics of professional competence, it is widely believed that these influences tend to be less obvious at the beginning of the university study time than later in professional practice. For this reason, the evaluation study focuses on the central aspects of the knowledge in mathematics and didactics of mathematics of the first two academic years for future mathematics teachers for lower and upper secondary levels, including the related beliefs referring to the fundamental aspects of professional knowledge of teachers as outlined by Shulman (1986) and Bromme (1992, 1997) (see Fig. 5.1).

For the evaluation study the dimensions of professional competence have been subdivided and operationalized as follows:

- Academic mathematical knowledge in the area calculus and linear algebra/analytic geometry
- Elementary mathematics from an advanced standpoint
- Pedagogical content knowledge of mathematics or didactics of mathematics referring to upper secondary level
- Pedagogical knowledge focusing on action-related aspects, such as the structuring of teaching, motivation, classroom management, assessment and dealing with heterogeneity
- Beliefs on mathematics as a science and on learning and teaching of mathematics

In this connection it must be noted that elementary mathematics from an advanced standpoint is a subarea of mathematics, but, simultaneously, it also creates basic elements for an interlocking of academic mathematical knowledge and didactics of mathematics in the meaning of the approaches of Klein (1932) which have been developed further by Kirsch (1987). These subdomains then are differentiated further with respect to cognitive aspects, by evaluating the respective declarative knowledge and the repertoire of pedagogical acting. In order to identify the different qualities of cognitive requirements to be met by the prospective teachers for solving the test items, Bloom's taxonomy of cognitive processes, as revised and extended by Anderson and Krathwohl (2001), was applied in connection with the test items. The focus was on three dimensions of cognitive processes: memorizing, understanding/analyzing, and creating (see Blömeke et al. 2011).

These mathematical and mathematics-didactical-related items of the study have successfully been developed in accordance with two interrelated content-based frameworks of reference: on the one hand, the largely canonical contents of the lectures at the beginning of the university study courses for mathematics teacher education for the upper secondary level and, on the other hand, the respective recommendations for the structure and design of the study from the *Standards for mathematics teacher education* as suggested by the German Society of Mathematics (DMV), the German Society for Didactics of Mathematics (GDM) and the Union for the Advancement of Mathematics and Science Teaching (MNU) (DMV, GDM & MNU 2008) by considering the central ideas and approaches of the innovative concept of mathematics teacher education of the universities of Giessen and Siegen.

As far as the items are not taken from the TEDS-M study, mathematical and mathematics didactical items which had been created for TEDS-Telekom were developed further by the mathematics didactical working group at the University of Hamburg guided by Gabriele Kaiser in cooperation with Hans-Dieter Rinkens from the University of Paderborn and then refereed in workshops by further experts of mathematics didactics from universities which are also participating in the study. Then, based on that expertise, the items were revised again. The items related to pedagogy have been developed by the working group Systematic Didactics and Instructional Research at Humboldt University of Berlin directed by Sigrid Blömeke in cooperation with Johannes König.

The test also contained items from the TEDS-M study so that later the results of the evaluation study can be evaluated and interpreted with reference to an external standard. Like the initial development of items, the TEDS-M items were selected with respect to the above-described content-based frameworks of reference (canonical contents of the respective university courses and DMV–GDM–MNU suggestions).

To illustrate the items used in the study, we describe in the following one of the TEDS-M 2008 items that has been used in the TEDS-Telekom study with the respective solution frequencies. But attention should be given to the fact that performance on the level of individual items can vary due to chance and thus should not be over-interpreted.

The task US25 (see Fig. 5.2) comes from the content area of the academic mathematical knowledge of linear algebra and analytic geometry and requires basic knowledge of the geometry of the plane and the space. The amount of points that satisfies the equation $3x = 6$ in the plane is a straight line, but in space it is a plane.

US25) We know that there is only one point on the number line that satisfies the equation $3x = 6$, namely $x = 2$.

Let us now transfer the equation to a plane with coordinates x and y, and then to space, with coordinates x, y and z. What is the set of points that satisfy the equation there?

Tick one box per row.

		A point	**A straight line**	**A plane**	**Else**
A)	The solution of $3x = 6$ in the plane	☐	☐	☐	☐
B)	The solution of $3x = 6$ in space	☐	☐	☐	☐

Fig. 5.2 TEDS-M 2008-item

Seventy-two percent of the German prospective teachers in TEDS-M 2008 were able to solve item A correctly; for item B of the figure, the proportion is still 68 %. The student teachers of the University of Giessen solve both items with 75 % at approximately the same height, as well as the comparative teacher training group (71.4 % for item A and 61.9 % for item B). For being able to make a statement about the achievements and the achievement development of first-year-student cohorts, the TEDS-Telekom study is designed as a real longitudinal study. The evaluation of the students by means of a 90-minute paper-and-pencil test took place at the beginning of the first semester (December 2008), the end of the second semester (July 2009), and at the end of the fourth semester (July 2010). Central assumptions for the evaluation of the test results were measurable success of achievements from the first to the third point of measurement, thus from the beginning of the first semester until the end of the fourth semester, as well as that the degree of success of achievement varies depending on the level of achievement at the beginning, the students' learning preconditions, and the learning opportunities provided by the universities—thus the innovative potential of the study programs (integration of domains of knowledge, extent of learning opportunities, etc.). Meanwhile, the results of the longitudinal measurements from all three evaluations are available (Buchholtz and Kaiser 2013, in print).

Going beyond the borders of the project *Thinking Mathematics in a New Way*, in order to investigate the influence of various aspects of institutional conditions and aspects of didactics of higher education on the individual acquisition of competence from a different, more qualitatively oriented point of view, additional problem-centered guided interviews according to Witzel (1982) were carried out with 19 prospective teachers from all participating universities. Within the scope of these interviews, the prospective teachers were asked about their perceptions and their estimations about learning opportunities and aspects of didactics of higher education in connection with their studies. Among these 19 prospective teachers, who participated voluntarily and were chosen randomly, there were four students from the Justus-Liebig-University of Giessen. The interviews were conducted by using a guideline which contains the following aspects of perception and estimation of university teaching within the introductory phase of their experienced university studies:

- Integration of visualization, examples and example-bound argumentations, and real-world applications in mathematical lectures
- Integration of elementary mathematics from an advanced standpoint in mathematical lectures
- Interweaving of mathematical and mathematics didactical content in university courses
- Beliefs about teaching and learning of mathematics

Currently, the interviews are systematically evaluated by means of the method of qualitative content analysis of Mayring (2000) so that now only preliminary results are available. A mixed-method design has been chosen due to the fact that empirical studies dealing with the efficiency of teacher education are mostly either grounded in the qualitative or the quantitative paradigm (as example see Blömeke et al. 2010a, b; Eilerts 2009; Schwarz 2013). The decisive advantage of a combination of qualitative and quantitative methods is that in this way characteristic weaknesses of one tradition of methods can be balanced by the strengths of others (Tashakkori and Teddlie 2003, p. 16). Johnson and Turner (2003, p. 299) even call it a fundamental principle "that has complementary strengths and non-overlapping weaknesses."

5.3 Development of Cognitive Dispositions of Achievement

Over all three measurement points, after the sample had been revised, altogether 128 students participated in the TEDS-Telekom study. Thirty-two students of them are from the Justus-Liebig-University. Because the subsamples of the control universities decreased over the study time of four semesters, and for particular methodical reasons and reasons of confidentiality, the results of the universities of Bielefeld, Paderborn, and Essen will not be compared to those of the University of Giessen

Table 5.1 Comparison of average *Abitur* grades; the grades can differ from 1.0 (best grade) to 4.0 (worst grade)

Group	*Abitur* grade M1	*Abitur* grade M2	*Abitur* grade M3	Deviation
University of Giessen	2.20	2.20	2.15	−0.05
Control group teaching	2.37	2.26	2.24	−0.13
Control group nonteaching	2.21	2.05	2.01	−0.20

on the level of universities. Instead of comparing small samples and heterogeneous groups, the groups of the students studying to become teachers and students not studying for the teaching profession will be treated separately, whereat these groups will be aggregated as “control groups,” or “reference groups,” containing students from all three universities: Bielefeld, Paderborn, and Essen. For this reason the composition of the control groups is extending across locations and consists of 39 prospective teachers (*Control Group Teaching*) and 30 nonteacher students (*Control Group Non-Teaching*). The results of the group of prospective teachers from the University of Siegen, in which also a subproject of the German Telekom foundation was funded, are not mentioned.

To give an overview about the individual preconditions of the students participating in the study, at each point of measurement the average grade of the general qualification for university entrance (the so-called *Abitur)* of the students still remaining in the sample was compared to the average grade at the beginning. At the beginning, at measurement point M1 the groups did nearly not show any differences in comparison, but at measurement point M3 the control group of nonteacher students had significantly better average grades of the general qualification for university entrance (the so-called *Abitur* grades) than all groups of prospective teachers. This points to possible selection processes, e.g., dropout of weaker students with lower grades in their *Abitur* grades during the first four semesters. However, it cannot be excluded that students with lower grades could not be reached anymore by the tests.

Likewise, the two universities promoted by the program of the German Telekom Foundation show a similar characteristic improvement of the average *Abitur* grade. However, the grade of improvement of the University of Giessen is the lowest. This suggests that the introductory selection procedure is less noticeable in this group (to compare see Table 5.1).

Further, data were collected about the kind of courses that were attended by students at school in the upper secondary level. For this, the answer options *basic course* (courses at basic mathematical level), *advanced course* (courses at higher mathematical level), and *neither basic nor advanced course* (optional in some federal states of Germany) were given. The comparison of the results of the samples of M1 and M3 is shown in Table 5.2. First, it shows that at the control universities the percentage of students—prospective teachers and nonteacher students as well—who had attended advanced mathematics courses during their schooltime increases

Table 5.2 Comparison of school-related preconditions

M	Kind of course	University of Giessen (%)	Control group teaching (%)	Control group nonteaching (%)
1	Advanced course	71.9	68.5	89.3
	Basic course	28.1	29.2	5.4
	Neither basic nor advanced course	0	2.2	5.4
3	Advanced course	71.9	76.9	96.7
	Basic course	28.1	20.5	3.3
	Neither basic nor advanced course	0	2.6	0

from measurement point 1 (M1) to measurement point 3 (M3), for which there might be a connection with effects from the introductory selection. However, at the University of Giessen the percentage remains the same. These results might indicate that the University of Giessen was more successful than the control universities in keeping students with less good preconditions over a longer period of time, at a minimum during the first four semesters.

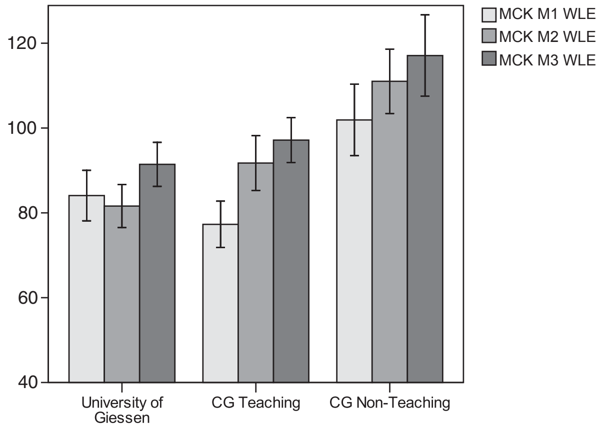
The collected data from the tests have been scaled by IRT models (see Rost 2004), for which scales had to be distinguished according to the domains of knowledge as described in Sect. 5.2. For the estimation and presentation of individual abilities, *weighted likelihood estimates* (WLEs, Warm 1989) were applied. Each scaling has been executed by using the software ConQuest (see Wu and Adams 2007), a software for fitting item response and latent regression models. As the second and third collection of data were intended to measure development, it was necessary to "anchor" the various kinds of tests at all three measurement points in all domains of knowledge with a respective number of items. This means that the same items had to be calculated over all kinds of tests and measurement points by referring to the same parameters of difficulty. As it is very difficult to equalize item parameters in ConQuest, an approach has been chosen, which also Hartig and Kühnbach (2006) reverted to. For an estimation of the item difficulties, first a one-dimensional scaling for all items with so-called "virtual persons" has been carried out.

Then, for an estimation of the person's abilities the item difficulties of the anchor items of measurement point 1 (M1) have been imported into a three-dimensional scaling of all items, for which the single measurement points are indicating the three latent dimensions. For this, the difficulty parameters of the anchor items from the scaling with virtual persons have been taken and been fixed for all three measurement points. Thus the anchor items show the same difficulty at each measurement point. Then, based on this model, the person parameters have been estimated.

The scales' reliability in the three-dimensional model ranges from sufficient to good at all three measurement points (scale reliability from 0.63 to 0.83).

In the following, the ability parameters are presented graphically and subdivided according to the different domains of knowledge that were tested. The WLEs of all

Fig. 5.3 Ability parameters (WLE) in the area content knowledge of calculus and linear algebra/analytic geometry



measurement points were transformed to an average value of M = 100 and a standard deviation of SD = 20. Finally, all results will be interpreted separately according to the respective domains.

5.4 Development of Performance in Cognitive Domains

5.4.1 Academic Mathematical Content Knowledge (Calculus & Linear Algebra/Analytic Geometry) (Fig. 5.3)

At all three measurement points, significant differences between the groups of the prospective teachers and the control group of nonteacher students exist with the group of the nonteacher students performing significantly better. However, altogether the achievements of all first semester prospective teachers increase significantly too. Therefore, the expectations of observable learning success have clearly been met by this study. After four semesters (M3) the prospective teacher groups have reached a similar level as the means of the two groups do not differ significantly, although they developed differently.

The stagnation between the first two measurement points can be balanced by the University of Giessen up to the third measurement point. The group reached the same level of ability as the control group "Teaching" by means of higher learning success from the second to the third testing, while the control group "Teaching" showed strong success from the first to the second testing. The stagnation of the University of Giessen between the first and second measurement point can easily be explained. According to their study program it was not planned that the prospective teachers of the sample attend the calculus courses. Instead of that they attended courses on linear algebra and analytic geometry. Therefore, no increase in the area

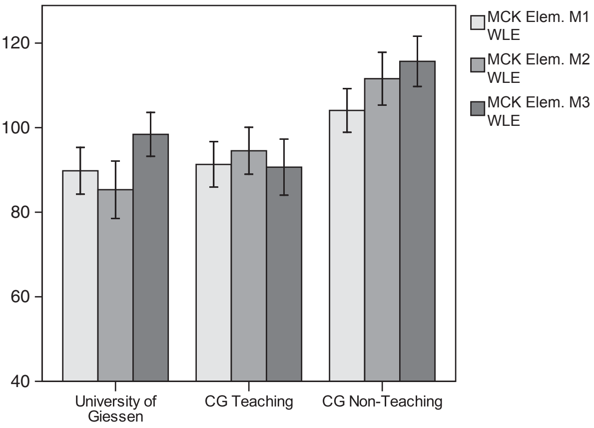


Fig. 5.4 Ability parameters (WLE) in the area content knowledge elementary mathematics from an advanced standpoint

of content knowledge could be expected, because many items of the tests refer to calculus which at the University of Giessen is part of the curriculum for the third and fourth semester. The high success of the Giessen group in the last testing can be explained by this different organization of the study structure and the timing of the offered learning opportunities.

5.4.2 Elementary Mathematical Content Knowledge (Elementary Mathematics from an Advanced Standpoint) (Fig. 5.4)

Best performances in the area of elementary mathematics from an advanced standpoint at the third measurement point are achieved again by the control group of the students not aiming for teaching profession (nonteacher students). In contrast to that, the group of the prospective teachers performs significantly lower. The control group of prospective teachers shows, at no point of measurement, any significant change of knowledge in the area of elementary mathematics, whereas the control group of nonteacher students improved their performance from an average level at measurement point M1 to a significantly higher level at measurement point M3. Among the prospective teachers, only the group of students from the University of Giessen achieved significantly better results at the third measurement point compared to that before, so that the University of Giessen holds the leading position among all participating groups of the prospective teachers.

At first sight, the outstanding performance of the nonteacher students in the area of elementary mathematics from an advanced standpoint is surprising, as especially one would assume the elementary mathematics from an advanced standpoint to be a domain where prospective teachers carve out their leading role in mathematics. A possible explanation might be that nonteacher students do not have problems with the school-related university mathematics on which the questions are based on. Further, generally this cohort of students does not study another academic subject, so that they can spend more hours per semester studying mathematics than prospective teachers. Compared to all subsample groups, the control group of nonteacher students (Non-Teaching Group) contains the biggest share of students who had attended advanced mathematics courses at school. Another reason may be the test structure and the test itself. The respective test differs from the tests for the other domains of knowledge insofar as the major part consists of TEDS-M 2008 items that aim at testing mathematical school knowledge. Only a few new and more difficult items are added in order to test an increase of knowledge. Altogether, the items of TEDS-M 2008 are structured in a slightly different way, sometimes more narrow in their questioning or stronger oriented to declarative knowledge. For this reason, the result suggests that these items tend to test other facets of knowledge which are structured differently compared to the other test parts, possibly more reproductive abilities. Anyhow, this predominant advantage of the group of students studying for nonteaching professions especially in this domain and the stagnation of the control group of prospective teachers compared to the students of the University of Giessen at all measurement points is a surprising and quite unexpected result.

The increase of performance of students from the University of Giessen is surprising because of the time it happens, namely between the second and third testing. As most of the activities in Giessen, which are supported by the German Telekom foundation, concentrate on the area of linear algebra, one would have expected an increase of performance in the area of elementary mathematics from an advanced standpoint between the first and second measurement point. But at that time the results stagnated. This result might be explained as a delayed effect of the support programs or caused by specific learning opportunities determined by the characteristics of the curricular content. In the third and fourth semester the students attend calculus courses and the items from the domain of knowledge of elementary mathematics from an advanced standpoint are more algebraic and algorithm oriented, which runs quite contrary to the more visual-based orientation of the mathematical content-based courses of the first two semesters at the University of Giessen, so that the numeric orientation of calculus might play a key role for understanding elementary mathematics.

5.4.3 Pedagogical Content Knowledge (Fig. 5.5)

The distinct pedagogical content knowledge of nonteacher students at M1 has been developed further by this group so that also until the third measurement point, they showed the best results in this field of knowledge. This indicates that the items which are mainly subject-matter based could obviously be solved well using mathematical-content knowledge. As expected, from all first-year students the students from the nonteaching group produced the lowest increase of performance. The prospective teacher groups showed generally a higher, sometimes even a clearly higher increase of performance. This result also refers to the fact that pedagogical content knowledge is an independent domain of knowledge.

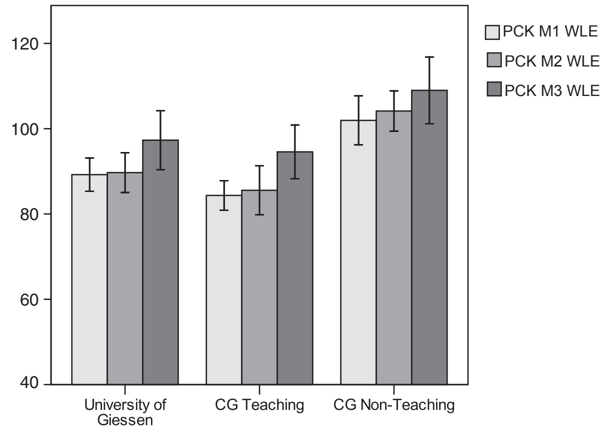


Fig. 5.5 Ability parameters (WLE) in the area pedagogical content knowledge

Students of the control group of prospective teachers and of the group of the University of Giessen showed a significant development of performance in pedagogical content knowledge from the second to the third measurement point. According to the study, regulations of the University of Giessen students must attend one pedagogical content course during the first four semesters, but not at a fixed time. This result indicates that the majority of the prospective teachers tend to choose a later point of time for attending these courses (Fig. 5.5).

5.5 Learning Opportunities

To get an overview about the students' learning opportunities, it is necessary to examine the specific university curricula for teacher education, concerning type and scope of courses of the participating universities (intended curriculum) among others. These data then need to be crosschecked with data about the students' perception about the recommended courses (implemented curriculum) (for differentiation of intended and implemented curriculum see McDonnell 1995). In TEDS-Telekom the analysis of the intended curriculum was based on the official study and examination regulations for upper secondary-level teacher education valid at that time at the participating universities. Besides that, module manuals and study plans taken from the internet which are more or less regulating the students' course of study were considered. The area of content knowledge is determined by a canonical uniformity of the study structure of the first four semesters (according to traditional teacher education terminology called "basic studies" or Grundstudium). During the first four semesters, students have to attend the classic introductory courses, which in the first two semesters traditionally are courses on calculus, and in the third and fourth semester on linear algebra. In contrast to the other universities which are participating in the study, at the University of Giessen the introductory courses are

Table 5.3 Number of students having attended the listed courses

Group	University of Giessen				n	CG-Teaching				n	CG-Non-Teaching				n
Semester	1	2	3	4		1	2	3	4		1	2	3	4	
course of MCK															
Linear Algebra 1	32				32	27	5	1		33	18	3	2		23
Linear Algebra 2		32			32		29	3	1	33		18		2	20
Calculus 1			30		30	28	2	4	1	35	23	2	4		29
Calculus 2				22	22		28	2		30		21	1	4	26
Calculus 3								10		10			11	2	13
Proseminar								1	1	2				1	1
Number theory							2	4	4	10			4		4
Numerics				1	1				1	1			5	1	6
Stochastics							1	11	10	22			7	3	10
Algebra								6	1	7			4	2	6
Geometry				9	9			5	2	7				2	2
Computers	6	2	2	4	14				8	8		1		2	3
else						1		3	6	10	2	1	8	6	17
course of PCK															
Didactics for low. Secondary		1	3		4	1	1	11	1	14	1				1
Didactics of Calculus								1	2	3					
Didactics of Linear Algebra	1	1	21		23				1	1					
Didactics of Geometry		2	21		23				1	1					
Didactics of Algebra			3		3			1	3	4					
Didactics of Arithmetic								1	2	3					

arranged in the reverse order by giving priority to linear algebra/analytic geometry. Some universities suggest students to attend calculus and linear algebra courses simultaneously in one and the same semester. These regulations for the first four semesters are completed by a compulsory mathematical elementary or an advanced seminar, or at some universities by a course on the usage of computers in mathematics, and in some cases even by more advanced courses on calculus such as differential equations, stochastics, numerics, number theory, or algebra. The amount of pedagogical content courses is relatively small, generally only up to maximal 20 % of mathematics courses. In the TEDS-Telekom study, data on the attendance of these courses referring to study regulations were sampled. As the range of courses concerning mathematical content and didactical content differs on a large scale between the participating universities and the instrument could not be changed for local deviance, the study had to restrict to a common nomenclature of elementary and advanced courses, which differs from the local nomenclature of the respective university. The suggested courses were presented to the students in a list in which they had to set the corresponding crosses, but the students also had the opportunity to note further courses they had attended. Table 5.3 gives an overview on the answers of the students, indicating courses which had been attended by the students.

Compared to the University of Giessen, the courses attended by students of the control universities appear broadly dispersed. Of course, this is caused by the fact that the control group consists of groups from three universities. However, as a part of the students of the control group attend courses on calculus and linear algebra in the same semester, they tend to take offered advanced courses already in the third and fourth semester, while students of the University of Giessen are limited to attend the courses in the strictly predetermined order. In addition, it is striking that over the whole time of four semesters, content courses on computer-application training are multiply mentioned by the students, which does not occur with students from the other universities. Reasons for high attendance of the geometry course in the fourth semester at the University of Giessen, which is not part of the studies' curriculum, lie probably in the overall high relevance of geometry at the University of Giessen.

Starting out from this background, the picture of the knowledge acquirement in various areas of knowledge gives a totally different impression. Concerning the acquisition of knowledge within the framework of the introductory courses, it obviously does not make any difference whether students take introductory courses simultaneously or one after another, because at the end of the fourth semester the prospective teachers have reached nearly the same ability level. Only the process of the development of content knowledge is different, and therefore no advantages or disadvantages of the study concept at the University of Giessen become evident. Effects might occur more strongly in the area of a working overload of the students and their perception of it.

Likewise, in the area of pedagogical content knowledge the groups differ significantly. Generally, students in Giessen attend during the second and third semester one or two pedagogical content courses which are focusing on geometry and linear algebra. Courses on pedagogical content knowledge were attended only sporadically at the control universities, but for most of the prospective teachers a kind of an introductory and overview-providing lecture exists. As expected, courses on pedagogical content knowledge were nearly not at all chosen by nonteacher students.

By looking at the increase of performance, the same effects from attending the respective course become evident. The performance of prospective teachers increases significantly from the second to the third measurement point. A differentiation of effects from attending introductory courses on pedagogical content knowledge and from specialized courses cannot be detected on the testing level.

5.6 First Results on the Evaluation of the Influence of Institutional Conditions

In the interviews the prospective teachers were asked to make estimations about the general institutional conditions. They should describe from which kind of learning opportunities they had gained most during their study time, and in addition which issues need to be addressed in order to ensure a more comprehension-oriented university teaching. In the following, only the aspect *integration of visualization in mathematical lectures* tackled in the guided interviews will be discussed, because the interviews are actually still being evaluated further, but this aspect is deeply related to the transformation of mathematics within academic teaching as Gustav Grüner formulated in his work on didactical transformation in general the importance of analogies, metaphors, and examples for illustrating a scientific statement in order to concretize it (Grüner 1967). In addition, especially in the work of Felix Klein we find the important role of models for visualizing abstract mathematical concepts (cf. Klein 1939). Below, we refer to the statements of the four prospective teachers from the University of Giessen, who—anonymously—give an insight into the impact of the project *Thinking Mathematics in a New Way*, which aims at a new orientation of teacher education for mathematics teachers for the upper secondary level at the University of Giessen. Within the statements, the students describe their involvement in the teacher training program and their perceptions about the mathematical courses on a very personal level. Biographical aspects as well as the individual development of professional knowledge for teaching form the background for these statements. (For a related approach to analyze professional identity of a student teacher going through teacher education and building up professional experience on a narrative way, see Grevholm 2013, this volume.)

The prospective teachers have repeatedly perceived the discontinuity described at the beginning of this chapter as a problem they are also struggling with during their studies. In Giessen, most of the statements refer to the calculus course which is taken only in the third semester and has not become part of the restructuring measures of mathematics teacher education: that means the calculus course is taught in the traditional abstract way attended by the prospective teachers and the mathematics students not aiming for the teaching profession.

> If now I‘m thinking back on school, [in calculus] we had a bit the evaluation of functions, a bit derivations. And actually in calculus, the time at school, now I do not remember anymore, but I think there were hardly any proofs. That is precisely the opposite at the university, there are definitions, proofs … And calculus in the university context actually consists only of proofs and definitions. […] No, there were clearly dropouts. Due to calculus there were clearly dropouts, but I can imagine, if we have had that in the first semester, the dropout rate would have been even higher. (Prospective teacher, female, 26)

This shows especially that a higher grade of abstraction causes obstacles for understanding and that what has been learned will be forgotten immediately, which shows that pure transmission of factual knowledge is not sustainable.

> The terminology of calculus remained abstract, 75 % of it I would even not know what to do with, what does it mean, for what I am actually doing that … That was just stupefying learned by heart and simply written down, that what the professor wanted to hear. (Prospective teacher, male, 21)

> It extends, the learning process extends. At home I sit down and work upon it by exemplification, so that I can understand it by myself. And so it takes much longer. If that would have been contributed by the lecture, one would not need to work on exemplification afterwards on one’s own. (Prospective teacher, female, 26)

The prospective teachers depend on an illustrative way of teaching mathematical content, which much too often cannot be realized in university courses which are attended by prospective teachers together with mathematics students not aiming for teaching profession.

> At the end actually, because there are so many prospective teachers, I personally think it would be wonderful if it would really be possible to separate Bachelor [(i.e. non-teacher) students] and prospective teachers, completely, and not only for selected courses. And the Bachelor [students] do not need these references of reality, the exemplification as strongly as prospective teachers need it. I think, because, the Bachelor [students] do not teach that later. (Prospective teacher, female, 26)

Obviously, the Justus-Liebig-University was successful with a course on Linear Algebra for students aiming for the teaching profession and succeeded to overcome, at least partly, comprehension problems through embedding exemplification into the mathematical content courses.

> In algebra, concerning vector spaces, it was beautifully made clear, that a vector is not just an arrow which is just drawn, but that it has a direction and which properties it has. Because, one has quasi developed an imagination of it, how it looks like. And therefore later it is good for the students, one can better explain it. (Prospective teacher, male, 21)

The students describe that indeed the exemplification is given a key role for their own understanding. The idea of transformation of mathematics in university mathematics courses via analogies, metaphors, and examples for illustrating and visualizing is even recognized in its exemplary function for the later demands of the teaching profession.

> I now also try to apply the exemplification in the private tutoring center, where I work. I try to put this also into the foreground. Because the experience, the short experience, that I could make now, has shown that the more exemplifying the beginning is, the more the pupils are willing to get to work on theory. (Prospective teacher, male, 21)

The teaching of mathematical content in an understanding-oriented way is on the one hand fostering learning, but on the other hand very time-consuming because the pace of learning may be reduced. But students do not consider that as impairing.

> Yes, exemplification I think is quite important, in order to have reference, so that one knows what one is doing there. If you have an image right in front of your eyes, then the theory remains more rooted in your head, later it is like this at school. And yes, then it is okay for me, if then in only one week lecture can be worked on only the half, but one knows: the students do understand it now. (Prospective teacher, female, 20)

> Yes, I just say, I personally think it makes more sense to work on less content, but to understand it, instead of working on more content of which one does not know anything at the end after having struggled through. (Prospective teacher, male, 20)

5.7 Conclusions and Further Prospects

The described analyses about the development of the average *Abitur* grades and the kind of the attended mathematics course at school of the students of each cohort demonstrate that at the University of Giessen during the first four semesters obviously no strong selection is prevailing. With the programs supported by the German Telekom foundation, the University of Giessen succeeds in keeping the grade of selectivity low, whereas a high grade of selectivity is characteristic for mathematics teacher education at the beginning of their study, which means that they succeed to keep students from the lower-achieving sector in their studies. Nevertheless, prospective teachers of the University of Giessen achieved comparably high results in the areas of calculus, linear algebra and analytic geometry, elementary mathematics, and pedagogical content knowledge. Concerning the learning increase, the students' performance stagnates between the first and second measurement point in the domains of content knowledge and pedagogical content knowledge, due to the structure of the curriculum of the University of Giessen. At the third measurement point, the students of the University of Giessen showed a remarkably high increase in the area of academic mathematical content knowledge, elementary mathematical, and pedagogical content knowledge.

We assume, and not at last based on the results of the interviews, that the special lectures for prospective teachers in Giessen separated from mathematics students not aiming for the teaching profession have a strong influence on the knowledge development of prospective teachers. One reason might be that a slower pacing and the empathic, exemplifying way of teaching applied in the courses has developed a "teacher-specific self-efficacy," which on the one hand might explain the high grade of identification with the project and on the other hand might have a positive influence on the acquisition of knowledge. Equally from the students' perspective it has been affirmed that this style of teaching in the mathematics courses has a motivating effect. Therefore, the students do not regard the sometimes slower pace of learning as an obstacle, but on the contrary, as a strengthening of their efforts of learning. Nevertheless, in the area of mathematical knowledge they do not show significant performance deficits. The results of this study give reason to ponder whether the improvement of teacher training can be achieved by restructuring the mathematical courses. Should teacher training in the future be more based on future practices, the integration of elements promoting understanding in mathematical lectures such as visualization or applications makes sense. The transformation of the mathematical content in the sense of being directed at understanding is related to curriculum changes and may, but not necessarily, need to be associated with a reduction of the teaching content. It lies in the responsibility of the universities to realize teacher-specific teaching, for example, by systematically building on the knowledge of elementary mathematics. According to the expressions of the students, in many mathematical lectures still a form of didactics is supposed to be dominating that conveys the systematics of the science subject on the learning process of student teachers in a rather unreflective way, which means without considering the perspective and the learning of student teachers. If the teacher education really should be improved, this thought of didactics should be discarded. Following Werner Jank and Hilbert Mayer (1991) in the general understanding of such an image didactics as a concept in which the professional scientific structures are transferred without changes to the process of selecting, structuring, and justifying the curriculum, the idea of transformation of academic content here means the exact opposite. The idea of transforming mathematical content as done by the Justus-Liebig University within the scope of the project *Thinking Mathematics in a New Way* can make a fruitful contribution to the sustainable improvement of teacher education.

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Chapter 6
Mathematical Moments in a Human Life: Narratives on Transformation

Barbro Grevholm

6.1 Introduction

In this chapter a sequence of narratives will illustrate episodes that contribute to the creation of the mathematical perception in a human being. Each episode contributes to the transformation of the view of mathematics held by that person. Added life experiences of mathematics will be part of the formation of the mathematical identity of the person who goes through the experiences. The chapter presents episodes from the life of Lisa, such episodes that she herself considers important for the development of her mathematical identity.

6.2 Theoretical Framework

To learn about the transformation of a human being in relation to her evolving views of mathematics, it seems most appropriate to use narratives as tools in research. Narratives and stories have been common data in qualitative research, and different understandings of narratives can be found in qualitative inquiry (see, for example, Richardson 1990; Connelly and Clandinin 1990, 2000; Kaasila 2007). The data from Lisa could be seen as a life story interview (Långström and Viklund 2010; Atkinson 1998) by which is meant an oral life story that the person in question offers to the interviewer. It does not cover the whole life of the person but some aspects of life, in this case transformations of views of mathematics which the person judges as important. The analysis of the data is then narrative/biographical (Bruner 1996). In the analysis I try to keep the whole story of Lisa and try to make sense of her trajectory of identity, her cultural history, her present experiences and imagined futures (Williams and Wake 2007b). According to Williams et al. (2008,

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p. 6), Bruner (1996) sketched "the essential elements of the narrative form of construal of cultural reality: temporal structure, generic particularity, reasons, hermeneutic composition, canonicity, ambiguity, 'troubles', negotiability, and historical extensibility" (pp. 133–143). Williams et al. argue that it is possible to understand how aspirations can evolve, how identities grow and how key moments are said to deflect trajectories in significant ways for the students.

The subjective knowledge of an individual cannot be reached without help from that individual to open windows into the mental images of the person. The episodes told need not be seen as the truth or objective reality; they are the memories the person has of an experience of importance. Justification of the story is given through the claim of the individual that it has been important. The value of the episode is decided by the person who had lived this experience.

6.2.1 The Concept Transformation

According to dictionaries the meaning of transformation is "radical change of some fundamental property (for example, form, expression in phenomenon and thought product etc, often a change possible to describe in scientific terms)" (Svensk ordbok 2009). In this chapter I will use the word transformation for changes or development of a person's views, beliefs, knowledge and attitudes (in this case in mathematics). This is mainly how the word has been used in mathematics education research. However, the word transformation certainly has precise meanings in other connections as, for example, in mathematics, in linguistics, in law and many other areas of society. The word transformation is used here for the development or changes that take place, for example, in the professional identity of a student teacher going through teacher education and building up professional experience.

6.2.2 Views of Mathematics

In research literature many different notions are used when talking about an individual's view of mathematics. Pehkonen uses the word conception or belief (Pehkonen 2003), while Mura (1993) for example talks about images, views or definitions. In this chapter when I use "view of mathematics" it is understood as the individual's personal conception of mathematics in broad sense, including all aspects that can be inferred by experiences or observations of mathematics in different contexts.

6.2.3 Images of Mathematics

In order to characterise the views or images of mathematics that Lisa encountered as she grew up I will use the classification offered by Mura based on a study of university mathematicians' views of mathematics. Roberta Mura (1993) investigated images of mathematics held by university teachers of mathematical sciences. The question "How would you define mathematics?" was answered in a questionnaire by 173 university teachers. The analysis led Mura to the identification of the following themes (and here she calls them themes, not views or images): (1) the study of formal axiomatic systems, of abstract structure and object, of their properties and relationships; (2) logic, rigour, accuracy, reasoning, especially deductive reasoning and the application of laws and rules; (3) a language, a set of notations and symbols; (4) design and analysis of models abstracted from reality and their application; (5) reduction of complexity to simplicity; (6) problem solving; (7) the study of patterns; (8) an art, a creative activity, a product of imagination, harmony and beauty; (9) a science, the mother, the queen, the core and a tool of other sciences; (10) truth and (11) reference to specific mathematical topics (number, quantity, shape, space, algebra etc). Mura concludes that there is considerable variety in the images of mathematics held by university teachers (ibid, p. 384). Additionally, individuals hold composite views. And many seem to have little interest in reflecting on the nature of mathematics.

6.2.4 Impact on an Individual's View on Mathematics

When interpreting Lisa's development to become a mathematics teacher, I will use the model below indicating the development of a mathematics teacher's professional identity. Especially, I search for the transformed views of mathematics that Lisa is developing as she builds up knowledge in mathematics and reveals her personal view and beliefs in relation to this knowledge. The concept map model was created in this form in 1998 (Grevholm 2006), and is based on the findings in a longitudinal study of student teachers, showing how teacher education can be perceived as development of a professional identity (see Fig. 6.1). This professional identity development is complementing the private identity of the teacher and it is governed by social demands, culture and the national identity (Grevholm 2006). The professional identity is reflecting who the person is in performing the professional activities and the private identity reflects who the person is as a private person. The two parts of the identity ought to be well integrated. The five main elements in the model (knowledge in mathematics related to teaching; competence to judge and diagnose pupils' learning in mathematics; knowledge about classroom management, methods and material; a personal view on and beliefs about knowledge and learning in mathematics and a professional language for a mathematics teacher) constitute core parts of the professional identity that is developed in teacher education. These five elements are interrelated and closely linked to each other. The model also indicates the sources or springs for the five areas and the sources for the knowledge and competencies, and how they are interrelated in a complex system. Student teachers' experiences, earlier knowledge, observations, reflections, practice, research and theoretical studies during the education contribute to the development of the five aspects of the teacher identity.

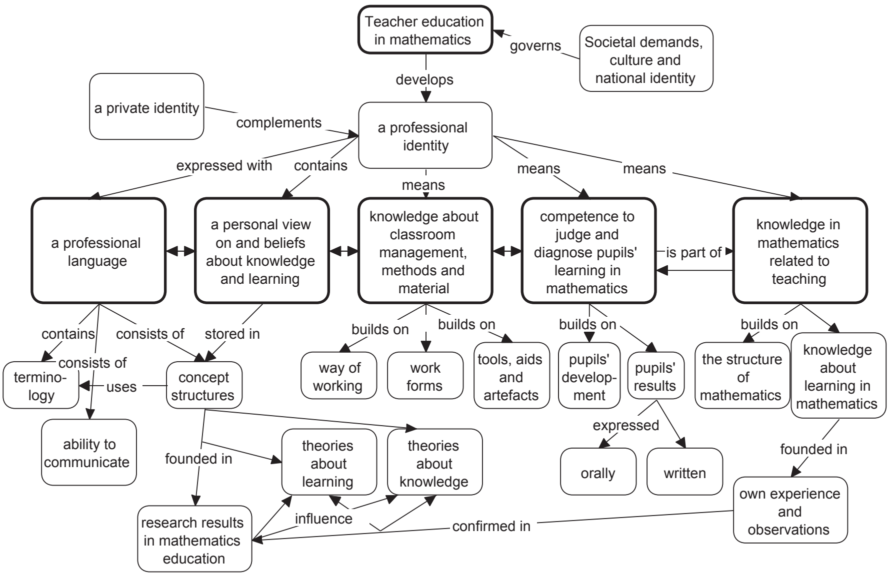


Fig. 6.1 A concept map showing how mathematics teacher education can be seen as the development of a professional identity, with five main elements and their sources. (Grevholm 2006)

6.3 Research Question

What characterises the transformations of views on mathematics that influence and impact an individual's life choices and later professional development to become a mathematics teacher?

6.4 Methods

A mature aged woman, Lisa, was interviewed by the author about what episodes of encounters with mathematics she could remember as having had impact on her. Lisa chose to study mathematics and she became a mathematics teacher, and the events that could have been of importance for her choice of professional career and identity are in focus of the investigation. Lisa's stories are presented as short narratives written by the author based on Lisa's memories. The intention is to find what transformations of views of mathematics become visible through these episodes.

6.4.1 Narrative Number One, Grandmother's Work

Grandmother Hulda was sitting at the kitchen table with the big important book in front of her. Outside the window children were playing and the sun was shining. It was summer. The book had blue horizontal lines and red vertical lines giving room for notes and numbers. Lisa's grandmother was keeping the book for the taxi company belonging to her grandfather and his partner Axel. Lisa was standing next to her grandmother at the edge of the table trying to see what she was writing. Lisa was 5 years old and could just reach above the top of the table to see and follow the writing process, which was going on. The letters and numbers were beautiful and it seemed so attractive to do the work of her grandmother. How beautifully she could write. Now and then she stopped and took an extra careful look at the handwritten receipt from where she seemed to fetch her information for the book. After all the numbers for the day were included her grandmother started to calculate the sums. Most of the days everything seemed to be fine and she could close the book with a satisfied sigh. But now and then it happened that the columns did not show the result her grandmother wanted. She started to check her calculations and redo the sums. Maybe it still did not come out as expected? Then she called for Elof, her husband, and asked if there were receipts missing or if any one of them had been written incorrectly. Sometimes they had long discussions before they could solve the problem. But they did always solve the problem and then her grandmother could close her work for that day. There was always a positive outcome of her grandmother's work.

6.4.2 Narrative Number Two, Grandfather's Work

In the cellar, Lisa's grandfather had a workshop and in rainy days when he was not working in the garden he used to be in the workshop and do some carpentry. Lisa liked to sneak in and watch him when he was working. Sometimes she became curious and asked him what he was doing. He always gave the same reply. I am doing a "himpajimpa for a vaermoella". This did of course not make sense to a small child (or anyone) and her grandfather did not want to explain what a "himpajimpa" was. The thing he worked on could turn out to be a small chair for her to use in the playhouse or a bird's house for the garden, a doll's house or something similar. When her grandfather worked he took measures on the pieces of wood with a carpenter's rule and made a mark with a flat pen. He used to say the numbers from the carpenter's rule aloud before he went on and wrote them down on a piece of wood. He then sawed a piece of the wood in the size he had marked. It happened that the pieces did not fit exactly when he put them together and then he had to file off a little of the wood. Lisa's grandfather had a small model of the object he was creating in full size in reality. It looked so exciting to see the pieces of plank being puzzled together into a nice object. After the pieces were screwed or glued together and had dried for some time her grandfather polished it and painted the new item nicely. The carpenter's rule, which was made of wood, was folded together so it could fit into the special pocket in grandfather's trousers. Lisa liked that instrument. Children were not allowed to play with it as it could easily break. It was two metres long and had one kind of measures on one side and another on the other side. One could measure in centimetre or in inches. Her grandfather's carpenter's rule was a different tool than her grandmother's measuring tape, although they both could measure lengths.

6.4.3 Narrative Number Three, the Tailoring Business

Lisa's mother Maja was educated as a tailor and had her own tailoring business at home. When Lisa grew up she was often a silent observer when her mother had customers to serve. Maja used a measuring tape to measure the customers' bodies after a specific pattern and the numbers where written down in a little black notebook. On top of the page was the name and telephone number of the customer and then, for example, the sequence 106–80–110 could mean bust, waist, and hip length. 14–35–30 could mean shoulder-over-arm-under-arm and finally 44–45–65 was width of back, length of back and length of skirt. Thus, the body of each customer was represented by three number sequences in the book. The most exciting moment to observe was when Maja was adjusting the paper patterns to the customer's sizes and then placing them on the piece of cloth to try to get as much as possible out of the given material. The forms or patterns had to be lined up along the threads in the cloth according to certain marks. She drew the patterns onto the material with the help of a white flat chalk crayon and the result was beautiful graphs, lines, angles, half-circles, parabolas and other curves. The patterns held different geometrical forms in combinations and to enlarge or diminish them to fit the right size seemed to be so exciting. Lisa heard her mother calculate aloud when she was working with the paper models. If we use four breadths in the skirt and the waist-length is 80 we need each breadth to be 20 at the top. And then add 2 on each side for seams, which gives 24 for each breadth. Maja seemed to be good in mental calculations. Lisa saw numbers, calculations, patterns and forms being used in the tailoring work. Enlargement or diminishing of patterns and forms were crucial aspects of the work in addition to the artistic and aesthetic work when Maja was sketching the dress trying to implement elements from latest fashion. Maja was very able and even if she happened to make a small mistake when she cut the cloth, she always knew how to repair it by adding some creative features to the dress, like an extra fold, an extra pocket or unexpected decorative seam.

6.4.4 Narrative Number Four, Lisa Goes to School

So far, Lisa had observed others use mathematics but at the age of six she started school and was invited to start to write numbers and do sums herself. The teacher probably invited pupils to approach numbers through two different representations. One was images of sets of dots or objects and the other the numerals (see Fig. 6.2).

Lisa still very vividly remembers how she in her mind marked each numeral with the same number of points as the number which the numeral indicated. Lisa combined the two representations into one in her own mind. When she was doing sums she very quickly counted all dots instead of using number facts as she could have done (see Lisa's own combined image in Fig. 6.3). She could go back to this way of counting dots when she was tired long after she knew all number facts. She easily saw the invisible dots on the numerals.

Fig. 6.2 Two representations of numbers

•	••	•••	••••	•••••	••••••
1	2	3	4	5	6

Fig. 6.3 Lisa's combined image of numerals and "invisible" dots

The first years in school did not leave Lisa with many explicit memories of mathematics. The change came in school-year five, when Lisa entered what was called "Realskolan". Only a minor part of pupils went to Realskolan, the rest continued for another 3 years in "folkskolan". A new teacher of mathematics, Oscar Brange, became her favourite and he obviously aroused her interest in the subject. Lisa can remember starting with Euclidean geometry and how fond she was of the systematic presentation with definitions, axioms, theorems and proofs. She liked the smell of the little book telling about Euclidean geometry (see Fig. 6.4). She liked much to draw the geometrical figures very carefully (see Fig. 6.5) and to learn the constructions and logical steps in the proofs. Other parts of mathematics were more difficult to learn. For example, fractions seemed to cause some confusion before the understanding entered. At this stage, Lisa met with algebra for the first time and found it very attractive. The algebraic rules and forms gave a certainty to her work. To use logic and reasoning systematically suited her. Lisa's grades in mathematics became better and better each semester. But her teacher Brange was demanding. When Lisa did not obey his advice to write larger numerals in her book, he punished her by a fee of 5 öre. But fair as he was she got the money back on the final day of the term together with a nice apple from his garden. And Lisa learnt the value of writing in a clear and readable manner.

6.4.5 *Narrative Number Five, Lisa Receives Vocational Advice*

At the time when Lisa went to school only about 10% of the entire year-group of pupils advanced to upper secondary school. As a preparation for the choice of educational direction in upper secondary school, Lisa went with her mother to a specialist for vocational advice. When this expert saw Lisa's grades, he suggested she should specialise in mathematics and science rather than language, humanistic subjects or economy. He claimed this would give her a safe future and easy access to good jobs. In his reasoning Lisa's results in mathematics played a decisive role. The fact that Lisa's mother was an able user of mathematics made it natural for her to support the given advice. Thus, following this advice Lisa applied for a school that offered "reallinjen", which means studying all subjects but specialising in mathematics and science. This specific school was a former boy's school and only two years earlier had girls been allowed to enter. When Lisa started there, the school was dominated by boys and in her class there were only seven girls out of 32

Fig. 6.4 Lisa's geometry book

pupils. Lisa could now imagine using this education to enter into further education to become a mathematics teacher.

6.4.6 *Narrative Number Six, Lisa Goes to Upper Secondary School*

Thus in school-year nine Lisa entered upper secondary school in a four-year programme with emphasis on mathematics, physics and chemistry. What Lisa remembers vividly from this time is that almost all calculations in both mathematics and science had to be carried out by using logarithms. The book with logarithm tables grew worn from all the use of it. In year two in upper secondary, though, the pupils were introduced to the slide rule. But even after the use of the slide rule was learnt, pupils could only use it in rare occasions as the precision was normally not good enough. The teacher Lisa admired most was her chemistry teacher Hall (who also taught mathematics but not to Lisa's class) and she now became Lisa's role model. In mathematics Lisa had several different teachers during these years. Lisa liked the connections that were made between algebra and geometry and the use of constructions in the mathematics work. She was especially fond of constructing graphs and calculating equations of circles, parabolas, hyperbolas and ellipses. She even created her own little plastic model of an ellipse, which she could use as template for drawings. In the last two years in school Lisa specialised in mathematics and had 7 lectures of mathematics each week. Mathematics was one of her favourite subjects and she had good grades. After twelve years of education Lisa finalised school with a higher certificate and received her white student cap.

Fig. 6.5 Lisa's drawing kept in the geometry book

6.4.7 Narrative Number Seven, Lisa Is Working as Private Teacher

As Lisa was successful in mathematics in school, the neighbours asked if she could teach their children. Already at the age of 16, Lisa started to work as a private teacher in mathematics for several other young students, who struggled with the learning of mathematics. This meant that Lisa had to explain a number of phenomena in mathematics and had to analyse what these students needed to learn by interviewing them and finding out what could possibly be tested in the next examination of them. She tried to convince her students to analyse the task properly and write down a plan for the solution before carrying it out. She was a patient and supportive teacher and she emphasised the good explanations for pupils. Lisa talked much with her pupils and thus developed a language for mathematical reasoning. Lisa found that she liked to teach others and that the students had made progress in their studies with her help. She went on working as a private teacher for many years. The extra income, although small, helped financing her studies. Her wish already then was to become a teacher of mathematics.

6.4.8 Narrative Number Eight, Lisa Starts University Studies

After four years in upper secondary school, Lisa graduated with very good results. She wanted to study at university to become a teacher of mathematics and chemistry. At the age of 18, she entered mathematical studies at tertiary level. Lisa liked the studies much and soon was part of a group of four friends who used to work as study group outside the mathematics lectures. Later Lisa learnt that such a group could be seen as a natural study group (Treisman 1992). The cooperative learning in this group assisted the development of a language for discussing mathematics and problem solving. In this group many heated discussions took place on how the tasks were best solved and what was the correct answer. Alternative methods were investigated, evaluated and compared. Now Lisa met with formal mathematics and theorems and proofs became even more important than in upper secondary school. Lisa took three full semesters of mathematics as the start of her academic career. She studied linear algebra, calculus, several variable analysis, algebra, differential equations, theory of analytic functions, non-Euclidean geometry and history of mathematics. But then when she wanted to turn to chemistry, it was not possible to start with chemistry in the fourth semester so she took physics instead and went on with physics, theoretical physics and astronomy. Before she finished her master's degree, she was asked if she wanted to take a job at the mathematics department as an assistant. This meant teaching halftime as senior lecturer and the rest of the time devoting yourself to further mathematics studies. Lisa thought she had nothing to lose and all to win so she took up this offer. Thus, she became a mathematics teacher at university for future civil engineers at the age of 20. Lisa took up her doctoral studies in mathematics as soon as she fulfilled all the prerequisites to enter, which was one more semester of fulltime mathematical studies. We will leave Lisa here for the sake of space in this chapter but her further stories about how she encountered mathematics in life later are also interesting and will be dealt with in another paper.

6.5 Analysis of the Episodes

Different kinds of transformations can be found in Lisa's stories. A number of them will be discussed below and will be related to the theoretical constructs mentioned above.

6.5.1 From Observer to Consumer, Transformation from Passive to Active

As a child Lisa met mathematics in everyday and vocational situations (accountant, carpenter and tailor) and obviously got a constructive and positive image of mathematics as something useful, important and beautiful. Lisa's grandmother's attractive writing of numbers in the book for Lisa's grandfather was admired and understood as meaningful and necessary for the taxi company. Lisa's grandfather's carpentry included measurement, drawings and calculations and led to the production of attractive and appreciated objects. Her mother Maja's tailoring work was a necessary source of income for the family and resulted in beautiful clothes for the customers. Lisa saw the beautiful image of mathematics and also how it could be valuable and important for everyday life. Research has shown that such early childhood experiences are influencing the concept development of the child in a sustainable and continuous way (Helldén 2003). It is interesting to ponder upon how much of Lisa's experiences are available for children today. The division of work and home life is more marked nowadays and children are not so often allowed to enter their parents' or other relatives' working life. The workplace mathematics is often claimed to be hidden in black boxes (Williams and Wake 2007a). Strässer (2000) even claims that this tendency is stronger in a time of increased technology use. Hundeland (2010) has shown that teachers commonly refer to their own early experiences as learners of mathematics for explanations and justification of their decisions in teaching. In these early observation situations, Lisa was an observer, passively learning about what mathematics can be. She saw aspects that she admired and found beautiful but also much meaning in the work and necessity for the daily outcome. Her early observations had left her with a positive view or image of mathematics.

From the observer situation, Lisa entered into a more active phase with mathematics when she started to learn how to write numbers and calculate. What Lisa emphasises here is actually her own way of doing sums early, when she did not use number facts but simply calculated all dots for the numbers. Could this mean that Lisa did not listen to the teacher's advice but went her own way in the solution? Research has shown the importance of allowing pupils to use their own methods before they are taught algorithms (Carpenter and Fennema 1988). Lisa seems to have taken this chance and used her own way of thinking.

It seems as if mathematics did not play a major role in the first years in school as Lisa remembers few episodes from there. The important episodes seem to come in year 5 or later. Here again she is not obedient to the teacher but writes too small numerals in her book. Many pupils have bad memories from school mathematics but for Lisa it seems as if the only negative feature came from her own behaviour, for example, in writing too small numerals and keeping her "counting-all" method longer than necessary. On the contrary, she does not describe the teacher as being negative or threatening or the subject as difficult in general. The different parts of mathematics are perceived as something important, positive and easy to learn. Lisa has a teacher as role model in both secondary and upper secondary school.

Lisa was through her own early observations and experiences laying a foundation for later learning of mathematics and building up knowledge for teaching (see Fig. 6.1). These observations and experiences combined with her knowledge about learning of mathematics from her own work and studies and from her early private teaching supported the development of knowledge of mathematics for teaching.

6.5.2 The Importance of the Role Model, Transformation from Child to Adult in Mathematics

In Lisa's episodes it is very clear for her who is the admired role model. We meet her grandmother, grandfather, mother and the teachers Brange and Hall. These persons were important to Lisa and she has kept some bright memories from them where mathematics is crucial. They have contributed to the fact that Lisa perceived a positive image of mathematics and came to admire and like it. For all of her role models mathematics was something useful, beautiful and important, something they worked with regularly, sometimes every day. No one in Lisa's surroundings claimed that mathematics was difficult or unnecessary. Quite the opposite was true. But Lisa's brothers who could have been transformed by the same experiences as Lisa did in fact encounter problems in school while learning mathematics. As we have not heard their stories, it is impossible to give any explanation for this difference. The teacher Brange probably influenced her wish to become a teacher of mathematics. Lisa admired him as a person and as a teacher. Lisa wanted to become a chemistry teacher because she admired her teacher Hall, who taught both chemistry and mathematics.

Later in her tertiary studies, Lisa had other role models, for example, her friends in the study group. They all worked hard to solve all problems, were good at arguing for their solutions, could admit that there were several possible solutions and could assess the quality of the methods. Uri Treisman (1992) has shown the value of such informal study groups for the quality of student learning. To work in such a group outside lectures also added to the total time for learning mathematics, which is another important factor for the learning outcome (Walberg 1988).

6.5.3 The Importance of Success, Transformation to Experienced User of Mathematics

Research has convinced us that pupils want to understand mathematics when they learn it (Kislenko 2011). Kislenko writes that the most striking result in her investigation is that it gives evidence for pupils wish to learn mathematics by understanding, and they like reasoning and to do things in mathematics, and to carry out activities. Nardi and Steward (2003) concluded that English pupils want to experience relevance, excitement, variety, challenge and deeper understanding. We also know that if pupils meet challenges and problems on appropriate levels and perceive that they succeed and master it they often come to like mathematics and are willing to learn more. Kislenko (2011) refers to Hoskonen (2007) and Stodolsky et al. (1991), who found that pupils liked mathematics when they were successful in it and when it was considered to be easy.

Another aspect emphasised in Kislenko (2011) is that the most important feature for the teacher to be liked is the ability to explain and to support understanding. Lisa probably developed her ability to explain mathematics and support understanding as she worked as a private mathematics teacher. In Lisa's case we see how this worked and she was allowed to experience success and wanted to learn more. As she felt herself that she understood, she was brave enough to start to teach others and help them to gain understanding. That choice probably strengthened her possibilities. Additionally, the choice of special mathematics in upper secondary school contributed to further experiences of joy and success. The experience of success in helping others was a support to the idea of choosing the profession of a mathematics teacher.

6.5.4 The Transformation from Learner to Teacher of Mathematics

Lisa took an early step towards being a teacher of mathematics from being a learner, although she started as an autodidact private teacher. This was paid work and thus Lisa took great responsibility in her task. This might even have helped her own development in mathematics as explaining to others also leads to development of her own deeper understanding. The experience of being able to help others reinforced her wish to become a teacher of mathematics. The five crucial elements in the model of development of a professional identity (see Fig. 6.1) can be seen to be impacted by Lisa's early experiences. She created for herself opportunities to develop a professional language by being a private teacher of mathematics, and she remembers many opportunities where she could develop a personal view and beliefs about knowledge and learning of mathematics. As a private teacher she also built some competence to judge and diagnose pupils' learning in mathematics and she built knowledge in mathematics related to teaching. Thus, when Lisa entered higher education, she had already a steady foundation of her own experiences and observations on which she could build further during her studies to become a teacher.

6.5.5 The Transformation from Teacher of Mathematics to Researcher of Mathematics

Before we leave Lisa she has taken the step to start as a doctoral student in mathematics. For her this transformation was a huge step. No one in her family had graduated from university before and then going on to a higher level was extremely exceptional. Additionally, there were very few women in mathematics in her country. Most male professors at that time were not eager to supervise women in mathematics. Less than a handful of women had taken a higher degree in mathematics before her. But she did not know this at the time when she took her studies. Not until many years later did Lisa learn that she was the third woman ever to take a higher degree in mathematics at that university. The first one graduated as doctor of philosophy in 1911 (Petrén 1911) and was not allowed to take up a position at the university (legal rules at that time). Lisa luckily approached a professor, who accepted to supervise women in mathematics.

6.5.6 Transformation Caused by Seeing Different Views or Images of Mathematics

Which of the images that Mura presented did Lisa encounter during her early years? Her first memory is from numbers and calculations done by her grandmother. This image contains both elements of category 11, namely the topic numbers, and elements of category 3, the notions and symbols used in mathematics that she experienced as so beautifully written by her grandmother that she longed to do it herself. In the situation where she learnt from her grandfather we can find the categorisation 4, design and analysis of models abstracted from reality and applications and also from category 7 the study of patterns when her grandfather used a pattern for the model of his work. Both of these categories are also present in her mother's work as a tailor. Here additionally we find mathematics used as an art, a creative activity and the development and design of fashion. Harmony and beauty became visible in the end product of that work when the customer collected the dress which was suitable as expected.

In school Lisa met with problem solving and other topics as geometry and algebra. She expresses her satisfaction with Euclidean geometry and the logic system of definitions, axioms and proof. Thus, categories 6 and 1 are partly visible in her experiences. Here she also was subjected to demands concerning the formal presentation of mathematics through the teacher, who demanded nice, clear writing of solutions in the book. She came to appreciate beautiful geometric constructions.

When Lisa entered her studies of mathematics at the tertiary level, she met it in a more formal axiomatic way than in school. The abstract structure and objects and their properties and relationships became important. In the study group of four friends, she learnt how to reason properly and to deduce and apply laws and rules. Rigour and accuracy were needed in all the solutions of problems that were produced in the group. Finally, Lisa enters doctoral studies and has the opportunity to experience mathematics as a science and a creative activity. Thus, almost all of the faces or images of mathematics seem to have been present more or less in the experiences of Lisa. She was offered the opportunity to meet all the images, but of course it is still not obvious which of these images are dominating her own conception.

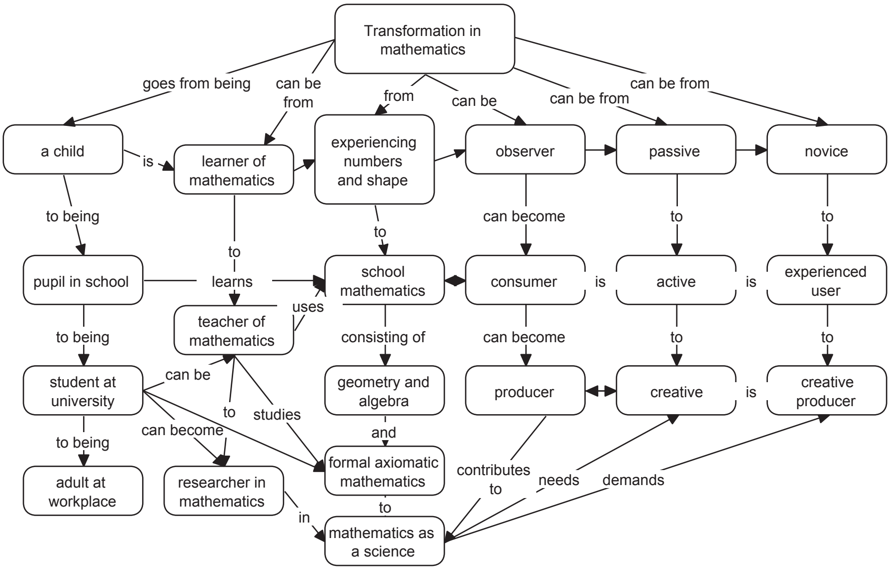
The transformations caused by the images were suitable for her conviction to specialise in mathematics and to become a mathematics teacher. The transformations being so varied and multifaceted gave Lisa a broad view of mathematics.

6.5.7 The Socio-economic Transformation of Lisa Through Mathematics

Lisa's grandparents and parents are working-class people or middle-class people. Thanks to her choices of mathematics, for her vocation as a teacher and her success leading to an employment at university, Lisa will have to leave her social roots and enter into upper class in the sense of having advanced academic education and safe employment with a good salary and working conditions. How will this influence Lisa? Will she remove herself from her social roots and enter into another world with a different life style or will she continue to be a hard working, careful employee? This we will have to learn from another story where Lisa tells us the continuation of her life story.

6.6 A Concept Map Model of the Transformations in Mathematics

Based on the transformations presented and discussed in this chapter, the following model is constructed. In relation to the model I now discuss how other researchers have illuminated the transformations that are included in the model.



In their model of pedagogical thinking and action, Wilson et al. (1987) describe six components of teaching: comprehension, transformation, instruction, evaluation, reflection and new comprehension. The transformation relates to the process where the teacher moves from a personal comprehension of the ideas to be taught to an understanding of how to support students to a similar comprehension. In our model this refers to the links showing that the teacher is a user of school mathematics but has studied formal axiomatic mathematics. In the earlier presented concept map on teacher identity development (Fig. 6.1), this refers to the element "mathematical knowledge related to teaching". In the French school of didactics, this is most often called the didactical transposition, referring to how scholarly academic mathematics is transposed into the mathematics taught in schools (Chevallard 1985; Bosch and Gascon 2006).

In his handbook chapter about advanced mathematical thinking, Tall (1992) writes: "The move to more advanced mathematical thinking involves a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions." (p. 495). In the model above this refers to the link from school mathematics to formal axiomatic mathematics. This transition or transformation is most often taken when the student progresses from secondary school to university mathematics. Many other researchers have investigated this transition, obviously as it is experienced as most often being a difficult passage (Wood 2001; Kajander and Lovric 2005; Brandell et al. 2008; Vollstedt et al., Chap. 2; Deiser and Reiss, Chap. 3). In the case of Lisa this transition from secondary to tertiary mathematics education is not reported as problematic. One explanation for this fact could be that Lisa went to secondary school during the 1950s when very few students did so and thus the level of teaching at secondary classes was supposedly much higher than later. The widening gap discussed by Brandell et al. did not yet exist.

The transformation from novice to expert has been studied by several researchers. Koehler and Grouws (1992) discuss the expert-novice paradigm and give an overview of research in this spirit. Expert teachers are found to have richer agendas and their plans contain more detailed information. They spend less time on transitions and distribute their time amongst other lesson components. The explanation part of the lessons showed that experts gave better explanation of new material. They emphasised more critical features and made fewer errors than novices, who often did not complete their explanations. These observations can be seen in the light of the fact that pupils expect teachers to give good explanations (Kislenko 2011). In Lisa's case she was a novice when she worked as a private teacher but later transformed into an expert teacher after her formal teacher education. This transformation took place over more than ten years, showing that the development from novice to expert may be a long-term process.

Included in the step from novice to expert or experienced user of mathematics as indicated in the concept map is the use of tools, artefacts and texts. In other chapters in this book, authors are analysing transformations concerning these objects. Thus, there is a discussion of dynamic and tangible representations by the Labordes (Chap. 11), the transformation related to signs and representations of mathematics is discussed by Kadunz (Chap. 8), Mariotti (Chap. 9) treats the construction and transformation of signs in a technology-integrated environment and finally Hölzl (Chap. 10) enlightens dynamical representations of conformal transformations. Further, Pepin (Chap. 4) argues that mathematical texts and materials can get transformed by working with them in teacher learning. Bessot (Chap. 13) designs a simulator in order to transform vocational education in her chapter.

Several chapters present work on didactical transposition on micro-domains of mathematics. Kuzniak (Chap. 18) argues that new meaning is given to visualisation and experimentation in the transformation of geometric knowledge to school context. Profke (Chap. 19) argues for a promotion of mathematical literacy via small steps in order to allow the big ideas of mathematics to determine the teaching.

In the model the long series of links showing the transformation from child to adult relates also to the work of Helldén (2003), where he has shown that the early experiences by the child give the basis for the conceptions that slowly develop as personal knowledge in the learner and stays with the learner into adulthood. Tall (1992) discussed the position where the concepts have an intuitive basis founded on experience and Helldén shows that such intuitive conceptions play a decisive role in later stages of the person's conceptual development. In Lisa's case one might interpret the clear and strong stories she can tell about early personal experience of mathematics as such important contributions to her development of a professional identity.

6.7 Conclusions

What Lisa has chosen to expose to us in her stories about memories that had an impact on her view of mathematics is highly personal. Her stories are situated in specific contexts and closely connected to the social and cultural opportunities of Lisa as a child and a young girl. It is clear that other persons have been important for her transforming views of mathematics. We can make no generalisations from the narratives. But they constitute an existence proof that this can be the life story of one person who decided to become a teacher of mathematics. Other teachers have different life stories.

When Lisa tells her life story related to mathematics it is clear that her view of mathematics has undergone several transformations and that she at the same time has transformed as an individual and developed her identity. Some of the important transformations are, for example, to become an active user of mathematics rather than an observer, to transform from a child to an adult aiming for a profession related to mathematics, to transform from a learner to a teacher and finally to a researcher of mathematics, and to transform her professional identity from the experience of success in mathematics. In these steps Lisa encountered positive experiences of mathematics, which following mentioned research results each could support her choice of mathematics at the next level, and finally support her choice to become a teacher of mathematics. Lisa is also able to tell stories reflecting many different images of mathematics and to experience mathematics in the workplace.

How have Lisa's experiences and observations from childhood and youth influenced her as professional? Will she be able to understand and support her pupils when they encounter problems in mathematics when she herself never had such experiences? Will she be able to listen to pupils' explanations even when they are idiosyncratic and not similar to her own way of thinking? As a teacher educator I have met several student teachers, who were weak in mathematics and claimed that this would help them to show understanding for their own pupils when they get stuck. This could be the issue for a future study. I hypothesise that being weak in mathematics will create difficulties in situations when you need to understand another persons' thinking. A teacher like Lisa, who is strong in mathematics, and has experienced several images of mathematics, will have a greater variety of modes of working, viewing, listening and thinking in the same mathematical situation.

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Chapter 7
Discussion of Part I

Transitions in Learning Mathematics as a Challenge for People and Institutions

Rolf Biehler

Abstract: The transition from school to university is a challenge for all students as the teaching-learning cultures and the types of mathematics are very different and require from students large efforts of adaptation. A deeper understanding and research into the features of this transition is necessary for informing institutions and their teachers to better support students in the transition phase. Vice versa, a backwards transition from university to school is part of every teachers' biography and includes particular challenges. On an institutional level, the backwards transition is concerned with updating school curricula by taking new developments of mathematics and science at university level into account. The paper elaborates these problems and provides an introduction into the set of papers that are concerned with transitions and transformations on a personal and institutional level.

7.1 Overview

The papers in this book are predominantly concerned with the transition between school and university. We can see this as a specific instance of the problem of transitions between different mathematical practices (Abreu et al. 2002).

When a student enters a university, he or she is already bearing a mathematical biography. The student has encountered mathematics as a subject to be learned in primary, middle and high school. The type of mathematics might have been heavily dependent on the respective school as an institution. Moreover, the student may have encountered implicit mathematics in other school subjects or everyday situations without being aware that he or she is encountering mathematics. Grevholm (Chap. 6) provides an illuminative case study of mathematical moments in the life of Lisa from her early childhood, where she encountered implicit mathematics in the vocational contexts of her parents, up to entering university. The mathematical development of a person can have discontinuities, partly due to discontinuities in

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the mathematical practices of the different institutions a person moves through. Earlier levels can disappear as such and become replaced or integrated in later levels or can still coexist in the mind of a student. For instance, a university student will have an opinion about the differences and commonalities between school and university mathematics, although it may not be trivial for him or her to switch back to the practice of high school mathematics or even lower levels of mathematics. Obviously, this “switching-back ability” is a fundamental qualification of teachers. Grevholm’s Lisa seems to be capable of doing this as she is successfully coping with the challenge of acting as a private teacher of school students during her university studies.

When mathematics students have finished with their university studies they face a further transition, namely from academic university mathematics into vocational contexts where, as a rule, mathematics is practiced in context. For future mathematics teachers, this second transition is a very specific one. Teachers go “back” to school and therefore may suffer in their mathematical biography from a “double discontinuity,” namely in the transition school-university and in the transition university-school. Felix Klein has coined this term in the introduction to his book “Elementary Mathematics from an Advanced Standpoint” (1st edition 1908, see Klein 1932) and he is often quoted in recent movements to “overcome” the double discontinuity (Ableitinger et al. 2013). Most papers in this book discuss either the first discontinuity or both from various perspectives.

The ability to see a single mathematical topic in the context of different mathematical practices seems to be one of the qualifications good mathematics teachers should have. Hefendehl-Hebeker (1996) describes this as the ability to see mathematics with a high depth of focus. Discontinuities in the transitions have been a concern in education for long. Bruner’s (1960) idea of orienting the whole school curriculum according to fundamental ideas, whose source is the respective university discipline, can be interpreted as avoiding or reducing discontinuities between levels within the school system and between school and university level. Teachers’ knowledge has to reflect this; Loewenberg Ball and Bass (2009) and Hill et al. (2008) put forward the notion of “horizon knowledge” as part of teachers’ knowledge; that is knowledge in and about mathematics and the mathematical practice of the next (upper) level in the educational system. Vice versa however, it seems to be equally helpful for teachers to have knowledge about the mathematical practices below the level he or she is teaching. A question is with which attitude and respect should mathematical practices of lower levels be regarded and taken into account at higher levels? The case study of a student that enters university that Pepin (Chap. 4) is presenting in her paper to this book reports of university teachers that favor a “confrontation approach” and tell students “forget the mathematics you have learnt in school.” Confrontation could be an adequate measure if it is unavoidable in making students aware of a discontinuity. Devaluating previous mathematical knowledge and experiences however, seems hardly to be a reasonable pedagogical strategy.

Pepin' s paper (Chap. 4) as well as the papers by Deiser and Reiss (Chap. 3) and Vollstedt et al. (Chap. 2) in this book analyze the transition from school to university from various perspectives for *all* mathematics major students, not just for future mathematics teachers. This transition has many general features that makes it difficult not only for all students but also for future mathematics teachers, who however may face specific motivational problems and have the specific concern of how university mathematics relates to the future school mathematics they have to teach. Kaiser and Buchholtz (Chap. 5) report on a particular innovative German project. This project was particularly devoted to first semester students who want to become mathematics teachers and it offered mathematics in a different way in order to smooth out the first of Klein's discontinuities.

Some transitions from school to university and within university are not discussed in the set of papers of this book. The papers that are concerned with teachers are looking at teachers who will teach at university or college bound schools. The transition has to be analyzed in quite a different way, if we think of mathematics courses for future primary or lower secondary teachers. The TEDS-M study (Blömeke et al. 2010a, b) reinforces the need of a careful specific curriculum design for these audiences. The LIMA project (*L*ehr*I*nnovation in der Studieneingangsphase "*Ma*thematik im Lehramtsstudium"—Hochschuldidaktische Grundlagen, Implementierung und Evaluation) (Biehler et al. 2012c) is one of the projects particularly devoted to improving the mathematics education of lower secondary student teachers in the first year of their studies. Moreover, school students who enter university studies in the STEM subjects (science, technology, engineering, and mathematics) but who are not mathematics majors encounter a different type of mathematics, which may vary in content and style among the different STEM subjects such as physics, biology, and the engineering sciences. Mathematics (including statistics) is present also in non-STEM subjects such as psychology, social and economic sciences, where the mathematical practices as well as students' attitudes and mathematical competences differ from the STEM subjects. In a very rough approximation, one can say that mathematics is more considered as a tool and language, whereas proof and formalization, the most distinguishing feature of all courses of mathematics majors, does not play a central role in these courses. These domains themselves often suffer from a clash of mathematical practices and cultures. The mathematical practice in an engineering course is quite different from the mathematical practice in course "mathematics for engineers." This can be a source of tensions between departments and within students' minds. Mathematical courses for economy students may be taught either by lecturers of the mathematics department (mathematics as a service subject) or by lecturers from the economy department. Some departments teach their mathematics courses themselves because they are convinced that lecturers from the mathematics department would import a mathematical culture that makes the transition to the uses of mathematics in the respective domain more difficult. Seeing this wider domain of mathematical practices including mathematics in vocational settings and in industry is a perspective that Rudolf Sträßer (2000) put forward in his work.

7.1.1 Transforming Backwards—From Universities to Schools

The paper of Biermann and Jahnke (Chap. 1) in this book reminds us of several historical instances where a new stance of university mathematics was used as a basis of school curricula reforms. University mathematics served as a major input for updating school mathematics. The authors point out how the eighteenth century conception of "algebraic analysis" systematically influenced the curricula in Prussian Grammar schools in the nineteenth century. The Meran curriculum reform of 1905, which was inspired by Felix Klein's ideas, intended to update school mathematics from a very different perspective, taking up major developments in nineteenth century mathematics as Klein and his companions conceived them. "Functional thinking" became a keyword of the reform movement. The concept of function was regarded from inside mathematics and at the same time seen as a bridging concept that relates mathematics to its applications in natural and engineering sciences (see Krüger 2000 for a deeper analysis). Klein was also concerned about the gap between pure mathematics and its applications. Klein's books on Elementary Mathematics from an Advanced Standpoint (Klein 1932, 1939) were written as books for teachers for supporting this specific backwards transition into schools. In the 1950s and 1960s a very strong international movement under the well-known name of the "new math reform" tried again to update school mathematics based on a specific view of university mathematics. The gap between school and university mathematics was supposed to be narrowed by changing school mathematics accordingly and adapt it closer to mathematics as a (university) discipline. More than 50 years later, we know much more about the complexities of the transformation from scientific knowledge to knowledge to be taught. Chevallard's (1985) book was a milestone in research in mathematics education that looks at these processes of transformation with an analytical stance (see Seeger et al. 1989 for a review). Rudolf Sträßer was among those, who took Chevallard's work into account in his work, particularly in geometry and particularly arguing for regarding not only university mathematics as a source for school mathematics but also vocational contexts and other non-university uses of mathematics (Sträßer 1992). This more general approach for reconstructing sources of meanings for concepts in school mathematics can be also exemplified by the concept of function (Biehler 2005).

In recent years, mathematical curricula seem to have developed in the direction of a more student-centered, application-oriented, and visual, less formal kind of mathematics allowing much more types of reasoning and argumentation than just formal proof. Due to these developments, the gap between school mathematics and the mathematics taught in university courses for mathematics majors seems to have become wider again. There is no easy solution with regard to the questions in which sense schools could readapt to university mathematics. We can frame it differently: How can we redefine what it means to mathematically "prepare" university bound students at school level for university courses with mathematical content. We have to take into account the problems of the school to university transition for gaining more insight.

7.1.2 The School to University Transition

The secondary–tertiary transition has become the object of theoretical analyses such as Gueudet's (2008) and of practical measures such as creating "bridging courses" (such as Biehler et al. 2011, 2012b). Redesigning the introductory university courses are further measures. The papers in this book contribute to this research and development domain from various perspectives. The paper by Kaiser and Buchholtz reports on an innovative German project that redesigned the introductory courses in analysis and in geometry and linear algebra at tertiary level in order that they better fit the needs of future Gymnasium teachers (Beutelspacher et al. 2012). Overcoming Klein's double discontinuity was one of the objectives. The courses itself were to reflect the relation between school and university mathematics by adequate examples and activities. Hereby, the students should appreciate and understand the need of a different kind of mathematics at university level, while at the same time understanding how this new mathematics is related to school mathematics, why it is different, and why it nevertheless has the potential of contributing to the development of students' mathematical competences in a way to make them useful for a qualified mathematics teaching at school level. Working in this way, the first discontinuity was regarded as laying foundations for smoothing out the second discontinuity at the transition from university to school. This is a very interesting experiment as it aims at maintaining and cultivating two views of mathematical practices at school and at university level from the beginning. In other words, it is cultivating the "switching back ability" as I called it earlier. The paper of Buchholtz and Kaiser reports on an empirical study that evaluated this innovative project in comparison to students who participated in more standard programs at different German universities. A quantitatively oriented evaluation has to develop adequate measurement instruments for mathematical competence. Based on knowledge conceptualizations and instruments developed in the context of the IEA directed TEDS-M project (Blömeke et al. 2010a, b), instruments were newly developed that distinguish between academic mathematical knowledge, knowledge in mathematics from an advanced standpoint, and mathematical pedagogical content knowledge and beliefs about mathematics. The study presents some slightly positive results but shows at the same time how difficult sustained educational reforms at the tertiary level are. A further theoretical modeling of the *development* of mathematical competencies at the secondary–tertiary transition and developing measurement instruments on this basis is still a challenge for the future.

The other papers of this section analyze the secondary–tertiary transition from various perspectives. Grevholm and Pepin provide holistic insights on how students experience a totally new culture including new people, tools, and learning and mathematical practices. Grevholm's narratives on transformation focus on the mathematical biography of Lisa, beginning in her preschool age up to her first years at university. It shows the complexity of constituting a mathematical identity and coping with transformations of various kinds. Lisa's story is a success story in the sense that she mastered all the transformations and finally started her doctoral studies in mathematics. Many students give up their mathematical studies in the first year. Also, engineering students fail, among other reasons because they fail in mathematics. Dropout rates of 30–50 % are not an exception, at least in German universities. Dieter (2012) did a quantitative study concerning premature matriculation and its influencing factors. Qualitative studies similar to Grevholm's case study, done with first year university students may contribute to a deeper understanding of factors that affect a successful and a less successful mathematical biography in the process of transition.

Pepin's paper goes into this direction, based on the TransMath project at the University of Manchester. Pepin analyzes the fundamental differences between schools and universities with regard to providing feedback and the requirements of self-regulated learning. This general framework is also applicable for other subjects than mathematics and opens the perspective to general problems students of all subjects face when entering the secondary–tertiary transition. Specific problems related to mathematics as a subject can thus be placed into a broader perspective. Based on case studies, Pepin provides a very instructive detailed portrait of the mathematical teaching-learning culture with the elements of lecturers, tutorials, and self-study and the implicit values and views of students and lecturers, which may not fit well with each other. Innovative reforms have to address the whole teaching-learning system and must not focus on curriculum and mathematics alone. In line with this perspective, we changed the teaching-learning culture in the small group tutorials that are accompanying our lectures with big audiences by creating a specific program for supporting student mathematics tutors that improved the quality of the group tutorials. We focused on enabling the student tutors to present solutions of homework with a view to student difficulties and to the process of problem solving. Moreover, the quality of feedback given to the problem solutions that students submitted as part of their homework assignments was increased. A third domain was supporting the student tutors in moderating collaborative group work with minimal and strategic intervention types. As a further step, our study has revealed the need of directly supporting students' motivation and competence of dealing with feedback as not all of them make optimal use of the improved feedback provided to them (Biehler et al. 2012b, c)

The papers of Vollstedt et al. and of Deiser and Reiss are concerned with mathematical knowledge and with mathematical learning resources in the secondary–tertiary transition and open a further important dimension in studying the school–university transition. Vollstedt et al. focus on a comparison of mathematical textbooks for upper secondary level and for school level. Learning resources are an important part of the knowledge and learning practice (Sträßer 2009), therefore comparisons can contribute to a deeper understanding of the major differences. Based on research on text book structure and use (among others, see Rezat 2009), the authors develop a new instrument for comparative analysis of secondary and tertiary text books along the dimensions of motivation, structure and visual representation, development and understanding of concepts, development and understanding of theorems, presentation of the proving process and proofs, and type of tasks. Although the categories were as yet only applied in a feasibility study, the system of categories provides orientation for future comparative text book research.

The paper by Deiser and Reiss focuses on specific elements of mathematical knowledge. Key differences between school and the mathematics of mathematics majors can be related to the different conceptions of proofs and definitions: “The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on definitions” (Tall 1991, p. 20). Deiser and Reiss’ paper fits well into the tradition to analyze students’ difficulties with definitions (Edwards and Ward 2004; Kintzel et al. 2011) as key part of the transition difficulties but also gives new insights into difficulties with seemingly basic definitions in a first semester analysis course. Deiser and Reiss provide first results of a larger research project that will study the development of mathematical competences of student teachers of mathematics within the first years of their university studies.

The papers of this book show that the transitions and transformations mathematical learners have to face have become the object of promising research and development studies in mathematics education, aiming at a deeper understanding and theoretical basis for educational innovations in the future.

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Part II
Transformations Related to Representations of Mathematics

Introduction

> *If you do mathematics every day, it seems the most natural thing in the world. If you stop to think about what you are doing and what it means, it seems one of the most mysterious. How are we able to tell about things no one has ever seen, and understand them better than the solid objects of daily life?*
> *(Davis and Hersh 1981, p. 318)*

In this quote Davis and Hersh paraphrase one of the most popular beliefs of the nature of mathematics: mathematics is abstract. Even from a philosophical point of view this remains true no matter what philosophical stance on the nature of mathematics is taken—a Platonist, a constructivist, a formalist position. Therefore, any encounter with mathematics is mediated through materializations of abstract mathematical objects and their relations. Taking up a Platonist point of view on the nature of mathematics, this issue is aptly expressed by Drijvers et al.: "Because of its epistemological nature any immediate relationship with mathematics is impossible; any relation passes through a mediation process. Ideal, immaterial, non-perceivable entities such as numbers or figures acquire existence, can be thought of and shared, only through their materialization in a concrete perceivable entity, generally referred to as representation" (Drijvers et al. 2009, p. 133).

Talking about representations of mathematics raises the question of the ontological status of what is being represented. From our perspective the word representation seems to be very closely related to a Platonist or realist view of mathematics, where objects that exist beyond or even within our perception are represented. In order to not enter into a philosophical discussion of this kind we relate to Wartofsky (1979) who suggests the term 'artefact' instead of 'representation', because artefacts are capable of representing human activity:

> *[...] what I took to be the crucial feature of human cognitive practice, namely the ability to make representations. This I traced to the primary production of artifacts—in the first place, tools and weapons, but more broadly, in good Aristotelian fashion, anything which human beings create by the transformation of nature and of themselves: thus, also language, forms of social organization and interaction, techniques of production, skills.*

> *The production of such artifacts for use, I argued, was at the same time the production of representations, in that such artifacts not only have a use, but also are understood as representing the mode of activity in which they are used, or the mode of their own production. Thus, spears and axes are not only made for the sake of hunting and cutting, but at the same time represent both the method of their manufacture and the activities of hunting animals or chopping wood. (Wartofsky 1979, pp. XIII–XIV)*

Defining mathematics very generally as a human activity, we prefer the more general term ‘artefact’ rather than to talk of representations to relate to materializations of mathematical objects and their relations.

Since the scientific discipline “didactics of mathematics” might be defined as the “the ‘sum’ of scientific activities to describe, analyse and better understand peoples’ joy, tinkering and struggle for/with mathematics” (Sträßer 2009, pp. 1–68) it seems natural that an important strand of this discipline tries to unveil this “mystery” as David and Hersh call it and thinks about what we are doing when we deal with mathematical objects and their relations through artefacts. The chapters in this part all relate to these efforts and address different kinds of transformations that are related to the way mathematics is mediated through artefacts.

The advent of technological artefacts has majorly increased the means to mediate mathematics. The opportunities these artefacts provide have changed the encounter with mathematics in a way that Sträßer (2001) even raises the question if there is a special mathematics incorporated in the technological artefacts. This means that there exists a dynamic relationship between mathematics and the mediating artefacts: depending on the view of what mathematics is the mediation of mathematics might even transform mathematics itself. However, this is not the only transformation that is related to the mediation of mathematics. The chapters in this part discuss the fact that the mediating artefacts are likely to transform the whole encounter with mathematics.

Kadunz (Chap. 8) shows that not only in technological environments the way mathematics is mediated is a crucial aspect in the construction of mathematical knowledge. Even reading a mathematical text involves transformations of representations of mathematics. The author’s language has to be transformed into the reader’s language. Kadunz argues that this transformation is often accompanied by writing, and fosters the construction of knowledge. This claim is supported by conceptualizing these transformations in terms of diagrammatic reasoning in the context of the semiotics of Charles Sanders Peirce.

Mariotti (Chap. 9) is concerned with how the construction of knowledge is fostered through instrument-mediated transformations of mathematical objects with a technological artefact. She claims that “dragging modalities (of Dynamic Geometry Systems (DGS)) offer a semiotic potential that can be exploited by the teacher to make the mathematical meaning of conditional statement evolve from haptic experience of direct and indirect movements, and the related different status of invariant properties” (Mariotti, Chap. 9). This claim is substantiated by an analysis of the semiotic potential of the dragging modalities based on the Theory of Semiotic Mediation (TSM). TSM is concerned with “the relationship between the accomplishment of a task and student’s learning” (Mariotti, Chap. 9). Based on a Vygotskian notion of semiotic mediation TSM conceptualizes this relationship in terms of a double semiotic relationship "(1) between a tool and meanings emerging in the accomplishment of the task (2) between the tool and the meanings related to specific mathematical content evoked by that use and recognizable by an expert" (Mariotti Chap. 9). In this case, the transformational possibilities of the artefact support the construction of mathematical knowledge because of the way these transformations elicit relations of mathematical objects.

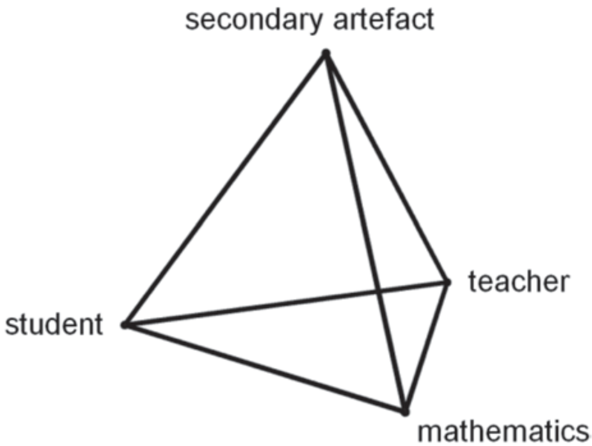
In Chap. 10 Hoelzl also focuses on how transformations of mathematical objects with the help of technological artefacts might support understanding in mathematics. His main concern is how to achieve a deep understanding of at least school mathematics in teacher education. The answer he offers is that mathematics should be represented in many different ways in order to foster understanding. Dynamic representations of complex numbers in a DGS exemplify how his ideas about the teaching of mathematical content knowledge in teacher education programs could be implemented.

Laborde and Laborde (Chap. 11) address three different dimensions of transformations that are closely connected to the dynamic representation of mathematical objects and their relations offered by DGS: an epistemological, a cognitive and a didactical dimension. The epistemological dimension relates to the problem of transforming the behaviour of abstract mathematical objects into a materialized form. The cognitive dimension is closely related to the Theory of Semiotic Mediation as introduced by Mariotti (Chap. 9). Laborde and Laborde argue, that the dynamic character of the objects in a DGS fosters a transformation of the students' cognition of mathematical concepts. The didactical dimension addresses transformations of tasks in order to unfold the potential of the dynamic representations. This third dimension already points to the interrelatedness of the epistemological, the cognitive and the didactical dimension of learning. The transformation of one aspect, e.g. the nature of representations used, will cause transformations of other aspects, e.g. the tasks implemented in order to stimulate a cognitive activity.

The issue of interrelatedness of different dimensions of teaching and learning such as the artefacts, the tasks, and the human actors and related transformations is also taken up by Geiger (Chap. 12). His focus is not on the mediation of mathematics through artefacts, but on social aspects related to the transformation of the encounter with mathematics through technological artefacts. Geiger draws attention to the role of social setting for technology-mediated interaction. In two episodes he provides evidence that social aspects are likely to transform the mathematical experience. Whereas, one episode shows how knowledge is transformed through the interplay of different social settings the other episodes relates to the transformation of identities of participation and non-participation in classroom interaction. The episodes are analysed using the tetrahedron-model of artefact mediated learning proposed by Rezat (2006) and generalized by Sträßer (2009) in order to unveil transformative powers of different learning arrangements. As a consequence of his analysis of social aspects in technology mediated interactions Geiger enhances the tetrahedron-model of artefact-mediated learning by including social aspects in surrounding interacting social spheres.

Whereas, all the previous papers addressing transformations of the mathematical experience associated with the use of technological artefacts related to the use of DGS, Chap. 13 by Bessot focuses on another technological tool. She introduces a simulator for reading–marking out activities in building work. In her chapter, she elaborates on how this simulator transforms the relationships of the worker with space.

In order to summarize the various transformations that might occur related to mediation of mathematics through artefacts we use Rezat's and Sträßer's (2012) tetrahedron-model as an analytical tool and relate the transformations to this model.



The chapters by Kadunz (Chap. 8), Mariotti (Chap. 9), Hoelzl (Chap. 10) and Laborde and Laborde (Chap. 11) all draw attention to the transformations of representations of mathematics within the mediating artefact and their relation to the construction of knowledge. Different theoretical frameworks are used to grasp how these transformations relate to the construction of knowledge. Kadunz uses Pierce and his semiotic theory to better understand how the transformations within the artefact relate to the construction of knowledge. Laborde and Laborde argue that the instrumental approach provides a tool to better understand how knowledge is constructed by the user of an artefact. The instrumental approach offers concepts that relate the use of artefacts to the understanding of mathematical concepts.

Mariotti also draws attention to the fact that a double semiotic relationship is recognizable between the artefact and meanings emerging in the accomplishment of the task at hand and between the artefact and meanings related to specific mathematical content evoked by that use and recognizable by an expert. Thus, it seems to be a shortcoming of the tetrahedron model to subsume mathematical knowledge and didactical aspects of didactical knowledge under one vertex. This does not take into account that the mathematical situation is usually characterized by students working on tasks. Mariotti argues that the meanings that emerge from working on the task might be different from the intended mathematical meanings. Therefore, it seems appropriate to separate the task from the mathematical knowledge in the tetrahedron model. This is supported by Laborde and Laborde's discussion of relations between mathematics and its materialization in the artefact. By providing examples from the design of Cabri 3D, Laborde and Laborde exemplify that the relation between mathematics and its materialization in the artefact is not simple. The design of an artefact is likely to create a different kind of mathematics. In the case of 2D DGS, Sträßer (2001) already illustrated that with his discussion of 'monsters'. Therefore, the relation between mathematics and its materialization in an artefact needs careful consideration.

Geiger addresses another shortcoming of the tetrahedron model. Referring to Sträßer's assertion "that it may be worthwhile to think of something surrounding this tetrahedron, e.g. all those persons and institutions interested in the teaching and learning of mathematics, the 'noosphere'" (Sträßer 2009, p. 75), he pays attention to the role of social setting for technology-mediated interaction. Drawing on Borba and Villarreal, who argue that an inter-shaping relationship exists between learners and technology in which both are transformed, he shows how this interaction might be conceptualized and how the interaction is depended on the social setting in the classroom. Therefore, he argues that it is not sufficient to surround the tetrahedron-model with only the noosphere, but to distinguish between different social settings represented by different spheres.

Seeger and Sutherland discuss the chapters in part II from different perspectives. Seeger embeds and comments on the idea of transformation from the so called "embodied" perspective on human activity. By discussing the difficulties of 'meaning', 'sense' and 'existence' of mathematical objects in a wider sense, he points to the socio-cultural and historical embeddedness of any artefact. In the following, he compares the idea of transformation to the idea of 'meaning of meaning' which is always reflexive and argues that the single idea of transformation is therefore not an appropriate motto to guide research in mathematics education.

Sutherland comments on the chapters in part II from a technological point of view. She frames most of the chapters by the socio-cultural approach of Vygotsky and distinguishes several more specific theories as Rabardel's instrumental approach to describe the work with tools adequately. In addition, she focuses on constraints by implementing new technologies in mathematics classrooms and gives examples how these technologies are already used by students at home without any instruction in school. In a further step, Sutherland gives research-based ideas for design-based approaches through processes of professional development to exploit researcher–practitioner synergies.

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Chapter 8
Constructing Knowledge by Transformation, Diagrammatic Reasoning in Practice

Gert Kadunz

8.1 Reading and Writing

The reading of mathematical texts is a vital part of learning mathematics. These texts can be notes from a lesson at school or university, or some part of a textbook[1]. In a similar way, a mathematician, as a researcher, updates his/her mathematical knowledge by reading journals and monographs. However, reading a mathematical text does not mean that all the mathematics presented in such a text is embedded into the mathematical knowledge of the reader. Students in school, in particular, quite often have difficulty reading a mathematical text successfully. At the same time, reading of mathematics is very often also writing of mathematics (see Heintz 2000, p. 162). Heintz's view on writing as a fundamental activity, when doing mathematics describes the purpose of this chapter. When reading a mathematical text, one very often has to write mathematics in order to transform the text or at least certain parts into something written. Though we use the word "writing" this does not mean the use of words alone when describing mathematics. On the contrary, mathematical text uses, e.g., formulas, geometrical figures, charts etc., as they will be used in the following deliberations. Looking for recent publications in the field of mathematics education concerning the use of diagrams, the reader finds quite a number of papers (cf. Hoffmann 2005; Dörfler 2006; Kadunz 2006). In the following deliberations, these diagrams are used as as a means of presenting a text by reader to the reader himself to gain insight into the presented text. These deliberations are not only backed by these papers having their origin within mathematics education but are supported by a number of publications presented by linguists and philosophers. Their views on the use of the written focus on the value of the visible for gaining new insight and new knowledge. To name just a few, I refer to Harris (2001), Coulmas (2003), or Krämer (2003). They all explain

[1] On the use of textbooks in mathematics, see Rezat 2010.

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the use of the written and of writing as essential parts of developing new knowledge. Similar ideas can be found within the semiotics of Charles S. Peirce. In addition to, e.g., Krämer's ideas, Peirce's semiotics will be used to describe the learning of mathematics. To do so, Peirce's concept of the diagram will be explained in the following. Using this concept, the transformation from a mathematical text that is only read to one, which the reader fully understands, will be shown.

This gaining of insight is explicated using the example of reading a text from Euclidian geometry. In his book "Analytische Geometrie der Ebene" (plane analytical geometry), Max Simon[2] (1915) presented, among others, properties of the parabola. The reason for choosing this text is to reveal its nature in presenting certain geometrical theorems. In this respect, Simon's text is a product of his age. This can be seen not only by the language he used but also by the comparatively slight number of diagrams and their use. This slight number—compared to a textbook today—may have been caused by the greater effort required to produce graphics for textbooks at the end of the nineteenth century. These two facts may be the reasons why the reading of this text is challenging. In particular, the need to read this kind of antiquated language and its use of references, which are very difficult to follow as other books Simon used are no more available. Second, the existing figures have to be complemented by new figures.

We close this introduction with a view on one theorem Simon presents on page 91 (see also Fig. 8.1):

> Theorem 5. Draw rays from the focus of a parabola to all tangents of this parabola, these rays *has all having the same* constant angle with the tangents. Then we conclude that the vertices of these angles determine one *tangent of the parabola which has this* constant angle with its touching ray.

Figure 8.1 illustrates this proposition.

Theorem 5 presents an example of Euclidian geometry which could be proved using methods from synthetic geometry. This was often the case when authors in the nineteenth century tried to prove a theorem from geometry. However, today's readers are not familiar with all the theorems and explanations used at that time. Hence, it is rather difficult to read this kind of a book. To read it successfully, the modern reader has to transform the "old fashioned" kind of presentations into his/her own knowledge. These transformations will be presented in Chaps. 3 and 4. However, before this transformation is demonstrated, some theoretical remarks on the use of the semiotical concepts are necessary.

8.2 Semiotics and Mathematics Education

In his "Handbook of Semiotics" (Nöth 2000), Wilfried Nöth presents a thorough and comprehensive review of the mainstreams of modern semiotics. Nöth's handbook demonstrates that there is no universal semiotics, but a number of quite differ-

[2] Max Simon (born in Kolberg in 1844, died in Strassburg in 1918), mathematics teacher and historian of mathematics, PhD 1867 by Weierstrass and Kummer.

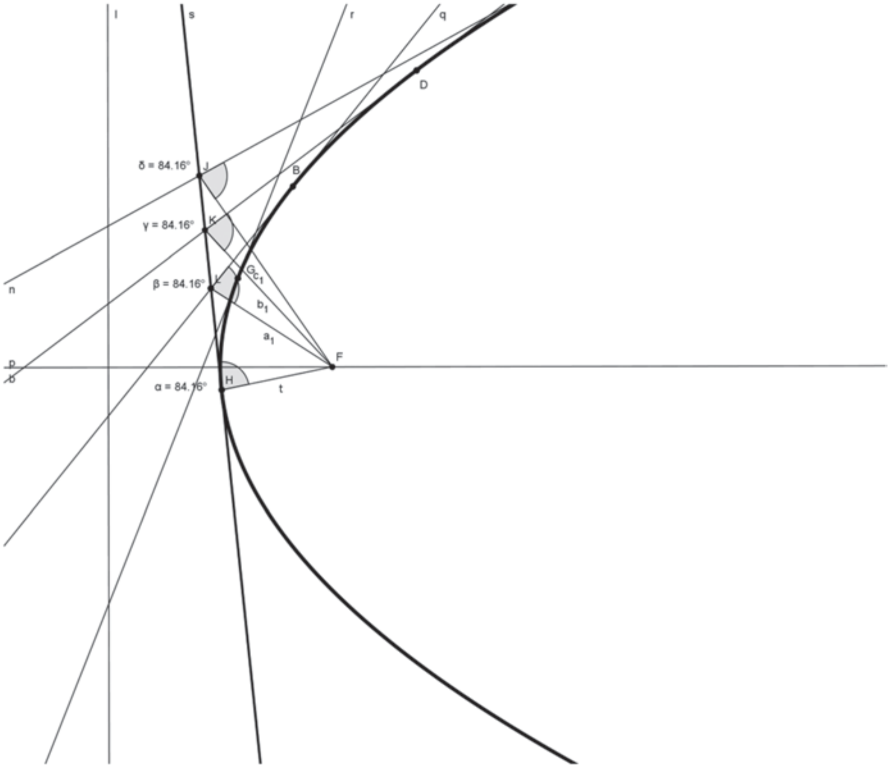

Fig. 8.1 A certain tangent of the parabola

ent ones. In addition to his presentation of well-known semioticians—from Peirce to Eco—Nöth also shows their semiotics to be valuable tools in different research areas. We can find semiotics, for example, in the field of linguistics, in aesthetics, or in media theory, to name only a few. Semiotics seem to be a very "broad" concept. The use(s) of semiotics in mathematics education seem similarly "broad" as we will now see.

If we look at papers in mathematics education, we can find numerous articles in journals and edited books treating questions from a semiotical point of view (Cobb 2000; Anderson 2003; Hoffmann 2003, 2005; Educational Studies in Mathematics Education (ESM), special issue 2006). Because of its topicality, I have chosen the special issue of ESM on semiotics from this list. Norma Presmeg and Adalira Saenz-Ludlow (Saenz-Ludlow and Presmeg 2006), the editors of this special issue, note the founding of a Psychology of Mathematics Education (PME) "discussion group on semiotics" at the 25th PME conference at Utrecht, and being continued in the following PME conferences (Norwich 2002; Honolulu 2003; Bergen 2004). The outcome of this discussion group was a basis for this special issue. Among other topics presented in this issue the reader finds semiotics as a means of studying epistemological questions, or of planning mathematics lessons (in a very wide sense), or of interpreting classroom communication. As diverse as these research questions are the semiotics used (Charles S. Peirce, Ferdinand de Saussure, or Michael Halliday) to mention only a few.

In a reviewing paper, Michael Hoffmann (2006) presents the highlights of all articles in this ESM special issue. He closes his text with an answer to the following question:

> … is there a shared conception of "semiotics" behind all the "semiotic perspectives" delivered here, and should there be one? (Hoffmann 2006, p. 290)

Hoffmann denies that one universal semiotics can be established and warns against blending different semiotics.

> This variety, however, is not necessarily a problem. As long as the terminology is consistently defined and used so that communication and understanding are possible, several semiotic approaches can be used side by side. … If we are interested in epistemological problems of learning and communicating mathematics, and if we need a highly differentiated semiotic terminology that allows very precise discussions of problems such as meaning, cognition, interaction, and interpretation in mathematics, Peirce's semiotics is by far the best tool. (Hoffmann 2006, p. 290)

In order to achieve a thorough and precise deliberation of a single problem in mathematics education, I will focus on one semiotic approach. Here, I not only follow Hoffmann (2006) but also Saenz-Ludlow and Presmeg (2006) or Schreiber (2010). Hence, I will concentrate in the following text on Peirce's semiotics as he developed not only a differentiated semiotic terminology but also used his semiotics to answer epistemological questions from mathematics. I will give a view on Peirce's famous classification of signs into icons, indices, and symbols. In particular, the role of icons and diagrams in constructing new knowledge will be investigated.

8.3 Diagrams as Means for Thinking

Like other sciences, mathematics education also deals with the concept of "representations". As a paradigmatic example, I refer to papers and research reports presented by researchers like Gerald G. Goldin or James J. Kaput (Goldin 1998a, b). Generally speaking, they investigate internal or mental representation and external or physical representation. This kind of separation between the mental and the physical representations brings up some epistemological and psychological difficulties I will not discuss here. For a detailed explanation, I refer to Falk Seeger (2000) or Michael Hoffmann (2005, 4th chapter). A remarkable development, at least of the last 20 years describes, a *turn* where those "representations" which are perceptible to our senses step into the center of interest.

> Before attributing any special quality to the mind or to the method of people, let us examine first the many ways through which inscriptions are gathered, combined, tied together and

> sent back. Only if there is something unexplained once the networks have been studied shall we start to speak of cognitive factors. (Latour 1987, p. 258)

More from art theory than from sociology, Thomas Mitchell (1994) diagnosed a *pictorial turn* and Gottfried Boehm introduced in 1994 his *iconic turn* (Boehm 1994, p. 13). With these *turns* Mitchell, Boehm, and other researchers express their interest in the epistemological importance of "representations" available to our senses. Similarly, Frederik Stjernfelt formulated the importance of icons in semiotics:

> …this return of the iconic in semiotics is probably the main event in semiotic scholarship during the recent decades…. (Stjernfelt 2000, p. 357)

In my deliberations, I will concentrate on such perceptible signs on icons and on diagrams. These diagrams will be introduced later in detail.

With his semiotics, Charles S. Peirce introduced a far-reaching project to demonstrate the importance of signs. I will point at a "trademark feature" of this semiotics. First, I only mention Peirce's view of signs as a triadic relation. This relation consists of an object, a representamen, and an interpretant[3]. They are the corners of Peirce's semiotic triangle. " … a 'sign' is *integrated* in a triadic relation whose most important feature is what he called the sign's 'interpretant'." (Bakker and Hoffmann 2005, p. 336). As I will concentrate on the second "trademark feature," I refer to papers which elaborate this triadic concept of sign (e.g., Hoffmann 2003; Bakker and Hoffmann 2005; Saenz-Ludlow and Presmeg 2006).

Peirce's "trademark feature," which I will examine, is his famous classification of signs into iconic, indexical, and symbolic.

Icon An icon is a sign which represents relations. By definition, it is a sign which is similar to its object. This similarity can lead to some misunderstanding (Stjernfelt 2000, p. 358). Critical remarks dealing with the concept of similarity can also be found in Nelson Goodman's *Language of Art* (Goodman 1976). As Stjernfelt indicates, it seems that Peirce himself had recognized some of the difficulties connected with similarity. Icons are not in themselves similar. The impression of similarity comes into existence from possible activities we can do with the icon.

> The icon is not only the only kind of sign involving a direct representation of qualities pertaining to its object; it is also—and this amounts to the same—the only sign by the contemplation of which more can be learnt than lies in the directions for its construction. (Stjernfelt 2000, p. 358)

These constructions and the activities with them may be the source of new knowledge, as I will show in the following using diagrams, which are intimately related to icons.

Index Following Peirce, a sign is an index, which focuses the attention of a person using this sign. We can find indexes in our everyday language when we use words

[3] To give an example of a sign we can think of a barometer as representamen, which gives some information about its object, the air pressure. A person recognizing a change of the air pressure can interpret this change. It is remarkable that the representamen itself can be called a sign. For more information see the previously cited literature.

indicating something. If we think of geometrical drawings then the labels on these drawings are indexes as they point to certain parts of the construction.

Symbol A symbol is a sign, the use of which is given by definition. We can find symbols in words of a language as the meaning of a word, which has to be learned by definition. In mathematics, symbols are widely used. We can think of e or pi to name the most famous ones. Also, letters used as variables in an equation are symbols in this sense.

Diagram According to Charles S. Peirce, icons can be further classified into images, diagrams, and metaphors. From these three, the diagram will have the greatest importance for the rest of my chapter. Diagrams are icons, which are *constructed* following certain rules, and may thereby show relations. When we look for diagrams, we can find them in geometry. Every drawing obeying the rules of geometry is a diagram. In the same sense, a written sentence is a diagram if it follows the grammar. On the other hand, the reader reading this sentence has to know the grammar to decide whether it is a diagram. Therefore, a diagram is not a diagram by itself!

However, diagrams are in most cases very complex signs. If we again take a diagram from geometry, we see in it symbols, indexes, and even other diagrams. As an example, we can imagine the drawing of a triangle and its circumscribed circle. The labels of its corners are indexes and symbols too. If we label the circle with "solution" then we have another symbol. The triangle itself is a diagram, as it is constructed using segments connecting three points in a special way.

Alongside this use of rules in constructing diagrams, the operational view on diagrams I mentioned previously for icons (Stjernfelt) will now be discussed. This operational view will be made use of in the interpretation of readers' activities to be presented in Sect. 8.4. With diagrams as a special kind of icons, we can perform experiments when learning mathematics. Doing experiments and constructing new knowledge is called diagrammatic reasoning (Hoffmann 2003; Bakker and Hoffman 2005). How can we imagine such reasoning when learning mathematics?

In a first step, a diagram has to be constructed. To give some examples, this may be an equation from algebra, a geometrical drawing using software, or pencil and paper, or designing a graph to solve a problem from graph theory. Once construction has been finished, we can start experimenting. The algebraic equation may be transformed following the rules from algebra. If we have used a software (DGS) for constructing the geometrical drawing, we can use the drag mode (Arzarello et al. 2002) to change the construction without destroying the geometrical relations of the drawing. However, we could also implement a new line or segment or even a new label into the drawing to gain a new view. This also means that when performing experiments we have to obey the rules governing the system.

> What makes experimenting with diagrams important is the rationality that is immanent to them… The rules define the possible transformations and actions, but also constraints of operations on diagrams. (Bakker and Hoffmann 2005, p. 340)

In a final third step, the results of the experiment are explored. In front of the observers' eyes new relations can become visible. A new configuration may show "itself." A new pattern (Oliveri 1997) may be visible within the algebraic equation. Making use of the DGS drag mode, the continuous movement of parts of the drawing may raise the idea of the equality of areas.

> As Peirce wrote, the diagram constructed by a mathematician 'puts before him an icon by the observation of which he detects relations between the parts of a diagram other than those which were used in the construction' (NEM III, 749). (Bakker and Hoffmann 2005, p. 341)

With this citation, I close my remarks on diagrams and diagrammatic reasoning. I will finish this part with some hints on two further concepts Peirce presented. I will use them as a tool to "measure" the creativity of our students. In his semiotics, Peirce introduced two interesting concepts to describe logical deduction.

> There are two kinds of Deduction; and it is truly significant that it should have been left for me to discover this. I first found, and subsequently proved, that every Deduction involves the observation of a Diagram (whether Optical, Tactical, or Acoustic) and having drawn the diagram (for I myself always work with Optical Diagrams) one finds the conclusion to be represented by it. Of course, a diagram is required to comprehend any assertion. My two genera of Deductions are first those in which any Diagram of a state of things in which the premises are true represents the conclusion to be true and such reasoning I call Corollarial because all the corollaries that different editors have added to Euclid's Elements are of this nature. To the Diagram of the truth of the Premises something else has to be added, which is usually a mere May-be, and then the conclusion appears. I call this Theorematic reasoning because all the most important theorems are of this nature. (Peirce 1998, A Letter to William James, EP 2:502, 1909)

As we see, the corollarial deduction is the more simple form of deduction. It describes those logical activities we have to do when we draw a conclusion from observing a diagram without changing this diagram. Take, for instance, an isosceles triangle with its axis of symmetry drawn in. Then we can deduce corollarially that the base angles of this isosceles triangle are equal. If we draw a new or change a given diagram and we deduce a conclusion then we have done a theorematic deduction. Mathematical argumentations or the proving of theorems are in most cases examples of theorematic deduction. I return now to Max Simon's analytical geometry and its theorems.

8.4 The Text: Transforming Diagrams into Diagrams

In Chap. 8 of his book, Simon introduces basic theorems of the parabola. Within this chapter, § 23 is dedicated to features of the focus of the parabola. To assist the reader, Simon refers to a list of theorems he had already presented and also proved. Some of these proofs were done with the aid of analytical geometry, while others rely on methods from synthetic geometry. This kind of blending of methods may have been common at the end of the nineteenth century; however, it does not

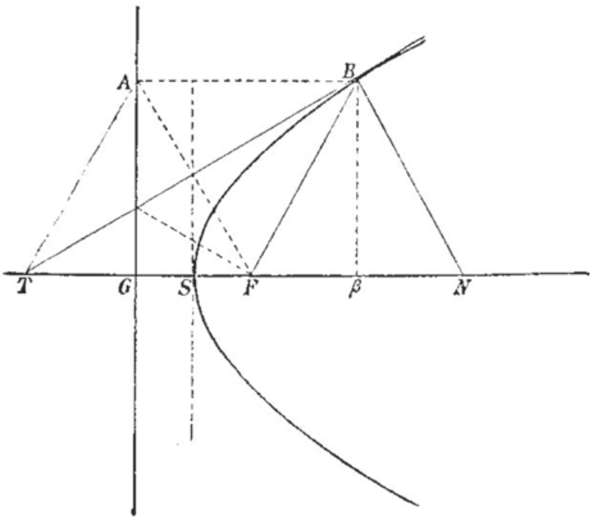

Fig. 8.2 Simons's figure 16

support a successful reading of this text. Such a blending contradicts, at least in my view, Freudenthal's advice that a proof of a theorem should, among other tasks, also locally organize the theorem (cf. Freudenthal 1973). A merging of methods from synthetic geometry with methods of analytical geometry may show a student reader, as a learner, the relations between these two methods but it seems not to be very supportive for understanding what is written.

What are Simon's arguments to prove the previously mentioned theorem 5? The theorem itself can be found as number 5 in a list of several theorems (cf. Simon 1915, p. 91). This list is headed by further properties of the focus. To accept these properties some fundamental characteristics of the parabola are presented. Some of them are relevant to understanding the proof of theorem 5. In particular, we can read relations between the direction of the ray (through the focus F and the point B on the parabola) and the tangent at point B (cf. Fig. 8.2, which shows Simon's Fig. 16).

Figure 16 is Simon's proof of how to draw a tangent at a given point of the parabola. He refers to the diamond ABFT and claims that the tangent at point B bisects the angle of the ray FB and the perpendicular through B to the directrix of the parabola. This is a well-known property. Simon continues with some more characteristics of the tangent. He ends with the answer for how to draw a tangent from a point P outside the parabola. He explains:

> Draw a circle with PF as its radius around P. This circle intersects the directrix at A and A1. The bisectors of FA or FA1 are the tangents through P, where B and B1 are boundary points. (Simon 1915, p. 90)

To explain this algorithm Simon offers Fig. 17 (see Fig. 8.3). An experienced geometrician would not ask any questions. However, a learning student has some difficulty seeing all the steps from Simon's Fig. 16 to Fig. 17 (cf. Fig. 8.3). Within Fig. 16, properties of the tangent are shown, whereas Fig. 17 shows the construction

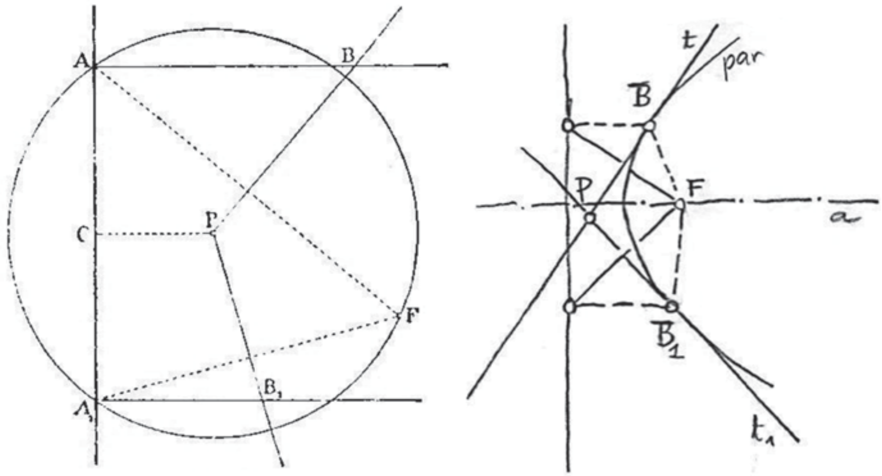

Fig. 8.3 How to draw a tangent

of the tangents to a parabola where no parabola is visible. Therefore, a sketch shown in Fig. 8.3 illustrates the tangents through P and the given parabola.

Let us follow Simon's deliberation. However, before we continue we have to point to the main didactic interest. To be able to understand Simon, a certain amount of specific geometrical knowledge is necessary. Otherwise, the student reading Simon's text has to translate this text together with the geometrical diagrams into a new text with some new and maybe simpler diagrams. With such a transformation he/she should be able to follow Simon successfully.

If a learning student accepts the tangent as an angle bisector then he/she could accept (together with Simon's Fig. 17) the equations AP = PF and PF = PA1. Therefore A1, A, and F are all on a circle with centre P. This circle is used by Simon to claim that the triangles A1PF and ABF (cf. Fig. 17 in Fig. 8.3) are similar. Simon writes:

> Complete Fig. 17 by connecting F with B and with B1. Draw the axis of symmetry of AA1 which passes through P. Remove AF and A1F. With these steps Fig. 18 is drawn. The central angle A1PF is twice as large as the peripheral angle A1AF, which is the same size as ABP. This holds as both of them add the angle BAF to 90°, hence angle(A1PF) = angle(ABF). Therefore the triangles A1PF and ABF are similar and furthermore their halves are similar. (cf. Simon 1915, p. 90)

Finally referring to Fig. 18 (see Fig. 8.4) Simon concludes

> Due to the fact that the angles B1PF and PBF are equal, this angle does not depend on the position of P on the tangent PB. … Draw rays from the focus of a parabola to all tangents of this parabola, these rays all having the same constant angle with the tangents. Then we conclude that the vertices of these angles determine one tangent of the parabola which has this constant angle with its touching ray. (cf. Simon 1915, p. 91)

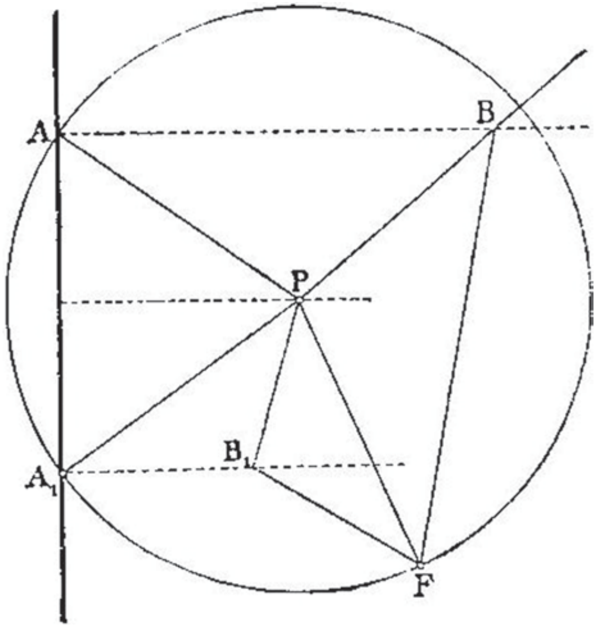

Fig. 8.4 Simon's final diagram

This quite long list of conclusions is backed by a few sparsely executed geometrical diagrams. Consequently, the comprehension of all these conclusions would seem to be quite difficult for a student. However, as indicated in Fig. 8.3, all these conclusions can be explained by sketches or using Perice's concept diagrams. With their help these conclusions may be embedded into the mathematical experience of the student. In this sense, mathematical knowledge is constructed by the transformation of the "spoken"—cf. the list of arguments within the earlier paragraph—into a list of diagrams. What does this mean for Simon's theorem? Let us start with the relation of the central angle to the peripheral angle.

Figure 8.5 presents the diagram. After drawing the segment A1F the claim becomes immediately visible. The next conclusion to explain is the proposition angle(A1PF) = angle(ABF). Which good arguments can we see? At first, we look back to the sketch in Fig. 8.3. The direction of BA is orthogonal to the directrix (line through A1A). Hence, the equation angle(A1AF) + angle(FAB) = 90° is valid. At the same time, BP is the tangent at point B where B is a point on the parabola. The triangle ABF is isosceles and the tangent at B is the axis of symmetry of this triangle. Therefore, the direction of AF is orthogonal to BP. Observing this relation between these directions and observing the diagram in Fig. 8.5 we conclude angle(ABP) + angle(FAB) = 90°. Hence, the triangles A1PF and ABF are similar.

But the proof of theorem 5 needs the independence of point P on the tangent. Another diagram may help (see Fig. 8.6). The lines through BP and PB1 are tangents to the parabola at B and B1. Hence, they are axes of symmetry of triangle ABF and triangle A1B1F. We conclude that the triangles BPF and PB1F are similar. If

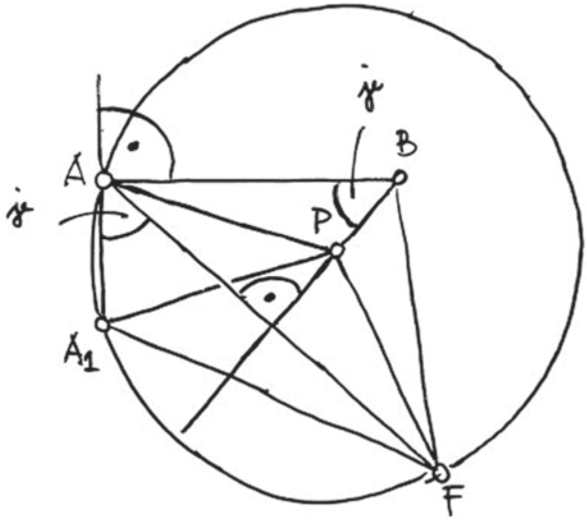

Fig. 8.5 Similar triangles

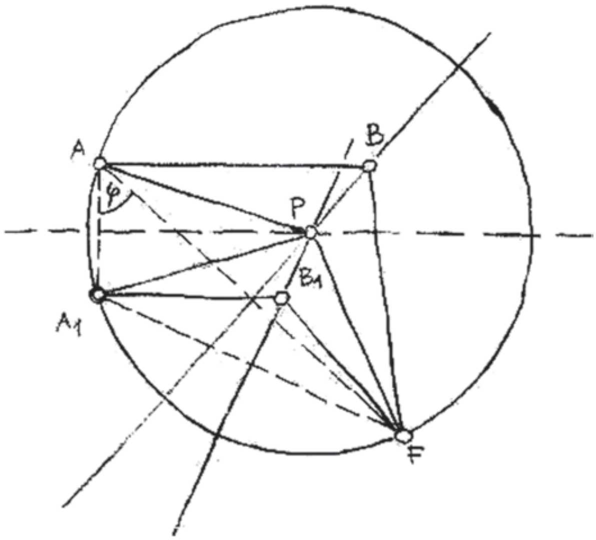

Fig. 8.6 Point P is independent

point P is moved along its own tangent BP then we are always able to determine the second tangent through P. There is always a point B1 at the parabola where PB1 is the tangent and the triangle A1PF is similar to triangle ABF. Therefore, the theorem holds. Figure 8.1 shows three of these similar triangles. At this point of my deliberations, I should like to point out that all "handmade" diagrams which were presented above mirror the knowledge of one single reader of Simon's text. In this sense, they are idiosyncratic and mirror the geometrical knowledge of one single student.

However, I wanted to demonstrate the more important fact of inventing diagrams to transform a mathematical text.

8.5 Conclusion

Let us reexamine this kind of reading. The understanding of Simon's deliberations is determined by the use of diagrams which follow the rules of geometry. On the one hand, Simon's figures were supplemented by lines and labels. This results in a valuable support for the reader. The diagram in Fig. 8.3 assists balancing Simon's Fig. 17 with his text as the sketched parabola and the tangents appear together with his more schematically executed diagram. The reader recognizes the property of the tangent as the axes of the isosceles triangle and the transitivity of the equals sign guarantees the use of the circle around P. No more deliberations are necessary. The existence of this circle is a corollarial deduction in Peirce's sense.

One step beyond this corollarial deduction seems to be the use of the diagram in Fig. 8.5. This diagram is used to explain to the reader of the given text the similarity between triangles. Several geometrical properties can be read into this diagram. There is, e.g., the orthogonality of BA with the directrix of the parabola or the already-mentioned property of the tangent as the axis of symmetry. The reader transforms the "printed" diagram into his/her own diagram. However, this is not enough to see the similarity. Another kind of diagram is used. Two equations are written beside each other and as a result the reader can see immediately and plainly that angle(ABP) equals angle(A1AF). The diagram becomes the source of certainty. In other words, the source of certainty is based on the construction of new diagrams from given diagrams, which are drawn by the reader himself/herself.

To sum up, this (re)construction of the reading of a mathematical text illustrated the intensive collaboration of reading and writing, whereby writing mainly consisted of the construction of geometrical diagrams. These had to explain the given text step by step. The reading of this text can be seen as a transformation from predominantly written diagrams to mostly constructed diagrams.

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Chapter 9
Transforming Images in a DGS: The Semiotic Potential of the Dragging Tool for Introducing the Notion of Conditional Statement

Maria Alessandra Mariotti

9.1 Introduction

The advent of dynamic geometry systems (DGSs) has dramatically changed the possible scenario of geometrical experiences at school. The transition from the traditional graphic environment based on paper and pencil, to the virtual graphic environment based on figures on the screen, realized by graphical tools and transformed by acting through the mouse, has the potential of deeply affecting the way students conceive and reason upon geometrical figures.

> Line segments that stretch and points that move relative to each other are not trivially the same objects that one treats in the familiar synthetic geometry, and this suggests new styles of reasoning. (Goldenberg 1995, p. 220)

DGSs for computers and calculators, such as the *Geometer's Sketchpad* (Jackiw 2009) and *Cabri géométre* (Laborde and Bellemain 1995), have been at the core of a number of studies claiming the potential to impact the teaching and learning of school geometry (Healy and Hoyles 2001; Hölzl 2001; Jones 2000; Laborde 2000; Mariotti 2000; Sträßer 2001; and for an extensive review, see Battista 2007; Laborde et al. 2006). Since the very beginning of their appearance research studies have highlighted the potential offered by DGSs in supporting students' solution of geometrical problems:

> […] the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning. (Laborde and Laborde 1991, p. 185)

Specifically, studies have investigated the support provided by a DGS in the solution of open problems that require the formulation of a conjecture. These types of tasks have been discussed to claim their didactic potential, not only with respect to the use of a DGS (Hadas et al. 2000; Boero et al. 2007; Pedemonte 2008).

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The use of a DGS, like Cabri, in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control that involves converting perceptual data into a conditional statement. During the conjecturing process, the way of transforming and observing screen images is directed by the intention of revealing a significant relationship between geometric properties, a relationship that may be formulated in the statement of a conjecture.

Elaborating on the seminal work of Arzarello, Olivero, and colleagues (Arzarello et al. 2002; Olivero 2001, 2002), the potential offered by a DGS not only in supporting conjecturing processes but also in mediating the mathematical meaning of *conjecture* and specifically of *conditional statement* in the geometry context is discussed.

The following discussion is framed within the theory of semiotic mediation (TSM) as it has been introduced by Bartolini Bussi and Mariotti (2008). In this specific theoretical framework, the semiotic potential of particular modalities of dragging with respect to the notion of *conditionality* is discussed. Any dragging mode can be considered as a specific artifact used to solve an open problem, meanings emerging from this use may be referred to the mathematical meaning of conjecture, that is, of a conditional statement expressing the logical dependency between a *premise* and a *conclusion*.

A brief outline of the TSM and specifically of the notion of *semiotic potential* and that of *didactic cycle* is given, then specific dragging modalities explaining how they can be related to the mathematical meanings of *premise*, *conclusion*, and *conditional link* between them are analyzed. Some illustrative examples are given, drawn from a recent study carried out at the upper secondary level, focusing on a particular process of conjecture generation (Baccaglini-Frank 2010a, 2010b; Baccaglini-Frank et al. 2009; Baccaglini-Frank and Mariotti 2010).

9.2 The Theory of Semiotic Mediation

In relation to the use of particular tools, specifically in relation to the use of new computer-based technologies in school practice, the term *mediation* has become widely present in the current mathematic education literature (Meira 1998; Radford 2003; Noss and Hoyles 1996; Borba and Villarreal 2005). However, the term "mediation" has been often employed in an unclear way, mixing up two interrelated potentialities of a given tool. On the one hand, the tool may be used and it may successfully contribute to the accomplishment of a task, and on the other hand, the use of the tool may foster learning processes concerning mathematical ideas.

What remains unaddressed is the epistemological issue concerning the relationship between the accomplishment of a task and the student's learning. This leaves implicit the key elements of the mediation process[1], which is triggered by the use of a specific tool and that is related to particular mathematical knowledge.

[1] Someone who mediates, i.e., a *mediator*; something that is mediated, i.e., a *content/force/energy* released by mediation; someone/something subjected to mediation, i.e., the "*mediatee*" to whom/

The TSM elaborated in Bartolini Bussi and Mariotti (2008) addresses this issue, combining a semiotic and an educational perspective and considering the crucial role of *human mediation* (Kozulin 2003, p. 19) in the teaching–learning process.

Starting from the notion of *semiotic mediation* introduced by Vygotsky (1978), we analyze the role of tools and of their functioning in the solution of specific tasks, and outline a model that describes how a specific tool can be exploited by the teacher as a means to enhance the teaching–learning process seen as effect of social and cultural interaction.

In the following, a short introduction of the model is provided, strictly finalized to clarifying the analysis of the dragging tool and the subsequent discussion of the examples that constitute the core of this contribution (for a full discussion and more references, see Bartolini Bussi and Mariotti 2008 and Mariotti 2009).

Following Vygotsky, we used the semiotic lens to describe individual knowledge construction in terms of *internalization* (Vygotsky 1981, p. 162) that constitutes the unifying element of description (Wertsch and Addison Stone 1985). The basic assumption concerns the claim that internalization is essentially a social process based on the communication dimension and on the asymmetric role played by the main interlocutors: the teacher and the students.

Specifically, the social use of a certain tool in accomplishing a task makes meanings emerge and these are shared via different semiotic means (verbal, gestural, etc.). Such meanings directly refer to the tool and its use in the context of the task; however, if observed from the point of view of an expert, they may also be related to specific mathematical content. Following Hoyles (1993), we can consider the relation between the tool's use and mathematics as *evoked knowledge*: for an expert, the teacher for instance, the use of a tool may evoke the mathematics knowledge that one has to resort to in order to solve a specific task. In this sense, one says that a DGS may evoke the classic "ruler and compass" geometry or the abacus may evoke the positional notation for numbers.

Hence, a *double semiotic relationship* is recognizable: (1) between a tool and meanings emerging in the accomplishment of the task and (2) between the tool and the meanings related to specific mathematical content evoked by that use and recognizable by an expert. We define (Bartolini Bussi and Mariotti 2008) this double semiotic link as the *semiotic potential* specific to the tool.

> A double relationship may occur between an artifact and on the one hand the personal meanings emerging from its use to accomplish a task (instrumented activity), and on the other hand the mathematical meanings evoked by its use and recognizable as mathematics by an expert. (op. cit., p. 754)

On the basis of the distinction between meanings emerging from the use of the tool and shared in a social interaction and mathematical meanings related to specific mathematical content, we can interpret the teaching–learning activity as organized

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which mediation makes some difference; the circumstances for mediation, viz, (a) the means of mediation, i.e., *modality*; (b) the location, i.e., *site* in which mediation might occur. For a full discussion, see Hasan (2002).

around the goal of making students' personal meanings evolve into mathematical meanings. In other terms, we can see the educational intervention as a way of exploiting the semiotic potential of a specific tool. On the one hand, the teacher organizes didactic situations where students use the tool and consequently are expected to generate specific personal meanings. On the other hand, the teacher organizes social interactions in order to support the transformation of the personal meanings that emerged in the artifact-centered activities into the mathematical meanings that constitute the teaching objectives.

> Thus any artifact will be referred to as tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. (Bartolini Bussi and Mariotti 2008, p. 758)

The complex semiotic processes of the emergence and transformation of personal meanings, evolving toward mathematical meanings can be developed through the design and implementation of the so-called *didactical cycle* (op. cit., p. 754 ff). Because of the specific focus of this contribution, the description of any more detail on this part of the model will not be dealt with.

Of course, in order to make use of a tool as a *tool of semiotic mediation*, the teacher needs to be aware of its semiotic potential both in terms of personal meanings that are expected to emerge when students are involved in specific tasks, and of the mathematical meanings that may be evoked by these activities. This asks for a careful a priori analysis, from both a cognitive and an epistemological perspective, of the use of a specific tool with respect to specific mathematical meanings that are educationally significant.

The discussion developed in the following section concerns the semiotic potential of a DGS with respect to the didactic goal of introducing students to conjecturing and developing the mathematical meaning of conditional statement, i.e., a logical dependency between premise and conclusion.

9.3 Transforming Figures in a DGS: Dragging and Invariants

Dynamic geometry provides, besides the traditional classic graphic representation, a new dimension—*dynamism*—that leads to a potentially quite powerful representation. The basic rationale behind dynamic geometry is that geometrical objects and properties can be presented in a dynamic format, which means that any figure that has been constructed using specific primitives can be *acted upon* by using the mouse.

This last action, generally referred to as "dragging" modality, constitutes the true novelty of such environments, and determines the well-known phenomenon of *moving figures* that gave the name to this category of softwares. After a construction is accomplished, the user may activate the dragging tool and determine the movement of the figure on the screen.

Perceiving the movement of the figure actually originates from the visual effect produced by a rapid sequence of images that are produced one after the other by the system according to the variation of the input (a new position of the dragged point) and the construction procedure given by the original sequence of the graphical tools. The perception of a *moving figure* comes from the fact that there is something that changes and something that is preserved: what is preserved—that is *invariant*—constitutes the identity of the object/figure in contrast to the changes that determines its transformation and consequently its movement.

> Dynamic geometry exteriorizes the duality invariant/variable in a tangible way by means of motion in the space of the plane. (Laborde 2005, p. 22)

The invariants correspond to the properties that are preserved and allow the observer to recognize the sequence of images as the same element in movement. The interplay between variation and invariants is the core of the process of categorization, what allows us to recognize quite different objects as belonging to the same category, somewhat like recognizing a friend's face over time.

But which are the "invariants" that are at the basis of the movement of figures in a DGS?

Actually, in a DGS like Cabri, there are two kinds of invariants appearing simultaneously as the dynamic figure is acted upon, and therefore "moves": first, there are the invariants determined by the geometrical relations defined by the commands used to accomplish the construction, we call them *direct invariants*, and second, there are the invariants that are derived—*indirect invariants*—as a consequence within the theory of Euclidean geometry (Laborde and Sträßer 1990).

The relationship of logical dependency between the two types of invariants corresponds to an asymmetry between the two types of invariants, an asymmetry that can also be recognized in the relative movement of the different elements of a figure. Dragging is accomplished by acting on the basic points, those from which the construction originates, but their movement will determine the motion of the other elements of the figure obtained through the construction. Thus, there are fundamentally two different types of movements that, as we will see in the following, are worth distinguishing and analyzing carefully: direct motion and indirect motion.

The *direct motion* of a basic element (for instance, a point) consists in the variation of this element in the plane under the direct control of the mouse.

The *indirect motion* of an element (a point of any other element of a figure) consists in the variation of this element as a consequence of the direct variation that can occur after a construction has been accomplished.

Therefore, the experience of dragging constructed figures allows the user to distinguish between *direct invariants* and *indirect invariants*, because the action of dragging can allow the user to "feel" motion dependency, which can be interpreted in terms of logical dependency. The distinction between direct and indirect invariants can be interpreted in terms of logical consequence between properties within the geometrical context. Consistently with this analysis, previous studies showed how the semiotic potential of dragging and constructions in Cabri could be exploited with the aim of introducing pupils to a theoretical perspective (Mariotti 2000, 2001, 2007, 2009).

Starting from this phenomenological analysis of the dragging tool, we now focus on the specific task of conjecture generation to analyze the semiotic potential of specific dragging modalities that can be used to solve it. Our discussion is consistent with classic results coming from previous studies where the dragging strategies were described (Arzarello 2000; Hölzl 1996; Leung and Lopez-Real 2000; Leung and Lopez-Real 2002; Lopez-Real and Leung 2006; Healy 2000), and aims to elaborate on them within the frame of the theory of semiotic mediation.

Specifically, we discuss how different dragging modalities can be used to produce a conditional statement. First, we consider the case of exploring the consequences of a certain set of premises, then we consider the case of finding under which conditions a given configuration takes on a certain property (as in Arzarello et al. 2002; Olivero 2002).

9.4 Dragging to Produce a Conditional Statement

The term "open problem" is common in mathematics education literature (Arsac and Mante 1983; Silver 1995) to express a task that poses a question without revealing or suggesting the expected answer. In the geometry context, open problems can consist of tasks requiring the formulation of a conjecture starting from a given configuration, i.e., a figure of which specific properties are given. The solver is let free to explore the possible significant properties that are compatible with the original configuration and to formulate a conditional statement linking the given properties and their possible consequences. As the previous analysis highlights, dragging for producing a conjecture requires a complex interpretation of perceptual data coming from the screen; clearly, such an interpretation presents a higher complexity as compared to, for example, dragging to test the correctness of a construction. It is not enough to observe the figure and its movement globally, but it is also necessary to analyze and decompose the image appearing on the screen, according to its elements and their properties, in order to recognize a geometrically significant relationship between them.

9.4.1 Dragging to Search for Consequences

In this case, the statement of a conjecture is generated by the interpretation of perceived invariants taking into account their logical hierarchy induced by the original construction. Consider the following example.

ABCD is a quadrilateral in which D is chosen on the parallel line to BC through A, and the perpendicular bisectors of AD and BC are constructed (Fig. 9.1).

Dragging any of the free points, the constructed parallelism and perpendicularity are preserved, but it also happens that the parallelism between the two perpen-

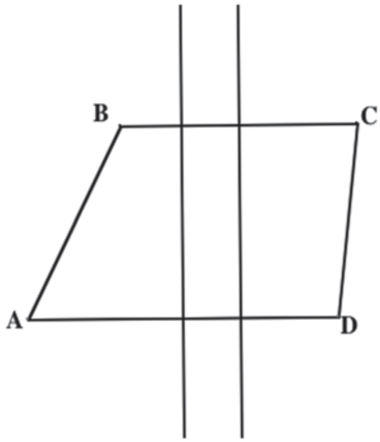


Fig. 9.1 ABCD as a result of the construction described in the earlier example

dicular bisectors is preserved. Complexity resides in being aware of the hierarchy induced on the different invariants, in spite of the fact that they are simultaneously perceived, and interpreting such a hierarchy as a logical dependency between properties of the "geometric figure." In the earlier example, the exploration by dragging can lead to the following conjecture: "if two sides of the quadrilateral are parallel, then the corresponding perpendicular bisectors are parallel" (Fig. 9.2).

In other words, what appears on the screen while dragging, that is, the fact that a specific relationship between invariants is preserved, corresponds to the general validity of a logical implication between properties of a geometrical figure.

The distinction between direct and indirect movements plays a key role in identifying and discerning the given properties and their consequences. As far as dragging is concerned, a dynamic figure moves when its basic points are acted upon. In the earlier example, A, B, and C are basic points of the dynamic figure and they can be dragged to any place on the screen (in this case, we speak of free dragging), while D can only be dragged along the parallel line to BC through A. The perpendicular bisectors, as dependent elements of the construction, cannot be directly acted upon, but they will move indirectly and their parallelism will be an indirect invariant. The different status of the elements of the figure, basic or dependent, as can be experienced through dragging, corresponds to the different logical status of the geometrical properties, premises, or consequences of a conditional statement. Therefore, we claim that the dragging tool used in solving a conjecture-production task has a *semiotic potential* recognizable in the relationship between:

- *Direct and indirect invariants and premise and conclusion of a conditional statement and*
- *The dynamic sensation of dependence between the two types of invariants and the geometrical meaning of logical dependence between premise and conclusion.*

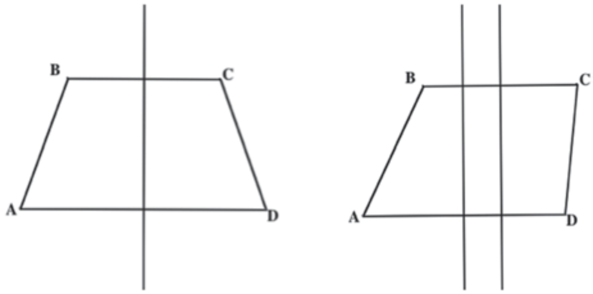


Fig. 9.2 The figures show the effect of dragging ABCD's basic point C while trying to maintain the coincidence of the perpendicular bisectors

9.4.2 Dragging to Search for a Premise

As shown by many studies in the literature, exploring the consequences of a certain set of premises is not the only possible use of dragging for generating conjectures. A different way involves the induction of a specific property by a "constrained" type of dragging. This way of dragging produces a new kind of invariant, which has been classified as a *soft* invariant, as opposed to a *robust* invariant that refers to direct invariants induced by a construction.

A soft invariant is a property "purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student" (Healy 2000; Laborde 2005). The use of soft invariants in the solution of conjecture problems has been observed in previous studies and described with different names, like *lieu muet* or *dummy locus dragging* (Arzarello et al. 2002). In a recent study, it has been referred to as *maintaining dragging* (Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2010).

As explained earlier, the control of the status of the different kinds of invariants is based on an enacted distinction between direct and indirect movements. This distinction leads to consider the new type of invariant, emerging from the specific goal-oriented dragging as *indirectly induced invariant*: A property occurs because of the movement of a basic point. Such a movement is direct, but controlled by the objective of causing a specific property to occur[2] (Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2010).

Consider the following example. Given a quadrilateral, construct the bisector of its sides, their intersections generate a new quadrilateral. Dragging freely, one observes that in some circumstances the internal quadrilateral collapses in one point. This may represent an interesting property for the solver, thus he/she decides to explore under which circumstances this may happen. In the earlier example, it is possible to try to induce the soft invariant "coinciding perpendicular bisectors" and search for a specific *condition under which* such property occurs.

[2] Referring to the intentionality of the action, Baccaglini-Frank (2010) calls this kind of invariant the intentionally induced invariant. For the objective of this contribution, it is not necessary to introduce the terminology elaborated by Baccaglini-Frank.

Using maintaining dragging, the special movement of a basic point intentionally induces the occurrence of a selected property and makes the figure assume a specific configuration as the consequence of a geometrical condition corresponding to the goal-oriented movement.

Once again we have the *simultaneity* of two invariants; however, they have a different status that comes from the *different control* exerted by the solver acting upon the figure. The haptic sensation of causality can be referred to the *conditionality* relating the unknown condition realized by the direct movement and the selected property indirectly induced as invariant, the first corresponding to the premise and the second to the conclusion of a conditional statement.

The different status of the two invariants is clearly discernible by their characteristics related to the exploration process carried out by the solver.

On the one hand, the indirectly induced invariant that will become the conclusion of the conjecture has the following characteristics that make it clearly recognizable:

> It is a property that is intentionally selected and may be induced indirectly as (soft) invariant by moving a basic point.

On the other hand, the condition destined to originate the premise has the following characteristics:

> It is searched for in response to the questions "what might cause the Indirectly Induced Invariant?", it is recognizable in the constrained movement performed during the maintaining dragging.

In summary, the asymmetry of the relationship between invariants in a DGS offers a great potential with respect to distinguishing the logical status of the properties that determines their belonging to the premise or the conclusion. Thus, according to our analysis it is possible to outline the following *semiotic potential of the maintaining dragging tool* in solving a conjecture-production task, recognizable in the relationship between

- The indirectly induced invariant, i.e., the property the solver intends to achieve, and the conclusion of the conjecture statement.
- The invariant constrained by the specific goal-oriented movement, i.e., the property that must be assumed in order to obtain the induced invariant, and the premise of the conjecture statement.
- The haptic sensation of causality relating the direct and indirect movements and the geometrical meaning of logical dependence between premise and conclusion.

9.5 The Teaching Experiment

As part of a broader research study (Baccaglini-Frank 2010), a teaching experiment was carried out with students of scientific high schools (licei scientifici) (a total of 31: 14 pairs and 3 single students.). The students had been using a DGS, specifically Cabri, for at least 1 year prior to the study. Different dragging modalities were explicitly introduced during two 1-hour introductory lessons (for details, see

Baccaglini-Frank 2010). Specifically, we introduced the maintaining dragging in relation to exploring a configuration to formulate conjectures. Subsequently, pairs of students were observed during a problem-solving session where four different open problems were proposed.

During these interview sessions, we intended to observe and describe the relationship between students' use of the different modalities of dragging, in particular of the maintaining dragging, and the production of conjectures.

Data collected included: audio and video tapes and transcriptions of the introductory lessons; Cabri files worked on by the instructor and the students during the classroom activities; audio and video tapes, screenshots of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews.

Among other results (for a full discussion, see Baccaglini-Frank 2010), the analysis of the collected data provides evidence supporting the previous analysis concerning the semiotic potential of the dragging tool and in particular of maintaining dragging with respect to conjecture tasks. In the following section, we discuss two examples drawn from this corpus of data.

9.6 The Unfolding of the Semiotic Potential

During the interviews, pairs of students were observed while solving open problems in which the production of a conjecture was required. The use of different modalities of dragging and specifically the maintaining dragging modality was promoted so that we could observe the relationship between the enactment of dragging and the emergence of meanings related to it. The following examples illustrate students' actions and their interpretations of their experiences with the images on the screen. The analysis of students' behavior will be related to the mathematical meanings discussed earlier. We will observe how the semiotic potential of dragging modalities unfolds. For the reader's convenience, the examples that we present will refer to the same open problem, which is reported subsequently.

Problem

- Draw three points: A, M, and K.
- Construct point B as the symmetric image of A with respect to M and C as the symmetric image of A with respect to K.
- Construct the parallel line l to BC through A.
- Construct the perpendicular to l through C, and construct D as the point of intersection of these two lines.
- Consider the quadrilateral ABCD.

Make conjectures about the types of quadrilaterals that can emerge, and try to describe all the ways in which it can become a particular type of quadrilateral.

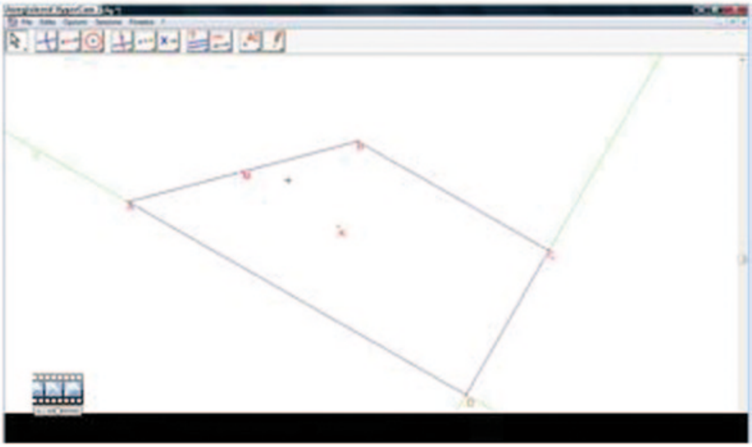


Fig. 9.3 A screen shot from the students' exploration in Cabri

9.6.1 Dragging to Search for Consequence

The following excerpt concerns the kind of exploration carried out at the very beginning of the solving process. We can observe how two students (Pie and Ale) notice and describe an indirect invariant, after free dragging of a basic point.

In the following excerpt, students are referring to the screen image which is shown in Fig. 9.3.

Excerpt 1

Transcript	Analysis
[1] Pie: the segment BC … if it varies what does it depend on?	Student's attention focuses on one of the elements of the figure—segment BC—and he asks himself what does its variation depend on
[2] Pie: So, point B is the symmetric image of A …	In order to identify dependency, the student comes back to the given properties that determine the original configuration
[3] Ale: I think that the segment [pointing to BC] is fixed [4] Pie: …and C is the symmetric image of A with respect to K. Therefore if I vary A, C varies too [5] Pie: because…they are…I mean A has influence over both B and C [6] Ale: But the distance between B and C always stays the same	In the exchange between the two students, the main elements come out: the identification of an invariant and the explicit expression of the dependence between basic points and constructed points. Ale identifies an invariant, Pie tries to link such invariance to the given properties, at the same time expressing such link in terms of dependency on the variation of basic points: "Therefore if I vary A, C varies too. […] I mean A has influence over both B and C"

Transcript	Analysis
[7] Pie: Here there is basically AK and KC, which are the same and AM and BM are always the same [8] Ale: Yes, try to move it? [9] Pie: yes [10] Ale: Hmm [11] I: What are you looking at? [12] Ale: No, nothing, just that…I wanted to… now we can also put [Ale refers to the statement that they are asked to write as a conjecture] that the distance between B and C always stays the same…in any case it does not vary	More invariants are identified. It seems that Ale is looking for geometrical reasons for their appearance, though he does not immediately see them. The control by dragging is invoked to verify them and decide to include them in the conjecture However, dragging a basic point is invoked to check whether a certain property is a consequence; provided that it is invariant, a property should appear in the conjecture ("we can also put …")

The main elements come out: the identification of an invariant and the explicit expression of the dependence between basic points and constructed points. When Ale identifies an invariant Pie tries to link it to the given properties [3–4], and expresses such links in terms of dependency on the variation of basic points: "Therefore if I vary A, C varies too. […] I mean A has influence over both B and C."

This is a good example of how dragging is combined with the control over the direct and indirect movements of points, and of how this combination may be associated with logical dependency between properties, orienting the recognition of the status of premises and derived properties.

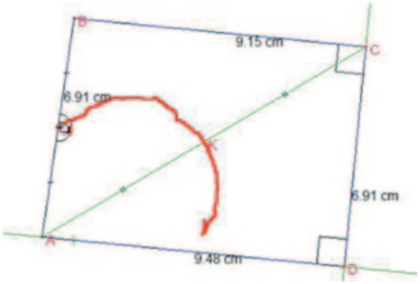
9.6.2 Dragging to Search for a Premise

Let us now consider an example of using the maintaining dragging modality. Here, after a first phase of exploration by free dragging, a pair of students start a more systematic investigation using the maintaining dragging mode, as it was introduced in the classroom. As we will see, they start searching for a condition to make the quadrilateral become a rectangle. It is possible to observe how this use of maintaining dragging allows the students to distinguish the status of the different properties involved.

The exploration of Fab and Gus The problem after dealing with a first conjecture concerning the fact that the "quadrilateral is always a right trapezium," the students notice the particular configuration "ABCD is a rectangle" and start dragging the basic point M trying to maintain this property, which we therefore refer to as the intentionally induced invariant. At the same time, Fab and Gus start to search for the condition that might be the cause that makes this property occur.

Excerpt 2.a

Transcript	Analysis
10 Fab: Ok, should we drag M now? 11 Fab: Let's try to get a rectangle first 12 Fab: How was it?	The solvers decide to activate the maintaining dragging on point M to induce the property "ABCD rectangle"
13 Fab: Like this. Now let's try to maintain … 14 Interviewer: rectangle 15 Fab: The property rectangle	It is clear that for the students the direct motion has to *cause* the invariance of the property "ABCD rectangle"
16 Gus: Eh, going up 17 I: Slowly, slowly … 20 Fab: No, no, it changes. It moves too much…	Students are searching for regularities in the direct motion and in order to detect them the students decide to activate trace on M, as they say "so we can see"
21 Fab: Should we try to do trace so we can see? …	The haptic sensation of dependency appears in the utterances 16–21, when modifications of the action of dragging is invoked after perceiving that maintaining is violated
28 Fab: It looks like a curve 29 Gus: It would look like a nice circle again 30 Fab: Like this	After the use of trace, the direct motion—causing the property of being a rectangle to occur—is reified in the *trajectory* produced by the trace tool. The solvers recognize such trajectory as a circle and specifically as the circle of diameter AK



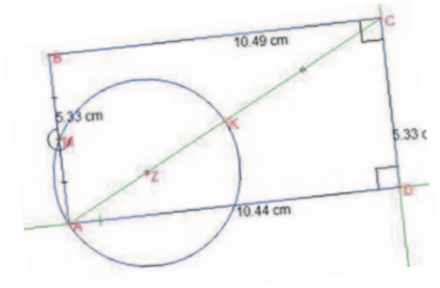
Transcript	Analysis
31 Fab: It is not a straight line, for sure! … 42 Fab: It looks like it goes through A 43 Gus: and through K 44 Fab: where? 45 Gus: It looks like a circle with diameter AK 46 Fab: Yes, it looks like a circle with diameter AK! ….	

The first property "ABCD is a rectangle" is identified and fixed at the very beginning, while the condition slowly emerges through the enacted goal-oriented dragging process. In this case, as in others, the use of the *trace tool* is combined with the maintaining dragging; this leads to the reification of the direct movement of the basic point, facilitating the recognition of the searched condition. A trace of the haptic sensation of causality is expressed by the students' verbal commentary to the dragging (16–21).

Both the properties in focus ("ABCD rectangle" and "M on circle") constitute soft invariants; the first is controlled "by eye" with a certain tolerance, the second emerges and is recognized after its reification in the trajectory produced by the trace tool. It is expressed as “the point M moves on the circle of diameter AK.”

In the following episode, we can find confirmation of students’ awareness of the different status of the two soft invariants.

Excerpt 2.b

Transcript	Analysis
50 Fab: So let’s draw the circle AK… 51 Interviewer: that you thought appeared 52 Fab: exactly, but first we need to…I mean I need to give it a centre, right? 53 Fab & Gus: So let’s construct the midpoint of AK. (They label it Z) …	Fab suggests to construct robustly the circle identified in the product of the trace
64 Fab: I need to link M to the circle. … 66 Fab: Because I am trying to maintain the property rectangle dragging M along this circle…  	Point M is linked to the constructed circle through the command “redefine object,” and Fab explains his goal “66 I’m trying to maintain ….”. It is clear that this robust property, obtained by construction, is recognized as the cause of the property “ABCD is rectangle”
69 Fab: That means, if M belongs to the circle with radius AZ and center Z … 70 Fab: then ABCD is a rectangle	Finally, the conjecture is given (69–70) in the form “if... then...”

The students construct the circle that was identified in the product of the trace, then point M is linked to it. Fab summarizes the interpretation of their phenomenological experience referring to the initial activity of exploration “66. Fab: I’m trying to maintain ….” It seems clear that they interpret the new directly induced invariant “M on the circle” as a cause of the indirectly induced invariant “ABCD rectangle.” All that finally (69–70) is expressed in a conditional statement, a conjecture of the form “if ... then ..”

Immediately after that, the conjecture is tested through constructing the premise and accomplishing a free dragging; the simultaneous appearance of the two robust invariants, confirms the conjecture.

We advance the hypothesis that as the solvers induce the invariants, the type of control that they experience over them can help them to perceive the asymmetry of their status in spite of the fact that they appear simultaneously. This may lead the students to interpret the dynamic relationship between the invariants as a *conditional relationship* between properties. Thus, perception of invariants together with the sensation of the causal relationship between them may be transformed into a conditional statement relating geometrical properties.

9.7 Conclusions

Though one should not under-evaluate the difficulty that students face in perceiving and interpreting dynamic phenomena occurring on the screen, the deep educational value of such activities motivates the effort requested in fostering it through adequate teaching interventions (Talmon and Yerushalmy 2004; Restrepo 2008; Baccaglini-Frank et al. 2009; Mariotti and Maracci 2010).

However, with this contribution I want to go a step further: it is claimed that not only the use of different dragging modalities may lead students to successfully solve conjecture-generation tasks, supporting them in producing conditional statements, but also the dragging modalities offer a semiotic potential that can be exploited by the teacher to make the mathematical meaning of *conditional statement* evolve from haptic experience of direct and indirect movements, and the related different status of invariant properties.

Simultaneity, combined with the control of direct and indirect movements, makes the different status of each kind of invariant emerge as well as the counterpart of the logical dependency between a *premise*, corresponding to the constructed invariants and a *conclusion*, corresponding to the derived invariants. Specifically, the two kinds of invariants can be characterized referring to their specific status in the exploratory activity, their specific characteristics make them clearly recognizable by the students and they can be used by the teacher to exploit the semiotic potential of maintaining dragging.

In summary, different meanings emerging from the semantic of the DGS can be exploited to mediate the mathematical meaning of *premise* and *conclusion* and generally speaking, the mathematical notion of *conditional statement*. Specifically, the two kinds of invariants can be characterized referring to their specific status in the exploratory activity, their specific characteristics make them clearly recognizable by the students and they can be used by the teacher to exploit the semiotic potential of maintaining dragging.

The model elaborated in the TSM (Bartolini Bussi and Mariotti 2008) describes the main components of the process that starts with the student's use of an artifact to accomplish a task and leads to the student's appropriation of a particular mathematical content. Taking a semiotic perspective, such a description is provided in terms of transformation of signs: personal signs, referring to meanings emerging from students' activities with the artifact, are expected to be transformed into mathematical signs. Such a transformation is not spontaneous, rather it has to be fostered by the teacher through organizing specific social activities designed to exploit the semiotic potential of the artifact. Collective mathematical discussions (Bartolini Bussi 1998) constitute the core of these activities, on which teaching and learning is based. The whole class is engaged: taking into account individual contributions and exploiting the semiotic potentialities coming from the use of a particular artifact, the teacher's action aims at fostering the transformations from personal meanings to mathematical meanings (Mariotti 2009; Mariotti and Maracci 2010).

Further investigations are necessary to explore the whole process of semiotic mediation related to the different dragging modalities. Teaching experiments designed according to a specific didactic organization (didactical cycle) described in the TSM (Bartolini Bussi and Mariotti 2008) have been planned, and are in progress, in order to collect evidence of how the semiotic potential of different dragging modalities can be exploited in classroom activities and how the transformation of signs might actually be realized. However, this will be the theme of another paper.

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Chapter 10
Dynamic Representations of Complex Numbers

Opportunities to Learn in Teacher Training

Reinhard Hölzl

10.1 Introduction

It seems a reasonable claim that students who will be future mathematics teachers are provided with appropriate opportunities to actively engage in mathematical topics and tasks in the course of their training. This should not only involve more or less difficult routine tasks following the lecture, but should also include open and self-differentiating tasks. In other words, their training at the teacher college or university should deliver and reflect, as far as possible, what is to be expected in their future careers.

Empirical findings support the view that teachers are better able to promote and facilitate the mathematical thinking of their students if they themselves have learned and engaged in mathematical activities.

Many issues concerning pedagogical content knowledge (Shulman 1986) and good practice are directly linked to teachers' own expertise in the subject matter (COACTIV, TEDS-M). For instance, knowledge of and in "differentiation," "individual learning paths," "multiple solution tasks," "basic ideas and misconceptions," "visualizing abstract concepts," "consistent image of mathematics," etc.

To be clear, for the sake of this chapter "professional proficiency of lower secondary teachers" does not refer to any *inert* knowledge of mathematical topics cumulatively acquired at the university, but the *ability* to independently explore mathematical topics or domains within a more or less delimited range. Asking questions of one's own mathematical practice, trying solutions to problems (subjective though they may be) could be regarded as necessary ingredients of such proficiency.

This paper reports on a course at our training institution for prospective mathematics teachers. It links content-related aspects with didactical perspectives: It is about functional relationships between complex numbers and their process-oriented acquisition by our students. In linking both aspects together, it is believed that the

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representation of the mathematical content by means of a dynamic geometry software (DGS) plays a key role.

10.2 High Hopes, Slow Pace

Since the advent of the so-called "dynamic geometry software" (DGS) in the late 1980s and the early 1990s there has been considerable discussion about its possibilities and potentials in the learning and teaching of geometry (Laborde & Sträßer 1990). Following the then general optimism, if not enthusiasm, within the community of mathematics educators, dynamic geometry could play a distinctive role in mediating geometric concepts and heuristic processes while supporting less teacher-centered styles of instruction (Laborde 2001).

However, it became clear that any technology-mediated mathematics is not the same as paper-and-pencil mathematics (Sträßer 2001). In the case of dynamic geometry, this was true especially for the very essence of DGS, namely its so-called "drag mode," the defining feature which allows certain parts of a figure to be moved around the screen preserving its geometrical relationships.

As Sträßer (1991) noted, distinguishing "drawing" from "figure" becomes essential when investigating geometric situations with DGS; while "drawing" refers to the visual aspects of, say a triangle, "figure" refers to its underlying relationships (for example, an isosceles triangle). This difference is crucial because "dragging" operates on the drawing but not on the figure. With this distinction, other dilemmas of dynamic representations also arise: In a paper-and-pencil environment, there is no principal difference between simply a point and a point of intersection, the latter being constructed by using ruler and compass. However, with DGS the difference between points and intersection points can become poignant in the eyes of students as the former are freely moveable whereas the latter are not. The consequences of this can be crucial when it comes to heuristic processes (Hölzl 1995, 1996; Sträßer 1992).

Given the fact that DGS has been in use for some 20 years now, one can ask what has been achieved in relation to the learning of geometry or in mathematics in general. Laborde & Sträßer (2010) conclude, in their review of 25 years of International Commission on Mathematical Instruction (ICMI) activities concerning new technology, that problems of implementation are far from being solved and "the discrepancy between intentions, suggestions and potentials to use new technology and the actual use of it is still wide" (p. 131). They suggest that, notwithstanding the abundance of well-designed software and well-meaning suggestions on how to use new technology in the classroom, the most important innovations seem to be in the arena of research as different theoretical frameworks had to be developed. Indeed, while the majority of research projects focused on various aspects of students' learning in or interacting with "microworlds" (Healy and Kynigos 2010), little attention had been paid to teachers and teaching (Lagrange et al. 2003). To explain the relatively slow pace of technology integration in the classroom, one had to take into account teachers' conceptions, beliefs, and knowledge (Lavicza 2010), and it is reasonable to assume that these variables are influenced by what the teachers experienced themselves in their own schooling and training.

To put it simply: Mathematics teachers who have not experienced a meaningful problem-based style of mathematics supported by certain facets of ICT are less likely to employ new technologies later in their own teaching. Thus, I see it as a challenge to give our teacher students opportunities to (a) find their own ways of dealing with mathematical content and (b) use, in our case, dynamic geometry as a tool to support an investigative style of learning and teaching.

10.3 Course Contents

Our course content is based on elementary properties of complex numbers and functions. In teacher education, we deem complex numbers to be a rewarding topic in many ways. On one hand, the epistemological problems associated with its development show that mathematical concepts are not simply defined, as the prevailing mode of lecturing at universities still suggests, but may have been developed over centuries. On the other hand, complex numbers provide ample opportunities for a rich interplay between geometric and algebraic ideas. In addition, for prospective mathematics teachers, the history of ideas about complex numbers can be very instructive because of the distinctive parallels to the development of negative numbers.

Complex numbers or, as they are formerly called, “imaginary quantities” appeared during the Renaissance. Cardano (1501–1576) set up and solved the problem of dividing 10 into two parts whose product is 40. The roots for the corresponding equation are $5+\sqrt{-15}$ and $5-\sqrt{-15}$ which are, in the words of Cardano, “sophistic quantities”—ingenious but of no value. However, “putting aside the mental tortures involved” Cardano proceeds to multiply both roots, as is common with real numbers, obtaining the perfectly natural number 40. His comment in Chap. 37 of his *Ars magna* (1545) states: “So progresses arithmetic subtlety, the end of which, as is said, is as refined as it is useless.” (cited in Kline 1980, p. 116)

Descartes (1596–1650) stresses the difference between *real* vs. *imaginary*. He says with each polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ ... \ + a_1 x + a_0 = 0$ one could *imagine* as many roots as its degree *n* indicates. At times, however, these imaginary roots do not correspond with real quantities.

Leibniz (1646–1716) enriches his doctrine of imaginary quantities with the surprising relation $\sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}} = \sqrt{6}$ and views the square root of a negative number as a “sublime outlet of the Divine Spirit, almost an amphibian between being and not-being” (cited in Ebbinghaus 1991, p. 48).

Euler (1707–1783) calculates at ease with complex numbers and with superb intuition. He already knows the relation

$$\sqrt{-1} \cdot \log \sqrt{-1} = -\frac{\pi}{2}$$

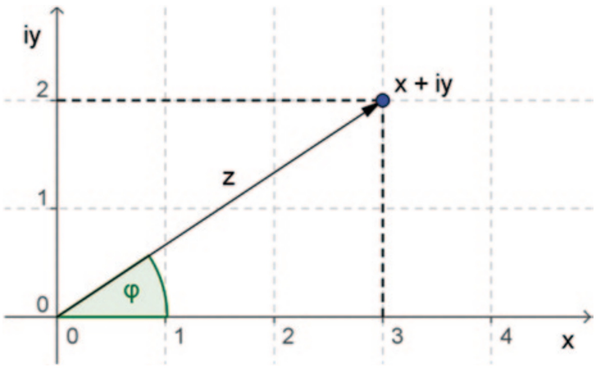


Fig. 10.1 Representing a complex number in the plane

but finds himself on slippery grounds by calculating $\sqrt{-1}\cdot\sqrt{-4}=\sqrt{4}=2$ thus using $\sqrt{a}\sqrt{b}=\sqrt{ab}$, which in general is no longer valid with complex numbers (indeed, even the $\sqrt{\ }$ sign is ambiguous with complex numbers).

The views on complex numbers start to change with the work of Gauss (1777–1855). He knows the geometric interpretation of complex numbers as points in a two-dimensional Cartesian coordinate system called the complex plane (cf. Fig. 10.1); he uses it albeit somewhat in disguise, in his dissertation in 1799, where he proves the "fundamental theorem of algebra". By 1815, Gauss was in full possession of the geometric theory but it was not until another of Gauss' treatises in 1831 that the complex plane gained broad dissemination (Ebbinghaus 1991, p. 50). Interestingly, the geometric and thus visual representation of complex numbers as points in a plane earned them a legitimate place in mathematics.

Complex numbers are characterized by two components: a real and an imaginary part, which are interpreted as Cartesian coordinates. The formal expression $x+iy$ is known as the algebraic (or rectangular) form of a complex number, where the symbol i denotes the "imaginary unit" with the characteristic property $i^2=-1$.

As long as calculations such as multiplication or division are not involved, one can comfortably switch between interpretations of complex numbers as numbers, points, or position vectors, as is practiced throughout the text.

A powerful way of interpreting calculations with complex numbers arises from the *polar form,* that is, instead of using the real and imaginary parts to determine a point in the complex plane, one measures its distance r to the origin (0, 0) and its angle φ to the real axis (counterclockwise). r defines the *absolute value* of a complex number z which by using Pythagoras' theorem is

$$r=|z|=|x+iy|=\sqrt{x^2+y^2}$$

The value of φ, the so-called *argument* of z, can change by any multiple of 360° (or 2π if radians are used).

Basic trigonometry shows that $z = x + iy = r(\cos\varphi + i\sin\varphi)$, the latter sometimes being abbreviated to r cis φ.

10.4 Course Arrangements and Opportunities to Learn

Instead of solely lecturing, our approach to complex numbers was twofold. Although there was some input to provide basic orientation, introducing key concepts and giving illustrations of using DGS, the main features of the course were nine "topic cards" that opened up space for exploratory learning about complex numbers. Topics ranged from complex numbers as a field, complex roots (sequences, functions) to non-Euclidian geometry (Coxeter 1969). Students could *choose* according to their own interests the card they wanted to engage with. This is crucial because there is sometimes (or often) a mismatch between the professional demand of taking heterogeneity in classes into account and what teacher students actually experience in their training. At least as far as mathematics for secondary teachers is concerned, there is, in my view, still scope for improvement.

Each topic card contained a *central theme* as well as *key concepts* to outline the territory that was to be explored. We asked our students to

1. …*write a mathematical report* in a reflective style which documents their personal questions and insights they came across on their path of learning. Clarity of text was to be supported and enhanced by giving examples, and appropriate forms of visualizing, specializing, or generalizing;
2. …*choose at least one specific question*, *thought*, *or problem*, etc. and develop some form of substantial preparation by using ICT, for instance, *Excel* or *GeoGebra*; and
3. …*work on a* "*research question*." The criteria for the "research question" were clearly subjective in nature. A valid question or problem was what students regarded as (personally) meaningful and to which there was (for them) no obvious answer or solution.

Multiple topic cards could be chosen and worked on cooperatively. However, reports had to be individual and cooperative work explicitly referenced.

10.5 Dynamic Representations of Complex Numbers

The geometric representation of complex numbers sheds new light on the basic calculations with those numbers. Take multiplication as an example:

$$z \cdot w = (x + iy) \cdot (u + iv) = (xu - yv) + i(xv + yu)$$

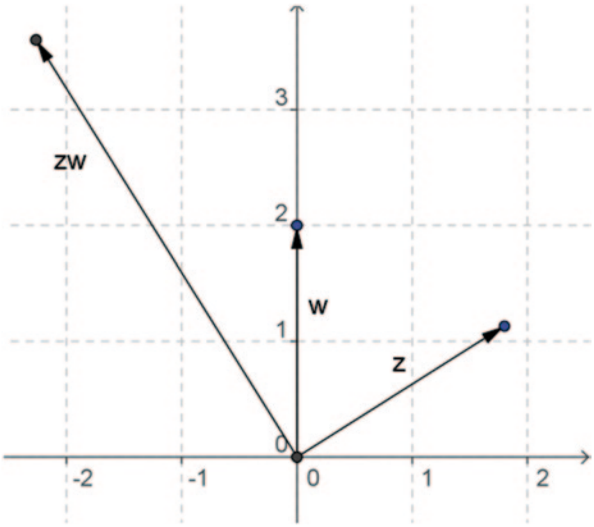


Fig. 10.2 Representing the complex multiplication dynamically

The procedural course seems quite straightforward—the result follows from the distributive law taking into account $i^2 = -1$. However, what is the "meaning" of the expressions $xu - yv$ and $xv + yu$, respectively?

Representing the complex multiplication $z \cdot w$ dynamically, we can study the effect of the multiplication from a geometrical viewpoint in various ways: We can set both the direction and length of the vector that represents the complex number w, then vary the vector representing z and study the changes to $z \cdot w$.

As an example, for $w \geq 0$, that is $w = u + iv$ with $u \geq 0$ and $v = 0$, it happens that z is dilated at the origin O, with the factor being of value w. Whereas if w has only an imaginary part $v \geq 0$ $(u = 0)$, then z appears to be not only dilated but rotated at O counterclockwise by 90° (cf. Fig. 10.2).

Similar special cases enable analogous observations. In general, the multiplication

$$z \xrightarrow{\cdot w} z \cdot w$$

can be interpreted geometrically as a dilative rotation about the origin O. The factor k of dilation equals the absolute value of w, the rotation angle φ is in accordance with the argument of w.

This geometric interpretation can be recognized in symbolic form too. Switching to polar coordinates and taking

$$z \cdot w = r_1 \left(\cos \varphi_1 + i \sin \varphi_1\right) \cdot r_2 \left(\cos \varphi_2 + i \sin \varphi_2\right)$$

as a starting point, we can factor out and apply appropriate trigonometric rules thus gaining

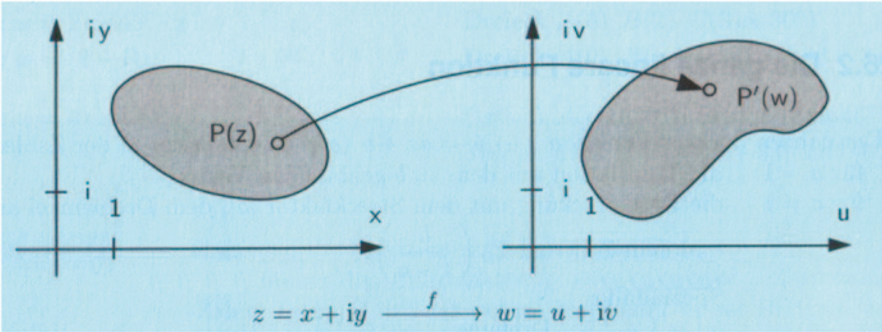


Fig. 10.3 Mapping a subset of the "z-plane" onto the "w-plane"

$$z \cdot w = r_1 \cdot r_2 \left[\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)\right]$$

Therefore, the complex product $z \cdot w$ is of magnitude $r_1 \cdot r_2$ and has the argument $\varphi_1 + \varphi_2$ (modulo 360°).

10.6 Functional Relations

There is a clear analogy between the real and complex numbers with respect to functional relations: A variable w changes depending on a variable z, e.g.,

$$w = 2i \cdot z + 1, w = z^2, w = 1/z, \text{ in general } w = f(z) \text{ or } z \xrightarrow{f} w\ .$$

In contrast to real numbers, the familiar graphic interpretation of a function f is no longer possible as its graph

$$G_f = \{(z \mid w) : w = f(z) \text{ for z, } w \text{ complex}\}$$

forms a four-dimensional real subset, because of $(z \mid w) = (x \mid y \mid u \mid v)$ for $z = x + iy$ und $w = u + iv$. However, visualizing can be powerful at least as long as there are pedagogical intentions involved; hence, graphical interpretations of complex functions exist. It is common, for instance, to draw "niveau curves," that is, curves where the absolute value $|w|$ does not change.

As an alternative way of visualizing, we can investigate the effects a function has on certain subsets of the complex plane (cf. Fig. 10.3). The basic idea is that a complex function somehow "distorts" the plane and in doing so shows its characteristics.

Example. How does the function $w = f(z) = -2z + 2i$ transform a special subset of the complex plane, say the triangle represented by its vertices $A(-2-i), B(-1-i)$ und $C(-1+i)$?

Fig. 10.4 Transforming a triangle by a linear complex function

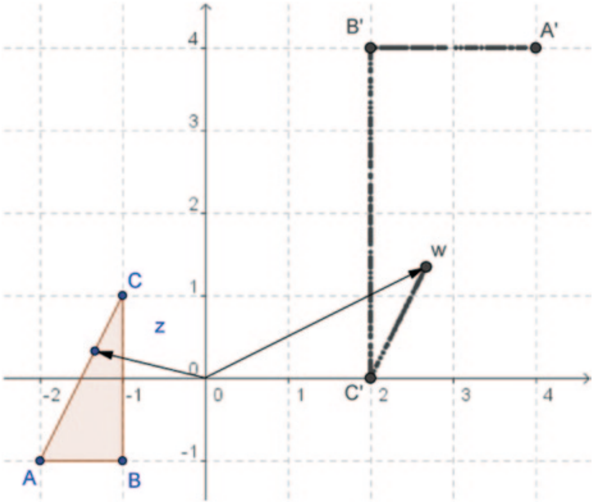
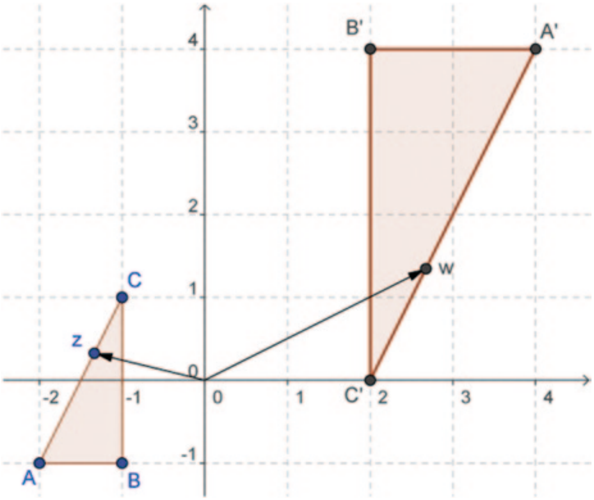


Fig. 10.5 The image of the triangle as a locus.



Each vertex transforms into

$$\begin{aligned} -2-i &\xrightarrow{f} 4+4i \\ -1-i &\xrightarrow{f} 2+4i \\ -1+i &\xrightarrow{f} 2 \end{aligned}$$

The dynamic representation of this situation within GeoGebra would be as follows: The complex number z represents a point of triangle ABC. While z goes around the

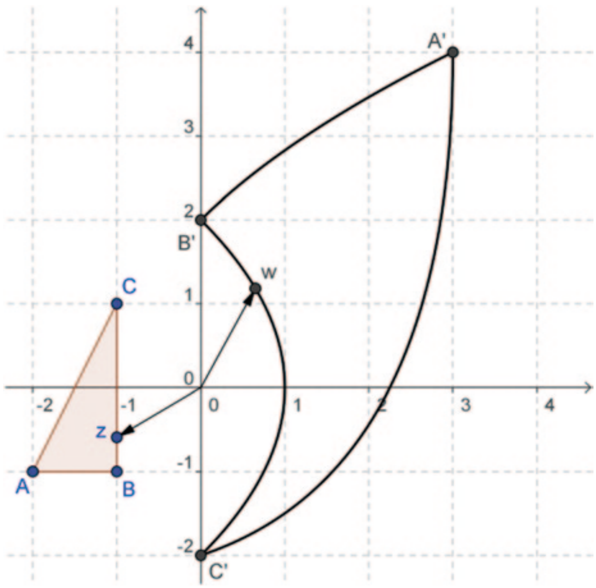


Fig. 10.6 Transforming a triangle by z^2

triangle *ABC*, $w = f(z)$ traces the image of the triangle (cf. Fig. 10.4). Mathematically speaking, the image is generated pointwise.

How can the result be explained? As mentioned above

$$z \xrightarrow{\cdot(-2)} -2 \cdot z \xrightarrow{+2i} -2 \cdot z + 3i$$

amounts to a dilative rotation about the origin by factor 2 and rotation angle $\arg(-2) = 180^{\circ}$, followed by a translation with $+3$ units parallel to the imaginary axis. Hence, three geometric operations are involved: *dilating*, *rotating*, and *shifting*.

As far as the dynamic representation within GeoGebra is concerned, it is worth mentioning that it is possible to generate the image of the triangle not only point wise as a trace (in this case, as a trace of *w*) but also as a locus (cf. Fig. 10.5).

Such loci already behave, in some respect, like other objects created with the software. For example, if the original triangle *ABC* changes, so does the image triangle *A'B'C'*.

More surprising is the dynamic representation of the complex function $w = z^2$: The intuitive notion that the triangle *ABC* would somehow be "dilated by a quadratic magnitude," does not stand the test because the sides of the triangle are obviously bent (cf. Fig. 10.6).

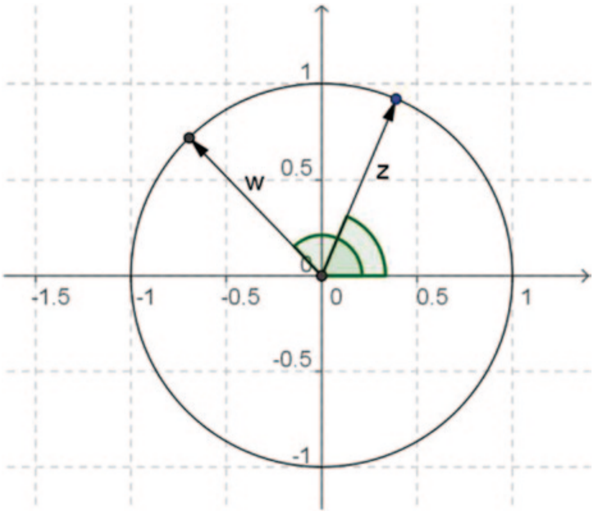


Fig. 10.7 Position vector of z and $w = z^2$

10.7 Impressions of Student's Work

Which topic cards did our students prefer? Mainly those where unusual, surprising insights are likely during the course of experimenting; topics where dynamic representations of the underlying mathematical situation offer various routes of investigation—qualities that are linked to the cards' "complex roots" or "complex functions" in particular.

To give some impressions as to which questions or insights our students came across during their work on the topic cards, we briefly illustrate some points.

Students often encounter the ambiguity of square roots (or any nth root, $n > 1$) when dealing with the GeoGebra model of the above-mentioned quadratic function. The image of the unit circle under the transformation $z \to z^2$ obviously remains the unit circle because of $\left|z^2\right| = |z|^2 = 1$. However, this fact has more facets within a dynamic representation.

Figure 10.7 indicates that the position vector of $w = z^2$ goes around the unit circle twice as fast as the position vector of z. This is because of $\arg(w) = \arg(z^2) = 2\arg(z)$.

In other words *one* round for z on the unit circle means *two* rounds for $w = z^2$. Conversely, for each position of w on the unit circle there must be *two* distinct positions of z such that $z^2 = w$, which differ in their argument by 180°.

The GeoGebra model in Fig. 10.8 serves as a visualization of the (ambiguous) complex square root $\sqrt{w}$.

Students who used this model were likely to generalize and attempted to model $\sqrt[n]{\ }$ for $n = 3, 4, 5$, too. A rather sophisticated form of dynamic representation of complex roots shows up in this work: A slider can be used to control n in $w^n = z$, while the position vector of z can be moved around, thus visualizing all possible roots $\sqrt[n]{z}$ in the complex plane (cf. Fig. 10.9).

Fig. 10.8 Complex square roots

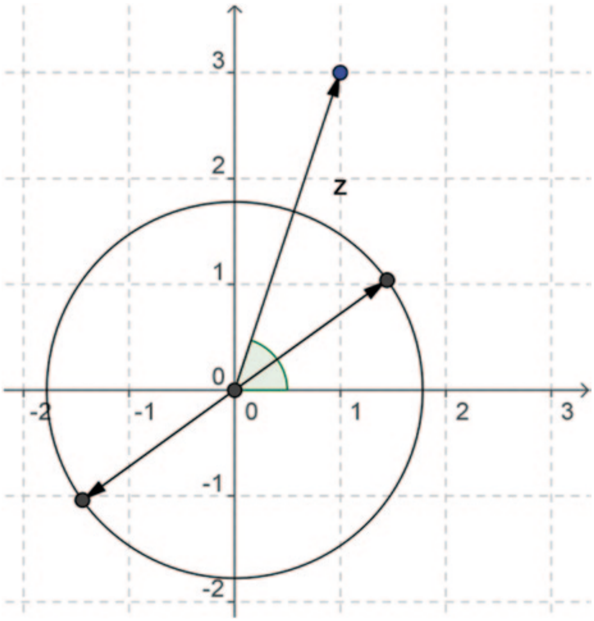
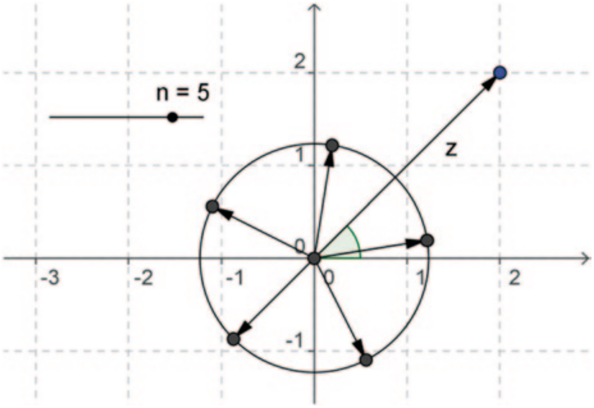


Fig. 10.9 Representing $\sqrt[n]{z}$, here $\sqrt[5]{2+2i}$



Dealing with complex roots poses the question, with some students, whether there is such a thing as $\sqrt[2.5]{\ }$, and if so, how many and what results could be expected, thus raising the question of complex powers in general.

In the case of complex functions, students mainly focused on the geometric properties of z^2, z^3, $1/z$. Not surprisingly, instances where the original figure (triangle, square, circle…) was transformed or even distorted gained a lot of attention. Thierry, for example, explores the effect the transformation $z \to z^2$ has on a horizontal line in Fig. 10.10. The line seems to be transformed into a parabola.

It should be mentioned that students very seldom resorted to analytical explanations for such phenomena, instead, intuitive geometric reasoning was invoked such

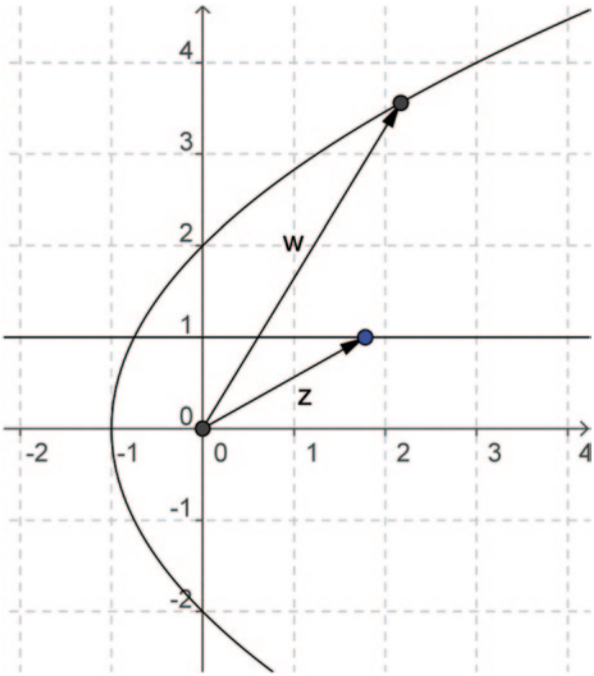


Fig. 10.10 Transforming a horizontal line

as "At the point i or near it, the absolute value of z equals approximately 1, consequently $\left|z^2\right| \approx 1$. Because changes of $\arg(z)$ exert near i a greater influence than changes of $|z|$, a curvature results."

Similar to the above case of a horizontal line, a vertical line would be transformed into a parabola open to the left. Viewed as a whole, the Cartesian grid transforms into a parabolic grid (with angles preserved) (cf. Fig. 10.11).

10.8 Summary

Recent studies on teacher cognition, such as COACTIV (Kunter et al. 2007), focus on teacher knowledge and skills. Among those aspects of teacher competence regarded as essential, the dimension "knowledge" plays a prominent role. Research shows that the meaningful interplay between "content knowledge," "pedagogical content knowledge," and "pedagogical knowledge" (Shulman 1986) is fundamental for good teaching. (Questions of organizational knowledge as well as counseling skills are not to be neglected, of course.)

Whereas the category "pedagogical content knowledge" is relatively well documented in the literature, the description of the category "content knowledge" remains somewhat traditional. To be sure, as far as school curricula are concerned, fundamental ideas are mentioned, useful forms of representing those ideas are

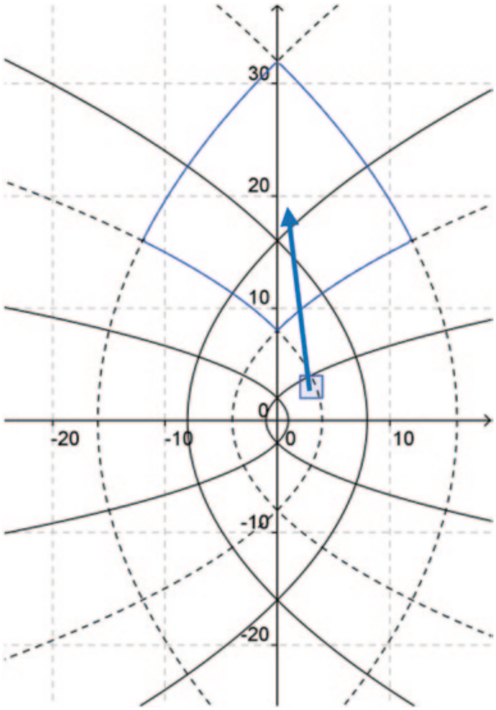


Fig. 10.11 Parabolic grid, the little square transforms into the parabolic quadrilateral as indicated by the arrow

suggested, as are good analogies, illustrations, examples, explanations, and demonstrations. In short, various ways of representing and formulating the subject matter that make it comprehensible to others are known and can be taught at a teacher college. However, when it comes to "content knowledge" the common phrase is that teacher students need a sufficient, preferably "deep understanding" of (at least) school mathematics. The question is how this is achieved. The normal route is via lectures in mathematics, which often still follow a transmission model: Presenting definitions, assertions, and proofs (cyclically in this order). We claim however that the central emphasis should not be on *how much* mathematics teacher students encounter and even *what mathematics*, but *in what different ways*.

If prospective teachers are to lead their future students to what makes the essence of mathematical thinking, namely recognizing (ir)regularities, exploring patterns, and finding arguments to back up or refute assumptions, then it is vital in our view that they encounter activities that reinforce an open and self-differentiating approach during their time at the college or university.

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Chapter 11
Dynamic and Tangible Representations in Mathematics Education

Colette Laborde and Jean-Marie Laborde

11.1 Introduction

As so often stated since the time of ancient Greece, the nature of mathematical objects is, by essence, abstract. Mathematical objects are only indirectly accessible through representations (D'Amore 2003, pp. 39–43) and this contributes to the paradoxical character of mathematical knowledge: "The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But at the same time, mathematical objects must not be confused with the used semiotic representations" (Duval 2000, p. 60). Other researchers have stressed the importance of these semiotic systems under various names. Duval calls them registers. Bosch and Chevallard (1999) introduce the distinction between ostensive and nonostensive objects and argue that mathematicians have always considered their work as dealing with non-ostensive objects, and that the treatment of ostensive objects (expressions, diagrams, formulas, and graphical representations) plays just an auxiliary role for them. Moreno Armella (1999) claims that every cognitive activity is an action mediated by material or symbolic tools.

Through digital technologies, new representational systems were introduced with increased capabilities in manipulation and processing. The dragging facility in dynamic geometry environments (DGEs) illustrates very well this transformation that technology can bring in the kind of representations offered for mathematical activity and consequently for the meaning of mathematical objects. A diagram in a DGE is no longer a static diagram representing an instance of a geometrical object, but a class of drawings representing invariant relationships among variable elements: The

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dynamic parallelogram *ABCD*, constructed on variable points *A*, *B*, and *C* represents two relationships of parallelism between two opposite sides, *AB* and *CD* on one hand and *AD* and *BC* on the other. It is because the relationships of parallelism are invariant in the dragging, while points and sides vary, that they constitute the mathematical essence of this figure. The dynamic representation expresses, in this example, the generality of the parallelogram. The distinction between *drawing* and *figure* is clearly illuminated by DGEs and a discussion (initiated by Parzysz in 1988 independently of DGE) took place at the international level, about the complexity of relationships between diagrams and figures, in particular, in a workshop organized by Straesser in August 1990 at IDM in Bielefeld (Germany) (Strässer 1991; Laborde 1991). A more recent synthesis is presented in Kadunz and Straesser (2007, pp. 39–46).

The role of representations in the use of digital technologies is essential. Hoyles and Noss (2003) consider digital technologies as “dynamic manipulable and interactive representational forms” that “mediate and are mediated by mathematical thinking and expression” (p. 326). As they stress it, the systems we use to present or represent our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation, constitute an essential part of our cultural infrastructure.

The paper addresses three dimensions of transformations brought about by these new kinds of representations:

- An epistemological dimension: the problems faced by software designers when working on the features of direct manipulation of representations of variable mathematical objects;
- a cognitive dimension: the way students learn mathematics using this new kind of representation offers a window on their conceptualizations; and
- a didactic dimension: how transforming the tasks by taking into account the features of these dynamic representations, in particular of the drag mode, may impact students’ learning.

11.2 Designing the Features of Direct Manipulation: the Case of Cabri 3D

Direct manipulation has proven to be a key feature to facilitate creative user interactions with a computer and has slowly generalized to most computer platforms. For educational software, nevertheless, direct manipulation cannot be designed by chance and has to follow some additional principles. One of them is called epistemic fidelity: the representations of mathematical objects have to avoid any contradiction with the abstract object they are supposed to represent; and this has to be true at the graphical level as at the level of their behavior under direct manipulation. Let us elaborate on the difference between interactivity and direct manipulation.

When interacting with a modern computer, the interface is essentially interactive in the sense that the user is “asking” the software to perform something, and after the reaction of the computer, he or she asks something again for a next step. The most basic interactivity is offered by the so-called "interactive books" giving essentially the possibility to display pages and, by pressing buttons, to turn the pages of the books.

This kind of interactivity (unfortunately still widely spread, especially through the Internet) is easy to develop and leads to a form of "impoverishment" of the user interface, with the generalization of the use of Internet. By contrast, authentic direct manipulation software is mainly not driven by the press of buttons, or by the filling of dialogs (or forms), or by typing command lines. It offers an interface where the user is invited to directly act on the mathematical objects. Actually, the action is on the representation of an object or an abstract entity; nevertheless, if the implementation is sophisticated enough and if the interaction turns out to be of direct engagement type (Schneidermann 1983), eventually the user perceives the representation of the object and the abstract object itself, as already noticed by the five main designers of the Star Machine (Smith et al. 1982).

The need for extending the benefit of direct manipulation present in many two-dimensional (2D) environments just followed the first introduction of direct-manipulation geometry software of Cabri type. Recall that Cabri actually stands for "Cahier de BRouillon Interactif," somehow "Interactive Sketchpad."

The need for extending direct manipulation from 2D to 3D cannot be achieved without a deep transformation of the environment under several aspects: the mathematical problems to be solved for the representing dynamic 3D objects are different and, for most of them, even still open.

Two kinds of new problems arise: representing 3D objects with a meaningful depth and representing their behavior in the drag mode; the manipulation of 2D objects with a mouse in a 2D screen is natural but becomes problematic for 3D objects: how can the mouse capture the depth of the space?

11.2.1 Complexity

A very common idea, about extending a 2D environment to 3D, is to think that this could be achieved (somehow) merely by adding an additional coordinate to the internal representation of the objects at the level of their data structure: essentially this would be then a trivial task. Actually, as it is well known by mathematicians, 3D objects are "essentially" more complex than 2D objects: in most of the cases, augmenting the dimension leads to some increase in complexity, even if eventually in a higher dimension the situation might show more regularities. In 3D, many "basic problems" are still open. Let us mention the classification of quadrics. In 2D, there are only three conics: ellipses, parabolas, and hyperbolas. In 3D, i.e., for quadrics, nobody has yet found any really "elegant" classification that would satisfy everybody. Another example could be the conjecture about the probable existence of unfolding of any convex polyhedron as a net of not overlapping connected (convex) polygons. In 2D, "to follow," in a reasonable way, the intersection of two conics is not an easy task (actually many of the DGEs cloned from the main ones

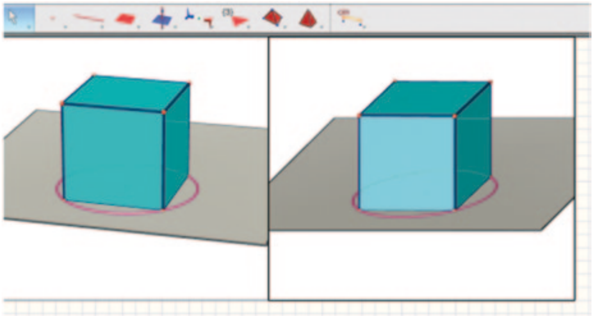


Fig. 11.1 On the left a cube in natural perspective, on the right the same cube in cavalier perspective

fail in trying to dynamically follow, in rather simple cases, the intersection points of a circle with a straight line). In 3D, nobody knows (yet and apparently) how to dynamically follow the intersections of a quadric with a line and, even less, the intersection curves of two quadrics.

11.2.2 Representing Dynamic 3D Objects

The first new problem arising when moving from 2D to 3D deals with the choice of the perspective.

In classrooms, the so-called cavalier perspective has been the most popular way of representing 3D scenes. The main reason for this is that it is easy to create perspective drawings using the rules of the cavalier perspective. Recall that this perspective is governed by the fact that it is a parallel (yet not orthogonal) perspective (the observer is at infinity and parallel lines are still parallel in the perspective drawing). In addition, there is a plane in which objects are in real size, and actually lines perpendicular to this plane are represented as oblique lines in the drawing. This is really different from the "natural" perspective as introduced by the painters at the time of the Renaissance (e.g., Alberti) and which today can be considered as realized by high-quality camera lenses.

Figure 11.1 shows a cube, in "natural perspective" vs. the same cube in "cavalier perspective": if we animate the cube to rotate around its vertical axis of rotational symmetry, the cube in natural perspective would keep its "cubic" shape, but, in cavalier perspective, the cube would actually somehow pulse as its shape would change during the various phases of the rotation. Cavalier perspective is thus in conflict with the user's perception about the cube as a solid object.

Therefore, we assumed that the natural perspective favors the appropriation of the figure by students and users more generally. This is the reason why in Cabri 3D the default perspective is actually a natural perspective. Its characteristics match the view of an object of approximately 40 cm in size held (as in the hand of the user) at a distance of 50 cm. This is in contrast to some other software that, by exaggerating the perspective effect (for some artistic purpose), apparently does not favor the appropriation just mentioned. In Fig. 11.1, one can clearly understand how, somehow,

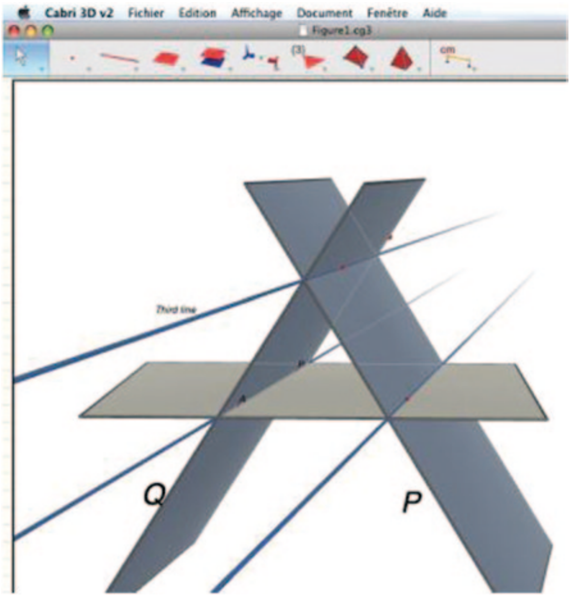


Fig. 11.2 Two planes intersecting in a third parallel line

the cavalier perspective is an attempt to look at an object from two points of view at once: from the front and from the right. Actually, this kind of representation has developed in many cultures, ranging from ancient Egypt to China and Japan.

Let us stress that this choice is not shared by other 3D environments. The latter generally favor either the parallel perspective, for easiness of the computations, or a strong perspective effect by taking advantage of the facilities of computer graphical cards.

11.2.3 Rendering 3D Mathematical Objects

There is not much space here to address all specificities of the rendering of 3D mathematical objects (lines, planes, spheres, cylinders…), some of them being “in nature” infinite. Let us just mention the case of the plane—an infinite object. In textbooks, planes are most of the time represented using a rectangle to display a “portion” of a plane. It is also worth to note that textbooks present only a really limited number of figures. Space geometry textbooks display hardly more than ten different types of 3D situations we could consider as stereotypic. Among them, one is the illustration of the famous “théorème du toit,” stating that if two planes intersect a plane along two parallels lines, their intersection is a third parallel line (Fig. 11.2).

In textbooks, which display only a static image, one can easily agree about “the good rectangle” taken to represent each plane. When things turn out to be dynamic, there is no “natural” way to “follow” the plane as it evolves. This is the reason why some 3D geometry software considering that planes are essentially infinite do not limit their representations. A plane (up to the special case of being viewed as a line, in French “de bout”) covers the whole screen and, in such environments, practically cannot be “seen” and so is not directly represented. We do not consider that this is a good idea for learners. In Cabri, after various attempts we decided to represent a plane as a rectangle (in some earlier version as a parallelogram) presenting a certain amount of thickness, in the very same spirit as when Hilbert and Cohn-Vossen designed their 3D figures for their famous “Anschauliche Geometrie” (Hilbert and Cohn-Vossen 1952).

11.2.4 Manipulating 3D Objects

One of the first things to be considered in order to directly manipulate objects in space is to have a mouse that can drive a point in 3D. Ordinary mice are essentially 2D, though expensive 3D mice have existed for a while. Because such 3D devices are not expected to be available soon, neither in the regular classroom nor at home, we have been looking for various solutions based on an ordinary mouse combined with modifiers (at keyboard level). Actually, in Cabri 3D a metaphor of the old “typewriter” is used: as long as the shift key is not pressed, the mouse simulates a displacement in some horizontal plane and if the user presses the shift key (implying a vertical movement of the carriage on antique typewriters) the mouse movement is interpreted as a movement along a vertical axis. Note that other environments may have employed another choice, like Archimedes Geo 3D, in which pressing the shift key provokes a move orthogonal to the screen plane. For reasons of making sense to the users, Cabri 3D does not permit any arbitrary rotation of the scene (as in looking at the scene in some upside-down way) and the horizontal reference plane always stays horizontal and consequently verticality is preserved.

All these choices made on the basis of epistemic and ergonomic reasons must then be confronted with the real use by teachers and students. Let us mention here a thorough study (Hattermann 2011) analyzing the use of Archimedes Geo 3D and Cabri 3D by university students, giving evidence of various aspects of the ways of using these environments for solving different kinds of problems. It is worth mentioning that students solved problems more rapidly in Cabri 3D than in Archimedes Geo 3D (p. 164) without a clear evidence of an effect due to a particular feature of the interface of both software environments. It may be the combination of several aspects of the whole interface that played a role.

11.3 A Cognitive Dimension: Dynamic Diagrams as a Window on Students’ Ideas

Interacting with dynamic diagrams transforms the usual ways of acting on mathematical objects into new ones. Because students cannot directly apply the usual paper-and-pencil routines in DGEs, they have to make decisions about actions.

These decisions are influenced by their conceptions about mathematical objects. Therefore, placing students in unfamiliar conditions may reveal their own ideas and conceptions about mathematical objects.

Many research works have been carried out on students solving tasks in DGEs. In particular, the way in which students drag as they solve geometry problems in DGEs was investigated by several researchers. Hölzl (1996) identified the "drag and link approach" in students' construction processes of Cabri diagrams. Students relax one condition to do the construction and then drag to satisfy the last condition. They obtain a visually correct diagram and want to secure the diagram by using the redefinition facility of Cabri. However, often it does not work because of hidden dependencies that are not considered by the students. Although Hölzl does not refer to instrumentation, this "drag and link approach" would be called an instrumentation scheme in terms of Vérillon and Rabardel (1995). The students constructed an instrumentation scheme incompatible with the functioning of Cabri.

Arzarello et al. (1998a, b) identified different kinds of dragging modalities that were not all referring to an organized experimentation: "wandering dragging", "lieu muet" dragging, and dragging to test hypotheses. Wandering is just moving without a predefined aim for searching for regularities, while "lieu muet" dragging refers to dragging in such a way that some regularity in the drawing is preserved. The dragging to test hypotheses obviously presupposes that regularities have already been detected which are not systematically tested. Goldenberg (1995) notes that often students do not know how to conduct experiments and are unsure what to vary and what to keep fixed. Thus, a student's purposeful move from "wandering dragging" to "lieu muet" dragging represents a cognitive shift. Restrepo (2008) who investigated, in depth, the instrumental genesis of the drag mode by sixth graders over one school year, concluded that the genesis lasts over a long time made of several steps.

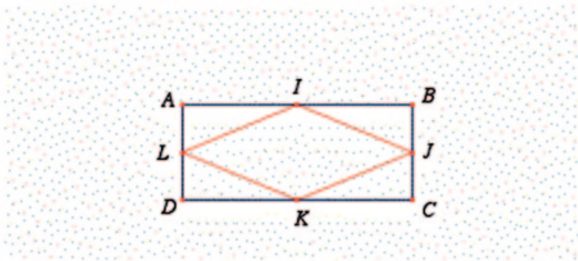
From all these investigations, it appears that the power of the drag mode in exploration is not spontaneously mastered by students because the mathematical meaning of the drag mode itself is not yet constructed by students. As claimed by Strässer (1992), dragging offers a mediation between drawings and figures and can only be used as such at the cost of an explicit introduction and analysis organized by the teacher. Transforming the interaction between student and geometric figures turned out to reveal that the notion of geometric figures, as a set of relationships between variable elements, is not appropriated by students. It may imply that students do not necessarily recognize the mathematics they learned in a paper-and-pencil environment. Mathematics itself may be changed in students' eyes by a DGE.

Below, an investigation on the construction of a proof by ninth graders is reported (Abd El All 1996) giving evidence that even a known theorem is transformed in a DGE for students.

11.3.1 Students' Conception of a Theorem

All students of a class (ninth graders) were given the following tasks. They worked in pairs. The work of four pairs was observed and audio-recorded.

Fig. 11.3 Rhombus in a rectangle



Task 1 (Fig. 11.3) Students were given a rectangle *ABCD* and the quadrilateral *IJKL* of the midpoints of the sides of *ABCD* in a paper-and-pencil environment. They had to determine the nature of *IJKL* and justify their answer. All students found that it is a rhombus.

Task 2 Then, they had to predict whether *IJKL* would remain a rhombus in any movement of *B* that does not preserve *ABCD* as a rectangle. All students predicted that *IJKL* would not be any longer a rhombus.

Task 3 They were given a rectangle *ABCD* in Cabri. Then, they had to construct the circle with center *D* and radius *AC* and to redefine *B* as belonging to the circle. They were asked "Is *IJK*L still a rhombus?" (Fig. 11.4).

The sequence of questions was designed with the intention to favor the need of having recourse to proof. In a computer environment, the need for proof cannot any longer be favored by the uncertainty of the result. It may arise for intellectual motives because the student wants to know why a phenomenon takes place. As pointed out by the Piagetian perspective, a means of provoking this intellectual curiosity may be caused by a conflict between what the learner believes or predicts and what actually happens. Such a conflict may be achieved by asking the students to predict properties of the diagram before allowing them to check on the computer, as in this problem. In task 1, we expected that students would prove that *IJLK* is a rhombus by using the specific properties of a rectangle (theorem of Pythagoras, properties of reflection, and congruence of right-angle triangles) rather than using the more general property of the midpoint segment that is valid even if *ABCD* is no longer a rectangle. In task 2, we expected them to predict that *IJKL* is no longer a rhombus, as they probably would have justified in task 1 that *IJK*L is a rhombus by using properties of a rectangle. In task 3, we expected them to be very surprised upon observing that *IJKL* remains a rhombus and that they would be eager to understand why. This is why they were not asked to justify what they had observed. We expected that from the strength of the contradiction would arise the need for justifying.

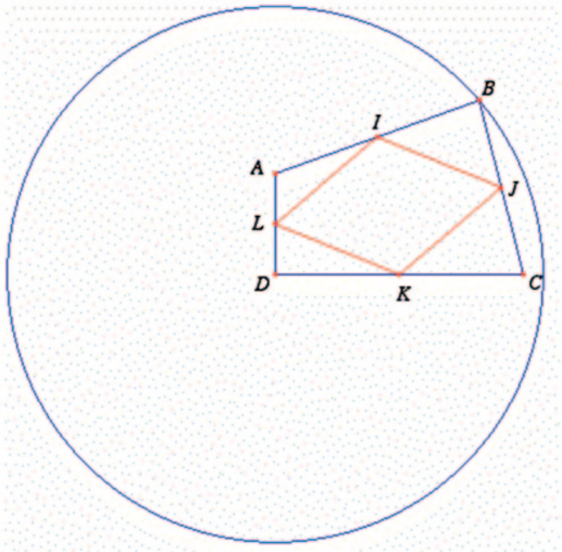


Fig. 11.4 Rhombus in a quadrilateral with congruent diagonals

This is exactly what happened. Students were so surprised to discover that *IJKL* was a rhombus, although *ABCD* was not a rectangle that they became eager to prove why without being asked in an explicit way to do so. However, it took time for them to construct a justification. We could observe that the variability of the diagram created several difficulties for students. We comment here on the effect of variability on the use of a theorem. Some students did recognize that *IJ* was the segment joining the midpoints and evoked the property of this segment, but they were not sure about the validity of using the theorem when the diagram moved. V. and L., for example, evoked the theorem of the midpoint segment but did not dare using it. Pushed by the observer, they selected a triangle and V. looked carefully at the triangle and the midpoint segment when point *B* was dragged. She expressed her satisfaction:

> "The theorem of midpoints moves, yes it moves. It works even if we move"
> L. confirmed: "the midpoint theorem it works"
> V. "it works the same way"
> V. even tried to justify the invariance of the property in the drag mode:
> "they are all the same because there is always the same length. AC it is two times that. It is always two times that. It is always two times that and it works there all the time even if we move anyway."

A student of another pair wrote at the end of their proof: "As *DB* is always the radius, this proof is always right" and then the partner added: "for any position of *B*."

For these students, a proof seems to be carried out only for a particular instance of the diagram. From the work in Cabri, the problem of the shift from proving one instance to proving all instances arose for them. According to Netz (1999), Greek proof was rather done on a generic example than in a general case. The validity of the general statement was claimed at the end of the proof in the final part called Sumperasma. The expression of the validity claimed by students for all instances obtained by the drag mode can be compared with the expression of the Sumperasma in the Greek proof.

In this example, Cabri provided a window (Noss and Hoyles 1996) on the conceptions of students about proof, but the complexity introduced by the variability of the diagram acted as a catalyst for change in this conception for students such as V. and L. who became aware of the fact that a theorem may be valid for a moving diagram as the relations between elements remain unchanged. Questioning the validity of the theorem under the drag mode led the students to focus their attention to the relationships between elements of the figure. They learned from the complexity brought by the computer environment that offered to the students another window on mathematics (Noss and Hoyles 1996). This point of view was supported by several researches on Computer AlgebraSystems (CAS) used as a lever to promote work on the syntax of algebraic expressions (Artigue 2002, p. 265; Lagrange 2002, p. 171, or DGE assisting pupils to distinguish the properties of a rhombus from those of a square, Hoyles and Jones 1998).

11.4 A Didactic Dimension: Transforming the Tasks by Making Use of the Features of the Software

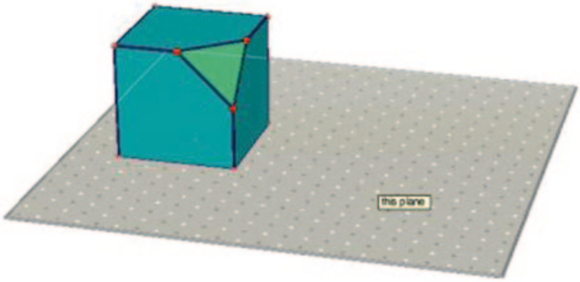
11.4.1 The Adidactical Milieu

The idea of a technological environment that has to be explored and that is interacting with the learner can be linked to the notion of "adidactical milieu" in the theory of didactic situations by Brousseau (1997). In this latter theory, knowledge is constructed by the student as a solution to a problem for which the constructed knowledge item provides an efficient solving strategy. The student does not solve the problem for satisfying the expectations of the teacher but because it is a genuine problem for him, a problem of the same kind as problems encountered in real life outside the classroom. The only difference is that real-life problems are not organized by a teacher in order to promote learning. Although designed by the teacher, a problem in an adidactical situation is experienced by the student as a real-life problem. In the core of the notion of adidactical situations is the notion of "adidactical milieu". An "adidactical milieu" offers information and a means of action to the student and reacts by providing feedback to his/her actions. It can be of material nature as well as of intellectual nature.

We do not claim that dynamic environments like Cabri II Plus and Cabri 3D provide an "adidactical milieu" but it can be organized and based on them for at least two main reasons:

- the available tools allow the user to perform mathematical operations on the representations of the mathematical objects; and
- the feedback offered by the drag mode allows the user to check whether his/her constructions are done by using mathematical properties and relations, and are not simply visually done.

Fig. 11.5 A truncated cube



Numerous examples of construction tasks with Cabri I, Cabri II, or Cabri II Plus are given in the literature and show how the first solving strategies of students are visual and evolve toward more geometrical constructions through the drag mode playing a double role. The drag mode invalidates purely visual constructions and also provides information about the geometrical behavior of objects (Noss and Hoyles 1996, p. 125; Jones 1998, pp. 79–82). In these construction tasks, geometrical knowledge is efficient as it is the only way to build a construction that is "drag mode proof". As it is possible to configure the software and to make available a restricted range of default tools or new tools obtained as macro-constructions, the designers of the construction tasks can thus promote the use of specific properties by the students and contribute to learning through the organized "adidactical milieu". An eloquent example is given by the task of drawing a perpendicular line to a line without the tool "Perpendicular" but with transformation tools, in particular the "Reflection" tool.

11.4.2 Example of an Adidactical Milieu in Cabri 3D

A more recent example (Mithalal 2010) is given here about a construction task in Cabri 3D. Cabri 3D is used to create an adidactical milieu fostering the move by students from a pure global visualization of a solid object, called *iconic visualization* by Duval (2005), to an analytical breaking down of a solid object into parts interrelated through geometrical relationships called *dimensional deconstruction* by Duval (2005).

Grade-10 students of two classrooms using Cabri 3D for learning 3D geometry (Mithalal 2010) were faced with the following activity at the beginning of the teaching of 3D geometry and after an introduction to the use of Cabri 3D: A cube with a triangular cross section was given on the screen and the students had to reconstruct the missing vertex so that it remains a vertex even when the cube is enlarged or dragged (Fig. 11.5).

In order to foster learning, students were asked to find several solving strategies, and tools were withdrawn from Cabri 3D once visual strategies appeared and once it was observed that these strategies did not produce robust vertices against dragging.

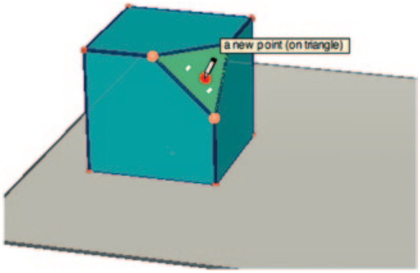


Fig. 11.6 A point visually located

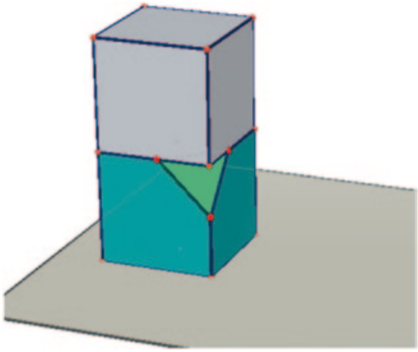


Fig. 11.7 Adjusting a second cube

It was expected that the added constraints would lead students to move to geometric characterizations of the missing vertex.

Producing a vertex by a purely visual strategy is not as easy in Cabri 3D as in a paper-and-pencil environment. If the user attempts to put a point by eye at the desired location, Cabri 3D proposes to create the point on the triangular crosssection (Fig. 11.6).

Visual strategies must be a little more elaborated and include some geometric components. An example of such a semi-visual strategy is creating a cube on the top of the original cube by visually placing its center at the center of the squared top face and putting one of its vertices to a vertex of the original cube and then visually adjusting this second cube so that its bottom face is coinciding with the top face of the original cube (Fig. 11.7). Of course, a vertex reconstructed in this way is not "dragging resistant." It was decided to withdraw the tool Cube from Cabri 3D after this strategy was proposed by the students.

Thus, strategies conceiving the vertex as an intersection of straight lines or of planes were expected in a second phase after semi-visual strategies. Students proposing such strategies were asked to find other strategies without using the tool "point."

The attentive observation of 30 pairs of students shows that a few of them resorted to a first phase to visual or semi-visual strategies. Some of those students tried to create a tetrahedron based on the triangular cross section with a fourth vertex providing the missing vertex of the cube. Noniconic visualization clearly underlies such a strategy. Students intended to reconstruct the entire cube as a material entity.

The most prevailing spontaneous strategy was to construct the missing vertex as the intersecting point of the three straight lines supporting the segments adjacent to the cross section (although two lines would be enough). The fact that often three lines, and not two, were constructed can be interpreted as a strategy inherited from paper-and-pencil environment mixed with iconic visualization. The three lines allowed students to restore the original representation of the whole cube in a paper-and-pencil environment. Some students then moved to the construction of the vertex as the intersecting point of planes or of a plane and a line. They took advantage of the possibility of using 2D objects in Cabri 3D and extended the intersection strategy. The use of a plane supporting a face moved them away from a purely iconic visualization of the cube and very often when using a plane and a line, the vertex was constructed only with one plane and one line and not with two planes and one line, and two lines and one plane. Through the instrumental deconstruction of the cube made possible by Cabri 3D, students moved toward a noniconic visualization.

Finally, it must be stressed that after the tool "point" was withdrawn, some students constructed the vertex using geometric transformations like point symmetry or translation: the vertex was constructed as the reflected image of another vertex with respect to the center of a cut face of the cube, or it is the image of a vertex in a translation with the vector which is defined by a side of the cube.

Construction tasks in DGEs are thus transformed. They become, in a way, more demanding as they require the use of geometrical knowledge to be solved and can be more difficult for students. However, the drag mode invalidating strategies may encourage the students.

11.4.3 Tasks Specific to Dynamic Geometry

Over the 20 years of existence of the Cabri technology, the design of "adidactical milieu" led to experiments with new kinds of tasks that cannot exist in paper-and-pencil environments, in particular:

- "black box" tasks in which dynamic constructions are given to students who must reconstruct them again so that the reconstructions have the same behavior in the drag mode as the original ones and
- prediction tasks in which students must predict without dragging as to what will happen if a specific point is dragged.

These tasks of a new kind are based on the same idea of taking advantage of the transformation of the nature of diagrams in DGEs, namely taking advantage of the variable nature of the diagrams controlled by the mathematical model underlying the software program.

In these tasks, students explore and interact with the environment. Through feedback and the available tools, they will develop strategies involving geometrical knowledge. The fact that it is the environment, and not the teacher who reacts, contributes to make the problem analogous to a genuine problem for students. In a black-box task, students can experiment on the given construction by adding elements and dragging, in order to find the relationships between its elements. The nature of the mathematical activity of the student is changed and becomes more of an experimental activity in which hypotheses are made and checked by experimenting. In prediction tasks, students must resort to geometrical knowledge to be able to predict the behavior of the construction in the drag mode. Then the predictions can be checked by dragging a point of the construction.

With the extension of Cabri tools to algebraic and graphing tools, the design of tasks making use of the Cabri features went beyond geometry. For example, Falcade (Falcade et al. 2007) designed a milieu for constructing the notion of graph of a function as expressing the covariation of two variables, the first one independent and the second one depending on the first one. Moreno (2006) designed a milieu in which the students had to find the ordinary differential equation of a family of dynamic curves, by exploring the variation and the invariant elements of this family in the drag mode, a kind of black-box task in calculus.

11.5 Transformations

In the nineteenth century, human knowledge led to the design of tools for the mechanization of human activity. At the end of the twentieth century, human knowledge could be embarked in technology-modeling domains of theory. In dynamic mathematics environments, human knowledge is embarked in the representations of theoretical objects, which behave according to the theoretical model underlying the technology, independent of the wishes of the user as soon as the latter has constructed them. One could say that a transformation of a new kind took place. It does not lie only in the creation of artefacts embarking knowledge but also in the creation of artifacts offering a dynamic model of theoretical objects. In terms of the Vygotskian perspective, the creation of artifacts embodying theoretical knowledge at a higher degree than before may affect the nature of the psychological tool constructed by humans. It may extend the role of these psychological tools on the mental activity of the individual. This is very apparent in the reaction of students L. and V. when they discovered that the theorem "moved", i.e., was valid for every occurrence of the diagram. The use of a dynamic construction led them to consider a theorem as an invariant statement about variable objects.

The teaching of mathematics can take advantage of the transformation of the offered representations of mathematical objects by changing the kind of tasks given to students for fostering learning as described in Sect. 11.3. However, the role of the teacher is still essential. The students may not be able to solve new kinds of tasks that are more demanding in terms of knowledge and need help from the teacher. Once students have solved the task, the teacher may contribute to an internalization process by organizing social interactions and collective discussions in the classroom, intervening in order to transform the meaning of what has been done on the computer into a meaning that can be related to the "official" mathematical meaning. This process of *semiotic mediation* based on this new nature of representations was theorized and extensively experimented by Mariotti (2001) (see also Mariotti, Chap. 9).

The industrial revolution deeply affected the human society. The transformations brought about by knowledge technology also affected the society. However, it influenced schools in a minor way. One missing link between this deep transformation of technology and schools is certainly pre- and in-service teacher education and accompanying measures for taking advantage in everyday teaching of this transformation of representations of mathematical objects.

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Chapter 12
The Role of Social Aspects of Teaching and Learning in Transforming Mathematical Activity: Tools, Tasks, Individuals and Learning Communities

Vince Geiger

12.1 Introduction

Developments in technology, including all types of hardware, applications and the Internet, have led to an expanding number and range of studies that focus on how digital tools can transform teaching and learning in mathematics (Hoyles and Lagrange 2010). This research now represents a substantial corpus of knowledge with increasing activity in many aspects of the field. Many studies have focussed on improving approaches to the development of content knowledge or concept development including: number (e.g. Kieran and Guzman 2005); algebra and calculus (e.g. Ferrara et al. 2006); and geometry (e.g. Laborde et al. 2006). Other studies have attempted to explore the role of digital technologies in enhancing particular types of mathematical activity, for example, problem solving (e.g. Lesh and English 2005) and mathematical modelling and applications (e.g. Geiger et al. 2010), or to investigate how specific digital tools can be used in the process of teaching and learning, such as hand-held technologies (e.g. Drijvers and Weigand 2010), computer algebra systems (e.g. Pierce et al. 2009), and dynamic geometry systems (e.g. Laborde 2002).

Fewer studies, however, have attempted to theorise the nature of the types of transformation that take place when teachers, students, digital tools, and tasks interact. Of relevance here, are studies which explore the way artefacts, such as digital technologies and mathematical tasks and human actors, students, and teachers, are all transformed through multiple and concurrent interactions. There is a reflexive, transformative relationship between each of these elements, each a component of a learning complex that mediates change in the other. While research in this area is developing, to date, it has tended to focus on individualistic point-to-point relationships, for example, the relationship between a single student and a single teacher or between a single student and a digital tool. As Strässer (2009) points out, the social

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aspects of learning and the potential for learning communities to move forward have often been neglected when attempting to explain the relationships between humans, as individuals and as collectives, physical and representational resources, and mathematical knowledge.

In this chapter I discuss research that seeks to theorise the relationships between humans, mathematical knowledge, digital tools, tasks, and other resources that interact in the processes of learning and teaching mathematics. I will then attempt to explore the role of social aspects in the process of learning, illustrated through selective examples drawn from my research program.

12.2 Technological Tools in Mathematics Education

Prominent in literature about how interaction between user and tool changes the nature of digital tools are authors such as Guin et al. (2005) and Artigue (2002), who have attempted to explain how digital artefacts such as computer algebra systems (CAS) are transformed into instruments for learning through interaction with teachers and students. According to Verillon's and Rabardel's (1995) distinction between an artefact and an instrument, an artefact has no intrinsic meaning of its own but meaningful relationships develop between an artefact and a user when both combine to work on a specific task. Different tasks will require different relationships between the user and the artefact, and the development of these relationships is referred to as the *instrumental genesis*. This instrumental genesis has two components (as described by Artigue 2002). First, the transformation of the artefact itself into an instrument is known as *instrumentalisation*. In this process, the potentialities of the artefact for performing specific tasks are recognised. Second, the process that takes place within the user in order to use the instrument for a particular task is known as *instrumentation*. Here, we see the development of schemas of instrumented action, which are developed either personally or through the appropriation of pre-existing schemas. Thus, an *instrument* is composed of the artefact, along with its affordances and constraints, and the user's task-specific schemas. In concert, these elements provide direction for the use of the instrument in a given context. Finally, the process of instrumental genesis is two-dimensional in that the possibilities and constraints shape the conceptual development of the user, while at the same time, the user's conceptualisation of the artefact and, thus, its instrumentation leads, in some cases, to the user changing the instrument (Drijvers and Gravemeijer 2005).

A teacher's activity in promoting a student's instrumental genesis is known as *instrumental orchestration* (Trouche 2003, 2005). Social aspects of learning are recognised within this process and take the form of student activity that makes explicit the schemas that individuals have developed within a small group or whole class. These schemas are then available for appropriation by other class members through careful and selective questioning by the class teacher, that is, the teacher orchestrates the interaction so that a new individual scheme is shared with others. While this perspective attempts to incorporate the role of social interaction into the instrumental approach, utilisation schemas are essentially individual, even though

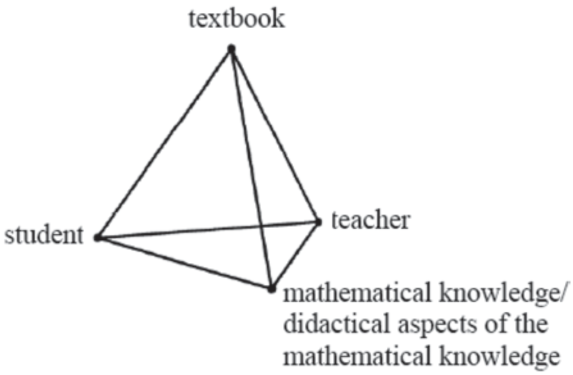


Fig. 12.1 Rezat's (2006) tetrahedral model

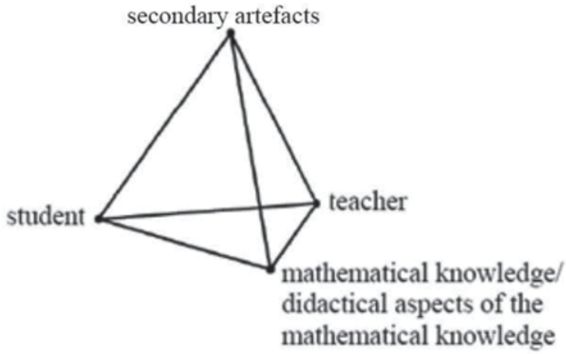
instrumental genesis may take place through a social process (Drijvers and Gravemeijer 2005). As a result, this conceptualisation of the role of social aspects of learning is limited to the contributions of individuals to a larger group and does not accommodate genuinely shared approaches to learning and teaching where mutuality is an accepted norm.

More recently, Gueudet and Trouche (2009) have extended the concept of the instrumental approach to include other artefacts in addition to CAS-enabled technologies when considering the professional work of teachers. They introduce the term *resources* to identify any artefact with the potential to promote semiotic mediation in the process of learning, including computer applications, student worksheets or discussions with a colleague. In their view, *resources* must undergo a process of genesis, parallel to that of the *instrumental approach*, in which a resource is appropriated and reshaped by a teacher in a way that reflects their professional experience in relation to the use of resources, to form a *scheme of utilisation*. The combination of the *resource* and the *scheme of utilisation* is called a *document*. The process of *documental genesis* is an ongoing one as *utilisation schemes* will be reshaped as the teacher gains more experience through the use of a *resource*. While this approach extends the theory of the instrumental approach to accommodate a broader range of material and representation resources, it still does not provide a clear role for social interaction in the process of developing a document.

Theorising about the role of artefacts in mediating transformative learning experiences from a different perspective, Rezat (2006) draws on activity theory to assign artefacts, such as textbooks, a position in a tetrahedral model (Fig. 12.1), which uses the so-called didactical triangle as a base. The faces of the tetrahedron represent activity sub-systems of textbook use, in which one element mediates between the other two elements of the triangular face. For example, the face that is bounded by the elements student–teacher–textbooks represents activity in which the teacher mediates the use of the textbook in order for the student to learn.

Strässer (2009) draws on Wartofsky's (1979) classification of artefacts to extend Rezat's (2006) tetrahedral model to combine the theoretical constructs of semiotic mediation (for more on this construct see Mariotti, Chap. 9), instrumental genesis, and a broader view of artefacts. He argues that Wartofsky's class of "secondary"

Fig. 12.2 Strässer's (2009) tertrahedral model



artefacts, which are "used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out" (Wartofsky 1979, pp. 202–209), can replace textbooks in Rezat's tetrahedron to create a more inclusive model for the use of instruments in teaching and learning mathematics (Fig. 12.2). In this new model, the edges, which join the elements, allow for a number of processes. The edges that join student and artefact and teacher and artefact, for example, allow for the process of instrumental genesis. The vertices of secondary artefact and mathematical knowledge are joined by an edge that represents semiotic mediation.

Strässer points out, however, that one of the shortcomings of this model is that it does not clearly theorise a role for social aspects of teaching and learning mathematics and that further development is needed in order to account for the influence of institutional and social influences in the process of instrumental genesis. While he does not attempt to address this gap in theory himself, he points to the concept of Chevallard's (1985/1991) "noosphere"—a space that accommodates "all persons and institutions interested in the teaching and learning of mathematics" (Strässer 2009, p. 75) and suggests his model for teaching and learning in mathematics education could accommodate social aspects of this activity by surrounding the model in such a noosphere. Strässer's suggested extension highlights the need to develop frameworks which describe, theorise, and interpret the ways that teachers and students engage with mathematical knowledge and secondary artefacts within learning communities. This implies that sociocultural perspectives must be considered when developing frameworks which accommodate the noosphere.

12.3 A Sociocultural Perspective on Teaching and Learning with Technology

Sociocultural perspectives on learning emphasise the socially and culturally situated nature of mathematical activity and view learning as a collective process of enculturation into the practices of mathematical communities. The classroom, as a community of mathematical practice, supports a culture of sense making, where students learn by immersion in the practices of the discipline.

A central claim of sociocultural theory is that human action is mediated by cultural tools and is fundamentally transformed in the process (Wertsch 1985). Within particular knowledge communities, then, tools are cultural resources that reorganise, as well as amplify, cognitive processes through their integration into human practices.

There is now an emerging body of literature that explores the idea that human thought and action, within social environments, are mediated through many aspects of a situation, including interactions between human actors and both material and representational resources. Pea (1985, 1993a, b), for example, draws on a Vygotskian view of learning to argue that learning and reasoning should now be considered an activity system, which involves minds, social contexts, and digital tools such as computers, that is, thinking is *distributed* among and between these elements.

This position is paralleled in Hutchins' (1995) account of the process of navigation on a naval vessel (as described by Cobb 2007), which considers the whole navigation team, including all physical and symbolic tools, as the reasoning system that provides for the safe piloting of the vessel into port. Further, this reasoning system is constituted by elements that exist not only in the moment of the act, for example, the navigator and the ship's guidance system, but also by elements that preceded the event that led to the process of navigation and the artefacts used to navigate. This is because traces of the intelligence of the other minds that developed relevant tools and procedures to guide navigation remain in those tools and procedures.

Pea (1985) further argues that tools, and in particular, digital tools can be used to reorganise mental processes, which in turn alter a task as it was originally conceived.

> Computers are commonly believed to change how effectively we do traditional tasks, amplifying or extending our capabilities, with the assumption that these tasks stay fundamentally the same. The central point I wish to make is quite different, namely, that a primary role for computers is changing the tasks we do by reorganizing our mental functioning, not only by amplifying it. (Pea 1985, p. 168)

Thus, in Pea's view, there is a reflexive, transformative relationship between digital tools, tasks and human players. He goes on to add:

> … human nature is not a product of environmental forces, but is of our own making as a society and is continually in the process of "becoming." Humankind is reshaped through a dialectic, or "conversation" of reciprocal influences: Our productive activities change the world, thereby changing the ways in which the world can change us. By shaping nature and how our interactions with it are mediated, we change ourselves. (Pea 1987, p. 93, original emphasis)

Other authors have also speculated on the nature of transformations that take place when humans and technology interact with the intention of learning and doing mathematics. Consistent with Pea's view of the interactive relationship between digital tools, tasks and human agents, Borba and Villarreal (2006) see learning as a collaborative act between collectives of humans and technology.

> They (computers) interact and are actors in knowing. They form part of a collective that thinks, and are not simply tools which are neutral or have some peripheral role in the production of knowledge. (p. 5)

From this perspective, Borba and Villarreal (2006) argue that an intershaping relationship exists between learners and technology in which both are transformed. In this relationship, technology mediates the way in which students learn and come to know while, at the same time, students interact with technology in ways unanticipated by the designers of the digital tools. Thus, learner/knower and technology shape each other. Knowledge is produced through the efforts of collectives of humans-with-media or humans-with-technologies. They further argue that this collective also produces different mathematical knowledge and so the discipline itself is influenced to change.

While studies, such as those discussed above, argue that there are transformative relationships between all agents, digital tools, tasks, and humans, when brought together in the act of learning, this research does not make clear how interactions between collectives of humans, when working on mathematical tasks, are influenced by and, in turn, influence digital tools. An important question which arises is how do peers, through the meditative influence of technology and tasks, assist each other to move forward in their knowing and doing of mathematics?

This issue is addressed by Manouchehri (2004), who identifies four ways in which the computer application NuCalc supported productive interaction among undergraduate pre-service teachers when studying a course in algebra. She found that there was a greater level of interaction when NuCalc was in use, compared to sessions when no technology was available, to support learning and concluded that this digital tool promoted interaction by:

1. Assisting peers in constructing more sophisticated mathematical explanations;
2. Motivating engagement and increased participation in group inquiry;
3. Mediating discourse, resulting in a significant increase in the number of collaborative explanations constructed and;
4. Shifting the pattern of interaction from teacher directed to peer driven.

This form of interaction also supported a culture of conjecturing, testing and verifying, formalising mathematics and collaboration that shifted the locus of power from the teacher to the students.

In a series of sociocultural informed studies conducted by myself and colleagues (see for example, Galbraith et al. 1999; Geiger 2005, 2006; Goos et al. 2000, 2003), ways in which productive interactions between students, teachers and secondary artefacts led to mathematical learning have been explored. Through these studies we documented instances of not only the most widely known definition of the zone of proximal development (ZPD), which considers the potential of an individual to learn with the support of a more knowledgeable and/or experienced other, for example, a teacher, but also the conceptualisation of the *ZPD in egalitarian partnerships* (Galbraith et al. 2001). This view of the ZPD involves equal status relationships wherein students have incomplete but relatively equal expertise—each partner possessing some knowledge and skill but requiring the contributions of the others in order to make progress. In our research context, this feature becomes relevant through the collaborative activity of students in bringing technology to bear on mathematical tasks with varying levels of individual technological and mathematical expertise and within different social settings.

As a second extension of the ZPD concept we have theorised the ways in which classroom participants and both material and representational resources interact within learning communities. These ways of reasoning, thinking and acting within such a community have been described through a typology (see, for example, Goos et al. 2003) in which four metaphors, *Master, Servant, Partner,* and *Extension-of-self,* are used to describe patterns of student–student–technology behaviour, where the boundaries between human and technological agents are blurred during the process of learning and using mathematics. Implicit in these interactions are the use of other resources such as mathematical tasks on which student–student–technology interactions are brought to bear. The blurring of the boundaries between human and non-human participants in learning means that it is possible to attribute agency to digital tools in a way not previously considered as part of the concept of the ZPD. This is particularly evident where digital tools are used by students as partners or extensions-of-self to assist them to move forward within their ZPDs. More recently, Geiger (2009) has developed a framework based around these metaphors that also differentiates between types of student–student–technology interactions on the basis of the social setting in which these interactions take place—as individuals, within small groups and within whole groups.

In the case of individual's interactions with digital tools, learning may be transformed, for example, because of the way tasks can be explored through viewing and manipulating a range of different representations, algebraic, numerical, graphical, and geometrical. Within small groups, digital tools may act as a means to transform the way a group coordinates its efforts when working on a task, for example, when all group members view a digital representation of a possible solution to a problem on the same computer display and negotiate how the solution might be improved. When students present their work on a task to a whole-class group, learning is transformed from a private activity into one that is brought into the public realm for scrutiny. Digital tools, in this social context, can provide the opportunity for collaborative and supportive critique to take place in order to lead the presenter forward in their learning.

As each of these social contexts represents possibilities for the ways in which resources, students, teachers, and mathematical knowledge interact, Geiger's portrayal of the role of social setting for technology-mediated interaction offers possibilities for the elements of Strässer's foreshadowed structure of the noosphere that surrounds his model for teaching and learning mathematics. Aspects of this structure will now be explored with reference to classroom episodes drawn from our research.

12.4 Episodes of Resource Use in Social Contexts

The episodes reported here, are drawn from a 2-year longitudinal case study based on a single mathematics classroom located in a large Australian city. Student participants were in years 11 and 12, the final 2 years of secondary schooling, and were 16–17 years of age. They were studying a subject, Mathematics C, which is designed for students who have a high interest in mathematics or who have intentions to study mathematically intensive courses at a tertiary level. The curriculum document associated with this course strongly encouraged, but did not mandate, the use of digital technologies, and schools in this educational jurisdiction were allowed to choose, without restriction, whatever form of digital technologies they believed offered the greatest potential to enhance student learning. In this study, students had unrestricted access to a wide range of digital tools including, CAS-enabled calculators, computer software applications, and the Internet.

The teacher in this study had based his classroom around a community of practice principles where learning was structured around approaches to knowing and meaning making, which included the public scrutiny of conjectures and ideas in order to provide for the opportunity of supportive critique from all classroom participants. In this way, the teacher attempted to initiate students into the socially constructed practices of knowledge creation and validation as is practised within the discipline of mathematics.

Since the aim of the study was to explore the types of interaction that took place between students, teachers and both material and representational resources in holistic classroom settings, *in situ*, ethnographic techniques for data collection were employed. These included: participant observation; semi-structured interviews with students as individuals, in small groups, and as a whole class; survey instruments; and video- and audio-taped records.

Consistent with a naturalistic methodology, data collection and analysis were conducted simultaneously with the development of theory. An iterative approach was utilised where initial observations of phenomena were used to formulate theoretical propositions which are in turn tested, revised and refined against further data. Instances of emergent behaviour were documented and categorised. Where emergent phenomena were noted and documented, the researcher made use of follow-up interviews with the relevant participant(s) in order to triangulate the occurrence of the identified phenomena and to discuss possible explanations for what was observed (Lincoln and Guba 1985). In this way, observed phenomena and explanations were concurrently incorporated into developing theory. This process led to the emergence of four categories of technology use: master; servant; partner; and extension-of-self. These metaphors are described in the following paragraphs.

Technology as Master Students may be subservient to the technology if their knowledge and usage are limited to a narrow range of operations over which they have technical competence. In the case of students, subservience may become dependence if the lack of mathematical understanding prevents them from evaluating the accuracy of the output generated by the calculator or computer.

Table 12.1 Metaphor/social setting framework array

Setting/metaphor	Master	Servant	Partner	Extension-of self
Individual				
Small group				
Whole group				

Technology as Servant Here, technology is used as a fast, reliable replacement for mental or pen and paper calculations, but the tasks of the classroom remain unchanged, that is, technology is a supplementary tool that amplifies cognitive processes but is not used in creative ways to change the nature of activities.

Technology as Partner In this case, technology is used creatively to increase the power students exercise over their learning; for example, by providing access to new kinds of tasks or new ways of approaching existing tasks. This cognitive reorganisation effect may involve using technology to facilitate understanding or to explore different perspectives.

Technology as Extension-of-Self The most sophisticated mode of functioning, this involves users incorporating technological expertise as a natural part of their mathematical and/or pedagogical repertoire. Students may integrate a variety of technological resources into the construction of a mathematical argument so that powerful use of computers and calculators forms an extension of the individual's mathematical prowess.

(selected from Goos et al. 2003, pp. 77–80)

Each of these categories was also differentiated into behaviour sets based on three types of social setting, individual, small group, and whole group, to form a 4×3 array (Table 12.1).

Within this framework, the ways in which human actors and material and representation resources interact vary in individual, small-group, and whole-class settings. The following is a section of the framework associated with the partner metaphor and its expression in each type of social setting (Table 12.2).

The following episodes have been selected to illustrate elements of the category of partner across the social settings of individual, small group, and whole group, with the aim of identifying possible expressions of Strässer's tetrahedral model.

12.5 Peer-Based Interactions in Small-Group and Whole-Group Social Settings

12.5.1 Episode 1

In this episode, students (Year 11) were asked to use the geometry facility on their TI-92 calculators (a version of Cabri Geométrè) to draw a line $\sqrt{45}$ units long. The

Table 12.2 Partner metaphor against social setting (Geiger 2009)

Social setting	Partner metaphor
Individual	Here, rapport has developed between the user and technology, which is used creatively to increase the power that students have over their learning. Students often appear to interact directly with the technology (e.g. graphical calculator), treating it almost as a human partner that responds to their commands—for example, with error messages that demand investigation. The calculator acts as a surrogate partner as students verbalise their thinking in the process of locating and correcting such errors
Small group	Technology is used to explore and investigate a problem but there is also evidence of the technology playing a part in the facilitation of collaborative processes. Calculator or computer output also provides a stimulus for peer discussion as students cluster together to compare their screens, often holding up graphical calculators side by side or passing them back and forth to neighbours to emphasise a point or to compare their work. Work can be progressed "live" on any group member's display. Technology may provide support that facilitates students' engagement in a group interaction
Whole group	Technology is used to explore and investigate a problem or an idea in a public forum. Technology is used to focus the intellectual resources of the community to help explore ideas, offer public critique of existing work or suggest improvements to work where faults are identified. Technology is used to provide support for engagement in this community of inquiry

teacher's aim was to encourage students to think about the geometric representation of irrational numbers, the topic being studied at that time. It was anticipated that students would solve the problem by making use of the geometric facilities of their calculators to explore possible Pythagorean relationships that would provide a solution to the problem. It was hoped that students would eventually realise the relationship $6^2 + 3^2 = (\sqrt{45})^2$ was a basis for the construction of a right-angled triangle with a hypotenuse equal to $\sqrt{45}$. This meant that the other two sides must be 3 and 6 units, respectively. This is illustrated via the calculator screen-shot in Fig. 12.3.

Students had been previously provided with experience in working with this facility through assignment work earlier in the year, although not all had become confident users as a result. The class was set to work on the task with few further instructions in the use of this application.

The teacher allowed some time for students to explore the problem. This excerpt concerns three students, Susie, Keira and Gena, working in a small group. Initially, Keira and Gena worked together, while Susie worked independently. Later, Susie joined the conversation as Keira and Gena raised issues that were also troublesome to Susie. During the discussion of the problem, Keira and Gena made use of a TI-92 calculator in a variety of ways. Each used the calculator to perform procedural calculations, such as taking the square root of numbers they wished to evaluate, as part of their exploration of the problem. After each calculation, however, they passed their calculators to each other as a way of sharing what they had found. Thus, the calculator was used as a means of communicating the findings between the students. The openness of this process of sharing was evident when Gena passed her calculator to Susie, who was not immediately next to Gena but on the other side

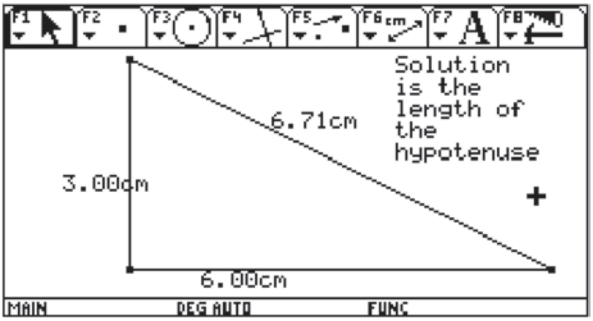


Fig. 12.3 Model solution to the length of $\sqrt{45}$ problem

of Keira. The passing of a calculator from one student to another was not just about the simple transmission of results. On a number of occasions, each student in this small group accepted the calculator of another and modified or added to what was being displayed. In this way, the calculator was acting as a medium for the progress of the thoughts and ideas of these students and so the calculator played the role of *Partner* in transforming initial conjectures into more developed ideas. Within the working cluster, each student's calculator display appeared to be public property on which ideas were offered up for comment and critique and were then transformed through the modification of existing ideas or the addition of new ones by the group as a collective. After allowing some time for investigation of the problem, the teacher called for volunteers to present preliminary results of the investigation. Gena offered the solution, developed by her group, to the class in a whole-group context. She moved to the front of the room, plugged her calculator into the viewscreen and entered $\sqrt{45}$ followed by the "enter" key. This produced a result with 10 decimal places which Gena assumed was a terminating decimal because of the calculator's known capacity to display up to 12 decimal places. Other students in class, however, pointed out that $\sqrt{45}$ is irrational and so could not terminate. They, then, offered counter-examples for Gena to input and display "live" in order to illustrate the misconception. The problem was identified by Gena, with the assistance of the class, to lie with a setting on the calculator that fixed the results of calculations to 10 decimal places. At this stage, Gena acknowledged the error although she was unable to suggest any improvements to her approach to the original task.

Gena's group initially used their calculators to perform procedural operations such as finding the $\sqrt{45}$ and secondly, to communicate their individual findings to other members of the group. The passing of calculators from one to another provided the opportunity and the medium to progress ideas or try new directions by altering the display and return it to an original group member for further consideration in the same way that a scratch pad might be used by a group of people working on a common problem. Here, technology was used as a *Partner* to make public individual contributions to the problem-solving endeavour and also as the canvas on which all members of the small group worked together as a collective in scaffolding each other's efforts towards finding a solution. In terms of Strässer's tetrahedron, student, secondary artefacts and mathematical knowledge have interacted to transform the original task into a proto-solution. However, Strässer's model does not accommodate the type of partnership that takes place between multiple students and technology as human and non-human participants work in concert.

The initial, faulty solution is a result of a misconception, clearly shared by all group members and was related to the way irrational numbers are represented on a calculator. The episode, described earlier, illustrates the capacity of technology to act as a *Partner* in whole-group settings as the combination of calculator and viewscreen permitted Gena to present her group's findings which, in turn, allowed other members of the class to identify an error and to help Gena correct the source of the problem. This instance again shows how student, secondary artefacts and mathematical knowledge interact, but this time in a whole-group context, to transform a proto-solution into a valid result. The episode also highlights the important role of a collective of learners in assisting a member of the class to overcome a misconception which was limiting her capacity to find an appropriate solution. This type of whole-class interaction is also not accommodated by Strässer's model. Further, this episode also represents a meta-interaction between two types of social settings. Findings that were produced via the interaction within the small group were held up for scrutiny by the whole group, where supportive critique was provided. The critique facilitated the improvement of the small group's initial result. This means that the interaction between the two social settings has resulted in the transformation of the mathematical knowledge of at least one member of the group who was assisted to overcome a misconception.

12.5.2 Episode 2

This section presents a series of excerpts from an episode that took place over two classroom sessions. Students (Year 12) were asked to design programs for their TI-83 or TI-92 calculators programs capable of taking inputs for vector pairs, either two-dimensional, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$, or three-dimensional, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$, in order to make use of the equations $\theta = \dfrac{\cos^{-1}(a \times c + b \times d)}{\sqrt{a^2 + b^2} \times \sqrt{c^2 + d^2}}$ or $\theta = \dfrac{\cos^{-1}(a \times d + b \times e + c \times f)}{\sqrt{a^2 + b^2 + c^2} \times \sqrt{d^2 + e^2 + f^2}}$ to determine the angle between relevant vector pairs.

The teacher provided only minimal instruction in basic programming techniques, and expected students to work as individuals or with consult peers, within small groups, for assistance. Students were given one lesson to work on their programs and advised that they would be expected to present their results at the next meeting of the class. During the following session, two volunteers offered to demonstrate their programs to the whole group via a calculator viewscreen. The presentations of the two volunteers are reported in the following excerpts.

Excerpt 1 Demi was the first of the volunteers to demonstrate her program to the whole group. She felt confident her program, which she developed as an individual, performed the required calculation effectively.

Demi: What do you want me to tell you about it?
Teacher: Just what each of those bits do (pointing to the command lines of the program displayed through the OHT).
Demi: OK. The Prompt feature … like when you want to do a vector in 3D it just goes A, B, C, etc. so that A, B and C are the first vector and D, E and F are the second one … and then it just.
Francis: (shouting out assistance) Does it!
Demi: Yeah! Does it!
Teacher: Do you want to show us an example?
Demi: Sure.
Demi began the program and entered the components of two vectors.
Francis: (reading out what Demi is entering) −1, 2, 3, 3, −1, 2.
Demi activated the program but recognized immediately her output was in radians, not in the form she wanted, which is degrees.
Demi: It's not in the right mode.
Demi opened up the mode menu and made the appropriate adjustment.
Teacher: Will it work in degrees?
Demi completed the adjustments.
Demi: Yeah.
Demi re-entered her example and calculator displayed the desired output.
Demi: Is it right now? (to audience)
Francis: Yeeaaaahhh!

Demi produced a program that functioned without error. She made minor adjustments "on the fly" in order to produce an output of the form required. This was a confident display in which she received few prompts or advice from the class. Information was presented in a clear and well-structured manner and the class seemed satisfied, as they did not ask questions for clarification or any other purpose. As the task itself did not require the exploration of deep mathematical concepts, the calculator and viewscreen were used purely for presentation purposes and were not required as a medium for the stimulation of discussion and debate, even though the presentation was in a public setting. Thus, in this case, Demi's work as an individual, based on the instruction she had already received from her teacher, was enough to effect a valid solution to the problem. In terms of Strässer's model, the faces defined by student–secondary artefacts–mathematical knowledge and student–teacher–mathematical knowledge appear to offer an appropriate framework for capturing the elements that have led to Demi's results. This episode demonstrates that transformation of an individual's or group's mathematical understanding does not happen simply because technology is available within a whole-group context. In this case, the nature of the task did not provide enough challenge for Demi to engage the assistance of others.

Excerpt 2 The second volunteer was a student who consistently rejected the teacher's invitations to participate in whole-class discussion and to contribute to thinking with his peers. It was quite surprising when Geoffery offered to present in relation to this task, as he had often been resistant to working in public forums or to contributing to any endeavour that required his active contribution. For example, prior to an earlier assessment item, the teacher asked class members to assist in the development of a revision sheet for the upcoming exam. This involved each student writing a question with a model solution for a revision sheet that the teacher offered to edit, and then print for each class member. Geoffery attempted to avoid participation in the activity.

Teacher: So…, what's your contribution going to be?
Geoffery: Not much?

In response to this obvious reluctance to assist with the classes' assessment preparation, the teacher decided to assign a section of work to Geoffery in order that he made a fair contribution to the classes' revision sheet. Geoffery expressed his opposition to this request.

Geoffery: But I wouldn't have a bloody clue.
Teacher: But that's part of the point…this is probably a really good way of revising.
Geoffery: Yes but me revising one thing isn't going to help me much!
After considering what was being asked of him further, Geoffery decided to make one more attempt to avoid the activity.
Geoffery: Does that mean if I choose not to take a revision sheet I don't have to write a question?
Several other students: Aw just grow up!

There appears to be at least two reasons for this student's reluctance to contribute to the class revision sheet. First, Geoffery did not believe that learning could be a collaborative activity and, as a result, he viewed preparation for assessment as an individual responsibility. Second, Geoffery seems to operate from a system of beliefs about learning that ascribes the role of a student to that of a passive receiver of knowledge from an expert source—in this case, the teacher. From Geoffery's perspective, his only responsibilities as a learner are to attend classes, to listen to the instruction provided by the teacher and to consolidate knowledge through exemplars provided by the teacher. In this instance, Geoffery believed the creation of a revision sheet was the teacher's responsibility and found it difficult to accept that this role was to be shared by the students. Thus, Geoffery does not seem to believe that there is a social aspect to learning.

Geoffery's participation in classroom events began to change when he was drawn into the activity described above. He offered to present his program, to the whole class, which included the initial screen illustrated in Fig. 12.4 (Screen 1) and also two screens which contained a question that appeared after vector components were entered but before the final calculations were displayed (Screens 2 and 3). If option 1—YES—was chosen (from Screen 3), the user received an answer to the problem (Screen 4). The selection of option 2—No—resulted in the display of a taunt (Screen 5).

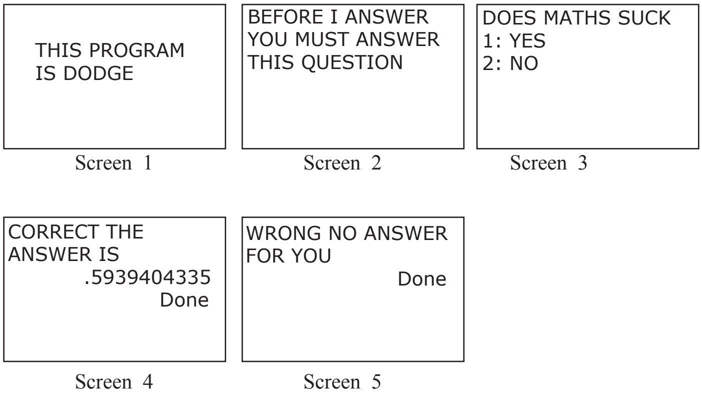


Fig. 12.4 Geoffery's program for solving a three-dimensional vector problem

Geoffery used the task to demonstrate dissent in relation to the culture the teacher had established in this mathematics classroom. This was a clever use, by Geoffery, of the very methods the teacher was using to encourage students' participation in a classroom community of inquiry in order to record a protest. Despite such an open challenge, the teacher did not issue a reprimand of any type as he recognised that this would only be counter-productive. Instead, Geoffery was complimented on his ingenuity and praised for the quality of his program.

Geoffery responded, over subsequent lessons, by increasing his involvement in classroom presentations whenever technology was used to mediate discussion about a mathematical task. This included presentations to the whole class of improved, and increasingly sophisticated, versions of his initial program.

After some weeks, Geoffery asked if he could present an animated program he had created that depicted the adventures of mathematical objects (various irrational numbers) as human-like characters—*Dodge: The Movie*. The enthusiastic and admiring response to his "movie" (and the sequel—*Dodge II: The Revenge of Dodge*) was significant in drawing this student into the kind of mathematical discussion he had previously resisted, and he became a willing participant in subsequent discussions both technology-focused and otherwise.

Geoffery had initially used a method of working within the class he had previously resisted to register dissent in relation to the way his mathematics classes were conducted. However, after receiving positive reinforcement from his peers (and no negative feedback from the teacher) for his initial and subsequent presentations, he was slowly drawn into the norms of interaction practised by his learning community. In this excerpt, Geoffery attempted to stand outside of the social contexts for learning set up by the teacher—small group and whole group—and work within a framework which consisted of only digital tools, tasks, mathematical knowledge, teacher and himself as a student. The resulting interactions are consistent with the faces of Strässer's model defined by student–teacher–secondary artefacts and student–teacher–mathematical knowledge. It should also be noted that his view of student–teacher interactions was unilateral, that is, from teacher to student and rarely the other way around. This began to change when Geoffery decided to use the combination of technology and task as a *Partner* to express the personal frustration he felt over a conflict between his view of how to learn and do mathematics and the social and cultural norms for doing so in his particular classroom. Technology and task had acted as a *Partner* "in crime" in this instance. It was this attempt to show dissent, however, that marked his appropriation of the modes of reasoning and meaning making that the teacher had established within this learning community. Resources, in the form of technology and task, had not only transformed ways of reasoning and making meaning in a way that was now inclusive of other members of the class but also transformed him into a member of a community from which he had previously isolated himself. Material and representation resources in this case were used as a supportive *Partner*, a "go-between", that encouraged him to move from the fringes of his learning community into the mainstream and so transforming his identity within this group.

12.6 Conclusion

This chapter began with a discussion of research into how technological tools in combination with mathematical tasks can be used to transform learning and teaching. In attempting to theorise the nature of this transformation, researchers such as Artigue (2002), Guin et al. (2005) and Gueudet and Trouche (2009) have proposed an instrumental approach which characterises the act of learning and knowing as a series of interactions between material and representational resources, learners, and teachers, within which all of these elements are transformed. Absent from this approach, however, is a clear description and interpretation of the role of social aspects of learning. Strässer (2009) has attempted to point a way forward in relation to this gap by extending the definition of resources to that of Wartofsky's (1979) secondary artefacts and situating a tetrahedral model of learning, knowing, and teaching that incorporates the elements of student, teacher, mathematical knowledge, and secondary artefacts, within a Chevallard (1985/1991) inspired notion of a noosphere in order to account for social aspects that influence teaching and learning. In this chapter, I have attempted to provide insight into the nature of the types of social interaction that take place within such a noosphere through examples drawn from research in authentic classroom settings. These social interactions, in concert with available secondary artefacts, influence the transformation of students' understanding of mathematical knowledge, as well as their modes of reasoning and meaning making, in different ways according to the particular social setting in which learning is situated.

The episodes, presented here, have provided evidence that these social settings have a range of different forms. Students' work can take place in individual settings with little input from the teacher or other class members, as in the case where Demi

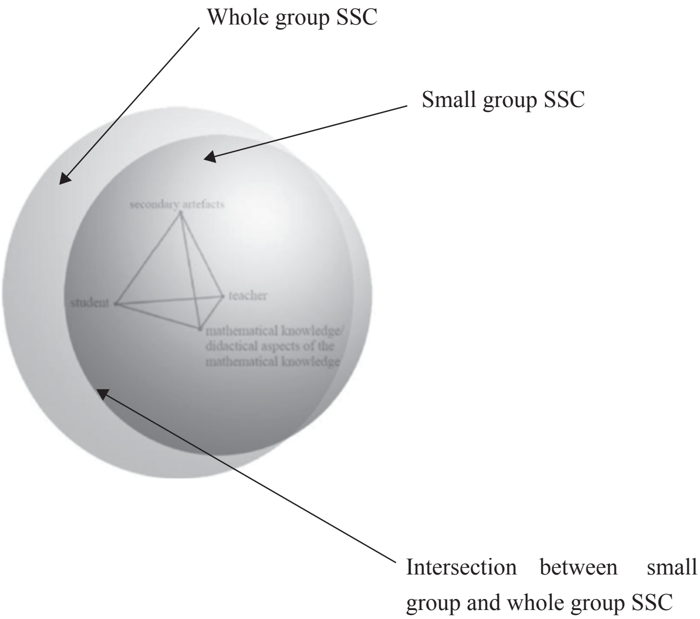


Fig. 12.5 Strässer's tetrahedral model surrounded by the spheres of social context

presented an error-free solution to a problem. In this case, the interaction between Demi and the whole class was really only for the sake of a presentation, not for the furthering of her knowledge or understanding.

When Keira and Gena worked as a small group on the length of $\sqrt{45}$ problem, both small-group and whole-group interactions were vital in developing their understanding of the concept under focus. Within the small group, digital tools and tasks mediated students' interactions with mathematical knowledge and with each other. This was particularly evident in the use of a CAS-enabled calculator as a dynamic scratchpad to share and develop ideas and initial solutions to the problem. Technology and task also mediated this small group's interaction with the whole group via Gena's presentation that was shared through a calculator viewscreen. The public display of the work they had completed within their small group provided for the scrutiny of their ideas by a larger learning community and offered up the opportunity for supportive critique to identify a misconception and to improve their mathematical knowledge and understanding. This type of interaction could be described as an interactive intersection between two spheres which represent social contexts. While inspired by Chevallard's (1985/1991) noosphere, these spheres of social context (SSCs) differ as they are specific to the types of interaction that take place in individual, small-group, and whole-group settings (illustrated in Fig. 12.5). In Gena's case, it would appear that a dynamic transformation took place in her mathematical knowledge and understanding, and it might be speculated, also that of her group, in relation to the problem and its solution. Because Gena and her group did not appear to know that an irrational number could not be represented by a terminating decimal, they presented a faulty solution to the $\sqrt{45}$ task to the class. After pointing out the misconception, the class acted as a collective to provide relevant counter-examples and to lead Gena to a valid solution. In doing so, the class transformed her understanding of the characteristics of an irrational number—all within a very public context. In this episode, a number of transformations have taken place within and between different SSCs. First, social interaction, mediated by the task and digital tools, and in concert with the group's collective mathematical knowledge, led to the transformation of the task into what the group believes is a solution. The group's solution is then presented to the whole class by Gena. This represents an interaction between the small-group SSC and the whole-group SSC, with the small group's solution to the task providing the impetus to the instantiation of this interaction. This interaction leads to the transformation of the initial solution into a valid one and also the transformation of Gena's individual mathematical knowledge. As the group assisted Gena to find a valid solution to the task, they challenged her understanding of the concept of irrational numbers. This is a transformation that takes place as a result of an interaction between an individual SSC and a whole-group SSC.

The final excerpt illustrates a different type of transformation. In Geoffery's case, this transformation is related to his relationship with the sociocultural norms that the teacher has established in his classroom. Initially, Geoffery was resistant to the appropriation of the ways of working and knowing that his fellow classmates had adopted over time. He seems almost antagonistic towards the whole class SSC. In attempting to express his dissent, however, through the use of a digital tool and an associated resource (his movie), Geoffery became captured by the very social way of working he had attempted to ridicule. In this situation, the secondary artefacts mediated Geoffery's migration from the fringes of his learning community to its centre as it provided the means for him to feel, first, comfortable, and eventually valued, within the SSCs, which represented the sociocultural ways of working within his classroom.

The examples described and interpreted in this chapter are by no means an exhaustive list of possibilities for what might constitute a plethora of social settings in which learning might take place within mathematics classrooms, but they do provide support for and extend Strässer's concept of a noosphere that surrounds his tetrahedral model of learning and teaching through the idea of SSCs. In addition, these examples suggest that SSCs are not concentric and independent entities but rather that SSCs interact. These places of intersection may also be sites that provide for dynamic learning possibilities. The evidence presented here suggests that there is still much research to be done on how the role of social aspects of learning and teaching can be theorised together with the roles of students, teachers, secondary artefacts and mathematical knowledge.

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Chapter 13
Designing a Simulator in Building Trades to Transform Vocational Education

Annie Bessot

13.1 Introduction

This chapter reports on a part of a research work included in a more general project aimed at changing vocational education. In the first part, we present the characteristics of a fundamental professional situation in building work that involves conceptualizations of a geometric nature. In the second part, we describe how this situation is first changed in the current teaching process and second in the computer simulation. The main design choices of this simulation are described. Eventually, we provide a specific example of the use of the simulator by students to illustrate how some relationships with space are transformed.

13.2 The Reading–Marking Out of Boxing Out: A Professional Situation to Observe Geometrical Conceptualization

What is a reading–marking out activity in a building work? Most building tasks are based on reading plans for marking out on the building site. We call this kind of task a reading–marking out task. In a building site, setting out elements takes into account what will be set out later. For example, when a floor is to be laid down, the marking out of the floor must leave holes for water pipes and electric cables. Setting out a wall must include a plan for the location for windows and doors by marking out their contour. Such marking out is called "boxing out." Generally speaking, a boxing out is a formwork placed in the middle of a structure before concrete casting, used to set aside an area in which additional equipment can be added at a later date.

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This task of reading information from a plan to mark out contours and boxing out on the building site is usual for workers in building trades.

Two types of controls can be distinguished in the marking out of boxing out:

- Controls coming from reading information on the plan and
- Subsequent and effective controls at the moment of putting the additional elements (pragmatic controls), the first type of controls being oriented toward the second type of controls.

The first type of controls is the focus of our attention. In the absence of pragmatic controls, only controls guided by knowledge about space and instruments can take place. The activity of setting out boxing out can allow researchers to observe geometrical conceptualization and help them answer questions such as: what is the nature of knowledge involved in this activity? How is such knowledge organized and what relationship does it have with the artifacts available at the building site?

The observation of students of a vocational school gave evidence of a discrepancy between the procedures of students and of professionals in this reading–marking out activity at a building site from reading a plan. Two types of analyses were carried out in order to better understand this discrepancy and its reasons: an analysis of the geometry in action underlying the students' activity in reading–marking out tasks in a workshop (Bessot and Laborde 2005) and an analysis of the didactical transposition of the professional activity in vocational education (Chevallard 1985) were needed.

The first analysis shows that the geometry in action in reading–marking out tasks cannot be described only by means of Euclidean geometry. This geometry in action must also account for knowledge of the individuals related to the material aspects of the situation and of perceptual knowledge.

Theoretical objects of Euclidean geometry constitute, nevertheless, efficient modeling tools of this geometry in action: Euclidean geometry partly originates from modeling the material space, which would be difficult to do without its basic concepts and its attached terminology (perpendicular, parallel, etc.). It provides a means for evaluating not only the validity but also the consistency of the two elements of geometry in action: concepts in action and theorems in action (Vergnaud 1991) identified for the same individual. As Vergnaud (2000) said,

> activity is usually intelligent and well adapted, […] different situations are handled by the same scheme, and above all that new situations, never met before, may be met with some success, owing to the decombination and recombination of existing operational invariants and rules, and the discovery of fresh ones.

Theorems in action and concepts in action are the two principal operational invariants.

> A theorem-in-action is a sentence that is held to be true in action.
> A concept-in-action is held to be relevant: it cannot be true or false, only relevant or irrelevant. (Vergnaud 2000)

For example, in the reading–marking out of a boxing out situation, the concept in action of a point can be a segment or an intersection of two segments: the first type of point may be a "parallel point."

The complex relationship between theoretical geometry and geometry in action of reading–marking out activity in building trades questions the nature and the place of theoretical geometry in situations preparing for the reading–marking out practice.

The second analysis focused on the place and status of reading–marking out activities in vocational education, in particular, when preparing students to a certification of qualified workers for building trades (in French, BEP: see in appendix an organizational diagram of the French education system).

This second analysis (Metzler 2006) has shown that the reading–marking out situation is fragmented and almost absent from the vocational education institution. This reflects the separation mentioned by Sträßer (2000a, p. 69) between "learning at the workplace versus classroom instructions."

More precisely, in French vocational education, knowledge about space belongs to the learning aims of three different teaching domains: in the teaching of mathematics (in particular geometry), in the teaching of construction, and in the teaching of practice in a workshop. *Reading is separated from marking out*: reading is present in mathematics and construction teaching; marking out is the aim of the practice workshop but reading is not taken into account.

How is it possible to restore the unity of the reading–marking out activity in vocational education? An answer is to use simulated situations "as a mediator between the trainee and the work situation (reference situation)" (Samurçay and Rogalski 1998).

From this point of view, a simulator is a means

- Of designing situations restoring the unity of reading–marking out activity in the three teaching domains of French vocational education and
- For posing critical problems of the professional practice.

A simulator is thus an artifact that "offers an opportunity to explore the inherent, implemented relations […], workplace reality would never allow because of the risk of material, financial and time losses" (Sträßer 2000, p. 244).

Finally, according to a key design choice, the simulator was meant as *an open-ended environment offering the possibility of constructing didactic situations* based on problems previously identified in the analysis of professional situations.

13.3 Fundamental Problems Involved in Reading–Marking Out Professional Situations

Previous research on different types of space (Bessot and Vérillon 1993; Brousseau 1983; Berthelot and Salin 1992; Samurcay 1984; Weill-Fassina and Rachedi 1993) as well as the analysis of professional practices (Bessot and Laborde 2005) allowed us to identify three types of problems related to the specific invariants of reading–marking out situations. The first two types are related to mesospace, the third type to the instruments of the building site (as set square, ruler, measuring tape, etc.).

The first type is the problem of locating the local space in which marking out takes place within the mesospace of the building site.

> […] the idea which students have of geometrical objects, the way in which they approach them, depends on their size […]. The 'straight lines' and angles appear during the process of surveying the macro-space. […] The micro-space is the context in which small objects may be manipulated. […] The meso-space is the space in which the observer is able to gain different viewpoints of objects by moving around. […]. (Brousseau 1986, pp. 467–471; translated from French by the author)

Two types of space are involved in the mesospace: the local spaces in which the actual marking out of the lines takes place, and the global space of moves that allows the worker to move from one local working space to another.

Locating the local space requires coordinating three frames of reference (Samurçay 1984):

- The frame of reference attached to the subject (egocentric reference frame): in front of, behind, on the left, and on the right of a subject;
- The frame of reference of the lines marked on the building site (allocentric reference frame) to construct from fixed existing objects of the mesospace that may also be lines already marked on the building site; and
- The frame of reference of the plan that is given by the dimension system.

The second problem related to the mesospace deals with the coordination of local spaces (Brousseau 1983; Galvez Perez 1985) that may be distant from each other. This coordination is needed in the process of obtaining the expected global set of marked lines of the mesospace.

The third problem is related to the use of instruments: transferring measures requires taking into account the features of the instruments.

13.4 Choices for Simulating the Mesospace

In order *to decouple* the problem of local marking out from that of moving and orienting, two different windows were created: the *first window* allows the worker to have access to various local spaces but never to the entire space; the second one provides access to the visual field of the worker within the global space and his/her movements in this global space. In the *second window* (global space), one can only move; in the former, one can mark out by means of instruments and one can move without a general view (through the scrolling bars).

The features of these two windows are presented in the following section.

13.4.1 Window Simulating the Local Space for Marking Out

The visual field of the worker with its real dimensions 1.50 m by 1.10 m is simulated in one window on the computer screen. It provides a representation of the real visual field *on a scale from 1 to 5* (Fig. 13.1).

One can perform measurements and marking out activities with the simulated instruments (see later). This window is located within the global space for marking

Fig. 13.1 Window simulating the marking out local space

out which is not entirely visually accessible. One can move in the global space from one local space to another by using the scrolling bars of the window (Fig. 13.1) but with only a partial view at a time making the linking up of local spaces difficult.

We wanted to simulate the change of viewpoint when the worker is moving away from or closer to the lines marked on the site. Zoom out (Zoom−) and zoom in (Zoom+) possibilities have been set up to simulate these moves, moving away and moving closer. Zoom facilities are limited in order to avoid a global view of the space for marking out. In addition, it is not possible to perform marking out when the zoom tool is active but it is possible to move the instruments. At any time, it is possible to come back to marking out by pressing the key "Zoom 0." This zooming possibility makes an accurate reading of the marks of the measuring tape and the move from one marking out local space to another one at a small distance easier.

13.4.2 Window Simulating the Global Space

In order to locate the current marking out local space within the mesospace, it is possible to have access to the simulation of the global space at any time by pressing the F9 key. The global space window is simulated by a squared vignette with a 7.5-cm-long side representing a real squared space with a 5-m-long side (Figs. 13.2, 13.3, 13.4, and 13.5).

When opening the window, a gray hard hat appears that represents the worker with its visual field represented by a rectangle. The size of the rectangle corresponds to the scale of the image on the screen (marking out local space). When opening the window, the gray hard hat is always oriented vertically below the rectangle (Figs. 13.3, 13.4, and 13.7).

It was chosen to simulate the moves of the worker (*gray hard hat) and not its position* (Figs. 13.5 and 13.7). Two moves are possible: shifts and rotations which are multiples of a quarter turn. Shifts are performed by directly moving the rectangle

Fig. 13.2 Window “marking out local space”

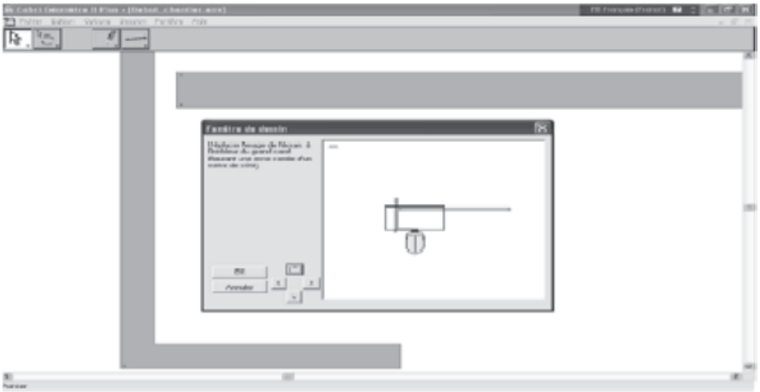


Fig. 13.3 Window global space in the screen after pressing F9 key

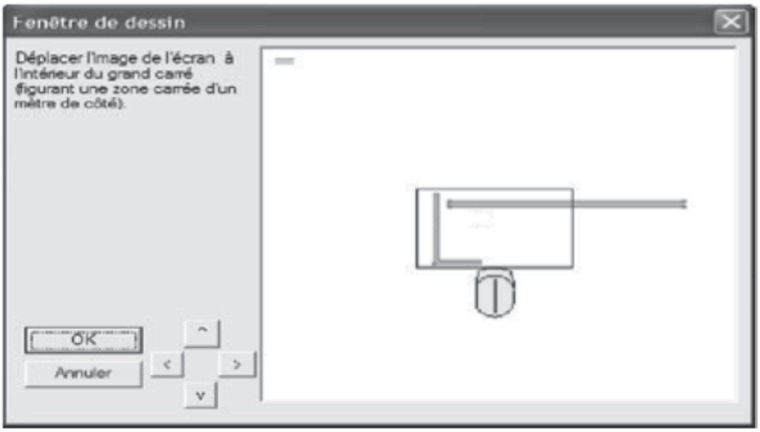


Fig. 13.4 Local space in the global space window

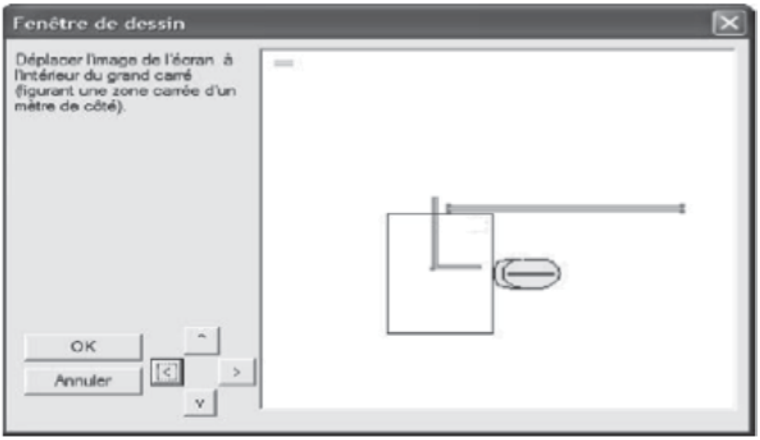


Fig. 13.5 After pressing button "<"

Fig. 13.6 Back to local space after pressing "OK"

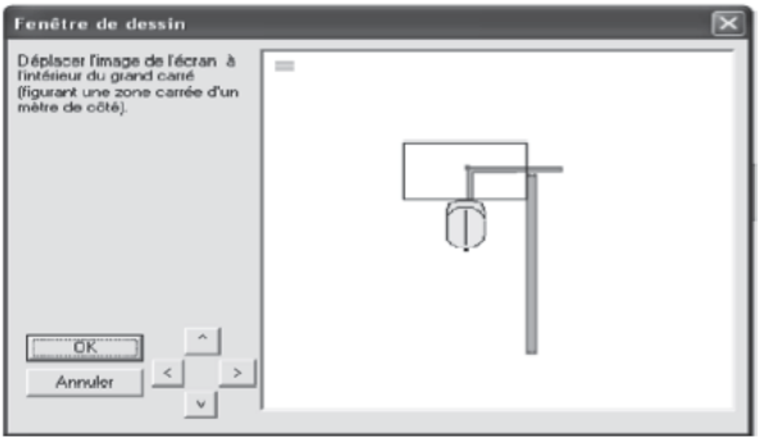


Fig. 13.7 Back to global space after pressing "F9"

through the mouse. Rotations are egocentric and are performed by pressing one of the three buttons “>,” “<,” “∨”: to get to the marking out local space on the right of the worker press button “>,” to get to the marking out local space on the left of the worker press button “<,” and to get to the marking out local space behind the worker press button “∨.” Back to the local space (Fig. 13.6), the worker sees the lines oriented in a way that results from the move performed in the global space window. In this way, the decision of moving and the effect of the move on the visual field are decoupled. If one comes back from the marking out local space to the global view (F9 key), when opening the window, the gray hard hat is always below the rectangle representing the local space (Fig. 13.7). Without a fixed frame of reference, the change of position cannot be inferred from the position of the gray hard hat with respect to the fixed border of the screen.

13.5 Choices for Simulating Objects

13.5.1 Choices for Simulating the Prefabrication Table

The prefabrication table, in which the slab is poured, is simulated by three rectangles with the same width of 0.05 m joined in a U shape: the table is 4 m long and 2.5 m wide. When opening the simulator, the borders of the table may have various directions with respect to the borders of the screen: parallel to the screen borders (see Fig. 13.8) or not (see Fig. 13.10). The U shape can be oriented in various directions (see Figs. 13.8 and 13.9).

The table is not entirely visible in the local space although, as a fixed object of this space, it can serve as a frame of reference of the mesospace for locating lines in coordination with the plan. The table is only entirely visible in the global space window (F9 key).

13.5.2 Choices for Simulating the Use of Instruments

The choices for simulating instruments deal with their appearance, their accessibility, their movements, and their use. We decided that all instruments should look like real instruments. In particular, their dimensions should be proportional to the real dimensions. The 2.5-m-long ruler and the 3-m-long tape even partly unwound exceed beyond the visual field (see Figs. 13.11 and 13.12).

Marking out instruments, namely the pen and the “blue” line, are permanently visible as icons at the top of the screen.

Instruments for measuring and transferring geometric properties read from the plan (set square, ruler, and tape) can be found in three boxes labeled with their names, which are simulated by rectangles located in a corner of the global space and are only accessible by moving in this space. Once an instrument is in use by clicking on its box, the worker may have to move to find it again in his/her visual field

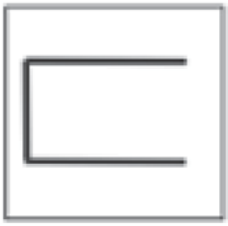


Fig. 13.8 Prefabrication table parallel to the screen borders: "open on the right, closed on the left"

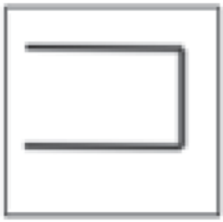


Fig. 13.9 Prefabrication table parallel to the screen borders, in another orientation: "open on the left, closed on the right"

Fig. 13.10 Prefabrication table nonparallel to the screen borders

(resorting to the global space window or to zoom) and to shift it in the screen (local space) to the adequate location in order to perform a marking out.

The materiality of the instruments was not preserved in that simulated instruments can overlap. However, seeking to make the edge of an instrument coincide with the prefabrication table or with the edge of another instrument partly replaces its materiality. However, note that the simulated tape is also retractable into a pink-squared case as in reality.

13.6 Conclusion about the Design of the Simulator

One of the important contributions of simulators lies in the possibility of being freed of the constraints of reality, like the irreversibility of some actions or the time passage.

It is clear that the simulator transforms the relationships of the worker with space. But what is lost in fidelity can be gained in terms of problems and control. Indeed, separating local and global spaces in the use of the simulator requires the subject to seek information in the global space. To this end, the subject must leave the local space in order to be and move in the global space, and then has to come back in order to perform the marking out. These conscious back-and-forth moves do not occur in reality.

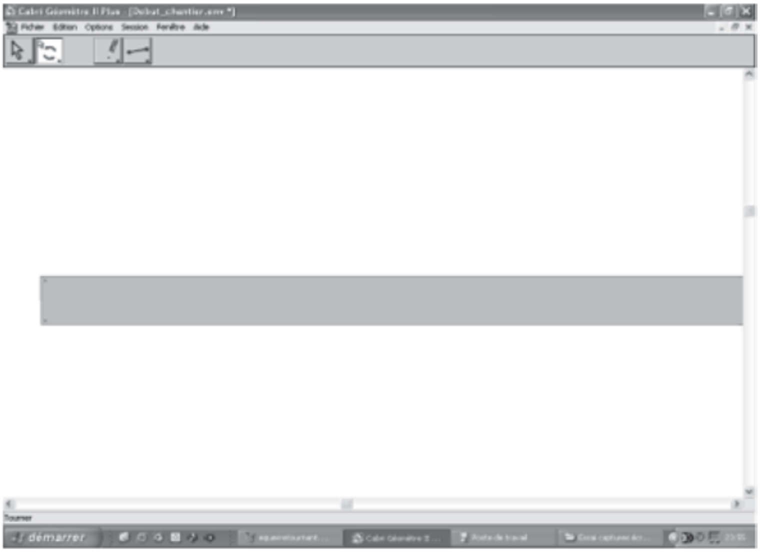


Fig. 13.11 The ruler cannot be seen entirely

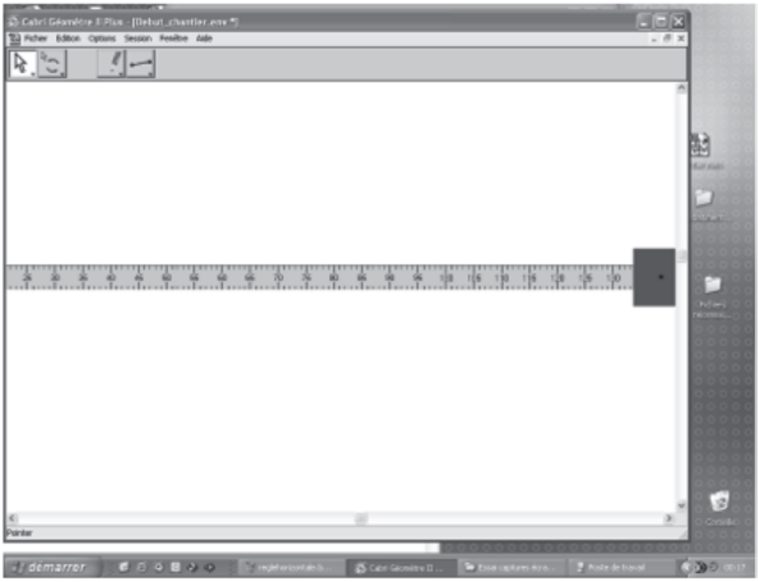
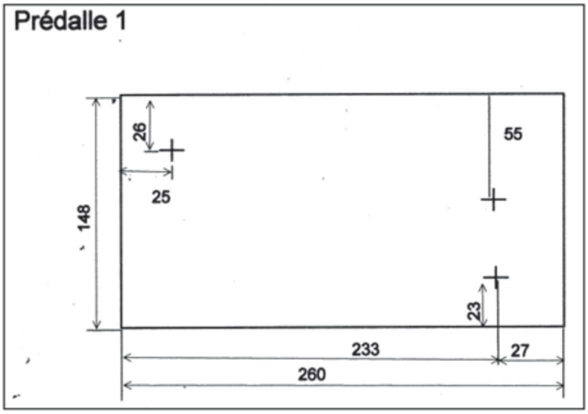


Fig. 13.12 A part of the measuring tape

As Sträßer (2000b) said, the simulator "can be used in learning processes to foster understanding of the professional use of mathematics by explicitly modeling the hidden mathematical relations […]," like the relations between local and global spaces in the reading–marking out activity.

Fig. 13.13 Plan of slab 1



As a result of this separation, the subject is certainly faced with a problem of coordination between the frames of reference of the two spaces: this favors the awareness of the necessity of an allocentric frame of reference.

The additional action of moving back and forth between the two spaces is tedious. It transforms the reading–marking out strategies and favors predictions to decrease the number of back-and-forth moves. However, it gives rise to observations for the subject and the educator and consequently can become an object of a reflexive work analyzing strategies in real and simulated situations.

Another contribution of the simulator is the possibility of controlled variation offered to the educator. The same simulator can give rise to different uses in vocational education. The educator has the command of the type of use and of tasks given to the students. An example of a didactical situation is briefly presented later.

13.7 Example of a Didactical Situation Making Use of the Simulator in Vocational Education

The situation reported here raises the problem of continuing a marking out already done without transmitting information on what has been already set out to the worker. This situation simulates a usual professional problem. Solving this problem requires that the worker identifies the local space within the mesospace by coordinating various frames of reference including the frame of reference of the plan.

13.7.1 Instructions for Students

The plan of slab 1 with three boxings out is given (Fig. 13.13) to the students.

1. Open the file "slab 1."
2. As is visible, the contour of slab 1 and one boxing out have already been marked.

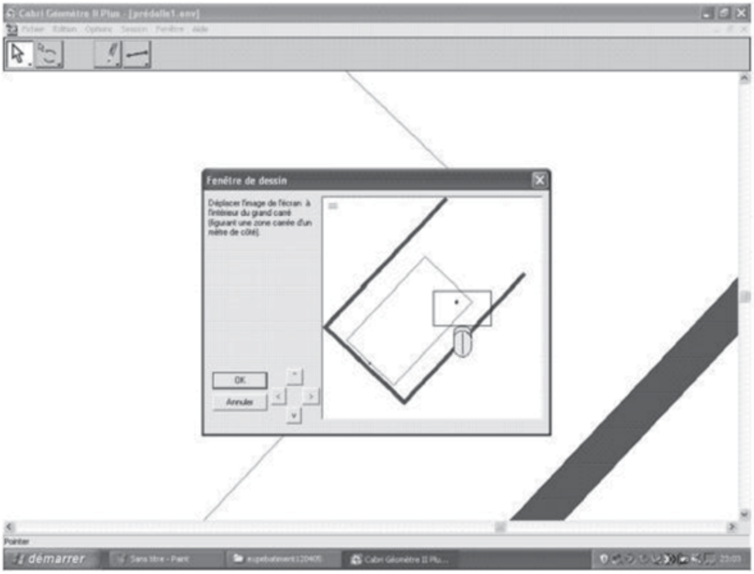


Fig. 13.14 At the opening of file “slab 1”

3. Mark out the two other boxings out of slab 1.

The plan of slab 1 provided to the students as well as the windows, local space and global space are given in Fig. 13.13, and Fig. 13.14.

The plan is oriented according to the orientation of the writing (from left to right and from top to bottom) and consequently imposes a position for reading. It is represented in this position in Fig. 13.13. When opening the file “slab 1,” part of the prefabrication table, part of the lines, and *the boxing out with dimensions 25 cm, 26 cm* (denoted by R(25; 26)) are visible in the local space (Fig. 13.15).

In Fig. 13.14, it is visible that the slab is rotated through 180° with respect to the frame of reference of the plan.

13.7.2 A Priori Analysis of the Situation

In the marking out activity, the worker’s aim is to reproduce the image of the drawing of the fabrication plan in the mesospace. The continuation of the marking out requires interpretation of the boxing out already marked in the mesospace as corresponding to a boxing out in the plan.

Two cases are possible:

- Either the plan and its (*unfinished*) image in the working local space have a similar orientation and the boxing out is erroneously considered as R(27; 23)

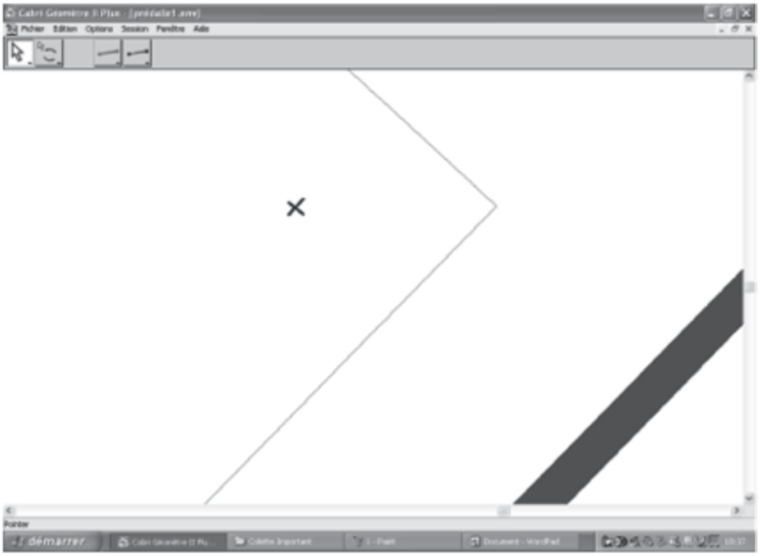


Fig. 13.15 The two windows

- Or measures are taken in order to identify the already drawn boxing out with a boxing out in the plan.

The choice of the dimensions of boxings out in slab 1 is deliberate. The distances to the border of the two boxings out R(25; 26) and R(27; 23) are visually close, thus favoring the choice of the (wrong) first case in the absence of the professional gesture of taking information on what already has been set out.

13.7.2.1 Incorrect Interpretation of the Already Marked Boxing Out Without Measuring: R(27; 23)

Two other boxings out must be marked. Here, the case of boxing out R(27; 55) is considered as the only one likely to lead to feedback. Two procedures for marking out R(27; 55) are possible:

- Through an alignment with R(27; 23) by resorting to the only measure 55: *no feedback*.
- By resorting to two measures 27 and 55 without making use of the alignment. Once the marking out is done, the absence of alignment of the two marked boxings out provides feedback that leads to reject the interpretation of the existing boxing out as R(27; 23). This leads to the second case which is analyzed below.

13.7.2.2 Correct Interpretation of the Already Marked Boxing Out Through Checking by Measuring: R(25; 26)

The coordination between the plan and its unfinished image can be achieved in two ways.

- Real or mental half turn of the plan of slab 1

The plan is rotated through 180° effectively or is thought to superimpose the image on the screen with the rotated plan: the marking out is performed with a prefabrication table in the position “open on the right, closed on the left.”

- Move in the mesospace through resorting to the global space window.

To keep the prefabrication table in its privileged position and make it coinciding with its image on the screen, it is possible to use the F9 key to get access to the global space in order to simulate a half turn in this space: the table is then in the position “open on the left, closed on the right.” When back in the local space, the boxing out already marked is the image of R(25; 26). Boxings out can be marked in the same position as they appear in the plan.

The situation aims to provide multiple opportunities in which checking measures of marked objects in mesospace (prefabrication table and lines) lead to an economy in marking out. Checking is a critical gesture of building trade as claimed by the educators in vocational education.

> Check what is left by the others… The guy is arriving. He is told to put this… Work starts… It must be checked (exchanges between two workshop teachers at the technological and vocational school of Sassenage; translated from French by the author).

13.7.3 A Posteriori Analysis of the Situation

The experiment took place in the technological and vocational school of Sassenage. The BEP class was comprised of 12 students working in pairs. We collected:

- The simulator screenshots of all student pairs taken every second and
- Audio and video recording.

In the following, an excerpt of the a posteriori analysis regarding pairs 1, 2, 4, 5, and 6 is presented.

As displayed in Table 13.1, only three pairs out of five resort to measuring on the marking out, in order to identify the boxing out.

Let us analyze the checking procedures of the three pairs 4, 5, and 6.

Pair 4 made two checks by measuring the dimensions of the slab and the dimension of the already marked boxing out (26 cm) which is sufficient for identifying the boxing out.

Pair 5 checked only one measure (26 cm) and did a half turn of the plan to make the screen match the plan.

Table 13.1 Checking procedures of already marked boxing out

Interpreting the already marked boxing out	Without measuring	With measuring	
	R(27; 23) *Pairs 1 and 2*	R(20; 21) then R(27; 23) *Pair 6*	R(25; 26) *Pairs 4 and 5*

Pair 6 drew surprising conclusions: the already marked boxing out is first considered as not being in the plan, and then as the erroneous boxing out R(27; 23). Verbal interactions among students V and N of this pair allow us to understand those successive conclusions. Similar to pairs 1 and 2, V immediately identified the already marked boxing out as R(27; 23), but N insisted on measuring. He measured one of the dimensions of the boxing out and obtained 20 cm as a result of a wrong use of the measuring tape: the distance is measured by making the centre of the boxing out coinciding with the border of the case of the measuring tape (with width 5 cm in real size). Then, he measured the second dimension in the same way and obtained 21 cm. Surprised not to find any boxing out of the plan, he resumed each measuring twice or three times.

V: It fits nothing. It means that it is already marked, then we must mark out the three others. We make one more, that's it.

N doubts that there can exist 4 boxings out and asks questions about the use of the measuring tape to observer O. He admits that he never used a measuring tape!

N: The end of the tape, is it at the black mark (corresponding to the clip of the real tape) or at the other end?

O: It is at the black mark as on a real tape… do you know, don't you?

N: No, I don't know, I never used a tape.

V: Didn't you? (Translated from French by the author)

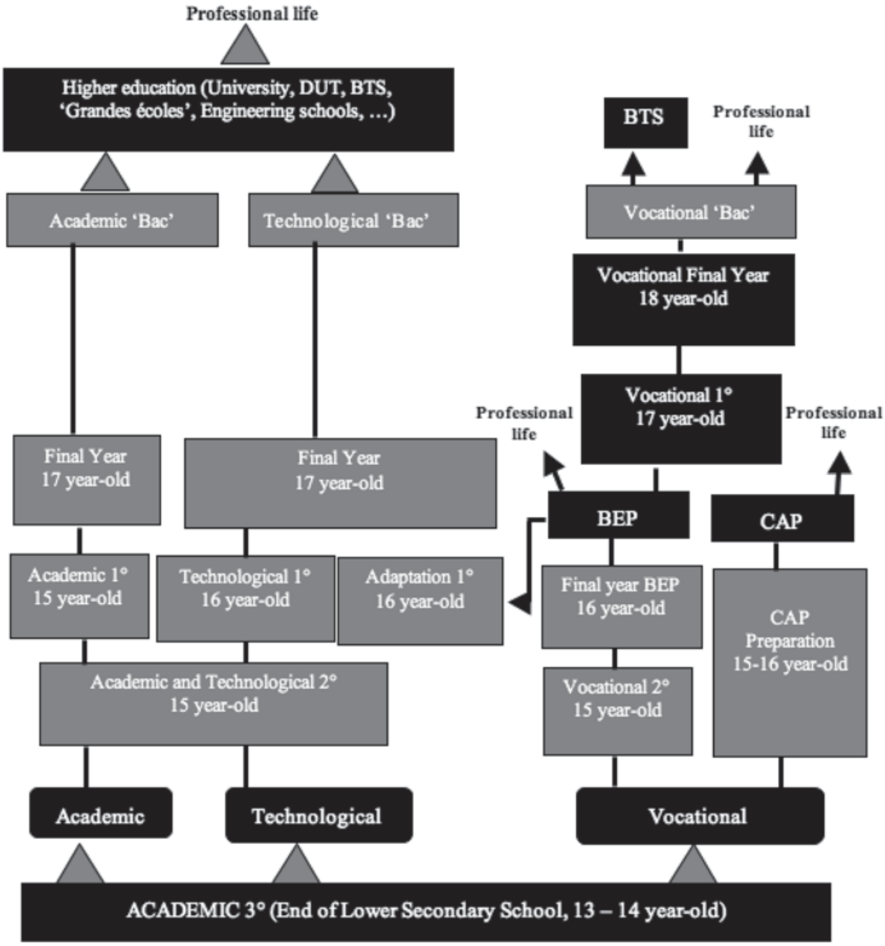
The doubt about the correct use of the tape as well as the cost of its use in the simulator lead them to give up checking the correspondence between measures and dimensions on the plan. They come back to the first opinion of V, i.e., identifying the already marked boxing out as R(27; 23).

13.8 Conclusion

The simulator made it possible to confront the students with the usual professional problem of continuing a marking out, which is a fundamental issue of the professional activity, as claimed by the teachers. The simulator revealed that, even at the end of the vocational training, almost half of the observed students do not resort to checking and among those who checked, the use of instruments may cause difficulties. This professional gesture of checking is not available to all students at the end of the school year.

The hidden mathematical knowledge underlying the professional activity of checking consists of metric and analytical geometrical properties related to the three systems of reference. As the simulator decouples the local marking out space from the global moving and orienting space, the student is compelled to take into account those geometrical properties in order to move from one subspace to the other. The use of the simulator aims at transforming the relationships of the learner to the mesospace. As said earlier, the simulator "can be used in learning processes to foster understanding of the professional use of mathematics by explicitly modelling the hidden mathematical relations and offering software tools to explore and better understand the underlying mathematical models" (Sträßer 2000, pp. 244–245).

Appendix: Organizational Diagram of the French Education System



Acknowledgments I am grateful to Colette Laborde for her valuable contribution.

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Chapter 14
Discussion I of Part II

Representing and Meaning-Making: The Transformation of Transformation

Falk Seeger

Abstract: This commentary simultaneously offers a broad and a narrow perspective. It is narrow in that it does not attempt to synthesize or digest the chapters in this section. It can be called broad as it attempts to sketch some salient features of the future development of and learning geometry, of the transformation of transformation. Four such strands of reasoning are discussed: the paramount significance of meaning-making, the role of artefacts as socially and culturally embedded, embodiment and enactment, and, finally, emotions, meaning-making, and triangulation.

All chapters in this section circle around the question how (technical) artefacts can be thought of as mediating mathematical meaning, especially how mathematical meaning emerges in the interaction of a subject with a (technical) artefact embedded in an educational situation, said shortly. In all chapters, geometry is the kind of representation that is investigated. I will not go into the question of the special kind of representations that geometry is incorporating. I will restrict my commentary to a discussion of multiple forms of psychosocial and semiotic triangulation that are salient in the teaching and learning of geometry, in meaning-making, emotion, and development. By triangulation I mean those meaning-making relationships that include three instances: a subject—an object—another subject, or a subject—an artefact—an object, or simply three subjects.

In my commentary, I will discuss the consequences of what has been called an "embodied" perspective on human activity and thinking following the seminal volume of Varela et al. (1991). They put this fundamentally different perspective on cognition as embodied action as follows:

> By using the term embodied we mean to highlight two points: first, that cognition depends upon the kinds of experience that come from having a body with various sensorimotor capacities, and second, that these individual sensorimotor capacities are themselves embedded in a more encompassing biological, psychological, and cultural context. By using the term action we mean to emphasize once again that sensory and motor processes, perception and action, are fundamentally inseparable in lived cognition. Indeed, the two are not merely contingently linked in individuals; they have also evolved together (Varela et al. 1991, p. 172–173).

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Reading this quotation again today, one is surprised to see that emotion is missing from this perspective—just as it is missing from the whole volume by Varela et al. It seemed easier then to think of cognition as having a bodily basis, leaving emotion unmentioned. It comes as no surprise that emotion is practically missing in a recent volume on the state of the art of mathematics education (Sriraman and English 2010), although today, emotion appears as the psychological function paradigmatic for a perspective of embodiment.

In a sense and finally, my commentary is also meant to be a discussion and critical reflection of the main concept of this volume, transformation.

14.1 Making Meaning

Meaning is the key. It has already been the key to the *New Math* movement that had been searching for an answer to fundamental changes in the society, the culture, and the life of the individual in the twentieth century. It has been central in the description of the ever growing faster and wider transformations of life. These transformations of the twentieth century had forced mathematics educators to respond, if they were followers of the *New Math* or not, and their common denominator was the problem of meaning. As Thom has expressed it:

> The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects (Thom 1973, p. 202).

Today, the problem in the center of mathematics education is still meaning—whatever we wish to call the societal, cultural, and psychosocial transformations under way in the twenty-first century. But now it is meaning with a discursive or dialogical and a distributed meaning (see Otte 2011).

In his famous speech on the equally famous 1972 *International Congress on Mathematical Education* in Exeter, Thom could still get away with a conception of meaning in a strictly object-related fashion. Forty years later, so it seems, the issue of meaning in mathematics education is not only related to the mathematical objects but also to the personal sense they make to students.

In this paper, I tend to use "meaning" in a rather unspecified way. Basically, most of the times when it is spoken of meaning, one could replace it with "sense". How are meaning and personal sense mutually related? A. N. Leont'ev (1981, 1978) has put it concisely in this way: "… sense is expressed in meanings (like motive in aim), but not meaning in senses" (Leont'ev 1981, p. 229). Meaning has a quasi-objective meaning, and meaning has a personal meaning, sense.

> Meaning is the reflection of reality irrespective of man's individual, personal relation to it. Man finds an already prepared, historically formed system of meanings and assimilates it just as he masters a tool, the material prototype of meaning (Leont'ev 1981, p. 227–228)

As it were, meaning has a general form mostly culturally organized; and meaning has a personal form: personal sense.

In what follows, I will begin to discuss the use of artefacts, not their creation and invention, that is, I will talk about technical artefacts "in use" and not "put down", as Stewart (2010, p. 19) has described the two modes of using tools and technical artefacts.

14.2 Artefacts as Socially and Culturally Embedded

The interaction with the artefact is not necessarily restricted to the interaction of an artefact and a person. The meaning of an artefact can only be accessed through the analysis of its socio-cultural (and historical) embeddedness. This is equally true for simple as for highly sophisticated technical artefacts.

Now, the fate of artefacts is that they "disappear" in the ongoing interaction as mastering the artefact is growing: when a human learns how to chop wood with an axe, the axe as technical artefact is the object of activity, when she or he has learned how to handle an axe with a sufficient degree of mastery, chopping wood, keeping a stove burning etc. is the object of activity and the artefact "disappears." Leont'ev (1978, p. 66) has described this very nicely as actions "sinking down" to operations as a very general case.

Now, this all is not true for artefacts that produce something automatically. Automata, in general, do not allow for any intervening actions of a human user: they produce a result that is either useful for a user or not. Correspondingly, the list is long with successful automata in mechanical systems governing and regulating output processes. However, the list of failed attempts to insert automata into human-machine-interactions or in user-supported processes is equally long (see, e.g., Latour 1993; Engeström and Escalante 1995).

It is interesting that the mastery of technical artefacts in any culture is often in itself an object—and here the artefact is not at all "disappearing", in the contrary. Gaining and showing mastery over the artefact is not only the core of the matter in wood-chopping tournaments, but in all kinds of sports like sailing, pole-jumping, motor-racing, darts tournaments, tennis, and so on and so forth. In many cultures, the mastery of artefacts has been and still is connected to weapons and other devices for survival. Often these tournaments have a religious embedding and ornamentation. But some, actually, also refer to mathematics and calculating like the *Soroban* contests held in Japan and the USA.

Learning to master a (technical) artefact, as it were, is a process deeply embedded into the culture. It is not the interaction with the artefact that determines the ultimate goal of the artefact-mediated activity, it is a goal and a motive that goes beyond this interaction. As Norbert Elias (1994) has shown for the cultural evolution of emotional and self-control in Europe from 800 to 1900, the control of artefacts is closely related to this process: mastering the artefact, like spoon and fork, entails mastering one's feeling and self-control.

In the interaction with the artefact, the cultural embeddedness is not something that is only "surrounding" the human-artefact interaction. It is, rather, the basic mechanism (see, e.g., Cole 1996; Hutchins 2010). In the learning sciences this has

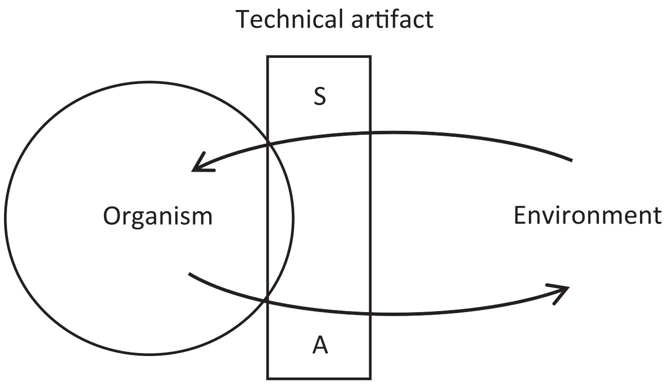

Fig. 14.1 Sensorimotor coupling of organism and environment mediated by the artefact (S = sensory processes; A = Actions; from Stewart 2010)

lead to an approach that focuses not on the direct teaching but on the indirect teaching that has its source in the culturally defined situation (see Lave 1988; Lave and Wenger 1991).

In order to make clearer what I am talking about, I will use the following figure taken from Stewart (2010) as an example (see Fig. 14.1).

Figure 14.1 is meant to give an idea of how an organism is related to an environment through artefacts. The artefact is not just "mediating" while remaining in a rather neutral position and status, the artefact is enacted into the organism as it is, in fact, constituting the world. It is of crucial importance that the processes of action and the sensory processes are of qualitative different nature. Neither can actions be explained through sensory processes nor can sensory processes be explained through actions. They remain in a sense incompatible. Only when looking at the complete circle as actions and the results of actions become sensory inputs do we realize that they belong together, much in the sense that Hutchins (2010, p. 446) has put it: "… perception is something we do, not something that happens to us."

Making a difference between sensory and motor processes is, of course, not a new idea. It has been a topic in the physiology of excitatory processes and in ethology as the difference between afference and efference including the principle of reafference (see, e.g., von Holst and Mittelstaedt 1950). However, already von Uexküll (2010) has elaborated in his theoretical model of environmental biology that the environment is not out there, outside the organisms, but that it is as well inside, as it is symbolically constructed. Insofar, the clear distinction of efference from afference is not tenable because it basically leads again to a great divide between organism and environment.

Uexküll's theoretical model made it clear that the unity, or complementarity, of organism and environment has to be grounded into a semiotic and symbolic approach. His attempt to resolve the uncomfortable duality of mind and body, of organism and environment, following from the Cartesian split, is reflected in current approaches to embodiment and enactment.

Thus, efference and afference will not be conceptually satisfactory to reveal the basic mechanisms of humans using technical artefacts. What would a satisfactory approach have to embrace that includes the role of the body in the use of technical artefacts?[1]

14.3 Embodiment and Enactment

As I said at the beginning, the role of the body appears of such great importance because in a unique way cognitive and emotional functions of human activity are no longer divided but can be seen as two fundamental features. In addition, this lens makes it absolutely essential to take record of what is happening at each and every moment of an ongoing activity, to understand it as a unity of lived experience and to try to reconstruct it accordingly.

As put in the initial quote by Varela et al., it is also essential to reconstruct the cultural embeddedness of such an ongoing activity. That is, to reconstruct the environment not as a loose collection of surrounding factors, but as a system of cultural meaning. In his paper *The problem of environment*, written in 1934 L. S. Vygotsky has discussed the issue why an identical learning environment does not "produce" identical learning results. The answer seems to suggest itself: different students experience the same situation differently. Vygotsky concludes that this means for educational research to look through a certain prism at the student and the situation:

> It ought to always be capable of finding the particular prism through which the influence of the environment on the child is refracted, i.e. it ought to be able to find the relationship which exists between the child and its environment, the child's emotional experience (*perezhivanie*), in other words how a child becomes aware of, interprets, and emotionally relates to a certain event (Vygotsky 1934/1994, p. 341).

This shift of focus on the concrete experience of the students entails some consequences in research on mathematics learning as attentional, cognitive-emotional, and social processes have to be considered in detail. Research has to highlight the interfaces where these different psychological functions meet and regulate the ongoing activity. One good example for such an interface is the research on *mathematical beliefs* as components that regulate student activity and action (see, e.g., Leder et al. 2002; Maas and Schlöglmann 2009; Goldin et al. 2011). Other examples

[1] In what follows, I will not discuss the specific details of embodied or enactment approaches. First, because these paradigms are still very much „under construction" and exhibit a great diversity; second, because space would not allow to go into the details. I will also not discuss the brain-focused approaches that sometimes label themselves as approaches to embodied cognition (for a short overview, see, e.g., Di Paolo and De Jaegher, 2012). Interesting and fascinating as these approaches may be, they reduce the role of the body to excitatory patterns in the brain that since recently can be traced with brain-scanners. For an overview and discussion of cognitive neuroscience see Campbell (2010). Dehaene, e.g., has presented an interesting tripartite model of the development of number (1992), but his numerous attempts to show how the components of this model are processed by and localized in the brain have not been very conclusive (see, e.g., Dehaene 1997).

of such interfaces could include the concepts of *student engagement* (see, e.g., Fredricks et al. 2004), *self-efficacy* (Bandura 1997, 2001) or *recognition* (see, e.g., Honneth 1995).

Another interface between the cognitive and the emotional, between thinking and communicating are gestures as movement and expression signs. Especially in mathematics education, gesturing has been found to be an interesting area of research (see, e.g., Maschietto and Bartolini Bussi 2009; Radford 2003; Radford et al. 2005; Robutti 2006; Sabena 2007; Roth 2001; Roth and Welzel 2001 as examples from other areas)[2].

14.4 Emotions, Meaning-Making, and Triangulation

In the center of meaning, body, and artefact we find emotions—or, to be more precise, emotions as they are related to cognition, and cognitions as they are related to emotions. Emotions have an important dual relation to body processes and to signs and symbols. In a unique way, emotions offer a possibility to access the two, now felt as basic, constituents of meaning-making: the embodiment of thinking and acting and the symbolic ground of meaning. Emotions control and express meaning and personal sense of one's own activity and actions and the activity and actions of others (Holodynski 2006). This, of course, brings the pivotal importance of emotions for learning right to the point.

We arrive at this pivotal point if we follow the trails of two great schools of thinking on the problem of meaning: the semiotic tradition after Peirce and the developmental approach to the ontogenesis of intersubjectivity, reciprocity, empathy, and cooperation—with L.S. Vygotsky somehow mediating the semiotic and the developmental approach. The development of meaning making under the perspective of reciprocity has to do with the complicated interplay between the social and the individual which has been the dominant theme in Vygotsky's developmental psychology[3].

Control and self-control have been an important motive for Peirce in formulating the pragmatic maxim. Equally, Vygotsky has taken great efforts in giving a vivid account of the transition from other-regulated to self-regulated control as it can be demonstrated in the development of volition and sign operations (see, e.g., Vygotsky 1997, 1999). To gain self-control is one of the great accomplishments

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[2] This issue would deserve an extended discussion because the relation of gestures to language is still very much in need of clarifications. Gestures are often seen as developmental precursors to language (see, e.g., Tomasello 2008). At the same time, they seem so important because gestures potentially express what cannot be expressed through language. For an interesting account of gestural language see Sacks (1989). Goldin-Meadow (2003) has discussed intensely the gesture-speech mismatches, and Sinclair (2010) has elaborated the idea of overt and covert forms of knowing in mathematics education, gestures indicating covert knowing.

[3] For reasons of space, I will not present of Vygotsky's ideas in more detail, also because this has been done extensively elsewhere over the past years (see, e.g., van der Veer and Valsiner 1991, 1994; Daniels et al. 2007).

in human ontogeny. It is especially remarkable on the background of the fact that no infant and child can develop normally if attachment and secure base are not provided by the mother and other caretakers. It is as if we have again two paradoxically opposing poles where development has to find its way—the self being neither completely attached nor completely self-directed. It is quite clear that real self-determination and autonomy are not in a steady state but in a process of becoming, in a developmental process.

When we look at what research has found out about this development of self-control we find again that already at a very early age the infant is not a passive vessel controlled by the mother. Rather, the infant starts meaning making from birth on—and even earlier. What could be mistaken for a genetic predisposition turns out to be, at closer scrutiny, a result of interaction and preverbal communication. Beginning in the late 1960s, gradually intensified research has accumulated evidence of the fundamental nature of reciprocal interaction in early development (see, e.g., Bullowa 1979). The work of Hanus Papousek (Papousek and Papousek 1974, 1977, 1981), Andrew Meltzoff (Meltzoff 2002, 2007; Meltzoff and Moore 1977; Meltzoff et al. 2009), Colwyn Trevarthen (Trevarthen 1979, 1980, 1994; Trevarthen and Hubley 1978), Daniel Stern (1971, 1985), to name only a few, has paved the way to a new understanding of the "competent infant."[4]

This whole work in developmental psychology has been tremendously extended and amplified through developmental research in non-human primates. Here the work of Tomasello and his co-workers on the evolutionary transition field of the great apes and human infants (see, e.g., Tomasello 1999, 2005) has been immensely stimulating for theoretical advances in our understanding of the genesis of symbol-formation and meaning-making as the result of social-interactive processes of sharing attention, of pointing and gesturing, as semiotic exchange in a very general sense. And finally, as the result of some emotional grounding, of belonging, of sharing, of empathy—all lead into an enriched understanding of the sociogenesis of the self.

It is also remarkable that new research into the etiology and the development of autism has fundamentally added to our understanding of what it means to be human (see, e.g., Hobson 1993, 2002; Dornes 2005). It may seem odd to mention this here—but actually the mathematical experience has sometimes been described by outsiders as tending to be rather "autistic" and research indicates at least a certain tendency (Baron-Cohen et al. 1998, 2007). However, this tendency is noteworthy not from the epidemiological perspective. Rather, the study of autism reveals that the sometimes amazing capacities of autistic persons seem so isolated because they are not embedded into the natural art of relating to other persons, into empathy and cooperation, into the emotional experience. Wing has brought this perspective to the point: "The key to autism is the key to the essence of humanity" (Wing 1996, p. 225). The key feature of this essence is triangulation in the sociogenesis of the self.

Triangulation is in a particular way appropriate to capture the specific quality of human thinking and acting, of the human mind and human activity, be they used in

[4] A term coined already in a 1973 volume by Stone and others (Stone et al. 1973; see also Dornes 1993).

Table 14.1 Firstness, Secondness, and Thirdness according to Peirce (after Trevarthen 1994)

Firstness			
Sign as such	Quality	Icon	Emotion *in* subject
Secondness			
Sign and relation to Object	Actuality	Index	Object *of* subject in intended action
Thirdness			
Sign and relation to Interpretant	Potentiality	Symbol	Cooperation, self, and value *between* subjects

semiotics, in education, in psychoanalysis, in developmental psychology, or in the analysis of educational situations.[5]

One enduring issue in teaching and learning and the underlying assumptions on how learning can be organized is the credo that it has to proceed from the simple to the complex and complicated, from the concrete to the abstract, from trying out to planned and reflected learning, from unreflected drill to understanding, from emotional to rational regulation of action. Even though some of these principles certainly have their justification, it must be questioned whether they support meaning-making in mathematics education. It is not solely the question whether symbols do describe or construct reality as Steinbring (2005) has put it. It is the problem that human activity is situated within a universe of meaning that is discursive and distributed at the same time, it is the old problem of how meaning as generated between subjects and with the help of artefacts is becoming meaning within the self, making personal sense.

In Table 14.1, Trevarthen who has done seminal research in the genesis of intersubjectivity and interiorisation (see, e.g., Trevarthen and Hubley 1978), has tried to bring together a semiotic perspective after Peirce and a developmental perspective in order to show how *Firstness, Secondness, and Thirdness* could be understood in terms of concepts from developmental psychology. While this is in itself a good summary and presentation of the different approaches seen together, this table can also help to make the above idea much clearer about the progression of development and the progression of learning. It turns out a developmental perspective changes the place and function of the signs as they are presented in the classical semiotic model. The progression from Firstness to Secondness to Thirdness is prone to be interpreted as a developmental sequence from the assumed simple and unmediated to the complex and mediated. However, research in developmental psychology has amply demonstrated that it is, in fact, only the triangulation between these three forms of signs and meanings can account for the development of meaning. Thirdness, as the assumed highest form of meaning, being the result of interaction between subjects is actually the precondition for a sound development of emotion—as research has shown (see, e.g., Fonagy et al. 2002). Conversely, the quality of emotion as a potentiality seems like a late accomplishment of development that needs the relation to the object as well as the relation to other subjects. In addition, developmental research has demonstrated that the relation to the objects is fundamentally mediated through other persons, primarily through caregivers during infancy.

[5] Already Hegel expressed this specific feature of triangulation in his "Quadratum est lex naturae, triangulum mentis" (Hegel 1801, p. 533).

14.5 Transformation and Communication

In their path-breaking book on the linguistics and philosophy of meaning, Ogden and Richards (1923) chose the title "The meaning of meaning." This nicely expressed that meaning is always reflexive and that the development of meaning is forming some sort of a never-ending spiral: one cannot get behind meaning and one cannot pin down some beginning, some starting point. Varela et al. (1991) have extended our understanding of this situation with the claim that in order to understand meaning-making one has to go beyond cognitive processes and include lived experience. So there is meaning that we make and meaning that is already there.

In a similar vein, there is also transformation that we make and transformation that appears to us as if it has always been there. It seems that we have largely lost the sense of authorship of the transformation going on. Looking at transformation from a global perspective, the current impression certainly reveals a picture that finds us largely alienated from the optimistic perspective on transformation: Gideon Rachman (2011) has presented a succession from the Age of *Transformation* (1978–1991), the Age of *Optimism* (1991–2008) to today's Age of *Anxiety* into what he calls today's Zero-Sum world. We do not have to take this analysis too seriously.[6] It is quoted only to show that transformation may not be a convenient motto for today's mathematics education, and it indicates that we may have to lower our sights

In search of a new motto, a nice story told by George Dyson (2010) might be amusing and whetting the appetite to search for alternatives:

> In the North Pacific Ocean, there were two approaches to boatbuilding. The Aleuts (and their kayak-building relatives) lived on barren, treeless islands and built their vessels by piecing together skeletal frameworks from fragments of beach-combed wood. The Tlingit (and their dugout canoe-building relatives) built their vessels by selecting entire trees out of the rainforest and removing wood until there was nothing left but a canoe.
> The Aleut and the Tlingit achieved similar results—maximum boat/mini-mum material—by opposite means. The flood of information unleashed by the Internet has produced a similar cultural split. We used to be kayak builders, collecting all available fragments of information to assemble the framework that kept us afloat. Now, we have to learn to become dugout-canoe builders, discarding unnecessary information to reveal the shape of knowledge hidden within.
> I was a hardened kayak builder, trained to collect every available stick. I resent having to learn the new skills. But those who don't will be left paddling logs, not canoes.

The problem for mathematics education seems to be that the *piecemeal* and the *dugout* style do coexist—and they coexist in different ways for students and for teachers at different moments of the teaching-learning process. Navigating the territory of meaning that is there and making meaning while we do this, certainly needs whatever works.

[6] Rachman is a leading figure in the *Financial Times*. Although his intention is to present an account of today's global situation, it must be doubted if he really can leave his fixation on the *investment* and money-making perspective behind.

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Chapter 15
Discussion II of Part II

Digital Technologies and Transformation in Mathematics Education

Rosamund Sutherland

Abstract: This chapter is a commentary on a collection of chapters that focus on the transformational potential of digital technologies for learning mathematics. I suggest that the theoretical perspectives represented within the collection cohere around theories that predominantly derive from sociocultural theory, with a focus on the mediating role of technologies in human activity. All of the chapters acknowledge the role of the teacher, and the importance of designing activities to exploit the semiotic potential of digital technologies for learning mathematics. However I argue that the chapters do not adequately take into account students' out-of-school uses of digital technologies which are likely to impact on their in-school use of 'mathematical' technologies, and also the societal and institutional factors that structure the use of technologies in schools. I also argue for the importance of scaling-up the design based studies represented in the collection and developing a model of professional development that exploits the potential of networked communities of mathematics teachers in order to initiate large-scale transformation in mathematics classrooms.

15.1 Introduction

This chapter is a commentary on a collection of chapters entitled Transformations Related to Representations of Mathematics, within the book Transformation—a Fundamental Idea of Mathematics Education. All of the chapters focus on the transformational potential of digital technologies as representational systems, and demonstrate how dynamic digital technologies both add to the available mathematical representational systems, and augment existing static representational systems. Dynamic representational systems offer the potential for transforming and democratising the teaching and learning of mathematics (Kaput et al. 2008), and the chapters in this book have provoked me to re-examine this potential in order to understand why changes at the level of the classroom have not been as dramatic as many of us had predicted.

My own involvement in mathematics education research started in 1983 with a research project that investigated the potential of Logo programming for learning

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mathematics and in particular algebra (Hoyles and Sutherland 1989; Sutherland 1989), and developed into a more general interest in the potential of computers and technology for learning mathematics (for example Sutherland and Rojano 1993; Sutherland 2007). More recently I have focused on approaches to professional development as I became aware that teachers need support to take the risk of experimenting with using digital technologies in the classroom (Sutherland et al. 2009). I mention this history because it feels as if I have lived through many "waves of optimism" about how "digital technologies" will transform mathematics education, yet despite extensive research in this area (see for example Hoyles and Lagrange 2010) it is widely recognized that teachers are generally not exploiting the potential of digital technologies for the teaching and learning of mathematics (Assude et al. 2010).

Over and over again it is the newest technology which excites teachers in schools, provoking them to think that the latest wave of technology will make a difference to teaching and learning. For example many schools in my local area are buying class sets of ipads, accompanied by a belief that the mere introduction of this technology into the classroom, together with the use of the internet is all that is needed to transform teaching and learning. It is difficult not to go along with this enthusiasm and the confidence that simply by making a technological system available, people will more or less automatically take advantage of the opportunities that it offers. It is a challenge to find ways of convincing school leaders and teachers that it is how the technology is used that is important, and that a seemingly "mathematical" technology can be used for non mathematical purposes. The theoretical ideas that are raised in this collection of chapters address this issue, providing frameworks for understanding the use of digital technologies and the role of the teacher in orchestrating such use for mathematical purposes.

Within this chapter I start by explaining why I believe it is important to consider the policy and institutional constraints on innovation at the level of the classroom. I then discuss the theoretical perspectives represented within this collection of chapters. I go on to argue that young people's out-of-school uses of digital technologies are likely to impact on their classroom learning of mathematics. I claim that whereas technologies can potentially be used to transform mathematics education, teachers and students have to learn to use them in mathematically purposeful ways. Finally I discuss why I believe that professional development is key to transformation in mathematics education.

15.2 Constraints on Innovation in Mathematics Classrooms

The research represented in this collection relates to what could be called bottom-up change at the level of the classroom. For example, the project by Bessot (Chap. 13) which designed and evaluated a computer-based mathematical simulator for vocational construction students to learn about geometry-related aspects of their professional practice, or the longitudinal study carried out by Geiger (Chap. 12) which investigated the dynamics of classroom interaction in which 16–17-year-old mathematics students had unrestricted access to a wide range of digital technologies. Both of these studies are design-based studies, the former influenced by the work of Brousseau (1997), and the later drawing on Sträßer's (2009) tetrahedral model for teaching and learning mathematics.

For most of my research career, I have also been involved in bottom-up research and development projects. However more recently I led a research project (the InterActive Education project) which examined learning at both the level of learner and classroom, as well as taking into account the institutional and societal factors which structure learning (Sutherland et al. 2009). If you take such an holistic perspective you begin to understand the challenges that teachers face when considering using digital technologies in the classroom. For example as we reported in the InterActive Education project, the mandate for ICT in education (in England) has overwhelmingly been interpreted by schools as a license to acquire equipment. Such a focus on acquiring equipment detracts from an emphasis on the professional development that teachers need in order to change established practices of teaching and learning.

When we examine the societal and institutional factors that structure the use of technologies in schools, we can begin to appreciate why mathematics teachers might not be embracing digital technologies for teaching and learning. For example in England many schools have recently invested in Virtual Learning Systems (VLEs) and this widespread adoption of VLEs is getting in the way of bottom-up innovation at the level of the classroom:

> "……far from being a source of enabling 'bottom-up' change, these institutional technologies appear to be entwined in a multiplicity of 'top down' relationships related to the concerns of school management and administration. It could be argued that the use of these systems is shaped more often by concerns of institutional efficiency, modernisation and rationalisation, rather than the individual concerns of learners or teachers. Indeed despite the connotations of the 'Learning Platform' and 'virtual learning environment' it would seem that the primary concern of these technologies is – at best – with a limited bureaucratic 'vision of academic success' based around qualifications and grades (Pring 2010, p. 84). With these issues in mind, we therefore need to approach institutional technologies in terms of enforcing the bureaucratic interests of the institution rather than expanding the educational interests of the individual" (Selwyn 2011, p. 477).

As Selwyn suggests it is important to understand the policy and institutional context in which digital technologies are being introduced into schools and this is likely to vary from country to country and change over time (Assude et al. 2010). Without such an understanding we may attribute lack of change in classrooms to, for example, lack of training of teachers, or to teachers' resistance to change, whereas there may be more complex and interrelated factors that need to be understood if we are going to be able to use digital technologies to innovate at the level of the classroom.

Engaging with the chapters in this book reminds me that there tends to be a divide in the education literature between those who focus on the more sociological aspects of learning in schools and those who focus on the more psychological aspects of learning. With notable exceptions (for example, Chevallard 1992) and the more recent work of Cobb (Cobb and McClain 2011), there is very little mathematics education research that situates teachers' classroom practices within the institutional and policy contexts in which they work. However, as Selwyn (2011) has pointed out digital technologies have not only been introduced into schools for educational purposes, with "many countries perceiving a close relationship between success in global economic markets and the increased use of technology in educational institutions" (p. 60).

Engaging with the political realities of schooling is a long way from the focus of the chapters in this collection, which are all concerned with classroom-based research that expands the potential of students to learn mathematics. I agree with the views of the authors, namely that digital technologies can potentially transform classroom mathematical practices. I also agree with the authors that transformation of learning mathematics needs to be informed by theory and evidence-informed research, and in the next section I discuss the theoretical perspectives represented in this collection.

15.3 Theory as a Way of Seeing

"Humans are irrepressible theorisers. We cannot help but note similarities among diverse experiences, to see relationships among events, and to develop theories that explain these relationships (and that predict others)" (Davis et al. 2000, p. 52).

The introductory chapter to this book starts by raising the issue of the diverse theoretical approaches that have evolved within the mathematics education community (Introduction). In this respect new researchers and practicing teachers could easily become confused by the plethora of theories related to the use of digital technologies for learning mathematics. However, it seems to me that many of the perspectives represented in this collection cohere around theories that predominantly derive from sociocultural theory and the work of Vygotsky (1978).

Sociocultural theory is predicated on the view that humans as learning, knowing, reasoning, feeling subjects are situated in social and cultural practices. Participation in these practices provides the fundamental mechanism for learning and knowing. Furthermore, human activity and practices must be understood as products of history, with artefacts and tools being fundamental parts of this history. A key concept within socio-cultural theory is the idea that all human activity is mediated by tools. These tools, invented by people living in particular cultures, are potentially transformative, that is they enable people to do things which they could not easily do without such tools. Within this framework the idea of person-acting-with-mediational-means (Wertsch 1991) expands the view of what a person can do and also suggests that a person will be constrained by their situated and mediated actions as they take place in various kinds of settings. In this respect as discussed in the previous section, learning events in school have to be understood as embedded in institutions, linked to the historical and political dynamics of the classroom.

The theoretical focus on tools is relevant when considering the role of the digital in mediating and potentially transforming mathematical activity. Both Mariotti (Chap. 9) and Geiger (Chap. 12) draw on the theory of instrumental genesis (Rabardel 2001) which derives in part from the work of Vygotsky. This conceptual approach allows us to understand more about the ways in which people interact differently with the same tool, and over time learn how to use it in different ways. This framework distinguishes between two aspects of a tool—the artefact and the instrument, separating what relates to the intention of the designer and what occurs in practice. From this perspective the instrument is made up of both artefact—type components and schematic components, associated with both the object/artefact and the subject/person. The instrument is constructed by the individual and relates to the context of use (utilisation process), which relates to the mathematical task to be solved as well as other contextual, institutional and policy-related factors. The particular instrument constructed by a student with respect to a particular artefact or technology (for example dynamic geometry software) may not be consistent with the intention of the teacher. To make the situation even more complex the instrument constructed by the student may not be consistent with the intentions of the designer. The theory of instrumental genesis has been used to explain the discrepancy between the students' behavior and the teacher's intentions with respect to the use of technology.

Acknowledging the role of the teacher in guiding instrumental genesis, Drijvers et al. (2010) have developed the idea of instrumental orchestration. This is defined as the intentional and systematic organisation and use of the various tools available to the teacher in a given mathematical situation, in order to guide students' instrumental genesis. This includes decisions about the way a mathematical task is introduced to students and worked on in the classroom, decisions about which tools to use (both digital and non digital), and on the schemes and techniques to be developed by the students. Mariotti also emphasises that the transformation process is not spontaneous and has to be "fostered by the teacher, through organizing specific social activities, designed to exploit the semiotic potential of the artefact" (Mariotti, Chap. 9). Bartollini-Bussi and Mariotti (2008) use the phrase "didactical cycle" to refer to the organisation of classroom activity incorporating the use of technologies. From a different theoretical antecedent Laborde and Laborde (Chap. 11) also emphasise the importance of designing mathematical teaching and learning situations, discussing the idea of the adidactical milieu which derives from the theory of Brousseau (1997).

Laborde and Laborde also discuss the perspective of the designer in terms of designing dynamic geometry environments and in particular Cabri 3D. They suggest that "the dragging facility in dynamic geometry environments illustrates very well the transformation technology can bring in the kind of representations offered for mathematical activity and consequently for the meaning of mathematical objects. A diagram in DGE is no longer a static diagram representing an instance of a geometrical object, but a class of drawings: representing invariant relationships among variable elements" (Chap. 11). They also emphasise that although the designer (Jean-Marie Laborde is the designer of Cabri) has clear intentions, the ways in which the technology is used may not relate to such intentions.

Whereas, appreciating that in the section above I have very much oversimplified the perspectives of the authors, I suggest that there are more similarities than differences in the theoretical perspectives represented in this collection of chapters. Work has already begun to connect these theoretical frames (Artigue and Cerulli 2008) and in the future more work could be carried out to develop an accessible framework that could inform mathematics teachers about the complex issues involved in using digital technologies to transform mathematics education.

15.4 Mathematics and Out-Of-School Use of Digital Technologies

Sociocultural theory recognises that a student's history of learning, what they learn out-of-school and what they have learned in previous schooling impacts on their ongoing learning experiences in school. From this perspective all students actively construct and make sense of a particular mathematical activity in terms of their previous learning, developing their own personal theories, or theories in action (Vergnaud 1994). In order to illustrate this I present an example from an interview with a 15-year-old student who was struggling with school mathematics. When interviewed about the meaning she gave to the use of letters in mathematics she told the interviewer that the value of a letter related to its position in the alphabet. When probed further she provided the following explanation:

Int: Does L have to be a larger number than A?
Eloise: Yes because A starts off as 1 or something.
Int: What made you think that [L has to be a larger number than A?]
Eloise: Because when we were little we used to do a code like that...in junior school...A would equal 1, B equals 2, C equals 3.....there were possibilities of A being 5 and B being 10 and that lot.....but it would come up too high a number to do it.........it was always in some order...

Eloise had developed her own theory about the meaning of letters, which derived from her work in primary school, and made sense to her in the context of the problems she was solving at the time. This personal knowledge had not been intentially taught by the teacher and was no longer appropriate (or correct) in the context of secondary school mathematics. Eloise's theory about the role of letters in mathematics, influenced how she made sense of letters when she encountered them in secondary school algebra. What this example illustrates is that each student brings to the classroom his/her own history of learning and when faced with a new situation makes sense of this from his/her own particular experience and way of knowing.

Another example derives from an interview with Anthony when he was a 10 year old primary school student. Anthony had not met algebraic symbols in school mathematics, yet when asked the question:

Which is larger, 2n or n + 2? He responded:

> "You can't say that because it wouldn't always be right….if n was 6 that would be 12…. and that would be 8 so that would be right….but if n was one then 2n would be 2 and n = 2 would be 3."

This response was surprising given that research has shown that this question is only answered correctly by 6% of 14 year olds (Küchemann 1981). When asked why he was able to answer the interview questions correctly he said:

> "It might be partly because of BASIC, where I've learned to use things like variables and things….like p is a number and you can use any letter for a number…."

This is an example in which a primary school student learned from out-of-school computer programming ideas that are related to the "scientific concepts" of school mathematics. The idea of "scientific concepts" draws attention to the importance of a systematic organised body of knowledge, knowledge that can be separated from the community that produced it. Vygotsky discussed the difference between informal and scientific concepts, and claimed that there is a dialectic relationship between the development of informal and scientific concepts:

> "the dividing line between these two types of concepts turns out to be highly fluid, passing from one side to the other in an infinite number of times in the actual course of development. Right from the start it should be mentioned that the development of spontaneous and academic concepts turn out as processes which are tightly bound up with one another and which constantly influence one another" (Steiner and Mahn 1996, p. 365).

In my research I continue to find examples of young people's out-of-school use of digital technologies impacting on their learning of mathematics in schools. For example, the following is an interview with two 8-year olds from the InterActive Education Project.

Int: Do either of you use Excel at home (Alan shakes head)?
Ray: Sometimes. My Dad uses it for his paper work.
Int: And when you use it what do you use it for?
Ray: Umm, he uses it, cos when he's got paper calculations and some are hard like for him, he puts it in Excel and then he puts, he circles it and then presses the equal button and it tells him what the sums are.
Int: What do you use it for?
Ray: Maths homework.
Alan: Cheat.

From sitting alongside his father at home Ray had observed him using a spreadsheet for his work. Ray's explanation shows that he understands how a spreadsheet can carry out "hard" calculations which are related to mathematics. Interestingly until this interview was carried out by a researcher the class teacher was not aware of this "fund of knowledge" (Moll and Greenberg 1990), illustrating the way in which home learning out-of-school is often not recognised by teachers at school.

Nowadays, the vast majority of young people engage with digital technologies in their lives outside school, and these experiences can impact negatively or positively on their mathematical learning with digital technologies in school. For example young people's experience of playing games out-of-school can impact on the ways in which they make sense of digital technologies in school and this can detract from the intended or “scientific” learning (Sutherland et al. 2009).

One of the research results from the InterActive Education project was that teachers often underestimate the impact of students’ past experiences on their learning in the classroom, and in particular their out-of-school experiences of using digital technologies. The theory of instrumental genesis discussed earlier explains why such out-of-school learning is likely to impact on the student’s construction of a particular digital instrument, that is how they make sense of the potential of the digital technology for learning mathematics. It is perhaps surprising therefore, that none of the authors in this collection of chapters appear to take such factors into account in their research. I suggest that mathematics education researchers tend to underestimate the impact of students’ out-of-school uses of ICT on their in-school learning of mathematics with digital technologies. As out-of-school uses of mobile devices become ubiquitous it will be even more important to consider the interrelationships between young people’s construction of the digital from their learning out-of-school and the mathematical concepts which teachers intend them to learn in school. Raising such issues presents a challenge to the use of digital technologies for transforming the teaching and learning of mathematics. In the next section of the paper, I explain why I believe that professional development is the way forward.

15.5 A Way Forward: Transformation Through Professional Development

As I argued earlier, a sociocultural approach to learning enables us to see the potential transformative nature of tools and artefacts that have been designed to enable us to do things that would be difficult to do without them. For example, the long multiplication algorithm enables us to perform calculations that would be difficult to perform mentally, dynamic geometry software enables us to visualise the invariant and variable properties of geometrical figures that are difficult to see in paper-and-pencil constructions, spreadsheets enable us to construct financial models that would be very difficult or impossible to develop on paper. However a focus on the transformative potential of digital technologies can fall into the trap of deterministic thinking, that is a belief that the mere use of such tools is sufficient for transformation to occur, and as I have discussed already the authors of this collection of chapters provide ample evidence for why this is not the case. Such deterministic thinking gets in the way of the productive use of ICT for teaching and learning, because from such a perspective there is no acknowledgement of the complexities and challenges involved in embedding digital technologies into mathematical classroom practices.

In our everyday lives we learn about the transformative potential of a particular digital technology through experimentation and discussion with colleagues and friends. However, as academics we are also aware that within the institutional setting of the University we may be resistant to using a technology that is being imposed on us to transform our everyday work practices. For example, I am resisting using the digital calendar that I am supposed to be using, and continue to use a paper diary which I argue is more transformative for me personally than a digital diary.

In order to start to use a digital diary to transform my time-management practices I would have to learn to use it in a transformative way. Similarly, teachers have to learn how to use "mathematical" digital technologies in a transformative way. Here the challenge is much greater than the challenge for me personally of learning how to use a digital diary. Teachers firstly have to learn how to use the chosen digital technology to transform both their own mathematical practices and their teaching of mathematics. Teachers then have to "teach" students to learn how to use digital technologies in transformative and mathematically appropriate ways.

Most of the authors of this collection of chapters carry out what could be called design-based research (Brown 1992). In my opinion the challenge is to scale-up such design-based (or didactical engineering) approaches through processes of professional development. The InterActive project showed that a successful model for professional development is to create networked communities in which teachers and researchers work in partnership to design and evaluate learning initiatives which use digital technologies as a tool for transforming learning. We argue that such professional development requires people to break out of set roles and relationships in which researchers are traditionally seen as knowledge generators and teachers as knowledge translators or users. For meaningful researcher-practitioner communities to emerge, trading zones are needed where co-learning and the co-construction of knowledge take place (Triggs and John 2004). Within such communities design can be informed by: theory and research-informed evidence; the craft knowledge of teachers; curriculum knowledge; policy and management constraints and possibilities and young people's use of digital technologies in their everyday lives. The focus is on iterative design and evaluation and a dynamic record of classroom activity and learning can be created from video and audio recording, screen-capture, observation, student interviews, and students' work.

Such design-based professional development should also pay attention to areas of tension that emerge through the process of classroom-based innovation (Sutherland et al. 2012). For example, as discussed earlier, there may be an area of tension around the ways in which senior management in a school intend to use technology to improve the qualifications and grades of students and the ways in which mathematics teachers intend to use digital technology to transform students' understanding of mathematical concepts.

In summary, we know from research on the use of digital technologies in schools that there is a dominant belief that simply by making a technological system available, teachers and students will more or less automatically take advantage of the opportunities it offers. We also know that despite many years of investing in technology in schools mathematics teachers are not taking advantages of the opportunities such technology offers for transforming the teaching and learning of mathematics. Whereas, theories of teaching and learning mathematics are a necessary part of opening up new ways for teachers to see what is possible, I suggest that the way forward is to focus attention on developing a model of networked communities of mathematics teachers that can be scaled-up in order to initiate large-scale transformation in mathematics classrooms.

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Part III
Transformation Related to Concepts and Ideas

Introduction

Part III deals with transformations related to five genuine didactical concepts. On the one hand new concepts are introduced in order to grasp aspects of students' cognitive state or activity: Dreyfus and Kidron (Chap. 16) propose the notion of proof image as an intermediate stage in a learner's production of a proof, Stanja and Steinbring (Chap. 17) introduce the notion of elementary stochastic seeing in order to describe aspects of elementary school students' stochastic thinking. Drawing on a definition from architecture Kuzniak (Chap. 18) introduces the notion of Geometric Work Space which is characterized by different geometrical paradigms and interaction between epistemological and cognitive levels. On the other hand Profke (Chap. 19) addresses the question of how didactical concepts can be translated into actions in order to help students in achieving a desired ability, competence or state of mind expressed in a didactical concept like mathematical literacy. Finally, Klep (Chap. 20) addresses a similar problem like Profke. However, instead of dealing with the problem of deducing actions from the definition of an abstract competency he tackles the problem differently. Klep models arithmetical competence on the basis of students' concrete actions.

All five chapters illustrate how the idea of transformation is inseparably linked to didactical concepts. Dreyfus and Kidron investigate transformations occurring within the process of the construction of knowledge. In order to describe the transformations that take place within the construction process they use *Abstraction in Context* as a theoretical framework. *Abstraction in Context* defines "abstraction as a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner" (Dreyfus and Kidron, Chap. 16). In other words, construction of knowledge is thought of as a process in which mathematical constructs available to the learner are transformed into new constructs. In this context, Dreyfus and Kidron pay particular attention to justifying, because "in the process of justifying a mathematical phenomenon, learners frequently need to expand their knowledge and to construct new knowledge" (Dreyfus and Kidron, Chap. 16). In order to grasp a particular intermediary stage in this transformation process they introduce the notion of *proof image* which “contains an entire mathematical situation as an image” (Dreyfus and Kidron, Chap. 16). A *proof image* is an insight into why a claim is true and how this truth can be argued. It contains different mathematical constructs and their combinations in order to justify a mathematical claim without being necessarily formal. According to Dreyfus and Kidron, a need for a more formal reasoning may appear as a consequence of the learner’s satisfaction from obtaining a proof image. The question remains, how this transformation from proof image to formal proof can be characterized. Is it possible to conceptualize this transformation process by means of the three epistemic actions “recognizing”, “building-with”, and “constructing” of the RBC-model on another level?

Stanja and Steinbring also introduce a new genuine didactical concept. They discuss the notion of *elementary stochastic seeing i*n order to describe aspects of elementary students’ stochastic thinking. According to Stanja and Steinbring *elementary stochastic seeing* has two dimensions: an epistemological and a semiotic dimension. The epistemological dimension refers to the children’s conceptions of knowledge and their ways of knowing related to random experiments. It is characterized by an understanding of stochastic predictions either being dichotomic, i.e., right or wrong, or relative, i.e., implying “a qualitative reference to a more or less probable occurrence or non-occurrence of a future event” (Stanja and Steinbring, Chap. 17). The semiotic dimension “concerns the interplay between recorded empirical observations of random experiments and symbolical interpretations of artefacts of elementary stochastics such as spinners and diagrams at the elementary theoretical level” (Stanja and Steinbring, Chap. 17).

Unlike the notion of *proof image* the concept of *elementary stochastic seeing* does not grasp a stage in a transformation process, but is a construct to describe a cognitive disposition. Nevertheless, transformation is related to the concept *elementary stochastic seeing* in terms of the following questions: How does *elementary stochastic seeing* develop? How do interpretations of given artefacts related to *elementary stochastic seeing* change? Which ideas about artefacts and stochastic contexts do children develop? (Stanja and Steinbring, Chap. 17).

Kuzniak addresses the issue of different geometrical paradigms that are coexistent in school geometry. Geometrical paradigms characterize the *Geometric Work Space*—a notion which is supposed to describe “a place organized to enable the work of people resolving geometric problems” (Kuzniak, Chap. 18).

The intention of the concept of *Geometric Work Space* is to conceptualize how geometric knowledge emerges and evolves in geometry classes and how it is reliant on different geometric paradigms.

According to Kuzniak a *Geometric Work Space* connects two levels: The epistemological level and the cognitive level. The epistemological level relates to geometrical activity into its purely mathematical dimension.

The second level is centred on the cognitive way that groups, and also particular individuals, use and appropriate the geometrical knowledge in their practice of the domain.

Transformational aspects of the notion of *Geometric Work Space* are based on the idea of a geometric work considered as a process involving creation, development, and transformation. To describe this complex process, Kuzniak uses a general meaning of the notion of genesis which is not only focused on origin but also on development and transformation of interactions.

The Geometrical Work Space requires various geneses in particular three fundamental types which articulate cognitive and epistemological levels.

- An instrumental genesis which transforms artefacts in tools within the construction process.
- A figural and semiotic genesis which provides the tangible objects their status of operating mathematical objects.
- A discursive genesis of proof which gives a meaning to properties used within mathematical reasoning.

With this specific framework, it is possible to describe ways in which students solve geometry problems and to understand how they form and transform their work in geometry within the education system.

Profke (Chap. 19) and Klep (Chap. 20) address a third aspect of transformations related to didactical concepts. Both chapters deal with the relation of mathematical competencies and mathematical activity. Profke approaches the question how a concept like *mathematical literacy* might be transformed into classroom activities that teachers and students can carry out in order to achieve this learning goal or competency. Thus, the transformation process exemplified by Profke is confronted with the problem of deducing tasks or activities from learning goals or competencies which is an open/unsolvable problem (Meyer 1971, 1974). Profke addresses the problem in the way that he describes mathematical activities which from his point of view are likely to foster the development of *mathematical literacy*. This chapter once more illustrates that the transformation of competencies into actions are an open problem, because precise justifications for the transformations seem to be vague.

The transformation process of learning goals or competencies like *mathematical literacy* into classroom activities addressed by Profke is comparable to the problem of any form of assessment: Any form of assessment needs to specify what knowledge of mathematics or what mathematical skills should be measured and how the tasks relate to both (Webb 2007).

The two transformation problems—transforming intended goals or competencies on the one hand into classroom activities and on the other hand into assessment items—are actually two sides of the same coin. The one relates to classroom instruction and the other to assessment of the same concept.

Klep (Chap. 20) seems to offer a solution to this problem by avoiding the problem of deducing tasks and actions from defined competencies. He tackles the problem in the way that he defines *arithmetical competence* based on student's actions. He is able to mirror the learning process of students by modelling the transformations of arithmetical structures students are potentially capable to carry out. Thus, his approach is not only about transforming potential arithmetic actions into a model and a measure of arithmetical competence, but mathematical transformation is at the core of his computational model of arithmetical competence.

Prediger discusses the meta-theoretical construct of ‘fundamental ideas’ in connection with mathematics and mathematics education and takes these results as a background to reflect the chapters of part III theoretically with regard to the idea of transformation. By working out and comparing the different approaches to transformation in each chapter, she tries to test the suitability of the notion of ‘transformation’ as a fundamental idea. In a further step Prediger gives ideas how to connect three chapters of part III concerning transformation of students’ cognitions and claims at the end of her discussion that the meta-theoretical construct ‘fundamental idea’ should be widened to a landscape of ideas to have the choice what epistemological level is the most promising to work with.

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Chapter 16
From Proof Image to Formal Proof—A Transformation

Tommy Dreyfus and Ivy Kidron

16.1 Introduction

The idea to investigate the notions of proof image and formal proof emerged within our research on construction of knowledge and of justification as a specific case of construction of knowledge. Indeed, justification is a central and crucial component of mathematical reasoning. In the process of justifying a mathematical phenomenon, learners frequently need to expand their knowledge and to construct new knowledge. The aim in our previous research was to elucidate the intricate relationships between processes of justification and the emergence of new (to the learner) knowledge constructs. In a previous study (Dreyfus and Kidron 2006), processes of knowledge construction of a solitary learner whom we call L were investigated. The learner was constructing knowledge about bifurcations of dynamic processes. While we were acutely aware that the core of the constructing process is justification, it was only later (Kidron and Dreyfus 2010a) that we paid attention to the question of what justification means to the learner and analyzed the relationship of this meaning of justification for the constructing actions and the patterns of knowledge construction. It is indeed important to elucidate what we mean by justification. Fischbein (1982) identified a gap between mathematical proof and justification in everyday life. Rather than starting from formal mathematics, justification takes into account the learner's point of departure with its intuitive thinking, visual intuitions, and verbal descriptions. For example, the solitary learner in our previous studies wanted to gain more insight into the phenomena causing the second bifurcation point. The term enlightenment, introduced by Rota (1997), seems appropriate to

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express her interpretation of the word justification. Rota also pointed out that, contrary to mathematical proof, enlightenment is a phenomenon that admits degrees.

The aim of this chapter is to elaborate the notion of proof image and to focus on the transformation from proof image to formal proof. In order to investigate this transformation, we use microanalytic methods related to knowledge constructing. For this purpose, we have chosen Abstraction in Context (AiC) as an appropriate theoretical framework. In the following section, we describe AiC and how we used it to analyze justification as a dynamic process of construction of knowledge. Then, we present the attempt of a mathematician whom we call K, to prove that under given conditions a function attains its minimum, and we show how the notion of proof image emerges from and can be discerned in the story of K. After presenting the story of L about bifurcation, we discuss, in more detail, the notion of proof image based on both stories. In particular, we point to a parallel between the well-known double-strand concept definition and concept image and the double-strand formal proof and proof image. We then come back to the transition from proof image to formal proof. We end the chapter with concluding remarks and explain how the transition from proof image to formal proof relates directly to the theme of this book—transformation.

16.2 Abstraction in Context

Abstraction in Context (AiC) is a theoretical framework that allows the researcher to describe and analyze, at a microanalytic level, processes of mathematical abstraction as they occur in their mathematical, historical, social, and learning context. Here, we give a short and partial introduction to AiC; we refer the reader to the literature for details and examples (see Schwarz et al. 2009, and references therein).

Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular, and this has led his collaborators to the idea of "vertical mathematization." Vertical mathematization points to a process of constructing by learners that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means. This process interweaves previous constructs and leads to a new construct.

AiC adopts this view and defines abstraction as a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. The genesis of an abstraction passes through a three-stage process, which includes the emergence of a need for learning, the emergence of a new construct, and its consolidation. This view of abstraction follows Van Oers (2001) in negating the role of decontextualization in abstraction and embraces Davydov's dialectic approach (1990); in that, it proceeds from an initial unrefined first form to a final coherent construct in a dialectic two-way relationship between the concrete and the abstract (see also Ozmantar and Monaghan 2007).

Activity theory proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social, and learning contexts in which these processes occur. In this, AiC follows Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artifacts in further ones, a feature, which is crucial to trace the genesis and the development of abstraction throughout a succession of activities. The kinds of actions that are relevant to abstraction are *epistemic actions*—actions that pertain to the knowing of the participants and that are observable by participants and researchers. The observability is crucial since other participants (teacher or peers) may challenge, share, or construct on what is made public.

The three epistemic actions that were found relevant and useful in order to model the central second stage of the process of abstraction are recognizing, building-with, and constructing. *Recognizing* (R) takes place when learners recognize that a specific previous knowledge construct is relevant to the problem they are dealing with. *Building-with* (B) is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* (C) consists of assembling and integrating previous constructs by vertical mathematization in a specific context in such a way that a new (to the learner) construct emerges. Constructing refers to the vertical mathematization process up to the first time the new construct is used or expressed by the learner, either through verbalization or through nonverbal action. Constructing does not refer to later stages of consolidation during which the construct becomes freely and flexibly available to the learner.

C-actions depend on R- and B-actions with previous constructs; while R- and B-actions are building blocks of the C-action, C-action is more than the collection of all the R- and B-actions in the same sense as the whole is more than the sum of its parts. The power of the C-action depends on the mathematical connections, which the learner establishes to link these building blocks and make them into a single whole unity—the new construct. It is in this sense that we say that R- and B-actions with previous constructs are nested within the C-action of a new construct. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognizing this construct, at least implicitly.

The model has been called the dynamically nested epistemic actions (RBC) model for AiC, or briefly, the RBC-model. It constitutes a methodological tool used for realizing the ideas of AiC. It has been developed and validated in a sequence of research studies showing a great variety of processes of constructing knowledge in terms of mathematical content, age of the learners, and the context in which the learning took place. In particular, it has been applied in several studies on justification as a dynamic process of constructing knowledge (Dreyfus and Kidron 2006; Kidron and Dreyfus 2010a, b); this research is briefly reviewed in the section about the bifurcation story presented below (16.5). For more detail on AiC and the RBC-model, we refer the reader to the review chapter by Schwarz et al. (2009).

16.3 The Story of K

A mathematician, whom we will call K, was offered the following problem. We found this problem in Scataglini-Belghitar and Mason's (2011) work; as they state, it was given as a weekly assignment in an analysis course at the University of Oxford.

> Problem: Show that a continuous function defined on R which tends to plus infinity as x tends to plus or minus infinity must have a minimum value.

K worked on the problem, in the presence of one of the researchers, for about half an hour. During this time, he checked graphically with an explicit example (with a continuous function with many oscillations) whether the claim was correct or not. Being unsuccessful in finding a counterexample, he tried to prove that the claim is correct. After he felt satisfied that he knew how to prove the claim, he was asked to describe his way of thinking. He talked to the researcher, and the researcher took notes, which were then immediately shaped into the following report. We are aware that the resulting report is a doubly indirect set of data, having undergone K's self-interpretation and the researcher's interpretation of what K said; however, having in mind our aim of examining K's thought processes while constructing the proof, we negated the option of interacting with K during the time when he was constructing the proof since this would have substantially disturbed the crucial phases of his thinking. We also submit that asking K to fluently describe the process orally immediately after the event allows for less reflection on his part than asking him to write down what the process was, and is therefore likely to give a more accurate picture of his thought processes. The following report resulted from the above procedure.

> I did not know what the function looks like. I imagined that the function is $|x|$ and then I thought about $x^2+\sin(x)$ in order to prevent myself from working only with monotonous functions. I tried to imagine some disturbing cases; for example, instead of adding $\sin(x)$, I added in my imagination $1{,}000\sin(x)$. I wanted to imagine a function with some uncontrollable oscillations. I saw in my imagination that at the end, in spite of the large oscillations the function should increase (not in a monotonic way) and there should be a clear upward tendency. It was clear to me that I should use the fact that in each finite closed interval the function attains its minimum. It was clear to me as well that I have to use the theorem (stating that a continuous function on a closed and bounded interval is bounded and attains its extreme values). It was also clear to me that the proof should be a proof by contradiction. I devoted some time to analyze what is the practical meaning that the function does not have a global minimum and then I realized that I need to use intervals that are nested in each other. Then, I formulated to myself what is the meaning that no global minimum exists: a function does not have a global minimum if to each interval $[-i,i]$ there exists an interval $[-i',i']$ such that $i'>i$ in which the minimum is smaller than the previous minimum. Now I needed the exact definition of tending to infinity. I felt I forgot it for a moment (I was not sure about it). Visually I saw in my imagination that from a specific x onward, the function

should have values bigger than a given value, specifically bigger than the minimum value in the interval $[-i,i]$.

In this situation I realized that if all the values I get are bigger than a given N, let us take N the minimum of the function in the interval $[-i,i]$, then it is clear that we cannot get a smaller minimum. I was still disturbed by the large oscillations that I saw in my imagination. At this moment I tried to reconstruct the definition of the limit and it is the formal definition of the limit and the meaning of the continuity of the function that helped me to overcome the problem. At the moment I had the formal definition, I realized that the problem is solved. At each stage, some picture accompanied the formal proof. The main difficulty was that it was clear to me that the claim is correct for a monotonous continuous increasing function but I was not sure of its correctness for a general function. Nevertheless when I added $\sin(x)$ to x^2, the function still increases despite the oscillations. Through the interplay between the definition of the limit and the image of the too large waves (that would not permit to get a contradiction), the formal definition saved me from having to picture additional "crazy" functions like $x^2 + 1{,}000{,}000\sin(x)$.

With the intention of keeping terminology uniform, we will refer to K as a learner; indeed, in the situation he was put, K needed to construct some new (to him) knowledge, namely the proof of the claim given in the problem.

Our interpretation of the report is that K constructed the proof in two stages. In the first stage (first paragraph), he collected ideas, previous knowledge constructs, and examples that seemed useful and connected to the problem in front of him; these previous constructs include "the theorem" (and subconstructs such as continuity and boundedness of a function on an interval of the real line), the notion of limit in the special case of a function tending to infinity as the independent variable tends to infinity, proof by contradiction, and a construct relating to the fact that a general claim needs to be valid in all cases, especially in extreme (or in K's words "disturbing") cases. Obviously, K has many other mathematical constructs available but he recognized these as relevant in the present problem situation, and selected them to build-with them in attempting to come up with a proof. Indeed, his selection was successful and helped him bring to light a contradiction between the ever-expanding intervals needed if the function was not to have a minimum, given the assumption that the function tends to infinity as x does.

We, as researchers and readers, can only conclude from the report on what might have been in K's mind at the moment he was actually conceiving of the proof. Although the report is, of course, verbal, the image in K's mind at the time was not or not necessarily verbal. We submit that K had in his mind a complex image of how the selected previous constructs resonate (interact and collaborate) in order to provide the contradiction he was looking for, and hence the proof of the claim. It is this image, which we propose to call K's *proof image*.

The second part of the report describes how K built-with his proof image, as well as with the formal definition of the limit in order to overcome his fear that maybe some "crazy" function could be found that escapes the contradiction. This part describes the transition from proof image to formal proof, to which the title of our paper refers. This transition, modulo the proof image on which it builds, is far more easily accessible to the researcher than the proof image itself. Specifically, we can identify here that K constructed a formal proof by recognizing that the definition of limit (the limit of a function as x tends to infinity) is relevant in the present situation and by building-with this definition of limit and with his formulation of the meaning of (absence of a) global minimum, an airtight argument from which he could infer the statement of the theorem. Hence, there is clear evidence for a constructing action by K, in which recognizing and building-with actions are nested. However, given that we collected the data by means of an after-the-effect report by K, it is difficult for the researchers to analyze this constructing process in more detail than has been done here.

K's demarche was, in part, determined by characteristics of the problem: Though not explicitly, the problem in fact challenges the learner to create his own examples in order to begin the justification process, examples that obviously fit the claim as well as "nasty" or "disturbing" examples. Nevertheless, by means of creating one's own examples, a problem arises concerning the generality of the justification because one might consider particular cases, which do not constitute an appropriate justification. It is interesting to note that it is the consolidation of the definition of the limit, which enabled the generalization and saved the learner from having to consider more examples, which might disturb his proof image.

16.4 What is a Proof Image?

At the most elementary level, a proof image may be the mental image held by a learner—student or mathematician—who can see why a claim is true but finds it difficult to articulate or express this in formal mathematical terms. Hence, each proof image is specific to a specific learner in a specific case.

A proof image can but need not be spatial. In the story of K, the graphical image is only of a function; however, the image of the proof includes a dynamic, developing process, namely the successively smaller minima of the function as one considers successively longer intervals on the real axis. Being the image of a proof, any proof image is also likely to have logical aspects. In the case of K, the logical aspect is the incompatibility between a function that tends to infinity and the dynamically developing function with successively smaller minima.

We see the justification process as a process of suitably combining selected previous constructs. Hence, recognizing and selecting relevant previous constructs is a preliminary stage of the justification process. Examining how they might combine, as well as possibly discarding some of the selected constructs and selecting others, is the first stage that may lead to a proof image. A proof image contains an entire mathematical situation as an image. Even if the proof image does not include a formal representation of the logical steps, it usually will include some intuitive threads that have the potential to link the selected previous constructs. The entire process of construction of the formal proof will come at a later, second stage. In this sense, we may find some analogies and some differences between the double-strand "concept image, concept definition" and the double-strand "proof image, formal proof." These analogies and differences will be discussed subsequently.

The proof image might appear as a flash of insight after working for a while trying to prove a statement. This was the case for some mathematicians like Nash. In her book*, A Beautiful Mind,* Nasar (1998) wrote:

> Nash always worked backward in his head. He will mull over a problem and, at some point, have a flash of insight, an intuition, a vision of the solution he was seeking. (p. 129)

She added that the solution did not present itself as a rigorous proof but as:

> ... a bunch of intuitive threads that have to be woven together and some of the early ones present themselves visually. (p. 129)

Nevertheless, we do not reduce the proof image to a "flash of insight." We rather see it as the process that leads to this "flash of insight." The flash of insight as a first degree of enlightenment demonstrates the existence of a proof image but is not necessarily identified with the proof image. Hence, we do not discount the possibility of a flash of insight, but neither do we postulate it as obligatory. Neither the story of K nor the following one are clearly flashes of insight.

16.5 The Bifurcation Story

A *bunch of intuitive threads that have to be woven together* is exactly what happened to a learner whom we will call L in a learning experience, which we will call the bifurcation story. We describe the parts of the bifurcation story that relate to combining the threads, how L's proof image emerged as a consequence, and how it supported her in the construction of a justification.

The bifurcation story is described in Dreyfus and Kidron (2006), where we considered a solitary learner dealing with bifurcations in a logistic dynamical system and attempting to justify the occurrence and position of the second bifurcation point (the transition point from 2-periodic to 4-periodic behavior of the system). Kidron and Dreyfus (2010a) analyzed the construction of the justification for the second bifurcation point.

L is an experienced mathematician and her motivation for finding a justification drove her entire learning process. She had a keen interest and strong motivation in learning about the period doublings that occur in dynamical systems, specifically in the system defined by the logistic equation $f(x)=x+rx(1-x)$, as r increases from $r=1$ through $r=2$ and beyond.

Gathering data about the learning process of a solitary learner presents great challenges because there is usually no need for the learner to report about her learning. The researchers were in the fortunate position to have reports of L's learning process. L was preparing a lecture on bifurcations in dynamical systems while being homebound for a period of 2 weeks. Like many mathematicians, L wrote, graphed, drew, and sketched a lot, some by hand and some by computer. As is her habit, she carefully dated and kept these notes as well as all computer files and printouts. These documents later served as a window into her thinking for the researchers and allowed them to infer her epistemic actions. The researchers then constructed

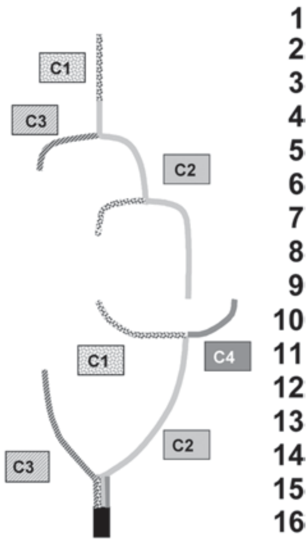


Fig. 16.1 The interacting parallel constructions diagram

a report of the learning process, following an elaborate procedure of several cycles of description by the learner and challenges by one of the researchers. The accuracy of the report was verified by observing its close correspondence with the raw data, some of which have been published (Dreyfus and Kidron 2006).

Considering the mathematical content and the development of L's thinking, Dreyfus and Kidron (2006) found an overarching constructing action, within which four secondary constructing actions were nested. These secondary constructing actions relate to different modes of thinking: numerical (C_1), algebraic (C_2), analytic (C_3), and visual (C_4). They are not linearly ordered but took place in parallel and interacted (see Fig. 16.1 for a representation of these interacting parallel constructions; in the diagram, the time axis runs from top to bottom; the numbers denote episodes). Interactions included branching of a new constructing action from an ongoing one (such as C_1 branching from C_2 at the beginning of episode 7), combining or recombining of constructing actions (such as C_1 and C_4 combining at the end of episode 10), and interruption and resumption of constructing actions. L aimed to justify, in the sense set forth in the introduction, results obtained empirically from her interaction with a computer. In L's learning experience, combining C-actions indicate steps in the justification process that lead to enlightenment.

In L's learning experience, we observe three successive degrees of enlightenment. They occur at three points in time when C-actions combine, and each combining point was characterized by the integration of different C-actions and different modes of thinking. For example, the combining of C_1 and C_4 at the end of episode 10 expresses the connection in L's thinking of the numerical mode and a graphical mode, first a static graphical mode, and then a dynamic graphical mode and thus reinforces and concretizes her view of the dynamics.

We claim that at this first combining point, L had a proof image and that this proof image helped her constructing the justification. In order to better understand what happened in episode 10 we briefly describe the previous episodes: Using information from several web-based resources, L learned that the fourth-order equation $f^2(x)-x=0$ will yield the 2-periodic points, and the quadratic equation $p(x)=\dfrac{f^2(x)-x}{f(x)-x}=0$ will yield those 2-periodic points that are not fixpoints. She solved $p(x)=0$ for general r. She found that the discriminant $D=0$ for $r=2$ and checked that for $r>2$ there are two real solutions—the 2-period. These results made it clear to her that $D=0$ where period doubling occurs and that this happens at $r=2$. Assuming an analogy, L set up the corresponding equation $p(x)=\dfrac{f^4(x)-x}{f^2(x)-x}=0$ for the 4-periodic points. The computer showed that this equation is of order 12 (episode 3) and cannot be solved for general r. The strategies that worked for the previous transition became inapplicable (episode 4). Web resources led L to the notion of discriminant for a general polynomial (episode 5), which she used with the help of the computer (episode 6) to find the numerical value $r=\sqrt{6}$ of the transition point to the 4-period. Encouraged by this numerical success, she began to search for the connection between multiple roots, $D=0$, and transition points.

In order to analyze the interaction pattern of combining constructions in episode 10, we describe in more detail the two constructions C_1 and C_4:

C_1: The process of constructing the solution of the polynomial equation.

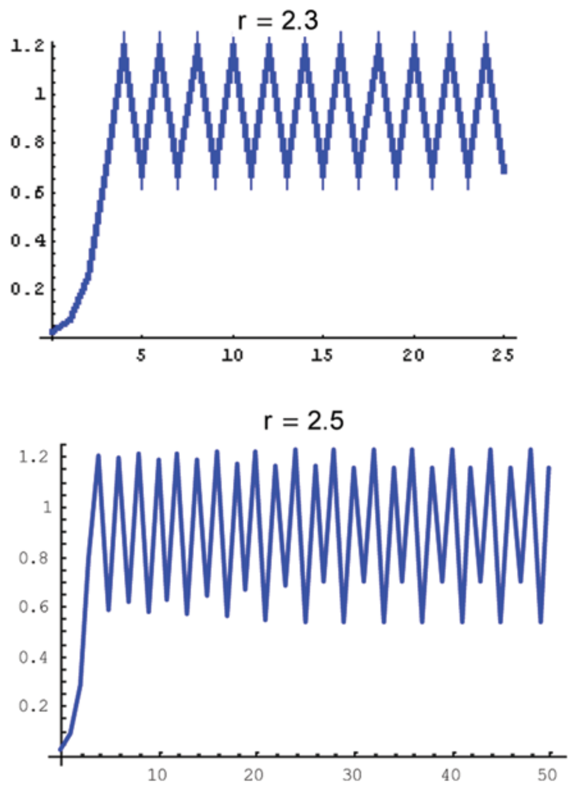
$p(x)=0$ in order to find the 4-periodic points. The solution process is considered algebraically and numerically. The focus is on the solutions for each value of the parameter r and relationships between the solutions for different values of r.

C_4: The process of constructing a dynamic view of the bifurcation in which the final state values of x are considered as functions of r.

The data we used for the analysis of L's learning experience consisted of a carefully constructed and double-checked report of the learning experience, as mentioned above. We cite part of episode 10 of this report.

> 10e The transition point is a single value of r ($r=\sqrt{6}$). I focused on the set of x-values at $r=\sqrt{6}$ (two that became four). I looked at the transition point and from there to the right. While I observed the bifurcation map, I noticed its structure: the branches that split off and my attention was focused to the points in which this change is detected: the bifurcation points.
>
> 10f I looked at this fork-like shape and associated its splitting point to the fact that the discriminant vanishes. At once, the bifurcation map seemed different, endowed with a new meaning. I looked at it and I could not understand how it could be that I did not see it this way before.
>
> 10g Now, it seemed to me intuitively clear that at the bifurcation points there must be double solutions and therefore the discriminant should equal zero.

Fig. 16.2 Time series plots for $r = 2.3$ and for $r = 2.5$



Our interpretation of this excerpt is that a mental image of the fork-like shape emerged and that this indicates the first degree of enlightenment (Kidron and Dreyfus 2010a). This occurs at the end of episode 10, at the first combining point in the construction diagram that marks the connection between the constructing actions C_1 and C_4.

As in the story of K, we see the justification process in the bifurcation story as a process of combination of selected previous constructs. The proof image includes a *bunch of intuitive threads that have to be woven together* and that have the potential to link the selected previous constructs. Indeed, the deep learning that takes place in episode 10 is characterized by the establishment of new connections between the numerical aspects of C_1 and the dynamic view of C_4. The selected previous constructs include the result of the constructing action C_1 with its numerical and algebraic potential as well as the graphic representations of several time series plots (Fig. 16.2) as well as the bifurcation diagram (Fig. 16.3).

The links between these previous constructs emerged while L focused, in this order, on:

- The two repeating solutions for $r = 2.3$,
- The double real solution for $r = \sqrt{6}$, and
- The four repeating solutions for $r = 2.5$,

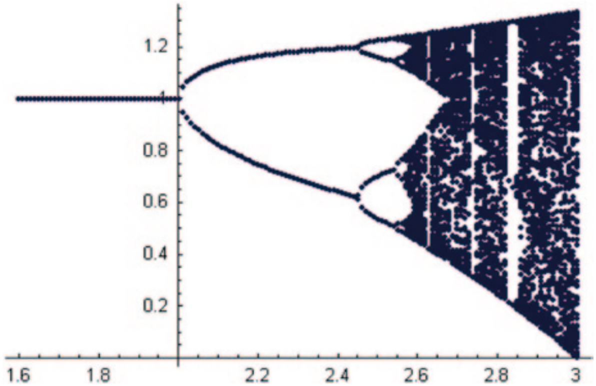


Fig. 16.3 The bifurcation diagram

and at the same time moved with her eyes along the *r*-axis of the bifurcation diagram.

Here, the dynamic nature of the transition from 2-period to 4-period became prominent. The connection between C_1 and C_4 is expressed by the transition from the numerical mode to the graphical mode, first to a static graphical mode, and then to a dynamic graphical mode. The numerical aspects of C_1 and the dynamic view of C_4 are thus complementary with the more powerful graphic dynamic view supplanting the static numerical one as the episode progresses. In fact, C_4 was not very powerful at the beginning of episode 10 when L vaguely remembered the bifurcation diagram as a static graphical representation. Only later, within numerical considerations (C_1) the static graphical mode turned to a dynamic graphical mode and C_4 became stronger. Thus, connections between the fragile weak branch C_4 and the more established knowledge of the stronger branch C_1 reinforced the weak branch and contributed to a combination of the two branches. This positive interaction between the two branches enabled a change of the view of the nature of the parameter *r* in the bifurcation map from discrete to continuous. This was the first degree of enlightenment. The proof image that was created in this stage is a result of the way the intuitive threads link the selected previous constructs. In order to understand what L's proof image was, we should point out that the transition from episode 9 to episode 10 was crucial because at that moment, all L's resources had been exhausted. The natural thing to do was to return to C_1. However, this was unlikely to yield more than it had already yielded: The numerical mode was not sufficient to understand the structure of the double solutions at the bifurcation points. L could not find an algebraic expression but was still looking for some connection between the *x*-values for different values of the parameter *r*, before and after the bifurcation point. She needed a new view, a different type of thinking. This need initiated C_4, leading to a graphic equivalent of the requested algebraic expression and thus to the consideration of the numerical values in a graphical framework. The graphic equivalent had to show the *x*-values as they depended on *r*. L found this view in the bifurcation diagram.

Like in K's case the report is, of course, verbal but the image in L's mind was not verbal. We submit that L had in her mind an image of how the selected previous constructs C_1 and C_4 resonate in order to provide the graphic equivalent of the requested algebraic expression she was looking for: the view of the dynamic behavior of the *x*-values, as *r* varies and the realization that at the bifurcation points, a branching of the *x*-values occurs. It is this image, which we call L's proof image. This first degree of enlightenment consisted of a mental image, the movement of the *x*-values through the fork-like shape, and as a consequence, L's drive to understand the structure of the double solutions at the bifurcation points was satisfied at least intuitively, visually.

This proof image supported the construction of the justification in the sense that L's background was now different. As a consequence of the dynamic view of the transition, she found the analogy with the first transition point, which she had been looking for but she wanted to know more, especially about the mathematics behind the computations with the help of the computer. She realized that her knowledge of double solutions at bifurcation points was still intuitive.

> 11a I was interested in a mathematical explanation why the transition point from 2-period to 4-period is obtained by comparing the discriminant to zero.
>
> 11b Therefore, I wanted to know more about the term discriminant. Especially, what is the meaning of the discriminant of an equation of higher degree?

Wanting to know more about the discriminant was a first step in the transformation of L's proof image into the formal proof or to the justification she was looking for. In Kidron and Dreyfus (2010a), we describe in detail how the proof image in episode 10 supports L's additional modes of thinking which permit the transition from the proof image to the justification.

16.6 Discussion of Proof Image

In this section, we will illuminate the notion of proof image by discussing its relationship to related notions as well as to the literature in which mathematicians refer to their experiences. We will then discuss characteristic aspects of the notion of proof image and conclude with remarks on the transition from proof image to formal proof, using information from both cases (K and L).

16.6.1 Proof Image and Proof Idea

A first question that arises concerns the difference between a proof image and a proof idea. We relate to the distinction between proof image and proof idea only in connection with the two cases described here. In distinction from a proof idea, a proof image is not necessarily the emergence of a first idea or of a single idea in the process of justifying. A proof image is rather oriented from the end to the beginning: an image of the entire mathematical justification. In "simple" cases, in which the proof image may emerge at the very beginning of the justification process, it might be difficult to make the distinction but we are interested in more complex cases, in which the person who wants to prove needs to select suitable previous constructs and establish links between them. This was the case in K's story. K's proof image with the intuitive threads that link his selected previous constructs contains an entire mathematical situation as an image. Indeed, in K's proof image, we discern the intuitive threads that link between K's previous constructs like the use of nested closed intervals, the application of the theorem on a continuous function in a closed and bounded interval, and the meaning K attributed to the words "the function tends to infinity." Similarly, L's proof image at the first combining point in episode 10 contains an entire mathematical situation. Here too, we discern the linkages between the numerical considerations offered by C_1 and the intuitive threads that link the different graphical static and dynamic representations.

A proof idea is communicable and hence can be made available to others. This is not necessarily the case for a proof image. People may have a proof image in their mind but be unable to write it down or explain it in words to somebody else or even to themselves; a proof image may be definite but transitory and disappear again, and the person may be worried about that possibility. This is what happened to L in episode 10 of the bifurcation story. Her proof image consisted of mental objects that were intimately connected and clear to her but are not fully expressed in the report. L looked at the fork-like shape and associated its splitting with the fact that the discriminant vanishes and, at once, the bifurcation diagram seemed different, endowed with a new meaning. L felt the urge to sit down and to write her notes about what happened before at this moment. In the case of K, we are not able to similarly point out the ephemeral nature of the proof image from the data, but as mentioned, this is due to methodological constraints. We submit that at the time he was actually constructing the proof, K's proof image was similarly transitory; the fact that he later, when reporting, recalled his doubts about further disturbingly oscillating functions may be taken as a sign for that.

16.6.2 Proof image and Rota's View of Proof

As already mentioned, our view of justification as enlightenment is compatible with the view of proof as held by Rota. Rota wrote: "we say that a proof is beautiful when it gives away the secret of the theorem, when it leads us to perceive the inevitability of the statement being proved" (p. 132).

This might be connected to proof image in the sense that when we realize the links between the previous constructs we have a feeling of the inevitability of the statement being proved. A proof image is an intermediate stage on the way to a more communicable and more detailed form of proof (which may but need not be formal). While being a mental image rather than a communicable text, and while possibly lacking detail, a proof image nevertheless is complete in the sense of giving the learner the clear feeling of clearly grasping the reasons why the claim is true, even inevitable (even though from an expert mathematician's point of view, the argument may not be complete or even correct). This is what happened in the first degree of enlightenment in which L did feel this kind of inevitability of the statement.

> 10f I looked at this fork-like shape and associated its splitting point to the fact that the discriminant vanishes. At once, the bifurcation map seemed different, endowed with a new meaning. I looked at it, and I could not understand how it could be that I did not see it this way before.

16.6.3 Proof Image and Thurston's View of Proof

Thurston (1994) claims that human thinking and understanding are organized into a variety of separate powerful facilities that work together loosely at high levels of organization. These facilities include language, vision, spatial sense, kinesthetic sense (motion sense), logic and deduction, intuition, association, metaphor, stimulus–response, process, and time. On this basis, he claimed that

> A group of mathematicians interacting with each other can keep a collection of mathematical ideas alive for a period of years, even though the recorded version of their mathematical work differs from their actual thinking, having much greater emphasis on language, symbols, logic and formalism. (p. 7)

He goes on to claim that in some areas of mathematics, "it is often pretty hard to have a document that reflects well the way people actually think" (p. 9). Expressed in the terms we introduced in this paper, Thurston stresses the importance of proof images over proof ideas. He teaches us that when doing research, professional mathematicians may rely more on their proof images than on their formally written proofs. Proof images of professional mathematicians used in the manner implied by Thurston should be expected to be more elaborate than proof images typically held by learners but they will usually not be communicable and therefore very difficult to ascertain. In summary, Thurston wrote that more than knowledge, people want *personal understanding*. A proof image is more related to personal understanding than a proof idea, which we can more easily transmit as knowledge to another person.

16.6.4 The Concept Image/Concept Definition Analogy

The notion of proof image as a stage on the way to formal proof emerged in the framework of our research on construction of knowledge and more particularly on justification as a specific case of construction of knowledge. Fischbein (1982) identified a gap between justification in everyday life and mathematical proof. Rather than starting from formal mathematics, justification takes into account the learner's point of departure with his intuitive thinking, visual intuitions, and verbal descriptions. We therefore encounter a similar situation as in the well-known gap between concept definition and concept image: Vinner and Hershkowitz (1980) have coined the term concept image to denote the set of all pictures that have ever been associated in a person's mind with the concept together with the set of properties associated with the concept. It has later been more explicitly characterized as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall and Vinner 1981, p. 152); the term *concept definition* is *a* "form of words used to specify that concept" (Tall and Vinner 1981, p. 152). Dealing with the learner's intuitive thinking, we are interested to explore the analogy between the double-strand *concept image/concept definition* and the double-strand *proof image/formal proof*. A main component of this analogy is that the concept image and the proof image are more personal and less objective than concept definition and formal proof. While the latter are, at least in most cases, explicit formulations largely agreed upon by the community of mathematicians, the former are mental images: A proof image consists of the cognitive structure in the learner's mind that is associated with the given proof. A limitation of the analogy is that a proof image usually contains some elements of the logic that underlies the formal proof whereas the concept image may be free of formal aspects. There is some interest in the transition from concept image to concept definition. There is also some interest in the transition from proof image to formal proof. In section 16.7, we will refer to this transition.

16.6.5 Logical Aspects of Proof Image

As pointed out earlier, and because of the nature of a proof image as a form of justification, logical aspects will invariably form part of any proof image—not their formal version but some intuitive representations of them. Even if the proof image does not include a formal representation of the logical steps, it does include some intuitive threads that link the selected previous constructs. In this respect, the notion of proof image differs from that of concept image, which does not necessarily include logical aspects.

Intuitive representations of the logical aspects appear in L's story in the links between the numerical aspects of C_1 and the dynamic view of C_4. The connection between C_1 and C_4 is expressed by the transition from finding the solutions (numerical aspects of C_1) to a graphical mode of thinking about the solutions, first a static one and then a dynamic one, which allowed L to begin constructing a dynamic view of the bifurcation (C_4). The intuitive representations of the logical aspects are well demonstrated by means of L's experimentation in a new setting (graphical) with information obtained in a previous setting (numerical). These linkages enabled the integration of the two modes of thinking associated with C_1 and C_4.

Among the logical aspects that accompany the development of the proof image in K's story, we notice K's awareness of the incompatibility between the successively smaller minima as one considers successively longer intervals on the real axis and the function that tends to infinity.

16.6.6 Visual Aspects of Proof Image

Here, we discuss the question whether a proof image needs to be related to a visual proof. For example, when proving the quadratic formula, we might think about quadratic completion with or without the graphical version of quadratic completion.

We suggest that a proof image may but need not be related to a visual proof. L's proof image is clearly visual, and the visual aspects include the crucial features of what makes the image into a proof image. However, in the case of a proof that is not inherently visual, a proof image does not need to be related to the visualization of the proof. The story of K is a case where the proof is not inherently visual; the proof image is only incidentally related to a visualization of the elements involved in the proof such as specific examples of functions or sequences of functions. Hence, K's proof image is not related to a visual proof. If we consider proofs in domains other than analysis (because in analysis, we have a tendency to visualize the objects involved) further questions arise: Could we imagine someone's proof image for Euclid's proof of the infinity of the number of primes or someone's proof image of Lagrange's theorem that the order of a subgroup is a divisor of the order of the group? We think we could but we have no empirical evidence for this yet. There is a need to further elaborate the notion of proof image with respect to examples not from the analysis.

16.6.7 Static Versus Dynamic Proof Image

We could have seen the proof image as a static overview of the mathematical situation with the links between relevant previous constructs as an image but we prefer to consider the proof image as a dynamic process which accompanies the learner for a certain time like a sequence of images or, maybe better, a single developing image with different foci of attention. This more dynamic description corresponds to the fact that we consider the justification itself as a dynamic process of construction of knowledge. It also contributes to the sharpness of the distinction between proof idea and proof image.

Moreover, this description of the proof image as a dynamic process also marks a difference between proof image and concept image, since a concept image is generally considered as something static. This does, of course, not mean that a concept image may not have dynamic components—it may, if it is a concept image of a dynamic object. For example, a learner's concept image of function may include a point moving along a graph in a Cartesian coordinate system. In the same sense, L's proof image has dynamic components, moving from left to right in the bifurcation diagram. But this is not the sense in which we are using the term dynamic in this subsection: A proof image is inherently dynamic since it represents a process, the process of justification, and this is the case independently of whether the mathematical objects that form part of the image have dynamic aspects or not (in L's story, they do, while in K's story they do not).

Considering proof image as a dynamic process we might ask: Could it be that the dynamics show how the various linkages between the selected previous constructs appear as the proof develops? This leads us to see the emergence of a proof image as a case of constructing an abstract mathematical notion.

16.6.8 Proof Image as a Construct

The answer to the question whether a proof image should be considered as a construct in the sense of AiC depends, in our opinion, on the situation. A proof image might be a construct, especially if it emerges as a consequence of an effort of thinking toward the selection and combination of relevant previous constructs, which are linked in the proof image. But considering the proof image as a construct turns it into a rather static outcome of the constructing process, in contradiction to the discussion in the previous subsection. The question thus arises whether the proof image is a construct or the constructing process. This question leads us back to the two descriptions of the proof image as a static overview of the mathematical situation or as a dynamic process, a dynamic image with different foci of attention. It might be both.

The proof image can be the result of serious efforts rather than appearing at the very beginning of the justification process. In such cases, we can analyze its emergence with AiC. We then use AiC not only for the second stage of the transition from the proof image to the formal proof but also for the first stage of the emergence of the proof image. This is, what we did for the story of L, where we had sufficiently reliable data for carrying out such an analysis. We used AiC to analyze the selection of the constructs C_1 and C_4 in episode 10 as well as the linkages between these two constructs (Kidron and Dreyfus 2010a). In particular, AiC was used to analyze the new meaning that L attributed to the bifurcation diagram in 10e, close to the combination of C_1 and C_4, which was of crucial importance in the learning process. We wrote that the proof image contains selected previous constructs and some intuitive threads that link these selected previous constructs. We then used AiC to analyze both:

- The selection of relevant previous constructs and
- The intuitive threads that link these selected previous constructs.

We found that the intuitive threads with the potential links are already present in the process of selection of the relevant previous constructs. We surmise that something similar happened in the story of K but our data are not sufficiently detailed to unambiguously support this claim, which is why we leave it to stand as a speculation.

16.6.9 Proof Image and Davydov's View of Abstraction

In L's story, we described a progressive mutual approach of two constructing actions, one of which was initially weak and strengthened in the process; this matches the genesis of abstraction as expressed by Davydov's (1990) method of ascent to the concrete, according to which abstraction starts from an initial, simple, undeveloped first form, which need not be internally and externally consistent, and ends with a consistent and elaborate final form. In the bifurcation case, for example, at the beginning of episode 10 the dynamics shows the progressive mutual approach between the weak branch C_4 and the strong branch C_1. At the end of episode 10, C_4 is the strong branch and C_1 is the weak branch.

It is interesting to note that in the story of K we also have two branches that combine, like in the story of L. We also have connections (linkages) between a fragile weak branch and a strong branch in which the more established knowledge of the stronger branch reinforced the weak branch and contributed to a combination of the two branches. The strong branch was K's mode of thinking in which he realizes that he should use nested closed intervals and apply the theorem stating that a continuous function on a closed and bounded interval is bounded and attains its extreme values. The weak branch refers to the meaning he attributed to the sentence "the function tends to plus infinity." The intuitive meaning that the function grows infinitely was enough for functions that behave nicely in a monotonic way but K was not satisfied with this fragile knowledge for functions with oscillations. For such functions, K needed more. There was a need to reinforce the weak branch.

Referring to Davydov's notion of abstracting, according to which abstraction proceeds from an unrefined and vague form to a final coherent construct begs the question whether the proof image is this unrefined vague form of the formal proof. Davydov describes how a process of abstraction develops. We do not think that a proof image is necessarily a vague initial notion, which need not be internally and externally consistent. While the proof image will usually not include a formalization of the logic involved, it will usually not be vague. Nevertheless, if we consider a proof image as a process in progress, the description of Davydov might apply to the initial step of the process. Therefore, Davydov's view might apply to the transition from the initial stages of proof image to its final stages as well as to the transition from proof image to formal proof.

16.7 The Transition from Proof Image to Formal Proof

The transition from proof image to formal proof in the case of L, including the important role of Davydov's view of abstraction in this transition, has been analyzed in detail in our previous articles (Dreyfus and Kidron 2006, Kidron and Dreyfus 2010a). That analysis was carried out in the framework of AiC. In particular, we have shown how L's proof image has played a crucial role in the constructing process leading to her more formal conception of the proof, a process that included building-with actions using additional constructs from analysis and algebra that were necessary for the transition but were present only as background (the algebraic ones) or not at all (the analytic ones) in the proof image.

The transition from proof image to formal proof in the case of K has been briefly analyzed above. We have exhibited it as a constructing action in the sense of AiC that can be usefully analyzed by means of the recognizing, building-with, and constructing epistemic actions. While this is not surprising, it lends credence to the claim that in cases in which learners' proof image can be identified, an analysis of the transition from proof image to formal proof as a process of AiC is not only possible but also promising and may lead to insight into ways in which learners construct proofs.

The process of constructing the proof image leads to some satisfaction for the learner. For example, Kidron and Dreyfus (2010a) noted how L's drive to understand the structure of the double solution at the bifurcation points was satisfied at least intuitively. More importantly, it is interesting to note that, for both, K and L, the process of constructing the proof image led to a need for more formal reasoning. K explicitly expressed this with his need for a formal definition of a function that tends to infinity. K needed the formal definition of the limit to ensure that there can be no strange examples, which disturb his proof image.

The need for more formal reasoning was explicitly expressed by L as well. Indeed, after the first degree of enlightenment or the flash of insight, expressed by the combining of constructions at the end of episode 10, the two constructions merge and L realized that her knowledge that there must be double solutions at bifurcation points was still intuitive. Her drive to understand the structure of the double solutions at bifurcation points was satisfied intuitively but her work was not finished. She was interested in a more formal explanation why the transition point from 2-period to 4-period is obtained by equating the discriminant to zero. As a consequence, in episode 11, she renewed and supported her quest for algebraic connections and wanted to know more about the term discriminant, especially what is the meaning of a discriminant of an equation of higher degree and how it can be computed. This was the first part of the transition from the proof image to the formal proof.

The need for more formal reasoning is exquisitely expressed in Nasar's (1998) description of Nash's thinking as a "bunch of intuitive threads that have to be woven together" (see above).

16.8 Concluding Remarks

In this chapter, we have introduced the notion of proof image as we see it in two cases: K's story about a function attaining its minimum and L's story about bifurcation. We found similarities between the two cases concerning the first part of the transition from proof image to formal proof. In both cases, there is a strong branch and a weak branch. The weak branch is vague in the beginning and is being elaborated during the episode under consideration in agreement with Davydov's view of abstraction (1990). This elaboration, which is a part of the process of constructing the proof image, is a constructing action that leads to a strengthening of the weak branch and, consequently, to the first part of the transition from proof image to formal proof. An obvious question is how specific is the role played by the dissymmetry between the strength of the two branches at stake in the emergence of the proof image to the two cases investigated in the present paper. Further research is required to answer this question.

We expect that in the near future, we will be in a position to add to the cases of K and L, taking into account other cases and other mathematical domains positioning order to enrich our description of the notion of proof image and of the transition from proof image to formal proof.

Transformational aspects are central during the two phases of constructing a proof. Transformational aspects during the phase of the emergence of a proof image have been discussed in the discussion of proof image above, especially in Sects. 16.6.7, 16.6.8, and 16.6.9. For example, we have pointed out and shown in the examples how various linkages between selected previous constructs appear as the proof image develops and hence is being continuously transformed. Transformational aspects are equally central in the phase of transition from proof image to formal proof discussed in the preceding section.

The transition from proof image to formal proof relates directly to the theme of this book—transformation—in the following sense: The learner who constructs a proof typically (at least this was the case for K and L) feels a need to transform her or his personal proof image into a communicable form of proof that can be presented to others, discussed in a classroom, and evaluated as to its power of conviction and mathematical validity. This transformation from proof image into a formal proof requires transformations in the learner's human cognition, just as the learning of concepts and strategies does. The transformations are likely to include changes of representation since the proof image often involves visual aspects and formal proof usually strives for independence from visual aspects, such as in the case of K. In summary, the transition from proof image to formal proof involves transformations of knowledge whose investigation is an important and promising avenue for further research.

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Chapter 17
Elementary Stochastic Seeing in Primary Mathematics Classrooms—Epistemological Foundation and Empirical Evaluation of a Theoretical Construct

Judith Stanja and Heinz Steinbring

17.1 Introduction

A central focus of our contribution is on the specific characteristics of an elementary access to "stochastic thinking" in primary mathematics teaching. We will elaborate that, right from the beginning, stochastic thinking is a complex human knowledge perception and requires adequate means and artefacts necessary for activating this scientific kind of thinking. As a central component of stochastic thinking, we concentrate on "making and understanding stochastic predictions" in elementary chance experiments. The essential question of investigation is how young students could modify their understanding about a prediction being simply right or wrong/true or false to a new conception of a stochastic prediction that implies a qualitative reference to a more or less probable occurrence or non-occurrence of a future event. This key idea is fundamental for the development of stochastic thinking, and as a basic component of mathematical thinking, it should be an early element of mathematical teaching in school.

In 2002, the working group "Stochastik" of the "Association of Didactics of Mathematics" (GDM) of German-speaking mathematics educators has approved a joint statement and recommendations concerning stochastics in school starting with the first grade at the primary level. According to the statement, stochastics should already be a part of primary mathematics based on the following arguments: Young students are confronted in their everyday life with stochastic phenomena, and the development of a stochastic general education is a fundamental and a long-range goal and therefore needs a propaedeutic treatment in primary teaching (see Engel 2002, pp. 75–83). Meanwhile, stochastics has been incorporated in the educational standards for the elementary school (see KMK 2004).

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Primary school children have heterogeneous experiences close to stochastics (as board games, dice, etc.) and in uncertain everyday situations that have a personal meaning (as birthday wishes) and where the child is confronted with an uncertain event and can only wait until the birthday to know what will happen, i.e. which wishes have come true.

The intention of this contribution is to make clear that children need an intervention with more structured stochastic situations and that they have to be provided with appropriate tools to develop a more differentiated perspective and more sophisticated interpretations of prognoses.

17.2 The Nature of Stochastic Knowledge and Stochastic Prognoses

We understand stochastics as part of mathematics integrating statistics and probability. Basically, this characterization of stochastics implies that it is seen as a kind of experimental science in which random experiments are performed and data are collected as well as elementary ideal models are developed for theoretically describing, evaluating and predicting the outcomes of the experiments. In this way, we get a dualistic formation of stochastics as the interplay between random experiments and elementary probability theory—with the conceptual counterparts of (relative) frequency and (relative) parts of probability (Laplace probability) (for further explanation, see Steinbring 1991, pp. 135–167).

Having this interplay of "random experiment ↔ elementary probabilistic model" as a first foundation we take into consideration the following epistemological perspective.

The experimental and the theoretical side of stochastic concepts are related to each other—they can be seen as explaining referential contexts for the other side. Elementary theoretical notions can gain explanations and understanding by reference to notions on the experimental side, and new notions on the experimental side can get specific stochastic interpretations with the help of concepts in the elementary probability theory. This dualistic use and reciprocal explaining of the reference of signs and symbols in stochastics can be explained with the help of the epistemological triangle (Steinbring 2009, see Fig. 17.1).

"The development of the probability concept can be used as a paradigmatic example for explaining essential features of the epistemological triangle. In the early history of the probability concept, the sign system is given by "fraction numbers" and an accompanying reference context is given by the "ideal die". Later in history, the reference context changed to "independent collectives" (cf. von Mises 1972), and the sign system to "limit of relative frequency". And, at the beginning of the twentieth century, the reference context changed to "stochastically independent/dependent structures" and the sign system to "implicitly defined axioms" (cf. Steinbring 1980). A central characteristic of this semiotic structure is the fact that the object or the reference context cannot be a fixed and definite point, but that it is interpreted by the learner more and more as a structural domain during the development

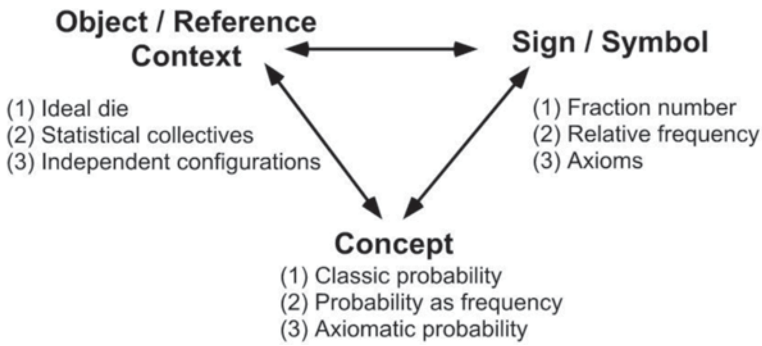


Fig. 17.1 The epistemological triangle applied to the concept of probability

of mathematical knowledge. Accordingly, mathematical meaning is produced in the interplay between a reference context and a sign system, by means of transferring possible meanings from a relatively familiar, or partly known, reference context to a new, still meaningless, sign system" (Steinbring 2009, pp. 24–25).

What makes stochastics different from other fields in school mathematics is the central meaning of application—as it is already basically inherent in the dualistic interplay of "random experiment ↔ elementary probabilistic model". Applications play an important role throughout the curriculum. Contrary to the academic field of probability theory, the conditions for application, such as independence, are mostly not considered in school mathematics. What makes stochastics so special is the relationship between the mathematical model and the part of reality being described by it. The modelling aims to describe a certain part of reality as good as possible, but right from the beginning there is a (never avoidable) gap between the model and the considered part of reality. Students often struggle with discrepancies between what probability theory tells them and the outcomes they observe in experiments.

In the case of random experiments, from a probability perspective, we try to describe experiments that have not been carried out yet. Of course, we cannot look into the future and say what will happen exactly—according to a deterministic world view. Still, we are able to make reasonable prognoses that are more or less certain and describe the outcomes of random experiments.

The outlined nature of stochastic knowledge can serve as orientation for primary school stochastics. We see stochastics in primary school as a preparation for later stochastic instructions and not as an antedated instruction of topics of the secondary level. This propaedeutics aims not primarily at the quantification of stochastic phenomena, nor should it at all be a course with axiomatic structure, that includes early calculations with however formalized probabilities (fractions, percentages, etc.). A course in primary stochastics should rather include adequate stochastic predictions of results of random experiments and the relationship between the prognoses and the outcomes. We see this as one possibility for a first approach in primary school to grasp the particularity of stochastic knowledge and to learn that there could be different qualities of what scientific knowledge can express. The importance of prognoses as part of instructional programmes was already outlined by Fischbein (1975). We give some examples of studies—without making claims of being complete—in which aspects of the understanding of prognoses may be found. We categorize these studies according to the role prognoses played such as:

1. Something that has to be constructed by the child,
2. Something that has to be justified or
3. The focus lies at the interplay between the prognoses and outcomes of an experiment.

For example, children have to predict single outcomes of random experiments with the highest or lowest probability (Way 2003, Nikiforidou and Pange 2010) or most likely events (Jones et al. 1997). In some studies, children are asked for predictions of frequencies of random events in symbolic forms as relative frequencies (mostly orally given for example Fischbein 1975) or absolute frequencies (symbolically for example Watson and Callingham 2003). The framework for assessing probabilistic thinking developed by Jones et al. (1997, 1999) includes prediction of events and their justification as one aspect in their stage model. Other studies focus on distributions instead of single events or probabilities. Thus, children have to predict the whole distribution (orally for example Abrahamson 2009; Bakker and Gravemeijer 2004). Several aspects of the interplay between prognoses and outcomes of an experiment can be found, for example, in the stage model of Jones et al. (1997, 1999) who state that a child can be assigned to the informal quantitative level (level 3 of 4) if the child is able to understand the differences between experimental and theoretical probability or as reported in Fischbein (1975) where experiments served for "verification" of predicted relative frequencies. A different attempt is carried out by Kazak and Confrey (2007) who let children predict, realize and interpret distributions of real objects as well as let them create iconic representations of these distributions antecedent to mathematical distributions.

In all of these studies, stochastic prognoses were not in the main focus. We are particularly interested in how students initially understand stochastic prognoses, and how they may develop their understanding. In our research about the development of children's emerging stochastic thinking, a decisive feature is taken into consideration, i.e. the completely different epistemological nature of stochastic knowledge vs. deterministic scientific knowledge. This central difference firstly appears for young students when being exposed to making prognoses for the outcomes of elementary random experiments.

17.3 Artefacts and Signs in the Learning of Stochastics

From a semiotic perspective on mathematics, signs and symbols are needed to make abstract notions accessible (see for example Duval 2008). Following Hoffmann (2003), thinking and reasoning in mathematics are not possible without signs. Therefore, they represent necessary tools for thinking and reasoning. Furthermore, with signs we can make abstract notions manipulable and communicable—so they can also be understood as tools for argumentation and the articulation of ideas.

Signs and symbols—in the general form we understand them here—also create the possibility to develop new ideas (e.g. representation of empirical distributions in diagrams) and build the possibility to adopt different perspectives. This corresponds to Sfard (2000), who states that introducing a new symbol is a crucial step to the generation of a concept and symbol and meaning constitute each other.

When introducing a new symbol, the artefact comes first. To become a symbol, it still has to be related to something it refers to. To understand this *transformation* from an artefact to a symbol, the following perspectives on artefacts might be helpful. First, we take the perspective that we look at the artefacts as *knowledge-oriented tools*. From this perspective, an artefact might be *modified* or *used* to do something. The structure of the associated mathematical knowledge produces additional constraints for the usage of the artefact as a sign, through the linkage of the meaning of the signs. Furthermore, a sign does not stand alone. There are relations to other signs—meaning the sign is located or embedded in a sign system. A second perspective might consider the *epistemological use* of the artefact. Here, the artefact may serve as a tool in order to explain something else (as part of the reference context) or it might be the artefact itself that has to be explained or interpreted (as part of a sign).

The outside appearance of mathematical signs is mostly and firstly conventional. However, having agreed on the signs to be used, in mathematics the signs and symbols then are the basis for constructing structures, patterns and relations.

In contrast to that, in stochastics, we often run the risk, that because of properties of the associated signs/symbols, we may successfully work on stochastic problems without any stochastic understanding—for example, when we use tree diagrams to solve Bayesian problems or fractions to determine or work with probabilities. Therefore, these sign systems allow us to use substituting strategies or to say more than we know (compare with Kazak and Wertsch 2005). For the teaching and learning of stochastics, as well as in research on stochastic thinking, we have to be aware of this fact, since it makes it difficult to see whether or not someone has a stochastic understanding. We stress that understanding stochastics means more than the mastery of the associated sign systems.

Both sides of the outlined interplay of "random experiment ↔ elementary probabilistic model" need many signs and symbols of different forms for coding the elementary stochastic concepts. In order to describe the outcomes of random experiments with spinners, one can record them in forms such as diagrams or lists, etc. In our study, we use templates (as shown in Figs. 17.2 and 17.3) that simplify the notation and that are adapted to the use in primary school (considering only absolute frequencies for example). These artefacts are used to code and interpret outcomes of experiments as well as to record prognoses.

Recording the outcomes of an experiment means to interpret the given artefact as a recording instrument and to modify it according to the outcomes. So, a symbol is created for each realisation of turns indicating the outcomes that have occurred. The way of recording the outcomes—the notation—allows different interpretations. If one records the outcomes in the form of a list, according to the order of their

Fig. 17.2 Template for a list

1	2	3	4	5	6	7	8	9	10

occurrence, one can use the list and read the information about the sequence of single outcomes, and one can further take the list as a database to study questions concerning the absolute or relative frequencies of the single events, patterns in the sequences, the length of runs, etc.

Compared to the data list, the diagrams aggregate data in different ways. So, the raw data are no longer available but this form of notation allows a direct comparison of the absolute frequencies of outcomes. However, the notations we described can be considered as abstract descriptions of the random experiments giving us information about the outcomes but ignoring other facts like who was carrying out the experiment and how this was done.

The understanding and use of such symbols could make it possible to compare the functioning of different random generators and maybe, later, to put aside the random generators used and consider the different distributions as new mathematical objects.

Bartolini Bussi et al. (2005) state that the presence of an artefact does not mechanically determine its use and understanding. In exploratory interviews, we could observe that the templates for diagrams, for example, were used in very different ways. In Fig. 17.4, we present the concrete usages of the given artefact in the recording contexts of the interviews. The third possibility is the one we will see again later in the example of Jule. In the prognosis context as well as in the experimenting context, single turns were coded by a cross or a hatched box. Sometimes, colours were used to indicate, additionally, the outcome of the turns.

In general, there were two different ideas for recording a prognosis that we could observe up to now during interviews. The first idea includes a mark for each single turn that would be carried out in an experiment. In the second, just the "final result" is recorded by marking the corresponding box counted from the bottom of the diagram. In both ideas, the global position of these markings (left/right) stands for the corresponding colour.

In recording outcomes, again, the global position indicates the outcome. All children started from the bottom of the template. However, there are differences in the local positioning of the marks. One idea was to mark the box above the preceding one for the next outcome. When a change of colour occurred, some children wanted to change the side and continue marking at the same height. Using this rule, one could reconstruct the sequence of the outcomes but the absolute frequencies could not immediately be read off. By contrast, a different idea of continuation when a change of colour occurred was to mark the next box above the last marked on that side. Using this strategy, one can easily read off the absolute frequencies, but the sequence of outcomes is no longer accessible.

When looking at the ways children recorded their prognoses and the outcomes of experiments, two things are striking. First, the spontaneous usages of the given

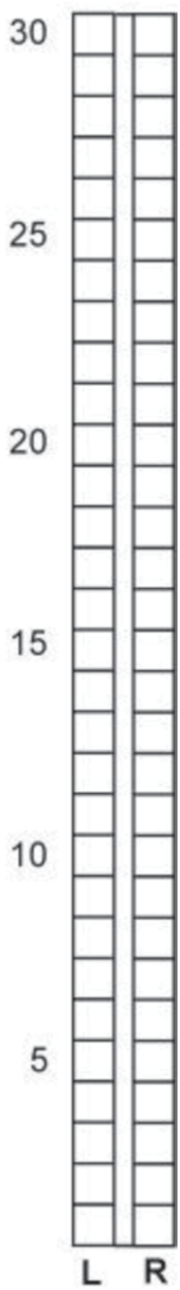


Fig. 17.3 Template for a diagram

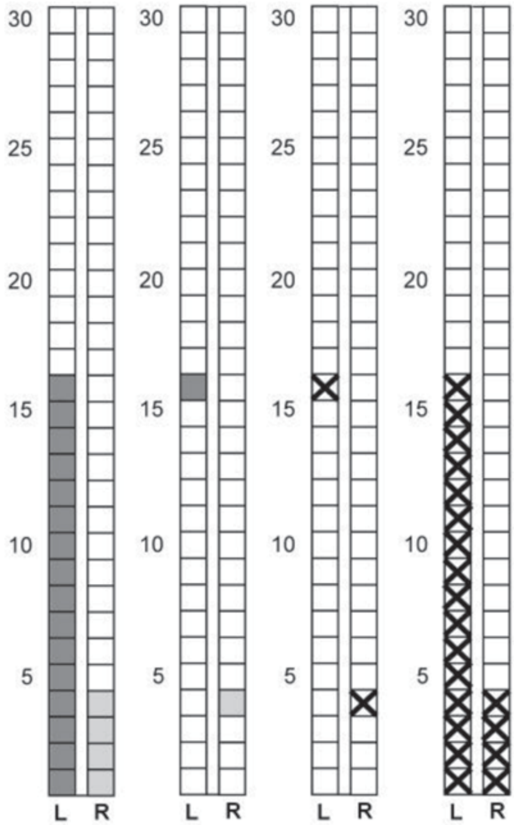


Fig. 17.4 Typical ways of recording a prognosis

artefact before any intervention might well differ from the intended usage and provide several advantages and disadvantages. This makes clear that the appropriate usage is something that has to be learnt by the child.

Second, and even more important, we can almost say nothing about the children's understanding of stochastic prognoses from their recorded prognosis alone. From the contexts of the interviews, we know that the same way of notation may be interpreted in various ways. This gives us a reason to design a qualitative study with interviews that allows us to gain more insight into the understanding of stochastic prognosis through the way children use and interpret recorded prognoses.

17.4 The Conception of Elementary Stochastic Seeing

At the beginning of this section we would like to explain what we mean by the expression *elementary stochastic seeing* (else) and how it might be described.

In general, *else* serves as an expression for a theoretical construct describing aspects of stochastic thinking in elementary school students. It differs from other descriptions in the way that we do not take a particular mathematical concept (as for instance distribution, expectation, etc.) into focus in order to describe which aspects of this concept are understood by a child. We also do not look at specific levels of quantification for these concepts. Instead, we take as an orientation the nature of stochastic knowledge and our understanding of mathematical thinking as *thinking with tools* to study elementary school students' thinking in connection with random experiments. What meaning do they find? What do they perceive of the stochastic situation? How do they understand stochastic prognoses?

For the development of elementary stochastic seeing, we emphasize that a stochastic prognosis and the actual outcomes of an experiment usually will not match exactly, and we investigate how one can deal with this fact as an important condition. Furthermore, we have already outlined the necessity of tools for thinking in mathematics (Hoffmann 2003; Sfard 2000) and the role of artefacts as sources for the construction of signs (material sign vehicles). Usually, children in primary school do not have appropriate tools or verbal means (see also Wollring 1993) yet to describe and study random phenomena.

For the development of an elementary sense of random phenomena, a child has to be introduced in an elementary stochastic culture with established ways of speaking and using/interpreting cultural tools. The enculturation would include the introduction to the usage of tools, ways of speaking and tasks, as well as examples of interpretations that serve as an orientating support and provide a frame that is a sound basis for the development of stochastic thinking. In order to cope with that demand, the development of appropriate materials and tasks is required.

How does the *elementary stochastic seeing* develop? How do the interpretations of the given artefacts change and what ideas do the children develop?

We designed a research project that consists of two parts: one part concerns the elaboration and empirical testing of the theoretical construct *else* describing stochastic thinking in primary school children and the other part concerns the design of materials that are an integral component of an intervention that will be carried out. Both parts interrelate with each other. In order to elaborate the theoretical construct, we design a qualitative study consisting of a first series of pre-interviews, an intervention and a second series of post-interviews. In the first series of interviews, prior to the intervention, we afford children with no or little pre-experience in stochastics an opportunity to make sense of random experiments. This series gives us some insight into spontaneous usages and interpretations of the given artefacts as spinners and diagrams. We assume, that young students' scientific comprehension at hand will not change on their own because of the complex and strongly different way of stochastic thinking and arguing; therefore, it becomes necessary to conduct an intervention that gives children the chance to develop their own *elementary stochastic seeing*. For this purpose, we design materials and tasks to work with in the third grade. Third-grade students have learnt in mathematics teaching the necessary elementary arithmetic (for instance, additive decompositions of natural numbers) and first geometric notions (as area and equal parts of area). These form a basis for understanding and using mathematical ideas for making connections between different semiotic representations that are a basis for stochastic interpretations. The intervention includes experimenting, provision with tools and language to describe and study random situations and tasks related to the interplay of prognoses and outcomes of experiments. The intervention aims at giving the students an orientation and opportunities to take possibly new perspectives.

The second series of interviews after the intervention gives us important information about changes in the way children use and interpret the given artefacts. This might serve as a contrast to the perspectives in the first series and it explores what is possible after the intervention. So, methodologically, this series serves as a basis for further development of the construct *else*. For the second series, we expect children to give a more detailed prognoses and more sophisticated justifications and hope that they develop first ideas concerning different qualities of knowledge.

With the help of the considerations of the previous sections and some exploratory interviews, we identified two dimensions to characterize *elementary stochastic seeing*—an epistemological dimension and a semiotic dimension. The epistemological dimension deals with questions such as: Do children think that it would be possible at all to say something about the outcomes of a random experiment and that they themselves are able to state something and, if so, what do they say? What do they use for a valid *justification* of the stated prognoses? How do they evaluate prognoses presented to them? These questions could be connected to the children's conception of knowledge and ways of knowing. Therefore, it might be stimulating to look more closely at the *epistemological beliefs* literature to further elaborate this dimension (see e.g. Hofer and Pintrich 1997).

The epistemological dimension reaches from a *dichotomic* to a *relativizing* perspective. A child is said to take a dichotomic perspective if she/he just considers whether the prognosis is right or wrong—say whether there is a perfect match between a prognosis and the outcomes. With a relativizing perspective, a child considers a prognosis as quite good or bad with respect to the outcomes of a random experiment having possible deviations of the given prognosis in mind.

The semiotic dimension concerns the interplay between recorded empirical observations of random experiments and symbolical interpretations of artefacts of elementary stochastics, such as spinners and diagrams at the elementary theoretical level. It deals with questions such as: How do children interpret recorded outcomes and prognoses and what relations do they construct? What interpretations do young students make of the different sorts of the broad types of signs, symbols and diagrams? Thus, the semiotic dimension is about the original semiotic understanding: investigating the student's idea about what the sign/symbol he/she uses stands for, or to what signified a used signifier refers. An example given in the following section shall further illustrate how we understand *elementary stochastic seeing*.

17.5 Example from the Exploratory Phase of the Project

In 2010, we conducted eight exploratory interviews with four third graders from each of two schools (4, 4). In both schools, the children, chosen by their teacher, usually have different achievements in mathematics. All children had no or almost

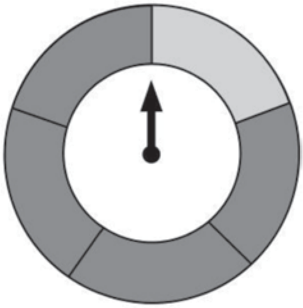


Fig. 17.5 Spinner

no experiences with stochastics in school. These exploratory interviews are not part of the actual study. Their purpose was twofold. On the one hand, we wanted to explore by which means we could investigate the children's (partly spontaneous) perspectives and understanding of prognoses. Apparently, it is useful to make a distinction between phenomenological and conceptual understanding. Children at this age, and when confronted with this subject, might have difficulties to articulate their understanding. For in going beyond a pure phenomenological understanding to a more conceptual comprehension, they need appropriate artefacts and semiotic tools. On the other hand, we evaluate our materials and tasks for further development.

For this contribution, we focus on the following question: How does a child use the given artefacts and how does he/she interpret them?

17.5.1 Setting of the Interview

The interview was carried out without a fixed guide but orientating questions and problems concentrating on the key idea of *elementary stochastic seeing*. In the following, we describe the artefacts that have been used in a twofold manner—in the prognosis and the experiment contexts in the interview.

17.5.2 Artefacts

(1) Spinner We take a "good" random generator with symmetric properties (see Fig. 17.5). With a "good" random generator, we mean a generator that is as unbiased as possible. The spinner here is understood not only as the object that has to be described but also as a tool for thinking and learning. Thus, the conditions for correctly applying mathematical theory are not an issue here. Due to the assumption, that the children do not have a mathematical notion of distribution and the various types of distributions, and for reasons such as length of an experiment and the relation between the empirical and theoretical distribution, we chose to use a "non-equal-distributed" random generator.

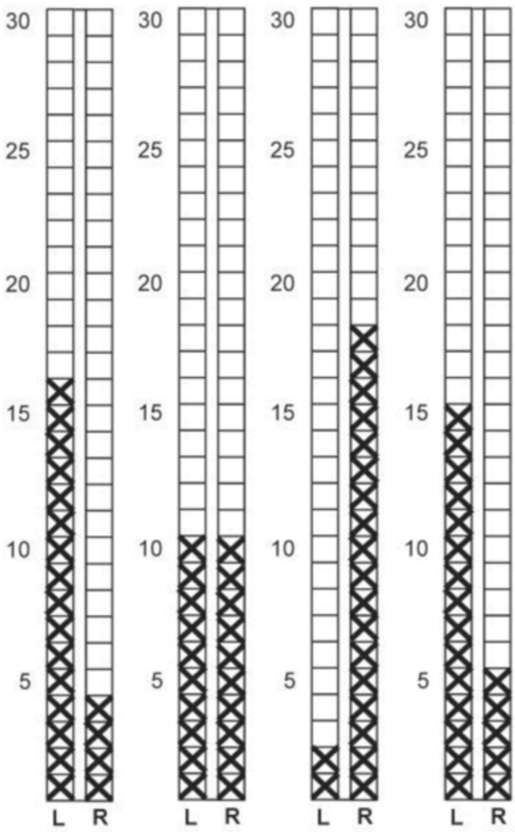


Fig. 17.6 Diagrams

(2) Diagrams We use empty templates and filled diagrams as shown in Figs. 17.2 and 17.6 (where L corresponds to blue and R corresponds to yellow). These diagrams are adapted for the use in primary school: the filling in and recording is facilitated by the use of boxes, the scale is given and absolute frequencies can be easily read off. (Though we know that this is not always done right from the beginning.) The diagram serves as a tool in two different ways. Firstly, the diagram functions as a tool to record prognoses. The idea is that the child has to give a precise prognosis in a definite way in order to get away from pure guessing and to make the prognosis more accessible for justifications, discussions and comparisons with actual outcomes of random experiments and the construction of the spinner. Secondly, we use it as a protocol for outcomes of random experiments. The child records outcomes in a certain way, the absolute frequencies of the events “yellow” and “blue” become “visible” and the raw data of the experiment are no longer available to the child. We offer the child a possibility to look at random experiments in a certain way—what aspects are considered in what ways. The protocol helps to “memorize” how often each colour occurred and how often the spinner was turned. The filled diagram might serve as a basis for interpreting the experiments and as a source for one's own prognoses for experiments.

The filled diagrams chosen for the interview include an ideal prognosis, one good prognosis close to the ideal prognosis and two bad prognoses—one including the idea of fairness and one opposite to the ideal prognosis.

17.5.3 Course of the Interview

In the introduction phase, the interviewer shows the spinner and asks the child if the he/she already has experiences with spinners. Both the interviewer and the child take a close look at the spinner and, if necessary, the interviewer explains how the spinner works.

The main phase of the interview starts with an oral prediction of how often blue and yellow would occur if one would turn the spinner 20 times. The child is asked to justify his/her prediction and to give an evaluation of the prediction in the sense of saying whether the results of an experiment would be exactly like the prediction. Before the child carries out any experiment (of several trials), he/she is asked to record the prediction in an empty diagram.

The child carries out one or two experiments and records the results in an empty diagram template. It might occur during the first experiment wherein the child tries to memorize every single result without recording and just records the final results. In that case, children usually realize that it is difficult to memorize every single result and to know when the experiment is done at the same time.

After the child has carried out the experiment, he/she has to compare the outcomes and the prediction and give a new prognosis.

In a next step, the interviewer introduces already-filled diagrams as ideas for prognoses of other children and lets the child evaluate them. The last task in the main part of the interview considers the question: with what spinner is it possible to get what outcome, and a justification is requested for each choice or non-choice.

At the end of the interview, the problem context is changed to a game context with two imagined players, that is still closely related to the experiment situation before. The child is asked to predict the winner in one and in more rounds of playing and to give a justification. Then the interviewer asks what colour the child would choose if he/she would play the game.

17.5.4 Jule

The following example is part of an interview carried out in the first half-year of the third grade. Jule is a 9-year-old girl. According to her teacher, she has good marks in mathematics. In her class, the only experience with stochastics is a teaching unit at the end of the second grade. The topic of this unit was the empirical distribution of the sum of the outcomes of two dice.

The chosen episode starts just after the beginning of the interview. The interviewer already introduced the spinner, they took a close look and the interviewer explained how it works. Jule tried to turn the spinner. The proportion of blue and yellow was not a subject of the introduction. The spinner remains on the table in front of Jule during the following episode. Now the interviewer asks Jule to make a prognosis for an experiment.

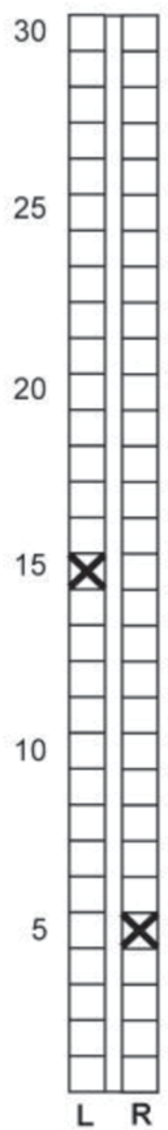
1 I: Now, imagine you would turn the spinner 20 times, just as you did.
2 J: Mhm.
3 I: What do you believe, how often will the pointer *(points at the pointer of the spinner)* (Fig. 17.5) stop at blue and how often will it stop at yellow?
4 J: Hmm. I guess[a]…(..) Hmm.(..) I guess[b] 15 times at blue and 5 times at yellow.
5 I: Mhm. What gives you that idea? What did you think?
6 J: I was just guessing.
7 I: Mhm. And—eh, if you would carry out this/such an experiment. If we would try and turn 20 times, does it come out exactly like that? What do you think?
8 J: Well, it could be not quite right now, I think. But maybe so (…) so in this piece, such that it is only one, two numbers away.
9 I: What do you mean by in this piece?
10 J: Well, that it would not lie at for example…15 times at yellow and 5 times at blue now. That this is not the other way around now, or something like that. I believe…, that it is also, eh, close.
11 I: Close? Is this, eh, close to what?
12 J: To the outcomes I've said.
13 I: Mhm. So 15 times blue and 5 times yellow.
14 J: Mhm.

[a] German: "schätze"
[b] Rate

The interviewer gives Jule an empty diagram and asks her to record her prognosis. Jule first shows how she would record her prognosis by pointing at the 15th box at the left side and the 5th box at the right side and then fills the empty diagram (Fig. 17.7).

15 I: Now, you've already said, that if we would carry out an experiment, then it comes about that area. Could you explain this to me again with the picture, how did you mean that? (*points at the diagram*)
16 J: Well, in the area of the 15 I mean about till the 10 or until the 20. (*points at the area between 10 and 20 at the left side*) That this is not full (*points at 30 and pauses*), full, eh…(..) That blue would not come out like 3 times now. *(marks the area between 10 and 20 with two fingers of her left hand*)
17 I: Mhm. Why wouldn't blue come 3 times?
18 J: This is what I think, because (…) there is plenty of blue (*points with a finger at the blue area of the spinner*) and yellow is just a little field. (*taps the yellow field*)
19 I: Mhm. So you mean that's why this is/has to be like that.
20 J: Yes, I think so.

Fig. 17.7 Jule's prognosis



17.5.4.1 Detailed Analysis of Lines 1–14

The interviewer asks for a prediction for 20 turns of the spinner (lines 1 and 3). Jule thinks for a while and answers in line 4: "Hmm. I guess[1]…(..) Hmm.(..) I guess[2] 15 times at blue and 5 times at yellow." She may have thought about decompositions of the number 20 and/or taking the single turns and/or the construction of the spinner as a basis for an estimation or calculation. Up to that point, there is no evidence for one or the other interpretation. From the transcript, we can only understand that she is thinking about something but not how she gets to her prognosis or how she understands it. To the question what she was thinking about her answer is "I was just guessing" (line 6). It might be that Jule means this as justification or it may be an expression of not being able to give a justification. How is "guessing" in lines 4 and 6 understood? It is possible that "guessing" is used as a description of uncertainty whereby this may relate to her answer or to the actual occurrence of her predicted outcome. There is also the possibility that she has difficulties in articulating her thoughts. In line 7, the interviewer asks Jule to interpret her prognosis in relation to a hypothetical experiment. Jule takes a relativizing point of view by saying that "it could be not quite right now", but "maybe so (…) so in this piece, such that it is only one, two numbers away" (line 8). It remains unclear whether she refers to her given answer or to hypothetical outcomes of an experiment. The expression "not quite right" could be a hint for a non-dualistic conception of knowledge. The usage of "maybe" reinforces the impression of uncertainty. The meaning of "in this piece" is questioned by the interviewer. Here, Jule refers to her prognosis and gives a counter-example ("15 times at yellow and 5 times at blue") and an additional explanation "that it is also, eh, close" and more precise "to the outcomes I've said" (lines 10 and 12). By "outcomes", she may mean just her prognosis or include the deviations she spoke about in lines 8 and 10.

Then Jule had to record her prognosis. For that, she had to interpret the empty diagram and modify the artefact by entering markings according to her prognosis. Jule records her prognosis by marking the 15th box from the bottom at the left side for 15 times blue and the 5th box from the bottom at the right side for 5 times yellow each with a cross.

17.5.4.2 Detailed Analysis of Lines 15–20

Jule is asked to explain what she meant with "in that area" with the help of the diagram (line 15). Here, the word "area" is introduced by the interviewer and Jule picks it up in her following explanation (line 16). She interprets "in the area of 15" by referring to her recorded prognosis. She says "about till the 10 or until the 20", "not full" and "blue would not come like 3 times" and points with her finger at the area between the number 10 and 20 at the left side of the diagram. This indicated

[1] German: "schätze".

[2] German: "rate".

area refers to a larger deviation than the one she talked about in the first part of the episode ("one, two numbers away", line 8). The words "not full" can be interpreted in different ways. It is possible that Jule is referring to the whole diagram as an area for possible outcomes and then restricts this area to make clear that not every outcome comes into question here.

The second possibility would be that the number 30 is given as a counter-example—as one outcome outside the mentioned area between 10 and 20. The given counter-example "blue would not come like 3 times" is consistent with both interpretations. For the first possibility, Jule refers to the area that ranges not till 3, for possibility 2 Jule gives a counter-example outside the mentioned area. When asked why "blue would not come like 3 times" she refers to the size of the blue and yellow area of the spinner indicating that "there is plenty of blue" and "yellow is just a little field". These verbal expressions are combined with pointing and tapping at the respective areas (line 18).

Jule is already taking a rather relativizing perspective while giving her prognosis. In the presented episode, we can see that she is having possible deviations of her prognosis in mind. Jule also shows aspects of a theoretical interpretation of the diagram referring to other possible outcomes.

17.5.4.3 Analysis by the Role of the Artefacts

In what ways do artefacts play a role in the presented episode? In our episode, we have two artefacts to consider—the spinner and the diagram. The spinner is not obviously used until the end of the episode where Jule refers to it in order to explain why "blue would not come out like 3 times". The diagram instead is used in several ways. It first appears as an empty given form that has to be used in order to record the prognosis. Now, the artefact has to be interpreted itself—where and how may the recording be done? Jule marks the 15th box at the left side and the 5th box at the right side with a cross and so modifies the given diagram. In what follows, the diagram is used for explaining the possible deviation from the given prognosis ("the area of 15").

From an epistemological perspective, we could, as an example, analyse Jule's statement (line 16) in the following way: the interviewer questions the meaning Jule presumes for the notion of "area", "Could you explain area to me again with the picture" (line 15). In this way, a new symbol/sign is questioned and the new symbol "area" has to be explained. In her statement (line 16) Jule gives at least four aspects of explanation. Further, these explanations are not restricted to concrete properties, but Jule characterizes important stochastic relations and connections. With the help of the epistemological triangle we can argue that the diagram with the blue and yellow columns of 30 boxes, in which Jule has marked with a cross the 15th blue and the 5th yellow box, serves Jule in this short interaction as a reference context in which she brings in new views. The new sign/symbol "area" the interviewer wants to be explained is constructed by Jule with the help of own descriptions and gestures: area is the space around 15, from 10 to 20, and it is not up to 30 nor near

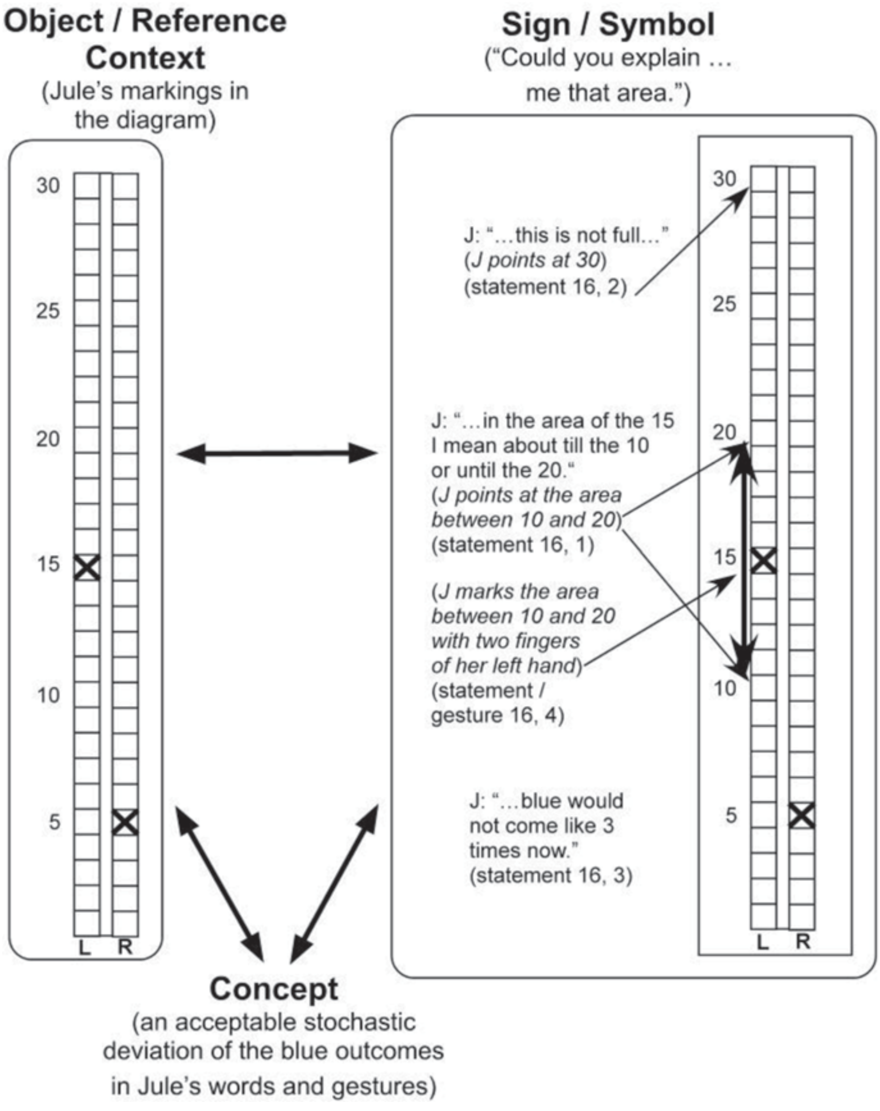


Fig. 17.8 Epistemological triangle for analysing the notion of "Area"

to 3. In this way, Jule expresses in her own communicative manner an acceptable stochastic deviation for possible outcomes of the experiment with the spinner in question. This deviation is an essential element for the new requirement of understanding and of developing an idea of *stochastic* prognosis.

The epistemological triangle (see Fig. 17.8) shows the essential epistemological relations and connections Jule constructs for explaining the symbol "area" (line 16).

It is indeed a productive epistemological perspective because Jule does not simply use the diagram as a protocol with the given or fixed data, but she elaborates structures and relations in the diagram in the sense of a symbolic understanding of this artefact.

In summary, Jule gives a prognosis close to the ideal one and verbalizes ideas of deviation from her given prognosis. She creates a sign for her prognosis through modification of the artefact of the template for a diagram. The modified template—the diagram—is used by her to clarify her idea of possible deviations from the prognosis. She gives a counter-example for an outcome that is not possible and justifies it with reference to the spinner comparing the size of the areas in a qualitative manner.

17.6 Final Remarks

What can we learn from the interview with Jule and examples of the other exploratory interviews?

From the presented episode, we mostly get information about the prognosis Jule gives and how she justifies it. We also get a first hint for her understanding of the prognoses. Jule is an example of a child who is surprisingly successful in the spontaneous usage of the new artefacts and semiotic tools. Here, the setting of the interview gives occasions for her to develop productive ideas for dealing with the relation of her prognosis and hypothetical outcomes of a random experiment. She is able to express aspects of *elementary stochastic seeing* with her own words in quite a good way.

However, from other interviews we know that this is not always the case. These interviews show that *elementary stochastic seeing* does not normally arise spontaneously and that an intervention offering children a possibility to enter and participate in the stochastic culture is a necessary condition for the development of stochastic thinking. The example of Jule helps to give an idea of what should be the aim of an intervention supporting the development of *elementary stochastic seeing*.

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Chapter 18
Understanding Geometric Work through Its Development and Its Transformations

Alain Kuzniak

18.1 Introduction

The influence of tools, especially drawing tools, on geometry development at school has recently improved greatly due to the appearance of dynamic geometry software (DGS). As Straesser (2001) suggested, we need to think more about the nature of geometry embedded in tools, and reconsider the traditional opposition between the practical and theoretical aspects of geometry. It is well-known that we can approach geometry through two main routes:

1. A concrete approach which tends to reduce geometry to a set of spatial and practical knowledge based on the material world.
2. An abstract approach oriented towards well-organized discursive reasoning and logical thinking.

With the social cynicism of the bourgeoisie in the mid-nineteenth century, the first approach was for a long time reserved for children coming from the lower class and the second was introduced to train the elite who needed to think and manage society.

Today, in France, with the "college unique", this conflict between both approaches stays more hidden in mathematics education, but such discussions have reappeared with the social expectation supported by the Organisation for Economic Co-operation and Development (OECD) and its "bras armé" Programme for International Student Assessment (PISA) with the opposition between "mathematical literacy" and "advanced mathematics".

The present paper leaves aside sociological and ideological aspects and focuses on what could be a didactical approach, keeping in mind a possible scientific approach to a more practical geometry referring to approximation and measure, in the sense Klein used when he suggested a kind of approximated Pascal's theorem on conics:

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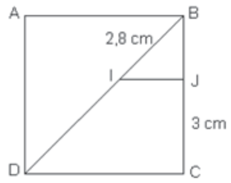


Fig. 18.1 Problem from French examination at grade 9 in 1991

> Habe ich sechs Punkte, die ungefähr auf einem Kegelschnitt liegen, ziehe deren ungefähre Verbindungsgeraden und bringe diese in a, b, c zum Schnitt, dann liegen diese Punkte ungefähr auf einer geraden Linie[1] (Klein 1903, III, p. 172).

The present discussion is supported by a first example showing what kind of contradiction exists in the French education system where no specific work on approximation exists during compulsory school. This contradiction appears as a source of confusion and misunderstandings between teachers and students. We were led to introduce some theoretical perspectives aiming at understanding and solving this issue. In the following, our theoretical framework for studies in geometry is introduced and is used to launch some perspectives.

18.2 Complexity of the Geometric Work

Mathematical domains are constituted by the aggregation and organization of knowledge. As Brousseau (2002) emphasized, this organization will not inevitably be the same as the actual implementation in a classroom. A mathematical domain is the object of various interpretations when it is transformed to be taught. These interpretations will also depend on school institutions. The case of geometry is especially complex at the end of compulsory school, as we show in the following.

The following problem was given for the French examination at grade 9 in 1991 and was used in a study we conducted (Houdement and Kuzniak 2003a).

Construct a square ABCD with side 5 cm Fig. 18.1.

1. Compute BD.
2. Draw the point I on [BD] such that BI = 2.8 cm, and then the point J on [BC] such that JC = 3 cm.
   Is the line (IJ) parallel to the line (DC)?

The intuitive evidence (the lines are parallel) contradicts the conclusion expected from a reasoning based on properties (the lines are not parallel). Students are faced

[1] Let six points be roughly located on a conic: if we draw the lines roughly joining points and they intersect at a, b and c, then these points are roughly aligned.

with a variety of tasks referring to different, somewhat contradictory, conceptions and the whole forms a fuzzy landscape:

1. In the first question, a real drawing is requested. Students need to use some drawing and measure tools to build the square and control and validate the construction.
2. Students then have to compute a length BD using the Pythagorean theorem and not measure it with drawing tools. However, which is the nature of the numbers that students have to use to give the result: An exact value with square roots or an approximate one with decimal numbers that is well adapted to using constructions and that allows students to check the result on the drawing?
3. In the third question—are the lines parallel?—students work again with constructions and have to place two points (I and J) by measuring lengths. Moreover, giving the value of 2.8 could suggest that the length is known up to one digit and could encourage students to use approximated numbers rounded to one digit. In that case, $\sqrt{2}$ is equal to 1.4 and both ratios are equal, which implies the parallelism by Thales' theorem (Strahlensatz) related to similarity. If students keep exact values and know that $\sqrt{2}$ is irrational, the same theorem implies that the lines are not parallel.

With grade 9 students The problem was given in a grade 9 class (22 students), 1 week after a lecture on exact value with square roots and its relationships to length measurement. After the students had spent 30 min working on the problem, half of them answered that the lines were parallel and the other half answered that they were not. On the teacher's request, they used the problem of approximated values to explain the differences among them. At the teacher's invitation, they started again to think about their solutions. At the end, 12 concluded the lines were not parallel, eight that they were and two hesitated.

Indeed, after studying their solutions and their comments on the problem, we can conclude that students' difficulties did not generally relate to a lack of knowledge on geometric properties but to their interpretations of the results. They had trouble with the conclusions to be drawn from Thales' theorem. Even after discussion, students expressed their perplexity about the result and its fluctuation. One student said, "I don't know if they are parallel for when I round off, the ratios are equal and so the lines are parallel, but they are not parallel when I take the exact values." For students, one answer is not more adequate than another. This gives birth to a geometric conception where some properties could be sometimes true or false. How to make students overcome the contradiction? A first possibility is to force the entrance into the didactical contract expected by the class-teacher, who explained us that at this moment in grade 9, it must be clear that "a figure is not a proof."

Working on approximations and thinking about the nature of geometry taught during compulsory school open a second way that we explore with geometrical paradigms in the following.

18.3 Geometrical Paradigms and Three Elementary Geometries

The previous example and numerous others of the same kind show that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend on both the evolution of mathematics and of school institutions. At the same time, we saw that students are strongly disturbed by this diversity of approaches. Geometrical paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Houdement and Kuzniak 1999, 2003b).

The idea of geometrical paradigms was inspired by the notion of paradigm introduced by Kuhn (1962) in his work on the structure of scientific revolutions. In a global view, one paradigm consists of all the beliefs, techniques and values shared by a scientific group. It indicates the correct way for putting and starting the resolution of a problem. Within the restricted frame of the teaching and learning of geometry, our study is limited to elementary geometry, and the notion of paradigm is used to pinpoint the relationships between geometry and belief or mathematical theories.

With the notion of paradigms, Kuhn has enlarged the idea of a theory to include the members of a community who share a common theory.

A paradigm is what the members of a scientific community share, and a scientific community consists of men who share a paradigm (Kuhn 1962, p. 180).

When people share the same paradigm, they can communicate very easily and in an unambiguous way. By contrast, when they stay in different paradigms, misunderstandings are frequent and can lead, in certain cases, to a total lack of comprehension. For instance, the use and meaning of figures in geometry depend on the paradigm. Sometimes it is forbidden to use a drawing to prove a property by measuring and only heuristic uses of figures are allowed.

To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms.

18.3.1 Geometry I: Natural Geometry

Natural geometry has the real and sensible world as a source of validation. In this geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and all arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs are also allowed. The development of this geometry was historically motivated by practical problems.

The perspective of Geometry I is of a technological nature.

18.3.2 Geometry II: Natural Axiomatic Geometry

Geometry II, whose archetype is classic Euclidean geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms in order to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition, even if this definition is often based on some characteristics of real and existing objects.

Both geometries have a close link to the real world even if it is in different ways.

18.3.3 Geometry III: Formal Axiomatic Geometry

To these two approaches, it is necessary to add a third geometry (formal axiomatic geometry), which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach.

In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications in the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness. Moreover, axioms are organized into families that structure geometrical properties: affine, Euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspectives are different and so the nature and the handling of problems change from one to the next what is important here is the idea of three different approaches of geometry: Geometry I, II and III.

Back to the example If we look again at our example, students—and teachers—are not explicitly aware of the existence of two geometrical approaches to the problem, each coherent and possible. Moreover, students generally think within the paradigm which seems natural to them and close to perception and instrumentation, Geometry I. However, in this geometry, measurement is approximated and known only over an interval. The parallelism of lines depends on the degree of approximation. Teachers insist on a logical approach—Geometry II—which leads the students to conclude blindly that the lines are not parallel, against what they see.

It could be interesting to follow Klein's ideas and introduce a kind of "approximated" theorems, more specifically here an "approximated" Thales' theorem: If the ratios are "approximately" equal, then the lines are "almost" parallel. In that case, it would be possible to reconcile what is seen on the drawing and what is deduced based on properties.

Developing thinking on approximation in geometry can be supported by DGS, which favours a geometric work into Geometry I, but with a better control of the degree of approximation. It is the case, for instance, with the CABRI version we used during the session with students. In this version, an "oracle" is available which can confirm, or not, the validity of a property seen on the drawing. Here, the parallelism of both lines was confirmed by the "oracle" according to the approach of the problem based on approximation.

Many problems allow discussion of the validity of a theorem or property in relationship to numerical fields. For instance, the CABRI oracle asserts that (EF) and (BC) are parallel lines in a triangle ABC when E and F are respectively defined as the midpoint of [AB] and [AC]. However, if E is defined as the midpoints of [AB], when we drag a point F on [AC], it is possible that CABRI oracle never concludes that (EF) and (BC) are parallel for any position of F. These variations in the conclusion need an explanation and provoke a discussion among students, which can be enriched by the different perspectives on geometry introduced by geometrical paradigms.

To discuss the question in depth and think about new routes in the teaching and learning of geometry, we introduce some details about the notion of the geometric work space (GWS; Kuzniak 2008, 2010; Kuzniak and Rauscher 2011).

18.4 The Notion of GWS within the Framework of Didactics of Geometry

At school, geometry is not a disembodied set of properties and objects reduced to signs manipulated by formal systems: It is at first and mainly a human activity. Considering mathematics as a social activity that depends on the human brain leads to understanding how a community of people and individuals use geometrical paradigms in the everyday practice of the discipline. When specialists try to solve geometric problems, they go back and forth between the paradigms and they use figures in various ways, sometimes as a source of knowledge and, at least for a while, as a source of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each one of these conclusions.

When students perform the same task, we are not sure about their ability to use the knowledge and techniques related to geometry. That requires an observation of geometric practices set up in a school frame, and, more generally, in professional and everyday contexts, if we aim to know common uses of mathematics tools. The whole work is summarized in the notion of *GWS,* a place organized to enable the work of people solving geometric problems. They can be experts (the mathematicians) or students or senior students in mathematics. Problems are not a part of the work space but they justify and motivate it.

Architects define work spaces as places built to ensure the best practice of a specific work (Lautier 1999). To conceive a work space, Lautier suggests thinking of it according to three main issues: a material device, an organization the designers are responsible for, and finally a representation which takes into account the way the users integrate this space. We do not intend to take up this structure oriented to productive work without any modifications, but it seems to us necessary to keep in mind these various dimensions, some more material and the others intellectual.

To define the GWS, two levels connected to each other have been introduced: the epistemological level and the cognitive level.

18.4.1 The Epistemological Level

Geometrical activity, in its purely mathematical dimension, can be characterized by three components. These three interacting components are the following ones:

- a real and local space as a material support with a set of concrete and tangible objects;
- a set of artefacts such as drawing instruments or software; and
- a theoretical frame of reference based on definitions and properties.

These components are not simply juxtaposed but must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name *epistemological plane* given to this first level. In our theoretical frame, the notion of paradigms brings together the components of this epistemological plane. The components are interpreted through the reference paradigm and, in return, through their different functions, the components specify each paradigm. When a community can agree on one paradigm, they can then formulate problems and organize their solutions by favouring tools or thought styles described in what we name the reference GWS. To know this GWS, it will be necessary to bring these styles out by describing the geometrical work with rhetoric rules of discourse, treatment and presentation.

18.4.2 The Cognitive Level

We introduced a second level, centred on the cognitive articulation of the GWS components, to understand how groups, and also particular individuals, use and appropriate the geometrical knowledge in their practice of the domain. From Duval (2005), we adapted the idea of three cognitive processes involved in geometrical activity:

- a visualization process connected to the representation of space and material support;
- a construction process determined by instruments (ruler, compass, etc.) and geometrical configurations; and
- a discursive process which conveys argumentation and proofs.

From Gonseth (1945–1952), we retained the idea of conceiving geometry as the synthesis between different modes of knowledge: intuition, experiment and deduction (Houdement and Kuzniak 1999).

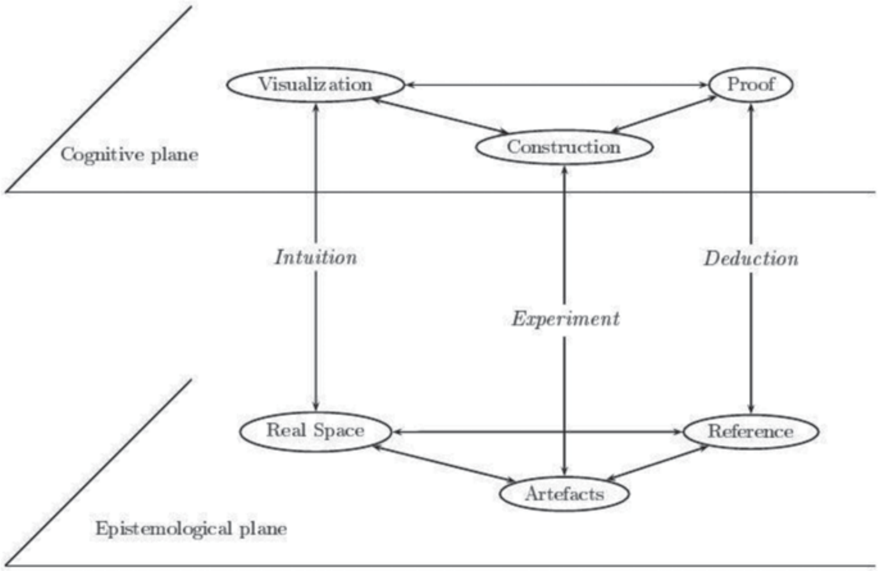


Fig. 18.2 The geometric work space

The real space will be connected to visualization by intuition, artefacts to construction by experiment, and the reference model to the notion of proof by deduction. This can be summarized in Fig. 18.2.

18.5 Building a GWS: A Transformation Process

18.5.1 On the Meaning of Genesis

In the following section, we will consider the formation of a GWS by teachers and students within the educational system. Our approach aims to better understand the creation and development of all components and levels shown in Fig. 18.2. The geometric work will be considered as a process involving creation, development and transformation. The whole process will be studied through the notion of genesis, used in a general meaning which focuses not only on the origin but also on the development and transformation of interactions. Through the transformation process, a structured space, the GWS is formed.

18.5.2 Various GWS Levels

In a particular school institution, the resolution of geometric tasks implies that one specific GWS has been developed and organized well to allow students to enter into the problem-solving process. This GWS has been termed *appropriate* and the *appropriate* GWS needs to meet two conditions: it enables the user to solve the problem within the right geometrical paradigm, and it is well built, in the sense in which its various components are organized in a valid way. Here, the designers play a role similar to architects conceiving a working place for prospective users. When the problem is put to an actual individual (young student, student or teacher), the problem will be treated in what we have termed a personal GWS. The geometric work at school can be described according to three GWS levels: geometry intended by the institution is described in the reference GWS, which must be fitted out in an appropriate GWS, enabling an actual implementation in a classroom where every student works within his or her personal GWS.

18.5.3 Various Geneses of the GWS

As we have seen, geometrical work is framed through the progressive implementation of various GWSs. Each GWS, and specifically the personal GWS, requires a general genesis which will lean on particular geneses connecting the components and cognitive processes essential to the functioning of the whole GWS. The epistemological plane of the GWS needs to be structured and organized through a process oriented by geometrical paradigms and mathematical considerations. This process has been named "epistemological genesis". In the same way, the cognitive plane needs a cognitive genesis when it is used by a generic or particular individual. Specific attention is due for some cognitive processes such as visualization, construction and discursive reasoning.

Both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. These three fundamental geneses relate to three kinds of transformation (Fig. 18.3) that occur in this process:

- an instrumental genesis which transforms artefacts into tools within the construction process;
- a figural and semiotic genesis which gives the tangible objects the status of operating mathematical objects; and
- a discursive genesis of proof which gives a meaning to properties used within mathematical reasoning.

We will examine how it comes into geometrical work by clarifying each genesis involved into the process.

On figural genesis The visualization question came back recently to the foreground of concerns in mathematics and didactics after a long period of ostracism

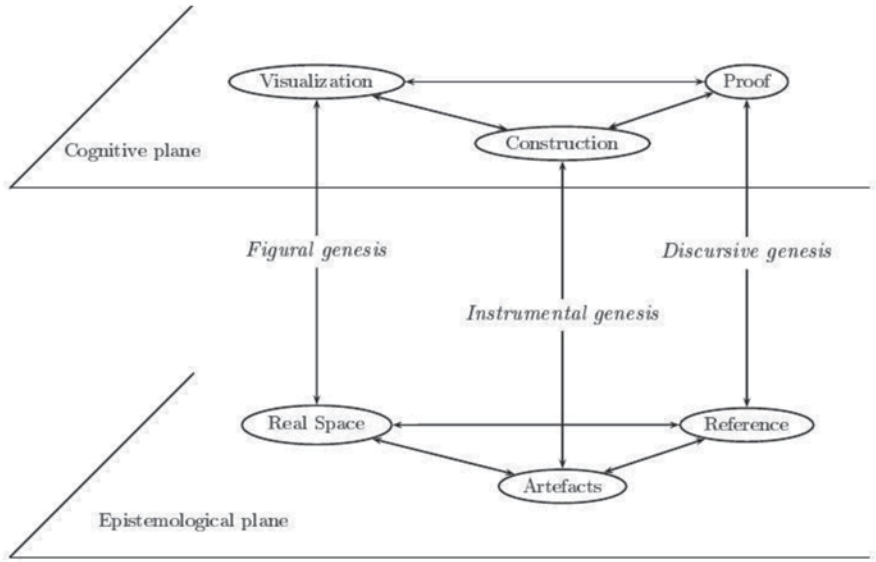


Fig. 18.3 Geneses into the geometric work space

and exclusion for suspicion. In geometry, figures are the visual supports favoured by geometrical work. This led us, in a slightly restrictive way, to introduce a figural genesis within the GWS framework to describe the semiotic process associated with visual thinking and involved in geometry. This process has been especially studied by Duval (2005) and Richard (2004). Duval has given some perspectives to describe the transition from a drawing seen as a tangible object to the figure conceived as a generic and abstract object. For instance, he spoke of a biologist's viewpoint when it is enough to recognize and classify geometric objects such as a triangle or Thales' configurations often drawn in a prototypical way. He also introduced the idea of dimensional deconstruction to explain the visual work required on a figure to guide the perceptive process. In that case, a figure needs to be seen as a two-dimensional (2D) object (a square as an area), a set of 1D objects (sides), or 0D objects (vertices). Conversely, Richard stresses the downward[2] process from the abstract and general object to a particular drawing.

On instrumental genesis A viewpoint on traditional drawing and measuring instruments depends on geometrical paradigms. These instruments are usually used for verifying or illustrating some properties of the studied objects. The appearance of computers has completely renewed the question of the role of instruments in mathematics by facilitating their use and offering the possibility of dynamic proofs. This aspect is related to the question of proof mentioned in the preceding paragraph,

[2] In the following, upward and downward refer to the diagram and do not have a positive or negative meaning.

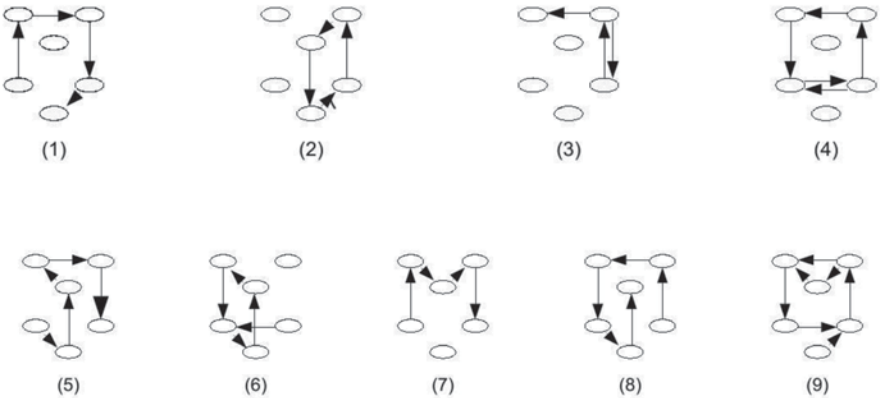
but the ability to drag elements adds a procedural dimension which further increases the strength of proof in contrast to the static perception engaged in paper and pencil environments. However, students do not master easily the use of artefacts. At the same time, teachers need to develop specific knowledge for implementing software in a classroom. Based on Rabardel's works on ergonomics, Artigue (2002) stressed the necessity of an instrumental genesis with two main phases that can be inserted in our frame. The upward transition, from the artefacts to the construction of geometric configurations, is called instrumentation and describes users' manipulation and mastery of the drawing tools. The downward process, from the configuration to the adequate choice and the correct use of one instrument, is related to geometric construction procedures and is called instrumentalization. In this second process, geometric knowledge is engaged and developed.

On discursive genesis of reasoning The geometrization process, which combines geometric shapes and mathematical concepts, is central to mathematical understanding. We saw the strength of images or experiments in developing or reinforcing certainty in the validity of an announced result. However, how can we make sure that students understand the logic of proof when they do not express their argumentation in words but instead base it on visual reconstructions that can create illusions? A discursive explanation with words is necessary to argue and to convince others.

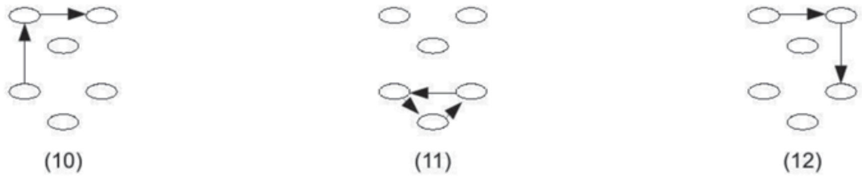
The nature and importance of written formulations differ from one paradigm to another. In most axiomatic approaches, it is possible to say that mathematical objects exist only in and by their definition. This is obviously not the case in the empiricist approach, where mathematical objects are formed from a direct access to more or less prototypical concrete objects. As for artefacts, we can pinpoint two geneses. The upward sense relates to a proof process based on initial properties (Balacheff 1982) and the other sense could be seen as a defining process (Ouvrier-Buffet 2007) and relates to institutionalization for Coutat and Richard (2011).

18.6 Towards a Coherent Geometric Work at the End of Compulsory School

Using the theoretical framework introduced above, we emphasize here some contradictory ways we encountered in French geometry education and highlight what could be a coherent approach using both geometric paradigms. For that, we draw some conclusions from a work of Lebot (2011), who has studied ways of teaching the introduction for the notion of angles at grades 6 to 8. Using the GWS diagram, it is possible to describe possible routes students may take when they use software or drawing tools to construct figures and solve problems (1 to 9). Lebot has observed interesting differences visible in the following diagrams, and we discuss some among them:



Besides these complex routes, he had observed some very incomplete schemas like these:



A coherent GI work space Generally, a geometric task begins with a construction performed using either traditional drawing tools or digital geometric software. Each time, the construction is adjusted and controlled by gesture and vision.

In this approach to geometry, the trail into the GWS diagram is like the one of Diagram 5 and done in a first sense (Instrumental (I)—Figural (F), and then Discursive (D)), which characterizes an empirical view on geometric concepts.

A coherent way to work theoretically in Geometry I would be to use "approximated" theorems in the sense we introduced (Sect. 18.3), where the numerical domain is based on decimal numbers rather than on real numbers. Theoretical discourse must justify what we see and not contradict it. This approach has been developed by Hjelmsev (1939), among others.

A coherent GII Work Space In the Geometry II conception, the focus is first on the discourse that structures the figure and controls its construction. This time, the route is trailed (Diagram 8) in an opposite sense (D—F—I) and the figure rests on its definition: All properties could be derived from the definition without surprises.

The inverse circulation of the geometric work in Geometry I and Geometry II can lead to a break in the geometric work that forms, when only one approach is explicitly privileged. As we can observe in the traditional teaching and learning of geometry, students are frequently asked to start geometric problems with the construction of real objects. This leads them to work in the sense (I—F—D) of the Diagram 5. However, for the teacher, the actual construction of an object is not really important: The discursive approach is preferred and expected, as in the Diagram 8 covered in the sense (D—F—I). In this pedagogical approach and for teachers, elements coming from Geometry I only support students' intuition for working in Geometry II, leading the formation of a (GII/gI)[3] work space. However, at the same time, students may believe that they work in a (GI/gII) work space where the objective is to think about real objects using some properties coming from Geometry II (Thales and Pythagorean theorems) to avoid direct measurement on the drawing. The geometric work made by students can be incomplete, as in Diagram 6, where students stay in an experimental approach without any discursive conclusion. They have paid too much attention to the construction task which requires time and care, but this work was neglected in the proof process expected by the teacher, where figures play only a heuristic supporting role. That can lead to another form of incomplete work but this time favoured by teachers, as in Diagram 4 where there exists only interaction between proof and figure.

We support the idea that both geometric paradigms must be included in geometry learning to develop a coherent (GI / GII) work space where both paradigms have the same importance. Only when this condition is met, an approximation can have both a numerical and a geometrical meaning, and a GWS can be created suitable for introducing "almost parallel" lines in relationship to decimal numbers where "strictly parallel" lines relate to real numbers. That would help resolve problems of mathematical coherency such as those experienced by students who asserted that they did not know whether the lines were parallel because "the lines (IJ) and (DC) are parallel if we round off, but they are not if we take the exact value".

18.7 Beyond the GWS

How can the notion of GWS be extended beyond geometry? First, the context in which the geometric work is developed can be taken into account. This context can be of a social and technological nature such as within Geiger's studies (Chap. 12), where ways in which productive social interactions between students, teachers and artefacts that led to mathematical learning have been explored. Another extension could consider the cognitive dimension in the teaching and learning processes. Arzarello (2008) took this approach by introducing the "Space of Action, Production and Communication" that he viewed as a metaphorical space where the student's cognitive processes mature through a variety of social interactions. Within these frameworks, it is clear that the notion of GWS can operate on and pinpoint what, at the end, is the goal of an educational approach in mathematics: to make an ad-

[3] We use capital letters to insist on the dominant paradigm and small letters for the supporting paradigm.

equate mathematical work. This assertion leads us to another kind of generalization related to the nature of mathematical work. In this direction, we have started some investigations with researchers interested in calculus, probability or algebra. A third symposium on this topic has been held held in Montreal in 2012 and some elements on this approach are given in Kuzniak (2011). The generalization supposes an in-depth epistemological study of the specific mathematical domain and of its relationships to other domains. Indeed, each domain relates to a particular class of problems and the crucial question is to find an equivalent to the role that real space plays in geometry. Variations and functions for calculus, and chance and data for probability and statistics, can play the same role as space and figures in geometry. It seems that the two planes, epistemological and cognitive, would keep the same importance as in geometric work, but figural genesis and visualization should be changed and reinterpreted through semiotic and representation processes in relation to the mathematical domain concerned.

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Chapter 19
Small Steps to Promote "Mathematical Literacy"

Lothar Profke

19.1 Mathematical Literacy and Big Ideas

There is no standard definition of "mathematical literacy." Some descriptions are specified in the following section.

19.1.1 Descriptions by OECD

In the work by the Organisation for Economic Co-operation and Development (OECD 1999, p. 41) "mathematical literacy" is defined as

> an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen,

and predominantly described within the definition of "mathematical competencies" as

> a non-hierarchical list of general mathematical skills which are relevant and pertinent to all levels of education. This list includes the following elements (cf. p. 43):
>
> 1. *Mathematical thinking skill*. This includes posing questions characteristic of mathematics […]; knowing the kinds of answers that mathematics offers to such questions; distinguishing between different kinds of statements […]; and understanding and handling the extent and limits of given mathematical concepts.
> 2. *Mathematical argumentation skill*. This includes knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; possessing a feel for heuristics […]; and creating mathematical arguments.

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3. *Modelling skill*. This includes structuring the field or situation to be modelled; "mathematising" […]; "de-mathematising" […]; working with a mathematical model; validating the model; reflecting, analyzing and offering a critique of a model and its results; communicating about the model and its results […]; and monitoring and controlling the modelling process.
4. *Problem posing and solving skill*. This includes posing, formulating, and defining different kinds of mathematical problems […]; and solving different kinds of mathematical problems in a variety of ways.
5. *Representation skill*. This includes decoding, interpreting and distinguishing between different forms of representation of mathematical objects and situations and the interrelationships between the various representations; choosing, and switching between, different forms of representation, according to situation and purpose.
6. *Symbolic, formal and technical skill*. This includes: decoding and interpreting symbolic and formal language and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; using variables, solving equations and undertaking calculations.
7. *Communication skill*. This includes expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters.
8. *Aids and tools skill*. This includes knowing about, and being able to make use of, various aids and tools […] that may assist mathematical activity, and knowing about the limitations of such aids and tools.

Here, some essential aspects are not included in the notion of "mathematical literacy," but are tacitly implied (cf. p. 42):

> Attitudes and emotions, such as self-confidence, curiosity, a feeling of interest and relevance, and a desire to do or understand things, to name but a few, are not components of the OECD/PISA definition of mathematical literacy but nevertheless are important prerequisites for it. In principle it is possible to possess mathematical literacy without harbouring such attitudes and emotions at the same time. In practice, however, it is not likely that mathematical literacy, as defined above, will be put into practice by someone who does not have self-confidence, curiosity, a feeling of interest, or the desire to do or understand things that contain mathematical components.

Mathematical competencies are solely acquired while doing mathematics (if at all). Thus, the question is, whether someone possesses such competencies. The extent to which competencies are applied in mathematical activities allows drawing inferences on the initial possession of such competencies.

The OECD (1999, p. 48) does not regard the traditional curriculum content strands as a major dimension of the mathematical literacy domain, because

> mathematics is the language that describes patterns, both patterns in nature and patterns invented by the human mind. In order to be mathematically literate, students must recognise these patterns and see their variety, regularity and interconnections.

Therefore, the mathematics to be assessed (and to be learned previously) should be organized around *big mathematical ideas*.

> A large number of big ideas can be identified. In fact, the domain of mathematics is so rich and varied that it would not be possible to draw up an exhaustive list of big ideas. For the purpose of focusing the OECD/PISA mathematical literacy domain, however, […] a

> selection of big ideas […] encompasses sufficient variety and depth to reveal the essentials of mathematics.

The OECD (1999) uses the following list of mathematical big ideas to meet this requirement:

> chance; change and growth; space and shape; quantitative reasoning; uncertainty; dependency and relationships.

19.1.2 Big Ideas in the German Didactics of Mathematics

Within the German didactics of mathematics, the outlined notion of *mathematical literacy* in the Programme for International Student Assessment (PISA) framework is slightly extended, modified (*mathematische Grundbildung*), and embedded in a broader discussion of general education (*Allgemeinbildung*; e.g., Neubrand 2001, p. 46).

Besides the notion of *big ideas*, the notion of *fundamental (mathematical) ideas* (sometimes also *central* and *leading ideas*) is used. Führer (1997, p. 83) prefers the notion of *fundamental concepts*, because it does not only relate to ideas but also includes typical questions, modes of marking, manners of structuring and reasoning, heuristics, methods, etc. for a discipline (mathematics). Following Schweiger (1992), Führer describes *fundamental concepts* (cf. p. 84):

> Fundamental concepts in mathematics instruction should only be understood as concepts (a question, an intellectual structural approach, an action scheme, etc.) which are meaningful, intuitively effective, insightful, relieving, and mathematically legitimate, i.e. which have at least several of the following features:
>
> - The concept can contribute to the discourse on the questions, what mathematics really is or means.
> - The concept has an archetype according to language or action concerning everyday language, acting and thinking.
> - The concept is revealing, i.e. it makes mathematical problems of very different levels more transparent.
> - The concept clears the short-term as well as the long-term memory, and hereby improves the flexibility of instruction.
> - The concept is useful for a vertical fibre in a spiral curriculum, i.e. it repetitively clarifies and bundles up essential contents of mathematics instruction.
> - The concept has proven itself to be successful in the historical development of mathematics.
>
> (Translated by L. P.)

Führer (1997) especially recommends the following *fundamental concepts*:

> functional variation, induction, approximation, algorithmisation, invariance, symmetry and symmetrisation, reflection (cf. p. 84), and efficiency (personal information, 2010).

The search for *fundamental concepts* (*ideas*) has already been going on for a long time. Each curriculum and each book which approaches the question "what is mathematics?" always sets up that task (Führer 1997, p. 83).

Vohns (2010) puts forward some very critical theses about the didactical use of *fundamental ideas* for the construction of curricula and their efficiency in mathematics lessons.

19.1.3 Mathematics Curricula in Germany

In German curricula and their preambles, the terms *mathematical literacy* and *big ideas* were usually mentioned only indirectly and in a different phrasing than the above cited definitions. However, after Trends in International Mathematics and Science Study (TIMSS) and PISA, they appear in curricula and in teaching aids for mathematics instruction more obviously and officially.

For example, Hessisches Kultusministerium (2011, p. 5) provides a list of competencies that students are supposed to acquire at certain stages of schooling. Here

> competencies are understood as compositions of knowledge and skills […].Thus the perspective focuses not only on pieces of knowledge that shall merge together into a general understanding but also on further conditions of successful mastery of cognitive demands. They comprise strategies for the acquisition of knowledge and for its use and application as well as personal and social dispositions, attitudes, and behaviour. By the interplay of these components the demand for general education of personality comes true; […]. (Translated by L. P.)

This *core curriculum* attributes particular importance to setting up competencies that are relevant for several subjects (cf. pp. 8, 11):

- *Personal competence:* self-perception, self-concept, self-regulation;
- *Social competence:* social perception abilities, consideration and solidarity, cooperation and teamwork abilities, dealing with conflicts, social responsibility, intercultural communication;
- *Learning competence:* problem-solving competence, working competence, media competence;
- *Linguistic competence:* reading competence, writing competence, communication competence.

> […] Together with other subjects, the subject of mathematics provides the foundations for the development of the learner's education […]. Mathematical education is revealed in a number of competencies, which are developed in the process of mathematical thinking and working.
> (Translated by L. P.)

Categories of competencies in mathematics education are (cf. pp. 12, 16):

Representing Learners

- Choose the type of representation that is appropriate for the addressees and adequate to the context, and appropriately prepare it for presentation;
- Develop representations;

- Detect connections and vary between different types of representations;
- Interpret and assess representations.

Communicating Learners

- Describe processes;
- Present, explain, compare, and assess multiple procedures to solve a task;
- Document reflections, solution strategies and results, describe them adequately to addressees, and present them, also using suitable media;
- Use adequate technical and formal language to addressees.

Arguing Learners

- Ask questions related to generalizations and specializations of mathematical facts, and evaluate their correctness;
- Express well-founded conjectures on mathematical connections and make comparisons;
- Analyze mathematical statements and procedures, explain, and justify them by multistep chains of arguments;
- Understand mathematical argumentations, evaluate them, and give reasons to meet the facts.

Dealing with symbolic, formal, and technical elements Learners

- Work formally with variables, terms, and equations;
- Translate technical language into colloquial language and vice versa, using appropriate symbols;
- Create tables and diagrams, and extract data and values;
- Conduct solution and checking methods;
- Use mathematical tools reasonably and smartly, like formulary, pocket calculator, software, measuring instruments. They select the tools according to criteria of accuracy, economy of time, and proneness to mistakes.

Problem-solving Learners

- Conceive possible mathematical questions, phrase them in their own words, and develop solution ideas in problem situations;
- Select appropriate heuristic aids, strategies, and principles for problem-solving, apply them, and evaluate solution strategies;
- Use different types of representations and approaches to solve problems;
- Identify relevant quantities and describe their interdependences;
- Interpret results with regard to the problem;
- Reflect on solution strategies.

Modeling Learners

- Gather information in complex, unfamiliar situations, and from different sources;
- Translate the context or the situation into familiar mathematical structures and connections using mathematical concepts and consider factors of influence and dependences;
- Work with the selected mathematical model and translate the results back into the real situation;
- Check and interpret results within the real situation including a critical evaluation of the selected model;
- Evaluate the selected model;
- Provide typical real-life situations for mathematical models.

(Translated by L. P.)

Standards are more specific about these competencies. Basic and indispensable pieces of mathematical knowledge are specified in content areas and their focal points. Furthermore, concepts of content are organized around *leading ideas* (*Leitideen*; cf. p. 14):

> Number and operation; space and form; quantities and measuring; functional variation; data and chance.

19.2 Small Steps to Promote Mathematical Literacy: Examples

Although the details of and the differences between the descriptions of *mathematical literacy*, *big ideas*, etc. may be important for didactical theories as well as for the construction of curricula and teaching materials, they are rather irrelevant in everyday mathematics lessons:

A "mediocre" teacher (i.e., a teacher who is not as proficient as expected by theorists) suffers from a lot of pressure in terms of the amount of content, the lack of time, and the daily challenge to support the students of lower performance levels. In this respect, most proposals provided in didactical papers require too much effort to apply. Such a teacher is looking for basic examples, which are relevant to the topic the teacher has to deal with "(the day after) tomorrow in his/her class," which can easily be prepared, and which take only little time of instruction. For this purpose, the teacher usually decides to use the textbook.

Furthermore, we have to consider that many teachers have to teach mathematics without a genuine qualification in the subject (cf. Törner and Törner 2010).

In order to provide examples to promote *mathematical literacy* and to demonstrate the references to *big ideas*, it appears reasonable to follow the lists of *mathematical competencies* (OECD 1999, p. 43), of *fundamental concepts* (Führer 1997, p. 84), or the *fields of competencies* (Hessisches Kultusministerium 2011, pp. 12, 16).

However, as it has just been argued, it is more useful for many teachers to illustrate that they can promote *mathematical literacy* with little additional effort, just by dealing with the topics prescribed in curricula or local syllabuses. For this reason, this paper provides hints and suggestions on how this can be achieved with reference to selected topics of secondary school mathematics.

The intention is to connect

- General descriptions of mathematical competencies, e.g., *general mathematical skills* in Sect. 19.1.1 and the *fields of competencies* in Sect. 19.1.3,
- General descriptions of *fundamental concepts*, for instance, those in Sect. 19.1.2, and
- Subject matters prescribed by curricula, (e.g., Hessisches Kultusministerium 2010, pp. 14, 18, 26),

as well as to give substance to such "catchwords" with methods for regular mathematics lessons at school.

Which arguments can be pointed out for the (additional) methodical proposals? As already mentioned, mathematical competencies are only acquired by taking part in mathematical activity. Therefore, mathematics instruction has to guide, or, if necessary, oblige students to carry out desirable mathematical activities with the help of suitable instruction, questions, and demonstrations provided by the teacher.

The proposals do not require special methodical classroom arrangements but are appropriate for "normal" traditional mathematical instruction, with the teacher often in front of the class, developing the subject matter with questions, impulses, and information (*fragend-entwickelnder Unterricht*). Even this kind of instruction does not discharge the students from learning.

Students (can) learn to think for themselves (selbst nachdenken) if they follow the teacher's thoughts (vordenken) and think with him (mitdenken).

The examples are chosen more or less randomly. The hints are supposed to illustrate a way of proceeding what can and should be applied in other topics. Most of them (if not all) can probably be found in textbooks, monographs, or booklets for mathematics education.

19.2.1 Concept Formation

Concepts are agreements, more or less expedient, usually related to some kind of (pre-) experience and mediate meaning, which is not isolated, but rather part of a system of concepts. Although mathematical concepts are determinately "defined," students can learn about all of them.

19.2.1.1 Prime Numbers

Prime numbers are often introduced by decomposing the natural numbers into components with as few divisors as possible.

Teachers may ask:

- Consider the definition:
  The natural number n is a prime number, if the only divisors of n are 1 and n?
  Should the number 1 also be regarded as a prime number? Why (not)?
- Can we define
  The natural number n is a prime number if and only if n has exactly two divisors?
  May we omit the word 'exactly' in this definition?

19.2.1.2 Common and Decimal Fractions

Common fractions are often introduced by dividing a cake, a pizza, etc. *equitably* into three, four, five, or more pieces and taking together a certain amount of these pieces.

- However, what does "equitable" mean, if the dimensions of the cake or the pizza are irregular or some ingredients are not distributed evenly?
  We have to ignore all irregularities and imagine absolute perfect cakes or pizzas.

Further activities:

- Compare: *2/3* of a cake, half a cake, *3/4* of a cake; half a 100 g bar of chocolate and *1/3* of a 200 g bar of chocolate.
- Which criterion should be used for comparisons: size, weight, taste, or price?
  In the case that not all cakes are identical, comparisons can turn out very differently.
- What could be the meaning of: *2/3* of an apple pie + *3/4* of a strawberry cake?
- Musical rhythms: Is three-four-time = six-eight-time?
- Look at the following notions. Can you determine the whole things? Quarterfinal, semifinal of sports competitions; lodgings and half-board (when booking a room on holiday); half-truth; 7/8 pants; half-pipe for children with rollerblades.
- Which numbers are decimal numbers? Why (not)?
  0.375, 3/8, 9/24, 37.5 %, 375 ‰, 5/3, 4.010 010 001 000 1…,
  2.999 999…
  How can you distinguish decimal *numbers* from decimal *fractions*?

19.2.1.3 Geometric Figures

In everyday life, students and we (including mathematicians) use phrases which are, strictly speaking, not correct:

Quadrangular box, hexagonal tower, circular flowerpot, rectangular room, walking line, etc.

However, we (also mathematicians) understand what is meant by such phrases almost every time.

In an analogous way, we use certain geometric terms often restrictively in everyday life:

Rectangles are not quadratic, and squares are no rectangles. Cylinders are thought to be like tins, rotational symmetrical and finite. Planes and flat areas are imagined horizontally.

Teachers can and should address the differences between the colloquial and the mathematical language.

- Evaluate the definition:
  A rectangle is a quadrilateral with four right angles and adjacent sides of different lengths.

- Compare and evaluate:
  A quadrilateral with three right angles is called a rectangle. Each quadrilateral with four right angles is called a rectangle. A quadrilateral with four right angles and opposite sides of equal length is a rectangle. Each rectangle is a quadrilateral with four right angles and adjacent sides of different length.
  Are some of these statements incorrect? Why and for whom? If any statement is incorrect, correct it and explain your changes.
- Is a rectangle also a parallelogram, a cube also a cuboid, etc.? Why (not)?

19.2.1.4 Geometric Symmetrical Figures

- Are the outcomes of the activities below consistently axisymmetric or mirror pictures?
  Producing patterns and colored blots after folding a paper along a straight line; making imprints with hands and feet onto sand or snow; stamps, seals, and their prints; mirror writings; looking through a transparent paper from the backside.
  Where do we have to place or to imagine a mirror to be placed?
- Symmetries in plane, in space geometry, and in real life can be studied using the following:
  Ornaments and friezes (one- or two-dimensional, on plane surfaces or on vases, plates)
- More general symmetries with similarities, oblique reflections, etc.
- Fundamental ideas of symmetry can be studied thus:
  A symmetrical figure can be dissected into equal parts; a pattern is repeated over and over.
  In both cases, how do we have to specify the terms “equal” and “repeated”?

19.2.1.5 Nets of Solids

The nets of boxes for shoes, cosmetics, foodstuffs, etc., of paper tubes, etc., often differ from nets of polyhedrons, cylinders, and cones presented in mathematics textbooks that are covered in mathematics lessons.

Teachers could animate students:

- To find out how a net has to be folded to generate the surface of the resulting solid figure;
- To compare both sorts of nets mentioned above and to transform one into the other;
- To analyze why real nets have their specific shape (production, transport of many flat nets, and easy folding to solid figures);
- To discuss whether nets of solid figures might be connected or not (e.g., building the surface of a solid figure with synthetic material) and whether a net should be made of the nondivided parts of the surface of the solid figure;

- To clarify which nets of a solid figure are different and how they differ; and
- To investigate whether there are nets that do determine several solid figures.

19.2.1.6 Sizes, Quantities, and Measures

Some tasks for students:

- What is the meaning of the following statements? How can we imagine the quantities described?
  Basel 334 km (*sign at highway 5*)*; Marburg 25.7 km* (*sign at a cycling track*)*; average distance between earth and sun* = 1.5×10^8 *km;*
  3 kg of potatoes; hundredweight-man; 40 t truck.
  2,500 ha (farm size); 357,000 km^2 *(country size);*
  80 L rain/m^2 *within 3 h; 202,106* m^3 *capacity of a reservoir dam; 82 millions* (*population of a country*)
  Some objects are bigger than others of the same kind (houses, countries, animals, etc.).
- Are the following invariable (invariant) or not?
  Will you become smaller, if you squat down? Does a piece of wire become shorter while it is coiled up? How does the size of a sheet of paper change, if it is crumpled or if it gets wet?
  Which geometric statements stay the same, if the orientation of the plane or the space changes?
- Illustrating quantities:
  Which quantities illustrate a 1 L paper bag for drinks? How do we have to use it to illustrate volume?
  Can a rhinoceros give you an idea of the mass of 1.5 t?
  How can you illustrate a great speed?
  Can you imagine the distance of 2.5 light years to the Andromeda galaxy, or the size of 10–300 nm of soot particles?
- How can you define and measure sizes?
  Distances between points, lines, curves, planes, surfaces; distances between villages, countries, stars, etc.; angles of lines in a two-dimensional or three-dimensional space; angles between two planes, between a plane and a line; angles in surveying and in astronomy.
  Can we always connect distance and orthogonality? Are some of the angles mentioned above extreme? What is the possible range of angles between lines or directions in space?
- How can we determine 23 % of an entity?
  As 23/100 of the entity; 23 of each 100 of the entity (sometimes it is necessary to imagine the entity to be a multiple of hundred); the x which solves the proportion, 100: entity = 23: x.

19.2.1.7 Functions, Mappings, etc.

After functions are introduced, the teacher might elaborate on this concept:

- There are not only those functions that assign real numbers to other real numbers. Which of the following concepts are also included in the concept of functions:
  Sequences, series; tables; geometric mappings (translation, reflection, dilatation, similarity, projection, etc.); movements (i.e., processes); arithmetic operations (e.g., addition, subtraction, multiplication, division); measuring; probabilities of events; parametric descriptions (equations) of curves or surfaces; parameterized sets of functions (Funktionenschar); differentiation, integration; certain buttons of an electronic calculator; ticket machine; macros of computer algebra systems and of dynamic geometry software
- The coordinate system determines what the graph of a function (which assigns real numbers to others) looks like.
  Compare the graphs of functions in Cartesian coordinate systems and in a polar coordinate system. Can a circular line be the graph of a function?
- Usually, in mathematics instruction we deal with geometric mappings of the plane into itself.
  How can we define and handle congruence or similarity mappings from one plane into another?
  Hint: Copy a picture with help of square grids in the plane of the picture and in the plane of its copy as well. What happens if the square grid in the second plane is replaced by a parallelogram grid?

19.2.1.8 Limits

- Consider and explain the meaning of the following:
  Decimal expansion of 6:7, non-terminating decimal fractions; calculating the perimeter and the area of a circle; determining the volume of a pyramid; Cavalieri’s theorem.
- Compare the following concepts:
  Limit of a sequence or a series of numbers; limit(s) of a function; limit of a sequence of geometric figures or solid figures; limits in probability theory.

19.2.1.9 Examples and Counterexamples

- Give examples and counterexamples of a concept and justify your choices:
  natural, integer, (ir)rational numbers, real numbers, and common and decimal fractions;
  square, rectangle, parallelogram, trapezoid, and rhombus;
  cube, cuboid, prism, pyramid, cylinder, and cone;

convergent or divergent sequences; and
continuous (differentiable and integrable) functions.

- How can we check whether we have found an example, a counterexample, or perhaps none?

Students might learn by using the following activities (probably the teacher has to demonstrate these before the students can apply them):

- Mathematical definitions (also definitions in other domains) are not "given by God" and therefore unalterable, but definitions fulfill certain criteria which can be stated.
- Each mathematical concept has several characterizations. After we have chosen one for defining the concept, the others become provable theorems of the concept.
- A definition does not have to be minimal, but if so, we can identify (counter-) examples more easily.
- As mathematics also serves to describe certain aspects of "the world," we can decide on one of the following: Either a mathematical definition should follow the characteristics of everyday life or we have to adapt these criteria to the traditional mathematical definition.
- It is sometimes claimed that it is necessary to use the technical terms in everyday life in order to be understood. However, often the technical terminology does not serve this purpose better than the colloquial expressions.
- The assignments right and wrong, concerning statements (even with mathematical content), depend on the context of the statement.

19.2.2 Producing and Ensuring Knowledge, Argumentation

Deepening insights into already acquired concepts, or new knowledge, can arise from connecting different subjects and embedding subject matters in greater contexts (cf. Vollrath 1995 for *long-term learning*). To receive new ideas you should know about the features and relations of the situation that is taken into account, which are remarkable, unexpected, and surprising.

The field of *proving or disproving conjectures and statements* has often been ploughed. Therefore, I will only provide a short reference.

19.2.2.1 Connecting Concepts, which Were Dealt with Formerly, and which are More or Less Different.

- Subtraction and division
  We subtract a number "b", either by taking it away from another number "a" or by adding a suitable number "d" to "b" so that, $b+d=a$.

Dividing a number "a" by a number "b" could be done by subtracting "b" from "a" as often as possible. Apply the adding idea of subtraction to the dividing concept.

- Geometric figures
  See Sect. 19.2.1.3. Apply the formula for calculating the areas of trapezoids to triangles, rectangles, and parallelograms.
  Bars, postcards, tiles, cuboids, cylinders, etc. can all be understood to be prisms; sometimes, it is necessary to imagine them to be perfect.
- Functions: see Sect. 19.2.1.7
- Coordinate systems
  Signs for outdoor water- and gas stopcocks; Cartesian systems in plane and in space geometry; rectangular systems for functions; geodesic and geographic systems (also clockwise-oriented or left-threaded); and longitude, latitude, and altitude of geography or topographic maps.
  Parametrical descriptions (equations) of curves or surfaces provide the figures with coordinate systems.
- Differentiation and integration
  The fundamental theorem of calculus combines differentiation and integration of functions. By using this theorem, one can define and tabulate nonelementary functions.
  The rule of integration by parts can be derived from the product rule of differential calculus. Which integration rule follows the quotient rule of differential calculus?

19.2.2.2 Proving Statements

- How can we understand statements that can be found in textbooks, formula books, etc. or ones that are guessed?
  Separate prerequisites and assertions. Explain all details. Transform colloquial language into mathematical correct phrasings such as: Can two objects be identical? Different geometric figures cut each other and not themselves. Parallelism, orthogonality, etc. are not features of single objects but relations between several ones. The words "a" and "the" could mean "all." "Some," "several," "none," "few," "many," "at least," "at best," etc. are also to be defined or rather clarified. The inversion of an if–then theorem results provisionally in a statement, perhaps later on in a theorem.
- Transfer a theorem together with a proof to an analogous situation.
  Formulas for calculating the areas of polygons; congruence statements for rectangles, circles, cuboids, etc.; perpendicular bisectors of the sides of a triangle to its angle bisectors.
  Does this also work for orthocenters or centroids of triangles and quadrilaterals? How does the product rule of calculus differ from the quotient rule?
- Can special facts prove general theorems?

Yes, if all calculations and considerations with specific facts show a general strategy regarding how to deal with other specific facts: Commutative law of number multiplication; paper and pencil calculations of the four rules (all numbers); divisibility tests for natural numbers; power rules; associative law of matrix multiplication; rule of three.

- Which reasons are allowed?
  Mathematical theorems are statements "if p, then q." To prove such a statement, we do not have to know whether p is true. To prove the statement, we may assume that p is true; otherwise, the statement is always true. Remember in a similar way, how to prove the validity of a for-all statement.
  In mathematics, at school we usually do not start with a system of axioms and then proceed by deduction. At first and often still later, students find and learn mathematical concepts and facts by induction. When proving (or deriving) a theorem, the teacher may interpret (for himself/herself) those concepts and facts as a system of axioms.
- Provide steps of proofs emphasizing with easily memorizable phrasings, also by writing on the blackboard and in students' exercise books.
  For example, proving a general rule of divisibility:

We have to show:	"*if* $a \mid b$ *and* $a \mid c$, *then* $a \mid (b+c)$"
We might assume:	*actually* a divides *both* b *and* c.
Now, we have to show:	$a \mid (b+c)$ *is also valid*.

19.2.2.3 Detecting and Guessing Mathematical Facts

- Which statements are we looking for?
  How to add, subtract, multiply, divide fractions or positive and negative numbers; theorems for working with vectors or matrices; differentiation rules or integration rules
- Which statements may be remarkable?
  Exceptional points and lines of triangles: Any three straight lines of a plane do not have to pass the same point; any three points do not have to be situated on the same straight line. However, each triangle has an inscribed circle, a circumcircle, all perpendicular bisectors of its sides pass through the same point, as do all angle bisectors, etc. Is that remarkable? Examine quadrilaterals in a similar manner. Can we assert analogous statements for some solid figures?
- Check a supposition. Test it with special data; examine extreme cases of the supposition. In geometry: Make and measure accurate drawings or spatial models.
- Experimental mathematics. Observe and follow scientific standards:
  Decide which data and how much you should collect; pay attention to reasonable accuracy of measuring; clearly arrange the data in tables and graphs; make assertions and test them; try to understand hypotheses that have proven to be valid in tests.

19.2.3 Modeling

There are many publications about teaching modeling in mathematics education. Not very often, you will find encouragements to teach students modeling activities while they have to deal with mathematics that is prescribed by curricula or local agreements.

19.2.3.1 In Daily Life We Need only One Arithmetic

- Which statements could be meaningful?
  Sum of zip codes; *3 workers* times *5 hours*; *4 apples* plus *5 pears*; mean *value* of test points or credits; *April 42nd*.
- Explain your answers:
  9.15 a.m. plus *55 min*; the day before yesterday minus *2 weeks*. When does a decade, or century, or millennium start and when does it come to an end?

19.2.3.2 Assigning Numbers to Facts

- Calculate costs of goods and services.
  Should the prize of a unit stay the same in a bulk purchase? May we get some discount or rather some extra charge for a bulk purchase or should the prizes be proportional to the prize of a single unit, respectively? What is of advantage to sellers and customers?
- Assess results.
  Sports: highest, widest, fastest, or the best mean value of several attempts. Which mean value should be used?
- Performances at school; pay for work; assigned punishment by justice.

19.2.3.3 Geometry in Real Life

- Look for geometric instruments used by craftsmen, engineers, or amateurs. How are these instruments used to draw lines, circles, right angles, and other angles or to produce planes and other surfaces, etc.?
- Carry out the following geometrical constructions.
  Find the central point of a tin cap. Construct a straight line on a football field with 100 m length. Imagine you have to mark a circular arc with radius 3.7 km in a hilly landscape as part of a traffic line.
  Scenery in theatre is designed on a reduced scale. How can you draw the sketch of the original scale?

- Describe geometric construction problems (cf. Holland 2007, Chap. 3; Profke 1986):
  Initial state, target state, allowed construction tools; practicability and correctness of the construction plan, number of different results (in which sense?); theoretical exact or approximate construction plans versus practical accuracy.
  Do mechanical plotters and electronic geometry software produce mathematically exact or practically accurate drawings?

19.2.3.4 Stochastics

- What are the main domains of probability theory, descriptive, and inference statistics? Compare their statements.
- The following characterize force and helplessness of the statistician:
  Influence of the client (prescribing a universal set and attributes to investigate, purpose of the investigation); guarantee of representativeness and significance; realization of an investigation that is dispassionate and impartial; use of (which?) special characteristic data (means, deviation, etc.); representing the results of an investigation in an understandable way; misunderstanding of an investigation by the consumers
- Some understandings or misinterpretations of stochastic statements are discussed: Whether a mathematical random experiment grasps a real event (more or less) adequately cannot be proven, rather disproven, and calls for competent interpretation. Realizing a mathematical random experiment is a technical and not a mathematical problem. What is the meaning (frequency interpretation) of hypothesis tests on a significance level of 95 %? Does such a hypothesis test prove anything scientifically (as often claimed in media)? The single test may lead to a misjudgment of the results, because of errors of the first or second kind. A rare event points to the invalidity of the zero hypothesis; *but the test result does not appear to have occurred accidentally under the zero hypothesis or not accidentally according to an opposite hypothesis.*

19.2.3.5 Supplement

See also Sects. 19.2.1.2, 19.2.1.4–19.2.1.6, examples (2), (4), and (5). Some more examples, concerning the application of mathematics in real life, can be found in Profke 2010.

Students can perform modeling activities from the beginning of mathematics instruction. As mathematics concepts are extracted from real-life situations, the acquired mathematics knowledge should be applied to real life. Students might learn (probably the teacher has to prompt this directly) by the following activities:

- Not everything that can be done with mathematics makes sense in real life. Mathematical correctness does not always result in practical applicability.
- Often, modeling a real life situation by mathematics can be done in several ways.

19.2.4 Procedures Planning, Carrying out, and Documenting

For working on exercises and problems, exploring situations, etc., it is very often helpful to make a plan. Here are some examples:
Finding rules to add or to multiply common or decimal fractions; four rules with positive and negative numbers; calculating areas of complex geometric figures or volumes of complex solid figures; kinds of problems in a domain (percentage or interest calculus, (inverse) proportionality, etc.); working on real-life situations (Sachrechnen); congruent or similar triangles; constructing or calculating triangles, quadrangles, etc.; transforming algebraic terms; and how to solve quadratic equations.

In order to teach students not to work without a plan, the teacher should discuss the following procedures together with the students. Hopefully, this makes the ideas for the preparation of the lesson(s) explicit.

19.2.4.1 Gathering Information

- Revising mathematical concepts and procedures. Search in students' textbooks: index, table of contents, and markings in the text. Are there treatments in the textbook or in exercise books that can serve as patterns?
- Getting an idea how to proceed.
  Is there a standard method? Look for an analogous problem and its solution. Approach the problem with a simpler collection of data. Identify and solve special cases. Are there other ideas to work with on the problem?
- Getting general ideas of dealing with a problem.
  In space geometry: deciding on mutual positions of points, straight lines, and planes; calculating distances between points, straight lines, and planes; determining specific distances with elementary geometric reflections.

19.2.4.2 Carrying Out a Plan

- Write down the meaning of and the reason for each step.
- How can we check whether the results are correct?
  Look for and apply specific tests (not very often available, and not always completely accurate); check each step (risky because we often make the same mistakes again). Apply the result to special and extreme cases of the original problem. If possible, work on the same problem in another way.

19.2.4.3 Documenting Plans and Procedures

- Write down all ideas and reflections as brief outlines in your exercise book.

Gather information and document where they were found including questions, answers, references, reasons, and plan(s) together with hints on how to follow the plan(s); reasons for a plan, the choice or the rejection of a mathematical concept or statement; and incorrect considerations and reasons for their incorrectness.

- The teacher has to guide the students on how to write down considerations.
  Students' exercise books mirror a teacher's blackboard in a more or less distorted manner. Therefore, the teacher should not be afraid of detailed, clearly arranged texts (also in brief outlines), which the teacher and students formulate together (practically possible). Do worksheets and work folders fulfill that intention?

See also Kaune et al. (2010).

19.2.5 Sum of Angles of Triangles

This section shows how a teacher can put some *small steps for promoting mathematical literacy* into action during treating a standard topic of secondary mathematics instruction. Only one outline is given, according to Walsch (1972, p. 136); see also Profke (2009, pp. 102–110).

Some more examples, concerning applying mathematics in real life, can be found in Profke 2010.

19.2.5.1 Introduction to the Topic and Posing a Problem

See Sect. 19.1.1.1 no. 1 and 19.2.2.3.

- Repetition
  How can we categorize triangles? Which characteristics can we use to distinguish the triangles? *Possible answers:* size, form, lengths of sides, and angles (right, acute, and obtuse).
  Constructing a triangle: What can we choose without restrictions, and what is the result of the choice?
  One expected answer: Two angles of a triangle determine the third angle.
- Problem posing

Heading:	Are the angles of a triangle interdependent?
Problem:	Given a triangle ABC with angle sizes α or β or γ respectively. Can we calculate γ, if we know (e.g.) α and β.

Write down the problem, questions, answers, etc. on the blackboard, and also should do the students into their exercise books (see Sect. 19.1.1.1 no. 7; 19.2.4.3).

Table 19.1 Triangles' data

Side lengths			Angle sizes		
a	b	c	α	β	γ
8 cm	6 cm	5 cm	49°	39°	93°
5 cm	4 cm	8 cm	25°	123°	32°
4 cm	6 cm	8 cm	47°	104°	30°
…	…	…	…	…	…

19.2.5.2 Making a Plan

(See Sect. 19.1.1.1 no. 7; and Sect. 19.2.4.)

- Experimenting, collecting information, and hypothesizing.
  - Search for patterns by
    a. Drawing some triangles and measuring their angles;
    b. Investigating special triangles, all of whose angles can be determined.
  - Test all supposed patterns with other general triangles.
- Asserting a statement for all triangles.
- Proving the asserted statement.
  Give reasons for the need for proof, first for some special triangles, then for all triangles (if possible by means of the first cases).
- Establishing a theorem on triangles.

Now, let's carry out the plan.

19.2.5.3 Experimenting, Collecting data, and Guessing

(See Sect. 19.1.1.1 no. 8; and Sect. 19.2.2.3).

- Drawing triangles and measuring their angles.
  Students may be uncertain which triangles to draw. Together with the teacher, students can set some data: the length of one side of the triangle and the sizes of the adjacent angles. Place the given and the measured data into a table (see Table 19.1):

  The students might not detect the invariance of the angle sum.

- Investigating special triangles whose angles can all be determined.
  Triangle ABC with angle sizes α or β and γ (see Table 19.2):

  All angles of an equilateral triangle have the same size. Six congruent equilateral triangles obviously can be joined together to a regular hexagon (see Fig. 19.1). From this, we can see that $\alpha=\beta=\gamma=60°$.

  We can complete each rectangular triangle ABC with $\alpha=90°$ to a rectangle. For any rectangle, this is evident (see Fig. 19.2): The angles sum up to 360°.

Table 19.2 Angle sizes of special triangles

α	β	γ
≈ 0°	≈ 0°	≈ 180°
≈ 0°	≈ 180°	≈ 0°
≈ 90°	≈ 90°	≈ 0°
→ 0°	fix	?

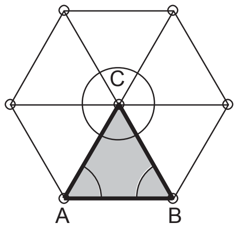

Fig. 19.1 Six congruent equilateral triangles joined together to form a regular hexagon

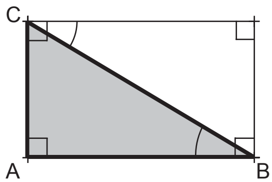

Fig. 19.2 A rectangular triangle ABC completed to form a rectangle

Each diagonal dissects the rectangle in two congruent (equal) rectangular triangles. (Instead of the second we can argue with the theorem on alternate angles at parallel straight lines).
We deduce and calculate $\gamma = 90° - \beta$.

- Detecting a pattern.
  The teacher might have to help the students to see the relation $\gamma = 180° - (\alpha + \beta)$ for the special triangles that are investigated.
  We also have to check this pattern with more general triangles, accurately drawn, and confirm the equation within the scope of the drawing precision and measurement precision.

19.2.5.4 Asserting a Statement for All Triangles

Our drawings and investigations suggest:
For all triangles, the following is valid (if we denote angle sizes as usual):
$\gamma = 180° - (\alpha + \beta)$ or $\alpha + \beta + \gamma = 180°$

19.2.5.5 Proving the Asserted Statement

(See Sect. 19.1.1.1 no. 1, 7; and Sect. 19.2.2.2.)

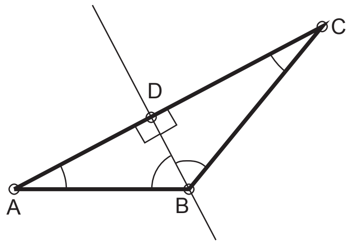

Fig. 19.3 Dissection of triangle ABC into two triangles ABD and BCD by an altitude

- Do we need to provide a proof for the asserted statement?
  Here are two reasons for this:

  a. We cannot produce perfect triangles and measure their angles precisely.
  b. We cannot check the statement for very (astronomic) big and very (microscopic) small triangles.

- Make a plan for proving the statement.
  Identify different cases, e.g., right, acute, and obtuse triangles and look for proof of each case. Look for a special case, in which you can prove the statement and then try to reduce the general case to the special case.

- Prove the statement.

  1. We already have strictly proven the statement for rectangular triangles.
  2. Now, given a non-rectangular triangle ABC with angle measures α, β, and γ.
  3. We have to prove: $\gamma = 180° - (\alpha + \beta)$ or equivalently: $\alpha + \beta + \gamma = 180°$.
  4. For the proof we can use: If a triangle is rectangular, then the sum of its angle measures equals 180°.
  5. Therefore, we dissect ABC by a suitable altitude line into two rectangular triangles, ABD and BCD. (If necessary, we have to change the names of the vertices and angles of the triangle.) See Fig. 19.3.
  6. We apply the valid case of the statement to ABD and BCD.
  7. In fact, we get: $\alpha + \beta + \gamma = 180°$ (short calculation).

19.2.5.6 Establishing a Theorem on Triangles

In the previous step we have proven:

For each triangle, the sizes of its angles amount to 180°:
$\alpha + \beta + \gamma = 180°$ (if we denote its angle sizes as usual by α, β, and γ)

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Chapter 20
Transformation as a Fundamental Concept in Arithmetical Competence Modelling: An Example of Informatical Educational Science

Joost Klep

20.1 Introduction

This chapter is an unusual approach to arithmetical competence. The starting point is the idea that students often produce their own strategies, which might differ from how their teacher teaches them, to solve prescribed exercises. To cope with these different strategies, the teacher has to make a great effort in understanding and solving them. Differences in strategy, in the use of facts and automatisms in sub-solutions, in speed and number of steps can be observed by an experienced teacher; but in a group of 25 children, these observations cannot be systematically used in noticing, giving support and planning new exercises. The given volume of data cannot be handled within the constraints of working in the classroom. It would then be a good idea to provide a computer program for the teacher, which can then assist in such assessments.

This kind of a computer program should be compatible with educational ideas, but it should not be an extension of existing educational theories. The new element lies in the change of constraints: what is hard to handle for a teacher might be an easy job for a computer and vice versa. The discipline of informatical educational science focusses on modelling educational data and processes, like the fostering of arithmetical competence. It focusses on computer programs as instruments in education.

Modelling learning processes might provide new ideas about learning and teaching because computer programs like dynamic models of arithmetical competence provide new ways of understanding and scientific research. In the case of Arithmeticus, some aspects of flexible calculation can be better understood.

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20.1.1 The Structure of this Chapter

The example of informatical educational science presented in this chapter bases itself and develops on several mathematical, psychological and educational theories about the learning of elementary mathematics. The central theme is problem solving with a focus on skill attainment. The themes of this chapter can be illustrated by a short description of an elder computer program for the fostering of elementary multiplications. These themes are discussed in Sect. 20.2.

Section 20.3 focusses on how 'interpreting students' work' can be understood. This section ends with the question, how to make interpretations of calculations which are unknown for the teacher or the system. The theme of Sect. 20.3 leads to the deeper mathematical and informatical question of how to think of 'all possible solutions for an exercise'. In Sect. 20.4, some informatical elements of the production of calculations by Arithmeticus are discussed. Here, the concept 'transformation' plays an important role.

Following these mathematically oriented sections, the fostering of the students' problem solving and skill attainment is revisited in Sect. 20.5. Here, the concept of transformation plays an important role in the model that deals with solving arithmetical problems.

In Sect. 20.6, several educational aspects of the informatical modelling of arithmetical competence are discussed in connection with actual psychological literature. Finally, in Sect. 20.7, a conclusion is presented.

20.2 Backgrounds

In this section the understanding and fostering of arithmetical competence are discussed: psychological backgrounds, solving arithmetical problems as a basis for competence, educational consequences and an example of educational software is used to illustrate some important concepts finally in this section.

20.2.1 Psychological Backgrounds

Theories about the learning and teaching of the basic arithmetical operations addition, subtraction, multiplication and division have made fundamental progress. Ebbinghaus (1885) has formulated his theory of learning and forgetting and his theory of saving effect, which says: although someone has forgotten some piece of information, relearning the same information seems to be faster than the initial learning. Repeated learning gives a good lasting result because the more the information has been relearned the more unlikely it is to be forgotten. This theory of learning elementary arithmetical operations has been very popular in education.

A long sequence of investigators and educationalists, however, proposed that information which is in any sense meaningful for the learner can be learnt better (Müller 1911; Bartlett 1932; Atkinson and Shiffrin 1971; Neisser 1982) than meaningless data, and mathematical educationalists emphasise that a mathematical understanding of facts improves a child's learning (Gerlach 1914; Erlwanger 1973, Skemp 1978, Davis 1978; Thornton 1978; Baroody 1983; Ashcraft 1983, 1985; Ginsburg 1977; Radatz and Schipper 1983; ter Heege 1986; Wittmann and Müller 1993). Factual knowledge is not necessarily agile or flexible knowledge (e.g. van Parreren 1960). Research has focussed more and more on children's development of meaningful strategies and their learning of facts.

20.2.2 Solving Arithmetical Problems as a Basis for Competence

For most of these authors, the learning of elementary mathematics is a mix of understanding strategies, learning to apply them and rote learning of arithmetical facts like elementary additions and multiplications. Students can attain an understanding by explanations and by problem solving. Problem solving can be fostered by offering appropriate problems and material, inviting students to model problem situations or offering them appropriate models and having them reflect on their mathematical activity. Students can learn facts by heart in meaningful application contexts, by repetition and in games.

When talking about a solution for a problem, teachers and students often focus on the 'how' (Skemp 1978, Instrumental understanding) and/ or the 'why' of the solution (Skemp 1978, Relational understanding). The 'how' and the 'why' do not explain 'how to find the solution'.

Although a teacher proposes a solution and the students are able to see each other's solutions, they are still able to produce their own alternative solutions (e.g. Ashcraft 1983, 1985; ter Heege 1986).

In the problem solving activity, students develop thinking activity, which is not always necessarily discussed or highlighted: This can be fostered by providing them with several educational impulses and inputs (Treffers 1987 and van den Heuvel-Panhuizen 2008), e.g., appropriate contexts, questions, models and applications.

Not only the 'how' and the 'why this way?' of solution strategies are important for teachers, but also the stimulation of students to produce their own queries, which is the secret of an effective educational program.

20.2.3 Educational Consequences

Teaching mathematics implies thinking about how students might produce, perform and justify/prove their strategies. A teacher has to make well-tailored educational offers for the students: Problems, exercises and help have to be chosen as stepping

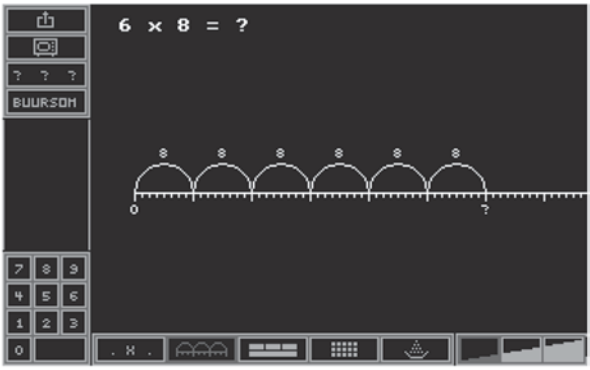


Fig. 20.1 6 · 8 as 6 jumps of 8 units on a number line. At the left side, icons are offered to start an animation of the jumps in the number line representation

stones for the students' learning and for their mathematical activities. The better teachers know the students and their actual competences, the better they can tailor the educational offer, by providing a rich learning environment for the students, where they can find what they might need.

The teacher has to think of how the students' learning might develop. Gravemeijer et al. (2003) call this 'hypothetical learning trajectories'. This thinking in advance about individual or group learning is only possible when the teacher has an appropriate student model available. The more a learning environment has to be individually tailored for specific learning the more detailed the student model should be. Relevant informatical educational questions include, how the student model could look like, how educational decisions in designing and organising learning environments could be made and how the student model has to be maintained. Deciding on and maintaining the student model can be done by the teacher and by a computer system. How these two can work together is the next question in the informatical educational approach.

20.2.4 Educational Software

To support the meaningful learning of multiplication facts, Klep and Gilissen (1986, 1987) developed a computer program that offers children the opportunity of changing between a formal representation of products and four dynamic multiplication models: a number line, a grid and bars and sets (Figs. 20.1, 20.2, 20.3 and 20.4). The icons at the bottom of the window can be used to change the model. At the left side, a list of neighbour exercises (Dutch: *buursom*) fitting to the current exercise can be displayed on demand: These neighbours can be displayed with the main exercise in the window. On demand, the relation between the main exercise and the neighbour can be explained by an animation. This learning environment offers the possibility of using strategies and of producing personal favourite strategies. In this learning environment, a student finds his personal knowledge and fitting strategies indicated as a help to find strategies that fit to his personal level and appreciations.

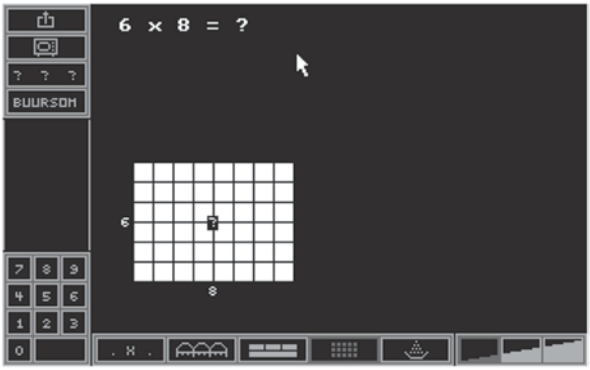


Fig. 20.2 6 · 8 as a grid

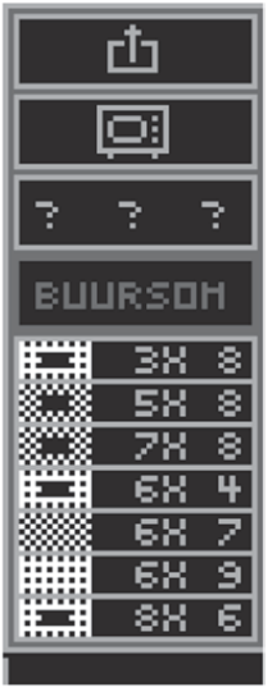


Fig. 20.3 Relevant strategies for 6 · 8, which are represented as 'neighbour' products (in Dutch: Buursom). The grids at the left of the neighbours are indicators telling the child how good he knows this neighbour on behalf of the child's personal learning history in the program

The student model contains a performance history over all the sessions a child had with the program. The system 'knows' the trends in response time and correctness for each individual exercise. This information is represented in the little indicators at the left of the neighbours, which remind the student of which neighbours are 'well known' to him. Subsequent sessions are planned by the program: In the first part of a session, well-known exercises are presented without a model; in the second part, exercises which are not yet well exercised in changing model representations

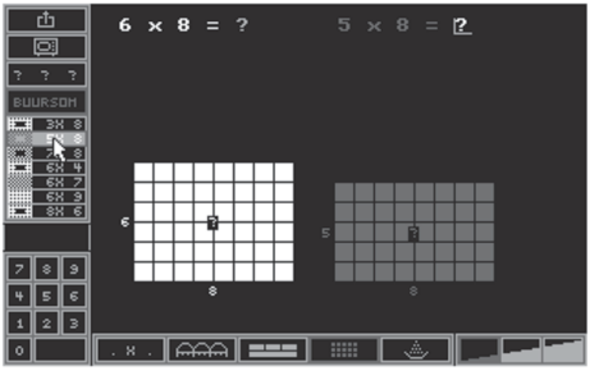


Fig. 20.4 $6 \cdot 8$ and its well known neighbor, $5 \cdot 8$

are presented; in the third part, new exercises are offered; and in the last part, the student can freely choose the exercises he/she wants to ‘play with’. A teacher program informs the teacher about the student’s level and the trend. Other details will not be discussed here.

This program offers the children a learning environment, where they can solve elementary multiplications using their preferred strategy. The program has a small student model that describes the level of a student in a rote-learning perspective. This student model is a basis for hinting at what might be an appropriate neighbour strategy, for planning subsequent sessions and for informing the teacher about the students’ progress.

During the development of this program and after the program was published, a mixture of research questions arose: Would it be possible to create:

- A program with a flexible user interface in which children can express their thinking easily,
- An interface which offers several representations,
- A program that can make interpretations of a child’s strategies and
- A program that can maintain a student model as a basis for educational decisions about further learning and for the dynamical composition of the mathematical user interface?

Would it be possible to comment on a child’s strategy related to his learning history?

A positive answer to these questions is only possible when there is a detailed student model, which describes at least the students’ strategies and factual knowledge.

In the next section, the interpretation of calculations and the underlying strategies is analyzed. The next section starts with a short discussion of intelligent tutoring systems (ITSs) because the kind of program under discussion has some features in common with the ITS. However, the idea of a student who is mathematically active and who constructs ‘new’ strategies goes beyond the traditional ITS. This ‘beyond’ is the central theme of the next section.

20.3 Towards an Informatical Model of Interpretation of Calculating/Doing Arithmetic

To build a computer program with features as discussed in the last section, it is important to decide what kind of a program it should be. At the end of the 1980s, ITSs were popular. These ITSs were generally considered for assessing the child's solutions by comparing these solutions with expert solutions. In fact, some kind of reproductive learning paradigm is the basis of these ITSs. An example is a schoolbook-related computer program by Hennecke et al. (2002), *Mathematik heute*—Bruchrechnung, which provides precise diagnosis of errors, tips and examples of solutions.

In the concept of realistic mathematics (Treffers 1987; Freudenthal 1991; Gravemeijer 1994; van den Heuvel-Panhuizen 1996), and in wider constructivist literature (e.g. Cobb et al. 1992), the fundamental idea is that children can reinvent mathematical ideas and solutions themselves. These solutions should be improved in classroom discussions and in processes like progressive mathematisation and progressive algorithmisation (Treffers 1987; Gravemeijer 1994).

The educational requirements, which are connected with these realistic and constructivistic ideas, led to a new kind of ITS. Freudenthal (1984) formulated the requirements clearly and simply: when a child is solving $7 \cdot 8$, he/she should be free to write $3 \cdot 8 = 24$ as the first step. The computer should nod, to give its approval, and the child should be free to write $6 \cdot 8 = 48$. Again, the computer ought to nod to encourage and the child could write $48 + 6 = 54$. Alternatively, the child may type $48 + 2 = 50$, and then 56. Freudenthal thought it would not be possible to create a program that could evaluate the complete calculation. It turned out, however, that it is possible.

20.3.1 A Nodding Computer?

The metaphor of a nodding computer could be a teacher looking at and listening to what a child does and who creates an interpretation of what the child does. Such an interpretation is something like finding a calculation which could be written down or stated in the steps the child produces. Of course, a teacher cannot be completely sure of a child's thought process. Sometimes, he/she has to ask what the child has done. The teacher may get an authentic answer; or the teacher may get an answer in which the child reconstructs his or her original thoughts, because the child does not know what he or she has thought anymore. The more flexible a teacher is, the better his/her chances are to understand the child and to be more sensible to alternative interpretations. In German there is a verb *nachvollziehen*, which has the approximate meaning of 'to re-enact', building a calculation or an argument that seems to match with the calculation or solution the child proposed.

These features lead to some design requirements for a 'nodding computer', which I will formulate here in single sentences:

- The first step in the program design should be to design a system, Arithmeticus, producing all possible solutions—like the flexible teacher can produce—for any exercise (a mathematical expression to be calculated) or mathematical expression which has to be solved.
- The second would be to build a program, solution matcher, which can match the child's solution with the possible solutions of Arithmeticus, like the observing teacher.
- The third step is the design of a program, assessor, which can evaluate the interpretations.
- The fourth step is to build a storing system, some kind of a database, which stores the interpreted children's solutions.
- A virtual teacher, planner, can provide comments and further questions for the child and it can produce changes in the user interface.

These design questions were the leading points in the Information Systems for (the support of) Mathematical Activity (ISMA) project . The ISMA project is documented by Klep (1998).

20.3.2 Central Hypothesis in Competence Modelling

The central hypothesis in the ISMA design is: Every correctly produced (written or spoken) calculation can be represented as a sequence of mathematical steps. This hypothesis is not a psychological one; it is a mathematical approach to calculations, as Turing (1948) and Kleene (1952) have proposed. This hypothesis does not say anything about a student's thinking; it states that it is possible to image a calculation in a sequence of mathematical steps. Later in this section, I will explain how these sequences of mathematical steps can be produced by Arithmeticus, which behaves as a flexible teacher as far as flexible production of calculations is concerned. In the diagram in Fig. 20.5 the functions of Arithmeticus are represented.

This approach of the ISMA project provides the opportunity for research on the learning of arithmetic, because psychological and educational theories, as mentioned in the introduction of this chapter, are operationalised in or can be related to the program.

20.3.3 Informatical Education Science: Programs are Theories

In fact, the program group in Fig. 20.5 is a theory which can be approved by falsification. Falsification questions could include:

- 'Can the program make interpretations for any calculation?'
- 'Are the interpretations made by the programs valid for educational experts?'
- 'Are the competences in the program model compatible with, for instance, psychometric test results?'

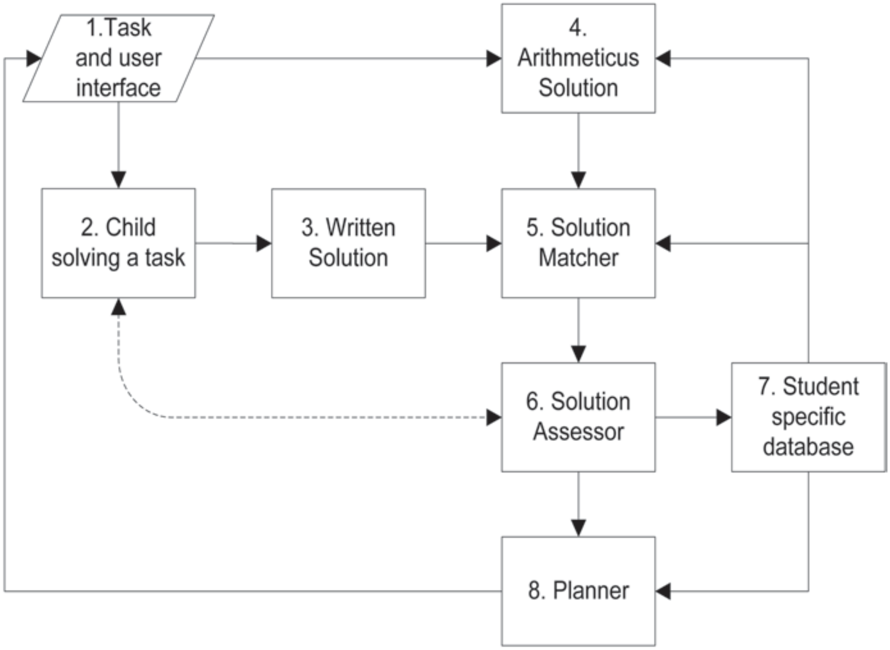


Fig. 20.5 Some relations between program components in the ISMA project: *steps 1 and 2:* A child solves a task in a learning environment; *step 3:* The written solution is documented in the user interface; *step 4:* Arithmeticus produces calculations, which are sequences of mathematical steps; *step 5:* The solution matcher tries to re-enact (nachvollziehen), in a mathematical way, the child's calculations; *step 6:* The assessor adds psychological concepts to the mathematical interpretation of the child's calculation; *step 7:* By storing the interpretations of the child's solutions in the database, a learning history is built; Arithmeticus, the solution matcher and the assessor can make new interpretations of what the child does; *step 8:* The planner either produces new tasks or gives the child freedom to experiment

This kind of theory has very special features. The kernel of the theory is a dynamic model: it is dynamic because its behaviour changes by 'the students' and the systems' 'learning' and its behaviour depends on parameters, which can be changed.

This is a new kind of educational research, which I call: Informatical educational science.

Important parts of the dynamic model in the ISMA project, e.g. Arithmeticus, will be discussed further in the next sections.

20.3.4 From Strategy to Transformations

The central hypothesis put forward in the last section is not a usual one in the psychology of the learning of mathematics. Usually, different calculations in school are considered as strategies (e.g. Ashcraft 1990). The concept of strategy is not a good basis to understand the production[1] of calculations by children or by Arithmeticus, because strategies—as I mentioned earlier—are the result of the production of calculations, not the production itself.

The core of this chapter is the idea of modelling arithmetical solutions as algorithms, to be defined as sequences of subsequent arithmetical transformations. This is a furthering hypothesis to the central hypothesis: in arithmetic, the mathematical steps can be represented by arithmetical transformations.

This approach starts from the mathematical idea of Turing machines (Turing 1948) and leads to a computational model for arithmetical competence, which can be used for making very precise interpretations of any arithmetical solution and for recording and planning individual learning paths for the children.

Solutions for elementary arithmetical tasks are usually discussed in terms of 'arithmetical strategies', which are defined by characteristic arithmetical steps in a solution. A strategy covers many different calculations and is not appropriate for discussing important details in a child's calculation. Nevertheless, strategies will be discussed as categories of calculations.

I will explain how the concept of algorithms as sequences of subsequent arithmetical transformations can be represented in a dynamic computer model of arithmetical competence. This cohesive approach makes sense because such a computer model opens five interesting fields of didactical research:

1. Analysing mathematical aspects of children's computations by matching them with sequences of subsequent arithmetical transformations;
2. Modelling psychological concepts like (the development of) routines, automatisms and facts as a basis for actual dynamic student models;
3. Analysing mathematical and psychological qualities of children's computational competence and modelling the concept of 'zone of proximal development of a child';
4. Developing user interfaces which can collect data for an individual dynamic computer model of arithmetical competence for each child; and
5. Developing user interfaces which can reflect actual child knowledge from the model as a problem-solving support for a child and can propose solutions based on the student model: reproductive proposals as well as new proposals in the zone of proximal development.

In the next section Arithmeticus, a dynamic model of producing calculations, will be explained. Arithmeticus plays a central role in the informatical educational theory of arithmetical competence.

[1] In this chapter 'production' is not meant in an industrial or technical meaning but in the original Latin meaning of 'to create and to reveal something', 'bringing out something', or 'to bring up something'.

20.4 Arithmeticus: A Model for 'all Possible Arithmetical Calculations'

20.4.1 'Algorithms' as an Alternative for 'Strategies'

In educational literature (Schipper 2009, Padberg and Benz 2011), some strategies for addition up to 100 are discussed. Two of them are:

a. complete decimal decomposition: 34+27=(30+20)+(4+7)=50+11=61 and
b. partial decimal decomposition: 34+27=(34+20)+7=54+7=61.

Partial decimal decomposition (strategy b) seems to be less complex than complete decimal decomposition (strategy a), because only two additions have to be performed and only three numbers (34, 20 and 7) are involved and not four (30, 20, 4 and 7) as in strategy a. The complexity of these two calculations depends not only on the number of numbers involved in the calculation and the number of sub-additions but also on how the sub-calculations like (30+20) and (34+20) are solved or should be solved.

In the complete decimal decomposition (strategy a), for instance, the addition of tens (30+20) could be done parallel to the addition of ones:

30+20=3 tens+2 tens=5 tens=50 or short:

30+20 (reminds of 3+2=5)=50.

Another solution for 30+20 is counting forward in steps of 10: 30, 40, 50.

In the partial decimal decomposition (strategy b), the addition (34+20) could involve counting forward from 34 in steps of ten:

34, 44, 54 or a short version of counting forward where 34+20=54 could be done parallel to 30+20=50, 34+20=30+20 (mind the 4)=50, 'and the 4' makes 54: The 4 of 34 is removed before and added after the addition 30+20=50.

Although the strategies a and b seem to offer two different approaches, each strategy can be performed in several actual calculations using several sub-calculations.

The calculation 34+27=30+20+4+7=50+4+7=54+7=61 cannot be attributed clearly to strategy a or b: the step 50+4+7 might be the result of a complete decomposition as in strategy a or of a strategy b where the analogy between 34+20 and 30+2 (with 4) is completely formulated as 50+4.

To understand calculations, involving sub-calculations seems to be necessary. When taking into account sub-calculations, the disadvantage is that the researcher has to cope with a large collection of slightly different calculations. A richer but still well-structured model for calculations and sub-calculations is needed.

Maybe thinking of calculations in terms of algorithms—sequences of mathematical transformations—offers a basis for such a model. Threlfall (2009) proposes, from a psychological perspective, a similar idea of 'transformation' for a better understanding of flexibility in calculation. Threlfall (2009, p. 545) discusses the idea of strategic choice, which is necessary for understanding why children do apply a specific strategy for a problem. This strategy thinking is not applicable for understanding the massive variety of calculations children produce, because it focusses on the solution as a whole, not on the small differences between different calculations. Threlfall (2009, p. 547) proposes the psychological concept of 'zeroing in' to explain the finding of fitting transformations. This 'zeroing in' can explain the variety of solutions and flexibility in calculation. From a scientific perspective, it is positive that the informatical–mathematical thinking (Klep 1998) and the psychological thinking (Threlfall 2009) match: the congruence provides some mutual validation of both approaches.

20.4.2 Production Rules for Calculations

A different approach to understand calculations is to concentrate on production rules for (all possible) calculations instead of categorising them. In the theory of (primitive) recursive functions, Kleene (1952) proposed thinking of production rules for all possible arithmetical expressions. It might be a good scientific project to investigate whether this approach can be used for understanding all possible calculations too. Turing (1948) proposed the concept of an abstract machine, which should produce all possible proofs (and calculations). I will leave aside the mathematical and philosophical discussions related to this approach and will use the idea of production rules for a better understanding of elementary calculations.

Two fundamental questions for modelling computational competence by means of sequences of subsequent transformations (algorithms) are:

- Is it possible to understand calculations as compositions of elementary steps?
- Is it possible to define a (finite) set of elementary steps which can be used to produce all possible arithmetical calculations?

In the theory of primitive recursive functions, five elementary functions are defined:

- The constant function,
- The successor function,
- The projection function (to take an argument of a function),
- Composition function and
- A primitive recursion function, which enables recursive definitions, which can be used to define addition as a recursion of counting and multiplication as a recursion of addition.

A subset of arithmetical calculations, as used in elementary education, can be rewritten by this set of construction rules. The theory of primitive recursive functions does not deal with the decimal structure of numbers. However, maybe Turing's and Kleene's ideas can be used for producing all calculations.

20.4.3 Transformation of Mathematical Expressions

In this section, I will explain how a calculation can be understood as a sequence of steps. In each step, a mathematical expression is produced and the final step leads to a single number or a desired mathematical expression format. For example, the format '…remainder…' defines a desired mathematical expression format for divisions:

$45:6 = 7$ remainder 3. In this example, $45:6$ is 'transformed' in hidden steps into the expression '7 remainder 3'. Often, like in the illustrated example, these subsequent mathematical expressions are connected with equal signs.

Sometimes special notations are used, like in the case of the complete decimal decomposition in calculation a:

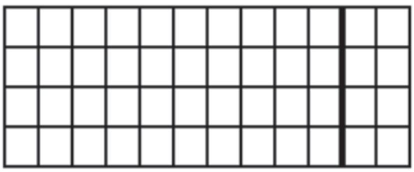
$$\begin{array}{r} \underline{34+27:} \\ 30+20=50 \\ \underline{4+7=11} \\ 34+27=61 \end{array}$$

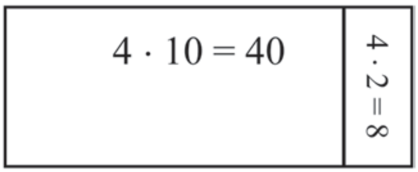
This scheme represents a format for the necessary steps in this calculation: decomposition of the addenda and the vertical addition of the two sums.

Strategy b (partial decimal decomposition) can be represented by a sequence of steps:

$$34 \xrightarrow{+20} 54 \xrightarrow{+7} 61$$

Models sometimes offer a more complex representation of a calculation format. An example for multiplication is the decomposition of $4 \cdot 12$ in $4 \cdot 10 + 4 \cdot 2$ and represented in a grid or a rectangle model:



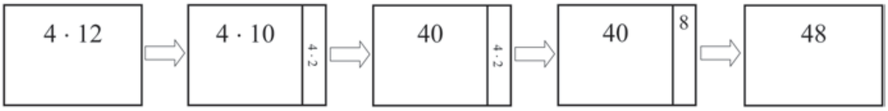


These different representations of calculations can be understood as 'tools'[2] (iconic representations) to organise the calculation.

The grid representation of $4 \cdot 12$ can be observed as a structure, which, in a second step, is reconstructed in two elements, which can be calculated easily: The $4 \cdot 12$ rectangle is reconstructed or transformed in two parts: $4 \cdot 10 + 4 \cdot 2$.

In the case of the rectangle, a calculation can be understood as a sequence of transformations of the rectangle and the mathematical expressions in the rectangles:

[2] In German, the nouns 'Bearbeitungshilfe' and 'Veranschaulichungsmittel'.



In this sequence, the transformation of $4 \cdot 12$ into $4 \cdot 10 + 4 \cdot 2$ is not yet clearly represented. In fact, the number 12 is decomposed and following the distribution rule the multiplication $4 \cdot (10+2)$ is distributed in $4 \cdot 10 + 4 \cdot 2$.

The next important aspect is how both terms $4 \cdot 10$ and $4 \cdot 2$ can be treated as mathematical expressions themselves and can be replaced by their values.

20.4.4 Calculations as Sequences of Transformations

These observations lead to the next step in thinking about calculations.

A calculation can be understood as a sequence of transformations of an initial structure. Transformation steps can be:

- Application of decomposition and composition rules,
- Application of algebraic rules,
- Taking apart, transforming and replacing terms (e.g. projection and composition in the theory of primitive recursive functions) and
- Application of well-known, more elementary calculations.

In an abstract representation, it looks like the sequence in calculation I. This sequence represents minutely, the transformation steps in the rectangle. The two strategies, a and b, for $34+27$ are represented in calculations II and III. Calculation IV starts as calculation III and goes on like calculation II; it is a mix of strategies a and b.

Calculation I: 4·12 as a sequence of transformations.	Calculation II: Complete decompoistion, along strategy a).	Calculation III: Partial decomposition, along strategy b).	Calculation IV: Calculation starting with a partial decomposition.
a. 4 · 12	a. 34 + 27	a. 34 + 27	a. 34 + 27
b. 4 · (10 + 2)	b. (30 + 4) + (20 + 7)	b. 34 + (20 + 7)	b. 34 + (20 + 7)
c. 4 · 10 + 4 . 2	c. 30 + 4 + 20 + 7	c. 34 + 20 + 7	c. 34 + 20 + 7
d. 4 · 10	d. 30 + 20	d. 34 + 20	d. 34 + 20
e. 40	e. 50	e. 54	e. 30 + 4 + 20
f. 40 + 4 · 2	f. 50 + 4 + 7	f. 54 + 7	f. 30 + 20
g. 4 · 2	g. 4 + 7	g. 61	g. 50
h. 8	h. 11		h. 50 + 4
i. 40 + 8	i. 50 + 11		i. 50 + 4 + 7
j. 48	j. 61		j. 4 + 7
			k. 11
			l. 50 + 11
			m. 61

- In calculation IV.d, the direct transformation $34+20=54$ is not used as in calculation III.d, but $34+20$ is solved by decomposition itself.
- The calculations in II.f. and IV.i. are both $50+4+7$: the difference is that in calculation II the expression comes from $30+4+20+7=50+4+7$ and in calculation IV it comes from an incompletely performed calculus along the decomposition of $34+20=50+4$.

A first glance tells us that, in these sequences of transformations, small differences in calculations can be represented.

20.4.5 Algorithms as Sequences of Transformations

Each subsequent mathematical expression in a calculation can be annotated by a description of the transformation on that mathematical expression:

Calculation V: A sequence of mathematical expressions and the subsequent transformations.
(This is the same calculation as in calculation I.)

a.	$4 \cdot 12$	initial mathematical expression
b.	$4 \cdot (10 + 2)$	decomposition of second number
c.	$4 \cdot 10 + 4 \cdot 2$	distribution of multiplication
d.	$4 \cdot 10$	take first mathematical sub-expression
e.	40	apply basic knowledge
f.	$40 + 4 \cdot 2$	replace sub-expression $4 \cdot 10$ by 40
g.	$4 \cdot 2$	take second sub-expression
h.	8	apply basic knowledge
i.	$40 + 8$	replace sub-expression $4 \cdot 2$ by 8
j.	48	apply basic knowledge

Taking apart the descriptions of the subsequent steps brings:

Algorithm 1: a sequence of transformations.

a. initial formula
b. decomposition of second number
c. distribution of multiplication
d. take first sub-expression a
e. apply basic knowledge
f. replace sub-expression a
g. take second sub-expression b
h. apply basic knowledge
i. replace sub-expression b
j. apply basic knowledge

A mathematical expression like $4 \cdot 11$ or $4 \cdot 13$ might be solved with a similar sequence of transformations with the same description. Hence, this list of transformations can be applied to several mathematical expressions and therefore this sequence of transformations can be interpreted as an algorithm (Algorithm 1), which can be successfully applied to at least the mathematical expressions $4 \cdot 12$ and $4 \cdot 11$.

A further example is given in calculation VI:

Calculation VI: Application of **Algorithm 1** to the formula 4·22

a.	$4 \cdot 22$	initial formula
b.	$4 \cdot (20 + 2)$	decomposition of second number
c.	$4 \cdot 20 + 4 \cdot 2$	distribution of multiplication
d.	$4 \cdot 20$	take first sub-expression $4 \cdot 20$
e.	80	apply basic knowledge
f.	$80 + 4 \cdot 2$	replace sub-expression $4 \cdot 20$
g.	$4 \cdot 2$	take second sub-expression $4 \cdot 2$
h.	8	apply basic knowledge
i.	$80 + 8$	replace sub-expression $4 \cdot 2$
j.	88	apply basic knowledge

If decomposition of a number, distributivity, taking and replacing mathematical sub-expressions are general—or at least on certain domains—applicable transformations, and if $4 \cdot 20 = 80$ and $80 + 8 = 88$ are basic knowledge, then Algorithm 1 can be applied to $4 \cdot 22$ too. If $4 \cdot 70 = 280$ and $280 + 8 = 288$ are basic knowledge, then, algorithm 1 can be applied to $4 \cdot 72$ as well.

Applying an algorithm to any mathematical expression might be successful or might fail. Hence, Algorithm 1 might be applied successfully to $4 \cdot 12$ and $4 \cdot 22$. Testing algorithm 1 on $4 \cdot 123$ might fail, because the decomposition of 123 is $100 + 20 + 3$ or $120 + 3$ or $100 + 23$. Maybe this decomposition and the following distribution are not available. If this decomposition and distribution are successful, then the availability of basic knowledge for $4 \cdot 120$, $4 \cdot 100$, or $4 \cdot 23$ might be a problem. Therefore, the steps e., h. and j. might fail.

The idea of testing algorithms makes the idea of the domain of an algorithm meaningful. On some mathematical expressions, the algorithm will succeed, on others it will fail, because certain transformations might fail or some basic knowledge is not available.

20.4.6 Domains of Algorithms

The concepts are expressed in detail within this section; this might prove to be a tedious read for the reader but from a mathematical and academic viewpoint it is essential in formulating the hypothesis.

The approach of testing algorithms as mentioned in the last section leads to two different interpretations of the concept of 'the domain of all mathematical expression an algorithm can be applied to'.

Before venturing into these two interpretations, there is a need for defining the idea of ‘set of mathematical expressions an algorithm A can be tested on’. Note the difference in the terms ‘domain’ and ‘set’. The set contains mathematical expressions which have to be tested; the domain contains expressions which were successfully tested. In a computer system, it is not a good idea to have infinite sets, like the set of all mathematical expressions. A computer cannot cope with infinite sets. Therefore, limited sets are needed and we need an appropriate limitation like this one: A set of mathematical expressions can be defined as a current universe of all mathematically possible expressions, which is of interest in an actual situation. Examples of this kind of current universes are: the additions up to 20 or the multiplications up to 100.

Given:
- an algorithm A,
- a current universe U and
- a set of actual basic knowledge ABK of a person P and
- a set of actually available transformations AAT of a person P,

then:
the actual domain of algorithm A for person P can be found by testing the algorithm A on universe U using the actual basic knowledge ABK and the actually available transformations AAT of person P.

Definition 1: Actual Personal Domain

This definition of actual domain defines a ‘student’s personal domain of an algorithm’. Later, this kind of domain will be discussed further.

A second interpretation of domain of A is a mathematical one:

Test algorithm A on all mathematical expressions ME of the current universe U and collect for each ME

a. The basic knowledge and
b. The transformations,

which are needed to enable a successful performance of A.

The result of this testing of A on the current universe U leads to a set of triples of a mathematical expression ME of U and the basic knowledge and the transformations which are necessary to perform A on that mathematical expression.

Given:
- an algorithm A and
- a current universe U

then:
A can be tested on each Mathematical Expression ME from U.
Either A fails on ME or the triple (ME , necessary basic knowledge, necessary transformations) is created.
The Maximum Domain of A on U is the collection of all these triples.

Definition 2: Maximum Domain

This second interpretation will be called the maximum domain of A in the current Universe because it does not depend on the knowledge of a person P. The maximum domain is congruent with the actual domain of algorithm A for a person P who has all necessary basic knowledge and all necessary transformations available.

(In Sect. 20.5.2, combinations of an algorithm A and an actual domain of A will be interpreted as (dynamic) concepts.)

This maximum domain can be used by the planning program to find out which expressions can nearly be solved using a certain algorithm A by a student. The planning-program can find out for which expressions and algorithm A a student knows the transformations involved but not all basic knowledge, or for which expressions and algorithm A, a student knows the basic knowledge but not all transformations. Offering these expressions as exercises might result in a good exercise in the zone of proximal development.

20.4.6.1 Domains as a Model for a Student's 'Zone of Proximal Development'

The availability of 'personal actual domains' and 'maximum domains' opens a highly sensitive way of thinking about flexible calculation; sensitive to 'what a child is able to' or 'a child's arithmetical competence'. It is an interpretation of what a 'nodding' teacher tries to do when observing and coaching a working child. The actual basic knowledge ABK and a set of actually available transformations (AAT) of an individual child are essential in this conceptualisation.

The traditional thinking of phases in the learning of arithmetic and the traditional thinking of more or less fixed strategies is left behind. Calculating has been modelled as the generation of sequences of transformations in a student. This generation has constraints, two of which are the actual basic knowledge and the set of actually available transformations of a student.

20.4.7 Arithmeticus: A Computer Model for Generating Algorithms

In the last section, algorithms—sequences of subsequent transformations—are proposed as a model for calculations generated by a student. In this section, Arithmeticus will be discussed. Arithmeticus is a computer program for the generation of algorithms.

Arithmeticus is a computer program in which mathematical expressions, transformations and algorithms have been implemented. Due to the paucity within the subject of this chapter, I will not discuss the informatical definitions of 'mathematical expression', 'transformation' and 'algorithm'. The important message in this section is: Algorithms can be tested by a computer. The maximum domain and the actual domain of an algorithm like Algorithm 1 are dynamic objects, which can be (re-)calculated at any moment for any universe and other sets of basic knowledge.

In Arithmeticus, transformations are operationalised at a computational level by procedures, which can transform a mathematical expression. Arithmeticus can successfully perform a transformation procedure on a mathematical expression or the transformation procedure fails. If one transformation fails, the whole algorithm fails. An algorithm is tested on a mathematical expression by applying the subsequent transformations on the generated successive mathematical expressions. The last mathematical expression of the calculation should be an acceptable result: e.g. a number.

20.4.8 A Generative Model for Algorithms

'Is it possible to define a (finite) set of elementary steps which can be used to produce all possible arithmetical calculations?' is the second question that arises in Sect. 20.4.2.

In Sect. 20.4.4, a number of transformations were presented. This Sect. 20.4.8 focusses on the production of algorithms based on a chosen set S of transformations.

The idea of generating algorithms is to test successive transformations on an initial mathematical expression getting sequences, like in calculation I to calculation IV. Although this might be a straightforward and attractive idea, it is a computational monster: Systematically testing six different transformations to produce algorithms of 6 to 12 steps would result in over $6^6=46{,}656$ to $6^{12}=2{,}176{,}782{,}336$ evaluations of transformations. If there are 20 different transformations in S, the number of tests to find all algorithms will be discouragingly large: even for computers.

To prevent the search tree from exploding, several further concepts are implemented in Arithmeticus. I will discuss some of them to give the reader an idea of how Arithmeticus can generate algorithms.

Problem Space Given any mathematical expression, a child might produce calculations by using:

- 'Personal transformations' like well-known facts, well-exercised automatisms and fairly known algorithms and
- Mathematical transformations, e.g. decomposition, distribution or taking a mathematical sub-expression.

The set of these personal or mathematical transformations which a child might use can be interpreted as a 'dynamic problem space' (Newell and Simon 1972; Klep 1998; Klep 2000). A problem space to a mathematical expression F for a child or a system S is the set of alternatives that S might be aware of when thinking about how to transform F. There are three reasons why I use the concept 'dynamic problem space':

- Because a child is not aware of everything he/she knows, the problem space changes from one mathematical expression to another while he/she works on a task.
- As a result of his/her learning process, his/her problem space will change in time: the actual problem space to a mathematical expression F will change by learning.
- While working on a task, a child might remember some actual solutions or results and might apply these in a next step. For example, when calculating $66\cdot 88$, the result $6\cdot 88=528$ leads to $66\cdot 88=5280+528$ by calculating $60\cdot 88$ as $10\cdot 528=5280$.

When reducing a mathematical expression, Arithmeticus can use *different* dynamic problem spaces:

- A dynamic problem space based on a database of specific child knowledge,
- A dynamic problem space with mathematical transformations and without (specific) facts or algorithms,

- A dynamic problem space with a restricted set of transformations, facts and automatisms and
- Combinations of these.

Arithmeticus can solve exercises with child-specific dynamic problem spaces: It brings shortcuts in the calculation trees by pruning mathematically necessary sequences of transformations, as ‘the child has fast solutions available’.

This pruning of well-known sub-calculations is a fundamental feature in generating calculations in Arithmeticus. This pruning leads to the simple and effective idea of Arithmeticus as a system that learns together with a child. The more the child learns and the more the student model grows, the more well-known calculations Arithmeticus has available, the more Arithmeticus can prune in new or long calculations and the more short calculations Arithmeticus can produce. This learning effect seems to be a good model for the learning of students in the usual mathematics education.

(Pseudo) Orthogonality An interesting idea is that sub-problems can be treated more or less independently from a main problem. For instance, $27 \cdot 45$ can be transformed (decomposition and distribution) in $20 \cdot 45 + 7 \cdot 45$. In a next step, $20 \cdot 45$ might be solved independently of the main calculation of $27 \cdot 45$. There are different possible calculations to solve the sub-problem $20 \cdot 45$, for instance:

$$20 \cdot 45 = 20 \cdot 40 + 20 \cdot 5 = 800 + 100 = 900, \text{ or}$$
$$20 \cdot 45 = 2 \cdot 450 = 900 \text{ or}$$
$$20 \cdot 45 = 20 \cdot (50 - 5) = 1000 - 100 = 900.$$

All these different calculations for $20 \cdot 45$ lead to different calculations of $27 \cdot 45$. If there exists one solution for $20 \cdot 45$, it can be used as a sub-solution in $27 \cdot 45$ and it can be replaced by any other solution of $20 \cdot 45$.

While producing calculations for $27 \cdot 45$, the search tree for $20 \cdot 25$ is independent of the search tree for $7 \cdot 45$. They seem to be perpendicular or orthogonal to each other.

Search trees for sub-expressions are often orthogonal to each other. Still, there are problems with orthogonality of sub-calculations, which became clear in the first prototypes of Arithmeticus:

Calculation VII: An infinite loop of transformations.

a. $7 \cdot 10$
b. 10
c. $5 \cdot 2$
d. $7 \cdot (5 \cdot 2)$
e. $2 \cdot (7 \cdot 5)$
f. $7 \cdot 5$
g. 5
h. $10 : 2$
i. $7 \cdot (10 : 2)$
j. $(7 \cdot 10) : 2$
k. $7 \cdot 10$
l. ...

leading to an infinite loop.

Unfortunately, sub-calculations are not always as independent of the main calculation as one would like them to be. It is necessary while generating a sub-calculation, to keep an eye on 'preceding' steps, to avoid infinite loops. For that reason, the term 'pseudo orthogonality' is used.

Context of a Mathematical Expression in an Evaluation To solve the problem of infinite loops like in calculation VII, the concept of context of an expression in a calculation is important.

In the example of $7 \cdot 5$ over here there is a need to check in all preceding steps if the mathematical expression F which has to be solved is used in any preceding step of the current calculation. All preceding calculation steps belong to the (broader) context of F.

A smaller context of F is the rest mathematical expression, which remains when F is taken out from a more complex mathematical expression.

$3 \cdot 45 + 27 + 3$,
$3 \cdot 45$ (a mathematical sub-expression) with context $27 + 3$.

Sometimes, it is effective to take the context into account when developing a calculation:

$6 \cdot 14 + 3 \cdot 13$
  $6 \cdot 14$ (context $+ 3 \cdot 13$)
  $2 \cdot 3 \cdot 14$
    $2 \cdot 14$
    28
  $3 \cdot 28$ (which will not be evaluated because it can be taken together with $+3 \cdot 13$ in the context)
$3 \cdot 28 + 3 \cdot 13$
$3 \cdot (28 + 13)$
  $28 + 13$
  41
$3 \cdot 41$
123

The idea to stop the evaluation of $6 \cdot 14$ at $3 \cdot 28$ is useful, because $3 \cdot 13$ (in the context of $6 \cdot 14$) can be contracted with $3 \cdot 28$, leading to $3 \cdot 41$, which can be evaluated with an automatism.

Arithmeticus, therefore, always looks into the contexts, to find combinations, which are 'easy to evaluate'.

The context of a calculation is interesting in a psychological sense. While generating a calculation, the stepwise changing context has to be remembered. This relates in some sense to the workload in the short-term memory. The psychological question is whether the number and the complexity of contexts in calculation steps make tasks more difficult. These observations are important when valuing calculations in a psychological perspective like 'heavy short-term-memory load'. In the TAL project (van den Heuvel-Panhuizen 2008), the researchers preferred the calculation

57 + 26
57 + 20 + 6
    57 + 20   (context +6)
    77   (context +6)
77 + 6
83

in place of the total distribution:

57 + 26
50 + 7 + 20 + 6
    50 + 20 (context +7+6)
    70   (context +7+6
70 + 7 + 6
    7 + 6   (context +30)
    13   (context +30)
70 + 13
83

The second solution demands more context management than the first one. However, these two algorithms demand different basic knowledge and automatisms.

Actual Solutions In $218 \cdot 345$, the number 218 can be decomposed into $200+18$. The number 18 is in the neighbourhood of 20: $18=20-2$. Therefore, $218=200+20-2$ and $218 \cdot 345=200 \cdot 345+20 \cdot 345-2 \cdot 345$.

Once the solution of $2 \cdot 345=690$ is available, solving $20 \cdot 345$ and $200 \cdot 345$ might be easy.

When Arithmeticus comes across a mathematical expression F in a calculation, it checks whether a solution of F is already available. That can be the case when F has been solved in another mathematical sub-expression in the same calculation or in a preceding calculation. Therefore, Arithmeticus stores solutions and looks in this store for actual solutions, to prune the search tree.

Set of Permitted Transformations Some transformations lead to complication of a mathematical expression, like the decomposition in: $34+17=34+10+7$.

Other complicating transformations are: replacing a number by a neighbour $17+19=17+(20-1)$, or distribution $34\cdot(40+5)=34\cdot40+34\cdot5$.

These transformations are used to change a mathematical expression in parts which are easier to solve.

These transformations should not be followed by their antagonists: $34\cdot45+34\cdot5=34\cdot(45+5)$ to get $34\cdot50$ or $17+(20-1)=17+19$.

When using a transformation, it might be a good idea to disable other transformations. Therefore, the set of permitted transformations is dynamically changing with the subsequent steps in a calculation.

Stop Criteria (to Stop Meaningless Searches) Identifying meaningless searches and stopping the evaluation process is important for effectively producing 'all possible evaluations of a mathematical expression'. In (pseudo-)orthogonal solutions, when using actual solutions, facts or automatisms or when using the context of a mathematical expression, stopping might often be useful.

Stopping the search process when an automatism is available is often effective, because a child will not choose an alternative when an automatism is available. Sometimes the context of the solution offers 'smart' combinations without using an available automatism.

Stop criterions are important and complex. This complexity has been solved in Arithmeticus by defining search processes with different stop criteria.

These six points in this section give an impression of how the production of algorithms in a search tree can be pruned. After the modelling of the search process, the following psychological questions come up:

- Do 'dynamic problem spaces' exist in students?
- Does a richer dynamic problem space of a student predict more flexibility in calculations?
- Are the concepts of pseudo-orthogonality and context helpful to understand why certain calculations are more difficult than others?
- Under what circumstances do children use actual solutions and when do experts use more actual solutions than poor calculators?
- Do children have the stop criteria to cut off useless search paths?

These questions might be answered when children could write down their calculations in a learning environment supervised by Arithmeticus, in its role as a nodding teacher.

In the next section, a first front end will be presented, in which students can express their calculations.

20.5 Transformation and Metamorphosis of Structures

In mathematics, problem solvers use symbols (numbers, operators and relation symbols) and models like set diagrams, a number line, a grid or bars, like in Fig. 20.6.

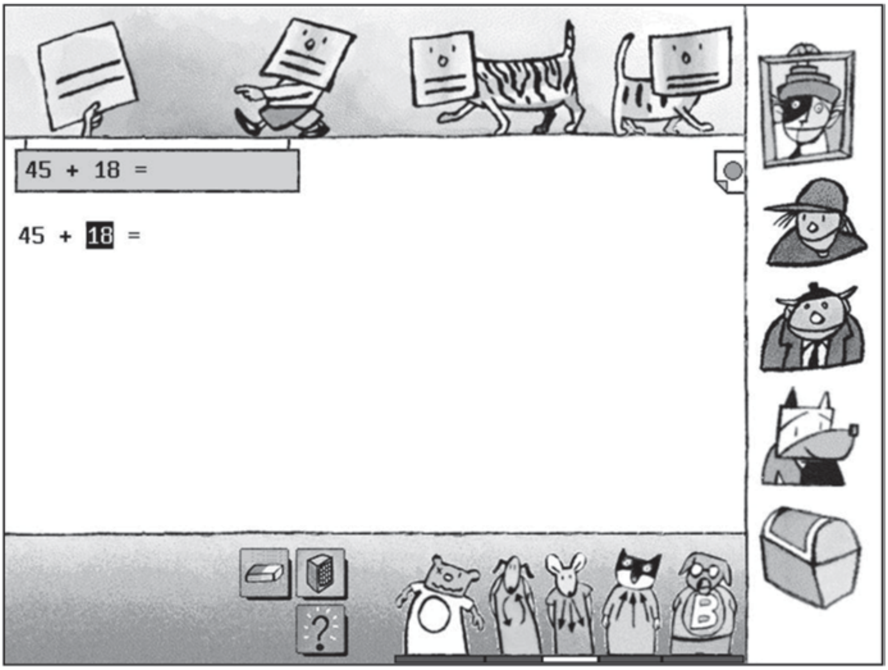


Fig. 20.6 45 + 18 (**a**)

In Sect. 20.4.3, some examples were given regarding how a calculation can be expressed in these symbols and diagrams. In the next section, the computer program *Plato en de rekenspiegel* (English ‘Plato and the math mirror’) (Klep and Spekken 1998) is be presented, which offers children several opportunities to transform mathematical expressions and which offers an individual child a reflection of its dynamic problem space connected with a current mathematical expression.

20.5.1 Plato and the Math Mirror

Plato en de rekenspiegel (Klep and Spekken 1998) is a computer program in which children can transform expressions.

Figure 20.6 shows a mathematical expression representation of 45 + 18. In the user front end, the number 18 is selected, marked in black; the third animal at the bottom, the one who can make decimal decompositions, has a bright marker, indicating it can propose a transformation. With a mouse click on this animal, the following decomposition transformation can be performed:

45 + 18 = 45 + 10 + 8. In a next step, the child can select the mathematical subexpression, 4 + 10 (Fig. 20.7), and he/she can replace 45 + 10 by 55.

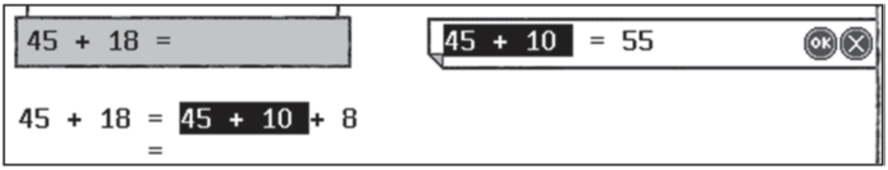


Fig. 20.7 45 + 18 (**b**)

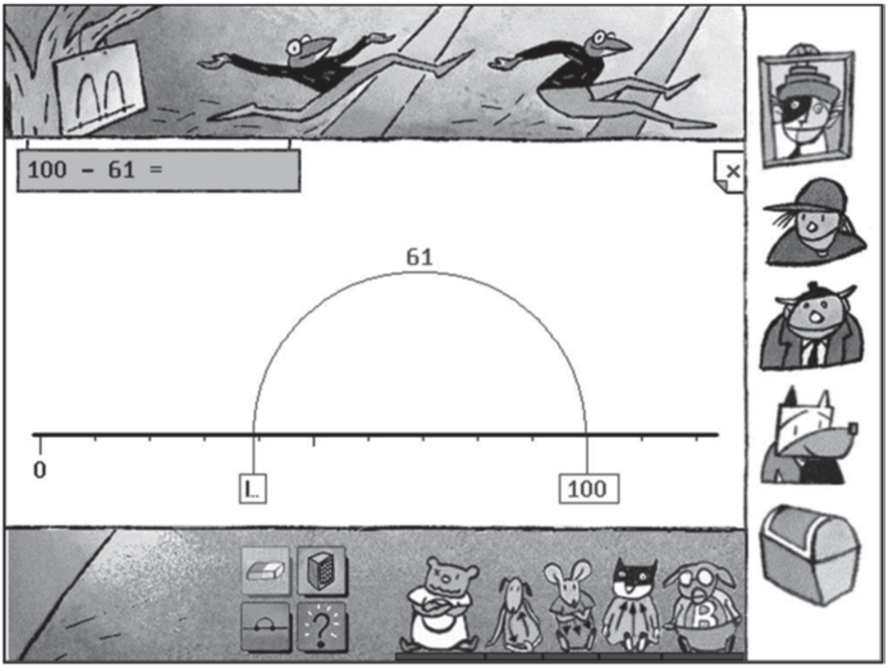


Fig. 20.8 100 − 61 (**a**)

While typing, a small window is opened and the child can see what he/she is replacing and the resulting new expression. Finally, the small window disappears and 45 + 10 + 8 is replaced by 55 + 8.

Figure 20.8 shows a number line representation of 100 − 61. The arc with the number 61 and the number 100 are given and can be selected by the child. Using the decomposition animal, two arcs are obtained: −60 − 1. In the small fields in Fig. 20.9, the numbers 40 and 39 can be filled in.

In Fig. 20.10, the decomposition animal proposes two alternatives for splitting 19: 19 = 10 + 9 or 19 = 20 − 1. The child can choose one of them, and that decomposition will be performed.

At any moment, the child can select a (sub-)expression and replace that (sub-) expression by means of the small worksheet as in Fig. 20.7. A child can select

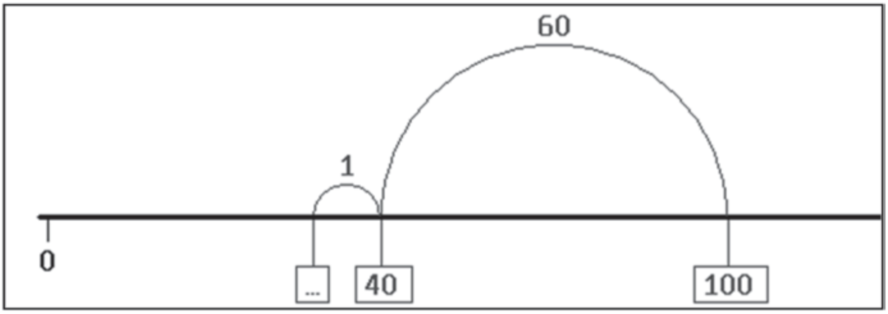


Fig. 20.9 100 − 61 (**b**)

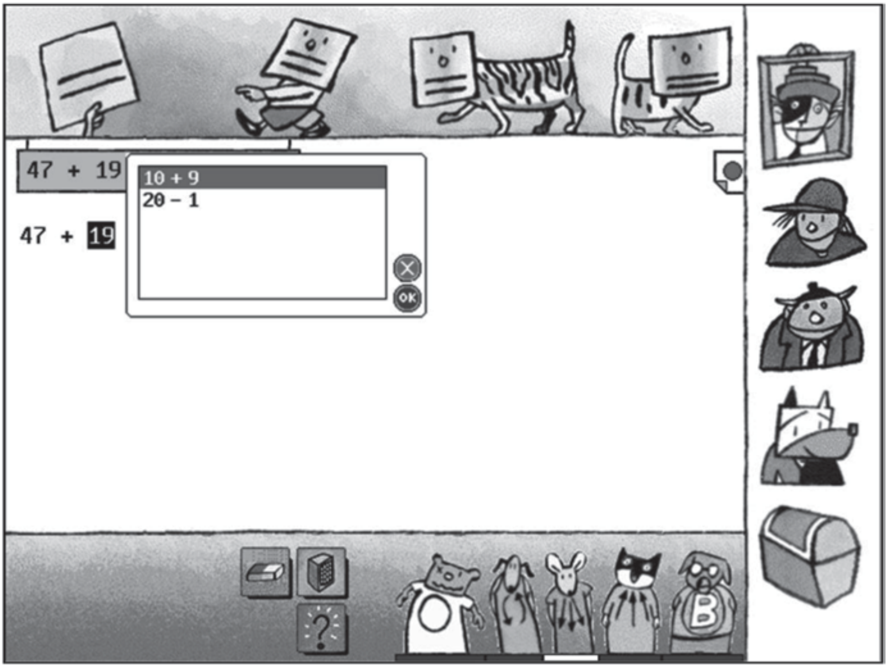


Fig. 20.10. 47 + 19 (**a**)

mathematical sub-expressions or a sub-graph to transform them. Each of the five animals at the bottom of the screen can propose several specific transformations on a selected (sub-)expression.

All selections and replacements by a child are tested for mathematical correctness and eventually feedback is given. All proposals of the animals at the bottom line are before presenting them, tested using the actual student model. The decomposition 19 = 20 − 1 is only meaningful if a child knows the difference between 19 and 20.

Other transformations of the animals -numbered from the left to the right at the bottomline- are:

1. Several kinds of rounding: $28+31=30-2+30+1=60-2+1$;
2. Commutation: $5+6=6+5$;
3. All kinds of decompositions like decimal decompositions, rounding up, distributions like $7\cdot(20-1)=7\cdot 20-7$ and transforming a product in a repeated addition;
4. Contractions like inversion of distribution and converting repeated addition in a product; and
5. Neighbours like $5+6=5+5+1$, $6\cdot 7=5\cdot 7+7$ or $6\cdot 7=2\cdot 3\cdot 7$.

At the right side in the screen, four further characters are presented: a sorcerer, a coach, Plato and a grandfather. The sorcerer can change the representation of the current problem between mathematical expression, number line and arrow representations. The coach comments on the performance of the child on the basis of the actual childs' input and of the student model. The grandfather can explain calculations, which the child is familiar with, somewhat familiar with and completely unfamiliar with. Plato is the child's friend and gives helpful remarks and suggestions in case the child remains inactive or makes errors.

20.5.2 Transformation of Structures: SITs

In Sect. 20.4.3, the multiplication $4\cdot 12$ is represented as a grid. A representation in symbolic mathematical expressions could be: $4\cdot 12=4\cdot 10+4\cdot 2=40+8=48$.

Both the grid and the mathematical expressions can be interpreted with the algorithm as presented in Calculation V. This is not the only possible interpretation, because 'written' and 'drawn' solutions do not tell the whole sequence of transformations generated by the problem solver. There are several interpretational problems:

- In informally written solutions, it is hard to find out the sequence of transformations that have been followed.
- Not all transformations are represented; some are 'hidden'.
- It remains unclear how long it has taken to perform a 'hidden' transformation.

In *Plato en de rekenspiegel*(Klep and Spekken 1998), all expressed steps in a calculation are annotated with time and eventually the use of animals. Because of these annotations, *Plato en de rekenspiegel* enables more precise interpretations by Arithmeticus.

When a child replaces a mathematical (sub-)expression by another mathematical expression or a number, intermediate stages might remain hidden. Arithmeticus solves this problem by testing whether there are automatisms or facts available in the actual student model that cover the gap. Another solution is to open a communication between Arithmeticus and the child, where Arithmeticus asks the child to explain what his hidden steps are. This was not implemented in *Plato en de rekenspiegel* (Klep and Spekken 1998), because computers at the time in primary schools were slow and had little memory.

A collection of calculations for $4\cdot 12$ in different representations like a number line, arrows or symbolic mathematical expressions can be interpreted as one algorithm. Not all steps have to be given explicitly by the child. Therefore, we see that:

- There are several multiplications like $4 \cdot 12$ (Calculation V), $4 \cdot 21$, $3 \cdot 12$ and others, which can be interpreted with Algorithm 1 and
- There are several models and symbolic mathematical expression representations (modes), in which Algorithm 1 can be expressed.

Alternatively, given an algorithm A, the domain of this algorithm A can be calculated and by a proper specification (Klep 1998, p. 151 (Klep 1998, p. 151) it can be performed in different representations (models). In the dynamic context of Arithmeticus, the actual domain of an algorithm is dependent on a child's knowledge base, because the algorithm may contain automatisms and facts.

An algorithm as a whole can be regarded as a transformation by itself. Automatisms and facts are the kind of (sub-)algorithms, which are performed in 'one step'. Therefore, it is possible to create an algorithm A with an unspecified algorithm B as a transformation step. Until now, B can be unspecified if at least one instance for B can be found, which makes A successful. B is some kind of a wildcard which will be specified when A needs a 'fitting' algorithm.

In a set of algorithms, there are relations between algorithms:

- Sub-algorithms,
- Competitive algorithms (same starting mathematical expressions, different algorithms, same result) and
- Regressions, where facts or automatisms are replaced by 'full' algorithms.

20.5.2.1 Defining an Algorithm Calculus

An open mathematical research question is whether it is possible to create an 'algorithm calculus', which may produce new algorithms, not by testing the subsequent steps to generate algorithms, but by transforming algorithms as structures itself. The meaning of this question is whether it is possible to make a model creating 'new' algorithms by transforming well-known algorithms and thereby creating some kind of meta-algorithms.

This then leads, in the next section, to the concept of a SIT and a connection with ideas of concepts, as formulated by other researchers.

20.5.2.2 Definition of a SIT

Let us think of a domain (a set) of expressions (e.g. multiplications) in different modes, which can be transformed by one (maybe partially unspecified) algorithm A. I propose to define a SIT, as a pair <A, Domain>. This concept of SIT is related to the concept of 'actual domain' in Sect. 20.4.6.

This SIT is a mathematical object, which is an informatical representation (Klep 1998) of the concepts *mental object* (Freudenthal 1987), *Erfahrungsbereich* (Bauersfeld 1985) and *Grundvorstellung* (vom Hofe 1995). The fundamental connecting idea is that concepts are defined by objects and operations, like in the theories of Piaget (1972, 1992) and Aebli (1963). This leads to further informatical modelling of mathematical problem solving, by an algorithm-generating or SIT-generating program.

In fact, some elements of this way of thinking are implemented in Arithmeticus. By these features, Arithmeticus can be used in a mode of 'self-employed' learning, because it can create new exercises in the zone of proximal development of itself: Arithmeticus can be its own teacher.

Arithmeticus is an informatical model of arithmetical competence based on:

- The idea of transformation of mathematical structures;
- Algorithms: sequences of transformations;
- SITs;
- Production rules, not description or categorisation of algorithms or strategies; and
- Psychological annotations of child algorithms (facts, automatisms and routines).

20.5.3 Some Results of the Idea: Transformation as a Fundamental Concept in Arithmetical Competence Modelling

In this chapter transformation is the fundamental concept for understanding calculations as sequences of transformations. This understanding leads to Arithmeticus: an informatical rule-driven dynamic model for 'all possible calculations'. In Arithmeticus, algorithms can be produced by applying mathematical transformations. The search process is pruned by applying psychological concepts like automatisms and facts.

Arithmeticus is a self-learning model: The more algorithms it produces, the more new algorithms it can produce. Arithmeticus is also a model for learning: It can be used to represent learning processes.

Arithmeticus can be connected with an interactive front end like in *Plato en de rekenspiegel* (Klep and Spekken 1998), and receive from the front end annotated child's calculations. The program can generate algorithmic interpretations of those calculations and store those interpretations.

In the next step, it can generate further algorithms representing classes of calculations 'a child might produce'. The fundamental hypothesis in the whole ISMA project is that Arithmeticus can make interpretations of all calculations the child makes. This it does by generating every possible calculation a child is able to produce.

At the front end, the child can make calculations by:

- Simply writing or drawing what he/she wants to express,
- Transforming an initial mathematical expression by means of replacing mathematical (sub-)expression or

- Transforming an initial mathematical expression by means of actors (animals), who propose meaningful transformations to the child.

In this context, ‘meaningful’ connotes testing whether the child could perform that transformation on behalf of his/her personal student model, which is maintained by interpreting his/her former activities in the front end. The algorithms and transformations, already used by the child, are mirrored to the child by the actors (hence the name Plato and the Math Mirror) and are used for planning new exercises.

The child in turn generates new algorithms and exercises them. Arithmeticus makes interpretations, enriches the student model and is able to support the further learning of the child; it is a transformation-based reproductive and productive model for a child’s arithmetical competence: a model for the child’s actual and feasible repertoire.

Arithmeticus can produce algorithms a child might be able to perform. In other words, it can calculate a zone of proximal development for individual children. This feature opens an important opportunity in education: it is possible to create a planning for an individual learning path for each child. It provides an alternative for traditional, mostly linear schoolbook-defined learning paths. A planner that was built on the basis of Arithmeticus is an example of a generative curriculum (Klep 1998; Klep 2002): a curriculum type not defined by a sequence of contents and levels, but defined by planning rules, stating which priorities are to be set in the actual zone of proximal development of a child. If a child performs calculations which are ‘out of order’ or ‘unexpected’ for the system, Arithmeticus can learn these strategies from the child and the planner can take them into account for planning future learning.

20.6 Discussion

In this section, I will find a common ground with some of the ideas presented by the authors mentioned in the section ‘Backgrounds’ and these I will combine with the points I have presented. I will find common ground also with certain theories mentioned in an issue of *ZDM—The International Journal on Mathematics Education* about flexible and adaptive use of strategies and representations, edited by Heinze et al. (2009).

20.6.1 Forgetting Facts and Automatisms

In Sect. 20.2, the theory of Ebbinghaus (1885) is mentioned. The concept of learning and forgetting is implemented in the Arithmeticus student model by maintaining a history of individual facts and individual algorithms of a child. Accuracy and response time are used to dynamically assess individual facts as well as more or less stable routines or more or less well-automated algorithms. The idea of forgetting is implemented by supposing a time-related forgetting process: facts or algorithms which are not—explicitly or implicitly—used for a long period might be less reliable for the child and will not be chosen by the child that quickly unlike in an earlier period, in which they were used frequently. The forgetting of products from the multiplication tables was observed when testing the program *Een wereld rond tafels* (Klep and Gilissen 1987). After the summer holidays, the whole experimental group had slower and less accurate results in the program. It took some time to return to the 'old level'.

For each fact or algorithm, a 'trend' is calculated and in the front end there is the 'coach' who comments on some of the results, e.g., 'this is better than your previous attempts', or 'it seems you know this kind of exercise quite well', or 'c'mon, I think you can do this better, because I remember you scoring very well on similar exercises!'

20.6.2 Competition Between Automatisms and Memory Retrieval

Ashcraft (1983) and Baroody (1983) have discussed the question whether facts are retrieved from (some kind of declarative) memory or are recalculated 'very fast'. When I conducted training in multiplication tables, I found evidence for both: on asking very well-memorised multiplication exercises to a child, he/she seemed to rely on a memory-retrieval mode, and after a 'difficult' multiplication exercise the same child seemed to remain within a re-calculating mode. In Arithmeticus, this theme is modelled by maintaining learning histories: exercises with stable correct answers, a stable net (corrected for typing and reading) response time of 2 s or less and a stable trend, which are considered to be well memorised. Solutions with intermediate steps or long response times are considered to be the result of calculation. Results without intermediate steps, with fair response times and a positive trend are considered to be automated. In fact, there is no difference between facts and highly automated algorithms in Arithmeticus.

When Arithmeticus produces new algorithms or when it tests algorithms, it takes the best elements: Arithmeticus prefers facts, then automatisms, then routines and it creates new sub-algorithms if necessary. An important hypothesis is that facts, automatisms and new algorithms are always in competition.

An important effect in the assessing of calculations is this one: If a child has automated calculations like (a) $7 \cdot 18 = 7 \cdot 10 + 7 \cdot 8$ and then tries to solve the exercise $7 \cdot 18$ by (b) $7 \cdot 20 - 7 \cdot 2$, the use of intermediate steps and the increasing response time will not be assessed as regression of the automated algorithm but as a result of the new algorithm b. When a child does not give any signals of using a new algorithm, the assessor in the system can start a communication with the question: 'can you explain what you have done?' to complete the interpretation. Unfortunately, we could not implement this option of the Assessor in *Plato en de rekenspiegel* (Klep and Spekken 1998): it only works in the experimental version of Arithmeticus.

20.6.3 The Nature of Automatisms and Zeroing In

Threlfall (2009) makes a difference between:

- Approach strategies which are 'the general form of mathematical cognition used for the problem—for example counting, or recall, or application of a learned method, or visualisation of a procedure, or exploiting known number relations' (Threlfall 2009, p. 541),
- Number-transformation strategies: 'the detailed way in which the numbers have been transformed' (Threlfall 2009, p. 542). 'Number-transformation-strategies each reflect an approach, but it is noticeable that some approach-strategies in effect determine the number-transformation-strategy, whereas others do not.' (Threlfall 2009, p. 542),
- Calculation strategies, number-transformation strategies that arises from an approach strategy based on exploiting known number relations. A calculation strategy in mental calculation is when a problem is answered by exploiting known number relations having adopted an approach to do so (Threlfall 2009, p. 542).

Threlfall states, 'In any particular case, it may not be possible to say from its form whether a number-transformation-strategy was an example of purposeful calculation based on number knowledge or the 'blind' application of a learned method. […] There is some psychological reality in the difference described, but the difficulty of the diagnosis does have an impact on educational implications' (Threlfall 2009, p. 542).

Calculations I to VII, as discussed above, are contained in Threlfall's psychological approach resulting from 'blind' number-transformation strategies or result of calculation strategies. Therefore, there is the option for opening communication between Arithmeticus and the child to find out what a child could have meant.

The idea in *Plato en de Rekenspiegel* (Klep and Spekken 1998) is that children can start with calculations through approach strategies and that these strategies will coagulate to fixed procedures which are to be automated and in some cases will be shortened to pairs of a mathematical expression and a number ($3+4=7$) and remembered as facts.

A further idea in the Assessor program is to ask children every now and then—this asking is a time-related procedure in the assessor—to explain their calculation strategy) behind a specific automatism or fact. There are other applications of this regarding automatisms. On behalf of the student model:

- The planner can ask a child to re-produce the algorithms/ calculation strategies it has developed before;
- The system can advise a child when it hesitates 'how a strategy was'; and
- the system can ask for alternative strategies: One of the possible responses of the assessor is: 'I suppose you can produce a better calculation using everything you have learnt so far'.

I agree with Threlfall that the nature of a well-performed number-transformation strategy is not always clear. Therefore, we need communication with children and eventually make them aware of possible argumentations. This will help in 'grading up' their strategy to a well-understood calculation strategy.

When going on in their learning process, children will automate procedures and forget about their argumentations and mathematical background. As discussed above, this is a necessary step, to free up mental energy for more complex new exercises. As Threlfall discusses, 'where an approach-strategy is adopted that involves the visualisation of the problem as a written 'sum', the number-transformation-strategy echoes the written procedure, and is more or less the same each time' (Threlfall 2009, p. 542) and 'other approach-strategies, such as the use of mental recall, or the imagining of the use of manipulatives, also lead directly to a number-transformation-strategy' (Threlfall 2009, p. 542). Number-transformation strategies in the sense of Threlfall or automated algorithms, as above, might be of a different nature than sequences of transformation steps. In many schoolbooks, some exercises present sequences as:

$8+7=15$
$18+7=25$
$28+7=35$ and so on, leading to some kind of pattern manipulation like $38+7$ ends with 5 because $8+7=15$ and the 3 increases by 1 because of crossing the 10. This is pure number transformation, nearly some kind of pattern-manipulation: either a visual pattern or a language pattern. In the language-pattern manipulation, it might be this way: thirty-eight and seven (sounds like) eight and seven,… are fifteen. This last step is an acoustic completion by memory retrieval of the known sentence 'eight and seven is fifteen'. The next step in the strategy is, in the sequence of tens that sounds like thirty, forty, resulting in the composition 'forty and five' or 'forty-five'. An alternative might be the sequence of sentences,

thirty-	eight	and	seven		
	eight	and	seven	is	fifteen
next to three-ty		and			five (is)
four-ty		and			five.

This sequence of sentences does not have much to do with mathematics, although it can be represented as an algorithm. It is a sequence of transforming expressions in an (acoustic) language representation.

My conclusion is, doing arithmetic can be psychologically understood as building sequences of subsequent transformations of expressions in an (acoustic) language representation. These transformations are not of a mathematical nature, although they can be matched with mathematical transformations, but they are associations of parts of sentences retrieved from memory.

I proposed (Klep 1998) the German word Anklang to characterise this effect: one sentence finds an echo in another sentence or in a more active description; one sentence generates a stimulus for the brain the way the conductor gives a 'beat-up' as a stimulus to the orchestra. The brain can react with sentences, which can be unified with the stimulus. One of them is used as a next step. From this set of competitive responses, one next step has to be selected. One of the selecting mechanisms is mathematical logic in its socio-constructivist meaning: some connections of sentences are more acceptable in the classroom community than others. Another selecting mechanism is the feeling: this transformation or this strategy might be successful for me (the student). It is some kind of appreciation mechanism that is discussed later in Sect. 20.6.4.

The concepts of facts, automatisms and routines together with the metaphor or mechanism of Anklang and its logical and appreciative selection mechanisms provide an explanatory set of ideas to understand the generation of calculations.

In Arithmeticus, this mechanism of Anklang is modelled in a mathematical way: A mathematical expression is represented in a structured manner, including information about the characteristics of that mathematical expression:

- To use the standard unification mechanisms in the Prolog language (Merritt 1995–2010) and
- To make a relevant description of a domain of an algorithm.

The Prolog unification mechanism can take an incomplete structure and try to unify it with other complete or incomplete sentences (clauses in the database), which themselves can call for further clauses. A call is completed or successful or true, when the original structure is completed. I have used this mechanism with some modifications as a technical representation of Anklang.

This idea of Anklang seems to have some parallels with the idea of zeroing in used by Threlfall (2009, p. 547).

20.6.4 Flexibility and Appreciation of Transformations

When an Anklang generates a set of competitive responses in a student, there seems to be a selection mechanism to choose one of them. I use the word ‘appreciation’ for this mechanism. In Arithmeticus, this appreciation was modelled keeping in mind:

1. Availability of necessary facts and automatisms (including pattern manipulation);
2. Availability of mathematical understanding;
3. Appreciation on behalf of energy saving:
    a. Facts are appreciated more than automated algorithms;
    b. Automated algorithms more than routines;
    c. Routines more than generating new algorithms;
4. Appreciation on behalf of risk for errors. Important characteristics are, e.g.:
    a. The number of steps in the algorithm;
    b. The recursion depth (mathematical sub-expressions in mathematical sub-expressions);

   c. Complexity of mathematical expressions involved: number of numbers and operators;
   d. Complexity of numbers involved;
   e. The complexity of the algorithm itself: the number of branches, complexity of the context when operating on mathematical sub-expressions;

5. Memory load:

   a. The use of actual knowledge;
   b. The stack of the algorithm (all procedural and declarative things that have to be kept in mind in the subsequent steps):

      – The number of numbers and operators in the subsequent stages;
      – The context;
      – The state in the flow of the algorithm;

6. The personal history:

   a. Is this algorithm stable for me;
   b. Is it reliable (few errors);
   c. Can I manage the work (working memory, stack control)?

Although this model for what a child feels to be ‘difficult’ seems to be relevant, the appreciation in the mind might be of another kind: not only will the algorithm itself be remembered, but also the feelings accompanying former calculations with this algorithm.

Nevertheless, Arithmeticus generates and uses this kind of characteristics for individual calculations, algorithms and SITs.

The concepts of facts, automatisms and routines together with the metaphor or mechanism of Anklang and its logical and appreciative selection mechanisms give a good basis to understand strategy flexibility as discussed by Heinze et al. (2009) and Threlfall (2009). The informatical model of arithmetical competence brings a set of precise and operational concepts and the opportunity to investigate different settings. Changes in the large set of parameters in Arithmeticus lead to different behaviour of the system. In the Amzi-Prolog user interface, it is possible to make simulations of the long-term learning processes: the researcher being a teacher and Arithmeticus being a learning child. The points 1–6 influence the ability of Arithmeticus to generate and apply algorithms.

20.7 Conclusion

Arithmeticus is an informatical model (artificial intelligence) for arithmetical competence, based on:

- The idea of transformation of mathematical structures,
- The idea of primitive recursive functions and Turing machines,

- The concept of algorithms as a sequence of mathematical transformations and dynamic transformations like facts and automatisms and
- A child's learning history as interpreted and maintained by an Assessor in interaction with Arithmeticus.

Arithmeticus can generate new algorithms and can 'learn' new algorithms as produced by children. The more Arithmeticus has learnt the more flexibly it can 'think'. Basically, it can learn every arithmetical strategy, which is mathematically valid.

Arithmeticus can be related to psychological and educational concepts and theories. The model seems to be rich enough to simulate in a meaningful way arithmetical learning processes.

Arithmeticus is a basis for planning systems, providing dynamic educational offers for a child with consideration paid to its learning history and the actual individual zone of proximal development of a child. The assessor can keep track of increasing or decreasing competence and can make comments on a child's performance.

Research and development on the competence modelling are continuing: A new version of Arithmeticus with a new production system of algorithms is under construction. The scope of Arithmeticus 3 is extended to arithmetic, algebra and applications of arithmetic and algebra. Readers who are interested in this project, are invited to contact the author.

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Chapter 21
Discussion of Part III

Fundamental Ideas of Didactics—Reactions to the Suggested Meta-theoretical Construct for Reflecting and Connecting Theories

Susanne Prediger

Abstract: The editors of this book have introduced the meta-theoretical construct ‘fundamental ideas of mathematics didactics’ as a very interesting approach for reflecting on the identity of the scientific discipline didactics of mathematics and for dealing with the diversity of theories in the field. This chapter discusses how and under which conditions the construct can be used for these purposes. For founding this discussion, some main aspects of the construct fundamental ideas, their meanings and functions are shortly revised and related to the editors’ suggestions for the fundamental ideas in didactics of mathematics (Section 21.1). After that, the concrete suggestion for a first idea, namely ‘transformation’ will be discussed with respect to contributions of part III (Section 21.2).

21.1 Fundamental Ideas as Meta-theoretical Constructs

21.1.1 Fundamental Ideas in Mathematics for Didactics of Mathematics

The construct of fundamental ideas has been suggested by the psychologist Jerome Bruner (1960) as a construct that allows to specify the essential cores of each scientific discipline and then use them to build up coherency within a spiral curriculum of each school subject. Although the construct itself is often used and cited, there is little agreement on its exact meaning, even within a single discipline like mathematics (Schweiger 2006; Vohns 2010).

While some authors try to figure out the *interfaces* between mathematics and the reality it describes (e.g. Heymann 2003 with his catalogue number, measuring, spatial structuring, functional dependency, algorithm, modeling), other researchers specify the main phenomena (e.g. Halmos 1981 with size for algebra, shape for geometry and change for analysis) or activities (e.g. MacLane 1985 with moving,

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measuring, shaping, counting). The first group emphasizes fundamental ideas as different *perspectives* of the discipline on reality, whereas the second group focuses on the *phenomena***,** *activities and concepts* themselves. The main phenomena (being the objects of specific fields of mathematics) have been termed *leading ideas* in the German discussion (i.e. Tietze et al. 1997; KMK 2004).

Tietze et al. (1997, p. 37 ff) have suggested to distinguish three interdepending epistemological levels to which fundamental ideas refer. All of them are important, although for different purposes and functions:

1. Leading ideas as characterizing the main phenomena (e.g. space and shape, quantities and measures, dependencies and change, etc.)
2. Interfaces for mathematization (e.g. number, systems of linear equations, approximation, …)
3. Domain-specific ideas and strategies (e.g. exhaustion, algebraization and geometrization, …)

Notwithstanding the variety of catalogues, Schweiger (2006, p. 68) cites four dimensions that can characterize fundamental ideas: *time dimension*, *horizontal dimension*, *vertical dimension*, and *human dimension* (cf. Introduction). These dimensions implicitly relate to different *functions* of fundamental ideas, which implies different prioritizations of dimensions as conducted by different authors. For example Vohns (2010) emphasizes establishing coherency as the most important function of fundamental ideas, he thus focuses on the time, horizontal, and vertical dimension. In contrast, Heymann (2003) emphasizes the horizontal dimension in its function to mark the specificity of mathematics and different mathematical domains, compared to everyday thinking. The different levels of ideas, however, seem to address different functions and different dimensions, and so, there seems to be no logical deduction of suitable levels from a choice of dimensions or functions.

The common essence of the construct fundamental idea in mathematics is captured by Vohn's definition that refers to different epistemological levels but emphasizes the idea behind:

> A mathematical idea signifies the main thought, that one can try to specify behind certain strategies, techniques, patterns of thinking and acting, it is the attempt to answer the question on the crucial point that allows to understand the core of a subject. (Vohns 2010, p. 230, translated by SP)

21.1.2 Fundamental Ideas in Didactics of Mathematics for Reflecting on Theories in Didactics of Mathematics

Rezat (2012) suggests the interesting and creative idea to transfer the meta-theoretical construct fundamental ideas to the reflection on theories *in* didactics of mathematics. By this transfer, he intends to contribute first to the search of the disciplines' identity (Biehler et al. 1994; Sierpinska and Kilpatrick 1998), and second, to the exploitation of the existing diversity of theories in the didactics of mathematics as a scientific discipline towards *more reflection and connection* (Prediger et al. 2008a; Sriraman and English 2010).

Following Rezat's main idea, the editors of this book formulate different functions to which the meta-theoretical construct 'fundamental ideas of didactics of mathematics' should contribute:

1. *Focus on core issues of the discipline:* contribute 'to the discipline's search for focus and identity'
2. *Systematize the scientific discipline:* 'provide a means to organize theories in terms of being related to a similar idea'
3. *Specify connecting points:* 'find theories that are worthwile connecting'
4. *Construct curricula for teacher education:* 'contribute to the development of curricula for teacher education'.

For these purposes, the editors suggest 'transformation' as a first fundamental idea that they locate on the general level of overarching phenomena in the sense of a leading idea. Before discussing 'transformation' in Sect. 21.2, these four purposes shall be related to the dimensions and complemented by a fifth purpose.

Focus on core issues of the discipline For the *focus on core issues* of didactics as a scientific discipline, the time dimension and the vertical dimension are to be taken into account for specifying relevant fundamental ideas in Vohn's sense. But also the horizontal dimension can contribute to finding of the core issues in different areas.

The time dimension applies only partly: Some didactical ideas have been discussed for many decades and thus seem to be (historically) important for the scientific discipline, for example Comenius' "omnes omnia omnino" (Comenius 1657) which was recalled by Bruner's (1960) idea of a spiral curriculum and his hypothesis that every subject can be taught on every level. However, the time dimension alone cannot be crucial for a young scientific discipline like didactics in which not all essential ideas have already been established. In contrast, many substantial new ideas arise in the research and are powerful drivers for the discipline's development (cf. ontological innovations, DiSessa and Cobb 2004). Hence, the time dimension should be extended to the following: either the idea is very old or has offered a historical contribution to the scientific discipline (like for example Cobb's construct of sociomathematical norms, discussed as an example in DiSessa and Cobb 2004).

For the vertical dimension, Rezat (2012) suggests to interpret the different levels as different discourses (that he calls contexts) on didactics, namely classroom practice, teachers' practically oriented discourse, teacher education discourses in university and seminars, and theoretical and research discourses in the scientific discipline (similar in the Introduction, p. 5). A fundamental idea that reflects the vertical dimension can contribute to the specification of the focus since this idea seems to be relevant in different discourse levels. However, it must be clear that the vertical dimension addresses not only the continuities (what is similar within the different discourse levels?), but also the discontinuities (how does an idea change from discourse to discourse?), as Vohns (2010) has emphasized for fundamental ideas in mathematics. The identity of the scientific discipline is shaped not only by similarities, but also by the differences to the purely practical discourses and practices.

The search for coherences seems to address mainly quite general fundamental ideas, i.e. the leading ideas in horizontal or vertical dimension; the search for differences mainly the more concrete levels on which the differences become more visible.

Systematize the discipline For providing means to systematize the discipline, ideas are needed that mostly satisfy the horizontal dimension (or its explicit absence—for marking differences). For organizing the field, the chosen ideas should cover more than one branch while not being too general.

Specify connecting points between different theories In contrast, the specification of connecting points needs very concrete ideas, most ideally those that serve as interfaces for interpreting phenomena. These ideas sometimes neither address horizontal nor vertical dimensions, but the human dimension.

Construct curricula for teacher education There is no doubt that Bruners' original construct of fundamental ideas can be applied to the curriculum construction also within didactics as a scientific discipline. Hence, the purpose of contributing to the curriculum construction for teacher education addresses all four dimensions, as explained by Schweiger (2006).

Beyond teacher education (mentioned in the Introduction), fundamental ideas should also guide the construction of curricula for novices in the scientific communities, namely the curricula for young researchers. The curricula for young researchers partly overlap with the one for teachers, but obviously not completely.

A further function: Identify black holes A fifth function should be added that emphasizes the human dimension: If Rezat (2012) suggests to interpret the 'everyday patterns' addressed in the human dimension by teachers' practices and concerns, then this dimension of fundamental ideas might give the opportunity to identify black holes in the scientific community: Some ideas seem to be very crucial for school practices but missing on the scientific discourse levels. In order to realize a better match between human dimension and vertical dimension, they might be considered as a starting point to identify black holes and initiate a further theory development.

These general considerations on fundamental ideas in didactics, their possible levels, functions, and dimensions serve as a background for discussing the concrete suggestion of one fundamental idea in Sect. 21.2.

21.2 'Transformation' as Fundamental Idea?

21.2.1 'Transformation' as a (Non-)Shared Fundamental Idea of five Contributions

Part III of this volume comprises five contributions around the fundamental idea 'transformation' that provided the title for the volume. Each of the five contributions is very interesting itself; but in sum, they offer a challenging field for testing the suitability of 'transformation' as a fundamental idea.

Three of the five contributions, Dreyfus and Kidron (Chap. 16), Stanja and Steinbring (Chap. 17), and Kuzniak (Chap. 18), explicitly deal with the transformation of students' cognitions in a large sense. They are in a certain way comparable, and the idea of transformation might inform this comparison.

The other two contributions use the term 'transformation' in a completely different sense: Klep (Chap. 20) mainly intends to model arithmetical competence (hence basically a learning *state*, not a learning process). Within this model, he uses the term 'transformation' not for the learning processes, but for the elementary steps in calculation procedures or strategies. Although his explicit focus is mostly on these learning states, his work might also contribute to modeling transformation of students' cognitions, as the technical model of students' calculations can develop in parallel to students' cognitions. The fundamental idea in his work that contributes to the overarching idea of transformation of students' cognitions is Vygotsky's zone of proximal development: Although not too explicit in the text, I suppose that beyond the author's claim that he can derive the zone of proximal development from the modeled arithmetical competence, he must have a model of how the competencies progress and follow each other. However, the systematizing of theories would be easier if he used the term transformation for this part of the theory.

Profke (Chap. 19) is the only one who does not aim at developing descriptive theories for *describing* change in students' cognitions, but emphasizes the *constructive* part of didactics. He illuminates design challenges while transforming overarching normative goals of mathematical literacy into concrete learning opportunities for everyday classrooms. For this, he chooses to give examples without being explicit on the theoretical conceptualization of the different discourse levels or on the applied techniques of transposing goals into concrete activities. Hence, the articles' use of the term 'transformation' seems to be nearer to Chevallard's (1985) idea of 'didactical transposition between praxeologies' than to 'transformation of students' cognitions'.

The comparison between these five contributions offers a good example on the meta-level for the claim that a fundamental idea can hardly serve any of the formulated purposes if its meanings are too divergent. Although a certain vagueness of the fundamental idea might be important for being applicable in different contexts (Schweiger 2006), the divergence between these concrete cases is too large.

In contrast, the comparison of these five papers might be interesting in the light of fundamental ideas on overall goals of didactic research (e.g. design science versus fundamental descriptive research, cf. Wittmann 1995) and on conditions for design for being scientific (Gravemeijer and Cobb 2006), for which especially Kuzniak's construct of 'geometric work space' gives interesting profiles.

21.2.2 Suitability for Specifying Focus and Structure of the Discipline

Whereas the general idea of 'transformation' seems to be too vague (see Sect. 21.2.1), the more specified idea 'transformation of students' cognitions' appears to be insightful for describing one typical phenomenon that is a central object of empirical investigation in didactical research, it could easier be called 'learning processes'.

For systematizing different theoretical approaches to the same large phenomenon 'learning processes', it might however be fruitful to be even more precise on the specific conceptualizations of learning processes that underlie the different theoretical approaches. The distinction between the ideas 'transmission', 'internalization', 'construction' and 'transformation' might illuminate different theoretical perspectives that conceptualize students' learning in different ways (Sierpinska and Lerman 1996). While 'transmission' addresses the traditional idea that learning could take place by transmitting contents from the teacher to the students, socio-cultural theories conceptualize learning mostly as 'internalization' of culturally shared practices. In contrast, 'construction' is the constructivist term that emphasized the active individual mental constructions as driving forces for learning.

The three contributions by Dreyfus and Kidron (Chap. 16), Stanja and Steinbring (Chap. 17), and Kuzniak (Chap. 18), all conceptualize learning processes as transformation processes, i.e. they consider learning not as starting from "empty sheets" but as changing existing cognitive structures and experiences. This interface between the real phenomena and its scientific conceptualization can be fruitfully called a shared fundamental idea. However, neither of the contributions would be completely modeled by this characterization. Instead, they differ in essential other points like the epistemological background for Stanja and Steinbring, the theories of proof to which Dreyfus and Kidron refer, or the large corpus of theory connected to instrumental genesis in Kuzniak's contribution. Hence, for providing "means to organize theories in terms of being related to a similar idea" (Introduction), a whole landscape of fundamental ideas is needed, not only one single idea.

21.2.3 Suitability of the Fundamental Idea 'Transformation' for Finding Connecting Points between Theories

It is inherent in the nature of the volume that its contributions do not aim primarily at connecting with each other. However, since all contributions are referred to the same fundamental idea, some should at least have the potential to be connected. For testing the suitability of the fundamental idea 'transformation', these reflections refer again to the small "data set" of three articles that all share the idea 'transformation of students' cognitions' and compare its potential to other possible connecting points.

The editors of this volume are right to mention that any networking strategy (Prediger et al. 2008b) can only be applied when connecting points between the theories are found (Introduction). Hence, other papers have suggested different meta-theoretical constructs as starting points for different activities of connecting theories (cf. Prediger et al. 2008b for an overview):

- Research practices have often been compared and contrasted by their *empirical methods*. Comparing and contrasting methods that are applied in different theoretical approaches address another area of reflection than fundamental ideas, since methods alone only cover the forms of research, not its content in the sense of central problems, questions, conceptualizations and results.
- Different authors have used a *common data set* as starting point for comparing and contrasting or partly integrating theories. A common piece of data is a very instructive starting point since the reflections can be very concrete and become hence substantial. However, the specification of data heavily depends on the research practice and its underlying theories, so the connection of theories usually has to be planned before the data collection.
- The integration or synthesizing of different theories often starts from a common phenomenon but is driven by the complementarity of different theoretical constructs that serve as interfaces for conceptualizing interconnected phenomena.
- Prediger and Ruthven (2008) report on a case study in which a *common sense problem or phenomenon* of everyday classrooms served as a starting point for comparing and contrasting how different theories conceptualized and investigated this problem. The case study showed that different research practices come to very different approaches which are not completely determined by the explicit parts of their theoretical approaches, in contrast, many further implicit aspects of the research traditions guide the researchers' decisions (e.g. what counts as interesting research question?). Comparing and contrasting these processes of conceptualization and developing research designs is hence very interesting.

This list of different connecting points shows that it makes a difference whether researchers have a common phenomenon and different perspectives or a common perspective and add several phenomena. For testing the suitability of fundamental ideas as a connecting point, we must thus specify the epistemological level.

- For finding similarities between theories, the *leading ideas* (like transformation of students' cognition in the large sense) give a first clue as it helps to restrict the search space.
- For contrasting the theories in more detail, it is interesting to consider the level of *specific strategies and techniques*.
- For partial integration or synthesizing, the most important epistemological level seems to be the theoretical constructs as interfaces for specific, perhaps complementarity conceptualization of specific aspects of the large phenomenon.

21.3 Outlook

The meta-theoretical construct fundamental idea seems promising for all postulated purposes. However, for further development of this idea, it appears to be necessary to specify not only one single idea but a landscape of ideas. This landscape will never be complete. Like for the fundamental ideas of mathematics, it will depend on the concrete purpose which epistemological level of ideas is the most promising. I wish the editors good luck in continuing the search!

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Epilogue

Rudolf Sträßer

Fundamental Idea: Transformation

At the end of a whole bunch of challenging chapters in a book full of information and ideas, it is impossible to write an epilogue which gives full credit to all the creative and innovative deliberations of the texts. As a consequence, I will restrict myself to a few features of the book, which—for me personally—make the essence of the book. Furthermore, I will also describe some aspects of a discussion on fundamental ideas and transformations, which seem not to be present in this book.

I want to start with a comment on the idea of the book: Discussing the idea and the potential of 'fundamental ideas' in didactics of mathematics (as I prefer to name this emerging scientific discipline instead of research in mathematics education) is an innovative and most promising attempt to create something like an overall and structured landscape of the emerging discipline didactics of mathematics. At present, the 'standard' way of creating the structure of a discipline by identifying the object of research and its methodology (or methodologies) seems to be unsuccessful (to say the least) in the case of didactics of mathematics as can be seen in volumes like *Mathematics education as a research domain. A search for identity* (Sierpinska and Kilpatrick 1998) and *Theories of Mathematics Education. Seeking New Frontiers* (Sriraman and English 2010). Consequently, didactics of mathematics can profit from a different approach to create something like an identity of the discipline. The book tries to create this identity with an approach borrowed from Bruner: By identifying 'fundamental ideas' of didactics of mathematics, it may be possible to develop a distinctive and accepted conception of what the discipline is about. By taking 'transformation' as a prototype of such a fundamental idea, the approach does not remain abstract and vague, but attempts to put the approach to practice—showing strengths and weaknesses of the enterprise.

As a consequence, an epilogue can comment on two levels: (1) What can we learn about fundamental ideas of didactics of mathematics from discussing the prototype

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'transformation'? I give my personal answer to this question in Sect. 2 of the epilogue, and (2) what about the approach as a whole, or: What to learn about taking 'fundamental ideas' as a possible answer to the diversity in the field of didactics of mathematics? My perspective on this issue can be found in Sect. 3 of the epilogue.

On Transformation

Transformation of Institutions

The title of part I of this book reads *Transformations at Transitions in Mathematics Education* and—using different methodologies—concentrates on the transformations of mathematical ideas and concepts, which result from their use in different institutions. As a complement and in the discussion of part II of the book, Sutherland reminds us of a major characteristic of this transformation: Institutions alone do not provoke transformations. Change (or transformation?) is an affair enacted and promoted by human beings, who have to take a personal interest and active role in the transformation of institutions. There seems to be always a 'human factor' in transformations—even if they can be described as changes inside institutions or between different institutions.

As one can easily see, part I of the book somehow concentrates on what Felix Klein has called the double discontinuity between school mathematics and university mathematics. From the history of mathematics education in Germany, with university mathematicians and mathematics teachers from 'Gymnasium' being one major source of didactics of mathematics, this does not come as a surprise (for a historical sketch of German didactics of mathematics, see Griesel and Steiner 1992). Two of the papers in part I analyse an important part of institutional transformations, but do not cover institutional change as a whole.

The discussion by Biehler correctly mentions an omission, which nevertheless is statistically more important: the transformation from school to work. This transformation is more important under the perspective of persons involved, because it is still a minority of young adults who directly move from upper secondary schools into university. Students aiming at becoming mathematics teachers are again a minority of beginning students. The small number of future mathematics teachers is outnumbered by persons directly going into the labour market, and mathematics is an important subject in nearly all types of vocational training (for an international perspective see Sträßer 2014, in print). The transformation, which school mathematics has to undergo to become vocational mathematics, may be less dramatic than the one for university mathematics, but the problems with mathematics in engineering studies already show that 'applied' or 'industry' mathematics also faces difficult problems (for examples see the ICMI-study on 'Educational Interfaces between Mathematics and Industry (EIMI)' with its discussion document Damlamian et al. (2009) and the forthcoming Study Book of this ICMI/ICIAM-study no. 20: Damlamian et al. 2013). The transformation of school mathematics to industrial mathematics (with 'industry' in the broad sense of the EIMI-study) must be a major issue in didactics of mathematics, hence an important aspect linked with transformation as a fundamental idea.

Transformation of (Mathematical) Content

It may come as a surprise that—in my epilogue—I do not distinguish between transformations of representations (of mathematics)—as discussed in part II—and transformations of ideas and concepts—as in part III of this book. In doing so, I just want to give a hint that at least for some approaches in didactics of mathematics, there is no distinction between a mathematical inscription (as Dörfler 2008 and Kadunz 2006 have called it) and mathematical concepts and/or ideas. This approach is, to say the least, quite tempting if one is interested in overcoming the separation of material representations of mathematics and the immaterial structure of patterns, the material/immaterial divide of realism or platonism in the epistemology of mathematics.

Looking back to the actual contents of the book, Seeger, in an aside in the beginning of his discussion of part II, correctly mentions a characteristic which is quite surprising: All papers in part II discuss topics from geometry. The text from Klep in part III shows that 'transformation' can also be used in the content area of arithmetic in order to better understand the calculations of children. Transformations of equations and formulae are one of the major topics from lower secondary school mathematics till the end of upper secondary school mathematics and beyond. Consequently, analysing the concept of transformation in (school) mathematics also has to look into algebraic transformations. Also, a very helpful candidate for the analysis of transformation in algebra can be looking into different 'representations' of the equations under study in order to find the solution of a given equation—like analysing appropriate function graphs or a table of values. Do the transformations of algebraic expressions serve the same purposes as do those in geometry? What about transformation in Arithmetic? What is the role of transformation(s) across different parts of school, industrial and university—and maybe other forms of—mathematics?

For transformation as a fundamental idea of didactics of mathematics, one can learn from this excursion into mathematics as such that fundamental ideas may be fundamental for a discipline only in parts of a discipline or change their appearance when the 'same' fundamental idea is looked after for different areas of a discipline.

On the Methodology of Research on Transformation

Looking through the texts in this volume—especially in the first section, a major feature stands out: When trying to empirically grasp a transformation, the research has to somehow mirror the flow of time. There are two basic ways of doing so. One method is comparing the state of something being transformed. This is often done by looking into the same thing at two or more different points in time. This implies a comparative approach and can be best illustrated by the text from Kaiser and Buchholtz (Chap. 5), who report on a longitudinal study with three consecutive data gathering points within 2 years altogether. In this approach, data is gathered based on questionnaires. The answers are classified according to a pre-defined evaluation manual. This approach allows for advanced data analysis methods including procedures based on Rasch-models and complemented with additional comparisons to cohorts from different populations to further interpret the data gathered at the different points in time. Qualitative approaches are used to find better interpretations of the quantitative data.

The other approach to transformation is a qualitative one, which uses the analysis of more open empirical data gathering methods at one point in time and tries to find out about the way individuals or groups of persons see their transformations of feelings, concepts and ideas in retrospective. In empirical studies, we find only one data gathering point in time (often by means of an interview—if one can identify data gathering at all). An extreme case in this book is the text by Grevholm (Chap. 6), presenting the narrative of a woman who describes her transformations of mathematical ideas and concepts and triggering an interpretation in terms of the development of a concept map from the author. This more or less qualitative approach can also be seen in the rest of the chapters in part I and all chapters in the other parts—with the first chapter by Biermann and Jahnke (Chap. 1) showing that it also makes sense to take the history of concepts in a 'national' curriculum as a way to analyse transformations of concepts and ideas. This text convincingly shows that 'data' can be more than instances gathered from individuals and/or groups, but have to be seen in the broad 'definition' given by Beck and Maier (1994). Additionally, the texts in this volume show the variety of types of 'data' or texts in didactics of mathematics.

As a consequence of this observation of methods to analyse transformations, it should be obvious that the idea of one and only one method for didactics of mathematics is obsolete. Looking only into a small sample of research on transformations shows a whole variety of methodological paradigms. A 'one method fits all' approach for didactics of mathematics is obviously doomed to failure.

On Fundamental Ideas

In her discussion of part III of the book, Prediger raises an issue which deserves to be mentioned at a prominent place: 'However, for further development of this idea, it appears to be necessary to specify not only one single idea but a landscape of ideas.' In an informative survey in another book, Schweiger (2006) shows that it may be difficult to arrive at such a landscape of fundamental ideas in the case of mathematics. The end of Schweiger's chapter can be even read as denying the possibility of arriving at a general catalogue of fundamental ideas. Instead and nearly at the end of the chapter, Schweiger takes catalogues of fundamental ideas as a reflection of a 'personal view of mathematics', which should be 'open to revision' and developed in communications between 'student teachers, teachers and teacher educators' (Schweiger 2006, p. 71). In this respect, the present book is a look around the personal landscapes of didactics of mathematics of the authors when being asked about the surroundings of a common concept, namely transformations. The book is a documentation of what comes to mind, when a set of didacticians of mathematics is asked about their personal neighbourhood of the concept of 'transformation'.

With transformation as a 'prototype' and for fundamental ideas in general, this can be read as a confirmation of the idea of a landscape of fundamental ideas. If one wants to cater for didactics of mathematics as a whole, a hopefully complete landscape of fundamental ideas is necessary. Paragraph 2 of this epilogue already mentioned results from an analysis of just one fundamental idea in didactics of mathematics (like the different facets of transformation in different areas of mathematics and didactics of mathematics and the absence of a methodological uniformity in didactics of mathematics). A single fundamental idea—like the prototype 'transformation'—cannot cover the whole variety of didactics of mathematics as a scientific discipline. I personally doubt that a 'one-fits-all' approach is appropriate in any scientific discipline whatsoever.

To end my epilogue, I want to bring to the forefront an issue which is not very explicitly treated in the chapters of this book so far, but was clearly identified in the introduction: Are we looking for fundamental ideas within the discipline mathematics or are we looking for fundamental ideas within didactics of mathematics? Or are they the same? If they are not the same, what makes the difference? Are there fundamental ideas common to both mathematics and didactics of mathematics? The case of 'transformation' seems to show that—at least for some authors—mathematics and didactics of mathematics share some fundamental ideas. In part I of the book, transformation is nearer to didactics of mathematics, because the transformation is clearly linked to institutions or—as Sutherland correctly puts forward—linked to human beings not fully dependant on institutional constraints. On the other hand, transformations in geometry illustrate that 'transformations' can even be taken as a fundamental idea within mathematics, which is relevant in the whole discipline if 'transformations' in geometry is the synonym for 'functions' in mathematics in general. In addition to this line of thought and in conformity to the introduction of the whole book, part I of the book shows that 'transformations' can have a different meaning as a fundamental idea in didactics of mathematics. Here, the meaning of 'transformation' can be explained by 'change of mathematical ideas influenced by human beings and/or institutions'. The text from Biermann and Jahnke illustrates that 'transformations' can also be 'change in time'. The text from Grevholm describes a 'change in the course of a biography'. In all and for a fundamental idea, it is important to decide which discipline is concerned when looking for fundamental ideas. Fundamental ideas in didactics of mathematics can be different from fundamental ideas in mathematics—even if the same word is used to describe them. If one takes up the issue raised in the paragraph before, it must be mentioned that a proposal for a catalogue of fundamental ideas for didactics of mathematics is still to be identified—if possible at all.

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