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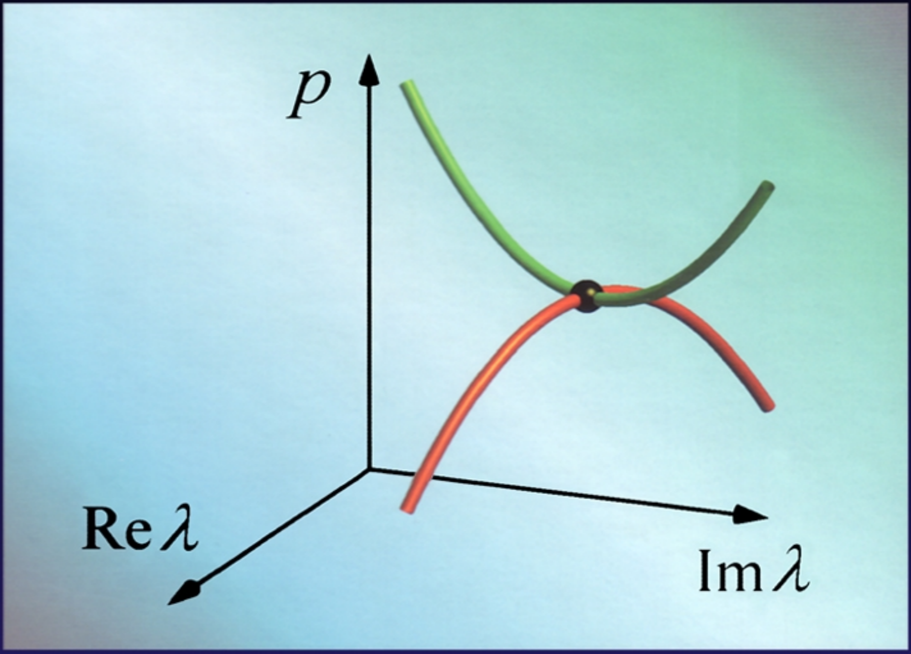
Multiparameter Stability Theory with Mechanical Applications

A. P. Seyranian & A. A. Mailybaev



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A. P. Seyranian
A. A. Mailybaev
Moscow State Lomonosov University, Russia

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MULTIPARAMETER STABILITY THEORY WITH MECHANICAL APPLICATIONS

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Preface

Stability theory is one of the most interesting and important fields of applied mathematics having numerous applications in natural sciences as well as in aerospace, naval, mechanical, civil and electrical engineering. Stability theory was always important for astronomy and celestial mechanics, and during last decades it is applied to stability study of processes in chemistry, biology, economics, and social sciences.

Every physical system contains parameters, and the main goal of the present book is to study how a stable equilibrium state or steady motion becomes unstable or vice versa with a change of problem parameters. Thus, the parameter space is divided into stability and instability domains. It turns out that the boundary between these domains consists of smooth surfaces, but can have different kind of singularities. Qualitatively, typical singularities for systems of ordinary differential equations were classified and listed in [Arnold (1983a); Arnold (1992)]. One of the motivations and challenges of the present book was to bring some qualitative results of bifurcation and catastrophe theory to the space of problem parameters making the theory also *quantitative*, i.e., applicable and practical. It is shown in the book how the stability boundary and its singularities can be described using information on the system.

Behavior of the eigenvalues near the stability boundary with a change of parameters determines stability or instability of the system. Fig. 0.1 reproduced from [Thompson (1982)] shows interaction of eigenvalues for a specific mechanical system, a pipe conveying fluid, depending on a single parameter p. As we can see, the eigenvalues approach each other, collide and diverge with exciting loops and pirouettes making the system stable or unstable. Looking at this and similar figures several questions appear: What are the rules for movements of eigenvalues on the complex plane de-

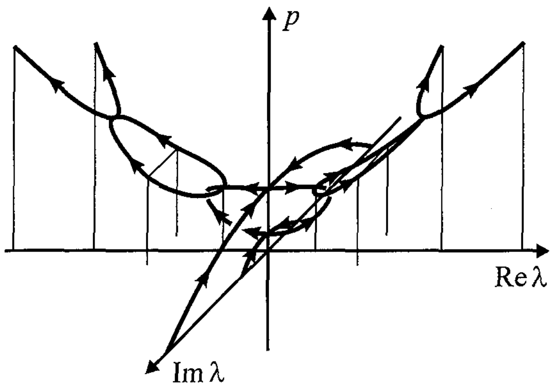


Fig. 0.1 Interaction of eigenvalues for a pipe conveying fluid.

pending on problem parameters? What kind of collisions are possible and which of them are typical? Are there some special properties for behavior of eigenvalues of mechanical systems with symmetry like gyroscopic or conservative systems? What is the relation between eigenvalues and properties of the stability boundary in the parameter space?

In concluding remarks to his book [Bolotin (1963)] pointed out that non-conservative stability problems are closely related to linear non-selfadjoint operators, and it is necessary to develop methods for studying dependence of their eigenvalues on one or more parameters. He also mentioned that general stability properties of linear systems with non-conservative positional (circulatory) forces were not fully investigated, and recalled that in the classical results by [Thomson and Tait (1879)] on stability of mechanical systems circulatory forces were not involved. Bolotin suggested to put more attention to the unexpected effect of destabilization of a circulatory system by small dissipative forces.

It is remarkable that [MacKay (1991)], who derived a formula for movement of simple eigenvalues of a Hamiltonian matrix under non-Hamiltonian perturbation, suggested to generalize the above result for the case of multiple eigenvalues and movement of Floquet multipliers as well as to apply the results to some non-trivial problems.

In this book we present a new multi-parameter bifurcation theory of eigenvalues answering the formulated questions and suggestions. Two important cases of strong and weak interactions (collisions) are distinguished and geometrical interpretation of these interactions is given. First publications on this subject were [Seyranian (1990a); Seyranian (1991a);

Seyranian (1993a)] and here we present an extended and advanced version of the theory. The presence of several parameters and the absence of differentiability of multiple eigenvalues constitute the main mathematical difficulty of the analysis. We could overcome this difficulty studying bifurcations of eigenvalues along smooth curves in the parameter space emitted from the singular points and then analyzing the obtained relations. For the study of bifurcations the perturbation theory of eigenvalues developed in [Vishik and Lyusternik (1960)] turned out to be very useful.

The presented multi-parameter bifurcation theory of eigenvalues is a key point for stability and instability studies. With this theory we analyze singularities of stability boundaries and give a consistent description and explanation for several interesting mechanical effects like gyroscopic stabilization, flutter and divergence instabilities, transference of instability between eigenvalue branches, destabilization and stabilization by small damping, disappearance of flutter instability, parametric resonance in periodically excited systems *etc.*

A significant part of the book is devoted to difficult stability problems of periodic systems dependent on multiple constant parameters. This subject has been a challenge for more than one hundred years since [Mathieu (1868); Floquet (1883); Hill (1886); Rayleigh (1887); Liapunov (1892); Poincaré (1899)]. From the very beginning these problems were multi-parameter. For example, finding stable solutions to famous Mathieu–Hill equation is a two-parameter problem. In the present book, with the bifurcation theory of multipliers, geometrical description of the stability boundary and its singularities for periodic systems is given. Then we formulate and solve parametric resonance problems for one- and multiple degrees of freedom systems in three-parameter space of physical parameters: excitation frequency Ω and amplitude δ, and viscous damping coefficient γ under assumption that the two last parameters are small. It is supposed that the unperturbed system is conservative. The main result obtained here is that we find the instability (parametric and combination resonance) domains as half-cones in the three-parameter space with the use of eigenfrequencies and eigenmodes of the corresponding conservative system, see Fig. 0.2. Finally, stability boundaries for non-conservative systems under small periodic excitation are investigated.

As applications of the presented theory, we consider a number of mechanical stability problems including pipes conveying fluid, beams and columns under different loading conditions, rotating shafts and systems

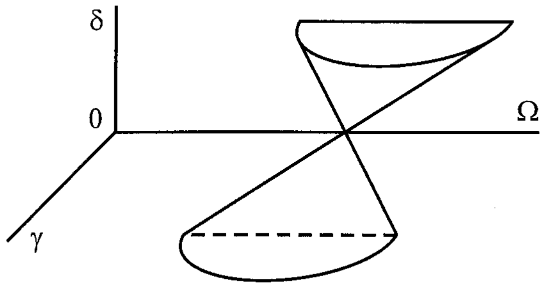


Fig. 0.2 Parametric resonance domain as a half-cone in three-dimensional space of physical parameters.

of connected bodies, panels and wings in airflow *etc.* For these systems we perform the detailed multi-parameter stability analysis showing how the developed bifurcation and singularity theory works in specific problems.

Among previous studies on the stability theory and applications we should mention the books [Liapunov (1892); Chetayev (1961); Bolotin (1963); Bolotin (1964); Panovko and Gubanova (1965); Malkin (1966); Ziegler (1968); Huseyin (1978); Thompson (1982); Huseyin (1986); Leipholz (1987); Yakubovich and Starzhinskii (1987); Troger and Steindl (1991); Kounadis and Kratzig (1995); Merkin (1997); Thomsen (1997); Rumyantsev and Karapetyan (1998)].

The book is based mostly on the authors' personal research, and the relevant papers are given in the list of references. The main results of the book were presented at numerous International Conferences and Symposia. Basic results of the book are given as a one-year course on stability and catastrophes of mechanical systems in Moscow State Lomonosov University by A. P. Seyranian. For the first time this course was presented in Technical University of Denmark and Aalborg University (Denmark) in 1991, see [Seyranian (1991b)]. This course was also given in Bauman Moscow State Technical University in 1993–1994 and Dalian University of Technology (China) in 1994. In 2001 A. P. Seyranian presented six lectures on bifurcations of eigenvalues and stability problems in mechanics at International Centre for Mechanical Sciences in Udine (Italy) [Seyranian and Elishakoff (2002)]. The course on singularities of stability boundaries was given by A. A. Mailybaev in the Institute of Pure and Applied Mathematics IMPA (Brazil) in 2001.

The book is divided into 12 Chapters. Chapter 1 presents an introduction to the stability theory. Chapter 2 is devoted to bifurcation analysis of eigenvalues depending on parameters. This important chapter is used in all parts of the book. In Chapter 3 the stability boundary and its singularities for general systems of ordinary differential equations smoothly dependent on parameters are analyzed. It is shown how to describe singularities in the parameter space using information on the system at the singular point. Chapter 4 presents general bifurcation theory of roots of characteristic polynomials dependent on parameters with application to analysis of stability boundaries. In Chapter 5 we consider linear conservative systems. Change of simple and multiple frequencies depending on several parameters is studied. Multi-parameter stability analysis reveals an interesting relation of singularities of stability boundaries to the so-called bimodal solutions in structural optimization problems. Chapter 6 provides detailed explanation of the effect of gyroscopic stabilization in terms of bifurcation theory of eigenvalues. Chapter 7 studies linear Hamiltonian systems, which are characterized by rich and sophisticated set of different kind of singularities on the stability boundary. Chapter 8 investigates several interesting mechanical phenomena and paradoxes associated with bifurcations and singularities. In Chapter 9 we give an introduction to multi-parameter stability theory of periodic systems. Results of this chapter are used in Chapter 10 for analysis of stability boundaries of general periodic systems. Chapter 11 studies systems with small damping under small periodic excitation, and Chapter 12 considers non-conservative systems under small parametric excitation.

The authors are much indebted to Professors V. I. Arnold, T. M. Atanackovic, V. V. Beletsky, I. I. Blekhman, V. V. Bolotin, Cheng Gengdong, F. L. Chernousko, G. G. Chernyi, S. S. Grigoryan, A. Yu. Ishlinskii, W. Kliem, V. V. Kozlov, V. B. Lidskii, F. Niordson, N. Olhoff, M. L. Overton, M. P. Paidoussis, Ya. G. Panovko, P. Pedersen, V. V. Rumyantsev, V. A. Sadovnichii, Y. Sugiyama, V. V. Svetlitskii, H. Troger, M. I. Vishik for their interest, support, and fruitful discussions. Effective collaboration with Pauli Pedersen, Frederic Solem, Wolfhard Kliem, Samvel S. Grigoryan, and Alexander V. Sharanyuk is highly appreciated and acknowledged. The authors express their gratitude to Oleg N. Kirillov for reading and commenting on the manuscript of the book. The second author is grateful to the Institute of Pure and Applied Mathematics (IMPA) for the support of his visit to Brazil, during which a part of the work on this book was done.

The book is addressed to graduate students and university professors interested in the stability theory and applications, as well as to researchers and industrial engineers. We hope that the book will promote studies of new problems, effects, and phenomena associated with instabilities and catastrophes, and give a fresh view to classical problems.

Alexander P. Seyranian and Alexei A. Mailybaev

Moscow – Rio de Janeiro, February 2003

Contents

Chapter 1

Introduction to Stability Theory

Concept of stability in common and engineering sense reflects necessity to keep response of a disturbed system within acceptable limits. If deviations describing response of the system from a given regime (e.g. state of equilibrium) lie within prescribed limits, the system is called stable. Otherwise, the system is called unstable. Disturbances, response, and prescribed limits can be specified in each case in different ways. In this book we mostly deal with dynamical problems for multiple degrees of freedom systems, and stability of motion is understood in the Liapunov sense.

1.1 Definition of stability

Consider a dynamical system described by ordinary differential equations written in a vector form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t). \tag{1.1}$$

Here it is assumed that $\mathbf{y} = (y_1, y_2, \ldots, y_m)^T$ is a real state vector, the dot over a symbol means differentiation with respect to time t, and $\mathbf{f} = (f_1, \ldots, f_m)^T$ is a real vector-function smoothly dependent on its variables providing existence and uniqueness of a solution with the initial condition $\mathbf{y}(t_0) = \mathbf{y}_0$ on the semi-infinite interval of time $t \geq t_0$.

When the vector-function $\mathbf{f}$ does not depend on time explicitly, the system is called *autonomous*. Otherwise, the system is called *non-autonomous* or *non-stationary*.

Considering a partial solution $\widetilde{\mathbf{y}}(t)$ of equation (1.1) as undisturbed motion and other solutions $\mathbf{y}(t)$ as disturbed motions, we observe evolution of disturbances $y_i(t_0) - \widetilde{y}_i(t_0), i = 1, \ldots, m$, taken at the initial instant

$t = t_0$, in time. For such solutions [Liapunov (1892)] introduced the well-known definition of stability.

Definition 1.1 The undisturbed motion (solution) $\widetilde{\mathbf{y}}(t)$ of system (1.1) is called *stable* with respect to the variables $y_1, y_2, \ldots, y_m$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $\mathbf{y}(t)$ of (1.1), satisfying the condition $\|\mathbf{y}(t_0) - \widetilde{\mathbf{y}}(t_0)\| < \delta$, the inequality

$$\|\mathbf{y}(t) - \widetilde{\mathbf{y}}(t)\| < \varepsilon \tag{1.2}$$

takes place for all $t \geq t_0$.

If, in addition,

$$\|\mathbf{y}(t) - \widetilde{\mathbf{y}}(t)\| \to 0 \quad \text{as} \quad t \to +\infty, \tag{1.3}$$

then the solution $\widetilde{\mathbf{y}}(t)$ is called *asymptotically stable.*

This definition means that small deviations of the initial conditions remain bounded in time for stable motions (solutions) and tend to zero for asymptotically stable solutions. The restrictions of the Liapunov definition of stability are that the disturbances are taken only at the initial instant of time and they are small. Besides, the state vectors for undisturbed and disturbed motions are compared at the same time, and stability is observed on the infinite interval of time. Nevertheless, the given definition of stability is very useful and practical for many physical problems.

Definition 1.2 The undisturbed motion (solution) $\widetilde{\mathbf{y}}(t)$ of system (1.1) is called *unstable* if there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists a solution $\mathbf{y}(t)$, satisfying the condition $\|\mathbf{y}(t_0) - \widetilde{\mathbf{y}}(t_0)\| < \delta$, that for some $t_* > t_0$ the inequality

$$\|\mathbf{y}(t_*) - \widetilde{\mathbf{y}}(t_*)\| > \varepsilon \tag{1.4}$$

takes place.

1.2 Equations for disturbed motion

It is convenient to write equations for disturbed motion in the deviations

$$x_i(t) = y_i(t) - \widetilde{y}_i(t). \tag{1.5}$$

Inserting (1.5) into (1.1) and expanding the right-hand side into Taylor series, we obtain

$$\dot{\tilde{y}}_i + \dot{x}_i = f_i(\tilde{y}_1, \ldots, \tilde{y}_m, t) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j} x_j + \eta_i(x_1, \ldots, x_m, t), \tag{1.6}$$

where η_i are the terms of order higher than one with respect to $x_1, \ldots, x_m$. Since for the undisturbed motion we have

$$\dot{\tilde{y}}_i = f_i(\tilde{y}_1, \ldots, \tilde{y}_m, t), \tag{1.7}$$

equation (1.6) yields

$$\dot{x}_i = \sum_{j=1}^{m} a_{ij}(t) x_j + \eta_i(x_1, \ldots, x_m, t), \quad i = 1, \ldots, m, \tag{1.8}$$

where the coefficients

$$a_{ij}(t) = \left(\frac{\partial f_i}{\partial y_j} \right)_{\tilde{\mathbf{y}}(t)} \tag{1.9}$$

are evaluated at $\mathbf{y} = \tilde{\mathbf{y}}(t)$. These are the equations for disturbed motion, which can be given in a vector form as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \boldsymbol{\eta}(\mathbf{x}, t) \tag{1.10}$$

with the real vector $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)^T$ and the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}. \tag{1.11}$$

The linear equation for the vector $\mathbf{x}$

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{1.12}$$

is called the *equation of first approximation* or *linearized equation* for disturbed motion.

Generally, differential equations for disturbed motion contain time t explicitly. However, there are important cases, when these equations are independent on time. This happens when stability of an *equilibrium state* $\tilde{\mathbf{y}}(t) \equiv \mathbf{y}_0$ of an autonomous system is studied. In this case all the functions $\tilde{y}_i(t)$ are constant and the functions f_i do not depend explicitly on time t. That is why the equations for disturbed motion do not contain time explicitly and the coefficients a_{ij} are constant. Independence on time for the equations of disturbed motion can also occur when stability of a specific motion $\tilde{\mathbf{y}}(t)$ of an autonomous system is studied.

We will call the undisturbed motion $\tilde{\mathbf{y}}(t)$ *steady* if the corresponding equation (1.10) does not contain time t explicitly. The case of steady motion is one of the simplest for the stability study. Another rather simple case is when the coefficients a_{ij} in (1.8) are periodic functions of time t.

1.3 Linear autonomous system

In this section we consider linear autonomous systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1.13}$$

with a constant real $m \times m$ matrix $\mathbf{A}$. Seeking solution to this problem as

$$\mathbf{x} = \mathbf{u} \exp \lambda t, \tag{1.14}$$

we substitute (1.14) into (1.13) and come to the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}, \tag{1.15}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector.

A non-trivial solution to (1.15) exists if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{1.16}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix. This is the *characteristic equation* for eigenvalues λ, which can be represented in a polynomial form

$$\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0 = 0 \tag{1.17}$$

with the coefficients $a_0, a_1, \ldots, a_{m-1}$ dependent on elements of the matrix $\mathbf{A}$. From equation (1.17) we find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, which are real or complex conjugate numbers. The eigenvalues can be simple or multiple as the roots of characteristic equation (1.17). Assuming that all the roots of equation (1.17) are simple with the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$, a general solution to (1.13) takes the form

$$\mathbf{x}(t) = c_1\mathbf{u}_1 \exp \lambda_1 t + \cdots + c_m\mathbf{u}_m \exp \lambda_m t, \tag{1.18}$$

where $c_1, c_2, \ldots, c_m$ are constant coefficients to be found from the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

If we admit multiple eigenvalues as the roots of characteristic equation (1.17), the number of linearly independent eigenvectors can be less than m. The form of general solution (1.18) remains valid only for so-called *semi-simple* eigenvalues, when in spite of multiplicity the number of linearly independent eigenvectors is equal to m. But generally multiple eigenvalues lead to *secular* terms proportional to $t^\beta \exp \lambda t$ in the general solution, where the integer exponent β is less than the multiplicity of λ as a root of the characteristic equation.

Consider, for example, a multiple eigenvalue λ with a *Jordan chain* of vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{r-1}$ satisfying the equations

$$\begin{aligned} \mathbf{A}\mathbf{u}_0 &= \lambda \mathbf{u}_0, \\ \mathbf{A}\mathbf{u}_1 &= \lambda \mathbf{u}_1 + \mathbf{u}_0, \\ &\vdots \\ \mathbf{A}\mathbf{u}_{r-1} &= \lambda \mathbf{u}_{r-1} + \mathbf{u}_{r-2}. \end{aligned} \tag{1.19}$$

The vector $\mathbf{u}_0$ is the eigenvector, and $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are called *associated* vectors corresponding to λ. Then the terms in the general solution corresponding to λ are the following, see [Lancaster (1966)]:

$$\begin{aligned} &c_0 \mathbf{u}_0 \exp \lambda t + c_1(\mathbf{u}_0 t + \mathbf{u}_1) \exp \lambda t + \\ &\cdots + c_{r-1} \left(\frac{\mathbf{u}_0 t^{r-1}}{(r-1)!} + \frac{\mathbf{u}_1 t^{r-2}}{(r-2)!} + \cdots + \mathbf{u}_{r-1} \right) \exp \lambda t. \end{aligned} \tag{1.20}$$

The general solution may contain several terms of type (1.20) with different Jordan chains corresponding to the same eigenvalue λ. This happens when there are several eigenvectors for the same λ.

From (1.18) and (1.20) it is obvious that if real parts of all the eigenvalues are negative, $\operatorname{Re} \lambda < 0$, the norm of the general solution $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This property means the asymptotic stability of the trivial solution $\mathbf{x}(t) \equiv 0$. And if there exists at least one eigenvalue with a positive real part, $\operatorname{Re} \lambda > 0$, then there are infinitely growing solutions $\mathbf{x}(t)$ for arbitrarily small initial conditions, which means instability of the trivial solution. Notice that the terms in the general solution corresponding to purely imaginary or zero eigenvalues (with $\operatorname{Re} \lambda = 0$) remain bounded only for simple or semi-simple eigenvalues, otherwise due to the presence of secular terms in (1.20) we get $\|\mathbf{x}(t)\| \to \infty$ as $t \to +\infty$. Now we can formulate the statements for stability and instability of linear systems.

Theorem 1.1 *Linear system (1.13) is asymptotically stable if and only if all the eigenvalues of the matrix* **A** *have negative real part* $\operatorname{Re}\lambda < 0$.

System (1.13) is stable if and only if all the eigenvalues of the matrix **A** *have negative or zero real part* $\operatorname{Re}\lambda \leq 0$ *with all the purely imaginary and zero eigenvalues being simple or semi-simple.*

Finally, linear system (1.13) is unstable if and only if there exists an eigenvalue of the matrix **A** *with a positive real part* $\operatorname{Re}\lambda > 0$, *or an eigenvalue with zero real part* $\operatorname{Re}\lambda = 0$, *which is neither simple nor semi-simple.*

1.4 Introduction of parameters

We assume now that elements of the matrix **A** smoothly depend on a vector of real parameters $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. With a change of parameters stability of system (1.13) can change to instability. This happens when one or several eigenvalues λ cross the imaginary axis, see Fig. 1.1.

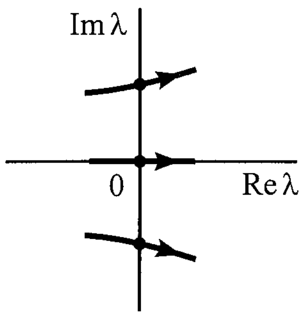


Fig. 1.1 How stability is changed to instability.

The case when a pair of complex conjugate eigenvalues crosses the imaginary axis with a frequency $\omega = \operatorname{Im}\lambda \neq 0$ is known in technical literature as *flutter instability*, and the case when a real negative eigenvalue λ crosses zero and becomes positive is called *divergence instability.* Flutter and divergence are dynamic and static forms of instability, respectively. According to Theorem 1.1, the parameter space can be divided into the stability and instability domains, see Fig. 1.2. The asymptotic stability domain is determined by the condition $\operatorname{Re}\lambda < 0$ satisfied for all the eigenvalues, and the instability domain is defined by the condition $\operatorname{Re}\lambda > 0$ for at least one eigenvalue. The boundary between the stability and instability domains is determined by the condition $\operatorname{Re}\lambda = 0$ satisfied for at least one eigenvalue

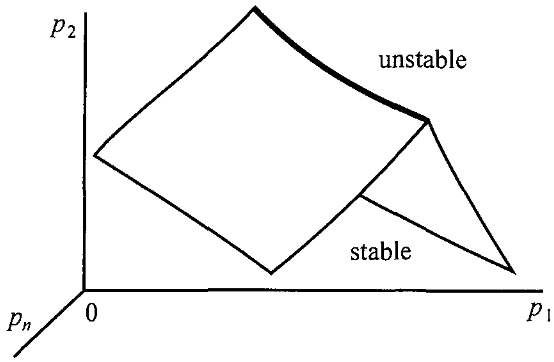


Fig. 1.2 Stability and instability domains.

while for others the condition $\mathrm{Re}\,\lambda < 0$ is fulfilled.

1.5 Stability theorems based on first approximation

Let us consider an autonomous system (1.1) and assume an equilibrium state or steady motion $\tilde{\mathbf{y}}$. Equation for disturbed motion (1.10) takes the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \eta(\mathbf{x}), \quad \mathbf{x} = \mathbf{y} - \tilde{\mathbf{y}}. \tag{1.21}$$

Liapunov formulated and proved two important theorems on stability of an equilibrium state or steady motion $\tilde{\mathbf{y}}$ of an autonomous system, based on linear approximation (1.13) of equation (1.21), see [Liapunov (1892); Chetayev (1961)].

Theorem 1.2 *If linearized system (1.13) is asymptotically stable, i.e., all the eigenvalues of the matrix* $\mathbf{A}$ *have negative real part* $\mathrm{Re}\,\lambda < 0$, *then the equilibrium state or steady motion* $\tilde{\mathbf{y}}$ *is asymptotically stable independently of the nonlinear term* $\eta(\mathbf{x})$ *in (1.21).*

Theorem 1.3 *If linearized system (1.13) has an eigenvalue with a positive real part* $\mathrm{Re}\,\lambda > 0$, *then the equilibrium state or steady motion* $\tilde{\mathbf{y}}$ *is unstable independently of the nonlinear term* $\eta(\mathbf{x})$ *in (1.21).*

These are the main theorems for stability and instability of non-linear systems based on the analysis of the first (linear) approximation. Notice that the case when some of the eigenvalues or all of them have zero real part is not covered by these theorems. Those cases are called *special cases* of Liapunov, see [Liapunov (1892); Chetayev (1961); Malkin (1966)]. Stability and instability of an equilibrium state or steady motion in these special cases depend on the non-linear terms.

We assume now that the right-hand side of equation for disturbed motion (1.21) smoothly depends on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. Then the parameter space can be divided into the stability and instability domains based on properties of eigenvalues of the matrix $\mathbf{A}(\mathbf{p})$. The boundary between these domains is characterized by the conditions $\operatorname{Re}\lambda = 0$ for some eigenvalues and $\operatorname{Re}\lambda < 0$ for the others. In Chapter 3 it is shown that the boundary between the stability and instability domains is a smooth surface, which can have different kind of singularities. We note that generally nothing can be said about stability or instability of the solution $\widetilde{\mathbf{y}}$ on the boundary between the stability and instability domains based on Liapunov's Theorems 1.2 and 1.3. However, in many physical problems this is not so important, since the boundary is a set of zero measure in the parameter space.

Example 1.1 Let us consider a rigid body moving inertially about a fixed point (the Euler case). Equations of motion are the following, see [Malkin (1966)]:

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= 0, \\ B\dot{\omega}_y + (A - C)\omega_z\omega_x &= 0, \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= 0, \end{aligned} \tag{1.22}$$

where ω_x, ω_y, and ω_z are the projections of the vector of angular velocity on the principal axes of inertia x, y, z of the body, and A, B, C are the moments of inertia about those axes.

System (1.22) admits a partial solution

$$\omega_x = \omega_0 = const, \quad \omega_y = \omega_z = 0, \tag{1.23}$$

which is rotation about the axis x with the constant angular velocity ω_0 (steady motion). There are also similar solutions describing rotations about the axes y and z.

Let us study stability of motion (1.23). Introducing the variables

$$x_1 = \omega_x - \omega_0, \quad x_2 = \omega_y, \quad x_3 = \omega_z \tag{1.24}$$

and substituting them into (1.22), we obtain the equations for disturbed motion

$$\begin{aligned}
\dot{x}_1 &= \frac{B-C}{A} x_2 x_3, \\
\dot{x}_2 &= \frac{C-A}{B} \omega_0 x_3 + \frac{C-A}{B} x_1 x_3, \\
\dot{x}_3 &= \frac{A-B}{C} \omega_0 x_2 + \frac{A-B}{C} x_1 x_2.
\end{aligned} \tag{1.25}$$

The characteristic equation for the linearized problem

$$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \dfrac{C-A}{B}\omega_0 \\ 0 & \dfrac{A-B}{C}\omega_0 & -\lambda \end{pmatrix} = 0 \tag{1.26}$$

gives the eigenvalues

$$\lambda_{1,2} = \pm\omega_0 \sqrt{\frac{(C-A)(A-B)}{BC}}, \quad \lambda_3 = 0. \tag{1.27}$$

If $C < A < B$ or $C > A > B$, i.e., if rotation takes place about the axis corresponding to the intermediate moment of inertia, then eigenvalues (1.27) are real, one of them being positive. Thus, according to Theorem 1.3 rotation about this axis is unstable.

However, stability of rotation about the axis corresponding to the extremal (smallest or largest) moment of inertia can not be established with the use of Theorem 1.2 because in these cases two eigenvalues (1.27) are purely imaginary and the third eigenvalue is zero. Stability of rotation in those cases can be proven using integrals of motion, see [Malkin (1966)].

1.6 Mechanical systems

Consider a scleronomic mechanical system with holonomic constraints having m degrees of freedom. This means that position of the system is specified by a vector of generalized coordinates $\mathbf{q} = (q_1, \ldots, q_m)^T$, generalized forces do not depend on time t explicitly, and constraints imposed on the system depend only on generalized coordinates. Motion of the system is governed by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = g_i(\mathbf{q}, \dot{\mathbf{q}}), \quad i = 1, \dots, m. \tag{1.28}$$

In these equations, the *kinetic energy* T is a quadratic form with respect to generalized velocities $\dot{q}_1, \dots, \dot{q}_m$:

$$T = \frac{1}{2} \sum_{i,j=1}^{m} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} \tag{1.29}$$

with the positive definite mass matrix $\mathbf{M} = [m_{ij}] > 0$ dependent only on the generalized coordinates $q_1, \dots, q_m$; $\mathbf{g} = (g_1, \dots, g_m)^T$ is the vector of generalized forces.

Let $\mathbf{q}(t) \equiv 0$ be an equilibrium state of the system defined by the condition $\mathbf{g}(0, 0) = 0$. Then the linear approximation of the generalized forces near the equilibrium state yields

$$\mathbf{g} = -\mathbf{B}\dot{\mathbf{q}} - \mathbf{C}\mathbf{q} \tag{1.30}$$

Generally, the matrices $\mathbf{B}$ and $\mathbf{C}$ are non-symmetric. They can be represented as the sum of symmetric and skew-symmetric matrices

$$\mathbf{B} = \mathbf{D} + \mathbf{G}, \quad \mathbf{C} = \mathbf{P} + \mathbf{N} \tag{1.31}$$

with

$$\begin{aligned} \mathbf{D} &= \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^T), \quad \mathbf{G} = \tfrac{1}{2}(\mathbf{B} - \mathbf{B}^T), \\ \mathbf{P} &= \tfrac{1}{2}(\mathbf{C} + \mathbf{C}^T), \quad \mathbf{N} = \tfrac{1}{2}(\mathbf{C} - \mathbf{C}^T). \end{aligned} \tag{1.32}$$

The force $-\mathbf{P}\mathbf{q}$ with the symmetric matrix $\mathbf{P}$ is called *potential* or *conservative*, and the quadratic form

$$\Pi(\mathbf{q}) = \frac{1}{2} \mathbf{q}^T \mathbf{P} \mathbf{q} \tag{1.33}$$

is the *potential energy.* Potential forces are widely known in mechanics, e.g. gravitational and elastic forces are conservative.

The quadratic form

$$R(\dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{D} \dot{\mathbf{q}} \tag{1.34}$$

with the symmetric positive semi-definite matrix $\mathbf{D} \geq 0$ is called the *Rayleigh's dissipative function*, and the force $-\mathbf{D}\dot{\mathbf{q}}$ is called *dissipative.* In case of positive definite matrix $\mathbf{D} > 0$ dissipation is *complete*, and the case

$\mathbf{D} \geq 0$ corresponds to *incomplete* dissipation. Dissipative forces describe viscous friction and resistance of medium appearing in physical systems.

The force $-\mathbf{G}\dot{\mathbf{q}}$ with the skew-symmetric matrix $\mathbf{G}^T = -\mathbf{G}$ is called *gyroscopic*. Power of a gyroscopic force (the work done by the force per unit time) is zero. Indeed, we have

$$(-\mathbf{G}\dot{\mathbf{q}})^T \dot{\mathbf{q}} = -\dot{\mathbf{q}}^T \mathbf{G}^T \dot{\mathbf{q}} = \dot{\mathbf{q}}^T \mathbf{G} \dot{\mathbf{q}} = 0 \tag{1.35}$$

since the vector $\dot{\mathbf{q}}$ is real and the matrix $\mathbf{G}$ is skew-symmetric. Gyroscopic forces appear in rotating systems, Coriolis and Lorentz forces are gyroscopic too.

The force $-\mathbf{N}\mathbf{q}$ is called *non-conservative positional* or *circulatory*. Notice that this force is orthogonal to the vector of generalized coordinates $\mathbf{q}$ since $\mathbf{q}^T\mathbf{N}\mathbf{q} = 0$. Circulatory forces appear as components of aerodynamic and follower forces (e.g. jet thrust). The presence of circulatory and dissipative forces means that the system can gain energy from the environment or lose energy, depending upon the ratio between the forces and their magnitudes for all the types of forces involved.

Using (1.29)–(1.32) in Lagrange equations (1.28), we obtain in a linear approximation

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{P} + \mathbf{N})\mathbf{q} = 0. \tag{1.36}$$

This is the linearized equation for disturbed motion near the equilibrium state $\mathbf{q} = 0$. Seeking solution to equation (1.36) in the form

$$\mathbf{q} = \mathbf{u} \exp \lambda t, \tag{1.37}$$

we come to the eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u} = 0. \tag{1.38}$$

Here λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. The eigenvalues are found from the characteristic equation

$$\det\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right) = 0. \tag{1.39}$$

This is an algebraic equation of $2m$th order for λ. There exist $2m$ roots $\lambda_1, \ldots, \lambda_{2m}$ (the eigenvalues), and corresponding eigenvectors should be found from equation (1.38).

Equation (1.36) can be transformed to a system of first order differential equations of double dimension:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1.40}$$

with

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{P}+\mathbf{N}) & -\mathbf{M}^{-1}(\mathbf{D}+\mathbf{G}) \end{pmatrix}. \tag{1.41}$$

It is easy to see that the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for the matrix $\mathbf{A}$ in (1.41) is equivalent to equation (1.39) implying that the eigenvalues in these two problems coincide.

If all the eigenvalues $\lambda_1, \ldots, \lambda_{2m}$ of (1.39) are simple or semi-simple, the general solution to equation (1.36) takes the form

$$\mathbf{q}(t) = c_1\mathbf{u}_1 \exp\lambda_1 t + \cdots + c_{2m}\mathbf{u}_{2m} \exp\lambda_{2m} t. \tag{1.42}$$

If the number of eigenvectors corresponding to a multiple eigenvalue λ is less than its multiplicity (as a root of the characteristic equation), secular terms appear in the general solution. Using transformation (1.40), (1.41) and the results of Section 1.3, we find that those terms are of the form

$$\begin{gathered} c_0\mathbf{u}_0 \exp\lambda t + c_1(\mathbf{u}_0 t + \mathbf{u}_1)\exp\lambda t + \\ \cdots + c_{r-1}\left(\frac{\mathbf{u}_0 t^{r-1}}{(r-1)!} + \frac{\mathbf{u}_1 t^{r-2}}{(r-2)!} + \cdots + \mathbf{u}_{r-1}\right)\exp\lambda t, \end{gathered} \tag{1.43}$$

where $\mathbf{u}_0$ is the eigenvector, and $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are associated vectors constituting the *Keldysh chain* of length r, see [Keldysh (1951)] and Section 2.13:

$$\begin{gathered} \left(\lambda^2\mathbf{M} + \lambda(\mathbf{D}+\mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u}_i + (2\lambda\mathbf{M} + \mathbf{D} + \mathbf{G})\mathbf{u}_{i-1} + \mathbf{M}\mathbf{u}_{i-2} = 0, \\ i = 0, \ldots, r-1 \quad \text{and} \quad \mathbf{u}_{-1} = \mathbf{u}_{-2} = 0. \end{gathered} \tag{1.44}$$

From (1.42), (1.43) it is obvious that system (1.36) is asymptotically stable if all the eigenvalues have negative real part, and it is unstable if at least one eigenvalue has positive real part.

Example 1.2 Let us consider vibrations of a pendulum with a linear viscous damping described by the equation

$$ml\ddot{\varphi} + \gamma l\dot{\varphi} + mg\sin\varphi = 0, \tag{1.45}$$

where φ is the angle of the pendulum measured from the vertical axis; m, l, and γ are the mass, length, and damping coefficient of the pendulum, respectively; and g is the acceleration of gravity, see Fig. 1.3.

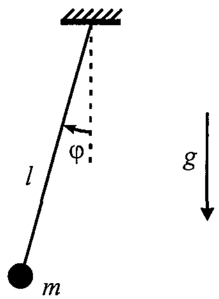


Fig. 1.3 Vibrating pendulum.

We introduce new variables $y_1 = \varphi$, $y_2 = \dot{\varphi}$, and rewrite (1.45) as the system of first order equations

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -\frac{g \sin y_1}{l} - \frac{\gamma y_2}{m}. \end{aligned} \tag{1.46}$$

To find stationary solutions we equate the right-hand sides of (1.46) zero and obtain two solutions

$$y_1 = 0, \quad y_2 = 0; \tag{1.47}$$

$$y_1 = \pi, \quad y_2 = 0. \tag{1.48}$$

These solutions correspond to lower and upper equilibrium states, respectively.

First, we consider equilibrium state (1.47). For this case the disturbances x_1 and x_2 coincide with the variables y_1 and y_2. Expanding the right-hand sides of equation (1.46) in Taylor series and replacing the variables, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g x_1}{l} - \frac{\gamma x_2}{m} + \frac{g x_1^3}{6l} + o(x_1^3). \end{aligned} \tag{1.49}$$

Thus, the matrix $\mathbf{A}$ of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{pmatrix}. \tag{1.50}$$

The characteristic equation for this matrix

$$\lambda^2 + \frac{\gamma}{m}\lambda + \frac{g}{l} = 0 \tag{1.51}$$

yields the roots

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{g}{l}}, \tag{1.52}$$

which always have negative real part for $\gamma > 0$. Thus, the lower equilibrium state is asymptotically stable.

For the case of upper equilibrium state (1.48) we have

$$x_1 = y_1 - \pi, \quad x_2 = y_2. \tag{1.53}$$

Substituting relations (1.53) into (1.46) and expanding the right-hand sides, we get the equations for disturbed motion as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{gx_1}{l} - \frac{\gamma x_2}{m} - \frac{gx_1^3}{6l} + o(x_1^3). \end{aligned} \tag{1.54}$$

It is easy to see that the linearization of equation (1.54) gives the eigenvalues

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} + \frac{g}{l}}, \tag{1.55}$$

one of them always having a positive real part. Thus, the upper vertical equilibrium state is unstable.

1.7 Asymptotic stability criteria for mechanical systems

Let us investigate stability of a linear mechanical system

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{P} + \mathbf{N})\mathbf{q} = 0 \tag{1.56}$$

with the corresponding eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{P} + \mathbf{N}\right)\mathbf{u} = 0. \tag{1.57}$$

System (1.56) is asymptotically stable if all the eigenvalues λ of problem (1.57) have negative real part.

We pre-multiply equation (1.57) by the complex-conjugate transposed eigenvector $\mathbf{u}^* = \overline{\mathbf{u}}^T$ and obtain the relation

$$M\lambda^2 + (D + iG)\lambda + P + iN = 0 \tag{1.58}$$

with the coefficients

$$\begin{gathered} M = \mathbf{u}^*\mathbf{M}\mathbf{u}, \quad D = \mathbf{u}^*\mathbf{D}\mathbf{u}, \quad P = \mathbf{u}^*\mathbf{P}\mathbf{u}, \\ iG = \mathbf{u}^*\mathbf{G}\mathbf{u}, \quad iN = \mathbf{u}^*\mathbf{N}\mathbf{u}, \end{gathered} \tag{1.59}$$

where M, D, P, G, and N are real quantities. Additionally, we assume that the eigenvector is normalized as

$$\mathbf{u}^*\mathbf{u} = 1. \tag{1.60}$$

Considering (1.58) as a quadratic equation for λ with complex coefficients, we demand that both roots of equation (1.58) have negative real part. Here we can use a theorem on stability properties of a polynomial with complex coefficients, see [Bilharz (1944)]. Applied to equation (1.58), the theorem states that both roots λ have negative real part if and only if the two determinants satisfy the relations

$$\det \begin{pmatrix} M & G \\ 0 & D \end{pmatrix} > 0, \quad \det \begin{pmatrix} M & G & -P & 0 \\ 0 & D & N & 0 \\ 0 & M & G & -P \\ 0 & 0 & D & N \end{pmatrix} > 0. \tag{1.61}$$

Since the matrix $\mathbf{M}$ is positive definite the quantity $M > 0$, and then (1.61) is equivalent to two inequalities

$$D > 0, \tag{1.62}$$

$$MN^2 - GDN < D^2P. \tag{1.63}$$

Metelitsyn was the first who derived inequality (1.63), assuming that (1.62) is satisfied, as a criterion for asymptotic stability of system (1.56), see [Metelitsyn (1952)].

Notice that an eigenvalue of (1.57) is one of the two roots of (1.58), the other root does not need to be an eigenvalue of (1.57). This important fact was pointed out in [Seyranian (1994b)]. Actually, it is more an exception than a rule that both roots of equation (1.58) are the eigenvalues of

(1.57). Metelitsyn made a mistake, also done in [Huseyin (1978)], believing that both roots are always eigenvalues of (1.57). This mistake led to the conclusion that inequality (1.63) is a necessary and sufficient condition for asymptotic stability. We emphasize that inequalities (1.62) and (1.63) taken for all eigenvalues λ are *sufficient* for asymptotic stability, but not necessary.

Example 1.3 Let the system be described by equation (1.56) with the 2×2 matrices

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 5.8186 & 0 \\ 0 & 0.1814 \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} 0 & 3.6667 \\ -3.6667 & 0 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 2.25 \\ -2.25 & 0 \end{pmatrix}. \tag{1.64}$$

The eigenvalues are $\lambda_{1,2} = -1 \pm 0.5i$ and $\lambda_{3,4} = -2 \pm 0.5i$, and therefore the system is asymptotically stable. Computing the corresponding eigenvectors $\mathbf{u}$, coefficients (1.59) of quadratic equation (1.58) can be determined. The roots of this equation (one equation for each eigenvalue) are, of course, the four already found eigenvalues $\lambda_{1,2}$ and $\lambda_{3,4}$, but additionally also $0.0625 \pm 0.875i$ and $0.1785 \pm 0.2859i$. Those roots have positive real part and, therefore, in spite of the system is asymptotically stable, Metelitsyn's inequality (1.63) is not satisfied since it demands that both roots of equation (1.58) have negative real part.

If we want to investigate the asymptotic stability for a given system by checking inequalities (1.62) and (1.63) as sufficient conditions, we face the following problem. The eigenvectors $\mathbf{u}$, which are used for finding coefficients (1.59) and for checking inequalities (1.62) and (1.63), are unknown. They can only be determined by solving eigenvalue problem (1.57), but then the stability analysis would be complete.

However, the statement known as the Thomson-Tait-Chetayev theorem, see [Chetayev (1961)], follows directly from inequalities (1.62) and (1.63).

Theorem 1.4 *If system (1.56) containing only potential forces is stable* ($\mathbf{P} > 0$)*, then addition of arbitrary gyroscopic forces and dissipative forces with complete dissipation* ($\mathbf{D} > 0$) *makes the system asymptotically stable.*

Indeed, the system possessing only potential forces with a positive definite matrix $\mathbf{P}$ is stable since all the eigenvalues are purely imaginary and semi-simple, see [Gantmacher (1998); Merkin (1997)]. In case of $\mathbf{P} > 0$, $\mathbf{D} > 0$, and $\mathbf{N} = 0$, i.e., $P > 0$, $D > 0$, and $N = 0$, inequality (1.63) reduces to $D^2P > 0$, guaranteeing asymptotic stability.

We are interested in obtaining a practical sufficient stability condition, which can be verified using extremal eigenvalues of the system matrices. Hermitian matrices like $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{P}$ (real symmetric in our case) have only real eigenvalues. The corresponding quantities M, D, and P in (1.59), known as Rayleigh quotients, are therefore limited by the minimal and maximal eigenvalues of the matrices $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{P}$, respectively, see e.g. [Lancaster and Tismenetsky (1985)]

$$\begin{aligned} M_{\min} = \lambda_{\min}(\mathbf{M}) \leq M \leq \lambda_{\max}(\mathbf{M}) = M_{\max}, \\ D_{\min} = \lambda_{\min}(\mathbf{D}) \leq D \leq \lambda_{\max}(\mathbf{D}) = D_{\max}, \\ P_{\min} = \lambda_{\min}(\mathbf{P}) \leq P \leq \lambda_{\max}(\mathbf{P}) = P_{\max}. \end{aligned} \tag{1.65}$$

We emphasize that these limits depend only on the system matrices and do not depend on the eigenvector $\mathbf{u}$.

Since the matrices $\mathbf{G}$ and $\mathbf{N}$ are real skew-symmetric, the matrices $i\mathbf{G}$ and $i\mathbf{N}$ are Hermitian. Notice that spectrum of a real skew-symmetric matrix consists of purely imaginary $\pm i\omega$ and zero eigenvalues. Therefore, the quantities G and N being real are limited by $-G_{\max}$ and $G_{\max}$, and $-N_{\max}$ and $N_{\max}$, respectively, where $G_{\max} = \lambda_{\max}(i\mathbf{G})$ and $N_{\max} = \lambda_{\max}(i\mathbf{N})$ are the maximal eigenvalues of the corresponding matrices. So, we have

$$-G_{\max} \leq G \leq G_{\max}, \quad -N_{\max} \leq N \leq N_{\max}. \tag{1.66}$$

If we assume

$$\mathbf{M} > 0, \quad \mathbf{D} > 0, \quad \mathbf{P} > 0, \tag{1.67}$$

then it is easy to see with the help of (1.65) and (1.66) that (1.63), rewritten in the form $D(DP + GN) - MN^2 > 0$, is satisfied for an arbitrary eigenvector $\mathbf{u}$ if

$$D_{\min}(D_{\min}P_{\min} - G_{\max}N_{\max}) - M_{\max}N_{\max}^2 > 0. \tag{1.68}$$

Here we took the smallest values of the first and second terms and the largest value of the third term of the inequality. Under assumption (1.67), inequality (1.68) is a practical sufficient condition for asymptotic stability of system (1.56), which can be checked knowing only the extreme eigenvalues of the system matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{P}$, and $\mathbf{N}$ [Kliem *et al.* (1998)].

From (1.68) we deduce the following stability statement.

Theorem 1.5 *Any mechanical system (1.56) can be stabilized by sufficiently large dissipative and/or potential forces* ($\mathbf{D} > 0, \mathbf{P} > 0$).

Here "large forces" means that the minimal eigenvalues $D_{\min}$ and/or $P_{\min}$ of the corresponding matrices are large enough.

Proof of the theorem follows from the observation that inequality (1.68) is satisfied by making the term $D_{\min}^2 P_{\min}$ sufficiently large. This result was first reported in [Seyranian (1994b)]. Another consequence of inequality (1.68) is that a stable conservative system with dissipative forces with complete dissipation (assumption (1.67)) can not be destabilized by adding rather small gyroscopic and/or positional non-conservative forces.

Example 1.4 The simplest model of a rotor consists of a massless shaft of circular cross-section with an elastic coefficient k rotating with a constant angular velocity Ω and carrying a single disk of mass m, see Fig. 1.4. External and internal damping coefficients are denoted by $d_e > 0$ and $d_i > 0$, respectively. With respect to an inertial frame, the equations of motion for the center of mass of the disk moving in the plane perpendicular to the shaft are given by (1.56) with the matrices [Bolotin (1963)]

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_e + d_i & 0 \\ 0 & d_e + d_i \end{pmatrix}, \quad \mathbf{G} = 0,$$
$$\mathbf{P} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & d_i\Omega \\ -d_i\Omega & 0 \end{pmatrix}. \tag{1.69}$$

Quantities (1.65) and (1.66) evaluated for system matrices (1.69) are

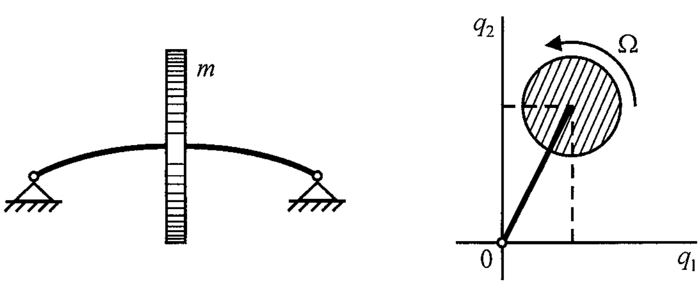


Fig. 1.4 Rotating shaft with a disk.

equal to

$$M_{\min} = M_{\max} = m, \quad D_{\min} = D_{\max} = d_e + d_i,$$
$$P_{\min} = P_{\max} = k, \quad G_{\max} = 0, \quad N_{\max} = d_i \Omega. \tag{1.70}$$

For system (1.56), (1.69) inequality (1.68) results in

$$\Omega^2 m d_i^2 < k(d_e + d_i)^2. \tag{1.71}$$

This inequality gives a lower bound for the critical angular velocity

$$\Omega_* = \sqrt{\frac{k}{m}} \left(1 + \frac{d_e}{d_i}\right). \tag{1.72}$$

Let us compare this estimate with the exact critical velocity. For this purpose, we introduce the complex variable $z = q_1 - iq_2$ and rewrite equations (1.56), (1.69) in a complex form as

$$m\ddot{z} + (d_e + d_i)\dot{z} + (k + id_i\Omega)z = 0. \tag{1.73}$$

The corresponding characteristic equation

$$m\lambda^2 + (d_e + d_i)\lambda + (k + id_i\Omega) = 0 \tag{1.74}$$

is a quadratic equation for λ with complex coefficients. Applying now inequalities (1.62) and (1.63), we obtain that the stability condition is the same as (1.71). Thus, estimate (1.72) is the exact critical velocity. This is one of those rare cases when sufficient stability condition (1.68) yields the exact stability boundary.

Chapter 2

Bifurcation Analysis of Eigenvalues

Behavior of simple and multiple eigenvalues with a change of parameters is a problem of general interest for applied mathematics and natural sciences. This problem has many important applications in aerospace, mechanical, civil, and electrical engineering. One-parameter perturbation theory of eigenvalues for nonsymmetric matrices and differential operators was developed in [Vishik and Lyusternik (1960)], and a constructive method for determining leading terms in eigenvalue expansions was given by [Lidskii (1965)]. These works study regular types of bifurcations, when perturbation satisfies a specific nondegeneracy condition. For the analysis of some non-regular cases see [Moro *et al.* (1997)]. The multi-parameter bifurcation theory for eigenvalues of nonsymmetric matrices was developed in [Seyranian (1990a); Seyranian (1991a); Seyranian (1993a); Seyranian (1994a); Mailybaev and Seyranian (1999b); Mailybaev and Seyranian (2000a); Seyranian and Kirillov (2001)], where perturbations along different directions or curves in the parameter space were studied. Recent achievements of the theory of interaction of eigenvalues in multi-parameter problems are given in [Kirillov and Seyranian (2002a); Seyranian and Mailybaev (2003)].

In this chapter we present general results on bifurcation theory of multiple eigenvalues for matrices dependent on several parameters. Strong and weak interactions of eigenvalues on the complex plane are distinguished and studied. Extensions to generalized eigenvalue problem and eigenvalue problem corresponding to vibrational systems are presented. The results of this chapter represent the main tool for the multi-parameter stability analysis and are used in all parts of this book.

2.1 Eigenvalue problem

Let us consider an eigenvalue problem

$$\mathbf{Au} = \lambda \mathbf{u}, \tag{2.1}$$

where $\mathbf{A}$ is an $m \times m$ real matrix, λ is an *eigenvalue*, and $\mathbf{u}$ is a corresponding *eigenvector*. The eigenvalues are determined from the *characteristic equation*

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{2.2}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix. Since $\det(\mathbf{A} - \lambda \mathbf{I})$ is a polynomial of degree m with respect to λ, there are m eigenvalues, counting multiplicities. Since $\mathbf{A}$ is a real matrix, its eigenvalues and corresponding eigenvectors are real or appear in complex conjugate pairs. Multiplicity of an eigenvalue as a root of the characteristic equation is called *algebraic multiplicity.*

The eigenvalue λ is called *simple* if its algebraic multiplicity is equal to one. There is a single eigenvector, up to a scaling factor, corresponding to a simple eigenvalue.

2.2 Multiple eigenvalues and the Jordan canonical form

A multiple eigenvalue λ of algebraic multiplicity k can have one or several corresponding eigenvectors. The maximal number of linearly independent eigenvectors k_g is called *geometric multiplicity* of the eigenvalue, which is less or equal to the algebraic multiplicity:

$$k_g \leq k. \tag{2.3}$$

If the algebraic and geometric multiplicities are equal ($k_g = k$), then the eigenvalue is called *semi-simple.* If there is a single eigenvector corresponding to λ ($k_g = 1$), then the eigenvalue is called *nonderogatory.*

First, let us consider a nonderogatory eigenvalue λ. There exist linearly independent vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ satisfying the equations

$$\begin{aligned}
\mathbf{Au}_0 &= \lambda \mathbf{u}_0, \\
\mathbf{Au}_1 &= \lambda \mathbf{u}_1 + \mathbf{u}_0, \\
&\vdots \\
\mathbf{Au}_{k-1} &= \lambda \mathbf{u}_{k-1} + \mathbf{u}_{k-2}.
\end{aligned} \tag{2.4}$$

The vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ are called the *Jordan chain of length k*, where $\mathbf{u}_0$ is the eigenvector and the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ are *associated vectors.* Equations (2.4) can be written in the matrix form as

$$\mathbf{A}\mathbf{U}_\lambda = \mathbf{U}_\lambda \mathbf{J}_\lambda(k), \tag{2.5}$$

where

$$\mathbf{U}_\lambda = [\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}] \tag{2.6}$$

is an $m \times k$ matrix, which is real or complex depending on the eigenvalue λ, and

$$\mathbf{J}_\lambda(k) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \tag{2.7}$$

is a $k \times k$ matrix called the *Jordan block.*

Let us consider an eigenvalue λ having several linearly independent eigenvectors (the *derogatory* eigenvalue), i.e., $k_g > 1$. In this case there are integers $1 \le l_1 \le \cdots \le l_{k_g}$ such that

$$l_1 + \cdots + l_{k_g} = k, \tag{2.8}$$

and linearly independent vectors $\mathbf{u}_0^{(i)}, \ldots, \mathbf{u}_{l_i-1}^{(i)}$, $i = 1, \ldots, k_g$, satisfying the Jordan chain equations

$$\begin{aligned} \mathbf{A}\mathbf{u}_0^{(i)} &= \lambda \mathbf{u}_0^{(i)}, \\ \mathbf{A}\mathbf{u}_1^{(i)} &= \lambda \mathbf{u}_1^{(i)} + \mathbf{u}_0^{(i)}, \\ &\vdots \\ \mathbf{A}\mathbf{u}_{l_i-1}^{(i)} &= \lambda \mathbf{u}_{l_i-1}^{(i)} + \mathbf{u}_{l_i-2}^{(i)}. \end{aligned} \tag{2.9}$$

The numbers $l_1, \ldots, l_{k_g}$ are unique and called *partial multiplicities* of the eigenvalue λ, and the vectors $\mathbf{u}_0^{(i)}, \ldots, \mathbf{u}_{l_i-1}^{(i)}$ are called the Jordan chain of length l_i. The vectors $\mathbf{u}_0^{(1)}, \ldots, \mathbf{u}_0^{(k_g)}$ are linearly independent eigenvectors corresponding to λ. Notice that $l_1 = \cdots = l_{k_g} = 1$ for a semi-simple eigenvalue, and $l_1 = k$ for a nonderogatory eigenvalue. Equations (2.9) can be written in the matrix form

$$\mathbf{A}\mathbf{U}_\lambda = \mathbf{U}_\lambda \begin{pmatrix} \mathbf{J}_\lambda(l_1) & & \\ & \ddots & \\ & & \mathbf{J}_\lambda(l_{k_g}) \end{pmatrix}, \tag{2.10}$$

where

$$\mathbf{U}_\lambda = [\mathbf{u}_0^{(1)}, \ldots, \mathbf{u}_{l_1-1}^{(1)}, \ldots, \mathbf{u}_0^{(k_g)}, \ldots, \mathbf{u}_{l_{k_g}-1}^{(k_g)}] \tag{2.11}$$

is an $m \times k$ matrix.

Computing Jordan chains for all the eigenvalues, we can form an $m \times m$ matrix $\mathbf{U}$ satisfying the equation

$$\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{J}, \tag{2.12}$$

or, equivalently,

$$\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}, \tag{2.13}$$

where $\mathbf{J}$ is a block-diagonal matrix with Jordan blocks on the diagonal.

Theorem 2.1 *Let* $\mathbf{A}$ *be a real* $m \times m$ *matrix. Then there is a nonsingular* $m \times m$ *matrix* $\mathbf{U}$ *consisting of eigenvectors and associated vectors, which transforms the matrix* $\mathbf{A}$ *to the form (2.13), where* $\mathbf{J}$ *is a block-diagonal matrix with Jordan blocks on the diagonal. The matrix* $\mathbf{J}$ *is unique up to the permutation of diagonal blocks and is called the Jordan canonical form of the matrix* $\mathbf{A}$.

Proof of this classical theorem can be found in [Gantmacher (1998)].

Eigenvectors and Jordan chains (as well as the transformation matrix $\mathbf{U}$) are not uniquely determined. In particular, any nontrivial linear combination of eigenvectors corresponding to the eigenvalue λ is an eigenvector. If λ is a nonderogatory eigenvalue and $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ is a corresponding Jordan chain, then following (2.4) all the Jordan chains can be expressed in the form

$$\begin{aligned} \mathbf{u}_0' &= c_0\mathbf{u}_0, \\ \mathbf{u}_1' &= c_0\mathbf{u}_1 + c_1\mathbf{u}_0, \\ &\vdots \\ \mathbf{u}_{k-1}' &= c_0\mathbf{u}_{k-1} + c_1\mathbf{u}_{k-2} + \cdots + c_{k-1}\mathbf{u}_0, \end{aligned} \tag{2.14}$$

where $c_0, \ldots, c_{k-1}$ are arbitrary numbers with $c_0 \neq 0$.

Example 2.1 Let us consider the matrix

$$\mathbf{A} = \begin{pmatrix} -5 & -4 & 8 \\ 0 & 7 & 0 \\ -12 & -4 & 15 \end{pmatrix}. \tag{2.15}$$

Characteristic equation (2.2) for this matrix takes the form

$$\lambda^3 - 17\lambda^2 + 91\lambda - 147 = 0. \tag{2.16}$$

This equation has the simple root $\lambda = 3$ and the double root $\lambda = 7$. The simple eigenvalue $\lambda = 3$ has the eigenvector determined by equation (2.1) and equal to

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.17}$$

The double eigenvalue $\lambda = 7$ is semi-simple and has two linearly independent eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{2.18}$$

Any nontrivial linear combination of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$. Taking vectors (2.17) and (2.18) as columns of the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \tag{2.19}$$

we find the Jordan canonical form

$$\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}. \tag{2.20}$$

Example 2.2 Let us consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ -2 & 4 & -3 \\ 1 & -1 & 3 \end{pmatrix}. \tag{2.21}$$

Its characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0. \tag{2.22}$$

This equation has the triple root $\lambda = 2$. The eigenvalue $\lambda = 2$ possesses only one eigenvector

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.23}$$

determined up to a nonzero scaling factor. Hence, this eigenvalue is nonderogatory with algebraic multiplicity $k = 3$ and geometric multiplicity $k_g = 1$. There is a Jordan chain $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ of length 3 corresponding to $\lambda = 2$. Solving equations (2.4), we find the associated vectors

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.24}$$

In general, the vectors $c_0\mathbf{u}_0$, $c_0\mathbf{u}_1 + c_1\mathbf{u}_0$, $c_0\mathbf{u}_2 + c_1\mathbf{u}_1 + c_2\mathbf{u}_0$ with a nonzero c_0 and arbitrary coefficients c_1, c_2 form another Jordan chain corresponding to the eigenvalue $\lambda = 2$. Taking vectors (2.23) and (2.24) as columns of the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \tag{2.25}$$

we find the Jordan canonical form

$$\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.26}$$

2.3 Left eigenvectors and Jordan chains

Let us consider the eigenvalue problem for the transposed matrix

$$\mathbf{A}^T\mathbf{v} = \lambda\mathbf{v}. \tag{2.27}$$

Since

$$\det(\mathbf{A}^T - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I}), \tag{2.28}$$

the characteristic equations for the matrices $\mathbf{A}$ and $\mathbf{A}^T$ coincide. The matrix $\mathbf{A}^T$ has the same Jordan canonical form as the matrix $\mathbf{A}$, see [Gantmacher (1998)]. Therefore, eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{A}^T$ are equal together with their algebraic, geometric, and partial multiplicities.

The eigenvalue problem for the matrix $\mathbf{A}^T$ after the transposition takes the form

$$\mathbf{v}^T\mathbf{A} = \lambda\mathbf{v}^T, \tag{2.29}$$

where $\mathbf{v}$ is called the *left eigenvector* corresponding to the eigenvalue λ, in contrast to the eigenvector $\mathbf{u}$ called the *right eigenvector.*

If an eigenvalue λ is simple, then the left eigenvector is determined up to a nonzero scaling factor. Assuming that the right eigenvector $\mathbf{u}$ is given, we can define the left eigenvector of the simple eigenvalue uniquely by means of the normalization condition

$$\mathbf{v}^T\mathbf{u} = 1. \tag{2.30}$$

Notice that if $\mathbf{u}'$ is a right eigenvector for an eigenvalue λ', and $\mathbf{v}$ is a left eigenvector for an eigenvalue $\lambda \neq \lambda'$, then [Gantmacher (1998)]

$$\mathbf{v}^T\mathbf{u}' = 0. \tag{2.31}$$

Equations of the Jordan chain (2.4) for a nonderogatory eigenvalue λ of the matrix $\mathbf{A}^T$ after transposition take the form

$$\begin{aligned}
\mathbf{v}_0^T\mathbf{A} &= \lambda\mathbf{v}_0^T, \\
\mathbf{v}_1^T\mathbf{A} &= \lambda\mathbf{v}_1^T + \mathbf{v}_0^T, \\
&\vdots \\
\mathbf{v}_{k-1}^T\mathbf{A} &= \lambda\mathbf{v}_{k-1}^T + \mathbf{v}_{k-2}^T.
\end{aligned} \tag{2.32}$$

The vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ are called the *left Jordan chain* for the eigenvalue λ as opposed to the *right Jordan chain* $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$. Right and left Jordan chains have the properties [Gantmacher (1998)]

$$
\begin{aligned}
\mathbf{v}_0^T\mathbf{u}_0 &= 0,\\
\mathbf{v}_1^T\mathbf{u}_0 = \mathbf{v}_0^T\mathbf{u}_1 &= 0,\\
&\vdots\\
\mathbf{v}_{k-2}^T\mathbf{u}_0 = \mathbf{v}_{k-3}^T\mathbf{u}_1 = \cdots = \mathbf{v}_0^T\mathbf{u}_{k-2} &= 0,\\
\mathbf{v}_{k-1}^T\mathbf{u}_0 = \mathbf{v}_{k-2}^T\mathbf{u}_1 = \cdots = \mathbf{v}_0^T\mathbf{u}_{k-1} &\neq 0,\\
\mathbf{v}_{k-1}^T\mathbf{u}_1 = \mathbf{v}_{k-2}^T\mathbf{u}_2 = \cdots &= \mathbf{v}_1^T\mathbf{u}_{k-1},\\
&\vdots\\
\mathbf{v}_{k-1}^T\mathbf{u}_{k-2} &= \mathbf{v}_{k-2}^T\mathbf{u}_{k-1},
\end{aligned}
\tag{2.33}
$$

which follow from equations (2.4), (2.32), and the relation

$$
\mathbf{v}_i^T\mathbf{u}_j = \mathbf{v}_i^T(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_{j+1} = \mathbf{v}_{i-1}^T\mathbf{u}_{j+1} \tag{2.34}
$$

valid for any $1 \le i \le k-1$ and $0 \le j \le k-2$. Assuming that the right Jordan chain is given, we can define the unique left Jordan chain satisfying the normalization conditions

$$
\mathbf{v}_0^T\mathbf{u}_{k-1} = 1, \quad \mathbf{v}_1^T\mathbf{u}_{k-1} = \cdots = \mathbf{v}_{k-1}^T\mathbf{u}_{k-1} = 0. \tag{2.35}
$$

Finally, let us consider a semi-simple eigenvalue λ, which has $k = k_g$ linearly independent right eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and left eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Any nontrivial linear combination of the left (or right) eigenvectors is a left (or right) eigenvector. Assuming that the right eigenvectors are given, we can define the left eigenvectors uniquely by means of the normalization conditions

$$
\mathbf{v}_i^T\mathbf{u}_j = \delta_{ij}, \quad i, j = 1, \ldots, k, \tag{2.36}
$$

where δ_{ij} denotes the Kronecker delta.

Example 2.3 Let us consider matrix (2.15). The left eigenvector, corresponding to the simple eigenvalue $\lambda = 3$ and satisfying normalization condition (2.30) for the right eigenvector given by expression (2.17), is equal to

$$
\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}. \tag{2.37}
$$

The left eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, corresponding to the semi-simple double eigenvalue $\lambda = 7$ and satisfying normalization conditions (2.36) with the right eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ given by expressions (2.18), are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.38}$$

Example 2.4 Let us consider matrix (2.21). The left Jordan chain, corresponding to the triple nonderogatory eigenvalue $\lambda = 2$ and satisfying normalization conditions (2.35) with the right Jordan chain given by expressions (2.23), (2.24), is

$$\mathbf{v}_0 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}. \tag{2.39}$$

2.4 Perturbation of simple eigenvalue

Let us assume that the matrix $\mathbf{A}$ smoothly depends on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. The function $\mathbf{A}(\mathbf{p})$ is called a multi-parameter *family of matrices.* Eigenvalues of the matrix family are continuous functions of the parameter vector. In this section we study behavior of a simple eigenvalue of the matrix family $\mathbf{A}(\mathbf{p})$.

Let $\lambda(\mathbf{p})$ be a simple eigenvalue of the matrix $\mathbf{A}(\mathbf{p})$. Since λ is a simple root of characteristic equation (2.2), we have

$$\frac{\partial}{\partial \lambda} \det(\mathbf{A}(\mathbf{p}) - \lambda \mathbf{I}) \neq 0. \tag{2.40}$$

Using inequality (2.40) and the implicit function theorem applied to characteristic equation (2.2), we find that the simple eigenvalue $\lambda(\mathbf{p})$ of the matrix family $\mathbf{A}(\mathbf{p})$ smoothly depends on the parameter vector, and its derivatives with respect to parameters are equal to

$$\frac{\partial \lambda}{\partial p_i} = -\frac{\partial}{\partial p_i} \det(\mathbf{A}(\mathbf{p}) - \lambda \mathbf{I}) \Big/ \frac{\partial}{\partial \lambda} \det(\mathbf{A}(\mathbf{p}) - \lambda \mathbf{I}), \quad i = 1, \ldots, n. \tag{2.41}$$

The eigenvector $\mathbf{u}(\mathbf{p})$ corresponding to $\lambda(\mathbf{p})$ is determined up to a nonzero scaling factor. This eigenvector determines a one-dimensional null-subspace of the matrix operator $\mathbf{A}(\mathbf{p}) - \lambda(\mathbf{p})\mathbf{I}$ smoothly dependent on $\mathbf{p}$. Hence, the eigenvector $\mathbf{u}(\mathbf{p})$ can be chosen as a smooth function of the parameter vector.

Let us consider a point $\mathbf{p}_0$ in the parameter space and assume that λ_0 is a simple eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$. Taking the derivative with respect to p_i of both sides of eigenvalue problem (2.1), we find

$$\frac{\partial \mathbf{A}}{\partial p_i}\mathbf{u}_0 + \mathbf{A}_0\frac{\partial \mathbf{u}}{\partial p_i} = \frac{\partial \lambda}{\partial p_i}\mathbf{u}_0 + \lambda_0\frac{\partial \mathbf{u}}{\partial p_i}, \tag{2.42}$$

where $\mathbf{u}_0$ is the eigenvector corresponding to λ_0 and the derivatives are taken at $\mathbf{p}_0$. Equation (2.42) can be transformed to the form

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\frac{\partial \mathbf{u}}{\partial p_i} = \left(\frac{\partial \lambda}{\partial p_i}\mathbf{I} - \frac{\partial \mathbf{A}}{\partial p_i}\right)\mathbf{u}_0. \tag{2.43}$$

This is a linear algebraic system for the unknown derivatives $\partial\lambda/\partial p_i$ and $\partial\mathbf{u}/\partial p_i$, where the matrix operator $\mathbf{A}_0 - \lambda_0\mathbf{I}$ is singular with rank$\,(\mathbf{A}_0 - \lambda_0\mathbf{I}) = m - 1$. It is known that solution of (2.43) exists if and only if

$$\mathbf{v}_0^T\left(\frac{\partial \lambda}{\partial p_i}\mathbf{I} - \frac{\partial \mathbf{A}}{\partial p_i}\right)\mathbf{u}_0 = 0, \tag{2.44}$$

where $\mathbf{v}_0$ is the left eigenvector corresponding to λ_0. Expression (2.44) can be obtained by means of pre-multiplying (2.43) by $\mathbf{v}_0^T$ and using equation (2.29). Therefore, we find another expression for the first order derivatives of the simple eigenvalue as follows

$$\frac{\partial \lambda}{\partial p_i} = \mathbf{v}_0^T\frac{\partial \mathbf{A}}{\partial p_i}\mathbf{u}_0 \Big/ (\mathbf{v}_0^T\mathbf{u}_0), \quad i = 1,\ldots,n. \tag{2.45}$$

If the left and right eigenvectors satisfy normalization condition (2.30), then the denominator in formula (2.45) equals one.

Under condition (2.45), equation (2.43) has a solution $\partial\mathbf{u}/\partial p_i$, which is determined up to an additive term $c\mathbf{u}_0$, where c is an arbitrary scalar. It is convenient to impose the normalization condition

$$\mathbf{v}_0^T\mathbf{u}(\mathbf{p}) = \text{const} \tag{2.46}$$

for the perturbed eigenvector $\mathbf{u}(\mathbf{p})$, which yields

$$\mathbf{v}_0^T\frac{\partial \mathbf{u}}{\partial p_i} = 0, \quad i = 1,\ldots,n. \tag{2.47}$$

Multiplying equation (2.47) by complex conjugate left eigenvector $\overline{\mathbf{v}}_0$ from the left and adding the result to equation (2.43), we obtain

$$\mathbf{G}_0 \frac{\partial \mathbf{u}}{\partial p_i} = \left(\frac{\partial \lambda}{\partial p_i} \mathbf{I} - \frac{\partial \mathbf{A}}{\partial p_i} \right) \mathbf{u}_0, \tag{2.48}$$

where

$$\mathbf{G}_0 = \mathbf{A}_0 - \lambda_0 \mathbf{I} + \overline{\mathbf{v}}_0 \mathbf{v}_0^T. \tag{2.49}$$

In expression (2.49) the product $\overline{\mathbf{v}}_0 \mathbf{v}_0^T$ represents the $m \times m$ matrix, which changes the singular operator $\mathbf{A}_0 - \lambda_0 \mathbf{I}$ on its null-space. As a result, the matrix $\mathbf{G}_0$ becomes nonsingular, and we can find a solution of equation (2.48) using the inverse matrix $\mathbf{G}_0^{-1}$ [Yakubovich and Starzhinskii (1987)]. Hence, we find the derivative of the eigenvector $\mathbf{u}(\mathbf{p})$ at $\mathbf{p}_0$ as

$$\frac{\partial \mathbf{u}}{\partial p_i} = \mathbf{G}_0^{-1} \left(\frac{\partial \lambda}{\partial p_i} \mathbf{I} - \frac{\partial \mathbf{A}}{\partial p_i} \right) \mathbf{u}_0. \tag{2.50}$$

Taking the partial derivative $\partial^2 / \partial p_i \partial p_j$ of both sides of eigenvalue problem (2.1), we find

$$\begin{aligned} (\mathbf{A}_0 - \lambda_0 \mathbf{I}) \frac{\partial^2 \mathbf{u}}{\partial p_i \partial p_j} = {} & \frac{\partial^2 \lambda}{\partial p_i \partial p_j} \mathbf{u}_0 + \frac{\partial \lambda}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} + \frac{\partial \lambda}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \\ & - \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j} \mathbf{u}_0 - \frac{\partial \mathbf{A}}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} - \frac{\partial \mathbf{A}}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i}. \end{aligned} \tag{2.51}$$

Again, the matrix $\mathbf{A}_0 - \lambda_0 \mathbf{I}$ is singular, and equation (2.51) has a solution if and only if the vector $\mathbf{v}_0^T$ multiplied by the right-hand side of (2.51) is zero. Hence, we find the second order derivative of the simple eigenvalue

$$\begin{aligned} \frac{\partial^2 \lambda}{\partial p_i \partial p_j} = \mathbf{v}_0^T & \left(\frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j} \mathbf{u}_0 + \frac{\partial \mathbf{A}}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} + \frac{\partial \mathbf{A}}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \right. \\ & \left. - \frac{\partial \lambda}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} - \frac{\partial \lambda}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \right) \Big/ (\mathbf{v}_0^T \mathbf{u}_0). \end{aligned} \tag{2.52}$$

Normalization condition (2.46) yields

$$\mathbf{v}_0^T \frac{\partial^2 \mathbf{u}}{\partial p_i \partial p_j} = 0. \tag{2.53}$$

Multiplying this relation by $\overline{\mathbf{v}}_0$ from the left and adding the result to equation (2.51), we find the second order derivative of the eigenvector satisfying normalization condition (2.46) as follows

$$\frac{\partial^2 \mathbf{u}}{\partial p_i \partial p_j} = \mathbf{G}_0^{-1}\left(\frac{\partial \lambda^2}{\partial p_i \partial p_j}\mathbf{u}_0 + \frac{\partial \lambda}{\partial p_i}\frac{\partial \mathbf{u}}{\partial p_j} + \frac{\partial \lambda}{\partial p_j}\frac{\partial \mathbf{u}}{\partial p_i}\right.$$
$$\left. - \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j}\mathbf{u}_0 - \frac{\partial \mathbf{A}}{\partial p_i}\frac{\partial \mathbf{u}}{\partial p_j} - \frac{\partial \mathbf{A}}{\partial p_j}\frac{\partial \mathbf{u}}{\partial p_i}\right). \tag{2.54}$$

This procedure can be continued to get higher order derivatives of the simple eigenvalue and the corresponding eigenvector. In order to give a general expression for the derivatives, we introduce the notation

$$\mathbf{A}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{A}}{\partial p_1^{h_1}\cdots \partial p_n^{h_n}}, \quad \mathbf{u}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{u}}{\partial p_1^{h_1}\cdots \partial p_n^{h_n}}, \quad \lambda^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\lambda}{\partial p_1^{h_1}\cdots \partial p_n^{h_n}}, \tag{2.55}$$

where $\mathbf{h} = (h_1, \ldots, h_n)$ is a vector with integer nonnegative components $h_i \geq 0$ and $|\mathbf{h}| = h_1 + \cdots + h_n$. Differentiating equation (2.1), we find

$$\sum_{\mathbf{h}_1+\mathbf{h}_2=\mathbf{h}} \frac{\mathbf{h}!}{\mathbf{h}_1!\mathbf{h}_2!}\left(\mathbf{A}^{(\mathbf{h}_1)}\mathbf{u}^{(\mathbf{h}_2)} - \mathbf{u}^{(\mathbf{h}_2)}\lambda^{(\mathbf{h}_1)}\right) = 0, \tag{2.56}$$

where $\mathbf{h}! = h_1!\cdots h_n!$ and the sum is taken over all the sets of the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ such that $\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h}$. The normalization condition (2.46) for the derivative $\mathbf{u}^{(\mathbf{h})}$ yields

$$\mathbf{v}_0^T\mathbf{u}^{(\mathbf{h})} = 0. \tag{2.57}$$

Formulae for the derivatives $\lambda^{(\mathbf{h})}$ and $\mathbf{u}^{(\mathbf{h})}$, found by solving equations (2.56) and (2.57), are given in the following theorem.

Theorem 2.2 *A simple eigenvalue λ of the matrix $\mathbf{A}$, smoothly dependent on the parameter vector $\mathbf{p}$, is a smooth function of $\mathbf{p}$ and its derivatives with respect to parameters at $\mathbf{p} = \mathbf{p}_0$ are given by the expression*

$$\lambda^{(\mathbf{h})} = \left(\mathbf{v}_0^T\mathbf{A}^{(\mathbf{h})}\mathbf{u}\right.$$
$$\left. + \mathbf{v}_0^T \sum_{\substack{\mathbf{h}_1+\mathbf{h}_2=\mathbf{h} \\ |\mathbf{h}_1|>0,\, |\mathbf{h}_2|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1!\mathbf{h}_2!}\left(\mathbf{A}^{(\mathbf{h}_1)}\mathbf{u}^{(\mathbf{h}_2)} - \mathbf{u}^{(\mathbf{h}_2)}\lambda^{(\mathbf{h}_1)}\right)\right)\Big/(\mathbf{v}_0^T\mathbf{u}_0), \tag{2.58}$$

where the sum is taken over all the sets of nonzero vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ such that $\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h}$. The corresponding eigenvector $\mathbf{u}$ can be chosen as a *smooth function of* $\mathbf{p}$ *with the derivatives at* $\mathbf{p} = \mathbf{p}_0$ *given by the expression*

$$\mathbf{u}^{(\mathbf{h})} = \mathbf{G}_0^{-1} \sum_{\substack{\mathbf{h}_1+\mathbf{h}_2=\mathbf{h} \\ |\mathbf{h}_1|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1!\mathbf{h}_2!}\left(\mathbf{u}^{(\mathbf{h}_2)}\lambda^{(\mathbf{h}_1)} - \mathbf{A}^{(\mathbf{h}_1)}\mathbf{u}^{(\mathbf{h}_2)}\right), \tag{2.59}$$

where the nonsingular matrix $\mathbf{G}_0$ *is given by (2.49). For the first and second order derivatives expressions (2.58) and (2.59) yield formulae (2.45), (2.50), (2.52), and (2.54).*

Example 2.5 Let us consider the two-parameter matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} p_1 - 5 & p_2 - 4 & 8 \\ p_1 & 7 & p_2 \\ p_2 - 12 & p_1 - 4 & 15 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{2.60}$$

At $\mathbf{p}_0 = 0$ the matrix $\mathbf{A}$ takes the form (2.15) and has the simple eigenvalue $\lambda_0 = 3$. The right and left eigenvectors corresponding to λ_0 are given by expressions (2.17) and (2.37), respectively. Then using Theorem 2.2, we find derivatives of the simple eigenvalue λ_0 and the corresponding eigenvector $\mathbf{u}_0$ at $\mathbf{p}_0$ as follows

$$\begin{gathered}
\frac{\partial \lambda}{\partial p_1} = 4, \ \frac{\partial \lambda}{\partial p_2} = -1, \ \frac{\partial^2 \lambda}{\partial p_1^2} = 7, \ \frac{\partial^2 \lambda}{\partial p_1 \partial p_2} = -\frac{7}{4}, \ \frac{\partial^2 \lambda}{\partial p_2^2} = -\frac{3}{2}, \\
\frac{\partial \mathbf{u}}{\partial p_1} = \begin{pmatrix} 3/4 \\ -1/4 \\ 1 \end{pmatrix}, \quad \frac{\partial \mathbf{u}}{\partial p_2} = \begin{pmatrix} -1/4 \\ -1/4 \\ -1/2 \end{pmatrix}, \\
\frac{\partial^2 \mathbf{u}}{\partial p_1^2} = \begin{pmatrix} 23/8 \\ -7/8 \\ 31/8 \end{pmatrix}, \ \frac{\partial^2 \mathbf{u}}{\partial p_1 \partial p_2} = \begin{pmatrix} -3/4 \\ -3/8 \\ -21/16 \end{pmatrix}, \ \frac{\partial^2 \mathbf{u}}{\partial p_2^2} = \begin{pmatrix} -1/8 \\ 3/8 \\ 0 \end{pmatrix}.
\end{gathered} \tag{2.61}$$

2.5 Bifurcation of double eigenvalue with single eigenvector

Let us consider the one-parameter matrix family

$$\mathbf{A}(p) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}. \tag{2.62}$$

Its eigenvalues are

$$\lambda = \pm\sqrt{p}, \tag{2.63}$$

which are real for positive p, complex conjugate for negative p, and double zero at $p = 0$. It is easy to see that at $p = 0$ the double eigenvalue $\lambda = 0$ is nonderogatory, i.e., it has a single eigenvector. Expression (2.63) shows that the eigenvalues are not differentiable functions of the parameter at $p = 0$, where the double eigenvalue appears; and derivatives of the eigenvalues tend to infinity as p approaches zero. Therefore, perturbation of a nonderogatory double eigenvalue is singular and needs special analysis.

Let us consider an arbitrary family of matrices $\mathbf{A}(\mathbf{p})$. Let $\mathbf{p}_0$ be a point in the parameter space, where the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has a double nonderogatory eigenvalue λ_0. Let $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ be, respectively, the right and left Jordan chains of length 2 corresponding to λ_0 and satisfying the equations

$$\begin{aligned} \mathbf{A}_0\mathbf{u}_0 &= \lambda_0\mathbf{u}_0, & \mathbf{v}_0^T\mathbf{A}_0 &= \lambda_0\mathbf{v}_0^T, \\ \mathbf{A}_0\mathbf{u}_1 &= \lambda_0\mathbf{u}_1 + \mathbf{u}_0, & \mathbf{v}_1^T\mathbf{A}_0 &= \lambda_0\mathbf{v}_1^T + \mathbf{v}_0^T, \end{aligned} \tag{2.64}$$

and the normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_1 = 1, \quad \mathbf{v}_1^T\mathbf{u}_1 = 0. \tag{2.65}$$

Recall that these Jordan chains have the properties

$$\mathbf{v}_0^T\mathbf{u}_0 = 0, \quad \mathbf{v}_1^T\mathbf{u}_0 = \mathbf{v}_0^T\mathbf{u}_1 = 1. \tag{2.66}$$

The right Jordan chain is not unique. The vectors $c_0\mathbf{u}_0$ and $c_0\mathbf{u}_1 + c_1\mathbf{u}_0$ with arbitrary coefficients $c_0 \neq 0$ and c_1 form a right Jordan chain, which can be easily verified by the substitution into equations (2.64). If the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ are given, then the left Jordan chain $\mathbf{v}_0$, $\mathbf{v}_1$ is uniquely determined by normalization conditions (2.65).

Our aim is to study behavior of two eigenvalues $\lambda(\mathbf{p})$ that merge to λ_0 at $\mathbf{p}_0$. For this purpose, we consider a perturbation of the parameter vector along a smooth curve

$$\mathbf{p} = \mathbf{p}(\varepsilon), \tag{2.67}$$

where $\varepsilon \geq 0$ is a small real perturbation parameter and

$$\mathbf{p}(0) = \mathbf{p}_0. \tag{2.68}$$

Curve (2.67) starts at $\mathbf{p}_0$ and has the initial direction $\mathbf{e} = (e_1, \ldots, e_n)$ in the parameter space determined by the expression

$$\mathbf{e} = \frac{d\mathbf{p}}{d\varepsilon} \tag{2.69}$$

with the derivative evaluated at $\varepsilon = 0$; see Fig. 2.1. The second order derivative of $\mathbf{p}(\varepsilon)$ taken at $\varepsilon = 0$ is denoted by

$$\mathbf{d} = (d_1, \ldots, d_n) = \frac{d^2\mathbf{p}}{d\varepsilon^2}. \tag{2.70}$$

An example of such a curve is the ray

$$\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e} \tag{2.71}$$

starting at $\mathbf{p}_0$ with the direction $\mathbf{e}$ and zero second order derivative vector $\mathbf{d} = 0$.

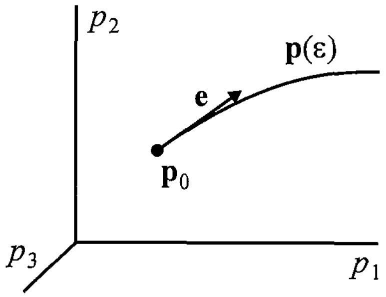


Fig. 2.1 Perturbation along a curve in the parameter space.

Along the curve $\mathbf{p}(\varepsilon)$ we have a one-parameter matrix family $\mathbf{A} = \mathbf{A}(\mathbf{p}(\varepsilon))$, which can be represented in the form of Taylor expansion

$$\mathbf{A}(\mathbf{p}(\varepsilon)) = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \cdots \tag{2.72}$$

with the matrices

$$\mathbf{A}_1 = \sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i} e_i, \quad \mathbf{A}_2 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i} d_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j} e_i e_j, \tag{2.73}$$

where the derivatives are evaluated at $\mathbf{p}_0$.

The perturbation theory of eigenvalues [Vishik and Lyusternik (1960)] tells us that the double nonderogatory eigenvalue λ_0 generally splits into a pair of simple eigenvalues λ under perturbation of the matrix $\mathbf{A}_0$. These eigenvalues λ and the corresponding eigenvectors $\mathbf{u}$ can be represented in the form of series in fractional powers of ε (called *Newton-Puiseux series*):

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \cdots, \\ \mathbf{u} &= \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{w}_1 + \varepsilon\mathbf{w}_2 + \varepsilon^{3/2}\mathbf{w}_3 + \cdots \end{aligned} \tag{2.74}$$

Substituting expansions (2.72) and (2.74) into eigenvalue problem (2.1) and comparing coefficients of equal powers of ε, we find the chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots$

$$\begin{aligned} \mathbf{A}_0\mathbf{u}_0 &= \lambda_0\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_1 &= \lambda_0\mathbf{w}_1 + \lambda_1\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_2 + \mathbf{A}_1\mathbf{u}_0 &= \lambda_0\mathbf{w}_2 + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_3 + \mathbf{A}_1\mathbf{w}_1 &= \lambda_0\mathbf{w}_3 + \lambda_1\mathbf{w}_2 + \lambda_2\mathbf{w}_1 + \lambda_3\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_4 + \mathbf{A}_1\mathbf{w}_2 + \mathbf{A}_2\mathbf{u}_0 &= \lambda_0\mathbf{w}_4 + \lambda_1\mathbf{w}_3 + \lambda_2\mathbf{w}_2 + \lambda_3\mathbf{w}_1 + \lambda_4\mathbf{u}_0, \\ &\vdots \end{aligned} \tag{2.75}$$

To determine the eigenvector $\mathbf{u}$ uniquely, it is convenient to choose the normalization condition

$$\mathbf{v}_1^T\mathbf{u} = 1, \tag{2.76}$$

where $\mathbf{v}_1$ is the left associated vector. Recall that $\mathbf{v}_1^T\mathbf{u}_0 = 1$ by equalities (2.66). Substituting expansion (2.74) for the eigenvector into equation (2.76), we find the chain of normalization conditions

$$\mathbf{v}_1^T\mathbf{w}_i = 0, \quad i = 1, 2, \ldots \tag{2.77}$$

Let us solve equations (2.75) and (2.77). The first equation in (2.75) is satisfied, since $\mathbf{u}_0$ is the eigenvector. Comparing the second equation in (2.75) with the equation for the associated vector $\mathbf{u}_1$ in (2.64) and using normalization condition for the vector $\mathbf{w}_1$ (2.77), we find

$$\mathbf{w}_1 = \lambda_1\mathbf{u}_1. \tag{2.78}$$

Using (2.78), the third equation in (2.75) can be written in the form

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_2 = \lambda_1^2\mathbf{u}_1 + \lambda_2\mathbf{u}_0 - \mathbf{A}_1\mathbf{u}_0. \tag{2.79}$$

The matrix operator $\mathbf{A}_0 - \lambda_0\mathbf{I}$ is singular and its rank equals $m - 1$, since λ_0 is a nonderogatory eigenvalue. System (2.79) has a solution $\mathbf{w}_2$ if and only if the right-hand side satisfies the orthogonality condition [Gantmacher (1998)]

$$\mathbf{v}_0^T\left(\lambda_1^2\mathbf{u}_1 + \lambda_2\mathbf{u}_0 - \mathbf{A}_1\mathbf{u}_0\right) = 0. \tag{2.80}$$

This equation can be obtained multiplying (2.79) by $\mathbf{v}_0^T$ from the left and using equalities (2.64). With properties (2.66) we find

$$\lambda_1^2 = \mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0. \tag{2.81}$$

Two values of $\lambda_1 = \pm\sqrt{\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0}$ determine leading terms in expansions for two different eigenvalues λ that bifurcate from the double eigenvalue λ_0.

Let us assume that

$$\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0 \neq 0, \tag{2.82}$$

i.e., $\lambda_1 \neq 0$. Multiplying equation (2.79) by the vector $\mathbf{v}_1^T$ from the left and using equations (2.64) and properties (2.65), (2.66), we find

$$\mathbf{v}_0^T\mathbf{w}_2 = \lambda_2 - \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_0. \tag{2.83}$$

Multiplying the fourth equation in (2.75) by $\mathbf{v}_0^T$ from the left and using expressions (2.64), (2.66), (2.78), (2.83) yields

$$2\lambda_1\lambda_2 = \lambda_1\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_1 + \lambda_1\mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_0. \tag{2.84}$$

Since $\lambda_1 \neq 0$, we have

$$\lambda_2 = \frac{\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_0}{2}. \tag{2.85}$$

Finally, adding normalization condition for $\mathbf{w}_2$ (2.77), multiplied by $\overline{\mathbf{v}}_0$ from the left, to equation (2.79), we find the vector

$$\mathbf{w}_2 = \lambda_2\mathbf{u}_1 + \mathbf{G}_1^{-1}(\lambda_1^2\mathbf{u}_1 - \mathbf{A}_1\mathbf{u}_0), \tag{2.86}$$

where $\mathbf{G}_1 = \mathbf{A}_0 - \lambda_0\mathbf{I} + \overline{\mathbf{v}}_0\mathbf{v}_1^T$. The $m\times m$ matrix $\overline{\mathbf{v}}_0\mathbf{v}_1^T$ redefines the singular matrix operator $\mathbf{A}_0 - \lambda_0\mathbf{I}$ on its null-space such that the matrix $\mathbf{G}_1$ becomes nonsingular [Yakubovich and Starzhinskii (1987)]. In derivation, we have used the equality $\mathbf{G}_1\mathbf{u}_1 = \mathbf{u}_0$ following from (2.64), (2.65).

Expansions (2.74) with the use of expressions (2.73), (2.78), (2.81), (2.82), (2.85), and (2.86) yield

Theorem 2.3 *Let λ_0 be a double nonderogatory eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ with the right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ satisfying normalization conditions (2.65). Consider a perturbation of* *the parameter vector along curve (2.67) starting at* $\mathbf{p}_0$ *with the direction* $\mathbf{e}$ *satisfying the condition*

$$\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i \neq 0. \tag{2.87}$$

Then, the double eigenvalue λ_0 *bifurcates into two simple eigenvalues given by the relation*

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + o(\varepsilon) \tag{2.88}$$

with the coefficients

$$\lambda_1 = \pm\sqrt{\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i}, \quad \lambda_2 = \frac{1}{2}\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i. \tag{2.89}$$

The corresponding eigenvectors are given by the expansions

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/2}\lambda_1\mathbf{u}_1 + \varepsilon \left(\lambda_2\mathbf{u}_1 + \lambda_1^2\mathbf{G}_1^{-1}\mathbf{u}_1 - \mathbf{G}_1^{-1}\sum_{i=1}^{n} \left(\frac{\partial \mathbf{A}}{\partial p_i}\mathbf{u}_0 \right) e_i \right) + o(\varepsilon), \tag{2.90}$$

where

$$\mathbf{G}_1 = \mathbf{A}_0 - \lambda_0\mathbf{I} + \overline{\mathbf{v}}_0\mathbf{v}_1^T \tag{2.91}$$

is a nonsingular matrix.

Now, let us consider the degenerate case

$$\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0 = 0. \tag{2.92}$$

Then $\lambda_1 = 0$, $\mathbf{w}_1 = 0$, and the fifth equation in (2.75) takes the form

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_4 = \lambda_2\mathbf{w}_2 + \lambda_4\mathbf{u}_0 - \mathbf{A}_1\mathbf{w}_2 - \mathbf{A}_2\mathbf{u}_0. \tag{2.93}$$

Multiplying (2.93) by $\mathbf{v}_0^T$ from the left and using equations (2.64), we find

$$\mathbf{v}_0^T(\lambda_2\mathbf{w}_2 + \lambda_4\mathbf{u}_0 - \mathbf{A}_1\mathbf{w}_2 - \mathbf{A}_2\mathbf{u}_0) = 0, \tag{2.94}$$

which, using properties (2.66), yields

$$\lambda_2\mathbf{v}_0^T\mathbf{w}_2 - \mathbf{v}_0^T\mathbf{A}_1\mathbf{w}_2 - \mathbf{v}_0^T\mathbf{A}_2\mathbf{u}_0 = 0. \tag{2.95}$$

Using expressions (2.83), (2.86), (2.91), and (2.92), we obtain the quadratic equation for the unknown λ_2

$$\lambda_2^2 - (\mathbf{v}_0^T \mathbf{A}_1 \mathbf{u}_1 + \mathbf{v}_1^T \mathbf{A}_1 \mathbf{u}_0)\lambda_2 + \mathbf{v}_0^T (\mathbf{A}_1 \mathbf{G}_1^{-1} \mathbf{A}_1 - \mathbf{A}_2)\mathbf{u}_0 = 0. \qquad (2.96)$$

Two roots of this equation describe bifurcation of the double eigenvalue λ_0.

Expansions (2.74) with the use of expressions (2.73), (2.78), (2.81), (2.86), (2.92), and (2.96) yield

Theorem 2.4 *Let λ_0 be a double nonderogatory eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ with the right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ satisfying normalization conditions (2.65). Let us consider a perturbation of the parameter vector along curve (2.67) starting at $\mathbf{p}_0$ with the direction $\mathbf{e}$ satisfying the degeneration condition*

$$\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i = 0. \qquad (2.97)$$

Then, bifurcation of the double eigenvalue λ_0 is given by the expansion

$$\lambda = \lambda_0 + \varepsilon \lambda_2 + o(\varepsilon), \qquad (2.98)$$

where two values of λ_2 are determined by the quadratic equation

$$\lambda_2^2 + \alpha_1 \lambda_2 + \alpha_2 = 0 \qquad (2.99)$$

with the coefficients

$$\begin{aligned}
\alpha_1 &= -\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i, \\
\alpha_2 &= \sum_{i,j=1}^{n} \left[\mathbf{v}_0^T \left(\frac{\partial \mathbf{A}}{\partial p_i} \mathbf{G}_1^{-1} \frac{\partial \mathbf{A}}{\partial p_j} - \frac{1}{2} \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j} \right) \mathbf{u}_0 \right] e_i e_j \qquad (2.100) \\
&\quad - \frac{1}{2} \sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) d_i.
\end{aligned}$$

The corresponding eigenvectors are given by the expansions

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \left(\lambda_2 \mathbf{u}_1 - \mathbf{G}_1^{-1} \sum_{i=1}^{n} \left(\frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i \right) + o(\varepsilon), \qquad (2.101)$$

where $\mathbf{G}_1$ is the nonsingular matrix given by (2.91).

Example 2.6 Let us consider the two-parameter matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 4 & p_1 - 5 & p_2 - 3 \\ p_1 + 1 & p_2 + 2 & p_1 - 1 \\ 3p_1 + 2p_2 & p_2 - 4 & 1 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{2.102}$$

At $\mathbf{p}_0 = 0$ the matrix $\mathbf{A}_0$ has the double nonderogatory eigenvalue $\lambda_0 = 3$. The corresponding right and left Jordan chains satisfying normalization conditions (2.65) are

$$\mathbf{u}_0 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}. \tag{2.103}$$

Let us consider a perturbation of the parameter vector along the ray

$$\mathbf{p} = \varepsilon(e_1, e_2). \tag{2.104}$$

By Theorem 2.3, we find that if

$$7e_1 - 2e_2 \neq 0, \tag{2.105}$$

then bifurcation of the double eigenvalue is described by the formula

$$\lambda = 3 \pm \sqrt{(7e_1 - 2e_2)\varepsilon} - (6e_1 + e_2)\varepsilon + o(\varepsilon), \tag{2.106}$$

and the corresponding eigenvectors take the form

$$\mathbf{u} = \begin{pmatrix} -1 \pm \sqrt{(7e_1 - 2e_2)\varepsilon} - (4e_1 + 5e_2/2)\varepsilon \\ 1 \pm \sqrt{(7e_1 - 2e_2)\varepsilon} + 4(e_1 - e_2)\varepsilon \\ -2 \mp \sqrt{(7e_1 - 2e_2)\varepsilon} - (12e_1 - 11e_2/2)\varepsilon \end{pmatrix} + o(\varepsilon). \tag{2.107}$$

The degeneration condition yields

$$7e_1 - 2e_2 = 0. \tag{2.108}$$

Using Theorem 2.4, we find perturbed eigenvalues and eigenvectors in the degenerate case:

$$\lambda = 3 + \lambda_2\varepsilon + o(\varepsilon),$$

$$\mathbf{u} = \begin{pmatrix} -1 + (\lambda_2 - 13e_1/4)\varepsilon \\ 1 + (\lambda_2 - e_1/2)\varepsilon \\ -2 - (\lambda_2 + 9e_1/4)\varepsilon \end{pmatrix} + o(\varepsilon), \tag{2.109}$$

where two values of λ_2 are determined from quadratic equation (2.99) in the form

$$\lambda_2 = \frac{-38 \pm \sqrt{1974}}{4} e_1. \tag{2.110}$$

In calculations we have used the relation $e_2 = 7e_1/2$ due to condition (2.108).

2.6 Strong interaction of two eigenvalues

Let us consider matrix family (2.62). Its eigenvalues plotted in the three-dimensional space $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p)$ and the complex plane are shown in Fig. 2.2, where the arrows indicate motion of the eigenvalues with an increase of p. The interaction is described by two identical parabolae lying in perpendicular planes. With an increase of p the eigenvalues approach along the imaginary axis on the complex plane, collide, and then diverge along the real axis in different directions. Such interaction is typical for a double eigenvalue λ_0 with a single eigenvector. We call it *strong interaction.*

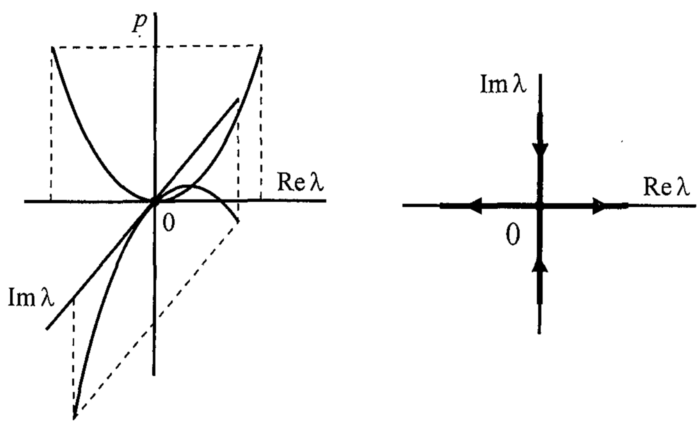


Fig. 2.2 Strong interaction of eigenvalues for matrix family (2.62).

Let us consider an arbitrary matrix family $\mathbf{A}(\mathbf{p})$. Let λ_0 be a double nonderogatory eigenvalue λ_0 of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ with corresponding right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ satisfying equations (2.64) and normalization conditions (2.65). Our aim is to study behavior of two eigenvalues λ, which are coincident and equal to λ_0 at $\mathbf{p}_0$, with a change of the vector of parameters $\mathbf{p}$ in the vicinity of the initial point $\mathbf{p}_0$. For this purpose we assume a variation $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$, where $\mathbf{e} = (e_1, \ldots, e_n)$ is a vector of variation, and $\varepsilon \geq 0$ is a small parameter. As a result, the eigenvalue λ_0 and the corresponding eigenvector $\mathbf{u}_0$ take increments given by Theorem 2.3 (page 37). For the sake of convenience, we introduce the notation

$$
\begin{aligned}
a_j + ib_j &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j}\mathbf{u}_0, \\
c_j + id_j &= \frac{1}{2}\left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j}\mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_j}\mathbf{u}_0\right), \\
\Delta p_j &= p_j - p_j^0 = \varepsilon e_j, \quad j = 1, 2, \ldots, n,
\end{aligned}
\tag{2.111}
$$

where a_j, b_j, c_j, d_j are real constants, and i is the imaginary unit. Then, using expressions (2.88) and (2.89), we obtain

$$
\lambda = \lambda_0 \pm \sqrt{\sum_{j=1}^{n}(a_j + ib_j)\Delta p_j} + \sum_{j=1}^{n}(c_j + id_j)\Delta p_j + o(\varepsilon). \tag{2.112}
$$

Equation (2.112) describes the increments of two eigenvalues λ, when the parameters $p_1, \ldots, p_n$ are changed under the assumption that ε is small. The inequality

$$
||\Delta \mathbf{p}|| = \varepsilon ||\mathbf{e}|| \ll 1 \tag{2.113}
$$

implies that all the increments $\Delta p_1, \ldots, \Delta p_n$ are small for their absolute values.

From expression (2.112) we see that when only one parameter, say the parameter p_1, is changed while the others remain unchanged ($\Delta p_j = 0$, $j = 2, \ldots, n$), then the speed of interaction $d\lambda/dp_1$ is infinite at $p_1 = p_1^0$. Indeed, following (2.112) we have

$$
\frac{d\lambda}{dp_1} = \pm\frac{1}{2}\sqrt{\frac{a_1 + ib_1}{p_1 - p_1^0}} + O(1). \tag{2.114}
$$

Since $a_1 + ib_1$ is a complex number, which is generally nonzero, $d\lambda/dp_1 \to \infty$ as $p_1 \to p_1^0$. We note that such behavior is typical for catastrophes, see [Arnold (1992)].

2.6.1 *Real eigenvalue* λ_0

Let us consider the case of a real double eigenvalue λ_0. In this case the eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$ and associated vectors $\mathbf{u}_1$, $\mathbf{v}_1$ can be chosen real, and in (2.112) we have $b_j = d_j = 0$, $j = 1, \ldots, n$. Let us fix the increments $\Delta p_2, \ldots, \Delta p_n$ and consider behavior of the interacting eigenvalues depending on the increment Δp_1. Then, formula (2.112) can be written in the form

$$\lambda = \lambda_0 + X + iY + o(\varepsilon), \tag{2.115}$$

where

$$X + iY = \pm\sqrt{a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)} + c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{\alpha c_1}{a_1} + \beta, \tag{2.116}$$

and α and β are small real numbers

$$\alpha = \sum_{j=2}^{n} a_j \Delta p_j, \quad \beta = \sum_{j=2}^{n} c_j \Delta p_j. \tag{2.117}$$

The real quantities X and Y describe, respectively, the real and imaginary parts of the leading terms in eigenvalue perturbation (2.112). If $a_1(\Delta p_1 + \alpha/a_1) > 0$, then equation (2.116) yields

$$X = \pm\sqrt{a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)} + c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{\alpha c_1}{a_1} + \beta, \quad Y = 0. \tag{2.118}$$

If $a_1(\Delta p_1 + \alpha/a_1) < 0$, then separating the real and imaginary parts in equation (2.116), we get

$$\begin{aligned} X &= c_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right) - \frac{\alpha c_1}{a_1} + \beta, \\ Y &= \pm\sqrt{-a_1\left(\Delta p_1 + \frac{\alpha}{a_1}\right)}. \end{aligned} \tag{2.119}$$

Eliminating $\Delta p_1 + \alpha/a_1$ from equations (2.119), we obtain the parabola

$$X + \frac{c_1}{a_1} Y^2 = \beta - \frac{\alpha c_1}{a_1} \tag{2.120}$$

in the plane (X, Y) symmetric with respect to the X-axis. Since α and β are small numbers dependent on Δp_j, $j = 2, \ldots, n$, parabola (2.120) gives trajectories of λ on the complex plane with a change of the parameter

p_1 while the other parameters remain fixed. Notice that the constants a_1 and c_1 involved in equation (2.120) are taken at the initial point $\mathbf{p}_0$ in the parameter space.

First, let us assume that $a_1 > 0$ and $\Delta p_2 = \cdots = \Delta p_n = 0$, which implies $\alpha = \beta = 0$. We deduce from equations (2.118)–(2.120) that with an increase of Δp_1 the eigenvalues come together along parabola (2.120), merge to λ_0 at $\Delta p_1 = 0$, and then diverge along the real axis in opposite directions. The general picture of strong interaction for $a_1 > 0$ is shown in Fig. 2.3a, where the arrows indicate motion of the eigenvalues as Δp_1 increases. The case $a_1 < 0$ implies the change of direction of motion for the eigenvalues.

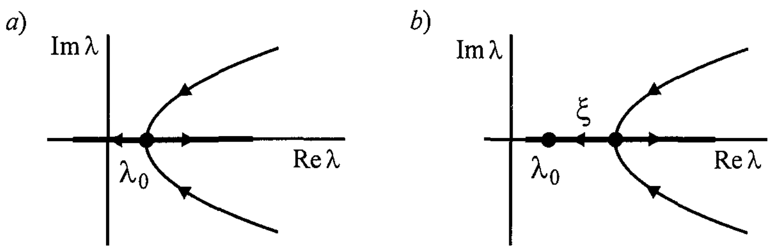


Fig. 2.3 Strong interaction of eigenvalues for real λ_0.

If $\Delta p_2, \ldots, \Delta p_n$ are nonzero and fixed, then the constants α and β are generally nonzero. This means the shift of parabola (2.120) along the real axis by $\xi = \beta - \alpha c_1/a_1$; see Fig. 2.3b. We see that the double eigenvalue does not disappear. It changes to $\lambda_0 + \xi + o(\varepsilon)$ and appears at $p_1 = p_1^0 - \alpha/a_1 + o(\varepsilon)$.

2.6.2 *Complex eigenvalue* λ_0

Let us consider a complex eigenvalue λ_0. In this case the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are complex and, hence, the constants b_j and d_j in expression (2.112) are generally nonzero. Keeping the term of order $\varepsilon^{1/2}$ in (2.112), we obtain

$$\lambda = \lambda_0 + X + iY + o(\varepsilon^{1/2}), \tag{2.121}$$

where

$$X + iY = \pm\sqrt{\sum_{j=1}^{n}(a_j + ib_j)\Delta p_j}. \tag{2.122}$$

Taking square of (2.122) and separating real and imaginary parts, we derive the equations

$$\begin{aligned} X^2 - Y^2 &= \sum_{j=1}^{n} a_j \Delta p_j, \\ 2XY &= \sum_{j=1}^{n} b_j \Delta p_j. \end{aligned} \tag{2.123}$$

Expressing the increment Δp_1 from one of equations (2.123) and substituting it into the other equation, we obtain

$$b_1 X^2 - 2a_1 XY - b_1 Y^2 = \gamma, \tag{2.124}$$

where γ is a small real constant

$$\gamma = \sum_{j=2}^{n} (a_j b_1 - a_1 b_j)\, \Delta p_j. \tag{2.125}$$

In equation (2.124) we assume that $a_1^2 + b_1^2 \neq 0$, which is the nondegeneracy condition for the complex eigenvalue λ_0. Equation (2.124) describes trajectories of the eigenvalues λ, when only Δp_1 is changed and the increments $\Delta p_2, \ldots, \Delta p_n$ are fixed.

If $\Delta p_j = 0$, $j = 2, \ldots, n$, or if they are nonzero but satisfy the equality $\gamma = 0$, then equation (2.124) yields two perpendicular lines

$$b_1 X - \left(a_1 \pm \sqrt{a_1^2 + b_1^2}\right) Y = 0, \tag{2.126}$$

which intersect at the origin $X = Y = 0$ corresponding to $\lambda = \lambda_0$. Two eigenvalues $\lambda = \lambda_0 + X + iY + o(\varepsilon^{1/2})$ approach along one of lines (2.126), merge to λ_0 at $\Delta p_1 = 0$, and then diverge along another line (2.126), perpendicular to the line of approach; see Fig. 2.4, where the arrows show motion of λ with increasing Δp_1. Strong interaction in the three-dimensional space $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p_1)$ is given by two identical parabolae lying in perpendicular planes. Equations for these parabolae are obtained by substituting expressions (2.126) into (2.123).

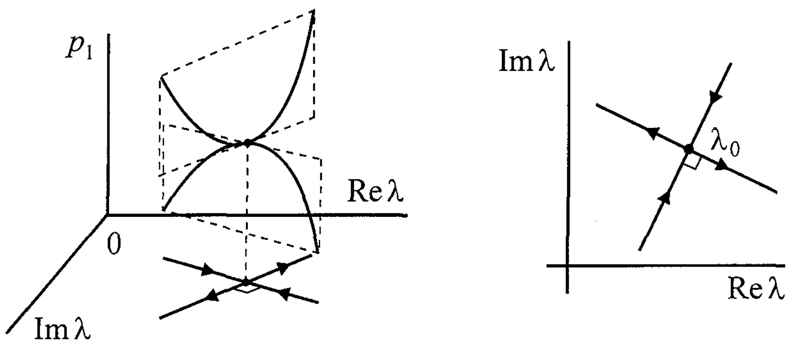


Fig. 2.4 Strong interaction of eigenvalues for complex λ_0 and $\gamma = 0$.

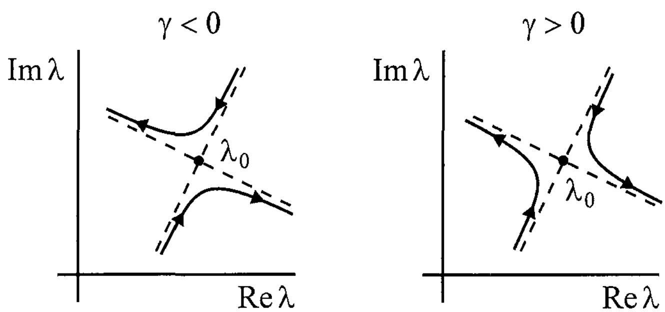


Fig. 2.5 Strong interaction of eigenvalues for complex λ_0 and $\gamma \neq 0$.

If $\gamma \neq 0$, then equation (2.124) defines two hyperbolae in the plane (X,Y) with asymptotes (2.126). As Δp_1 increases, two eigenvalues come closer, turn, and diverge; see Fig. 2.5. When γ changes the sign, the quadrants containing hyperbola branches are changed to the adjacent.

Example 2.7 As an example, we consider stability of vibrations of a rigid panel of infinite span in airflow. It is assumed that the panel is maintained on two elastic supports with the stiffness coefficients k_1 and k_2 per unit span. The panel has two degrees of freedom: a vertical displacement y and a rotation angle φ, Fig. 2.6. It is supposed that the aerodynamic lift force L is proportional to the angle of attack φ, the dynamic pressure of airflow, and the width b of the panel:

$$L = c_y \frac{\rho v^2}{2} b\varphi. \tag{2.127}$$

Here, c_y is the aerodynamic coefficient, ρ and v are the density and speed of the flow, respectively. It is assumed that m is the mass of the panel per unit surface. Damping forces are not involved in the model.

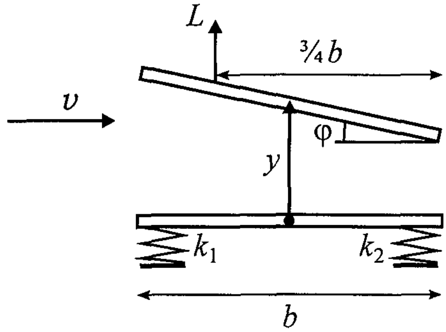


Fig. 2.6 A panel on elastic supports vibrating in airflow.

Small vibrations of the panel in airflow are described by the differential equations [Panovko and Gubanova (1965)]

$$\begin{aligned} \ddot{y} + a_{11}y + a_{12}\varphi &= 0, \\ \ddot{\varphi} + a_{21}y + a_{22}\varphi &= 0, \end{aligned} \tag{2.128}$$

where

$$\begin{aligned} a_{11} &= \frac{k_1 + k_2}{mb}, \quad a_{12} = \frac{k_1 - k_2}{2m} - c_y\frac{\rho v^2}{2m}, \\ a_{21} &= \frac{6(k_1 - k_2)}{mb^2}, \quad a_{22} = \frac{3(k_1 + k_2)}{mb} - 3c_y\frac{\rho v^2}{2mb}. \end{aligned} \tag{2.129}$$

We introduce the non-dimensional variables

$$k = \frac{k_1 - k_2}{2(k_1 + k_2)}, \; q = \frac{c_y \rho v^2}{2(k_1 + k_2)}, \; \widetilde{y} = \frac{y}{b}, \; \tau = t\sqrt{\frac{k_1 + k_2}{mb}}, \tag{2.130}$$

where k is a relative stiffness parameter changing in the interval $-1/2 \leq k \leq 1/2$, and $q \geq 0$ is a load parameter. Using these variables in equations (2.128) and separating the time $(\widetilde{y}, \varphi)^T = \mathbf{u}\exp(i\omega\tau)$, we arrive at the eigenvalue problem (2.1) with

$$\mathbf{A} = \begin{pmatrix} 1 & k - q \\ 12k & 3 - 3q \end{pmatrix}, \quad \lambda = \omega^2. \tag{2.131}$$

The stability problem of motion of the panel depending on two parameters $\mathbf{p} = (q, k)$ was studied in [Seyranian and Kirillov (2001)]. The characteristic equation yields

$$\lambda^2 + (3q - 4)\lambda + 12kq - 3q - 12k^2 + 3 = 0. \tag{2.132}$$

Motion of the panel is stable if all the eigenvalues λ are positive and simple. Stability of the panel can be lost by divergence or by flutter, the boundaries of which are given by $\lambda = 0$ or double positive eigenvalues with single eigenvectors, respectively. Setting the discriminant of (2.132) equal to zero, we find

$$q_f = \frac{2}{3}\left(1 + 4k - 2\sqrt{k(k+2)}\right). \tag{2.133}$$

This is the boundary between flutter and stability domains; see Fig. 2.7, where S and F denote the stability and flutter domains, respectively. It follows from (2.133) that the flutter domain belongs to the half-plane $k \geq 0$. The other branch of the solution, with plus before the square root in (2.133), corresponds to the boundary between flutter and divergence domains [Seyranian and Kirillov (2001)], shown in Fig. 2.7 by dashed line.

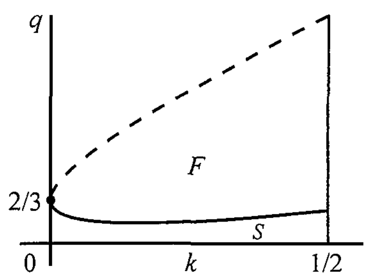


Fig. 2.7 The stability and flutter regions for the panel vibrating in airflow.

Let us take a point (k, q_f) on the flutter boundary (2.133). For this point we solve the characteristic equation (2.2) with (2.131), (2.133) and find the double eigenvalue $\lambda_0 = 2 - 3q_f/2$ with the corresponding eigenvectors and associated vectors satisfying normalization conditions (2.65):

$$\mathbf{u}_0 = \begin{pmatrix} \dfrac{2(q_f - k)}{3q_f - 2} \\ 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ \dfrac{2}{2 - 3q_f} \end{pmatrix},$$

$$\mathbf{v}_0 = \begin{pmatrix} 12k \\ \dfrac{2 - 3q_f}{2} \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} \dfrac{24k}{3q_f - 2} \\ 0 \end{pmatrix}.$$

Then, according to (2.111) we find the quantities

$$a_1 = -\frac{3}{2}(8k - 3q_f + 2), \; a_2 = 12(2k - q_f), \; c_1 = -\frac{3}{2}, \; c_2 = 0, \\ b_1 = b_2 = 0, \quad d_1 = d_2 = 0, \tag{2.134}$$

and write approximate formula (2.112) for the eigenvalues in the form

$$\lambda = \frac{4 - 3q_f}{2} \pm \sqrt{-\frac{3}{2}(8k - 3q_f + 2)\Delta q + 12(2k - q_f)\Delta k} - \frac{3}{2}\Delta q. \tag{2.135}$$

This formula coincides with that of obtained from characteristic equation (2.132) with first order Taylor expansion of the terms under and out of the square root. Equation of parabola (2.120) takes the form

$$Y^2 + (8k - 3q_f + 2)X = 12(q_f - 2k)\Delta k. \tag{2.136}$$

Due to expressions (2.133) and (2.134) the constant a_1 is negative for $0 \leq k \leq 1/2$. This means that for $\Delta k = 0$ with an increase of q in the vicinity of the flutter boundary two positive eigenvalues λ approach each other, merge to $\lambda_0 = 2 - 3q_f/2$, become complex conjugate (flutter) and diverge along parabola (2.136). If $\Delta k \neq 0$ is small and fixed, then there is a shift of the double eigenvalue by $\xi = -a_2 c_1 \Delta k / a_1 = 12(q_f - 2k)\Delta k/(8k - 3q_f + 2)$ and a shift of the parameter q at which the eigenvalue becomes double (flutter boundary)

$$q_f(k + \Delta k) \approx q_f(k) - a_2 \Delta k / a_1 = q_f(k) + \frac{8(2k - q_f)}{8k - 3q_f + 2}\Delta k. \tag{2.137}$$

Notice that these shifts are negative or positive depending on the sign of $2k - q_f$, which is negative for $0 \leq k < 2/\sqrt{3} - 1$ and positive for $2/\sqrt{3} - 1 <$

$k \leq 1/2$. At $k = 2/\sqrt{3} - 1$ the function $q_f(k)$ takes the minimum, see Fig. 2.7.

2.7 Bifurcation of nonderogatory eigenvalue of arbitrary multiplicity

Let $\mathbf{p}_0$ be a point in the parameter space, where the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has a nonderogatory eigenvalue of algebraic multiplicity k. The geometric multiplicity of this eigenvalue equals $k_g = 1$ (there is a single eigenvector). The corresponding right and left Jordan chains $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ and $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ satisfy equations (2.4), (2.32) and normalization conditions (2.35). Under perturbation of the parameter vector along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, the eigenvalue λ_0 and eigenvector $\mathbf{u}_0$ take increments that can be expressed in the form of Newton-Puiseux series

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/k}\lambda_1 + \varepsilon^{2/k}\lambda_2 + \varepsilon^{3/k}\lambda_3 + \cdots, \\ \mathbf{u} &= \mathbf{u}_0 + \varepsilon^{1/k}\mathbf{w}_1 + \varepsilon^{2/k}\mathbf{w}_2 + \varepsilon^{3/k}\mathbf{w}_3 + \cdots \end{aligned} \tag{2.138}$$

These expansions are valid if the leading term λ_1 is nonzero (nondegenerate case) [Vishik and Lyusternik (1960)]. To choose the eigenvector $\mathbf{u}$ uniquely, it is convenient to impose the normalization condition as

$$\mathbf{v}_{k-1}^T\mathbf{u} = 1, \tag{2.139}$$

which is satisfied at $\varepsilon = 0$ due to conditions (2.33) and (2.35).

Substituting expansions (2.72) and (2.138) into eigenvalue problem (2.1) and collecting coefficients of equal powers of ε, we get the chain of equations

$$\begin{aligned} \mathbf{A}_0\mathbf{u}_0 &= \lambda_0\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_1 &= \lambda_0\mathbf{w}_1 + \lambda_1\mathbf{u}_0, \\ &\vdots \\ \mathbf{A}_0\mathbf{w}_{k-1} &= \lambda_0\mathbf{w}_{k-1} + \cdots + \lambda_{k-2}\mathbf{w}_1 + \lambda_{k-1}\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{w}_k + \mathbf{A}_1\mathbf{u}_0 &= \lambda_0\mathbf{w}_k + \cdots + \lambda_{k-1}\mathbf{w}_1 + \lambda_k\mathbf{u}_0, \\ &\vdots \end{aligned} \tag{2.140}$$

Normalization condition (2.139) with expansion (2.138) for the eigenvector yield

$$\mathbf{v}_{k-1}^T\mathbf{w}_i = 0, \quad i = 1, 2, \ldots \tag{2.141}$$

First k equations in (2.140) can be solved using equations of the Jordan chain (2.4) and relations (2.33), (2.35), (2.141). As a result, we find

$$\begin{aligned}\mathbf{w}_1 &= \lambda_1 \mathbf{u}_1,\\ \mathbf{w}_2 &= \lambda_1^2 \mathbf{u}_2 + \lambda_2 \mathbf{u}_1,\\ &\vdots\\ \mathbf{w}_{k-1} &= \lambda_1^{k-1} \mathbf{u}_{k-1} + \cdots,\end{aligned} \tag{2.142}$$

where dots in the last expression denote a linear combination of the vectors $\mathbf{u}_{k-2}, \ldots, \mathbf{u}_1$. Substituting relations (2.142) into the $(k+1)$th equation of (2.140), multiplying it by $\mathbf{v}_0^T$ from the left, and using relations (2.33) and normalization conditions (2.35), we find

$$\lambda_1^k = \mathbf{v}_0^T \mathbf{A}_1 \mathbf{u}_0. \tag{2.143}$$

With the use of expression (2.73) for the matrix $\mathbf{A}_1$, we get

Theorem 2.5 *Let λ_0 be a nonderogatory eigenvalue of the matrix $\mathbf{A}_0$ with multiplicity k. Then bifurcation of the eigenvalue λ_0 and corresponding eigenvector $\mathbf{u}_0$ under perturbation of the parameter vector along curve (2.67)- (2.69) is given by*

$$\begin{aligned}\lambda &= \lambda_0 + \varepsilon^{1/k}\lambda_1 + o(\varepsilon^{1/k}),\\ \mathbf{u} &= \mathbf{u}_0 + \varepsilon^{1/k}\lambda_1 \mathbf{u}_1 + o(\varepsilon^{1/k}),\end{aligned} \tag{2.144}$$

where λ_1 takes k different complex values of the root

$$\lambda_1 = \sqrt[k]{\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i}. \tag{2.145}$$

Remark 2.1 *It is easy to see that expression (2.145) can be written in the form*

$$\lambda_1 = \sqrt[k]{\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i \Big/ (\mathbf{v}_0^T \mathbf{u}_{k-1})}, \tag{2.146}$$

which is independent on the normalization condition for the left eigenvector $\mathbf{v}_0$.

In one-parameter case formulae (2.144), (2.145) yield

$$\lambda = \lambda_0 + \sqrt[k]{\left(\mathbf{v}_0^T \frac{d\mathbf{A}}{dp}\mathbf{u}_0\right) \Delta p} + o(\Delta p^{1/k}), \tag{2.147}$$

where $\Delta p = p - p_0$. With a monotonous increase of the parameter p, k eigenvalues collide at the point λ_0 on the complex plane at equal angles $2\pi/k$, and then diverge along bisectors of these angles; see Fig. 2.8. The angles between adjacent directions of approach and divergence of the eigenvalues are equal to π/k. Derivatives of the eigenvalues with respect to p are of the order $O(\Delta p^{(1-k)/k})$ and tend to infinity as $p \to p_0$.

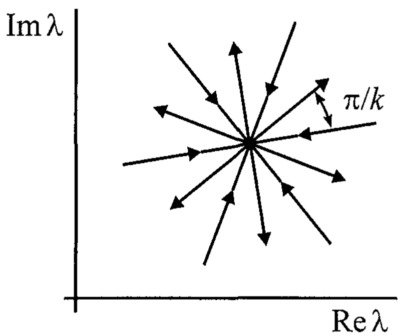


Fig. 2.8 Bifurcation of a nonderogatory eigenvalue with multiplicity k.

If λ_1 is nonzero, we can continue solving equations (2.140) and (2.141) to get higher order approximations of the eigenvalues and eigenvectors.

Example 2.8 Let λ_0 be a triple nonderogatory eigenvalue and assume that the direction $\mathbf{e}$ in the parameter space satisfies the nondegeneracy condition

$$\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i}\mathbf{u}_0\right) e_i \neq 0. \tag{2.148}$$

Then, the eigenvalue λ_0 splits into three simple eigenvalues under perturbation along curve (2.67). These eigenvalues are given by the formula

$$\lambda = \lambda_0 + \varepsilon^{1/3}\lambda_1 + \varepsilon^{2/3}\lambda_2 + \varepsilon\lambda_3 + o(\varepsilon), \tag{2.149}$$

where λ_1 takes three different complex values of the cubic root

$$\lambda_1 = \sqrt[3]{\sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i}. \tag{2.150}$$

If we continue solving equations (2.140), (2.141), we find coefficients λ_2 and λ_3 in the form [Mailybaev and Seyranian (1999b)]

$$\begin{aligned} \lambda_2 &= \frac{1}{3\lambda_1} \sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i, \\ \lambda_3 &= \frac{1}{3} \sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_2 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 + \mathbf{v}_2^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i. \end{aligned} \tag{2.151}$$

If $\lambda_1 = 0$ (degenerate case), then expansions (2.138) are, in general, invalid. In this case the eigenvalues are given by Newton-Puiseux series in different fractional powers of ε. Particular type of the series can be determined by means of the Newton diagram applied to the characteristic equation of the matrix $\mathbf{A}(\mathbf{p})$; see Section 4.5 for more details.

Example 2.9 Let λ_0 be a triple nonderogatory eigenvalue and assume that the direction $\mathbf{e}$ in the parameter space satisfies the conditions

$$\begin{aligned} \mathbf{v}_0^T \mathbf{A}_1 \mathbf{u}_0 &= \sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i = 0, \\ \mathbf{v}_0^T \mathbf{A}_1 \mathbf{u}_1 + \mathbf{v}_1^T \mathbf{A}_1 \mathbf{u}_0 &= \sum_{i=1}^{n} \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_0 \right) e_i \neq 0. \end{aligned} \tag{2.152}$$

Then, expansion (2.149) is invalid, since $\lambda_1 = 0$ is standing in the denominator of the coefficient λ_2 (2.151). In this case the triple eigenvalue λ_0 splits into three simple eigenvalues such that two eigenvalues are given by the expansions in powers of $\varepsilon^{1/2}$

$$\lambda = \lambda_0 + \varepsilon^{1/2}\mu_1 + \varepsilon\mu_2 + \varepsilon^{3/2}\mu_3 + \cdots, \tag{2.153}$$

and the third eigenvalue is represented by expansion in powers of ε

$$\lambda = \lambda_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \cdots \tag{2.154}$$

The corresponding eigenvectors are given by analogous expansions. Substituting these series into the eigenvalue problem, we get a chain of equations for the unknown coefficients. Solving these equations yields [Mailybaev and Seyranian (2000a)]

$$
\begin{aligned}
\nu_1 &= \frac{\mathbf{v}_0^T(\mathbf{A}_1\mathbf{G}_2^{-1}\mathbf{A}_1 - \mathbf{A}_2)\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_0}, \\
\mu_1 &= \pm\sqrt{\mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_0}, \\
\mu_2 &= \frac{1}{2}(-\nu_1 + \mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_2 + \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_1 + \mathbf{v}_2^T\mathbf{A}_1\mathbf{u}_0),
\end{aligned}
\tag{2.155}
$$

where $\mathbf{G}_2 = \mathbf{A}_0 - \lambda_0\mathbf{I} + \overline{\mathbf{v}}_0\mathbf{v}_2^T$ is a nonsingular matrix.

2.8 Bifurcation of double semi-simple eigenvalue

Let us consider the two-parameter matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 0 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{2.156}$$

Eigenvalues of the matrix $\mathbf{A}(\mathbf{p})$ are

$$\lambda = p_1 \pm \sqrt{p_1^2 + p_2^2}. \tag{2.157}$$

At $\mathbf{p}_0 = 0$ the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has the semi-simple double zero eigenvalue. We see that eigenvalues (2.157) are not differentiable functions of the parameters at $\mathbf{p}_0$. Nevertheless, leaving a single parameter, for example, setting $p_2 = 0$, we get two differentiable functions for the eigenvalues

$$\lambda = 0, \quad \lambda = 2p_1. \tag{2.158}$$

Thus, directional derivatives of the semi-simple eigenvalue exist.

Now let us consider an arbitrary family of matrices $\mathbf{A}(\mathbf{p})$. Let $\mathbf{p}_0$ be a point in the parameter space, where the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has a semi-simple double eigenvalue λ_0. There are two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying the equations

$$\mathbf{A}_0\mathbf{u}_1 = \lambda_0\mathbf{u}_1, \quad \mathbf{A}_0\mathbf{u}_2 = \lambda_0\mathbf{u}_2. \tag{2.159}$$

Two left eigenvectors satisfy the equations

$$\mathbf{v}_1^T\mathbf{A}_0 = \lambda_0\mathbf{v}_1^T, \quad \mathbf{v}_2^T\mathbf{A}_0 = \lambda_0\mathbf{v}_2^T, \tag{2.160}$$

and can be uniquely determined for given $\mathbf{u}_1$, $\mathbf{u}_2$ by the normalization conditions

$$\mathbf{v}_1^T\mathbf{u}_1 = \mathbf{v}_2^T\mathbf{u}_2 = 1, \quad \mathbf{v}_1^T\mathbf{u}_2 = \mathbf{v}_2^T\mathbf{u}_1 = 0. \tag{2.161}$$

Assuming perturbation of the parameter vector along curve (2.67), we can express perturbations of the eigenvalue λ_0 and corresponding eigenvectors in the form of power series [Vishik and Lyusternik (1960)]

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \\ \mathbf{u} &= \mathbf{w}_0 + \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + \cdots \end{aligned} \tag{2.162}$$

Notice that any linear combination of the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ is also an eigenvector. This means that the zero order term $\mathbf{w}_0$ in the expansion for the eigenvector $\mathbf{u}$ (the limit value of the eigenvector $\mathbf{u}$ as $\varepsilon \to 0$) is unknown *a priori*. Substituting expansions (2.72) and (2.162) into eigenvalue problem (2.1), we find

$$\begin{aligned} \mathbf{A}_0\mathbf{w}_0 &= \lambda_0\mathbf{w}_0, \\ \mathbf{A}_0\mathbf{w}_1 + \mathbf{A}_1\mathbf{w}_0 &= \lambda_0\mathbf{w}_1 + \lambda_1\mathbf{w}_0, \\ &\vdots \end{aligned} \tag{2.163}$$

From the first equation in (2.163) we see that $\mathbf{w}_0$ is an eigenvector corresponding to λ_0 and, hence, it can be represented in the form

$$\mathbf{w}_0 = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2, \tag{2.164}$$

where γ_1 and γ_2 are scalar coefficients. Multiplying the second equation in (2.163) by $\mathbf{v}_1^T$ and $\mathbf{v}_2^T$ from the left, we find two equations

$$\begin{aligned} \mathbf{v}_1^T\mathbf{A}_1\mathbf{w}_0 &= \lambda_1\mathbf{v}_1^T\mathbf{w}_0, \\ \mathbf{v}_2^T\mathbf{A}_1\mathbf{w}_0 &= \lambda_1\mathbf{v}_2^T\mathbf{w}_0, \end{aligned} \tag{2.165}$$

where relations (2.160) were used. Substituting expression (2.164) into (2.165), we find

$$\begin{pmatrix} \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_2 \\ \mathbf{v}_2^T\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_2^T\mathbf{A}_1\mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{2.166}$$

A nontrivial solution γ_1, γ_2 of equation (2.166) exists if and only if λ_1 is an eigenvalue of the 2×2 matrix standing in the left-hand side.

The obtained results provide the following description of bifurcation of a double semi-simple eigenvalue.

Theorem 2.6 *Let λ_0 be a semi-simple double eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$. Then bifurcation of the eigenvalue λ_0 under perturbation of the parameter vector along curve (2.67) is given by*

$$\lambda = \lambda + \varepsilon\lambda_1 + o(\varepsilon), \tag{2.167}$$

where two values of λ_1 are the eigenvalues of the 2×2 matrix

$$\begin{pmatrix} \sum_{i=1}^{n} \left(\mathbf{v}_1^T \dfrac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 \right) e_i & \sum_{i=1}^{n} \left(\mathbf{v}_1^T \dfrac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_2 \right) e_i \\ \sum_{i=1}^{n} \left(\mathbf{v}_2^T \dfrac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_1 \right) e_i & \sum_{i=1}^{n} \left(\mathbf{v}_2^T \dfrac{\partial \mathbf{A}}{\partial p_i} \mathbf{u}_2 \right) e_i \end{pmatrix}. \tag{2.168}$$

The corresponding eigenvectors are found in the form

$$\mathbf{u} = \gamma_1 \mathbf{u}_1 + \gamma_2 \mathbf{u}_2 + o(1), \tag{2.169}$$

where $(\gamma_1, \gamma_2)^T$ are the eigenvectors of matrix (2.168) corresponding to λ_1.

If matrix (2.168) has two distinct eigenvalues, then the double λ_0 splits into two simple eigenvalues under the perturbation. The eigenvalue λ_0 may remain double, which implies that λ_1 is a double eigenvalue of matrix (2.168). In the latter case, if the double λ_1 is nonderogatory (has a single eigenvector $(\gamma_1, \gamma_2)^T$), then perturbed double eigenvalue (2.167) becomes nonderogatory with single eigenvector (2.169). If the double eigenvalue λ_1 is semi-simple (matrix (2.168) is equal to the 2×2 identity matrix multiplied by λ_1), then the perturbed eigenvalue (2.167) remains double and semi-simple.

Example 2.10 Let us consider the two-parameter matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} p_2 - 5 & p_1 - 4 & 8 \\ 2p_1 - p_2 & 7 & 3p_2 \\ p_1 - 12 & p_2 - 4 & 15 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{2.170}$$

At $\mathbf{p}_0 = 0$ the matrix $\mathbf{A}_0$ has the semi-simple double eigenvalue $\lambda_0 = 7$ with the eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \tag{2.171}$$

satisfying normalization conditions (2.161). Then, by Theorem 2.6, the eigenvalue λ_0 takes increment (2.167), where two values of λ_1 are the eigenvalues of the matrix

$$\begin{pmatrix} -4e_2 & 2e_1 + 5e_2 \\ 2e_2 - 2e_1 & 0 \end{pmatrix}. \tag{2.172}$$

If matrix (2.172) has distinct eigenvalues, then the double $\lambda_0 = 7$ splits into two simple eigenvalues

$$\lambda = 7 + \varepsilon\lambda_1 + o(\varepsilon), \tag{2.173}$$

where

$$\lambda_1 = -2e_2 \pm \sqrt{14e_2^2 - 6e_1e_2 - 4e_1^2}. \tag{2.174}$$

To keep the eigenvalue double under the perturbation, it is necessary that matrix (2.172) has a double eigenvalue λ_1. Equating the discriminant of the characteristic equation for matrix (2.172) zero, we find

$$14e_2^2 - 6e_1e_2 - 4e_1^2 = 0. \tag{2.175}$$

This equation has two solutions

$$14e_2 = (3 \pm \sqrt{65})e_1, \tag{2.176}$$

which determine directions in the parameter space, along which the perturbed eigenvalue remains double. It can be checked that matrix (2.172) has a double eigenvalue with a single eigenvector for directions (2.176). Therefore, though the semi-simple eigenvalue λ_0 remains double under the perturbation along directions (2.176), it becomes nonderogatory (only one eigenvector remains).

2.9 Weak interaction of eigenvalues

In this section we study multi-parameter behavior of two eigenvalues that merge and form a semi-simple double eigenvalue λ_0 at $\mathbf{p}_0$. The eigenvalue λ_0 has two right eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and two left eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ satisfying normalization conditions (2.161). Let us consider a perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$, where $\Delta\mathbf{p} = \varepsilon\mathbf{e}$ with a direction $\mathbf{e}$ in the parameter space and a small perturbation parameter ε. By Theorem 2.6, the eigenvalue λ_0 and corresponding eigenvector $\mathbf{u}_0$ take increments, which can be given in the form of expansions

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \\ \mathbf{u} &= \mathbf{w}_0 + \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + \cdots, \end{aligned} \tag{2.177}$$

where

$$\mathbf{w}_0 = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2, \tag{2.178}$$

and the coefficients γ_1, γ_2 are determined from the equation

$$\begin{pmatrix} \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_1^T\mathbf{A}_1\mathbf{u}_2 \\ \mathbf{v}_2^T\mathbf{A}_1\mathbf{u}_1 & \mathbf{v}_2^T\mathbf{A}_1\mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{2.179}$$

The coefficient λ_1 is an eigenvalue of the 2×2 matrix standing in the left-hand side. Two eigenvalues λ_1 of this matrix and the corresponding eigenvectors $(\gamma_1, \gamma_2)^T$ determine leading terms in expansions (2.177) for two eigenvalues λ and corresponding eigenvectors $\mathbf{u}$, which appear due to bifurcation of the double semi-simple eigenvalue λ_0.

Introducing the notation

$$X + iY = \varepsilon\lambda_1, \tag{2.180}$$

where X and Y are, respectively, the real and imaginary parts of the term $\varepsilon\lambda_1$, expansion for the eigenvalue (2.177) can be written in the form

$$\lambda = \lambda_0 + X + iY + o(\varepsilon). \tag{2.181}$$

According to relations (2.73) and (2.179), $X + iY$ is an eigenvalue of the 2×2 matrix

$$\sum_{j=1}^{n} \begin{pmatrix} f_j^{11} & f_j^{12} \\ f_j^{21} & f_j^{22} \end{pmatrix} \Delta p_j, \tag{2.182}$$

where

$$f_j^{kl} = \mathbf{v}_k^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_l \tag{2.183}$$

with the derivative evaluated at $\mathbf{p}_0$. Solving the characteristic equation for matrix (2.182), we find

$$X + iY = \sum_{j=1}^{n} g_j \Delta p_j \pm \sqrt{\sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k}, \tag{2.184}$$

where

$$g_j = \frac{f_j^{11} + f_j^{22}}{2}, \quad h_{jk} = \frac{(f_j^{11} - f_j^{22})(f_k^{11} - f_k^{22})}{4} + \frac{f_j^{12} f_k^{21} + f_j^{21} f_k^{12}}{2}. \tag{2.185}$$

Notice that $h_{jk} = h_{kj}$ for any j and k. Expression (2.184) determines approximation of eigenvalues (2.181) as the parameter vector $\mathbf{p}$ is changing under the assumption that $\|\Delta\mathbf{p}\|$ is small. The coefficients g_j and h_{jk} in this expression depend on the left and right eigenvectors, corresponding to the eigenvalue λ_0, and first order derivatives of the matrix $\mathbf{A}$ with respect to the parameters taken at $\mathbf{p}_0$.

2.9.1 *Real eigenvalue* λ_0

Let us consider a real semi-simple eigenvalue λ_0. In this case we can always choose real eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ and, hence, the coefficients f_j^{kl}, g_j, and h_{jk} in expressions (2.183), (2.185) can be chosen real. Expressing the square root from equality (2.184) and taking square of the obtained relation, we find two equations for the real and imaginary parts as follows

$$\begin{gathered}\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right)^2 - Y^2 = \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k, \\ 2\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right) Y = 0.\end{gathered} \tag{2.186}$$

The second equation requires that $X = \sum_{j=1}^{n} g_j \Delta p_j$ or $Y = 0$. Therefore, we get two independent systems

$$\left(X - \sum_{j=1}^{n} g_j \Delta p_j\right)^2 = \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k, \quad Y = 0 \tag{2.187}$$

and

$$X = \sum_{j=1}^{n} g_j \Delta p_j, \quad Y^2 = -\sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k. \tag{2.188}$$

Let us study behavior of eigenvalues depending on the parameter p_1, when other parameters $p_2, \ldots, p_n$ are fixed. First, let us put the increments $\Delta p_2 = \cdots = \Delta p_n = 0$. Then, equations (2.187) and (2.188) take the form

$$(X - g_1 \Delta p_1)^2 = h_{11} \Delta p_1^2, \quad Y = 0, \tag{2.189}$$

$$X = g_1 \Delta p_1, \quad Y^2 = -h_{11} \Delta p_1^2. \tag{2.190}$$

Assuming that $h_{11} \neq 0$ (the nondegenerate case), only one of systems (2.189) or (2.190) has nonzero solutions. If $h_{11} > 0$, then system (2.190) has only zero solution $X = Y = \Delta p_1 = 0$, and system (2.189) yields

$$X = (g_1 \pm \sqrt{h_{11}}) \Delta p_1, \quad Y = 0. \tag{2.191}$$

Expressions (2.191) describe two real eigenvalues (2.181), which cross each other at the point λ_0 on the complex plane as Δp_1 changes from negative to positive values; see Fig. 2.9 (r'), where the arrows show motion of the eigenvalues with increasing Δp_1. If $h_{11} < 0$, then system (2.189) has only zero solution, and system (2.190) yields

$$X = g_1 \Delta p_1, \quad Y = \pm\sqrt{-h_{11}} \Delta p_1. \tag{2.192}$$

These formulae describe two complex conjugate eigenvalues crossing at the point λ_0 on the real axis with a change of Δp_1; see Fig. 2.9 (r'').

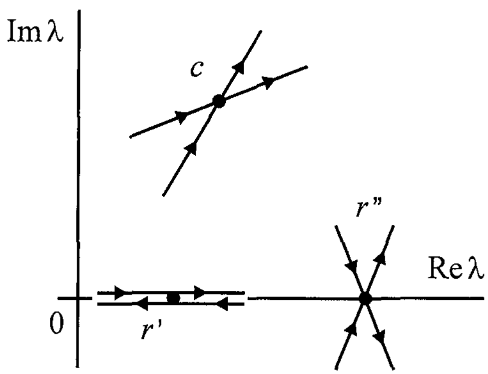


Fig. 2.9 Weak interaction of eigenvalues for $\Delta p_2 = \cdots = \Delta p_n = 0$.

From expressions (2.191) and (2.192) we see that the interaction occurs at a plane in the three-dimensional space $(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda, p_1)$, and the speed of interaction $d\lambda/dp_1$ remains finite, see Fig. 2.10. Such interaction of two eigenvalues, caused by the appearance of a double semi-simple eigenvalue, we call *weak*.

Now, let us consider the case when the increments $\Delta p_2, \ldots, \Delta p_n$ are

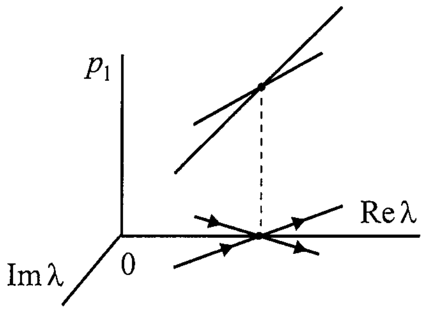


Fig. 2.10 Weak interaction of eigenvalues in the space $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p_1)$.

small and fixed. Then, using the notation

$$\sum_{j=1}^{n} g_j \Delta p_j = g_1 \Delta p_1 + \sigma, \quad \sum_{j,k=1}^{n} h_{jk} \Delta p_j \Delta p_k = h_{11}(\Delta p_1 - \delta)^2 + \psi, \tag{2.193}$$

where σ, δ, and ψ are small real constants dependent on $\Delta p_2, \ldots, \Delta p_n$

$$\sigma = \sum_{j=2}^{n} g_j \Delta p_j, \quad \delta = -\sum_{j=2}^{n} \frac{h_{1j}}{h_{11}} \Delta p_j, \quad \psi = \sum_{j,k=2}^{n} h_{jk} \Delta p_j \Delta p_k - h_{11}\delta^2, \tag{2.194}$$

equations (2.187) and (2.188) take the form

$$(X - \sigma - g_1 \Delta p_1)^2 - h_{11}(\Delta p_1 - \delta)^2 = \psi, \quad Y = 0 \tag{2.195}$$

and

$$Y^2 + h_{11}(\Delta p_1 - \delta)^2 = -\psi, \quad X = \sigma + g_1 \Delta p_1. \tag{2.196}$$

Solutions of systems (2.195) and (2.196) depend qualitatively on the signs of the constants h_{11} and ψ. Under the nondegeneracy conditions $h_{11} \neq 0$ and $\psi \neq 0$ there are four possibilities.

Case r'_+ $(h_{11} > 0,\ \psi > 0)$. System (2.195) determines two hyperbolae in the plane $(\Delta p_1, X)$; system (2.196) has no solutions; see Fig. 2.11. Two simple real eigenvalues approach, and then diverge as Δp_1 is changed; a double eigenvalue does not appear; see Fig. 2.12.

Case r'_- $(h_{11} > 0,\ \psi < 0)$. System (2.195) determines two hyperbolae in the plane $(\Delta p_1, X)$; system (2.196) defines an ellipse in the plane $(\Delta p_1, Y)$;

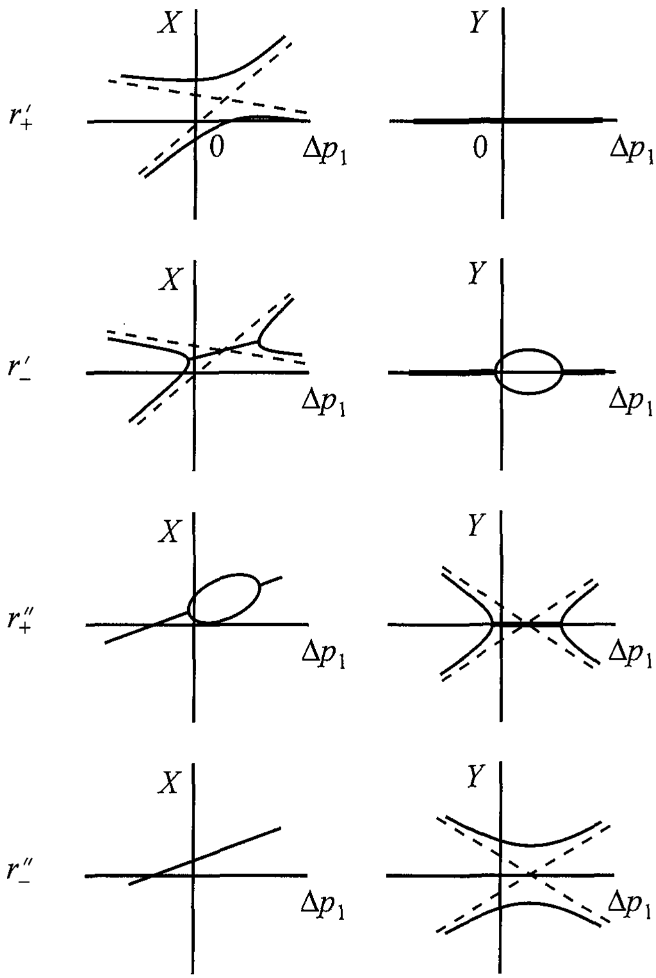


Fig. 2.11 Weak interaction of eigenvalues for small $\Delta p_2, \ldots, \Delta p_n$.

see Fig. 2.11. The hyperbolae and ellipse have two common points

$$\Delta p_1^{\pm} = \delta \pm \sqrt{-\frac{\psi}{h_{11}}}, \quad X^{\pm} = \sigma + g_1 \Delta p_1^{\pm}, \quad Y^{\pm} = 0. \tag{2.197}$$

With increasing Δp_1 two simple real eigenvalues approach, interact strongly at $\Delta p_1^- = \delta - \sqrt{-\psi/h_{11}}$, become complex conjugate, interact strongly again at $\Delta p_1^+ = \delta + \sqrt{-\psi/h_{11}}$, and then diverge along the real axis; see Fig. 2.12.

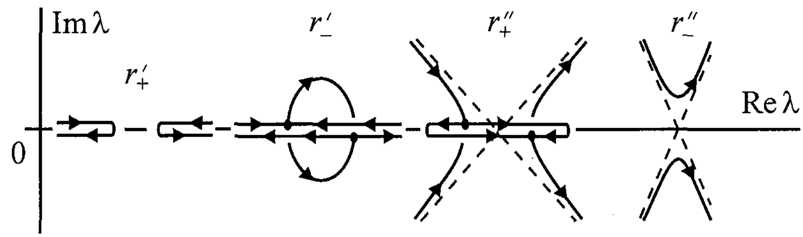


Fig. 2.12 Weak interaction of eigenvalues on the complex plane for small $\Delta p_2, \ldots, \Delta p_n$.

Eliminating Δp_1 from equation (2.196), we obtain the ellipse on the complex plane

$$Y^2 + h_{11}\left(\frac{X - \sigma - g_1\delta}{g_1}\right)^2 = -\psi \tag{2.198}$$

shown in Fig. 2.12.

If we plot the eigenvalues in the $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p_1)$ space, we observe a small elliptic bubble appearing from the point $(\lambda_0, 0, p_1^0)$; see Fig. 2.13. This bubble is placed in the plane perpendicular to the plane of the original interaction.

At points (2.197) the double real eigenvalues

$$\lambda^\pm = \lambda_0 + X^\pm + o(\varepsilon) \tag{2.199}$$

appear. It is easy to show that each of these eigenvalues has a single eigenvector. Indeed, if eigenvalues (2.199) were semi-simple, then $X^\pm$ have to be semi-simple eigenvalues of 2×2 matrix (2.182) at $\Delta p_1 = \Delta p_1^\pm$. Hence, the matrix (2.182) becomes

$$\begin{pmatrix} X^\pm + f_1^{11}(\Delta p_1 - \Delta p_1^\pm) & f_1^{12}(\Delta p_1 - \Delta p_1^\pm) \\ f_1^{21}(\Delta p_1 - \Delta p_1^\pm) & X^\pm + f_1^{22}(\Delta p_1 - \Delta p_1^\pm) \end{pmatrix}. \tag{2.200}$$

Using this matrix with expressions (2.185) and (2.193), we find

$$\begin{aligned} h_{11}(\Delta p_1 - \delta)^2 + \psi &= \sum_{j,k=1}^{n} h_{ij}\Delta p_j \Delta p_k \\ = \left(\frac{(f_1^{11} - f_1^{22})^2}{4} + f_1^{12}f_1^{21}\right)&(\Delta p_1 - \Delta p_1^\pm)^2 = h_{11}(\Delta p_1 - \Delta p_1^\pm)^2 \end{aligned} \tag{2.201}$$

and, hence, $\psi = 0$. But this is the contradiction to the assumption that $\psi < 0$. Therefore, two interactions at points (2.197) are strong and follow

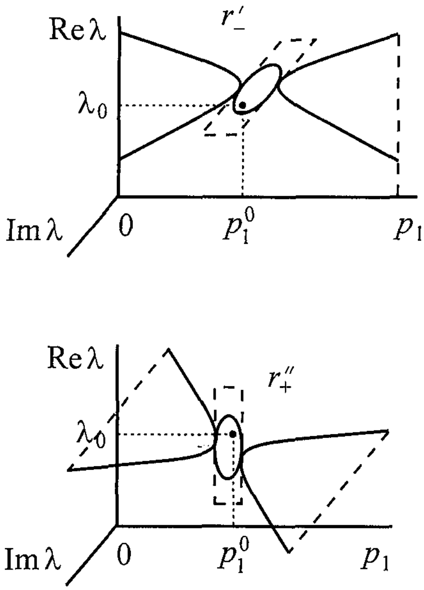


Fig. 2.13 Weak interaction of eigenvalues for small $\Delta p_2, \ldots, \Delta p_n$ in cases r'_- and r''_+.

the scenarios described in Section 2.6.

Case r''_+ ($h_{11} < 0$, $\psi > 0$). System (2.195) determines an ellipse in the plane $(\Delta p_1, X)$; system (2.196) defines two hyperbolae in the plane $(\Delta p_1, Y)$; see Fig. 2.11. The hyperbolae and ellipse have two common points (2.197), where double real eigenvalues (2.199) appear and cause strong interactions of eigenvalues. Therefore, with a monotonous change of Δp_1 two complex conjugate eigenvalues approach, interact strongly at $\Delta p_1^- = \delta - \sqrt{-\psi/h_{11}}$, become real, interact again at $\Delta p_1^+ = \delta + \sqrt{-\psi/h_{11}}$, then become complex conjugate and diverge; see Fig. 2.12. For this case equation (2.198) gives hyperbolae on the complex plane. The behavior of eigenvalues in the three-dimensional space $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p_1)$ is shown in Fig. 2.13, where we can see a small elliptic bubble appearing in the plane $\operatorname{Im}\lambda = 0$ perpendicular to the plane of the original interaction.

Case r''_- ($h_{11} < 0$, $\psi < 0$). System (2.195) has no solutions; system (2.196) determines two hyperbolae in the plane $(\Delta p_1, Y)$ symmetric with respect to the Δp_1-axis; see Fig. 2.11. Two complex conjugate eigenvalues approach, and then diverge with increasing Δp_1; a double eigenvalue does not appear; see Fig. 2.12. Notice that hyperbolae (2.198) change the vertical angles, where they appear, compared to the case r''_+.

We see that variations of the parameters $\Delta p_2, \ldots, \Delta p_n$ change a picture of weak interaction in two ways: either the double semi-simple real eigenvalue λ_0 disappears and simple eigenvalues move along hyperbolae as Δp_1 changes, or the double semi-simple eigenvalue λ_0 splits in two double eigenvalues with single eigenvectors, which leads to a couple of successive strong interactions with appearance of a small bubble in the space $(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda, p_1)$.

2.9.2 *Complex eigenvalue* λ_0

Finally, we consider the case when a double semi-simple eigenvalue λ_0 is complex. In this case the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ and the coefficients f_j^{kl}, g_j, h_{jk} are complex. If $\Delta p_2 = \cdots = \Delta p_n = 0$, then expression (2.184) yields

$$X + iY = (g_1 \pm \sqrt{h_{11}})\Delta p_1, \tag{2.202}$$

where g_1 and h_{11} are complex numbers. With a change of Δp_1 two eigenvalues (2.181) cross each other at the point λ_0 on the complex plane; see Fig. 2.9 (*c*).

Assuming that the increments $\Delta p_2, \ldots, \Delta p_n$ are small and fixed, we find from (2.184):

$$X + iY = \sigma + g_1\Delta p_1 \pm \sqrt{h_{11}}\sqrt{(\Delta p_1 - \delta)^2 + \psi/h_{11}}, \tag{2.203}$$

where σ, δ, and ψ are small complex numbers defined by expressions (2.194). If we assume that the second term under the square root in (2.203) is much smaller than the first term, we deduce the formula

$$\begin{aligned} X + iY &= \sigma + g_1\delta + g_1(\Delta p_1 - \delta) \pm \sqrt{h_{11}}\left((\Delta p_1 - \delta) + \frac{\psi}{2h_{11}(\Delta p_1 - \delta)}\right) \\ &= \sigma + g_1\delta + (g_1 \pm \sqrt{h_{11}})(\Delta p_1 - \delta) + o(\Delta p_1 - \delta) \end{aligned} \tag{2.204}$$

showing that the main directions of eigenvalues on the complex plane before and after the weak interaction remain the same as for unperturbed case (2.202).

The expression under the square root in (2.203)

$$z = (\Delta p_1 - \delta)^2 + \psi/h_{11} \tag{2.205}$$

defines a parabola on the complex plane with the implicit parameter Δp_1; see Fig. 2.14 (in the case $\mathrm{Im}\,\delta = 0$ the parabola degenerates to a ray).

Computing points z_1 and z_2 of the parabola belonging to the imaginary axis, which is perpendicular to the axis of the parabola, we find

$$\eta = z_1 z_2 = 4(\operatorname{Im}\delta)^4 - 4(\operatorname{Im}\delta)^2 \operatorname{Re}\frac{\psi}{h_{11}} - \left(\operatorname{Im}\frac{\psi}{h_{11}}\right)^2 \in \mathbb{R}. \tag{2.206}$$

We assume that $\eta \neq 0$, which is a nondegenerate case. This means that $z \neq 0$ for all Δp_1 and, hence, two values of $X + iY$ given by expression (2.203) are different. As a result, eigenvalues (2.181) are different and the double eigenvalue disappears.

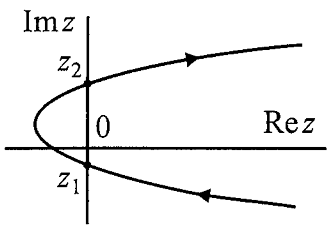


Fig. 2.14 Image of the function $z(\Delta p_1)$ with monotonous change of Δp_1.

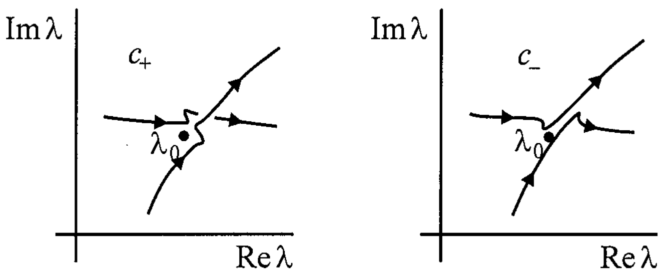


Fig. 2.15 Weak interaction of eigenvalues for small $\Delta p_2, \ldots, \Delta p_n$.

If $\eta > 0$, then two purely imaginary points z_1 and z_2 lie at different sides of the origin, i.e., the origin belongs to the interior of the parabola. In this case z makes a turn around the origin as Δp_1 changes. This means that eigenvalues (2.181) approach, and then diverge without a change of direction as shown in Fig. 2.15 (c_+). If $\eta < 0$, then the origin lies outside the parabola (this condition remains valid, when the parabola does not intersect the imaginary axis). As a result, eigenvalues (2.181) approach, and then diverge with a change of direction as shown in Fig. 2.15 (c_-).

We see that variations of the parameters $\Delta p_2, \ldots, \Delta p_n$ destroy a double semi-simple complex eigenvalue. A picture of weak interaction can change in two ways: either eigenvalues follow the same directions after passing the neighborhood of λ_0, or the eigenvalues interchange their directions. Behavior of the eigenvalues in the neighborhood of λ_0 can be rather complicated due to the square root of the complex expression in formula (2.203).

Example 2.11 Let us consider a linear conservative system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \tag{2.207}$$

where $\mathbf{q} \in \mathbb{R}^m$ is a vector of generalized coordinates; $\mathbf{M}$ and $\mathbf{P}$ are symmetric positive definite real matrices of size $m \times m$ smoothly dependent on a vector of two real parameters $\mathbf{p} = (p_1, p_2)$. Seeking a solution of this system in the form $\mathbf{q} = \mathbf{u}\exp(i\omega t)$, we obtain the eigenvalue problem

$$\mathbf{P}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u}, \tag{2.208}$$

where $\omega > 0$ is a frequency and $\mathbf{u}$ is a mode of vibration. Denoting

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{P}, \quad \lambda = \omega^2, \tag{2.209}$$

we can write equation (2.208) in standard form (2.1).

Let us consider a point $\mathbf{p}_0$ in the parameter space, where the matrix $\mathbf{A}_0 = \mathbf{M}_0^{-1}\mathbf{P}_0$ has a double eigenvalue $\lambda_0 = \omega_0^2$. Since the matrices $\mathbf{M}_0$ and $\mathbf{P}_0$ are symmetric, the multiple eigenvalue λ_0 is always semi-simple. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the right eigenvectors (modes) corresponding to the eigenvalue λ_0. It is easy to see that the left eigenvectors are

$$\mathbf{v}_1 = \mathbf{M}_0\mathbf{u}_1, \quad \mathbf{v}_2 = \mathbf{M}_0\mathbf{u}_2. \tag{2.210}$$

Normalization conditions (2.161) take the form

$$\mathbf{u}_1^T\mathbf{M}_0\mathbf{u}_1 = \mathbf{u}_2^T\mathbf{M}_0\mathbf{u}_2 = 1, \quad \mathbf{u}_1^T\mathbf{M}_0\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{M}_0\mathbf{u}_1 = 0. \tag{2.211}$$

Using expressions (2.209) and (2.210) in formula (2.183), we get

$$f_j^{kl} = \mathbf{u}_k^T \left(\frac{\partial \mathbf{P}}{\partial p_j} - \omega_0^2 \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{u}_l, \tag{2.212}$$

where $f_j^{12} = f_j^{21}$ due to the symmetry of the matrices $\mathbf{M}$ and $\mathbf{P}$. With expressions (2.185) we obtain

$$g_1 = \frac{f_1^{11} + f_1^{22}}{2}, \quad h_{11} = \frac{(f_1^{11} - f_1^{22})^2}{4} + \left(f_1^{12}\right)^2 \geq 0. \tag{2.213}$$

Assuming that $h_{11} \neq 0$, bifurcation of the double eigenvalue λ_0 is given by expression (2.191) for the case $\Delta p_2 = 0$. This bifurcation is of the type r', see Fig. 2.9, where λ_0 splits into two real eigenvalues for small increments Δp_1. This agrees with the general theory, which says that all the frequencies of the conservative system under consideration are real.

If Δp_2 is nonzero and small, then from expression (2.194) we have

$$\sigma = \frac{f_2^{11} + f_2^{22}}{2}\,\Delta p_2,$$

$$\delta = -\frac{(f_1^{11} - f_1^{22})(f_2^{11} - f_2^{22}) + 4f_1^{12}f_2^{12}}{4h_{11}}\,\Delta p_2, \tag{2.214}$$

$$\psi = \frac{\left((f_1^{11} - f_1^{22})f_2^{12} - (f_2^{11} - f_2^{22})f_1^{12}\right)^2}{4h_{11}}\,(\Delta p_2)^2 \geq 0.$$

Hence, behavior of the eigenvalues with a change of Δp_1 is described by two hyperbolae (2.195), see Fig. 2.11 and Fig. 2.12 (r'_+). Two real eigenvalues approach, turn at some distance from each other, and diverge with a monotonous change of Δp_1.

The frequencies

$$\omega = \sqrt{\lambda} = \sqrt{\omega_0^2 + X + iY + o(\varepsilon)} = \omega_0 + \frac{X + iY}{2\omega_0} + o(\varepsilon) \tag{2.215}$$

have the same type of behavior in the neighborhood of ω_0. We see that a small perturbation of the second parameter Δp_2 destroys a picture of weak interaction in such a way that the double frequency disappears. This agrees with the results by [Wigner and von Neumann (1929)], who studied crossing of energy levels in quantum mechanics.

2.10 Bifurcation of semi-simple eigenvalue of arbitrary multiplicity

Let λ_0 be a semi-simple eigenvalue of the matrix $\mathbf{A}_0$ with multiplicity k. There are k linearly independent right eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and left eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We assume that the left and right eigenvectors satisfy the normalization conditions

$$\mathbf{v}_i^T \mathbf{u}_j = \delta_{ij}, \quad i, j = 1, \ldots, k, \tag{2.216}$$

where δ_{ij} is the Kronecker delta. Under perturbation of the parameter vector along curve (2.67) the eigenvalue λ_0 bifurcates. The perturbed eigenvalues and corresponding eigenvectors can be represented in the form of series

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \\ \mathbf{u} &= \mathbf{w}_0 + \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + \cdots \end{aligned} \tag{2.217}$$

Since any linear combination of the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is also an eigenvector, we do not know *a priori* the vector $\mathbf{w}_0$, which is a limit value of the eigenvector $\mathbf{u}$ as $\varepsilon \to 0$. Substituting expansions (2.72) and (2.217) into the eigenvalue problem, we find the equations

$$\begin{aligned} \mathbf{A}_0\mathbf{w}_0 &= \lambda_0\mathbf{w}_0, \\ \mathbf{A}_0\mathbf{w}_1 + \mathbf{A}_1\mathbf{w}_0 &= \lambda_0\mathbf{w}_1 + \lambda_1\mathbf{w}_0, \\ &\vdots \end{aligned} \tag{2.218}$$

The first equation says that $\mathbf{w}_0$ is an eigenvector corresponding to λ_0, i.e.,

$$\mathbf{w}_0 = \gamma_1\mathbf{u}_1 + \cdots + \gamma_k\mathbf{u}_k \tag{2.219}$$

with some coefficients $\gamma_1, \ldots, \gamma_k$. Multiplying the second equation of (2.218) by $\mathbf{v}_i^T$ from the left and using equation for the left eigenvector $\mathbf{v}_i$, we find

$$\mathbf{v}_i^T\mathbf{A}_1\mathbf{w}_0 = \lambda_1\mathbf{v}_i^T\mathbf{w}_0. \tag{2.220}$$

Using expression (2.219) and normalization conditions (2.216) in equation (2.220) for $i = 1, \ldots, k$, we obtain

$$\mathbf{F}\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix} = \lambda_1\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \end{pmatrix}, \tag{2.221}$$

where $\mathbf{F} = [f_{ij}]$ is a $k \times k$ matrix with the elements

$$f_{ij} = \mathbf{v}_i^T\mathbf{A}_1\mathbf{u}_j = \sum_{l=1}^{n}\left(\mathbf{v}_i^T\frac{\partial\mathbf{A}}{\partial p_l}\mathbf{u}_j\right)e_l. \tag{2.222}$$

A nonzero solution $(\gamma_1, \ldots, \gamma_k)^T$ exists if and only if λ_1 is an eigenvalue of the matrix $\mathbf{F}$. Therefore, k eigenvalues of the matrix $\mathbf{F}$ determine expansions for k eigenvalues (2.217) appearing due to bifurcation of the multiple eigenvalue λ_0.

Theorem 2.7 *Let λ_0 be a semi-simple eigenvalue of multiplicity k for the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$. Then bifurcation of the eigenvalue λ_0 under perturbation of the parameter vector along curve (2.67) is given by*

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon), \tag{2.223}$$

where k values of λ_1 are the eigenvalues of the $k \times k$ matrix $\mathbf{F}$ with elements (2.222). The eigenvectors corresponding to eigenvalues (2.223) are

$$\mathbf{u} = \gamma_1\mathbf{u}_1 + \cdots + \gamma_k\mathbf{u}_k + o(1), \tag{2.224}$$

where $(\gamma_1, \ldots, \gamma_k)^T$ are the eigenvectors of the matrix $\mathbf{F}$ corresponding to the eigenvalues λ_1.

Remark 2.2 *Bifurcation of the semi-simple eigenvalue λ_0 can be described in the form independent on normalization conditions (2.216). In this case, the coefficients λ_1 and $\gamma_1, \ldots, \gamma_k$ are found as the eigenvalues and eigenvectors of the generalized eigenvalue problem*

$$\mathbf{F}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_k\end{pmatrix} = \lambda_1\mathbf{N}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_k\end{pmatrix}, \tag{2.225}$$

where $\mathbf{F} = [f_{ij}]$ and $\mathbf{N} = [n_{ij}]$ are $k \times k$ matrices with the elements (2.222) and $n_{ij} = \mathbf{v}_i^T\mathbf{u}_j$.

2.11 Bifurcation of multiple eigenvalues with arbitrary Jordan structure

Bifurcation of multiple eigenvalues and corresponding eigenvectors is more complicated in the case of an arbitrary Jordan structure. It should be noted that multiple eigenvalues that are neither nonderogatory nor semi-simple are very rare in matrix families describing real-world systems. Perturbation theory for multiple eigenvalues of an arbitrary Jordan structure in one-parameter case can be found in [Vishik and Lyusternik (1960); Lidskii (1965); Moro *et al.* (1997)], which can be applied to the multi-parameter case using the method of perturbation along smooth curves in the parameter space. This theory is well developed for perturbations satisfying a specific nondegeneracy condition, which is not valid for all directions in the multi-parameter space. The general consideration would require using the Newton diagram applied to the characteristic polynomial of the matrix $\mathbf{A}(\mathbf{p})$; see Section 4.5.

2.12 Generalized eigenvalue problem

In this section we consider a generalized eigenvalue problem

$$\mathbf{B}\mathbf{u} = \lambda \mathbf{C}\mathbf{u}, \tag{2.226}$$

where $\mathbf{B}$ and $\mathbf{C}$ are real nonsymmetric $m \times m$ matrices smoothly dependent on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$; λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. It is assumed that the matrix $\mathbf{C}$ is nonsingular. There are m eigenvalues of the generalized eigenvalue problem, counting multiplicities, which are roots of the characteristic equation

$$\det(\mathbf{B} - \lambda \mathbf{C}) = 0. \tag{2.227}$$

Multiplying equation (2.226) by the matrix $\mathbf{C}^{-1}$ from the left, we get the standard eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u} \tag{2.228}$$

for the matrix

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}. \tag{2.229}$$

Hence, eigenvalues and eigenvectors of the generalized eigenvalue problem coincide with those of the eigenvalue problem for matrix (2.229).

The left eigenvector for the generalized eigenvalue problem is defined by the equation

$$\mathbf{v}^T\mathbf{B} = \lambda \mathbf{v}^T\mathbf{C}. \tag{2.230}$$

Introducing the vector

$$\widetilde{\mathbf{v}} = \mathbf{C}^T\mathbf{v}, \tag{2.231}$$

we find

$$\widetilde{\mathbf{v}}^T\mathbf{A} = \lambda \widetilde{\mathbf{v}}^T. \tag{2.232}$$

Therefore, expression (2.231) connects the left eigenvectors of the generalized and standard eigenvalue problems.

2.12.1 *Simple eigenvalue*

Let us consider a simple eigenvalue λ_0 of the generalized eigenvalue problem at $\mathbf{p} = \mathbf{p}_0$ with corresponding right and left eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ satisfying the equations

$$\mathbf{B}_0\mathbf{u}_0 = \lambda_0\mathbf{C}_0\mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{B}_0 = \lambda_0\mathbf{v}_0^T\mathbf{C}_0, \tag{2.233}$$

where $\mathbf{B}_0 = \mathbf{B}(\mathbf{p}_0)$ and $\mathbf{C}_0 = \mathbf{C}(\mathbf{p}_0)$.

Theorem 2.8 *A simple eigenvalue λ_0 of generalized eigenvalue problem (2.226) is a smooth function of the parameter vector, and its derivative with respect to parameter p_i at $\mathbf{p}_0$ is given by the expression*

$$\frac{\partial\lambda}{\partial p_i} = \mathbf{v}_0^T\left(\frac{\partial\mathbf{B}}{\partial p_i} - \lambda_0\frac{\partial\mathbf{C}}{\partial p_i}\right)\mathbf{u}_0\Big/(\mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_0), \tag{2.234}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ are the right and left eigenvectors determined by equations (2.233). The corresponding eigenvector $\mathbf{u}(\mathbf{p})$ can be chosen as a smooth function of $\mathbf{p}$ with the derivative given by the expression

$$\frac{\partial\mathbf{u}}{\partial p_i} = \mathbf{G}_0^{-1}\left(\frac{\partial\lambda}{\partial p_i}\mathbf{C}_0 + \lambda_0\frac{\partial\mathbf{C}}{\partial p_i} - \frac{\partial\mathbf{B}}{\partial p_i}\right)\mathbf{u}_0, \tag{2.235}$$

where

$$\mathbf{G}_0 = \mathbf{B}_0 - \lambda_0\mathbf{C}_0 + \overline{\mathbf{v}}_0\mathbf{v}_0^T\mathbf{C}_0 \tag{2.236}$$

is a nonsingular matrix.

Theorem 2.8 is proven similarly to Theorem 2.2 (page 32). We differentiate equation (2.226) with respect to the parameter p_i:

$$(\mathbf{B}_0 - \lambda_0\mathbf{C}_0)\frac{\partial\mathbf{u}}{\partial p_i} = \frac{\partial\lambda}{\partial p_i}\mathbf{C}_0\mathbf{u}_0 - \left(\frac{\partial\mathbf{B}}{\partial p_i} - \lambda_0\frac{\partial\mathbf{C}}{\partial p_i}\right)\mathbf{u}_0. \tag{2.237}$$

Then expression (2.234) follows from (2.237) after multiplication by $\mathbf{v}_0^T$ from the left and use of (2.233). For determining the eigenvector $\mathbf{u}$ we impose the normalization condition

$$\mathbf{v}_0^T\mathbf{C}_0\mathbf{u} = const. \tag{2.238}$$

Differentiating (2.238) with respect to p_i, we obtain

$$\mathbf{v}_0^T\mathbf{C}_0\frac{\partial\mathbf{u}}{\partial p_i} = 0. \tag{2.239}$$

Then expression (2.235) is obtained when we add (2.239) multiplied by $\overline{\mathbf{v}}_0$ from the left to equation (2.237).

Example 2.12 Let us consider the one-parameter generalized eigenvalue problem (2.226) with the matrices

$$\mathbf{B}(p) = \begin{pmatrix} p-2 & 1 & 6 \\ 2p-3 & 2 & 2 \\ 2p-1 & 1 & 3p \end{pmatrix}, \quad \mathbf{C}(p) = \begin{pmatrix} p+1 & 2 & 3 \\ 2 & 3 & 2p+1 \\ 1 & 1 & 4p \end{pmatrix}. \tag{2.240}$$

At $p_0 = 0$ there is the simple eigenvalue $\lambda_0 = i$ with the right and left eigenvectors

$$\mathbf{u}_0 = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 1+i \\ -3-3i \\ 7+5i \end{pmatrix}. \tag{2.241}$$

By Theorem 2.8, we find that the eigenvalue λ_0 and corresponding eigenvector $\mathbf{u}_0$ depend smoothly on the parameter p, and their first order derivatives are equal to

$$\frac{d\lambda}{dp} = \frac{5}{2} + i, \quad \frac{d\mathbf{u}}{dp} = \frac{1}{40}\begin{pmatrix} -15+20i \\ -20+15i \\ -6+2i \end{pmatrix}. \tag{2.242}$$

2.12.2 *Semi-simple eigenvalue*

Using Theorem 2.7 (page 70) with matrix (2.229) and left eigenvector (2.231), we describe bifurcation of a semi-simple eigenvalue for the generalized eigenvalue problem.

Theorem 2.9 *Let λ_0 be a semi-simple eigenvalue of multiplicity k for generalized eigenvalue problem (2.226) at $\mathbf{p}_0$. We assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are the right and left linearly independent eigenvectors, respectively, satisfying the normalization conditions*

$$\mathbf{v}_i^T \mathbf{C}_0 \mathbf{u}_j = \delta_{ij}, \quad i,j = 1, \ldots, k. \tag{2.243}$$

Then, bifurcation of the eigenvalue λ_0 under perturbation of the parameter vector along curve (2.67) is given by

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon), \tag{2.244}$$

where k values of λ_1 are the eigenvalues of the $k \times k$ matrix $\mathbf{F}$ with the elements

$$f_{ij} = \sum_{l=1}^{n} \left(\mathbf{v}_i^T \left(\frac{\partial \mathbf{B}}{\partial p_l} - \lambda_0 \frac{\partial \mathbf{C}}{\partial p_l} \right) \mathbf{u}_j \right) e_l. \tag{2.245}$$

The eigenvectors corresponding to eigenvalues (2.244) are

$$\mathbf{u} = \gamma_1 \mathbf{u}_1 + \cdots + \gamma_k \mathbf{u}_k + o(1), \tag{2.246}$$

where $(\gamma_1, \ldots, \gamma_k)^T$ are the eigenvectors of the matrix $\mathbf{F}$ corresponding to the eigenvalues λ_1.

2.12.3 *Nonderogatory eigenvalue*

Finally, we consider a nonderogatory eigenvalue λ_0 of multiplicity k for generalized eigenvalue problem (2.226) at $\mathbf{p}_0$. The Jordan chain $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ corresponding to λ_0 is defined by the equations

$$\begin{aligned} \mathbf{B}_0 \mathbf{u}_0 &= \lambda_0 \mathbf{C}_0 \mathbf{u}_0, \\ \mathbf{B}_0 \mathbf{u}_1 &= \lambda_0 \mathbf{C}_0 \mathbf{u}_1 + \mathbf{C}_0 \mathbf{u}_0, \\ &\vdots \\ \mathbf{B}_0 \mathbf{u}_{k-1} &= \lambda_0 \mathbf{C}_0 \mathbf{u}_{k-1} + \mathbf{C}_0 \mathbf{u}_{k-2}. \end{aligned} \tag{2.247}$$

Multiplying these equations by the matrix $\mathbf{C}_0^{-1}$ from the left, we find that the vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ form the Jordan chain of the standard eigenvalue problem for the matrix $\mathbf{A}_0 = \mathbf{C}_0^{-1} \mathbf{B}_0$. The left Jordan chain $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ for the generalized eigenvalue problem is defined by the equations

$$\begin{aligned} \mathbf{v}_0^T \mathbf{B}_0 &= \lambda_0 \mathbf{v}_0^T \mathbf{C}_0, \\ \mathbf{v}_1^T \mathbf{B}_0 &= \lambda_0 \mathbf{v}_1^T \mathbf{C}_0 + \mathbf{v}_0^T \mathbf{C}_0, \\ &\vdots \\ \mathbf{v}_{k-1}^T \mathbf{B}_0 &= \lambda_0 \mathbf{v}_{k-1}^T \mathbf{C}_0 + \mathbf{v}_{k-2}^T \mathbf{C}_0. \end{aligned} \tag{2.248}$$

It is easy to see that the vectors

$$\widetilde{\mathbf{v}}_i = \mathbf{C}_0^T \mathbf{v}_i, \quad i = 0, \ldots, k-1, \tag{2.249}$$

form the left Jordan chain corresponding to the eigenvalue λ_0 of the matrix $\mathbf{A}_0$.

Equalities (2.33) for the vectors of right and left Jordan chains of the matrix $\mathbf{A}_0$ yield the properties

$$\begin{aligned}
\mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_0 &= 0,\\
\mathbf{v}_1^T\mathbf{C}_0\mathbf{u}_0 = \mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_1 &= 0,\\
&\vdots\\
\mathbf{v}_{k-2}^T\mathbf{C}_0\mathbf{u}_0 = \mathbf{v}_{k-3}^T\mathbf{C}_0\mathbf{u}_1 = \cdots = \mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_{k-2} &= 0,\\
\mathbf{v}_{k-1}^T\mathbf{C}_0\mathbf{u}_0 = \mathbf{v}_{k-2}^T\mathbf{C}_0\mathbf{u}_1 = \cdots = \mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_{k-1} &\neq 0,\\
\mathbf{v}_{k-1}^T\mathbf{C}_0\mathbf{u}_1 = \mathbf{v}_{k-2}^T\mathbf{C}_0\mathbf{u}_2 = \cdots &= \mathbf{v}_1^T\mathbf{C}_0\mathbf{u}_{k-1},\\
&\vdots\\
\mathbf{v}_{k-1}^T\mathbf{C}_0\mathbf{u}_{k-2} &= \mathbf{v}_{k-2}^T\mathbf{C}_0\mathbf{u}_{k-1}.
\end{aligned} \tag{2.250}$$

Normalization conditions for Jordan chains (2.35) can be written in terms of the generalized eigenvalue problem as follows

$$\mathbf{v}_0^T\mathbf{C}_0\mathbf{u}_{k-1} = 1, \quad \mathbf{v}_i^T\mathbf{C}_0\mathbf{u}_{k-1} = 0, \quad i = 1,\ldots,k-1. \tag{2.251}$$

These conditions define the left Jordan chain $\mathbf{v}_0,\ldots,\mathbf{v}_{k-1}$ uniquely for a given right Jordan chain $\mathbf{u}_0,\ldots,\mathbf{u}_{k-1}$. Using Theorem 2.5 (page 51) for matrix (2.229) with left Jordan chain (2.249), we describe bifurcation of a nonderogatory eigenvalue.

Theorem 2.10 *Let λ_0 be a nonderogatory eigenvalue of generalized eigenvalue problem (2.226) at $\mathbf{p}_0$. Then bifurcation of the eigenvalue λ_0 and corresponding eigenvector $\mathbf{u}_0$ under perturbation of the parameter vector along curve (2.67) is given by*

$$\begin{aligned}
\lambda &= \lambda_0 + \varepsilon^{1/k}\lambda_1 + o(\varepsilon^{1/k}),\\
\mathbf{u} &= \mathbf{u}_0 + \varepsilon^{1/k}\lambda_1\mathbf{u}_1 + o(\varepsilon^{1/k}),
\end{aligned} \tag{2.252}$$

where λ_1 takes k different complex values of the root

$$\lambda_1 = \sqrt[k]{\sum_{i=1}^{n}\left(\mathbf{v}_0^T\left(\frac{\partial\mathbf{B}}{\partial p_i} - \lambda_0\frac{\partial\mathbf{C}}{\partial p_i}\right)\mathbf{u}_0\right)e_i}. \tag{2.253}$$

Other results on bifurcation of eigenvalues for a matrix family $\mathbf{A}(\mathbf{p})$ can be written in terms of the generalized eigenvalue problem analogously.

2.13 Eigenvalue problem for vibrational system

The eigenvalue problem for a linear vibrational system has the form

$$(\lambda^2\mathbf{M} + \lambda\mathbf{B} + \mathbf{C})\mathbf{u} = 0, \tag{2.254}$$

where $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{C}$ are real $m \times m$ matrices smoothly dependent on the vector of parameters $\mathbf{p}$; λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. The matrix $\mathbf{M}$ is assumed to be nonsingular. Typically, $\mathbf{M}$ is a symmetric positive definite matrix, but we will not use this property here. There are $2m$ eigenvalues, counting multiplicities, of problem (2.254) determined from the characteristic equation

$$\det(\lambda^2\mathbf{M} + \lambda\mathbf{B} + \mathbf{C}) = 0, \tag{2.255}$$

where $\det(\lambda^2\mathbf{M} + \lambda\mathbf{B} + \mathbf{C})$ is a polynomial of order $2m$. Notice that the eigenvectors corresponding to different eigenvalues are not necessarily linearly independent, since the dimension of eigenvectors equals m, which is twice smaller than the number of eigenvalues. The left eigenvector $\mathbf{v}$ is defined by the equation

$$\mathbf{v}^T(\lambda^2\mathbf{M} + \lambda\mathbf{B} + \mathbf{C}) = 0. \tag{2.256}$$

Problem (2.254) can be reduced to the generalized eigenvalue problem

$$\widetilde{\mathbf{B}}\widetilde{\mathbf{u}} = \lambda\widetilde{\mathbf{C}}\widetilde{\mathbf{u}}, \tag{2.257}$$

where $\widetilde{\mathbf{u}}$ is the vector of double dimension $2m$

$$\widetilde{\mathbf{u}} = \begin{pmatrix} \mathbf{u} \\ \lambda\mathbf{u} \end{pmatrix}, \tag{2.258}$$

and the $2m \times 2m$ block matrices $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{C}}$ are

$$\widetilde{\mathbf{B}} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{C} & -\mathbf{B} \end{pmatrix}, \quad \widetilde{\mathbf{C}} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{M} \end{pmatrix}. \tag{2.259}$$

It is easy to see that the left eigenvector of generalized eigenvalue problem (2.257) equals

$$\widetilde{\mathbf{v}} = \begin{pmatrix} (\lambda\mathbf{M}^T + \mathbf{B}^T)\mathbf{v} \\ \mathbf{v} \end{pmatrix}. \tag{2.260}$$

2.13.1 *Simple eigenvalue*

For a simple eigenvalue of problem (2.254) we have

Theorem 2.11 *A simple eigenvalue λ of problem (2.254) is a smooth function of the parameter vector, and its derivative with respect to parameter p_i is given by the expression*

$$\frac{\partial \lambda}{\partial p_i} = -\mathbf{v}^T \left(\lambda^2 \frac{\partial \mathbf{M}}{\partial p_i} + \lambda \frac{\partial \mathbf{B}}{\partial p_i} + \frac{\partial \mathbf{C}}{\partial p_i} \right) \mathbf{u} \Big/ (\mathbf{v}^T (2\lambda \mathbf{M} + \mathbf{B}) \mathbf{u}), \qquad (2.261)$$

where $\mathbf{u}$ and $\mathbf{v}$ are the right and left eigenvectors determined by equations (2.254) and (2.256). The corresponding eigenvector $\mathbf{u}(\mathbf{p})$ can be chosen as a smooth function of $\mathbf{p}$ with the derivative given by the expression

$$\frac{\partial \mathbf{u}}{\partial p_i} = -\mathbf{G}_0^{-1} \left(\lambda^2 \frac{\partial \mathbf{M}}{\partial p_i} + \lambda \frac{\partial \mathbf{B}}{\partial p_i} + \frac{\partial \mathbf{C}}{\partial p_i} + 2\lambda \frac{\partial \lambda}{\partial p_i} \mathbf{M} + \frac{\partial \lambda}{\partial p_i} \mathbf{B} \right) \mathbf{u}, \qquad (2.262)$$

where $\mathbf{G}_0 = \lambda^2 \mathbf{M} + \lambda \mathbf{B} + \mathbf{C} + \overline{\mathbf{v}} \mathbf{v}^T (2\lambda \mathbf{M} + \mathbf{B})$ is a nonsingular matrix.

To prove Theorem 2.11, we take derivative of both sides of equation (2.254) with respect to p_i:

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{B} + \mathbf{C}) \frac{\partial \mathbf{u}}{\partial p_i} = -\frac{\partial \lambda}{\partial p_i} (2\lambda \mathbf{M} + \mathbf{B}) \mathbf{u} - \left(\lambda^2 \frac{\partial \mathbf{M}}{\partial p_i} + \lambda \frac{\partial \mathbf{B}}{\partial p_i} + \frac{\partial \mathbf{C}}{\partial p_i} \right) \mathbf{u}. \qquad (2.263)$$

Expression (2.261) follows from (2.263) after multiplication by the vector $\mathbf{v}^T$ from the left and using equation (2.256). Let us consider the normalization condition

$$\mathbf{v}^T (2\lambda \mathbf{M} + \mathbf{B}) \frac{\partial \mathbf{u}}{\partial p_i} = 0 \qquad (2.264)$$

for the derivative of the eigenvector $\mathbf{u}$, which is equivalent to normalization condition (2.239) for the generalized problem with matrices (2.259) and eigenvectors (2.258), (2.260). Adding equation (2.264) multiplied by the vector $\overline{\mathbf{v}}$ from the left to equation (2.263), we obtain expression (2.262).

2.13.2 *Semi-simple eigenvalue*

Analogously, we describe bifurcation of a semi-simple eigenvalue for vibrational problem (2.254).

Theorem 2.12 *Let λ_0 be a semi-simple eigenvalue of multiplicity k for problem (2.254) at $\mathbf{p}_0$. We assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are the* *right and left linearly independent eigenvectors, respectively, satisfying the normalization condition*

$$\mathbf{v}_i^T(2\lambda_0\mathbf{M}_0 + \mathbf{B}_0)\mathbf{u}_j = \delta_{ij}, \quad i,j = 1,\ldots,k, \tag{2.265}$$

where $\mathbf{M}_0 = \mathbf{M}(\mathbf{p}_0)$ *and* $\mathbf{B}_0 = \mathbf{B}(\mathbf{p}_0)$. *Then bifurcation of the eigenvalue* λ_0 *under perturbation of the parameter vector along curve (2.67) is given by*

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon), \tag{2.266}$$

where k *values of* λ_1 *are the eigenvalues of the* $k \times k$ *matrix* $\mathbf{F}$ *with the elements*

$$f_{ij} = -\sum_{l=1}^{n}\left[\mathbf{v}_i^T\left(\lambda_0^2\frac{\partial\mathbf{M}}{\partial p_l} + \lambda_0\frac{\partial\mathbf{B}}{\partial p_l} + \frac{\partial\mathbf{C}}{\partial p_l}\right)\mathbf{u}_j\right]e_l. \tag{2.267}$$

The eigenvectors corresponding to eigenvalues (2.266) are

$$\mathbf{u} = \gamma_1\mathbf{u}_1 + \cdots + \gamma_k\mathbf{u}_k + o(1), \tag{2.268}$$

where $(\gamma_1,\ldots,\gamma_k)^T$ *are the eigenvectors of the matrix* $\mathbf{F}$ *corresponding to the eigenvalues* λ_1.

2.13.3 *Nonderogatory eigenvalue*

Finally, we consider a nonderogatory eigenvalue λ_0 of multiplicity k with the Jordan chain $\tilde{\mathbf{u}}_0,\ldots,\tilde{\mathbf{u}}_{k-1}$ satisfying equations (2.247) with matrices (2.259) taken at $\mathbf{p}_0$. Using explicit form of matrices (2.259), it can be shown that the vectors of the Jordan chain have the form

$$\tilde{\mathbf{u}}_0 = \begin{pmatrix}\mathbf{u}_0\\ \lambda_0\mathbf{u}_0\end{pmatrix}, \quad \tilde{\mathbf{u}}_i = \begin{pmatrix}\mathbf{u}_i\\ \lambda_0\mathbf{u}_i + \mathbf{u}_{i-1}\end{pmatrix}, \quad i = 1,\ldots,k-1. \tag{2.269}$$

The vectors $\mathbf{u}_0,\ldots,\mathbf{u}_{k-1}$ are called the *Keldysh chain* and satisfy the equations

$$\begin{aligned}
\mathbf{L}_0\mathbf{u}_0 &= 0,\\
\mathbf{L}_0\mathbf{u}_1 &= -\mathbf{L}_1\mathbf{u}_0,\\
\mathbf{L}_0\mathbf{u}_2 &= -\mathbf{L}_1\mathbf{u}_1 - \mathbf{L}_2\mathbf{u}_0,\\
&\ \vdots\\
\mathbf{L}_0\mathbf{u}_{k-1} &= -\mathbf{L}_1\mathbf{u}_{k-2} - \mathbf{L}_2\mathbf{u}_{k-3}.
\end{aligned} \tag{2.270}$$

Here $\mathbf{L}_0$, $\mathbf{L}_1$, and $\mathbf{L}_2$ are the matrix operators

$$\mathbf{L}_0 = \lambda_0^2\mathbf{M}_0 + \lambda_0\mathbf{B}_0 + \mathbf{C}_0, \quad \mathbf{L}_1 = 2\lambda_0\mathbf{M}_0 + \mathbf{B}_0, \quad \mathbf{L}_2 = \mathbf{M}_0, \tag{2.271}$$

where $\mathbf{M}_0 = \mathbf{M}(\mathbf{p}_0)$, $\mathbf{B}_0 = \mathbf{B}(\mathbf{p}_0)$, and $\mathbf{C}_0 = \mathbf{C}(\mathbf{p}_0)$. We note that $\mathbf{L}_1$ and $\mathbf{L}_2$ can be expressed via derivatives of the matrix-function $\mathbf{L}(\lambda) = \lambda^2\mathbf{M}_0 + \lambda\mathbf{B}_0 + \mathbf{C}_0$ as

$$\mathbf{L}_1 = \frac{\partial \mathbf{L}}{\partial \lambda}, \quad \mathbf{L}_2 = \frac{1}{2}\frac{\partial^2 \mathbf{L}}{\partial \lambda^2}, \tag{2.272}$$

taken at $\lambda = \lambda_0$. Expression (2.269) for the linearly independent vectors $\widetilde{\mathbf{u}}_0, \ldots, \widetilde{\mathbf{u}}_{k-1}$ does not imply the linear independence of the vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$. The Keldysh chain ends up with the vector $\mathbf{u}_{k-1}$ if we cannot find the next vector $\mathbf{u}_k$ satisfying the Keldysh chain equations such that the vector

$$\widetilde{\mathbf{u}}_k = \begin{pmatrix} \mathbf{u}_k \\ \lambda_0\mathbf{u}_k + \mathbf{u}_{k-1} \end{pmatrix} \tag{2.273}$$

is linearly independent on $\widetilde{\mathbf{u}}_0, \ldots, \widetilde{\mathbf{u}}_{k-1}$.

For the left Jordan chain $\widetilde{\mathbf{v}}_0, \ldots, \widetilde{\mathbf{v}}_{k-1}$, satisfying equations (2.248) with matrices (2.259), we have

$$\begin{aligned} \widetilde{\mathbf{v}}_0 &= \begin{pmatrix} (\lambda_0\mathbf{M}_0^T + \mathbf{B}_0^T)\mathbf{v}_0 \\ \mathbf{v}_0 \end{pmatrix}, \\ \widetilde{\mathbf{v}}_i &= \begin{pmatrix} (\lambda_0\mathbf{M}_0^T + \mathbf{B}_0^T)\mathbf{v}_i + \mathbf{M}_0^T\mathbf{v}_{i-1} \\ \mathbf{v}_i \end{pmatrix}, \quad i = 1, \ldots, k-1, \end{aligned} \tag{2.274}$$

where the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ form the *left Keldysh chain* satisfying the equations

$$\begin{aligned} \mathbf{v}_0^T\mathbf{L}_0 &= 0, \\ \mathbf{v}_1^T\mathbf{L}_0 &= -\mathbf{v}_0^T\mathbf{L}_1, \\ \mathbf{v}_2^T\mathbf{L}_0 &= -\mathbf{v}_1^T\mathbf{L}_1 - \mathbf{v}_0^T\mathbf{L}_2, \\ &\vdots \\ \mathbf{v}_{k-1}^T\mathbf{L}_0 &= -\mathbf{v}_{k-2}^T\mathbf{L}_1 - \mathbf{v}_{k-3}^T\mathbf{L}_2. \end{aligned} \tag{2.275}$$

The orthogonality properties for the right and left Keldysh chains can be obtained by substitution of expressions (2.259), (2.269), and (2.274) into equations (2.250). In particular, the orthogonality property of eigenvectors is the following

$$\mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_0 = 0. \tag{2.276}$$

This equation satisfied for the right eigenvector $\mathbf{u}_0$ and an arbitrary left eigenvector $\mathbf{v}_0$ represents a criterion for the existence of the Keldysh chain starting with $\mathbf{u}_0$.

Analogously, using equations (2.251), we get the normalization conditions for the right and left Keldysh chains in the form

$$\begin{aligned} &\mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_{k-1} + \mathbf{v}_0^T \mathbf{L}_2 \mathbf{u}_{k-2} = 1, \\ &\mathbf{v}_i^T \mathbf{L}_1 \mathbf{u}_{k-1} + \mathbf{v}_i^T \mathbf{L}_2 \mathbf{u}_{k-2} + \mathbf{v}_{i-1}^T \mathbf{L}_2 \mathbf{u}_{k-1} = 0, \quad i = 1, \ldots, k-1. \end{aligned} \tag{2.277}$$

These conditions define the left Keldysh chain $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ uniquely for a given right Keldysh chain $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$.

Using expressions (2.259), (2.269), and (2.274) in Theorem 2.10 (page 75), we describe bifurcation of the nonderogatory eigenvalue and corresponding eigenvector in terms of the right and left Keldysh chains.

Theorem 2.13 *Let λ_0 be a nonderogatory eigenvalue of multiplicity k for problem (2.254) at $\mathbf{p}_0$. Then bifurcation of the eigenvalue λ_0 and eigenvector $\mathbf{u}_0$ under perturbation of the parameter vector along curve (2.67) is given by*

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/k}\lambda_1 + o(\varepsilon^{1/k}), \\ \mathbf{u} &= \mathbf{u}_0 + \varepsilon^{1/k}\lambda_1 \mathbf{u}_1 + o(\varepsilon^{1/k}), \end{aligned} \tag{2.278}$$

where λ_1 takes k different complex values of the root

$$\lambda_1 = \sqrt[k]{-\sum_{i=1}^{n} \left(\mathbf{v}_0^T \left(\lambda_0^2 \frac{\partial \mathbf{M}}{\partial p_i} + \lambda_0 \frac{\partial \mathbf{B}}{\partial p_i} + \frac{\partial \mathbf{C}}{\partial p_i} \right) \mathbf{u}_0 \right) e_i}. \tag{2.279}$$

Notice that though only the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{v}_0$ are used in Theorem 2.13, all the right Keldysh chain has to be found in order to satisfy normalization conditions (2.277). Expression (2.279) written in the form independent on normalization conditions (2.277) is

$$\lambda_1 = \sqrt[k]{-\frac{\sum_{i=1}^{n} \left(\mathbf{v}_0^T \left(\lambda_0^2 \frac{\partial \mathbf{M}}{\partial p_i} + \lambda_0 \frac{\partial \mathbf{B}}{\partial p_i} + \frac{\partial \mathbf{C}}{\partial p_i} \right) \mathbf{u}_0 \right) e_i}{\mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_{k-1} + \mathbf{v}_0^T \mathbf{L}_2 \mathbf{u}_{k-2}}}. \tag{2.280}$$

Example 2.13 Let us consider two-parameter eigenvalue problem (2.254) with the matrices

$$\mathbf{M}(\mathbf{p}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B}(\mathbf{p}) = \begin{pmatrix} 0 & -p_1 \\ p_1 & 0 \end{pmatrix},$$

$$\mathbf{C}(\mathbf{p}) = \begin{pmatrix} p_2 - 3 & 0 \\ 0 & p_1 + 4p_2 - \sqrt{3} \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{2.281}$$

At $\mathbf{p}_0 = (\sqrt{3}, 0)$ there is the nonderogatory eigenvalue $\lambda_0 = 0$ of multiplicity 4 with the right and left Keldysh chains

$$\mathbf{u}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -\sqrt{3}/3 \\ 0 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} -4\sqrt{3}/9 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \ \mathbf{v}_1 = \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \tag{2.282}$$

satisfying normalization conditions (2.277). Though the vectors in the right Keldysh chain are not linearly independent, one can easily check that the vectors of corresponding Jordan chain (2.269) are linearly independent. By Theorem 2.13, we find that the bifurcation of the quadruple zero eigenvalue λ_0 and corresponding eigenvector along curve (2.67) is given by the relations

$$\lambda = \varepsilon^{1/4}\lambda_1 + o(\varepsilon^{1/4}), \quad \mathbf{u} = \begin{pmatrix} -\sqrt{3}\varepsilon^{1/4}\lambda_1/3 + o(\varepsilon^{1/4}) \\ 1 + o(\varepsilon^{1/4}) \end{pmatrix}, \tag{2.283}$$

where four different complex values of the coefficient λ_1 are given by the expression

$$\lambda_1 = \sqrt[4]{3e_1 + 12e_2}. \tag{2.284}$$

Chapter 3

Stability Boundary of General System Dependent on Parameters

A wide range of practical problems in mechanics and physics requires stability analysis of a linear system of ordinary differential equations, which appear as a result of linearization of equations of motion near a stationary solution or steady motion. Any physical system depends on parameters, and values of the parameters at which the system is stable form the stability domain in the parameter space. It is clear that construction of the stability domain is closely related to finding its boundary.

Analysis of the stability domain and its boundary is a problem of great practical importance. A number of examples reveal complexity of the stability boundary, which consists of smooth parts and can have different singularities. The singularities are related to bifurcations of eigenvalues of the system operator. They reflect specific physical properties of the system and may lead to numerical difficulties of the analysis. Classification of singularities of the stability boundary for two- and three-parameter systems was done in [Arnold (1972); Arnold (1983a)], and the extension to a more general case was given in [Levantovskii (1980a); Levantovskii (1982)]. Quantitative methods of stability analysis near regular and singular points of the stability boundary were developed in [Seyranian (1982); Pedersen and Seyranian (1983); Burke and Overton (1992); Mailybaev (1998); Mailybaev and Seyranian (1998b); Mailybaev and Seyranian (1999b); Mailybaev (1999)].

This chapter is devoted to stability analysis of a general linear system of ordinary differential equations, whose coefficients are smooth functions of parameters. First, we introduce the concept of general position. This important notion coming from singularity theory allows selecting typical (generic) structures and concentrating attention on the most practical and observable situations. Then, we give qualitative description of the stability boundary in the parameter space, determine its regular part and classify generic singularities (of codimension 2 and 3). Using results of Chapter 2 on bifurcation of eigenvalues, we perform quantitative analysis of the stability domain in the neighborhood of regular and singular points of the boundary. As a result, we derive general and constructive formulae for local approximation of the stability domain using only information at the initial regular or singular boundary points.

3.1 Stability and dynamics of linear system

Let us consider a linear system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3.1}$$

where $\mathbf{x}$ is an m-dimensional vector of phase variables, $\mathbf{A}$ is an $m \times m$ nonsymmetric real matrix, and dot denotes the derivative with respect to time t. Looking for a solution of (3.1) in the form

$$\mathbf{x}(t) = \mathbf{u} \exp \lambda t, \tag{3.2}$$

we obtain the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}, \tag{3.3}$$

where λ is an eigenvalue and $\mathbf{u}$ is a corresponding eigenvector. There are m eigenvalues $\lambda_1, \ldots, \lambda_m$ of the matrix $\mathbf{A}$, counting multiplicities, that satisfy the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{3.4}$$

Let λ be a real eigenvalue. Then, the corresponding eigenvector $\mathbf{u}$ can be chosen real, and (3.2) represents a real solution of system (3.1). In case of complex λ, the eigenvector $\mathbf{u}$ has to be complex with linearly independent real and imaginary parts. Taking real and imaginary parts of expression (3.2), we find two real linearly independent solutions of system (3.1) in the form

$$\begin{aligned} \mathbf{x}_1(t) &= \exp \alpha t \left(\operatorname{Re} \mathbf{u} \cos \omega t - \operatorname{Im} \mathbf{u} \sin \omega t \right), \\ \mathbf{x}_2(t) &= \exp \alpha t \left(\operatorname{Re} \mathbf{u} \sin \omega t + \operatorname{Im} \mathbf{u} \cos \omega t \right), \end{aligned} \tag{3.5}$$

where $\lambda = \alpha + i\omega$. Notice that the couple of solutions (3.5) corresponds to the complex conjugate pair of eigenvalues $\lambda = \alpha \pm i\omega$.

If all the eigenvalues of the matrix $\mathbf{A}$ are simple or semi-simple, we have m linearly independent solutions (3.2) and (3.5) taken for all eigenvalues. A linear combination of these solutions forms a general solution of system (3.1), where real coefficients of the linear combination are determined by initial conditions.

If there are multiple eigenvalues with Jordan chains (a number of linearly independent eigenvectors is less than the algebraic multiplicity of the eigenvalue), solutions (3.2) and (3.5) are not sufficient to construct a general solution. Let λ be a real eigenvalue, and $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ be real vectors of the corresponding Jordan chain. It is easy to see that

$$\begin{aligned} \mathbf{x}_0(t) &= \mathbf{u}_0 \exp \lambda t, \\ \mathbf{x}_1(t) &= (\mathbf{u}_1 + t\mathbf{u}_0) \exp \lambda t, \\ &\vdots \\ \mathbf{x}_{k-1}(t) &= \left(\mathbf{u}_{k-1} + t\mathbf{u}_{k-2} + \cdots + \frac{t^{k-1}}{(k-1)!}\mathbf{u}_0 \right) \exp \lambda t \end{aligned} \tag{3.6}$$

are k linearly independent solutions of system (3.1). The terms with powers of time in these solutions are called *secular terms*. In case of a complex λ, real and imaginary parts of solutions (3.6) provide $2k$ linearly independent real solutions corresponding to a complex conjugate pair of eigenvalues $\lambda = \alpha \pm i\omega$. Taking these solutions for all the eigenvalues and Jordan chains, we obtain a set of m linearly independent solutions, whose linear combination is a general solution of system (3.1) for an arbitrary matrix $\mathbf{A}$.

We see that norms of solutions (3.2) and (3.6) grow (or decay) exponentially for positive (or negative) real eigenvalue λ. In case of a complex λ, these solutions oscillate with an amplitude growing (or decaying) in time if $\operatorname{Re}\lambda > 0$ (or $\operatorname{Re}\lambda < 0$). In case of a purely imaginary or zero eigenvalue, behavior of the solution depends on the corresponding Jordan structure. If λ is simple or semi-simple, then the solution remains bounded but does not tend to zero as $t \to +\infty$. But if a purely imaginary or zero eigenvalue has Jordan chains (a number of linearly independent eigenvectors is less than the algebraic multiplicity), then due to the presence of secular terms there are unbounded solutions growing as a power of time.

Linear system (3.1) is stable if all the solutions of the system are bounded as $t \to +\infty$, while the asymptotic stability implies that any solution of the system tends to zero as $t \to +\infty$. Instability means that there are solutions of system (3.1) unbounded as $t \to +\infty$. The above construction of a general solution of system (3.1) provides the stability criterion in terms of eigenvalues of the matrix $\mathbf{A}$. Linear system (3.1) is stable if and only if all the eigenvalues of the matrix $\mathbf{A}$ have negative or zero real part $\operatorname{Re}\lambda \leq 0$ with all purely imaginary and zero eigenvalues being simple or semi-simple. Linear system (3.1) is asymptotically stable if and only if all the eigenvalues of the matrix $\mathbf{A}$ have negative real part $\operatorname{Re}\lambda < 0$. Finally, linear system (3.1) is unstable if and only if there exists an eigenvalue of the matrix $\mathbf{A}$ with a positive real part $\operatorname{Re}\lambda > 0$, or an eigenvalue with zero real part $\operatorname{Re}\lambda = 0$ which is neither simple nor semi-simple. Fig. 3.1 shows three examples for distribution of eigenvalues in the cases of asymptotic stability, stability, and instability.

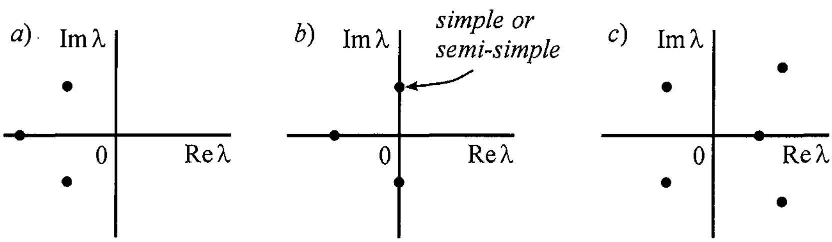


Fig. 3.1 Distribution of eigenvalues on the complex plane for a) asymptotic stability, b) stability, and c) instability.

Stability of a linear system is closely related to stability of a stationary solution for a nonlinear autonomous system of ordinary differential equations

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \tag{3.7}$$

where $\mathbf{f}(\mathbf{y})$ is a smooth function of the vector $\mathbf{y} \in \mathbb{R}^m$. The stationary solution

$$\mathbf{y}(t) \equiv \tilde{\mathbf{y}} \tag{3.8}$$

is determined by the condition

$$\mathbf{f}(\tilde{\mathbf{y}}) = 0. \tag{3.9}$$

Let us introduce the matrix

$$\mathbf{A} = \left[\frac{d\mathbf{f}}{d\mathbf{y}}\right]_{\tilde{\mathbf{y}}}, \tag{3.10}$$

which is the Jacobian matrix of the function $\mathbf{f}(\mathbf{y})$ at $\tilde{\mathbf{y}}$. According to Theorem 1.2 (page 7), if linear system (3.1) with matrix (3.10) is asymptotically stable, i.e., all the eigenvalues of matrix (3.10) have negative real part $\operatorname{Re}\lambda < 0$, then stationary solution (3.8) of nonlinear system (3.7) is asymptotically stable. If matrix (3.10) has an eigenvalue with a positive real part $\operatorname{Re}\lambda > 0$, then stationary solution (3.8) of nonlinear system (3.7) is unstable.

Notice that in the case, when $\operatorname{Re}\lambda \leq 0$ for all the eigenvalues of the matrix $\mathbf{A}$ and there are eigenvalues on the imaginary axis $\operatorname{Re}\lambda = 0$, stability for the nonlinear system is determined by nonlinear terms neglected in the linearization.

3.2 Stability domain and its boundary

Let us assume that the system matrix $\mathbf{A}$ depends smoothly on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. The function $\mathbf{A}(\mathbf{p})$ is called a multiparameter *family of matrices*. Then, values of the parameter vector such that system (3.1) is asymptotically stable ($\operatorname{Re}\lambda < 0$ for all the eigenvalues) form the *stability domain* in the parameter space. If $\mathbf{p}$ belongs to the stability domain, then sufficiently small perturbations of the parameter vector keep the system asymptotically stable (the eigenvalues stay in the left half-plane). The *instability domain* in the parameter space consists of the vectors $\mathbf{p}$ such that system (3.1) is unstable. A boundary of the stability domain is represented by values of $\mathbf{p}$ such that the matrix $\mathbf{A}(\mathbf{p})$ has some of eigenvalues on the imaginary axis $\operatorname{Re}\lambda = 0$, while the others belong to the left half-plane $\operatorname{Re}\lambda < 0$.

Multi-parameter stability analysis of system (3.1) implies construction of the stability domain in the parameter space, which requires finding a boundary of the stability domain. Simple examples show that the boundary of the stability domain (in short, the stability boundary) is a hypersurface with singularities. Singularities represent nonsmooth points on the boundary, like edges, angles *etc.* These singularities have strong influence on numerical and physical properties of the underlying system.

Let us consider a point $\mathbf{p}_0$ in the parameter space belonging to the stability boundary. This requires some of eigenvalues to be purely imaginary or zero. Under perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$ these eigenvalues change, some of them shift to the left part of the complex plane, others to the right part, or stay on the imaginary axis. Perturbation that

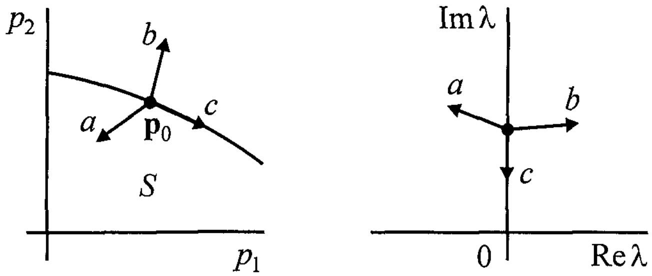


Fig. 3.2 Perturbation of the parameter vector and the eigenvalue on the stability boundary: a) stabilizing, b) destabilizing, c) along the stability boundary.

moves all the eigenvalues from the imaginary axis to the left is stabilizing (the perturbed vector $\mathbf{p}$ belongs to the stability domain). If at least one of the eigenvalues moves to the right, perturbation is destabilizing. The case, when some eigenvalues stay on the imaginary axis while the others move to the left, corresponds to perturbation along the stability boundary; see Fig. 3.2 (the stability domain is denoted by the letter S).

3.3 Case of general position

There is a wide variety of stability boundary points, which differ by the number of eigenvalues on the imaginary axis, their multiplicities and Jordan structures. Nevertheless, not all of them are typical and occur in the analysis of particular problems. Typical points of the stability boundary are structurally stable. This means that the stability boundary point of a certain type does not disappear if we take a small perturbation of the matrix family $\mathbf{A}(\mathbf{p}) + \delta\mathbf{B}(\mathbf{p})$, but may undergo a small shift in the parameter space. Such points are called *generic*. The case, when all the stability boundary points are generic, is called the *case of general position* [Arnold (1983a)]. In contrary, nongeneric boundary points may disappear under an arbitrarily small variation of the matrix family.

To illustrate the introduced notion on a simple example, we consider a family of 2×2 matrices

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} a_{11}(\mathbf{p}) & a_{12}(\mathbf{p}) \\ a_{21}(\mathbf{p}) & a_{22}(\mathbf{p}) \end{pmatrix}, \tag{3.11}$$

whose elements $a_{ij}(\mathbf{p})$ are smooth functions of the vector of two parameters $\mathbf{p} = (p_1, p_2)$. There are four possible types of a stability boundary point represented by a complex conjugate pair of purely imaginary eigenvalues $\lambda = \pm i\omega$, the simple zero eigenvalue, the double zero eigenvalue with the Jordan chain of length 2, or the double semi-simple zero eigenvalue. Following [Arnold (1983a)] we introduce short notation

$$\pm i\omega, \quad 0, \quad 0^2, \quad 00 \tag{3.12}$$

for these types, respectively. Eigenvalues of matrix (3.11) can be found explicitly by solving the characteristic equation

$$\lambda^2 - (a_{11}(\mathbf{p}) + a_{22}(\mathbf{p}))\lambda + a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p}) = 0. \tag{3.13}$$

Elementary analysis shows that cases (3.12) are realized at points $\mathbf{p}$ satisfying the following relations

$$\begin{aligned}
\pm i\omega: &\quad a_{11}(\mathbf{p}) + a_{22}(\mathbf{p}) = 0, \quad a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p}) > 0;\\
0: &\quad a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p}) = 0, \quad a_{11}(\mathbf{p}) + a_{22}(\mathbf{p}) < 0;\\
0^2: &\quad a_{11}(\mathbf{p}) + a_{22}(\mathbf{p}) = 0, \; a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p}) = 0, \; \mathbf{A}(\mathbf{p}) \neq 0;\\
00: &\quad a_{11}(\mathbf{p}) = a_{12}(\mathbf{p}) = a_{21}(\mathbf{p}) = a_{22}(\mathbf{p}) = 0.
\end{aligned} \tag{3.14}$$

If the gradient of the function $a_{11}(\mathbf{p}) + a_{22}(\mathbf{p})$ is nonzero then, by the implicit function theorem, boundary points of type $\pm i\omega$ form a smooth curve in the parameter plane. Therefore, boundary points of this type are generic, and we can expect appearance of these points in two-parameter families (3.11). Analogously, points of type 0 are generic and form a smooth curve in the parameter plane provided that the gradient of the function $a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p})$ is nonzero. A point of type 0^2 is determined by two equalities. Points of this type are isolated in the parameter plane if the gradients of the functions $a_{11}(\mathbf{p}) + a_{22}(\mathbf{p})$ and $a_{11}(\mathbf{p})a_{22}(\mathbf{p}) - a_{12}(\mathbf{p})a_{21}(\mathbf{p})$ are linearly independent at these points. Hence, points of type 0^2 are generic too. Finally, points of type 00 are determined by four independent equalities. Since there are only two parameters, these equalities have no solutions in the case of general position. If a point of type 00 appears in a particular family (3.11), it can be removed by an arbitrarily small change of the functions $a_{ij}(\mathbf{p})$. Therefore, points of type 00 are nongeneric in two-parameter families (3.11). Points of type 00 become generic only if we have four of more parameters.

We observe that a stability boundary point is generic or nongeneric depending on the type of a point and the number of parameters in the matrix family. Points of a given type are determined by several equalities and, in the case of general position, form a smooth surface of codimension d (i.e., dimension $n-d$) equal to a number of these equalities. The codimension d depends only on type of the point. If dimension of the parameter space is less than d, then a point of this type is nongeneric (it can disappear under an arbitrarily small perturbation of the matrix family). If the number of parameters is equal to d, then points of this type are isolated points. Finally, if the number of parameters is greater than d, then stability boundary points of this type form a smooth surface of codimension d. Codimensions for different types of stability boundary points are determined using the versal deformation theory, see [Arnold (1971); Arnold (1983a)], and depend only on the number of eigenvalues on the imaginary axis and their Jordan structures.

Let us introduce short notation to distinguish different types of stability boundary points. We denote the type symbolically by the product of determinants of Jordan blocks for all purely imaginary and zero eigenvalues. Notation (3.12) follows this rule, where 0^2 denotes the double nonderogatory eigenvalue $\lambda = 0$ (one Jordan block of size 2) and 00 denotes the double semi-simple eigenvalue $\lambda = 0$ (two Jordan blocks of size 1). As to more complicated types, $0^2(\pm i\omega)$ corresponds to points $\mathbf{p}$, where the matrix $\mathbf{A}(\mathbf{p})$ has a double zero eigenvalue with Jordan chain of length 2 and a complex conjugate pair of simple eigenvalues $\lambda = \pm i\omega$; $(\pm i\omega_1)(\pm i\omega_2)$ corresponds to matrices having two different pairs of simple complex conjugate eigenvalues on the imaginary axis.

Full list of types of stability boundary points for codimensions 1, 2, and 3 are as follows [Arnold (1983a)]

$$\begin{aligned}
\mathrm{cod}\,1: &\quad 0, \quad \pm i\omega; \\
\mathrm{cod}\,2: &\quad 0^2, \quad 0(\pm i\omega), \quad (\pm i\omega_1)(\pm i\omega_2); \\
\mathrm{cod}\,3: &\quad 0^3, \quad (\pm i\omega)^2, \quad 0^2(\pm i\omega), \\
&\quad 0(\pm i\omega_1)(\pm i\omega_2), \quad (\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3).
\end{aligned} \tag{3.15}$$

The number of different types increases with codimension. At the same time, types of higher codimension are rare in matrix families and need more parameters to be realized in structurally stable way.

Example 3.1 Let us consider the matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1 & p_2 & p_3 \end{pmatrix}. \tag{3.16}$$

The characteristic equation for this matrix is

$$\lambda^3 - p_3\lambda^2 - p_2\lambda - p_1 = 0. \tag{3.17}$$

The stability domain found using the Routh-Hurwitz condition has the form

$$p_1 + p_2p_3 > 0, \quad p_1 < 0, \quad p_2 < 0, \quad p_3 < 0, \tag{3.18}$$

shown in Fig. 3.3, where the stability domain is denoted by S. The boundary of the stability domain contains two smooth surfaces, one surface $p_1 + p_2p_3 = 0$, $p_2 < 0$, $p_3 < 0$, consists of points of type $\pm i\omega$, and the other surface $p_1 = 0$, $p_2 < 0$, $p_3 < 0$, consists of points of type 0. There are two edges of the stability boundary represented by the rays $p_2 < 0$, $p_1 = p_3 = 0$ and $p_3 < 0$, $p_1 = p_2 = 0$. These rays consist of points of types $0(\pm i\omega)$ and 0^2, respectively. Finally, there is an isolated point of type 0^3 at the origin. Every part of the stability boundary has codimension as given in list (3.15).

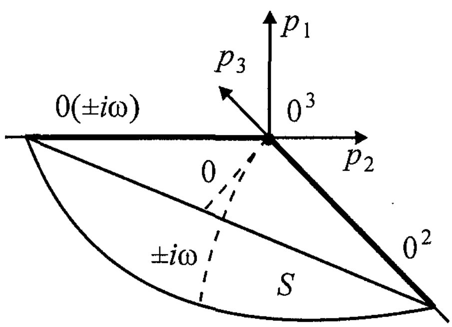


Fig. 3.3 Boundary of the stability domain and its types.

Notice that nongeneric points of the stability boundary can appear in the analysis of specific problems. This usually indicates special degeneracy or symmetry existing in the system under consideration. One of the properties frequently leading to the appearance of nongeneric structures is the conservation of energy, like it happens in Hamiltonian or gyroscopic systems. If we know the properties responsible for nongeneric structure, we can study a restricted class of systems possessing such property. Then the notion of general position specific for this class can be introduced. For linear conservative and Hamiltonian systems this analysis is done in Chapters 5 and 7.

3.4 Stability boundary: qualitative analysis

Form of the stability boundary in the neighborhood of its point $\mathbf{p}_0$ depends on the behavior of eigenvalues lying on the imaginary axis. As we showed in Chapter 2, this behavior strongly depends on multiplicities and Jordan structures of these eigenvalues. Stability domain is determined locally by the points $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$ such that all the eigenvalues are shifted to the left from the imaginary axis (stabilizing perturbations).

3.4.1 *Regular part*

According to (3.15) the most common points of the stability boundary are of types

$$0, \quad \pm i\omega. \tag{3.19}$$

Let us assume that $\mathbf{p}_0$ is a stability boundary point of type 0. The stability criterion in the neighborhood of $\mathbf{p}_0$ takes the form

$$\lambda(\mathbf{p}) < 0 \tag{3.20}$$

for the simple real eigenvalue that vanishes at $\mathbf{p}_0$. Since the simple eigenvalue $\lambda(\mathbf{p})$ is a smooth function of the parameter vector and its gradient is nonzero in the case of general position, the stability boundary is a smooth surface determined by the equation

$$\lambda(\mathbf{p}) = 0. \tag{3.21}$$

All the points of the stability boundary are of type 0 in the neighborhood of $\mathbf{p}_0$. When the parameter vector passes from the stability domain into the instability domain through $\mathbf{p}_0$, the real simple eigenvalue λ crosses the imaginary axis from the left to the right through the origin; see Fig. 3.4a. At the point $\mathbf{p}_0$ system (3.1) has the constant solution $\mathbf{x}(t) \equiv \mathbf{u}$, where $\mathbf{u}$ is the eigenvector corresponding to $\lambda = 0$. At the point $\mathbf{p}$ inside the instability domain there is a solution $\mathbf{x} = \mathbf{u}\exp\lambda t$, $\lambda > 0$, that exponentially grows in time; see Fig. 3.5a. This type of instability is called *static instability* or *divergence.*

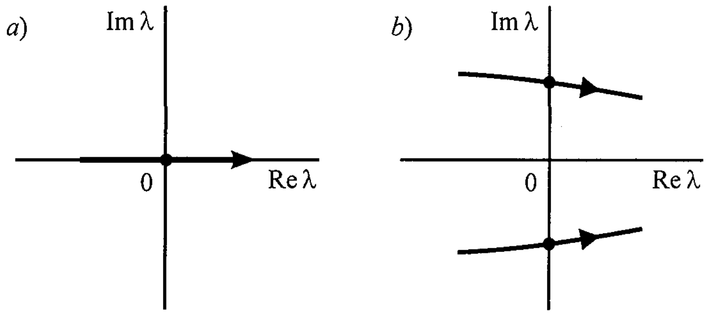


Fig. 3.4 Motion of eigenvalues for loss of stability: a) divergence and b) flutter.

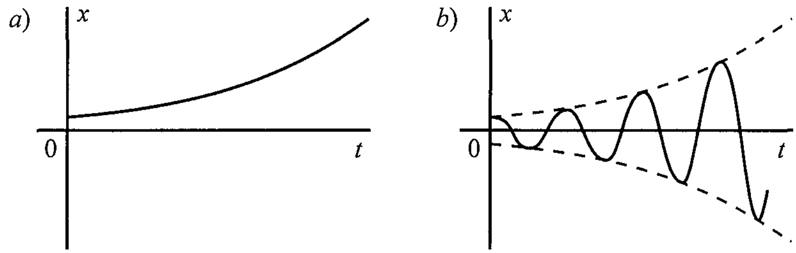


Fig. 3.5 Solution of the unstable system: a) divergence and b) flutter.

If $\mathbf{p}_0$ is a point of type $\pm i\omega$, then the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$ determined by the equation

$$\operatorname{Re}\lambda(\mathbf{p}) = 0, \tag{3.22}$$

provided that the gradient of the function $\operatorname{Re}\lambda(\mathbf{p})$ is nonzero. All the points of the stability boundary near $\mathbf{p}_0$ are of type $\pm i\omega$, where the frequency ω depends smoothly on $\mathbf{p}$. When the parameter vector passes from the stability domain into the instability domain through $\mathbf{p}_0$, two simple complex conjugate eigenvalues cross the imaginary axis from the left to the right through the points $\pm i\omega$; see Fig. 3.4b. At the point $\mathbf{p}_0$ system (3.1) has the periodic solutions $\mathbf{x}(t) = \operatorname{Re}\mathbf{u}\cos\omega t - \operatorname{Im}\mathbf{u}\sin\omega t$ and $\mathbf{x}(t) = \operatorname{Re}\mathbf{u}\sin\omega t + \operatorname{Im}\mathbf{u}\cos\omega t$. For the point $\mathbf{p}$ inside the instability domain there is an oscillating solution with exponentially growing amplitude;

see Fig. 3.5b. This type of instability is called *dynamic instability* or *flutter*.

Divergence and flutter boundaries are regular parts of the stability boundary. All other types of points on the stability boundary have higher codimensions and, hence, represent singularities on the stability boundary. Since the stability boundary of a generic one-parameter system consists only of points of codimension 1, we have

Theorem 3.1 *In the case of general position one-parameter system (3.1) loses stability either by flutter, represented by two complex conjugate simple eigenvalues on the imaginary axis, or by divergence, characterized by a simple zero eigenvalue.*

3.4.2 *Singularities of codimension 2*

First, let us consider the most typical singularities (of lowest codimension) of the stability boundary. These are the points of codimension 2 listed in (3.15). A point of type $0(\pm i\omega)$ represents intersection of divergence and flutter boundaries. This intersection forms an angle in the two-parameter plane or an edge in the three-parameter space, where the stability domain lies inside an angle (or an edge); see Figs. 3.6 and 3.7. Analogously, a stability boundary point of type $(\pm i\omega_1)(\pm i\omega_2)$ lies in the intersection of flutter boundaries corresponding to two different pairs of purely imaginary eigenvalues and represents an angle or edge in the two- and three-parameter spaces, respectively; see Figs. 3.6 and 3.7.

The remaining boundary point of type 0^2 is characterized by the double eigenvalue $\lambda = 0$ with a single eigenvector (Jordan chain of length 2). Stability of the system in the neighborhood of such a point depends on the behavior of this double eigenvalue. The system is asymptotically stable if and only if both eigenvalues λ_1 and λ_2 appearing due to bifurcation of $\lambda = 0$ move to the left side of the complex plane. We know that a double eigenvalue is a nonsmooth function of parameters. Nevertheless, the combinations $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are smooth functions of $\mathbf{p}$, see [Kato (1980)]. This means that the eigenvalues λ_1 and λ_2 are the roots of the quadratic equation

$$\lambda^2 - a_1(\mathbf{p})\lambda - a_2(\mathbf{p}) = 0, \tag{3.23}$$

where $a_1(\mathbf{p}) = \lambda_1 + \lambda_2$ and $a_2(\mathbf{p}) = -\lambda_1\lambda_2$ are smooth real functions of the parameter vector such that $a_1(\mathbf{p}) = a_2(\mathbf{p}) = 0$ at the point of singularity. Stability condition $\operatorname{Re}\lambda < 0$ for both roots of equation (3.23) takes the form

$$a_1(\mathbf{p}) < 0, \quad a_2(\mathbf{p}) < 0. \tag{3.24}$$

These inequalities determine an angle or edge in the two- and three-parameter spaces, respectively; see Figs. 3.6 and 3.7. Notice that the gradients of the functions $a_1(\mathbf{p})$ and $a_2(\mathbf{p})$ are linearly independent in the case of general position and, hence, the size of angle (3.24) is nonzero. The stability domain belongs to the interior of the angle. From equation (3.23) we see that the equality $a_1(\mathbf{p}) = 0$ for $a_2(\mathbf{p}) < 0$ determines the flutter boundary, while the equality $a_2(\mathbf{p}) = 0$ for $a_1(\mathbf{p}) < 0$ defines the divergence boundary. Therefore, the points of type 0^2 lie in the intersection of flutter and divergence boundaries. The flutter frequency ω tends to zero as we approach the singularity along the flutter boundary.

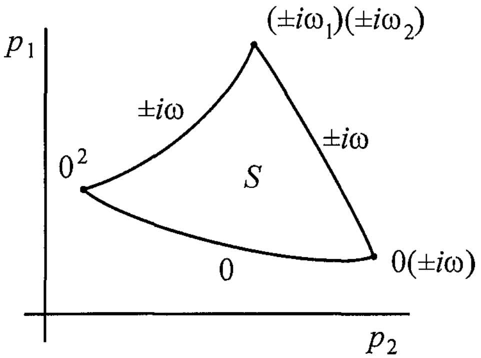


Fig. 3.6 Stability boundary and its singularities for two-parameter systems.

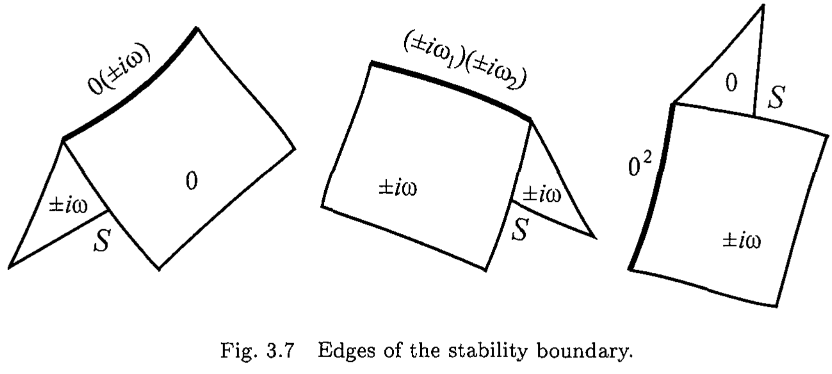


Fig. 3.7 Edges of the stability boundary.

Since the stability boundary of a generic two-parameter family of matrices consists of points of codimensions 1 and 2, we obtain

Theorem 3.2 *In the case of general position, the stability boundary of two-parameter system (3.1) consists of smooth curves corresponding to flutter and divergence instability, whose only singularities are angles of types* $0(\pm i\omega)$, $(\pm i\omega_1)(\pm i\omega_2)$, *and* 0^2; *see Fig. 3.6.*

Fig. 3.6 shows an example of the stability domain in the two-parameter plane with the boundary possessing singularities of all generic types. Notice that the stability domain always lies inside the angles (of size less than π). This reflects the *principle of fragility of all good things*, see [Arnold (1992)].

3.4.3 *Singularities of codimension 3*

Now, let us consider stability boundary points of codimension 3 listed in (3.15). The types $0(\pm i\omega_1)(\pm i\omega_2)$, $(\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3)$, and $0^2(\pm i\omega)$ are obtained from the types of codimension 2 by adding an extra pair of purely imaginary simple eigenvalues. This additional pair represents an extra stability condition and results in additional flutter boundary. Therefore, in the three-parameter space the singularities $0(\pm i\omega_1)(\pm i\omega_2)$, $(\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3)$, and $0^2(\pm i\omega)$ are trihedral angles as shown in Fig. 3.8. The stability domain lies inside the angles.

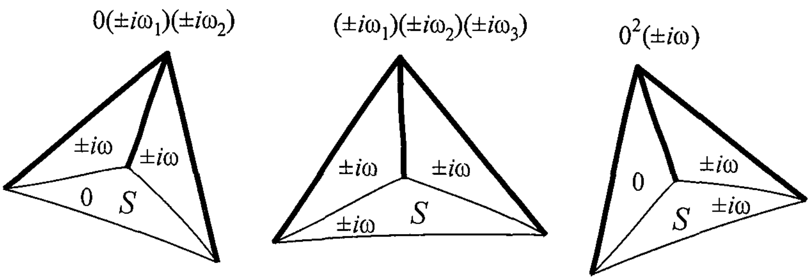


Fig. 3.8 Trihedral angles of the stability boundary.

The boundary point of type 0^3 is characterized by the triple eigenvalue $\lambda = 0$ with a single eigenvector (Jordan chain of length 3). In the neighborhood of such a point the system is asymptotically stable if and only if three eigenvalues λ_1, λ_2, and λ_3, appearing due to bifurcation of $\lambda = 0$, move to the left side of the complex plane. The eigenvalues λ_1, λ_2, and λ_3

are the roots of the cubic equation

$$\lambda^3 - a_3(\mathbf{p})\lambda^2 - a_2(\mathbf{p})\lambda - a_1(\mathbf{p}) = 0, \tag{3.25}$$

where $a_1(\mathbf{p})$, $a_2(\mathbf{p})$, and $a_3(\mathbf{p})$ are smooth real functions of the parameter vector such that $a_1(\mathbf{p}) = a_2(\mathbf{p}) = a_3(\mathbf{p}) = 0$ at the singularity point, see [Arnold (1983a)]. Notice that a similar equation was studied in Example 3.1. Stability condition $\operatorname{Re}\lambda < 0$ for three roots of equation (3.25) takes the form

$$a_1(\mathbf{p}) + a_2(\mathbf{p})a_3(\mathbf{p}) > 0, \quad a_1(\mathbf{p}) < 0, \quad a_2(\mathbf{p}) < 0, \quad a_3(\mathbf{p}) < 0. \tag{3.26}$$

In the space (a_1, a_2, a_3) the stability domain has the form shown in Fig. 3.9. Its boundary has two edges of types 0^2 and $0(\pm i\omega)$ intersecting at the point of type 0^3. Magnitudes of dihedral angles for both edges tend to zero as we approach the point 0^3. As a result, two surfaces (flutter and divergence boundaries) are tangent at the singularity. This singularity is called *break of an edge.* The stability domain belongs to the interior of the singularity. In the original parameter space $\mathbf{p}$ the stability boundary has the same form up to a smooth change of parameters.

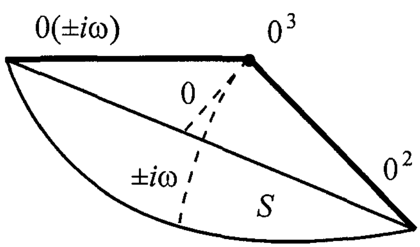


Fig. 3.9 Singularity "break of an edge" of the stability boundary.

The boundary point of type $(\pm i\omega)^2$ is characterized by a pair of double purely imaginary eigenvalues $\lambda = \pm i\omega$ with single eigenvectors (Jordan chains of length 2). In the neighborhood of such a point the system is asymptotically stable if and only if two eigenvalues λ_1 and λ_2 appearing due to bifurcation of $\lambda = i\omega$ belong to the left half of the complex plane $\operatorname{Re}\lambda < 0$ (behavior of the double eigenvalue $\lambda = -i\omega$ on the complex plane is symmetric with respect to the real axis). The eigenvalues λ_1 and λ_2 are the roots of the quadratic equation with complex coefficients smoothly dependent on $\mathbf{p}$ [Arnold (1983a)]. This equation can be given in the form

$$(\lambda - a_1(\mathbf{p}) - ib_1(\mathbf{p}))^2 - a_2(\mathbf{p}) - ib_2(\mathbf{p}) = 0, \tag{3.27}$$

where $a_1(\mathbf{p})$, $a_2(\mathbf{p})$, $b_1(\mathbf{p})$, and $b_2(\mathbf{p})$ are smooth real functions of the parameter vector such that $a_1(\mathbf{p}) = a_2(\mathbf{p}) = b_2(\mathbf{p}) = 0$ and $b_1(\mathbf{p}) = \omega$ at the singularity point. The stability condition $\operatorname{Re}\lambda < 0$ for both roots of equation (3.27) takes the form

$$(b_2(\mathbf{p}))^2 + 4(a_1(\mathbf{p}))^2\left(a_2(\mathbf{p}) - (a_1(\mathbf{p}))^2\right) < 0, \quad a_1(\mathbf{p}) < 0. \tag{3.28}$$

We see that stability conditions (3.28) do not depend on $b_1(\mathbf{p})$. In the space (a_1, a_2, b_2) the stability domain has the form shown in Fig. 3.10. The stability boundary has one edge of type $(\pm i\omega_1)(\pm i\omega_2)$, whose angle tends to zero as we approach the point $(\pm i\omega)^2$. The edge ends abruptly at the singularity point. This singularity is called *deadlock of an edge.* The stability domain belongs to the interior of the singularity. In the original parameter space $\mathbf{p}$ the stability boundary has the same form up to a smooth change of parameters.

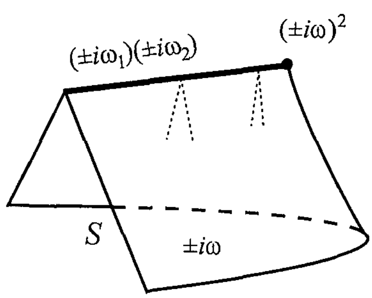


Fig. 3.10 Singularity "deadlock of an edge" of the stability boundary.

Since the stability boundary of a generic three-parameter family of matrices consists of points of codimensions 1, 2, and 3, we obtain

Theorem 3.3 *In the case of general position, the stability boundary of three-parameter system (3.1) consists of smooth surfaces corresponding to flutter and divergence, whose only singularities are edges of types* $0(\pm i\omega)$, $(\pm i\omega_1)(\pm i\omega_2)$, *and* 0^2, *trihedral angles of types* $0(\pm i\omega_1)(\pm i\omega_2)$, $(\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3)$, *and* $0^2(\pm i\omega)$, *"break of an edge" of type* 0^3, *and "deadlock of an edge" of type* $(\pm i\omega)^2$; *see Figs. 3.7–3.10.*

3.5 Quantitative analysis of divergence and flutter boundaries

Let us consider a point $\mathbf{p}_0$ of type 0 on the stability boundary (divergence instability). This point is characterized by a simple eigenvalue $\lambda_0 = 0$, while the other eigenvalues have negative real parts $\operatorname{Re}\lambda < 0$. The stability condition in the neighborhood of $\mathbf{p}_0$ is represented by the inequality

$$\lambda(\mathbf{p}) < 0, \tag{3.29}$$

where $\lambda(\mathbf{p})$ is the simple eigenvalue vanishing at the stability boundary $\lambda(\mathbf{p}_0) = 0$. Then, the stability boundary is determined by the equation

$$\lambda(\mathbf{p}) = 0. \tag{3.30}$$

By Theorem 2.2 (page 32), we know that $\lambda(\mathbf{p})$ is a smooth function of the parameter vector. Its gradient at $\mathbf{p}_0$ is given by the formula

$$\mathbf{f}_0 = \nabla\lambda = \left(\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_1}\mathbf{u}_0, \ldots, \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_n}\mathbf{u}_0\right) \Big/ (\mathbf{v}_0^T \mathbf{u}_0), \tag{3.31}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ are the right and left eigenvectors corresponding to $\lambda_0 = 0$. If $\mathbf{f}_0$ is nonzero, then we find the first order approximation of the stability domain

$$(\mathbf{f}_0, \Delta\mathbf{p}) < 0, \tag{3.32}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$, and $(\mathbf{f}_0, \Delta\mathbf{p}) = f_{01}\Delta p_1 + \cdots + f_{0n}\Delta p_n$ denotes the scalar product in $\mathbb{R}^n$. Hence, the stability boundary is a smooth surface with the tangent plane

$$(\mathbf{f}_0, \Delta\mathbf{p}) = 0 \tag{3.33}$$

at the point $\mathbf{p}_0$, and $\mathbf{f}_0$ is the normal vector to the stability boundary at $\mathbf{p}_0$ directed into the instability (divergence) domain; see Fig. 3.11. Notice that the vector $\mathbf{f}_0$ is nonzero in the case of general position.

Using the second order derivatives of the eigenvalue $\lambda(\mathbf{p})$ with respect to parameters given by expression (2.52), we find the second order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ as

$$\sum_{i=1}^{n} \frac{\partial\lambda}{\partial p_i}\Delta p_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\lambda}{\partial p_i \partial p_j}\Delta p_i \Delta p_j < 0. \tag{3.34}$$

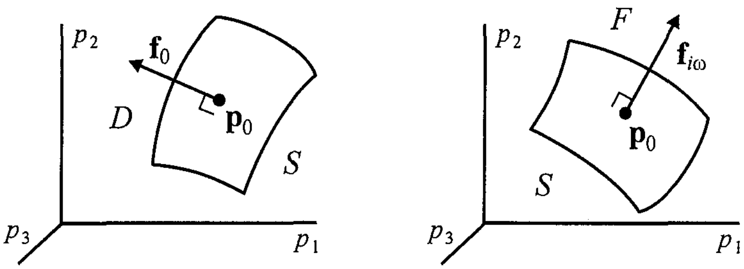


Fig. 3.11 Normal vectors to the divergence (D) and flutter (F) boundaries.

Theorem 3.4 *Let $\mathbf{p}_0$ be a point of type 0 on the stability boundary (divergence), and assume that the vector $\mathbf{f}_0$ given by expression (3.31) is nonzero. Then, the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$, and $\mathbf{f}_0$ is the normal vector to the stability boundary directed into the divergence instability domain. The second order approximation of the stability domain is given by (3.34).*

Now, let us consider a point $\mathbf{p}_0$ of type $\pm i\omega$ on the stability boundary (flutter instability). This point is characterized by a pair of complex conjugate simple eigenvalues $\pm i\omega$, while the other eigenvalues have negative real parts $\operatorname{Re}\lambda < 0$. The stability condition in the neighborhood of the point $\mathbf{p}_0$ takes the form

$$\operatorname{Re}\lambda(\mathbf{p}) < 0 \tag{3.35}$$

for the eigenvalue that equals $i\omega$ at $\mathbf{p}_0$. Since eigenvalues responsible for the loss of stability are complex conjugate, there is no need to check this condition for the other eigenvalue that equals $-i\omega$ at $\mathbf{p}_0$.

Let us introduce the vector

$$\begin{aligned}\mathbf{f}_{i\omega} &= \operatorname{Re}\nabla\lambda \\ &= \operatorname{Re}\left(\mathbf{v}_0^T\frac{\partial\mathbf{A}}{\partial p_1}\mathbf{u}_0\Big/(\mathbf{v}_0^T\mathbf{u}_0),\ldots,\mathbf{v}_0^T\frac{\partial\mathbf{A}}{\partial p_n}\mathbf{u}_0\Big/(\mathbf{v}_0^T\mathbf{u}_0)\right),\end{aligned} \tag{3.36}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ are the right and left eigenvectors corresponding to the eigenvalue $\lambda_0 = i\omega$. By Theorem 2.2 (page 32), $\mathbf{f}_{i\omega}$ is the gradient vector of the real part of $\lambda(\mathbf{p})$ at $\mathbf{p}_0$. If the vector $\mathbf{f}_{i\omega}$ is nonzero, then the first order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ is given by

$$(\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0. \tag{3.37}$$

The stability boundary is a smooth surface, whose tangent plane at $\mathbf{p}_0$ is represented by the equation

$$(\mathbf{f}_{i\omega}, \Delta\mathbf{p}) = 0. \tag{3.38}$$

Therefore, the vector $\mathbf{f}_{i\omega}$ is the normal vector to the stability boundary directed into the flutter region; see Fig. 3.11.

The second order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ is given by the inequality

$$\sum_{i=1}^{n} \operatorname{Re} \frac{\partial \lambda}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i,j=1}^{n} \operatorname{Re} \frac{\partial^2 \lambda}{\partial p_i \partial p_j} \Delta p_i \Delta p_j < 0, \tag{3.39}$$

where second order derivatives of the eigenvalue $\lambda(\mathbf{p})$ at $\mathbf{p}_0$ are given by expression (2.52).

Theorem 3.5 *Let $\mathbf{p}_0$ be a point of type $\pm i\omega$ on the stability boundary (flutter), and assume that the vector $\mathbf{f}_{i\omega}$ given by expression (3.36) is nonzero. Then, the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$, and $\mathbf{f}_{i\omega}$ is the normal vector to the stability boundary directed into the flutter instability domain. The second order approximation of the stability domain is given by (3.39).*

3.6 Quantitative analysis of singularities of codimension 2

Let us consider a point $\mathbf{p}_0$ of type $0(\pm i\omega)$ on the stability boundary. This point lies in the intersection of the divergence and flutter boundaries. Stability condition in the neighborhood of $\mathbf{p}_0$ requires both zero and purely imaginary eigenvalues to move to the left on the complex plane. Evaluating the gradient vectors $\mathbf{f}_0$ and $\mathbf{f}_{i\omega}$ for the eigenvalues $\lambda_0 = 0$ and $\lambda_0 = i\omega$ by formulae (3.31) and (3.36), respectively, we find the first order approximation of the stability domain in the form

$$(\mathbf{f}_0, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0. \tag{3.40}$$

If the vectors $\mathbf{f}_0$ and $\mathbf{f}_{i\omega}$ are linearly independent, then inequalities (3.40) define a (dihedral) angle in the parameter space; see Figs. 3.12 and 3.13. In the three-parameter space points $\mathbf{p}_0$ of type $0(\pm i\omega)$ form an edge of the

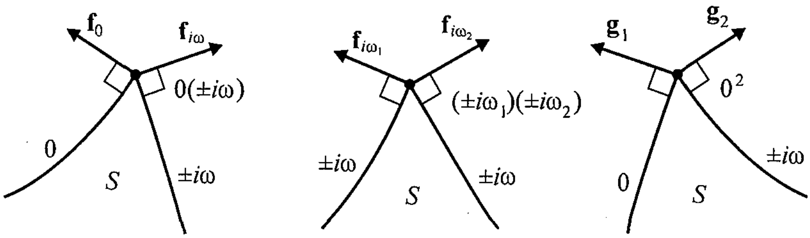


Fig. 3.12 Angles on the stability boundary and their approximation in two-parameter plane.

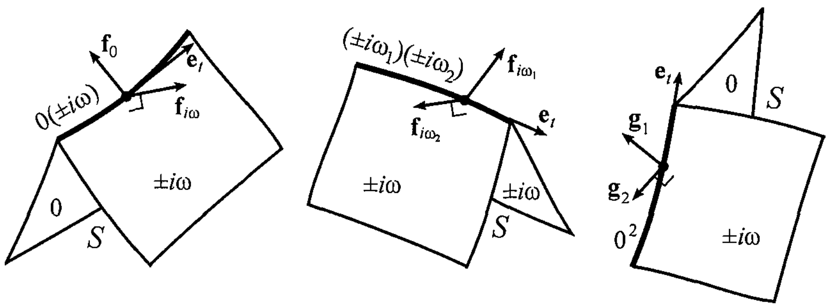


Fig. 3.13 Edges of the stability boundary and their approximation in three-parameter space.

stability boundary, and the tangent vector to the edge can be found as the cross product

$$\mathbf{e}_t = \mathbf{f}_0 \times \mathbf{f}_{i\omega}. \tag{3.41}$$

A singular point of the stability boundary $\mathbf{p}_0$ of type $(\pm i\omega_1)(\pm i\omega_2)$ belongs to the intersection of two flutter boundaries corresponding to different frequencies ω_1 and ω_2. The first order approximation for the stability domain is given by the inequalities

$$(\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0, \tag{3.42}$$

where $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ are gradients of real parts of the eigenvalues $i\omega_1$ and $i\omega_2$, respectively, evaluated by formula (3.36). If the vectors $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ are linearly independent, inequalities (3.42) define a (dihedral) angle in the parameter space; see Figs. 3.12 and 3.13. In the three-parameter space points $\mathbf{p}_0$ of this type form an edge of the stability boundary, and the tangent vector to the edge is given by the formula

$$\mathbf{e}_t = \mathbf{f}_{i\omega_1} \times \mathbf{f}_{i\omega_2}. \tag{3.43}$$

Finally, let us consider a singularity 0^2 of the stability boundary. This point is characterized by the double eigenvalue $\lambda_0 = 0$ with a single eigenvector. Let us introduce real vectors $\mathbf{g}_1 = (g_{11}, \dots, g_{1n})$ and $\mathbf{g}_2 = (g_{21}, \dots, g_{2n})$ with the components

$$\begin{aligned} g_{1j} &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0, \\ g_{2j} &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0, \quad j = 1, \dots, n, \end{aligned} \tag{3.44}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ are, respectively, the right and left Jordan chains of the eigenvalue $\lambda_0 = 0$ satisfying normalization conditions (2.65), and the derivatives are taken at $\mathbf{p}_0$. We assume that the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are linearly independent. By Theorem 2.3 (page 37), perturbation of the double nonderogatory eigenvalue $\lambda_0 = 0$ along the ray $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{e}$ is given by the expansion

$$\lambda = \pm\sqrt{(\mathbf{g}_1, \Delta\mathbf{p})} + \frac{1}{2}(\mathbf{g}_2, \Delta\mathbf{p}) + o(\varepsilon), \tag{3.45}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = \varepsilon\mathbf{e}$ denotes increment of the parameter vector. The stability condition for small ε requires $\operatorname{Re}\lambda < 0$ for both eigenvalues (3.45). Since the square root is taken with both positive and negative signs, the first term in the right-hand side of (3.45) is purely imaginary for stabilizing perturbation $\Delta\mathbf{p}$, which implies $(\mathbf{g}_1, \Delta\mathbf{p}) \leq 0$. Then, the second term in the right-hand side of expression (3.45) has to be negative to ensure stability. Therefore, we get two inequalities

$$(\mathbf{g}_1, \Delta\mathbf{p}) < 0, \quad (\mathbf{g}_2, \Delta\mathbf{p}) < 0, \tag{3.46}$$

which provide the first order approximation of the stability domain. Conditions (3.46) determine a (dihedral) angle in the parameter space; see Figs. 3.12 and 3.13. The tangent vector to the edge 0^2 of the stability boundary in the three-parameter space can be found by the formula

$$\mathbf{e}_t = \mathbf{g}_1 \times \mathbf{g}_2. \tag{3.47}$$

Notice that the conditions $(\mathbf{g}_1, \Delta\mathbf{p}) < 0$, $(\mathbf{g}_2, \Delta\mathbf{p}) = 0$, determining one side of the angle, correspond to a pair of purely imaginary eigenvalues

(3.45) and, hence, approximate the flutter boundary. For the other side of the angle $(\mathbf{g}_1, \Delta\mathbf{p}) = 0$, $(\mathbf{g}_2, \Delta\mathbf{p}) < 0$ expansion (3.45) is not valid, since nondegeneracy condition (2.87) of Theorem 2.3 is violated. In this case the double $\lambda_0 = 0$ splits into a pair of simple eigenvalues given by Theorem 2.4 (page 39) with one eigenvalue being zero along the stability boundary. Hence, this side of the angle is the divergence boundary; see Figs. 3.12 and 3.13.

Theorem 3.6 *Let $\mathbf{p}_0$ be a singular point of the stability boundary of one of the following types $0(\pm i\omega)$, $(\pm i\omega_1)(\pm i\omega_2)$, or 0^2. Depending on the type, we determine the vectors $\mathbf{f}_0$ and $\mathbf{f}_{i\omega}$, $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$, or $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, and assume that the vectors in a pair are linearly independent. Then, the point $\mathbf{p}_0$ represents a (dihedral) angle singularity of the stability boundary. The first order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ is given by the inequalities*

$$\begin{aligned}
0(\pm i\omega): &\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0; \\
(\pm i\omega_1)(\pm i\omega_2): &\quad (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0; \\
0^2: &\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0, \quad (\mathbf{g}_2, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{3.48}$$

Notice that the vector pairs in conditions (3.48) are linearly independent in the case of general position. Moreover, linear independence of these vectors provides a constructive criterion to recognize generic and nongeneric cases. In the nongeneric case, when the vectors are linearly dependent, conditions (3.48) are not valid, and we need to use higher order approximations of eigenvalues to determine a local form of the stability domain. In each particular case this task can be accomplished using bifurcation analysis of eigenvalues given in Chapter 2.

Example 3.2 As an example, let us consider the stability problem of equilibrium in a circuit consisting of a voltaic arc, resistor R, inductance L, and shunting capacitor C connected in series. Linearized differential equations near the equilibrium of the system have the form [Andronov *et al.* (1966)]

$$\begin{aligned}
\frac{d\xi}{dt} &= -\frac{\rho\xi}{L} + \frac{\eta}{L}, \\
\frac{d\eta}{dt} &= -\frac{\xi}{C} - \frac{\eta}{RC},
\end{aligned} \tag{3.49}$$

where $\xi(t)$, $\eta(t)$ are, respectively, the electric current and voltage in the voltaic arc, and ρ is the resistance of the arc.

System (3.49) depends on four parameters: three positive quantities L, C, R and the parameter ρ, which can take both positive and negative values. Assuming that the parameters L and C are fixed, we consider the stability problem on the plane of two parameters: $p_1 = R$ and $p_2 = \rho$. The matrix $\mathbf{A}$ corresponding to system (3.49) is

$$\mathbf{A} = \begin{pmatrix} -\dfrac{\rho}{L} & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix}. \tag{3.50}$$

The characteristic equation of the system takes the form

$$\lambda^2 + \left(\frac{1}{RC} + \frac{\rho}{L}\right)\lambda + \frac{1}{LC}\left(\frac{\rho}{R} + 1\right) = 0. \tag{3.51}$$

At the point $\mathbf{p}_0 = (R_0, \rho_0)$, where $R_0 = \sqrt{L/C}$ and $\rho_0 = -\sqrt{L/C}$, characteristic equation (3.51) has the double root $\lambda_0 = 0$ with a Jordan chain of length 2. Hence, $\mathbf{p}_0$ is the point of type 0^2 representing a vertex of an angle on the stability boundary. Equations for the right and left Jordan chains (2.64) with normalization conditions (2.65) yield at this point

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ -\sqrt{L/C} \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ L \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 1/\sqrt{LC} \\ 1/L \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.52}$$

Using vectors (3.52) and the matrix $\mathbf{A}$ from (3.50), we calculate the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ according to formulae (3.44) as follows

$$\mathbf{g}_1 = -\frac{1}{L\sqrt{LC}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{g}_2 = \frac{1}{L}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.53}$$

Thus, we find the angle at the singularity 0^2 of the stability boundary given by inequalities (3.48) at the point $R = R_0$, $\rho = \rho_0$; see Fig. 3.14. This angle is equal to $\pi/2$. The obtained result is in accordance with [Andronov *et al.* (1966)], where it was shown that the stability boundary consists of the segment $\rho = -R$, $0 \le R \le \sqrt{L/C}$, and the part of hyperbola $\rho = -L/(RC)$, $\sqrt{L/C} \le R$; see Fig. 3.14. Notice that the stability boundary to the left of the singularity corresponds to divergence instability, while the right part of the stability boundary corresponds to flutter instability.

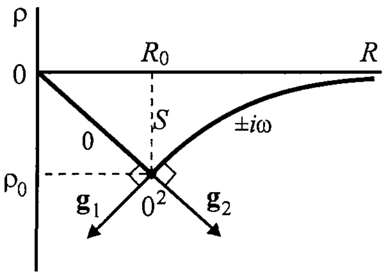


Fig. 3.14 Stability domain of equilibrium in electric circuit.

3.7 Quantitative analysis of singularities of codimension 3

Let us consider a singular point $\mathbf{p}_0$ on the stability boundary corresponding to one of the types

$$0(\pm i\omega_1)(\pm i\omega_2), \quad (\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3), \quad 0^2(\pm i\omega). \tag{3.54}$$

These types differ from those considered in the previous section by an extra pair of complex conjugate eigenvalues $\lambda = \pm i\omega$. This additional pair corresponds to the flutter boundary and leads to an extra condition of the form (3.37). Therefore, first order approximations of the stability domain in the neighborhood of the point $\mathbf{p}_0$ take the form

$$\begin{aligned} 0(\pm i\omega_1)(\pm i\omega_2): &\ (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;\\ (\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3): &\ (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_3}, \Delta\mathbf{p}) < 0;\\ 0^2(\pm i\omega): &\ (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0; \end{aligned} \tag{3.55}$$

where the vector $\mathbf{f}_0$ is determined by formula (3.31) for the simple zero eigenvalue, the vectors $\mathbf{f}_{i\omega}$, $\mathbf{f}_{i\omega_1}$, $\mathbf{f}_{i\omega_2}$, and $\mathbf{f}_{i\omega_3}$ are given by formula (3.36) for the eigenvalues $i\omega$, $i\omega_1$, $i\omega_2$, and $i\omega_3$, respectively, and the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are determined by expressions (3.44) for the double zero eigenvalue. Assuming that three vectors in each condition of (3.55) are linearly independent, we find that stability boundary points of types (3.54) are trihedral angle singularities whose sides are flutter or divergence boundaries, and the edges are of the types studied in Section 3.6. Vectors in conditions (3.55) are normal vectors to corresponding sides of the trihedral angle, directed opposite to the stability domain; see Fig. 3.15.

Now, let us consider a singularity "deadlock of an edge" $(\pm i\omega)^2$ determined by a pair of nonderogatory double eigenvalues $\pm i\omega$. Let us introduce

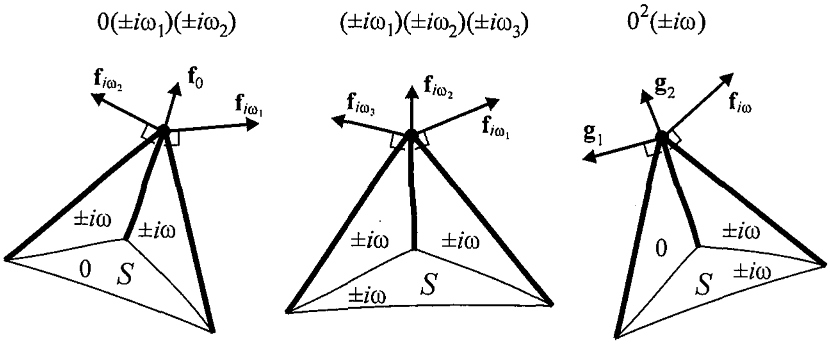


Fig. 3.15 Trihedral angle singularities of the stability boundary and their approximations.

complex vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ of dimension n with the components determined by expressions (3.44), where the right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$ are taken for the eigenvalue $\lambda_0 = i\omega$. Stability condition requires that four eigenvalues, which form two double eigenvalues on the imaginary axis at $\mathbf{p}_0$, move to the left half of the complex plane $\operatorname{Re}\lambda < 0$ under stabilizing perturbation. Considering perturbations of the parameter vector along a ray $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$, where $\Delta\mathbf{p} = \varepsilon\mathbf{e}$, and using Theorem 2.3 (page 37), we find bifurcation of the double eigenvalue $\lambda_0 = i\omega$ in the form

$$\lambda = i\omega \pm \sqrt{(\mathbf{g}_1, \Delta\mathbf{p})} + \frac{1}{2}(\mathbf{g}_2, \Delta\mathbf{p}) + o(\varepsilon), \tag{3.56}$$

where $(\mathbf{g}_j, \Delta\mathbf{p}) = (\operatorname{Re}\mathbf{g}_j, \Delta\mathbf{p}) + i(\operatorname{Im}\mathbf{g}_j, \Delta\mathbf{p})$, $j = 1, 2$. Since both positive and negative signs are taken before the square root, the second term in the right-hand side of expansion (3.56) must be purely imaginary for stabilizing perturbation. Hence, the expression under the square root must be real and negative. This yields two independent relations $(\operatorname{Re}\mathbf{g}_1, \Delta\mathbf{p}) \leq 0$ and $(\operatorname{Im}\mathbf{g}_1, \Delta\mathbf{p}) = 0$. Under these conditions the stability is determined by the third term in expansion (3.56), which yields $(\operatorname{Re}\mathbf{g}_2, \Delta\mathbf{p}) < 0$. Therefore, we find three conditions

$$(\pm i\omega)^2: \ (\operatorname{Re}\mathbf{g}_1, \Delta\mathbf{p}) < 0, \ (\operatorname{Im}\mathbf{g}_1, \Delta\mathbf{p}) = 0, \ (\operatorname{Re}\mathbf{g}_2, \Delta\mathbf{p}) < 0, \tag{3.57}$$

which provide the first order approximation of the stability domain. The real vectors $\operatorname{Re}\mathbf{g}_1$, $\operatorname{Im}\mathbf{g}_1$, and $\operatorname{Re}\mathbf{g}_2$ are assumed to be linearly independent, which is the case of general position.

In the three-parameter space conditions (3.57) define a plane angle. This angle describes a specific form of the stability domain near the singularity

$(\pm i\omega)^2$, which is very narrow such that only a zero-measure set of directions lead inside the stability domain; see Fig. 3.16. The singularity "deadlock of an edge" appears at the tip of an edge $(\pm i\omega_1)(\pm i\omega_2)$, when the size of dihedral angle becomes zero. From expansion (3.56) we see that two different purely imaginary eigenvalues $i\omega_1$ and $i\omega_2$ exist if

$$(\mathrm{Re}\,\mathbf{g}_1, \Delta\mathbf{p}) < 0, \;\; (\mathrm{Im}\,\mathbf{g}_1, \Delta\mathbf{p}) = 0, \;\; (\mathrm{Re}\,\mathbf{g}_2, \Delta\mathbf{p}) = 0. \tag{3.58}$$

Conditions (3.58) determine a ray with the direction $\mathbf{e}_t$, which is tangent to the edge $(\pm i\omega_1)(\pm i\omega_2)$; see Fig. 3.16.

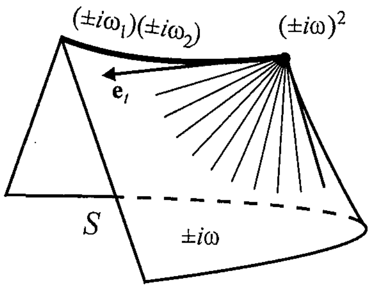


Fig. 3.16 Singularity "deadlock of an edge" of the stability boundary and its approximation.

Let us consider an arbitrary curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ with the direction $\mathbf{e}$. Then, bifurcation of the eigenvalue $\lambda_0 = i\omega$ takes the form (3.56) with the substitution of $\Delta\mathbf{p}$ by $\varepsilon\mathbf{e}$. Hence, any curve with the direction $\mathbf{e}$ satisfying the conditions

$$(\mathrm{Re}\,\mathbf{g}_1, \mathbf{e}) < 0, \;\; (\mathrm{Im}\,\mathbf{g}_1, \mathbf{e}) = 0, \;\; (\mathrm{Re}\,\mathbf{g}_2, \mathbf{e}) < 0 \tag{3.59}$$

belongs to the stability domain for small positive ε. At the same time any curve, whose direction does not satisfy the conditions

$$(\mathrm{Re}\,\mathbf{g}_1, \mathbf{e}) \leq 0, \;\; (\mathrm{Im}\,\mathbf{g}_1, \mathbf{e}) = 0, \;\; (\mathrm{Re}\,\mathbf{g}_2, \mathbf{e}) \leq 0 \tag{3.60}$$

lies in the instability domain for small $\varepsilon > 0$.

Finally, let us consider the singularity "break of an edge" 0^3 determined by the nonderogatory triple eigenvalue $\lambda_0 = 0$ at $\mathbf{p}_0$, while the other eigenvalues have negative real parts $\mathrm{Re}\,\lambda < 0$. For this purpose, we introduce real vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ of dimension n with the components

$$\begin{aligned} h_{1j} &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0, \\ h_{2j} &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0, \\ h_{3j} &= \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 + \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 + \mathbf{v}_2^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0, \quad j = 1, \ldots, n, \end{aligned} \tag{3.61}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ are the right and left Jordan chains corresponding to the eigenvalue $\lambda_0 = 0$ and satisfying equations (2.4), (2.32) with normalization conditions (2.35). Then, by Theorem 2.5 (page 51), the triple eigenvalue $\lambda_0 = 0$ takes the increment

$$\lambda = \varepsilon^{1/3} \sqrt[3]{(\mathbf{h}_1, \mathbf{e})} + o(\varepsilon^{1/3}) \tag{3.62}$$

along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ with the direction $\mathbf{e}$. Since the cubic root in (3.62) takes three different complex values, there is always one eigenvalue with a positive real part unless

$$(\mathbf{h}_1, \mathbf{e}) = 0. \tag{3.63}$$

Hence, directions of curves lying in the stability domain satisfy degeneration condition (3.63).

Directions $\mathbf{e}$ determined by (3.63) are degenerate in the sense that expansions of the eigenvalues along the curve $\mathbf{p}(\varepsilon)$ are not given in powers of $\varepsilon^{1/3}$. This case was studied in Example 2.9 (page 53), where it was shown that splitting of the triple eigenvalue $\lambda_0 = 0$ is given by two eigenvalues

$$\lambda = \pm \varepsilon^{1/2} \sqrt{(\mathbf{h}_2, \mathbf{e})} + \frac{\varepsilon}{2} \left(-\frac{(\mathbf{He}, \mathbf{e}) - (\mathbf{h}_1, \mathbf{d})}{2(\mathbf{h}_2, \mathbf{e})} + (\mathbf{h}_3, \mathbf{e}) \right) + o(\varepsilon), \tag{3.64}$$

and by the third eigenvalue

$$\lambda = \varepsilon \frac{(\mathbf{He}, \mathbf{e}) - (\mathbf{h}_1, \mathbf{d})}{2(\mathbf{h}_2, \mathbf{e})} + o(\varepsilon), \tag{3.65}$$

where $\mathbf{d} = d^2\mathbf{p}/d\varepsilon^2$ is evaluated at $\varepsilon = 0$, and $\mathbf{H} = [h_{ij}]$ is an $n \times n$ real matrix with the components

$$h_{ij} = \mathbf{v}_0^T \left(2 \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{G}_2^{-1} \frac{\partial \mathbf{A}}{\partial p_j} - \frac{\partial^2 \mathbf{A}}{\partial p_i \partial p_j} \right) \mathbf{u}_0, \quad \mathbf{G}_2 = \mathbf{A}_0 + \mathbf{v}_0 \mathbf{v}_2^T. \tag{3.66}$$

In formulae (3.64) and (3.65), $(\mathbf{He}, \mathbf{e})$ denotes the quadratic form

$$(\mathbf{He}, \mathbf{e}) = \sum_{i,j=1}^{n} h_{ij} e_i e_j. \tag{3.67}$$

Expressions (3.64)–(3.66) are obtained from expansions (2.153)–(2.155) in Example 2.9 using relations (2.73) and (3.61).

Since both negative and positive signs are taken before the square root in expansion (3.64), the first term in the right-hand side of (3.64) must be purely imaginary for stabilizing perturbation, which yields

$$(\mathbf{h}_2, \mathbf{e}) \leq 0. \tag{3.68}$$

Then, stability of the system is determined by the second term for eigenvalues (3.64) and the first term in expansion for the third eigenvalue (3.65). These terms are negative for stabilizing perturbations. Using (3.68) and assuming that $(\mathbf{h}_2, \mathbf{e}) \neq 0$, we find the stability condition for $\varepsilon > 0$ as

$$(\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_2, \mathbf{e})(\mathbf{h}_3, \mathbf{e}) < (\mathbf{h}_1, \mathbf{d}) < (\mathbf{He}, \mathbf{e}). \tag{3.69}$$

Since $(\mathbf{h}_2, \mathbf{e}) < 0$, solutions $\mathbf{d}$ of double inequality (3.69) exist if

$$(\mathbf{h}_3, \mathbf{e}) < 0. \tag{3.70}$$

Under this condition inequalities (3.69) provide a set of vectors $\mathbf{d}$ determining curvatures of the curves $\mathbf{p}(\varepsilon)$ lying in the stability domain for positive ε. Therefore, we find the first order approximation of the stability domain in the neighborhood of the singularity "break of an edge" as

$$0^3: \quad (\mathbf{h}_1, \Delta\mathbf{p}) = 0, \ (\mathbf{h}_2, \Delta\mathbf{p}) < 0, \ (\mathbf{h}_3, \Delta\mathbf{p}) < 0. \tag{3.71}$$

The vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ are linearly independent in the case of general position.

In the three-parameter space conditions (3.71) define a plane angle. This reflects a specific form of the stability boundary near the point $\mathbf{p}_0$, which is very narrow such that there is only a zero-measure set of directions leading to the stability domain; see Fig. 3.17. The singularity "break of an edge" appears at the intersection of two edges $0(\pm i\omega)$ and 0^2, when the sizes of dihedral angles for both edges become zero. From expansions (3.64) and (3.65) we see that a pair of purely imaginary eigenvalues $\pm i\omega$ and simple zero eigenvalue appear if

$$(\mathbf{h}_1, \mathbf{e}) = 0, \ (\mathbf{h}_2, \mathbf{e}) < 0, \ (\mathbf{h}_3, \mathbf{e}) = 0, \tag{3.72}$$

which determine the vector $\mathbf{e}_1$ tangent to the edge $0(\pm i\omega)$. The other side of the plane angle

$$(\mathbf{h}_1, \mathbf{e}) = 0, \ (\mathbf{h}_2, \mathbf{e}) = 0, \ (\mathbf{h}_3, \mathbf{e}) < 0 \tag{3.73}$$

determines the vector $\mathbf{e}_2$ tangent to the edge 0^2; see Fig. 3.17.

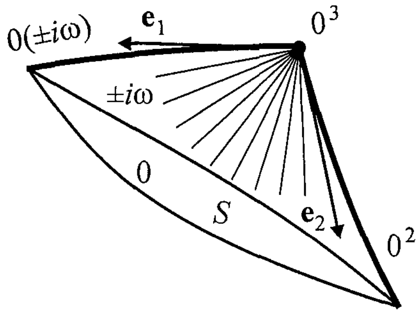


Fig. 3.17 Singularity "break of an edge" of the stability boundary and its approximation.

Unlike the "deadlock of an edge", in the case of the "break of an edge" singularity not all the curves with directions

$$(\mathbf{h}_1, \mathbf{e}) = 0, \ (\mathbf{h}_2, \mathbf{e}) < 0, \ (\mathbf{h}_3, \mathbf{e}) < 0 \tag{3.74}$$

belong to the stability domain for $\varepsilon > 0$. Stable curves are restricted by condition on second order derivatives (3.69). Therefore, all the curves with directions (3.74) and second order derivatives (3.69) belong to the stability domain for small positive ε. If the direction $\mathbf{e}$ does not belong to the set

$$(\mathbf{h}_1, \mathbf{e}) = 0, \ (\mathbf{h}_2, \mathbf{e}) \leq 0, \ (\mathbf{h}_3, \mathbf{e}) \leq 0, \tag{3.75}$$

or the second order derivatives vector $\mathbf{d}$ does not satisfy the nonstrict inequalities

$$(\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_2, \mathbf{e})(\mathbf{h}_3, \mathbf{e}) \leq (\mathbf{h}_1, \mathbf{d}) \leq (\mathbf{He}, \mathbf{e}), \tag{3.76}$$

then the curve $\mathbf{p}(\varepsilon)$ lies in the instability domain for small $\varepsilon > 0$. The curves such that

$$(\mathbf{h}_1, \mathbf{d}) = (\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_2, \mathbf{e})(\mathbf{h}_3, \mathbf{e}) \quad \text{or} \quad (\mathbf{h}_1, \mathbf{d}) = (\mathbf{He}, \mathbf{e}) \tag{3.77}$$

approximate the stability boundary.

Example 3.3 Let us consider a two degrees of freedom pendulum loaded by a follower force, see Fig. 3.18. It is assumed that at the joints of the pendulum visco-elastic restoring moments appear, and gravitational forces are neglected. This system was studied in [Ziegler (1952)] and in the present extended version with two different damping parameters in [Herrmann and Jong (1965)]. A boundary surface of the stability domain was plotted and studied in [Seyranian and Pedersen (1995)]. We consider this example from the point of view of singularities of the stability boundary and show that the effects known as destabilization due to damping [Ziegler (1952)] and uncertainty of the critical load as damping parameters tend to zero [Seyranian (1996)] are closely related to the "deadlock of an edge" singularity. This singularity takes place at the point of the critical force of the system with no damping. At lower values of the follower force we have the dihedral angle singularity.

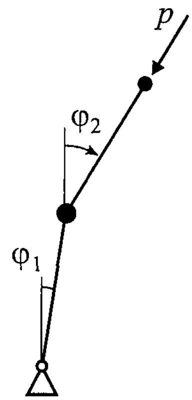


Fig. 3.18 Double pendulum with a follower force.

Linearized equations of motion of the pendulum in non-dimensional variables are [Herrmann and Jong (1965)]

$$\begin{aligned}&\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\ddot{\varphi}_1\\ \ddot{\varphi}_2\end{pmatrix}+\begin{pmatrix}\gamma_1+\gamma_2 & -\gamma_2\\ -\gamma_2 & \gamma_2\end{pmatrix}\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix}\\ &\quad+\begin{pmatrix}2-p & p-1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=0,\end{aligned} \tag{3.78}$$

where γ_1 and γ_2 are independent nonnegative damping parameters, and p

is magnitude of the follower force. Introducing the variables $\varphi_3 = \dot{\varphi}_1$ and $\varphi_4 = \dot{\varphi}_2$, equations (3.78) take the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3.79}$$

where

$$\mathbf{x} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p/2 - 3/2 & 1 - p/2 & -\gamma_1/2 - \gamma_2 & \gamma_2 \\ 5/2 - p/2 & p/2 - 2 & \gamma_1/2 + 2\gamma_2 & -2\gamma_2 \end{pmatrix}. \tag{3.80}$$

We investigate the stability domain of the system in the space of three parameters $\mathbf{p} = (\gamma_1, \gamma_2, p)$. The characteristic equation of system (3.79), (3.80) is

$$\lambda^4 + \frac{\gamma_1 + 6\gamma_2}{2}\lambda^3 + \frac{7 - 2p + \gamma_1\gamma_2}{2}\lambda^2 + \frac{\gamma_1 + \gamma_2}{2}\lambda + \frac{1}{2} = 0. \tag{3.81}$$

At $\gamma_1 = \gamma_2 = 0$ (system without damping) we find

$$\lambda^2 = \frac{1}{2}\left(p - \frac{7}{2} \pm \sqrt{\left(p - \frac{7}{2}\right)^2 - 2}\right). \tag{3.82}$$

Hence, at $0 \le p < 7/2 - \sqrt{2}$ we have two different pairs of purely imaginary simple eigenvalues corresponding to the edge $(\pm i\omega_1)(\pm i\omega_2)$ of the stability boundary. At $p_0 = 7/2 - \sqrt{2}$ there exists a pair of double eigenvalues $\lambda = \pm i\omega$ with Jordan chains of length 2. Hence, this point corresponds to the singularity "deadlock of an edge" $(\pm i\omega)^2$. Therefore, the segment $\gamma_1 = \gamma_2 = 0$, $0 \le p \le p_0$ is an edge of the stability boundary with the deadlock at the point $p = p_0$; see Fig. 3.19.

In the neighborhood of the point $\mathbf{p} = (0, 0, p)$, $0 \le p < p_0$, on the edge $(\pm i\omega_1)(\pm i\omega_2)$ an approximation of the stability domain is given by Theorem 3.6 (page 104) in the form

$$(\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0, \tag{3.83}$$

where the vectors $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ are determined by expression (3.36) for two different purely imaginary eigenvalues $i\omega_1$ and $i\omega_2$. Evaluating the vectors

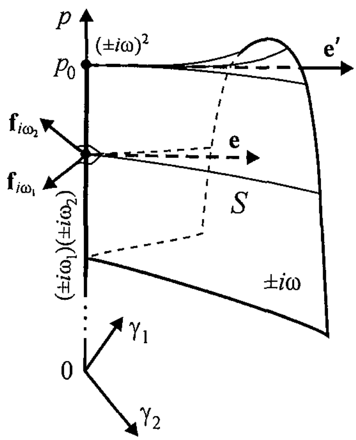


Fig. 3.19 Stability domain of the double pendulum.

$\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ for matrix (3.80), we find

$$\mathbf{f}_{i\omega_1} = \frac{1}{8}\begin{pmatrix} \dfrac{3/2-p}{\sqrt{(p-7/2)^2-2}} - 1 \\ \dfrac{19-6p}{\sqrt{(p-7/2)^2-2}} - 6 \\ 0 \end{pmatrix}, \quad \mathbf{f}_{i\omega_2} = \frac{1}{8}\begin{pmatrix} \dfrac{p-3/2}{\sqrt{(p-7/2)^2-2}} - 1 \\ \dfrac{6p-19}{\sqrt{(p-7/2)^2-2}} - 6 \\ 0 \end{pmatrix} \tag{3.84}$$

The angle between the vectors $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ (equal to the difference of π and the size of the dihedral angle) increases with an increase of p from zero and tends to π as $p \to p_0$. But at $p = p_0$ the vectors $\mathbf{f}_{i\omega_1}$ and $\mathbf{f}_{i\omega_2}$ become infinite, because denominators in (3.84) vanish. At this point the frequencies ω_1 and ω_2 merge, and the edge $(\pm i\omega_1)(\pm i\omega_2)$ ends up with appearance of the "deadlock of an edge" singularity $(\pm i\omega)^2$; see Fig. 3.19.

The first order approximation of the stability domain near the point

$$\gamma_1 = \gamma_2 = 0, \quad p = p_0 \tag{3.85}$$

of the "deadlock of an edge" singularity has the form (3.57), where the real vectors $\mathrm{Re}\,\mathbf{g}_1$, $\mathrm{Im}\,\mathbf{g}_1$, and $\mathrm{Re}\,\mathbf{g}_2$ are determined by expressions (3.44). For matrix (3.80) these vectors, up to a positive scaling factor, take the form

$$\mathrm{Re}\,\mathbf{g}_1 = (0,0,1), \quad \mathrm{Im}\,\mathbf{g}_1 = (1,-4-5\sqrt{2},0), \quad \mathrm{Re}\,\mathbf{g}_2 = (-1,-6,0). \tag{3.86}$$

The approximation of the stability domain in the neighborhood of point

(3.85) is represented by the plane angle

$$\gamma_1 = (4 + 5\sqrt{2})\gamma_2, \quad \gamma_2 > 0, \quad p < p_0. \tag{3.87}$$

Any curve $\mathbf{p}(\varepsilon)$ starting at point (3.85) with the direction $\mathbf{e} = ((4 + 5\sqrt{2})\zeta_1,\ \zeta_1,\ -\zeta_2)$, where ζ_1 and ζ_2 are arbitrary positive real numbers, belongs to the stability domains for small $\varepsilon > 0$. A curve with the direction $\mathbf{e}' = ((4+5\sqrt{2}),\ 1,\ 0)$ is tangent to the upper part of the stability boundary surface, see Fig. 3.19.

At fixed values of the damping parameters γ_1, γ_2, a critical force $p_{cr}(\gamma_1, \gamma_2)$ is defined as the smallest value of p at which the system becomes unstable. Let us consider damping in the form $\gamma_1 = e_1\varepsilon$, $\gamma_2 = e_2\varepsilon$, where ε is a small positive number. Since the segment $\gamma_1 = \gamma_2 = 0$, $0 \le p \le p_0$ is the edge of the stability boundary, the limit of the critical load as damping tends to zero,

$$p_d = \lim_{\varepsilon \to 0} p_{cr}(e_1\varepsilon, e_2\varepsilon) \tag{3.88}$$

for a fixed direction (e_1, e_2) is equal to the value of p at which the vector $\mathbf{e} = (e_1, e_2, 0)$ leaves the dihedral angle $(\pm i\omega_1)(\pm i\omega_2)$ with an increase of p from zero. In this case either the condition $(\mathbf{f}_{i\omega_1}, \mathbf{e}) = 0$ or $(\mathbf{f}_{i\omega_2}, \mathbf{e}) = 0$ is fulfilled. For example, considering $\gamma_1 = \varepsilon$ and $\gamma_2 = 0$, we have $\mathbf{e} = (1, 0, 0)$, $p_d = 2$, $\mathbf{f}_{i\omega_2} = (0, -5/2, 0)$, and $(\mathbf{f}_{i\omega_2}, \mathbf{e}) = 0$. From this argument we conclude that the limit of the critical force p_d is different for various directions (e_1, e_2). For all $(e_1, e_2) \ne \zeta(4 + 5\sqrt{2}, 1)$, $\zeta > 0$, this limit is less than p_0. At $(e_1, e_2) = \zeta(4 + 5\sqrt{2}, 1)$ we have $p_d = p_0$. This is related to the fact that the direction $\mathbf{e}' = (4 + 5\sqrt{2}, 1, 0)$ is tangent to the upper part of the stability boundary; see Fig. 3.19.

Degeneration of a dihedral angle at a "deadlock of an edge" singular point geometrically illustrates the effects of destabilization of a non-conservative system by small dissipative forces and uncertainty of the critical load as damping parameters tend to zero [Ziegler (1952); Seyranian and Pedersen (1995); Seyranian (1996)]. More detailed study of the effect of dissipative forces on stability of non-conservative systems is presented in Section 8.3.

Chapter 4

Bifurcation Analysis of Roots and Stability of Characteristic Polynomial Dependent on Parameters

In this chapter we consider a linear ordinary differential equation of mth order, whose coefficients smoothly depend on parameters. Stability analysis for such equation is reduced to the study of roots of the characteristic polynomial. Asymptotic stability corresponds to polynomials, whose roots have negative real parts. First, we describe general methods and results of bifurcation analysis for roots of polynomials dependent on parameters. Then, we analyze the stability domain in the parameter space, describe a regular part of the stability boundary and derive local approximations of the stability domain in the neighborhood of regular points of the boundary. Using bifurcation analysis of the roots, we describe generic singularities of the stability boundary (for codimensions 2 and 3). To study an arbitrary singular point of the stability boundary we apply the Weierstrass preparation theorem, which provides local multi-parameter factorization of the characteristic polynomial. This technique is used for qualitative and quantitative study of the stability domain near singularities. As a result, stabilizing directions in the parameter space are found explicitly using only values and derivatives of polynomial coefficients with respect to parameters at the stability boundary point under consideration.

One-parameter perturbation theory for multiple roots of a polynomial is based on the method of Newton diagrams, see [Newton (1860); Vainberg and Trenogin (1974); Baumgärtel (1984)]. Classification of singularities of the stability boundary for families of polynomials was done in [Levantovskii (1980b)]. Quantitative multi-parameter analysis of bifurcations of roots and stability analysis of polynomials presented in this chapter follow the papers by [Mailybaev and Seyranian (1999b); Mailybaev (2000a); Grigoryan and Mailybaev (2001)].

4.1 Stability of ordinary differential equation of mth order

Let us consider an ordinary differential equation of mth order

$$a_m x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_1 \dot{x} + a_0 x = 0, \tag{4.1}$$

where x is a real variable, $a_0, \ldots, a_m$ are real time-independent coefficients, and derivatives are taken with respect to time t. Looking for a solution of this problem in the form

$$x(t) = \exp \lambda t, \tag{4.2}$$

we obtain the characteristic equation

$$a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{4.3}$$

If $a_m \neq 0$, then equation (4.3) has m roots λ, counting multiplicities.

If λ is a real root of multiplicity k, then

$$\begin{aligned} x_1(t) &= \exp \lambda t, \\ x_2(t) &= t \exp \lambda t, \\ &\vdots \\ x_k(t) &= t^{k-1} \exp \lambda t \end{aligned} \tag{4.4}$$

are linearly independent solutions of equation (4.1). For a pair of complex conjugate roots λ and $\overline{\lambda}$ of multiplicity k, we find $2k$ linearly independent real solutions taking real and imaginary parts of expressions (4.4). A linear combination of these solutions, taken for all the roots, represents a general solution of equation (4.1).

We say that the system is stable if any solution of equation (4.1) is bounded as $t \to +\infty$. If, in addition, $x(t) \to 0$ as $t \to +\infty$ the system is asymptotically stable. The form of solutions (4.4) yields the following stability criterion.

Theorem 4.1 *System (4.1) is asymptotically stable if and only if all the roots of characteristic equation (4.3) have negative real parts* $\operatorname{Re}\lambda < 0$.

System (4.1) is stable if and only if $\operatorname{Re}\lambda \leq 0$ *for all the roots of characteristic equation (4.3) and the roots with zero real parts are simple.*

System (4.1) is unstable if and only if there exists a root with positive real part $\operatorname{Re}\lambda > 0$ *or a multiple root with zero real part* $\operatorname{Re}\lambda = 0$.

Checking condition of asymptotic stability $\operatorname{Re}\lambda < 0$ does not require evaluating the roots of characteristic equation (4.3), but needs only signs of their real part. This property is used in the well-known *Routh-Hurwitz conditions* for the polynomial coefficients guaranteeing asymptotic stability of the system, see for example [Chetayev (1961); Merkin (1997)]:

Theorem 4.2 *Let us consider the polynomial*

$$a_m\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0, \quad a_m > 0, \tag{4.5}$$

and introduce the Hurwitz matrix

$$\mathbf{H} = \begin{pmatrix} a_{m-1} & a_{m-3} & a_{m-5} & \cdots & 0 \\ a_m & a_{m-2} & a_{m-4} & \cdots & 0 \\ 0 & a_{m-1} & a_{m-3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}. \tag{4.6}$$

Then, all the roots of polynomial (4.5) have negative real part if and only if all principal minors of matrix (4.6) are positive:

$$\begin{aligned} \Delta_1 &= a_{m-1} > 0, \\ \Delta_2 &= \det\begin{pmatrix} a_{m-1} & a_{m-3} \\ a_m & a_{m-2} \end{pmatrix} > 0, \\ \Delta_3 &= \det\begin{pmatrix} a_{m-1} & a_{m-3} & a_{m-5} \\ a_m & a_{m-2} & a_{m-4} \\ 0 & a_{m-1} & a_{m-3} \end{pmatrix} > 0, \\ &\vdots \\ \Delta_m &= \det\mathbf{H} = a_0\Delta_{m-1} > 0. \end{aligned} \tag{4.7}$$

For example, in case of the characteristic polynomial of third order

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \quad a_3 > 0, \tag{4.8}$$

asymptotic stability conditions take the form

$$a_2 > 0, \quad a_1a_2 - a_0a_3 > 0, \quad a_0 > 0. \tag{4.9}$$

Routh-Hurwitz conditions (4.7) are useful for stability analysis of equation (4.1) with fixed coefficients. But the use of these conditions for multi-parameter stability analysis of high order equation (4.1) would be difficult due to the complicated way, in which the coefficients enter inequalities (4.7).

Introducing the vector of dimension m

$$\mathbf{x} = \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(m-1)} \end{pmatrix}, \tag{4.10}$$

we write equation (4.1) in the form of a system of m first order differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0/a_m & -a_1/a_m & -a_2/a_m & \cdots & -a_{m-1}/a_m \end{pmatrix}. \tag{4.11}$$

The $m \times m$ matrix $\mathbf{A}$ in (4.11) is called the *companion matrix.* The characteristic equation of the companion matrix $\mathbf{A}$ coincides with (4.3) after multiplication by $(-1)^m a_m$.

It is easy to show that any eigenvalue λ of the matrix $\mathbf{A}$ has a single eigenvector

$$\mathbf{u} = \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{pmatrix}. \tag{4.12}$$

Hence, eigenvalues of the matrix $\mathbf{A}$ are simple or multiple nonderogatory. This property connects stability Theorems 4.1 and 1.1 (page 6).

Let us consider a linear system of ordinary differential equations of rth order

$$\mathbf{A}_r\mathbf{x}^{(r)} + \mathbf{A}_{r-1}\mathbf{x}^{(r-1)} + \cdots + \mathbf{A}_1\dot{\mathbf{x}} + \mathbf{A}_0\mathbf{x} = 0, \tag{4.13}$$

where $\mathbf{x}$ is a real vector of dimension s, and $\mathbf{A}_0, \ldots, \mathbf{A}_r$ are real time-independent $s \times s$ matrices. Looking for a solution of (4.13) in the form

$$\mathbf{x}(t) = \mathbf{u} \exp \lambda t, \tag{4.14}$$

we get the eigenvalue problem

$$(\lambda^r \mathbf{A}_r + \cdots + \lambda \mathbf{A}_1 + \mathbf{A}_0)\mathbf{u} = 0, \tag{4.15}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. Eigenvalues λ can be found from the characteristic equation

$$\det(\lambda^r \mathbf{A}_r + \cdots + \lambda \mathbf{A}_1 + \mathbf{A}_0) = 0, \tag{4.16}$$

which is the polynomial equation of the form (4.3) with $m = rs$. Notice that the coefficient $a_m = \det \mathbf{A}_r$ of the leading term in characteristic equation (4.16) is nonzero if the matrix $\mathbf{A}_r$ is nonsingular. An example of system (4.13) is a linear vibrational system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = 0, \tag{4.17}$$

see Section 1.6.

If $\det \mathbf{A}_r \neq 0$, system (4.13) can be transformed to a system of first order differential equations $\dot{\widetilde{\mathbf{x}}} = \widetilde{\mathbf{A}}\widetilde{\mathbf{x}}$, if we introduce the vector $\widetilde{\mathbf{x}}$ of dimension $m = rs$ and the $m \times m$ block matrix $\widetilde{\mathbf{A}}$ as

$$\widetilde{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \vdots \\ \mathbf{x}^{(r-1)} \end{pmatrix},$$

$$\widetilde{\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{I} \\ -\mathbf{A}_r^{-1}\mathbf{A}_0 & -\mathbf{A}_r^{-1}\mathbf{A}_1 & -\mathbf{A}_r^{-1}\mathbf{A}_2 & \cdots & -\mathbf{A}_r^{-1}\mathbf{A}_{r-1} \end{pmatrix}, \tag{4.18}$$

where $\mathbf{I}$ and 0 denote the identity and zero $s \times s$ matrices, respectively.

Notice that an eigenvalue λ of matrix (4.18) may have several corresponding eigenvectors $\mathbf{u}$ and, therefore, in a solution of equation (4.13) secular terms can appear.

Theorem 4.3 *System (4.13) is asymptotically stable if and only if all the roots of characteristic equation (4.16) have negative real parts* $\mathrm{Re}\,\lambda < 0$.

System (4.13) is stable if and only if $\operatorname{Re}\lambda \le 0$ *for all the roots of characteristic equation (4.16) and the roots with zero real parts are simple or semi-simple as eigenvalues of problem (4.15).*

System (4.13) is unstable if and only if there exists a root with positive real part $\operatorname{Re}\lambda > 0$ *or a multiple root with zero real part* $\operatorname{Re}\lambda = 0$ *which is neither simple nor semi-simple as eigenvalue of problem (4.15).*

4.2 Stability domain for characteristic polynomial dependent on parameters

Let us consider the characteristic polynomial

$$P(\lambda, \mathbf{p}) = a_m(\mathbf{p})\lambda^m + a_{m-1}(\mathbf{p})\lambda^{m-1} + \cdots + a_1(\mathbf{p})\lambda + a_0(\mathbf{p}), \tag{4.19}$$

where the coefficients $a_i(\mathbf{p})$, $i = 0, \ldots, m$, smoothly depend on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. For a given value of the parameter vector $\mathbf{p}$ the polynomial $P(\lambda, \mathbf{p})$ has m roots provided that $a_m(\mathbf{p}) \neq 0$. If the leading coefficients vanish at $\mathbf{p}$ such that

$$a_m(\mathbf{p}) = a_{m-1}(\mathbf{p}) = \cdots = a_{M+1}(\mathbf{p}) = 0, \quad a_M(\mathbf{p}) \neq 0, \tag{4.20}$$

then there are only M roots of the polynomial $\mathbf{P}(\lambda, \mathbf{p})$. In this case we say that there is an *infinite root* $\lambda = \infty$ of multiplicity $m - M$. We assume that the polynomial $P(\lambda, \mathbf{p})$ is not identically zero for any value of $\mathbf{p}$. Taking into account the infinite root, the polynomial $P(\lambda, \mathbf{p})$ has m roots, counting multiplicities, at any point $\mathbf{p}$. The infinite root corresponds to equation (4.1) such that the coefficients of the highest order terms vanish and the order of equation decreases.

The polynomial $P(\lambda, \mathbf{p})$ is called *stable* for a given $\mathbf{p}$, if all the roots of this polynomial are finite and have negative real parts $\operatorname{Re}\lambda < 0$, i.e., corresponding equation (4.1) is asymptotically stable. Notice that stable polynomials are structurally stable, since small perturbation of their coefficients keeps the stability property.

The stability domain for a multi-parameter family of polynomials $P(\lambda, \mathbf{p})$ is defined as a set of values of the parameter vector $\mathbf{p}$ corresponding to stable polynomials. A boundary of the stability domain is characterized by the points $\mathbf{p}$ such that the polynomial $P(\lambda, \mathbf{p})$ has zero, purely imaginary, or infinite roots, while the other roots have negative real parts. We denote type of a stability boundary point as product of zero, purely imaginary, and infinite roots in powers of their multiplicities

$$0^{k_0}(\pm i\omega_1)^{k_1}\cdots(\pm i\omega_l)^{k_l}\infty^{k_\infty}, \tag{4.21}$$

where $k_1 \geq \cdots \geq k_l > 0$ are multiplicities for different pairs of purely imaginary roots, k_0 and $k_\infty = m - M$ are multiplicities of zero and infinite roots, respectively. For example, $(\pm i\omega)\infty^2$ denotes a point $\mathbf{p}$, where the polynomial $P(\lambda, \mathbf{p})$ has a pair of purely imaginary simple roots $\lambda = \pm i\omega$ and a double infinite root $\lambda = \infty$, while the other roots satisfy the condition $\operatorname{Re}\lambda < 0$.

Example 4.1 Let us consider the polynomial

$$P(\lambda, \mathbf{p}) = p_1\lambda^2 + \lambda + p_2 \tag{4.22}$$

dependent on a vector of two parameters $\mathbf{p} = (p_1, p_2)$. The stability domain for this polynomial is given by the conditions

$$p_1 > 0, \quad p_2 > 0. \tag{4.23}$$

The stability boundary consists of the points of types 0, ∞, and 0∞; see Fig. 4.1.

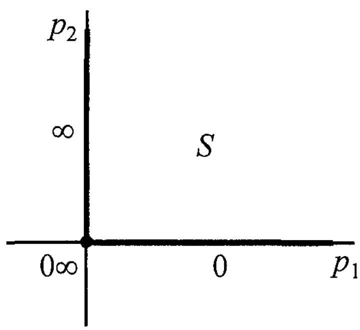


Fig. 4.1 Stability boundary for the polynomial $P(\lambda, \mathbf{p}) = p_1\lambda^2 + \lambda + p_2$.

4.3 Perturbation of simple roots

Let us consider a simple finite root λ of the polynomial $P(\lambda, \mathbf{p})$. By the implicit function theorem, this root is a smooth function of the parameter vector. Taking derivative of the equation

$$P(\lambda(\mathbf{p}), \mathbf{p}) = 0 \tag{4.24}$$

with respect to p_i, we find

$$\frac{\partial P}{\partial \lambda}\frac{\partial \lambda}{\partial p_i} + \frac{\partial P}{\partial p_i} = 0. \tag{4.25}$$

Since

$$\frac{\partial P}{\partial \lambda} = m a_m(\mathbf{p})\lambda^{m-1} + (m-1)a_{m-1}(\mathbf{p})\lambda^{m-2} + \cdots + a_1(\mathbf{p}) \neq 0 \tag{4.26}$$

for a simple root, we obtain the first order derivative of the root λ as follows

$$\frac{\partial \lambda}{\partial p_i} = -\frac{\partial P}{\partial p_i}\left(\frac{\partial P}{\partial \lambda}\right)^{-1}. \tag{4.27}$$

Taking the derivative $\partial^2/\partial p_i \partial p_j$ of equation (4.24), we find the second order derivative of the simple root in the form

$$\begin{aligned}\frac{\partial^2 \lambda}{\partial p_i \partial p_j} = -\Bigg(&\frac{\partial^2 P}{\partial p_i \partial p_j} + \frac{\partial^2 P}{\partial \lambda \partial p_i}\frac{\partial \lambda}{\partial p_j} + \frac{\partial^2 P}{\partial \lambda \partial p_j}\frac{\partial \lambda}{\partial p_i} \\ &+ \frac{\partial^2 P}{\partial \lambda^2}\frac{\partial \lambda}{\partial p_i}\frac{\partial \lambda}{\partial p_j}\Bigg)\left(\frac{\partial P}{\partial \lambda}\right)^{-1}.\end{aligned} \tag{4.28}$$

Analogously, higher order derivatives of a simple root with respect to parameters can be found.

If the root λ equals zero at the point $\mathbf{p}$ under consideration, expressions (4.27) and (4.28) take the form

$$\frac{\partial \lambda}{\partial p_i} = -\frac{1}{a_1}\frac{\partial a_0}{\partial p_i}, \tag{4.29}$$

$$\frac{\partial^2 \lambda}{\partial p_i \partial p_j} = -\frac{1}{a_1}\frac{\partial^2 a_0}{\partial p_i \partial p_j} + \frac{1}{a_1^2}\left(\frac{\partial a_0}{\partial p_i}\frac{\partial a_1}{\partial p_j} + \frac{\partial a_0}{\partial p_j}\frac{\partial a_1}{\partial p_i}\right) - \frac{2a_2}{a_1^3}\frac{\partial a_0}{\partial p_i}\frac{\partial a_0}{\partial p_j}. \tag{4.30}$$

Now, let us consider a point $\mathbf{p}$ in the parameter space, where the polynomial $P(\lambda, \mathbf{p})$ has a simple infinite root $\lambda = \infty$. At this point $a_m(\mathbf{p}) = 0$ and $a_{m-1}(\mathbf{p}) \neq 0$. Introducing a new variable

$$\mu = \frac{1}{\lambda}, \tag{4.31}$$

the polynomial $P(\lambda, \mathbf{p})$ is transformed to

$$\widetilde{P}(\mu, \mathbf{p}) = a_0(\mathbf{p})\mu^m + a_1(\mathbf{p})\mu^{m-1} + \cdots + a_{m-1}(\mathbf{p})\mu + a_m(\mathbf{p}), \tag{4.32}$$

which is the polynomial with the same coefficients taken in reverse order. The polynomials $P(\lambda, \mathbf{p})$ and $\widetilde{P}(\mu, \mathbf{p})$ are related by

$$P(\lambda, \mathbf{p}) = \frac{\widetilde{P}(\mu, \mathbf{p})}{\mu^m}. \tag{4.33}$$

The simple infinite root $\lambda = \infty$ is transformed to the simple zero root $\mu = 0$ of the polynomial $\widetilde{P}(\mu, \mathbf{p})$. Hence, the root μ is a smooth real function of the parameter vector with the derivatives

$$\frac{\partial \mu}{\partial p_i} = -\frac{1}{a_{m-1}} \frac{\partial a_m}{\partial p_i}, \tag{4.34}$$

$$\begin{aligned} \frac{\partial^2 \mu}{\partial p_i \partial p_j} = &-\frac{1}{a_{m-1}} \frac{\partial^2 a_m}{\partial p_i \partial p_j} + \frac{1}{a_{m-1}^2} \left(\frac{\partial a_m}{\partial p_i} \frac{\partial a_{m-1}}{\partial p_j} + \frac{\partial a_m}{\partial p_j} \frac{\partial a_{m-1}}{\partial p_i} \right) \\ &- \frac{2a_{m-2}}{a_{m-1}^3} \frac{\partial a_m}{\partial p_i} \frac{\partial a_m}{\partial p_j}. \end{aligned} \tag{4.35}$$

By relation (4.31), the corresponding root λ appears from infinity along the real axis as the parameter vector is changing.

4.4 Bifurcation analysis of multiple roots (nondegenerate case)

Let us consider a finite root λ_0 of the polynomial $P(\lambda, \mathbf{p})$ at a given point $\mathbf{p} = \mathbf{p}_0$ in the parameter space. We assume that the root λ_0 has multiplicity k, which implies

$$P = \frac{\partial P}{\partial \lambda} = \cdots = \frac{\partial^{k-1} P}{\partial \lambda^{k-1}} = 0, \quad \frac{\partial^k P}{\partial \lambda^k} \neq 0 \tag{4.36}$$

at $\lambda = \lambda_0$ and $\mathbf{p} = \mathbf{p}_0$. A multiple root is a nonsmooth function of the parameter vector. Under perturbation of parameters it generally splits into k simple roots. Let us consider variation of the parameter vector along the curve $\mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ and having a direction $\mathbf{e} = (e_1, \ldots, e_n) = d\mathbf{p}/d\varepsilon$ evaluated at $\mathbf{p}_0$, ε being a small positive parameter of the curve.

Then, the polynomial $\mathbf{P}(\lambda, \mathbf{p})$ takes the increment

$$
\begin{aligned}
P(\lambda, \mathbf{p}(\varepsilon)) = P(\lambda, \mathbf{p}_0) + \varepsilon \sum_{i=1}^{n} \frac{\partial P}{\partial p_i} e_i \\
+ \frac{\varepsilon^2}{2} \left(\sum_{i,j=1}^{n} \frac{\partial^2 P}{\partial p_i \partial p_j} e_i e_j + \sum_{i=1}^{n} \frac{\partial P}{\partial p_i} d_i \right) + \cdots ,
\end{aligned}
\tag{4.37}
$$

where derivatives are taken at λ_0 and $\mathbf{p}_0$, and $\mathbf{d} = (d_1, \ldots, d_n) = d^2\mathbf{p}/d\varepsilon^2$ evaluated at $\mathbf{p}_0$.

Under a nondegeneracy condition, that will be specified below, perturbation of the multiple root λ_0 can be represented in the form of the Newton-Puiseux series

$$
\lambda = \lambda_0 + \varepsilon^{1/k}\lambda_1 + \varepsilon^{2/k}\lambda_2 + \cdots \tag{4.38}
$$

Using expressions (4.36)–(4.38) in the characteristic equation $P(\lambda, \mathbf{p}) = 0$ and requiring the coefficients of each power of ε to be zero, we obtain a chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ as follows

$$
\begin{gathered}
P(\lambda_0, \mathbf{p}_0) = 0, \\
\frac{1}{k!} \frac{\partial^k P}{\partial \lambda^k} \lambda_1^k + \sum_{i=1}^{n} \frac{\partial P}{\partial p_i} e_i = 0, \\
\frac{1}{(k+1)!} \frac{\partial^{k+1} P}{\partial \lambda^{k+1}} \lambda_1^{k+1} + \frac{1}{(k-1)!} \frac{\partial^k P}{\partial \lambda^k} \lambda_1^{k-1}\lambda_2 + \sum_{i=1}^{n} \frac{\partial^2 P}{\partial \lambda \partial p_i} e_i \lambda_1 = 0, \\
\vdots
\end{gathered}
\tag{4.39}
$$

Solving the second equation in (4.39), we find

$$
\lambda_1 = \sqrt[k]{-k! \left(\frac{\partial^k P}{\partial \lambda^k} \right)^{-1} \sum_{i=1}^{n} \frac{\partial P}{\partial p_i} e_i}, \tag{4.40}
$$

where k different complex values of the root in (4.40) determine k different λ appearing due to bifurcation of λ_0. Assuming the condition

$$
\sum_{i=1}^{n} \frac{\partial P}{\partial p_i} e_i \neq 0, \tag{4.41}
$$

we solve the third equation in (4.39) as follows

$$\lambda_2 = -\frac{(k-1)!}{\lambda_1^{k-2}}\left(\frac{\partial^k P}{\partial \lambda^k}\right)^{-1}\left(\frac{1}{(k+1)!}\frac{\partial^{k+1} P}{\partial \lambda^{k+1}}\lambda_1^k + \sum_{i=1}^{n}\frac{\partial^2 P}{\partial \lambda \partial p_i}e_i\right). \quad (4.42)$$

This procedure can be continued to get the coefficients $\lambda_3, \lambda_4, \ldots$ in series (4.38). Inequality (4.41) represents the *nondegeneracy condition* for a curve direction $\mathbf{e}$.

For zero root $\lambda_0 = 0$, expressions (4.40) and (4.42) take the form

$$\lambda_1 = \sqrt[k]{-\frac{1}{a_k}\sum_{i=1}^{n}\frac{\partial a_0}{\partial p_i}e_i}\,, \quad (4.43)$$

$$\lambda_2 = \frac{1}{k a_k \lambda_1^{k-2}}\sum_{i=1}^{n}\left(\frac{a_{k+1}}{a_k}\frac{\partial a_0}{\partial p_i} - \frac{\partial a_1}{\partial p_i}\right)e_i \quad (4.44)$$

with the nondegeneracy condition as

$$\sum_{i=1}^{n}\frac{\partial a_0}{\partial p_i}e_i \neq 0. \quad (4.45)$$

In case of the infinite root $\lambda_0 = \infty$ we can study perturbation of a root $\mu_0 = 1/\lambda_0 = 0$ of multiplicity k for the reverse polynomial $\widetilde{P}(\mu, \mathbf{p})$ determined by expression (4.32). Formulae for perturbation of $\mu_0 = 0$ along the curve $\mathbf{p}(\varepsilon)$ can be obtained from (4.38), (4.43)–(4.45), if we substitute $a_0, a_1, \ldots, a_m$ and λ, $\lambda_0, \lambda_1, \ldots$ by $a_m, a_{m-1}, \ldots, a_0$ and μ, $\mu_0, \mu_1, \ldots$, respectively. As a result, we find the expansion

$$\mu = \mu_0 + \varepsilon^{1/k}\mu_1 + \varepsilon^{2/k}\mu_2 + \cdots, \quad (4.46)$$

where

$$\mu_1 = \sqrt[k]{-\frac{1}{a_{m-k}}\sum_{i=1}^{n}\frac{\partial a_m}{\partial p_i}e_i}, \quad (4.47)$$

$$\mu_2 = \frac{1}{k a_{m-k}\mu_1^{k-2}}\sum_{i=1}^{n}\left(\frac{a_{m-k-1}}{a_{m-k}}\frac{\partial a_m}{\partial p_i} - \frac{\partial a_{m-1}}{\partial p_i}\right)e_i, \quad (4.48)$$

and the nondegeneracy condition is

$$\sum_{i=1}^{n} \frac{\partial a_m}{\partial p_i} e_i \neq 0. \tag{4.49}$$

4.5 Bifurcation analysis of multiple roots (degenerate case)

For perturbation along a curve $\mathbf{p}(\varepsilon)$ with a direction $\mathbf{e}$ satisfying the *degeneration condition*

$$\sum_{i=1}^{n} \frac{\partial P}{\partial p_i} e_i = 0 \tag{4.50}$$

expansion (4.38) is, in general, invalid. Expansions for bifurcating roots are taken in fractional powers of ε, but the fractions can be different for different roots.

Let us consider a finite root λ_0 of multiplicity k of the polynomial $P(\lambda, \mathbf{p})$ (the infinite root is studied analogously by means of the substitution $\lambda = 1/\mu$; see Section 4.3). Perturbation of parameters along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ yields

$$P(\lambda, \mathbf{p}(\varepsilon)) = c_m(\varepsilon)\Delta\lambda^m + c_{m-1}(\varepsilon)\Delta\lambda^{m-1} + \cdots + c_1(\varepsilon)\Delta\lambda + c_0(\varepsilon), \tag{4.51}$$

where $\Delta\lambda = \lambda - \lambda_0$ and the coefficients are

$$c_i(\varepsilon) = \frac{1}{i!}\frac{\partial^i P}{\partial \lambda^i}, \quad i = 0, \ldots, m, \tag{4.52}$$

with the derivatives taken at $\lambda = \lambda_0$ and $\mathbf{p} = \mathbf{p}(\varepsilon)$. Let

$$c_i(\varepsilon) = \hat{c}_i \varepsilon^{\alpha_i} + \cdots, \quad i = 0, \ldots, m, \tag{4.53}$$

where α_i is the leading exponent of $c_i(\varepsilon)$, i.e., $\hat{c}_i \neq 0$ and no term of order lower than α_i appears in the expansion of $c_i(\varepsilon)$.

The roots of (4.51) are given by expansions in fractional powers of ε. The leading exponents can be found by the following geometric construction: we plot the points $A_0 = (m, \alpha_0)$, $A_1 = (m-1, \alpha_1)$, ..., $A_m = (0, \alpha_m)$ on a plane (if $c_i(\varepsilon) \equiv 0$, the corresponding point A_i is disregarded). Then we draw the segments on the lower boundary of the convex hull of the plotted points. These segments constitute the so-called *Newton diagram* associated with polynomial (4.51); see Fig. 4.2. Slopes of the segments on the Newton diagram are precisely the leading powers of the ε-expansions for the roots

$\Delta\lambda = \Delta\lambda(\varepsilon)$ of (4.51); see [Newton (1860); Vainberg and Trenogin (1974); Moro *et al.* (1997)]. The number of roots corresponding to each slope equals the length of the projection on the horizontal axis of the segment with that particular slope.

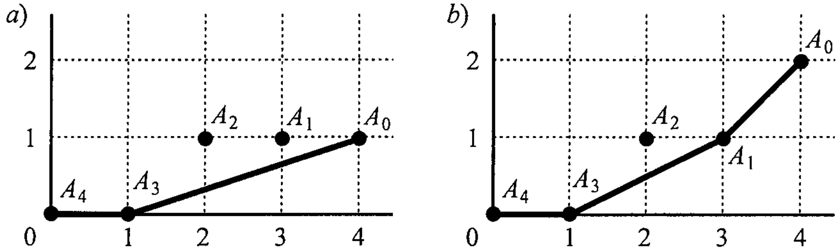


Fig. 4.2 Newton diagrams associated with the polynomials: a) $\lambda^4 + \lambda^3 + \varepsilon\lambda^2 + 2\varepsilon\lambda + 3\varepsilon + \varepsilon^2$, b) $\lambda^4 + \lambda^3 + \varepsilon\lambda^2 + 2\varepsilon\lambda + \varepsilon^2$.

By conditions (4.36), the point $A_k = (m - k, 0)$, while the points $A_0, \ldots, A_{k-1}$ lie over the horizontal axis. In the nondegenerate case (4.41) the point $A_0 = (m, 1)$, which determines a segment A_kA_0 of slope $1/k$ and horizontal projection of length k; see for example Fig. 4.2a, where $\lambda_0 = 0$ and $k = 3$. Therefore, there are k roots $\Delta\lambda$ of polynomial (4.51) whose expansions start with $\varepsilon^{1/k}$. These expansions determine bifurcation of the multiple root λ_0 along the curve $\mathbf{p}(\varepsilon)$ as described in the previous section. In degenerate case (4.50) we have $\alpha_0 > 1$, i.e., the point $A_0 = (m, \alpha_0)$ is push up and the form of the Newton diagram depends on the exponents $\alpha_0, \ldots, \alpha_{k-1}$; see Fig. 4.2b.

The underlying idea of the Newton diagram if the following. Let us consider expansion of $\Delta\lambda$ in fractional powers of ε with the leading term

$$\Delta\lambda = \mu_1\varepsilon^{\beta_1} + \cdots, \tag{4.54}$$

where μ_1 and β_1 are to be determined. Substituting expansions (4.53) and (4.54) into (4.51), we find

$$\hat{c}_m\mu_1^m\varepsilon^{\alpha_m + m\beta_1} + \hat{c}_{m-1}\mu_1^{m-1}\varepsilon^{\alpha_{m-1}+(m-1)\beta_1} + \cdots + \hat{c}_1\mu_1\varepsilon^{\alpha_1+\beta_1} + \hat{c}_0\varepsilon^{\alpha_0}$$
$$+ \cdots = 0. \tag{4.55}$$

Every point $A_i = (m - i, \alpha_i)$ corresponds to the term $\varepsilon^{\alpha_i + i\beta_1}$ in equation (4.55). If $\Delta\lambda(\varepsilon)$ is a root of (4.51), all the terms we obtain from (4.55) must cancel each other. Hence, at least two terms of the lowest order in ε must be present. This lowest order is clearly to be found among the exponents

$$\alpha_m + m\beta_1,\ \alpha_{m-1} + (m-1)\beta_1,\ \dots,\ \alpha_1 + \beta_1, \alpha_0. \tag{4.56}$$

Consider the segment S of the Newton diagram with the slope s and choose $\beta_1 = s$ in (4.54). All the points A_i lying on S give rise to terms with the same exponents since $\alpha_i + is$ is constant on S. The fact that no point A_i lies below S implies that no other term of expansion (4.55) can be of lower order in ε.

The leading coefficients μ_1 are determined as the solutions of

$$\sum_{(m-i,\alpha_i)\in S} \mu_1^i \hat{c}_i = 0, \tag{4.57}$$

where the sum is taken over all the points $A_i = (m-i, \alpha_i)$ lying on the segment S. A number of nonzero roots μ_1 of equation (4.57), counting multiplicities, is equal to the length of the horizontal projection of the segment S. Taking one of these roots μ_1, we can find the next term of the expansion

$$\Delta\lambda = \mu_1 \varepsilon^{\beta_1} + \mu_2 \varepsilon^{\beta_2} + \cdots, \tag{4.58}$$

where $\beta_2 > \beta_1$ and μ_2 are to be determined by substitution into equation (4.51) and comparison of coefficients for terms of the lowest order in ε.

Starting with slopes $\beta_1 > 0$, this procedure can be continued to find k expansions describing bifurcation of the multiple root $\lambda = \lambda_0$ of the polynomial $P(\lambda, \mathbf{p})$ along the curve $\mathbf{p}(\varepsilon)$.

Example 4.2 Let λ_0 be a triple root of the polynomial $P(\lambda, \mathbf{p})$. Consider perturbation along a curve $\mathbf{p}(\varepsilon)$ with degenerate direction (4.50) and assume that

$$\begin{aligned}
\alpha_0 &= 2, \quad \hat{c}_0 = \frac{1}{2}\frac{d^2 c_0}{d\varepsilon^2} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 P}{\partial p_i \partial p_j} e_i e_j + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial P}{\partial p_i} d_i \neq 0, \\
\alpha_1 &= 1, \quad \hat{c}_1 = \frac{dc_1}{d\varepsilon} = \sum_{i=1}^{n} \frac{\partial^2 P}{\partial \lambda \partial p_i} e_i \neq 0, \\
\alpha_2 &= 1, \quad \hat{c}_2 = \frac{dc_2}{d\varepsilon} = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^3 P}{\partial \lambda^2 \partial p_i} e_i \neq 0
\end{aligned} \tag{4.59}$$

with $d_i = d^2 p_i / d\varepsilon^2$ evaluated at $\varepsilon = 0$. Conditions (4.36), (4.50), and (4.59) determine the points $A_0 = (m,\ 2)$, $A_1 = (m-1,\ 1)$, $A_2 = (m-2,\ 1)$, and $A_3 = (m-3, 0)$; see Fig. 4.3. There are two segments on the Newton diagram with positive slopes: $S_1 = A_3A_1$ with slope 1/2 and length 2 of the horizontal projection, and $S_2 = A_1A_0$ with slope 1 and length 1 of the horizontal projection. Solving equation (4.57) for these two segments, we find the expansions

$$\begin{aligned} \lambda &= \lambda_0 + \mu_1 \varepsilon^{1/2} + \dots, \\ \lambda &= \lambda_0 + \nu_1 \varepsilon + \dots, \end{aligned} \tag{4.60}$$

where

$$\nu_1 = -\frac{\hat{c}_0}{\hat{c}_1} = -\frac{1}{2}\left(\sum_{i,j=1}^{n} \frac{\partial^2 P}{\partial p_i \partial p_j} e_i e_j + \sum_{i=1}^{n} \frac{\partial P}{\partial p_i} d_i\right)\left(\sum_{i=1}^{n} \frac{\partial^2 P}{\partial \lambda \partial p_i} e_i\right)^{-1} \tag{4.61}$$

and μ_1 takes two different values

$$\mu_1 = \pm\left(-\frac{\hat{c}_1}{\hat{c}_3}\right)^{1/2} = \pm\left(-6\left(\frac{\partial^3 P}{\partial \lambda^3}\right)^{-1} \sum_{i=1}^{n} \frac{\partial^2 P}{\partial \lambda \partial p_i} e_i\right)^{1/2}. \tag{4.62}$$

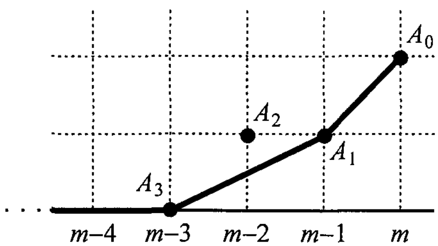


Fig. 4.3 Newton diagram for perturbation of parameters along degenerate direction.

Expressions (4.60)–(4.62) describe bifurcation of the triple root λ_0. In order to find the next term of the first expansion in (4.60), we substitute $\Delta\lambda = \mu_1\varepsilon^{1/2} + \mu_2\varepsilon^{\beta_2} + \cdots$ into equation (4.51). Using conditions (4.59) and (4.62), we find the lowest order terms

$$c_4(0)\mu_1^4\varepsilon^2 + 3\hat{c}_3\mu_1^2\mu_2\varepsilon^{1+\beta_2} + \hat{c}_2\mu_1^2\varepsilon^2 + \hat{c}_1\mu_2\varepsilon^{1+\beta_2} + \hat{c}_0\varepsilon^2 = 0. \tag{4.63}$$

Hence, the exponent $\beta_2 = 1$ and the first expansion in (4.60) is extended as

$$\lambda = \lambda_0 + \mu_1\varepsilon^{1/2} + \mu_2\varepsilon + \dots, \tag{4.64}$$

where the coefficient μ_2 is found from (4.63) with the use of expressions (4.61) and (4.62) in the form

$$\begin{aligned}\mu_2 &= \frac{1}{2}\left(\frac{\hat{c}_0}{\hat{c}_1} + \frac{c_4(0)\hat{c}_1}{\hat{c}_3^2} - \frac{\hat{c}_2}{\hat{c}_3}\right) \\ &= \frac{1}{2}\left(-\nu_1 + 3\left(\frac{\partial^3 P}{\partial \lambda^3}\right)^{-2} \sum_{i=1}^{n} \left(\frac{1}{2}\frac{\partial^4 P}{\partial \lambda^4}\frac{\partial^2 P}{\partial \lambda \partial p_i} - \frac{\partial^3 P}{\partial \lambda^3}\frac{\partial^3 P}{\partial \lambda^2 \partial p_i}\right) e_i\right).\end{aligned} \tag{4.65}$$

In case of zero triple root $\lambda_0 = 0$, expressions (4.61), (4.62), and (4.65) take the form

$$\nu_1 = -\frac{1}{2}\left(\sum_{i,j=1}^{n} \frac{\partial^2 a_0}{\partial p_i \partial p_j} e_i e_j + \sum_{i=1}^{n} \frac{\partial a_0}{\partial p_i} d_i\right)\left(\sum_{i=1}^{n} \frac{\partial a_1}{\partial p_i} e_i\right)^{-1}, \tag{4.66}$$

$$\mu_1 = \pm\left(-\frac{1}{a_3}\sum_{i=1}^{n} \frac{\partial a_1}{\partial p_i} e_i\right)^{1/2}, \tag{4.67}$$

$$\mu_2 = \frac{1}{2}\left(-\nu_1 + \frac{1}{a_3^2}\sum_{i=1}^{n}\left(a_4 \frac{\partial a_1}{\partial p_i} - a_3 \frac{\partial a_2}{\partial p_i}\right) e_i\right). \tag{4.68}$$

If $P(\lambda, \mathbf{p})$ is the characteristic polynomial of the matrix family $\mathbf{A}(\mathbf{p})$ and λ_0 is a triple nonderogatory eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$, degeneracy condition (4.50) takes the form of the first equation in (2.152). Then expansions (4.60)–(4.62), (4.64), (4.65) can be written in terms of the right and left Jordan chains of λ_0 and derivatives of the matrix $\mathbf{A}(\mathbf{p})$ with respect to parameters; see Example 2.9 (page 53).

4.6 Regular part of stability boundary

Let us consider stability boundary points $\mathbf{p}$ of types

$$0, \quad \pm i\omega, \quad \infty \tag{4.69}$$

represented by simple roots $\lambda = 0$, $\lambda = \pm i\omega$, and $\lambda = \infty$, respectively. In each case we introduce the vectors

$$\begin{aligned} 0: \quad & \mathbf{f}_0 = -\frac{\nabla a_0}{a_1}, \\ (\pm i\omega): \quad & \mathbf{f}_{i\omega} = -\operatorname{Re} \frac{\nabla a_m (i\omega)^m + \cdots + \nabla a_1 i\omega + \nabla a_0}{m a_m (i\omega)^{m-1} + \cdots + 2a_2 i\omega + a_1}, \\ \infty: \quad & \mathbf{f}_\infty = -\frac{\nabla a_m}{a_{m-1}}, \end{aligned} \tag{4.70}$$

where

$$\nabla = \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \right) \tag{4.71}$$

is the gradient operator evaluated at the stability boundary point under consideration. From expressions (4.27), (4.29), and (4.34) we see that the vector $\mathbf{f}_0$ is the gradient of zero root $\lambda = 0$, $\mathbf{f}_{i\omega}$ is the gradient of the real part of the root $\lambda = i\omega$, and $\mathbf{f}_\infty$ is the gradient of the inverse of infinite root $\mu = 1/\lambda = 0$.

The local stability condition is given by the inequality

$$\operatorname{Re} \lambda(\mathbf{p}) < 0 \tag{4.72}$$

for the zero or purely imaginary root, and by the inequality

$$\mu(\mathbf{p}) < 0 \tag{4.73}$$

in case of the infinite root. Using instead of the functions $\operatorname{Re} \lambda(\mathbf{p})$ and $\mu(\mathbf{p})$ their linear approximations with vectors (4.70), we formulate the following statement.

Theorem 4.4 *Let* $\mathbf{p}$ *be a stability boundary point of one of the types listed in (4.69), and assume that corresponding vector (4.70) is nonzero. Then, the stability boundary is a smooth surface in the neighborhood of the point* $\mathbf{p}$ *with corresponding vector (4.70) being the normal vector to the stability boundary directed into the instability domain; see Fig. 4.4. The stability boundary of types* 0 *and* ∞ *corresponds to divergence instability, while the type* $\pm i\omega$ *corresponds to flutter instability.*

Types listed in (4.69) represent a regular part of the stability boundary. There are two types of stability boundary points corresponding to divergence instability. For the first type 0, the development of instability

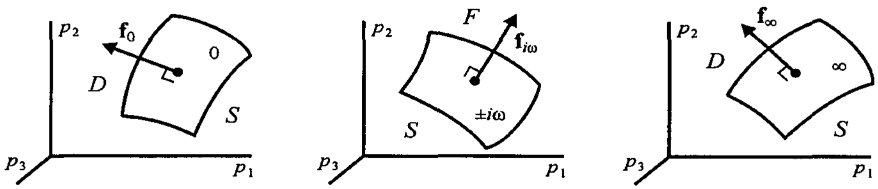


Fig. 4.4 Regular part of the stability boundary and its normal vectors.

is governed by a simple real root crossing the imaginary axis through the origin as we cross the stability boundary. For the second type ∞, a simple real root tends to $-\infty$ as we approach the stability boundary, and then comes from $+\infty$ after crossing the stability boundary. The type $\pm i\omega$ corresponds to flutter instability, when two complex conjugate roots cross the imaginary axis.

Evaluating second order derivatives of the roots $\lambda = 0$, $\lambda = i\omega$, and $\mu = 0$ by formulae (4.28), (4.30), and (4.35), respectively, we find the second order approximation of the stability domain and its boundary by means of inequalities (4.72) and (4.73).

4.7 Singularities of stability boundary (codimension 2 and 3)

Bifurcation analysis of roots of the characteristic equation allows quantitative study of the stability domain in the neighborhood of singular points of its boundary. Expansions for simple, double, and triple roots have the form analogous to those for simple, double, and triple nonderogatory eigenvalues for the matrix family $\mathbf{A}(\mathbf{p})$. The principal difference between the polynomial and matrix cases is the infinite root, which can appear at the stability boundary point. Nevertheless, perturbation of the inverse variable $\mu = 1/\lambda$ for the infinite root is similar to perturbation of simple or nonderogatory zero eigenvalue of the same multiplicity.

In the case of general position, there are following types of singular points of the stability boundary for codimensions 2 and 3 [Levantovskii

(1980b)]:

$$\begin{aligned}
\mathrm{cod}\,2: \quad & 0^2,\ \ 0(\pm i\omega),\ \ 0\infty,\ \ (\pm i\omega_1)(\pm i\omega_2),\ \ (\pm i\omega)\infty,\ \ \infty^2; \\
\mathrm{cod}\,3: \quad & 0^2(\pm i\omega),\ \ 0^2\infty,\ \ 0(\pm i\omega_1)(\pm i\omega_2),\ \ 0(\pm i\omega)\infty, \\
& (\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3),\ \ (\pm i\omega_1)(\pm i\omega_2)\infty, \\
& 0\infty^2,\ \ (\pm i\omega)\infty^2,\ \ (\pm i\omega)^2,\ \ 0^3,\ \ \infty^3.
\end{aligned} \tag{4.74}$$

Considering zero, purely imaginary, and infinite roots of multiplicities 2 and 3, we introduce the vectors

$$\begin{aligned}
0^2: \quad & \mathbf{g}_1 = -\frac{\nabla a_0}{a_2}, \quad \mathbf{g}_2 = \frac{a_3 \nabla a_0 - a_2 \nabla a_1}{a_2^2}; \\
(i\omega)^2: \quad & \mathbf{g}_1 = -2\left(\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1} \nabla P, \\
& \mathbf{g}_2 = 2\left(\frac{1}{3}\frac{\partial^3 P}{\partial \lambda^3}\nabla P - \frac{\partial^2 P}{\partial \lambda^2}\nabla \frac{\partial P}{\partial \lambda}\right)\left(\frac{\partial^2 P}{\partial \lambda^2}\right)^{-2}; \\
\infty^2: \quad & \mathbf{g}_1' = -\frac{\nabla a_m}{a_{m-2}}, \quad \mathbf{g}_2' = \frac{a_{m-3}\nabla a_m - a_{m-2}\nabla a_{m-1}}{a_{m-2}^2}; \\
0^3: \quad & \mathbf{h}_1 = -\frac{\nabla a_0}{a_3}, \quad \mathbf{h}_2 = \frac{a_4 \nabla a_0 - a_3 \nabla a_1}{a_3^2}, \\
& \mathbf{h}_3 = \frac{(a_3 a_5 - a_4^2)\nabla a_0 + a_3 a_4 \nabla a_1 - a_3^2 \nabla a_2}{a_3^3}; \\
\infty^3: \quad & \mathbf{h}_1' = -\frac{\nabla a_m}{a_{m-3}}, \quad \mathbf{h}_2' = \frac{a_{m-4}\nabla a_m - a_{m-3}\nabla a_{m-1}}{a_{m-3}^2}, \\
& \mathbf{h}_3' = \frac{(a_{m-3}a_{m-5} - a_{m-4}^2)\nabla a_m + a_{m-3}a_{m-4}\nabla a_{m-1}}{a_{m-3}^3} \\
& \qquad - \frac{\nabla a_{m-2}}{a_{m-3}}.
\end{aligned} \tag{4.75}$$

Vectors (4.75) determine expansions of double and triple roots along a curve $\mathbf{p}(\varepsilon)$; see Sections 4.4 and 4.5. In terms of vectors (4.75) these expansions coincide with those for double and triple nonderogatory eigenvalues of matrices; see Sections 3.6 and 3.7. Therefore, stability analysis in the neighborhood of singularities of the stability boundary is carried out in the same way as in Sections 3.6 and 3.7. As a result, we find the following description of singularities of the stability boundary for codimensions 2 and 3.

Theorem 4.5 *In the case of general position the stability boundary of a family of polynomials* $P(\lambda, \mathbf{p})$ *has singularities of codimension 2:* 0^2, $0(\pm i\omega)$, 0∞, $(\pm i\omega_1)(\pm i\omega_2)$, $(\pm i\omega)\infty$, *and* ∞^2, *which are (dihedral) angles, and singularities of codimension 3: trihedral angles* $0^2(\pm i\omega)$, $0^2\infty$, $0(\pm i\omega_1)(\pm i\omega_2)$, $0(\pm i\omega)\infty$, $(\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3)$, $(\pm i\omega_1)(\pm i\omega_2)\infty$, $0\infty^2$, $(\pm i\omega)\infty^2$, *"deadlock of an edge"* $(\pm i\omega)^2$, *and "breaks of an edge"* 0^3 *and* ∞^3. *First order approximations of the stability domain in the neighborhood of singular points are given by the relations*

$$\begin{aligned}
0^2 :&\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2, \Delta\mathbf{p}) < 0;\\
0(\pm i\omega) :&\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0;\\
0\infty :&\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_\infty, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)(\pm i\omega_2) :&\quad (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;\\
(\pm i\omega)\infty :&\quad (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_\infty, \Delta\mathbf{p}) < 0;\\
\infty^2 :&\quad (\mathbf{g}'_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}'_2, \Delta\mathbf{p}) < 0;\\
0^2(\pm i\omega) :&\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0;\\
0^2\infty :&\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_\infty, \Delta\mathbf{p}) < 0;\\
0(\pm i\omega_1)(\pm i\omega_2) :&\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;\\
0(\pm i\omega)\infty :&\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_\infty, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)(\pm i\omega_2)(\pm i\omega_3) :&\quad (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_3}, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)(\pm i\omega_2)\infty :&\quad (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_\infty, \Delta\mathbf{p}) < 0;\\
0\infty^2 :&\quad (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{g}'_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}'_2, \Delta\mathbf{p}) < 0;\\
(\pm i\omega)\infty^2 :&\quad (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0,\ (\mathbf{g}'_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}'_2, \Delta\mathbf{p}) < 0;\\
(\pm i\omega)^2 :&\quad (\mathrm{Re}\,\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathrm{Im}\,\mathbf{g}_1, \Delta\mathbf{p}) = 0,\\
&\quad (\mathrm{Re}\,\mathbf{g}_2, \Delta\mathbf{p}) < 0;\\
0^3 :&\quad (\mathbf{h}_1, \Delta\mathbf{p}) = 0,\ (\mathbf{h}_2, \Delta\mathbf{p}) < 0,\ (\mathbf{h}_3, \Delta\mathbf{p}) < 0;\\
\infty^3 :&\quad (\mathbf{h}'_1, \Delta\mathbf{p}) = 0,\ (\mathbf{h}'_2, \Delta\mathbf{p}) < 0,\ (\mathbf{h}'_3, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{4.76}$$

Vectors for each approximation in (4.76) are linearly independent in the case of general position.

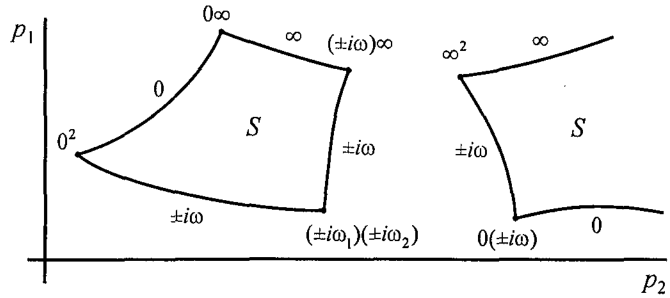


Fig. 4.5 Singularities of the stability boundary in the two-parameter space.

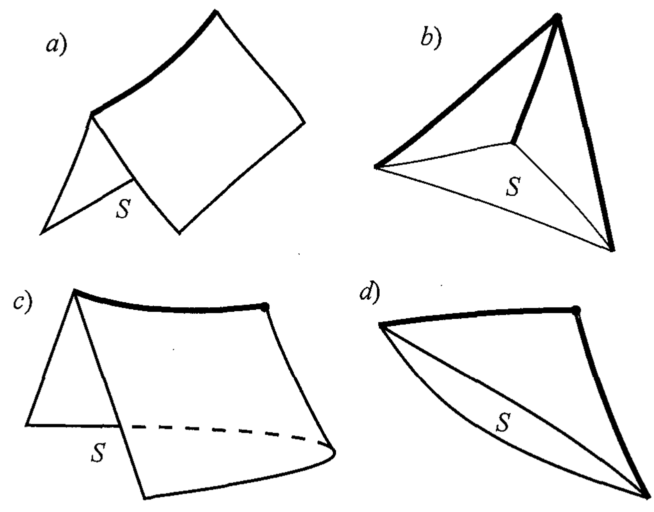


Fig. 4.6 Singularities of the stability boundary in the three-parameter space: a) dihedral angle (edge), b) trihedral angle, c) deadlock of an edge, d) break of an edge.

Singularities appearing on the stability boundary in the two- and three-parameter spaces are shown in Figs. 4.5 and 4.6. Notice that all the curves with directions satisfying inequalities

$$(\operatorname{Re}\mathbf{g}_1, \Delta\mathbf{p}) < 0, \quad (\operatorname{Im}\mathbf{g}_1, \Delta\mathbf{p}) = 0, \quad (\operatorname{Re}\mathbf{g}_2, \Delta\mathbf{p}) < 0 \tag{4.77}$$

for the "deadlock of an edge" singularity $(\pm i\omega)^2$ lie in the stability domain for small $\varepsilon > 0$. In case of the "break of an edge" singularity 0^3 or ∞^3 the curves satisfying, respectively, the conditions

$$\begin{gathered}(\mathbf{h}_1, \Delta\mathbf{p}) = 0, \quad (\mathbf{h}_2, \Delta\mathbf{p}) < 0, \quad (\mathbf{h}_3, \Delta\mathbf{p}) < 0, \\ (\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_2, \mathbf{e})(\mathbf{h}_3, \mathbf{e}) < (\mathbf{h}_1, \mathbf{d}) < (\mathbf{He}, \mathbf{e})\end{gathered} \tag{4.78}$$

or

$$\begin{gathered}(\mathbf{h}'_1, \Delta\mathbf{p}) = 0, \quad (\mathbf{h}'_2, \Delta\mathbf{p}) < 0, \quad (\mathbf{h}'_3, \Delta\mathbf{p}) < 0, \\ (\mathbf{H}'\mathbf{e}, \mathbf{e}) - 2(\mathbf{h}'_2, \mathbf{e})(\mathbf{h}'_3, \mathbf{e}) < (\mathbf{h}'_1, \mathbf{d}) < (\mathbf{H}'\mathbf{e}, \mathbf{e})\end{gathered} \tag{4.79}$$

belong to the stability domain for small $\varepsilon > 0$, where elements of the $n \times n$ matrices $\mathbf{H} = [h_{ij}]$ and $\mathbf{H}' = [h'_{ij}]$ of quadratic forms are determined by the expressions

$$h_{ij} = \frac{1}{a_3} \frac{\partial^2 a_0}{\partial p_i \partial p_j}, \quad h'_{ij} = \frac{1}{a_{m-3}} \frac{\partial^2 a_m}{\partial p_i \partial p_j}. \tag{4.80}$$

If conditions (4.78) or (4.79), where all the inequalities are taken as non-strict, are not satisfied, then the curve belongs to the instability domain for small $\varepsilon > 0$.

Example 4.3 Let us consider an automatic control system consisting of an integrating, oscillatory, and two aperiodic elements connected as shown in Fig. 4.7. The characteristic equation for this system takes the form [Feldbaum and Butkovskii (1971)]

$$\lambda(T_0\lambda^2 + T_1\lambda + 1)(T_2\lambda + 1)(T\lambda + 1) + kk_1k_2 = 0. \tag{4.81}$$

Assuming that parameters of the aperiodic elements are fixed and equal to $T = T_2 = 1$, $k = 2$, $k_2 = 1$, we study stability of the system depending

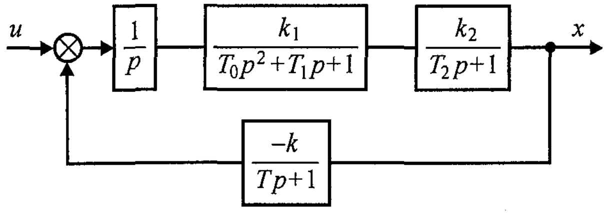


Fig. 4.7 Automatic control system.

on the vector of three parameters $\mathbf{p} = (T_0, T_1, k_1)$ corresponding to the oscillatory element:

$$P(\lambda, \mathbf{p}) = T_0\lambda^5 + (2T_0 + T_1)\lambda^4 + (T_0 + 2T_1 + 1)\lambda^3 + (T_1 + 2)\lambda^2 + \lambda + 2k_1. \quad (4.82)$$

Let us consider a point $\mathbf{p}_0 = (0, 0, 1)$ in the parameters space, corresponding to the system without the oscillatory element (a corresponding transfer function is reduced to unity). At $\mathbf{p} = \mathbf{p}_0$ we find

$$P(\lambda, \mathbf{p}_0) = \lambda^3 + 2\lambda^2 + \lambda + 2. \quad (4.83)$$

Finite roots of this polynomial are simple and equal to $\lambda = \pm i$ and $\lambda = -2$. In addition, there is the double infinite root, since the order of the polynomial is decreased by two at $\mathbf{p}_0$. Hence, the point $\mathbf{p}_0$ belongs to the stability boundary of polynomial (4.82) and has the type $(\pm i\omega)\infty^2$, where $\omega = 1$. By Theorem 4.5, the first order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ is given by

$$(\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0, \quad (\mathbf{g}'_1, \Delta\mathbf{p}) < 0, \quad (\mathbf{g}'_2, \Delta\mathbf{p}) < 0, \quad (4.84)$$

where the vectors $\mathbf{f}_{i\omega}$, $\mathbf{g}'_1$, and $\mathbf{g}'_2$ are given by formulae (4.70) and (4.75). For polynomial (4.82) these vectors are equal to

$$\mathbf{f}_{i\omega} = \frac{1}{5}(1, 2, 1), \quad \mathbf{g}'_1 = (-1, 0, 0), \quad \mathbf{g}'_2 = (0, -1, 0). \quad (4.85)$$

Inequalities (4.84) determine a trihedral angle in the parameter space, see Fig. 4.8a. For comparison, Fig. 4.8b shows the stability boundary found numerically by solving the characteristic equation and checking the stability

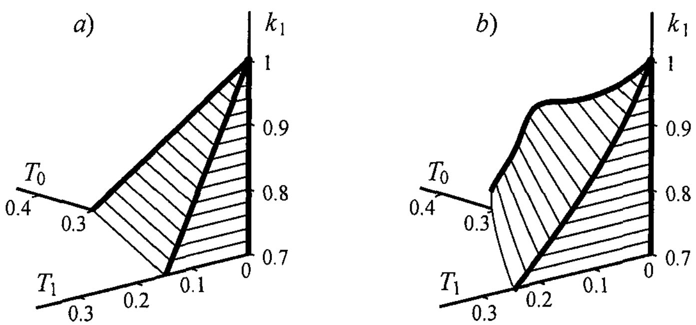


Fig. 4.8 Stability boundary for a control system found by means of a) first order approximation, b) numerical calculation.

condition $\operatorname{Re}\lambda < 0$. Calculations confirm existence of the trihedral angle at the point $\mathbf{p}_0$ with first order approximation (4.84), (4.85).

4.8 Reduction to polynomial of lower order by the Weierstrass preparation theorem

Stability of a characteristic polynomial in the neighborhood of a stability boundary point depends on behavior of zero, purely imaginary, and infinite roots. Multiplicities of these roots are typically low, while the order of the characteristic polynomial can be high in practical problems. The principle difficulty of stability analysis is appearance of multiple roots, which are non-smooth functions of parameters. The following theorem (called the Weierstrass preparation theorem for analytic functions [Weierstrass (1895); Chow and Hale (1982)] with extension to smooth functions called the Malgrange preparation theorem [Malgrange (1964)]) applied to the polynomial case allows reduction of stability analysis in the presence of multiple roots to study of a low order polynomial, whose coefficients smoothly depend on the parameter vector.

Theorem 4.6 *Let λ_0 be a root of multiplicity k for the polynomial $P(\lambda, \mathbf{p})$ at the point $\mathbf{p} = \mathbf{p}_0$ of the parameter space, i.e., conditions (4.36) are satisfied at λ_0 and $\mathbf{p}_0$. Then, in the neighborhood of $\mathbf{p}_0$ the polynomial $P(\lambda, \mathbf{p})$ can be represented in the form*

$$P(\lambda, \mathbf{p}) = (\Delta\lambda^k + b_{k-1}(\mathbf{p})\Delta\lambda^{k-1} + \ldots + b_1(\mathbf{p})\Delta\lambda + b_0(\mathbf{p}))Q(\lambda, \mathbf{p}), \quad (4.86)$$

where $\Delta\lambda = \lambda - \lambda_0$ and $Q(\lambda, \mathbf{p})$ is a polynomial of order $m - k$. The functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ and coefficients of the polynomial $Q(\lambda, \mathbf{p})$ are real-valued or complex-valued smooth functions of $\mathbf{p}$ in case of a real or complex λ_0, respectively, such that $b_0(\mathbf{p}_0) = \cdots = b_{k-1}(\mathbf{p}_0) = 0$ and $Q(\lambda_0, \mathbf{p}_0) \neq 0$.

In case of the infinite root $\lambda_0 = \infty$ of multiplicity k we have

$$P(\lambda, \mathbf{p}) = (b_0(\mathbf{p})\lambda^k + b_1(\mathbf{p})\lambda^{k-1} + \ldots + b_{k-1}(\mathbf{p})\lambda + 1)Q(\lambda, \mathbf{p}), \quad (4.87)$$

where $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ and coefficients of the polynomial $Q(\lambda, \mathbf{p})$ are smooth real functions of the parameter vector such that $b_0(\mathbf{p}_0) = \cdots = b_{k-1}(\mathbf{p}_0) = 0$ and the coefficient of the leading term λ^{m-k} of the polynomial $Q(\lambda, \mathbf{p}_0)$ is nonzero.

Therefore, all information on bifurcation of a multiple root λ_0 is given by the polynomial

$$R(\lambda, \mathbf{p}) = \Delta\lambda^k + b_{k-1}(\mathbf{p})\Delta\lambda^{k-1} + \ldots + b_1(\mathbf{p})\Delta\lambda + b_0(\mathbf{p}) \tag{4.88}$$

of the lowest possible order k. The functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ can be found in the form of Taylor series in the neighborhood of $\mathbf{p}_0$. For this purpose, we need to know their partial derivatives with respect to parameters.

Let us introduce the notation

$$b_{i,\mathbf{h}} = \frac{1}{\mathbf{h}!}\frac{\partial^{|\mathbf{h}|} b_i}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}},$$

$$P_{i,\mathbf{h}} = \frac{1}{i!\mathbf{h}!}\frac{\partial^{i+|\mathbf{h}|} P}{\partial \lambda^i \partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \quad Q_{i,\mathbf{h}} = \frac{1}{i!\mathbf{h}!}\frac{\partial^{i+|\mathbf{h}|} Q}{\partial \lambda^i \partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \tag{4.89}$$

$$\Delta\mathbf{p}^{\mathbf{h}} = \prod_{i=1}^{n}(p_i - p_{0i})^{h_i}, \quad |\mathbf{h}| = h_1 + \cdots + h_n, \quad \mathbf{h}! = h_1! \cdots h_n!,$$

where derivatives are evaluated at λ_0 and $\mathbf{p}_0$, and $\mathbf{h} = (h_1, \ldots, h_n)$ is a vector with nonnegative integer components. Notice that the zero order derivative in the notation means that we do not take the derivative with respect to a corresponding variable, for example,

$$P_{i,0} = \frac{1}{i!}\frac{\partial^i P}{\partial \lambda^i}.$$

Then, the functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ are given by the Taylor series

$$b_i(\mathbf{p}) = \sum_{\mathbf{h}} b_{i,\mathbf{h}} \Delta\mathbf{p}^{\mathbf{h}}, \tag{4.90}$$

where the sum is taken over all the vectors $\mathbf{h}$ with nonnegative integer components. Recall that $b_{i,0} = b_i(\mathbf{p}_0) = 0$, $i = 0, \ldots, k-1$.

Analogously, perturbation of the infinite root $\lambda_0 = \infty$ is given by the polynomial

$$R(\lambda, \mathbf{p}) = b_0(\mathbf{p})\lambda^k + b_1(\mathbf{p})\lambda^{k-1} + \ldots + b_{k-1}(\mathbf{p})\lambda + 1, \tag{4.91}$$

whose coefficients $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ can be taken in the form of Taylor series (4.90) in the neighborhood of $\mathbf{p}_0$.

Values of the coefficients $b_{i,\mathbf{h}}$ in Taylor series (4.90) are found using the following recurrent formulae given in [Grigoryan and Mailybaev (2001)].

Theorem 4.7 *Let λ_0 be a finite root of multiplicity k of the polynomial $P(\lambda, \mathbf{p}_0)$. Then, values of $b_{i,\mathbf{h}}$ and $Q_{i,\mathbf{h}}$ for the functions $b_i(\mathbf{p})$, $i = 0, \ldots, k-1$, and $Q(\lambda, \mathbf{p})$ in factorization (4.86) satisfy the following recurrent formulae*

$$b_{i,\mathbf{h}} = \sum_{j=0}^{i} \alpha_{i-j} \left(P_{j,\mathbf{h}} - \sum_{l=0}^{j} \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h} \\ \mathbf{h}'\neq 0,\ \mathbf{h}''\neq 0}} b_{l,\mathbf{h}'} Q_{j-l,\mathbf{h}''} \right), \tag{4.92}$$

$$Q_{i,\mathbf{h}} = P_{k+i,\mathbf{h}} - \sum_{j=0}^{k-1} \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h} \\ \mathbf{h}'\neq 0}} b_{j,\mathbf{h}'} Q_{k+i-j,\mathbf{h}''}, \tag{4.93}$$

where the coefficients α_i are

$$\alpha_0 = \frac{1}{P_{k,0}}, \quad \alpha_i = -\frac{1}{P_{k,0}} \sum_{j=0}^{i-1} P_{k+i-j,0}\alpha_j, \quad i = 1, \ldots, k-1. \tag{4.94}$$

Proof of Theorem 4.7 is based on the differentiation of equation (4.86) with respect to λ and parameters, and solution of the obtained equations for the unknowns $b_{i,\mathbf{h}}$ and $Q_{i,\mathbf{h}}$.

In case of the infinite root $\lambda_0 = \infty$ we can consider the reverse polynomial $\widetilde{P}(\mu, \mathbf{p})$ given by expression (4.32). The functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ in factorization (4.87) for $\lambda_0 = \infty$ are the same as in factorization (4.86) for zero root $\mu_0 = 1/\lambda_0 = 0$ of the reverse polynomial $\widetilde{P}(\mu, \mathbf{p})$. Hence, values of $b_{i,\mathbf{h}}$ can be found using formulae of Theorem 4.7 for zero root of the polynomial $\widetilde{P}(\mu, \mathbf{p})$.

Evaluating first order derivatives of the functions $b_i(\mathbf{p})$ by Theorem 4.7, we obtain

Corollary 4.1 *Gradient vectors of the functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ in factorization (4.86) for a finite root λ_0 are*

$$\nabla b_i = \sum_{j=0}^{i} \frac{\alpha_{i-j}}{j!} \nabla \frac{\partial^j P}{\partial \lambda^j}. \tag{4.95}$$

For the infinite root $\lambda_0 = \infty$, we have

$$\nabla b_i = \sum_{j=0}^{i} \widetilde{\alpha}_{i-j} \nabla a_{m-j}, \tag{4.96}$$

where

$$\widetilde{\alpha}_0 = \frac{1}{a_{m-k}(\mathbf{p}_0)}, \quad \widetilde{\alpha}_i = -\sum_{j=0}^{i-1} \frac{a_{m-k-i+j}(\mathbf{p}_0)}{a_{m-k}(\mathbf{p}_0)} \widetilde{\alpha}_j, \quad i = 1, \ldots, k-1. \quad (4.97)$$

Example 4.4 Let us consider the two-parameter family of polynomials

$$P(\lambda, \mathbf{p}) = \lambda^4 + (-1+p_2+p_1^2)\lambda^3 + (-1+p_1p_2)\lambda^2 + (1-2p_1)\lambda + p_1 + p_2^2. \quad (4.98)$$

At $\mathbf{p}_0 = 0$ the polynomial $P(\lambda, \mathbf{p}_0)$ has the double root $\lambda_0 = 1$. By Theorem 4.6, polynomial (4.98) has the local representation

$$P(\lambda, \mathbf{p}) = (\Delta\lambda^2 + b_1(\mathbf{p})\Delta\lambda + b_0(\mathbf{p}))Q(\lambda, \mathbf{p}), \quad (4.99)$$

where $\Delta\lambda = \lambda - 1$ and $Q(\lambda, \mathbf{p})$ is a polynomial of order 2 such that $Q(1, \mathbf{p}_0) \neq 0$. Using Theorem 4.7, we find the functions

$$\begin{aligned} b_0(\mathbf{p}) &= \frac{-p_1 + p_2}{2} + \frac{13p_1^2 + 4p_1p_2 + 7p_2^2}{16} + o(\|\Delta\mathbf{p}\|^2), \\ b_1(\mathbf{p}) &= \frac{-p_1 + 3p_2}{4} + \frac{8p_1^2 + 3p_1p_2 - 13p_2^2}{16} + o(\|\Delta\mathbf{p}\|^2). \end{aligned} \quad (4.100)$$

Bifurcation of the double root $\lambda_0 = 1$ in the neighborhood of $\mathbf{p}_0$ is given by the formula

$$\lambda = 1 + \frac{1}{2}\left(-b_1(\mathbf{p}) \pm \sqrt{(b_1(\mathbf{p}))^2 - 4b_0(\mathbf{p})}\right) \quad (4.101)$$

obtained from the equation $R(\lambda, \mathbf{p}) = \Delta\lambda^2 + b_1(\mathbf{p})\Delta\lambda + b_0(\mathbf{p}) = 0$.

4.9 Approximation of stability domain near singularities (general case)

Let us consider a point $\mathbf{p}_0$ on the stability boundary for the polynomial $P(\lambda, \mathbf{p})$. At this point there are zero, purely imaginary, or infinite roots, while the other roots have negative real parts. Let us consider zero root $\lambda_0 = 0$ of multiplicity k. By Theorem 4.6, in the neighborhood of $\mathbf{p}_0$ the polynomial $P(\lambda, \mathbf{p})$ can be represented in the form (4.86), where $\Delta\lambda = \lambda$. Hence, values of the parameter vector $\mathbf{p}$, such that the polynomial $P(\lambda, \mathbf{p})$ has zero root of multiplicity k, are determined by the equations

$$b_0(\mathbf{p}) = \cdots = b_{k-1}(\mathbf{p}) = 0. \quad (4.102)$$

If the gradient vectors $\nabla b_0, \ldots, \nabla b_{k-1}$ are linearly independent, then equations (4.102) determine a smooth surface of codimension k in the parameter space. The linear independence condition is satisfied in the case of general position. The same considerations hold in case of the infinite root $\lambda_0 = \infty$ of multiplicity k. For a purely imaginary root $\lambda_0 = i\omega$ of multiplicity k the functions $b_0(\mathbf{p}), \ldots, b_{k-1}(\mathbf{p})$ are complex-valued. Hence, conditions (4.102) determine $2k$ real equations. If we do not fix the frequency ω, then the values of $\mathbf{p}$, such that the polynomial $P(\lambda, \mathbf{p})$ has a pair of purely imaginary roots $\pm i\omega$ of multiplicity k, form a smooth surface of codimension $2k-1$ in the parameter space. The type of a stability boundary point is determined by roots lying on the imaginary axis and their multiplicities. To find codimension for a set of stability boundary points $\mathbf{p}$ of a certain type, we take a sum of codimensions for each of the roots 0, $\pm i\omega$, and ∞. As a result, we get

Theorem 4.8 *In the case of general position, a set of stability boundary points* $\mathbf{p}$ *of type*

$$0^{k_0}(\pm i\omega_1)^{k_1} \cdots (\pm i\omega_l)^{k_l} \infty^{k_\infty} \tag{4.103}$$

for the polynomial family $P(\lambda, \mathbf{p})$ *is a smooth surface of codimension*

$$k_0 + (2k_1 - 1) + \cdots + (2k_l - 1) + k_\infty \tag{4.104}$$

in the parameter space.

For codimensions 2 and 3 all the types of stability boundary points are listed in (4.74). Recall that the notion of general position means that the property under consideration persists under a small variation of the polynomial family $P(\lambda, \mathbf{p})$. In the case, when $P(\lambda, \mathbf{p})$ is the characteristic polynomial of a matrix family $\mathbf{A}(\mathbf{p})$, not all variations of $P(\lambda, \mathbf{p})$ are possible. For example, considering a matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ with semi-simple double eigenvalue $\lambda_0 = 0$, the coefficient $a_0(\mathbf{p})$ of the characteristic polynomial $P(\lambda, \mathbf{p})$ has the order $O(\|\mathbf{p} - \mathbf{p}_0\|^2)$ for any matrix family $\mathbf{A}(\mathbf{p})$. Semi-simple as well as other derogatory eigenvalues (when there are several eigenvectors corresponding to a multiple eigenvalue) make the main difference between generic structures in the matrix and polynomial cases. For nonderogatory eigenvalues of a matrix, consideration of the matrix and polynomial families is essentially similar.

Let us consider the polynomial

$$\lambda^k + b_{k-1}\lambda^{k-1} + \cdots + b_1\lambda + b_0. \tag{4.105}$$

For $b_0 = \cdots = b_{k-1} = 0$, polynomial (4.105) has zero root of multiplicity k. Considering smooth perturbation of the coefficients

$$b_0(\varepsilon), \ \ldots, \ b_{k-1}(\varepsilon), \tag{4.106}$$

where $\varepsilon \geq 0$ is a small real parameter and

$$b_0(0) = \cdots = b_{k-1}(0) = 0, \tag{4.107}$$

roots of the polynomial change. As a result, the polynomial is stabilized or destabilized. The following propositions describe stabilizing one-parameter perturbations.

Proposition 4.1 *If polynomial (4.105) with real coefficients (4.106), (4.107) is stable for $\varepsilon > 0$, i.e., all the roots satisfy the condition* $\operatorname{Re}\lambda < 0$, *then*

$$\frac{db_0}{d\varepsilon} = \cdots = \frac{db_{k-3}}{d\varepsilon} = 0, \quad \frac{db_{k-2}}{d\varepsilon} \geq 0, \quad \frac{db_{k-1}}{d\varepsilon} \geq 0, \tag{4.108}$$

where derivatives are taken at $\varepsilon = 0$. For any values of derivatives $db_i/d\varepsilon$, $i = 0, \ldots, k-1$, satisfying conditions (4.108) there exist functions $b_0(\varepsilon), \ldots, b_{k-1}(\varepsilon)$, such that polynomial (4.105) is stable for $\varepsilon > 0$ [Levantovskii (1980a)].

Proposition 4.2 *If polynomial (4.105) with complex coefficients (4.106), (4.107) is stable for $\varepsilon > 0$, then*

$$\operatorname{Re}\frac{db_0}{d\varepsilon} = \cdots = \operatorname{Re}\frac{db_{k-3}}{d\varepsilon} = 0,$$

$$\operatorname{Im}\frac{db_0}{d\varepsilon} = \cdots = \operatorname{Im}\frac{db_{k-3}}{d\varepsilon} = \operatorname{Im}\frac{db_{k-2}}{d\varepsilon} = 0, \tag{4.109}$$

$$\operatorname{Re}\frac{db_{k-2}}{d\varepsilon} \geq 0, \quad \operatorname{Re}\frac{db_{k-1}}{d\varepsilon} \geq 0.$$

For any derivatives $db_i/d\varepsilon$, $i = 0, \ldots, k-1$, satisfying conditions (4.109) there exist functions $b_0(\varepsilon), \ldots, b_{k-1}(\varepsilon)$ such that polynomial (4.105) is stable for $\varepsilon > 0$ [Levantovskii (1980a)].

Notice that the statement of Proposition 4.2 can be enforced as follows: for an arbitrary smooth real function $\operatorname{Im} b_{k-1}(\varepsilon)$, such that $\operatorname{Im} b_{k-1}(0) = 0$, and derivatives $\operatorname{Re} db_0/d\varepsilon, \ldots, \operatorname{Re} db_{k-1}/d\varepsilon$, $\operatorname{Im} db_0/d\varepsilon, \ldots, \operatorname{Im} db_{k-2}/d\varepsilon$, satisfying conditions (4.109), there exist functions $b_0(\varepsilon), \ldots, b_{k-1}(\varepsilon)$ such that polynomial (4.105) is stable for $\varepsilon > 0$ [Mailybaev (2000a)].

By the implicit function theorem, Propositions 4.1 and 4.2 can be extended to the case of polynomial (4.105) whose coefficients are smooth functions of the parameter vector $\mathbf{p}$. Then, perturbation of the coefficients $b_0(\mathbf{p}(\varepsilon)), \ldots, b_{k-1}(\mathbf{p}(\varepsilon))$ along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, such that $\varepsilon \geq 0$, $\mathbf{p}(0) = \mathbf{p}_0$, and $b_0(\mathbf{p}_0) = \cdots = b_{k-1}(\mathbf{p}_0) = 0$, is considered. In this case conditions (4.108) for stability along the curve $\mathbf{p}(\varepsilon)$ take the form

$$(\nabla b_0, \mathbf{e}) = \cdots = (\nabla b_{k-3}, \mathbf{e}) = 0, \ (\nabla b_{k-2}, \mathbf{e}) \geq 0, \ (\nabla b_{k-1}, \mathbf{e}) \geq 0, \quad (4.110)$$

where ∇ is the gradient operator at $\mathbf{p}_0$, and $\mathbf{e} = d\mathbf{p}/d\varepsilon$ is the direction of the curve at $\mathbf{p}_0$. Analogously, conditions (4.109) are written in the form

$$\begin{aligned} &(\mathrm{Re}\,\nabla b_0, \mathbf{e}) = \cdots = (\mathrm{Re}\,\nabla b_{k-3}, \mathbf{e}) = 0, \\ &(\mathrm{Im}\,\nabla b_0, \mathbf{e}) = \cdots = (\mathrm{Im}\,\nabla b_{k-3}, \mathbf{e}) = (\mathrm{Im}\,\nabla b_{k-2}, \mathbf{e}) = 0, \\ &(\mathrm{Re}\,\nabla b_{k-2}, \mathbf{e}) \geq 0, \quad (\mathrm{Re}\,\nabla b_{k-1}, \mathbf{e}) \geq 0. \end{aligned} \quad (4.111)$$

Let $\mathbf{p}_0$ be a point on the stability boundary for the polynomial $P(\lambda, \mathbf{p})$. By the Weierstrass preparation theorem (Theorem 4.6, page 140), bifurcation of a multiple root λ_0 of $P(\lambda, \mathbf{p}_0)$ in the neighborhood of $\mathbf{p}_0$ is given by polynomial (4.88) or (4.91) for a finite or infinite λ_0, respectively. The polynomial $P(\lambda, \mathbf{p})$ is stable in the neighborhood of $\mathbf{p}_0$, if all polynomials (4.88) and (4.91) taken for zero, purely imaginary, and infinite roots are stable. Using Propositions 4.1 and 4.2 with their extension to the multi-parameter case given above, we obtain

Theorem 4.9 *Let $\mathbf{p}_0$ be a point of type (4.103) on the stability boundary for the polynomial $P(\lambda, \mathbf{p})$. If a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ lies in the stability domain for $\varepsilon > 0$, then the direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ of the curve satisfies conditions (4.110) for zero ($k = k_0$) and infinite ($k = k_\infty$) roots and conditions (4.111) for each purely imaginary root $i\omega_1, \ldots, i\omega_l$ ($k = k_1, \ldots, k_l$), where the gradients $\nabla b_0, \ldots, \nabla b_{k-1}$ are given by Corollary 4.1. If vectors in these conditions (all together) are linearly independent, then for any direction $\mathbf{e}$ satisfying the necessary conditions there exists a curve $\mathbf{p}(\varepsilon)$ lying in the stability domain for $\varepsilon > 0$.*

Theorem 4.9 provides a local description of the stability domain in the neighborhood of an arbitrary boundary point in terms of stabilizing and destabilizing perturbations of parameters along a curve. Notice that the linear independence condition in Theorem 4.9 is satisfied in the case of general position [Mailybaev (2000a)]. Nevertheless, this condition may be violated if $P(\lambda, \mathbf{p})$ is a characteristic polynomial of the matrix family $\mathbf{A}(\mathbf{p})$

in the presence of derogatory eigenvalues on the imaginary axis. In any case, Theorem 4.9 provides necessary conditions for stabilizing perturbations $\mathbf{p}(\varepsilon)$.

For regular points of the stability boundary and its singularities of codimension 2 and 3 the results of Theorem 4.9 agree with first order approximations of the stability domain given in Theorems 4.4 and 4.5 (pages 133 and 136). Notice that the approximations in Theorem 4.5 contain strict inequalities, showing that the stability domain is an open set in the parameter space. Inequalities in conditions of Theorem 4.9 are nonstrict, since there are stabilizing perturbations $\mathbf{p}(\varepsilon)$ along directions $\mathbf{e}$ tangent to the stability boundary.

Chapter 5

Vibrations and Stability of Conservative System

Analysis of vibrations and stability of a conservative system is a classical question. It is of great importance due to many applications in physics and mechanics. Determination of frequencies and modes of vibration is a typical requirement in the design of buildings, bridges, and machines. In many cases modification of frequencies and modes by changing design parameters is necessary to avoid resonances and noise. Stability problems for conservative systems appear in studying elastic structures under action of potential forces like stationary loads, gravity forces *etc.* In many practical problems there is a specific parameter F describing load of the system. The minimal value of the load parameter, at which the system becomes unstable, is called the *critical load* F_{cr}. Avoiding instability, required in the design of structures, implies that the loads F must be less than F_{cr}. In the presence of several parameters, graph of the critical load in the parameter space represents a boundary of the stability domain. Analysis of the stability boundary allows changing design parameters in order to modify (increase or decrease) the critical load of the system.

Sensitivity analysis of simple and multiple frequencies with application to stability optimization problems was done in [Bratus and Seyranian (1984); Seyranian *et al.* (1994); Seyranian (1997)]. Multiple eigenvalues of multi-parameter symmetric and Hermitian matrices were studied in [Wigner and von Neumann (1929); Arnold (1978)]. Stability domains for conservative systems linearly dependent on parameters were investigated in [Papkovich (1963); Huseyin (1978)]. A general method for multi-parameter stability analysis of a linear conservative system was presented in [Seyranian and Mailybaev (2001a)].

In this chapter, we consider a linear conservative system dependent on parameters. Sensitivity analysis of simple and multiple frequencies of the system is given. It is shown that dependence of frequencies on parameters near a point with double frequency is described by a cone in the frequency-parameter space. Then we investigate the stability boundary of a conservative system including its regular part and singularities. First and second order approximations of the stability domain near regular points of the boundary are given. First order approximations of the stability domain near singular points of the boundary are derived for all types of singularities. It is shown that the stability boundary of a two-parameter conservative system has no singularities in the case of general position, while the only generic singularity of the stability boundary for a three-parameter conservative system is a cone. A simple model of elastic column loaded by an axial force is considered. The cone singularity appears on the stability boundary and determines optimal shape of the column corresponding to the maximal value of the critical force. A specific property of the optimal column is *bimodality*, which means that there are two linearly independent modes (eigenvectors) corresponding to the the same (double) buckling load. At the end of the chapter, we analyze the effect of small dissipative forces on eigenvalues and stability of a conservative system.

5.1 Vibrations and stability

Vibrations of a linear multiple degrees of freedom conservative system is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \tag{5.1}$$

where $\mathbf{q} \in \mathbb{R}^m$ is a vector of generalized coordinates, $\mathbf{M}$ is a positive definite symmetric *mass matrix*, and $\mathbf{P}$ is a symmetric *stiffness matrix.* The *kinetic energy* T and *potential energy* Π of the system are defined as

$$T(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}, \quad \Pi(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\mathbf{P}\mathbf{q}. \tag{5.2}$$

The total energy of the system

$$E(\mathbf{q}, \dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) + \Pi(\mathbf{q}) \tag{5.3}$$

is a quadratic form with respect to the phase variables $\mathbf{q}$ and $\dot{\mathbf{q}}$. The total energy $E(\mathbf{q}, \dot{\mathbf{q}})$ is conserved since

$$\dot{E} = \dot{T} + \dot{\Pi} = \dot{\mathbf{q}}^T(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{P}\mathbf{q}) = 0. \tag{5.4}$$

Therefore, dynamical behavior of the system is restricted to a given energy level $E(\mathbf{q}, \dot{\mathbf{q}}) = \text{const}$ in the phase space $(\mathbf{q}, \dot{\mathbf{q}})$.

Looking for a solution of system (5.1) in the form

$$\mathbf{q}(t) = \mathbf{u} \exp \lambda t, \tag{5.5}$$

we come to the eigenvalue problem

$$(\lambda^2 \mathbf{M} + \mathbf{P})\mathbf{u} = 0, \tag{5.6}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. Introducing a new variable

$$\mu = -\lambda^2, \tag{5.7}$$

we obtain the generalized eigenvalue problem

$$\mathbf{Pu} = \mu \mathbf{Mu}. \tag{5.8}$$

It is always possible to change the basis

$$\mathbf{q} = \mathbf{Uy}, \tag{5.9}$$

where $\mathbf{U}$ is a nonsingular $m \times m$ real matrix, such that quadratic forms (5.2) take the form

$$T(\dot{\mathbf{y}}) = \frac{1}{2}\dot{\mathbf{y}}^T \dot{\mathbf{y}}, \quad \Pi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T \mathbf{D} \mathbf{y}, \tag{5.10}$$

where $\mathbf{D}$ is a real diagonal matrix, see [Gantmacher (1998)]. Matrices of quadratic forms (5.2) and (5.10) are related by

$$\mathbf{U}^T \mathbf{M} \mathbf{U} = \mathbf{I}, \quad \mathbf{U}^T \mathbf{P} \mathbf{U} = \mathbf{D}. \tag{5.11}$$

Using expressions (5.11) in (5.8), it is easy to show that diagonal elements of the matrix

$$\mathbf{D} = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_m \end{pmatrix} \tag{5.12}$$

are eigenvalues of (5.8), and the columns of the matrix $\mathbf{U}$ are corresponding eigenvectors. Therefore, all the eigenvalues of (5.8) are real and simple or semi-simple, and corresponding eigenvectors can be chosen real.

If $\mu > 0$, then there are two solutions of system (5.1) of the form

$$\mathbf{q}_1(t) = \mathbf{u}\cos\omega t, \quad \mathbf{q}_2(t) = \mathbf{u}\sin\omega t, \tag{5.13}$$

where $\omega = \sqrt{\mu}$. Solutions (5.13) describe harmonic motion with the frequency ω and mode of vibration $\mathbf{u}$. If $\mu = 0$, then there are two solutions

$$\mathbf{q}_1(t) = \mathbf{u}, \quad \mathbf{q}_2(t) = t\mathbf{u}, \tag{5.14}$$

where $\mathbf{q}_1(t)$ is a constant and $\mathbf{q}_2(t)$ is a linearly growing solution. For $\mu < 0$ there are two solutions

$$\mathbf{q}_1(t) = \mathbf{u}\exp\alpha t, \quad \mathbf{q}_2(t) = \mathbf{u}\exp(-\alpha t), \tag{5.15}$$

where $\alpha = \sqrt{-\mu}$. Solutions (5.15) grow and decay exponentially in time.

Since all the eigenvalues μ are simple or semi-simple, there are m linearly independent eigenvectors $\mathbf{u}$ that give rise to $2m$ solutions (5.13)–(5.15). These are particular solutions, and their linear combination represents a general solution of equation (5.1). It is clear that system (5.1) is stable (any solution remains bounded) if and only if all the eigenvalues μ are positive. Condition $\mu > 0$ implies that the matrix $\mathbf{D}$ and, hence, the matrix $\mathbf{P}$ are positive definite. Therefore, stability property is equivalent to positive definiteness of the stiffness matrix $\mathbf{P}$ and does not depend on the mass matrix $\mathbf{M}$.

Theorem 5.1 *Conservative system (5.1) is stable if and only if all the eigenvalues μ of (5.8) are positive or, equivalently, the stiffness matrix* $\mathbf{P}$ *is positive definite.*

The energy $E(\mathbf{q}, \dot{\mathbf{q}})$ of a stable system is a positive definite quadratic form, and the energy level is a bounded surface in the phase space (precisely, $E(\mathbf{q}, \dot{\mathbf{q}}) = \text{const}$ is an ellipsoid with the center at the origin). A solution of system (5.1) lies on this ellipsoid and is represented by a linear combination of harmonic solutions (5.13). Therefore, system (5.1) with the positive definite stiffness matrix $\mathbf{P}$ is stable, but not asymptotically stable.

Let us consider a nonlinear conservative system with the kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and potential energy $\Pi(\mathbf{q})$, where the mass matrix $\mathbf{M}(\mathbf{q})$ and potential energy $\Pi(\mathbf{q})$ smoothly depend on the vector of generalized coordinates $\mathbf{q}$. Equations of motion are given by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i}, \quad i = 1, \ldots, m. \tag{5.16}$$

The total energy of the system given by

$$E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + \Pi(\mathbf{q}) \tag{5.17}$$

is conserved, i.e., $E(\mathbf{q}, \dot{\mathbf{q}}) = \text{const}$ for any solution of system (5.16).

A point $\mathbf{q} = \mathbf{q}_0$ is the equilibrium state of the system if it is a critical point of the potential energy

$$\frac{\partial \Pi}{\partial q_i} = 0, \quad i = 1, \dots, m. \tag{5.18}$$

The equations of motion linearized near the equilibrium take the form

$$\mathbf{M}_0 \Delta\ddot{\mathbf{q}} + \mathbf{P}_0 \Delta\mathbf{q} = 0, \tag{5.19}$$

where $\Delta\mathbf{q} = \mathbf{q} - \mathbf{q}_0$, $\mathbf{M}_0 = \mathbf{M}(\mathbf{q}_0)$, and $\mathbf{P}_0 = d^2P/d\mathbf{q}^2 = [\partial^2 P/\partial q_i \partial q_j]$ is the matrix of second order derivatives evaluated at $\mathbf{q}_0$. Relation of stability properties for linear and nonlinear conservative systems can be seen through the energy conservation criterion. Indeed, solution of stable linearized system (5.19) is restricted to the ellipsoid

$$\frac{1}{2}\Delta\dot{\mathbf{q}}^T \mathbf{M}_0 \Delta\dot{\mathbf{q}} + \frac{1}{2}\Delta\mathbf{q}^T \mathbf{P}_0 \Delta\mathbf{q} = \text{const} \tag{5.20}$$

surrounding the stationary point $\mathbf{q} = \mathbf{q}_0$, $\dot{\mathbf{q}} = 0$. In the neighborhood of the equilibrium total energy (5.17) is given by the expansion

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \Pi(\mathbf{q}_0) + \frac{1}{2}\Delta\dot{\mathbf{q}}^T \mathbf{M}_0 \Delta\dot{\mathbf{q}} + \frac{1}{2}\Delta\mathbf{q}^T \mathbf{P}_0 \Delta\mathbf{q} + o(\|\Delta\mathbf{q}\|^2 + \|\Delta\dot{\mathbf{q}}\|^2). \tag{5.21}$$

For small $\|\Delta\mathbf{q}\|$ and $\|\Delta\dot{\mathbf{q}}\|$, energy level surfaces $E(\mathbf{q}, \dot{\mathbf{q}}) = \text{const}$ are small deformations of ellipsoids (5.20). Therefore, the energy level surfaces remain bounded, which preserves the stability property. This yields the following theorem.

Theorem 5.2 *If linearized conservative system (5.19) is stable, then the stationary solution* $\mathbf{q}(t) \equiv \mathbf{q}_0$ *of nonlinear conservative system (5.16) is stable.*

Theorem 5.2 is a corollary of the Lagrange theorem saying that the equilibrium $\mathbf{q} = \mathbf{q}_0$ of a conservative system is stable if the potential energy $\Pi(\mathbf{q})$ has local minimum at $\mathbf{q}_0$, see [Arnold (1978); Merkin (1997)].

5.2 Sensitivity of simple and multiple frequencies

Let us consider linear conservative system (5.1) with the matrices $\mathbf{M}$ and $\mathbf{P}$ smoothly dependent on a vector of real parameters $\mathbf{p}$. Perturbation theory of eigenvalues and eigenvectors for generalized eigenvalue problem (5.8) is given in Section 2.12. Let us fix a value of the parameter vector $\mathbf{p} = \mathbf{p}_0$ and consider a simple eigenvalue μ_0 with the corresponding eigenvector $\mathbf{u}_0$ satisfying the equation

$$\mathbf{P}_0\mathbf{u}_0 = \mu_0\mathbf{M}_0\mathbf{u}_0, \tag{5.22}$$

where $\mathbf{M}_0 = \mathbf{M}(\mathbf{p}_0)$ and $\mathbf{P}_0 = \mathbf{P}(\mathbf{p}_0)$. Since the matrices $\mathbf{M}_0$ and $\mathbf{P}_0$ are symmetric, the right and left eigenvalue problems coincide. Therefore, we can take the left eigenvector $\mathbf{v}_0 = \mathbf{u}_0$. By Theorem 2.8 (page 72), the simple eigenvalue μ_0 smoothly depends on parameters with the derivative

$$\frac{\partial \mu}{\partial p_j} = \mathbf{u}_0^T\left(\frac{\partial \mathbf{P}}{\partial p_j} - \mu_0\frac{\partial \mathbf{M}}{\partial p_j}\right)\mathbf{u}_0\Big/(\mathbf{u}_0^T\mathbf{M}_0\mathbf{u}_0). \tag{5.23}$$

If $\mu_0 > 0$, then the frequency $\omega_0 = \sqrt{\mu_0}$ smoothly depends on parameters with the derivative

$$\frac{\partial \omega}{\partial p_j} = \frac{1}{2\omega_0}\frac{\partial \mu}{\partial p_j} = \frac{1}{2\omega_0}\mathbf{u}_0^T\left(\frac{\partial \mathbf{P}}{\partial p_j} - \omega_0^2\frac{\partial \mathbf{M}}{\partial p_j}\right)\mathbf{u}_0\Big/(\mathbf{u}_0^T\mathbf{M}_0\mathbf{u}_0). \tag{5.24}$$

For the derivative of the eigenvector (mode) $\mathbf{u}_0$ we get

$$\frac{\partial \mathbf{u}}{\partial p_j} = (\mathbf{P}_0 - \mu_0\mathbf{M}_0 + \mathbf{u}_0\mathbf{u}_0^T\mathbf{M}_0)^{-1}\left(\frac{\partial \mu}{\partial p_j}\mathbf{M}_0 + \mu_0\frac{\partial \mathbf{M}}{\partial p_j} - \frac{\partial \mathbf{P}}{\partial p_j}\right)\mathbf{u}_0. \tag{5.25}$$

Let us consider an eigenvalue μ_0 of multiplicity k. Since μ_0 is semi-simple, there are k linearly independent eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ corresponding to μ_0. We assume that the eigenvectors satisfy the normalization conditions

$$\mathbf{u}_i^T\mathbf{M}_0\mathbf{u}_j = \delta_{ij}, \quad i, j = 1, \dots, k. \tag{5.26}$$

Right and left eigenvalue problems coincide for symmetric matrices $\mathbf{M}$ and $\mathbf{P}$ and, therefore, the left eigenvectors can be taken equal to the right eigenvectors. Consider perturbation of the parameter vector along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ with a direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$. By Theorem 2.9 (page 73), the eigenvalue μ_0 takes the increment

$$\mu = \mu_0 + \varepsilon\mu_1 + o(\varepsilon), \tag{5.27}$$

where k values of μ_1 are the eigenvalues of the $k \times k$ matrix $\mathbf{F}$ with the elements

$$f_{ij} = \sum_{l=1}^{n} \left(\mathbf{u}_i^T \left(\frac{\partial \mathbf{P}}{\partial p_l} - \mu_0 \frac{\partial \mathbf{M}}{\partial p_l} \right) \mathbf{u}_j \right) e_l. \tag{5.28}$$

Introducing vectors of dimension n

$$\mathbf{f}_{ij} = \left(\mathbf{u}_i^T \left(\frac{\partial \mathbf{P}}{\partial p_1} - \mu_0 \frac{\partial \mathbf{M}}{\partial p_1} \right) \mathbf{u}_j, \ldots, \mathbf{u}_i^T \left(\frac{\partial \mathbf{P}}{\partial p_n} - \mu_0 \frac{\partial \mathbf{M}}{\partial p_n} \right) \mathbf{u}_j \right), \tag{5.29}$$

expression (5.28) takes the form

$$f_{ij} = (\mathbf{f}_{ij}, \mathbf{e}). \tag{5.30}$$

Since the matrices $\mathbf{M}$ and $\mathbf{P}$ are symmetric, $\mathbf{f}_{ij} = \mathbf{f}_{ji}$ for any i and j. Hence, the matrix $\mathbf{F}$ is symmetric. All the eigenvalues μ_1 of the matrix $\mathbf{F}$ are real, which is a natural consequence since eigenvalues μ are real.

Let us consider a double positive eigenvalue $\mu_0 > 0$. The corresponding double frequency is $\omega_0 = \sqrt{\mu_0}$. Using expression (5.27), we find perturbation of the frequency ω_0 along the ray $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{e}$ as follows

$$\omega = \sqrt{\mu} = \omega_0 + \frac{\mu_1}{2\omega_0} \varepsilon + o(\varepsilon). \tag{5.31}$$

Two eigenvalues μ_1 of the 2×2 matrix

$$\mathbf{F} = \begin{pmatrix} (\mathbf{f}_{11}, \mathbf{e}) & (\mathbf{f}_{12}, \mathbf{e}) \\ (\mathbf{f}_{12}, \mathbf{e}) & (\mathbf{f}_{22}, \mathbf{e}) \end{pmatrix} \tag{5.32}$$

are

$$\mu_1 = \frac{1}{2}(\mathbf{f}_{11} + \mathbf{f}_{22}, \mathbf{e}) \pm \frac{1}{2}\sqrt{(\mathbf{f}_{11} - \mathbf{f}_{22}, \mathbf{e})^2 + 4(\mathbf{f}_{12}, \mathbf{e})^2}. \tag{5.33}$$

Substituting (5.33) into (5.31) and neglecting a higher order term $o(\varepsilon)$, we find the first order approximation for bifurcation of the double frequency

$$\omega = \omega_0 + \frac{(\mathbf{f}_{11} + \mathbf{f}_{22}, \Delta\mathbf{p}) \pm \sqrt{(\mathbf{f}_{11} - \mathbf{f}_{22}, \Delta\mathbf{p})^2 + 4(\mathbf{f}_{12}, \Delta\mathbf{p})^2}}{4\omega_0}, \tag{5.34}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = \varepsilon\mathbf{e}$. Expressing the square root from equation (5.34) and taking square of the obtained expression, we find

$$(4\omega_0(\omega - \omega_0) - (\mathbf{f}_{11} + \mathbf{f}_{22}, \Delta\mathbf{p}))^2 = (\mathbf{f}_{11} - \mathbf{f}_{22}, \Delta\mathbf{p})^2 + 4(\mathbf{f}_{12}, \Delta\mathbf{p})^2. \tag{5.35}$$

In case of two parameters $\mathbf{p} = (p_1, p_2)$, equation (5.35) determines a cone in the three-dimensional space (p_1, p_2, ω); see Fig. 5.1. There are two simple frequencies ω at each value of the parameter vector $\mathbf{p}$, except for the point $\mathbf{p}_0$, where two frequencies merge and form a double semi-simple frequency ω_0. Fixing the parameter p_2 and considering dependence of the frequencies on the parameter p_1, we obtain two intersecting lines or two hyperbolae in the plane (p_1, ω) describing the weak interaction of two frequencies; see also Example 2.11 in Section 2.9.

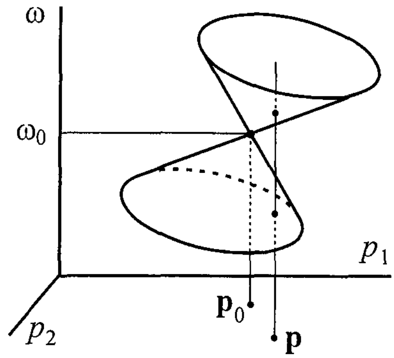


Fig. 5.1 Perturbation of a double frequency ω_0 for a two-parameter conservative system.

5.3 Stability domain and its boundary

The stability domain of a multi-parameter conservative system consists of points $\mathbf{p}$ in the parameter space, where the system is stable, i.e., the stiffness matrix $\mathbf{P}(\mathbf{p})$ is positive definite. Recall that stability property does not depend on the positive definite mass matrix $\mathbf{M}$. Boundary of the stability domain is represented by the points $\mathbf{p}$, where the matrix $\mathbf{P}(\mathbf{p})$ is singular and positive semi-definite, that is, the matrix $\mathbf{P}(\mathbf{p})$ has zero eigenvalue while other eigenvalues are positive.

Let us consider a point $\mathbf{p}_0$ on the stability boundary. We assume that the matrix $\mathbf{P}_0 = \mathbf{P}(\mathbf{p}_0)$ has a simple zero eigenvalue $\lambda_0 = 0$ with a corresponding real eigenvector $\mathbf{u}_0$ satisfying the equation

$$\mathbf{P}_0\mathbf{u}_0 = 0. \tag{5.36}$$

The right and left eigenvalue problems for the symmetric matrix $\mathbf{P}_0$ coincide. The stability domain in the neighborhood of the point $\mathbf{p}_0$ is described by the condition

$$\lambda(\mathbf{p}) > 0 \tag{5.37}$$

for the eigenvalue $\lambda(\mathbf{p})$ of the matrix $\mathbf{P}(\mathbf{p})$ vanishing at $\mathbf{p}_0$. By Theorem 2.2 (page 32), we find that the simple eigenvalue $\lambda(\mathbf{p})$ is a smooth function of $\mathbf{p}$ with the derivative

$$\frac{\partial \lambda}{\partial p_j} = \mathbf{u}_0^T \frac{\partial \mathbf{P}}{\partial p_j} \mathbf{u}_0 \Big/ (\mathbf{u}_0^T \mathbf{u}_0). \tag{5.38}$$

Introducing the gradient vector of $\lambda(\mathbf{p})$

$$\mathbf{f} = \nabla \lambda = \left(\frac{\partial \lambda}{\partial p_1}, \ldots, \frac{\partial \lambda}{\partial p_n} \right), \tag{5.39}$$

we find the first order approximation of the stability domain as

$$(\mathbf{f}, \Delta \mathbf{p}) > 0, \quad \Delta \mathbf{p} = \mathbf{p} - \mathbf{p}_0. \tag{5.40}$$

If $\mathbf{f} \neq 0$, then the stability boundary is a smooth surface with the tangent plane

$$(\mathbf{f}, \Delta \mathbf{p}) = 0, \tag{5.41}$$

where the vector $\mathbf{f}$ is a normal vector to the stability boundary.

Theorem 5.3 *Let $\mathbf{p}_0$ be a point on the stability boundary for conservative system (5.1), and assume that there is a simple zero eigenvalue of the matrix $\mathbf{P}_0 = \mathbf{P}(\mathbf{p}_0)$. If the vector $\mathbf{f}$ determined by expressions (5.38), (5.39) is nonzero, then the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$ with the normal vector $\mathbf{f}$ directed into the stability domain; see Fig 5.2.*

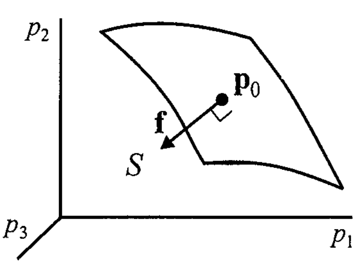


Fig. 5.2 Normal vector to the stability boundary of a conservative system.

Points $\mathbf{p}_0$ considered in Theorem 5.3 form a regular part of the stability boundary. Let us find the second order approximation of the stability domain in the neighborhood of a regular boundary point $\mathbf{p}_0$. By Theorem 2.2 (page 32), the second order derivative of the eigenvalue $\lambda(\mathbf{p})$ equals

$$\frac{\partial^2 \lambda}{\partial p_i \partial p_j} = \mathbf{u}_0^T \left(\frac{\partial^2 \mathbf{P}}{\partial p_i \partial p_j} \mathbf{u}_0 + \frac{\partial \mathbf{P}}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} + \frac{\partial \mathbf{P}}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} - \frac{\partial \lambda}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \right) \Big/ (\mathbf{u}_0^T \mathbf{u}_0), \tag{5.42}$$

where the derivative $\partial \mathbf{u}/\partial p_i$ is

$$\frac{\partial \mathbf{u}}{\partial p_i} = (\mathbf{P}_0 + \mathbf{u}_0 \mathbf{u}_0^T)^{-1} \left(\frac{\partial \lambda}{\partial p_i} \mathbf{I} - \frac{\partial \mathbf{P}}{\partial p_i} \right) \mathbf{u}_0. \tag{5.43}$$

Let us consider a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$ and tangent to the stability boundary. The direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ of the curve satisfies the orthogonality condition

$$\sum_{i=1}^{n} \frac{\partial \lambda}{\partial p_i} e_i = (\mathbf{f}, \mathbf{e}) = 0. \tag{5.44}$$

Increment of the eigenvalue $\lambda(\mathbf{p})$ along the curve takes the form

$$\lambda = \varepsilon \sum_{i=1}^{n} \frac{\partial \lambda}{\partial p_i} e_i + \frac{\varepsilon^2}{2} \left(\sum_{i=1}^{n} \frac{\partial \lambda}{\partial p_i} d_i + \sum_{i,j=1}^{n} \frac{\partial^2 \lambda}{\partial p_i \partial p_j} e_i e_j \right) + o(\varepsilon^2), \tag{5.45}$$

where $d_i = d^2 p_i / d\varepsilon^2$ is evaluated at $\mathbf{p}_0$. Using expressions (5.42)–(5.45) in equation (5.37), we find

$$\frac{\varepsilon^2}{2} \big((\mathbf{f}, \mathbf{d}) + (\mathbf{G}\mathbf{e}, \mathbf{e}) \big) + o(\varepsilon^2) > 0, \tag{5.46}$$

where $\mathbf{G} = [g_{ij}]$ is an $n \times n$ matrix with the elements

$$g_{ij} = \mathbf{u}_0^T \left(\frac{\partial^2 \mathbf{P}}{\partial p_i \partial p_j} - 2 \frac{\partial \mathbf{P}}{\partial p_i} (\mathbf{P}_0 + \mathbf{u}_0 \mathbf{u}_0^T)^{-1} \frac{\partial \mathbf{P}}{\partial p_j} \right) \mathbf{u}_0 \Big/ (\mathbf{u}_0^T \mathbf{u}_0), \tag{5.47}$$

and $(\mathbf{G}\mathbf{e}, \mathbf{e})$ denotes the quadratic form

$$(\mathbf{G}\mathbf{e}, \mathbf{e}) = \sum_{i,j=1}^{n} g_{ij} e_i e_j. \tag{5.48}$$

Equation (5.46) with the use of relation (5.44) can be written in the form

$$(\mathbf{f}, \Delta\mathbf{p}) + \frac{1}{2}(\mathbf{G}\Delta\mathbf{p}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|^2) > 0, \tag{5.49}$$

where $\Delta\mathbf{p} = \mathbf{p}(\varepsilon) - \mathbf{p}_0 = \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d}/2 + \cdots$

Equation (5.49) represents the second order approximation of the stability domain. The stability domain is convex at $\mathbf{p}_0$ if the matrix $\mathbf{G}$ is negative definite, and concave if $\mathbf{G}$ is positive definite. By Papkovich's theorem [Papkovich (1963); Huseyin (1978)] the stability domain of a conservative system linearly dependent on parameters is convex. In this case second order derivatives of the matrix $\mathbf{P}$ with respect to parameters are all zeros and

$$\mathbf{G} = -\frac{2}{\mathbf{u}_0^T\mathbf{u}_0}\mathbf{W}^T(\mathbf{P}_0 + \mathbf{u}_0\mathbf{u}_0^T)^{-1}\mathbf{W}, \quad \mathbf{W} = \left[\frac{\partial\mathbf{P}}{\partial p_1}\mathbf{u}_0, \ldots, \frac{\partial\mathbf{P}}{\partial p_n}\mathbf{u}_0\right]. \tag{5.50}$$

The matrices $\mathbf{P}_0$ and $\mathbf{u}_0\mathbf{u}_0^T$ are positive semi-definite. Their sum $\mathbf{P}_0+\mathbf{u}_0\mathbf{u}_0^T$ is a nonsingular matrix according to Theorem 2.2 (page 32) and, hence, it is positive definite. Then the inverse matrix $(\mathbf{P}_0+\mathbf{u}_0\mathbf{u}_0^T)^{-1}$ is positive definite too. Using this property, it is easy to check that the matrix $\mathbf{G}$ determined by expression (5.50) is negative definite or negative semi-definite depending on the matrix $\mathbf{W}$. This implies convexity of the stability domain at $\mathbf{p}_0$, which confirms the Papkovich's theorem.

5.4 Singularities of stability boundary

Singular points of the stability boundary (points where the boundary surface is nonsmooth) are determined by matrices $\mathbf{P}(\mathbf{p})$ having multiple zero eigenvalues. In the case of general position a set of singular points associated with the zero eigenvalue of multiplicity k forms a smooth surface of codimension $k(k+1)/2$ in the parameter space [Wigner and von Neumann (1929); Arnold (1978)]. In particular, this means that in case of two parameters there are no singularities and the stability boundary is a smooth curve; see Fig. 5.3.

The simplest singular point $\mathbf{p}_0$ corresponds to the double ($k = 2$) eigenvalue $\lambda_0 = 0$ and appears if we have $k(k+1)/2 = 3$ or more parameters. The eigenvalue $\lambda_0 = 0$ is semi-simple and has two linearly independent real eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, which can be chosen satisfying the normalization

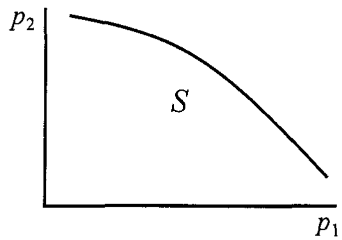


Fig. 5.3 Stability boundary of a generic two-parameter conservative system is smooth.

conditions

$$\mathbf{u}_i^T \mathbf{u}_j = \delta_{ij}, \quad i, j = 1, 2. \tag{5.51}$$

Let us consider perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{e}$, where ε is small. By Theorem 2.6 (page 56), the semi-simple double eigenvalue $\lambda_0 = 0$ of the matrix $\mathbf{P}_0$ splits into two simple eigenvalues

$$\lambda = \varepsilon \lambda_1 + o(\varepsilon). \tag{5.52}$$

Two different values of λ_1 are the eigenvalues of the matrix

$$\mathbf{F} = \begin{pmatrix} (\mathbf{f}_{11}, \mathbf{e}) & (\mathbf{f}_{12}, \mathbf{e}) \\ (\mathbf{f}_{21}, \mathbf{e}) & (\mathbf{f}_{22}, \mathbf{e}) \end{pmatrix}, \tag{5.53}$$

where

$$\mathbf{f}_{ij} = \left(\mathbf{u}_i^T \frac{\partial \mathbf{P}}{\partial p_1} \mathbf{u}_j, \ldots, \mathbf{u}_i^T \frac{\partial \mathbf{P}}{\partial p_n} \mathbf{u}_j \right). \tag{5.54}$$

Due to the symmetry of the matrix $\mathbf{P}$, we have $\mathbf{f}_{12} = \mathbf{f}_{21}$. For stability we need both perturbed eigenvalues λ to be positive. This implies that the 2×2 matrix $\mathbf{F}$ is positive definite. Using Sylvester's conditions, we obtain

$$(\mathbf{f}_{11} + \mathbf{f}_{22}, \Delta\mathbf{p}) > 0, \quad (\mathbf{f}_{11}, \Delta\mathbf{p})(\mathbf{f}_{22}, \Delta\mathbf{p}) - (\mathbf{f}_{12}, \Delta\mathbf{p})^2 > 0, \tag{5.55}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = \varepsilon \mathbf{e}$. Inequalities (5.55) provide the first order approximation of the stability domain.

In case of three parameters inequalities (5.55) determine a cone in the vicinity of the point $\mathbf{p}_0$ in the parameter space. Indeed, after introduction of new variables

$$x = \frac{1}{2}(\mathbf{f}_{11} - \mathbf{f}_{22}, \Delta\mathbf{p}), \quad y = (\mathbf{f}_{12}, \Delta\mathbf{p}), \quad z = \frac{1}{2}(\mathbf{f}_{11} + \mathbf{f}_{22}, \Delta\mathbf{p}), \tag{5.56}$$

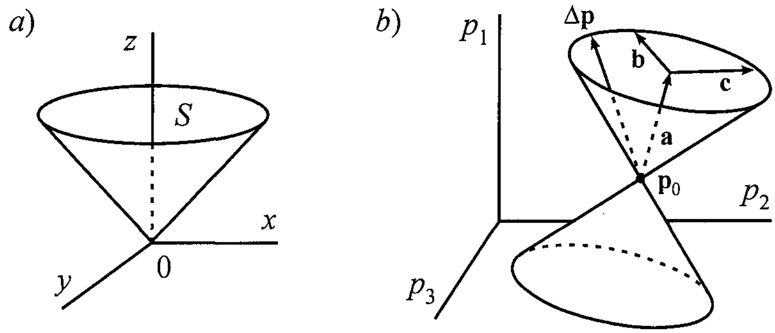


Fig. 5.4 Cone singularity of the stability boundary: a) in the parameter space (x, y, z), b) in the parameter space $\mathbf{p}$ (upper part of the cone for $\gamma > 0$ or lower part for $\gamma < 0$).

approximation of the stability domain (5.55) takes the form of a circular cone (see Fig. 5.4a)

$$x^2 + y^2 < z^2, \quad z > 0. \tag{5.57}$$

The conical surface $x^2 + y^2 = z^2$, $z \geq 0$, can be parameterized as follows

$$x = z\cos\alpha, \quad y = z\sin\alpha, \quad z \geq 0, \tag{5.58}$$

where $0 \leq \alpha < 2\pi$. Using expressions (5.56) in (5.58), we get the equations

$$\begin{aligned} (\mathbf{f}_{11} - \mathbf{f}_{22} - (\mathbf{f}_{11} + \mathbf{f}_{22})\cos\alpha, \Delta\mathbf{p}) &= 0, \\ (2\mathbf{f}_{12} - (\mathbf{f}_{11} + \mathbf{f}_{22})\sin\alpha, \Delta\mathbf{p}) &= 0, \\ (\mathbf{f}_{11} + \mathbf{f}_{22}, \Delta\mathbf{p}) &\geq 0. \end{aligned} \tag{5.59}$$

Then the vector $\Delta\mathbf{p}$ can be found as the cross product

$$\begin{aligned} \Delta\mathbf{p} &= \beta(\mathbf{f}_{11} - \mathbf{f}_{22} - (\mathbf{f}_{11} + \mathbf{f}_{22})\cos\alpha) \times (2\mathbf{f}_{12} - (\mathbf{f}_{11} + \mathbf{f}_{22})\sin\alpha) \\ &= \beta(2(\mathbf{f}_{11} - \mathbf{f}_{22}) \times \mathbf{f}_{12} - (\mathbf{f}_{11} - \mathbf{f}_{22}) \times (\mathbf{f}_{11} + \mathbf{f}_{22})\sin\alpha \\ &\quad -2(\mathbf{f}_{11} + \mathbf{f}_{22}) \times \mathbf{f}_{12}\cos\alpha), \end{aligned} \tag{5.60}$$

where β is a real number. Using (5.60) in the third inequality of (5.59) yields

$$2\beta(\mathbf{f}_{11} + \mathbf{f}_{22},\, (\mathbf{f}_{11} - \mathbf{f}_{22}) \times \mathbf{f}_{12}) \geq 0. \tag{5.61}$$

Hence, $\beta \geq 0$ or $\beta \leq 0$ for positive or negative values of the quantity

$$\gamma = (\mathbf{f}_{11} + \mathbf{f}_{22},\, (\mathbf{f}_{11} - \mathbf{f}_{22}) \times \mathbf{f}_{12}), \tag{5.62}$$

respectively. This provides the following representation of the cone surface in the parameter space

$$\Delta\mathbf{p} = \beta(\mathbf{a} + \mathbf{b}\sin\alpha + \mathbf{c}\cos\alpha), \quad \gamma\beta \geq 0, \quad 0 \leq \alpha < 2\pi, \tag{5.63}$$

where the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are defined by

$$\begin{aligned} \mathbf{a} &= 2(\mathbf{f}_{11} - \mathbf{f}_{22}) \times \mathbf{f}_{12}, \\ \mathbf{b} &= (\mathbf{f}_{11} + \mathbf{f}_{22}) \times (\mathbf{f}_{11} - \mathbf{f}_{22}), \\ \mathbf{c} &= 2\mathbf{f}_{12} \times (\mathbf{f}_{11} + \mathbf{f}_{22}). \end{aligned} \tag{5.64}$$

Vector (5.63) runs through the cone surface as β and α are changing, see Fig. 5.4b, where depending on the sign of γ its upper or lower part should be taken (stability domain is inside the corresponding cone part).

Finally, let us consider a general case, when $\mathbf{p}_0$ is a singular point of the stability boundary represented by zero eigenvalue $\lambda_0 = 0$ of multiplicity k. Recall that in the case of general position this singularity can appear if we have $k(k+1)/2$ or more parameters. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be linearly independent eigenvectors corresponding to $\lambda_0 = 0$:

$$\mathbf{P}_0\mathbf{u}_i = 0, \quad i = 1, \ldots, k, \tag{5.65}$$

and satisfying the normalization conditions

$$\mathbf{u}_i^T\mathbf{u}_j = \delta_{ij}, \quad i, j = 1, \ldots, k. \tag{5.66}$$

Consider perturbation of the parameter vector in the form $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$. By Theorem 2.7 (page 70), the eigenvalue $\lambda_0 = 0$ takes increment (5.52), where k values of λ_1 are the eigenvalues of the $k \times k$ matrix $\mathbf{F} = [f_{ij}]$ with the elements

$$f_{ij} = (\mathbf{f}_{ij}, \mathbf{e}), \quad i, j = 1, \ldots, k, \tag{5.67}$$

and the vectors $\mathbf{f}_{ij}$ defined by (5.54). Since the matrix $\mathbf{P}$ is symmetric, $\mathbf{f}_{ij} = \mathbf{f}_{ji}$ for any i and j. Hence, the matrix $\mathbf{F}$ is symmetric. Stability condition requires $\lambda > 0$ for all perturbed eigenvalues. As a result, we obtain a first order approximation of the stability domain in the form

$$\begin{pmatrix} (\mathbf{f}_{11}, \Delta\mathbf{p}) & \cdots & (\mathbf{f}_{1k}, \Delta\mathbf{p}) \\ \vdots & \ddots & \vdots \\ (\mathbf{f}_{k1}, \Delta\mathbf{p}) & \cdots & (\mathbf{f}_{kk}, \Delta\mathbf{p}) \end{pmatrix} > 0 \quad \text{(positive definite)}. \tag{5.68}$$

The condition of positive definiteness of matrix (5.68) can be written in the form of inequalities called Sylvester's conditions, see [Korn and Korn (1968)].

5.5 Buckling problem of column loaded by axial force

Let us consider a finite dimensional model of a column consisting of four equal links of length l and loaded by an axial force F; see Fig. 5.5. A bending moment in the ith node is proportional to $a_i^2\theta_i$, where a_i is a dimensionless cross-section area of the column at the ith node, and θ_i is the angle between links in the ith node. Taking into account the boundary conditions $q_0 = q_4 = 0$, the system possesses three degrees of freedom determined by components of the vector of generalized coordinates $\mathbf{q} = (q_1, q_2, q_3)^T$, where q_i is deflection of the ith node. The stiffness matrix $\mathbf{P}$ of the system in non-dimensional coordinates takes the form [Choi and

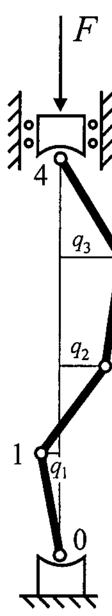


Fig. 5.5 Simple model of column loaded by axial force.

Haug (1981)]

$$\mathbf{P} = \begin{pmatrix} a_0^2 + 4a_1^2 + a_2^2 - 2F & -2a_1^2 - 2a_2^2 + F & a_2^2 \\ -2a_1^2 - 2a_2^2 + F & a_1^2 + 4a_2^2 + a_3^2 - 2F & -2a_2^2 - 2a_3^2 + F \\ a_2^2 & -2a_2^2 - 2a_3^2 + F & a_2^2 + 4a_3^2 + a_4^2 - 2F \end{pmatrix}, \tag{5.69}$$

where F is the dimensionless axial force. We assume that the cross-section areas at the ends of the column are equal and given by $a_0 = a_4 = \sqrt{3}$, and the total volume is fixed by the condition $a_1 + a_2 + a_3 = 7/2$. Then the stiffness matrix $\mathbf{P}$ depends on a vector of three parameters $\mathbf{p} = (a_1, a_3, F)$ with the natural constraints $a_1 > 0$, $a_3 > 0$, and $a_2 = 7/2 - a_1 - a_3 > 0$.

Let us consider stability of the system in the vicinity of the point $\mathbf{p}_0 = (1, 1, 7/2)$. At this point the matrix $\mathbf{P}(\mathbf{p})$ takes the form

$$\mathbf{P}_0 = \begin{pmatrix} 9/4 & -3 & 9/4 \\ -3 & 4 & -3 \\ 9/4 & -3 & 9/4 \end{pmatrix}. \tag{5.70}$$

The matrix $\mathbf{P}_0$ possesses the double zero eigenvalue $\lambda = 0$ and the simple eigenvalue $\lambda = 17/2$. Thus, at $\mathbf{p} = \mathbf{p}_0$ we have the cone singularity.

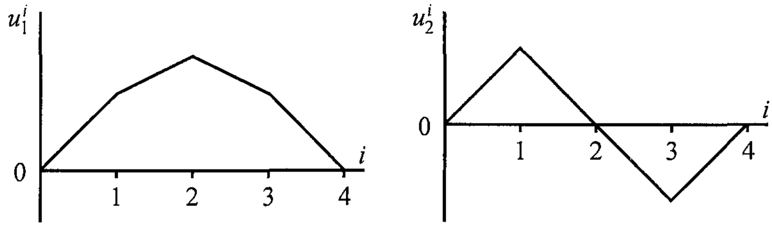


Fig. 5.6 Symmetric and anti-symmetric buckling modes of the optimal column.

The eigenvectors corresponding to $\lambda_0 = 0$ and satisfying normalization conditions (5.51) are

$$\mathbf{u}_1 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{5.71}$$

Eigenvectors (5.71) represent the symmetric and anti-symmetric buckling modes of the column, see Fig. 5.6. The vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and the constant γ evaluated by formulae (5.54), (5.62), and (5.64) are

$$\mathbf{f}_{11} = \frac{10}{17}(-1,-1,-1), \quad \mathbf{f}_{12} = \frac{4}{\sqrt{34}}(1,-1,0), \quad \mathbf{f}_{22} = (4,4,-2),$$

$$\mathbf{a} = \frac{48\sqrt{2}}{17\sqrt{17}}(2,2,13), \quad \mathbf{b} = \frac{120}{17}(-1,1,0), \quad \mathbf{c} = \frac{16\sqrt{2}}{17\sqrt{17}}(11,11,29),$$

$$\gamma = -\frac{480\sqrt{2}}{17\sqrt{17}} < 0. \tag{5.72}$$

The first order approximation of the stability boundary in the neighborhood of the singular point $\mathbf{p}_0$ is given by cone (5.63), where $\Delta\mathbf{p} = (\Delta a_1, \Delta a_3, \Delta F)$.

Fig. 5.7 Optimal design of the column.

For fixed parameters a_1 and a_3, the minimal positive value of the force F, at which the system becomes unstable, is called the *critical force* F_{cr}. The third component of the vector $\Delta\mathbf{p}$ in (5.63) is equal to

$$\Delta F = \beta\,\frac{16\sqrt{2}}{17\sqrt{17}}(39 + 29\cos\alpha), \tag{5.73}$$

which is negative for all α and $\beta < 0$. Hence, increment of the force ΔF inside the stability domain is negative for all small perturbations of the geometric parameters Δa_1 and Δa_3. As a result, the critical force F_{cr} attains the maximum at $a_1 = a_3 = 1$. The parameters $a_1 = a_3 = 1$, $a_2 = 3/2$ yield the optimal design of the column, see Fig. 5.7, which is called bimodal since there are two linearly independent eigenvectors (modes) corresponding to zero eigenvalue at F_{cr} [Seyranian *et al.* (1994)]. The stability boundary calculated numerically is shown in Fig. 5.8. Numerical analysis confirms existence of the cone singularity. The numerical results are in a good agreement with first order approximation (5.63) of the stability boundary near $\mathbf{p}_0$.

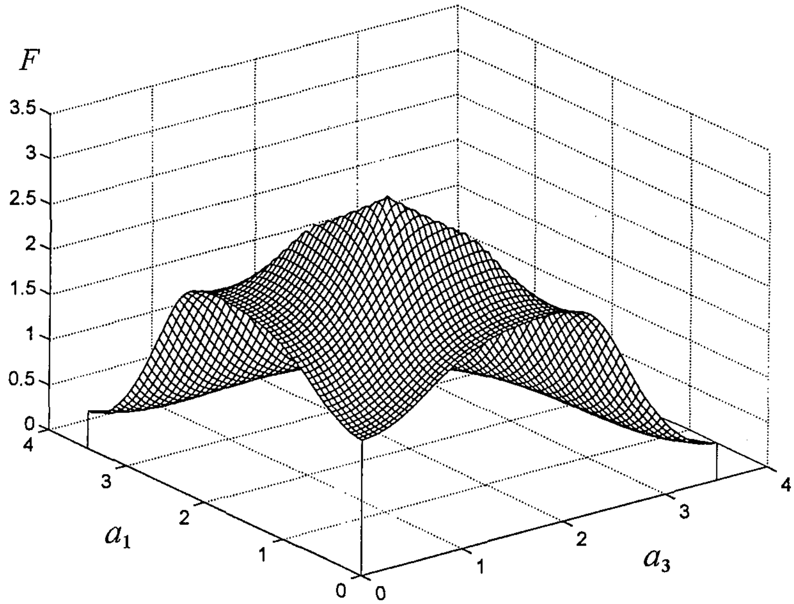


Fig. 5.8 Stability boundary for the column in the parameter space $\mathbf{p} = (a_1, a_3, F)$.

We note that the problem considered is a simplified version of the famous Lagrange problem on optimal column [Lagrange (1868); Seyranian and Privalova (2003)]. It is interesting that the bimodal optimal design shown in Fig. 5.7 qualitatively agrees with the solution of the problem in continuous formulation.

5.6 Effect of small dissipative forces on eigenvalues and stability of conservative system

By Thomson–Tait–Chetayev theorem [Chetayev (1961)], addition of dissipative forces to stable conservative system (5.1) makes the system asymptotically stable. Work done by dissipative forces is negative and decreases the energy of the system. As a result, position of the system tends to the equilibrium $\mathbf{q} = 0$ as $t \to +\infty$.

Let us consider the system

$$\mathbf{M}\ddot{\mathbf{q}} + \varepsilon\mathbf{D}\dot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \tag{5.74}$$

where $\mathbf{D}$ is a symmetric positive definite matrix describing dissipative forces, and ε is a small positive parameter. The eigenvalue problem corresponding to (5.74) takes the form

$$(\lambda^2\mathbf{M} + \lambda\varepsilon\mathbf{D} + \mathbf{P})\mathbf{u} = 0. \tag{5.75}$$

At $\varepsilon = 0$ we have conservative system (5.1) with eigenvalue problem (5.6). Assuming that the conservative system is stable and the frequencies $\omega_1, \dots, \omega_m$ are simple, the eigenvalues λ are

$$\lambda = \pm i\omega_j, \quad j = 1, \dots, m. \tag{5.76}$$

By Theorem 2.11 (page 77), eigenvalues (5.76) smoothly depend on ε, and the derivative of the eigenvalue $\lambda = i\omega$ at $\varepsilon = 0$ is equal to

$$\frac{d\lambda}{d\varepsilon} = -\frac{\mathbf{u}^T\mathbf{D}\mathbf{u}}{2\mathbf{u}^T\mathbf{M}\mathbf{u}}, \tag{5.77}$$

where $\mathbf{u}$ is the corresponding right eigenvector (mode of vibration). Recall that the right and left eigenvectors coincide due to the symmetry of the matrices $\mathbf{M}$ and $\mathbf{P}$. By the assumption that the matrices $\mathbf{M}$ and $\mathbf{D}$ are positive definite, derivative (5.77) is negative. Therefore, derivatives of eigenvalues (5.76) are real and negative.

We see that addition of small dissipative forces $\varepsilon\mathbf{D}\dot{\mathbf{q}}$ to conservative system (5.1) pushes all simple eigenvalues off the imaginary axis to the left, parallel to the real axis; see Fig. 5.9. This result can be easily extended to the case of multiple eigenvalues. Thus, addition of small damping makes the stable conservative system asymptotically stable.

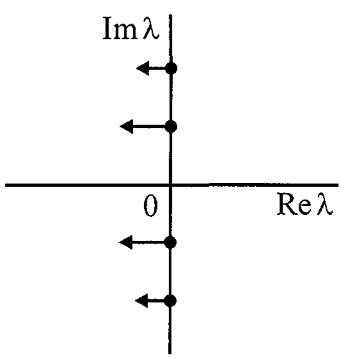


Fig. 5.9 Perturbation of eigenvalues of a conservative system due to dissipative forces.

Chapter 6

Gyroscopic Stabilization

The theory of gyroscopic systems has a history which is more than one hundred years old. The possibility of stabilization of a statically unstable conservative system by gyroscopic forces is well known in mechanics for all kinds of rotating bodies such as tops, elastic shafts, satellites, spacecrafts *etc.* We also notice that for some boundary conditions Coriolis forces appearing in elastic pipes conveying fluid are of gyroscopic nature.

We restrict ourselves to mentioning only a few of the numerous books and papers on this subject. One of the first important contributions in this field is [Thomson and Tait (1879)]. This topic has been also treated in the books [Chetayev (1961); Lancaster (1966); Müller (1977); Huseyin (1978); Merkin (1997)] and articles [Hagedorn (1975); Lakhadanov (1975); Barkwell and Lancaster (1992); Seyranian (1993b); Seyranian *et al.* (1995); Kliem and Seyranian (1997); Mailybaev and Seyranian (1999a); Seyranian and Kliem (2001)]. In this literature one can find more references.

In this chapter we study stability of linear gyroscopic systems, i.e., conservative systems with gyroscopic forces. We discuss general properties of gyroscopic systems and behavior of eigenvalues with a change of parameters. It is shown that strong interaction of eigenvalues is the mechanism of gyroscopic stabilization as well as loss of stability. As mechanical examples we consider stability problems for rotating shafts.

6.1 General properties of gyroscopic system

Let us consider a linear gyroscopic system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \tag{6.1}$$

where $\mathbf{M}$ and $\mathbf{P}$ are $m \times m$ real symmetric mass and stiffness matrices with $\mathbf{M} > 0$, $\mathbf{G}$ is a real skew-symmetric matrix of the same size representing gyroscopic forces, and $\mathbf{q}$ is a vector of generalized coordinates. Separating the time with $\mathbf{q} = \mathbf{u} \exp \lambda t$, we arrive at the eigenvalue problem

$$\left(\lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{P}\right) \mathbf{u} = 0. \tag{6.2}$$

Here λ is an eigenvalue and $\mathbf{u}$ is a corresponding eigenvector. The eigenvalues are found from the characteristic equation

$$\det \left(\lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{P}\right) = 0. \tag{6.3}$$

Since the determinants of transposed matrices are equal and $\mathbf{M}^T = \mathbf{M}$, $\mathbf{P}^T = \mathbf{P}$, $\mathbf{G}^T = -\mathbf{G}$, it follows from (6.3) that

$$\det \left(\lambda^2 \mathbf{M} + \lambda \mathbf{G} + \mathbf{P}\right)^T = \det \left(\lambda^2 \mathbf{M} - \lambda \mathbf{G} + \mathbf{P}\right) = 0. \tag{6.4}$$

Hence, $-\lambda$ is also a root of the characteristic equation, i.e., it is an eigenvalue. Moreover, since all the system matrices are real, the complex conjugate $\overline{\lambda}$ is an eigenvalue of (6.2) too. Thus, if λ is an eigenvalue, then $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are also eigenvalues. This means that the eigenvalues of gyroscopic system (6.2) are mirror symmetrical with respect to the real and imaginary axes on the complex plane, see Fig. 6.1. This property indicates that stability of the gyroscopic system can be achieved only when all the eigenvalues λ are purely imaginary and simple or semi-simple, and the asymptotic stability can not take place.

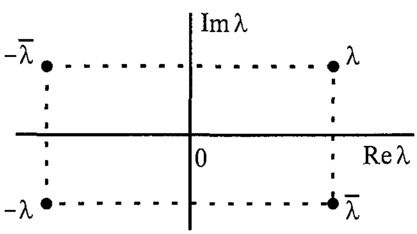


Fig. 6.1 Mirror symmetry of eigenvalues.

6.2 Positive definite stiffness matrix

Consider now the case when the matrix of potential forces is positive definite $\mathbf{P} > 0$. Then it is easy to show that all the eigenvalues of (6.2) are purely imaginary. Indeed, pre-multiplying equation (6.2) by the vector $\mathbf{u}^* = \overline{\mathbf{u}}^T$, we obtain a quadratic equation for λ

$$M\lambda^2 + iG\lambda + P = 0 \tag{6.5}$$

with the coefficients

$$M = \mathbf{u}^*\mathbf{M}\mathbf{u}, \quad P = \mathbf{u}^*\mathbf{P}\mathbf{u}, \quad iG = \mathbf{u}^*\mathbf{G}\mathbf{u}. \tag{6.6}$$

Notice that M, P, and G are real numbers, and $M > 0$, $P > 0$ since $\mathbf{M} > 0$, $\mathbf{P} > 0$. Solving quadratic equation (6.5), we find

$$\lambda = \frac{-iG \pm \sqrt{-G^2 - 4PM}}{2M}. \tag{6.7}$$

At least one of the roots (6.7) is the eigenvalue of (6.2) with the corresponding eigenvector $\mathbf{u}$. Due to the properties $M > 0$, $P > 0$ both roots in (6.7) are purely imaginary. Thus, for a positive definite matrix $\mathbf{P}$ all the eigenvalues of (6.2) are purely imaginary $\lambda = \pm i\omega$; $\omega > 0$ being vibration frequencies.

Besides, we can prove that for $\mathbf{P} > 0$ all the eigenvalues of (6.2) are simple or semi-simple, i.e., the Keldysh chain can not appear, see Section 2.13. To prove this property we assume that the Keldysh chain exists and will show that this can not take place. Let us consider the right eigenvector $\mathbf{u}$ corresponding to an eigenvalue λ with the Keldysh chain of order $k \geq 2$. Then, the eigenvector $\mathbf{u}$ must satisfy the orthogonality condition

$$\mathbf{v}^T(2\lambda\mathbf{M} + \mathbf{G})\mathbf{u} = 0 \tag{6.8}$$

for any left eigenvector $\mathbf{v}$, see Section 2.13. The operator $\mathbf{L} = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{P}$ is Hermitian for $\lambda = i\omega$. Indeed, in this case we have $\lambda = i\omega = -\overline{\lambda}$ and then

$$\mathbf{L}^* = \overline{\lambda}^2\mathbf{M}^T + \overline{\lambda}\mathbf{G}^T + \mathbf{P}^T = \lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{P} = \mathbf{L}. \tag{6.9}$$

Taking complex-conjugate transpose of equation $\mathbf{v}^T\mathbf{L} = 0$ for the left eigenvalue problem, we have

$$(\mathbf{v}^T\mathbf{L})^* = \mathbf{L}^*\overline{\mathbf{v}} = \mathbf{L}\overline{\mathbf{v}} = 0. \tag{6.10}$$

So, we find the left eigenvector as $\mathbf{v} = \overline{\mathbf{u}}$. Substituting the vector $\mathbf{v}$ into equation (6.8) and multiplying it by $\lambda = i\omega$, we get

$$\mathbf{u}^*(2\lambda^2\mathbf{M} + \mathbf{G}\lambda)\mathbf{u} = 2M\lambda^2 + iG\lambda = 0. \tag{6.11}$$

Using now equation (6.5), we obtain

$$2M\lambda^2 + iG\lambda = M\lambda^2 - P = -M\omega^2 - P = 0, \tag{6.12}$$

which contradicts to the properties $M > 0$ and $P > 0$.

Hence, for $\mathbf{P} > 0$ the Keldysh chain can not appear, and all the eigenvalues $\lambda = \pm i\omega$ are simple or semi-simple. This implies stability of gyroscopic system (6.1).

6.2.1 *Sensitivity analysis of vibration frequencies*

We assume now that the system matrices $\mathbf{M}$, $\mathbf{G}$, and $\mathbf{P}$ smoothly depend on a vector of real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. To find variation of a simple eigenvalue $\Delta\lambda$ due to a change of parameters $\Delta\mathbf{p} = (\Delta p_1, \ldots, \Delta p_n)$ we use the results of Section 2.13 and obtain

$$\Delta\lambda = -\frac{\mathbf{v}^T(\lambda^2\Delta\mathbf{M} + \lambda\Delta\mathbf{G} + \Delta\mathbf{P})\mathbf{u}}{\mathbf{v}^T(2\lambda\mathbf{M} + \mathbf{G})\mathbf{u}} + o(||\Delta\mathbf{p}||) \tag{6.13}$$

with

$$\Delta\mathbf{M} = \sum_{j=1}^{n} \frac{\partial\mathbf{M}}{\partial p_j}\Delta p_j, \quad \Delta\mathbf{G} = \sum_{j=1}^{n} \frac{\partial\mathbf{G}}{\partial p_j}\Delta p_j, \quad \Delta\mathbf{P} = \sum_{j=1}^{n} \frac{\partial\mathbf{P}}{\partial p_j}\Delta p_j. \tag{6.14}$$

We multiply both the numerator and denominator of (6.13) by λ and substitute $\lambda = i\omega$ and $\mathbf{v} = \overline{\mathbf{u}}$. Then, using (6.2) we get variation of the frequency as

$$\Delta\omega = \frac{\mathbf{u}^*(-\omega^3\Delta\mathbf{M} + i\omega^2\Delta\mathbf{G} + \omega\Delta\mathbf{P})\mathbf{u}}{\omega^2\,\mathbf{u}^*\mathbf{M}\mathbf{u} + \mathbf{u}^*\mathbf{P}\mathbf{u}} + o(||\Delta\mathbf{p}||). \tag{6.15}$$

Notice that the numerator and denominator of this expression are real, and the denominator is positive due to the properties $\mathbf{M} > 0$ and $\mathbf{P} > 0$. If variation of the vector of parameters $\Delta\mathbf{p}$ is such that $\Delta\mathbf{M} > 0$, $\Delta\mathbf{P} = \Delta\mathbf{G} = 0$ (only the mass matrix is changed), then from (6.15) we find $\Delta\omega < 0$ for rather small $||\Delta\mathbf{p}||$. And if the variation $\Delta\mathbf{p}$ is such that $\Delta\mathbf{P} > 0$, $\Delta\mathbf{M} = \Delta\mathbf{G} = 0$ (only the stiffness matrix is changed), then from (6.15) we obtain $\Delta\omega > 0$ for rather small $||\Delta\mathbf{p}||$. Thus, we have proven the statement.

Theorem 6.1 *Vibration frequencies of gyroscopic system (6.1) with a positive definite stiffness matrix* $\mathbf{P}$ *increase with an increase of stiffness, and decrease with an increase of mass.*

The increase of stiffness and mass are treated here in the sense $\Delta\mathbf{P} > 0$ and $\Delta\mathbf{M} > 0$, respectively. This theorem was formulated first in [Metelitsyn (1963)] and was proven in [Zhuravlev (1976)]. It is generalization of Rayleigh's theorem on the behavior of frequencies of a conservative system to gyroscopic system. The case of multiple frequencies was considered in [Seyranian and Sharanyuk (1987)].

We should point out that local relation (6.15) enables us to make not only qualitative but also quantitative estimates for the sensitivity of vibration frequencies of gyroscopic systems dependent on parameters. For this purpose, using (6.14) equation (6.15) can be written in the gradient form as

$$\Delta\omega = (\nabla\omega, \Delta\mathbf{p}) + o(||\Delta\mathbf{p}||), \quad \nabla\omega = \left(\frac{\partial\omega}{\partial p_1}, \ldots, \frac{\partial\omega}{\partial p_n}\right), \tag{6.16}$$

where

$$\frac{\partial\omega}{\partial p_j} = \mathbf{u}^* \left(-\omega^3 \frac{\partial\mathbf{M}}{\partial p_j} + i\omega^2 \frac{\partial\mathbf{G}}{\partial p_j} + \omega \frac{\partial\mathbf{P}}{\partial p_j}\right) \mathbf{u} \Big/ \left(\omega^2\, \mathbf{u}^*\mathbf{M}\mathbf{u} + \mathbf{u}^*\mathbf{P}\mathbf{u}\right). \tag{6.17}$$

This form of equation (6.15) is more practical for estimation of changes of vibration frequencies with a change of system parameters.

Example 6.1 As an example, we consider small oscillations of an uniaxial gyrostabilizer with elastic compliance of the elements described by the equations [Ishlinskii (1976)]

$$\begin{aligned} &A\ddot{\alpha} + H\dot{\beta} - K(\psi - \alpha) = 0, \\ &A\ddot{\beta} - H\dot{\alpha} = 0, \\ &\Psi\ddot{\psi} + K(\psi - \alpha) - N(\theta - \psi) = 0, \\ &\Theta\ddot{\theta} + N(\theta - \psi) = 0. \end{aligned} \tag{6.18}$$

Here A, Ψ, Θ are the moments of inertia of the system; K and N are the stiffness parameters; H is the angular momentum of the gyroscope rotor; α, β, ψ, and θ are the angles describing disturbed motion of the system. With the vector of generalized coordinates $\mathbf{q} = (\alpha, \beta, \psi, \theta)^T$, the matrices

$\mathbf{M}$, $\mathbf{G}$, and $\mathbf{P}$ according to equations (6.1) and (6.18) take the form

$$\mathbf{M} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \Psi & 0 \\ 0 & 0 & 0 & \Theta \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & H & 0 & 0 \\ -H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathbf{P} = \begin{pmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K+N & -N \\ 0 & 0 & -N & N \end{pmatrix}. \tag{6.19}$$

Thus, gyroscopic system (6.18) has $m = 4$ degrees of freedom, and the vector of parameters $\mathbf{p} = (A, \Psi, \Theta, H, K, N)$ is of dimension $n = 6$. The matrix $\mathbf{M}$ is positive definite for positive values of the parameters A, Ψ, Θ, while the matrix $\mathbf{P}$ is positive semi-definite for positive values of K and N.

Calculations were performed for the following values of parameters taken from [Ishlinskii (1976)]: $H/A = 200sec^{-1}$, $K/A = 10^4 sec^{-2}$, $K/\Psi = 1600sec^{-2}$, $N/\Theta = 6400sec^{-2}$, $\Theta/\Psi = 0.2$, and $A = 10g \cdot cm^2$. System (6.18) possesses three nonzero vibration frequencies and one zero frequency. The vibration frequencies ω, the corresponding eigenvectors $\mathbf{u}$, and the gradients $\nabla\omega$ were evaluated using (6.2), (6.16), and (6.17). The results of calculations are given below:

$$\begin{aligned}
&\omega_1 = 32.067, \\
&\nabla\omega_1 = (-0.331, -0.199, -0.282, 0.332 \times 10^{-2}, 0.122 \times 10^{-3}, 0.708 \times 10^{-5}); \\
&\omega_2 = 88.957, \\
&\nabla\omega_2 = (-0.053, -0.155, -2.78, 0.443 \times 10^{-3}, 0.114 \times 10^{-4}, 0.536 \times 10^{-3}); \\
&\omega_3 = 224.36, \\
&\nabla\omega_3 = (-20, -0.0128, -0.271 \times 10^{-3}, 0.0885, 0.236 \times 10^{-3}, 0.333 \times 10^{-6}).
\end{aligned} \tag{6.20}$$

Thus, for example, the first order increment of the third frequency is

$$\begin{aligned}
\Delta\omega_3 = &-20\Delta A - 0.0128\Delta\Psi - 0.271 \times 10^{-3}\Delta\Theta + 0.0885\Delta H \\
&+ 0.236 \times 10^{-3}\Delta K + 0.333 \times 10^{-6}\Delta N.
\end{aligned} \tag{6.21}$$

As is to be expected, increase of the mass characteristics A, Ψ, Θ corresponds to decrease of the frequencies ω_1, ω_2, and ω_3, while increase of the stiffness parameters K and N corresponds to increase of those frequencies. Notice that increase of the angular momentum H (for given values of the parameters) corresponds to increase of all three frequencies. The frequency ω_3 is mostly affected by the parameter A. The first frequency ω_1 is affected by the parameters A, Ψ, and Θ in roughly the same order. As for ω_2, it is mostly affected by the parameter Θ, while the angular momentum H has an appreciable effect on the third frequency. The effect of the stiffness parameters K and N is relatively slight. The parameter N exerts the greatest effect on ω_2.

6.3 Loss of stability

In this section we consider the case when the positive definite matrix of potential forces $\mathbf{P}$ loses this property with a change of parameters. As a simple example, we consider the matrix $\mathbf{P}$ in the form

$$\mathbf{P} = \mathbf{C} - p\mathbf{B}. \tag{6.22}$$

The matrices $\mathbf{C}$ and $\mathbf{B}$ are assumed to be symmetric and positive definite, and $p \geq 0$ is a single real parameter.

According to (6.17) for a simple eigenvalue $\lambda = i\omega$ we obtain

$$\frac{d\omega}{dp} = -\frac{\omega \mathbf{u}^* \mathbf{B} \mathbf{u}}{\omega^2 \mathbf{u}^* \mathbf{M} \mathbf{u} + \mathbf{u}^* (\mathbf{C} - p\mathbf{B}) \mathbf{u}}. \tag{6.23}$$

As one might expect, when p increases from 0 all the frequencies decrease. Then a pair of the eigenvalues $\lambda = \pm i\omega$, smallest for their absolute values, merge to zero. Taking in equation (6.2) $\lambda = 0$ and using (6.22), we get the eigenvalue problem for p:

$$\mathbf{C}\mathbf{u} = p\mathbf{B}\mathbf{u}. \tag{6.24}$$

This problem with the symmetric positive definite matrices $\mathbf{C}$ and $\mathbf{B}$ possesses m real eigenvalues $0 < p_1 \leq \ldots \leq p_m$.

Thus, the first meeting of the eigenvalues $\lambda = \pm i\omega$ takes place at $p = p_1$. At this point relation (6.23) is not valid since the eigenvalues become multiple zero.

6.3.1 *Strong interaction*

If the first eigenvalue p_1 of problem (6.24) is simple (possessing one eigenvector), then the strong interaction takes place, see Section 2.6. As a result of the strong interaction, the gyroscopic system loses stability for $p > p_1$, see Fig. 6.2. This is the static form of instability (divergence).

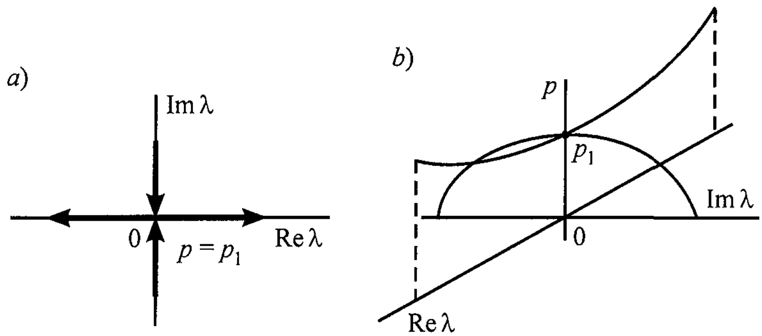


Fig. 6.2 Static loss of stability via strong interaction of eigenvalues.

To describe bifurcation of the double eigenvalue $\lambda = 0$ at $p = p_1$ in case of the strong interaction we use the results of Section 2.13. For gyroscopic system (6.2), (6.22) we obtain

$$\lambda = \pm\lambda_1\sqrt{\Delta p} + O(\Delta p), \quad \Delta p = p - p_1, \tag{6.25}$$

with the coefficient λ_1 given by the equality

$$\lambda_1^2 = \mathbf{v}_0^T\mathbf{B}\mathbf{u}_0. \tag{6.26}$$

The right eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$ corresponding to $\lambda = 0$ are real and satisfy the equations

$$(\mathbf{C} - p_1\mathbf{B})\mathbf{u}_0 = 0, \tag{6.27}$$

$$(\mathbf{C} - p_1\mathbf{B})\mathbf{u}_1 = -\mathbf{G}\mathbf{u}_0, \tag{6.28}$$

and the left eigenvector $\mathbf{v}_0$ satisfies the equation and normalization condition

$$\mathbf{v}_0^T(\mathbf{C} - p_1\mathbf{B}) = 0, \quad \mathbf{v}_0^T(\mathbf{G}\mathbf{u}_1 + \mathbf{M}\mathbf{u}_0) = 1. \tag{6.29}$$

Due to the symmetry of the matrix $\mathbf{C} - p_1\mathbf{B}$, the vector $\mathbf{v}_0$ is found as

$$\mathbf{v}_0 = \frac{\mathbf{u}_0}{\mathbf{u}_0^T\mathbf{M}\mathbf{u}_0 + \mathbf{u}_0^T\mathbf{G}\mathbf{u}_1}. \tag{6.30}$$

Using (6.28) and (6.30), we can write (6.26) in the form

$$\lambda_1^2 = \frac{\mathbf{u}_0^T\mathbf{B}\mathbf{u}_0}{\mathbf{u}_0^T\mathbf{M}\mathbf{u}_0 + \mathbf{u}_1^T(\mathbf{C} - p_1\mathbf{B})\mathbf{u}_1}. \tag{6.31}$$

According to Rayleigh's principle

$$\mathbf{u}_1^T(\mathbf{C} - p_1\mathbf{B})\mathbf{u}_1 > 0, \tag{6.32}$$

since p_1 is the first eigenvalue of self-adjoint problem (6.24) with the eigenvector $\mathbf{u}_0$. Hence, the numerator and denominator in expression (6.31) are positive, and $\lambda_1^2 > 0$. The strong interaction of eigenvalues at p_1 described by (6.25), (6.31) is shown in Fig. 6.2.

In the three-dimensional space $(\operatorname{Re}\lambda, \operatorname{Im}\lambda, p)$ the strong interaction of eigenvalues is characterized by change of the plane of interaction, which causes instability. The curves in Fig. 6.2b at $p \approx p_1$ according to (6.25) behave like parabolae with equal curvatures lying in the orthogonal planes.

Notice that at p equal to the other eigenvalues of (6.24) we have $\lambda = 0$, and the strong interaction can also take place. However, the denominator in (6.31) is not necessarily positive because inequality (6.32) holds only for the first eigenvalue p_1. This means that the quantity λ_1 in (6.25) may be purely imaginary. This case corresponds to approach of two real λ as p increases, merging to zero, and then diverging in opposite directions along the imaginary axis. Thus, the strong interaction of eigenvalues is the mechanism of losing as well as gaining stability of a gyroscopic system.

6.3.2 *Weak interaction*

This is the case when p_1 is a double root of the characteristic equation $\det(\mathbf{C} - p\mathbf{B}) = 0$. Due to symmetry of the matrices $\mathbf{B}$ and $\mathbf{C}$ there are two linearly independent real eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. We assume that the eigenvalue $\lambda = 0$ of problem (6.2) at $p = p_1$ is semi-simple possessing two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. The corresponding left eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, satisfying the normalization conditions

$$\mathbf{v}_i^T\mathbf{G}\mathbf{u}_j = \delta_{ij}, \quad i, j = 1, 2, \tag{6.33}$$

are

$$\mathbf{v}_1 = \frac{\mathbf{u}_2}{\mathbf{u}_2^T \mathbf{G} \mathbf{u}_1}, \quad \mathbf{v}_2 = -\frac{\mathbf{u}_1}{\mathbf{u}_2^T \mathbf{G} \mathbf{u}_1}. \tag{6.34}$$

Here, the condition $\mathbf{u}_2^T \mathbf{G} \mathbf{u}_1 \neq 0$ is satisfied, since it is equivalent to the absence of the Keldysh chain for $\lambda = 0$.

Using vectors (6.34) in Theorem 2.12 (page 77), we get the perturbed eigenvalues as

$$\lambda = \lambda_1 \Delta p + o(\Delta p), \tag{6.35}$$

where

$$\lambda_1 = \pm \frac{i\sqrt{\mathbf{u}_1^T \mathbf{B} \mathbf{u}_1 \mathbf{u}_2^T \mathbf{B} \mathbf{u}_2 - (\mathbf{u}_1^T \mathbf{B} \mathbf{u}_2)^2}}{\mathbf{u}_2^T \mathbf{G} \mathbf{u}_1}. \tag{6.36}$$

Notice that the expression under the square root in (6.36) is positive since it represents the determinant of Gram matrix for two linearly independent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ with the scalar product defined by the positive definite matrix $\mathbf{B}$. Hence, λ_1 is purely imaginary.

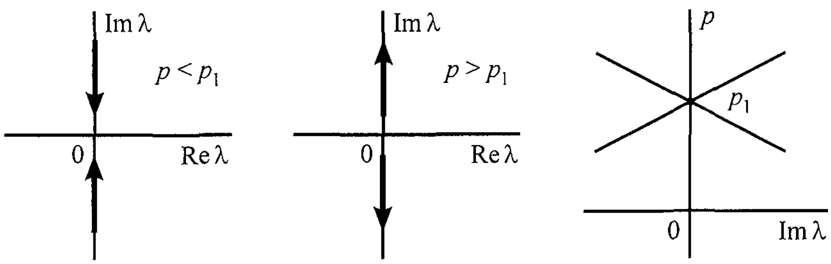


Fig. 6.3 Weak interaction of eigenvalues.

The weak interaction of eigenvalues is shown in Fig. 6.3. The eigenvalues pass zero along the imaginary axis, and the gyroscopic system does not lose stability. The system remains stable also at the critical point p_1, because the double eigenvalue $\lambda = 0$ is semi-simple.

This means that gyroscopic stabilization at $p > p_1$ can be achieved by an appropriate choice of the matrix $\mathbf{P}(p)$.

6.3.3 *Further increase of parameter p*

With further increase of the parameter $p > p_1$, at $p = p_2, p_3, \ldots, p_m$, the pairs of eigenvalues $\lambda = \pm i\omega$ pass zero. At those points their interaction may be strong or weak. Nevertheless, we can show that for rather large values of p system (6.1), (6.22) becomes unstable. To prove this statement we use expression (6.7). The system is unstable if the inequality

$$-G^2 - 4PM > 0 \tag{6.37}$$

is valid. According to (6.6) and (6.22), $P = C - pB$ with $C = \mathbf{u}^*\mathbf{C}\mathbf{u} > 0$ and $B = \mathbf{u}^*\mathbf{B}\mathbf{u} > 0$. Using this expression in (6.37), we obtain the instability condition as

$$p > p_c, \quad p_c = \frac{G^2 + 4CM}{4BM}. \tag{6.38}$$

If this inequality holds for at least one eigenvector $\mathbf{u}$, the system becomes unstable.

Let us estimate the critical value p_c. First of all, we have

$$p_c = \frac{G^2 + 4CM}{4BM} \geq \frac{C}{B} \geq p_1. \tag{6.39}$$

The last inequality holds due to Rayleigh's principle. Relation (6.39) provides the lower bound for p_c.

We use the estimates of Section 1.7 to obtain the upper bound

$$\begin{aligned} p_c &= \frac{G^2 + 4CM}{4BM} = \frac{G^2}{4BM} + \frac{C}{B} \\ &\leq \frac{G^2_{\max}}{4B_{\min}M_{\min}} + \max\frac{C}{B} \leq \frac{G^2_{\max}}{4B_{\min}M_{\min}} + p_m = p_*, \end{aligned} \tag{6.40}$$

where $G_{\max}$ is the maximal absolute value for the eigenvalues of the matrix $\mathbf{G}$; $B_{\min}$ and $M_{\min}$ are the minimal eigenvalues of the symmetric matrices $\mathbf{B}$ and $\mathbf{M}$, respectively; and p_m is the maximal eigenvalue for the symmetric problem (6.24).

Combining estimates (6.39) and (6.40), we obtain $p_1 \leq p_c \leq p_*$. For all $p > p_*$ system (6.1), (6.22) is unstable. The loss of stability can be static (through multiple $\lambda = 0$) or dynamic (through multiple $\lambda = \pm i\omega$, $\omega \neq 0$).

Example 6.2 As an example we consider interaction of eigenvalues for an elastic shaft of non-circular cross-section with a mounted disk rotating with a constant angular velocity p, see Fig. 6.4.

The linearized equations of motion of the disk in the coordinate system, rotating uniformly with the angular velocity p, are the following [Ziegler

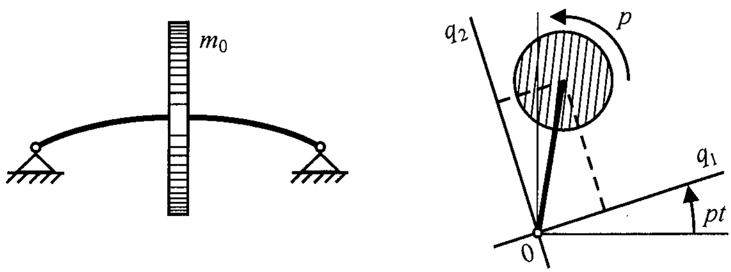


Fig. 6.4 Rotating shaft with a disk.

(1968)]

$$\begin{aligned} &\ddot{q}_1 - 2p\dot{q}_2 + \left(\frac{c_1}{m_0} - p^2\right) q_1 = 0, \\ &\ddot{q}_2 + 2p\dot{q}_1 + \left(\frac{c_2}{m_0} - p^2\right) q_2 = 0. \end{aligned} \tag{6.41}$$

In these equations m_0 is the mass of the disk, c_1 and c_2 are the elastic constants, representing the stiffnesses of the shaft with respect to its two principle axes, and we assume $c_1 < c_2$. Dissipative forces are neglected.

Finding solution to (6.41) in the form $q_1 = u_1 \exp \lambda t$, $q_2 = u_2 \exp \lambda t$, we obtain

$$\begin{aligned} &\left(\lambda^2 + \frac{c_1}{m_0} - p^2\right) u_1 - 2p\lambda u_2 = 0, \\ &2p\lambda u_1 + \left(\lambda^2 + \frac{c_2}{m_0} - p^2\right) u_2 = 0. \end{aligned} \tag{6.42}$$

Here λ is an eigenvalue, and $\mathbf{u} = (u_1, u_2)^T$ is an eigenvector. A nontrivial solution to equations (6.42) exists if λ satisfies the characteristic equation

$$\lambda^4 + \left(\frac{c_1 + c_2}{m_0} + 2p^2\right)\lambda^2 + \left(\frac{c_1}{m_0} - p^2\right)\left(\frac{c_2}{m_0} - p^2\right) = 0. \tag{6.43}$$

It is easy to see that the system is stable ($\lambda^2 < 0$) if the angular velocity p satisfies the inequalities $0 \le p < \sqrt{c_1/m_0}$ or $p > \sqrt{c_2/m_0}$, and it is unstable if $\sqrt{c_1/m_0} \le p \le \sqrt{c_2/m_0}$.

The double eigenvalue $\lambda = 0$ appears at the critical values $p_1 = \sqrt{c_1/m_0}$ and $p_2 = \sqrt{c_2/m_0}$. It is characterized by a single eigenvector, which implies the strong interaction. The eigenvalues λ for the parameter p close to p_1

and p_2 are given in the first approximation by the expressions

$$\lambda = \pm\sqrt{\frac{(c_2 - c_1)\Delta p^2}{3c_1 + c_2}} + O(\|\Delta p^2\|), \quad \Delta p^2 = p^2 - \frac{c_1}{m_0},$$
$$\lambda = \pm i\sqrt{\frac{(c_2 - c_1)\Delta p^2}{c_1 + 3c_2}} + O(\|\Delta p^2\|), \quad \Delta p^2 = p^2 - \frac{c_2}{m_0}. \tag{6.44}$$

The picture of the strong interaction of eigenvalues is presented in Fig. 6.5. The interaction takes place in the planes $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda = 0$. As p increases from zero, two purely imaginary eigenvalues, minimal in the absolute value, approach each other. Then, at $p_1 = \sqrt{c_1/m_0}$, the strong interaction occurs. The eigenvalues become real and diverge making the system unstable. With further increase of p those eigenvalues begin to come together. At $p_2 = \sqrt{c_2/m_0}$ the second strong interaction takes place. The eigenvalues become purely imaginary and diverge, and the system attains stability. As to the second pair of purely imaginary eigenvalues, with an increase of p they grow monotonically in the absolute value without affecting stability.

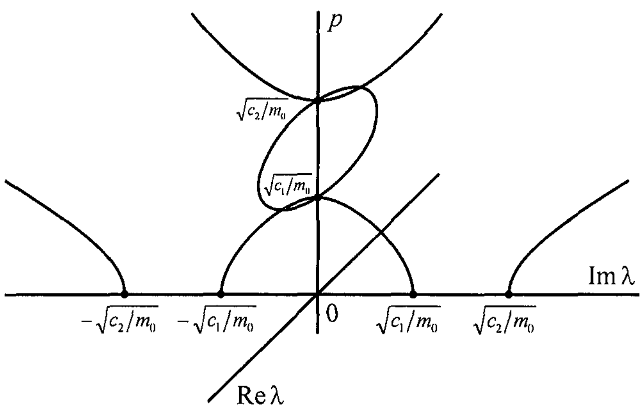


Fig. 6.5 Strong interaction of eigenvalues for rotating shaft.

In the case of $c_1 = c_2 = c$ the double eigenvalue $\lambda = 0$ appears at $p = \sqrt{c/m_0}$. It is semi-simple and possesses two linearly independent eigenvectors $\mathbf{u}_1 = (1, 0)^T$ and $\mathbf{u}_2 = (0, 1)^T$ of problem (6.42). Hence, at $p = \sqrt{c/m_0}$ the interaction of eigenvalues is weak, and the system does not lose stability. Indeed, in this case according to (6.43) we have

$$\lambda = \pm i \left(\sqrt{\frac{c}{m_0}} \pm p \right). \tag{6.45}$$

This formula means linear dependence of four eigenvalues λ on the parameter p, see Fig. 6.6.

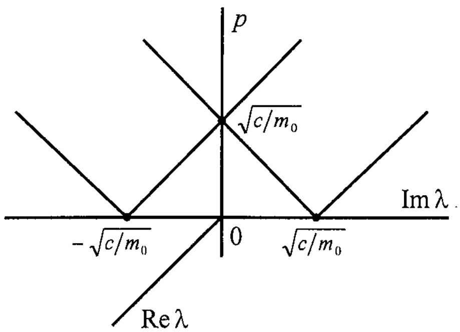


Fig. 6.6 Weak interaction of eigenvalues for rotating shaft.

The interaction of the eigenvalues occurs in the plane $\mathrm{Re}\,\lambda = 0$. At $p = 0$ the eigenvalues $\lambda_1 = i\sqrt{c/m_0}$ and $\lambda_2 = -i\sqrt{c/m_0}$ are double and semi-simple. With an increase of p they diverge along the imaginary axis. Two eigenvalues, minimal in the absolute value, approach and become zero at $p = \sqrt{c/m_0}$. Change of the plane of interaction does not occur (weak interaction), and the system remains stable for all $p \geq 0$.

Example 6.3 We study stability of a massless shaft with a mounted disk of mass m_0 rotating with a constant angular velocity p, see Fig. 6.7. The shaft is assumed to be rigid with two torsional springs with the stiffness coefficients C_1 and C_2, and is subjected to a constant vertical compression force F. The linearized equations of motion of the disk in a coordinate system, rotating uniformly with the angular velocity p, are the following [Huseyin (1978)]

$$\begin{aligned} &\ddot{q}_1 - 2p\dot{q}_2 + \left(c_1 - \eta - p^2\right) q_1 = 0, \\ &\ddot{q}_2 + 2p\dot{q}_1 + \left(c_2 - \eta - p^2\right) q_2 = 0. \end{aligned} \tag{6.46}$$

In these equations the constants $c_1 = C_1/(m_0 l^2)$ and $c_2 = C_2/(m_0 l^2)$ represent stiffnesses of the shaft, and $\eta = F/(m_0 l)$ represents the vertical force.

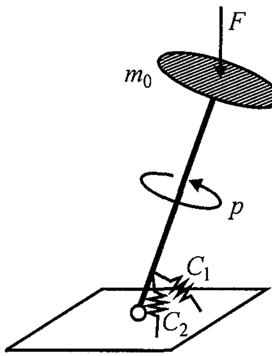


Fig. 6.7 Rotating shaft loaded by vertical force.

The characteristic equation of system (6.46) is

$$\lambda^4 + \lambda^2(c_1 + c_2 - 2\eta + 2p^2) + (c_1 - \eta - p^2)(c_2 - \eta - p^2) = 0. \qquad (6.47)$$

The system is stable if the two roots λ^2 of the biquadratic equation (6.47) are negative. This implies that the coefficients and discriminant of (6.47) must be positive. Thus, we have the stability conditions

$$\begin{aligned} & c_1 + c_2 - 2\eta + 2p^2 > 0, \\ & (c_1 - \eta - p^2)(c_2 - \eta - p^2) > 0, \\ & (c_1 - c_2)^2 + 8(c_1 + c_2 - 2\eta)p^2 > 0. \end{aligned} \qquad (6.48)$$

Let us study behavior of eigenvalues and stability of the system with a change of the angular velocity p and fixed c_1, c_2, and η. We assume that $0 < c_1 < c_2$.

If $0 \leq \eta < c_1$, then the system is stable for $p = 0$. The four eigenvalues are purely imaginary and simple. With an increase of p the eigenvalues $\lambda = \pm i\omega$, lowest for their absolute values, approach, interact strongly at the origin for $p_1 = \sqrt{c_1 - \eta}$, and diverge along the real axis. The system becomes unstable (divergence). With further increase of p these eigenvalues return to the origin, interact strongly at $p_2 = \sqrt{c_2 - \eta}$, and enter the imaginary axis. The system is stabilized. The stability region consists of two intervals $0 < p < p_1$ and $p > p_2$. Behavior of eigenvalues is similar to that of shown in Fig. 6.5.

If $c_1 < \eta < (c_1 + c_2)/2$, then there is one real pair and one purely imaginary pair of eigenvalues for $p = 0$ (the system is unstable). With an increase of p the real eigenvalues approach, interact strongly at the origin for

$p_2 = \sqrt{c_2 - \eta}$, and diverge along the imaginary axis. The system becomes stable. The region of gyroscopic stabilization is $p > p_2$.

If $(c_1 + c_2)/2 < \eta < (c_1 + 3c_2)/4$, then there is one real pair and one purely imaginary pair of eigenvalues for $p = 0$ (the system is unstable). With an increase of p the real eigenvalues approach, interact strongly at the origin for $p_2 = \sqrt{c_2 - \eta}$, and diverge along the imaginary axis. The system is stabilized. With further increase of p, two pairs of purely imaginary eigenvalues approach, interact strongly at $p_f = (c_2 - c_1)/\sqrt{8(2\eta - c_1 - c_2)}$, and become complex $\lambda = \pm\alpha \pm i\omega$. At the interaction point the double non-derogatory eigenvalues $\lambda = \pm i\sqrt{c_1/2 + c_2/2 - \eta + p_f^2}$ appear. The system becomes unstable (flutter). The region of gyroscopic stabilization in this case is the finite interval $p_2 < p < p_f$. Behavior of eigenvalues depending on p is shown in Fig. 6.8.

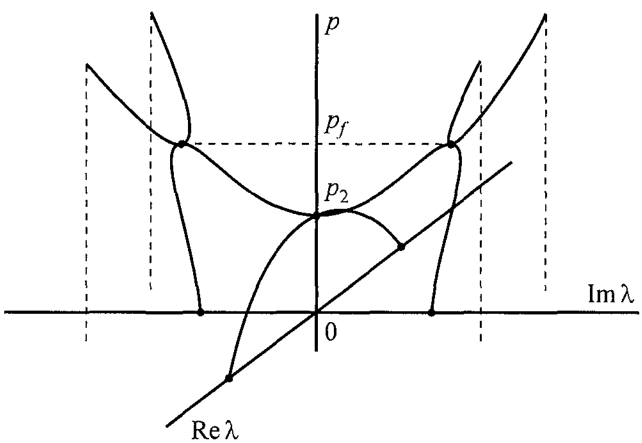


Fig. 6.8 Behavior of eigenvalues for rotating shaft.

Finally, if $\eta > (c_1 + 3c_2)/4$, then the system is unstable for all values of the angular velocity p.

Following [Inman (1988)] and [Inman (1989), page 84] we fix the values of η and p, and ask for the stability regions in the plane (c_1, c_2). For $\eta = 3$ and $p = 2$, Fig. 6.9 shows the result of the investigation of conditions (6.48) with the natural restrictions $c_1 > 0$ and $c_2 > 0$. S, F, and D in Fig. 6.9 indicate stability, flutter, and divergence regions, respectively. The flutter boundary is given by the parabola

$$(c_1 - c_2)^2 + 32(c_1 + c_2) - 192 = 0. \tag{6.49}$$

Notice that the presented picture does not entirely agree with [Inman (1988)] and is more complete than the picture given in [Inman (1989)].

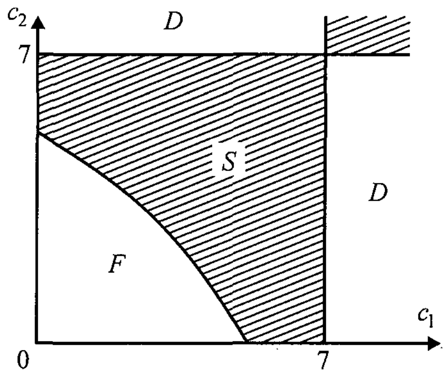


Fig. 6.9 Stability map of rotating shaft for $\eta = 3$ and $p = 2$.

6.4 Gyroscopic stabilization problem

Let us consider gyroscopic system (6.1), in which the matrix $\mathbf{P}$ is symmetric and negative definite, $\mathbf{P} = -\mathbf{C}$, $\mathbf{C} > 0$; the gyroscopic forces $p\mathbf{G}$ are proportional to a real parameter $p \geq 0$, and $\det \mathbf{G} \neq 0$. As a result, we get the eigenvalue problem

$$\left(\lambda^2 \mathbf{M} + p\lambda \mathbf{G} - \mathbf{C}\right) \mathbf{u} = 0. \tag{6.50}$$

We are interested in the behavior of eigenvalues λ with a change of the parameter p. At $p = 0$ we obtain from (6.50)

$$\mathbf{Cu} = \lambda^2 \mathbf{Mu}. \tag{6.51}$$

Since $\mathbf{M} > 0$ and $\mathbf{C} > 0$, this problem has $2m$ real eigenvalues $\lambda = \pm\alpha_i$, $i = 1, \ldots, m$, where $0 < \alpha_1 \leq \ldots \leq \alpha_m$. Thus, at $p = 0$ all the eigenvalues lie on the real axis symmetrically with respect to zero, implying instability of the system.

Along with (6.50) we consider the eigenvalue problem for the left eigenvector $\mathbf{v}$:

$$\mathbf{v}^T \left(\lambda^2 \mathbf{M} + p\lambda \mathbf{G} - \mathbf{C}\right) = 0. \tag{6.52}$$

By means of equations (6.50), (6.52) and the results of Section 2.13 we find the first derivative of a simple eigenvalue λ with respect to p:

$$\frac{d\lambda}{dp} = -\frac{\lambda \mathbf{v}^T \mathbf{G} \mathbf{u}}{2\lambda \mathbf{v}^T \mathbf{M} \mathbf{u} + p \mathbf{v}^T \mathbf{G} \mathbf{u}}. \tag{6.53}$$

At $p = 0$ we have $\mathbf{v} = \mathbf{u}$. Since $\mathbf{G}^T = -\mathbf{G}$ and the eigenvector $\mathbf{u}$ is real, $\mathbf{u}^T \mathbf{G} \mathbf{u} = 0$. Hence, according to (6.53), $d\lambda/dp = 0$ at $p = 0$ for all the simple eigenvalues λ.

Consider now the case of an eigenvalue λ_0 with algebraic multiplicity r at $p = 0$. Since problem (6.51) is symmetric, the eigenvalue λ_0 is semi-simple (weak interaction). According to the results of Section 2.13, the eigenvalue expansion for small p takes the form $\lambda = \lambda_0 + \lambda_1 p + o(p)$, where r coefficients λ_1 are found from the equation

$$\det\left(\lambda_0[\mathbf{u}_i^T \mathbf{G} \mathbf{u}_j] + \lambda_1 \mathbf{I}\right) = 0, \quad i, j = 1, \ldots, r, \tag{6.54}$$

where the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ corresponding to λ_0 are normalized as

$$2\lambda_0 \mathbf{u}_i^T \mathbf{M} \mathbf{u}_j = \delta_{ij}, \quad i, j = 1, \ldots, r. \tag{6.55}$$

Since the $r \times r$ matrix $[\mathbf{u}_i^T \mathbf{G} \mathbf{u}_j]$ is skew-symmetric, all the roots λ_1 are either purely imaginary or zero. For example, if $r = 2$ we get

$$\lambda_1 = \pm i \lambda_0 \mathbf{u}_2^T \mathbf{G} \mathbf{u}_1. \tag{6.56}$$

This means that for small $p \geq 0$ the eigenvalues become complex conjugate and diverge in opposite directions parallel to the imaginary axis.

In [Lakhadanov (1975)] it is shown that $\mathbf{C} > 0$ and $\det \mathbf{G} \neq 0$ are sufficient conditions for stabilization of a gyroscopic system at rather large values of $p \geq p_*$, and the estimate for the critical value p_* is given. As we have already noted, along with λ the quantities $-\lambda$, $\overline{\lambda}$, $-\overline{\lambda}$ are also the eigenvalues. Thus, stabilization of the gyroscopic system means that all λ come to the imaginary axis. We notice that gyroscopic stabilization cannot be achieved through $\lambda = 0$ since $\mathbf{C} > 0$.

Let us study evolution of eigenvalues λ with a change of p. At $p = 0$ all λ are real. With an increase of p the simple eigenvalues λ start moving along the real axis, see Fig. 6.10. Simple eigenvalues cannot leave the real axis, since otherwise the additional eigenvalues $\overline{\lambda}$ appear, which is impossible. Therefore, the eigenvalues can leave the real axis only at the points R, R', where two of them collide and the strong interaction occurs, see Fig. 6.10.

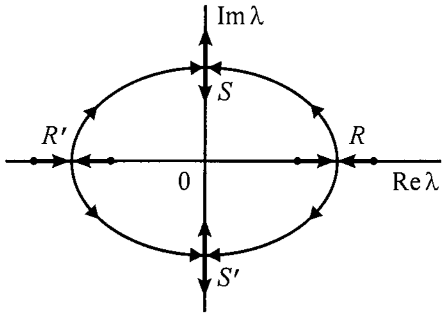


Fig. 6.10 Mechanism of gyroscopic stabilization.

With further increase of p, transition of λ to the imaginary axis occurs when the pairs λ, $-\overline{\lambda}$ and $-\lambda$, $\overline{\lambda}$ meet at the points S, S', see Fig. 6.10, the strong interaction takes place, and then the eigenvalues diverge along the imaginary axis in opposite directions. Stabilization of the gyroscopic system is attained when all λ reach the imaginary axis.

Gyroscopic stabilization in the three-dimensional space is illustrated in Fig. 6.11, where behavior of four eigenvalues with a change of p is presented.

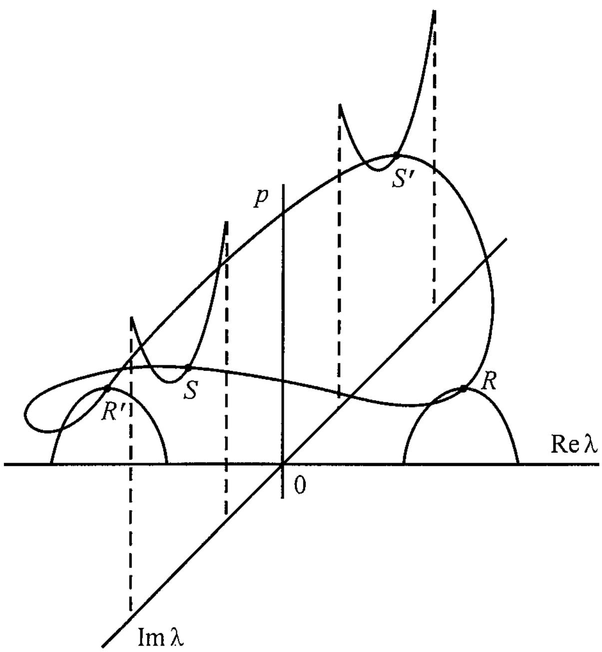


Fig. 6.11 Gyroscopic stabilization in the three-dimensional space.

So, we have proven the statement.

Theorem 6.2 *Strong interaction of eigenvalues is the mechanism of gyroscopic stabilization.*

Notice that according to one of the Thomson-Tait-Chetayev theorems, see for example [Merkin (1997)], stabilization of a gyroscopic system is possible only for even degrees of instability of the potential system (the number of positive eigenvalues of the matrix $\mathbf{C}$). Since in our case $\mathbf{C} > 0$, the number of degrees of freedom m must be even. This condition is satisfied due to the assumption $\det \mathbf{G} \neq 0$, because for an odd number m any skew-symmetric matrix $\mathbf{G}$ has the property $\det \mathbf{G} = 0$.

Thus, we can treat the Thomson–Tait–Chetayev theorem in the following way: *gyroscopic stabilization is provided by the strong interaction of pairs of eigenvalues, see Fig. 6.10. When m is an odd number, there is no pair for one of λ to interact with and leave the real axis.*

If λ is real, the corresponding eigenvector $\mathbf{u}$ is also real. Then, multiplying (6.50) by $\mathbf{u}^T$ from the left and using $\mathbf{u}^T\mathbf{G}\mathbf{u} = 0$, we get

$$\lambda^2 = \frac{\mathbf{u}^T\mathbf{C}\mathbf{u}}{\mathbf{u}^T\mathbf{M}\mathbf{u}}. \tag{6.57}$$

Hence, squares of the real eigenvalues are bounded by $\alpha_1^2 \leq \lambda^2 \leq \alpha_m^2$, where α_1^2 and α_m^2 are the minimum and maximum eigenvalues of problem (6.51).

Near the point p_0 of the strong interaction R, R' or S, S' the following bifurcation for the double eigenvalue λ_0 is valid:

$$\lambda = \lambda_0 \pm \lambda_1\sqrt{\Delta p} + O(\Delta p), \quad \Delta p = p - p_0, \tag{6.58}$$

where the coefficient λ_1 is given by Theorem 2.13 (page 80) as

$$\lambda_1^2 = -\lambda_0 \mathbf{v}_0^T\mathbf{G}\mathbf{u}_0. \tag{6.59}$$

In this equation the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ are determined by (6.50) and (6.52), respectively. Besides, the left eigenvector $\mathbf{v}_0$ satisfies the normalization condition

$$\mathbf{v}_0^T(2\lambda_0\mathbf{M} + p_0\mathbf{G})\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{M}\mathbf{u}_0 = 1, \tag{6.60}$$

where the associated vector $\mathbf{u}_1$ is found from the equation

$$\left(\mathbf{M}\lambda_0^2 + p_0\mathbf{G}\lambda_0 - \mathbf{C}\right)\mathbf{u}_1 = -(2\lambda_0\mathbf{M} + p_0\mathbf{G})\mathbf{u}_0. \tag{6.61}$$

The eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ satisfy the orthogonality condition

$$\mathbf{v}_0^T \left(2\lambda_0\mathbf{M} + p_0\mathbf{G}\right)\mathbf{u}_0 = 0, \tag{6.62}$$

see Section 2.13. Multiplying condition (6.62) by λ_0 and using (6.50), we find

$$\lambda_0^2\mathbf{v}_0^T\mathbf{M}\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{C}\mathbf{u}_0 = 0. \tag{6.63}$$

At the points S, S' in Fig. 6.10, where the eigenvalues λ reach the imaginary axis, we have $\lambda_0 = \pm i\omega_0$. In this case the left eigenvector $\mathbf{v}_0$ takes the form

$$\mathbf{v}_0 = \frac{\overline{\mathbf{u}}_0}{\mathbf{u}_0^* \left(2\lambda_0\mathbf{M} + p_0\mathbf{G}\right)\mathbf{u}_1 + \mathbf{u}_0^*\mathbf{M}\mathbf{u}_0}. \tag{6.64}$$

Substituting (6.64) into (6.59), we find

$$\lambda_1^2 = -\frac{\lambda_0\mathbf{u}_0^*\mathbf{G}\mathbf{u}_0}{\mathbf{u}_0^* \left(2\lambda_0\mathbf{M} + p_0\mathbf{G}\right)\mathbf{u}_1 + \mathbf{u}_0^*\mathbf{M}\mathbf{u}_0}. \tag{6.65}$$

We can show that the numerator in expression (6.65) is nonzero. Indeed, using (6.64) in orthogonality condition (6.62), we get

$$\lambda_0\mathbf{u}_0^*\mathbf{G}\mathbf{u}_0 = \frac{2\omega_0^2\mathbf{u}_0^*\mathbf{M}\mathbf{u}_0}{p_0} > 0. \tag{6.66}$$

This means that $\lambda_1 \neq 0$ and, hence, the strong interaction does not degenerate at the points S, S'.

Let us estimate the frequency ω_0 at the points S and S'. Using expression (6.64) in condition (6.63), we have

$$\omega_0^2 = \frac{\mathbf{u}_0^*\mathbf{C}\mathbf{u}_0}{\mathbf{u}_0^*\mathbf{M}\mathbf{u}_0}. \tag{6.67}$$

Hence, $\alpha_1^2 \leq \omega_0^2 \leq \alpha_m^2$, where α_1^2 and α_m^2 are the minimum and maximum eigenvalues of problem (6.51).

When the eigenvalues λ reach the imaginary axis at the points S, S', they diverge along this axis in opposite directions, see Fig. 6.10. The derivative (6.53) of a simple eigenvalue $\lambda = i\omega$ with the use of equation (6.50) and $\mathbf{v} = \overline{\mathbf{u}}$ yields

$$\frac{d\omega}{dp} = -\frac{\omega(C + \omega^2 M)}{p(C - \omega^2 M)}, \tag{6.68}$$

where $C = \mathbf{u}^*\mathbf{C}\mathbf{u} > 0$ and $M = \mathbf{u}^*\mathbf{M}\mathbf{u} > 0$. The derivative (6.68) can not change the sign unless the denominator becomes zero, which is equivalent to the orthogonality condition (6.63). At this point two eigenvalues collide with derivatives (6.68) tending to infinity. So, the strong interaction takes place, and the system loses stability.

Consider the situation, when all the eigenvalues reach the imaginary axis and for $p > p_*$ gyroscopic stabilization takes place [Lakhadanov (1975)]. Then with further increase of the parameter p half of the eigenvalues move monotonically towards the origin, and the other half of the eigenvalues move to infinity. The presented scenario agrees with the asymptotic behavior of frequencies at large values of p given in [Merkin (1974)].

Example 6.4 As an example, we consider a gyroscopic system with two degrees of freedom

$$\begin{aligned} \ddot{x}_1 + p\dot{x}_2 - c_1 x_1 &= 0, \\ \ddot{x}_2 - p\dot{x}_1 - c_2 x_2 &= 0. \end{aligned} \tag{6.69}$$

The constants c_1 and c_2 are assumed to be positive and $c_2 > c_1$. The system is unstable when only potential forces are present ($p = 0$), and for rather large values $p > p_S$ the system is stabilized by the gyroscopic forces.

The characteristic equation for this system is

$$\lambda^4 + (p^2 - c_1 - c_2)\lambda^2 + c_1 c_2 = 0. \tag{6.70}$$

At $p = 0$, $\lambda_1 = \pm\sqrt{c_1}$ and $\lambda_2 = \pm\sqrt{c_2}$. As p increases, the real eigenvalues λ_1 and λ_2 approach, merge at $p_R = \sqrt{c_2} - \sqrt{c_1}$ to a double $\lambda = \pm\sqrt[4]{c_1 c_2}$, and then leave the real axis. At $p_S = \sqrt{c_1} + \sqrt{c_2}$, the eigenvalues become double again, $\lambda = \pm i\sqrt[4]{c_1 c_2}$, the second strong interaction occurs, and for $p > p_S$ the system gets stable, see Figs. 6.10 and 6.11. We note that when p is within the limits $\sqrt{c_2} - \sqrt{c_1} \le p \le \sqrt{c_1} + \sqrt{c_2}$, all the eigenvalues λ lie on the circle of the radius $\sqrt[4]{c_1 c_2}$.

Chapter 7

Linear Hamiltonian Systems

Hamiltonian systems model a number of important problems in theoretical physics, celestial mechanics, fluid dynamics *etc.* The stability problems, being significant for Hamiltonian systems, gave rise to sophisticated mathematical methods, see for example the books by [Guckenheimer and Holmes (1983); Arnold *et al.* (1996); Arnold and Givental (2001)]. Properties of Hamiltonian systems like conservation of energy (represented by a Hamiltonian function) and conservation of volume in the phase space lead to specific dynamical features.

In this chapter we study multi-parameter linear Hamiltonian systems with finite degrees of freedom. These systems are represented by Hamiltonian matrices, whose eigenvalues are situated on the complex plane symmetrically with respect to the real and imaginary axes. Due to this property, the stability of a linear Hamiltonian system can be only marginal, and the asymptotic stability can not take place. That is why the stability analysis of an equilibrium for a linearized Hamiltonian system does not guarantee the stability of a non-linear system. Nevertheless it provides the necessary stability condition. Qualitative analysis of multi-parameter linear Hamiltonian systems was done in [Galin (1975); Patera *et al.* (1982); Koçak (1984)]. Bifurcation theory of eigenvalues and methods of multi-parameter stability analysis in case of linear Hamiltonian systems was given in [Mailybaev and Seyranian (1998c); Mailybaev and Seyranian (1999a); Seyranian and Mailybaev (1999)], and our presentation is mostly based on these papers.

We start the chapter with the brief introduction to dynamics and stability of Hamiltonian systems and their relationship to gyroscopic systems. Then bifurcation theory for multiple eigenvalues of Hamiltonian matrices dependent on parameters is presented. The stability boundaries of linear

Hamiltonian systems in the parameter space are described. Analysis of singularities of the stability boundary is based on the bifurcation theory of eigenvalues and versal deformations of Hamiltonian matrices. As examples, stability of two gyroscopic systems: an elastic simply supported pipe conveying fluid and a multi-body rotating system are investigated.

7.1 Stability and dynamics of Hamiltonian system

Let us consider an autonomous Hamiltonian system with m degrees of freedom described by canonical variables: *generalized coordinates* $\hat{q}_1, \ldots, \hat{q}_m$ and *generalized impulses* $\hat{p}_1, \ldots, \hat{p}_m$. Dynamics of the system is governed by a scalar *Hamiltonian function* $H = H(\hat{q}_1, \ldots, \hat{q}_m, \hat{p}_1, \ldots, \hat{p}_m)$ smoothly dependent on the canonical variables. Equations of motion are given by the *Hamiltonian equations*

$$\begin{aligned} \frac{d\hat{q}_j}{dt} &= \frac{\partial H}{\partial \hat{p}_j}, \\ \frac{d\hat{p}_j}{dt} &= -\frac{\partial H}{\partial \hat{q}_j}, \quad j = 1, \ldots, m. \end{aligned} \tag{7.1}$$

Using equations (7.1), we find

$$\frac{dH}{dt} = \sum_{j=1}^{m} \left(\frac{\partial H}{\partial \hat{q}_j} \frac{d\hat{q}_j}{dt} + \frac{\partial H}{\partial \hat{p}_j} \frac{d\hat{p}_j}{dt} \right) = \sum_{j=1}^{m} \left(\frac{\partial H}{\partial \hat{q}_j} \frac{\partial H}{\partial \hat{p}_j} - \frac{\partial H}{\partial \hat{p}_j} \frac{\partial H}{\partial \hat{q}_j} \right) = 0. \tag{7.2}$$

Therefore, the Hamiltonian function is constant for any solution of equations (7.1). In many physical systems the Hamiltonian function represents the energy function, which is conserved in time.

7.1.1 *Linearization near equilibrium*

Let $\hat{q}_j(t) \equiv \hat{q}_{j0}$, $\hat{p}_j(t) \equiv \hat{p}_{j0}$, $j = 1, \ldots, m$, be a stationary solution of the Hamiltonian system, that is,

$$\frac{\partial H}{\partial \hat{q}_j} = \frac{\partial H}{\partial \hat{p}_j} = 0, \quad j = 1, \ldots, m, \tag{7.3}$$

at $\hat{q}_j = \hat{q}_{j0}$, $\hat{p}_j = \hat{p}_{j0}$, $j = 1, \ldots, m$. Linearization of Hamiltonian equations (7.1) near the stationary solution yields the linear Hamiltonian system

$$\begin{aligned} \frac{d\Delta\hat{q}_j}{dt} &= \sum_{k=1}^{m} \frac{\partial^2 H}{\partial\hat{q}_k \partial\hat{p}_j} \Delta\hat{q}_k + \sum_{k=1}^{m} \frac{\partial^2 H}{\partial\hat{p}_j \partial\hat{p}_k} \Delta\hat{p}_k, \\ \frac{d\Delta\hat{p}_j}{dt} &= -\sum_{k=1}^{m} \frac{\partial^2 H}{\partial\hat{q}_j \partial\hat{q}_k} \Delta\hat{q}_k - \sum_{k=1}^{m} \frac{\partial^2 H}{\partial\hat{q}_j \partial\hat{p}_k} \Delta\hat{p}_k, \quad j = 1, \ldots, m, \end{aligned} \tag{7.4}$$

where $\Delta\hat{q}_k = \hat{q}_k - \hat{q}_{k0}$ and $\Delta\hat{p}_k = \hat{p}_k - \hat{p}_{k0}$. Introducing the vector $\mathbf{x}$ of dimension $2m$ with the components

$$x_j = \Delta\hat{q}_j, \quad x_{m+j} = \Delta\hat{p}_j, \quad j = 1, \ldots, m, \tag{7.5}$$

we can write linear Hamiltonian system (7.4) in the form

$$\dot{\mathbf{x}} = \mathbf{JAx}, \tag{7.6}$$

where $\mathbf{A}$ is the $2m \times 2m$ symmetric block matrix

$$\mathbf{A} = \begin{pmatrix} \left[\dfrac{\partial^2 H}{\partial\hat{q}_j \partial\hat{q}_k}\right] & \left[\dfrac{\partial^2 H}{\partial\hat{q}_j \partial\hat{p}_k}\right] \\ \left[\dfrac{\partial^2 H}{\partial\hat{q}_k \partial\hat{p}_j}\right] & \left[\dfrac{\partial^2 H}{\partial\hat{p}_j \partial\hat{p}_k}\right] \end{pmatrix} \tag{7.7}$$

consisting of second order derivatives evaluated at the stationary point, and $\mathbf{J}$ is a $2m \times 2m$ skew-symmetric block matrix of the form

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \tag{7.8}$$

The matrix $\mathbf{JA}$ is called the *Hamiltonian matrix*. Notice that the matrix $\mathbf{J}$ satisfies the conditions

$$\mathbf{J}^T = \mathbf{J}^{-1} = -\mathbf{J}. \tag{7.9}$$

7.1.2 *Stability and instability*

Seeking a solution of system (7.6) in the form $\mathbf{x} = \mathbf{u} \exp \lambda t$, we get the eigenvalue problem

$$\mathbf{JAu} = \lambda\mathbf{u}, \tag{7.10}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. Multiplying both sides of equation (7.10) by $\mathbf{J}^{-1}$ from the left and using conditions (7.9), we obtain the generalized eigenvalue problem of the form

$$\mathbf{A}\mathbf{u} = -\lambda \mathbf{J}\mathbf{u}, \tag{7.11}$$

or, equivalently,

$$(\mathbf{A} + \lambda \mathbf{J})\mathbf{u} = 0. \tag{7.12}$$

There are $2m$ eigenvalues λ, counting multiplicities, determined by the characteristic equation

$$\det(\mathbf{A} + \lambda \mathbf{J}) = 0. \tag{7.13}$$

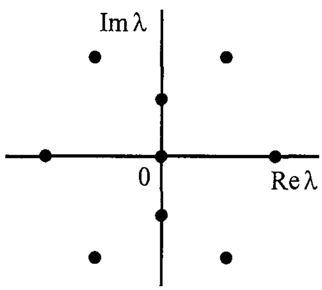


Fig. 7.1 Distribution of eigenvalues of a Hamiltonian matrix.

Since the matrix $\mathbf{A}$ is symmetric and the matrix $\mathbf{J}$ is skew-symmetric, we have

$$\det(\mathbf{A} + \lambda \mathbf{J}) = \det(\mathbf{A} + \lambda \mathbf{J})^T = \det(\mathbf{A} - \lambda \mathbf{J}) = 0. \tag{7.14}$$

Therefore, if λ is an eigenvalue then $-\lambda$ is an eigenvalue too, and characteristic equation (7.13) contains λ in even powers only. Since the matrices $\mathbf{A}$ and $\mathbf{J}$ are real, $\overline{\lambda}$ and $-\overline{\lambda}$ are eigenvalues as well. We see that eigenvalues of the Hamiltonian matrix $\mathbf{JA}$ are placed on the complex plane symmetrically with respect to both real and imaginary axes, i.e., they appear in complex quadruples $\pm\alpha \pm i\omega$, real pairs $\pm\alpha$, purely imaginary pairs $\pm i\omega$, or zero eigenvalue of even algebraic multiplicity, see Fig. 7.1. Jordan structure of eigenvalues is the same in complex quadruples $\pm\alpha \pm i\omega$, real pairs $\pm\alpha$, and purely imaginary pairs $\pm i\omega$, see [Arnold and Givental (2001)]. Notice that the number of Jordan blocks of odd size corresponding to zero eigenvalue is even. Theorem 1.1 (page 6) applied to linear Hamiltonian system (7.6) yields

Theorem 7.1 *Linear Hamiltonian system (7.6) is stable if and only if all the eigenvalues of the Hamiltonian matrix* $\mathbf{JA}$ *have zero real part* $\operatorname{Re}\lambda = 0$ *and are simple or semi-simple.*

Stability of a linear Hamiltonian system is not asymptotic, since existence of the eigenvalue λ with negative real part ($\operatorname{Re}\lambda < 0$) implies existence of the eigenvalue $-\lambda$ with positive real part, i.e., instability. There is a complicated relation of stability for linearized and nonlinear Hamiltonian systems. If the linearized system has an eigenvalue with positive real part, then the nonlinear system is unstable. If the matrix $\mathbf{A}$ is positive definite, then both the linearized and nonlinear systems are stable. In other cases stability of the nonlinear system is influenced by nonlinear terms and does not necessarily follow from the stability of the linearized system, see [Arnold and Givental (2001)].

7.1.3 *Relation to gyroscopic system*

Let us consider a linear gyroscopic system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \tag{7.15}$$

where $\mathbf{q} \in \mathbb{R}^m$ is a vector of generalized coordinates; $\mathbf{M} > 0$ and $\mathbf{P}$ are symmetric matrices and $\mathbf{G}$ is a skew-symmetric matrix; see Section 6.1. Introducing the vector of dimension $2m$

$$\mathbf{x} = \begin{pmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{M}\dot{\mathbf{q}} + \mathbf{G}\mathbf{q}/2 \end{pmatrix}, \tag{7.16}$$

we can write equation (7.15) in the form of linear Hamiltonian equations (7.6) with the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{P} - \mathbf{G}\mathbf{M}^{-1}\mathbf{G}/4 & \mathbf{G}\mathbf{M}^{-1}/2 \\ -\mathbf{M}^{-1}\mathbf{G}/2 & \mathbf{M}^{-1} \end{pmatrix}. \tag{7.17}$$

Therefore, linear gyroscopic systems possess the properties of linear Hamiltonian systems. Notice that transformation (7.16), (7.17) represents the Legendre transformation from the Lagrange to Hamiltonian equations for the case of linear systems, see for example [Arnold (1978)]. The Hamiltonian function

$$H = \frac{1}{2}\,\mathbf{x}^T\mathbf{A}\mathbf{x} = \frac{1}{2}\,\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \frac{1}{2}\,\mathbf{q}^T\mathbf{P}\mathbf{q} \tag{7.18}$$

represents the energy of the gyroscopic system.

7.2 Bifurcation of purely imaginary and zero eigenvalues of Hamiltonian matrix

A stable Hamiltonian system is characterized by eigenvalues lying on the imaginary axis. In this section we study a change of simple and double purely imaginary and zero eigenvalues for a Hamiltonian matrix $\mathbf{JA}(\mathbf{p})$ smoothly dependent on a vector of real parameters $\mathbf{p} \in \mathbb{R}^n$.

Let $\lambda_0 = i\omega$ be a purely imaginary eigenvalue of a Hamiltonian matrix $\mathbf{JA}_0 = \mathbf{JA}(\mathbf{p}_0)$ at a point $\mathbf{p}_0$ of the parameter space. The corresponding right eigenvector $\mathbf{u}_0$ is defined by the eigenvalue problem

$$\mathbf{JA}_0\mathbf{u}_0 = i\omega\mathbf{u}_0, \tag{7.19}$$

and the left eigenvector $\mathbf{v}_0$ is found from the equation

$$\mathbf{v}_0^T\mathbf{JA}_0 = i\omega\mathbf{v}_0^T. \tag{7.20}$$

Taking the complex conjugate transpose of (7.20) and pre-multiplying the result by the matrix $-\mathbf{J}$, we find

$$\mathbf{JA}_0\mathbf{J}\overline{\mathbf{v}}_0 = i\omega\mathbf{J}\overline{\mathbf{v}}_0. \tag{7.21}$$

Comparing equations (7.19) and (7.21), we get

$$\mathbf{u}_0 = c\mathbf{J}\overline{\mathbf{v}}_0, \tag{7.22}$$

where c is an arbitrary nonzero scalar, or equivalently

$$\mathbf{v}_0 = c\mathbf{J}\overline{\mathbf{u}}_0. \tag{7.23}$$

Expression (7.23) relates the right and left eigenvectors of the purely imaginary eigenvalue $\lambda_0 = i\omega$. In the case of zero eigenvalue $\lambda_0 = 0$ relation between the real eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ takes the form $\mathbf{v}_0 = c\mathbf{J}\mathbf{u}_0$.

7.2.1 *Simple eigenvalue*

Let us consider a simple eigenvalue $\lambda_0 = i\omega$. By Theorem 2.2 (page 32), this eigenvalue is a smooth function of the parameter vector $\mathbf{p}$ with the derivative

$$\frac{\partial \lambda}{\partial p_j} = \mathbf{v}_0^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0 \Big/ (\mathbf{v}_0^T \mathbf{u}_0). \tag{7.24}$$

Substituting expression (7.23) into (7.24) and using properties (7.9), we find

$$\frac{\partial \lambda}{\partial p_j} = -\mathbf{u}_0^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_0 \Big/ (\mathbf{u}_0^* \mathbf{J} \mathbf{u}_0), \tag{7.25}$$

where $\mathbf{u}_0^*$ denotes the complex conjugate transpose $\overline{\mathbf{u}}_0^T$. Since $\mathbf{A}$ is a symmetric matrix and $\mathbf{J}$ is a skew-symmetric matrix, the numerator and denominator in expression (7.25) are real and purely imaginary, respectively. Hence, derivative (7.25) of the purely imaginary eigenvalue $\lambda_0 = i\omega$ is purely imaginary. Under perturbation of the parameter vector the simple eigenvalue $\lambda_0 = i\omega$ moves along the imaginary axis. A purely imaginary eigenvalue cannot leave the imaginary axis unless it becomes multiple. Otherwise, a pair of simple purely imaginary eigenvalues becomes a complex quadruple, which is impossible since the total number of the eigenvalues is fixed.

7.2.2 *Double eigenvalue with single eigenvector*

Now, let us consider a nonderogatory double eigenvalue $\lambda_0 = i\omega$ (having a single eigenvector). The eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$ are determined by the Jordan chain equations

$$\begin{aligned} \mathbf{J}\mathbf{A}_0\mathbf{u}_0 &= i\omega\mathbf{u}_0, \\ \mathbf{J}\mathbf{A}_0\mathbf{u}_1 &= i\omega\mathbf{u}_1 + \mathbf{u}_0. \end{aligned} \tag{7.26}$$

The left eigenvector $\mathbf{v}_0$ and associated vector $\mathbf{v}_1$ are given by the equations

$$\begin{aligned} \mathbf{v}_0^T\mathbf{J}\mathbf{A}_0 &= i\omega\mathbf{v}_0^T, \\ \mathbf{v}_1^T\mathbf{J}\mathbf{A}_0 &= i\omega\mathbf{v}_1^T + \mathbf{v}_0^T, \end{aligned} \tag{7.27}$$

and we imply the normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_1 = 1, \quad \mathbf{v}_1^T\mathbf{u}_1 = 0. \tag{7.28}$$

Then using relation (7.23), we find the eigenvector $\mathbf{v}_0$ satisfying the first condition in (7.28) as

$$\mathbf{v}_0 = -\frac{\mathbf{J}\overline{\mathbf{u}}_0}{\mathbf{u}_0^*\mathbf{J}\mathbf{u}_1}. \tag{7.29}$$

The denominator in (7.29) can be transformed with the use of the second equation in (7.26) as follows

$$\mathbf{u}_0^*\mathbf{J}\mathbf{u}_1 = \mathbf{u}_1^*(\mathbf{J}\mathbf{A}_0 - i\omega\mathbf{I})^*\mathbf{J}\mathbf{u}_1 = \mathbf{u}_1^*(\mathbf{A}_0 + i\omega\mathbf{J})\mathbf{u}_1. \tag{7.30}$$

Hence, we have

$$\mathbf{v}_0 = -\frac{\mathbf{J}\overline{\mathbf{u}}_0}{\mathbf{u}_1^*(\mathbf{A}_0 + i\omega\mathbf{J})\mathbf{u}_1}, \tag{7.31}$$

where the denominator $\mathbf{u}_1^*(\mathbf{A}_0 + i\omega\mathbf{J})\mathbf{u}_1$ is a real number since the matrix $\mathbf{A}_0 + i\omega\mathbf{J}$ is Hermitian.

By Theorem 2.3 (page 37), bifurcation of the double eigenvalue $\lambda_0 = i\omega$ along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, is given by the relation

$$\lambda = i\omega + \lambda_1\varepsilon^{1/2} + o(\varepsilon^{1/2}), \tag{7.32}$$

where two values of λ_1 are

$$\lambda_1 = \pm\sqrt{\sum_{j=1}^{n}\left(\mathbf{v}_0^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_j}\mathbf{u}_0\right)e_j} \tag{7.33}$$

and $e_j = dp_j/d\varepsilon$ with the derivatives evaluated at $\mathbf{p} = \mathbf{p}_0$ and $\varepsilon = 0$. Substituting expression (7.31) into (7.33) and using relations (7.9), we find

$$\lambda_1 = \pm\sqrt{(\mathbf{f}, \mathbf{e})}, \tag{7.34}$$

where $\mathbf{e} = (e_1, \ldots, e_n)$, and $\mathbf{f} = (f_1, \ldots, f_n)$ is a real vector with the components

$$f_j = -\left(\mathbf{u}_0^*\frac{\partial\mathbf{A}}{\partial p_j}\mathbf{u}_0\right) \Big/ \left(\mathbf{u}_1^*(\mathbf{A}_0 + i\omega\mathbf{J})\mathbf{u}_1\right), \quad j = 1, \ldots, n. \tag{7.35}$$

The values of λ_1 are real or purely imaginary depending on the sign of the scalar product $(\mathbf{f}, \mathbf{e})$. Therefore, under perturbation of the parameter vector the double eigenvalue $\lambda_0 = i\omega$ splits into two purely imaginary eigenvalues if $(\mathbf{f}, \mathbf{e}) < 0$ and into two complex eigenvalues with positive and negative real parts if $(\mathbf{f}, \mathbf{e}) > 0$.

Let us consider the double eigenvalue $\lambda_0 = 0$ with the corresponding real eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$. Bifurcation of the eigenvalue $\lambda_0 = 0$ along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ is described by the expansion

$$\lambda = \lambda_1 \varepsilon^{1/2} + o(\varepsilon^{1/2}), \tag{7.36}$$

where two values of λ_1 are given by expressions (7.34) and (7.35). The double eigenvalue $\lambda_0 = 0$ splits into a pair of purely imaginary eigenvalues $\pm i\omega$ if $(\mathbf{f}, \mathbf{e}) < 0$ and into a pair of real eigenvalues $\pm\alpha$ if $(\mathbf{f}, \mathbf{e}) > 0$.

7.2.3 *Double semi-simple eigenvalue*

Finally, let us consider a double semi-simple eigenvalue $\lambda_0 = i\omega$, $\omega \neq 0$, with two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. These eigenvectors can be chosen such that the following orthogonality condition is satisfied

$$\mathbf{u}_1^* \mathbf{J} \mathbf{u}_2 = 0. \tag{7.37}$$

This choice ensures that

$$\mathbf{u}_1^* \mathbf{J} \mathbf{u}_1 \neq 0, \quad \mathbf{u}_2^* \mathbf{J} \mathbf{u}_2 \neq 0. \tag{7.38}$$

Otherwise, $\mathbf{v}_1^T \mathbf{u}_1 = \mathbf{v}_1^T \mathbf{u}_2 = 0$ or $\mathbf{v}_1^T \mathbf{u}_2 = \mathbf{v}_2^T \mathbf{u}_2 = 0$ for the left eigenvectors $\mathbf{v}_1 = c_1 \mathbf{J}\bar{\mathbf{u}}_1$ and $\mathbf{v}_2 = c_2 \mathbf{J}\bar{\mathbf{u}}_2$, but this implies the existence of a Jordan chain.

According to (7.23), the left eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be taken in the form

$$\mathbf{v}_1 = -\frac{\mathbf{J}\bar{\mathbf{u}}_1}{\mathbf{u}_1^* \mathbf{J} \mathbf{u}_1}, \quad \mathbf{v}_2 = -\frac{\mathbf{J}\bar{\mathbf{u}}_2}{\mathbf{u}_2^* \mathbf{J} \mathbf{u}_2}. \tag{7.39}$$

The scalar quantities in the denominators of (7.39) are chosen such that the right and left eigenvectors satisfy the normalization conditions

$$\mathbf{v}_j^T \mathbf{u}_k = \delta_{jk}, \quad j, k = 1, 2. \tag{7.40}$$

By Theorem 2.6 (page 56), bifurcation of the double semi-simple eigenvalue $\lambda_0 = i\omega$ along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, is given by the expansion

$$\lambda = i\omega + \lambda_1 \varepsilon + o(\varepsilon), \tag{7.41}$$

where two values of λ_1 are the eigenvalues of the matrix

$$\begin{pmatrix} \sum_{j=1}^{n} \left(\mathbf{v}_1^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 \right) e_j & \sum_{j=1}^{n} \left(\mathbf{v}_1^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right) e_j \\ \sum_{j=1}^{n} \left(\mathbf{v}_2^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 \right) e_j & \sum_{j=1}^{n} \left(\mathbf{v}_2^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right) e_j \end{pmatrix}. \tag{7.42}$$

Using relations (7.39), we write the characteristic equation for matrix (7.42) in the form

$$\lambda_1^2 - i \left(\frac{a_1}{b_1} + \frac{a_2}{b_2} \right) \lambda_1 + \frac{a_3^2 + a_4^2 - a_1 a_2}{b_1 b_2} = 0, \tag{7.43}$$

where a_1, a_2, a_3, a_4, and b_1, b_2 are real numbers determined by the expressions

$$\begin{gathered} a_1 = \sum_{j=1}^{n} \left(\mathbf{u}_1^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 \right) e_j, \quad a_2 = \sum_{j=1}^{n} \left(\mathbf{u}_2^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right) e_j, \\ a_3 + i a_4 = \sum_{j=1}^{n} \left(\mathbf{u}_1^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right) e_j, \\ b_1 = -i \mathbf{u}_1^* \mathbf{J} \mathbf{u}_1, \quad b_2 = -i \mathbf{u}_2^* \mathbf{J} \mathbf{u}_2. \end{gathered} \tag{7.44}$$

The constants b_1 and b_2 are nonzero due to conditions (7.38). Solving the characteristic equation, we find

$$\lambda_1 = \frac{i(a_1 b_2 + a_2 b_1) \pm \sqrt{-(a_1 b_2 - a_2 b_1)^2 - 4 b_1 b_2 (a_3^2 + a_4^2)}}{2 b_1 b_2}. \tag{7.45}$$

Let us define real vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ of dimension n with the components

$$\begin{gathered} g_{1j} = b_2 \left(\mathbf{u}_1^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1 \right) - b_1 \left(\mathbf{u}_2^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right), \\ g_{2j} + i g_{3j} = 2\sqrt{|b_1 b_2|} \left(\mathbf{u}_1^* \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2 \right), \quad j = 1, \dots, n. \end{gathered} \tag{7.46}$$

Then, expression (7.45) takes the form

$$\lambda_1 = \frac{i(a_1 b_2 + a_2 b_1) \pm \sqrt{D}}{2 b_1 b_2}, \tag{7.47}$$

where the discriminant D is equal to

$$D = -(\mathbf{g}_1, \mathbf{e})^2 - \operatorname{sign}(b_1 b_2) \big((\mathbf{g}_2, \mathbf{e})^2 + (\mathbf{g}_3, \mathbf{e})^2 \big). \tag{7.48}$$

Under perturbation of the parameter vector, the double semi-simple eigenvalue $\lambda_0 = i\omega$ splits into two purely imaginary eigenvalues if $D < 0$, and into two complex eigenvalues with nonzero real part if $D > 0$. Notice that $D \le 0$ for any perturbation if $b_1 b_2 > 0$. In this case any perturbation of the parameter vector leaves eigenvalues (7.41) on the imaginary axis.

In case of the double semi-simple eigenvalue $\lambda_0 = 0$, we can choose the linearly independent real eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying the relation

$$\mathbf{u}_1^T \mathbf{J} \mathbf{u}_2 = 1. \tag{7.49}$$

Then the corresponding left eigenvectors satisfying normalization conditions (7.40) are

$$\mathbf{v}_1 = \mathbf{J}\mathbf{u}_2, \quad \mathbf{v}_2 = -\mathbf{J}\mathbf{u}_1, \tag{7.50}$$

where the equalities $\mathbf{u}_1^T \mathbf{J} \mathbf{u}_1 = \mathbf{u}_2^T \mathbf{J} \mathbf{u}_2 = 0$ are valid for the skew-symmetric matrix $\mathbf{J}$ and real vectors $\mathbf{u}_1$, $\mathbf{u}_2$. Substituting (7.50) into expression (7.42) and solving the characteristic equation, we find

$$\lambda_1 = \pm\sqrt{D}, \tag{7.51}$$

where

$$D = (\mathbf{h}_3, \mathbf{e})^2 - (\mathbf{h}_1, \mathbf{e})(\mathbf{h}_2, \mathbf{e}), \tag{7.52}$$

and $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ are real vectors of dimension n with the components

$$h_{1j} = \mathbf{u}_1^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_1, \quad h_{2j} = \mathbf{u}_2^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2, \quad h_{3j} = \mathbf{u}_1^T \frac{\partial \mathbf{A}}{\partial p_j} \mathbf{u}_2, \qquad j = 1, \ldots, n. \tag{7.53}$$

Bifurcation of the double eigenvalue $\lambda_0 = 0$ along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, is given by the expansion

$$\lambda = \pm\sqrt{D}\,\varepsilon + o(\varepsilon). \tag{7.54}$$

The double semi-simple eigenvalue $\lambda_0 = 0$ splits into two purely imaginary eigenvalues $\pm i\omega$ if $D < 0$ and into two real eigenvalues $\pm\alpha$ if $D > 0$.

7.3 Versal deformation of Hamiltonian matrix

A change of coordinates

$$\mathbf{x} = \mathbf{S}\mathbf{y}, \tag{7.55}$$

where $\mathbf{y} \in \mathbb{R}^{2m}$ and $\mathbf{S}$ is a nonsingular real $2m \times 2m$ matrix, transforms equation (7.6) to the form

$$\dot{\mathbf{y}} = \mathbf{S}^{-1}\mathbf{JASy}. \tag{7.56}$$

In order to keep the Hamiltonian form of equations, we require

$$\mathbf{S}^T\mathbf{JS} = \mathbf{J}, \tag{7.57}$$

or, equivalently,

$$\mathbf{S}^{-1} = -\mathbf{JS}^T\mathbf{J}. \tag{7.58}$$

Under condition (7.58), equation (7.56) can be written as

$$\dot{\mathbf{y}} = \mathbf{JA}'\mathbf{y}, \tag{7.59}$$

where $\mathbf{A}'$ is the symmetric matrix

$$\mathbf{A}' = \mathbf{S}^T\mathbf{AS}. \tag{7.60}$$

The matrix $\mathbf{S}$ satisfying condition (7.57) is called *symplectic.*

Let us consider a Hamiltonian matrix smoothly dependent on a vector of parameters $\mathbf{p} \in \mathbb{R}^n$ in the neighborhood of a point $\mathbf{p} = \mathbf{p}_0$. Change of coordinates (7.55) given by a symplectic matrix $\mathbf{S}(\mathbf{p})$ smoothly dependent on $\mathbf{p}$ yields the Hamiltonian matrix

$$\mathbf{JA}'(\mathbf{p}) = \mathbf{JS}^T(\mathbf{p})\mathbf{A}(\mathbf{p})\mathbf{S}(\mathbf{p}) \tag{7.61}$$

of the equivalent multi-parameter Hamiltonian system. In this way we can transform the family of Hamiltonian matrices to a simple form in the neighborhood of the point $\mathbf{p}_0$. Such forms, called *versal deformations*, have been studied in [Galin (1975); Patera *et al.* (1982); Koçak (1984)]. Versal deformations depend only on the matrix $\mathbf{JA}_0 = \mathbf{JA}(\mathbf{p}_0)$ at the initial point $\mathbf{p}_0$.

Versal deformations obtained in [Galin (1975)] provide decomposition of Hamiltonian equations to a set of independent subsystems. Each subsystem corresponds to a complex quadruple $\pm\alpha\pm i\omega$, real pair $\pm\alpha$, purely imaginary pair $\pm i\omega$, or zero eigenvalue of the matrix $\mathbf{JA}_0$. For example, subsystems corresponding to quadruples or pairs of simple eigenvalues are given by the Hamiltonian equations

$$\dot{\mathbf{z}} = \mathbf{JBz}, \tag{7.62}$$

where the Hamiltonian matrices $\mathbf{JB}$ are

$$\pm\alpha \pm i\omega: \ \mathbf{JB}(\mathbf{q}) = \begin{pmatrix} -\alpha + q_1 & \omega & 0 & 0 \\ -\omega + q_2 & -\alpha & 0 & 0 \\ 0 & 0 & \alpha - q_1 & \omega - q_2 \\ 0 & 0 & -\omega & \alpha \end{pmatrix}, \ \mathbf{q} = (q_1, q_2);$$

$$\pm\alpha: \ \mathbf{JB}(\mathbf{q}) = \begin{pmatrix} -\alpha + q_1 & 0 \\ 0 & \alpha - q_1 \end{pmatrix}, \ \mathbf{q} = (q_1);$$

$$\pm i\omega: \ \mathbf{JB}(\mathbf{q}) = \begin{pmatrix} 0 & \sigma\omega^2 \\ -\sigma - 2q_1 & 0 \end{pmatrix}, \ \sigma = \pm 1, \ \mathbf{q} = (q_1); \tag{7.63}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{p})$ is a smooth function of the parameter vector $\mathbf{p}$ determined in the neighborhood of $\mathbf{p}_0$ such that $\mathbf{q}(\mathbf{p}_0) = 0$. In the case of general position the Jacobian matrix $[d\mathbf{q}/d\mathbf{p}]$ has maximal rank. Two different signs $\sigma = \pm 1$ in the matrix $\mathbf{JB}$ for a purely imaginary pair of simple eigenvalues $\pm i\omega$ determine different versal deformations that are not equivalent under symplectic transformation (7.61). Eigenvalues of matrices (7.63) are

$$\begin{aligned} \pm\alpha \pm i\omega: &\quad \lambda = \pm\left(\alpha - \frac{q_1}{2}\right) \pm i\sqrt{\omega^2 - q_2\omega - \frac{q_1^2}{4}}; \\ \pm\alpha: &\quad \lambda = \pm(\alpha - q_1); \\ \pm i\omega: &\quad \lambda = \pm i\omega\sqrt{1 + 2\sigma q_1}. \end{aligned} \tag{7.64}$$

Expressions (7.64) show that the structure of a complex quadruple, real pair, or purely imaginary pair of simple eigenvalues is kept under perturbation of parameters. The matrices $\mathbf{JB}$ in (7.63) can be given through corresponding Hamiltonian functions $H = \frac{1}{2}\mathbf{z}^2\mathbf{B}\mathbf{z}$, which take the form [Galin (1975)]

$$\begin{aligned} \pm\alpha \pm i\omega: &\quad H = -\alpha(z_1 z_3 + z_2 z_4) - \omega(z_1 z_4 - z_2 z_3) \\ &\qquad\quad + q_1 z_1 z_3 + q_2 z_1 z_4, \ \mathbf{z} \in \mathbb{R}^4; \\ \pm\alpha: &\quad H = -\alpha z_1 z_2 + q_1 z_1 z_2, \ \mathbf{z} \in \mathbb{R}^2; \\ \pm i\omega: &\quad H = \frac{\sigma}{2}\left(z_1^2 + \omega^2 z_2^2\right) + q_1 z_1^2, \ \sigma = \pm 1, \ \mathbf{z} \in \mathbb{R}^2. \end{aligned} \tag{7.65}$$

For multiple eigenvalues of the matrix $\mathbf{JA}_0$ Hamiltonian functions

$H = \frac{1}{2}\mathbf{z}^T\mathbf{B}\mathbf{z}$, determining subsystems in a versal deformation, take more complicated form. Below we give a list of these functions for several particular cases important for the stability analysis [Galin (1975)]:

$$
\begin{aligned}
0^2: &\quad H = \mp\frac{1}{2}z_1^2 + q_1 z_2^2, \ \mathbf{z} \in \mathbb{R}^2, \ \mathbf{q} \in \mathbb{R};\\
00: &\quad H = q_1 z_1^2 + q_2 z_1 z_2 + q_3 z_2^2, \ \mathbf{z} \in \mathbb{R}^2, \ \mathbf{q} \in \mathbb{R}^3;\\
0^4: &\quad H = \pm\frac{1}{2}(z_3^2 - 2z_1 z_2) - z_2 z_3 + q_1 z_3 z_4 + q_2 z_4^2,\\
&\qquad \mathbf{z} \in \mathbb{R}^4, \ \mathbf{q} \in \mathbb{R}^2;\\
0^6: &\quad H = \pm\frac{1}{2}(2z_4 z_5 - 2z_1 z_3 - z_2^2) - z_2 z_4 - z_3 z_5 + q_1 z_4 z_6\\
&\qquad + q_2 z_5 z_6 + q_3 z_6^2, \ \mathbf{z} \in \mathbb{R}^6, \ \mathbf{q} \in \mathbb{R}^3;\\
(\pm i\omega)^2: &\quad H = \pm\frac{1}{2}\left(\frac{1}{\omega^2}z_1^2 + z_2^2\right) - \omega^2 z_2 z_3 + z_1 z_4 + q_1 z_2 z_3 + q_2 z_3^2,\\
&\qquad \mathbf{z} \in \mathbb{R}^4, \ \mathbf{q} \in \mathbb{R}^2;\\
(\pm i\omega)(\pm i\omega): &\quad H = \pm\frac{1}{2}\left(z_1^2 + \omega^2 z_3^2\right) \pm \frac{1}{2}(z_2^2 + \omega^2 z_4^2) + q_1 z_1^2 + q_2 z_1 z_2\\
&\qquad + q_3 z_2^2 + q_4 z_1 z_4, \ \mathbf{z} \in \mathbb{R}^4, \ \mathbf{q} \in \mathbb{R}^4;\\
(\pm i\omega)^3: &\quad H = \pm\frac{1}{2}\left(z_2^2 - 2z_1 z_3 + \omega^2(z_5^2 - 2z_4 z_6)\right) - z_2 z_4\\
&\qquad - z_3 z_5 + q_1 z_1^2 + q_2 z_1 z_3 + q_3 z_1 z_5, \ \mathbf{z} \in \mathbb{R}^6, \ \mathbf{q} \in \mathbb{R}^3;\\
(\pm i\omega)^4: &\quad H = \pm\frac{1}{2}\left(2z_2 z_4 + \frac{2}{\omega^2}z_1 z_3 - \omega^2 z_7^2 - z_8^2\right)\\
&\qquad - \omega^2(z_2 z_5 + z_4 z_7) + z_1 z_6 + z_3 z_8 + q_1 z_2 z_5\\
&\qquad + q_2 z_4 z_5 + q_3 z_5^2 + q_4 z_5 z_7, \ \mathbf{z} \in \mathbb{R}^8, \ \mathbf{q} \in \mathbb{R}^4.
\end{aligned}
\tag{7.66}
$$

Here 0^k and $(\pm i\omega)^k$ denote nonderogatory zero and purely imaginary eigenvalues of multiplicity k, while 00 and $(\pm i\omega)(\pm i\omega)$ denote semi-simple double zero and purely imaginary eigenvalues. The vector $\mathbf{q}$ smoothly depends on $\mathbf{p}$. The function $\mathbf{q}(\mathbf{p})$ is determined by the family of Hamiltonian matrices $\mathbf{JA}(\mathbf{p})$ under consideration. Different signs in (7.66) determine different versal deformations, which are not equivalent under symplectic transformation (7.61).

Using (7.66), we find the Hamiltonian matrices in the cases 0^2 and $(\pm i\omega)^2$ as

$$0^2 : \mathbf{JB}(\mathbf{q}) = \begin{pmatrix} 0 & 2q_1 \\ \sigma & 0 \end{pmatrix},$$

$$(\pm i\omega)^2 : \mathbf{JB}(\mathbf{q}) = \begin{pmatrix} 0 & -\omega^2 + q_1 & 2q_2 & 0 \\ 1 & 0 & 0 & 0 \\ -\sigma/\omega^2 & 0 & 0 & -1 \\ 0 & -\sigma & \omega^2 - q_1 & 0 \end{pmatrix}, \tag{7.67}$$

where $\sigma = \pm 1$. It is easy to see that the first matrix in (7.67) has the double eigenvalue $\lambda = 0$ if $q_1(\mathbf{p}) = 0$. Hence, in the case of general position a set of points $\mathbf{p}$, such that the matrix $\mathbf{JA}(\mathbf{p})$ has the double nonderogatory eigenvalue $\lambda = 0$, is a smooth hypersurface in the parameter space. The second matrix in (7.67) has a pair of purely imaginary double eigenvalues for $q_2(\mathbf{p}) = 0$ and sufficiently small $q_1(\mathbf{p})$. In the case of general position a set of points $\mathbf{p}$, such that the matrix $\mathbf{JA}(\mathbf{p})$ has a purely imaginary pair of double nonderogatory eigenvalues, is a smooth hypersurface in the parameter space.

7.4 Stability domain and its boundary

Let us consider linear Hamiltonian system (7.6) with the symmetric matrix $\mathbf{A}(\mathbf{p})$ smoothly dependent on a vector of real parameters $\mathbf{p}$. The stability domain is defined as a set of values of the parameter vector $\mathbf{p}$ such that corresponding system (7.6) is stable. Recall that stability of a linear Hamiltonian system is not asymptotic, and all the eigenvalues of the stable system with the Hamiltonian matrix $\mathbf{JA}$ are simple or semi-simple and lie on the imaginary axis.

Let us consider a point $\mathbf{p}_0$ in the parameter space such that all the eigenvalues of the Hamiltonian matrix $\mathbf{JA}_0 = \mathbf{JA}(\mathbf{p}_0)$ are simple and purely imaginary. As we showed in Section 7.2, simple purely imaginary eigenvalues cannot leave the imaginary axis. The linear Hamiltonian system remains stable for any small perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$. Therefore, the point $\mathbf{p}_0$ is an internal point of the stability domain. Stability can be lost only if multiple purely imaginary or zero eigenvalues appear.

Let us consider a point $\mathbf{p}_0$ in the parameter space such that the matrix $\mathbf{JA}_0$ has a double nonderogatory eigenvalue $\lambda_0 = 0$ (having a single eigenvector), while other eigenvalues are simple and purely imaginary. We say that such a point $\mathbf{p}_0$ is of type 0^2. Bifurcation of the double eigenvalue $\lambda_0 = 0$ affects stability and instability of the system. Considering perturbation of the parameter vector as $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$, bifurcation of $\lambda_0 = 0$ is described by formulae (7.34)–(7.36). Since $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = \varepsilon\mathbf{e}$, we have

$$\lambda = \pm\sqrt{(\mathbf{f}, \Delta\mathbf{p})} + o(\varepsilon^{1/2}). \tag{7.68}$$

The system is stable for small ε if two eigenvalues (7.68) are simple and purely imaginary, which yields the condition

$$(\mathbf{f}, \Delta\mathbf{p}) < 0. \tag{7.69}$$

Inequality (7.69) provides the first order approximation of the stability domain. The stability boundary is a smooth surface in the neighborhood of the point $\mathbf{p}_0$ with the tangent plane

$$(\mathbf{f}, \Delta\mathbf{p}) = 0, \tag{7.70}$$

where $\mathbf{f}$ is the normal vector to the stability boundary at $\mathbf{p}_0$ directed into the instability domain; see Fig. 7.2a. Expression (7.68) shows that two purely imaginary eigenvalues $\pm i\omega$ come closer, interact strongly at the origin, and become real $\pm\alpha$ as we cross the stability boundary in any transversal direction from the stability to instability domain, see Fig. 7.3a. This mechanism of loss of stability is called *divergence*.

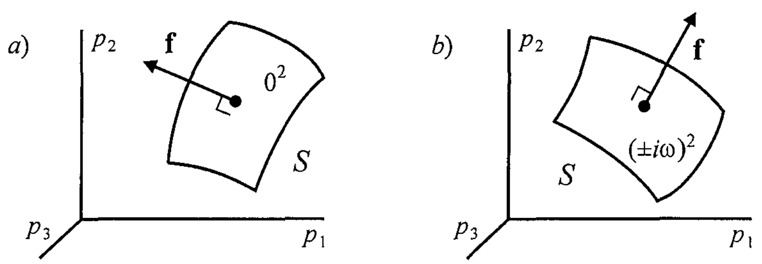


Fig. 7.2 Stability boundary and its normal vector: a) divergence, b) flutter.

Let us consider a point $\mathbf{p}_0$ of type $(\pm i\omega)^2$ such that the Hamiltonian matrix $\mathbf{JA}_0$ has a pair of double nonderogatory eigenvalues $\pm i\omega$, while other eigenvalues are simple and purely imaginary. From formulae (7.32), (7.34)

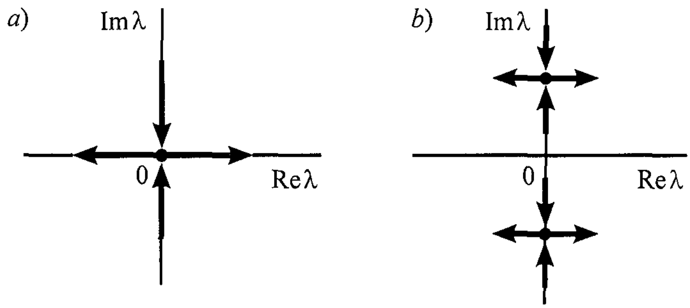


Fig. 7.3 Development of instability on the complex plane: a) divergence, b) flutter.

we know that bifurcation of double eigenvalues $\pm i\omega$ under perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$ is given by the expression

$$\lambda = \pm i\omega \pm \sqrt{(\mathbf{f}, \Delta\mathbf{p})} + o(\varepsilon^{1/2}), \tag{7.71}$$

where the real vector $\mathbf{f}$ is determined by expression (7.35) for the double eigenvalue $\lambda_0 = i\omega$. The system is stable for small ε if four eigenvalues (7.71) are purely imaginary. This condition provides the first order approximation of the stability domain (7.69). The stability boundary is a smooth surface with tangent plane (7.70), where $\mathbf{f}$ is the normal vector to the stability boundary at $\mathbf{p}_0$ directed into the instability domain; see Fig. 7.2b. Behavior of eigenvalues as we cross the stability boundary in any direction in the parameter space is shown in Fig. 7.3b: two pairs of purely imaginary eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$ interact strongly and become a complex quadruple $\pm\alpha \pm i\omega$. This mechanism of loss of stability is called *flutter*.

Points of types 0^2 and $(\pm i\omega)^2$ form a regular part of the stability boundary, which consists of smooth surfaces representing divergence and flutter boundaries, respectively.

7.5 Singularities of stability boundary

Let us denote types of stability boundary points by product of multiple eigenvalues in powers of sizes of the corresponding Jordan blocks. For example, 0^4 denotes type of the point $\mathbf{p}$ at which the matrix $\mathbf{JA}(\mathbf{p})$ has the nonderogatory eigenvalue $\lambda = 0$ of multiplicity 4, while $(\pm i\omega)(\pm i\omega)$ corresponds to the point $\mathbf{p}$ at which there is a pair of semi-simple double eigenvalues $\lambda = \pm i\omega$ (other eigenvalues are assumed to be simple and purely imaginary). Regular part of the stability boundary is represented by the points of types 0^2 and $(\pm i\omega)^2$. Types different from 0^2 and $(\pm i\omega)^2$ determine points, where the stability boundary is nonsmooth, i.e., has singularities. The codimension for a set of points of a particular type and a local form of the stability domain can be determined by means of the versal deformation theory described in Section 7.3.

Let us consider the singularity 0^4. Stability of the Hamiltonian system in the neighborhood of the point $\mathbf{p}_0$ of type 0^4 depends on behavior of the quadruple eigenvalue $\lambda_0 = 0$. Using normal forms (7.66), we find that bifurcation of $\lambda_0 = 0$ is given by the matrix

$$\mathbf{JB}(\mathbf{q}) = \begin{pmatrix} 0 & -1 & \sigma & q_1 \\ 0 & 0 & q_1 & 2q_2 \\ 0 & \sigma & 0 & 0 \\ \sigma & 0 & 1 & 0 \end{pmatrix}, \quad \sigma = \pm 1, \tag{7.72}$$

where $\mathbf{q}(\mathbf{p}) = (q_1(\mathbf{p}), q_2(\mathbf{p}))$ is a smooth function such that $\mathbf{q}(\mathbf{p}_0) = 0$; the sign of σ depends on the matrix $\mathbf{JA}_0$. Eigenvalues of matrix (7.72) are

$$\lambda = \pm\sqrt{\sigma q_1 \pm \sqrt{2\sigma q_2}}. \tag{7.73}$$

The eigenvalue $\lambda_0 = 0$ remains quadruple if and only if $q_1 = q_2 = 0$. In the case of general position the Jacobian of the mapping $\mathbf{q} = (q_1(\mathbf{p}), q_2(\mathbf{p}))$ has maximal rank and, hence, a set of points of type 0^4 is a smooth surface of codimension 2 in the parameter space. In the neighborhood of $\mathbf{p}_0$ the system is stable if

$$\sigma q_1 < 0, \quad 0 < 2\sigma q_2 < q_1^2. \tag{7.74}$$

Under conditions (7.74) four eigenvalues (7.73) are purely imaginary and simple. Stability domain (7.74) in the plane (q_1, q_2) is shown in Fig. 7.4 for $\sigma = 1$. The stability boundary has a cusp singularity at the origin with two sides being curves of types 0^2 and $(\pm i\omega)^2$. Hence, the singularity 0^4 of the stability boundary represents a cusp or cuspidal edge in the two- or three-parameter space, respectively. Geometry of the singularity in the original parameter space $\mathbf{p}$ is determined by the mapping $\mathbf{q} = \mathbf{q}(\mathbf{p})$.

Analogously, we can study singularities of other types finding corresponding codimensions and local forms of the stability domain. As a result, we find all the singularities of codimension 2 represented by the types

$$0^2(\pm i\omega)^2, \quad (\pm i\omega_1)^2(\pm i\omega_2)^2, \quad 0^4, \quad (\pm i\omega)^3, \tag{7.75}$$

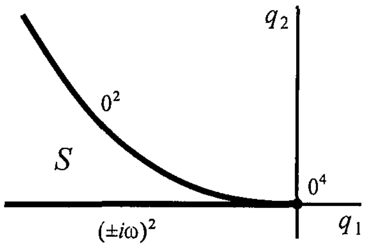


Fig. 7.4 Singularity 0^4 of the stability boundary in the plane (q_1, q_2).

and singularities of codimension 3 given by the types

$$\begin{gathered} 0^2(\pm i\omega_1)^2(\pm i\omega_2)^2, \quad (\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2, \\ 0^4(\pm i\omega)^2, \quad (\pm i\omega)^3 0^2, \quad (\pm i\omega_1)^3(\pm i\omega_2)^2, \\ 00, \quad (\pm i\omega)(\pm i\omega), \quad 0^6, \quad (\pm i\omega)^4. \end{gathered} \tag{7.76}$$

Form of the stability domain in the neighborhood of singular boundary points (7.75) and (7.76), up to a smooth change of parameters, is given by the formulae

$$\begin{aligned} 0^2(\pm i\omega)^2,\ (\pm i\omega_1)^2(\pm i\omega_2)^2 :& \quad p_1 > 0,\ p_2 > 0; \\ 0^4 :& \quad p_1 > 0,\ 0 < p_2 < p_1^2; \\ (\pm i\omega)^3 :& \quad p_1 > 0,\ p_2^2 < p_1^3; \\ 0^2(\pm i\omega_1)^2(\pm i\omega_2)^2,& \\ (\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2 :& \quad p_1 > 0,\ p_2 > 0,\ p_3 > 0; \\ 0^4(\pm i\omega)^2 :& \quad p_1 > 0,\ 0 < p_2 < p_1^2,\ p_3 > 0; \\ (\pm i\omega)^3 0^2,\ (\pm i\omega_1)^3(\pm i\omega_2)^2 :& \quad p_1 > 0,\ p_2^2 < p_1^3,\ p_3 > 0; \\ 00,\ (\pm i\omega)(\pm i\omega) :& \quad p_1^2 + p_2^2 < p_3^2 \ \text{ and } \ p_1 = p_2 = p_3 = 0; \end{aligned} \tag{7.77}$$

where the singularity point is $\mathbf{p}_0 = 0$. In case of the singularity 0^6, form of the stability domain is given by the condition that all the roots of the polynomial $\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ are simple, real, and negative. Finally, form of the stability domain in case of the singularity $(\pm i\omega)^4$, up to a smooth change of parameters, is given by the condition that all the roots of the polynomial $\lambda^4 + p_1\lambda^2 + p_2\lambda + p_3$ are simple and real. The stability boundary in the latter case represents a part of the well-known singularity *swallow tail*, see [Arnold (1983a)]. In both cases, the singularity is formed by one edge and two cuspidal edges starting at the singular point along the same direction. Therefore, we call this singularity *trihedral spire.*

Theorem 7.2 *In the case of general position, the stability boundary of two-parameter Hamiltonian system (7.6) consists of smooth curves of types* 0^2 *and* $(\pm i\omega)^2$*. Singularities of the stability boundary are angles* $0^2(\pm i\omega)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2$ *and cusps* 0^4, $(\pm i\omega)^3$*; see Fig. 7.5.*

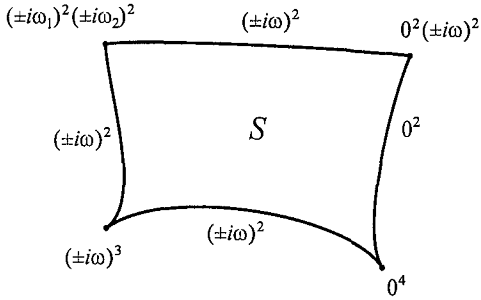


Fig. 7.5 Generic singularities of the stability boundary for two-parameter linear Hamiltonian system.

Theorem 7.3 *In the case of general position, the stability boundary of three-parameter Hamiltonian system (7.6) consists of smooth surfaces of types* 0^2 *and* $(\pm i\omega)^2$*. Singularities of the stability boundary are edges* $0^2(\pm i\omega)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2$, *cuspidal edges* 0^4, $(\pm i\omega)^3$, *trihedral angles* $0^2(\pm i\omega_1)^2(\pm i\omega_2)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2$, *truncated cuspidal edges* $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$, *cones* 00, $(\pm i\omega)(\pm i\omega)$, *and trihedral spires* 0^6, $(\pm i\omega)^4$*; see Fig. 7.6.*

Notice that the point of type $(\pm i\omega)(\pm i\omega)$ does not necessarily determine a cone singularity. There are different normal forms (7.66) corresponding to a pair of double semi-simple eigenvalues $\pm i\omega$. These forms differ by signs of the first and second terms in the Hamiltonian function. The cone singularity appears for different signs. If the signs are the same, the point of type $(\pm i\omega)(\pm i\omega)$ is an internal point of the stability domain.

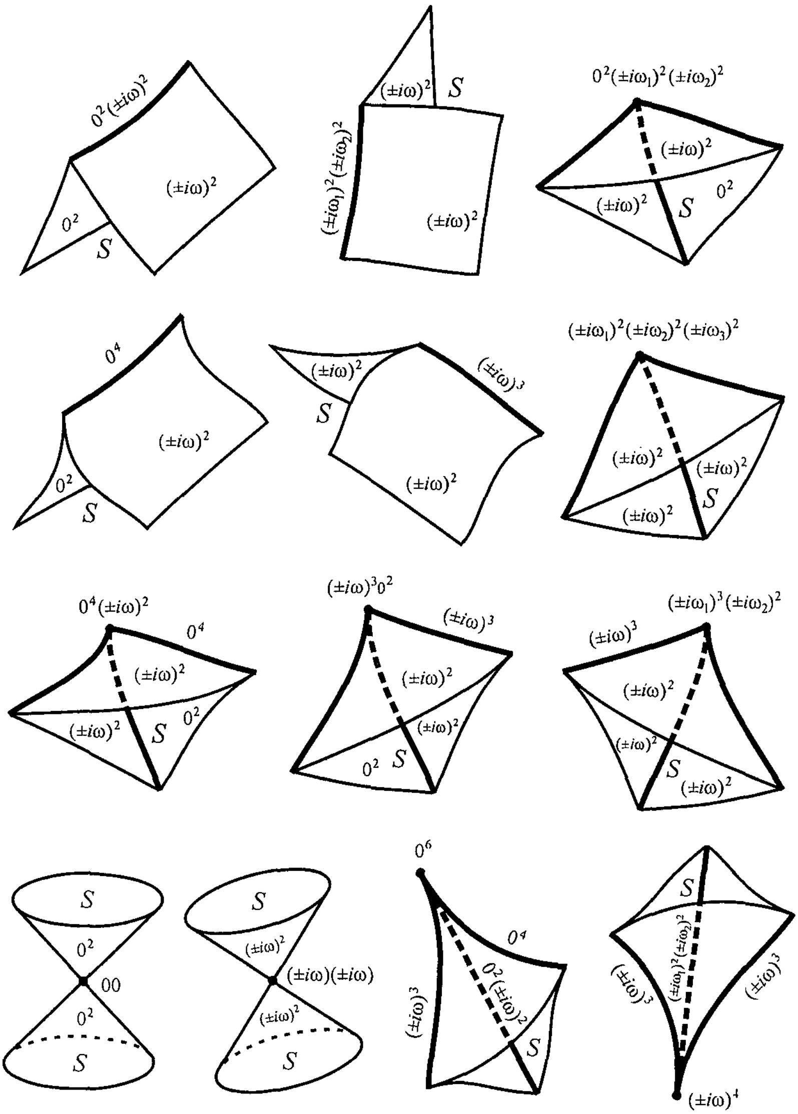


Fig. 7.6 Generic singularities of the stability boundary for three-parameter linear Hamiltonian system.

7.6 Stability analysis near singularities associated with double eigenvalues

In this section we study singularities of the stability boundary determined by double zero or purely imaginary eigenvalues, i.e., (dihedral) angles $0^2(\pm i\omega)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2$, trihedral angles $0^2(\pm i\omega_1)^2(\pm i\omega_2)^2$, $(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2$, and cones 00, $(\pm i\omega)(\pm i\omega)$.

As we have shown in Section 7.4, bifurcation of a double nonderogatory zero or purely imaginary eigenvalue into a pair of simple purely imaginary eigenvalues yields the condition

$$(\mathbf{f}, \Delta\mathbf{p}) < 0, \tag{7.78}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = \varepsilon\mathbf{e}$ is a perturbation of the parameter vector, and the real vector $\mathbf{f}$ is given by formula (7.35). To get stabilizing perturbations $\Delta\mathbf{p}$, we take condition (7.78) for each zero or purely imaginary pair of double nonderogatory eigenvalues. As a result, we find first order approximations of the stability domain near (dihedral) angle and trihedral angle singularities of the stability boundary:

$$\begin{aligned}
0^2(\pm i\omega)^2 &: (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)^2(\pm i\omega_2)^2 &: (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;\\
0^2(\pm i\omega_1)^2(\pm i\omega_2)^2 &: (\mathbf{f}_0, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)^2(\pm i\omega_2)^2(\pm i\omega_3)^2 &: (\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_3}, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{7.79}$$

Here, the subscript denotes the eigenvalue for which the vector $\mathbf{f}$ is evaluated. The vectors $\mathbf{f}_0$ and $\mathbf{f}_{i\omega}$ are normal vectors to the divergence and flutter boundaries, respectively.

Now, let us consider the cone singularity 00 represented by the semisimple double eigenvalue $\lambda_0 = 0$. Bifurcation of zero eigenvalue under perturbation of the parameter vector $\Delta\mathbf{p} = \varepsilon\mathbf{e}$ is described by formulae (7.52)–(7.54). The system remains stable if $\lambda_0 = 0$ splits into a pair of simple purely imaginary eigenvalues $\pm i\omega$, which requires

$$00: \quad (\mathbf{h}_3, \Delta\mathbf{p})^2 < (\mathbf{h}_1, \Delta\mathbf{p})(\mathbf{h}_2, \Delta\mathbf{p}), \tag{7.80}$$

where components of the vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ are determined by formulae (7.53). Condition (7.80) provides the first order approximation of the stability domain near the cone singularity 00. Introducing new variables

$$x = (\mathbf{h}_3, \Delta\mathbf{p}),\ y = \left(\frac{\mathbf{h}_2 - \mathbf{h}_1}{2}, \Delta\mathbf{p}\right),\ z = \left(\frac{\mathbf{h}_1 + \mathbf{h}_2}{2}, \Delta\mathbf{p}\right), \tag{7.81}$$

we write equation (7.80) as

$$x^2 + y^2 < z^2, \tag{7.82}$$

which shows that the stability domain corresponds to the interior part of the cone. The cone surface can be given in the form

$$x = z\cos\alpha, \quad y = z\sin\alpha, \tag{7.83}$$

where $0 \le \alpha < 2\pi$. Using relations (7.81) in (7.83), we obtain

$$\left(\mathbf{h}_3 - \frac{\mathbf{h}_1 + \mathbf{h}_2}{2}\cos\alpha, \Delta\mathbf{p}\right) = 0, \ \left(\mathbf{h}_2 - \mathbf{h}_1 - (\mathbf{h}_1 + \mathbf{h}_2)\sin\alpha, \Delta\mathbf{p}\right) = 0. \tag{7.84}$$

In the three-parameter space the vector $\Delta\mathbf{p}$ can be found in the form

$$\begin{aligned} \Delta\mathbf{p} &= \beta\left(\mathbf{h}_3 - \frac{\mathbf{h}_1 + \mathbf{h}_2}{2}\cos\alpha\right) \times \left(\mathbf{h}_2 - \mathbf{h}_1 - (\mathbf{h}_1 + \mathbf{h}_2)\sin\alpha\right) \\ &= \beta(\mathbf{a} + \mathbf{b}\sin\alpha + \mathbf{c}\cos\alpha), \quad \beta \in \mathbb{R}, \ 0 \le \alpha < 2\pi, \end{aligned} \tag{7.85}$$

where the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are given by the cross products

$$\begin{aligned} \mathbf{a} &= \mathbf{h}_3 \times (\mathbf{h}_2 - \mathbf{h}_1), \\ \mathbf{b} &= (\mathbf{h}_1 + \mathbf{h}_2) \times \mathbf{h}_3, \\ \mathbf{c} &= \mathbf{h}_2 \times \mathbf{h}_1. \end{aligned} \tag{7.86}$$

Expression (7.85) provides the parameterization of the cone surface; see Fig. 7.7.

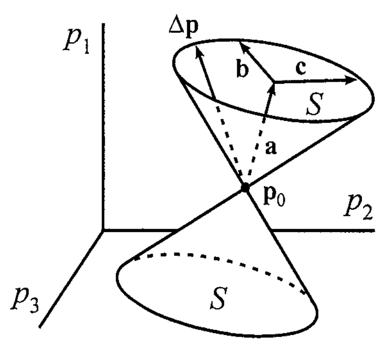


Fig. 7.7 Parameterization of the cone singularity.

Finally, let us consider the cone singularity $(\pm i\omega)(\pm i\omega)$ determined by a purely imaginary pair of double semi-simple eigenvalues. Bifurcation of the eigenvalue $\lambda_0 = i\omega$ is described by formulae (7.41), (7.47), (7.48). Nature of the bifurcation depends on the sign of the product

$$b_1 b_2 = -\mathbf{u}_1^* \mathbf{J} \mathbf{u}_1 \mathbf{u}_2^* \mathbf{J} \mathbf{u}_2, \tag{7.87}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linearly independent eigenvectors corresponding to the double eigenvalue $\lambda_0 = i\omega$ and satisfying orthogonality condition (7.37).

If $b_1 b_2 > 0$, then the eigenvalues remain purely imaginary for any small perturbation of the parameter vector and, hence, the system remains stable in the vicinity of the point $\mathbf{p}_0$. Therefore, the cone singularity does not appear if $b_1 b_2 > 0$. Notice that the product $b_1 b_2$ depends only on the matrix $\mathbf{J}\mathbf{A}_0$.

In case of $b_1 b_2 < 0$ the double eigenvalue $\lambda_0 = i\omega$ splits into two purely imaginary simple eigenvalues under perturbation of the parameter vector $\Delta\mathbf{p} = \varepsilon\mathbf{e}$ if

$$(\pm i\omega)(\pm i\omega): \quad (\mathbf{g}_1, \Delta\mathbf{p})^2 > (\mathbf{g}_2, \Delta\mathbf{p})^2 + (\mathbf{g}_3, \Delta\mathbf{p})^2, \tag{7.88}$$

where components of the vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ are defined by formulae (7.46). Condition (7.88) provides the first order approximation of the stability domain near the point $\mathbf{p}_0$. Analogously to (7.85), (7.86), we can give the cone surface in the parameterized form by the expression

$$\Delta\mathbf{p} = \beta(\mathbf{a}' + \mathbf{b}' \sin\alpha + \mathbf{c}' \cos\alpha), \quad \beta \in \mathbb{R},\ 0 \le \alpha < 2\pi, \tag{7.89}$$

where the vectors $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{c}'$ are the following

$$\mathbf{a}' = \mathbf{g}_2 \times \mathbf{g}_3, \quad \mathbf{b}' = \mathbf{g}_1 \times \mathbf{g}_2, \quad \mathbf{c}' = \mathbf{g}_3 \times \mathbf{g}_1. \tag{7.90}$$

Theorem 7.4 *First order approximations of the stability domain in the neighborhood of (dihedral) angle, trihedral angle, and cone singularities are given by expressions (7.79), (7.80), and (7.88), where the cone singularity $(\pm i\omega)(\pm i\omega)$ appears only if $b_1 b_2 < 0$. The vectors in conditions (7.79), (7.80), and (7.88) are linearly independent in the case of general position.*

7.7 Stability analysis near singularities associated with eigenvalues of multiplicity $k > 2$

In this section we study more complicated singularities: cusps (cuspidal edges) 0^4, $(\pm i\omega)^3$, truncated cuspidal edges $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$, and trihedral spires 0^6, $(\pm i\omega)^4$, see Figs. 7.5 and 7.6.

Let us consider a purely imaginary eigenvalue $\lambda_0 = i\omega$ of the matrix $\mathbf{JA}_0$ (in the case of zero eigenvalue $\omega = 0$). We assume that the eigenvalue λ_0 is nonderogatory and has multiplicity k. The corresponding right Jordan chain $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$ and the left Jordan chain $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ are defined by the equations

$$\begin{aligned}
\mathbf{JA}_0\mathbf{u}_0 &= i\omega\mathbf{u}_0, & \mathbf{v}_0^T\mathbf{JA}_0 &= i\omega\mathbf{v}_0^T, \\
\mathbf{JA}_0\mathbf{u}_1 &= i\omega\mathbf{u}_1 + \mathbf{u}_0, & \mathbf{v}_1^T\mathbf{JA}_0 &= i\omega\mathbf{v}_1^T + \mathbf{v}_0^T, \\
&\vdots & &\vdots \\
\mathbf{JA}_0\mathbf{u}_{k-1} &= i\omega\mathbf{u}_{k-1} + \mathbf{u}_{k-2}, & \mathbf{v}_{k-1}^T\mathbf{JA}_0 &= i\omega\mathbf{v}_{k-1}^T + \mathbf{v}_{k-2}^T,
\end{aligned} \tag{7.91}$$

with the normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_{k-1} = 1, \quad \mathbf{v}_1^T\mathbf{u}_{k-1} = \cdots = \mathbf{v}_{k-1}^T\mathbf{u}_{k-1} = 0. \tag{7.92}$$

Let us define vectors $\mathbf{f}^j = (f_1^j, \ldots, f_n^j)$, $j = 0, \ldots, k-1$, with the components

$$\begin{aligned}
f_s^0 &= -i^k\mathbf{v}_0^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0, \\
f_s^1 &= -i^{k-1}\left(\mathbf{v}_0^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0\right), \\
f_s^2 &= -i^{k-2}\left(\mathbf{v}_0^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_2 + \mathbf{v}_1^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_2^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0\right), \\
&\vdots \\
f_s^{k-1} &= -i\sum_{r=0}^{k-1}\mathbf{v}_r^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_{k-1-r}, \quad s = 1, \ldots, n,
\end{aligned} \tag{7.93}$$

where derivatives are taken at $\mathbf{p} = \mathbf{p}_0$, and i is the imaginary unit. General properties of these vectors are described in the following lemma.

Lemma 7.1 *The vectors $\mathbf{f}^0, \ldots, \mathbf{f}^{k-1}$ are real and do not depend on a choice of the Jordan chains.*

Proof. The general formula for the sth component of the vector $\mathbf{f}^j$ in

(7.93) can be written in the form

$$f_s^j = -i^{k-j}\mathrm{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}^T\mathbf{J}\frac{\partial \mathbf{A}}{\partial p_s}\mathbf{U}\right), \tag{7.94}$$

where $\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{k-1}]$ and $\mathbf{V} = [\mathbf{v}_{k-1}, \ldots, \mathbf{v}_1, \mathbf{v}_0]$ are $m \times k$ matrices, and

$$\mathbf{C}_j = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.95}$$

is the $k \times k$ matrix with ones of the jth diagonal to the right from the main diagonal and zeros elsewhere. Notice that the matrices $\mathbf{C}_j$ can be obtained as powers of the matrix $\mathbf{C}_1 = \mathbf{J}_0(k)$, which is the Jordan block of size k with zero eigenvalue:

$$\mathbf{C}_j = \mathbf{C}_1^j, \quad j = 0, \ldots, k-1, \tag{7.96}$$

where $\mathbf{C}_0 = \mathbf{I}$. Equations of Jordan chain (7.91) and normalization conditions (7.92) can be written in a matrix form as

$$\mathbf{J}\mathbf{A}_0\mathbf{U} = \mathbf{U}\mathbf{J}_{i\omega}(k), \quad \mathbf{V}^T\mathbf{J}\mathbf{A}_0 = \mathbf{J}_{i\omega}(k)\mathbf{V}^T, \quad \mathbf{V}^T\mathbf{U} = \mathbf{I}, \tag{7.97}$$

where $\mathbf{J}_{i\omega}(k)$ is the Jordan block of size k with the eigenvalue $\lambda_0 = i\omega$; see Sections 2.2 and 2.3.

An arbitrary Jordan chain $\widetilde{\mathbf{U}} = [\widetilde{\mathbf{u}}_0, \ldots, \widetilde{\mathbf{u}}_{k-1}]$ is given in the form

$$\widetilde{\mathbf{U}} = \mathbf{U}\mathbf{C}, \quad \mathbf{C} = c_0\mathbf{I} + c_1\mathbf{C}_1 + \cdots + c_{k-1}\mathbf{C}_{k-1}, \tag{7.98}$$

where $c_0, \ldots, c_{k-1}$ are arbitrary numbers with $c_0 \neq 0$; see Section 2.2. The corresponding left Jordan chain $\widetilde{\mathbf{V}} = [\widetilde{\mathbf{v}}_{k-1}, \ldots, \widetilde{\mathbf{v}}_0]$ satisfying normalization conditions (7.92) is

$$\widetilde{\mathbf{V}} = \mathbf{V}\left(\mathbf{C}^{-1}\right)^T. \tag{7.99}$$

Using expressions (7.98) and (7.99), we write formula (7.94) for arbitrary

Jordan chains as

$$\begin{aligned} f_s^j &= -i^{k-j}\text{trace}\left(\mathbf{C}_{k-1-j}\widetilde{\mathbf{V}}^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\widetilde{\mathbf{U}}\right) \\ &= -i^{k-j}\text{trace}\left(\mathbf{C}_{k-1-j}\mathbf{C}^{-1}\mathbf{V}^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{U}\mathbf{C}\right) \\ &= -i^{k-j}\text{trace}\left(\mathbf{C}_{k-1-j}\mathbf{C}\mathbf{C}^{-1}\mathbf{V}^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{U}\right) \\ &= -i^{k-j}\text{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}^T\mathbf{J}\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{U}\right). \end{aligned} \tag{7.100}$$

In (7.100) we used the permutability property of the matrices $\mathbf{C}_j$ and $\mathbf{C}$, which are polynomials of the matrix $\mathbf{C}_1$, and the property

$$\text{trace}\,(\mathbf{AB}) = \text{trace}\,(\mathbf{BA}) \tag{7.101}$$

of the trace function. Equation (7.100) proves the independence of the vectors $\mathbf{f}^0, \ldots, \mathbf{f}^{k-1}$ on the choice of Jordan chains.

Taking complex-conjugate transpose of equations (7.91) for the right Jordan chain and multiplying the result by $\mathbf{J}$ from the right, we obtain

$$\begin{aligned} (-\mathbf{J}\overline{\mathbf{u}}_0)^T\mathbf{J}\mathbf{A}_0 &= i\omega(-\mathbf{J}\overline{\mathbf{u}}_0)^T, \\ (\mathbf{J}\overline{\mathbf{u}}_1)^T\mathbf{J}\mathbf{A}_0 &= i\omega(\mathbf{J}\overline{\mathbf{u}}_1)^T + (-\mathbf{J}\overline{\mathbf{u}}_0)^T, \\ &\vdots \\ ((-1)^k\mathbf{J}\overline{\mathbf{u}}_{k-1})^T\mathbf{J}\mathbf{A}_0 &= i\omega((-1)^k\mathbf{J}\overline{\mathbf{u}}_{k-1})^T + ((-1)^{k-1}\mathbf{J}\overline{\mathbf{u}}_{k-2})^T, \end{aligned} \tag{7.102}$$

where the relations $\mathbf{J}^T = \mathbf{J}^{-1} = -\mathbf{J}$ are used. Therefore, the vectors $\widetilde{\mathbf{v}}_j = (-1)^{j+1}\mathbf{J}\overline{\mathbf{u}}_j$, $j = 0, \ldots, k-1$, form a left Jordan chain. Analogously, the vectors $\widetilde{\mathbf{u}}_j = (-1)^{k-j}\mathbf{J}\overline{\mathbf{v}}_j$, $j = 0, \ldots, k-1$, form a right Jordan chain. It is straightforward to verify that the new Jordan chains satisfy normalization conditions (7.92). Using these Jordan chains in formulae (7.93), we derive

$$
\begin{aligned}
f_s^j &= -i^{k-j} \sum_{r=0}^{j} \widetilde{\mathbf{v}}_r^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_s} \widetilde{\mathbf{u}}_{j-r} \\
&= (-i)^{k-j} \sum_{r=0}^{j} \mathbf{u}_r^* \frac{\partial \mathbf{A}}{\partial p_s} \mathbf{J} \overline{\mathbf{v}}_{j-r} \\
&= \left(-i^{k-j} \sum_{r=0}^{j} \mathbf{v}_{j-r}^T \mathbf{J} \frac{\partial \mathbf{A}}{\partial p_s} \mathbf{u}_r \right)^* = \overline{f_s^j}.
\end{aligned}
\tag{7.103}
$$

Equation (7.103) shows that all the components f_s^j are real. □

In case of zero eigenvalue $\lambda_0 = 0$, the Jordan chains can be chosen real. Due to the coefficient i^{k-j} the vectors $\mathbf{f}^j$ are zero for odd numbers $k - j$. Since zero eigenvalue of a Hamiltonian matrix has even multiplicity, we have $\mathbf{f}^1 = \mathbf{f}^3 = \cdots = \mathbf{f}^{k-1} = 0$.

In case of a double eigenvalue $\lambda_0 = i\omega$ (purely imaginary or zero), we take the left eigenvector in the form (7.31). Then expression (7.93) for components of the vector $\mathbf{f}^0$ yields

$$
f_s^0 = -\left(\mathbf{u}_0^* \frac{\partial \mathbf{A}}{\partial p_s} \mathbf{u}_0 \right) \Big/ \left(\mathbf{u}_1^* (\mathbf{A}_0 + i\omega \mathbf{J}) \mathbf{u}_1 \right). \tag{7.104}
$$

Hence, in case of a double eigenvalue the vectors $\mathbf{f}$ and $\mathbf{f}^0$ defined in (7.35) and (7.93), respectively, coincide.

The following theorem describes stabilizing perturbations of the parameter vector and, thus, gives the orientation of singularities in the parameter space.

Theorem 7.5 *First order approximations of the stability domain near the singularities of types* 0^4, $(\pm i\omega)^3$, $0^4(\pm i\omega)^2$, $(\pm i\omega)^3 0^2$, $(\pm i\omega_1)^3(\pm i\omega_2)^2$,

0^6*, and* $(\pm i\omega)^4$ *are given by the formulae*

$$
\begin{aligned}
0^4 &: (\mathbf{f}_0^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_0^2, \Delta\mathbf{p}) < 0;\\
(\pm i\omega)^3 &: (\mathbf{f}_{i\omega}^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_{i\omega}^1, \Delta\mathbf{p}) < 0;\\
0^6 &: (\mathbf{f}_0^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_0^2, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_0^4, \Delta\mathbf{p}) > 0;\\
(\pm i\omega)^4 &: (\mathbf{f}_{i\omega}^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_{i\omega}^1, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_{i\omega}^2, \Delta\mathbf{p}) < 0; \qquad (7.105)\\
0^4(\pm i\omega)^2 &: (\mathbf{f}_0^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_0^2, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0;\\
(\pm i\omega)^3 0^2 &: (\mathbf{f}_{i\omega}^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_{i\omega}^1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_0, \Delta\mathbf{p}) < 0;\\
(\pm i\omega_1)^3(\pm i\omega_2)^2 &: (\mathbf{f}_{i\omega_1}^0, \Delta\mathbf{p}) = 0,\ (\mathbf{f}_{i\omega_1}^1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0;
\end{aligned}
$$

where the subscript denotes the eigenvalue for which the corresponding vector is evaluated. The vectors appearing for each singularity in (7.105) are linearly independent in the case of general position.

Proof. Let us transform the family of Hamiltonian matrices $\mathbf{JA}(\mathbf{p})$ to the versal deformation; see Section 7.3. Expression for the versal deformation (7.61) with the use of (7.58) takes the form

$$\mathbf{JA}'(\mathbf{p}) = \mathbf{S}^{-1}(\mathbf{p})\mathbf{JA}(\mathbf{p})\mathbf{S}(\mathbf{p}), \qquad (7.106)$$

where $\mathbf{S}(\mathbf{p})$ is a nonsingular matrix smoothly dependent on the parameter vector.

Let $\lambda_0 = i\omega$ be a nonderogatory eigenvalue of multiplicity k for the matrix $\mathbf{JA}_0$ with the right and left Jordan chains $\mathbf{U} = [\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}]$ and $\mathbf{V} = [\mathbf{v}_{k-1}, \ldots, \mathbf{v}_0]$. Then the right and left Jordan chains $\mathbf{U}' = [\mathbf{u}'_0, \ldots, \mathbf{u}'_{k-1}]$ and $\mathbf{V}' = [\mathbf{v}'_{k-1}, \ldots, \mathbf{v}'_0]$ corresponding to the eigenvalue λ_0 for the matrix $\mathbf{JA}'_0$ are

$$\mathbf{U}' = \mathbf{S}_0^{-1}\mathbf{U}, \quad \mathbf{V}' = \mathbf{S}_0^T\mathbf{V}, \qquad (7.107)$$

where $\mathbf{S}_0 = \mathbf{S}(\mathbf{p}_0)$.

Using formulae (7.97), (7.106), and (7.107) in expression (7.94), we find

$$
\begin{aligned}
f_s^j &= -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}'^T\mathbf{J}\frac{\partial \mathbf{A}'}{\partial p_s}\mathbf{U}'\right) \\
&= -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}'^T\frac{\partial \mathbf{S}^{-1}}{\partial p_s}\mathbf{J}\mathbf{A}_0\mathbf{S}_0\mathbf{U}'\right) \\
&\quad -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}'^T\mathbf{S}_0^{-1}\mathbf{J}\frac{\partial \mathbf{A}}{\partial p_s}\mathbf{S}_0\mathbf{U}'\right) \\
&\quad -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}'^T\mathbf{S}_0^{-1}\mathbf{J}\mathbf{A}_0\frac{\partial \mathbf{S}}{\partial p_s}\mathbf{U}'\right) \qquad (7.108) \\
&= -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}^T\mathbf{J}\frac{\partial \mathbf{A}}{\partial p_s}\mathbf{U}\right) \\
&\quad -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}^T\left(\mathbf{S}_0\frac{\partial \mathbf{S}^{-1}}{\partial p_s}+\frac{\partial \mathbf{S}}{\partial p_s}\mathbf{S}_0^{-1}\right)\mathbf{U}\mathbf{J}_{i\omega}(k)\right) \\
&= -i^{k-j}\operatorname{trace}\left(\mathbf{C}_{k-1-j}\mathbf{V}^T\mathbf{J}\frac{\partial \mathbf{A}}{\partial p_s}\mathbf{U}\right).
\end{aligned}
$$

Here we have used relation (7.101) and the permutability property of the matrices $\mathbf{C}_{k-1-j}$ and $\mathbf{J}_{i\omega}(k)$. Hence, we proved that the vectors $\mathbf{f}^0, \ldots, \mathbf{f}^{k-1}$ evaluated for a purely imaginary or zero eigenvalue $\lambda_0 = i\omega$ by formulae (7.93) for the matrix families $\mathbf{JA}(\mathbf{p})$ and $\mathbf{JA}'(\mathbf{p})$ coincide.

Let us study the case of singularity 0^4 characterized by the eigenvalue $\lambda_0 = 0$ of multiplicity $k = 4$. The vectors $\mathbf{f}^0$ and $\mathbf{f}^2$ can be evaluated using the versal deformation $\mathbf{JA}'(\mathbf{p})$ (recall that $\mathbf{f}^1 = \mathbf{f}^3 = 0$). Hamiltonian equations corresponding to the versal deformation are decomposed into a set of independent subsystems; see Section 7.3. Hence, we can evaluate the vectors $\mathbf{f}^0$ and $\mathbf{f}^2$ for the matrix $\mathbf{JB}(\mathbf{q})$, $\mathbf{q} = \mathbf{q}(\mathbf{p})$, corresponding to the subsystem associated with the eigenvalue $\lambda_0 = 0$ and given by formula

(7.72). The right and left Jordan chains for the matrix $\mathbf{JB}_0 = \mathbf{JB}(0)$ are

$$\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{pmatrix} 0 & \sigma & -\sigma & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathbf{V} = [\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0] = \begin{pmatrix} 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7.109}$$

Using (7.109) in formulae (7.93), we obtain

$$\mathbf{f}^0 = -2\sigma\nabla q_2, \quad \mathbf{f}^2 = 2\sigma\nabla q_1, \tag{7.110}$$

where

$$\nabla = \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right) \tag{7.111}$$

is the gradient operator at $\mathbf{p} = \mathbf{p}_0$.

Stability domain in the neighborhood of the singularity point $\mathbf{p}_0$ is given by inequalities (7.74), see Section 7.5. Using linear approximations for the functions

$$q_i(\mathbf{p}) = (\nabla q_i, \Delta\mathbf{p}) + o(||\Delta\mathbf{p}||), \quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0, \quad i = 1, 2, \tag{7.112}$$

and vectors (7.110), we find the first order approximation of the stability domain in the form

$$(\mathbf{f}^0, \Delta\mathbf{p}) = 0, \quad (\mathbf{f}^2, \Delta\mathbf{p}) < 0, \tag{7.113}$$

which proves the first expression in (7.105).

For other types of singularities, expressions for first order approximations of the stability domain are found analogously. The main idea of the stability analysis is the transformation of the matrix family $\mathbf{JA}(\mathbf{p})$ to the versal deformation and evaluation of the gradients of the transformation function $\mathbf{q}(\mathbf{p})$ using the vectors $\mathbf{f}^0, \ldots, \mathbf{f}^{k-1}$.

Finding higher order derivatives of the function $\mathbf{q}(\mathbf{p})$, we can obtain higher order approximations of the stability domain near the singularity.

For general theory of transformation to versal deformations we refer to [Mailybaev (2001)]. □

7.8 Mechanical examples

7.8.1 *Elastic simply supported pipe conveying fluid*

Let us consider an elastic simply supported pipe conveying fluid, see Fig. 7.8. Oscillations of the pipe are described by the partial differential equation for the deflection function $w(x,t)$, $0 \le x \le l$, see [Thompson (1982)]:

$$(m + m_f)\frac{\partial^2 w}{\partial t^2} + 2v_f m_f \frac{\partial^2 w}{\partial x \partial t} + EJ\frac{\partial^4 w}{\partial x^4} + m_f v_f^2 \frac{\partial^2 w}{\partial x^2} = 0 \qquad (7.114)$$

with the boundary conditions

$$w(0,t) = \left(EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = 0, \quad w(l,t) = \left(EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=l} = 0. \qquad (7.115)$$

In these equations, m, EJ, and l are the mass per unit length, the bending stiffness, and the length of the pipe, respectively; m_f and v_f are the mass per unit length and the velocity of the fluid. The terms in equation (7.114) describe inertial, Coriolis, stiffness, and centrifugal forces, respectively. Dissipative forces are not taken into account.

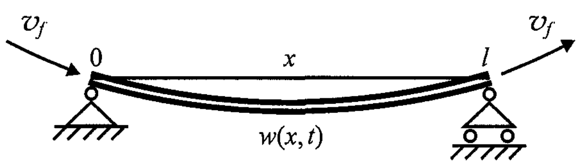


Fig. 7.8 Elastic pipe conveying fluid.

For $v_f = 0$ equations (7.114), (7.115) describe transverse vibrations of the pipe with immovable fluid. As the fluid velocity increases, the pipe can lose stability in static (divergence) or dynamic way (flutter).

We find approximate solution to equation (7.114) using Galerkin's method with two coordinate functions

$$w(x,t) = q_1(t)\sin\frac{\pi x}{l} + q_2(t)\sin\frac{2\pi x}{l}, \qquad (7.116)$$

where $q_1(t)$ and $q_2(t)$ are unknown functions of time. As a result, we obtain the linear gyroscopic system (7.15) with the matrices [Thompson (1982)]

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G} = \sqrt{\alpha\Lambda}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1-\Lambda & 0 \\ 0 & 16-4\Lambda \end{pmatrix}, \tag{7.117}$$

where

$$\Lambda = \frac{m_f v_f^2 l^2}{\pi^2 EJ}, \quad \alpha = \left(\frac{16}{3\pi}\right)^2 \frac{m_f}{m+m_f} \tag{7.118}$$

are the two dimensionless parameters. The parameter $\Lambda \geq 0$ describes the fluid velocity, and the parameter α, characterizing mass ratio of the pipe and fluid, changes in the interval $0 \leq \alpha \leq (16/3\pi)^2 \approx 2.882$. Using transformation (7.16), (7.17), we obtain linear system (7.6) with the Hamiltonian matrix

$$\mathbf{JA} = \begin{pmatrix} 0 & \sqrt{\alpha\Lambda}/2 & 1 & 0 \\ -\sqrt{\alpha\Lambda}/2 & 0 & 0 & 1 \\ -1+\Lambda-\alpha\Lambda/4 & 0 & 0 & \sqrt{\alpha\Lambda}/2 \\ 0 & -16+4\Lambda-\alpha\Lambda/4 & -\sqrt{\alpha\Lambda}/2 & 0 \end{pmatrix}. \tag{7.119}$$

Let us consider the point $\mathbf{p}_0 = (4,\, 3/4)$ in the parameter space $\mathbf{p} = (\Lambda, \alpha)$. At this point the matrix $\mathbf{JA}_0$ possesses the nonderogatory eigenvalue $\lambda_0 = 0$ of multiplicity $k = 4$. Hence, at $\mathbf{p}_0$ we have the cusp singularity 0^4 on the stability boundary.

The right and left Jordan chains (7.91), (7.92) of the zero eigenvalue are

$$\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{pmatrix} 0 & -\sqrt{3}/3 & 0 & -4\sqrt{3}/9 \\ 1 & 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & -5/(2\sqrt{3}) & 0 \\ 0 & 1/2 & 0 & 1/3 \end{pmatrix},$$

$$\mathbf{V} = [\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0] = \begin{pmatrix} 0 & \sqrt{3} & 0 & -3\sqrt{3}/2 \\ 5/2 & 0 & -3/2 & 0 \\ \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 0 & 4 & 0 & -3 \end{pmatrix}. \tag{7.120}$$

Using matrix (7.119) and vectors (7.120) in formulae (7.93), we find

$$\mathbf{f}_0^0 = (12,\ 0), \quad \mathbf{f}_0^2 = (17/4,\ -4). \tag{7.121}$$

First order approximation of the stability domain (7.105) yields the ray

$$0^4: \quad \Delta\mathbf{p} = (\Delta\Lambda, \Delta\alpha), \quad \Delta\Lambda = 0, \quad \Delta\alpha > 0. \tag{7.122}$$

This ray describes the orientation of the cusp singularity in the parameter space, see Fig. 7.9.

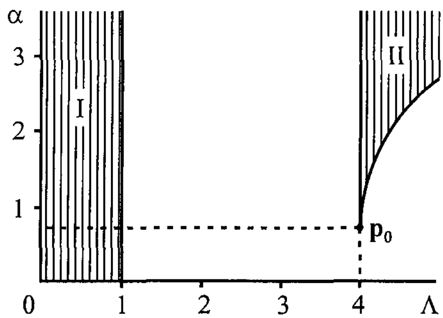


Fig. 7.9 Cusp singularity of the stability boundary for the pipe conveying fluid.

The characteristic equation for matrix (7.119) takes the form

$$\lambda^4 + \lambda^2(17 - 5\Lambda + \alpha\Lambda) + 4(1-\Lambda)(4-\Lambda) = 0. \tag{7.123}$$

This is a biquadratic equation for λ. Stability condition requires that the roots λ^2 are real and negative. This implies that the coefficients and discriminant of polynomial (7.123) must be positive:

$$\begin{gathered} 17 - 5\Lambda + \alpha\Lambda > 0, \quad 4(1-\Lambda)(4-\Lambda) > 0, \\ (17 - 5\Lambda + \alpha\Lambda)^2 > 16(1-\Lambda)(4-\Lambda). \end{gathered} \tag{7.124}$$

These conditions determine the stability zones I and II in the parameter space

$$\begin{aligned} &\mathrm{I}: \quad 0 < \Lambda < 1, \\ &\mathrm{II}: \quad \Lambda > 4, \quad \alpha > 5 - \frac{17}{\Lambda} + 4\sqrt{\left(\frac{1}{\Lambda} - 1\right)\left(\frac{4}{\Lambda} - 1\right)}, \end{aligned} \tag{7.125}$$

shown in Fig. 7.9. The boundary of the second zone has the cusp singularity 0^4 at the point $\mathbf{p}_0 = (4, 3/4)$, where two curves of the stability boundary are tangent to the same ray (7.122).

7.8.2 *Gyroscopic stabilization of statically unbalanced rotating system*

Let us consider a mechanical system in the field of gravity consisting of a disk of mass m and radius $2l$ connected by two massless rods with a vertical shaft rotating with a constant angular velocity Ω, see Fig. 7.10. Lengths of the rods are $2l$ and l, and the second rod is rigidly attached to the center of the disk perpendicular to the disk plane. The rods are connected to the rotor and to each other by elastic spherical hinges. Each hinge provides three degrees of freedom between the connected bodies.

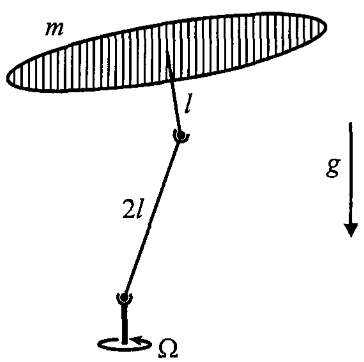


Fig. 7.10 Rotation of a disk on elastically connected rods.

The system has six degrees of freedom. As generalized coordinates we choose the Krylov angles α_i, β_i, $i = 1, 2$, determining position of each rod relative to the vertical axis in the reference frame rotating with the rotor, and angles γ_i, $i = 1, 2$, characterizing twist in the hinges. We study stability of rotation of the system about the vertical axis with $\alpha_i = \beta_i = \gamma_i = 0$, $i = 1, 2$.

The Lagrange function of the system written up to second order terms takes the form [Mailybaev and Seyranian (1999a)]

$$
\begin{aligned}
L = \frac{A}{2}\left(\left(\dot\alpha_2 - \Omega\beta_2\right)^2 + \left(\dot\beta_2 + \Omega\alpha_2\right)^2\right) \\
+\frac{ml^2}{2}\left(\left(2(\dot\alpha_1 - \Omega\beta_1) + (\dot\alpha_2 - \Omega\beta_2)\right)^2 + \left(2(\dot\beta_1 + \Omega\alpha_1) + (\dot\beta_2 + \Omega\alpha_2)\right)^2\right) \\
+\frac{B}{2}\left(\Omega^2(1-\alpha_2^2-\beta_2^2) + \Omega\left(\dot\alpha_2\beta_2 - \alpha_2\dot\beta_2\right) + \dot\gamma_2^2\right) \\
-mgl\left(\left(1-\frac{\alpha_2^2}{2}-\frac{\beta_2^2}{2}\right) + 2\left(1-\frac{\alpha_1^2}{2}-\frac{\beta_1^2}{2}\right)\right) \\
-\frac{C_1}{2}(\alpha_1^2+\beta_1^2) - \frac{C_2}{2}\left((\alpha_2-\alpha_1)^2 + (\beta_2-\beta_1)^2\right) \\
-\frac{C_1'}{2}\gamma_1^2 - \frac{C_2'}{2}(\gamma_2-\gamma_1)^2,
\end{aligned}
\tag{7.126}
$$

where $A = ml^2$, $B = 2ml^2$ are the principal moments of inertia of the disk; C_1, C_2 and C_1', C_2' are the bending and torsional stiffnesses of the hinges, respectively; and g is the acceleration of gravity. Using function (7.126) in the Lagrange equations, we find equations of motion of the system linearized near the vertical equilibrium position $\alpha_i = \beta_i = \gamma_i = 0$, $i = 1, 2$.

Equations of motion separate into two independent systems. One system depends only on γ_1, γ_2 and has the form

$$
\begin{aligned}
C_1'\gamma_1 - C_2'(\gamma_2-\gamma_1) = 0, \\
B\ddot\gamma_2 + C_2'(\gamma_2-\gamma_1) = 0.
\end{aligned}
\tag{7.127}
$$

After elementary transformations, we obtain

$$
\begin{aligned}
\gamma_1 = \frac{C_2'}{C_1'+C_2'}\gamma_2, \\
\ddot\gamma_2 + \frac{C_1'C_2'}{B(C_1'+C_2')}\gamma_2 = 0.
\end{aligned}
\tag{7.128}
$$

This system is stable for any positive elastic coefficients C_1' and C_2'.

The second system after introduction of dimensionless time $\tau = \Omega t$ takes the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0, \quad \mathbf{q} = (\alpha_1, \beta_1, \alpha_2, \beta_2)^T, \tag{7.129}$$

with the matrices

$$\mathbf{M} = \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & -8 & 0 & -4 \\ 8 & 0 & 4 & 0 \\ 0 & -4 & 0 & -2 \\ 4 & 0 & 2 & 0 \end{pmatrix},$$

$$\mathbf{P} = \frac{1}{\widetilde{\Omega}^2} \begin{pmatrix} c_1 + c_2 - 2 & 0 & -c_2 & 0 \\ 0 & c_1 + c_2 - 2 & 0 & -c_2 \\ -c_2 & 0 & c_2 - 1 & 0 \\ 0 & -c_2 & 0 & c_2 - 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \tag{7.130}$$

where

$$c_1 = \frac{C_1}{mgl}, \quad c_2 = \frac{C_2}{mgl}, \quad \widetilde{\Omega} = \Omega\sqrt{\frac{l}{g}} \tag{7.131}$$

are the dimensionless parameters. System (7.129), (7.130) is gyroscopic. Notice that the system is statically unbalanced, i.e., it is unstable in the absence of rotation.

Using transformation (7.16), we obtain equations of motion in Hamiltonian form (7.6), (7.17). Let us consider the point $\mathbf{p}_0 = (3/2,\ 2\sqrt{2}-5/2,\ 2-\sqrt{2})$ in the parameter space $\mathbf{p} = (c_1, c_2, \widetilde{\Omega})$. At this point the Hamiltonian matrix takes the form

$$\mathbf{J}\mathbf{A}_0 = -\frac{1}{4}\begin{pmatrix} 0 & -4 & 0 & -2 & -2 & 0 & 2 & 0 \\ 4 & 0 & 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -4 \\ -2 & 0 & -\kappa & 0 & 0 & -4 & 0 & 0 \\ 0 & -2 & 0 & -\kappa & 4 & 0 & 0 & 0 \\ -\kappa & 0 & -\kappa & 0 & 0 & -2 & 0 & 0 \\ 0 & -\kappa & 0 & -\kappa & 2 & 0 & 0 & 0 \end{pmatrix}, \tag{7.132}$$

where $\kappa = 1 + 2\sqrt{2}$. This matrix has two pairs of double nonderogatory eigenvalues $\lambda = \pm i\omega_1$ and $\lambda = \pm i\omega_2$ with

$$\omega_1 = \frac{2-\sqrt{2}}{4}, \quad \omega_2 = \frac{2+\sqrt{2}}{4}. \tag{7.133}$$

Hence, $\mathbf{p}_0$ is the point of type $(\pm i\omega_1)^2(\pm i\omega_2)^2$ on the edge of the stability boundary; see Theorem 7.3 (page 210). Jordan chains corresponding to the eigenvalues $\lambda = i\omega_1$ and $\lambda = i\omega_2$ are

$$\begin{aligned} \lambda = i\omega_1: \quad & \mathbf{u}_0 = (1,\, i,\, -\sqrt{2},\, -i\sqrt{2},\, -i,\, 1,\, -i\sqrt{2}/2,\, \sqrt{2}/2)^T, \\ & \mathbf{u}_1 = (-2i,\, 2,\, 2i,\, -2,\, 2-3\sqrt{2},\, i(2-3\sqrt{2}),\, -2\sqrt{2},\, -i2\sqrt{2})^T; \\ \lambda = i\omega_2: \quad & \mathbf{u}_0 = (1,\, i,\, \sqrt{2}-2,\, i(\sqrt{2}-2), \\ & \qquad -i(\sqrt{2}-1),\, \sqrt{2}-1,\, -i\sqrt{2}/2, \sqrt{2}/2)^T, \\ & \mathbf{u}_1 = (0,\, 0,\, i(2-2\sqrt{2}),\, 2\sqrt{2}-2, \\ & \qquad 4-\sqrt{2},\, i(4-\sqrt{2}),\, \sqrt{2},\, i\sqrt{2})^T. \end{aligned} \tag{7.134}$$

Using Jordan chains (7.134) in formula (7.35), we find the vectors

$$\begin{aligned} \mathbf{f}_{i\omega_1} &= -\frac{10+7\sqrt{2}}{8}\left(1,\, 3+2\sqrt{2},\, 2\right), \\ \mathbf{f}_{i\omega_2} &= -\frac{2+\sqrt{2}}{8}\left(3+2\sqrt{2},\, 9+4\sqrt{2},\, 6-4\sqrt{2}\right). \end{aligned} \tag{7.135}$$

By Theorem 7.4 (page 214), approximation of the stability domain in the neighborhood of the point $\mathbf{p}_0 = (3/2,\, 2\sqrt{2}-5/2,\, 2-\sqrt{2})$ is given by the dihedral angle

$$(\mathbf{f}_{i\omega_1}, \Delta\mathbf{p}) < 0, \quad (\mathbf{f}_{i\omega_2}, \Delta\mathbf{p}) < 0, \tag{7.136}$$

shown in Fig. 7.11 by bold lines. The magnitude of dihedral angle (7.136) is equal to 0.883π. The vector $\mathbf{e}_\tau$ tangent to the edge, and the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ tangent to the sides of the dihedral angle of the stability boundary are

$$\begin{aligned}
\mathbf{e}_\tau &= \mathbf{f}_{i\omega_1} \times \mathbf{f}_{i\omega_2} = \frac{1}{4}(-58 - 41\sqrt{2}, 24 + 17\sqrt{2}, -41 - 29\sqrt{2}),\\
\mathbf{e}_1 &= \mathbf{f}_{i\omega_1} \times \mathbf{e}_\tau = \frac{1}{32}(5712 + 4039\sqrt{2}, 1492 + 1055\sqrt{2}, -7204 - 5094\sqrt{2}),\\
\mathbf{e}_2 &= \mathbf{e}_\tau \times \mathbf{f}_{i\omega_2} = \frac{1}{32}(-2080 - 1471\sqrt{2}, 748 + 529\sqrt{2}, 3380 + 2390\sqrt{2}).
\end{aligned} \tag{7.137}$$

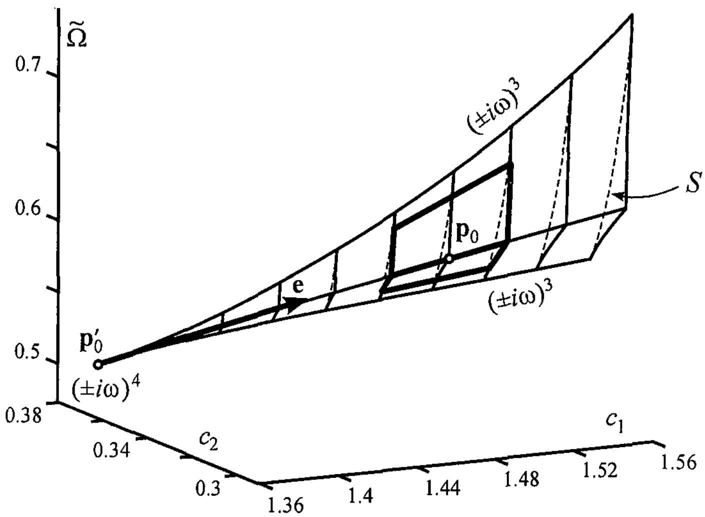


Fig. 7.11 Stability domain of rotation of the disk with singularities.

The obtained information is useful for construction of the stability boundary. Moving along the edge $(\pm i\omega_1)^2(\pm i\omega_2)^2$ in direction $\mathbf{e}_\tau$, we observe that the frequencies ω_1 and ω_2 come closer, and the size of the dihedral angle increases. The edge ends up at the point $\mathbf{p}_0' = (11/8, 3/8, 1/2)$, as which the frequencies ω_1 and ω_2 merge to $\omega = 1/2$. The size of the dihedral angle tends to π as we approach $\mathbf{p}_0'$.

At the point $\mathbf{p}_0' = (11/8,\ 3/8,\ 1/2)$ the Hamiltonian matrix $\mathbf{JA}_0' = \mathbf{JA}(\mathbf{p}_0')$ takes the form

$$\mathbf{JA}_0' = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 1 & 1 & 0 & -1 & 0 \\ -2 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 2 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 & -2 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 3 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{7.138}$$

This matrix possesses a pair of quadruple nonderogatory eigenvalues $\lambda = \pm i\omega$, where $\omega = 1/2$. The right and left Jordan chains for the eigenvalue $\lambda_0 = i\omega$ are found from equations (7.91), (7.92) in the form

$$\begin{aligned}
\mathbf{u}_0 &= (i,\ -1,\ -i,\ 1,\ 1,\ i,\ 1,\ i)^T, \\
\mathbf{u}_1 &= (0,\ 0,\ 1,\ i,\ i,\ -1,\ 0,\ 0)^T, \\
\mathbf{u}_2 &= (0,\ 0,\ 0,\ 0,\ 2,\ 2i,\ 2,\ 2i)^T, \\
\mathbf{u}_3 &= (0,\ 0,\ 1,\ i,\ -i,\ 1,\ 0,\ 0)^T, \\
\mathbf{v}_0 &= (1,\ -i,\ 1,\ -i,\ i,\ 1,\ -i,\ -1)^T/4, \\
\mathbf{v}_1 &= (i,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ -i)^T/4, \\
\mathbf{v}_2 &= (1,\ -i,\ 1,\ -i,\ -i,\ -1,\ i,\ 1)^T/4, \\
\mathbf{v}_3 &= (-i,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)^T/2.
\end{aligned} \tag{7.139}$$

Using vectors (7.139) in formulae (7.93), we find the vectors

$$\begin{aligned}
\mathbf{f}^0 &= (-2,\ -8,\ -1), \\
\mathbf{f}^1 &= (0,\ -8,\ -4), \\
\mathbf{f}^2 &= (-2,\ -10,\ -6), \\
\mathbf{f}^3 &= (0,\ 0,\ 0).
\end{aligned} \tag{7.140}$$

By Theorem 7.3 (page 210), the stability boundary has the singularity "trihedral spire" $(\pm i\omega)^4$ at the point $\mathbf{p}_0' = (11/8,\ 3/8,\ 1/2)$. First order approximation of the stability domain near $\mathbf{p}_0'$ is given by Theorem 7.5 (page 218) as

$$(\mathbf{f}^0, \Delta\mathbf{p}) = 0, \quad (\mathbf{f}^1, \Delta\mathbf{p}) = 0, \quad (\mathbf{f}^2, \Delta\mathbf{p}) < 0, \tag{7.141}$$

which determines the ray

$$\Delta\mathbf{p} = \varepsilon\mathbf{e}, \quad \mathbf{e} = (3,\ -1,\ 2), \quad \varepsilon > 0. \tag{7.142}$$

All three edges (two cuspidal edges of type $(\pm i\omega)^3$ and one edge of type $(\pm i\omega_1)^2(\pm i\omega_2)^2$) start at $\mathbf{p}_0' = (11/8,\ 3/8,\ 1/2)$ along the same direction $\mathbf{e}$, see Fig. 7.11. The stability boundary near the point $\mathbf{p}_0'$ represents a part of the singularity known as "swallow tail".

Checking stability condition numerically at the points of dense mesh in the parameter space, we find the stability domain as shown in Fig. 7.11. Numerical analysis confirms existence of singularities at the points $\mathbf{p}_0$, $\mathbf{p}_0'$ and their approximation given by (7.136), (7.142).

The stability domain determines the values of parameters, where gyroscopic stabilization takes place. We see that the stability domain enlarges with an increase of the first stiffness coefficient c_1 and decrease of the second stiffness coefficient c_2, and appears for the angular velocities $\widetilde{\Omega}$ higher than 1/2. The stability domain appears to be very narrow, which is typical for gyroscopic stabilization domains with singularities. The presence of singularities makes numerical analysis of the stability domain very difficult without knowledge on geometry of singularities presented in this chapter.

Chapter 8

Mechanical Effects Associated with Bifurcations and Singularities

In this chapter some mechanical effects associated with bifurcations of eigenvalues and singularities of the stability boundaries are studied. First, we analyze stability and catastrophes in one-parameter circulatory systems (with non-conservative positional forces). It is proven that flutter and divergence instabilities and transition of divergence to flutter are typical catastrophes for one-parameter circulatory systems.

Then two other interesting mechanical phenomena are considered. The first one is the phenomenon of transference of instability between eigenvalue branches. It turns out that a stable eigenvalue branch of a system subjected to non-conservative loading suddenly becomes unstable and vice versa with a change of problem parameters. The second phenomenon is the destabilization of a circulatory system by infinitely small damping. It turns out that the critical load parameter of the system with small damping is typically smaller than the critical load of the system with no damping. In this chapter, these two mechanical phenomena are explained from the point of view of behavior of eigenvalue branches in the vicinity of a double eigenvalue with a single eigenvector.

Finally, we discuss an exciting effect of disappearance of flutter instability in the problem of aeroelastic stability of an unswept wing braced by struts of two types. This problem was first considered by [Keldysh (1938)]. In this chapter it is shown that for one type of the strut the flutter instability is replaced by the static form (divergence) and the critical speed has a discontinuity: it jumps to a higher value. And for the second type of the strut, the critical speed turns out to be finite and continuous, reaching the maximum value that is almost four times greater than the critical speed of the unbraced wing. We show that the effect of disappearance of flutter instability can be explained from the point of view of convexity of the flutter domain in the two-parameter plane.

It should be noted that mechanical problems and effects are considered almost in all chapters of the present book. However, here we present the effects known in the literature and treat them from the point of view of bifurcations and singularities.

8.1 Stability and catastrophes in one-parameter circulatory systems

Stability problems for circulatory systems have been considered in several books, see for example [Bolotin (1963); Panovko and Gubanova (1965); Ziegler (1968); Huseyin (1978); Leipholz (1980); Thompson (1982); Merkin (1997)], as well as in many papers. Most of these problems study the stability of circulatory systems dependent on a load parameter: magnitude of a follower force, velocity or density of flow *etc.* In this sense these problems are one-parametric.

Let us consider a linear autonomous system with non-conservative positional forces

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \tag{8.1}$$

where $\mathbf{M}$ is a real symmetric positive definite mass matrix of size $m \times m$, $\mathbf{C}$ is a real non-symmetric matrix of the same size describing potential and circulatory forces, and $\mathbf{q}$ is a vector of generalized coordinates of dimension m. System (8.1) is usually termed as a *circulatory system*, see [Ziegler (1968)]. Finding solution to this equation in the form $\mathbf{q} = \mathbf{u}\exp\lambda t$, we obtain the eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \mathbf{C}\right)\mathbf{u} = 0. \tag{8.2}$$

Here λ is an eigenvalue, and $\mathbf{u}$ is a corresponding eigenvector. The eigenvalues are found from the characteristic equation

$$\det\left(\lambda^2\mathbf{M} + \mathbf{C}\right) = 0. \tag{8.3}$$

Since the matrices $\mathbf{M}$ and $\mathbf{C}$ are real it is easy to see that if λ is a solution to the characteristic equation, then the quantities $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are also solutions to this equation. This means that system (8.1) is stable only when all the eigenvalues λ belong to the imaginary axis on the complex plane and are simple or semi-simple.

Using the notation

$$\mu = -\lambda^2, \quad \mathbf{A} = \mathbf{M}^{-1}\mathbf{C} \tag{8.4}$$

in (8.2), we obtain the standard eigenvalue problem

$$\mathbf{A}\mathbf{u} = \mu\mathbf{u}. \tag{8.5}$$

The stability condition requires that all the eigenvalues μ are real and positive.

We assume that the matrices $\mathbf{M}$ and $\mathbf{C}$ smoothly depend on one parameter $p \in \mathbb{R}$. Then the matrix $\mathbf{A}(p)$ is smooth too. According to [Galin (1972); Arnold (1983a)] generic (typical) one-parameter family of real matrices is characterized by simple eigenvalues μ and, at some isolated values of the parameter $p = p_0$, by a double real μ_0 with a Jordan chain of second order. More complicated Jordan structures are not typical and can be destroyed by infinitely small change in the family of matrices $\mathbf{A}(p)$.

It is easy to see that if μ is a simple real eigenvalue, it remains real with a change of p. Indeed, in the other case the complex conjugate eigenvalue $\overline{\mu}$ also appears, which means increase of the total number of the roots of the characteristic equation. Therefore, with a change of p the eigenvalues are able to leave the real axis only when they meet and become multiple.

Now we study bifurcation of eigenvalues for circulatory systems using the theory presented in Chapter 2. Let at $p = p_0$ the matrix $\mathbf{A}_0 = \mathbf{A}(p_0)$ possess a double real eigenvalue μ_0 with a Jordan chain of second order. This means that there are the right eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$ satisfying the equations

$$\begin{aligned} \mathbf{A}_0\mathbf{u}_0 &= \mu_0\mathbf{u}_0, \\ \mathbf{A}_0\mathbf{u}_1 &= \mu_0\mathbf{u}_1 + \mathbf{u}_0. \end{aligned} \tag{8.6}$$

Along with (8.6), we consider the left Jordan chain with the eigenvector $\mathbf{v}_0$ and associated vector $\mathbf{v}_1$:

$$\begin{aligned} \mathbf{v}_0^T\mathbf{A}_0 &= \mu_0\mathbf{v}_0^T, \\ \mathbf{v}_1^T\mathbf{A}_0 &= \mu_0\mathbf{v}_1^T + \mathbf{v}_0^T. \end{aligned} \tag{8.7}$$

The vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, $\mathbf{v}_1$ in equations (8.6) and (8.7) are related by the conditions

$$\mathbf{v}_0^T\mathbf{u}_0 = 0, \quad \mathbf{v}_0^T\mathbf{u}_1 = \mathbf{v}_1^T\mathbf{u}_0 \neq 0, \tag{8.8}$$

and are chosen satisfying the normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_1 = 1, \quad \mathbf{v}_1^T\mathbf{u}_1 = 0. \tag{8.9}$$

If we take an increment $p = p_0 + \Delta p$, then the matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1\Delta p + \dots, \tag{8.10}$$

where $\mathbf{A}_1 = (d\mathbf{A}/dp)_{p=p_0}$. The matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ are related to the matrices $\mathbf{C}$ and $\mathbf{M}$ as

$$\begin{gathered}
\mathbf{A}_0 = \mathbf{M}_0^{-1}\mathbf{C}_0, \quad \mathbf{A}_1 = \mathbf{M}_0^{-1}\mathbf{C}_1 - \mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{C}_0, \\
\mathbf{C}_0 = \mathbf{C}(p_0), \quad \mathbf{M}_0 = \mathbf{M}(p_0), \\
\mathbf{C}_1 = \left(\frac{d\mathbf{C}}{dp}\right)_{p=p_0}, \quad \mathbf{M}_1 = \left(\frac{d\mathbf{M}}{dp}\right)_{p=p_0}
\end{gathered} \tag{8.11}$$

Due to variation Δp the eigenvalues and eigenvectors of (8.5) also take increments. In case of the Jordan chain of second order, the disturbed eigenvalues are expressed as series in square roots of the small parameter Δp, see Section 2.5:

$$\mu = \mu_0 \pm \sqrt{f\Delta p} + O(\Delta p) \tag{8.12}$$

with the constant

$$f = \mathbf{v}_0^T\mathbf{A}_1\mathbf{u}_0. \tag{8.13}$$

Due to the assumption that μ_0 is real, the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ can be chosen real. Hence, f is a real constant.

Bifurcation (8.12) is illustrated in Fig. 8.1. In Section 2.6 it is interpreted as *strong interaction* of two eigenvalues. The arrows in Fig. 8.1a show the direction of motion of μ as p increases for $f < 0$: two real eigenvalues come together, merge and then diverge along a straight line parallel to the imaginary axis. If $f > 0$, the direction of motion of μ changes to the opposite: two complex conjugate μ approach each other, merge to a real μ_0, and then diverge along the real axis in opposite directions.

Bifurcation (8.12) in the three-dimensional space (for $f < 0$) is shown in Fig. 8.1b. At $p \approx p_0$ the intersecting curves are quadratic parabolae of the same curvature lying in the orthogonal planes $\operatorname{Im}\mu = 0$ and $\operatorname{Re}\mu = \mu_0$. Due to the fact that the matrix $\mathbf{A}$ is real, the interaction pictures in Fig. 8.1 are symmetric with respect to the axis (plane) $\operatorname{Im}\mu = 0$.

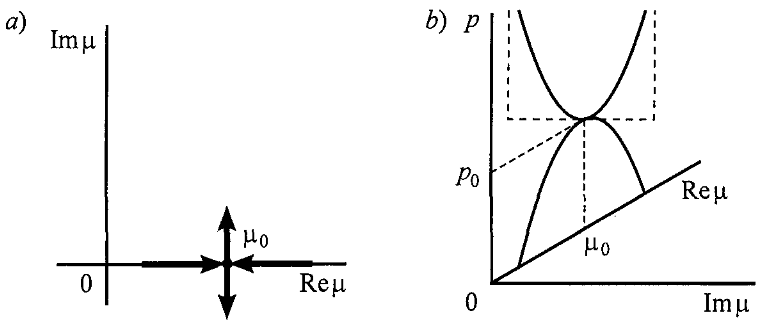


Fig. 8.1 Bifurcation of a double nonderogatory eigenvalue in circulatory systems.

Let us return to problem (8.2). Because $\mu = -\lambda^2$, positive eigenvalues μ correspond to purely imaginary pairs $\lambda = \pm i\sqrt{\mu}$, and negative μ correspond to real pairs $\lambda = \pm\sqrt{|\mu|}$. Thus, a positive $\mu_0 = -\lambda_0^2$ in bifurcation (8.12) means transition of stability of system (8.1) to dynamic instability (flutter), or vice versa (depending on the sign of f), and a negative μ_0 corresponds to transition of aperiodic instability (divergence) to dynamic instability (flutter), or vice versa.

Bifurcation (8.12) can be expressed through $\lambda = \pm i\sqrt{\mu}$. Substituting $p - p_0$ instead of Δp and using $\mu_0 = -\lambda_0^2 \neq 0$, we obtain

$$\lambda = \pm\lambda_0 \left(1 \pm \frac{\sqrt{f(p-p_0)}}{2\lambda_0^2}\right) + O(p-p_0). \tag{8.14}$$

Bifurcation (8.14) is illustrated in Fig. 8.2a,b for $f < 0$. Direction of the arrows corresponds to the increase of p. If $f > 0$, then the arrows should be reversed.

Besides catastrophes (8.12), (8.14), system (8.1) can lose stability statically, which means transition of positive eigenvalues μ to negative values through simple $\mu_0 = 0$ (divergence). Expansion for a simple eigenvalue in the vicinity of $\mu_0 = 0$ leads to the relation

$$\mu = d\Delta p + o(\Delta p), \quad d = \frac{\mathbf{v}_0^T \mathbf{A}_1 \mathbf{u}_0}{\mathbf{v}_0^T \mathbf{u}_0}, \tag{8.15}$$

see Section 2.4. Here $\mathbf{u}_0$ and $\mathbf{v}_0$ are the right and left eigenvectors corresponding to $\mu_0 = 0$. In terms of λ we obtain

$$\lambda = \pm i\sqrt{d(p-p_0)} + O(p-p_0). \tag{8.16}$$

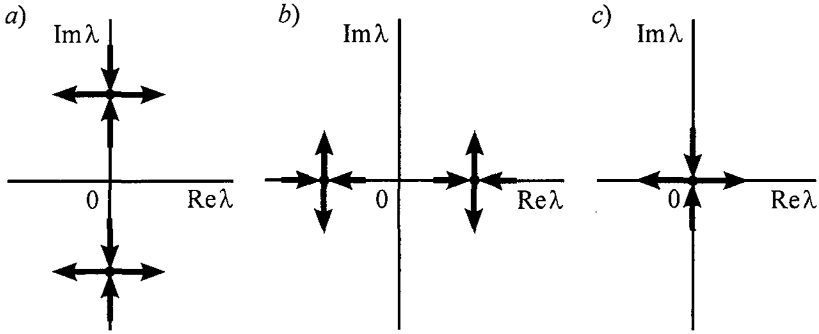


Fig. 8.2 Catastrophes of eigenvalues λ in circulatory systems: a) flutter, b) transition of divergence to flutter, c) divergence.

For $d < 0$, the behavior of λ with an increase of p is shown in Fig. 8.2c. If $d > 0$, the direction of motion of λ changes to the opposite. In the three-dimensional space $(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda, p)$ the plots $\lambda(p)$ are described by parabolae similar to that of shown in Fig. 8.1b.

According to (8.14) and (8.16), the derivative $d\lambda/dp$ tends to infinity as p tends to the critical value. This means *infinite velocity of growth of the catastrophe*, which is typical in the catastrophe theory, see [Arnold (1992)]. Due to fast growth of the increment $\mathrm{Re}\,\lambda$ proportional to $\sqrt{p - p_0}$, we observe rapidly growing amplitude of vibrations $\sim \exp(t\,\mathrm{Re}\,\lambda)$ as the critical load p_0 is passed. That is why it is often very difficult to damp the flutter and divergence instabilities.

The main result of this section can be formulated as [Seyranian (1994c)]:

Theorem 8.1 *One-parameter circulatory systems (8.1) in the case of general position are subjected to catastrophes of three types: flutter, transition of divergence to flutter (or vice versa), and divergence, described by relations (8.14), (8.16) and shown in Fig. 8.2a,b,c, respectively.*

In many books and papers, see for instance [Bolotin (1963); Panovko and Gubanova (1965); Ziegler (1968); Huseyin (1978); Leipholz (1987)], a typical load-frequency pattern $p(\omega)$, $\omega = \sqrt{\mu}$, reproduced in Fig. 8.3, is plotted. We emphasize that at $p > p_0$ the eigenvalues do not disappear, they just become complex conjugate.

Example 8.1 As an example, we consider a flutter problem of an airfoil in simplified formulation, see [Ziegler (1968)]. The airfoil is modeled by a rigid rectangular panel with two degrees of freedom: a vertical displacement z of the line passing through the point L and normal to the plane of the

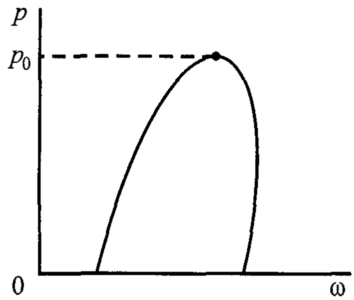


Fig. 8.3 A typical load-frequency curve.

figure, and a rotation angle θ about this line, see Fig. 8.4.

It is assumed that the aerodynamic lift force $p\theta$, applied to the point L, is proportional to the angle of attack θ with the coefficient p. The point O denotes the centroid of the section. The differential equations of motion are [Ziegler (1968)]

$$\begin{aligned} m_0\ddot{z} - m_0a_0\ddot{\theta} + c_1z - p\theta &= 0, \\ -m_0a_0\ddot{z} + m_0(i_0^2 + a_0^2)\ddot{\theta} + c_2\theta &= 0. \end{aligned} \tag{8.17}$$

In these equations m_0 is the mass of the panel per unit span, c_1 and c_2 are the stiffness coefficients, p is the load parameter proportional to the square of the flow velocity V, and i_0 is the radius of gyration of the airfoil about the normal vector to the plane of figure through the point O. The radius of gyration about the point L is $i_L^2 = i_0^2 + a_0^2$.

Regarding all the parameters of the airfoil as fixed quantities, let us analyze the stability of system (8.17) dependent on the parameter p. It is easy to see that system (8.17) is circulatory. The characteristic equation

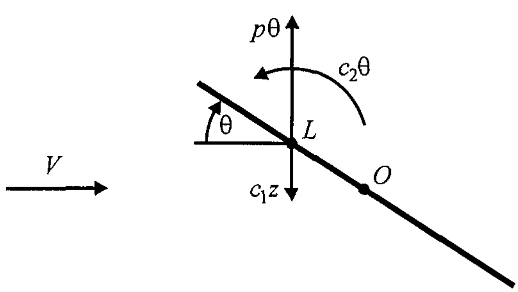


Fig. 8.4 Rigid airfoil vibrating in a flow.

takes the form

$$b_0\lambda^4 + b_2\lambda^2 + b_4 = 0 \tag{8.18}$$

with $b_0 = m_0^2 i_0^2$, $b_2 = c_2 m_0 + c_1 m_0 i_L^2 - p m_0 a_0$, and $b_4 = c_1 c_2$. The discriminant of this equation $D = b_2^2 - 4b_0 b_4$ is a quadratic polynomial with respect to the parameter p. It is easy to see that it has two positive roots

$$p_1 = \frac{c_1 a_0^2 + (\sqrt{c_2} - i_0\sqrt{c_1})^2}{a_0}, \quad p_2 = \frac{c_1 a_0^2 + (\sqrt{c_2} + i_0\sqrt{c_1})^2}{a_0}, \tag{8.19}$$

and $D < 0$ for $p_1 < p < p_2$. The coefficient $b_2(p)$ is a linear decaying function of p with $b_2(p) = 0$ at $p = p_3$, where $p_1 < p_3 < p_2$. Hence, the roots λ^2 of equation (8.18) are negative and different when $0 \le p < p_1$, complex conjugate when $p_1 < p < p_2$, and positive when $p > p_2$.

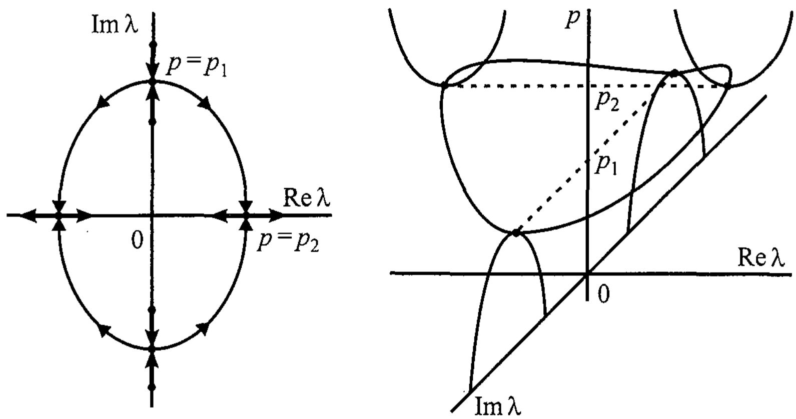


Fig. 8.5 Behavior of eigenvalues λ depending on p.

Behavior of the eigenvalues λ depending on p is shown in Fig. 8.5. The panel is stable for $0 \le p < p_1$. The interval $p_1 \le p < p_2$ is the flutter domain, and when p becomes greater than p_2 the flutter is changed to the divergence instability.

8.2 Transfer of instability between eigenvalue branches

Instability transference is one of those interesting phenomena that occurs very often in parametric studies of stability of non-conservative systems.

It turns out that a stable eigenvalue branch of a system subjected to non-conservative loading becomes unstable and vice versa with a change of problem parameters. Qualitatively, this effect is illustrated in Fig. 8.6, where behavior of eigenvalues with a change of the first (principal) parameter u is shown for different fixed values of the second parameter η. We see that the eigenvalue branch responsible for the loss of stability changes as η passes the value η_0. The major point in understanding this behavior is related to the existence of a double eigenvalue $\lambda_0 = \alpha_0 + i\omega_0$ ($\alpha_0 < 0$ and $\omega_0 \neq 0$) with a single eigenvector. At this point, the strong interaction between complex eigenvalues occurs, see Section 2.6.

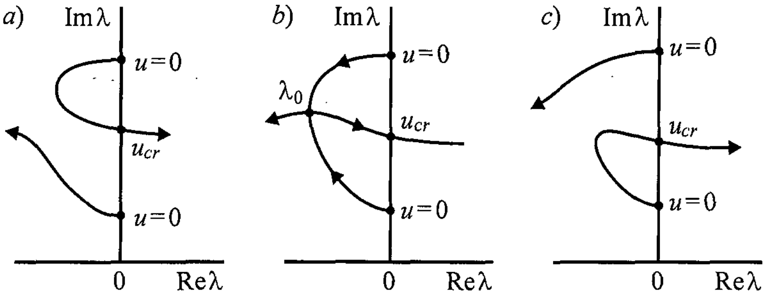


Fig. 8.6 Transference of instability between eigenvalue branches: a) $\eta < \eta_0$, b) $\eta = \eta_0$, c) $\eta > \eta_0$.

Transference of instability between eigenvalue branches was discussed in the papers by [Bun'kov (1969); Sugiyama and Noda (1981); Ryu *et al.* (2002)]. In [Bishop and Fawzy (1976)] it was suggested calling this phenomenon as "dynamic interference". Analytical description and explanation of this effect was given in [Seyranian and Pedersen (1995)], and classification of different possibilities for the transference of instability between eigenvalue branches was done in [Mailybaev (2000b)].

8.2.1 *Pipe conveying fluid*

In this section, we describe the transference effect in a specific mechanical system of a pipe conveying fluid, see Fig. 8.7. The dynamic behavior of this system has been studied in a number of papers, of which our main references are [Benjamin (1961); Bishop and Fawzy (1976)]. Further references on this and other models can be found in [Paidoussis and Li (1993); Paidoussis (1998)].

The system has two degrees of freedom described by the angles φ_1 and

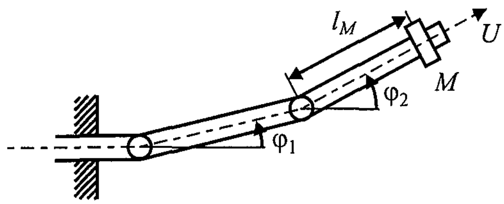


Fig. 8.7 Two degrees of freedom model of a pipe conveying fluid.

φ_2, and contains four non-dimensional parameters: a parameter describing relative fluid mass

$$\eta = \sqrt{\frac{m_f}{m + m_f}}, \quad 0 \leq \eta < 1, \tag{8.20}$$

a relative concentrated mass

$$\mu = \frac{M}{(m + m_f)l}, \quad \mu \geq 0, \tag{8.21}$$

a position of the concentrated mass

$$\xi = \frac{l_M}{l}, \quad 0 \leq \xi \leq 1, \tag{8.22}$$

and a damping coefficient

$$\gamma = \frac{c}{\sqrt{k(m + m_f)l^3}}, \quad \gamma \geq 0. \tag{8.23}$$

The quantities involved are: the mass per unit length m for the pipe and m_f for the fluid, the concentrated mass M, the length of half the pipe l, the distance to the concentrated mass l_M, the stiffness coefficient in hinges k, and the viscous damping coefficient c. The fluid speed U enters a non-dimensional speed parameter u as

$$u = Ul\sqrt{\frac{m_f}{kl}}, \quad u \geq 0, \tag{8.24}$$

and non-dimensional time τ is related to the absolute time t as

$$\tau = t\sqrt{\frac{k}{(m + m_f)l^3}}. \tag{8.25}$$

The equations of small vibrations as stated in [Sugiyama and Noda (1981)] are

$$\frac{1}{6}\begin{pmatrix} 8+6\mu & 3+6\mu\xi \\ 3+6\mu\xi & 2+6\mu\xi^2 \end{pmatrix}\begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} + \begin{pmatrix} \eta u+2\gamma & 2\eta u-\gamma \\ -\gamma & \eta u+\gamma \end{pmatrix}\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}$$

$$+\begin{pmatrix} 2-u^2 & u^2-1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8.26}$$

The three matrices are: the symmetric positive definite mass matrix, the matrix with gyroscopic and dissipative forces, and the matrix with potential and circulatory forces. The characteristic equation for system (8.26) takes the form

$$a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{8.27}$$

where the coefficients $a_0, \ldots, a_4$ are

$$\begin{aligned} a_0 &= 1, \quad a_1 = -u^3\eta + 5\eta u + 2\gamma, \\ a_2 &= -\mu\xi(u^2-2)(\xi+1) + (\eta u+\gamma)^2 + 3\eta u\gamma + \mu + 3 - 5u^2/6, \\ a_3 &= \mu\eta u(\xi-1)^2 + \gamma(2\mu\xi^2 + 2\mu\xi + \mu + 3) + 2\eta u/3, \\ a_4 &= -\mu\xi + (48\mu\xi^2 + 12\mu + 7)/36. \end{aligned} \tag{8.28}$$

8.2.2 *Flutter instability*

Let us find the critical fluid speed for the onset of flutter, represented by a pair of complex conjugate eigenvalues crossing the imaginary axis. Substituting $\lambda = i\omega$ into equation (8.27), we obtain two conditions for real and imaginary parts:

$$\begin{aligned} a_4\omega^4 - a_2\omega^2 + a_0 &= 0, \\ -a_3\omega^2 + a_1 &= 0. \end{aligned} \tag{8.29}$$

Expressing ω^2 from the second equation and substituting it into the first equation yields

$$H = a_1a_2a_3 - a_4a_1^2 - a_0a_3^2 = 0. \tag{8.30}$$

As noted in [Benjamin (1961)], the left-hand side of (8.30) constitutes the Hurwitz determinant, see Section 4.1. The function $H = H(\eta, \mu, \xi, \gamma, u)$ in condition (8.30) depends smoothly on five problem parameters.

In specific case of no concentrated mass $\mu = 0$ and no damping $\gamma = 0$ we have

$$H = \frac{\eta^2 u^2}{36}\left((13 - 24\eta^2)u^4 + (120\eta^2 - 102)u^2 + 169\right) = 0. \tag{8.31}$$

Solving (8.31) for u, we find the critical speed of flutter as

$$u_{cr} = \sqrt{\frac{51 - 60\eta^2 - 2\sqrt{900\eta^4 - 516\eta^2 + 101}}{13 - 24\eta^2}}. \tag{8.32}$$

The second equation in (8.29) gives the flutter frequency

$$\omega_{cr} = \sqrt{\frac{15 - 3u_{cr}^2}{2}}. \tag{8.33}$$

Fig. 8.8 shows the results as functions of the non-dimensional fluid mass η^2.

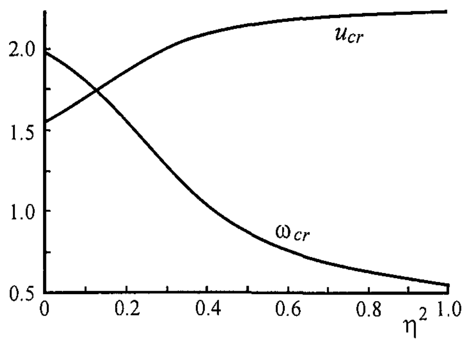


Fig. 8.8 Critical non-dimensional fluid speed u_{cr} and corresponding flutter frequency ω_{cr} as functions of non-dimensional fluid mass η^2.

8.2.3 *Strong interaction of complex eigenvalues*

Now, let us find a double nonderogatory eigenvalue $\lambda_0 = \alpha_0 + i\omega_0$. Since $\overline{\lambda}_0 = \alpha_0 - i\omega_0$ is also a double eigenvalue, we know all four roots of the characteristic equation

$$\begin{aligned}&(\lambda - \alpha_0 - i\omega_0)^2(\lambda - \alpha_0 + i\omega_0)^2\\ &= \lambda^4 - 4\alpha_0\lambda^3 + 2(3\alpha_0^2 + \omega_0^2)\lambda^2 - 4\alpha_0(\alpha_0^2 + \omega_0^2)\lambda + (\alpha_0^2 + \omega_0^2)^2 = 0.\end{aligned} \tag{8.34}$$

Considering the case $\mu = \gamma = 0$, we multiply characteristic polynomial (8.34) by $a_4 = 7/36$ and compare its coefficients with (8.27), (8.28). As a result, we find the single solution with positive u and η as

$$u_0 = \sqrt{\frac{5\sqrt{7}-4}{\sqrt{7}}} \approx 1.868, \quad \eta_0 = \sqrt{\frac{7\left(7\sqrt{7}-6\right)}{18\left(5\sqrt{7}-4\right)}} \approx 0.726, \tag{8.35}$$

which gives the double eigenvalue $\lambda_0 = \alpha_0 + i\omega$ with

$$\alpha_0 = -\sqrt{\frac{2(7\sqrt{7}-6)}{7\sqrt{7}}} \approx -1.163, \quad \omega_0 = \sqrt{\frac{2(27-7\sqrt{7})}{7\sqrt{7}}} \approx 0.957. \tag{8.36}$$

Notice that $\alpha_0 < 0$ and, thus, the system with parameters (8.35) is stable.

Using Theorem 2.13 (page 80), we describe behavior of eigenvalues near the point λ_0 by the approximate formula

$$\lambda \approx \lambda_0 + X + iY, \tag{8.37}$$

where

$$X + iY = \pm\sqrt{(a_1 + ib_1)\Delta u + (a_2 + ib_2)\Delta\eta} \tag{8.38}$$

with the coefficients

$$a_1 = 6.88318, \quad b_1 = 0.278831, \quad a_2 = 1.33044, \quad b_2 = -6.78555. \tag{8.39}$$

In formulae (8.37)–(8.39) the increments $\Delta u = u - u_0$ and $\Delta\eta = \eta - \eta_0$ are assumed to be small. Expression (8.38) for changing Δu and fixed $\Delta\eta$ yields two hyperbolae given by the equation

$$(X - 49.3919Y)(X + 0.02024Y) = 168.837\Delta\eta. \tag{8.40}$$

These hyperbolae describe local position of the eigenvalue branches near λ_0.

Behavior of eigenvalues with a change of u for $\Delta\eta = \pm 0.01$ is shown in Fig. 8.9, where the bold dashed lines represent local approximations given by hyperbolae (8.40). We see how the critical eigenvalue branch is changing. For $\eta < \eta_0$ the flutter instability corresponds to the second branch (mode). For $\eta > \eta_0$ the second branch gets stable, and the loss of stability corresponds to the first branch. This transference of instability between eigenvalue branches is the result of *strong interaction* of complex eigenvalues at λ_0.

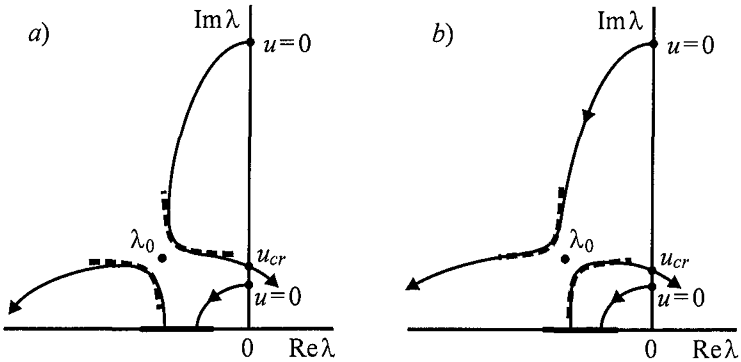


Fig. 8.9 Behavior of eigenvalue branches for pipe conveying fluid: a) $\Delta\eta = -0.01$, b) $\Delta\eta = 0.01$.

8.3 Destabilization of non-conservative system by small damping

Let us consider a non-conservative vibrational system with small dissipative forces

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma\mathbf{D}\dot{\mathbf{q}} + \mathbf{C}(p)\mathbf{q} = 0. \tag{8.41}$$

Here, $\mathbf{M}$ and $\mathbf{D}$ are symmetric positive definite matrices representing masses and dissipative forces, respectively; the non-symmetric matrix $\mathbf{C}(p)$ depends smoothly on a real load parameter p and describes non-conservative positional forces; $\mathbf{q}$ is a vector of generalized coordinates; and γ is a small positive damping parameter. Finding solution of (8.41) in the form $\mathbf{q} = \mathbf{u}\exp\lambda t$, we obtain the eigenvalue problem

$$\left(\lambda^2\mathbf{M} + \lambda\gamma\mathbf{D} + \mathbf{C}(p)\right)\mathbf{u} = 0, \tag{8.42}$$

where λ is an eigenvalue, and $\mathbf{u}$ is an eigenvector. The system depends on two parameters p and γ. The stability problem for system (8.41) was formulated and solved numerically for specific examples in [Bolotin and Zhinzher (1969)].

The undamped system ($\gamma = 0$) is circulatory. As we have shown in Section 8.1, this system loses stability through strong interaction of eigenvalues on the imaginary axis. Let us consider the case of flutter, when with an increase of p from zero two pairs of purely imaginary eigenvalues approach and merge at $p = p_0$. At this point, double eigenvalues $\lambda = \pm i\omega_0$ with single eigenvectors appear. We are interested in the effect of dissipative forces on behavior of eigenvalues and stability near this critical point.

Let $\mathbf{u}_0$ and $\mathbf{u}_1$ be the Jordan chain corresponding to the double $\lambda_0 = i\omega_0$:

$$\mathbf{L}_0\mathbf{u}_0 = 0, \quad \mathbf{L}_0\mathbf{u}_1 = -\mathbf{L}_1\mathbf{u}_0, \tag{8.43}$$

where the real matrix $\mathbf{L}_0$ and purely imaginary matrix $\mathbf{L}_1$ are

$$\mathbf{L}_0 = -\omega_0^2\mathbf{M} + \mathbf{C}(p_0), \quad \mathbf{L}_1 = 2i\omega_0\mathbf{M}, \quad \mathbf{L}_2 = \mathbf{M}. \tag{8.44}$$

The left eigenvector $\mathbf{v}_0$ is defined by the equation and normalization condition

$$\mathbf{v}_0^T\mathbf{L}_0 = 0, \quad \mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}_2\mathbf{u}_0 = 1. \tag{8.45}$$

It is easy to see that the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ can be chosen real and purely imaginary, respectively. Then the vector $\mathbf{v}_0$ is real.

Using results of Section 2.13, we find the following asymptotic expression for bifurcation of the double λ_0 as

$$\lambda \approx i\omega_0 \pm \sqrt{a\Delta p + ib\gamma}, \quad \Delta p = p - p_0, \tag{8.46}$$

where a and b are real numbers

$$a = -\mathbf{v}_0^T\frac{\partial \mathbf{C}}{\partial p}\mathbf{u}_0, \quad b = -\omega_0\mathbf{v}_0^T\mathbf{D}\mathbf{u}_0. \tag{8.47}$$

First, consider the case in which $\gamma = 0$ (no damping) and assume that $a > 0$. Then, with an increase of p two eigenvalues λ come together along the imaginary axis, merge at $p = p_0$ to the double $\lambda_0 = i\omega_0$, and then split along the line perpendicular to the imaginary axis as shown in Fig. 8.10. This means the onset of flutter with p_0 being the critical value.

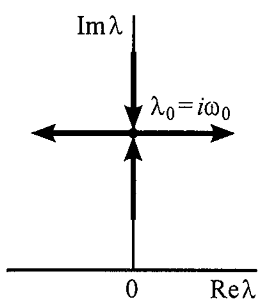


Fig. 8.10 Behavior of eigenvalues in case of no damping.

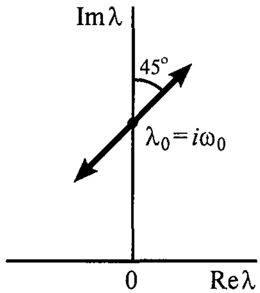


Fig. 8.11 Splitting of double eigenvalue with introduction of damping.

If $p = p_0$ and γ increases from zero, then the double eigenvalue λ_0 splits along the 45 degrees line passing through the point λ_0. The case in which b is a positive number is shown in Fig. 8.11. In case of negative b, the double root λ_0 splits along the line perpendicular to that of shown in Fig. 8.11.

Approximate expression (8.46) can be written in the form

$$\lambda \approx i\omega_0 + X + iY, \tag{8.48}$$

where the real quantities X and Y satisfy the equations

$$X^2 - Y^2 = a\Delta p, \quad 2XY = b\gamma. \tag{8.49}$$

If one of the parameters p or γ is fixed and the other one is changing, then equations (8.49) define hyperbolae in the plane (X, Y).

For small fixed damping parameter $\gamma > 0$ and increasing p we have the picture of interaction shown in Fig. 8.12. The picture of interaction when p is fixed and γ is increasing from zero is shown in Fig. 8.13, where the cases $b < 0$ and $b > 0$ are shown with solid and dashed lines, respectively.

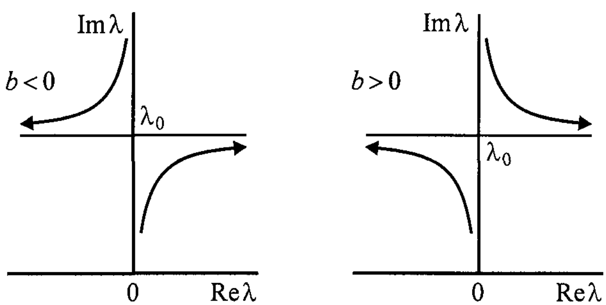


Fig. 8.12 Behavior of eigenvalues with small fixed damping.

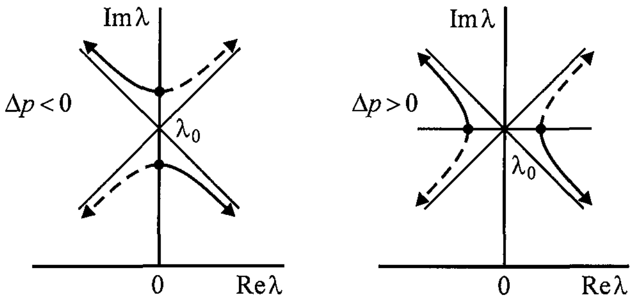


Fig. 8.13 Behavior of eigenvalues with small fixed load change.

The described behavior of eigenvalues shows that the system is destabilized if dissipative forces are introduced. As we see from Fig. 8.12, destabilization has the catastrophic character: the critical force of the system without damping p_0 decreases abruptly when arbitrarily small damping is introduced. This effect known as the destabilization paradox was intensively discussed in the literature, see [Ziegler (1952); Bolotin (1963); Herrmann and Jong (1965); Herrmann (1967); Bolotin and Zhinzher (1969); Huseyin (1978); Seyranian (1990b); Seyranian and Pedersen (1995); Seyranian (1996)].

8.3.1 *Double pendulum with follower force*

Let us illustrate the obtained theoretical results on a two degrees of freedom pendulum loaded by a follower force, see Fig. 8.14. Recall that this system was studied in Example 3.3 (page 112) from the point of view of singularities of the stability boundary.

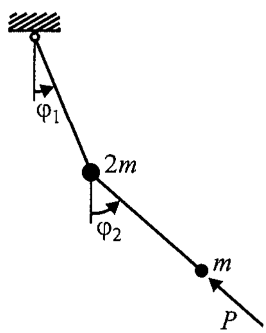


Fig. 8.14 Double pendulum with a follower force.

The equations of motion for the system are given by [Herrmann and Jong (1965)]

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} + \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_2 \\ -\gamma_2 & \gamma_2 \end{pmatrix}\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} + \begin{pmatrix} 2-p & p-1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8.50}$$

with the non-dimensional time $\tau = t\sqrt{k/(ml^2)}$ and non-dimensional parameters: the damping coefficients at the support and midpoint

$$\gamma_1 = \frac{c_1}{\sqrt{kml^2}} \geq 0, \quad \gamma_2 = \frac{c_2}{\sqrt{kml^2}} \geq 0, \tag{8.51}$$

and the follower force

$$p = \frac{Pl}{k}. \tag{8.52}$$

The dimensional quantities are: the concentrated masses $2m$ and m, the length of half the pendulum l, the stiffness coefficient at the hinges k, the viscous damping coefficients at the hinges c_1 and c_2, and the magnitude of the follower force P.

The characteristic equation for system (8.50) is

$$2\lambda^4 + (\gamma_1 + 6\gamma_2)\lambda^3 + (7 - 2p + \gamma_1\gamma_2)\lambda^2 + (\gamma_1 + \gamma_2)\lambda + 1 = 0. \tag{8.53}$$

If $\gamma_1 = \gamma_2 = 0$ (no damping), the critical load of the system is equal to

$$p_0 = \frac{7}{2} - \sqrt{2}, \tag{8.54}$$

which is found by equating the discriminant of biquadratic polynomial (8.53) with zero. If damping $\gamma_1 > 0$ and $\gamma_2 > 0$ is introduced, we obtain the stability condition using the Routh-Hurwitz criterion (see Section 4.1) as

$$(\gamma_1 + 6\gamma_2)(7 - 2p + \gamma_1\gamma_2)(\gamma_1 + \gamma_2) - 2(\gamma_1 + \gamma_2)^2 - (\gamma_1 + 6\gamma_2)^2 > 0. \tag{8.55}$$

Inequality (8.55) yields the critical value of the follower force p_{cr} for the damped system [Herrmann and Jong (1965)]

$$p_{cr} = \frac{4\gamma_1^2 + 33\gamma_1\gamma_2 + 4\gamma_2^2}{2(\gamma_1^2 + 7\gamma_1\gamma_2 + 6\gamma_2^2)} + \frac{1}{2}\gamma_1\gamma_2. \tag{8.56}$$

Expression (8.56) determines the surface (stability boundary) in the parameter space, see Fig. 8.15. It should be noted that p_{cr} as a function of two parameters is discontinuous at the point $\gamma_1 = \gamma_2 = 0$, since there is no limit of p_{cr} as γ_1, γ_2 tend to zero.

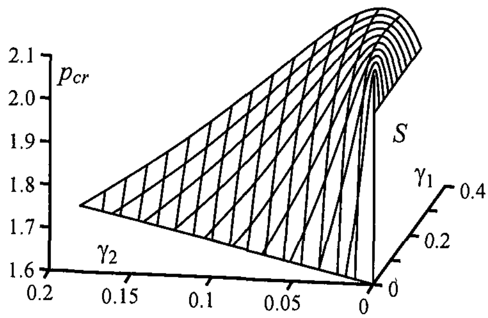


Fig. 8.15 Stability boundary in the parameter space.

Let us fix the ratio of the damping coefficients $d = \gamma_1/\gamma_2$, $d \geq 0$. In this case there exists a limit of the critical force as damping tends to zero:

$$p_d = \lim_{\gamma \to 0} p_{cr}, \quad \gamma_1 = d\gamma, \quad \gamma_2 = \gamma. \tag{8.57}$$

This limit satisfies the inequality

$$p_d \leq p_0, \tag{8.58}$$

where p_0 is the critical force of the system with no damping (8.54). For example, considering the specific case of [Ziegler (1952)] $\gamma_1 = \gamma_2 = \gamma$, we find from (8.56)

$$p_{cr} = \frac{41}{28} + \frac{\gamma^2}{2}. \tag{8.59}$$

So, we have

$$p_d = \frac{41}{28} \approx 1.46 < p_0 \approx 2.08. \tag{8.60}$$

This constitutes what is termed as destabilization due to infinitely small damping, see [Herrmann (1967)].

8.3.2 *Stabilization effect in case of two damping parameters*

Let us consider the damping parameters

$$\gamma_1 = d\gamma, \quad \gamma_2 = \gamma, \tag{8.61}$$

where $d \geq 0$ is a fixed ratio, and $\gamma \geq 0$ is a single damping parameter. At $p = p_0$ and $\gamma = 0$ system (8.50) possesses a double nonderogatory eigenvalue $\lambda_0 = i2^{-1/4}$. Using formulae (8.46) and (8.47), we describe bifurcation of the double eigenvalue as

$$\lambda \approx i2^{-1/4} \pm \sqrt{a\Delta p + ib\gamma}, \tag{8.62}$$

where

$$a = \frac{1}{4}, \quad b = \frac{\sqrt{2}-1}{8 \cdot 2^{1/4}}(d - 5\sqrt{2} - 4). \tag{8.63}$$

The system is destabilized by small damping if $b \neq 0$, i.e.,

$$d \neq d_0 = 5\sqrt{2} + 4. \tag{8.64}$$

Now, let us consider the case $d = d_0$. In this case the limit of the critical force with infinitely small damping p_d equals the critical force of the system with no damping p_0. Introduction of small damping with the ratio d_0 leads to stabilization, and the critical load of damped system p_{cr} increases as

$$p_{cr} = \frac{7}{2} - \sqrt{2} + \frac{5\sqrt{2}+4}{2}\gamma^2. \tag{8.65}$$

Solving the inequality $p_{cr} > p_0$ for expression (8.56), we find the region of stabilization due to damping in the plane (γ_1, γ_2). This region is shown in Fig. 8.16 (hatched). Boundaries of the stabilization region are described by the asymptotic curves

$$\gamma_1 = d_0\gamma_2 \pm \gamma_2^2\sqrt{50(133 + 94\sqrt{2})} + o(\gamma_2^2). \tag{8.66}$$

Comparing characteristic polynomial (8.53), divided by two, with polynomial (8.34), we find

$$\gamma_1 = d_0\gamma, \quad \gamma_2 = \gamma, \quad p = p_* = \frac{7}{2} - \sqrt{2} - \frac{59 + 30\sqrt{2}}{8}\gamma^2. \tag{8.67}$$

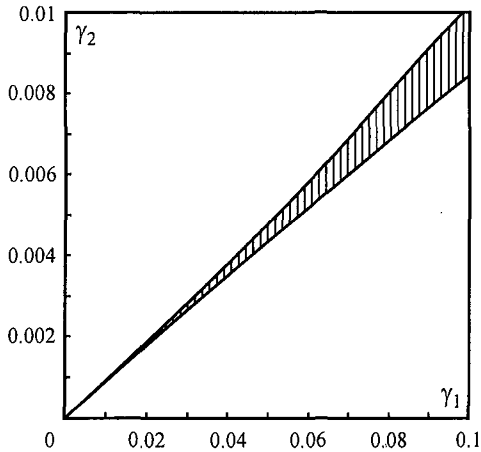


Fig. 8.16 Region of stabilization.

Parameters (8.67) determine the system possessing a double eigenvalue $\lambda_* = \alpha_* + i\omega_*$, where

$$\alpha_* = -\frac{5\sqrt{2}+10}{8}\gamma < 0, \quad \omega_* = \sqrt{\frac{\sqrt{2}}{2} - \frac{25}{32}(3+2\sqrt{2})\gamma^2}\,. \tag{8.68}$$

This means that the double eigenvalue $\lambda_0 = i2^{-1/4}$ is shifted to the left half-plane when small damping with the specific ratio $d_0 = 5\sqrt{2}+4$ is added.

At $p = p_*$ the strong interaction of eigenvalues takes place: with an increase of p the eigenvalues approach in the stable half-plane $\operatorname{Re}\lambda < 0$, merge to λ_* at $p = p_*$, and then split perpendicular to the line of approach, see Fig. 8.17. According to (8.54), (8.56), and (8.67) we have the relation

$$p_* < p_0 < p_{cr} \tag{8.69}$$

for $\gamma > 0$.

One of important consequences of the present example is that destabilization or stabilization of a non-conservative system due to small damping depends on the way how the damping is introduced. In multi-parameter case it may be possible to choose the damping parameters so that the system is stabilized. The other interesting feature when two independent damping parameters are considered is that the limit of p_{cr} as $\gamma_1 \to 0$, $\gamma_2 \to 0$ does not exist, while it exists for any fixed ratio between the damping coefficients.

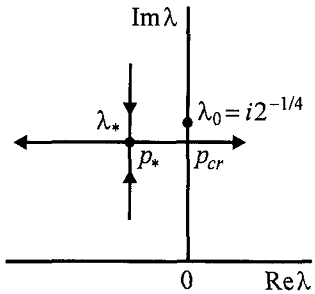


Fig. 8.17 Illustration of stabilization due to damping for the specific damping ratio.

8.4 Disappearance of flutter instability in the Keldysh problem

Keldysh was the first who considered the problem of aeroelastic stability of an unswept high-aspect-ratio wing braced by the struts, see [Keldysh (1938)]. The strut is supposed to be a rigid rod connecting the point P of the wing with the fuselage, see Fig. 8.18, where l is the halfspan of the wing, and h is the coordinate of the point P. The presence of the strut implies that P is a fixed point, which imposes extra boundary conditions on the functions describing vibration modes.

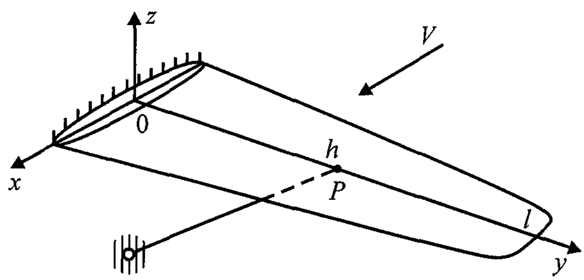


Fig. 8.18 Wing braced by a strut.

For particular case of a rectangular wing with a single strut attached to the stiffness axis of the wing (the A-type strut), Keldysh made calculations and came to the conclusion that "at about $h = 0.47l$, the critical speed becomes imaginary; consequently, vibrations of the wing with the strut turn out to be impossible at $h > 0.47l$" [Keldysh (1938)]. Evidently, this means that the wing becomes stable. Keldysh made a similar conclusion for the B-type strut (two struts bracing the cross-section h of the wing and holding it fixed during vibrations): "at $h/l > 0.8$, the critical speed for the wing with the B-type strut does not exist" [Keldysh (1938)].

These conclusions, which Keldysh made on the basis of the Bubnov-Galerkin method with one-term approximation for bending and torsional modes, appear to be incorrect. In this section, following the papers by [Mailybaev and Seyranian (1996); Mailybaev and Seyranian (1998a)], the problem of aeroelastic stability of the wing is reduced to the study of behavior of eigenvalues λ for linearized equations of motion of the wing on the complex plane as functions of the flow speed V and the coordinate h of the point, the strut is attached to. In this way, the critical speeds for the vibrational (flutter) and static (divergence) types of loss of stability are determined, and the domains of stability, flutter, and divergence are plotted on the plane (h, V). It is shown that in case of the A-type strut, the flutter instability is replaced by the divergence, and at $h = 0.47l$ the critical speed has a discontinuity: its value jumps from $V_f = 55.7$m/s to $V_d = 61.3$m/s. In case of the B-type strut, the critical speed turns out to be finite and continuous and, at $h = 0.76l$, reaches the maximum $V_{cr} = 119$m/s that is almost four times greater than the critical speed of the unbraced wing $V_{cr} = 30.3$m/s.

Extension of this problem to the case of the strut attached to an arbitrary point of the wing and the optimization problem for the strut position was investigated in [Mailybaev and Seyranian (1997)]. The performed calculations show that the use of struts can effectively improve the aeroelastic stability characteristics.

8.4.1 *Aeroelastic stability problem*

Let us consider vibrations of a thin high-aspect-ratio wing braced by an A-type strut in airflow, see Fig. 8.18. The wing is modeled by an elastic beam subjected to torsion and bending with a straight elastic axis Oy (stiffness axis) perpendicular to the fuselage. Deformation of the wing is described by the deflection function $w(y, t)$ and the angle of rotation $\theta(y, t)$ about the elastic axis, where t is the time. The linearized equations of motion of the wing in the flow have the form, see [Grossman (1937); Fung (1955)]:

$$\begin{aligned} \frac{\partial^2}{\partial y^2}\left(EI\frac{\partial^2 w}{\partial y^2}\right) + m\frac{\partial^2 w}{\partial t^2} - m\sigma\frac{\partial^2 \theta}{\partial t^2} &= L_a, \\ -\frac{\partial}{\partial y}\left(GJ\frac{\partial \theta}{\partial y}\right) - m\sigma\frac{\partial^2 w}{\partial t^2} + I_m\frac{\partial^2 \theta}{\partial t^2} &= M_a. \end{aligned} \tag{8.70}$$

In these equations EI and GJ are the bending and torsional stiffnesses of the wing, m and I_m are the specific (per unit length of the span) mass and moment of inertia with respect to the elastic axis, and σ is the distance between the center of stiffness and the center of gravity of the cross-section of the wing. The aerodynamic force L_a and moment M_a per unit span are determined on the basis of the quasisteady hypothesis, see [Grossman (1937); Fung (1955)]. The expressions for L_a and M_a have the form

$$\begin{aligned} L_a &= C_y^\alpha \rho V^2 b \left(\theta + \frac{b}{V} \left(\frac{3}{4} - \frac{x_0}{b} \right) \frac{\partial \theta}{\partial t} - \frac{1}{V} \frac{\partial w}{\partial t} \right), \\ M_a &= C_m^\alpha \rho V^2 b^2 \left(\theta + \frac{b}{V} \left(\frac{3}{4} - \frac{x_0}{b} - \frac{\pi}{16 C_m^\alpha} \right) \frac{\partial \theta}{\partial t} - \frac{1}{V} \frac{\partial w}{\partial t} \right), \end{aligned} \tag{8.71}$$

where b is the chord of the wing, x_0 is the distance between the front edge and elastic axis, V is the flow speed, ρ is the air density, and C_y^α and C_m^α are the aerodynamic coefficients.

Assuming that the wing is a cantilever clamped at the fuselage, we write the boundary conditions for the functions w and θ at the clamped ($y = 0$) and free ($y = l$) ends of the wing:

$$\begin{aligned} y = 0 : &\quad w = \frac{\partial w}{\partial y} = \theta = 0, \\ y = l : &\quad EI \frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y} \left(EI \frac{\partial^2 w}{\partial y^2} \right) = GJ \frac{\partial \theta}{\partial y} = 0. \end{aligned} \tag{8.72}$$

We assume that the wing is braced by the strut at the point P lying on the elastic axis. The continuity conditions for the deflection function, angle of torsion, slope of the deflection function, and torsional and bending moments in the h-section yield [Keldysh (1938)]

$$\begin{aligned} y = h : \quad w_- = w_+ = 0, \quad \theta_- = \theta_+, \quad \left(\frac{\partial w}{\partial y} \right)_- = \left(\frac{\partial w}{\partial y} \right)_+, \\ \left(GJ \frac{\partial \theta}{\partial y} \right)_- = \left(GJ \frac{\partial \theta}{\partial y} \right)_+, \quad \left(EI \frac{\partial^2 w}{\partial y^2} \right)_- = \left(EI \frac{\partial^2 w}{\partial y^2} \right)_+. \end{aligned} \tag{8.73}$$

Here $+$ and $-$ denote the right and left limits as y tends to h, respectively.

The system of equations (8.70)–(8.73) represents a linear homogeneous boundary value problem. We seek a solution of this system in the form

$$w(y, t) = f(y) \exp \lambda t, \quad \theta(y, t) = \varphi(y) \exp \lambda t, \tag{8.74}$$

where λ is an eigenvalue, and $f(y)$ and $\varphi(y)$ are eigenfunctions. Substituting (8.74) into (8.70)–(8.73), we obtain a system of ordinary differential equations for the functions $f(y)$ and $\varphi(y)$ as

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} f \\ \varphi \end{pmatrix} = 0, \tag{8.75}$$

where L_{ij} are the linear differential operators dependent on λ given by

$$\begin{aligned} L_{11} &= \frac{d^2}{dy^2}\left(EI\frac{d^2}{dy^2}\right) + m\lambda^2 + \lambda V C_y^\alpha \rho b, \\ L_{12} &= -m\sigma\lambda^2 - C_y^\alpha \rho V^2 b - C_y^\alpha \rho\lambda V b^2 \left(\frac{3}{4} - \frac{x_0}{b}\right), \\ L_{21} &= -m\sigma\lambda^2 + C_m^\alpha \rho\lambda V b^2, \\ L_{22} &= -\frac{d}{dy}\left(GJ\frac{d}{dy}\right) + I_m\lambda^2 - C_m^\alpha \rho V^2 b^2 \\ &\quad - C_m^\alpha \rho\lambda V b^3 \left(\frac{3}{4} - \frac{x_0}{b} - \frac{\pi}{16 C_m^\alpha}\right). \end{aligned} \tag{8.76}$$

Boundary conditions for the functions f and φ are the same as (8.72) and (8.73) for the functions w and θ, respectively.

Due to the fact that the problem is non-conservative (non-selfadjoint) the eigenvalues are, in general, complex quantities $\lambda = \alpha + i\omega$. Depending on the flow speed V, the amplitudes of solutions (8.74), as functions of time, can decrease ($\operatorname{Re}\lambda < 0$, stability), be constant ($\operatorname{Re}\lambda = 0$, stability boundary), or increase ($\operatorname{Re}\lambda > 0$, instability). The flutter critical speed V_f is defined by the relations $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda = \omega \neq 0$, where ω is the flutter frequency, and the divergence critical speed V_d is defined by the equality $\lambda = 0$. The critical speed of the loss of stability of the system V_{cr} is equal to the lowest of the speeds V_f and V_d.

The divergence critical speed can be found directly from equations (8.75), (8.76) with the boundary conditions (8.72) and (8.73) by setting $\lambda = 0$. As a result, we arrive at the following problem:

$$\frac{d}{dy}\left(GJ\frac{d\varphi}{dy}\right) + C_m^\alpha \rho V_d^2 b^2 \varphi = 0, \tag{8.77}$$

$$\begin{aligned} &\varphi(0) = 0, \quad \left(GJ\frac{d\varphi}{dy}\right)_{y=l} = 0, \\ y = h: \quad &\varphi_- = \varphi_+, \quad \left(GJ\frac{d\varphi}{dy}\right)_- = \left(GJ\frac{d\varphi}{dy}\right)_+ . \end{aligned} \tag{8.78}$$

This is a self-adjoint eigenvalue problem with V_d^2 as an eigenvalue. Hence, the squared critical divergence speed is the minimum eigenvalue of problem (8.77), (8.78). Using variational formulation, we find

$$V_d^2 = \min_{\varphi} \frac{\int_0^l GJ\,(d\varphi/dy)^2\,dy}{\int_0^l C_m^\alpha \rho b^2 \varphi^2 dy}, \tag{8.79}$$

where the trial function φ is continuously differentiable and satisfies the first boundary condition in (8.78). The remaining boundary conditions are natural for functional (8.79). From (8.79) it follows that V_d^2 is independent on the location of the strut h. This is a natural consequence since the bracing point on the wing lies on the elastic axis and, hence, does not affect the wing torsion.

To solve eigenvalue problem (8.75), (8.76) we use the Bubnov-Galerkin method. With this aim, we choose two systems of linearly independent coordinate functions $f_1, \dots, f_m$ and $\varphi_1, \dots, \varphi_m$ that are smooth on the intervals $(0, h)$ and (h, l), and satisfy boundary conditions (8.72) and (8.73) of the problem. According to the Bubnov-Galerkin method, the eigenfunctions f and φ of system (8.75) may be represented in the form of linear combinations of the coordinate functions with unknown coefficients a_j and b_j:

$$f(y) = \sum_{j=1}^m a_j f_j(y), \quad \varphi(y) = \sum_{j=1}^m b_j \varphi_j(y). \tag{8.80}$$

Substituting these expansions into equations (8.75), multiplying the left-hand sides of these equations by f_k and φ_k, $k = 1, \dots, m$, respectively, and integrating with respect to y from 0 to l, we obtain $2m$ linear homogeneous equations for the coefficients a_j and b_j, which constitute an algebraic eigenvalue problem of the form

$$(\lambda^2 \mathbf{M} + \lambda V \mathbf{B} + \mathbf{C}_1 + V^2 \mathbf{C}_2)\mathbf{u} = 0, \tag{8.81}$$

where $\mathbf{u} = (a_1, \dots, a_m, b_1, \dots, b_m)^T$ is a vector consisting of the unknown coefficients, and $\mathbf{M}$, $\mathbf{B}$, $\mathbf{C}_1$, $\mathbf{C}_2$ are square matrices of size $2m \times 2m$. A nontrivial solution to problem (8.81) exists only if

$$\det(\lambda^2\mathbf{M} + \lambda V\mathbf{B} + \mathbf{C}_1 + V^2\mathbf{C}_2) = 0. \tag{8.82}$$

This is the characteristic equation determining the eigenvalues λ as functions of the flow speed V.

8.4.2 *Behavior of eigenvalue branches on the complex plane*

Let us study stability of a rectangular wing braced with the A-type strut depending on the parameters V and h. For this purpose, we use the method described above with the input data given in [Keldysh (1938)]. From five to seven functions were used in expansions (8.80) that were chosen among the eigenfunctions of pure bending and torsional vibrations of the wing with a bracing strut in the vacuum (for $\sigma = 0$ and $V = 0$).

Fig. 8.19 shows the eigenvalue branches on the complex plane for different fixed values of h, when the speed V varies in the range from 0 to 155m/s. The arrows in Fig. 8.19 indicate the direction in which the speed increases. As the matrices in equation (8.81) are real, the eigenvalues lie symmetrically relative to the real axis on the complex plane. The modes are numbered according to the eigenfrequencies of the conservative system at $V = 0$. The branches corresponding to higher modes are located on the left half of the complex plane for all h, i.e., they are stable. The numbers to the right of the imaginary axis are the values of the critical speed for the corresponding mode.

For small h, the loss of stability occurs for the first, second, and fourth modes, Fig. 8.19a. The first mode is divergent: two complex conjugate eigenvalues approach each other, collide, and then diverge in opposite directions along the real axis (strong interaction). At the speed $V_d = 61.3$m/s, one of them crosses the imaginary axis through $\lambda = 0$. The second and fourth modes correspond to flutter: the appropriate eigenvalues cross the imaginary axis at the points with $\operatorname{Im}\lambda \neq 0$ at V_f equal to 28 and 128m/s, respectively. Thus, the second mode is a critical one. Fig. 8.19a shows that the second mode branch crosses the imaginary axis twice. This means that, starting with a certain value of the speed, this mode becomes stable again.

With an increase of h, the following changes are observed, see Fig. 8.19b: the fourth mode becomes stable and the third unstable ($V_f = 147$m/s). The second mode branch shifts to the left. The critical speed is still the flutter speed of the second mode $V_f = 25$m/s. With further increase of h, the second mode branch continues moving into the left half-plane, so

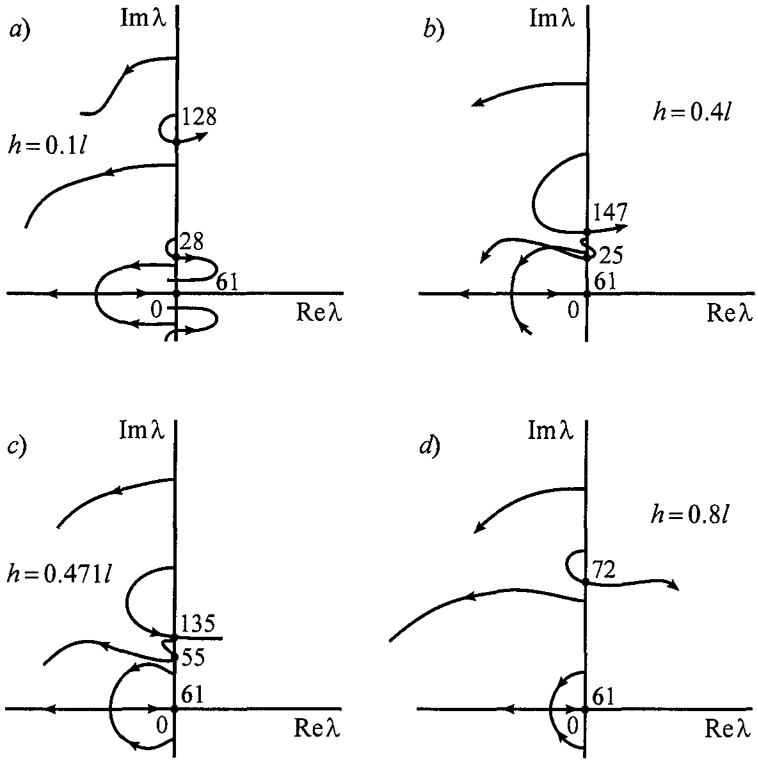


Fig. 8.19 Behavior of eigenvalue branches $\lambda(V)$ for different values of h.

that at $h = 0.471l$ (Fig. 8.19c) it only touches (but does not intersect) the imaginary axis, and then it becomes straighter (Fig. 8.19d). On the contrary, the branch corresponding to the third mode shifts to the right with the value of V_f decreasing from 135m/s at $h = 0.471l$ to 72m/s at $h = 0.8l$. Starting with the value $h = 0.471l$, that corresponds to the transition of the second mode branch into the region $\mathrm{Re}\,\lambda < 0$ (stability), the divergence speed of the first mode $V_d = 61.3$m/s becomes the critical one.

The results of calculations for the stability, flutter, and divergence domains on the plane of the parameters V and h/l are presented in Fig. 8.20. Here, the divergence instability corresponds to a system with only real eigenvalues in the right half-plane $\mathrm{Re}\lambda > 0$, while flutter means existence of at least one pair of complex conjugate eigenvalues $\lambda = \alpha \pm i\omega$, $\alpha > 0$. The flutter and divergence domains are hatched with horizontal and vertical

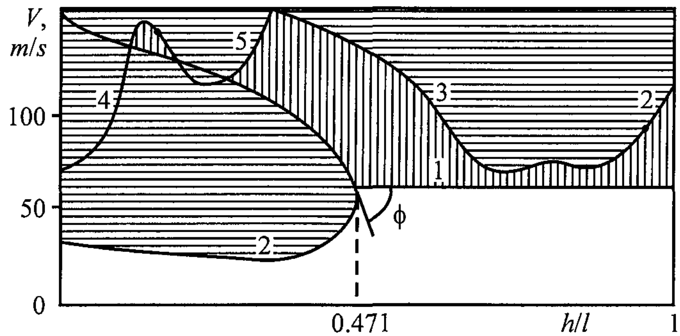


Fig. 8.20 Stability, flutter, and divergence domains for the bracing strut of type A.

lines, respectively. The numbers indicate the modes that become unstable on transition through the corresponding boundary.

The flutter domain is convex at $h \leq 0.471l$ and has a vertical tangent $h = 0.471l$. At the intersection, the flutter and divergence domains form a sharp angle ϕ in the stability domain. This implies discontinuity of the critical speed at $h = 0.471l$ and the existence of a small stability domain at super-critical flow speeds $V > V_{cr}$ within the small range $0.470l < h < 0.471l$, see Fig. 8.20. The discontinuity takes place at the transition of the second (critical) mode branch into the stable half-plane $\mathrm{Re}\,\lambda < 0$. The discontinuity point corresponds to the critical mode branch touching the imaginary axis at the critical flutter speed $V_f = 55.7$m/s for $h = 0.471l$, Fig. 8.19c.

Comparing our results with those of [Keldysh (1938), Fig. 4], we conclude that they are in a good agreement for $h < 0.471l$. Whereas at $h = 0.471l$ the critical speed has a discontinuity, and for $h > 0.471l$ becomes equal to the critical divergence speed rather than disappearing, as it was stated in [Keldysh (1938)].

Keldysh did not study the divergence instability. If the strut is attached to the elastic axis of the wing, the divergence critical speed does not depend on h and, in case of a rectangular wing, can be found analytically

$$V_d = \frac{\pi}{2bl}\sqrt{\frac{GJ}{C_m^\alpha \rho}} = 61.3m/s. \tag{8.83}$$

We note that the flutter domain also exists at $h > 0.471l$ for super-critical speeds, see Fig. 8.20.

8.4.3 *Wing with B-type strut*

Now, let us investigate stability of the wing braced with the B-type strut that fixes the cross-section of the wing at $y = h$. In this case, the boundary conditions take the form [Keldysh (1938)]

$$y = h: \quad w_- = w_+ = 0, \quad \theta_- = \theta_+ = 0,$$
$$\left(\frac{\partial w}{\partial y}\right)_- = \left(\frac{\partial w}{\partial y}\right)_+, \quad \left(EI\frac{\partial^2 w}{\partial y^2}\right)_- = \left(EI\frac{\partial^2 w}{\partial y^2}\right)_+. \tag{8.84}$$

The results of stability analysis for the specific rectangular wing are presented in Fig. 8.21. It turns out that for any position of the strut $0 \le h \le l$ the wing loses stability by flutter: for $0 \le h \le 0.65l$ and $0.74l \le h \le l$ the second mode is critical, and for $0.65l \le h \le 0.74l$ the stability is lost owing to the third mode. The transference of instability between branches, as described in Section 8.2, takes place at $h = 0.65l$ and $h = 0.74l$. With an increase of h from zero, the flutter critical speed increases reaching the maximum at $h = 0.76l$, and then monotonically decreases. The maximum of the critical speed is equal to $V_f = 119$m/s, which is about four times greater than the critical speed of the unbraced wing.

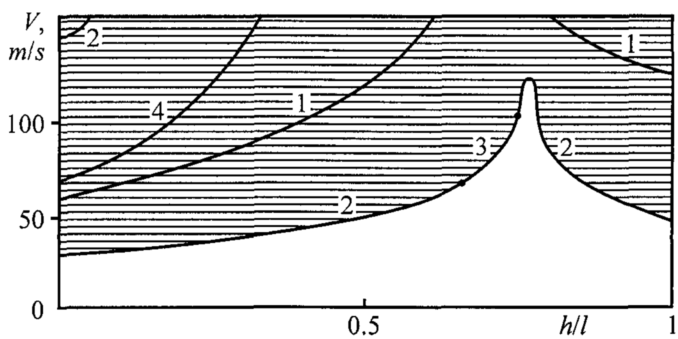


Fig. 8.21 Stability and flutter domains for the bracing strut of type B.

Comparison with [Keldysh (1938), Fig. 8] shows that the results are in satisfactory agreement for $0 \le h \le 0.7l$, but for $h > 0.7l$ the results disagree. So, the conclusion that "the wing becomes nonvibratory at $h > 0.8l$" [Keldysh (1938)] is not confirmed. The results obtained evidence that the use of the struts, especially of the type B, can effectively improve the aeroelastic stability characteristics of the wing.

8.4.4 *Generalization*

In this subsection, we give a general description and explanation of the effect of flutter disappearance with a jump of the critical speed presented above. For this purpose, we consider an eigenvalue problem for a real non-symmetric matrix $\mathbf{A}$ smoothly dependent on two real parameters h and V:

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}. \tag{8.85}$$

Let us consider behavior of an eigenvalue branch $\lambda(V)$ on the complex plane with a change of the parameter h, as shown in Fig. 8.22. Arrows indicate motion of the eigenvalue λ with an increase of V. We assume that at the point $h = h_0$, $V = V_0$ in the parameter space we have a simple purely imaginary eigenvalue $\lambda_0 = i\omega$, and the following conditions are satisfied

$$\operatorname{Re}\lambda_0 = 0, \quad \frac{\partial \operatorname{Re}\lambda}{\partial V} = 0, \quad \frac{\partial^2 \operatorname{Re}\lambda}{\partial V^2} < 0, \quad \frac{\partial \operatorname{Re}\lambda}{\partial h} < 0. \tag{8.86}$$

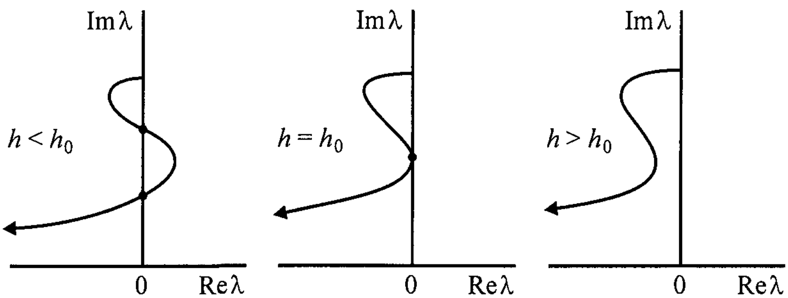


Fig. 8.22 Behavior of the eigenvalue branch $\lambda(V)$ near the critical point.

The first three conditions mean that at $h = h_0$ the eigenvalue branch touches the imaginary axis from the left side, and the last condition implies that the flutter instability disappears at $h > h_0$, see Fig. 8.22. Since λ_0 is a simple eigenvalue, it is a smooth function of h and V. Hence, we can write the Taylor expansion

$$\begin{aligned}\lambda = \lambda_0 &+ \frac{\partial \lambda}{\partial h}\Delta h + \frac{\partial \lambda}{\partial V}\Delta V \\ &+ \frac{1}{2}\frac{\partial^2 \lambda}{\partial h^2}(\Delta h)^2 + \frac{\partial^2 \lambda}{\partial h \partial V}\Delta h \Delta V + \frac{1}{2}\frac{\partial^2 \lambda}{\partial V^2}(\Delta V)^2 + \cdots,\end{aligned} \tag{8.87}$$

where $\Delta h = h - h_0$ and $\Delta V = V - V_0$. If we take the real part of both sides of (8.87), then at the stability boundary $\mathrm{Re}\,\lambda = 0$ we obtain

$$\mathrm{Re}\left(\frac{\partial\lambda}{\partial h}\Delta h + \frac{1}{2}\frac{\partial^2\lambda}{\partial h^2}(\Delta h)^2 + \frac{\partial^2\lambda}{\partial h\partial V}\Delta h\Delta V + \frac{1}{2}\frac{\partial^2\lambda}{\partial V^2}(\Delta V)^2 + \cdots\right) = 0, \tag{8.88}$$

where conditions (8.86) have been used.

Formula (8.88) is an approximate equation for the stability boundary in the vicinity of the point $h = h_0$, $V = V_0$. Assuming that $\Delta h = a(\Delta V)^k$ with a and k being unknown constants, we immediately find from (8.88) up to the terms of second order

$$\Delta h = a(\Delta V)^2, \tag{8.89}$$

$$a = -\frac{1}{2}\frac{\partial^2 \mathrm{Re}\,\lambda}{\partial V^2}\Big/\frac{\partial \mathrm{Re}\,\lambda}{\partial h}. \tag{8.90}$$

Relation (8.89) determines the parabola on the plane (h, V). Due to assumptions (8.86) the coefficient a is negative. It is easy to see from (8.87) and (8.88) that in the vicinity of the point (h_0, V_0) the corresponding flutter domain $\mathrm{Re}\,\lambda > 0$ is given approximately by the inequality $h < h_0 + a(V - V_0)^2$. This means convexity of the flutter domain with a vertical tangent at the boundary point (h_0, V_0), see Fig. 8.23a. This also shows that at this point a jump in the critical flutter speed can happen due to the presence of other eigenvalue branches.

Notice that if we assume that at the point (h_0, V_0) the derivative $\partial\mathrm{Re}\,\lambda/dh > 0$, then $a > 0$ and the flutter domain is given by the inequality $h > h_0 + a(V - V_0)^2$, Fig. 8.23b.

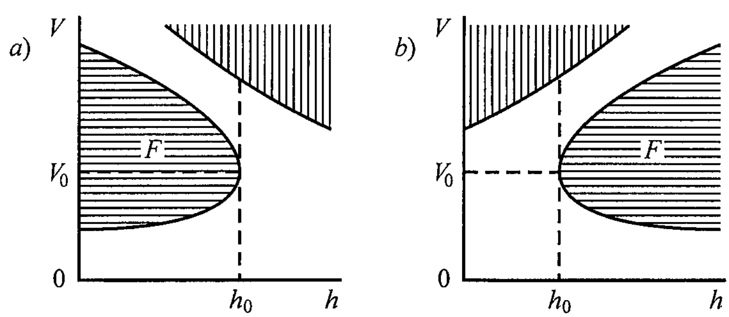


Fig. 8.23 Convexity of flutter domain with vertical tangent leading to discontinuity of the critical speed.

The only concern now is how to calculate the constant a. According to (8.90) we need the first and second derivatives of the eigenvalue λ with respect to the parameters taken at the point (h_0, V_0). These derivatives were derived in Section 2.4. To calculate them we have to find the left and right eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ corresponding to λ_0, as well as the first and second derivatives of the matrix $\mathbf{A}$ with respect to the parameters.

Note that the effect of discontinuity of the critical flutter load was revealed in different non-conservative stability problems, see for example [Claudon (1975); Hanaoka and Washizu (1980); Kounadis and Katsikadelis (1980); Kirillov and Seyranian (2002b)].

Chapter 9

Stability of Periodic Systems Dependent on Parameters

A large number of important stability problems are modeled by multi-parameter linear differential equations with periodic coefficients. As direct applications we may mention mechanical systems with periodically varying stiffness, mass, and load (parametric excitation). Other problems are from frequency modulation, warble tone room testing in acoustics, plasma physics *etc.* Finally, we mention the applications, which originated the study of periodic differential equations, including mean motion of the lunar perigee and wave propagation in stratified media.

The stability analysis for solutions to differential equations with periodic coefficients has been a challenge for more than hundred years. From a historical point of view, the important early studies are [Mathieu (1868); Floquet (1883); Hill (1886); Rayleigh (1887); Liapunov (1892); Poincaré (1899)]. For further development we refer to the books [Malkin (1966); Schmidt (1975); Yakubovich and Starzhinskii (1975); Yakubovich and Starzhinskii (1987); Nayfeh and Balachandran (1995)].

Different methods for analysis of stability are available: the classical Floquet method [Floquet (1883); Cesari (1971)], the method of infinite determinants [Bolotin (1964)], the perturbation method [Hsu (1963); Nayfeh and Mook (1979)], and the Galerkin method [Pedersen (1985)]. Few of these methods can from a practical point of view be extended to multiple degrees of freedom systems. For such extension we refer to [Lindh and Likins (1970); Fu and Nemat-Nasser (1972); Hansen (1985); Wu *et al.* (1995); Turhan (1998)]. It is concluded that the Floquet method is a general and practical method for systems with multiple degrees of freedom. Even with increasing computer power the large number of numerical integrations required in this method put limits to our possibilities. Research with the goal of carrying out these integrations in the most effective way has been performed in [Sinha and Wu (1991)].

The other challenge for stability problems of periodic systems is connected with multiple parameters. It should be noted that influence of parameters on the stability of periodic systems was studied from the very beginning. For example, the famous Mathieu–Hill equation contains two parameters: the frequency and amplitude of excitation. Probably, [Liapunov (1892)] was the first who introduced multiple parameters to a general system of linear differential equations with periodic coefficients. Methods for sensitivity analysis of stability characteristics of periodic systems with respect to parameters were developed in [Seyranian *et al.* (1999); Seyranian *et al.* (2000); Mailybaev and Seyranian (2000a)], and these results are used in this chapter.

We start this chapter with the introduction to the stability theory for periodic systems based on the Floquet method, where the decision on stability or instability is given upon the calculation of the Floquet matrix and its eigenvalues (multipliers). Considering multi-parameter periodic systems, we derive derivatives of the Floquet matrix with respect to parameters. Then formulae for derivatives of simple multipliers are given, and bifurcation of multiple multipliers is studied. This information allows studying stability of periodic systems depending on parameters in effective and constructive way.

To make clear the large amount of information included in the sensitivity analysis, we provide a number of analytical and numerical examples. Stability diagrams are studied analytically for the Mathieu and Meissner equations. As numerical applications, we consider the problem of optimal design of a beam loaded by a periodic axial force with constraints on stability requirements, stabilization of a system described by the Carson-Cambi equation, and numerical analysis of motion of multipliers on the complex plane in the cases of parametric and combination resonances.

9.1 Stability of periodic solution

Let us consider a periodic system described by a nonlinear system of ordinary differential equations

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t), \tag{9.1}$$

where $\mathbf{y}$ is a real vector of dimension m, and $\mathbf{g}(\mathbf{y}, t)$ is a periodic function of time t with a period $T > 0$, that is,

$$\mathbf{g}(\mathbf{y}, t) = \mathbf{g}(\mathbf{y}, t + T) \tag{9.2}$$

for any $\mathbf{y}$ and t. We assume that $\mathbf{g}(\mathbf{y}, t)$ is a smooth function of $\mathbf{y}$ and continuous function of t, ensuring existence and uniqueness of a solution with the initial condition $\mathbf{y}_0 = \mathbf{y}(t_0)$ on the semi-infinite interval of time $t \geq t_0$. A solution $\mathbf{y}(t)$ of system (9.1) is called *periodic* if

$$\mathbf{y}(t) = \mathbf{y}(t + T) \tag{9.3}$$

for any value of time t. It is sufficient to check condition (9.3) at a particular time value $t = t_0$. By uniqueness of the solution $\mathbf{y}(t)$ and periodicity of the function $\mathbf{g}(\mathbf{y}, t)$, if $\mathbf{y}(t_0) = \mathbf{y}(t_0 + T)$ then equality (9.3) is valid for any t. A periodic solution can be stable, asymptotically stable, or unstable; see Section 1.1 for definitions.

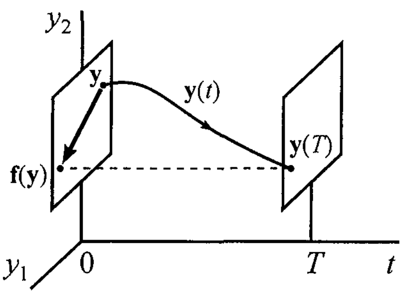


Fig. 9.1 Poincaré map of a periodic system.

Let us consider a map $\mathbf{f}(\mathbf{y})$, which transfers a point $\mathbf{y} = \mathbf{y}(0)$ in the state space to the point $\mathbf{f}(\mathbf{y}) = \mathbf{y}(T)$, where $\mathbf{y}(t)$ is a solution of system (9.1). The map $\mathbf{f}(\mathbf{y})$ is called the *Poincaré map*. The Poincaré map describes dynamics of the system over the period T; see Fig. 9.1. Successive action of the Poincaré map on a point $\mathbf{y}$ yields

$$\mathbf{f}^k(\mathbf{y}) = \underbrace{\mathbf{f}(\mathbf{f}(\cdots \mathbf{f}(\mathbf{y}) \cdots))}_{k \text{ times}} = \mathbf{y}(kT), \tag{9.4}$$

which is the value of the solution $\mathbf{y}(t)$ at $t = kT$. A point $\mathbf{y}_0$ is called *stationary* for the map $\mathbf{f}(\mathbf{y})$ if

$$\mathbf{f}(\mathbf{y}_0) = \mathbf{y}_0. \tag{9.5}$$

Relation of the Poincaré map to periodic system (9.1) is given by the following statement.

Theorem 9.1 *A point* $\mathbf{y}_0$ *is stationary for the Poincaré map* $\mathbf{f}(\mathbf{y})$ *if and only if the solution* $\mathbf{y}(t)$ *of system (9.1) with the initial condition* $\mathbf{y}(0) = \mathbf{y}_0$ *is periodic with the period* T.

The Poincaré map determines the discrete dynamical system

$$\mathbf{y} = \mathbf{f}^0(\mathbf{y}) \to \mathbf{f}^1(\mathbf{y}) \to \mathbf{f}^2(\mathbf{y}) \to \cdots \to \mathbf{f}^k(\mathbf{y}) \to \cdots, \tag{9.6}$$

where k represents the discrete (integer) time. The stationary point $\mathbf{y}_0$ remains unchanged under the action of the Poincaré map. Points $\mathbf{y}$, which are close to $\mathbf{y}_0$, can approach or move off the stationary point under multiple action of the map $\mathbf{f}(\mathbf{y})$. Such behavior determines stability properties of the stationary point $\mathbf{y}_0$ for discrete dynamical system (9.6).

Definition 9.1 The stationary point $\mathbf{y}_0$ of the map $\mathbf{f}(\mathbf{y})$ is called *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\mathbf{y}$, satisfying the condition $\|\mathbf{y} - \mathbf{y}_0\| < \delta$, the inequality

$$\|\mathbf{f}^k(\mathbf{y}) - \mathbf{f}^k(\mathbf{y}_0)\| < \varepsilon \tag{9.7}$$

takes place for any integer $k > 0$. If, in addition, $\|\mathbf{f}^k(\mathbf{y}) - \mathbf{f}^k(\mathbf{y}_0)\| \to 0$ as $k \to +\infty$, then the stationary point is called *asymptotically stable.*

Though the Poincaré map $\mathbf{f}(\mathbf{y})$ contains only a part of information on system (9.1) (we can not predict a state of the system in time $t \neq kT$ using this map), it is sufficient to make a decision on stability of a periodic solution of system (9.1).

Theorem 9.2 *A periodic solution* $\mathbf{y}(t)$ *of system (9.1) is stable (asymptotically stable) if and only if the stationary point* $\mathbf{y}_0 = \mathbf{y}(0)$ *of the Poincaré map* $\mathbf{f}(\mathbf{y})$ *is stable (asymptotically stable).*

Let $\tilde{\mathbf{y}}(t)$ be a periodic solution of system (9.1). Introducing the vector

$$\mathbf{x}(t) = \mathbf{y}(t) - \tilde{\mathbf{y}}(t) \tag{9.8}$$

describing deviation from the periodic solution and using equation (9.1), we obtain

$$\dot{\mathbf{x}} = \dot{\mathbf{y}} - \dot{\tilde{\mathbf{y}}} = \mathbf{g}(\tilde{\mathbf{y}}(t) + \mathbf{x}, t) - \mathbf{g}(\tilde{\mathbf{y}}(t), t). \tag{9.9}$$

Assuming that $\|\mathbf{x}\|$ is small and neglecting higher order terms, we find

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \tag{9.10}$$

where

$$\mathbf{G}(t) = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right)_{\widetilde{\mathbf{y}}(t)} \tag{9.11}$$

is the $m \times m$ Jacobian matrix of the function $\mathbf{g}(\mathbf{y}, t)$ evaluated at $\mathbf{y} = \widetilde{\mathbf{y}}(t)$. Since the functions $\mathbf{g}(\mathbf{y}, t)$ and $\widetilde{\mathbf{y}}(t)$ are periodic in time with the period T, the matrix $\mathbf{G}(t)$ is periodic with the same period. Linear periodic system (9.10) represents the *linearization* of system (9.1) near the periodic solution $\widetilde{\mathbf{y}}(t)$.

Analogously, we perform the linearization of the Poincaré map $\mathbf{y} \to \mathbf{f}(\mathbf{y})$ near the stationary point $\widetilde{\mathbf{y}}_0 = \widetilde{\mathbf{y}}(0)$ as follows

$$\mathbf{x} \to \mathbf{F}\mathbf{x}, \tag{9.12}$$

where $\mathbf{x} = \mathbf{y} - \widetilde{\mathbf{y}}_0$ and

$$\mathbf{F} = \left(\frac{d\mathbf{f}}{d\mathbf{y}}\right)_{\widetilde{\mathbf{y}}_0} \tag{9.13}$$

is the $m \times m$ Jacobian matrix of the mapping $\mathbf{f}(\mathbf{y})$ evaluated at the stationary point $\mathbf{y} = \widetilde{\mathbf{y}}_0$.

9.2 Floquet theory

Let us consider a linear periodic system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^m, \tag{9.14}$$

where $\mathbf{G}(t)$ is an $m \times m$ real matrix periodically dependent on time t with a period $T > 0$, i.e.,

$$\mathbf{G}(t) = \mathbf{G}(t + T). \tag{9.15}$$

Since system (9.14) is linear, a sum of its particular solutions is the solution. Let us define m linearly independent solutions $\mathbf{x}_1(t), \dots, \mathbf{x}_m(t)$ satisfying the initial conditions

$$\mathbf{x}_i(0) = \mathbf{e}_i, \quad i = 1, \dots, m, \tag{9.16}$$

where $\mathbf{e}_i$ is the ith column of the $m \times m$ identity matrix $\mathbf{I}$. Then a solution of system (9.14) satisfying the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{pmatrix} x_{01} \\ \vdots \\ x_{0m} \end{pmatrix} \tag{9.17}$$

can be found in the form

$$\mathbf{x}(t) = \sum_{i=1}^{m} x_{0i}\mathbf{x}_i(t). \tag{9.18}$$

Taking the vector-functions $\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)$ as columns of the $m \times m$ real matrix $\mathbf{X}(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)]$, we represent solution (9.18) as

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{x}_0. \tag{9.19}$$

The matrix $\mathbf{X}(t)$ satisfies the equation

$$\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X} \tag{9.20}$$

with the initial condition

$$\mathbf{X}(0) = \mathbf{I}, \tag{9.21}$$

and is called the *principal fundamental matrix* or *matriciant* of system (9.14). The matriciant $\mathbf{X}(t)$ taken at the period $t = T$ provides the matrix

$$\mathbf{F} = \mathbf{X}(T) \tag{9.22}$$

called the *Floquet matrix* or *monodromy matrix.* It is clear that $\mathbf{F}\mathbf{x}(0) = \mathbf{x}(T)$ for any solution $\mathbf{x}(t)$ of system (9.14). Therefore, the matrix operator $\mathbf{F}$ represents the Poincaré map $\mathbf{x} \to \mathbf{F}\mathbf{x}$ for periodic system (9.14).

Let us consider the eigenvalue problem for the Floquet matrix

$$\mathbf{F}\mathbf{u} = \rho\mathbf{u}, \tag{9.23}$$

where ρ is an eigenvalue and $\mathbf{u}$ is an eigenvector. Since columns of the Floquet matrix are linearly independent, $\det \mathbf{F} \neq 0$ and, hence, $\rho \neq 0$. Equation (9.23) yields a particular solution of the discrete dynamical system defined by the Floquet matrix

$$\widetilde{\mathbf{x}}(kT) = \mathbf{F}^k\mathbf{u} = \rho^k\mathbf{u}, \quad k = 0, 1, 2, \ldots \tag{9.24}$$

After each period, solution (9.24) is multiplied by ρ. Due to this property, eigenvalues ρ of the Floquet matrix are called *(Floquet) multipliers.* The norm of solution (9.24) exponentially increases or decreases with an increase of k if $|\rho| > 1$ or $|\rho| < 1$, respectively. If $|\rho| = 1$, then the norm of solution (9.24) does not change with an increase of k.

Let us consider a multiplier ρ possessing a Jordan chain of length $l > 1$:

$$\begin{aligned} \mathbf{F}\mathbf{u}_0 &= \rho\mathbf{u}_0, \\ \mathbf{F}\mathbf{u}_1 &= \rho\mathbf{u}_1 + \mathbf{u}_0, \\ &\vdots \\ \mathbf{F}\mathbf{u}_{l-1} &= \rho\mathbf{u}_{l-1} + \mathbf{u}_{l-2}. \end{aligned} \tag{9.25}$$

Then we find l linearly independent solutions of the discrete system

$$\begin{aligned} \widetilde{\mathbf{x}}_1(kT) &= \mathbf{F}^k\mathbf{u}_0 = \rho^k\mathbf{u}_0, \\ \widetilde{\mathbf{x}}_2(kT) &= \mathbf{F}^k\mathbf{u}_1 = \rho^k\mathbf{u}_1 + k\rho^{k-1}\mathbf{u}_0, \\ &\vdots \\ \widetilde{\mathbf{x}}_l(kT) &= \mathbf{F}^k\mathbf{u}_{l-1} = \sum_{i=0}^{\min(k,l-1)} C_k^i\rho^{k-i}\mathbf{u}_{l-i-1}, \quad C_k^i = \frac{k!}{i!(k-i)!}, \\ & k = 0, 1, 2, \ldots \end{aligned} \tag{9.26}$$

We see that the norms of all solutions (9.26) decrease for big k if and only if $|\rho| < 1$. If $|\rho| = 1$ or $|\rho| > 1$, then we observe infinite growth of the norms of solutions (9.26) for big k, respectively.

General solution of the discrete system defined by the Floquet matrix can be constructed taking a linear combination of solutions (9.24) and (9.26) for all the multipliers ρ and corresponding Jordan chains. As a result, we find the stability criterion for the discrete system. Using Theorem 9.2, we obtain the *Floquet theorem* for stability of linear periodic system (9.14):

Theorem 9.3 *Linear periodic system (9.14) is asymptotically stable ($\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$ for any solution $\mathbf{x}(t)$) if and only if $|\rho| < 1$ for all the multipliers of the Floquet matrix.*

Linear periodic system (9.14) is stable (all the solutions $\mathbf{x}(t)$ are bounded as $t \to +\infty$) if and only if $|\rho| \leq 1$ for all the multipliers of the Floquet matrix, and the multipliers with the unit absolute value $|\rho| = 1$ are simple or semi-simple.

Linear periodic system (9.14) is unstable (there is a solution $\mathbf{x}(t)$ unbounded as $t \to +\infty$) if and only if there is a multiplier of the Floquet matrix such that $|\rho| > 1$, or a multiplier with the unit absolute value $|\rho| = 1$ which is neither simple nor semi-simple.

Theorem 9.3 says that stability of a linear periodic system depends on distribution of its multipliers with respect to the unit circle on the complex plane; see Fig. 9.2.

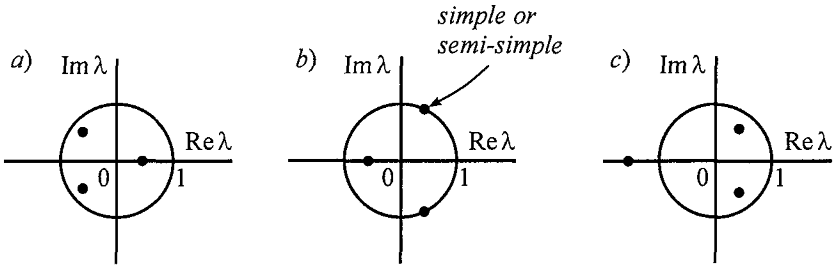


Fig. 9.2 Distribution of multipliers for a) asymptotically stable, b) stable, and c) unstable linear periodic system.

Let us denote

$$\lambda = \frac{1}{T} \ln \rho. \tag{9.27}$$

The quantities λ are called *characteristic exponents.* The characteristic exponents are determined up to additive terms $2\pi ki/T$, $k \in \mathbb{Z}$, where i is the imaginary unit. Since $\operatorname{Re}\lambda = \ln|\rho|/T$ and, hence, $|\rho| = \exp(T\operatorname{Re}\lambda)$, the stability criterion of Theorem 9.3 yields

Corollary 9.1 *Linear periodic system (9.14) is asymptotically stable if and only if* $\operatorname{Re}\lambda < 0$ *for all the characteristic exponents. If* $\operatorname{Re}\lambda > 0$ *for at least one characteristic exponent, then linear periodic system (9.14) is unstable.*

Corollary 9.1 is similar to the stability criterion for a linear autonomous system, where λ are eigenvalues of the system operator. This similarity is supported by the following relationship between periodic and autonomous linear systems.

Let us introduce the matrix

$$\mathbf{A} = \frac{1}{T} \ln \mathbf{F}; \tag{9.28}$$

see [Korn and Korn (1968)] for definition of the logarithm function for square matrices. Eigenvalues of the matrix $\mathbf{A}$ are the characteristic exponents λ. The change of coordinates

$$\mathbf{x} = \mathbf{X}(t) \exp(-\mathbf{A}t)\mathbf{z}, \tag{9.29}$$

where $\mathbf{X}(t)$ is the matriciant, in equation (9.14) yields

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{X}(t)\exp(-\mathbf{A}t)\mathbf{z} - \mathbf{X}(t)\exp(-\mathbf{A}t)\mathbf{A}\mathbf{z} + \mathbf{X}(t)\exp(-\mathbf{A}t)\dot{\mathbf{z}} \quad (9.30)$$

for the left-hand side and

$$\mathbf{G}\mathbf{x} = \mathbf{G}(t)\mathbf{X}(t)\exp(-\mathbf{A}t)\mathbf{z} \quad (9.31)$$

for the right-hand side. Using (9.30) and (9.31) in equation (9.14) and pre-multiplying by $(\mathbf{X}(t)\exp(-\mathbf{A}t))^{-1}$, we find

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}. \quad (9.32)$$

Using relations $\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{F}$ and $\mathbf{F} = \exp(\mathbf{A}T)$, we obtain

$$\mathbf{X}(t+T)\exp(-\mathbf{A}(t+T)) = \mathbf{X}(t)\mathbf{F}\mathbf{F}^{-1}\exp(-\mathbf{A}t) = \mathbf{X}(t)\exp(-\mathbf{A}t). \quad (9.33)$$

Hence, change of coordinates (9.29) is periodic and transforms linear periodic system (9.14) into linear autonomous system (9.32). Therefore, Corollary 9.1 can be obtained directly from the stability analysis of autonomous system (9.32). This statement is known as the *Liapunov reduction theorem*:

Theorem 9.4 *Periodic change of coordinates in the state space (9.29) transforms linear periodic system (9.14) into linear autonomous system (9.32), where the matrix* $\mathbf{A}$ *is given by expression (9.28).*

Remark 9.1 *Notice that the matrix* $\mathbf{A}$ *is complex unless all the multipliers are real and positive. In general, it is possible to find a real periodic change of coordinates with the double period* $2T$ *transforming a linear periodic system to a linear autonomous system, see [Yakubovich and Starzhinskii (1975)].*

General form of solution of linear autonomous system (9.32) is described in Section 3.1. Using Theorem 9.4, linear independent solutions of periodic system (9.14) corresponding to a real multiplier ρ with Jordan chain of length l can be expressed in the form

$$\begin{aligned}
\mathbf{x}_1^{(\rho)}(t) &= \varphi_1(t)\exp\rho t, \\
\mathbf{x}_2^{(\rho)}(t) &= (\varphi_2(t) + t\varphi_1(t))\exp\rho t, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.34) \\
\mathbf{x}_l^{(\rho)}(t) &= \left(\varphi_l(t) + t\varphi_{l-1}(t) + \cdots + \frac{t^{l-1}}{(l-1)!}\varphi_1(t)\right)\exp\rho t,
\end{aligned}$$

where $\varphi_1(t), \ldots, \varphi_l(t)$ are linearly independent periodic vector-functions. For a complex multiplier ρ real and imaginary parts of solutions (9.34) give $2l$ linearly independent solutions. A general solution is given by a linear combination of these solutions taken for all multipliers and Jordan chains.

Finally, we return to the analysis of stability of a periodic solution $\widetilde{\mathbf{y}}(t)$ for nonlinear system (9.1). Let (9.14) be the linearization of system (9.1) near the periodic solution $\widetilde{\mathbf{y}}(t)$, where the periodic matrix $\mathbf{G}(t)$ is given by formula (9.11). The corresponding Floquet matrix $\mathbf{F}$ determines the linearization of the Poincaré map $\mathbf{f}(\mathbf{y})$ near the stationary point $\widetilde{\mathbf{y}}_0 = \widetilde{\mathbf{y}}(0)$. In many practical cases stability of the periodic solution $\widetilde{\mathbf{y}}(t)$ can be determined from the analysis of the linearized system using the following theorem by [Liapunov (1892)].

Theorem 9.5 *If linearized periodic system (9.14) is asymptotically stable ($|\rho| < 1$ for all the multipliers), then the periodic solution $\widetilde{\mathbf{y}}(t)$ of nonlinear periodic system (9.1) is asymptotically stable.*

If linearized periodic system (9.14) is unstable and there is a multiplier lying outside the unit circle, $|\rho| > 1$, then the periodic solution $\widetilde{\mathbf{y}}(t)$ of nonlinear periodic system (9.1) is unstable.

The case when $|\rho| \leq 1$ for all the multipliers with some multipliers lying on the unit circle $|\rho| = 1$ is not covered by Theorem 9.5. In this case the stability property can be affected by nonlinear terms, and stability or instability of the linearized system does not necessarily lead to the same property for the nonlinear system.

Example 9.1 As an example, let us consider the Meissner equation

$$\ddot{x} + (a + qh(t))x = 0, \quad x \in \mathbb{R}, \tag{9.35}$$

where a and q are non-negative parameters and $h(t)$ is the 2π-periodic piecewise-constant function

$$h(t) = \begin{cases} 1, & 0 < t \leq \pi, \\ -1, & \pi < t \leq 2\pi, \end{cases}, \quad h(t + 2\pi) = h(t). \tag{9.36}$$

System (9.35), (9.36) can be transformed to the first order differential equation as follows

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 & 1 \\ -a - qh(t) & 0 \end{pmatrix}. \tag{9.37}$$

System (9.37) is autonomous in the intervals $0 < t \leq \pi$ and $\pi < t \leq 2\pi$ and, therefore, can be integrated analytically. Finding solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ with the initial conditions

$$\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9.38}$$

we determine the matriciant $\mathbf{X}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t)]$. As a result, we get the Floquet matrix $\mathbf{F} = \mathbf{X}(2\pi)$ in the form

$$\mathbf{F} = \begin{pmatrix} \cos \pi\omega_2 & \dfrac{1}{\omega_2} \sin \pi\omega_2 \\ -\omega_2 \sin \pi\omega_2 & \cos \pi\omega_2 \end{pmatrix} \begin{pmatrix} \cos \pi\omega_1 & \dfrac{1}{\omega_1} \sin \pi\omega_1 \\ -\omega_1 \sin \pi\omega_1 & \cos \pi\omega_1 \end{pmatrix} \tag{9.39}$$

for $a > q$;

$$\mathbf{F} = \begin{pmatrix} \cosh \pi\omega_2 & \dfrac{1}{\omega_2} \sinh \pi\omega_2 \\ \omega_2 \sinh \pi\omega_2 & \cosh \pi\omega_2 \end{pmatrix} \begin{pmatrix} \cos \pi\omega_1 & \dfrac{1}{\omega_1} \sin \pi\omega_1 \\ -\omega_1 \sin \pi\omega_1 & \cos \pi\omega_1 \end{pmatrix} \tag{9.40}$$

for $a < q$; and

$$\mathbf{F} = \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi\omega_1 & \dfrac{1}{\omega_1} \sin \pi\omega_1 \\ -\omega_1 \sin \pi\omega_1 & \cos \pi\omega_1 \end{pmatrix} \tag{9.41}$$

for $a = q$, where $\omega_1 = \sqrt{a + q}$ and $\omega_2 = \sqrt{|a - q|}$.

Evaluating determinants of matrices (9.39)–(9.41), we find

$$\det \mathbf{F} = 1. \tag{9.42}$$

Hence, $\rho^a \rho^b = 1$, where ρ^a and ρ^b are the multipliers and stability of the system requires both the multipliers to be complex conjugate and lie on the unit circle $|\rho^a| = |\rho^b| = 1$. If the multipliers ρ^a and ρ^b are real and different, then one of them lies outside of the unit circle (instability). Transference from stability to instability is possible only if the multipliers merge to 1 or -1 and become double. Therefore, the boundary between the stability and instability domains in the parameter space (a, q) is given by the condition

$$|\rho^a + \rho^b| = |\mathrm{trace}\, \mathbf{F}| = 2. \tag{9.43}$$

This condition takes the form

$$\left|2\cos\pi\omega_1\cos\pi\omega_2 - \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)\sin\pi\omega_1\sin\pi\omega_2\right| = 2 \tag{9.44}$$

for $a > q$,

$$\left|2\cos\pi\omega_1\cosh\pi\omega_2 - \left(\frac{\omega_1}{\omega_2} - \frac{\omega_2}{\omega_1}\right)\sin\pi\omega_1\sinh\pi\omega_2\right| = 2 \tag{9.45}$$

for $a < q$, and

$$|2\cos\pi\omega_1 - \pi\omega_1\sin\pi\omega_1| = 2 \tag{9.46}$$

for $a = q$.

Stability diagram in the space (a, q) found numerically using conditions (9.44)–(9.46) is shown in Fig. 9.3, where the instability domain is hatched. The instability domain consists of tongues touching the a-axis at the points $a = k^2/4$, $k = 1, 2, \ldots$

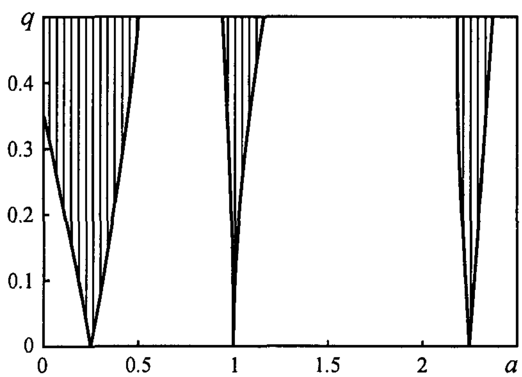


Fig. 9.3 Stability diagram for the Meissner equation.

9.3 Derivatives of Floquet matrix with respect to parameters

Let us consider a linear periodic system

$$\dot{\mathbf{x}} = \mathbf{G}(t, \mathbf{p})\mathbf{x}, \tag{9.47}$$

where the matrix operator $\mathbf{G}$ smoothly depends on a vector of real parameters $\mathbf{p} \in \mathbb{R}^n$. First, we assume that the period of the system T does not depend on parameters, i.e., $\mathbf{G}(t+T, \mathbf{p}) = \mathbf{G}(t, \mathbf{p})$ for any t and $\mathbf{p}$. Equation for the matriciant $\mathbf{X}(t)$ takes the form

$$\dot{\mathbf{X}} = \mathbf{G}(t, \mathbf{p})\mathbf{X} \tag{9.48}$$

with the initial condition

$$\mathbf{X}(0) = \mathbf{I}. \tag{9.49}$$

Stability of system (9.47) for a given value of the parameter vector $\mathbf{p}$ is determined by multipliers of the Floquet matrix $\mathbf{F} = \mathbf{X}(T)$. Both the matriciant and Floquet matrix are smooth functions of the parameter vector.

Let us introduce the $m \times m$ real matrix satisfying the adjoint equation

$$\dot{\mathbf{Y}} = -\mathbf{G}^T(t, \mathbf{p})\mathbf{Y} \tag{9.50}$$

with the initial condition

$$\mathbf{Y}(0) = \mathbf{I}. \tag{9.51}$$

Differentiating the product $\mathbf{Y}^T\mathbf{X}$ with respect to time and using equations (9.48) and (9.50), we find

$$(\mathbf{Y}^T\mathbf{X})^{\cdot} = \dot{\mathbf{Y}}^T\mathbf{X} + \mathbf{Y}^T\dot{\mathbf{X}} = -\mathbf{Y}^T\mathbf{G}(t, \mathbf{p})\mathbf{X} + \mathbf{Y}^T\mathbf{G}(t, \mathbf{p})\mathbf{X} = 0. \tag{9.52}$$

Together with initial conditions (9.49) and (9.51) equation (9.52) yields the relation

$$\mathbf{Y}^T(t) = \mathbf{X}^{-1}(t). \tag{9.53}$$

Taking the derivative of both sides of equation (9.48) with respect to the parameter p_i, we find

$$\frac{\partial \dot{\mathbf{X}}}{\partial p_i} = \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X} + \mathbf{G}\frac{\partial \mathbf{X}}{\partial p_i}. \tag{9.54}$$

Pre-multiplying (9.54) by the matrix $\mathbf{Y}^T(t)$ and integrating over the time interval $[0, t]$, we get

$$\int_0^t \mathbf{Y}^T \frac{\partial \dot{\mathbf{X}}}{\partial p_i} d\tau = \int_0^t \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X} d\tau + \int_0^t \mathbf{Y}^T \mathbf{G}\frac{\partial \mathbf{X}}{\partial p_i} d\tau. \tag{9.55}$$

Using integration by parts and equations (9.49), (9.50), we represent the left-hand side of (9.55) as

$$\begin{aligned}\int_0^t \mathbf{Y}^T \frac{\partial \dot{\mathbf{X}}}{\partial p_i} d\tau &= \mathbf{Y}^T \frac{\partial \mathbf{X}}{\partial p_i}\bigg|_0^t - \int_0^t \dot{\mathbf{Y}}^T \frac{\partial \mathbf{X}}{\partial p_i} d\tau \\ &= \mathbf{Y}^T(t) \frac{\partial \mathbf{X}(t,\mathbf{p})}{\partial p_i} + \int_0^t \mathbf{Y}^T \mathbf{G} \frac{\partial \mathbf{X}}{\partial p_i} d\tau.\end{aligned} \tag{9.56}$$

Substitution of relation (9.56) into equation (9.55) yields

$$\mathbf{Y}^T(t) \frac{\partial \mathbf{X}(t,\mathbf{p})}{\partial p_i} = \int_0^t \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau. \tag{9.57}$$

Using relation (9.53), we find the derivative of the matriciant with respect to the parameter p_i in the form

$$\frac{\partial \mathbf{X}}{\partial p_i} = \mathbf{X}(t) \int_0^t \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau. \tag{9.58}$$

Since $\mathbf{F} = \mathbf{X}(T)$, we find the first order derivative of the Floquet matrix as

$$\frac{\partial \mathbf{F}}{\partial p_i} = \mathbf{F} \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau. \tag{9.59}$$

We observe that derivatives of the Floquet matrix are determined by the matriciant $\mathbf{X}(t)$, its inverse $\mathbf{Y}^T(t) = \mathbf{X}^{-1}(t)$, and derivatives of the system matrix $\mathbf{G}(t,\mathbf{p})$ with respect to parameters.

Taking the second order derivative of equation (9.48) with respect to parameters and performing analogous transformations with the use of expression (9.58), we find

$$\begin{aligned}\frac{\partial^2 \mathbf{X}}{\partial p_i \partial p_j} = \mathbf{X}(t) \bigg(&\int_0^t \mathbf{Y}^T \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} \mathbf{X} d\tau \\ &+ \int_0^t \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} \left(\int_0^{\tau_1} \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X} d\tau_2 \right) d\tau_1 \\ &+ \int_0^t \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X} \left(\int_0^{\tau_1} \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau_2 \right) d\tau_1 \bigg).\end{aligned} \tag{9.60}$$

Taking expression (9.60) at $t = T$, we find the second order derivative of the Floquet matrix with respect to parameters as

$$\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} = \mathbf{F}\left(\int_0^T \mathbf{Y}^T \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j}\mathbf{X}d\tau \right.$$
$$+ \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X}\left(\int_0^{\tau_1} \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}d\tau_2\right)d\tau_1 \tag{9.61}$$
$$\left. + \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\left(\int_0^{\tau_1} \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X}d\tau_2\right)d\tau_1\right).$$

In the same way, higher order derivatives of the matriciant and Floquet matrix with respect to parameters can be found. In order to give a general expression for the derivatives, we introduce the notation

$$\mathbf{G}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{G}}{\partial p_1^{h_1}\cdots\partial p_n^{h_n}}, \quad \mathbf{X}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{X}}{\partial p_1^{h_1}\cdots\partial p_n^{h_n}}, \quad \mathbf{F}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{F}}{\partial p_1^{h_1}\cdots\partial p_n^{h_n}}, \tag{9.62}$$

where $\mathbf{h} = (h_1, \ldots, h_n)$ is a vector with integer non-negative components $h_i \geq 0$ and $|\mathbf{h}| = h_1 + \cdots + h_n$. We denote

$$\mathbf{H}_{\mathbf{h}}(t) = \left(\frac{\mathbf{Y}^T\mathbf{G}^{(\mathbf{h})}\mathbf{X}}{\mathbf{h}!}\right)_t, \tag{9.63}$$

where $\mathbf{h}! = h_1!\cdots h_n!$.

Theorem 9.6 *Derivative of the matriciant $\mathbf{X}(t, \mathbf{p})$ with respect to parameters for linear periodic system (9.47) has the form*

$$\mathbf{X}^{(\mathbf{h})}(t) = \mathbf{h}!\mathbf{X}(t) \sum_{\substack{\mathbf{h}_1+\cdots+\mathbf{h}_s=\mathbf{h} \\ s=1,\ldots,|\mathbf{h}|,\ |\mathbf{h}_i|>0}} \int_0^t \mathbf{H}_{\mathbf{h}_1}(\tau_1)\int_0^{\tau_1}\mathbf{H}_{\mathbf{h}_2}(\tau_2)\cdots \tag{9.64}$$
$$\cdots\int_0^{\tau_{s-1}}\mathbf{H}_{\mathbf{h}_s}(\tau_s)d\tau_s\cdots d\tau_2 d\tau_1,$$

where the sum is taken over all the sets of nonzero vectors $\mathbf{h}_1, \ldots, \mathbf{h}_s$ such that $\mathbf{h}_1 + \cdots + \mathbf{h}_s = \mathbf{h}$. If the period T does not depend on parameters, the derivative of the Floquet matrix $\mathbf{F}(\mathbf{p}) = \mathbf{X}(T, \mathbf{p})$ with respect to parameters *is*

$$\mathbf{F}^{(\mathbf{h})} = \mathbf{h}!\mathbf{F} \sum_{\substack{\mathbf{h}_1+\cdots+\mathbf{h}_s=\mathbf{h} \\ s=1,\ldots,|\mathbf{h}|,\ |\mathbf{h}_i|>0}} \int_0^T \mathbf{H}_{\mathbf{h}_1}(\tau_1) \int_0^{\tau_1} \mathbf{H}_{\mathbf{h}_2}(\tau_2) \cdots \tag{9.65}$$
$$\cdots \int_0^{\tau_{s-1}} \mathbf{H}_{\mathbf{h}_s}(\tau_s) d\tau_s \cdots d\tau_2 d\tau_1.$$

If the period $T = T(\mathbf{p})$ smoothly depends on parameters, then derivatives of the Floquet matrix $\mathbf{F}(\mathbf{p}) = \mathbf{X}(T(\mathbf{p}), \mathbf{p})$ can be obtained by differentiating the matrix $\mathbf{X}(T(\mathbf{p}), \mathbf{p})$ as a composite function. For the first order derivative we find

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial p_i} &= \left(\frac{\partial \mathbf{X}}{\partial p_i}\right)_T + \left(\frac{\partial \mathbf{X}}{\partial t}\right)_T \frac{\partial T}{\partial p_i} \\ &= \mathbf{F} \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau + \mathbf{G}(0, \mathbf{p}) \mathbf{F} \frac{\partial T}{\partial p_i}, \end{aligned} \tag{9.66}$$

where equation (9.48) and expressions (9.15), (9.58) were used. Another way to treat the case of a parameter-dependent period is to make the change of time

$$\widetilde{t} = \frac{T_0}{T(\mathbf{p})} t, \tag{9.67}$$

where T_0 is a positive constant. Then system (9.47) takes the form

$$\dot{\mathbf{x}} = \widetilde{\mathbf{G}}(\widetilde{t}, \mathbf{p}) \mathbf{x}, \tag{9.68}$$

where the dot denotes differentiation with respect to new time $\widetilde{t}$, and

$$\widetilde{\mathbf{G}}(\widetilde{t}, \mathbf{p}) = \frac{T(\mathbf{p})}{T_0} \mathbf{G}\left(\frac{T(\mathbf{p})}{T_0} \widetilde{t}, \mathbf{p}\right) \tag{9.69}$$

is a periodic matrix with the period T_0 independent on parameters. The Floquet matrix evaluated for system (9.68), (9.69) coincides with that for system (9.47), and its derivatives can be found by Theorem 9.6 applied to system (9.68), (9.69).

9.4 Stability analysis of the Mathieu equation

Let us consider the Mathieu equation

$$\ddot{x} + (a + q\cos t)x = 0, \quad x \in \mathbb{R}, \tag{9.70}$$

dependent on two parameters: the non-negative parameter a denoting the squared eigenfrequency of the unexcited system and the excitation amplitude q. Transforming equation (9.70) to the first order form, we obtain

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 & 1 \\ -a - q\cos t & 0 \end{pmatrix}, \tag{9.71}$$

where the matrix $\mathbf{G}(t)$ is periodic in time with the period $T = 2\pi$. Derivatives of the matrix $\mathbf{G}$ with respect to parameters are used to find derivatives of the Floquet matrix. From (9.71) we find

$$\frac{\partial \mathbf{G}}{\partial a} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{\partial \mathbf{G}}{\partial q} = \begin{pmatrix} 0 & 0 \\ -\cos t & 0 \end{pmatrix}. \tag{9.72}$$

Higher order derivatives of the matrix $\mathbf{G}$ with respect to parameters are all equal to zero.

The determinant of the matriciant $\mathbf{X}(t)$ for system (9.71) does not depend on time. Indeed,

$$\begin{aligned} \frac{d}{dt}\det \mathbf{X}(t) &= \frac{d}{dt}(x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t)) \\ &= x_{21}(t)x_{22}(t) - x_{11}(t)(a + q\cos t)x_{12}(t) \\ &\quad -x_{22}(t)x_{21}(t) + x_{12}(t)(a + q\cos t)x_{11}(t) = 0, \end{aligned} \tag{9.73}$$

where $x_{ij}(t)$ are the elements of the matrix $\mathbf{X}(t)$. This equality reflects the property of the system to preserve volume in the state space as the time is changing. As a result, we find the determinant of the Floquet matrix as

$$\det \mathbf{F} = \det \mathbf{X}(2\pi) = \det \mathbf{X}(0) = 1. \tag{9.74}$$

By condition (9.74), the product of the multipliers ρ^a and ρ^b of the Floquet matrix $\mathbf{F}$ equals one. Hence,

$$\rho^a = \frac{1}{\rho^b}. \tag{9.75}$$

The multipliers can be real and lie in different sides with respect to the unit circle on the complex plane (instability), or complex conjugate and lie on the unit circle (stability). The transference between stability and instability corresponds to the case of a double multiplier $\rho^a = \rho^b = \pm 1$ (stability boundary).

The multipliers ρ^a and ρ^b can be found in the form

$$\begin{aligned} \rho^a &= \frac{f_{11}+f_{22}}{2} + \sqrt{\frac{(f_{11}-f_{22})^2}{4} + f_{12}f_{21}}, \\ \rho^b &= \frac{f_{11}+f_{22}}{2} - \sqrt{\frac{(f_{11}-f_{22})^2}{4} + f_{12}f_{21}}, \end{aligned} \tag{9.76}$$

where f_{ij} are the elements of the Floquet matrix $\mathbf{F}$. We remind that according to (9.74) $\det \mathbf{F} = f_{11}f_{22} - f_{12}f_{21} = 1$. If

$$\frac{(f_{11}-f_{22})^2}{4} + f_{12}f_{21} > 0, \tag{9.77}$$

the multipliers are real and lie in different sides with respect to the unit circle (the system is unstable). In the case

$$\frac{(f_{11}-f_{22})^2}{4} + f_{12}f_{21} < 0 \tag{9.78}$$

the multipliers are complex conjugate and lie on the unit circle (the system is stable). Finally, the equality

$$\frac{(f_{11}-f_{22})^2}{4} + f_{12}f_{21} = 0 \tag{9.79}$$

implies that the multipliers are multiple and equal to $\rho^a = \rho^b = \pm 1$, which corresponds to the stability boundary.

If the excitation amplitude q is set zero, equation (9.71) is a linear autonomous system of ordinary differential equations, which can be solved analytically. The matriciant for system (9.71) when $a > 0$ and $q = 0$ is

$$\mathbf{X}(t) = \begin{pmatrix} \cos(t\sqrt{a}) & \dfrac{1}{\sqrt{a}}\sin(t\sqrt{a}) \\ -\sqrt{a}\sin(t\sqrt{a}) & \cos(t\sqrt{a}) \end{pmatrix}. \tag{9.80}$$

The matrix $\mathbf{Y}(t)$ for the adjoint system (9.50), (9.51) is

$$\mathbf{Y}(t) = \begin{pmatrix} \cos(t\sqrt{a}) & \sqrt{a}\sin(t\sqrt{a}) \\ -\dfrac{1}{\sqrt{a}}\sin(t\sqrt{a}) & \cos(t\sqrt{a}) \end{pmatrix}. \tag{9.81}$$

Notice that $\mathbf{Y}^T(t)\mathbf{X}(t) = \mathbf{I}$ as expected. As the period of the system equals $T = 2\pi$, the Floquet matrix is

$$\mathbf{F} = \mathbf{X}(2\pi) = \begin{pmatrix} \cos(2\pi\sqrt{a}) & \dfrac{1}{\sqrt{a}}\sin(2\pi\sqrt{a}) \\ -\sqrt{a}\sin(2\pi\sqrt{a}) & \cos(2\pi\sqrt{a}) \end{pmatrix}. \tag{9.82}$$

In case of $a = 0$ and $q = 0$ we have

$$\mathbf{X}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Y}(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}. \tag{9.83}$$

The multipliers of the Floquet matrix are

$$\rho^a = \cos(2\pi\sqrt{a}) + i\sin(2\pi\sqrt{a}), \quad \rho^b = \cos(2\pi\sqrt{a}) - i\sin(2\pi\sqrt{a}). \tag{9.84}$$

Thus, the multipliers are situated on the unit circle in the complex plane when $a \geq 0$ and $q = 0$.

Due to condition (9.75), simple multipliers can not leave the unit circle. Hence, development of instability is possible only near the points, where two multipliers coincide. From (9.84) it is seen that this happens when

$$\sin(2\pi\sqrt{a}) = 0 \tag{9.85}$$

or, equivalently,

$$a = \frac{k^2}{4}, \quad k = 0, 1, 2, \ldots \tag{9.86}$$

At points (9.86) multipliers (9.84) are double and equal to

$$\rho^a = \rho^b = (-1)^k. \tag{9.87}$$

Points (9.86) are called the *resonance points.*

Using matrices (9.72), (9.80)–(9.83) in formulae (9.59) and (9.61), we find first and second order derivatives of the Floquet matrix at resonance points (9.86). General formulae for these derivatives and their values for the first three resonance points are given in Tables 9.1 and 9.2, where, for example, $f_{12,aq}$ denotes the second order derivative of the element f_{12} of the Floquet matrix with respect to a and q.

We know that the stability diagram for the Mathieu equation in the parameter space (a, q) is symmetric with respect to the a-axis. This symmetry reflects the phase shift $t \to t + \pi$, which changes q to $-q$ in equation (9.70). Thus, one might suspect that $\mathbf{F}(a, q) = \mathbf{F}(a, -q)$. But this is not the case which is seen from the fact that the first order derivative of $\mathbf{F}$ with

Table 9.1 First order derivatives of elements of the Floquet matrix at the resonance points.

a	0	1/4	1
$f_{11,a} = -\frac{\pi}{\sqrt{a}}\sin(2\pi\sqrt{a})$	$-2\pi^2$	0	0
$f_{12,a} = \frac{\pi}{a}\cos(2\pi\sqrt{a}) - \frac{1}{2a\sqrt{a}}\sin(2\pi\sqrt{a})$	$-\frac{4}{3}\pi^3$	-4π	π
$f_{21,a} = -\pi\cos(2\pi\sqrt{a}) - \frac{1}{2\sqrt{a}}\sin(2\pi\sqrt{a})$	-2π	π	$-\pi$
$f_{22,a} = -\frac{\pi}{\sqrt{a}}\sin(2\pi\sqrt{a})$	$-2\pi^2$	0	0
$f_{11,q} = 0$	0	0	0
$f_{12,q} = \frac{2}{(1-4a)\sqrt{a}}\sin(2\pi\sqrt{a})$	4π	2π	0
$f_{21,q} = \frac{2\sqrt{a}}{1-4a}\sin(2\pi\sqrt{a})$	0	$\frac{1}{2}\pi$	0
$f_{22,q} = 0$	0	0	0

respect to q is nonzero. We observe that the derivatives of the element f_{11} with respect to a and q are equal to the corresponding derivatives of the element f_{22}. As a result, the first term in conditions (9.77)–(9.79) vanishes.

Using the derivatives of $\mathbf{F}$, we can approximate the Floquet matrix for small values of q and Δa, where $\Delta a = a - k^2/4$ is the distance from the resonant value. As both the first and second order derivatives of $\mathbf{F}$ are known, we expand the Floquet matrix into the Taylor series

$$\begin{aligned}\mathbf{F}(a,q) &= \mathbf{F}\left(\frac{k^2}{4},0\right) + \frac{\partial \mathbf{F}}{\partial a}\Delta a + \frac{\partial \mathbf{F}}{\partial q}q \\ &\quad + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial a^2}\Delta a^2 + \frac{\partial^2 \mathbf{F}}{\partial a \partial q}\Delta a\, q + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial q^2}q^2 + \cdots,\end{aligned} \tag{9.88}$$

where all the derivatives are taken at $a = k^2/4$ and $q = 0$. Using expansion (9.88), we determine the Floquet matrix for small Δa and q and, thus, find the boundaries between the stability and instability domains in the parameter space.

Let us study the first three resonance points. According to (9.86) these points are $a = 0$, $a = \frac{1}{4}$, and $a = 1$. The stability boundaries are given by equation (9.79). This equation can be solved approximately using expansion (9.88) as shown below.

Table 9.2 Second order derivatives of elements of the Floquet matrix at the resonance points.

a	0	1/4	1
$f_{11,aa} = -\frac{\pi^2}{a}\cos(2\pi\sqrt{a}) + \frac{\pi}{2a\sqrt{a}}\sin(2\pi\sqrt{a})$	$\frac{4}{3}\pi^4$	$4\pi^2$	$-\pi^2$
$f_{12,aa} = -\frac{3\pi}{2a^2}\cos(2\pi\sqrt{a}) + \frac{3-4a\pi^2}{4a^2\sqrt{a}}\sin(2\pi\sqrt{a})$	$\frac{8}{15}\pi^5$	24π	$-\frac{3}{2}\pi$
$f_{21,aa} = -\frac{\pi}{2a}\cos(2\pi\sqrt{a}) + \frac{1+4a\pi^2}{4a\sqrt{a}}\sin(2\pi\sqrt{a})$	$\frac{8}{3}\pi^3$	2π	$-\frac{1}{2}\pi$
$f_{22,aa} = -\frac{\pi^2}{a}\cos(2\pi\sqrt{a}) + \frac{\pi}{2a\sqrt{a}}\sin(2\pi\sqrt{a})$	$\frac{4}{3}\pi^4$	$4\pi^2$	$-\pi^2$
$f_{11,aq} = 0$	0	0	0
$f_{12,aq} = \frac{2\pi}{(1-4a)a}\cos(2\pi\sqrt{a}) - \frac{1-12a}{(1-4a)^2a\sqrt{a}}\sin(2\pi\sqrt{a})$	$16\pi - \frac{8}{3}\pi^3$	-6π	$-\frac{2}{3}\pi$
$f_{21,aq} = \frac{2\pi}{1-4a}\cos(2\pi\sqrt{a}) + \frac{1+4a}{(1-4a)^2\sqrt{a}}\sin(2\pi\sqrt{a})$	4π	$\frac{1}{2}\pi$	$-\frac{2}{3}\pi$
$f_{22,aq} = 0$	0	0	0
$f_{11,qq} = \frac{\pi}{(4a-1)\sqrt{a}}\sin(2\pi\sqrt{a})$	$-2\pi^2$	$-\pi^2$	0
$f_{12,qq} = \frac{\pi}{(1-4a)a}\cos(2\pi\sqrt{a}) - \frac{1-16a+24a^2}{2(1-a)(1-4a)^2a\sqrt{a}}\sin(2\pi\sqrt{a})$	$11\pi - \frac{4}{3}\pi^3$	$-\pi$	$\frac{1}{6}\pi$
$f_{21,qq} = -\frac{\pi}{(1-4a)}\cos(2\pi\sqrt{a}) - \frac{1+8a^2}{2(1-a)(1-4a)^2\sqrt{a}}\sin(2\pi\sqrt{a})$	-2π	$\frac{1}{4}\pi$	$\frac{5}{6}\pi$
$f_{22,qq} = \frac{\pi}{(4a-1)\sqrt{a}}\sin(2\pi\sqrt{a})$	$-2\pi^2$	$-\pi^2$	0

The first resonance point is $a = 0$. Using the results given in Tables 9.1 and 9.2, we write equation (9.79) in the form

$$\begin{aligned}&\Big(2\pi - \tfrac{4}{3}\pi^3\Delta a + 4\pi q + \tfrac{4}{15}\pi^5(\Delta a)^2 \\ &\quad + \left(16\pi - \tfrac{8}{3}\pi^3\right)\Delta a\, q + \left(\tfrac{11}{2}\pi - \tfrac{2}{3}\pi^3\right) q^2 + \cdots\Big) \\ &\times\Big(- 2\pi\Delta a + \tfrac{4}{3}\pi^3(\Delta a)^2 + 4\pi\Delta a\, q - \pi q^2 + \cdots\Big) = 0,\end{aligned} \tag{9.89}$$

which implies that

$$\Delta a = -\frac{1}{2}q^2 + o(q^2). \tag{9.90}$$

Instability condition (9.77) is satisfied up to second order terms for

$$\Delta a < -\frac{1}{2}q^2. \tag{9.91}$$

The second resonance point is at $a = \frac{1}{4}$. Using the results of Tables 9.1 and 9.2 in equation (9.79), we get

$$\begin{aligned}&\Big(-4\pi\Delta a + 2\pi q + 12\pi(\Delta a)^2 - 6\pi\Delta a\, q - \tfrac{1}{2}\pi q^2 + \cdots\Big)\\ &\times\Big(\pi\Delta a + \tfrac{1}{2}\pi q + \pi(\Delta a)^2 + \tfrac{1}{2}\pi\Delta a\, q + \tfrac{1}{8}\pi q^2 + \cdots\Big) = 0,\end{aligned} \tag{9.92}$$

and thus

$$\Delta a = \pm\frac{1}{2}q - \frac{1}{8}q^2 + o(q^2). \tag{9.93}$$

Instability condition (9.77) is satisfied up to the second order terms for

$$\begin{aligned}&-\frac{1}{2}q - \frac{1}{8}q^2 < \Delta a < \frac{1}{2}q - \frac{1}{8}q^2 \qquad \text{when} \quad q > 0,\\ &\frac{1}{2}q - \frac{1}{8}q^2 < \Delta a < -\frac{1}{2}q - \frac{1}{8}q^2 \quad \text{when} \quad q < 0.\end{aligned} \tag{9.94}$$

The third resonance point is at $a = 1$. Using Tables 9.1 and 9.2 in the stability boundary equation (9.79), we get

$$\begin{aligned}&\Big(\pi\Delta a - \tfrac{3}{4}\pi(\Delta a)^2 - \tfrac{2}{3}\pi\Delta a\, q + \tfrac{1}{12}\pi q^2 + \cdots\Big)\\ &\times\Big(-\pi\Delta a - \tfrac{1}{4}\pi(\Delta a)^2 - \tfrac{2}{3}\pi\Delta a\, q + \tfrac{5}{12}\pi q^2 + \cdots\Big) = 0,\end{aligned} \tag{9.95}$$

and thus

$$\Delta a = \left(\frac{1}{6} \pm \frac{1}{4}\right) q^2 + o(q^2). \tag{9.96}$$

The instability domain is given by the relation

$$-\frac{1}{12}q^2 < \Delta a < \frac{5}{12}q^2. \tag{9.97}$$

Approximations of the instability domains (9.91), (9.94), and (9.97) are shown in Fig. 9.4, where the approximate boundaries are indicated by bold lines, and the exact instability domains found numerically are hatched. Note that the third resonance zone has a sharp tongue (cusp) touching the

a-axis. Evaluating higher order derivatives of the Floquet matrix, approximations of the instability domains can be obtained near other resonance points.

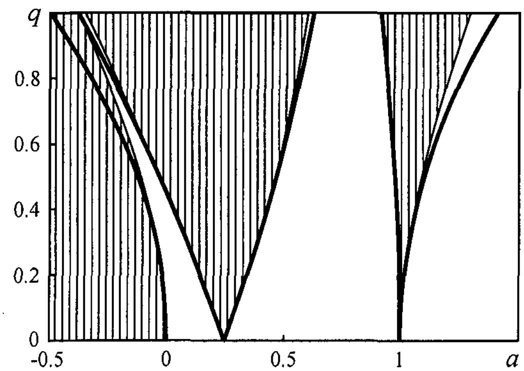


Fig. 9.4 Instability domains for the Mathieu equation and their approximations.

9.5 Sensitivity analysis of simple multipliers

Let us consider an eigenvalue problem

$$\mathbf{F}\mathbf{u} = \rho\mathbf{u} \tag{9.98}$$

for the Floquet matrix $\mathbf{F}(\mathbf{p})$ corresponding to multi-parameter periodic system (9.47). Let ρ be a simple eigenvalue (multiplier). Since $\mathbf{F}(\mathbf{p})$ is a smooth function of parameters, the multiplier $\rho(\mathbf{p})$ smoothly depends on $\mathbf{p}$. The left eigenvector corresponding to the multiplier ρ is defined by the equation

$$\mathbf{v}^T\mathbf{F} = \rho\mathbf{v}^T. \tag{9.99}$$

First and second order derivatives of the simple multiplier ρ with respect to parameters are given by Theorem 2.2 (page 32) in the form

$$\frac{\partial\rho}{\partial p_i} = \mathbf{v}^T\frac{\partial\mathbf{F}}{\partial p_i}\mathbf{u}\Big/(\mathbf{v}^T\mathbf{u}), \quad i = 1, \ldots, n, \tag{9.100}$$

$$\frac{\partial^2 \rho}{\partial p_i \partial p_j} = \mathbf{v}^T \left(\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} \mathbf{u} + \frac{\partial \mathbf{F}}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} + \frac{\partial \mathbf{F}}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \right.$$
$$\left. - \frac{\partial \rho}{\partial p_i} \frac{\partial \mathbf{u}}{\partial p_j} - \frac{\partial \rho}{\partial p_j} \frac{\partial \mathbf{u}}{\partial p_i} \right) \Big/ (\mathbf{v}^T \mathbf{u}), \quad i, j = 1, \ldots, n, \tag{9.101}$$

where the first order derivative of the eigenvector is given by

$$\frac{\partial \mathbf{u}}{\partial p_i} = (\mathbf{F} - \rho \mathbf{I} + \overline{\mathbf{v}} \mathbf{v}^T)^{-1} \left(\frac{\partial \rho}{\partial p_i} \mathbf{I} - \frac{\partial \mathbf{F}}{\partial p_i} \right) \mathbf{u}. \tag{9.102}$$

Using formula (9.66) for the first order derivative of the Floquet matrix and equations (9.98), (9.99), we obtain

$$\frac{\partial \rho}{\partial p_i} = \rho \mathbf{v}^T \left(\int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} d\tau + \mathbf{G}(0, \mathbf{p}) \frac{\partial T}{\partial p_i} \right) \mathbf{u} \Big/ (\mathbf{v}^T \mathbf{u}), \quad i = 1, \ldots, n. \tag{9.103}$$

Using expression (9.103), we can find derivatives of the absolute value of the multiplier $\rho = \alpha + i\omega$ as

$$\begin{aligned}
\frac{\partial |\rho|}{\partial p_i} &= \frac{\partial \sqrt{\alpha^2 + \omega^2}}{\partial p_i} \\
&= \frac{1}{\sqrt{\alpha^2 + \omega^2}} \left(\alpha \frac{\partial \alpha}{\partial p_i} + \omega \frac{\partial \omega}{\partial p_i} \right) \\
&= \frac{1}{|\rho|} \left(\operatorname{Re} \rho \operatorname{Re} \frac{\partial \rho}{\partial p_i} + \operatorname{Im} \rho \operatorname{Im} \frac{\partial \rho}{\partial p_i} \right) \\
&= \frac{1}{|\rho|} \operatorname{Re} \left(\overline{\rho} \frac{\partial \rho}{\partial p_i} \right).
\end{aligned} \tag{9.104}$$

Having derivatives of the multipliers, we know how the multipliers change with a variation of parameters. Since multipliers determine stability properties of a periodic system, their derivatives can be used for multi-parameter stability analysis and optimization problems under stability criteria.

Notice that it is convenient to choose the eigenvectors satisfying the following normalization condition

$$\mathbf{v}^T \mathbf{u} = 1. \tag{9.105}$$

Then the denominators in formulae (9.100), (9.101), and (9.103) are equal to 1.

9.6 Numerical applications

In this section, numerical examples involving the expressions for first and second order derivatives of the Floquet matrix are presented. First, a problem of optimizing the thickness distribution of an axially loaded beam, where the axial load is a periodic function of time, is considered. The objective of optimization is to make the beam more stable by changing the thickness distribution under the constant volume constraint. Transverse vibrations of the beam are described by a partial differential equation with boundary conditions. This equation is approximated by a finite degrees of freedom system using the finite difference method. As another numerical example, a problem of stabilization of a system described by the Carson-Cambi equation by changing problem parameters is studied.

9.6.1 *Axially loaded beam*

Consider a straight beam of length L, see Fig. 9.5. The beam is loaded by a periodic axial force $p\cos\omega t$. We assume that the beam is externally damped, and c is the external damping coefficient. Deflection of the beam at position x is $w(x,t)$. Then, the equation for transverse vibrations of the beam is

$$\rho A\,\frac{\partial^2 w}{\partial t^2} + c\,\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 w}{\partial x^2}\right) + p\cos\omega t\,\frac{\partial^2 w}{\partial x^2} = 0, \tag{9.106}$$

where E is Young's modulus, $I(x)$ is the cross-sectional moment of inertia, ρ is the density, and $A(x)$ is the cross-sectional area of the beam. We assume that the beam has a circular cross-section with the area and cross-sectional moment of inertia

$$A(x) = \pi r^2(x), \quad I(x) = \frac{\pi r^4(x)}{4}, \tag{9.107}$$

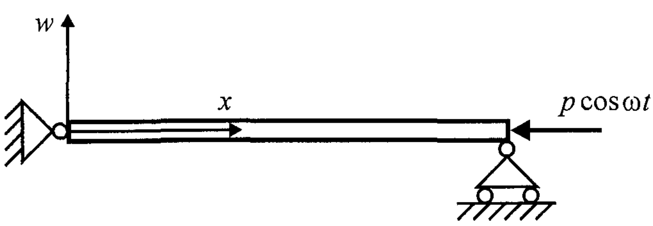


Fig. 9.5 Beam loaded by axial periodic force.

where $r(x)$ denotes the radius of the beam. As the beam is simply supported at both ends, the boundary conditions are

$$x = 0,\ L: \quad w = 0, \quad EI\frac{\partial^2 w}{\partial x^2} = 0. \tag{9.108}$$

Let

$$V = \int_0^L A(x)dx \tag{9.109}$$

be the volume of the beam. The critical buckling force and the first natural frequency of a simply supported uniform beam of circular cross-section are

$$p_{c,\,\mathrm{uniform}} = \frac{\pi E V^2}{4L^4}, \quad \omega_{c,\,\mathrm{uniform}} = \frac{\pi}{2L^2}\sqrt{\frac{EV\pi}{\rho L}}. \tag{9.110}$$

The dimensionless excitation amplitude q and excitation frequency Ω are given by

$$q = \frac{p}{p_{c,\,\mathrm{uniform}}}, \quad \Omega = \frac{\omega}{\omega_{c,\,\mathrm{uniform}}}. \tag{9.111}$$

The following dimensionless quantities are introduced

$$\tau = \omega t, \quad \zeta = \frac{x}{L}, \quad \nu = \frac{w}{L}, \quad R = r\sqrt{\frac{\pi L}{V}}, \quad \gamma = \frac{2cL^4}{\pi^2 V^2}\sqrt{\frac{\pi V}{\rho L E}}, \tag{9.112}$$

describing the time, beam coordinate, deflection, radius, and damping, respectively. Using expressions (9.110)–(9.112) in equation (9.106), we obtain

$$R^2\Omega^2\frac{\partial^2 \nu}{\partial \tau^2} + \gamma\Omega\frac{\partial \nu}{\partial \tau} + \frac{1}{\pi^4}\frac{\partial^2}{\partial \zeta^2}\left(R^4\frac{\partial^2 \nu}{\partial \zeta^2}\right) + \frac{q}{\pi^2}\cos\tau\frac{\partial^2 \nu}{\partial \zeta^2} = 0. \tag{9.113}$$

Boundary conditions (9.108) become

$$\zeta = 0,\ 1: \quad \nu = 0, \quad R^4\frac{\partial^2 \nu}{\partial \zeta^2} = 0. \tag{9.114}$$

Partial differential equation (9.113) with boundary conditions (9.114) describes transverse vibrations of the axially loaded beam in dimensionless coordinates. This equation can be reduced to a system of ordinary differential equations using the finite difference method [Iwatsubo *et al.* (1973)]. For this purpose, we consider a beam consisting of m elements of equal length, each element having a constant radius R_i, $i = 1, \ldots, m$. As a result, we obtain the finite degrees of freedom vibrational system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \tag{9.115}$$

where $\mathbf{M}$ and $\mathbf{B}$ are constant $m \times m$ matrices, and the matrix $\mathbf{C}(\tau)$ is the periodic function of time with the period $T = 2\pi$. Equation (9.115) can be written in the first order form

$$\dot{\mathbf{x}} = \mathbf{G}(\tau)\mathbf{x}, \tag{9.116}$$

where

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{B} \end{pmatrix}, \quad \mathbf{G}(\tau + T) = \mathbf{G}(\tau). \tag{9.117}$$

In this way the theory described in the previous sections can be applied to system (9.115). In the expressions for first and second order derivatives of the Floquet matrix, first and second order derivatives of the matrix $\mathbf{G}$ with respect to parameters are used. From equation (9.117) we see that this information can be obtained using corresponding derivatives of the matrix $\mathbf{C}$. Once derivatives of the Floquet matrix are found, we compute first and second order derivatives of simple multipliers.

9.6.2 *Optimization problem*

Fig. 9.6 shows the instability domains for the first two modes of the uniform beam, where the external damping coefficient is $\gamma = 0.2$. At the boundaries of the instability domains one of multipliers is equal to $\rho_c = -1$. The instability domain for the first mode occurs in the neighborhood of twice the first natural frequency of the beam, while the instability domain for the second mode occurs in the neighborhood of twice the second natural frequency of the beam. Due to the damping the instability domain starts at some positive value $q = q_c$, which is the minimal value of the excitation amplitude at which the system can be destabilized by the periodic axial force (minimum critical load level). The objective of optimization is to maximize the excitation amplitude q_c by changing the thickness distribution of the beam under the constraint of constant volume.

Let Φ denote the objective function in the optimization problem. That is, Φ is equal to the minimum value of the excitation amplitude on the

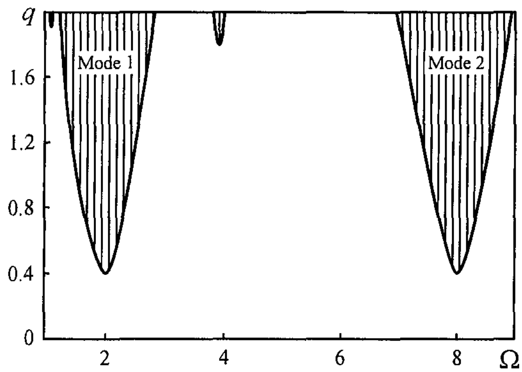


Fig. 9.6 Stability diagram for the uniform beam.

stability boundary

$$\Phi = q_c. \tag{9.118}$$

Objective function (9.118) makes sense only if the system is damped, because $\Phi = 0$ for the undamped system independently of the design parameters. The design is called optimal if Φ is maximized. Notice that the maximal Φ can be attained not only at one, but at two or more modes.

The objective function Φ is maximized by using the sequential linear programming and simplex method. In the optimization process, the sensitivities of the objective function Φ with respect to the design variables $R_1, \ldots, R_m$ are used. Let $\Omega = \Omega_c$ be the boundary frequency at $q = q_c$. At the point $(\Omega, q) = (\Omega_c, q_c)$ the multiplier ρ_c satisfies the relations

$$\rho_c = -1, \quad \frac{\partial \rho_c}{\partial \Omega} = 0, \quad \frac{\partial \rho_c}{\partial q} < 0. \tag{9.119}$$

The multiplier ρ_c depends on the excitation frequency Ω, the excitation amplitude q, and the design variables $R_1, \ldots, R_m$. A general variation of the multiplier takes the form

$$\delta \rho_c = \frac{\partial \rho_c}{\partial \Omega} \delta \Omega + \frac{\partial \rho_c}{\partial q} \delta q + \sum_{j=1}^{m} \frac{\partial \rho_c}{\partial R_j} \delta R_j. \tag{9.120}$$

This equation is valid due to differentiability of the simple multiplier at $(\Omega, q) = (\Omega_c, q_c)$. Let $\delta R_j = 0$ for all $j \neq i$. Using conditions (9.119) and

$\delta\rho_c = 0$ at (Ω_c, q_c), equation (9.120) yields

$$\frac{\partial \rho_c}{\partial q}\delta q_c + \frac{\partial \rho_c}{\partial R_i}\delta R_i = 0 \quad \text{at} \quad (\Omega, q) = (\Omega_c, q_c). \tag{9.121}$$

From equation (9.121) we find the sensitivity of the minimum critical load level with respect to the design parameter R_i as

$$\frac{\partial q_c}{\partial R_i} = -\left(\frac{\partial \rho_c/\partial R_i}{\partial \rho_c/\partial q}\right)_{(\Omega_c, q_c)}. \tag{9.122}$$

Let $\mathbf{F}$ be the Floquet matrix. Using expressions for derivatives of a simple multiplier, we have

$$\frac{\partial \rho_c}{\partial q} = \mathbf{v}^T\frac{\partial \mathbf{F}}{\partial q}\mathbf{u}\Big/(\mathbf{v}^T\mathbf{u}), \quad \frac{\partial \rho_c}{\partial R_i} = \mathbf{v}^T\frac{\partial \mathbf{F}}{\partial R_i}\mathbf{u}\Big/(\mathbf{v}^T\mathbf{u}), \tag{9.123}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the right and left eigenvectors of the matrix $\mathbf{F}$ corresponding to the multiplier $\rho_c = -1$. By substitution of expressions (9.123) into equation (9.122), the sensitivity of the minimum critical load level becomes

$$\frac{\partial q_c}{\partial R_i} = -\left(\frac{\mathbf{v}^T(\partial \mathbf{F}/\partial R_i)\mathbf{u}}{\mathbf{v}^T(\partial \mathbf{F}/\partial q)\mathbf{u}}\right)_{(\Omega_c, q_c)} \tag{9.124}$$

The sensitivity of the objective function is

$$\frac{\partial \Phi}{\partial R_i} = \frac{\partial q_c}{\partial R_i}, \tag{9.125}$$

where the sensitivities of the minimum critical load level are given by formula (9.124). If the design variables R_i are changed by amounts ΔR_i, the linear increment Δq_c of the minimum critical load level q_c is equal to

$$\Delta q_c = \sum_{i=1}^{m} \frac{\partial q_c}{\partial R_i}\Delta R_i. \tag{9.126}$$

The volume of the beam is kept constant during the optimization. This volume constraint becomes

$$\sum_{i=1}^{m} R_i^2 = m, \tag{9.127}$$

where m is the number of beam elements.

The problem of maximizing the objective function Φ is reduced to a sequence of linear optimal redesign problems, which are solved by using the simplex method. In each of the linear optimal redesign problems, the value of the objective function Φ is evaluated. This value can be determined by utilizing the fact that, at the point $(\Omega, q) = (\Omega_c, q_c)$, the multiplier ρ_c satisfies

$$\rho_c = -1, \quad \frac{\partial \rho_c}{\partial \Omega} = 0. \tag{9.128}$$

To find the minimal critical excitation amplitude q_c, the Newton-Raphson method is applied, which uses first and second order derivatives of the multiplier ρ_c with respect to q and Ω. In order to compute these derivatives, first and second derivatives of the Floquet matrix are evaluated. In the optimization process, first order derivatives of the objective function with respect to the design variables $R_1, \ldots, R_m$ are used, see equation (9.124).

9.6.3 *Results of optimization*

The results presented here are obtained for the beam divided into $m = 25$ elements. The design variables are constrained by

$$R_i \geq 0.5, \quad i = 1, \ldots, m, \tag{9.129}$$

and the uniform beam is taken as the initial design.

First, the beam is optimized with respect to the instability domain of the first mode (see Fig. 9.6) and the optimal design in Fig. 9.7 is obtained. Constraints (9.129) are active for the left and right elements of the beam. In Table 9.3 the values Ω_c and q_c for the instability domains for the first two modes of the beam in Fig. 9.7 are compared with those of the uniform beam. The objective function $\Phi^{mode\ 1}$ is 8.4% higher for the beam in Fig. 9.7 than for the uniform beam. When $\Phi^{mode\ 1}$ is maximized, the value of $\Phi^{mode\ 2}$ decreases below the value for the uniform beam; see Table 9.3.

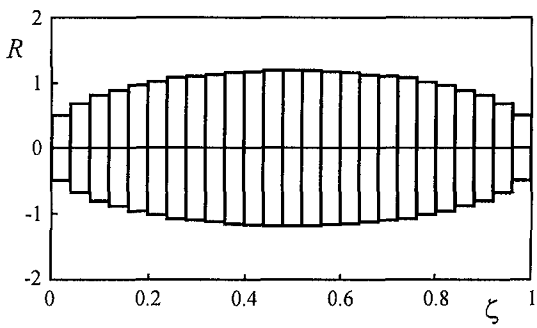


Fig. 9.7 Optimal design for the beam, where the objective function Φ is related to the instability domain for the first mode.

Table 9.3 Values of Ω_c and q_c for the instability domains of the first and second modes, when the beam is optimized with respect to the instability domain of the first mode.

Design	$\Omega_c^{mode\ 1}$	$\Phi^{mode\ 1} = q_c^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$\Phi^{mode\ 2} = q_c^{mode\ 2}$
Uniform	1.9826	0.3998	7.9542	0.4000
Optimal	2.1803	0.4332	7.9582	0.3850

Table 9.4 Values of Ω_c and q_c for the instability domains of the first and second modes, when the beam is optimized with respect to the instability domain of the second mode.

Design	$\Omega_c^{mode\ 1}$	$\Phi^{mode\ 1} = q_c^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$\Phi^{mode\ 2} = q_c^{mode\ 2}$
Uniform	1.9826	0.3998	7.9542	0.4000
Optimal	1.6877	0.3339	8.7200	0.4331

If the beam is optimized with respect to the instability domain for the second mode, the optimal design in Fig. 9.8 is obtained. Constraints (9.129) are active for the middle element of the beam. According to Table 9.4, the objective function $\Phi^{mode\ 2}$ is 8.3% higher for the beam in Fig. 9.8 than for the uniform beam. When $\Phi^{mode\ 2}$ is maximized, the value of $\Phi^{mode\ 1}$ decreases below the value for the uniform beam; see Table 9.4.

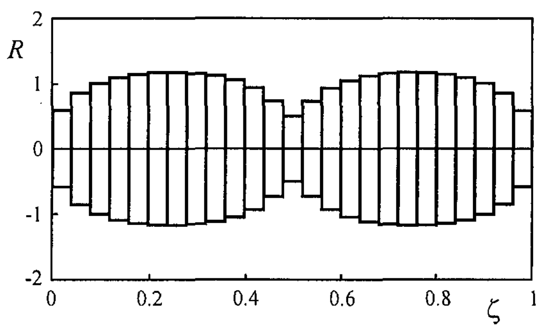


Fig. 9.8 Optimal design for the beam, where the objective function Φ is related to the instability domain for the second mode.

If the beam is optimized with respect to both the instability domains for the first and second modes, the optimal design in Fig. 9.9 is obtained. None of constraints (9.129) is active for the beam in Fig 9.9. According to

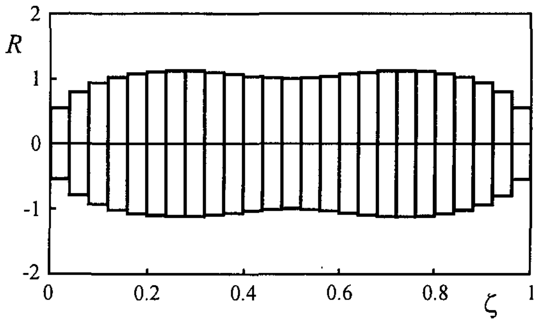


Fig. 9.9 Optimal design for the beam, where the objective function Φ is related to both the instability domains for the first and second modes.

Table 9.5 Values of Ω_c and q_c for the instability domains of the first and second modes, when the beam is optimized with respect to both the instability domains of the first and second modes.

Design	$\Omega_c^{mode\ 1}$	$\Phi^{mode\ 1} = q_c^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$\Phi^{mode\ 2} = q_c^{mode\ 2}$
Uniform	1.9826	0.3998	7.9542	0.4000
Optimal	2.0771	0.4177	8.4030	0.4177

Table 9.5, the objective functions $\Phi^{mode\ 1}$ and $\Phi^{mode\ 2}$ are raised by 4.5% and 4.4%, respectively, relative to the values for the uniform beam.

The optimal designs in Figs. 9.7 and 9.9 look similar to the optimal designs obtained in [Pedersen (1982–83)], where the volume of the beam is minimized while the first and two first natural frequencies of the beam, respectively, are kept constant. The optimal design in Fig. 9.8 looks similar to the optimal designs obtained in [Pedersen (1982–83)], where the second natural frequency is maximized while keeping the volume of the beam constant.

9.6.4 *Stabilization of unstable system: Carson-Cambi equation*

Sensitivity analysis of multipliers can be used for stabilization of an unstable periodic system by changing values of parameters. As an example, we consider the Carson-Cambi equation [Pedersen (1980)]

$$(1 + p_1 \cos t)\frac{d^2y}{dt^2} + p_2 y = 0, \quad |p_1| < 1, \tag{9.130}$$

with p_1 and p_2 as the problem parameters. If the system is unstable, then at least one multiplier is situated outside the unit circle, $|\rho| > 1$. To stabilize the system, all the multipliers outside the unit circle must be brought onto or inside the unit circle.

Let ρ be a multiplier situated outside the unit circle. The value of $|\rho|$ smoothly depends on the parameter vector $\mathbf{p} = (p_1, p_2)$. If parameter variations δp_i are chosen as

$$\delta p_i = -\alpha \frac{\partial |\rho|}{\partial p_i}, \quad i = 1, 2, \tag{9.131}$$

where α is a real positive constant, then we get

$$\delta |\rho| = -\alpha \sum_{i=1}^{2} \left(\frac{\partial |\rho|}{\partial p_i} \right)^2 \leq 0. \tag{9.132}$$

Thus, by choosing the change of parameters according to (9.131), the variation of $|\rho|$ is smaller or equal to zero, and the multiplier moves towards the origin. As a result, the system becomes "more" stable.

The necessary formulae for calculating the sensitivities are given by expressions (9.103), (9.104). The stability diagram of equation (9.130) is shown in Fig. 9.10, where the instability domain is hatched. Taking $\alpha = 0.001$, expression (9.131) is used to stabilize the system by changing the parameters starting at two different initial points $\mathbf{p} = (0.70, 0.19)$ and $\mathbf{p} = (0.70, 0.23)$. The paths shown in Fig. 9.10 follow the steepest descent of $|\rho|$. We emphasize that for systems with many degrees of freedom and

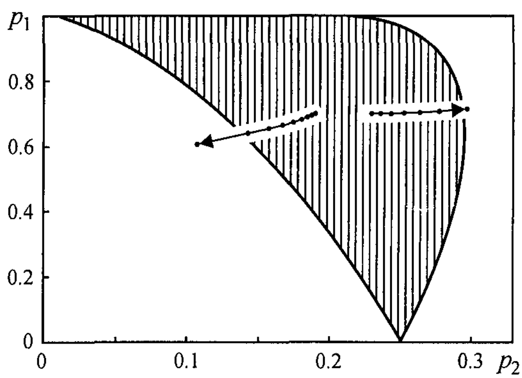


Fig. 9.10 Stability diagram for the Carson-Cambi equation, and the paths from the instability to stability domain for two initial points.

having multiple parameters the stabilization procedure will be the same as illustrated here.

9.7 Bifurcation of multipliers

Derivatives of the Floquet matrix with respect to parameters obtained in Section 9.3 allow using the bifurcation theory of Chapter 2 for analysis of multiple multipliers. In this section, we present the results for bifurcation of nonderogatory double and triple multipliers and semi-simple double multipliers.

9.7.1 *Nonderogatory double multiplier*

We assume that at $\mathbf{p} = \mathbf{p}_0$ the Floquet matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ possesses a double multiplier ρ_0 with a single eigenvector $\mathbf{u}_0$. The corresponding Jordan chain has length 2 and consists of the eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$ satisfying the equations

$$\begin{aligned} \mathbf{F}_0\mathbf{u}_0 &= \rho_0\mathbf{u}_0, \\ \mathbf{F}_0\mathbf{u}_1 &= \rho_0\mathbf{u}_1 + \mathbf{u}_0. \end{aligned} \tag{9.133}$$

Vectors of the corresponding left Jordan chain $\mathbf{v}_0$ and $\mathbf{v}_1$ satisfy the equations

$$\begin{aligned} \mathbf{v}_0^T\mathbf{F}_0 &= \rho_0\mathbf{v}_0^T, \\ \mathbf{v}_1^T\mathbf{F}_0 &= \rho_0\mathbf{v}_1^T + \mathbf{v}_0^T \end{aligned} \tag{9.134}$$

and normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_1 = 1, \quad \mathbf{v}_1^T\mathbf{u}_1 = 0. \tag{9.135}$$

Normalization conditions (9.135) determine the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ uniquely for a given Jordan chain $\mathbf{u}_0$ and $\mathbf{u}_1$.

Let us consider perturbation of the parameter vector along a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, where $\varepsilon \geq 0$ is a small real perturbation parameter. The curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starts at $\mathbf{p}_0 = \mathbf{p}(0)$ and has the initial direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ evaluated at $\varepsilon = 0$. By Theorem 2.3 (page 37), bifurcation of the double nonderogatory eigenvalue ρ_0 along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ is described by the expansion

$$\rho = \rho_0 + \varepsilon^{1/2}\rho_1 + \varepsilon\rho_2 + \cdots, \tag{9.136}$$

where the coefficients ρ_1 and ρ_2 are given by the formulae

$$\begin{aligned}
\rho_1 &= \pm\sqrt{\sum_{i=1}^{n}\left(\mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_i}\mathbf{u}_0\right)e_i},\\
\rho_2 &= \frac{1}{2}\sum_{i=1}^{n}\left(\mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_i}\mathbf{u}_1+\mathbf{v}_1^T\frac{\partial\mathbf{F}}{\partial p_i}\mathbf{u}_0\right)e_i.
\end{aligned}\tag{9.137}$$

Here e_i denotes the ith component of the direction vector $\mathbf{e}$. Recall that the second expression in (9.137) for ρ_2 is valid only if $\rho_1 \neq 0$.

Using expression (9.66) for first order derivatives of the Floquet matrix and equations (9.133), (9.134) in (9.137), we find

$$\rho_1 = \pm\sqrt{(\mathbf{g}_1,\mathbf{e})},\quad \rho_2 = \frac{1}{2}(\mathbf{g}_2,\mathbf{e}),\tag{9.138}$$

where $\mathbf{g}_1 = (g_1^1,\ldots,g_1^n)$ and $\mathbf{g}_2 = (g_2^1,\ldots,g_2^n)$ are vectors with the components

$$\begin{aligned}
g_1^i &= \rho_0\mathbf{v}_0^T\mathbf{W}_i\mathbf{u}_0,\\
g_2^i &= \mathbf{v}_0^T\mathbf{W}_i\mathbf{u}_0+\rho_0(\mathbf{v}_0^T\mathbf{W}_i\mathbf{u}_1+\mathbf{v}_1^T\mathbf{W}_i\mathbf{u}_0),
\end{aligned}\tag{9.139}$$

and $\mathbf{W}_i$ is the $m\times m$ real matrix

$$\mathbf{W}_i = \int_0^T \mathbf{Y}^T\frac{\partial\mathbf{G}}{\partial p_i}\mathbf{X}d\tau+\mathbf{G}(0,\mathbf{p}_0)\frac{\partial T}{\partial p_i}.\tag{9.140}$$

The scalar product in (9.138) is given by $(\mathbf{g}_i,\mathbf{e}) = g_i^1e_1+\cdots+g_i^ne_n$, $i=1,2$. The vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are real if the multiplier ρ_0 is real.

9.7.2 *Nonderogatory triple multiplier*

Let us consider a triple multiplier ρ_0 of the matrix $\mathbf{F}_0$ with the right and left Jordan chains satisfying the equations

$$\begin{aligned}
\mathbf{F}_0\mathbf{u}_0 &= \rho_0\mathbf{u}_0, & \mathbf{v}_0^T\mathbf{F}_0 &= \rho_0\mathbf{v}_0^T,\\
\mathbf{F}_0\mathbf{u}_1 &= \rho_0\mathbf{u}_1+\mathbf{u}_0, & \mathbf{v}_1^T\mathbf{F}_0 &= \rho_0\mathbf{v}_1^T+\mathbf{v}_0^T,\\
\mathbf{F}_0\mathbf{u}_2 &= \rho_0\mathbf{u}_2+\mathbf{u}_1, & \mathbf{v}_2^T\mathbf{F}_0 &= \rho_0\mathbf{v}_2^T+\mathbf{v}_1^T
\end{aligned}\tag{9.141}$$

with the normalization conditions

$$\mathbf{v}_0^T\mathbf{u}_2 = 1,\quad \mathbf{v}_1^T\mathbf{u}_2 = \mathbf{v}_2^T\mathbf{u}_2 = 0.\tag{9.142}$$

We consider perturbation of the parameter vector along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ satisfying conditions $\mathbf{p}_0 = \mathbf{p}(0)$ and $\mathbf{e} = d\mathbf{p}/d\varepsilon$ at $\varepsilon = 0$. Bifurcation of a nonderogatory triple eigenvalue was studied in Examples 2.8 and 2.9 (pages 52 and 53). Additionally, we use equations (9.141) and formula (9.66) for first order derivatives of the Floquet matrix.

Let us define the vectors $\mathbf{h}_j = (h_j^1, \ldots, h_j^n)$, $j = 1, 2, 3$, with the components

$$\begin{aligned} h_1^i &= \rho_0 \mathbf{v}_0^T \mathbf{W}_i \mathbf{u}_0, \\ h_2^i &= \mathbf{v}_0^T \mathbf{W}_i \mathbf{u}_0 + \rho_0(\mathbf{v}_0^T \mathbf{W}_i \mathbf{u}_1 + \mathbf{v}_1^T \mathbf{W}_i \mathbf{u}_0), \\ h_3^i &= \mathbf{v}_0^T \mathbf{W}_i \mathbf{u}_1 + \mathbf{v}_1^T \mathbf{W}_i \mathbf{u}_0 + \rho_0(\mathbf{v}_0^T \mathbf{W}_i \mathbf{u}_2 + \mathbf{v}_1^T \mathbf{W}_i \mathbf{u}_1 + \mathbf{v}_2^T \mathbf{W}_i \mathbf{u}_0), \end{aligned} \tag{9.143}$$

where the matrix $\mathbf{W}_i$ is given by (9.140). The vectors $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$ are real if ρ_0 is real. Then, for the direction vector $\mathbf{e}$ satisfying the nondegeneracy condition

$$(\mathbf{h}_1, \mathbf{e}) \neq 0, \tag{9.144}$$

bifurcation of the triple multiplier into three simple multipliers is given by the expansion

$$\rho = \rho_0 + \varepsilon^{1/3}\rho_1 + \varepsilon^{2/3}\rho_2 + \varepsilon\rho_3 + o(\varepsilon), \tag{9.145}$$

where

$$\rho_1 = \sqrt[3]{(\mathbf{h}_1, \mathbf{e})}, \quad \rho_2 = \frac{1}{3\rho_1}(\mathbf{h}_2, \mathbf{e}), \quad \rho_3 = \frac{1}{3}(\mathbf{h}_3, \mathbf{e}), \tag{9.146}$$

and ρ_1 takes three different complex values of the cubic root.

If the direction vector $\mathbf{e}$ satisfies the degeneracy conditions

$$(\mathbf{h}_1, \mathbf{e}) = 0, \quad (\mathbf{h}_2, \mathbf{e}) \neq 0, \tag{9.147}$$

then bifurcation of the triple multiplier is given by the expansion

$$\rho = \rho_0 + \varepsilon^{1/2}\mu_1 + \varepsilon\mu_2 + o(\varepsilon) \tag{9.148}$$

for the first two multipliers and the expansion

$$\rho = \rho_0 + \varepsilon\nu_1 + o(\varepsilon) \tag{9.149}$$

for the third multiplier, where

$$\nu_1 = \frac{1}{(\mathbf{h}_2, \mathbf{e})} \mathbf{v}_0^T (\mathbf{F}_1 \mathbf{G}_2^{-1} \mathbf{F}_1 - \mathbf{F}_2) \mathbf{u}_0,$$
$$\mu_1 = \pm\sqrt{(\mathbf{h}_2, \mathbf{e})}, \quad \mu_2 = \frac{1}{2}(-\nu_1 + (\mathbf{h}_3, \mathbf{e})), \tag{9.150}$$

and the matrices $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{G}_2$ are

$$\mathbf{F}_1 = \sum_{i=1}^{n} \frac{\partial \mathbf{F}}{\partial p_i} e_i, \quad \mathbf{F}_2 = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \mathbf{F}}{\partial p_i} d_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} e_i e_j,$$
$$\mathbf{G}_2 = \mathbf{F}_0 - \rho_0 \mathbf{I} + \overline{\mathbf{v}}_0 \mathbf{v}_2^T; \tag{9.151}$$

$d_1, \ldots, d_n$ are components of the second order derivative vector $\mathbf{d} = d^2\mathbf{p}/d\varepsilon^2$ taken at $\varepsilon = 0$.

9.7.3 *Semi-simple double multiplier*

Finally, let us consider a double multiplier ρ_0, which has two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying the equations

$$\mathbf{F}_0 \mathbf{u}_1 = \rho_0 \mathbf{u}_1, \quad \mathbf{F}_0 \mathbf{u}_2 = \rho_0 \mathbf{u}_2. \tag{9.152}$$

The left eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are uniquely determined by the equations

$$\mathbf{v}_1^T \mathbf{F}_0 = \rho_0 \mathbf{v}_1^T, \quad \mathbf{v}_2^T \mathbf{F}_0 = \rho_0 \mathbf{v}_2^T \tag{9.153}$$

and the normalization conditions

$$\mathbf{v}_i^T \mathbf{u}_j = \delta_{ij}, \quad i, j = 1, 2. \tag{9.154}$$

Consider perturbation of the parameters along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ starting at $\mathbf{p}_0 = \mathbf{p}(0)$ with the initial direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$. By Theorem 2.6 (page 56), bifurcation of a semi-simple double multiplier ρ_0 is described by the expansion

$$\rho = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \cdots, \tag{9.155}$$

where two values of the coefficient ρ_1 are eigenvalues of the matrix

$$\rho_0 \sum_{i=1}^{n} \begin{pmatrix} \mathbf{v}_1^T \mathbf{W}_i \mathbf{u}_1 & \mathbf{v}_1^T \mathbf{W}_i \mathbf{u}_2 \\ \mathbf{v}_2^T \mathbf{W}_i \mathbf{u}_1 & \mathbf{v}_2^T \mathbf{W}_i \mathbf{u}_2 \end{pmatrix} e_i, \tag{9.156}$$

and the matrix $\mathbf{W}_i$ is given by expression (9.140).

9.8 Numerical examples on interaction of multipliers

Let us consider the system of the form [Hansen (1985)]

$$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + \begin{pmatrix} 2c\omega & 0 \\ 0 & 2c\omega \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}$$
$$+ \begin{pmatrix} 12 + 24q\cos 2t & 4 + 8q\cos 2t \\ 4 + 4q\cos 2t & 20 + 16q\cos 2t \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.157}$$

Stability diagram for this system in the frequency-amplitude parameter space (ω, q) can be found numerically by evaluation of the Floquet matrix and checking the stability condition for its multipliers. The calculations were carried out for the ranges of parameters $2.5 \le \omega \le 5.5$ and $0 \le q \le 0.8$, where stability was checked at $200 \times 200 = 40000$ points.

Fig. 9.11 shows the stability diagram when the system is undamped, that is, $c = 0$. The instability domain shown in Fig. 9.11 (hatched) contains three parts marked 1, 2, and 3. For small values of q, these three regions can be clearly distinguished. For larger values of the excitation amplitude q, they unite to one large instability domain. Fig. 9.12 shows the stability diagram when the damping with the coefficient $c = 0.1$ is included. The length of each unstable frequency interval goes to zero as the amplitude parameter q attains some positive value. Now we will show how the multipliers move on the complex plane when the boundaries between the stability and instability domains are crossed.

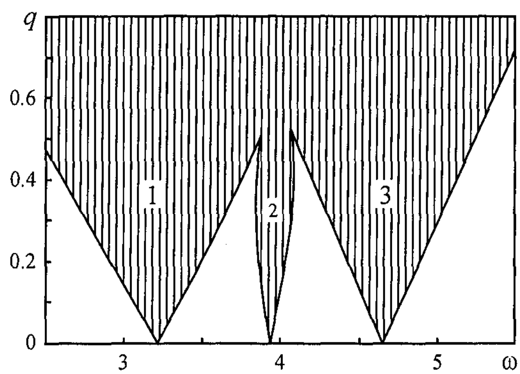


Fig. 9.11 Stability diagram on frequency-amplitude plane for undamped system.

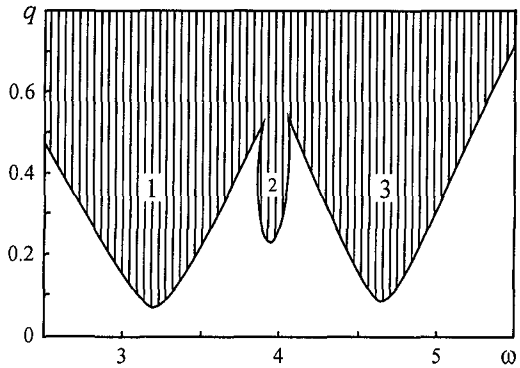


Fig. 9.12 Stability diagram on frequency-amplitude plane for system with damping.

9.8.1 *Parametric resonance*

By keeping the excitation amplitude fixed, $q = 0.4$, and varying the excitation frequency ω from 2.5 to 3.0, the left boundary of the first instability domain is crossed. In Figs. 9.13 and 9.14 the traces of four multipliers are plotted in the complex plane. In Fig. 9.13 the system is undamped. In Fig. 9.14 the damping is included.

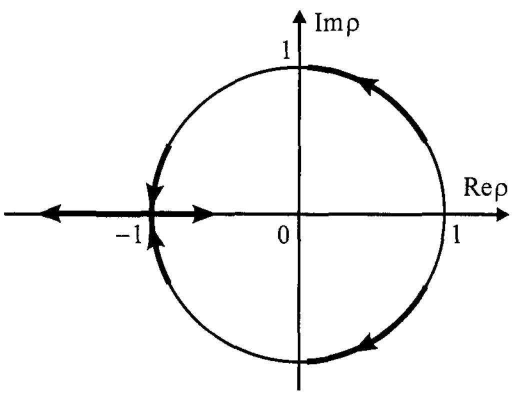


Fig. 9.13 Traces of multipliers for undamped system in case of parametric resonance.

In case of no damping, multipliers of the stable system lie on the unit circle. In Fig. 9.13 two complex conjugate multipliers situated on the unit circle collide at the point $\rho = -1$. This happens on the stability boundary. At this point there exists a double multiplier with a single eigenvector. Two multipliers interact strongly and leave the unit circle along the real axis, see

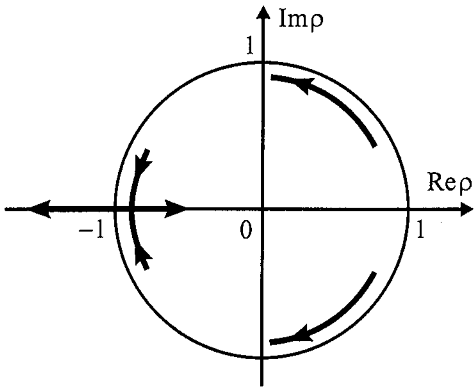


Fig. 9.14 Traces of multipliers for system with damping in case of parametric resonance.

Section 2.6. When the stability boundary is crossed, one multiplier becomes greater than -1 and another multiplier gets smaller than -1. Two other multipliers stay on the unit circle.

When the first part of the instability domain is left by crossing the right stability boundary, the change from instability to stability occurs. This change is caused by the return of two real multipliers to the point $\rho = -1$. The multipliers collide at $\rho = -1$, interact strongly, and branch out entering the unit circle. By passing the third part of the instability domain, the same process is observed.

If the double multiplier $\rho = -1$ is semi-simple (there are two linearly independent eigenvectors), two multipliers just pass each other and stay on the unit circle. Hence, the system is stable at both sides of the interaction point. This is termed as weak interaction, see Section 2.9. Weak interaction happens when the tip of the stability domain is passed with $q = 0$, see Fig. 9.11.

When damping is included, the multipliers move as shown in Fig. 9.14. Now multipliers of a stable system are situated inside the unit circle. Two multipliers interact strongly, but the interaction occurs inside the unit circle. The change from stability to instability occurs by passing of a simple real multiplier through the point $\rho = -1$.

The type of instability described by Figs. 9.13 and 9.14 is called the *parametric resonance*. In Figs. 9.13 and 9.14 the parametric resonance is related to the multiplier $\rho = -1$ and called the *subharmonic parametric resonance*. The parametric resonance can also be related to the multiplier $\rho = 1$. Then it is called the *harmonic parametric resonance*.

9.8.2 *Combination resonance*

By keeping the excitation amplitude fixed, $q = 0.4$, and varying the excitation frequency ω from 3.8 to 3.9, the left boundary of the second instability domain is crossed. In Figs. 9.15 and 9.16 the traces of four multipliers are plotted on the complex plane. Fig. 9.15 corresponds to no damping. In Fig. 9.16 the damping is included.

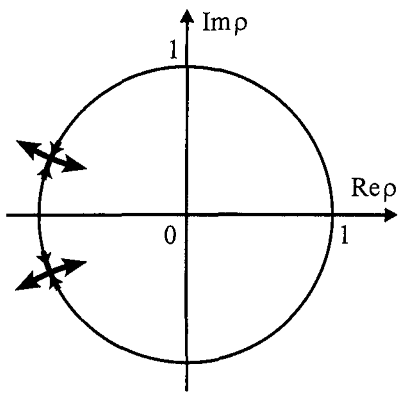


Fig. 9.15 Traces of multipliers for undamped system in case of combination resonance.

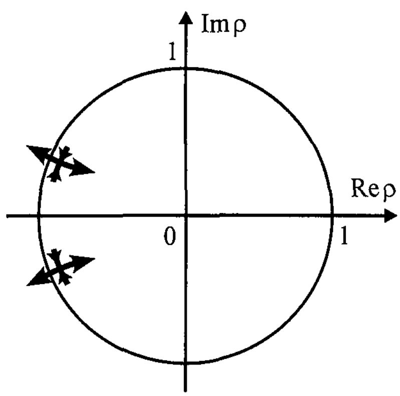


Fig. 9.16 Traces of multipliers for system with damping in case of combination resonance.

In Fig. 9.15 two complex conjugate pairs of multipliers situated on the unit circle coincide at the points ρ and $\overline{\rho}$. This happens on the boundary of the second instability domain. At this point, a complex conjugate pair of double multipliers with Jordan chains of second order appears. Then the multipliers branch out and leave the unit circle (strong interaction). In the instability domain two complex multipliers lie outside the unit circle and two other multipliers lie inside the unit circle. As the instability domain is left by crossing the right boundary, the multipliers collide, interact strongly, and branch out entering the unit circle again.

If the double complex conjugate multipliers are semi-simple, the multipliers pass each other and stay on the unit circle (weak interaction). This happens when the tip of the second instability domain is passed at $q = 0$, see Fig. 9.11.

When damping is included, the multipliers move as shown in Fig. 9.16. The multipliers interact as in the case of no damping. But now the strong interaction occurs inside the unit circle. The change from stability to instability occurs by passing of a complex conjugate pair of simple multipliers through the unit circle. The type of instability described by Figs. 9.15 and 9.16 is called the *combination resonance*.

Chapter 10

Stability Boundary of General Periodic System

Finding the stability and instability domains in the parameter space is the main problem for the stability theory of periodic systems. Usually this problem is solved by constructing the stability boundary and specifying the type of instability for each part of the boundary. We know that the stability boundary in the parameter space consists of smooth surfaces, but may have singularities. The simplest singularities are angles appearing in two-dimensional parameter space, but more complicated singularities can occur in multi-parameter spaces. These singularities reflect physical properties of the underlying system, and their study requires special treatment based on the bifurcation theory approach. Naturally, we are mostly interested in analyzing generic (typical) singularities of the stability boundary.

In this chapter, following [Mailybaev and Seyranian (2000a); Mailybaev and Seyranian (2000b)] we describe the stability boundary for a general linear system of ordinary differential equations with periodic coefficients dependent on real parameters. Regular part of the stability boundary corresponding to parametric and combination resonances is described, and its first and second order approximations are derived using derivatives of simple multipliers. Classification of generic singularities of the stability boundary for two- and three-parameter periodic systems is given, and the formulae for first order approximations of the stability domain near the singularities are derived. These formulae have a constructive form and require the information only at the singularity point: values of multipliers, eigenvectors, matriciants, and derivatives of the system matrix with respect to parameters. The suggested approach is useful for numerical construction and analysis of the stability boundary, and helps avoiding numerical difficulties associated with singularities.

As numerical application, we consider the stability problem for vibrations of two elastically attached pipes conveying pulsating flow. In the three-parameter space (mean velocity of the flow, amplitude and frequency of pulsations) we analyze the stability boundary, find and approximate the dihedral angle singularity.

10.1 Stability boundary of periodic system

Let us consider a linear periodic system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \tag{10.1}$$

where $\mathbf{x} \in \mathbb{R}^m$ is a state vector, $\mathbf{G}(t)$ is a nonsymmetric $m \times m$ matrix, whose components are continuous periodic functions of time t with a period T, $\mathbf{G}(t+T) = \mathbf{G}(t)$. According to the Floquet theory, we introduce the $m \times m$ matrix (the matriciant) $\mathbf{X}(t)$ as a solution of the equation

$$\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X} \tag{10.2}$$

with the initial condition

$$\mathbf{X}(0) = \mathbf{I}. \tag{10.3}$$

The Floquet matrix $\mathbf{F}$ is defined as

$$\mathbf{F} = \mathbf{X}(T). \tag{10.4}$$

Stability of system (10.1) is determined by the multipliers (eigenvalues) $\rho_1, \ldots, \rho_m$ of the Floquet matrix. If all the multipliers are inside the unit circle on the complex plane, $|\rho| < 1$, system (10.1) is asymptotically stable. If at least one multiplier lies outside the unit circle, $|\rho| > 1$, then the system is unstable. The transition from stability to instability occurs, when one or several multipliers cross the unit circle, $|\rho| = 1$, while others remain inside the unit circle.

Let us consider multi-parameter system (10.1), where the matrix $\mathbf{G} = \mathbf{G}(t, \mathbf{p})$ and the period $T = T(\mathbf{p})$ are smooth functions of the vector of parameters $\mathbf{p} \in \mathbb{R}^n$. In this case the Floquet matrix $\mathbf{F} = \mathbf{F}(\mathbf{p})$ corresponding to this system smoothly depends on the parameters. The stability criterion divides the parameter space into the stability and instability domains. We define the stability domain as a set of points $\mathbf{p}$ such that the corresponding system is asymptotically stable ($|\rho| < 1$ for all the multipliers of $\mathbf{F}(\mathbf{p})$). Therefore, the boundary of the stability domain is represented by the points $\mathbf{p}$ for which the matrix $\mathbf{F}(\mathbf{p})$ has multipliers on the unit circle, $|\rho| = 1$, while other multipliers are inside the unit circle.

There are two basic ways how a general periodic system can lose stability. The first way, called the parametric resonance, corresponds to a simple real multiplier leaving the unit circle. This multiplier can cross the unit circle at the points $\rho = 1$ or $\rho = -1$. These cases correspond to the harmonic and subharmonic parametric resonances, respectively, see Fig. 10.1a,b. The second way, called the combination resonance, corresponds to a complex conjugate pair of simple multipliers crossing the unit circle at the points $\rho = \exp(\pm i\omega)$, $0 < \omega < \pi$, see Fig. 10.1c. The parametric and combination resonances represent regular parts of the stability boundary. Other ways for losing stability of the system, for example, when the multipliers on the unit circle are multiple, correspond to singular points of the stability boundary.

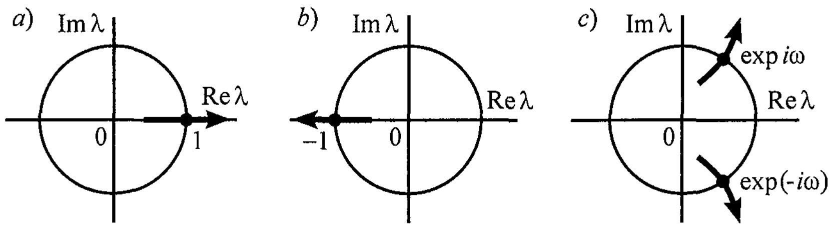


Fig. 10.1 Loss of stability for a general periodic system: a) harmonic parametric resonance, b) subharmonic parametric resonance, and c) combination resonance.

Let $\mathbf{p}_0$ be a regular point of the stability boundary, and $\rho(\mathbf{p})$ be a simple multiplier crossing the unit circle at $\mathbf{p} = \mathbf{p}_0$. Since the multipliers are complex conjugate, in case of combination resonance we consider only the multiplier having positive imaginary part. The stability boundary in the neighborhood of the point $\mathbf{p}_0$ is given by the equation

$$|\rho(\mathbf{p})| = 1, \tag{10.5}$$

and the stability domain is determined by the inequality

$$|\rho(\mathbf{p})| < 1. \tag{10.6}$$

Since a simple multiplier smoothly depends on parameters, equation (10.5) defines a smooth hypersurface in the parameter space, provided that the real gradient vector

$$\mathbf{f} = \left(\frac{\partial|\rho|}{\partial p_1}, \ldots, \frac{\partial|\rho|}{\partial p_n}\right) \tag{10.7}$$

evaluated at $\mathbf{p}_0$ is nonzero. Using results of Section 9.5, we find components of the gradient vector $\mathbf{f} = (f_1, \ldots, f_n)$ in the form

$$f_j = \frac{\partial|\rho|}{\partial p_j} = \frac{1}{|\rho_0|}\mathrm{Re}\left(\overline{\rho}_0 \frac{\partial \rho}{\partial p_j}\right), \quad j = 1, \ldots, n, \tag{10.8}$$

where $\rho_0 = \rho(\mathbf{p}_0)$ and the derivatives are evaluated at $\mathbf{p}_0$. In case of parametric resonance formulae (10.8) become

$$\begin{aligned} \rho_0 = 1: &\quad f_j = \frac{\partial \rho}{\partial p_j}, \\ \rho_0 = -1: &\quad f_j = -\frac{\partial \rho}{\partial p_j}, \end{aligned} \tag{10.9}$$

and in case of combination resonance we have

$$\rho_0 = \exp i\omega: \quad f_j = \mathrm{Re}\,\frac{\partial \rho}{\partial p_j}\cos\omega + \mathrm{Im}\,\frac{\partial \rho}{\partial p_j}\sin\omega. \tag{10.10}$$

Derivatives of the multiplier $\rho(\mathbf{p})$ at $\mathbf{p}_0$ are given by the formula

$$\frac{\partial \rho}{\partial p_j} = \rho_0 \mathbf{v}^T \left(\int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}dt + \mathbf{G}(0, \mathbf{p}_0)\frac{\partial T}{\partial p_j}\right)\mathbf{u}\Big/(\mathbf{v}^T\mathbf{u}), \tag{10.11}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the right and left eigenvectors of the matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ corresponding to the multiplier ρ_0, $\mathbf{X}(t)$ is the matriciant evaluated at $\mathbf{p}_0$, and $\mathbf{Y}(t) = (\mathbf{X}^{-1}(t))^T$ is the matrix satisfying adjoint equations (9.50), (9.51). The first order approximation of stability condition (10.6) is

$$(\mathbf{f}, \Delta\mathbf{p}) < 0, \quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0, \tag{10.12}$$

which shows that the vector $\mathbf{f}$ is normal to the stability boundary and directed into the instability domain, see Fig. 10.2 (the stability domain is denoted by S).

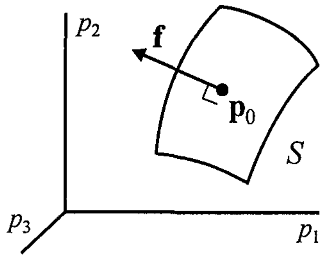


Fig. 10.2 Regular part of the stability boundary and its normal vector $\mathbf{f}$.

Second order derivatives of $|\rho|$ can be obtained as follows

$$\frac{\partial^2|\rho|}{\partial p_j \partial p_k} = \frac{\partial^2}{\partial p_j \partial p_k}\sqrt{(\mathrm{Re}\,\rho)^2 + (\mathrm{Im}\,\rho)^2}$$

$$= \frac{1}{|\rho_0|}\mathrm{Re}\left(\bar{\rho}_0 \frac{\partial^2 \rho}{\partial p_j \partial p_k} + \overline{\frac{\partial \rho}{\partial p_j}}\frac{\partial \rho}{\partial p_k}\right) - \frac{1}{|\rho_0|^3}\mathrm{Re}\left(\bar{\rho}_0 \frac{\partial \rho}{\partial p_j}\right)\mathrm{Re}\left(\bar{\rho}_0 \frac{\partial \rho}{\partial p_k}\right), \tag{10.13}$$

where first and second order derivatives of the multiplier $\rho(\mathbf{p})$ at $\mathbf{p}_0$ are given by formulae (9.100)–(9.102). Using expressions (10.8) and (10.13), we obtain the second order approximation for the absolute value of the multiplier as

$$|\rho(\mathbf{p})| = 1 + \sum_{j=1}^{n} \frac{\partial|\rho|}{\partial p_j}\Delta p_j + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2|\rho|}{\partial p_j \partial p_k}\Delta p_j \Delta p_k + o(\|\Delta \mathbf{p}\|^2), \tag{10.14}$$

where $\Delta\mathbf{p} = (\Delta p_1, \ldots, \Delta p_n) = \mathbf{p} - \mathbf{p}_0$. Substituting expansion (10.14) into conditions (10.5) and (10.6), we obtain the second order approximations of the stability boundary and stability domain in the neighborhood of the point $\mathbf{p}_0$.

Theorem 10.1 *Let $\mathbf{p}_0$ be a regular point of the stability boundary of periodic system (10.1). If the vector $\mathbf{f}$ given by expressions (10.9) or (10.10) is nonzero, then the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$, and $\mathbf{f}$ is the normal vector to the stability boundary directed into the instability domain. The second order approximation of the stability domain is given by (10.6), (10.14).*

10.2 Singularities of stability boundary

Singular points of the stability boundary differ by multipliers lying on the unit circle and their multiplicities (Jordan structures). We denote type of a point on the stability boundary by a product of multipliers lying on the unit circle in powers equal to sizes of corresponding Jordan blocks. For example, the type $1\exp(\pm i\omega_1)\exp(\pm i\omega_2)$ corresponds to a point $\mathbf{p}$ associated with the simple multipliers $\rho = 1$, $\rho = \exp(\pm i\omega_1)$, and $\rho = \exp(\pm i\omega_2)$ $(0 < \omega_1 < \omega_2 < \pi)$; the type $(-1)\,1^2$ corresponds to a point $\mathbf{p}$ associated with the simple multiplier $\rho = -1$ and the double nonderogatory multiplier $\rho = 1$ (with the Jordan chain of length 2). Other multipliers are assumed to be inside the unit circle.

The number of different types for singular points of the stability boundary, which can appear for a particular multi-parameter system, is very large. Among them, structurally stable types are the most important. Points of these types appear in the case of general position and cannot be removed by small changes of the system. Selection of generic (structurally stable) cases is a complicated task in case of periodic systems, since these systems are represented by periodic matrix-functions $\mathbf{G}(t, \mathbf{p})$. In order to avoid mathematical difficulties related to consideration of functional spaces, we consider a multi-parameter Floquet matrix $\mathbf{F}(\mathbf{p})$ corresponding to a periodic system. Then, we relate the notion of general position to the matrix family $\mathbf{F}(\mathbf{p})$ rather than to the family of periodic matrix-functions $\mathbf{G}(t, \mathbf{p})$.

For convenience, we introduce short notation for several specific types

$$B_1[1], \quad B_2[-1], \quad B_3[\exp(\pm i\omega)], \tag{10.15}$$

$$C_1[1^2], \quad C_2[(-1)^2], \tag{10.16}$$

$$D_1[1^3], \quad D_2[(-1)^3], \quad D_3[(\exp(\pm i\omega))^2], \tag{10.17}$$

and their combinations

$$\begin{gathered} B_{12}[1(-1)], \quad B_{13}[1\exp(\pm i\omega)], \\ B_{23}[(-1)\exp(\pm i\omega)], \quad B_{33}[\exp(\pm i\omega_1)\exp(\pm i\omega_2)], \end{gathered} \tag{10.18}$$

$$\begin{gathered} B_{123}[1(-1)\exp(\pm i\omega)], \\ B_{133}[1\exp(\pm i\omega_1)\exp(\pm i\omega_2)], \quad B_{233}[(-1)\exp(\pm i\omega_1)\exp(\pm i\omega_2)], \\ B_{333}[\exp(\pm i\omega_1)\exp(\pm i\omega_2)\exp(\pm i\omega_3)], \quad C_1B_2[1^2(-1)], \\ C_1B_3[1^2\exp(\pm i\omega)], \quad C_2B_1[(-1)^2 1], \quad C_2B_3[(-1)^2\exp(\pm i\omega)]. \end{gathered} \tag{10.19}$$

Here, for example, C_1B_3 denotes existence of the double nonderogatory multiplier $\rho = 1$ (with a Jordan chain of length 2) and a pair of simple multipliers $\rho = \exp(\pm i\omega)$ on the unit circle.

A set of points of the same type forms a smooth surface in the parameter space. In the case of general position, the codimension of this surface (dimension of the parameter space n minus dimension of the surface) depends only on the type. General formulae for these codimensions are given in [Arnold (1972); Galin (1972)]. It is clear that points of the stability boundary for a generic n-parameter system have only the types of codimension n or lower. The types B_1, B_2, and B_3 listed in (10.15) have codimension 1 and describe the regular part of the stability boundary (harmonic and subharmonic parametric resonances and combination resonance, respectively). Types of codimension 2 are listed in (10.16) and (10.18), while types of codimension 3 are given in (10.17) and (10.19).

Qualitative analysis of singularities of the stability boundary can be performed using the versal deformation theory developed in [Arnold (1983a)]. According to this theory, the stability domain in the neighborhood of a singular boundary point $\mathbf{p}_0$ can be determined by the analysis of a specific low-size matrix $\mathbf{F}'(\mathbf{a})$, which depends only on type of the point $\mathbf{p}_0$. The relation between the stability domains for the matrices $\mathbf{F}(\mathbf{p})$ and $\mathbf{F}'(\mathbf{a})$ is given by a smooth change of parameters $\mathbf{a} = (a_1(\mathbf{p}), \ldots, a_{n'}(\mathbf{p}))$, $\mathbf{a}(\mathbf{p}_0) = 0$. The Jacobian matrix $[d\mathbf{a}/d\mathbf{p}]$ has maximal rank in the case of general position. Depending on type of the point $\mathbf{p}_0$, the matrix $\mathbf{F}'(\mathbf{a})$ takes the form

$$\begin{gathered} B_1: \ (1 + a_1), \quad B_2: \ (-1 + a_1), \\ B_3: \ (\exp(a_1 + i\omega)), \end{gathered} \tag{10.20}$$

$$C_1: \begin{pmatrix} 1 & 1 \\ a_2 & 1 + a_1 \end{pmatrix}, \quad C_2: \begin{pmatrix} -1 & 1 \\ a_2 & -1 + a_1 \end{pmatrix}, \tag{10.21}$$

$$D_1: \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ a_3 & a_2 & 1+a_1 \end{pmatrix}, \quad D_2: \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ a_3 & a_2 & -1+a_1 \end{pmatrix}, \tag{10.22}$$

$$D_3: \begin{pmatrix} \exp(a_1+i\omega) & 1 \\ a_2+ia_3 & \exp(a_1+i\omega) \end{pmatrix}.$$

Matrices corresponding to types (10.18) and (10.19) have the block-diagonal form composed by matrices (10.20), (10.21), where the blocks contain independent parameters. For example, the matrix $\mathbf{F}'(\mathbf{a})$ corresponding to the type C_1B_3 (combination of the types C_1 and B_3) takes the form

$$\begin{pmatrix} 1 & 1 & 0 \\ a_2 & 1+a_1 & 0 \\ 0 & 0 & \exp(a_3+i\omega) \end{pmatrix}. \tag{10.23}$$

Stability analysis for the matrices $\mathbf{F}'(\mathbf{a})$ in cases (10.16)–(10.19) can be done analytically. As a result, we determine a local form of the stability domain and stability boundary (up to a smooth change of parameters) for singular boundary points of codimensions 2 and 3.

For illustration, let us consider the type C_1. Multipliers (eigenvalues) of the corresponding matrix $\mathbf{F}'(\mathbf{a})$ (10.21) are

$$\rho = 1 + \frac{a_1}{2} \pm \sqrt{a_2 + \frac{a_1^2}{4}}. \tag{10.24}$$

The maximal squared absolute value of multipliers (10.24) is equal to

$$\max|\rho|^2 = \begin{cases} 1 + a_1 - a_2, & a_2 + a_1^2/4 < 0, \\ \left(1 + a_1/2 + \sqrt{a_2 + a_1^2/4}\right)^2, & a_2 + a_1^2/4 \geq 0. \end{cases} \tag{10.25}$$

The stability domain, determined by the inequality $|\rho| < 1$, in the neighborhood of $\mathbf{a} = 0$ takes the form $a_1 < a_2$, $a_2 < 0$ shown in Fig. 10.3. The stability boundary has the angle singularity at the origin in the parameter space (a_1, a_2). The boundary consists of two lines of types B_1 and B_3 (the parametric and combination resonance boundaries) intersecting transversally. Other types of singularities of the stability boundary can be studied analogously. The results are stated in the following theorems.

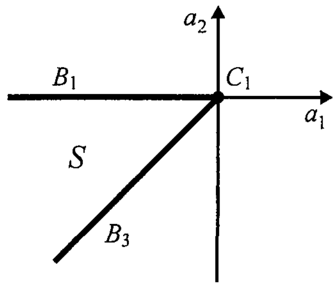


Fig. 10.3 Singularity C_1 of the stability boundary.

Theorem 10.2 *In the case of general position, the stability boundary of one-parameter periodic system (10.1) consists of isolated points of types* $B_1[1]$, $B_2[-1]$ *(parametric resonance) and* $B_3[\exp(\pm i\omega)]$ *(combination resonance).*

Theorem 10.3 *In the case of general position, the stability boundary of two-parameter periodic system (10.1) consists of smooth curves corresponding to parametric and combination resonances, whose only singularities are angles of types (10.16) and (10.18); see Fig. 10.4. The stability domain always lies inside the angles of size less than* π.

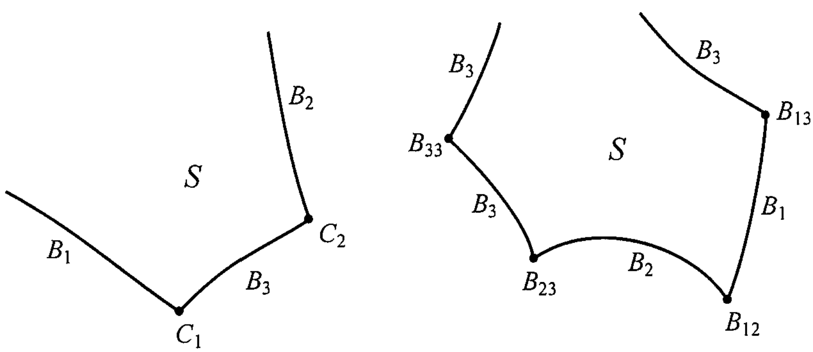


Fig. 10.4 Generic singularities of the stability boundary for two-parameter periodic system.

Theorem 10.4 *In the case of general position, the stability boundary of three-parameter periodic system (10.1) consists of smooth surfaces corresponding to parametric and combination resonances, whose only singularities are dihedral angles (edges) of types (10.16) and (10.18), trihedral angles* *of types (10.19), "breaks of an edge" of types $D_1[1^3]$ and $D_2[(-1)^3]$, and "deadlock of an edge" of type $D_3[(\exp(\pm i\omega))^2]$; see Fig. 10.5.*

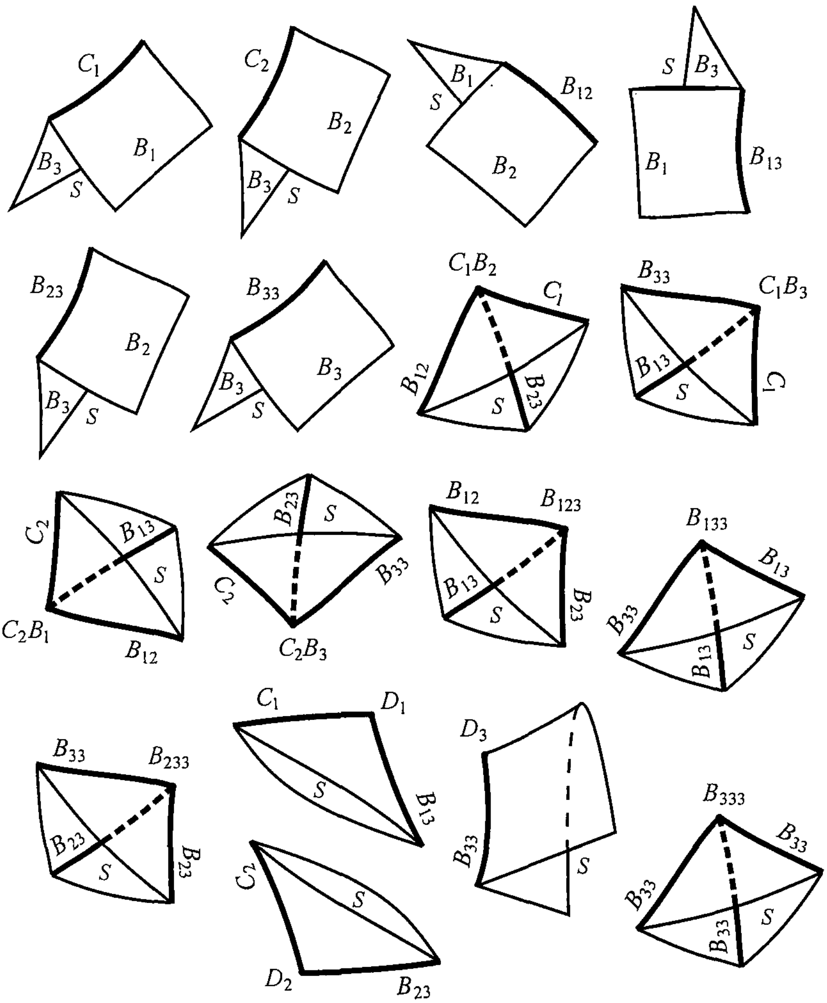


Fig. 10.5 Generic singularities of the stability boundary for three-parameter periodic system.

Figs. 10.4 and 10.5 show the qualitative form of the singularities (up to a smooth change of parameters), where the stability domain is denoted by the letter S.

Remark 10.1 *In the case D_3 (the "deadlock of an edge" singularity) the stability boundary, up to a smooth change of parameters, is given by the equation $xy^2 = z^2$, $x \geq 0$, $y \geq 0$, determining a part of the so-called Whitney–Cayley umbrella. In the cases D_1 and D_2 (the "break of an edge" singularities) the form of the stability boundary is described qualitatively by the equation $x^2y^2 = z^2$, $x \geq 0$, $y \geq 0$.*

Remark 10.2 *Generic singularities of the stability boundary for periodic system (10.1) dependent on two and three parameters are the same as for autonomous systems, see Section 3.4, though the stability criteria and types of stability boundary points are different. We note that the types B_{12}, B_{123}, C_1B_2, and C_2B_1 for periodic systems are essentially different compared to autonomous systems. If we use the transformation of a periodic system to an autonomous one by Theorem 9.4 (page 275), the multipliers $\rho = 1$ and $\rho = -1$ become zero eigenvalues for the autonomous system with two Jordan blocks. But this is not the case of general position for two- and three-parameter autonomous systems.*

10.3 Quantitative analysis of singularities

Quantitative analysis of singularities of the stability boundary for multi-parameter periodic systems is essentially similar to the case of autonomous systems. It uses the bifurcation theory for multipliers and formulae for derivatives of the Floquet matrix with respect to parameters. The main distinctive feature of this analysis is that the stability is characterized by absolute values of multipliers, $|\rho| < 1$, while in the autonomous case we consider real parts of eigenvalues, $\operatorname{Re}\lambda < 0$.

Let $\mathbf{p} = \mathbf{p}_0$ be a singular point of the stability boundary. Under perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$ the multipliers lying on the unit circle change. If all these multipliers move inside the unit circle, the perturbation is stabilizing. If at least one of the multipliers moves outside of the unit circle, the perturbation is destabilizing. Depending on the type of singularity, we have different sets of multipliers on the unit circle, whose behavior depends on their values (real or complex) and multiplicities.

First, let us consider the singularities associated with simple multipliers. These are B_{12}, B_{13}, B_{23}, B_{33}, B_{123}, B_{133}, B_{233}, B_{333}. Each of the multipliers smoothly depends on the parameters together with its absolute value. The first order approximation of the absolute value of a simple multiplier

ρ lying on the unit circle takes the form

$$|\rho(\mathbf{p})| = 1 + (\mathbf{f}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|), \tag{10.26}$$

where components of the real vector $\mathbf{f}$ are given by expressions (10.9) and (10.10). The first order approximation of the stability condition for this multiplier is

$$|\rho(\mathbf{p})| \approx 1 + (\mathbf{f}, \Delta\mathbf{p}) < 1, \tag{10.27}$$

or, simply,

$$(\mathbf{f}, \Delta\mathbf{p}) < 0. \tag{10.28}$$

Taking condition (10.28) for all the multipliers lying on the unit circle, we obtain first order approximations of the stability domain in the neighborhood of the singular point $\mathbf{p}_0$. Depending on the type these approximations are the following:

$$\begin{aligned}
B_{12}: &\quad (\mathbf{f}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{-1}, \Delta\mathbf{p}) < 0;\\
B_{13}: &\quad (\mathbf{f}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega}, \Delta\mathbf{p}) < 0;\\
B_{23}: &\quad (\mathbf{f}_{-1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega}, \Delta\mathbf{p}) < 0;\\
B_{33}: &\quad (\mathbf{f}_{\exp i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_2}, \Delta\mathbf{p}) < 0;\\
B_{123}: &\quad (\mathbf{f}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{-1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega}, \Delta\mathbf{p}) < 0;\\
B_{133}: &\quad (\mathbf{f}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_2}, \Delta\mathbf{p}) < 0;\\
B_{233}: &\quad (\mathbf{f}_{-1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_2}, \Delta\mathbf{p}) < 0;\\
B_{333}: &\quad (\mathbf{f}_{\exp i\omega_1}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_2}, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega_3}, \Delta\mathbf{p}) < 0;
\end{aligned} \tag{10.29}$$

where the subscripts denote the multipliers, for which the vector $\mathbf{f}$ is evaluated. We see that for the (dihedral) angle singularities B_{12}, B_{13}, B_{23}, B_{33} approximations (10.29) provide two inequalities, while the approximations for the trihedral angle singularities B_{123}, B_{133}, B_{233}, B_{333} are given by three inequalities. The vectors $\mathbf{f}$ in these inequalities are the normal vectors to corresponding sides of the stability boundary looking in opposite direction to the stability domain.

Now let us consider a double nonderogatory multiplier ρ_0 lying on the unit circle at $\mathbf{p} = \mathbf{p}_0$. Using results of Subsection 9.7.1, we find that bifurcation of this multiplier is given by the expansion

$$\rho = \rho_0 \pm \sqrt{(\mathbf{g}_1, \Delta\mathbf{p})} + \frac{1}{2}(\mathbf{g}_2, \Delta\mathbf{p}) + o(\varepsilon), \tag{10.30}$$

where $\Delta\mathbf{p} = \varepsilon\mathbf{e}$, and components of the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are determined by formulae (9.139), (9.140). Keeping two lowest order terms in (10.30), we find

$$|\rho|^2 = 1 \pm 2\mathrm{Re}\sqrt{(\overline{\rho}_0^2\mathbf{g}_1, \Delta\mathbf{p})} + o(\varepsilon^{1/2}). \tag{10.31}$$

Since the second term in the right-hand side is taken with both signs, the stability condition $|\rho| < 1$ requires it to be zero. This happens when the expression under the square root is real and negative, i.e.,

$$(\mathrm{Re}\,(\overline{\rho}_0^2\mathbf{g}_1), \Delta\mathbf{p}) < 0, \quad (\mathrm{Im}\,(\overline{\rho}_0^2\mathbf{g}_1), \Delta\mathbf{p}) = 0. \tag{10.32}$$

Using condition (10.32) in expansion (10.30), we obtain

$$|\rho|^2 = 1 + \big(\mathrm{Re}\,(\overline{\rho}_0\mathbf{g}_2 - \overline{\rho}_0^2\mathbf{g}_1),\, \Delta\mathbf{p}\big) + o(\varepsilon). \tag{10.33}$$

From (10.33) we find the third stability condition as

$$\big(\mathrm{Re}\,(\overline{\rho}_0\mathbf{g}_2 - \overline{\rho}_0^2\mathbf{g}_1),\, \Delta\mathbf{p}\big) < 0. \tag{10.34}$$

In case of real multipliers $\rho_0 = \pm 1$ and complex multiplier $\rho_0 = \exp i\omega$ stability conditions (10.32), (10.34) become

$$\begin{aligned}
\rho_0 = 1: &\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2 - \mathbf{g}_1, \Delta\mathbf{p}) < 0;\\
\rho_0 = -1: &\quad (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_1 + \mathbf{g}_2, \Delta\mathbf{p}) > 0;\\
\rho_0 = \exp i\omega: &\quad (\mathrm{Re}\,\mathbf{g}_1 \cos 2\omega + \mathrm{Im}\,\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) < 0,\\
&\quad (\mathrm{Im}\,\mathbf{g}_1 \cos 2\omega - \mathrm{Re}\,\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) = 0,\\
&\quad (\mathrm{Re}\,\mathbf{g}_2 \cos\omega + \mathrm{Im}\,\mathbf{g}_2 \sin\omega\\
&\quad\ -\mathrm{Re}\,\mathbf{g}_1 \cos 2\omega - \mathrm{Im}\,\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{10.35}$$

Using conditions (10.35), we obtain first order approximations of the stability domain for the (dihedral) angle singularities C_1, C_2 and the "deadlock of an edge" singularity D_3 as follows

$$\begin{aligned}
C_1: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2 - \mathbf{g}_1, \Delta\mathbf{p}) < 0; \\
C_2: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_1 + \mathbf{g}_2, \Delta\mathbf{p}) > 0; \\
D_3: \quad & (\operatorname{Re}\mathbf{g}_1 \cos 2\omega + \operatorname{Im}\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) < 0, \\
& (\operatorname{Im}\mathbf{g}_1 \cos 2\omega - \operatorname{Re}\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) = 0, \\
& (\operatorname{Re}\mathbf{g}_2 \cos \omega + \operatorname{Im}\mathbf{g}_2 \sin \omega \\
& -\operatorname{Re}\mathbf{g}_1 \cos 2\omega - \operatorname{Im}\mathbf{g}_1 \sin 2\omega, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{10.36}$$

The vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are evaluated for the double multiplier lying on the unit circle.

Stabilizing perturbations for the singularity "deadlock of an edge" D_3 in the first approximation form a plane angle in the three-dimensional space. This reflects a very narrow form of the stability domain near the singular point. Notice that all the smooth curves $\mathbf{p}(\varepsilon)$ starting at $\mathbf{p}(0) = \mathbf{p}_0$, whose initial direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ satisfies conditions (10.36) (being substituted instead of $\Delta\mathbf{p}$), lie inside the stability domain for small positive ε. The initial direction $\mathbf{e}$ of the edge B_{33} for the singularity "deadlock of an edge" D_3, see Fig. 10.5, is given by

$$\begin{aligned}
& (\operatorname{Re}\mathbf{g}_1 \cos 2\omega + \operatorname{Im}\mathbf{g}_1 \sin 2\omega, \mathbf{e}) < 0, \\
& (\operatorname{Im}\mathbf{g}_1 \cos 2\omega - \operatorname{Re}\mathbf{g}_1 \sin 2\omega, \mathbf{e}) = 0, \\
& (\operatorname{Re}\mathbf{g}_2 \cos \omega + \operatorname{Im}\mathbf{g}_2 \sin \omega \\
& -\operatorname{Re}\mathbf{g}_1 \cos 2\omega - \operatorname{Im}\mathbf{g}_1 \sin 2\omega, \mathbf{e}) = 0.
\end{aligned} \tag{10.37}$$

Analogously, for the trihedral angle singularities C_1B_2, C_1B_3, C_2B_1, C_2B_3 we get local approximations of the stability domain as

$$\begin{aligned}
C_1B_2: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2 - \mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{-1}, \Delta\mathbf{p}) < 0; \\
C_1B_3: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_2 - \mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{f}_{\exp i\omega}, \Delta\mathbf{p}) < 0; \\
C_2B_1: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_1 + \mathbf{g}_2, \Delta\mathbf{p}) > 0,\ (\mathbf{f}_1, \Delta\mathbf{p}) < 0; \\
C_2B_3: \quad & (\mathbf{g}_1, \Delta\mathbf{p}) < 0,\ (\mathbf{g}_1 + \mathbf{g}_2, \Delta\mathbf{p}) > 0,\ (\mathbf{f}_{\exp i\omega}, \Delta\mathbf{p}) < 0.
\end{aligned} \tag{10.38}$$

Finally, let us consider the triple nonderogatory multiplier $\rho_0 = 1$. Let $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon \geq 0$, be a smooth curve in the parameter space starting at $\mathbf{p}(0) = \mathbf{p}_0$ and having the initial direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ evaluated at $\varepsilon = 0$.

Using results of Subsection 9.7.2, we find that bifurcation of this multiplier along the curve $\mathbf{p}(\varepsilon)$ is given by the expansion

$$\rho = 1 + \varepsilon^{1/3}\sqrt[3]{(\mathbf{h}_1, \mathbf{e})} + o(\varepsilon^{1/3}), \tag{10.39}$$

where components of the real vector $\mathbf{h}_1$ are evaluated by formulae (9.140), (9.143). Since the cubic root in (10.39) takes three different complex values, there is always a multiplier lying outside the unit circle unless

$$(\mathbf{h}_1, \mathbf{e}) = 0. \tag{10.40}$$

Hence, directions of the curves lying in the stability domain satisfy equation (10.40).

Directions $\mathbf{e}$ determined by (10.40) are degenerate in the sense that expansions of the multipliers along the curve $\mathbf{p}(\varepsilon)$ are not given in powers of $\varepsilon^{1/3}$. In this case splitting of the triple multiplier $\rho_0 = 1$ is described by two multipliers

$$\rho = 1 \pm \varepsilon^{1/2}\sqrt{(\mathbf{h}_2, \mathbf{e})} + \frac{\varepsilon}{2}\left((\mathbf{h}_3, \mathbf{e}) - \frac{(\mathbf{He}, \mathbf{e}) - (\mathbf{h}_1, \mathbf{d})}{2(\mathbf{h}_2, \mathbf{e})}\right) + o(\varepsilon), \tag{10.41}$$

and by the third multiplier

$$\rho = 1 + \varepsilon\,\frac{(\mathbf{He}, \mathbf{e}) - (\mathbf{h}_1, \mathbf{d})}{2(\mathbf{h}_2, \mathbf{e})} + o(\varepsilon), \tag{10.42}$$

where the vector $\mathbf{d} = d^2\mathbf{p}/d\varepsilon^2$ is evaluated at $\varepsilon = 0$, the components of the real vectors $\mathbf{h}_2$ and $\mathbf{h}_3$ are given by formulae (9.140), (9.143), and $\mathbf{H} = [h_{ij}]$ is the $n \times n$ real matrix with the elements

$$h_{ij} = \mathbf{v}_0^T\left(2\frac{\partial \mathbf{F}}{\partial p_i}(\mathbf{F}_0 - \rho_0\mathbf{I} + \mathbf{v}_0\mathbf{v}_2^T)^{-1}\frac{\partial \mathbf{F}}{\partial p_j} - \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j}\right)\mathbf{u}_0, \quad i, j = 1, \ldots, n, \tag{10.43}$$

determining the quadratic form

$$(\mathbf{He}, \mathbf{e}) = \sum_{i,j=1}^{n} h_{ij}e_ie_j. \tag{10.44}$$

In expression (10.43), $\mathbf{u}_0$ is the real right eigenvector, $\mathbf{v}_0$ and $\mathbf{v}_2$ are the real eigenvector and second associated vector of the left Jordan chain corresponding to $\rho_0 = 1$, see equations (9.141) and (9.142). Expressions (10.41)–(10.43) are obtained using formulae (9.148)–(9.151) of Subsection 9.7.2.

For the absolute values of multipliers (10.41) we find

$$|\rho|^2 = 1 \pm \varepsilon^{1/2}2\,\mathrm{Re}\sqrt{(\mathbf{h}_2, \mathbf{e})} + o(\varepsilon^{1/2}). \tag{10.45}$$

Since both negative and positive signs are taken before the square root, the second term in the right-hand side of (10.45) must vanish for stabilizing perturbation, which yields

$$(\mathbf{h}_2, \mathbf{e}) < 0. \tag{10.46}$$

Using condition (10.46) in expansion (10.41), we find

$$|\rho|^2 = 1 + \varepsilon \left((\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e}) - \frac{(\mathbf{He}, \mathbf{e}) - (\mathbf{h}_1, \mathbf{d})}{2(\mathbf{h}_2, \mathbf{e})} \right) + o(\varepsilon). \tag{10.47}$$

Stabilizing perturbations are determined by the second term in the right-hand side of expression (10.47), which must be negative. Using (10.46), we obtain the inequality

$$(\mathbf{h}_1, \mathbf{d}) > (\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e})(\mathbf{h}_2, \mathbf{e}). \tag{10.48}$$

Expansion for the third eigenvalue (10.42) with the use of inequality (10.46) yields one more stability condition

$$(\mathbf{h}_1, \mathbf{d}) < (\mathbf{He}, \mathbf{e}). \tag{10.49}$$

Since $(\mathbf{h}_2, \mathbf{e}) < 0$, solutions $\mathbf{d}$ of two inequalities (10.48) and (10.49) exist if

$$(\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e}) < 0. \tag{10.50}$$

Using conditions (10.40), (10.46), and (10.50), we find the first order approximation of the stability domain in the neighborhood of the singularity "break of an edge" D_1 as

$$D_1: \quad (\mathbf{h}_1, \Delta\mathbf{p}) = 0, \; (\mathbf{h}_2, \Delta\mathbf{p}) < 0, \; (\mathbf{h}_3 - \mathbf{h}_2, \Delta\mathbf{p}) < 0. \tag{10.51}$$

This approximation represents a plane angle in the three-parameter space, which reflects a narrow form of the stability domain near the singular point. Under conditions (10.51) taken for the direction vector $\mathbf{e}$, the double inequality

$$(\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e})(\mathbf{h}_2, \mathbf{e}) < (\mathbf{h}_1, \mathbf{d}) < (\mathbf{He}, \mathbf{e}) \tag{10.52}$$

provides a set of second order derivative vectors $\mathbf{d}$ determining curvatures of the curves $\mathbf{p}(\varepsilon)$ lying in the stability domain for positive ε. Tangent vectors to the edges C_1 and B_{13} at the point $\mathbf{p}_0$ are given by

$$\begin{aligned} \text{edge } C_1: &\quad (\mathbf{h}_1, \mathbf{e}) = 0,\ (\mathbf{h}_2, \mathbf{e}) = 0,\ (\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e}) < 0; \\ \text{edge } B_{13}: &\quad (\mathbf{h}_1, \mathbf{e}) = 0,\ (\mathbf{h}_2, \mathbf{e}) < 0,\ (\mathbf{h}_3 - \mathbf{h}_2, \mathbf{e}) = 0. \end{aligned} \tag{10.53}$$

The singularity "break of an edge" D_2 associated with the triple nonderogatory multiplier $\rho_0 = -1$ is studied analogously. As a result, we get the first order approximation of the stability domain

$$D_2: \quad (\mathbf{h}_1, \Delta\mathbf{p}) = 0,\ (\mathbf{h}_2, \Delta\mathbf{p}) < 0,\ (\mathbf{h}_2 + \mathbf{h}_3, \Delta\mathbf{p}) > 0. \tag{10.54}$$

Under conditions (10.54) taken for the direction vector $\mathbf{e}$, the double inequality

$$(\mathbf{He}, \mathbf{e}) < (\mathbf{h}_1, \mathbf{d}) < (\mathbf{He}, \mathbf{e}) - 2(\mathbf{h}_2 + \mathbf{h}_3, \mathbf{e})(\mathbf{h}_2, \mathbf{e}) \tag{10.55}$$

provides a set of second order derivative vectors $\mathbf{d}$ for the curves $\mathbf{p}(\varepsilon)$, $\varepsilon \geq 0$, lying in the stability domain. Tangent vectors to the edges C_2 and B_{23} at the point of singularity are given by

$$\begin{aligned} \text{edge } C_2: &\quad (\mathbf{h}_1, \mathbf{e}) = 0,\ (\mathbf{h}_2, \mathbf{e}) = 0,\ (\mathbf{h}_2 + \mathbf{h}_3, \mathbf{e}) > 0; \\ \text{edge } B_{23}: &\quad (\mathbf{h}_1, \mathbf{e}) = 0,\ (\mathbf{h}_2, \mathbf{e}) < 0,\ (\mathbf{h}_2 + \mathbf{h}_3, \mathbf{e}) = 0. \end{aligned} \tag{10.56}$$

The cases considered cover all generic types of singularities having codimensions 2 and 3.

Theorem 10.5 *For singularities of the stability boundary of codimensions 2 and 3, local first order approximations of the stability domain are given by (10.29), (10.36), (10.38), (10.51), and (10.54). In the case of general position, the vectors determining the approximations are linearly independent.*

The approximations obtained in this section allow determining a local form of the stability domain near the singular point for a multi-parameter periodic system using only information at this point: multipliers, right and left eigenvectors and associated vectors, matriciants, and derivatives of the system matrix with respect to parameters. These approximations are very useful for constructing the stability boundary near singularities, and for stabilization and stability optimization problems.

10.4 Stability of two elastically attached pipes conveying pulsating flow

Let us consider in-plane vibrations of a pipe conveying flow, see Fig. 10.6. The pipe consists of two rigid parts of lengths l_1 and l_2, which are connected by means of hinges having elastic coefficients c_1 and c_2. The right end of the pipe is free. The pipe conveys flow with a mass per unit length m and pulsating velocity $u(t) = U(1 + \nu \sin \Omega t)$. The mass of the pipe per unit length is equal to M. The system has two degrees of freedom. As generalized coordinates we choose the angles φ and ψ, which describe deflection of the parts of the pipe from the horizontal axis.

The linearized equations of motion of the system in non-dimensional form are [Szabo *et al.* (1996)]

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \quad \mathbf{q} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{10.57}$$

with the matrices

$$\begin{gathered} \mathbf{M} = \begin{pmatrix} \lambda^3 + 3\lambda^2 & 1.5\lambda \\ 1.5\lambda & 1 \end{pmatrix}, \quad \mathbf{B} = v(\tau) \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & 1 \end{pmatrix}, \\ \mathbf{C} = \begin{pmatrix} \sigma + 1 - \lambda f(\tau) & -1 + \lambda f(\tau) \\ -1 & 1 \end{pmatrix}. \end{gathered} \tag{10.58}$$

The dimensionless variables in matrices (10.58) are

$$\begin{gathered} \tau = \alpha t, \ f(\tau) = \dot{v}(\tau) + \frac{v^2(\tau)}{\mu}, \ \lambda = \frac{l_1}{l_2}, \ \sigma = \frac{c_1}{c_2}, \ \mu = \frac{3m}{M+m}, \\ v(\tau) = V(1 + \delta \sin \omega \tau), \ V = \frac{U\mu}{\alpha l_2}, \ \omega = \frac{\Omega}{\alpha}, \end{gathered} \tag{10.59}$$

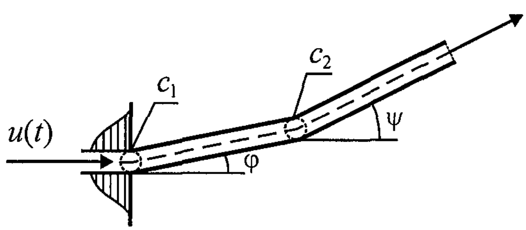


Fig. 10.6 Pipe conveying pulsating flow.

where $\alpha = \sqrt{\mu c_2/(ml_2^3)}$. The dots in equation (10.57) denote derivatives with respect to dimensionless time τ.

Let us write equation (10.57) in the first order form

$$\dot{\mathbf{x}} = \mathbf{G}(\tau)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{G}(\tau) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{B} \end{pmatrix}, \tag{10.60}$$

where the matrix operator $\mathbf{G}(\tau)$ is periodic with the period $T = 2\pi/\omega$ and smoothly depends on the dimensionless parameters λ, σ, μ, ω, δ, and V.

Let us fix the parameters $\lambda = \sigma = \mu = 1$, which correspond to equal lengths of the parts of the pipe, equal elastic coefficients in the hinges, and the fluid mass per unit length equal to half the pipe mass per unit length. We study stability of the system in the three-dimensional parameter space $\mathbf{p} = (\omega, \delta, V)$, where the parameters ω and δ describe the frequency and amplitude of pulsation, and V characterizes the mean velocity of the flow. At $\delta = 0$ we have $v(\tau) \equiv V = const$, i.e., the system is autonomous. In this case the critical velocity of the flow (the minimal velocity, when the system loses stability) is equal to $V_{cr} = \sqrt{6.2 - 0.4\sqrt{29}} \approx 2.0115$ [Szabo *et al.* (1996)].

Let us analyze the stabilizing effect of pulsation at super-critical velocities of the flow $V > V_{cr}$. We take the velocity $V = 2.8$ and the pulsation frequency $\omega = 8$. The pulsation amplitude δ corresponding to a point on the stability boundary can be found numerically. For this purpose, we check stability of the system by evaluating the Floquet matrix and its multipliers with δ increasing from zero. The Floquet matrix $\mathbf{F}$ is found by integration of equation (10.2) with initial condition (10.3) in the time interval $0 \le \tau \le T$ using the Runge-Kutta method. As a result, we find the value $\delta = 0.7366$ for the pulsation amplitude at which the system becomes stable. At the point $\mathbf{p}_0 = (8,\ 0.7366,\ 2.8)$ in the parameter space all the multipliers are simple and equal to

$$\exp(\pm 0.8807i), \quad 0.5350, \quad 0.1514. \tag{10.61}$$

Since the multipliers $\rho = \exp(\pm 0.8807i)$ lying on the unit circle are simple, the point $\mathbf{p}_0$ is a regular point of the stability boundary (the combination resonance). The stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$.

Let us find approximation of the stability domain in the neighborhood of $\mathbf{p}_0$. For this purpose, we calculate the matriciant $\mathbf{X}(t)$ by equation (10.2) with initial condition (10.3), and determine the right and left eigenvectors

$\mathbf{u}_0$ and $\mathbf{v}_0$ corresponding to the multiplier $\rho_0 = \exp 0.8807i$ of the Floquet matrix $\mathbf{F} = \mathbf{X}(T)$. Using this information, we evaluate components of the vector $\mathbf{f}$ by formulae (10.10) and (10.11). As a result, we get

$$\mathbf{f} = (0.0316, -2.0464, 0.5114). \tag{10.62}$$

The vector $\mathbf{f}$ is a normal vector to the stability boundary directed into the instability domain, see Theorem 10.1.

Now let us find the second order approximation of the stability boundary. First, we make the change of time $\tilde{\tau} = \omega\tau$ in system (10.60) in order to obtain a periodic system with the period $\widetilde{T} = 2\pi$ independent on parameters. Then, second order derivatives of the Floquet matrix are evaluated by formula (9.61) for the obtained system, see Section 9.3. Second order derivatives of the multiplier $\rho_0 = \exp 0.8807i$ are determined by formulae (9.100)–(9.102), see Section 9.5. Using this data in expression (10.13), we find second order derivatives of the absolute value of the multiplier $\rho_0 = \exp 0.8807i$. As a result, expansion (10.14) takes the form

$$\begin{aligned} |\rho(\mathbf{p})| &= 1 + 0.0316\Delta\omega - 2.0464\Delta\delta + 0.5114\Delta V \\ &-0.0102(\Delta\omega)^2 + 0.4011\Delta\omega\Delta\delta + 0.2529(\Delta\delta)^2 \\ &-2.1832\Delta\delta\Delta V + 0.1221(\Delta V)^2 - 0.0638\Delta\omega\Delta V + o(\|\Delta\mathbf{p}\|^2), \end{aligned} \tag{10.63}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 = (\Delta\omega, \Delta\delta, \Delta V)$. Substituting (10.63) into equation $|\rho(\mathbf{p})| = 1$, we find the local second order approximation of the stability boundary as

$$\begin{aligned} &0.0316\Delta\omega - 2.0464\Delta\delta + 0.5114\Delta V \\ &-0.0102(\Delta\omega)^2 + 0.4011\Delta\omega\Delta\delta + 0.2529(\Delta\delta)^2 \\ &-2.1832\Delta\delta\Delta V + 0.1221(\Delta V)^2 - 0.0638\Delta\omega\Delta V = 0. \end{aligned} \tag{10.64}$$

Approximation (10.64) and vector (10.62) are shown in Fig. 10.7, where the normal vector $\mathbf{f}$ is scaled in order to fit the figure size. We see that the system is stabilized by pulsation of the flow (the mean critical velocity of the flow V grows rapidly with an increase of the pulsation amplitude δ).

For comparison, the exact stability boundary found numerically is shown in Fig. 10.7 by dashed lines. Fig. 10.7 demonstrates very good agreement of the second order approximation and the exact stability boundary. Notice that the approximation was evaluated using only the information at the point $\mathbf{p}_0$: the matriciants, multipliers, right and left eigenvectors, and derivatives of the system matrix $\mathbf{G}(t, \mathbf{p})$ with respect to parameters.

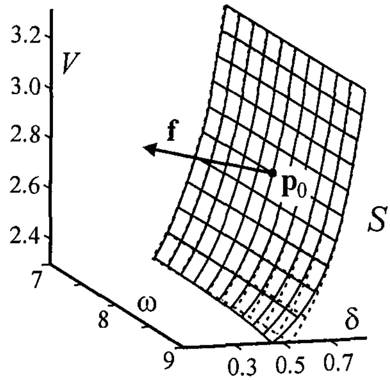


Fig. 10.7 Regular part of the stability boundary for the pipe conveying pulsating flow (second order approximation and normal vector $\mathbf{f}$).

Approximation of the stability boundary can be used for construction of the stability domain and its boundary, stabilization, and stability optimization. The variation of parameters $\delta\mathbf{p} = -\alpha\mathbf{f}$, where α is a small positive number, changes the absolute value of the multiplier as

$$\delta|\rho| = -\alpha||\mathbf{f}||^2 < 0, \tag{10.65}$$

which means that the system is stabilized. Approximation of the stability boundary determines the best variations of the pulsation frequency and amplitude increasing the critical mean velocity. It can be used in gradient methods for motion along the stability boundary. Doing the multi-parameter analysis, we can expect to arrive at a singularity of the stability boundary, for example, at an edge. Then, the results of Section 10.3 should be used for the stability analysis.

Let us consider the point $\mathbf{p}_0 = (3.643, 0.5555, 2.6)$ in the parameter space, where the Floquet matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ has the simple multipliers $\rho = 0.225$, $\rho = 0.026$ and the double multiplier $\rho = -1$. The double multiplier $\rho = -1$ has a single eigenvector. The corresponding right and left Jordan chains (eigenvectors and associated vectors) satisfying equations (9.133)–(9.135) are

$$\begin{array}{ll} \mathbf{u}_0 = (0.92,\, 0.7,\, -0.59,\, 2.49)^T, & \mathbf{u}_1 = (0.11,\, -0.35,\, 0.12,\, 1)^T, \\ \mathbf{v}_0 = (3.34,\, -2.27,\, 1.21,\, -0.3)^T, & \mathbf{v}_1 = (-0.57,\, 1.11,\, 0.4,\, 0.4)^T. \end{array} \tag{10.66}$$

The point $\mathbf{p}_0$ is of type C_2 and represents the dihedral angle singularity (edge) of the stability boundary, see Theorem 10.4.

Let us find the local approximation of the stability domain near $\mathbf{p}_0$ using Theorem 10.5. For this purpose, we evaluate the matriciant $\mathbf{X}(t)$ by equation (10.2) with initial condition (10.3) and use it, together with vectors (10.66), in formulae (9.139) and (9.140). This yields the vectors

$$\mathbf{g}_1 = (-5.15, 45.2, -7.77), \quad \mathbf{g}_2 = (4.49, -31.1, 3.16), \tag{10.67}$$

which determine the first order approximation of the stability domain

$$(\mathbf{g}_1, \Delta\mathbf{p}) < 0, \quad (\mathbf{g}_1 + \mathbf{g}_2, \Delta\mathbf{p}) > 0. \tag{10.68}$$

The vectors $\mathbf{g}_1$ and $-(\mathbf{g}_1 + \mathbf{g}_2)$ are the normal vectors to the sides of the stability boundary looking at the directions opposite to the stability domain. The vector tangent to the edge of the stability boundary at $\mathbf{p}_0$ can be found as the cross product

$$\mathbf{e}_\tau = (\mathbf{g}_1 + \mathbf{g}_2) \times \mathbf{g}_1 = (98.8, 18.6, 42.7). \tag{10.69}$$

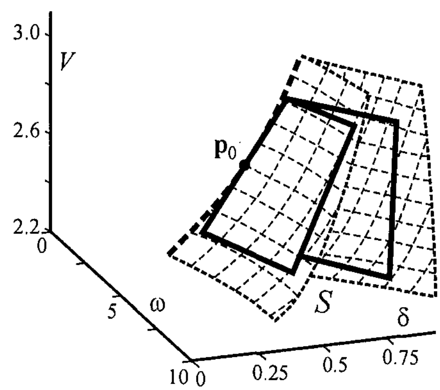


Fig. 10.8 Edge C_2 of stability boundary for pipe conveying pulsating flow.

Fig. 10.8 (bold solid lines) shows approximation of the stability boundary (10.67), (10.68). For comparison, the exact stability boundary found numerically is given by dashed lines. It confirms existence of the singularity and shows good agreement with the approximation. We note that finding the approximation of the stability domain requires only a single integration of equation (10.2) for finding the matriciant and taking integrals (9.140) for each of the parameters. The obtained information can be used for construction of the stability boundary and motion along the stability boundary with the purpose of optimization under stability criteria. For example, the fastest increase of the critical mean velocity of the flow corresponds to variation of the parameters along the edge, whose direction is given by expression (10.69).

Chapter 11

Instability Domains of Oscillatory System with Small Parametric Excitation and Damping

In this chapter we consider linear oscillatory systems with periodic coefficients dependent on three physical parameters: frequency Ω and amplitude δ of periodic excitation, and coefficient of dissipative forces γ. The parameters δ and γ are assumed to be small. It is supposed that the unexcited system without dissipative forces ($\delta = 0$ and $\gamma = 0$) is autonomous and conservative.

Systems of this type are subjected to the instability phenomenon called parametric resonance, which is of great significance for problems of mechanics and physics. Parametric resonance takes place in oscillatory systems with periodically varying parameters. In mechanical systems parametric excitation is realized in the form of periodically changing stiffnesses, masses, and geometric characteristics. It turns out that introduction of weak parametric excitation can destabilize the system if the value of the excitation frequency belongs to certain intervals, which are related in a specific way to natural frequencies of the system. Two types of parametric resonance are distinguished: simple resonance, when the excitation frequency is close to specific fractions of a natural frequency of the system, and combination resonance represented by combination of two different natural frequencies.

Dissipative forces are of great importance in the theory of parametric resonance. Generally, dissipative forces decrease critical intervals of the excitation frequency, making the parametric resonance impossible at sufficiently large level of dissipation. At the same time, better performance of mechanical and physical systems often requires dissipative forces to be small. In this case critical intervals of the excitation frequency are highly sensitive to the relation between the excitation amplitude and damping parameter. The strongest form of this dependence is realized in the phenomenon of destabilization of the system by infinitely small dissipative forces in case of combination resonance. A number of works are devoted to the study of influence of small periodic excitation and dissipative forces on stability of a conservative system, see the books by [Bolotin (1964); Yakubovich and Starzhinskii (1975); Schmidt (1975); Nayfeh and Mook (1979); Yakubovich and Starzhinskii (1987)] and many references therein. In this chapter we mostly follow the papers by [Seyranian (2001); Mailybaev and Seyranian (2001)].

The chapter starts with investigation of a one degree of freedom system described by the Hill equation with damping. Then, we present a general theory of parametric resonance for a multiple degrees of freedom system. The study is based on bifurcation theory of multipliers developed in Chapter 9 and provides universal approach to solution of parametric resonance problems. We emphasize that the important feature of this study is the formulation of the instability (parametric resonance) problem in the three-dimensional space of independent physical parameters.

11.1 Instability domains for the Hill equation with damping

Let us consider the Hill equation with damping

$$\ddot{y} + \gamma\dot{y} + (\omega^2 + \delta\varphi(t))y = 0, \tag{11.1}$$

where $\varphi(t)$ is a piecewise continuous periodic function of time t with the period $T = 2\pi$, δ is a constant describing the amplitude of parametric excitation, $\gamma \geq 0$ is a damping coefficient, and $\omega > 0$ is a natural frequency of the system without damping and parametric excitation. The Hill equation depends on a vector of three parameters $\mathbf{p} = (\omega, \delta, \gamma)$. We are going to study the instability domain of the trivial solution $y(t) \equiv 0$ for the system under the assumption that γ and δ are small, which means that the dissipative forces and the amplitude of parametric excitation are small.

Equation (11.1) can be written in the first order form as

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 - \delta\varphi(t) & -\gamma \end{pmatrix}. \tag{11.2}$$

Let us fix the value of the frequency $\omega = \omega_0$ and consider the system without damping and periodic excitation ($\gamma = 0$ and $\delta = 0$). In this case the matrix $\mathbf{G}$ is independent on time, and equation (11.2) can be solved analytically. As a result, we find the matricians $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ satisfying equations (9.48)–(9.51) as

$$\mathbf{X}(t) = \begin{pmatrix} \cos\omega_0 t & \dfrac{\sin\omega_0 t}{\omega_0} \\ -\omega_0 \sin\omega_0 t & \cos\omega_0 t \end{pmatrix},$$
$$\mathbf{Y}(t) = \begin{pmatrix} \cos\omega_0 t & \omega_0 \sin\omega_0 t \\ -\dfrac{\sin\omega_0 t}{\omega_0} & \cos\omega_0 t \end{pmatrix}. \tag{11.3}$$

It is easy to verify the property $\mathbf{Y}^T(t)\mathbf{X}(t) = \mathbf{I}$. The Floquet matrix takes the form

$$\mathbf{F}_0 = \mathbf{X}(2\pi) = \begin{pmatrix} \cos 2\pi\omega_0 & \dfrac{\sin 2\pi\omega_0}{\omega_0} \\ -\omega_0 \sin 2\pi\omega_0 & \cos 2\pi\omega_0 \end{pmatrix}. \tag{11.4}$$

Solving the characteristic equation for the Floquet matrix, we get two eigenvalues (multipliers)

$$\rho^{a,b} = \cos 2\pi\omega_0 \pm i \sin 2\pi\omega_0. \tag{11.5}$$

If $\omega_0 \neq k/2$ for any integer $k = 1, 2, \ldots$, then the multipliers ρ^a and ρ^b are simple, complex conjugate, and lie on the unit circle in the complex plane. At the values

$$\omega_0 = \frac{k}{2}, \quad k = 1, 2, \ldots, \tag{11.6}$$

the multipliers are double and equal to

$$\rho^a = \rho^b = (-1)^k. \tag{11.7}$$

The corresponding Floquet matrix is $\mathbf{F}_0 = (-1)^k\mathbf{I}$, which implies that double multiplier (11.7) is semi-simple (has two linearly independent eigenvectors). Values (11.6) are called the *resonance* (critical) values of the natural frequency.

Using the matrix $\mathbf{G}$ from (11.2), matriciants (11.3), and Floquet matrix (11.4) in formula for first order derivatives of the Floquet matrix (9.59), see Section 9.3, we find at $\mathbf{p}_0 = (\omega_0, 0, 0)$

$$\frac{\partial \mathbf{F}}{\partial \omega} = \frac{1}{2\omega_0}\mathbf{F}_0 \begin{pmatrix} 1 - \cos 4\pi\omega_0 & \dfrac{4\pi\omega_0 - \sin 4\pi\omega_0}{\omega_0} \\ -\omega_0(4\pi\omega_0 + \sin 4\pi\omega_0) & \cos 4\pi\omega_0 - 1 \end{pmatrix}, \tag{11.8}$$

$$\frac{\partial \mathbf{F}}{\partial \delta} = \frac{1}{2\omega_0}\mathbf{F}_0 \int_0^{2\pi} \begin{pmatrix} \sin 2\omega_0 t & \dfrac{1-\cos 2\omega_0 t}{\omega_0} \\ -\omega_0(1+\cos 2\omega_0 t) & -\sin 2\omega_0 t \end{pmatrix} \varphi(t)dt, \qquad (11.9)$$

$$\frac{\partial \mathbf{F}}{\partial \gamma} = \frac{1}{4\omega_0}\mathbf{F}_0 \begin{pmatrix} \sin 4\pi\omega_0 - 4\pi\omega_0 & \dfrac{1-\cos 4\pi\omega_0}{\omega_0} \\ \omega_0(1-\cos 4\pi\omega_0) & -\sin 4\pi\omega_0 - 4\pi\omega_0 \end{pmatrix}. \qquad (11.10)$$

Using formulae (11.8)–(11.10), the Floquet matrix can be found approximately in the neighborhood of the point $\mathbf{p}_0 = (\omega_0, 0, 0)$ as

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial \omega}\Delta\omega + \frac{\partial \mathbf{F}}{\partial \delta}\delta + \frac{\partial \mathbf{F}}{\partial \gamma}\gamma + o(\|\Delta\mathbf{p}\|), \qquad (11.11)$$

where $\Delta\omega = \omega - \omega_0$ and $\Delta\mathbf{p} = (\Delta\omega, \delta, \gamma)$. The system is asymptotically stable if and only if all the multipliers (eigenvalues) ρ of the Floquet matrix lie inside the unit circle, $|\rho| < 1$. If at least one eigenvalue lies outside the unit circle, $|\rho| > 1$, the system is unstable, see Theorem 9.3 (page 273).

First, let us consider a non-critical value of the frequency $\omega_0 \neq k/2$, $k = 1, 2, \ldots$ Simple multipliers of the undamped system ($\gamma = 0$) satisfy the relation $\rho^a = 1/\rho^b$, which can be shown by analogy with equations (9.73)–(9.75), see Section 9.4. Hence, simple multipliers can not leave the unit circle for small values of δ (the system remains stable). If damping is added then, using (11.8)–(11.11), we find

$$\det \mathbf{F}(\mathbf{p}) = 1 - 2\pi\gamma + o(\|\Delta\mathbf{p}\|). \qquad (11.12)$$

Since the multipliers ρ^a and ρ^b are complex conjugate, we have

$$|\rho^a|^2 = |\rho^b|^2 = \rho^a\rho^b = \det \mathbf{F}(\mathbf{p}). \qquad (11.13)$$

Therefore, addition of damping forces ($\gamma > 0$) pushes the multipliers inside the unit circle. As a result, the system becomes asymptotically stable. Notice that if $\gamma < 0$ (negative damping), then the multipliers move outside the unit circle, which leads to instability.

Now, let us consider the critical frequency $\omega_0 = k/2$. In this case there is the double semi-simple multiplier $\rho^a = \rho^b = (-1)^k$. Using matrices (11.8)–(11.10) in expansion (11.11) and neglecting higher order terms, we find approximate values of the multipliers as

$$\rho^{a,b} = (-1)^k(1 - \pi\gamma) \pm \pi\sqrt{D}, \qquad (11.14)$$

where

$$D = \frac{r_k^2}{k^2}\,\delta^2 - 4\left(\Delta\omega + \frac{c_0}{k}\,\delta\right)^2, \quad r_k = \sqrt{a_k^2 + b_k^2}, \tag{11.15}$$

with c_0 and a_k, b_k being the mean value and real Fourier coefficients of the periodic function $\varphi(t)$:

$$\begin{gathered} c_0 = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)dt, \\ a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(t)\cos kt\,dt, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(t)\sin kt\,dt. \end{gathered} \tag{11.16}$$

The system is unstable if the absolute value of at least one multiplier ρ^a or ρ^b is bigger than one. If $D < 0$, then the first order instability condition, when two complex conjugate multipliers leave the unit circle, takes the form $\gamma < 0$. This condition turns out to be exact, since the multipliers lie on the unit circle at $\gamma = 0$. Recall that in our problem the parameter γ is assumed to be positive. Therefore, the system can lose stability when a real multiplier leaves the unit circle. In this case $D > 0$ and the instability condition takes the form $D > \gamma^2$. Using (11.15), this condition yields

$$4\left(\Delta\omega + \frac{c_0}{k}\,\delta\right)^2 + \gamma^2 < \frac{r_k^2}{k^2}\,\delta^2. \tag{11.17}$$

Inequality (11.17) determines interior of a half-cone in the three-parameter space, see Fig. 11.1. This half-cone represents the first order approximation of the instability domain in the neighborhood of the point $\mathbf{p}_0 = (k/2, 0, 0)$.

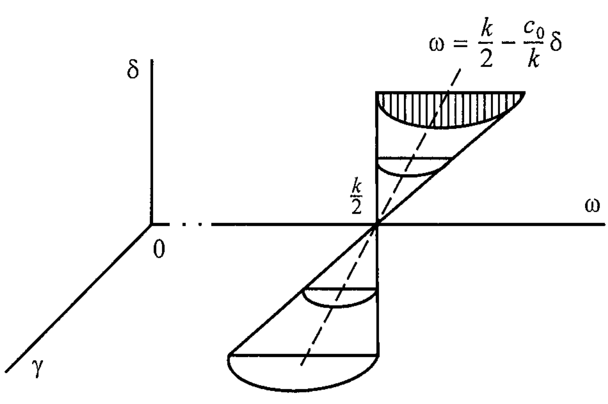


Fig. 11.1 Instability domain for the Hill equation with damping.

The cone is inclined if the mean value of the parametric excitation is nonzero, $c_0 \neq 0$. If $c_0 = 0$, then the cone is symmetric with respect to the plane $\delta = 0$. We see that if k increases (higher order resonances are considered), then the coefficients of δ and δ^2 in (11.17) decrease (the constant r_k^2 decreases as a sum of squared Fourier coefficients). This means that the cone axis approaches the vertical position, i.e., it tends to become parallel to the δ-axis, and the cone becomes narrower.

If the function $\varphi(t)$ is given by a finite Fourier series like, for example, $\varphi(t) = \cos t$ (the Mathieu equation), then the coefficient $r_k = 0$ for $k > k_0$, where k_0 is the number of the last nonzero harmonic in the Fourier series. In this case half-cone (11.17) degenerates to a single straight line $\Delta\omega = -c_0\delta/k$, $\gamma = 0$. This line provides the first order approximation of the instability domain, which means that the instability domain is very narrow near the resonance point, and destabilization of the system is only possible if the parameters are changed along a curve tangent to this line. The detailed analysis of these degenerate cases for the undamped Mathieu–Hill equation is given in [Arnold (1983b)].

Theorem 11.1 *First order approximation of the instability domains for the Hill equation with damping (11.1) in the neighborhood of the resonance points $\delta = \gamma = 0$, $\omega = k/2$, $k = 1, 2, \ldots$, is given by half-cones (11.17).*

Cross-sections of half-cone (11.17) by the plane $\delta = const$ provide half-ellipses, see Fig. 11.2. Centers of these ellipses belong to the line $\Delta\omega = -c_0\delta/k$. At $\gamma = 0$ the width of the instability range of the frequency ω is equal to $r_k\delta/k$. With an increase of the damping parameter γ, the width of the instability range of the frequency ω decreases and becomes zero at $\gamma = r_k\delta/k$. This means that the system can not be destabilized by the parametric excitation $\delta\varphi(t)$ if the damping coefficient is higher than $\gamma > r_k\delta/k$.

Cross-sections of half-cone (11.17) by the plane $\gamma = const > 0$ yield the instability domains in the frequency-amplitude plane confined by two hyperbolae, see Fig. 11.3. Asymptotes of the hyperbolae are given by the equations

$$\Delta\omega = \left(-c_0 \pm \frac{r_k}{2}\right)\frac{\delta}{k}. \tag{11.18}$$

The minimal absolute value of the excitation amplitude, such that the system can be destabilized, is equal to $|\delta| = k\gamma/r_k$. The corresponding frequencies are shifted from $\omega_0 = k/2$ to the values $\omega = k/2 \mp c_0\gamma/r_k$.

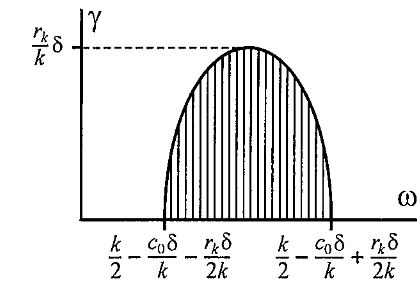


Fig. 11.2 Instability domain for the Hill equation with fixed excitation amplitude δ.

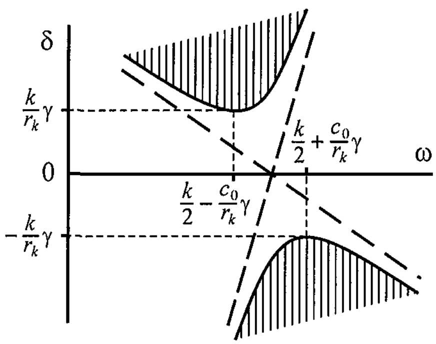


Fig. 11.3 Instability domain for the Hill equation with fixed damping coefficient $\gamma > 0$.

Finally, let us consider the case of no damping, $\gamma = 0$, when the multipliers satisfy the condition $\rho^a = 1/\rho^b$. Using approximate expression (11.14), we find that the multipliers are complex conjugate and lie on the unit circle if $D < 0$, which means that the system is stable but not asymptotically stable. If $D > 0$, then the multipliers are real and lie in opposite sides of the unit circle (the system is unstable). This yields the instability condition

$$D > 0 \tag{11.19}$$

or, using (11.15),

$$-\frac{c_0}{k} - \frac{r_k}{2k} < \frac{\Delta\omega}{\delta} < -\frac{c_0}{k} + \frac{r_k}{2k}. \tag{11.20}$$

Instability domain given by inequalities (11.20) coincides with the limit of the instability domain (11.17) as the damping parameter tends to zero, $\gamma \to 0$. Notice that in case of the undamped Mathieu equation ($\varphi(t) = \cos t$, $\gamma = 0$) inequalities (11.20) yield the first order approximation $(1 - \delta)/2 <$

$\omega < (1+\delta)/2$ for the instability domain near the first critical frequency $\omega_0 = 1/2$. This result agrees with that of given in Section 9.4, where $q = \delta$ and $a = \omega^2$.

Example 11.1 Let us consider a pendulum with a vertically vibrating hanging point, see Fig. 11.4. The equation of motion for the pendulum linearized near the low equilibrium state is [Merkin (1997)]

$$ml\frac{d^2\theta}{dt^2} + \beta l\frac{d\theta}{dt} + m\left(g - \frac{d^2z}{dt^2}\right)\theta = 0, \tag{11.21}$$

where θ is the angle between the pendulum and vertical axis, m and l are the mass and length of the pendulum, respectively, g is the acceleration of gravity, and $z = z(t)$ is the vertical coordinate of the hanging point. The function $z(t)$ is given by

$$z = a\phi(\Omega t), \tag{11.22}$$

where a and Ω are the amplitude and frequency of parametric excitation, respectively, and $\phi(\tau)$ is a 2π-periodic function.

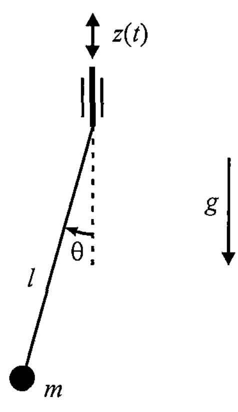


Fig. 11.4 Pendulum with vibrating support.

We study stability of the low equilibrium of the pendulum for small excitation amplitude a and damping coefficient β. Introducing the dimensionless time $\tau = \Omega t$ in equation (11.21), we obtain the Hill equation

$$\frac{d^2\theta}{d\tau^2} + \gamma\frac{d\theta}{d\tau} + (\omega^2 + \delta\varphi(\tau))\theta = 0, \tag{11.23}$$

where the dimensionless coefficients ω, δ, γ, and 2π-periodic function $\varphi(\tau)$ are given by

$$\omega = \frac{1}{\Omega}\sqrt{\frac{g}{l}}, \quad \delta = \frac{a}{l}, \quad \gamma = \frac{\beta}{\Omega m}, \quad \varphi(\tau) = -\frac{d^2\phi}{d\tau^2}. \tag{11.24}$$

By assumption, the quantities δ and γ are small. The instability occurs near the values $\omega = k/2$, which yields the critical excitation frequencies

$$\Omega_k = \frac{2}{k}\sqrt{\frac{g}{l}}, \quad k = 1, 2, \dots \tag{11.25}$$

Using expressions (11.24) in condition (11.17), we find

$$\left(1 - \frac{\Omega}{\Omega_k} + \frac{2c_0 a}{k^2 l}\right)^2 + \frac{\beta^2 l}{4gm^2} < \frac{r_k^2 a^2}{k^4 l^2}, \tag{11.26}$$

where the coefficients c_0 and r_k are determined by formulae (11.15), (11.16) for the function $\varphi(\tau)$. Condition (11.26) determines a half-cone in the space of parameters $\mathbf{p} = (\Omega, a, \beta)$, which is the first order approximation of the instability domain in the neighborhood of the point $\mathbf{p}_0 = (\Omega_k, 0, 0)$. From (11.26) one can see how fast the cone becomes vertical and narrow with an increase of k. Expression (11.26) is simplified if the excitation function has zero mean value $c_0 = 0$.

11.2 Oscillatory systems with m degrees of freedom: simple and combination resonance points

Let us consider a linear oscillatory system with periodic coefficients

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma\mathbf{D}\dot{\mathbf{q}} + (\mathbf{P} + \delta\mathbf{B}(\Omega t))\mathbf{q} = 0, \tag{11.27}$$

where $\mathbf{q} = (q_1, \dots, q_m)^T$ is a vector of generalized coordinates; $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{P}$ are symmetric positive definite $m \times m$ matrices describing inertial, damping, and potential forces, respectively; $\mathbf{B}(\tau)$ is a piecewise continuous 2π-periodic matrix function of parametric excitation. The system depends on the vector of three parameters $\mathbf{p} = (\Omega, \delta, \gamma)$, where Ω and δ are the frequency and amplitude of parametric excitation, respectively, and γ describes the magnitude of damping forces. We study stability of the system under assumption that the quantities δ and γ are small. This means that system (11.27) is close to an autonomous conservative system. The parameters Ω and γ satisfy the natural restrictions $\Omega > 0$ and $\gamma \geq 0$.

We write equation (11.27) in the first order form

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \tag{11.28}$$

where

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{P} + \delta\mathbf{B}(\Omega t)) & -\gamma\mathbf{M}^{-1}\mathbf{D} \end{pmatrix}. \tag{11.29}$$

The matrix $\mathbf{G}(t)$ is of size $2m \times 2m$ and depends periodically on time t with the period $T = 2\pi/\Omega$. Matriciant of system (11.28) is the $2m \times 2m$ matrix $\mathbf{X}(t)$ satisfying the equation and initial condition

$$\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}, \tag{11.30}$$

where $\mathbf{I}$ is the identity matrix. The value of the matriciant at $t = T$ is the Floquet matrix

$$\mathbf{F} = \mathbf{X}(T). \tag{11.31}$$

System (11.28) is asymptotically stable if and only if all the multipliers (eigenvalues) ρ of the Floquet matrix lie inside the unit circle, $|\rho| < 1$. If at least one multiplier lies outside the unit circle, $|\rho| > 1$, then the system is unstable, see Theorem 9.3 (page 273).

Consider the case $\delta = 0$, $\gamma = 0$, when there is no parametric excitation and damping. In this case system (11.27) is autonomous and conservative:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{P}\mathbf{q} = 0. \tag{11.32}$$

Seeking solution to system (11.32) in the form $\mathbf{q} = \mathbf{u}\exp i\omega t$, we get the eigenvalue problem

$$\mathbf{P}\mathbf{u} = \omega^2\mathbf{M}\mathbf{u}, \quad \mathbf{u}^T\mathbf{M}\mathbf{u} = 1, \tag{11.33}$$

where the second equality represents the normalization condition. Equations (11.33) determine real eigenfrequencies ω and corresponding eigenmodes $\mathbf{u}$. We assume that all the eigenfrequencies are simple and ordered as $0 < \omega_1 < \omega_2 < \cdots < \omega_m$. The corresponding eigenmodes are denoted by $\mathbf{u}_j$, $j = 1, \ldots, m$.

The matriciant and Floquet matrix for the case $\delta = \gamma = 0$ are found from equations (11.30) and (11.31) as

$$\mathbf{X}(t) = \exp \mathbf{G}_0 t, \quad \mathbf{F}_0 = \exp \mathbf{G}_0 T, \quad \mathbf{G}_0 = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{P} & 0 \end{pmatrix}. \tag{11.34}$$

Eigenvalues λ of the matrix $\mathbf{G}_0$ with corresponding right and left eigenvectors $\mathbf{w}$ and $\mathbf{v}$ are determined by the equations

$$\mathbf{G}_0\mathbf{w} = \lambda\mathbf{w}, \quad \mathbf{v}^T\mathbf{G}_0 = \lambda\mathbf{v}^T, \quad \mathbf{v}^T\mathbf{w} = 1, \tag{11.35}$$

where the last equation represents the normalization condition. From equations (11.33)–(11.35) it follows that the eigenvalues of the matrix $\mathbf{G}_0$ are

$$\lambda_j, \ \overline{\lambda}_j = \pm i\omega_j, \quad j = 1, \ldots, m. \tag{11.36}$$

The right and left eigenvectors corresponding to the eigenvalue λ_j take the form

$$\mathbf{w}_j = \begin{pmatrix} \mathbf{u}_j \\ i\omega_j\mathbf{u}_j \end{pmatrix}, \quad \mathbf{v}_j = \begin{pmatrix} \dfrac{1}{2}\mathbf{M}\mathbf{u}_j \\ \dfrac{1}{2i\omega_j}\mathbf{M}\mathbf{u}_j \end{pmatrix}. \tag{11.37}$$

From the matrix theory, see [Gantmacher (1998)], we know that the eigenvalues of the monodromy matrix $\mathbf{F}_0 = \exp\mathbf{G}_0T$ are

$$\rho_j, \ \overline{\rho}_j = \exp\left(\pm i\omega_jT\right) = \exp\left(\pm i\frac{2\pi\omega_j}{\Omega}\right), \quad j = 1, \ldots, m. \tag{11.38}$$

The right and left eigenvectors are the same as for the matrix $\mathbf{G}_0$. Therefore, the vectors $\mathbf{w}_j$ and $\mathbf{v}_j$ given by expressions (11.37) are the eigenvectors corresponding to the multiplier $\rho_j = \exp i\omega_jT$ and satisfy the equations

$$\mathbf{F}_0\mathbf{w}_j = \rho_j\mathbf{w}_j, \quad \mathbf{v}_j^T\mathbf{F}_0 = \rho_j\mathbf{v}_j^T, \quad \mathbf{v}_j^T\mathbf{w}_j = 1. \tag{11.39}$$

The complex conjugate multiplier $\overline{\rho}_j = \exp(-i\omega_jT)$ has the complex conjugate right and left eigenvectors $\overline{\mathbf{w}}_j$ and $\overline{\mathbf{v}}_j$, respectively.

The multipliers $\rho_j, \overline{\rho}_j$ from (11.38) lie on the unit circle $|\rho| = 1$. In the case of general position all the multipliers are simple. Multiple multipliers appear at the resonance (critical) values of the excitation frequency

$$\Omega = \frac{2\omega_j}{k}, \quad j = 1, \ldots, m; \ k = 1, 2, \ldots; \tag{11.40}$$

$$\Omega = \frac{\omega_j + \omega_l}{k}, \quad j, l = 1, \ldots, m; \ j > l; \ k = 1, 2, \ldots; \tag{11.41}$$

$$\Omega = \frac{\omega_j - \omega_l}{k}, \quad j, l = 1, \ldots, m; \ j > l; \ k = 1, 2, \ldots \tag{11.42}$$

The cases (11.40), (11.41), and (11.42) are called the simple, summed combination, and difference combination resonances, respectively. In the case (11.40) there is the double real multiplier

$$\rho_j = \overline{\rho}_j = (-1)^k, \tag{11.43}$$

the case (11.41) corresponds to the pair of double complex conjugate multipliers

$$\rho_j = \overline{\rho}_l = \exp\left(i\frac{2\pi k\omega_j}{\omega_j + \omega_l}\right), \quad \overline{\rho}_j = \rho_l = \exp\left(-i\frac{2\pi k\omega_j}{\omega_j + \omega_l}\right), \tag{11.44}$$

and, finally, the case (11.42) corresponds to the pair of double complex conjugate multipliers

$$\rho_j = \rho_l = \exp\left(i\frac{2\pi k\omega_j}{\omega_j - \omega_l}\right), \quad \overline{\rho}_j = \overline{\rho}_l = \exp\left(-i\frac{2\pi k\omega_j}{\omega_j - \omega_l}\right). \tag{11.45}$$

Double multipliers (11.43)–(11.45) are semi-simple. The eigenvectors corresponding to multiplier (11.43) are $\mathbf{w}_j$ and $\overline{\mathbf{w}}_j$. In the cases (11.44) and (11.45) the eigenvectors corresponding to the first multiplier are $\mathbf{w}_j$, $\overline{\mathbf{w}}_l$ and $\mathbf{w}_j$, $\mathbf{w}_l$, respectively.

As we will show below, instability of the system may occur if the excitation frequency is close to critical values (11.40)–(11.42). Therefore, resonance frequencies are of special interest. Multipliers of higher multiplicities appear only if there are rational relations among the quantities $\omega_j \pm \omega_l$, $j, l = 1, \ldots, m$, $j \geq l$ ($j \neq l$ in the case $\omega_j - \omega_l$). In the case of general position such relations do not appear. We will not study those nongeneric cases here, though they can be investigated analogously.

11.3 Behavior of simple multipliers

Let us consider a simple multiplier $\rho_j = \exp i\omega_j T_0$, $T_0 = 2\pi/\Omega_0$, corresponding to the Floquet matrix $\mathbf{F}_0$ at $\delta = \gamma = 0$ and some value of the excitation frequency $\Omega = \Omega_0$. The case of the complex conjugate multiplier $\overline{\rho}_j = \exp(-i\omega_j T_0)$ is studied analogously. The simple multiplier ρ_j depends smoothly on the vector of parameters $\mathbf{p} = (\Omega, \delta, \gamma)$. Its first order derivatives at $\mathbf{p}_0 = (\Omega_0, 0, 0)$ with the use of normalization condition (11.39) are given by the formula

$$\frac{\partial \rho_j}{\partial p_k} = \mathbf{v}_j^T \frac{\partial \mathbf{F}}{\partial p_k} \mathbf{w}_j, \tag{11.46}$$

where derivatives of the Floquet matrix are determined by

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F}_0 \int_0^{T_0} \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X} d\tau + \mathbf{G}_0 \mathbf{F}_0 \frac{\partial T}{\partial p_k} \tag{11.47}$$

with $\mathbf{Y}^T(t) = \mathbf{X}^{-1}(t) = \exp(-\mathbf{G}_0 t)$, see Theorem 2.2 (page 32) and equation (9.66) in Section 9.3. Using expression (11.29) for the matrix $\mathbf{G}$, right and left eigenvectors (11.37), and the relations

$$\begin{aligned} \mathbf{X}(t)\mathbf{w}_j &= \exp(\mathbf{G}_0 t)\mathbf{w}_j = \exp(i\omega_j t)\mathbf{w}_j, \\ \mathbf{v}_j^T \mathbf{Y}^T(t) &= \mathbf{v}_j^T \exp(-\mathbf{G}_0 t) = \exp(-i\omega_j t)\mathbf{v}_j^T, \end{aligned} \tag{11.48}$$

we find the derivatives of ρ_j with respect to parameters at the point $\mathbf{p}_0 = (\Omega_0, 0, 0)$ in the form

$$\begin{aligned} \frac{\partial \rho_j}{\partial \Omega} &= -\rho_j \frac{i2\pi\omega_j}{\Omega_0^2}, \\ \frac{\partial \rho_j}{\partial \delta} &= \rho_j \frac{i\pi c_0^{jj}}{\omega_j \Omega_0}, \quad c_0^{jj} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_j d\tau, \\ \frac{\partial \rho_j}{\partial \gamma} &= -\rho_j \frac{\pi \mathbf{u}_j^T \mathbf{D} \mathbf{u}_j}{\Omega_0}. \end{aligned} \tag{11.49}$$

Then, in the neighborhood of $\mathbf{p}_0$ the multiplier $\rho_j(\mathbf{p})$ is represented as

$$\begin{aligned} \rho_j(\mathbf{p}) &= \rho_j(\mathbf{p}_0) + \frac{\partial \rho_j}{\partial \Omega}(\Omega - \Omega_0) + \frac{\partial \rho_j}{\partial \delta}\delta + \frac{\partial \rho_j}{\partial \gamma}\gamma + o(\|\mathbf{p} - \mathbf{p}_0\|) \\ &= \left(1 - \frac{i2\pi\omega_j}{\Omega_0^2}(\Omega - \Omega_0) + \frac{i\pi c_0^{jj}}{\omega_j \Omega_0}\delta - \frac{\pi \mathbf{u}_j^T \mathbf{D} \mathbf{u}_j}{\Omega_0}\gamma\right) \exp i\omega_j T_0 + o(\|\mathbf{p} - \mathbf{p}_0\|). \end{aligned} \tag{11.50}$$

Expansion (11.50) yields the following expression for the absolute value of the multiplier

$$|\rho_j(\mathbf{p})| = 1 - \frac{\pi \mathbf{u}_j^T \mathbf{D} \mathbf{u}_j}{\Omega_0}\gamma + o(\|\mathbf{p} - \mathbf{p}_0\|). \tag{11.51}$$

From the assumption that $\mathbf{D}$ is a positive definite matrix it follows that the factor of γ in (11.51) is negative. Therefore, with addition of dissipative forces ($\gamma > 0$) the multiplier ρ_j moves inside the unit circle.

Let us put $\gamma = 0$ and consider the case, when $\mathbf{B}(\tau) = \mathbf{B}^T(\tau)$ or $\mathbf{B}(\tau_0 + \tau) = \mathbf{B}(\tau_0 - \tau)$, where τ_0 is a real number. Then system (11.27) is

Hamiltonian or reversible, respectively. Multipliers of such a system possess the following property: if ρ is a multiplier, then $1/\rho$ is a multiplier too, see [Yakubovich and Starzhinskii (1987)]. This property implies that the simple multiplier ρ_j stays exactly on the unit circle $|\rho| = 1$ for the parameter vector $\mathbf{p}$ belonging to the plane $\gamma = 0$ in the neighborhood of the point $\mathbf{p}_0$.

We see that introduction of small dissipative forces leads to the shift of all the simple multipliers inside the unit circle. This means that small dissipative forces stabilize the system with small parametric excitation if the excitation frequency Ω_0 is different from (11.40)–(11.42), i.e., stays away from the resonance frequencies. In the resonant cases the double multiplier splits up, and one of the multipliers can move outside the unit circle. Hence, only multiple multipliers need to be considered when studying stability of the system near the resonance frequencies.

11.4 Local approximation of stability domain for simple and combination resonances

Instability can occur if the excitation frequency is close to the critical values given by (11.40)–(11.42). At those values double multipliers (11.43)–(11.45) appear on the unit circle. Let us consider the excitation frequency $\Omega = \Omega_0$ satisfying the relation

$$\Omega_0 = \frac{\omega_j + \omega_l}{k}, \quad j \geq l, \tag{11.52}$$

for some eigenfrequencies ω_j and ω_l of conservative system (11.32) and a positive integer k. Condition (11.52) includes the case of simple resonance (11.40) for $j = l$ and the case of summed combination resonance (11.41) for $j > l$. Critical excitation frequencies (11.42) corresponding to difference combination resonance will be considered below analogously.

Condition (11.52) implies that two multipliers coincide

$$\rho_j = \overline{\rho}_l = \exp i\omega_j T_0, \quad T_0 = \frac{2\pi}{\Omega_0}. \tag{11.53}$$

For convenience, we denote $\rho_0 = \rho_j = \overline{\rho}_l$. If $j = l$ (simple resonance) we have $\rho_0 = (-1)^k$, and in the case $j > l$ (combination resonance) ρ_0 is a complex multiplier. In the second case there is also a complex conjugate double multiplier $\overline{\rho}_0$. But due to the symmetry of multipliers with respect to the real axis on the complex plane, it is enough to study behavior of the multiplier ρ_0. The double multiplier ρ_0 is semi-simple, since it has two linearly independent eigenvectors $\mathbf{w}_j$ and $\overline{\mathbf{w}}_l$, as well as two left eigenvectors $\mathbf{v}_j$ and $\overline{\mathbf{v}}_l$ given by expressions (11.37).

Under perturbation of parameters the double multiplier ρ_0 splits into two simple multipliers. Let us consider perturbation of the parameter vector in the form $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}$, $\varepsilon \geq 0$, where $\mathbf{p}_0 = (\Omega_0, 0, 0)$ and $\mathbf{e}$ is the direction vector in the parameter space. Then bifurcation of the multiplier ρ_0 is given by the expression

$$\rho = \rho_0(1 + \varepsilon\mu + o(\varepsilon)). \tag{11.54}$$

Two values of the quantity μ are found from the quadratic equation

$$\det \begin{pmatrix} \mathbf{v}_j^T \mathbf{F}_1 \mathbf{w}_j - \rho_0\mu & \mathbf{v}_j^T \mathbf{F}_1 \overline{\mathbf{w}}_l \\ \overline{\mathbf{v}}_l^T \mathbf{F}_1 \mathbf{w}_j & \overline{\mathbf{v}}_l^T \mathbf{F}_1 \overline{\mathbf{w}}_l - \rho_0\mu \end{pmatrix} = 0, \tag{11.55}$$

where

$$\mathbf{F}_1 = \frac{d\mathbf{F}(\mathbf{p}_0 + \varepsilon\mathbf{e})}{d\varepsilon} = \frac{\partial \mathbf{F}}{\partial \Omega} e_1 + \frac{\partial \mathbf{F}}{\partial \delta} e_2 + \frac{\partial \mathbf{F}}{\partial \gamma} e_3 \tag{11.56}$$

with the derivatives taken at $\varepsilon = 0$ and $\mathbf{p} = \mathbf{p}_0$, see Theorem 2.6 (page 56). Equation (11.55) can be written in the form

$$\mu^2 + (x_1 + ix_2)\mu + y_1 + iy_2 = 0, \tag{11.57}$$

where

$$\begin{aligned} x_1 + ix_2 &= -\frac{\mathbf{v}_j^T \mathbf{F}_1 \mathbf{w}_j + \overline{\mathbf{v}}_l^T \mathbf{F}_1 \overline{\mathbf{w}}_l}{\rho_0}, \\ y_1 + iy_2 &= \frac{\mathbf{v}_j^T \mathbf{F}_1 \mathbf{w}_j \, \overline{\mathbf{v}}_l^T \mathbf{F}_1 \overline{\mathbf{w}}_l - \mathbf{v}_j^T \mathbf{F}_1 \overline{\mathbf{w}}_l \, \overline{\mathbf{v}}_l^T \mathbf{F}_1 \mathbf{w}_j}{\rho_0^2}. \end{aligned} \tag{11.58}$$

Using (11.54), inequality $|\rho| < 1$ can be written in the form

$$|\rho| = |\rho_0(1 + \varepsilon\mu + o(\varepsilon))| = 1 + \varepsilon \mathrm{Re}\,\mu + o(\varepsilon) < 1. \tag{11.59}$$

Hence, the asymptotic stability condition in the first order approximation is reduced to the inequality

$$\mathrm{Re}\,\mu < 0 \tag{11.60}$$

for both roots of equation (11.57). This condition can be written in terms of the polynomial coefficients x_1, x_2, y_1, y_2 as (see Section 1.7)

$$x_1 > 0, \quad (x_1 y_1 + x_2 y_2) x_1 - y_2^2 > 0. \tag{11.61}$$

We substitute expressions (11.58) into inequalities (11.61). After elementary transformations using explicit expressions for the eigenvectors (11.37), derivatives of the Floquet matrix (11.47), and equations (11.29), (11.34), (11.35), (11.39), (11.48), (11.52), we obtain

$$\begin{gathered} \frac{\pi}{\Omega_0}(\eta_j + \eta_l) e_3 > 0, \\ \frac{\pi^4}{\Omega_0^4}\left[e_3^2(\eta_j + \eta_l)^2 \left(\eta_j \eta_l e_3^2 - \xi_1 e_2^2 + k^2 \left(e_1 + \frac{e_2 \sigma_+}{k}\right)^2\right)\right. \\ \left. - \left(\xi_2 e_2^2 + k(\eta_j - \eta_l)\left(e_1 + \frac{e_2 \sigma_+}{k}\right) e_3\right)^2\right] > 0. \end{gathered} \tag{11.62}$$

The real quantities η_j, η_l, ξ_1, ξ_2, and σ_+ are given by the expressions

$$\begin{gathered} \eta_j = \mathbf{u}_j^T \mathbf{D} \mathbf{u}_j, \quad \eta_l = \mathbf{u}_l^T \mathbf{D} \mathbf{u}_l, \\ \xi_1 + i\xi_2 = \frac{c_{-k}^{jl} c_k^{lj}}{\omega_j \omega_l}, \quad \sigma_+ = -\frac{\omega_j c_0^{ll} + \omega_l c_0^{jj}}{2\omega_j \omega_l}, \\ c_k^{lj} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{u}_l^T \mathbf{B}(\tau) \mathbf{u}_j \exp(ik\tau) d\tau, \end{gathered} \tag{11.63}$$

where c_k^{lj} is a complex Fourier coefficient of the scalar 2π-periodic function $\mathbf{u}_l^T \mathbf{B}(\tau) \mathbf{u}_j$. Notice that the constants η_j and η_l are positive due to the assumption of positive definiteness of the matrix $\mathbf{D}$. Expressing the vector $\mathbf{e}$ from the relation $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{e}$, we find

$$\mathbf{e} = (e_1, e_2, e_3) = \frac{\mathbf{p} - \mathbf{p}_0}{\varepsilon} = \frac{(\Delta\Omega, \delta, \gamma)}{\varepsilon}, \quad \Delta\Omega = \Omega - \Omega_0. \tag{11.64}$$

Substituting (11.64) into inequalities (11.62) and cancelling positive factors, we find the first order approximation of the stability domain in the neighborhood of the point $\mathbf{p}_0$ as follows

$$\gamma > 0, \tag{11.65}$$

$$\begin{aligned}&\gamma^2(\eta_j+\eta_l)^2\left(\eta_j\eta_l\gamma^2-\xi_1\delta^2+k^2\left(\Delta\Omega+\frac{\delta\sigma_+}{k}\right)^2\right)\\&-\left(\xi_2\delta^2+k(\eta_j-\eta_l)\left(\Delta\Omega+\frac{\delta\sigma_+}{k}\right)\gamma\right)^2>0.\end{aligned}\tag{11.66}$$

The first condition (11.65) implies existence of dissipative forces. The second condition (11.66) determines the local form of the stability domain in the space of parameters $\mathbf{p}=(\Omega,\delta,\gamma)$.

Finally, we consider the critical excitation frequency

$$\Omega_0=\frac{\omega_j-\omega_l}{k},\quad j>l,\tag{11.67}$$

(difference combination resonance). In this case there is a double complex multiplier $\rho_0=\rho_j=\rho_l$, which has two linearly independent eigenvectors $\mathbf{w}_j$ and $\mathbf{w}_l$ given by (11.37). Stability analysis in this case is performed analogously using substitution of ω_l by $-\omega_l$ in all the equations. As a result, we find the first order approximation of the stability domain in the neighborhood of the point $\mathbf{p}_0=(\Omega_0,0,0)$ as follows

$$\gamma>0,\tag{11.68}$$

$$\begin{aligned}&\gamma^2(\eta_j+\eta_l)^2\left(\eta_j\eta_l\gamma^2+\xi_1\delta^2+k^2\left(\Delta\Omega+\frac{\delta\sigma_-}{k}\right)^2\right)\\&-\left(\xi_2\delta^2-k(\eta_j-\eta_l)\left(\Delta\Omega+\frac{\delta\sigma_-}{k}\right)\gamma\right)^2>0,\end{aligned}\tag{11.69}$$

where the real quantities η_j, η_l, ξ_1, ξ_2 are determined in (11.63), and the real constant σ_- is equal to

$$\sigma_-=\frac{\omega_jc_0^{ll}-\omega_lc_0^{jj}}{2\omega_j\omega_l}.\tag{11.70}$$

Theorem 11.2 *First order approximations of the stability domain of system (11.27) near the critical points $\mathbf{p}_0=(\Omega_0,0,0)$ are given by inequalities (11.65), (11.66) for simple and summed combination resonances (11.52), and by inequalities (11.68), (11.69) for difference combination resonances (11.67).*

11.5 Special cases of parametric excitation

In this section we analyze geometry of the instability domain for two most typical cases of parametric excitation appearing in applications.

11.5.1 *Symmetric matrix of parametric excitation*

Let us consider the case, when the matrix of parametric excitation is symmetric, $\mathbf{B}(\Omega t) = \mathbf{B}^T(\Omega t)$. Then the quantities c_{-k}^{jl} and c_k^{lj} are complex conjugate. Hence, $\xi_2 = 0$ and the constant ξ_1 in (11.63) takes the form

$$\xi_1 = \frac{c_{-k}^{jl} c_k^{lj}}{\omega_j \omega_l} = \frac{\left(a_k^{jl}\right)^2 + \left(b_k^{jl}\right)^2}{4\omega_j \omega_l} \geq 0, \tag{11.71}$$

where a_k^{jl} and b_k^{jl} are real Fourier coefficients of the 2π-periodic function $\mathbf{u}_l^T \mathbf{B}(\tau)\mathbf{u}_j$ defined by

$$\begin{aligned} a_k^{jl} &= \frac{1}{\pi}\int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_l \cos(k\tau)d\tau, \\ b_k^{jl} &= \frac{1}{\pi}\int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_l \sin(k\tau)d\tau. \end{aligned} \tag{11.72}$$

In case of simple and summed combination resonances (11.40), (11.41), stability condition (11.66) after cancelling a positive factor and changing the inequality sign yields the instability domain as

$$\eta_j\eta_l\gamma^2 - \xi_1\delta^2 + 4k^2\frac{\eta_j\eta_l}{(\eta_j+\eta_l)^2}\left(\Delta\Omega + \frac{\delta\sigma_+}{k}\right)^2 \leq 0. \tag{11.73}$$

The quantities η_j, $\eta_l > 0$ and $\xi_1 \geq 0$. Assuming that $\xi_1 \neq 0$, condition (11.73) defines the interior of a half-cone in the space of three parameters $\mathbf{p} = (\Omega, \delta, \gamma)$, where $\gamma \geq 0$, see Fig 11.5. The cone axis formed by centers of the cone cross-sections (ellipses) by the planes $\delta = const$ is given by the formulae

$$\gamma = 0, \quad \Omega = \Omega_0 - \frac{\delta\sigma_+}{k}. \tag{11.74}$$

In case of parametric excitation with zero mean values

$$\int_0^{2\pi} \mathbf{u}_j^T \mathbf{B}(\tau)\mathbf{u}_j d\tau = 0, \quad \int_0^{2\pi} \mathbf{u}_l^T \mathbf{B}(\tau)\mathbf{u}_l d\tau = 0 \tag{11.75}$$

we have $c_0^{jj} = c_0^{ll} = \sigma_+ = 0$ and, hence, the cone axis is vertical (parallel to the δ-axis). The stability domain corresponds to the exterior of the half-cone.

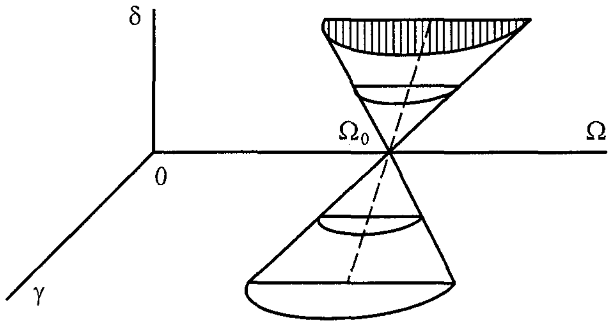


Fig. 11.5 Half-cone of the instability domain in the space of parameters.

Explicit form of instability domain (11.73) with coefficients given by (11.63), (11.71), (11.72) clearly shows how instability of the system depends on the frequency and amplitude of parametric excitation, dissipative forces, eigenfrequencies ω_j, ω_l, and number of resonance k.

Fixing the eigenfrequencies ω_j, ω_l and increasing the number of resonance k, the coefficient ξ_1 decreases as a sum of squared Fourier coefficients, see (11.71). As a result, with an increase of k the cone narrows down and its axis (11.74) tends to the vertical line. The factor k^2 in (11.73) forces the cone to be flattened in the direction of Ω-axis as the resonance number k increases. Absolute values of the quantities ξ_1 and σ_+ usually decrease with an increase of j and l (for bigger ω_j and ω_l). As a result, the instability cone narrows down and its axis tends to the vertical line for resonances corresponding to higher eigenfrequencies.

If the matrix function $\mathbf{B}(\tau)$ is represented by a Fourier series with finite number of terms, then starting with some k the coefficient ξ_1 vanishes. This means degeneration of the first order approximation for the instability domain (11.73) to straight line (11.74). Hence, destabilization of the system is possible only if we perturb the parameter vector along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ tangent to line (11.74). The instability domain in the degenerate case either does not exist or consists of narrow wedges tangent to line (11.74) at $\mathbf{p}_0 = (\Omega_0, 0, 0)$. Further analysis of the instability domain requires construction of higher order approximations. Degeneration of this type is well-known for systems without damping, where the instability domain has cusps of different order in the frequency-amplitude plane, see [Arnold (1983b)]. The results presented above show that such degeneration also takes place in the three-parameter space in the presence of damping forces.

In case of difference combination resonance (11.42) using stability condition (11.69), we find the first order approximation of the instability domain as

$$\eta_j \eta_l \gamma^2 + \xi_1 \delta^2 + 4k^2 \frac{\eta_j \eta_l}{(\eta_j + \eta_l)^2} \left(\Delta\Omega + \frac{\delta \sigma_-}{k} \right)^2 \leq 0. \tag{11.76}$$

Notice that inequalities (11.73) and (11.76) differ only by the sign of the second term and the coefficients σ_+ and σ_-. Hence, if $\xi_1 \neq 0$ (nondegenerate case) only one of the inequalities defines a cone of instability, and the other inequality yields a single point $\Delta\Omega = \delta = \gamma = 0$ (no instability domain). Since according to (11.71) ξ_1 is positive, the instability domain for difference combination resonance does not exist. Notice that a similar effect, the absence of the instability domain for difference combination resonance, is known for Hamiltonian systems (without dissipative forces), see [Yakubovich and Starzhinskii (1987)].

Theorem 11.3 *System (11.27) with a symmetric matrix* $\mathbf{B}(\Omega t) = \mathbf{B}^T(\Omega t)$ *undergoes only simple resonances (11.40) and summed combination resonances (11.41). First order approximations of the instability domain near the resonance points in the space of parameters* $\mathbf{p} = (\Omega, \delta, \gamma)$ *are described by half-cone (11.73).*

11.5.2 *Matrix of parametric excitation* $\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0$

Let us consider the matrix of parametric excitation in the form

$$\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0, \tag{11.77}$$

where $\mathbf{B}_0$ is an arbitrary constant matrix and $\varphi(\tau)$ is a 2π-periodic scalar function. In this case the product $c_{-k}^{jl} c_k^{lj}$ in expressions (11.63) is real. Hence, $\xi_2 = 0$ and the coefficient ξ_1 takes the form

$$\xi_1 = c_{jl} \frac{a_k^2 + b_k^2}{4\omega_j \omega_l}, \quad c_{jl} = \mathbf{u}_j^T \mathbf{B}_0 \mathbf{u}_l \, \mathbf{u}_l^T \mathbf{B}_0 \mathbf{u}_j, \tag{11.78}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} \varphi(\tau) \cos(k\tau) d\tau, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} \varphi(\tau) \sin(k\tau) d\tau \tag{11.79}$$

are real Fourier coefficients of the function $\varphi(\tau)$.

Stability condition (11.66) in case of simple resonance (11.40) and summed combination resonance (11.41) yields the first order approximation of the instability domain (11.73). In case of difference combination resonance (11.42), the first order approximation of the instability domain takes the form (11.76). In the nondegenerate case $\xi_1 \neq 0$, the sign of ξ_1 coincides with the sign of c_{jl}. For simple resonance $j = l$ we have $c_{jj} > 0$ and, hence, the instability domain exists and is described by half-cone (11.73). Existence of the instability domain for combination resonances depends on the sign of c_{jl}: if $c_{jl} > 0$ then there is only the instability domain for summed combination resonance, while if $c_{jl} < 0$ then there is only the instability domain for difference combination resonance. Form of the instability domain depends on the eigenfrequencies ω_j, ω_l and the number of resonance k in the same way as described above for the case of symmetric matrix $\mathbf{B}(\tau)$. If $\xi_1 = 0$ then the instability domain is either absent or degenerate (the first order approximation gives a line).

Theorem 11.4 *System (11.27) with the excitation matrix* $\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0$, *where* $\mathbf{B}_0$ *is a constant matrix and* $\varphi(\tau)$ *is a* 2π*-periodic scalar function, undergoes simple resonances (11.40) and either summed combination resonances (11.41) for* $c_{jl} > 0$ *or difference combination resonances (11.42) for* $c_{jl} < 0$. *First order approximations of the instability domain in the space of parameters* $\mathbf{p} = (\Omega, \delta, \gamma)$ *near the resonance points are described by half-cone (11.73) for simple and summed combination resonances, and by half-cone (11.76) for difference combination resonances.*

11.6 One degree of freedom system

Consider one degree of freedom system (11.27) with $\mathbf{M} = \mathbf{D} = 1$, $\mathbf{P} = \omega_0^2$, and $\mathbf{B}(\Omega t) = \varphi(\Omega t)$:

$$\ddot{y} + \gamma\dot{y} + (\omega_0^2 + \delta\varphi(\Omega t))y = 0. \tag{11.80}$$

Since there is a single eigenfrequency ω_0, only simple resonances

$$\Omega_0 = \frac{2\omega_0}{k}, \quad k = 1, 2, \dots, \tag{11.81}$$

exist. The instability domain described by cone (11.73) takes the form

$$\gamma^2 - \frac{a_k^2 + b_k^2}{4\omega_0^2}\delta^2 + k^2\left(\Delta\Omega - \frac{c_0\delta}{k\omega_0}\right)^2 \le 0, \quad c_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\tau)d\tau, \tag{11.82}$$

where a_k and b_k are the real Fourier coefficients, and c_0 is the mean value of the function $\varphi(\tau)$.

Equation (11.80) can be transformed to the form of the Hill equation (11.1) by means of introduction of the new time and parameters

$$\tau = \Omega t, \quad \widetilde{\gamma} = \frac{\gamma}{\Omega}, \quad \widetilde{\delta} = \frac{\delta}{\Omega^2}, \quad \widetilde{\omega} = \frac{\omega_0}{\Omega}. \tag{11.83}$$

It can be checked that approximation of the instability domain (11.82) is equivalent, up to higher order terms, to the expression obtained in Section 11.1.

11.7 Influence of dissipative forces on instability domain

In this section we study the instability domain in the frequency-amplitude parameter plane with the damping parameter being fixed. We consider the case when the condition $\xi_2 = 0$ is fulfilled, which corresponds to the types of parametric excitation studied in Section 11.5. In this case, the first order approximation of the instability domain is given by formula (11.73) for simple and summed combination resonances, and by formula (11.76) for difference combination resonances.

11.7.1 *Small dissipative forces*

Let us consider the instability domain for fixed $\gamma > 0$. Depending on the sign of ξ_1 the instability domain either does not exist or lies inside two hyperbolae (cone cross-sections by the plane $\gamma = const$) in the plane of parameters (Ω, δ), see Fig. 11.6. Asymptotes of these hyperbolae are given by the equations

$$\sqrt{|\xi_1|}\,\delta \pm 2k\frac{\sqrt{\eta_j\eta_l}}{\eta_j + \eta_l}\left(\Delta\Omega + \frac{\sigma_s\delta}{k}\right) = 0, \tag{11.84}$$

where the subscript s denotes $+$ or $-$ for the resonances (11.40), (11.41) or (11.42), respectively. Recall that $\sigma_+ = \sigma_- = 0$ if the parametric excitation matrix has zero mean values (11.75). If we increase the parameter γ determining the magnitude of dissipative forces, the instability zone in the parameter plane (Ω, δ) gets smaller. Using inequality (11.73) or (11.76), depending on the type of resonance, we find the first order approximation for the minimal (critical) amplitudes of parametric excitation, at which the system can be destabilized, as

$$\delta_{\pm} = \pm\gamma\sqrt{\frac{\eta_j\eta_l}{|\xi_1|}}. \tag{11.85}$$

The values $\delta_{\pm}$ correspond to the excitation frequencies

$$\Omega_{\pm} = \Omega_0 - \frac{\sigma_s\delta_{\pm}}{k}, \tag{11.86}$$

where Ω_0 is the critical frequency, see Fig. 11.6.

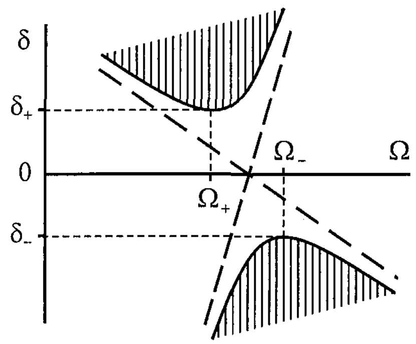


Fig. 11.6 Instability domains in the frequency-amplitude plane for a fixed value of the dissipation parameter $\gamma > 0$.

Theorem 11.5 *First order approximation of the stability boundary of system (11.27) for a small fixed value of the dissipation parameter $\gamma > 0$ in the case $\xi_2 = 0$ is described by hyperbolae (11.73) corresponding to simple and summed combination resonances ($\xi_1 > 0$), and by hyperbolae (11.76) corresponding to difference combination resonances ($\xi_1 < 0$). The minimal (critical) values of the parametric excitation amplitude and corresponding excitation frequencies are determined by formulae (11.85) and (11.86).*

Using relations (11.73) and (11.76), we can find the limit of the instability domain as $\gamma \to +0$, i.e., for the infinitely small damping. The first order approximation of this domain in case of simple resonance takes the form

$$\xi_1\delta^2 \geq k^2\left(\Delta\Omega + \frac{\delta\sigma_+}{k}\right)^2. \tag{11.87}$$

Inequality (11.87) defines two vertical angles in the plane (Ω, δ), which do not depend on the way the damping is introduced to the system. In case of summed combination resonance ($s = +$) or difference combination resonance ($s = -$) the limit instability domain is given by

$$s\xi_1\delta^2 \geq 4k^2 \frac{\eta_j\eta_l}{(\eta_j + \eta_l)^2}\left(\Delta\Omega + \frac{\sigma_s\delta}{k}\right)^2. \tag{11.88}$$

Inequality (11.88) defines two vertical angles in the plane (Ω, δ), which depend on the relation η_j/η_l of magnitudes of dissipative forces corresponding to the jth and lth eigenmodes of the unexcited system. If $\eta_j = \eta_l$ then the limit instability domain is minimal, while if $\eta_j \ll \eta_l$ or $\eta_j \gg \eta_l$ then the limit instability domain is maximal and is represented by two vertical angles approaching π as $\eta_j/\eta_l \to 0$ or $\eta_j/\eta_l \to \infty$, see Fig. 11.7.

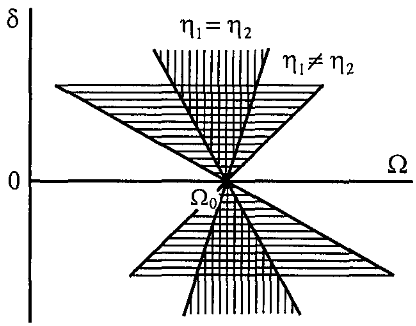


Fig. 11.7 Instability domains in the frequency-amplitude plane as $\gamma \to +0$ in case of combination resonance.

11.7.2 *Instability domain for system without dissipation*

Let us put $\gamma = 0$ (no damping) and consider the case, when $\mathbf{B}(\tau) = \mathbf{B}^T(\tau)$ or $\mathbf{B}(\tau_0 + \tau) = \mathbf{B}(\tau_0 - \tau)$ for some number τ_0. Then system (11.27) is Hamiltonian or reversible, respectively. Multipliers of such a system possess the following property: if ρ is a multiplier, then $1/\rho$ is a multiplier too, see [Yakubovich and Starzhinskii (1987)]. In this case the system is stable (but not asymptotically) if and only if all the multipliers are simple or semi-simple and lie on the unit circle $|\rho| = 1$.

Using equations (11.54) and (11.57), we find the stability condition in the form

$$\operatorname{Re}\mu = 0 \quad \Leftrightarrow \quad x_1 = 0, \quad y_2 = 0, \quad x_2^2 + 4y_1 > 0. \tag{11.89}$$

Substituting the values of x_1, x_2, y_1, and y_2 from (11.58), using relations

(11.29), (11.34)–(11.39), (11.47), (11.48) and the assumption $\xi_2 = 0$, we find the first order approximation of the instability domain

$$s\xi_1\delta^2 \geq k^2 \left(\Delta\Omega + \frac{\sigma_s\delta}{k}\right)^2, \tag{11.90}$$

where $s = +$ and $s = -$ for resonances (11.40), (11.41) and (11.42), respectively. Inequality (11.90) determines two vertical angles in the frequency-amplitude plane.

11.7.3 *Effect of destabilization by infinitely small dissipative forces*

Comparing inequalities (11.87) and (11.90), we see that the instability domains of the system with infinitely small damping ($\gamma \to +0$) and the system without damping ($\gamma = 0$) coincide in case of simple resonance. In case of combination resonances, the instability domains (11.88) and (11.90) coincide only if $\eta_j = \eta_l$. In the case $\eta_j \neq \eta_l$ the instability domain of the system with infinitely small damping is always larger. This effect of abrupt increase of the instability domain by introducing infinitely small damping has been detected in different periodic mechanical systems, see [Iwatsubo *et al.* (1974); Yakubovich and Starzhinskii (1987); Bolotin (1999)].

Let us try to understand the nature of the destabilization phenomenon. At $\gamma = 0$ (no dissipation) all the multipliers of a stable system lie on the unit circle $|\rho| = 1$, while an unstable system is characterized by a multiplier lying outside the unit circle $|\rho| > 1$. An unstable system is structurally stable, i.e., it remains unstable under small perturbation, in particular, if small damping is introduced. Considering a stable system, we have a different situation. Since all the multipliers belong to the unit circle, an arbitrarily small perturbation may destabilize the system if at least one multiplier moves outside the unit circle, or may leave it stable if all the multipliers move on or inside the unit circle. Hence, the stability domain may decrease discontinuously if infinitely small damping is introduced. Respectively, the instability domain abruptly increases. This phenomenon is similar to the effect of destabilization of a non-conservative (circulatory) system by infinitely small dissipative forces, see Section 8.3.

Now, let us analyze the difference between simple and combination resonances. Following calculations of Section 11.4 in case of simple and summed combination resonances (11.52), we find that the dissipative term enters matrix (11.55) as

$$\begin{pmatrix} \mathbf{v}_j^T \dfrac{\partial \mathbf{F}}{\partial \gamma} \mathbf{w}_j & \mathbf{v}_j^T \dfrac{\partial \mathbf{F}}{\partial \gamma} \overline{\mathbf{w}}_l \\ \overline{\mathbf{v}}_l^T \dfrac{\partial \mathbf{F}}{\partial \gamma} \mathbf{w}_j & \overline{\mathbf{v}}_l^T \dfrac{\partial \mathbf{F}}{\partial \gamma} \overline{\mathbf{w}}_l \end{pmatrix} e_3 = \rho_0 \frac{\pi \gamma}{\Omega_0 \varepsilon} \begin{pmatrix} -\eta_j & 0 \\ 0 & -\eta_l \end{pmatrix}. \tag{11.91}$$

In case of simple resonance we have $j = l$ and, hence, $\eta_j = \eta_l$. As a result, introduction of small dissipative forces shifts both multipliers (11.54) towards the origin, see Fig. 11.8a, where two multipliers are different due to variation of the parameters Ω and δ. That is why the destabilization phenomenon does not happen for simple resonances. In case of summed combination resonance the values of η_j and η_l are generally different. The mean value of two multipliers (11.54) is still shifted inside the unit circle, but the multipliers can turn around the mean value. This kind of behavior generates the destabilization phenomenon: though one multiplier always moves inside the unit circle, the other multiplier can leave the unit circle causing instability, see Fig. 11.8b. Only in the specific case $\eta_j = \eta_l$, when the dissipative forces applied to the jth and lth modes are equal, the multipliers do not turn about the mean value (matrix (11.91) is proportional to the identity matrix). In this case destabilization by infinitely small dissipative forces does not occur. Similar conclusions are valid for the case of difference combination resonance.

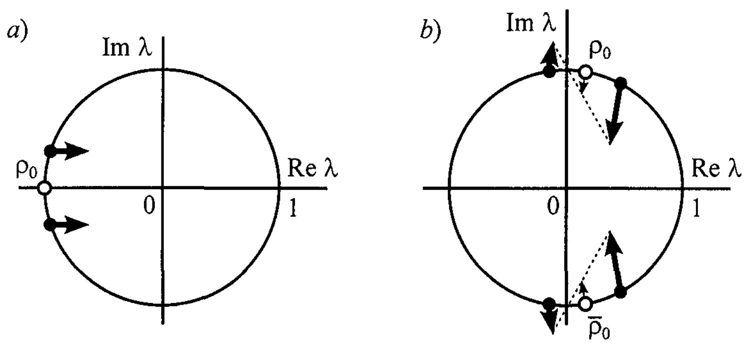


Fig. 11.8 Behavior of multipliers under introduction of small dissipative forces: a) near simple resonance, b) near combination resonance.

11.8 Applications

11.8.1 *Beam loaded by periodic bending moments (Bolotin's problem)*

Let us consider the stability problem of a plane form of a slender elastic beam loaded at the ends by bending moments acting in the plane on maximal stiffness, see Fig. 11.9. Magnitudes of the moments $M(\Omega t) = \delta\varphi(\Omega t)$ are periodic in time with the frequency Ω and amplitude δ, where $\varphi(\tau)$ is a 2π-periodic piecewise continuous function. Bending-torsional out-of-plane vibrations of the beam are described by the equations [Bolotin (1995)]

$$\begin{aligned} &m\frac{\partial^2 w}{\partial t^2} + \gamma m d_1 \frac{\partial w}{\partial t} + EJ\frac{\partial^4 w}{\partial x^4} + \delta\varphi(\Omega t)\frac{\partial^2 \theta}{\partial x^2} = 0, \\ &mr^2\frac{\partial^2 \theta}{\partial t^2} + \gamma m r^2 d_2 \frac{\partial \theta}{\partial t} + \delta\varphi(\Omega t)\frac{\partial^2 w}{\partial x^2} - GI\frac{\partial^2 \theta}{\partial x^2} = 0, \end{aligned} \tag{11.92}$$

where x is the longitudinal coordinate of the beam; $w(x,t)$ and $\theta(x,t)$ are the out-of-plane deflection and twist angle of the beam cross-section, respectively; EJ and GI are the bending and torsional stiffnesses; m is the mass of the beam per unit length; r is the radius of inertia of the beam cross-section; γ is the parameter of dissipative forces (viscous friction); d_1 and d_2 are fixed constants determining the magnitudes of friction forces with respect to bending and torsion. Assuming that the beam is simply supported, we write the boundary conditions as

$$x = 0, l: \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \theta = 0, \tag{11.93}$$

where l is the beam length.

Solution of equations (11.92), (11.93) can be found in the form of the

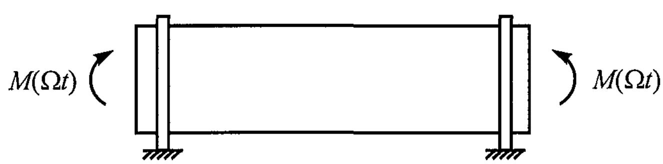


Fig. 11.9 Slender elastic beam loaded by periodic moments.

series [Bolotin (1995)]

$$w(x,t) = \sum_{n=1}^{\infty} W_n(t) \sin \frac{n\pi x}{l}, \quad \theta(x,t) = \sum_{n=1}^{\infty} \Theta_n(t) \sin \frac{n\pi x}{l}, \tag{11.94}$$

where $W_n(t)$ and $\Theta_n(t)$ are unknown functions of time. Substituting (11.94) into equations (11.92), we obtain a system of ordinary differential equations of the form (11.27) for the unknowns $W_n(t)$ and $\Theta_n(t)$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{pmatrix},$$

$$\mathbf{B}(\Omega t) = \varphi(\Omega t) \begin{pmatrix} 0 & -\dfrac{\pi^2 n^2}{l^2 m} \\ -\dfrac{\pi^2 n^2}{r^2 l^2 m} & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} W_n \\ \Theta_n \end{pmatrix}. \tag{11.95}$$

Here ω_{n1} and ω_{n2} are, respectively, the bending and torsional eigenfrequencies of free vibrations of the undamped beam equal to

$$\omega_{n1} = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EJ}{m}}, \quad \omega_{n2} = \frac{n\pi}{rl}\sqrt{\frac{GI}{m}}, \quad n = 1, 2, \ldots \tag{11.96}$$

The eigenvectors corresponding to the eigenfrequencies ω_{n1} and ω_{n2} are equal to $\mathbf{u}_{n1} = (1,0)^T$ and $\mathbf{u}_{n2} = (0,1)^T$.

Let us study stability of system (11.27), (11.95) assuming that the periodic moments and damping forces are small. Since $\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0$, where $\mathbf{B}_0$ is a constant matrix, the system is of the type studied in Subsection 11.5.2. The quantities c_{jl} evaluated by formula (11.78) take the form

$$c_{11} = c_{22} = 0, \quad c_{12} = \frac{\pi^4 n^4}{l^4 r^2 m^2} > 0. \tag{11.97}$$

Hence, there are no instability domains corresponding to difference combination resonances, and the instability domains for simple resonances are degenerate (analysis of these domains requires finding higher order approximations). According to (11.73), we find the instability domains for summed combination resonances

$$d_1 d_2 \gamma^2 - \frac{c_{12}(a_k^2 + b_k^2)}{4\omega_{n1}\omega_{n2}}\delta^2 + 4k^2 \frac{d_1 d_2}{(d_1 + d_2)^2}(\Omega - \Omega_0)^2 \leq 0, \tag{11.98}$$

corresponding to the excitation frequencies Ω close to the critical values

$$\Omega_0 = \frac{\omega_{n1} + \omega_{n2}}{k}, \quad k = 1, 2, \ldots \tag{11.99}$$

Note that a_k and b_k in (11.98) are the real Fourier coefficients of the function $\varphi(\tau)$, see (11.79).

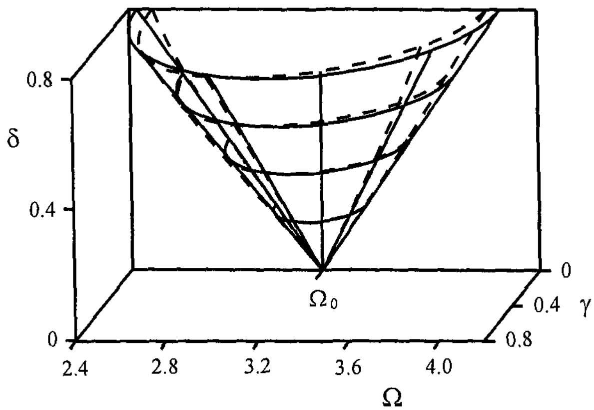


Fig. 11.10 Approximate and exact boundaries of the instability domain for the first summed combination resonance of the beam.

Let us compare the analytical results with numerical calculations. For this purpose, we consider the case $n = 1$, the excitation law $\varphi(\tau) = \cos\tau$, and the beam parameters $d_1 = d_2 = 1$, $\omega_{n1} = 1$, $\omega_{n2} = \sqrt{5}$, $l^2 m = \pi^2/4$, $r^2 = 4/\sqrt{5}$. In Fig. 11.10 solid lines show the boundary of the instability domain for the first summed combination resonance ($k = 1$) found using first order approximation (11.98). Dotted lines denote the boundary of the instability domain found numerically by evaluation of the Floquet matrix for different values of the parameters Ω, δ, and γ. Fig. 11.10 shows good agreement of the exact (computed numerically) and approximate boundaries of the instability domain for the excitation amplitudes up to $\delta \approx 0.8$.

11.8.2 *Beam of variable cross-section loaded by periodic axial force*

Consider an elastic beam of variable cross-section loaded by a periodic axial force $P = P_0 + \delta\varphi(\Omega t)$, where Ω and δ are the frequency and amplitude of parametric excitation, and P_0 is a fixed value of the force lower than the critical Euler force P_E. Equation for small vibrations of the beam takes the form [Bolotin (1999)]

$$m\frac{\partial^2 w}{\partial t^2} + \gamma s\frac{\partial w}{\partial t} + \left(P_0 + \delta\varphi(\Omega t)\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 w}{\partial x^2}\right) = 0. \quad (11.100)$$

In this equation x is the longitudinal coordinate of the beam; $w(x,t)$ is the beam deflection; $EJ(x)$, $m(x)$, and $s(x)$ are the bending stiffness, mass per unit length, and thickness of the beam, respectively; and γ is the coefficient of viscous friction. As the boundary conditions, we consider the case of a beam elastically clamped at both ends:

$$\begin{gathered} w(0,t) = w(l,t) = 0, \\ \left(-c_1\frac{\partial w}{\partial x} + EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = \left(c_2\frac{\partial w}{\partial x} + EJ\frac{\partial^2 w}{\partial x^2}\right)_{x=l} = 0, \end{gathered} \quad (11.101)$$

where $c_1 \geq 0$ and $c_2 \geq 0$ are the elastic coefficients of supports, and l is the length of the beam. The limit cases $c_1 = c_2 = 0$ and $c_1^{-1} = c_2^{-1} = 0$ correspond to simply supported and clamped-clamped beams, respectively.

Solution of system (11.100), (11.101) can be found in the form of the series

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)u_n(x), \quad (11.102)$$

where $w_n(t)$ are unknown functions of time, and $u_n(x)$ are the modes of free vibrations of the undamped beam loaded by the constant axial force $P = P_0$. The modes $u_n(x)$ are determined from the eigenvalue problem

$$\frac{d^2}{dx^2}\left(EJ\frac{d^2 u_n}{dx^2}\right) + P_0\frac{d^2 u_n}{dx^2} - \omega_n^2 m u_n = 0, \quad (11.103)$$

$$\begin{gathered} u_n(0) = u_n(l) = 0, \\ \left(-c_1\frac{du_n}{dx} + EJ\frac{d^2 u_n}{dx^2}\right)_{x=0} = \left(c_2\frac{du_n}{dx} + EJ\frac{d^2 u_n}{dx^2}\right)_{x=l} = 0, \end{gathered} \quad (11.104)$$

where ω_n are the frequencies of free vibrations. The modes satisfy the following normalization condition

$$\int_0^l m u_i u_j dx = \delta_{ij}, \quad (11.105)$$

where δ_{ij} is the Kronecker delta.

Substituting (11.102) into equations (11.100) and using (11.103), we find

$$\sum_{n=1}^{\infty}\left[m\frac{d^2w_n}{dt^2}u_n+\gamma s\frac{dw_n}{dt}u_n+\delta\varphi(\Omega t)w_n\frac{d^2u_n}{dx^2}+\omega_n^2 m w_n u_n\right]=0. \tag{11.106}$$

We multiply equation (11.106) by $u_j(x)$, $j = 1,2,\ldots$, and integrate with respect to x from 0 to l. After integration by parts with the use of conditions (11.104) and (11.105), we obtain a system of ordinary differential equations of the form (11.27) for the unknown functions $w_1(t), w_2(t), \ldots$ The corresponding matrices $\mathbf{M}$ and $\mathbf{P}$ are diagonal, and the matrices $\mathbf{D}$ and $\mathbf{B}(\Omega t)$ are symmetric:

$$\begin{gathered}\mathbf{M}=[\delta_{ij}],\quad \mathbf{P}=[\omega_i^2\delta_{ij}],\quad \mathbf{D}=[d_{ij}],\quad \mathbf{B}=\varphi(\Omega t)[b_{ij}],\\ d_{ij}=\int_0^l s u_i u_j dx,\quad b_{ij}=-\int_0^l \frac{du_i}{dx}\frac{du_j}{dx}dx.\end{gathered} \tag{11.107}$$

Notice that the diagonal elements d_{jj} are positive, while b_{jj} are negative.

Let us study stability of the beam for small values of δ and γ. According to Theorem 11.3, only simple and summed combination resonances are possible. The eigenvector $\mathbf{u}_i$ corresponding to the frequency ω_i has all zero components except for the ith component, which is equal to one. Using matrices (11.107) in the first order approximation of the instability domain (11.73) for simple and summed combination resonances $\Omega_0 = (\omega_j + \omega_l)/k$, we obtain

$$\begin{gathered}d_{jj}d_{ll}\gamma^2-\frac{b_{jl}^2(a_k^2+b_k^2)}{4\omega_j\omega_l}\delta^2\\ +4k^2\frac{d_{jj}d_{ll}}{(d_{jj}+d_{ll})^2}\left(\Delta\Omega-\frac{c_0(\omega_j b_{ll}+\omega_l b_{jj})}{2k\omega_j\omega_l}\delta\right)^2\le 0,\end{gathered} \tag{11.108}$$

where a_k and b_k are the real Fourier coefficients of the function $\varphi(\tau)$, see (11.79), and $c_0 = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\tau)d\tau$ is the mean value of $\varphi(\tau)$. Critical values of the excitation amplitude (11.85) and corresponding excitation frequencies (11.86) for a fixed value of the damping parameter $\gamma > 0$ are found as

$$\delta_\pm=\pm\gamma\sqrt{\frac{4\omega_j\omega_l d_{jj}d_{ll}}{b_{jl}^2(a_k^2+b_k^2)}},\quad \Omega_\pm=\Omega_0+\frac{c_0(\omega_j b_{ll}+\omega_l b_{jj})}{2k\omega_j\omega_l}\delta_\pm. \tag{11.109}$$

For the case of a uniform simply supported or clamped-clamped beam and the axial force $P = \delta\cos\Omega t$, equations (11.108), (11.109) yield the formulae for the first resonance zones ($k = 1$) found earlier in [Iwatsubo *et al.* (1974)].

Chapter 12

Stability Domains of Non-Conservative System under Small Parametric Excitation

In this chapter we continue the study of stability of systems under small parametric excitation. This time the non-excited system is assumed to be autonomous and essentially non-conservative. That is why it is subjected to the divergence and flutter instabilities. We are interested in knowing how stability characteristics and critical values of parameters change depending on the form and frequency of the parametric excitation.

A number of important mechanical problems, where non-conservative loading is essential, are modeled by non-selfadjoint differential equations. As examples we mention systems with fluid-structure interaction (pipes conveying fluid, structures moving in fluid or gas), systems under action of follower forces (e.g. jet thrust) *etc.* Parametric excitation is realized in the form of periodic variation of system parameters like pulsation of the flow velocity, periodic change of loading forces, geometric or elastic characteristics. Stability analysis of a non-conservative system includes determining critical values of the parameters (velocities, loads *etc.*), at which the system becomes unstable. Effect of small parametric excitation strongly depends on the type of instability of the corresponding non-excited system.

Analysis of the stability domain and its boundary in the multi-parameter space is carried out under assumption that periodic terms are small. For this purpose, we use the perturbation theory for a simple multiplier, when considering a smooth stability boundary, and the versal deformation theory for the analysis of singularities. It turns out that the stability boundary is smooth if the corresponding autonomous system is subjected to the divergence instability. But in the case when the autonomous system undergoes flutter instability, the stability boundary can have singularities. This happens when the frequency of parametric excitation is in a certain relation to the flutter frequency. Both qualitative and quantitative analysis is carried out, and local approximations of the stability domain of high order are obtained.

As an application, the stability problem for an elastic cantilever pipe conveying pulsating fluid is considered. Approximations of the stability domain in the regular and singular cases are derived. Comparison of the approximations with the stability domains found numerically confirms efficiency of the suggested approach. It is shown that singularities of the stability boundary provide geometric description of typical stability diagrams on the amplitude–frequency plane.

The material of this chapter is based on the paper by [Mailybaev (2002)]. For related stability studies of non-conservative systems of more specific form see [Fu and Nemat-Nasser (1972); Fu and Nemat-Nasser (1975); Ariaratnam and Sri Namachchivaya (1986)].

12.1 Stability of non-conservative periodic system

Motion of a linear multiple degrees of freedom non-conservative system is governed by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \tag{12.1}$$

where $\mathbf{q} = (q_1, \ldots, q_{m'})^T$ is a real vector of generalized coordinates; $\mathbf{M}$ is a positive definite mass matrix; the matrix $\mathbf{B}$ determines dissipative and gyroscopic forces; and the matrix $\mathbf{C}$ describes conservative and circulatory (non-conservative positional) forces, see Section 1.6. Transforming equation (12.1) to the system of first order, we obtain

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{B} \end{pmatrix}, \tag{12.2}$$

where the real vector $\mathbf{x}$ and the square real matrix $\mathbf{G}$ have dimension $m = 2m'$.

Let us consider autonomous system (12.2) determined by a time-independent matrix $\mathbf{G} = \mathbf{G}_0$. The corresponding eigenvalue problem takes the form

$$\mathbf{G}_0\mathbf{u} = \lambda\mathbf{u}, \tag{12.3}$$

where λ is an eigenvalue and $\mathbf{u}$ is an eigenvector. System (12.2) is asymptotically stable if and only if all the eigenvalues λ have negative real parts ($\operatorname{Re}\lambda < 0$).

Let us consider an n-parameter periodic system of the form

$$\dot{\mathbf{x}} = \mathbf{G}(t, \mathbf{p})\mathbf{x}, \quad \mathbf{G}(t, \mathbf{p}_0) \equiv \mathbf{G}_0. \tag{12.4}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ is the vector of real parameters. Equation (12.4) represents a non-autonomous linear system, which is autonomous at $\mathbf{p}_0 = (p_1^0, \ldots, p_n^0)$. It is assumed that the matrix $\mathbf{G}(t, \mathbf{p})$ is a smooth function of the parameter vector $\mathbf{p}$ and periodic piecewise continuous function of time, $\mathbf{G}(t + T, \mathbf{p}) = \mathbf{G}(t, \mathbf{p})$, where the period $T = T(\mathbf{p}) > 0$ smoothly depends on $\mathbf{p}$. Our aim is to study stability of system (12.4) in the neighborhood of the point $\mathbf{p}_0$, where periodic terms are small (small parametric excitation).

If the autonomous system at $\mathbf{p}_0$ is asymptotically stable, then periodic system (12.4) is stable for the parameter vectors $\mathbf{p}$ taken from some neighborhood of the point $\mathbf{p}_0$. If the autonomous system is unstable and has an eigenvalue with a positive real part, $\operatorname{Re}\lambda > 0$, then perturbed system (12.4) remains unstable for $\mathbf{p}$ sufficiently close to $\mathbf{p}_0$. These properties follow from the continuity of eigenvalues (multipliers) as functions of the parameters. Hence, the interesting cases, when small periodic terms can affect stability properties, correspond to the matrix $\mathbf{G}_0$ having eigenvalues on the imaginary axis. There are two basic cases to be considered: a simple zero eigenvalue or a pair of complex conjugate simple eigenvalues $\lambda = \pm i\omega$, $\omega \neq 0$, lying on the imaginary axis. These critical cases describe autonomous systems subjected to the static instability (divergence) or dynamic instability (flutter), respectively.

Let us fix the parameter vector $\mathbf{p}$. A matriciant of system (12.4) is the $m \times m$ matrix function $\mathbf{X}(t)$ satisfying the differential equation and initial condition

$$\dot{\mathbf{X}} = \mathbf{G}(t, \mathbf{p})\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}. \tag{12.5}$$

Equation (12.5) is equivalent to m equations (12.4) for the columns of $\mathbf{X}(t)$ with initial conditions being the columns of $\mathbf{I}$. The value of the matriciant at the period T gives the Floquet matrix

$$\mathbf{F} = \mathbf{X}(T). \tag{12.6}$$

The eigenvalue problem for the Floquet matrix is

$$\mathbf{F}\mathbf{u} = \rho\mathbf{u}, \tag{12.7}$$

where ρ is a multiplier and $\mathbf{u}$ is an eigenvector. System (12.4) is asymptotically stable if and only if all the multipliers lie inside the unit circle on the complex plane, $|\rho| < 1$. If for some multiplier $|\rho| > 1$, then system (12.4) is unstable, see Theorem 9.3 (page 273).

The Floquet matrix $\mathbf{F}(\mathbf{p})$ is a smooth function of the parameter vector. Derivatives of the Floquet matrix calculated at $\mathbf{p}_0$ can be found using the matriciant and derivatives of the functions $\mathbf{G}(t, \mathbf{p})$ and $T(\mathbf{p})$ at $\mathbf{p}_0$. Explicit formulae for these derivatives were obtained in Section 9.3. In particular, the first order derivative has the form

$$\frac{\partial \mathbf{F}}{\partial p_i} = \mathbf{F} \int_0^T \mathbf{X}^{-1} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X} dt + \mathbf{G}_0 \mathbf{F} \frac{\partial T}{\partial p_i}. \tag{12.8}$$

Notice that though system (12.4) is autonomous at $\mathbf{p}_0$, derivatives of the matrix $\mathbf{G}(t, \mathbf{p})$ at $\mathbf{p}_0$ are time-dependent periodic functions.

The stability criterion defines the stability and instability domains in the parameter space. A boundary of the stability domain consists of points $\mathbf{p}$ such that the Floquet matrix $\mathbf{F}(\mathbf{p})$ has multipliers lying on the unit circle, $|\rho| = 1$, while for other multipliers the inequality $|\rho| < 1$ holds.

Since system (12.4) is autonomous at $\mathbf{p}_0$, the matriciant and the Floquet matrix calculated at $\mathbf{p}_0$ have the form

$$\mathbf{X}(t, \mathbf{p}_0) = \exp(\mathbf{G}_0 t), \quad \mathbf{F}(\mathbf{p}_0) = \exp(\mathbf{G}_0 T_0), \tag{12.9}$$

where $T_0 = T(\mathbf{p}_0)$. From the second expression of (12.9) it follows that the multipliers ρ of the matrix $\mathbf{F}(\mathbf{p}_0)$ and the eigenvalues λ of the matrix $\mathbf{G}_0$ are connected by the relation

$$\rho = \exp \lambda T_0 \tag{12.10}$$

and the corresponding eigenvectors are equal. For two types of the matrix $\mathbf{G}_0$ under consideration, the Floquet matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ has a simple multiplier $\rho = 1$ or a pair of multipliers $\rho = \exp(\pm i\omega T_0)$ on the unit circle, while other multipliers lie inside the unit circle, $|\rho| = \exp(\operatorname{Re} \lambda T_0) < 1$. Hence $\mathbf{p}_0$ is a point on the stability boundary and system (12.4) can be stable or unstable for different points $\mathbf{p}$ in the neighborhood of $\mathbf{p}_0$.

12.2 Approximation of the stability domain in regular case

Let us consider the first case, when the matrix $\mathbf{G}_0$ has the simple zero eigenvalue $\lambda_0 = 0$ and the other eigenvalues have negative real parts. In this case, the corresponding autonomous system is subjected to the static instability (divergence). There are the right and left real eigenvectors $\mathbf{u}_0$

and $\mathbf{v}_0$ corresponding to $\lambda_0 = 0$ and satisfying the equations

$$\mathbf{G}_0\mathbf{u}_0 = 0, \quad \mathbf{v}_0^T\mathbf{G}_0 = 0, \quad \mathbf{v}_0^T\mathbf{u}_0 = 1, \tag{12.11}$$

where the last equality represents the normalization condition.

From relation (12.10) it follows that the Floquet matrix $\mathbf{F}_0$ has the simple multiplier $\rho_0 = 1$ on the unit circle, while other multipliers lie inside the unit circle. Stability of system (12.4) depends on behavior of the multiplier $\rho_0 = 1$ under perturbation of the parameter vector $\mathbf{p}$. The simple multiplier $\rho(\mathbf{p})$ depends smoothly on $\mathbf{p}$ and can be approximated in the neighborhood of $\mathbf{p}_0$ by the Taylor series

$$\rho(\mathbf{p}) = 1 + \sum_{i=1}^{n} \frac{\partial \rho}{\partial p_i}\Delta p_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial p_i \partial p_j}\Delta p_i \Delta p_j + \cdots, \quad \Delta p_i = p_i - p_i^0. \tag{12.12}$$

Derivatives of the multiplier $\rho(\mathbf{p})$ at $\mathbf{p}_0$ can be calculated using explicit formulae of Theorem 2.2 (page 32). For this purpose, only the eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$ and the derivatives of the matrix $\mathbf{F}(\mathbf{p})$ at $\mathbf{p}_0$ are needed. For example, the first order derivative of $\rho(\mathbf{p})$ takes the form

$$\frac{\partial \rho}{\partial p_i} = \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_i}\mathbf{u}_0, \tag{12.13}$$

where the first order derivative of $\mathbf{F}(\mathbf{p})$ is determined by equation (12.8).

Stability of the system in the neighborhood of $\mathbf{p}_0$ is determined by the inequality $|\rho(\mathbf{p})| < 1$ for multiplier (12.12). Since $\rho(\mathbf{p})$ is real and positive in the neighborhood of $\mathbf{p}_0$, the stability condition can be written with the use of expansion (12.12) in the form

$$\rho(\mathbf{p}) - 1 = \sum_{i=1}^{n} \frac{\partial \rho}{\partial p_i}\Delta p_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial p_i \partial p_j}\Delta p_i \Delta p_j + \cdots < 0. \tag{12.14}$$

Calculating derivatives of $\rho(\mathbf{p})$ at $\mathbf{p}_0$, we obtain approximation (12.14) of the stability domain with the accuracy up to the terms of any order. Taking the equality sign in (12.14), we get an approximation of the stability boundary.

According to (12.13), (12.14), the first order approximation of the stability domain has the form

$$(\mathbf{f}_0, \Delta\mathbf{p}) < 0, \quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0, \tag{12.15}$$

where $(\mathbf{f}_0, \Delta\mathbf{p})$ is the scalar product in $\mathbb{R}^n$, and the vector $\mathbf{f}_0$ is

$$\mathbf{f}_0 = \left(\mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_1}\mathbf{u}_0, \ldots, \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_n}\mathbf{u}_0\right). \tag{12.16}$$

If $\mathbf{f}_0 \neq 0$, then the stability boundary is a smooth surface in the neighborhood of $\mathbf{p}_0$, and $\mathbf{f}_0$ is the normal vector to the stability boundary directed into the instability domain. Notice that formula (12.16) for the normal vector coincides with that of obtained in Section 10.1 for general periodic systems. The stability boundary is associated with the simple multiplier $\rho = 1$ and, according to the traditional terminology, represents the harmonic parametric resonance boundary.

Let us consider the second case, when the matrix $\mathbf{G}_0$ possesses a pair of simple purely imaginary eigenvalues $\lambda_0, \overline{\lambda}_0 = \pm i\omega$, $\omega \neq 0$, while for other eigenvalues the inequality $\operatorname{Re}\lambda < 0$ holds. This means that the autonomous system is subjected to the dynamic instability (flutter). There are the right and left complex eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ corresponding to λ_0 and satisfying the equations

$$\mathbf{G}_0\mathbf{u}_0 = \lambda_0\mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{G}_0 = \lambda_0\mathbf{v}_0^T, \quad \mathbf{v}_0^T\mathbf{u}_0 = 1, \tag{12.17}$$

where the last equality represents the normalization condition. From (12.10) it follows that the Floquet matrix $\mathbf{F}_0$ has the multipliers $\rho_0, \overline{\rho}_0 = \exp(\pm i\omega T_0)$ lying on the unit circle, and for other multipliers $|\rho| < 1$. In this section we assume that $\omega T_0 \neq \pi k$ for any integer k (the regular case), which means that the multipliers ρ_0 and $\overline{\rho}_0$ are simple and complex.

The simple multipliers $\rho_0, \overline{\rho}_0$ are smooth functions of the parameters in the neighborhood of the point $\mathbf{p}_0$. Stability of the system depends on absolute values of these multipliers. Since they are complex conjugate, it is sufficient to study behavior of only one multiplier ρ_0. Then the stability condition takes the form

$$|\rho(\mathbf{p})| = \left|\rho_0 + \sum_{i=1}^{n} \frac{\partial \rho}{\partial p_i}\Delta p_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial p_i \partial p_j}\Delta p_i \Delta p_j + \cdots\right| < 1. \tag{12.18}$$

Calculating derivatives of $\rho(\mathbf{p})$ at $\mathbf{p}_0$ up to order s, we find approximation of the stability domain (12.18) up to small terms of order s. The stability boundary in this case is the combination resonance boundary associated with a pair of complex conjugate multipliers $\rho, \overline{\rho}$ lying on the unit circle.

Using expression (12.13) for the first order derivative of $\rho(\mathbf{p})$ and the relations $|\rho(\mathbf{p})| = |\overline{\rho}_0\rho(\mathbf{p})|$, $\rho_0 = \cos\omega T_0 + i\sin\omega T_0$, in the stability condition (12.18), we find the first order approximation of the stability domain in the form

$$(\mathbf{f}_{i\omega}, \Delta\mathbf{p}) < 0, \tag{12.19}$$

where components of the real vector $\mathbf{f}_{i\omega} = (f_1, \ldots, f_n)$ are given by the expression

$$\begin{aligned} f_j &= \operatorname{Re}\left(\overline{\rho}_0 \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_j} \mathbf{u}_0\right) \\ &= \cos(\omega T_0) \operatorname{Re}\left(\mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_j} \mathbf{u}_0\right) + \sin(\omega T_0) \operatorname{Im}\left(\mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_j} \mathbf{u}_0\right). \end{aligned} \tag{12.20}$$

If $\mathbf{f}_{i\omega} \neq 0$, then the stability boundary is a smooth surface in the vicinity of the point $\mathbf{p}_0$, and $\mathbf{f}_{i\omega}$ is a normal vector to the stability boundary directed into the instability domain. Note that formula (12.20) for the normal vector $\mathbf{f}_{i\omega}$ coincides with that of derived in Section 10.1 for the case of a general periodic system.

12.3 Local analysis of the stability domain in singular case

In the previous section we considered the smooth stability boundary in the neighborhood of $\mathbf{p}_0$. In this section a singular case, when the stability boundary is not smooth at $\mathbf{p}_0$, is investigated.

Let us consider the matrix $\mathbf{G}_0$ having a pair of complex conjugate simple eigenvalues on the imaginary axis λ_0, $\overline{\lambda}_0 = \pm i\omega$, $\omega \neq 0$, while for other eigenvalues $\operatorname{Re}\lambda < 0$. We assume that the frequency of vibrations of the autonomous system (flutter frequency) ω is related to the period of parametric excitation by the equality

$$\omega T_0 = \pi k \tag{12.21}$$

for some integer k. Let $\mathbf{u}_0$ and $\mathbf{v}_0$ be the right and left complex eigenvectors corresponding to λ_0 (12.17). Then the right and left eigenvectors corresponding to the complex conjugate eigenvalue $\overline{\lambda}_0$ are $\overline{\mathbf{u}}_0$ and $\overline{\mathbf{v}}_0$, respectively. From (12.10) it follows that the Floquet matrix $\mathbf{F}_0$ has a double multiplier $\rho_0 = (-1)^k$. Both $\mathbf{u}_0$ and $\overline{\mathbf{u}}_0$ are the eigenvectors corresponding to ρ_0. This means that the multiplier ρ_0 is semi-simple (there are two linearly independent eigenvectors). Since ρ_0 is real, we can choose two real eigenvectors as follows

$$\mathbf{u}_1 = \operatorname{Re}\mathbf{u}_0, \quad \mathbf{u}_2 = \operatorname{Im}\mathbf{u}_0 \tag{12.22}$$

and define the $m \times 2$ matrix $\mathbf{U}_0 = [\mathbf{u}_1, \mathbf{u}_2]$. Two left real eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be taken in the form

$$\mathbf{V}_0 = [\mathbf{v}_1, \mathbf{v}_2] = [\mathrm{Re}\,\mathbf{v}_0,\, \mathrm{Im}\,\mathbf{v}_0] \begin{pmatrix} \mathrm{Re}\,\mathbf{v}_0^T\mathbf{u}_1 & \mathrm{Im}\,\mathbf{v}_0^T\mathbf{u}_1 \\ \mathrm{Re}\,\mathbf{v}_0^T\mathbf{u}_2 & \mathrm{Im}\,\mathbf{v}_0^T\mathbf{u}_2 \end{pmatrix}^{-1} \tag{12.23}$$

Expression (12.23) gives the left eigenvectors satisfying the following normalization condition

$$\mathbf{V}_0^T\mathbf{U}_0 = \mathbf{I}, \tag{12.24}$$

where $\mathbf{I}$ is the 2×2 identity matrix.

For the vectors $\mathbf{p}$ in the vicinity of $\mathbf{p}_0$ the double multiplier ρ_0 splits into two simple multipliers $\rho_1(\mathbf{p})$ and $\rho_2(\mathbf{p})$. Stability of periodic system (12.4) depends on the absolute values $|\rho_{1,2}(\mathbf{p})|$. The multipliers $\rho_{1,2}(\mathbf{p})$ are generally nonsmooth functions of the vector $\mathbf{p}$ at $\mathbf{p}_0$. In this case we can apply the bifurcation theory for a double semi-simple eigenvalue described in Sections 2.8 and 2.9. But being interested in high order approximations of the stability domain, we will use another approach based on the versal deformation theory. The main idea of this approach is to consider the matrix operator $\mathbf{F}(\mathbf{p})$ restricted to the invariant subspace of the multipliers $\rho_{1,2}(\mathbf{p})$. Therefore, instead of analyzing the nonsmooth multipliers $\rho_{1,2}(\mathbf{p})$ of the $m \times m$ matrix $\mathbf{F}(\mathbf{p})$ we introduce a 2×2 matrix $\mathbf{F}'(\mathbf{p})$. This matrix is a smooth function of $\mathbf{p}$ and its eigenvalues are $\rho_{1,2}(\mathbf{p})$.

12.3.1 *Method of versal deformations*

According to the versal deformation theory, there exists a smooth 2×2 matrix-function $\mathbf{F}'(\mathbf{p})$ determined in the neighborhood of $\mathbf{p}_0$ by the equations

$$\mathbf{F}(\mathbf{p})\mathbf{U}(\mathbf{p}) = \mathbf{U}(\mathbf{p})\mathbf{F}'(\mathbf{p}), \quad \mathbf{F}'(\mathbf{p}_0) = \rho_0\mathbf{I}, \quad \mathbf{U}(\mathbf{p}_0) = \mathbf{U}_0, \tag{12.25}$$

where $\mathbf{U}(\mathbf{p})$ is an $m \times 2$ matrix smoothly dependent on $\mathbf{p}$ [Galin (1972); Arnold (1978)]. The matrix $\mathbf{F}'(\mathbf{p})$ represents a block of the so-called *versal deformation.* Notice that the matrix-functions $\mathbf{F}'(\mathbf{p})$ and $\mathbf{U}(\mathbf{p})$ are not uniquely determined.

Eigenvalues of the matrix $\mathbf{F}'(\mathbf{p})$ are equal to the multipliers $\rho_{1,2}(\mathbf{p})$ of the matrix $\mathbf{F}(\mathbf{p})$. In the neighborhood of $\mathbf{p}_0$ the matrix-function $\mathbf{F}'(\mathbf{p})$ can be approximated by the Taylor series

$$\mathbf{F}'(\mathbf{p}) = \rho_0\mathbf{I} + \sum_{i=1}^{n} \frac{\partial \mathbf{F}'}{\partial p_i}\Delta p_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{F}'}{\partial p_i \partial p_j}\Delta p_i \Delta p_j + \cdots \tag{12.26}$$

To find derivatives of the matrices $\mathbf{F}'(\mathbf{p})$ and $\mathbf{U}(\mathbf{p})$, we take the derivative of the first equation in (12.25), which yields

$$\frac{\partial \mathbf{F}}{\partial p_i}\mathbf{U}_0 + \mathbf{F}_0\frac{\partial \mathbf{U}}{\partial p_i} = \frac{\partial \mathbf{U}}{\partial p_i}\rho_0 + \mathbf{U}_0\frac{\partial \mathbf{F}'}{\partial p_i}. \tag{12.27}$$

Multiplying this equation by $\mathbf{V}_0^T$ from the left and using the equations $\mathbf{V}_0^T\mathbf{F}_0 = \rho_0\mathbf{V}_0^T$, $\mathbf{V}_0^T\mathbf{U}_0 = \mathbf{I}$, we obtain the first order derivative of $\mathbf{F}'(\mathbf{p})$ in the form

$$\frac{\partial \mathbf{F}'}{\partial p_i} = \mathbf{V}_0^T\frac{\partial \mathbf{F}}{\partial p_i}\mathbf{U}_0. \tag{12.28}$$

To find derivatives of the matrix $\mathbf{U}(\mathbf{p})$, we impose the normalization condition

$$\mathbf{V}_0^T\mathbf{U}(\mathbf{p}) = \mathbf{I}, \tag{12.29}$$

which for the first order derivatives yields

$$\mathbf{V}_0^T\frac{\partial \mathbf{U}}{\partial p_i} = 0. \tag{12.30}$$

Pre-multiplying (12.30) by the matrix $\overline{\mathbf{V}}_0$ and adding the result to equation (12.27), we obtain

$$\frac{\partial \mathbf{U}}{\partial p_i} = \widetilde{\mathbf{G}}_0^{-1}\left(\mathbf{U}_0\frac{\partial \mathbf{F}'}{\partial p_i} - \frac{\partial \mathbf{F}}{\partial p_i}\mathbf{U}_0\right), \tag{12.31}$$

where the matrix

$$\widetilde{\mathbf{G}}_0 = \mathbf{F}_0 - \rho_0\mathbf{I} + \overline{\mathbf{V}}_0\mathbf{V}_0^T \tag{12.32}$$

is nonsingular. Procedure for computing higher order derivatives of $\mathbf{F}'(\mathbf{p})$ and $\mathbf{U}(\mathbf{p})$ is the same as for a simple multiplier, since the corresponding equations are formally identical (if $\mathbf{u}$, $\mathbf{v}_0$, and λ are substituted by $\mathbf{U}$, $\mathbf{V}_0$, and $\mathbf{F}'$, respectively), see Section 2.4. The general expressions for derivatives of $\mathbf{F}'(\mathbf{p})$ and $\mathbf{U}(\mathbf{p})$ at $\mathbf{p} = \mathbf{p}_0$ take the form

$$\mathbf{F}'^{(\mathbf{h})} = \mathbf{V}_0^T \mathbf{F}^{(\mathbf{h})} \mathbf{U}_0 + \mathbf{V}_0^T \sum_{\substack{\mathbf{h}_1+\mathbf{h}_2=\mathbf{h} \\ |\mathbf{h}_1|>0,\, |\mathbf{h}_2|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1!\mathbf{h}_2!} \left(\mathbf{F}^{(\mathbf{h}_1)} \mathbf{U}^{(\mathbf{h}_2)} - \mathbf{U}^{(\mathbf{h}_2)} \mathbf{F}'^{(\mathbf{h}_1)} \right), \tag{12.33}$$

$$\mathbf{U}^{(\mathbf{h})} = \widetilde{\mathbf{G}}_0^{-1} \sum_{\substack{\mathbf{h}_1+\mathbf{h}_2=\mathbf{h} \\ |\mathbf{h}_1|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1!\mathbf{h}_2!} \left(\mathbf{U}^{(\mathbf{h}_2)} \mathbf{F}'^{(\mathbf{h}_1)} - \mathbf{F}^{(\mathbf{h}_1)} \mathbf{U}^{(\mathbf{h}_2)} \right). \tag{12.34}$$

In these expressions we use the notation

$$\mathbf{F}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{F}}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \quad \mathbf{F}'^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{F}'}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \quad \mathbf{U}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{U}}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \tag{12.35}$$

where $\mathbf{h} = (h_1, \ldots, h_n)$ is a vector with integer nonnegative components $h_i \geq 0$; $|\mathbf{h}| = h_1 + \cdots + h_n$ and $\mathbf{h}! = h_1! \cdots h_n!$

12.3.2 *Approximation of the stability domain*

The stability domain in the neighborhood of $\mathbf{p}_0$ is determined by the condition $|\rho_{1,2}(\mathbf{p})| < 1$ for both eigenvalues of $\mathbf{F}'(\mathbf{p})$. To write this condition in more convenient form, we can take the 2×2 matrix $\mathbf{H}(\mathbf{p}) = \ln(\rho_0 \mathbf{F}'(\mathbf{p}))$, where the logarithm of a matrix is defined as

$$\ln(\mathbf{I} + \mathbf{D}) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \mathbf{D}^i. \tag{12.36}$$

Since $\rho_0 = (-1)^k$ and $\rho_0 \mathbf{F}'(\mathbf{p}_0) = \mathbf{I}$, we have $\mathbf{H}(\mathbf{p}_0) = 0$. The inequality $|\rho_{1,2}(\mathbf{p})| < 1$ is equivalent to the condition $\operatorname{Re} \mu_{1,2}(\mathbf{p}) < 0$ for the eigenvalues $\mu_{1,2}(\mathbf{p}) = \ln(\rho_0 \rho_{1,2}(\mathbf{p}))$ of the matrix $\mathbf{H}(\mathbf{p})$. Using the Routh–Hurwitz conditions for the characteristic polynomial of the matrix $\mathbf{H}(\mathbf{p})$ (see Section 4.1), the stability criterion $\operatorname{Re} \mu_{1,2}(\mathbf{p}) < 0$ for both eigenvalues of $\mathbf{H}(\mathbf{p})$ can be written in the from

$$\begin{cases} h_{11}(\mathbf{p}) + h_{22}(\mathbf{p}) < 0, \\ h_{12}(\mathbf{p}) h_{21}(\mathbf{p}) - h_{11}(\mathbf{p}) h_{22}(\mathbf{p}) < 0, \end{cases} \tag{12.37}$$

where $h_{ij}(\mathbf{p})$ are elements of the matrix $\mathbf{H}(\mathbf{p})$. Using Taylor expansions (12.26) and (12.36) for the matrix $\mathbf{F}'(\mathbf{p})$ and the logarithm function, we find the Taylor series for the matrix $\mathbf{H}(\mathbf{p})$ as follows

$$\mathbf{H}(\mathbf{p}) = \begin{pmatrix} h_{11}(\mathbf{p}) & h_{12}(\mathbf{p}) \\ h_{21}(\mathbf{p}) & h_{22}(\mathbf{p}) \end{pmatrix} = \ln(\rho_0 \mathbf{F}'(\mathbf{p}))$$

$$= \rho_0 \sum_{i=1}^{n} \frac{\partial \mathbf{F}'}{\partial p_i} \Delta p_i + \frac{\rho_0}{2} \sum_{i,j=1}^{n} \left[\frac{\partial^2 \mathbf{F}'}{\partial p_i \partial p_j} - \rho_0 \frac{\partial \mathbf{F}'}{\partial p_i} \frac{\partial \mathbf{F}'}{\partial p_j} \right] \Delta p_i \Delta p_j + \cdots, \tag{12.38}$$

where any number of terms in the series can be found explicitly using symbolic computation software. Thus, evaluating derivatives of $\mathbf{F}'(\mathbf{p})$ at $\mathbf{p}_0$ by formulae (12.33), (12.34) and substituting them into expression (12.38), we find the approximation of the stability domain (12.37) up to small terms of arbitrary order.

For the first order approximation of the stability domain expressions (12.37) yield

$$\begin{cases} (\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) < 0, \\ (\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) < 0, \end{cases} \tag{12.39}$$

where the vectors $\mathbf{h}_{ij}$ are gradients of the functions $h_{ij}(\mathbf{p})$ at $\mathbf{p}_0$; the symmetric $n \times n$ matrix $\mathbf{Q} = [q_{ij}]$ has the form

$$\mathbf{Q} = \frac{1}{2}(\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}^T), \quad \tilde{\mathbf{Q}} = \mathbf{h}_{12}^T \mathbf{h}_{21} - \mathbf{h}_{11}^T \mathbf{h}_{22}; \tag{12.40}$$

and $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p})$ denotes the quadratic form

$$(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = \sum_{i,j=1}^{n} q_{ij} \Delta p_i \Delta p_j. \tag{12.41}$$

Using expressions (12.28), (12.38), the vectors $\mathbf{h}_{ij}$ are given as

$$\mathbf{h}_{ij} = \rho_0 \left(\mathbf{v}_i^T \frac{\partial \mathbf{F}}{\partial p_1} \mathbf{u}_j, \ldots, \mathbf{v}_i^T \frac{\partial \mathbf{F}}{\partial p_n} \mathbf{u}_j \right). \tag{12.42}$$

We have found approximation of the stability domain (12.37), (12.38). Since the stability condition consists of two equations, the stability boundary is generally nonsmooth at $\mathbf{p}_0$. In the following subsections we will analyze geometry of the stability boundary and classify its singularities using first order approximation (12.39).

12.3.3 *Two-parameter case*

The first inequality in (12.39) defines a half-plane in the parameter space $\mathbf{p} = (p_1, p_2)$. The second inequality in (12.39) gives different solutions for $\Delta\mathbf{p}$ depending on the type of the matrix $\mathbf{Q}$. In the case $\mathbf{Q} > 0$ (positive definite) the second inequality in (12.39) is not satisfied for all $\Delta\mathbf{p}$. If $\mathbf{Q} < 0$ (negative definite), then the second inequality in (12.39) is satisfied for all $\Delta\mathbf{p}$. Finally, in case of indefinite matrix $\mathbf{Q}$ ($\det\mathbf{Q} < 0$) this inequality gives two domains lying between two intersecting lines $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$. Equation for these lines can be written in the form

$$q_{11}\Delta p_1 + \left(q_{12} \pm \sqrt{q_{12}^2 - q_{11}q_{22}}\right)\Delta p_2 = 0, \tag{12.43}$$

where q_{ij} are elements of the matrix $\mathbf{Q}$. Expression (12.43) is found by solving the quadratic equation $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ with respect to Δp_1.

The first order approximation of the stability boundary is the intersection of the domains defined by two inequalities (12.39). For the case of indefinite matrix $\mathbf{Q}$ geometry of the stability domain depends on the mutual position of those domains. There are two typical cases: when the line $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) = 0$ lies inside or outside the domain $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) < 0$. This corresponds to the inequalities $(\mathbf{Qt}, \mathbf{t}) < 0$ or $(\mathbf{Qt}, \mathbf{t}) > 0$, respectively, where $\mathbf{t}$ is a nonzero vector satisfying the equation $(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{t}) = 0$.

The general result can be formulated as follows. The stability domain in the neighborhood of the point $\mathbf{p}_0$ has four typical forms corresponding to the cases: a) $\mathbf{Q} < 0$, b) $\mathbf{Q} > 0$, c) $\mathbf{Q}$ is indefinite, $(\mathbf{Qt}, \mathbf{t}) > 0$, and d) $\mathbf{Q}$ is indefinite, $(\mathbf{Qt}, \mathbf{t}) < 0$. In the case a) the stability boundary is a smooth curve; in the case b) the system is unstable for all $\mathbf{p}$ near $\mathbf{p}_0$; in the cases c) and d) the stability domain consists of one and two angles, respectively, with the vertices at $\mathbf{p}_0$; see Fig. 12.1. In Fig. 12.1 the curves $h_{12}(\mathbf{p})h_{21}(\mathbf{p}) - h_{11}(\mathbf{p})h_{22}(\mathbf{p}) = (\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|^2) = 0$ are denoted by α, and the curve $h_{11}(\mathbf{p}) + h_{22}(\mathbf{p}) = (\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|) = 0$ is denoted by β; the stability domain is denoted by S. Since the multipliers $\rho_{1,2} = \rho_0 \exp\mu_{1,2}$, where $\mu_{1,2}$ are eigenvalues of $\mathbf{H}(\mathbf{p})$, the lines α represent the parametric resonance boundary (corresponding to the multiplier $\rho = (-1)^k$), while the lines β are the combination resonance boundaries (determined by a pair of complex conjugate multipliers on the unit circle).

Other (degenerate) cases occur, when the matrix $\mathbf{Q}$ is singular, or $\mathbf{h}_{11} + \mathbf{h}_{22} = 0$, or $(\mathbf{Qt}, \mathbf{t}) = 0$ (if $\mathbf{Q}$ is indefinite). Then higher order approximations for the functions $h_{ij}(\mathbf{p})$ should be used to determine the form of the stability domain.

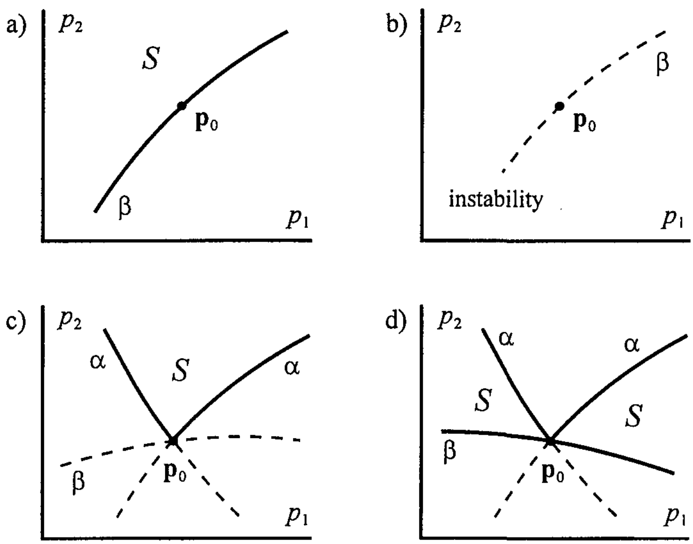


Fig. 12.1 Singularities of the stability domain in the two-parameter space.

12.3.4 *Three-parameter case*

In this case $\mathbf{Q}$ and $\mathbf{h}_{ij}$ are the real matrix and real vectors of dimension 3, respectively. It is easy to show that the matrix $\mathbf{Q}$ is always indefinite. Indeed, there exists a nonzero vector $\mathbf{e}$ satisfying the equations $(\mathbf{h}_{11}, \mathbf{e}) = (\mathbf{h}_{12}, \mathbf{e}) = 0$ such that $(\mathbf{Qe}, \mathbf{e}) = (\mathbf{h}_{12}, \mathbf{e})(\mathbf{h}_{21}, \mathbf{e}) - (\mathbf{h}_{11}, \mathbf{e})(\mathbf{h}_{22}, \mathbf{e}) = 0$. Therefore, for the nonsingular matrix $\mathbf{Q}$ equation $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ defines a cone surface [Korn and Korn (1968)]. Depending on the sign of $\det \mathbf{Q}$ the stability domain $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) < 0$ is placed inside or outside the cone (inside for $\det \mathbf{Q} < 0$). There exists a 3×3 nonsingular real matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$ transforming $\mathbf{Q}$ to the diagonal form [Korn and Korn (1968)]

$$\mathbf{Q} = \mathbf{W}\mathbf{D}\mathbf{W}^T, \tag{12.44}$$

where $\mathbf{D} = \mathrm{diag}(-1, -1, 1)$ or $\mathbf{D} = \mathrm{diag}(1, 1, -1)$. Then, the equation for the cone surface $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ can be written in the form

$$(\mathbf{w}_1, \Delta\mathbf{p})^2 + (\mathbf{w}_2, \Delta\mathbf{p})^2 = (\mathbf{w}_3, \Delta\mathbf{p})^2. \tag{12.45}$$

The cone (12.45) can be written in the parameterized form as

$$\begin{aligned} \Delta\mathbf{p} &= s(\mathbf{a} + \mathbf{b}\cos\alpha + \mathbf{c}\sin\alpha), \quad s \in \mathbb{R},\ 0 \le \alpha < 2\pi, \\ \mathbf{a} &= \mathbf{w}_1 \times \mathbf{w}_2, \quad \mathbf{b} = \mathbf{w}_3 \times \mathbf{w}_1, \quad \mathbf{c} = \mathbf{w}_2 \times \mathbf{w}_3, \end{aligned} \tag{12.46}$$

where the vectors **a**, **b**, and **c** describe geometry of the cone as shown in Fig. 12.2. Expression (12.46) can be checked by substitution into (12.45).

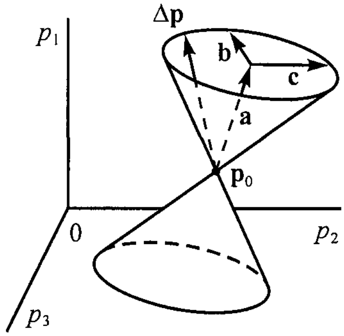


Fig. 12.2 Parameterization of the cone surface.

The form of the stability domain is determined by the mutual position of the cone $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) < 0$ and the half-space $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) < 0$. There are two typical cases: when the plane $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) = 0$ intersects the cone and when the only joint point of the cone and the plane is $\Delta\mathbf{p} = 0$, i.e., $\mathbf{p} = \mathbf{p}_0$. The first case occurs if there exists a nonzero vector $\Delta\mathbf{p}$, satisfying both equations $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ and $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) = 0$. Substituting $\Delta\mathbf{p}$ from (12.46), which is the solution of the first equation, into the second one, we obtain

$$(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a} + \mathbf{b}\cos\alpha + \mathbf{c}\sin\alpha) = 0,$$

$$(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b})\cos\alpha + (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c})\sin\alpha = -(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a}),$$

$$\sin(\alpha + \varphi_0) = \xi, \tag{12.47}$$

where

$$\begin{gathered}\xi = -(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a})/\sqrt{(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b})^2 + (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c})^2},\\ \tan\varphi_0 = (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b})/(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c}).\end{gathered} \tag{12.48}$$

Thus, two cases, corresponding to the plane $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) = 0$ intersecting or not intersecting the cone, are determined by the inequalities $|\xi| < 1$ and $|\xi| > 1$, respectively. If $|\xi| < 1$, then two different roots α of (12.47) after substitution into (12.46) give the tangents to the edges of the stability boundary.

The general result for the three-parameter case can be formulated as follows: the form of the stability domain in the neighborhood of $\mathbf{p}_0$ has four typical forms. These forms are determined by the conditions: a) $\det \mathbf{Q} < 0$, $|\xi| > 1$, b) $\det \mathbf{Q} > 0$, $|\xi| > 1$, c) $\det \mathbf{Q} < 0$, $|\xi| < 1$, and d) $\det \mathbf{Q} > 0$, $|\xi| < 1$, and shown in Fig. 12.3 (the stability domain is denoted by S). A part of the stability boundary corresponding to the cone is the parametric resonance boundary (harmonic or subharmonic depending on the sign of $\rho_0 = (-1)^k$), while the other part represents the combination resonance boundary.

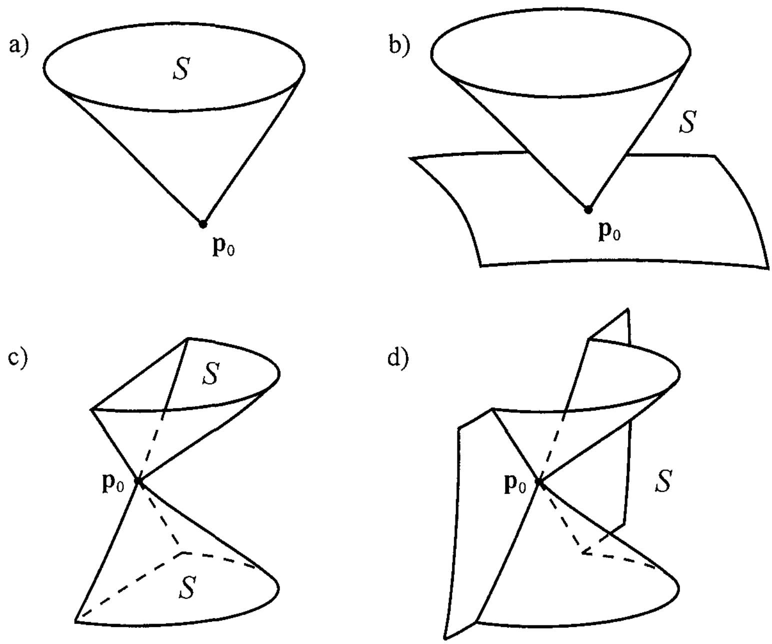


Fig. 12.3 Singularities of the stability domain in the three-parameter space.

Notice that in all the cases a) – d) the instability domain consists of two parts, where one part is the combination resonance domain (determined by complex multipliers lying outside the unit circle) and another part is the parametric resonance domain (determined by real multipliers with $|\rho| > 1$). The boundary between those domains is characterized by existence of a double multiplier $\rho_1(\mathbf{p}) = \rho_2(\mathbf{p})$. This happens, when $\mu_1(\mathbf{p}) = \mu_2(\mathbf{p})$, which means that the discriminant of the characteristic equation for $\mathbf{H}(\mathbf{p})$

is equal to zero:

$$\begin{aligned}(h_{11}(\mathbf{p}) - h_{22}(\mathbf{p}))^2 + 4h_{12}(\mathbf{p})h_{21}(\mathbf{p}) = \\ (\mathbf{h}_{11} - \mathbf{h}_{22}, \Delta\mathbf{p})^2 + 4(\mathbf{h}_{12}, \Delta\mathbf{p})(\mathbf{h}_{21}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|^2) = 0.\end{aligned} \tag{12.49}$$

Equation (12.49) defines a cone surface in the first order approximation, except the degenerate case when the 3×3 matrix $(\mathbf{h}_{11} - \mathbf{h}_{22})^T(\mathbf{h}_{11} - \mathbf{h}_{22}) + 4\mathbf{h}_{12}^T\mathbf{h}_{21}$ is singular. A part of this surface belonging to the instability domain divides the combination and parametric resonance domains.

To determine the form of the stability domain in the degenerate cases $\det \mathbf{Q} = 0$ or $|\xi| = 1$, we need to use higher order approximations of the functions $h_{ij}(\mathbf{p})$ in formulae (12.37).

12.3.5 *General case $n \geq 4$*

Let us consider the case of four or more parameters. If the vectors $\mathbf{h}_{ij}$, $i, j = 1, 2$ (gradients of the functions $h_{ij}(\mathbf{p})$ at $\mathbf{p}_0$) are linearly independent, then the stability domain (12.37), after a nonsingular smooth change of the parameters $\mathbf{p}' = \mathbf{p}'(\mathbf{p})$, $\mathbf{p}'(\mathbf{p}_0) = 0$, in the vicinity of $\mathbf{p}_0$, takes the form

$$\begin{cases} p_1' + p_2' < 0, \\ p_3'p_4' - p_1'p_2' < 0, \\ p_5', \ldots, p_n' \in \mathbb{R}, \end{cases} \tag{12.50}$$

where $p_1' = h_{11}(\mathbf{p})$, $p_2' = h_{22}(\mathbf{p})$, $p_3' = h_{12}(\mathbf{p})$, $p_4' = h_{21}(\mathbf{p})$, and other functions $p_i' = p_i'(\mathbf{p})$, $i = 5 \ldots, n$, are chosen such that the Jacobian matrix $[d\mathbf{p}'/d\mathbf{p}]$ at $\mathbf{p} = \mathbf{p}_0$ is nonsingular. Thus, in case $n \geq 4$ there is one possible form (12.50) of the stability domain. The case, when the vectors $\mathbf{h}_{ij}$ are linearly dependent, can be studied using higher order approximations of the functions $h_{ij}(\mathbf{p})$.

12.4 Stability of pipe conveying pulsating fluid

As an application of the presented theory, we consider a uniform flexible cantilever pipe of length L, mass per unit length m, and bending stiffness EI, conveying incompressible fluid. Let the mass of the fluid per unit length be M and the flow velocity be $U(t)$. The pipe hangs down vertically and the undeformed pipe axis coincides with the x-axis; see Fig. 12.4. We consider small lateral motions of the pipe $y(x, t)$ in the plane (x, y).

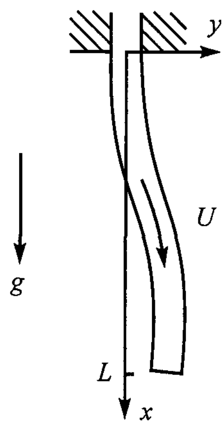


Fig. 12.4 Elastic pipe conveying pulsating fluid.

The dimensionless equation for small vibrations of the pipe has the form [Paidoussis and Issid (1974)]

$$\begin{aligned}\alpha\frac{\partial^5\eta}{\partial\xi^4\partial\tau}+\frac{\partial^4\eta}{\partial\xi^4}+\left(u^2+(\beta^{1/2}\frac{du}{d\tau}-\gamma)(1-\xi)\right)\frac{\partial^2\eta}{\partial\xi^2}\\+2\beta^{1/2}u\frac{\partial^2\eta}{\partial\xi\partial\tau}+\gamma\frac{\partial\eta}{\partial\xi}+\frac{\partial^2\eta}{\partial\tau^2}=0,\end{aligned}\tag{12.51}$$

and the boundary conditions are

$$\eta=\frac{\partial\eta}{\partial\xi}=0 \text{ at } \xi=0; \qquad \frac{\partial^2\eta}{\partial\xi^2}=\frac{\partial^3\eta}{\partial\xi^3}=0 \text{ at } \xi=1. \tag{12.52}$$

The dimensionless variables and parameters are given by

$$\begin{aligned}\xi=\frac{x}{L},\ \eta=\frac{y}{L},\ \tau=\frac{t}{L^2}\left(\frac{EI}{M+m}\right)^{1/2},\ \alpha=\frac{E^*}{L^2}\left(\frac{I}{E(M+m)}\right)^{1/2},\\ u=UL\left(\frac{M}{EI}\right)^{1/2},\ \beta=\frac{M}{M+m},\ \gamma=\frac{M+m}{EI}L^3g,\end{aligned}\tag{12.53}$$

where E^* is the coefficient of internal dissipation and g is the acceleration of gravity. The external viscous damping coefficient is assumed to be zero (the parameter $\chi=0$ in [Paidoussis and Issid (1974)]).

Let us study the case of pulsating flow velocity $u(t)=u_0(1+\mu\cos\Omega\tau)$, where $u_0=(M/EI)^{1/2}U_0L$, μ, and $\Omega=[(M+m)/EI]^{1/2}\widehat{\Omega}L^2$ are dimensionless parameters; U_0, μ, and $\widehat{\Omega}$ are the flow mean velocity, the amplitude, and the frequency of pulsation, respectively.

12.4.1 *Discretization by Galerkin's method*

For the numerical analysis of system (12.51), (12.52) we use Galerkin's method. For this purpose, the solution $\eta(\xi, \tau)$ is expressed as a linear combination of m' normalized coordinate functions $\varphi_j(\xi)$ with the coefficients $q_j(\tau)$, where the functions $\varphi_j(\xi)$ are the free-vibration eigenfunctions of a uniform cantilever beam. Application of Galerkin's method to equation (12.51) gives [Paidoussis and Issid (1974)]

$$\ddot{\mathbf{q}} + \mathbf{B}(\tau)\dot{\mathbf{q}} + \mathbf{C}(\tau)\mathbf{q} = 0, \tag{12.54}$$

where $\mathbf{q} = (q_1, \ldots, q_{m'})^T$ and the nonsymmetric $m' \times m'$ matrices $\mathbf{B}$ and $\mathbf{C}$ are

$$\begin{aligned} \mathbf{B}(\tau) &= \alpha\mathbf{\Lambda} + 2u\sqrt{\beta}\mathbf{K}, \\ \mathbf{C}(\tau) &= \mathbf{\Lambda} + (u^2 + \dot{u}\sqrt{\beta} - \gamma)\mathbf{L} + (\gamma - \dot{u}\sqrt{\beta})\mathbf{N} + \gamma\mathbf{K}, \end{aligned} \tag{12.55}$$

The elements λ_{sr}, k_{sr}, l_{sr}, and n_{sr} of the $m' \times m'$ matrices $\mathbf{\Lambda}$, $\mathbf{K}$, $\mathbf{L}$, and $\mathbf{N}$ are defined by the expressions

$$\begin{aligned} \lambda_{sr} &= \int_0^1 \varphi_s\varphi_r^{(4)}d\xi, \quad k_{sr} = \int_0^1 \varphi_s\varphi_r' d\xi, \\ l_{sr} &= \int_0^1 \varphi_s\varphi_r'' d\xi, \quad n_{sr} = \int_0^1 \xi\varphi_s\varphi_r'' d\xi. \end{aligned} \tag{12.56}$$

The values of integrals (12.56) are given in [Paidoussis and Issid (1974)].

Equation (12.54) can be transformed to the form

$$\dot{\mathbf{x}} = \mathbf{G}(\tau)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{G}(\tau) = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{C}(\tau) & -\mathbf{B}(\tau) \end{pmatrix}. \tag{12.57}$$

The matrix $\mathbf{G}(\tau)$ of dimension $m = 2m'$ is periodic with the period $T = 2\pi/\Omega$. Introducing new time $\tau' = \Omega\tau$, system (12.57) is transformed to the form $d\mathbf{x}/d\tau' = \mathbf{G}'(\tau')\mathbf{x}$, where $\mathbf{G}'(\tau') = \mathbf{G}(\tau'/\Omega)/\Omega$ is a periodic matrix with the constant period $T' = 2\pi$. This form is more convenient for computation of derivatives of the Floquet matrix. In what follows, $m' = 6$ functions in Galerkin's method are used. Hence, the matrix $\mathbf{G}$ has dimension $m = 12$.

12.4.2 *Flow with constant velocity*

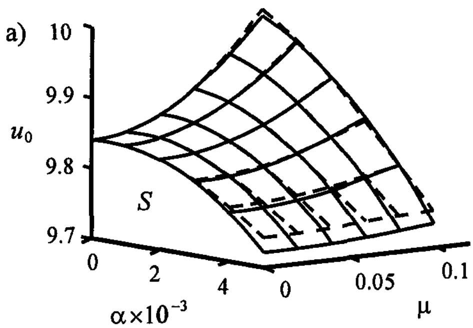
Let us consider the pipe with the parameters $\gamma = 10$, $\beta = 1/2$, $\mu = \alpha = 0$. This means that the mass of the fluid in the pipe is equal to the mass of the pipe, and there are no pulsations and internal dissipation. Then system (12.57) is autonomous. Increasing the dimensionless flow velocity u_0 starting from zero and checking numerically the stability condition $\mathrm{Re}\,\lambda < 0$ for all the eigenvalues of the matrix $\mathbf{G}$, we find the critical velocity of the fluid $u_0^c = 9.84$. At this velocity the autonomous system is subjected to the dynamic instability (flutter) associated with the third mode. The flutter frequency is equal to $\omega = 28.11$ and the matrix $\mathbf{G}_0$ at $u_0 = u_0^c$ has simple eigenvalues $\lambda = \pm i\omega$ on the imaginary axis.

12.4.3 *Stability domain in regular case*

Let us study a change of the critical flow velocity in the presence of small pulsation with the frequency $\Omega = 21$ and small internal damping $\alpha \geq 0$. For this purpose, we need to find the stability boundary in the three-parameter space $\mathbf{p} = (\mu, \alpha, u_0)$ in the neighborhood of the point $\mathbf{p}_0 = (0, 0, u_0^c)$. In this case $\omega T_0 = 2\pi\omega/\Omega = 2.677\pi \neq \pi k$ for any integer k. Using results of Section 12.2, we conclude that the stability boundary is a smooth surface being the boundary of combination resonance domain.

Computing derivatives of the Floquet matrix $\mathbf{F}(\mathbf{p})$ and of the simple multiplier $\rho(\mathbf{p})$, we find approximation (12.18) of the stability boundary up to small terms of any order. Accuracy of the approximation can be estimated numerically by comparison of approximations of different orders. In Fig. 12.5 the third and fifth order approximations of the stability boundary are shown. It can be concluded from Fig. 12.5 that pulsations stabilize the system (increase the critical mean velocity of the flow), while the internal damping has destabilizing effect, which becomes stronger for higher amplitudes of pulsation.

The dashed line in Fig. 12.5 shows the exact stability boundary. It was found by the calculation of the Floquet matrix at different values of the parameters, where differential equation (12.5) was solved using the Runge–Kutta method. It can be seen that the fifth order approximation of the stability boundary is almost identical to the exact one for the range of parameters under consideration; see Fig. 12.5b. Notice that the time spent for calculation of the third and fifth order approximations and the exact form of the stability domain relates as 1 : 6 : 360. This shows the efficiency of the developed method for numerical analysis.



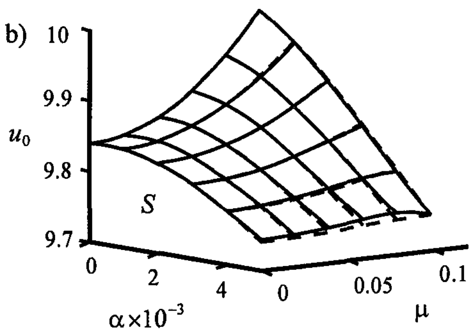


Fig. 12.5 Third (a) and fifth (b) order approximations of the stability boundary.

12.4.4 *Stability domain in singular case*

Let us study influence of pulsations with different amplitudes μ and frequencies Ω on stability of the pipe. We consider the point $\mathbf{p}_0 = (2\omega, 0, u_0^c)$ in the parameter space $\mathbf{p} = (\Omega, \mu, u_0)$. In this case $T_0 = 2\pi/\Omega_0 = \pi/\omega$ and $\omega T_0 = k\pi$, $k = 1$. Hence, this is the case when the stability boundary has a singularity.

The Floquet matrix $\mathbf{F}(\mathbf{p}_0) = \exp(\mathbf{G}_0 T_0)$ possesses a semi-simple double multiplier $\rho_0 = -1$. The first order approximation of the stability domain in the neighborhood of $\mathbf{p}_0$ is given by equation (12.39), where the vectors

$\mathbf{h}_{ij}$ and the matrix $\mathbf{Q}$ calculated by expressions (12.8), (12.40), (12.42) are

$$\begin{gathered}\mathbf{h}_{11} = (0, 1.63, 0.74), \quad \mathbf{h}_{12} = (-0.056, 0.24, 0.27),\\ \mathbf{h}_{21} = (0.056, 0.24, -0.27), \quad \mathbf{h}_{22} = (0, -1.63, 0.74),\\ \mathbf{Q} = \begin{pmatrix} -0.0031 & 0 & 0.015 \\ 0 & 2.72 & 0 \\ 0.015 & 0 & -0.62 \end{pmatrix}.\end{gathered} \tag{12.58}$$

Using (12.58), the first inequality in (12.39) takes the form

$$u_0 < u_0^c. \tag{12.59}$$

Since $\det \mathbf{Q} = 0.0047 > 0$, the second inequality in (12.39) defines the external part of cone (12.46) as

$$\begin{gathered}\Delta\mathbf{p} = s(\mathbf{a} + \mathbf{b}\cos\alpha + \mathbf{c}\sin\alpha), \quad s \in \mathbb{R},\ 0 \leq \alpha < 2\pi,\\ \mathbf{a} = (0, -0.042, 0),\ \mathbf{b} = (0.0021, 0, -0.087),\ \mathbf{c} = (-1.3, 0, 0.031).\end{gathered} \tag{12.60}$$

Intersection of the half-space (12.59) with the external part of the cone (12.60) gives the first order approximation of the stability domain; see Fig. 12.6. In Fig. 12.6 only the half-space $\mu \geq 0$ is shown (the other part $\mu \leq 0$ is symmetric with respect to the plane $\mu = 0$). The stability boundary has a singularity at $\mathbf{p}_0$ of the type d); see Fig. 12.3d. The instability domain consists of the subharmonic parametric resonance and combination resonance domains separated by the boundary (12.49) shown in Fig. 12.6 by dotted lines.

Calculating higher order derivatives of the matrices $\mathbf{F}(\mathbf{p})$ and $\mathbf{F}'(\mathbf{p})$ at $\mathbf{p}_0$, we can find higher order terms in the Taylor series of the functions $h_{ij}(\mathbf{p})$ (12.38). As a result, we obtain higher order approximations of the stability domain (12.37). In Fig. 12.7 the fourth order approximation of the stability boundary is shown. For comparison, the dashed lines in Fig. 12.7 denote the exact stability boundary calculated numerically. The computation of the first and fourth order approximations took 4 and 150 seconds, respectively, while numerical calculation of the stability boundary by the Floquet method needed 5-6 hours (the calculations were carried out on PC using a standard MATLAB package). Notice that even the first order approximation provides a good qualitative and quantitative description of the stability domain.

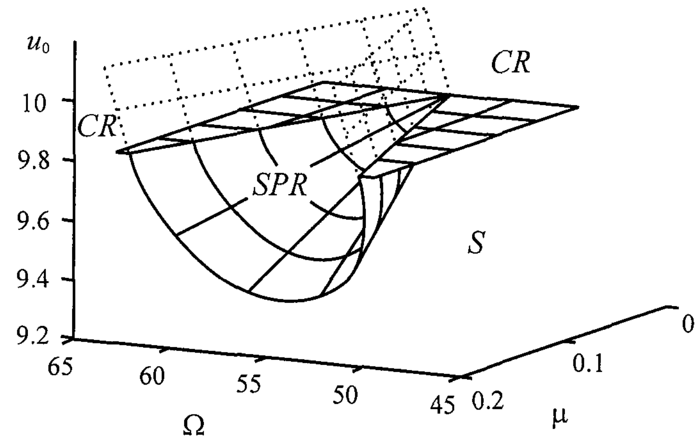


Fig. 12.6 First order approximation of the stability (S), subharmonic parametric resonance (SPR), and combination resonance (CR) domains.

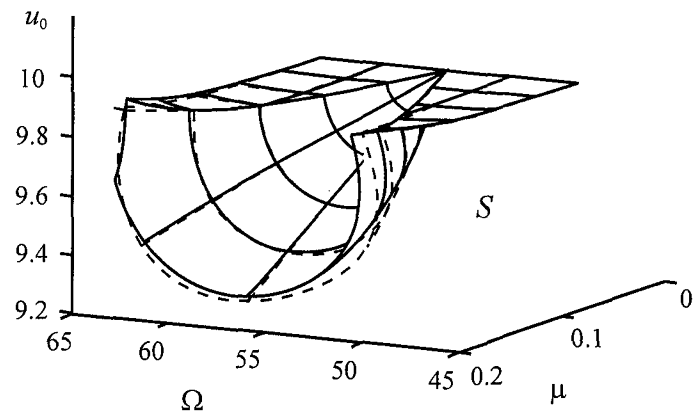


Fig. 12.7 Fourth order approximation of the stability domain.

It can be seen from Figs. 12.6 and 12.7 that small pulsations can destabilize or stabilize the system. For example, pulsations with the amplitude $\mu = 0.2$ and the frequency $\Omega = 2\omega$ change the critical mean velocity of the flow by $u_0 - u_0^c = -0.54$, which leads to about 5.5% decrease of the critical velocity compared with the autonomous system. Therefore, pulsations with the frequency $\Omega \approx 2\omega$ are dangerous for stability of the system. Notice that the relation $\Omega = 2\omega$ is similar to the condition of simple parametric resonance well known in the stability theory of oscillatory periodic systems, where ω is a natural frequency of a conservative autonomous system, see Chapter 11. Recall that in our problem ω is the flutter frequency of the non-conservative autonomous system.

12.4.5 *Stability diagrams on the amplitude-frequency plane*

In the papers devoted to the stability analysis of pipes conveying pulsating fluid the stability domains (stability diagrams) are plotted on the amplitude–frequency plane (μ, Ω) for fixed values of the flow velocity u_0, see [Paidoussis and Issid (1974); Paidoussis and Sundararajan (1975)]. Analyzing singularities of the stability boundary in the three-parameter space $\mathbf{p} = (\Omega, \mu, u_0)$, we can give qualitative description of typical stability diagrams.

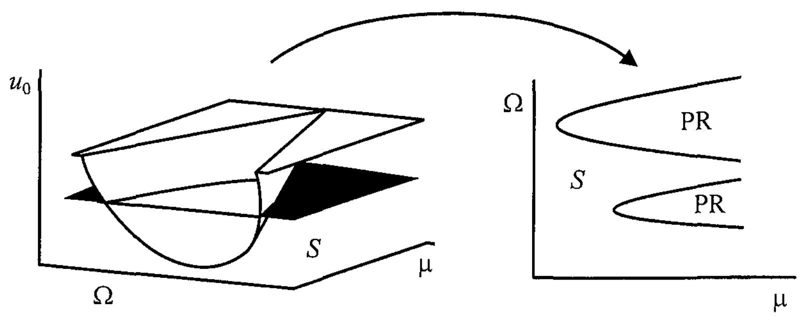


Fig. 12.8 Singularities of stability boundary and stability diagrams.

It is shown that singularities of the stability boundary arise at the points $\mathbf{p} = (\Omega_0, 0, u_0^c)$ with $\Omega_0 = 2\omega/k$, where ω is the flutter frequency of the autonomous system. The stability domain is symmetric with respect to the $\mu = 0$ plane; the upper part of this plane ($\mu = 0$, $u_0 > u_0^c$) belongs to the instability domain, while the lower part belongs to the stability domain. Using these properties we conclude that all the singularities of the stability boundary are of the fourth type, see Fig. 12.3d, where the cone axis is parallel to the μ-axis, and the surface determining the combination resonance boundary is tangent to the $u_0 = u_0^c$ plane, see Figs. 12.6 and 12.8. Hence, stability diagrams on the plane (μ, Ω) for $u_0 < u_0^c$ typically consist of several convex parametric resonance zones at the frequencies $\Omega \approx 2\omega/k$, being cross-sections of the cones, corresponding to different singular points, by the $u_0 = const$ plane, see Fig. 12.8. With a decrease of u_0 those domains appear for higher values of the pulsation amplitude

μ. Notice that for the pipe under consideration similar singularities appear at the super-critical value of the flow velocity $u_0 = 16.9$, when the second mode becomes unstable (the corresponding frequency is $\omega = 59.5$). These singularities lie inside the instability domain, but they also give rise to convex parametric resonance zones on the plane (μ, Ω) at $u_0 < u_0^c$. Since instability of the autonomous system is associated with the third and second modes, there are no parametric resonance zones corresponding to the first mode. For $u_0 > u_0^c$ the plane (μ, Ω) consists mostly of the instability domain including both parametric and combination resonance zones.

Concluding Remarks

In this book we have studied linear multi-parameter stability problems for multiple degrees of freedom systems, which can be treated as linearized equations near an equilibrium state or given motion. This study allows one to construct stability and instability domains in the space of multiple parameters. As further development of the analysis, we suggest to investigate stability problems for distributed parameter (continuous) systems described by partial differential equations. It would be interesting to investigate how the eigenvalues of those systems depend on several parameters, including possible changes from discrete to continuous spectrum. As another perspective direction of research, we suggest multi-parameter analysis of nonlinear effects and bifurcations, in particular, influence of nonlinearities on stability domains of marginally stable systems. We think that investigation of bifurcations and post-critical behavior of nonlinear systems near singularities of the stability boundaries will lead to discoveries of new physical effects.

It would be interesting to apply the theory and methods presented in this book to the study of complicated stability problems in different areas of natural sciences, e.g. quantum physics. One could expect new phenomena and effects associated with singularities and bifurcations in economics and social sciences.

We live in the world full of instabilities and catastrophes dependent on many parameters and circumstances. We hope that the book will help to find stable solutions providing safe operation of machines and devices in engineering, stable and expectable development of processes in natural and social sciences.

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