

Stability Analysis via Matrix Functions Method

Part I

A. A. Martynyuk



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Stability Analysis via Matrix Functions Method: Part I

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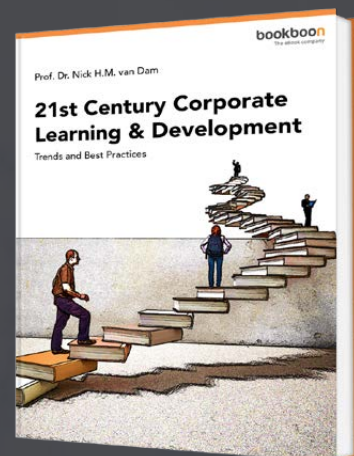
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PREFACE

One can hardly name a branch of natural science or technology in which the problems of stability do not claim the attention of scholars, engineers, and experts who investigate natural phenomena or operate designed machines or systems. If, for a process or a phenomenon, for example, atom oscillations or a supernova explosion, a mathematical model is constructed in the form of a system of differential equations, the investigation of the latter is possible either by a direct (numerical as a rule) integration of the equations or by its analysis by qualitative methods.

The direct Liapunov method based on scalar auxiliary function proves to be a powerful technique of qualitative analysis of the real world phenomena.

This volume examines new generalizations of the matrix-valued auxiliary function. Moreover the matrix-valued function is a structure the elements of which compose both scalar and vector Liapunov functions applied in the stability analysis of nonlinear systems.

Due to the concept of matrix-valued function developed in the book, the direct Liapunov method becomes yet more versatile in performing the analysis of nonlinear systems dynamics.

The possibilities of the generalized direct Liapunov method are opened up to stability analysis of solutions to ordinary differential equations, singularly perturbed systems, and systems with random parameters.

The reader with an understanding of fundamentals of differential equations theory, elements of motion stability theory, mathematical analysis, and linear algebra should not be confused by the many formulas in the book. Each of these subjects is a part of the mathematics curriculum of any university.

In view of the fact that beginners in motion stability theory usually face some difficulties in its practical application, the sets of problems taken from various branches of natural sciences and technology are solved at the end of each chapter. The problems of independent value are integrated in Chapter 5.

A certain contribution to the development of the Liapunov matrix function method has been made by the scientists and experts of Belgrade University, Technical University in Zurich, and Stability of Processes Department of Institute of Mechanics National Academy of Sciences of Ukraine.

The useful remarks by the reviewers of Marcel Dekker, Inc., have been taken into account in the final version of the book. Great assistance in preparing the manuscript for publication has been rendered by S.N. Rasshivalova, L.N. Chernetskaya, A.N. Chernienko, and V.I. Goncharenko. The author expresses his sincere gratitude to all these persons.

A. A. Martynyuk

NOTATION

- R the set of all real numbers
 $R_+ = [0, +\infty) \subset R$ the set of all nonnegative numbers
 R^k k -th dimensional real vector space
 $R \times R^n$ the Cartesian product of R and R^n
 $\mathcal{T} = [-\infty, +\infty] = \{t: -\infty \leq t \leq +\infty\}$ the largest time interval
 $\mathcal{T}_\tau = [\tau, +\infty) = \{t: \tau \leq t < +\infty\}$ the right semi-open unbounded interval associated with τ
 $\mathcal{T}_i \subseteq R$ a time interval of all initial moments to under consideration (or, all admissible t_0)
 $\mathcal{T}_0 = [t_0, +\infty) = \{t: t_0 \leq t < +\infty\}$ the right semi-open unbounded interval associated with t_0
 $\|x\|$ the Euclidean norm of x in R^n
 $\chi(t; t_0, x_0)$ a motion of a system at $t \in R$ iff $x(t_0) = x_0$, $\chi(t_0; t_0, x_0) \equiv x_0$
 $B_\varepsilon = \{x \in R^n: \|x\| < \varepsilon\}$ open ball with center at the origin and radius $\varepsilon > 0$
 $\delta_M(t_0, \varepsilon) = \max \{\delta: \delta = \delta(t_0, \varepsilon) \ni x_0 \in B_\delta(t_0, \varepsilon) \Rightarrow \chi(t; t_0, x_0) \in B_\varepsilon, \forall t \in \mathcal{T}_0\}$ the maximal δ obeying the definition of stability
 $\Delta_M(t_0) = \max \{\Delta: \Delta = \Delta(t_0), \forall \rho > 0, \forall x_0 \in B_\Delta, \exists \tau(t_0, x_0, \rho) \in (0, +\infty) \ni \chi(t; t_0, x_0) \in B_\rho, \forall t \in \mathcal{T}_\tau\}$ the maximal Δ obeying the definition of attractivity
 $\tau_m(t_0, x_0, \rho) = \min \{\tau: \tau = \tau(t_0, x_0, \rho) \ni \chi(t; t_0, x_0) \in B_\rho, \forall t \in \mathcal{T}_\tau\}$ the minimal τ satisfying the definition of attractivity
 \mathcal{N} a time-invariant neighborhood of original of R^n
 $f: R \times \mathcal{N} \rightarrow R^n$ a vector function mapping $R \times \mathcal{N}$ into R^n
 $C(\mathcal{T}_\tau \times \mathcal{N})$ the family of all functions continuous on $\mathcal{T}_\tau \times \mathcal{N}$
 $C^{(i,j)}(\mathcal{T}_\tau \times \mathcal{N})$ the family of all functions i -times differentiable on \mathcal{T}_τ and j -times differentiable on \mathcal{N}

- $D^+v(t, x)$ ($D^-v(t, x)$) the upper right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t, x)
- $D_+v(t, x)$ ($D_-v(t, x)$) the lower right (left) Dini derivative of v along $\chi(t; t_0, x_0)$ at (t, x)
- $D^*v(t, x)$ denotes that both $D^+v(t, x)$ and $D_+v(t, x)$ can be used
- $Dv(t, x)$ the Eulerian derivative of v along $\chi(t; t_0, x_0)$ at (t, x)
- $\lambda_i(\cdot)$ the i -th eigenvalue of a matrix (\cdot)
- $\lambda_M(\cdot)$ the maximal eigenvalue of a matrix (\cdot)
- $\lambda_m(\cdot)$ the minimal eigenvalue of a matrix (\cdot)

PRELIMINARIES

1.1 Introduction

Nonlinear dynamics of systems is a branch of science that studies actual equilibriums and motions of natural or artificial real objects. However it is known that hardly every state of a really functioning system is observed in practice that corresponds to a mathematically strict solution of either equilibrium or differential motion equations. It has been found out that only those equilibriums and motions of real systems are evident that possess certain “resistivity” to the outer perturbations. The equilibrium states and motions of this kind are referred to as stable while the others are called unstable.

The notion of stability had been clearly intuited but difficult to formulate and only Liapunov (see Liapunov [101]) managed to give accurate definitions (for the historical aspect see Moiseev [146]).

Section 1.2 presents recent strict definitions of stability of nonautonomous systems and other general information necessary for proper understanding of the monograph. Presently there is a series of monographs and textbooks that expose the direct Liapunov method of motion stability investigation based on auxiliary scalar function and provide a lot of many illustrative examples of its application. The books by Chetaev [19], Malkin [107], Lur’e [104], Duboshin [32], Demidovich [24], Krasovskii [89], Barbashin [10], Zubov [177], Letov [99], Bellman [15], Hahn [66], Harris and Milles [68], Yoshizawa [174], LaSalle and Lefschetz [98], Coppel [23], Lakshmikantham, Leela and Martynyuk [94] and others show the modern level of Liapunov method development in qualitative theory of equations.

Section 1.3 (subsection 1.3.1) gives a brief account of results obtained in this direction.

In 1962 it was proposed by Bellman [16], Martosov [132], and Melnikov [139] to apply Liapunov functions consisting of more than one component.

Such functions were referred to as vector Liapunov functions. A quick development of investigations in the field has been summarized in a series of monographs such as in Grujić [55], Michel and Miller [143], Šiljak [167], Rouche, Habets and Laloy [159], LaSalle [97], Grujić, Martynyuk and Ribbens-Pavella [57], Lakshmikantham, Matrosov and Sivasundaram [96], Abdullin, Anapolskii et al. [1].

Section 1.3 (subsection 1.3.2) provides a short survey of the direct Liapunov method development in terms of vector function.

The preliminary information and the survey of the direct Liapunov method development in terms of both scalar and vector auxiliary functions are cited here with the aim to prepare the reader to the study of a new method in qualitative theory of equations called the method of matrix Liapunov functions.

1.2 On Definition of Stability

1.2.1 Liapunov's original definition

Liapunov started his investigations with the following (see Liapunov [101], p.11):

Let us consider any material system with k degrees of freedom. Let q_1, q_2, \dots, q_k be k independent variables, which we use to determine its position.

We shall assume that quantities taking real values for all real system positions are taken for such variables.

Considering the mentioned variables as functions in time t we shall denote their first time derivatives by q'_1, q'_2, \dots, q'_k .

In every dynamic problem, in which forces are prespecified in a certain way, such functions will satisfy some k second order differential equations.

Let any particular solution for such equations be found

$$q_1 = f_1(t), \quad q_2 = f_2(t), \quad \dots, \quad q_k = f_k(t),$$

in which the quantities q_j are expressed as real functions in t , which at every t give only possible values to them.¹

¹It can happen that the quantities q_j by their choice do not take all real values but only those not greater than – and not less than certain bounds.

To that particular solution will correspond a definite motion of our system. Comparing it in a known sense with others, which are possible under the same forces, we shall call that motion *unperturbed*, and all others, with which it is compared, *perturbed*.

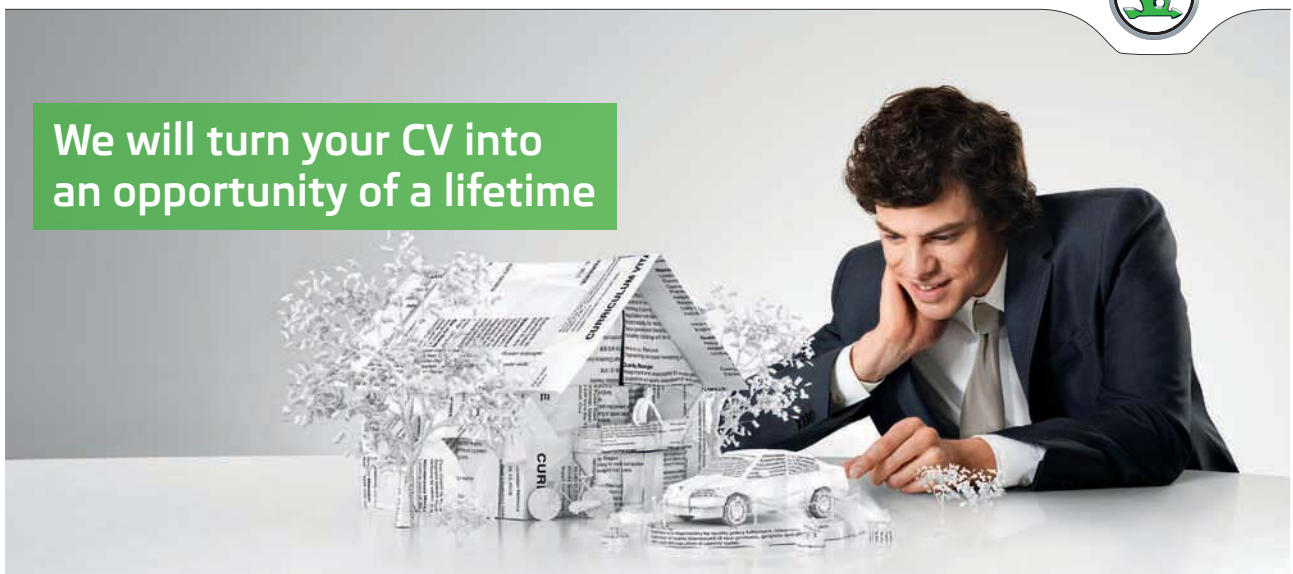
For t_0 understood a given instant, let us denote the values corresponding to it of quantities q_i , q'_i along any motion with q_{i0} , q'_{i0} .

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Let

$$q_{10} = f_1(t_0) + \varepsilon_1, q_{20} = f_2(t_0) + \varepsilon_2, \dots, q_{k0} = f_k(t_0) + \varepsilon_k, \\ q'_{10} = f'_1(t_0) + \varepsilon'_1, q'_{20} = f'_2(t_0) + \varepsilon'_2, \dots, q'_{k0} = f'_k(t_0) + \varepsilon'_k,$$

where $\varepsilon_j, \varepsilon'_j$ are real-valued constants.

Prespecifying the constants, which will be called *perturbations*, a perturbed motion is determined. We shall assume that we may prescribe them every number sufficiently small.

By speaking about perturbed motions, *close* to the unperturbed one, we shall comprehend motions, for which the perturbations are numerically small.

Let Q_1, Q_2, \dots, Q_n be any given continuous real-valued functions of quantities

$$q_1, q_2, \dots, q_k, \quad q'_1, q'_2, \dots, q'_k.$$

Along the unperturbed motion they become known functions of t , which will be denoted by F_1, F_2, \dots, F_n . Along a perturbed motion they will be functions of quantities

$$t, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_k.$$

When all $\varepsilon_j, \varepsilon'_j$ are equal to zero, then quantities

$$Q_1 - F_1, Q_2 - F_2, \dots, Q_n - F_n$$

will be equal to zero for every t . However, if the constants $\varepsilon_j, \varepsilon'_j$ are not zero, but all are infinitely small, then a question rises: is it possible to specify such the latter never become greater than their values?

A solution of the question, which is the topic of our investigations, depends on both a character of the considered unperturbed motion and a choice of the functions Q_1, Q_2, \dots, Q_n and the instant t_0 . Under a specific choice of the latter, the reply to the question,

respectively, will characterize in some sense the unperturbed motion, by determining a feature of the latter, which will be called *stability*, or that contrary to it, will be called *instability*.

We shall be exclusively interested in those cases in which the solution of the considered question does not depend on a choice of the instant t_0 , when perturbations are acting. Thus we accept the following definition.

Let L_1, L_2, \dots, L_n be arbitrary given positive numbers. If all L_s , regardless of how small they are, can be selected positive numbers $E_1, E_2, \dots, E_k, E'_1, E'_2, \dots, E'_k$ so that for all real $\varepsilon_j, \varepsilon'_j$, satisfying the conditions¹

$$|\varepsilon_j| \leq E_j, \quad |\varepsilon'_j| \leq E'_j \quad (j = 1, 2, \dots, k),$$

and for all t , greater than t_0 , the inequalities

$$|Q_1 - F_1| < L_1, \quad |Q_2 - F_2| < L_2, \quad \dots, \quad |Q_n - F_n| < L_n,$$

are satisfied, then the unperturbed motion is stable with respect to the quantities Q_1, Q_2, \dots, Q_n ; otherwise it is unstable with respect to the same quantities.

1.2.2 Comments on Liapunov's original definition

COMMENT 1.2.1. The inequalities on $|\varepsilon_j|$ and $|\varepsilon'_j|$ are weak and those on $|Q_j - F_j|$ are strong. This asymmetry is usually avoided imposing the same type of inequalities on all $|\varepsilon_j|, |\varepsilon'_j|$ and $|Q_j - F_j|$, which yields stability definitions equivalent to Liapunov's original definition. This equivalence can be easily proved.

COMMENT 1.2.2. Stability of the reference motion was defined by Liapunov with respect to arbitrary functions Q_j that are continuous in all q_i, q'_i . This has been very thoughtful and physically important because Q_j can represent energy or material flow. In this connection Liapunov introduced new variables x_i ,

$$x_i = Q_i - F_i, \quad i = 1, 2, \dots, n,$$

¹In general $|x|$ means the absolute value of a real-, or modulus of a complex quantity x .

and accepted the following (Liapunov [101], p.15):

We shall assume that the number n and the functions Q_s , are such, that the order of the system is n and that it is reducible to the normal form

$$(1) \quad \frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \dots, \quad \frac{dx_n}{dt} = X_n,$$

and everywhere in the sequel we shall consider these last equations, calling them the differential equations of a perturbed motion.

All X_s in the equations (1) are known functions of quantities

$$x_1, x_2, \dots, x_n, t,$$

vanishing for

$$x_1 = x_2 = \dots = x_n = 0.$$

COMMENT 1.2.3. Stability of the reference motion requires arbitrary closeness of the perturbed motions to the reference motion provided their sufficient closeness is assured at the initial instant t_0 .

COMMENT 1.2.4. The closeness of the perturbed motions to the reference motion is to be realized over unbounded time interval $\mathcal{T}_0^* = (t_0, +\infty]$, i.e. for *all* t greater than t_0 . This point has been commonly neglected in the literature. Namely, the closeness has been commonly required either on $\overline{\mathcal{T}}_0 = [t_0, +\infty]$ or on $\mathcal{T}_0 = [t_0, +\infty)$, i.e. for *all* t not less than t_0 . This difference can be crucial in cases when system motions are discontinuous at $t = t_0$.

COMMENT 1.2.5. A.M.Liapunov defined stability of the reference motion for cases when it is not influenced by t_0 . However, the initial moment can essentially influence stability of the reference motion in cases when system motions are not continuous in t . Besides, t_0 can essentially influence the maximal admissible values of all E_j and E'_j even when all system motions are continuous in t .

COMMENT 1.2.6. The stability of the reference motion was defined by A.M.Liapunov with respect to initial perturbations of the general coordinates q_j, q'_j , rather than with respect to persistent external disturbances.

COMMENT 1.2.7. The stability definition does not care about the values E_j and E'_j except that they must be positive. Hence, for large values of all L_j , the maximal admissible E_j and E'_j can be so small that they are not useful for engineering needs.

1.2.3 Relationship between the reference motion and the zero solution

Let $2k$ be the order of the system and $y_i, i = 1, 2, \dots, 2k,$ be its i -th state variable. Using basic physical laws (e.g. the law of the energy conservation and the law of the material conservation) we can for a large class of systems get state differential equations in the following scalar form

$$(1.2.1) \quad \frac{dy_i}{dt} = Y_i(t, y_1, \dots, y_{2k}), \quad i = 1, 2, \dots, 2k,$$

or in the equivalent vector form

$$(1.2.2) \quad \frac{dy}{dt} = Y(t, y),$$

where* $y = (y_1, y_2, \dots, y_{2k})^T \in R^{2k}$ and $Y = (Y_1, Y_2, \dots, Y_{2k})^T, Y: \mathcal{T} \times R^{2k} \rightarrow R^{2k}$. A motion of (1.2.2) is denoted by $\eta(t; t_0, y_0), \eta(t_0; t_0, y_0) \equiv y_0$, and the reference motion $\eta_r(t; t_0, y_{r0})$. From the physical point of view the reference motion should be realizable by the system. From the mathematical point of view this means that the reference motion is a solution of (1.2.2),

$$(1.2.3) \quad \frac{d\eta_r(t; t_0, y_{r0})}{dt} \equiv Y[t, \eta_r(t; t_0, y_{r0})].$$

Let the Liapunov transformation of coordinates be used,

$$(1.2.4) \quad x = y - y_r,$$

where $y_r(t) \equiv \eta_r(t; t_0, y_{r0})$. Let $f: \mathcal{T} \times R^{2k} \rightarrow R^{2k}$ be defined by

$$(1.2.5) \quad f(t, x) = Y[t, y_r(t) + x] - Y[t, y_r].$$

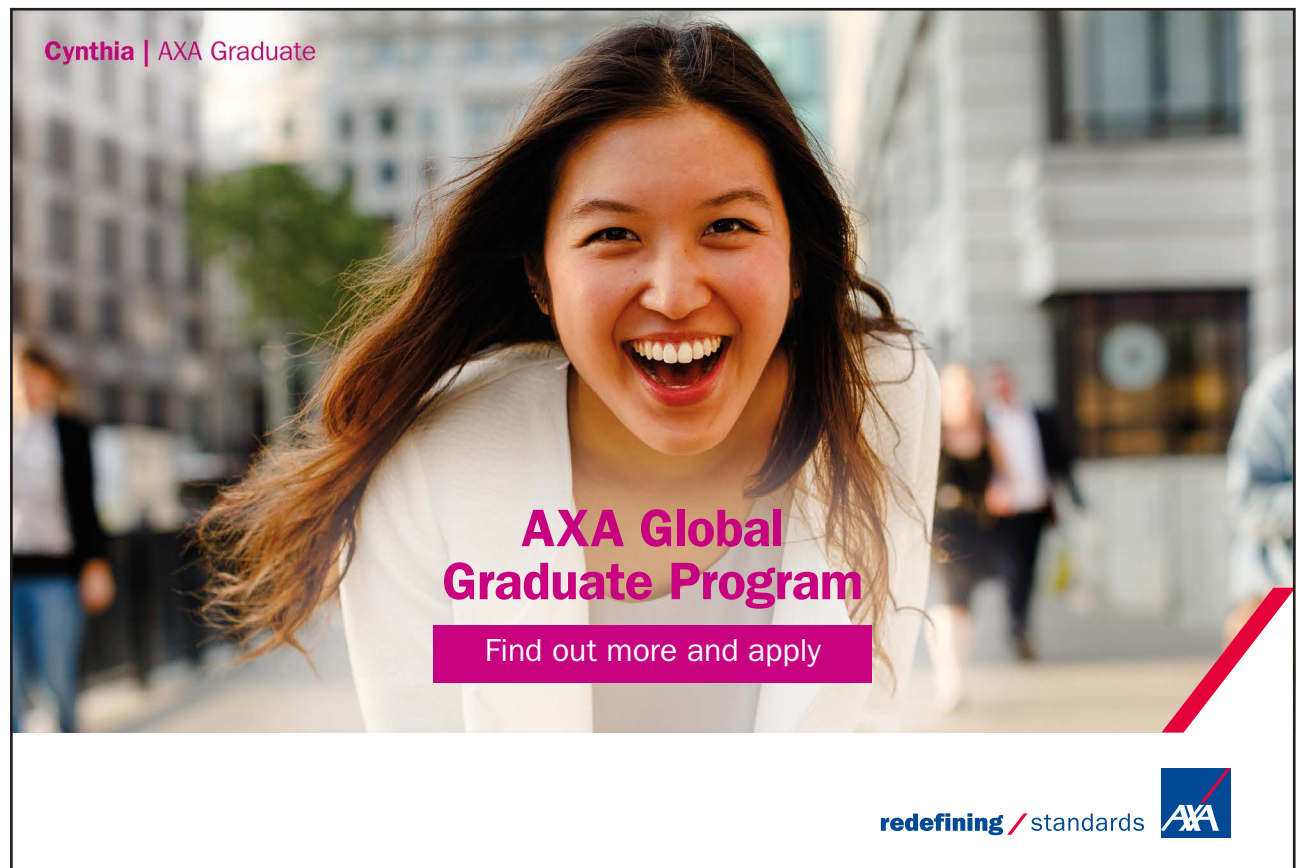
*In Liapunov's notation $y = (q_1, q_2, \dots, q_k, q'_1, q'_2, \dots, q'_k)^T$.

It is evident that

$$(1.2.6) \quad f(t, 0) \equiv 0.$$

Now (1.2.2) – (1.2.5) yield


$$(1.2.7) \quad \frac{dx}{dt} = f(t, x).$$



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In this way, the behavior of perturbed motions related to the reference motion (in total coordinates) is represented by the behavior of the state deviation x with respect to the zero state deviation. The reference motion in the total coordinates y_i is represented by the zero deviation $x = 0$ in state deviation coordinates x_i . With this in mind, the following result emphasizes complete generality of both Liapunov's second method and results represented in Liapunov [101] for the system (1.2.7). Let $Q: R^{2k} \rightarrow R^n$, $n = 2k$ is admissible but not required.

THEOREM 1.2.1. *Stability of $x = 0$ of the system (1.2.7) with respect to $Q = x$ is necessary and sufficient for stability of the reference motion η_r of the system (1.2.2) with respect to every vector function Q that is continuous in y .*

PROOF. *Necessity.* This part is true because $Q(y) = y$ is continuous in y and evidently stability of $x = 0$ with respect to x is implied by stability of η_r with respect to $Q(y) = y$.

Sufficiency. Let $L_i > 0$, $i = 1, 2, \dots, n$, be arbitrarily chosen. Continuity of Q in y implies existence of $l_i > 0$, $l_i = l_i(L, y_r)$, $L = (L_1, L_2, \dots, L_n)^T$, $i = 1, 2, \dots, n$, such that $|y_i - y_{ri}| < l_i$, $\forall i = 1, 2, \dots, 2k$, implies $|Q_i(y) - Q_i(y_r)| < L_i$, $i = 1, 2, \dots, n$. Stability of $x = 0$ of (1.2.7) (with respect to x) guarantees existence of $\delta_i > 0$, $\delta_i = \delta_i(l)$, $l = (l_1, l_2, \dots, l_{2k})^T$, such that $|x_{i0}| < \delta_i$, $i = 1, 2, \dots, 2k$, where $\chi(t; t_0, x_0)$, $\chi(t_0; t_0, x_0) \equiv x_0$, is the solution of (1.2.7), $\chi = (\chi_1, \chi_2, \dots, \chi_{2k})^T$. Finally, for every $L_i > 0$, $i = 1, 2, \dots, n$, there is $\delta_j^* > 0$, $\delta_j^* = \frac{1}{2}\delta_j$, $j = 1, 2, \dots, n$, such that $|y_{j0} - y_{rj0}| \leq \delta_j^*$, $j = 1, 2, \dots, n$, implies

$$|Q_i[\eta(t; t_0, y_0)] - Q_i[\eta_r(t; t_0, y_{r0})]| < L_i, \quad \forall t \geq t_0, \quad i = 1, 2, \dots, n.$$

This theorem reduced the problem of the stability of the reference motion of (1.2.2) with respect to Q to the stability problem of $x = 0$ of (1.2.7) with respect to x ; it is stated and proved herein for the first time.

1.2.4 Accepted definitions of stability

By the very definition, stationary (time-invariant) systems are those whose motions are not effected by (the choice of) the initial instant $t_0 \in R$. However, such property is not characteristic for nonstationary (time-varying) systems. It is therefore natural to consider the influence of t_0 on stability properties of nonstationary systems, which is motivation for accepting the next definitions.

DEFINITION 1.2.1. The state $x = 0$ of the system (1.2.7) is:

- (i) *stable with respect to \mathcal{T}_i* iff for every $t_0 \in \mathcal{T}_i$ and every $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$, such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies

$$\|\chi(t; t_0, x_0)\| < \varepsilon, \quad \forall t \in \mathcal{T}_0;$$

- (ii) *uniformly stable with respect to \mathcal{T}_0* iff both (i) holds and for every $\varepsilon > 0$ the corresponding maximal δ_M obeying (i) satisfies

$$\inf[\delta_M(t, \varepsilon): t \in \mathcal{T}_i] > 0;$$

- (iii) *stable in the whole with respect to \mathcal{T}_i* iff both (i) holds and

$$\delta_M(t, \varepsilon) \rightarrow +\infty \quad \text{as} \quad \varepsilon \rightarrow +\infty, \quad \forall t \in \mathcal{T}_i;$$

- (iv) *uniformly stable in the whole with respect to \mathcal{T}_i* iff both (ii) and (iii) hold;

- (v) *unstable with respect to \mathcal{T}_i* iff there are $t_0 \in \mathcal{T}_i$, $\varepsilon \in (0, +\infty)$ and $\tau \in \mathcal{T}_0$, $\tau > t_0$, such that for every $\delta \in (0, +\infty)$ there is $x_0, \|x_0\| < \delta$, for which

$$\|\chi(\tau; t_0, x_0)\| \geq \varepsilon.$$

The expression “with respect to \mathcal{T}_i ” is omitted from (i)–(v) iff $\mathcal{T}_i = R$. These stability properties hold as $t \rightarrow +\infty$ but not for $t = +\infty$.

EXAMPLE 1.2.1. (see Grujić [45]). Let $x \in R$ and $\dot{x} = (1 - t)^{-1}x$. Then,

$$\chi(t; t_0, x_0) = (t - 1)^{-1}(t_0 - 1)x_0 \quad \text{for} \quad t_0 \neq 1 \quad \text{and} \quad t \neq 1.$$

For $t_0 = 1$ the motion is not defined and

$$\|\chi(t; t_0, x_0)\| \rightarrow +\infty \quad \text{as } t \rightarrow (1-0), \quad \forall t_0 \in (-\infty, 1), \quad \forall (x_0 \neq 0) \in R.$$

Hence,

$$\delta_M(t, \varepsilon) = 0, \quad \forall \varepsilon > 0, \quad \forall t \in (-\infty, 1].$$

However,

$$\delta_M(t, \varepsilon) = \varepsilon, \quad \forall t \in (1, +\infty).$$

The state $x = 0$ is uniformly stable in the whole with respect to every $\mathcal{T}_i \subseteq (-1, +\infty)$, but it is not stable.

EXAMPLE 1.2.2. (see Grujić [45]). The first order nonstationary system is defined by

$$\frac{dx}{dt} = \frac{(1 + t \sin t + t^2 \cos t)x \cdot \exp\{-\frac{1}{2}\pi\}}{\frac{1}{2}\pi \cdot \exp\{-t \sin t\} + t \cdot \exp\{-\frac{1}{2}\pi\}}.$$

Solutions are found in the form

$$\chi(t; t_0, x_0) = \frac{\frac{1}{2}\pi + t_0 \exp\{-\frac{1}{2}\pi + t_0 \sin t_0\}}{\frac{1}{2}\pi + t \exp\{-\frac{1}{2}\pi + t \sin t\}} x_0, \quad t_0 \neq -\frac{\pi}{2}, \quad t \neq -\frac{\pi}{2},$$

so that

$$\begin{aligned} |\chi(t; t_0, x_0)| &\rightarrow +\infty \quad \text{as } t \rightarrow \left(\frac{\pi}{2}, -0\right), \\ &\forall t_0 \in \left(-\infty, -\frac{\pi}{2}\right), \quad \forall (x_0 \neq 0) \in R. \end{aligned}$$

This result and analysis of $\chi(t; t_0, x_0)$ yield

$$\delta_M(t, \varepsilon) = \begin{cases} 0, & t \in (-\infty, -\frac{\pi}{2}]; \\ \varepsilon, & t \in (-\frac{\pi}{2}, 0]; \\ \varepsilon \pi [\pi + 2t \cdot \exp\{-\frac{\pi}{2} + t \sin t\}]^{-1}, & t \in [0, +\infty). \end{cases}$$

The state $x = 0$ is stable in the whole with respect to $(-\frac{\pi}{2}, +\infty)$ and uniformly stable in the whole with respect to every bounded $\mathcal{T}_i \subset (-\frac{\pi}{2}, +\infty)$, but it is not stable.

In these examples, the motions χ are not continuous in all $t \in R$.

PROPOSITION 1.2.1. *If there is a time-invariant neighborhood $\mathcal{N} \subseteq R^n$ of $x = 0$ such that $\chi(t; t_0, x_0)$ is continuous in $(t; t_0, x_0) \in \mathcal{T}_0 \times R \times \mathcal{N}$, then stability of $x = 0$ of the system (1.2.7) with respect to some non-empty \mathcal{T}_i implies its stability.*

This result can be easily proved as well as the following:

PROPOSITION 1.2.2. *If $x = 0$ of (1.2.7) is stable (in the whole) then, respectively, it is uniformly stable (in the whole) with respect to every bounded $\mathcal{T}_i \subset R$.*

EXAMPLE 1.2.3. (see Grujić [45]). Solutions of the first order non-stationary system

$$\frac{dx}{dt} = -\frac{\beta + 2\gamma t}{\alpha + \beta t + \gamma t^2}x, \quad \alpha > 0, \quad \beta^2 < 4\alpha\gamma, \quad \gamma > 0$$

are given by

$$\chi(t; t_0, x_0) = (\alpha + \beta t_0 + \gamma t_0^2)(\alpha + \beta t + \gamma t^2)^{-1}x_0.$$

In this case

$$\delta_M(t, \varepsilon) = \frac{(4\alpha\gamma - \beta^2)\varepsilon}{8\gamma(\alpha + \beta t + \gamma t^2)} \left[1 - \text{sign} \left(t + \frac{\beta}{2\gamma} \right) \right] + \frac{\varepsilon}{2} \left[1 + \text{sign} \left(t + \frac{\beta}{2\gamma} \right) \right].$$

Hence,

$$\inf [\delta_M(t, \varepsilon) : t \in R] = 0, \quad \forall \varepsilon \in (0, +\infty),$$

and

$$\delta_M(t, \varepsilon) \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow +\infty, \quad \forall t \in R.$$

The state $x = 0$ is stable in the whole but not uniformly.

However, it is uniformly stable in the whole with respect to $\mathcal{T}_i = [\zeta, +\infty)$ for any $\zeta \in (-\infty, +\infty)$.

DEFINITION 1.2.2. The state $x = 0$ of the system (1.2.7) is:

- (i) *attractive with respect to \mathcal{T}_i* iff for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for every $\zeta > 0$ there exists $\tau(t_0; x_0, \zeta) \in [0, +\infty)$ such that $\|x_0\| < \Delta(t_0)$ implies $\|\chi(t; t_0, x_0)\| < \zeta, \forall t \in (t_0 + \tau(t_0; x_0, \zeta), +\infty)$;
- (ii) *x_0 - uniformly attractive with respect to \mathcal{T}_i* iff both (i) is true and for every $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for every $\zeta \in (0, +\infty)$ there exists $\tau_u[t_0, \Delta(t_0), \zeta] \in [0, +\infty)$ such that

$$\sup [\tau_m(t_0; x_0, \zeta) : x_0 \in \mathcal{T}_i] = \tau_u(\mathcal{T}_i, x_0, \zeta);$$

- (iii) t_0 – uniformly attractive with respect to \mathcal{T}_i iff (i) is true, there is $\Delta > 0$ and for every $(x_0, \zeta) \in B_\Delta \times (0, +\infty)$ there exists $\tau_u(\mathcal{T}_i, x_0, \zeta) \in [0, +\infty)$ such that

$$\sup [\tau_m(t_0); x_0, \zeta) : t_0 \in \mathcal{T}_i] = \tau_u(\mathcal{T}_i, x_0, \zeta);$$

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- (iv) *uniformly attractive with respect to \mathcal{T}_i* iff both (ii) and (iii) hold, that is, that (i) is true, there exists $\Delta > 0$ and for every $\zeta \in (0, +\infty)$ there is $\tau_u(\mathcal{T}_i, \Delta, \zeta) \in [0, +\infty)$ such that

$$\sup [\tau_m(t_0; x_0, \zeta) : (t_0, x_0) \in \mathcal{T}_i \times B_\Delta] = \tau(\mathcal{T}_i, \Delta, \zeta);$$

- (v) The properties (i) – (iv) hold “*in the whole*” iff (i) true for every $\Delta(t_0) \in (0, +\infty)$ and every $t_0 \in \mathcal{T}_i$.

The expression “*with respect to \mathcal{T}_i* ” is omitted iff $\mathcal{T}_i = R$.

EXAMPLE 1.2.4. For the system of Example 1.2.1 the following are found:

$$\Delta_M(t) = \begin{cases} 0, & t \in (-\infty, 1) \\ +\infty, & t \in (1, +\infty) \end{cases};$$

$$\tau_m(t, x, \zeta) = \begin{cases} +\infty, & t \in (-\infty, 1) \\ \frac{t-1}{\zeta}|x| + 1, & t \in (1, +\infty) \end{cases}.$$

The state $x = 0$ is:

- (a) attractive in the whole with respect to $\mathcal{T}_i = (1, +\infty)$,
- (b) t_0 – uniformly attractive in the whole with respect to any bounded $\mathcal{T}_i \subset (1, +\infty)$,
- (c) x_0 – uniformly attractive with respect to $\mathcal{T}_i = (1, +\infty)$,
- (d) uniformly attractive with respect to any bounded $\mathcal{T}_i \subset (1, +\infty)$,
- (e) not attractive.

The next results can be easily verified.

PROPOSITION 1.2.3. *If there is a time-invariant neighborhood $\mathcal{N} \subseteq R^n$ of $x = 0$ such that $\chi(t; t_0, x_0)$ is continuous in $(t; t_0, x_0) \in \mathcal{T}_0 \times R \times \mathcal{N}$, then attraction of $x = 0$ of the system (1.2.7) with respect to some nonempty \mathcal{T}_i implies its attraction.*

EXAMPLE 1.2.5. We consider the system of Example 1.2.3 once again and find:

$$\inf [\Delta_M(t) : t \in R] = +\infty,$$

$$\tau_m(t, \Delta, \zeta) = \begin{cases} \min [0, (2\gamma)^{-1} \{[\beta^2 - 4\alpha\gamma + 4\gamma\zeta^{-1} \Delta(\alpha + \beta t + \gamma t^2)]^{\frac{1}{2}} - \beta\}] \\ \quad \text{for } \Delta \geq (4\alpha\gamma - \beta^2)\zeta[4\gamma(\alpha + \beta t + \gamma t^2)]^{-1}, \\ 0, \quad \text{for } \Delta < (4\alpha\gamma - \beta^2)\zeta[4\gamma(\alpha + \beta t + \gamma t^2)]^{-1}. \end{cases}$$

Hence,

$$\sup [\tau_m(t, \Delta, \zeta) : t \in R] = +\infty \quad \text{for } \Delta \geq (4\alpha\gamma - \beta^2)\zeta[4\gamma(\alpha + \beta t + \gamma t^2)]^{-1}.$$

The state $x = 0$ is:

- (a) attractive in the whole,
- (b) x_0 – uniformly attractive in the whole,
- (c) t_0 – uniformly attractive in the whole with respect to any bounded $\mathcal{T}_i \subset R$,
- (d) uniformly attractive in the whole with respect to any bounded $\mathcal{T}_i \subset R$,
- (e) not uniformly attractive.

DEFINITION 1.2.3. The state $x = 0$ of the system (1.2.7) is:

- (i) *asymptotically stable with respect to \mathcal{T}_i* iff it is both stable with respect to \mathcal{T}_i and attractive with respect to \mathcal{T}_i ;
- (ii) *equi-asymptotically stable with respect to \mathcal{T}_i* iff it is both stable with respect to \mathcal{T}_i and x_0 -uniformly attractive with respect to \mathcal{T}_i ;
- (iii) *quasi-uniformly asymptotically stable with respect to \mathcal{T}_i* iff it is both uniformly stable with respect to \mathcal{T}_i and t_0 -uniformly attractive with respect to \mathcal{T}_i ;
- (iv) *uniformly asymptotically stable with respect to \mathcal{T}_i* iff it is both uniformly stable with respect to \mathcal{T}_i and uniformly attractive with respect to \mathcal{T}_i ;
- (v) the properties (i) – (iv) hold “*in the whole*” iff both the corresponding stability of $x = 0$ and the corresponding attraction of $x = 0$ hold in the whole;
- (vi) *exponentially stable with respect to \mathcal{T}_i* iff there are $\Delta > 0$ and real numbers $\alpha \geq 1$ and $\beta > 0$ such that $\|x_0\| < \Delta$ implies

$$\|\chi(t; t_0, x_0)\| \leq \alpha \|x_0\| \exp[-\beta(t - t_0)], \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$$

This holds *in the whole* iff it is true for $\Delta = +\infty$.

The expression “with respect to \mathcal{T}_i ” is omitted iff $\mathcal{T}_i = R$.

EXAMPLE 1.2.6. (see Grujić [45]). The second order system is described by

$$\frac{dx}{dt} = A(t)x, \quad A(t) = \frac{1}{1+t^2} \begin{bmatrix} -t, & 1 \\ -1, & -t \end{bmatrix},$$

and its solutions are found in the form

$$\chi(t; t_0, x_0) = \frac{1}{1+t^2} \begin{bmatrix} 1+t_0t, & t-t_0 \\ t_0-t, & 1+t_0t \end{bmatrix} x_0.$$

Hence,

$$\delta_M(t, \varepsilon) = \frac{\varepsilon}{2} [1 + (1+t^2)^{-1}(1 - \text{sign } t) + \text{sign } t],$$

which implies

$$\inf [\delta_M(t, \varepsilon) : t \in R] = 0, \quad \forall \varepsilon \in (0, +\infty),$$

and

$$\tau_m(t, \|x\|, \zeta) = \begin{cases} [(1+t^2)^{\frac{1}{2}} \|x\| \zeta^{-1} - 1]^{\frac{1}{2}}, & \text{for } \|x\| \geq \zeta(1+t^2)^{-\frac{1}{2}}, \\ 0, & \text{for } 0 < \|x\| \leq \zeta(1+t^2)^{-\frac{1}{2}}, \end{cases}$$

which yields

$$\sup [\tau_m(t, \Delta, \zeta) : t \in R] = +\infty \quad \text{for } 0 < \zeta \leq \Delta(1+t^2)^{\frac{1}{2}}, \quad \forall \Delta \in (0, +\infty).$$

Therefore, the state $x = 0$ is:

- (a) asymptotically stable in the whole,
- (b) equi-asymptotically stable,
- (c) uniformly asymptotically stable with respect to any bounded $\mathcal{T}_i \subset R$,
- (d) not equi-asymptotically stable in the whole,
- (e) not uniformly asymptotically stable in the whole with respect to any bounded $\mathcal{T}_i \subset R$.

Notice that the system is linear.

The next results are straightforward corollaries to Propositions 1.2.1 – 1.2.4.

PROPOSITION 1.2.5. *If there is a time-invariant neighborhood $\mathcal{N} \subset \mathbb{R}^n$ of $x = 0$ such that $\chi(t; t_0, x_0)$ is continuous in $(t; t_0, x_0) \in \mathcal{T}_0 \times \mathbb{R} \times \mathcal{N}$ then asymptotic stability of $x = 0$ of the system (1.2.7) with respect to some nonempty \mathcal{T}_i implies its asymptotic stability.*

PROPOSITION 1.2.6. *If $x = 0$ of (1.2.7) is asymptotically stable then it is uniformly asymptotically stable with respect to every bounded $\mathcal{T}_i \subset \mathbb{R}$.*

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1.2.5 Equilibrium states

For the sake of clarity we state

DEFINITION 1.2.4. State x^* of the system (1.2.7) is its *equilibrium state over \mathcal{T}_i* iff

$$(1.2.8) \quad \chi(t; t_0, x^*) = x^*, \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$$

The expression “over \mathcal{T}_i ” is omitted iff $\mathcal{T}_i = R$.

PROPOSITION 1.2.7. For $x^* \in R^n$ to be an equilibrium state of the system (1.2.7) over \mathcal{T}_i it is necessary and sufficient that both

- (1) for every $t_0 \in \mathcal{T}_i$ there is the unique solution $\chi(t; t_0, x^*)$ of (1.2.7), which is defined for all $t_0 \in \mathcal{T}_0$

and

- (2) $f(t, x^*) = 0, \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$

PROOF. *Necessity.* Necessity of (i) and (ii) for x^* to be an equilibrium state of (1.2.7) is evidently implied by (1.2.8).

Sufficiency. If x^* satisfies the condition (ii) then $x(t) = x(t; t_0, x^*) = x^*, \quad \forall t \in \mathcal{T}_0$ and $\forall t_0 \in \mathcal{T}_i$, obeys

$$\frac{dx(t)}{dt} = 0 = f(t, x^*) = f[t, x(t)], \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$$

Hence, $\chi(t; t_0, x^*) = x^*$ is a solution of (1.2.7) at (t_0, x^*) for all $t_0 \in \mathcal{T}_i$, which is unique due to the condition (i).

Hence (1.2.8) holds.

The conditions for existence and uniqueness of the solutions can be found in the books by Bellman [15], Hartman [69], Halanay [67] and Pontriagin [154] (see also Kalman and Bertram [80]).

PROPOSITION 1.2.8. *If $x = 0$ of the system (1.2.7) is stable with respect to \mathcal{T}_i then it is then it is an equilibrium state of the system over \mathcal{T}_i .*

PROOF. Let $x = 0$ of (1.2.7) be stable with respect to \mathcal{T}_i and $\varepsilon > 0$ be arbitrarily small. Then $\|\chi(t; t_0, 0)\| < \varepsilon$ for all $t \in \mathcal{T}_0$ and every $t_0 \in \mathcal{T}_i$ because $x_0 = 0$ and $\|x_0\| = 0 < \delta_M(t_0, \varepsilon)$. Let χ_1 and χ_2 be two solutions of (1.2.7) through $(t_0, 0)$, $t_0 \in \mathcal{T}_i$. Then,

$$(1.2.9) \quad \|\chi_1(t; t_0, 0) - \chi_2(t; t_0, 0)\| \leq \|\chi_1(t; t_0, 0)\| + \|\chi_2(t; t_0, 0)\| < \varepsilon_n$$

for all $t \in \mathcal{T}_0$ and every $t_0 \in \mathcal{T}_i$ because

$$\|x_0\| = 0 < \delta_M\left(t_0, \frac{\varepsilon_n}{2}\right).$$

Let $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. It now follows from (1.2.9) that $\|\chi_1(t; t_0, 0) - \chi_2(t; t_0, 0)\|$ is less than ε_n no matter how large integer n is taken. Hence,

$$\chi_1(t; t_0, 0) \equiv \chi_2(t; t_0, 0)$$

and

$$\|\chi_i(t; t_0, 0)\| < \varepsilon_n, \quad i = 1, 2,$$

for arbitrarily large integer n . It follows that $\chi(t; t_0, 0) \equiv 0$ is the unique solution of (1.2.7) on \mathcal{T}_0 for all $t_0 \in \mathcal{T}_i$, which proves that $x = 0$ is an equilibrium state of (1.2.7) over \mathcal{T}_i .

Let $g: R^n \rightarrow R^n$ define an autonomous system

$$(1.2.10) \quad \frac{dx}{dt} = g(x).$$

Every stability property of $x = 0$ of (1.2.10) is uniform in $t_0 \in R$. Besides, Proposition 1.2.8 yield the following.

COROLLARY 1.2.1. *If $x = 0$ of the system (1.2.10) is its equilibrium state over some nonempty interval $\mathcal{T}_i \subset R$ then it is an equilibrium state of the system.*

1.3 Brief Outline of Trends in Liapunov's Stability Theory

1.3.1 Of Liapunov's original results

Liapunov ([101], p.25) defined two essentially different approaches to solving stability problems as follows:

All ways, which we can present for solving the question we are interested in, we can divide in two categories.

With one we associate all those, which lead to a direct investigation of a perturbed motion and in the basis of which there is a determination of general and particular solutions of the differential equation (1.2.1).

In general the solutions should be searched in the form of infinite series, the simplest type of which can be considered from those in the preceding paragraph. They are series ordered in terms of integer powers of fixed variables. However we shall meet series of another character in the sequel.

The collection of all ways for the stability investigation, which are in this category, we call *the first method*.

With another one we associate all those, which are based on principles independent of a determination of any solution of the differential equations of a perturbed motion.

One such example is the well-known way for an investigation of equilibrium stability in the case that there is a force function.

All these ways can be reduced to a determination and an investigation of integrals of the equations (1.2.1), and in general in the basis of all of them, which we shall meet in the sequel, there will be always a determination of functions of variables x_1, x_2, \dots, x_n, t according to given conditions, which should be satisfied by their total derivatives in t , taken under an assumption that x_1, x_2, \dots, x_n are functions of t satisfying the equations (1.2.1).

The collection of all ways of such a category we shall call *the second method*.

In order to effectively develop the second method Liapunov introduced the concept of semi-definite and definite functions and the notion of de-

creasing functions as follows (Liapunov [101], p.59):

We shall consider herein real-valued functions of real variables

$$(39) \quad x_1, x_2, \dots, x_n, t,$$



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obeying conditions of the norm

$$(40) \quad t \geq T, \quad |x_s| \leq H \quad (s = 1, 2, \dots, n),$$

where T and H are constants, the former of which can be arbitrarily large and the latter may be arbitrarily small (but different than zero).

Then we shall consider only functions which are continuous and one-to-one under the conditions (40) and vanish at

$$x_1 = x_2 = \dots = x_n = 0.$$

Such properties will possess all functions considered by us (even if it were not mentioned). But, besides that, they can possess special features; for definitions we shall introduce several terms.

Consider a function V such that under the conditions T sufficiently large and H sufficiently small, it can take, apart from those equal to zero, only values of one arbitrary sign.

Such a function we shall call *signconstant*. When we wish to underline its sign, then we shall say that it is a *positive* or *negative function*.

In addition to that, if the function V does not depend on t , and the constant H can be chosen sufficiently small so that, under the conditions (40) the equation $V = 0$ can hold only for one set of values of the variables

$$x_1 = x_2 = \dots = x_n = 0,$$

then we shall call the function V *signdefinite* one, and wishing to underline its sign – *positive-definite* or *negative-definite*.

We shall use the last notions also with respect to functions depending on t . However, in such a case the function V will be called *signdefinite* only under the condition, if for it is possible

to find such a t -independent positive-definite function W , for which one of two expressions

$$V - W \quad - V - W$$

would represent a positive function.

Hence, each of functions

$$x_1^2 + x_2^2 - 2x_1x_2 \cos t, \quad t(x_1^2 + x_2^2) - 2x_1x_2 \cos t$$

is signconstant. However, the former is only signconstant, and the latter, if $n = 2$, is simultaneously signdefinite.

Every function V , for which the constant H can be chosen so small that for numerical values of that function under the conditions (40) there is an upper bound, will be called *bounded*.

In view of the properties which, under our assumption, possess all functions considered by us, will be such, for example, every function independent of t .

A bounded function can be such that for every positive ε , regardless how small, there is such nonzero number h , for which for all values of variables, satisfying conditions

$$t \geq T, \quad |x_s| \leq h \quad (s = 1, 2, \dots, n),$$

will hold the following:

$$|V| \leq \varepsilon.$$

This condition will satisfy, for example, every function independent of t . However functions depending on t , even bounded, can violate it. Such a case represents, for example, a function

$$\sin [(x_1 + x_2 + \dots + x_n)t].$$

When the function V fulfills the preceding requirement, then we shall say that it *admits infinitely small upper bound*.

Such an example is the function

$$(x_1 + x_2 + \dots + x_n) \sin t.$$

Let V be a function admitting infinitely small upper bound. Then, if we know that the variables satisfy a condition

$$t \geq T, \quad |V| \geq \ell,$$

where ℓ is a positive number, hence we conclude that there is another positive number λ , less than which cannot be the greatest quantity among $|x_1|, |x_2|, \dots, |x_n|$.

In order to examine behavior of the values of a definite function V along system motions without using the motions themselves Liapunov ([101], p.61) proposed the following:

Simultaneously with the function V we shall often consider an expression

$$V' = \frac{\partial V}{\partial x_1} X_1 + \frac{\partial V}{\partial x_2} X_2 + \dots + \frac{\partial V}{\partial x_n} X_n + \frac{\partial V}{\partial t},$$

representing its total time derivative, taken under the assumption that x_1, x_2, \dots, x_n are functions of t , which satisfy differential equations of a perturbed motion.

In such cases we shall always assume that the function V is such that V' as a function of the variables (39) would be continuous and one-to-one under the conditions (40).

Speaking further about the derivative of the function V , we shall mean that it is the total derivative.

These concepts have been the keystone of the second Liapunov method and for a solution of (uniform) stability of $x = 0$ (Liapunov [101], p.61):

THEOREM 1. If the differential equations of a perturbed motion are such that it is possible to find a signdefinite function V , the derivative V' of which in view of these equations would be either a signconstant function with the opposite sign to that of V , or identically equal to zero, then the unperturbed motion is stable.

In addition to this result Liapunov [101] made the “*Remark 2*” that has become the foundation of the asymptotic stability concept and for a solution of (uniform) asymptotic stability of $x = 0$.

In order to illustrate deepness, generality and importance of Liapunov’s results once again, let following his results be cited (Liapunov [101],

p.79–80):

THEOREM 1. When the roots k_1, k_2, \dots, k_n of the characteristic equation are such that for a given natural number m it is impossible to have any relationship of the form

$$m_1 k_1 + m_2 k_2 + \dots + m_n k_n = 0,$$

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in which all m_s are nonnegative integers, giving their sum equal to m , then it is always possible to find just one whole homogenous function V of the power m of the quantities k_s satisfying the equation

$$(9) \quad \sum_{s=1}^n (p_{s1}x_1 + p_{s2}x_2 + \dots + p_{sn}x_n) \frac{\partial V}{\partial x_s} = U$$

for arbitrarily given whole homogenous function U of the quantities x_s of the same power m .

THEOREM 2. When the real parts of all roots k_s are negative and when in the equation (9) there is the function U being sign-definite form of any even power m , then the form V of the power m satisfying that equation is also sign definite with the opposite sign to that of U .

Gantmakher [38] recognized the fundamental potential of these results and deduced the Liapunov matrix theorem (see Barnett and Storey [14]). This theorem is a fundamental theorem for stability theory. For its presentation the following is needed.

DEFINITION 1.3.1. A matrix $H = (h_{ij}) \in R^{n \times n}$ is:

- (i) *positive (negative) semi-definite* iff its quadratic form $V(x) = x^T H x$ is positive (negative) semi-definite, respectively;
- (ii) *positive (negative) definite* iff its quadratic form $V(x) = x^T H x$ is positive (negative) definite, respectively.

Let a k -th order principal minor of the matrix H be denoted by

$$H \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{bmatrix} = \begin{bmatrix} h_{i_1 i_1} & h_{i_1 i_2} & \dots & h_{i_1 i_k} \\ h_{i_2 i_1} & h_{i_2 i_2} & \dots & h_{i_2 i_k} \\ \dots & \dots & \dots & \dots \\ h_{i_k i_1} & h_{i_k i_2} & \dots & h_{i_k i_k} \end{bmatrix},$$

where

$$i_j \in \{1, 2, \dots, n\}, \quad i_j < i_{j+1}, \quad j = 1, 2, \dots, k, \quad k = 1, 2, \dots, n.$$

The leading principal minor of the k -th order of H is

$$H \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1k} \\ h_{21} & h_{22} & \dots & h_{2k} \\ \dots & \dots & \dots & \dots \\ h_{k1} & h_{k2} & \dots & h_{kk} \end{bmatrix}, \quad k = 1, 2, \dots, n.$$

The following criteria are well known (see Gantmacher [38]).

THEOREM 1.3.1. *Necessary and sufficient for a symmetric $n \times n$ matrix H to be:*

- (1) *positive semi-definite is that all its principal minors are non-negative*

$$H \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{bmatrix} \geq 0, \quad 1 \leq i_1 < i_2 < \dots < i_k \leq n, \quad k = 1, 2, \dots, n;$$

- (2) *negative semi-definite is that both all its even order principal minors are non-negative and all its odd order principal minors are non-positive*

$$H \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{bmatrix} \begin{cases} \geq 0, & k = 2, 4, \dots; \\ \leq 0, & k = 1, 3, \dots; \end{cases}$$

- (3) *positive definite is that all its leading principal minors are positive*

$$H \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix} > 0, \quad k = 1, 2, \dots, n;$$

- (4) *negative definite is that both its first order leading principal minor is negative and all its leading principal minors are alternatively negative and positive*

$$(-1)^k H \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix} > 0, \quad k = 1, 2, \dots, n.$$

Notice that a square matrix A with all real valued elements is (semi-) definite iff its symmetric part $A_s = \frac{1}{2}(A + A^T)$ is (semi-) definite, and a square matrix A with complex valued elements is (semi-) definite iff its Hermitian part $A_H = \frac{1}{2}(A + A^*)$ is (semi-) definite, where A^* is the transpose conjugate matrix of the matrix A .

Now, the fundamental theorem of the stability theorem – the Liapunov matrix theorem – can be stated as a corollary to the preceding Theorems 1 and 2 by Liapunov.

THEOREM 1.3.2. *In order that real parts of all eigenvalues of a matrix A , $A \in R^{n \times n}$, be negative it is necessary and sufficient that for any positive definite symmetric matrix G , $G \in R^{n \times n}$, there exists the unique solution H , $H \in R^{n \times n}$, of the (Liapunov) matrix equation*

$$A^T H + H A = -G,$$

which is also positive definite symmetric matrix.

For solving the Liapunov matrix equation, see for example Barnett and Storey [14], and Barbashin [10].

1.3.2 Classical and novel developments of the scalar Liapunov functions method

Following Liapunov [101], the classical development of his second method consists of a number of stability theorems providing stability conditions are imposed on appropriate *scalar function* V and its total time derivative along system motions over a *time-invariant* neighborhood of $x = 0$. Adequate expositions of the classic development of the Liapunov second method can be found in the books by Yoshizawa [174, 175] and Rouche, Habets and Laloy [159].

1.3.2.1 Comparison functions. Comparison functions are used as upper or lower estimates of the function V and its total time derivative. They are usually denoted by φ , $\varphi: R_+ \rightarrow R_+$. The main contributor to the investigation of properties of and use of the comparison functions is Hahn [66]. What follows is mainly based on his definitions and results.

DEFINITION 1.3.2. A function φ , $\varphi: R_+ \rightarrow R_+$, belongs to

- (i) the class $K_{[0, \alpha)}$, $0 < \alpha \leq +\infty$, iff both it is defined, continuous and strictly increasing on $[0, \alpha)$ and $\varphi(0) = 0$;
- (ii) the class K iff (i) holds for $\alpha = +\infty$, $K = K_{[0, +\infty)}$;
- (iii) the class KR iff both it belongs to the class K and $\varphi(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow +\infty$;
- (iv) the class $L_{[0, \alpha)}$ iff both it is defined, continuous and strictly decreasing on $[0, \alpha)$ and $\lim[\varphi(\zeta): \zeta \rightarrow +\infty] = 0$;
- (v) the class L iff (iv) holds for $\alpha = +\infty$, $L = L_{[0, +\infty)}$.

Let φ^I denote the *inverse function* of φ , $\varphi^I[\varphi(\zeta)] \equiv \zeta$. The next result was established by Hahn [66].

PROPOSITION 1.3.1.

- (1) If $\varphi \in K$ and $\psi \in K$ then $\varphi(\psi) \in K$;
- (2) If $\varphi \in K$ and $\sigma \in L$ then $\varphi(\sigma) \in L$;
- (3) If $\varphi \in K_{[0,\alpha]}$ and $\varphi(\alpha) = \xi$ then $\varphi^I \in K_{[0,\xi]}$;
- (4) If $\varphi \in K$ and $\lim [\varphi(\zeta): \zeta \rightarrow +\infty] = \xi$ then φ^I is not defined on $(\xi, +\infty]$;
- (5) If $\varphi \in K_{[0,\alpha]}$, $\psi \in K_{[0,\alpha]}$ and $\varphi(\zeta) > \psi(\zeta)$ on $[0, \alpha)$ then $\varphi^I(\zeta) < \psi^I(\zeta)$ on $[0, \beta]$, where $\beta = \psi(\alpha)$.

DEFINITION 1.3.3. A function $\varphi, \varphi: R_+ \times R_+ \rightarrow R_+$, belongs to:

- (i) the class $KK_{[0;\alpha,\beta]}$ iff both $\varphi(0, \zeta) \in K_{[0,\alpha]}$ for every $\zeta \in [0, \beta]$ and $\varphi(\zeta, 0) \in K_{[0,\beta]}$ for every $\zeta \in [0, \alpha]$;
- (ii) the class KK iff (i) holds for $\alpha = \beta = +\infty$;
- (iii) the class $KL_{[0;\alpha,\beta]}$ iff both $\varphi(0, \zeta) \in K_{[0,\alpha]}$ for every $\zeta \in [0, \beta]$ and $\varphi(\zeta, 0) \in L_{[0,\beta]}$ for every $\zeta \in [0, \alpha]$;
- (iv) the class KL iff (iii) holds for $\alpha = \beta = +\infty$.

DEFINITION 1.3.4. Two functions $\varphi_1, \varphi_2 \in K$ or $\varphi_1, \varphi_2 \in KR$ are said to be of the same order of magnitude if there exist positive constants $\alpha_i, \beta_i, i = 1, 2$, such that

$$\alpha_i \varphi_i(\zeta) \leq \varphi_j(\zeta) \leq \beta_i \varphi_i(\zeta), \quad i \neq j; \quad i, j \in [1, 2].$$

1.3.2.2 Some generalizations of the theory by Liapunov. We shall set out some generalizations of Liapunov theorems with regard to the results obtained by Zubov [178, 179].

DEFINITION 1.3.5. A function $v: R \times R^n \rightarrow R$ is positive definite on $\mathcal{T}_\tau, \tau \in R$, if and only if there is a time-invariant connected neighborhood \mathcal{N} of $x = 0, \mathcal{N} \subseteq R^n$ and $a \in K_{[0,\alpha]}$, where $\alpha = \sup \{\|x\|: x \in \mathcal{N}\}$ such that $v(t, 0) = 0, \forall t \in \mathcal{T}_\tau$, and $a(\|x\|) \leq v(t, x) \quad \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}$.

THEOREM 1.3.3. Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist

- (1) an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;
- (2) a positive definite function v on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{N}$) such that:
 - (a) $v(t, 0) = 0$ and for a fixed $t \in R (t \in \mathcal{T}_\tau)$ the function $v(t, x)$ is continuous at the point $x = 0$;

(b) $v(t, x)$ is definite on any integral curve $x = x(t; t_0, x_0)$ of the system (1.2.7) unless the curve leaves the definition domain of function $v(t, x)$ and on every such curve the function

$$v(t) = v(t, x(t; t_0, x_0))$$

does not increase when $t \in \mathcal{R}$ (for all $t \in \mathcal{T}_\tau$), then and only then the state $x = 0$ of the system (1.2.7) is stable (on \mathcal{T}_τ).

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The proof of sufficiency of the theorem conditions is a routine of the Liapunov functions method (see e.g. Liapunov [101], Demidovich [23], etc.).

In the proof of necessity of the Theorem 1.3.3 conditions one employs the function

$$v(t_0, x_0) = \begin{cases} \sup_{t \geq t_0} \|x(t; t_0, x_0)\|, & \text{if } \sup_{t \geq t_0} \|x(t; t_0, x_0)\| \leq 1, \\ 1, & \text{if } \sup_{t \geq t_0} \|x(t; t_0, x_0)\| > 1. \end{cases}$$

It is easy to verify that these functions satisfy all conditions of the Theorem 1.3.3.

DEFINITION 1.3.6. A function $v: R \times R^n \rightarrow R$ is *decreasing on \mathcal{T}_τ* , $\tau \in R$, if and only if there is a time-invariant neighborhood \mathcal{N} of $x = 0$ and a function $b \in K_{[0, \alpha]}$, such that

$$v(t, x) \leq b(\|x\|) \quad \forall t \in \mathcal{T}_\tau \times \mathcal{N}.$$

THEOREM 1.3.4. *In order that the solution $x = 0$ of the system (1.2.7) is t_0 -uniformly stable (on \mathcal{T}_τ), it is necessary and sufficient that the function $v(t, x)$ mentioned in Theorem 1.3.3 be decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) and all conditions of Theorem 1.3.1 be satisfied.*

THEOREM 1.3.5. *For the solution $x = 0$ of the system (1.2.7) to be asymptotically stable (on \mathcal{T}_τ), it is necessary and sufficient that the conditions of Theorem 1.3.3 be satisfied and along any integral curve $x(t; t_0, x_0)$ the function $v(t, x)$ tend to zero as $t \rightarrow +\infty$, i.e.*

$$v(t) = v(t, x(t; t_0, x_0)) \rightarrow 0 \quad \text{for } t \rightarrow +\infty, \\ \|x_0\| < \gamma(t_0), \quad t > t_0, \quad t_0 \in \mathcal{T}_i.$$

Theorems 1.3.3 – 1.3.5 have the condition associated with the function $v(t, x)$ nonincreasing or decreasing along the integral curves of the system (1.2.7). As the explicit representation of the integral curves $x(t; t_0, x_0)$ of the system (1.2.7) is not known, it is impossible to test this condition. Therefore, when these theorems actually are employed, various sufficient conditions of function $v(t, x)$ nonincreasing (decreasing) that are easier to check become of great importance.

THEOREM 1.3.6. *Let a vector function f in (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;*
- (2) *the function $v(t, x)$ satisfying condition (2)(a) of the Theorem 1.3.3;*
- (3) *the nonpositive function $w(t, x)$ that is a total derivative of the function $v(t, x)$ along the solutions of the system (1.2.7) such that*

$$w(t, x) \leq \varphi_\alpha(t) \leq 0 \quad \text{for } \|x\| \geq \alpha^2$$

and

(4)

$$\int_I \varphi(s) ds = -\infty,$$

where I is any infinite system of closed nonintersecting segments on the interval $[t_0, \infty)$, $t_0 \in \mathcal{T}_i$, such that the length of each one is not less than a fixed positive constant, then the state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ).

THEOREM 1.3.7. *For the solution $x = 0$ of the system (1.2.7) to be t_0 -uniformly asymptotically stable (on \mathcal{T}_τ), it is necessary and sufficient that the function $v(t, x)$ satisfy all conditions of Theorem 1.3.5 and be decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$).*

This theorem is an immediate corollary of Theorems 1.3.4 and 1.3.5.

THEOREM 1.3.8. *Let a vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;*
- (2) *the function $v(t, x)$ being positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) and decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (3) *the function $w(t, x)$ that is negative definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) and decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (4) *the correlation*

$$Dv(t, x) = w(t, x) \quad \text{for } (t, x) \in R \times \mathcal{G} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{G}),$$

then and only then the state $x = 0$ of the system (1.2.7) is uniformly asymptotically stable (on \mathcal{T}_τ) and uniformly attractive (on \mathcal{T}_τ).

COROLLARY 1.3.1. If the state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ), then an independent variable t can be transformed so that the zero solution of the newly obtained system is uniformly attractive (on \mathcal{T}_τ).

THEOREM 1.3.9. Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist

- (1) the function $v(t, x)$ decreasing on \mathcal{G} and taking negative values in the arbitrarily small semiaxis neighborhood (for any fixed $t > T$), $x_1 = x_2 = \cdots = x_n = 0$, $t \geq 0$;
- (2) an integrable function $\varphi_\alpha(t)$ such that

$$Dv(t, x) = w(t, x)$$

and for $\|x\|^2 \geq \alpha^2$, $w(t, x) \leq \varphi_\alpha(t)$,

$$\int_0^t \varphi_\alpha(t) dt \rightarrow -\infty \quad \text{for } t \rightarrow +\infty,$$

then the equilibrium state $x = 0$ of the system (1.2.7) is unstable.

COROLLARY 1.3.2. If the function $v(t, x)$ satisfies conditions (1)–(2) of Theorem 1.3.9, and the function $w(t, x)$ is negative definite, then the equilibrium state $x = 0$ of the system (1.2.7) is unstable.

This corollary is the first Liapunov theorem on instability (see Liapunov [101], p.65).

Following Krasovskii [89] it is easy to prove.

THEOREM 1.3.10. If χ is continuous on $\mathcal{T}_0 \times R \times \mathcal{N}$ (on $\mathcal{T}_0 \times \mathcal{T}_\tau \times \mathcal{N}$) then existence of a time-invariant neighborhood \mathcal{S} of $x = 0$, a function v , positive real numbers η_1 , η_2 and η_3 and a positive integer p such that $v(t, x) \in C(\mathcal{T}_0 \times \mathcal{N})$ and both, respectively,

$$(1) \quad \eta_1 \|x\|^p \leq v(t, x) \leq \eta_2 \|x\|^p, \quad \forall (t, x) \in R \times \mathcal{S} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{S}),$$

and

$$(2) \quad D^*v(t, x) \leq -\eta_3 \|x\|^p, \quad \forall (t, x) \in R \times \mathcal{S} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{S}),$$

is necessary and sufficient for exponential stability (on \mathcal{T}_τ) of $x = 0$ of the system (2.1.7).

THEOREM 1.3.11. *If χ is continuous on $\mathcal{T}_0 \times R \times R^n$ (on $\mathcal{T}_0 \times \mathcal{T}_\tau \times R^n$) then existence of a function v , positive real numbers η_1, η_2 and η_3 and a positive integer p such that $v(t, x) \in C(\mathcal{T}_0 \times R^n)$ and both, respectively,*

$$(1) \quad \eta_1 \|x\|^p \leq v(t, x) \leq \eta_2 \|x\|^p, \quad \forall (t, x) \in R \times R^n \quad (\forall (t, x) \in \mathcal{T}_\tau \times R^n),$$

and

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$$(2) \quad D^*v(t, x) \leq -\eta_3 \|x\|^p, \quad \forall (t, x) \in R \times R^n \quad (\forall (t, x) \in \mathcal{T}_\tau \times R^n),$$

is necessary and sufficient for exponential stability in the whole (on \mathcal{T}_τ) of $x = 0$ of the system (2.1.7).

1.3.2.3 Partial stability. We return back to the system (1.2.7) and represent the vector x of the system state as

$$x^T = (x_1^T, x_2^T)^T,$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $n_1 + n_2 = n$. Then we assume on system of the equations (1.2.7) that:

- (H_1). In domain $t \in R$, $\mathcal{T}_\tau \times \Omega(H) \times D$ the right-hand parts of the system (1.2.7) are continuous and locally Lipschitzian in x , i.e. $f \in C(R \times \Omega(H) \times D, R^n)$, where $\Omega(H) = \{x_1 \in R^{n_1} : \|x_1\| < H, H = \text{const} > 0\}$, $D = \{x_2 \in R^{n_2} : 0 < \|x_2\| < +\infty\}$.
- (H_2). The solution of the system (1.2.7) are x_2 -continuabile, i.e. any solution $x(t; t_0, x_0)$ of the system (1.2.7) is definite for all $t \geq 0$ ($t \in \mathcal{T}_\tau$) such that $\|x_1(t)\| \leq H$.

It was noted by Liapunov [102] that a more general problem on motion stability with respect to a part of variables may be studied.

The theory of motion stability with respect to a part of variables is exposed by Rumyantzev and Oziraner [161]. In this presentation we restrict ourselves to a few results obtained in this direction.

DEFINITION 1.3.7. The state $x = 0$ of the system (1.2.7) is x_1 -stable with respect to \mathcal{T}_τ , iff for every $t_0 \in \mathcal{T}_i$ and every $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta(t_0, \varepsilon)$ implies $\|x_1(t; t_0, x_0)\| < \varepsilon \quad \forall t \in \mathcal{T}_0$.

The other types of x_1 -stability are defined in the same way as Definition 1.3.7 taking into account Definitions 1.2.1–1.2.3.

Following the results by Rumyantzev [160] we shall set out the following result.

THEOREM 1.3.12. *Let the vector function f in system (1.2.7) satisfy conditions H_1 and H_2 . If there exist the function $v(t, x)$ and comparison functions a , b and c of class K such that*

- (1) $a(\|x_1\|) \leq v(t, x) \leq b(\|x_1\|) \quad \forall (t, x) \in R \times \Omega(H) \times D \quad (\forall (t, x) \in \mathcal{T}_\tau \times \Omega(H) \times D)$;
- (2) $D^+v(t, x) \leq -c(\|x_1\|) \quad \forall (t, x) \in R \times \Omega(H) \times D \quad (\forall (t, x) \in \mathcal{T}_\tau \times \Omega(H) \times D)$.

Then

- (a) any $\alpha > 0$ and any $(t_0, x_0) \in R \times (B_\alpha \cap \Omega(H)) \times D \quad ((t_0, x_0) \in \mathcal{T}_\tau \times (B_\alpha \cap \Omega(H)) \times D)$, the solution $x_1(t; t_0, x_0) \rightarrow 0$ uniformly relatively (t_0, x_0) as $t \rightarrow +\infty$;
- (b) the state $x = 0$ of the system (1.2.7) is uniformly asymptotically x_1 -stable (on \mathcal{T}_τ).

1.3.2.4 The development of Marachkov's idea. One of the trends in generalization of Liapunov's theorems is the establishment of conditions that could replace the condition of function v decreasing in the theorems on asymptotic stability. The Marachkov's theorem [108] is the first result in this direction.

THEOREM 1.3.13. *Let the vector function f in the system (1.2.7) be bounded on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) a positive definite function $v \in C^1(R \times \mathcal{N}, R_+) \quad (v \in C^1(\mathcal{T}_\tau \times \mathcal{N}, R_+))$, $v(t, 0) = 0, \quad \forall t \in R \quad (\forall t \in \mathcal{T}_\tau)$;
- (2) a function c of class K such that

$$Dv(t, x) \leq -c(\|x\|)$$

$$\forall (t, x) \in R \times \mathcal{N} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}),$$

then the equilibrium state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ).

The Marachkov's theorem was generalized by Salvadori [162] via the application of two auxiliary functions. We shall formulate this result in the following way.

THEOREM 1.3.14. *Let the vector function f in the system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) a positive definite function $v \in C^1(R \times \mathcal{N}, R_+) \quad (v \in C^1(\mathcal{T}_\tau \times \mathcal{N}, R_+))$, $v(t, 0) = 0, \quad \forall t \in R \quad (\forall t \in \mathcal{T}_\tau)$;

- (2) the function $w \in C^1(R \times \mathcal{N}, R_+)$ ($w \in C^1(\mathcal{T}_\tau \times \mathcal{N}, R_+)$), $w(t, x)$ is positive definite and $Dw(t, x)|_{(1.2.7)}$ is bounded from above or below on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$);
- (3) a function c of class K such that

$$Dv(t, x) \leq -c(w(t, x)) \quad \text{on } R \times \mathcal{N} \text{ (on } \mathcal{T}_\tau \times \mathcal{N}\text{)}.$$

Then the equilibrium state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ).

Below we shall cite a result showing that the positive definiteness condition in Theorem 1.3.13 may be replaced by the condition of positive semidefiniteness.

THEOREM 1.3.15. *Let in condition (1) of Theorem 1.3.13 the function $v \in C^1(R \times \mathcal{N})$, $v \in C^1(\mathcal{T}_\tau \times \mathcal{N})$, $v(t, x) \geq 0$ and $v(t, 0) = 0 \quad \forall t \in R$ ($\forall t \in \mathcal{T}_\tau$) and condition (2) be satisfied.*

Then the equilibrium state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ).

1.3.2.5 Generalized comparison principle. Further alongside the system (1.2.7) the equation

$$(1.3.1) \quad \frac{du}{dt} = g(t, u, x)$$

is considered, where $u \in R_+$, $g \in C(\mathcal{T}_\tau \times R_+ \times R^n, R)$, $g(t, 0, 0) = 0$ for all $t \in \mathcal{T}_\tau$.

We recall that equation (1.3.1) emerges as a result of estimation of the total derivative $D^+v(t, x)$ along a solution of the system (1.2.7) in terms of the inequality


$$(1.3.2) \quad \begin{aligned} D^+v(t, x) &\leq g(t, v(t, x), x) \\ (t, x) &\in R \times \mathcal{N} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}). \end{aligned}$$

Sometimes an obvious dependence of function g on vector x widens the possibility to apply the principle of comparison with scalar Liapunov function (cf. Corduneanu [20]).

THEOREM 1.3.16. *Let the vector function f in the system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist scalar function $v(t, x)$ and $g(t, u, x)$ and comparison functions a and b of class K such that*

- (1) $a(\|x\|) \leq v(t, x) \leq b(\|x\|) \quad \forall (t, x) \in R \times \mathcal{N} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N});$
- (2) $D^+v(t, x) \leq g(t, v(t, x), x) \quad \forall (t, x) \in R \times \mathcal{N} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}),$


then the property of u -stability of the extended system

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$$\begin{aligned}\frac{dx}{dt} &= f(t, x), \quad x(t_0) = x_0; \\ \frac{du}{dt} &= g(t, u, x), \quad u(t_0) = u_0 \geq 0,\end{aligned}$$

implies the corresponding property of stability of solution $x = 0$ to the system (1.2.7).

For the proof of this theorem when $\mathcal{T}_\tau = R$ see Hatvani [71] and for its generalization see Martynyuk [110].

We note that for the case when estimate (1.3.2) holds with an inverse inequality and the function $g(t, u, x) = g(t, u)$ the theorems on instability of solution $x = 0$ to system (1.2.7) are known (see Rouche, Habets, Laloy [159]) that are based on the principle of comparison with scalar Liapunov function.

1.3.3 A survey of development of the method of vector Liapunov functions

With the purpose to weaken the requirements to the Liapunov functions used in the theory of motion stability it was proposed by Duhem [33] in 1902 to apply several Liapunov functions instead of one.

In modern terms he discovered a multicomponent Liapunov function. After 60 years this idea of multicomponent function was developed by Bellman [16], Matrosov [132] and Melnikov [139]. The papers by Corduneanu [20, 21] where the scalar Liapunov function were applied together with differential inequalities and the works by Kamke [81] and Ważewski [171] have become a background for a series of important results in motion stability theory obtained via the principle of comparison with vector Liapunov function. This section reviews basic ideas and results developed lately while working out the method of vector Liapunov functions.

1.3.3.1 *Scalar approach.* We return back to the system (1.2.7) and consider also a *vector function*

$$(1.3.3) \quad V(t, x) = (v_1(t, x), v_2(t, x), \dots, v_m(t, x))^T,$$

where $v_s \in C(\mathcal{T}_0 \times R^n, R_+)$, $s = 1, 2, \dots, m$ and its total derivative along solutions of the system (1.2.7)

$$(1.3.4) \quad D^+V(t, x) = \limsup \{ [V(t+\theta, x+\theta f(t, x)) - V(t, x)] \theta^{-1} : \theta \rightarrow \theta^+ \}$$

for $(t, x) \in \mathcal{T}_0 \times R^n$.

The notion of the property of having a fixed sign of function (1.3.3) is introduced as follows. By means of a real vector $\alpha \in R^m$ one constructs a scalar function

$$(1.3.5) \quad v(t, x, \alpha) = \alpha^T V(t, x) \quad (t, x) \in \mathcal{T}_0 \times R^n.$$

DEFINITION 1.3.8. A vector function $V : \mathcal{T}_0 \times R^n \rightarrow R^m$ is

- (i) *positive semi-definite on* $\mathcal{T}_\tau = [\tau, +\infty)$, $\tau \in R$ iff there exist a connected time-invariant neighborhood \mathcal{N} of point $x = 0$, $\mathcal{N} \subseteq R^n$ and a real vector $\alpha \in R^m$ such that
 - (a) $v(t, x, \alpha)$ is continuous in $(t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;
 - (b) $v(t, x, \alpha)$ is nonnegative on \mathcal{N} ; $v(t, x, \alpha) \geq 0 \quad \forall (t, x, \alpha \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times R^m$;
 - (c) $v(t, x, \alpha)$ vanishes whenever $x = 0$ for any $(t, \alpha \neq 0) \in \mathcal{T}_\tau \times R^m$.

REMARK 1.3.1. Taking Definition 1.3.8 for the sample the other definitions for function (1.3.3) are introduced in a similar way.

The state vector x of system (1.2.7) is divided into m subvectors, i.e. $x = (x_1^T, \dots, x_m^T)^T$, where $x_s \in R^{n_s}$ and $n_1 + n_2 + \dots + n_m = n$.

Assume that

$$(1.3.6) \quad a_{i1} \psi_{i1}^{\frac{1}{2}}(\|x_i\|) \leq v_i(t, x) \leq a_{i2} \psi_{i2}^{\frac{1}{2}}(\|x_i\|), \quad i = 1, 2, \dots, m,$$

where a_{i1} and a_{i2} are some positive constants and ψ_{i1} and ψ_{i2} are of class $K(KR)$.

Actually the condition (1.3.6) means that the components $v_i(t, x)$ of the vector function (1.3.3) are positive definite and decreasing with respect to a part of variables.

Let us introduce designations

$$(1.3.7) \quad \begin{aligned} A_1 &= \text{diag} [a_{11}, a_{12}, \dots, a_{1m}], \\ A_2 &= \text{diag} [a_{21}, a_{22}, \dots, a_{2m}]. \end{aligned}$$

PROPOSITION 1.3.2. *For the vector function (1.3.3) to be positive definite and decreasing, it is necessary and sufficient that the bilateral inequalities*

$$(1.3.8) \quad u_1^T A_1 u_1 \leq v(t, x, \alpha) \leq u_2^T A_2 u_2$$

be satisfied, where

$$u_1 = \left(\psi_{11}^{\frac{1}{2}}(\|x_1\|), \dots, \psi_{1m}^{\frac{1}{2}}(\|x_m\|) \right)^T,$$

$$u_2 = \left(\psi_{21}^{\frac{1}{2}}(\|x_1\|), \dots, \psi_{2m}^{\frac{1}{2}}(\|x_m\|) \right)^T.$$

REMARK 1.3.2. If $\psi_{i1} = \psi_{i2} = \|x_i\|$, then the estimates (1.3.8) are known (see Krasovskii [89]) as the estimates characteristics of the quadratic forms.

Taking into account (1.3.4) we get for the function (1.3.5)

$$(1.3.9) \quad D^+V(t, x, \alpha) = \alpha^T D^+V(t, x).$$

Let for $(t, x) \in \mathcal{T}_0 \times R^n$ there exist an $m \times m$ matrix $S(t, x)$, for which

$$(1.3.10) \quad D^+V(t, x, \alpha) \leq \psi_3^T S(t, x) \psi_3,$$

where $\psi_3 = \left(\psi_{13}^{\frac{1}{2}}(\|x_1\|), \psi_{23}^{\frac{1}{2}}(\|x_2\|), \dots, \psi_{m3}^{\frac{1}{2}}(\|x_m\|) \right)^T$.

Estimates (1.3.8)–(1.3.10) allows us to establish stability conditions for the state $x = 0$ of system (1.2.7) as follows.

THEOREM 1.3.17. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;*
- (2) *the decreasing positive definite vector function V on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (3) *the $m \times m$ -matrix $S(t, x)$ on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) such that inequality (1.3.10) is satisfied.*

Then

- (a) *the state $x = 0$ of the system (1.2.7) is uniformly stable if the matrix $S(t, x)$ is negative semidefinite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (b) *the state $x = 0$ of the system (1.2.7) is uniformly asymptotically stable (on \mathcal{T}_τ) providing the matrix $S(t, x)$ is negative definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$).*

PROOF. Formula (1.3.5) and estimates (1.3.8) and (1.3.10) allow us to repeat all points of the proof of Theorems 8.1 and 8.3 by Yoshizawa [174] on uniform (asymptotic) stability. The theorem is proved.

REMARK 1.3.3. New points of the theorem resulting from the application of vector function (1.3.3) are

- (a) a possibility to apply the components $v_i(t, x)$, $i = 1, 2, \dots, m$ being of a fixed sign with respect to a part of variables;
- (b) a possibility to check the property of having a fixed sign of the matrix $S(t, x)$ via the algebraic method.



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A specific way of constructing $m \times m$ -matrix $S(t, x)$ enables us to derive from Theorem 1.3.17 the assertions found in the monographs by Michel and Miller [143], Šiljak [167] and Grujić, Martynyuk, Ribbens-Pavella [57]. Thus, Theorem 1.3.17 proves to be quite universal in the framework of the scalar approach of the vector Liapunov function application.

Also, within the scalar approach the application of the vector Liapunov function together with the comparison principle is developed.

THEOREM 1.3.18. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;*
- (2) *the vector function $V(t, x)$ and a vector $\alpha \in R^m$ for which inequalities (1.3.8) are satisfied;*
- (3) *the function $w \in C(\mathcal{T}_\tau \times R_+, R)$, $w(t, 0) = 0$ such that*

$$D^+v(t, x, \alpha) \leq w(t, v(t, x, \alpha)) \quad \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N};$$

- (4) *the solution $r^* = 0$ of the comparison equation*

$$(1.3.11) \quad \frac{dr}{dt} = w(t, r), \quad r(t_0) = r_0 \geq 0$$

existing for $t \geq t_0$.

Then

- (a) *the stability of state $r = 0$ of the equation (1.3.11) implies the stability of state $x = 0$ of the system (1.2.7);*
- (b) *the asymptotic stability of state $r = 0$ of (1.3.11) implies the asymptotic stability of state $x = 0$ of the system (1.2.7);*
- (c) *if, moreover, $v(t, x, \alpha) \rightarrow 0$ as $\|x\| \rightarrow 0$ uniformly on \mathcal{T}_τ , then the uniform stability or uniform asymptotic stability of state $r = 0$ of system (1.3.11) implies the corresponding stability of state $x = 0$ of system (1.2.7).*

For the analysis of various partial cases of inequality (1.3.11) or the same inequality in the integral form see Grujić, Martynyuk and Ribbens-Pavella [57].

One of the Theorem 1.3.18 generalizations is based on the application of a majorizing function $w \in C(\mathcal{T}_\tau \times R^n \times R_+, R)$, $w(t, r, x) = 0$ when $r = 0$ and $x = 0$.

Besides an extended system

$$(1.3.12) \quad \begin{aligned} \frac{dx}{dt} &= f(t, x), & x(t_0) &= x_0, \\ \frac{dr}{dt} &= w(t, r, x), & r(t_0) &= r_0 \geq 0 \end{aligned}$$

is treated for which certain type of r -stability of the zero solution $(x^T, r)^T = 0$ yields an appropriate type of stability of the state $x = 0$ of (1.2.7).

The theorem has been developed and applied for the cases when the function $w(t, r) = w(r)$, i.e. it is independent of $t \in \mathcal{T}_\tau$. These and other results obtained in this direction are set out by Grujić, Martynyuk and Ribbens-Pavella [57].

1.3.3.2 Vector approach. The combination of vector function (1.3.3) with the comparison system

$$(1.3.13) \quad \frac{du}{dt} = \Omega(t, u), \quad u(t_0) = u_0 \geq 0,$$

where $u \in R_+^m$, $\Omega \in C(\mathcal{T}_\tau \times R_+^m, R^m)$, $\Omega(t, 0) = 0$ for all $t \in \mathcal{T}_\tau$, leads to the following general result of the method of vector Liapunov functions.

THEOREM 1.3.19. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of point $x = 0$;*
- (2) *the vector function $V \in C(\mathcal{T}_\tau \times \mathcal{N}, R_+^m)$, $V(t, x)$ is locally Lipschitzian in x and a real vector $\alpha \in R^m$ such that function (1.3.5) satisfies bilateral inequality (1.3.8);*
- (3) *the function $\Omega \in C(\mathcal{T}_\tau \times R_+^m, R^m)$, $\Omega(t, 0) = 0$ and $\Omega(t, u)$ is quasimonotone nondecreasing in u when all $t \in \mathcal{T}_\tau$, so that*

$$D^+V(t, x) \leq \Omega(t, V(t, x)), \quad (t, x) \in \mathcal{T}_\tau \times \mathcal{N}.$$

Then certain stability properties of the state $u = 0$ of the system (1.3.13) imply appropriate stability properties of the state $x = 0$ of the system (1.2.7).

PROOF. We shall cite first an assertion that establishes a relationship between the vector function variation and maximal solution to comparison system (1.3.13).

PROPOSITION 1.3.3. Let $V \in C(\mathcal{T}_\tau \times \mathcal{N}, R_+^m)$ and $V(t, x)$ be locally Lipschitzian in x . Let the vector function $D^+V(t, x)$ specified by (1.3.4) satisfy the inequality

$$D^+V(t, x) \leq \Omega(t, V(t, x)), \quad \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N},$$

where $\Omega \in C(\mathcal{T}_\tau \times R_+^m, R_+^m)$ and the function $\Omega(t, u)$ be quasimonotone increasing in u .

Assume that the maximal solution $u_M(t; t_0, r_0)$ of the comparison system

$$\frac{du}{dt} = \Omega(t, u)$$

exists on the interval \mathcal{T}_τ and passes through the point $(t_0, r_0) \in \mathcal{T}_\tau \times R_+^m$. If $x(t; t_0, x_0)$ is any solution to system (1.2.7) defined on $[t_0, t_0 + \delta)$, $t_0 \in \mathcal{T}_\tau$ and passing through the point $(t_0, x_0) \in \mathcal{T}_\tau \times \mathcal{N}$, then the condition

$$(1.3.14) \quad V(t_0, x_0) \leq r_0$$

yields the estimate

$$(1.3.15) \quad V(t, x(t; t_0, x_0)) \leq u_M(t; t_0, r_0) \quad \forall t \in [t_0, t_0 + \delta).$$

Further the fact that function (1.3.5) satisfies bilateral inequality (1.3.8) implies that the vector function $V(t, x)$ is positive definite and decreasing.

Estimate (1.3.15) and the fact that the solution $u = 0$ of the system (1.3.15) possesses a certain type of stability allow the conclusion that the solution $x = 0$ of the system (1.2.8) has a corresponding type of stability (for further details see Lakshmikantham, Leela and Martynyuk [94], etc.).

In the case when system (1.3.15) is autonomous

$$(1.3.16) \quad \frac{du}{dt} = \Omega(u), \quad u \in R_+^m,$$

where $\Omega \in C(R_+^m, R^m)$, $\Omega(u)$ satisfies the quasimonotonicity condition and the solution of the system (1.3.16) is locally unique for any $u_0 \in R_+^m$ we establish a criterion of asymptotic stability of the state $u = 0$ of the system (1.3.16) as follows.

THEOREM 1.3.20. *Let for the system (1.3.16) there exist a neighborhood \mathcal{U} of state $u = 0$ such that for all $u \in \mathcal{U}$, $u \neq 0$, $\Omega(u) \neq 0$ and $\Omega(u) = 0$ when $u = 0$.*

The isolated equilibrium state $u = 0$ of the system (1.3.16) is asymptotically stable iff there exists a positive vector $u^0 = K^0 \cap \mathcal{U}$ such that the system of inequalities

$$\Omega_s(u_1^0, \dots, u_m^0) < 0 \quad \forall s \in [1, m]$$

is joint.

Besides, $K^0 = \text{int } K$ and $K = \{u \in R^m : u_s \geq, s = 1, 2, \dots, m\}$.

Under some additional conditions the theorem is proved as well for the case when the comparison system (1.3.16) has a nonisolated singular point (see Martynyuk and Obolenskii [129]).

Further we assume that the vector function $\Omega(t, u)$ has bounded partial derivatives in u .

Designate

$$\left. \frac{\partial \Omega}{\partial u} \right|_{u=0} = P(t), \quad \Phi(t, u) = \Omega(t, u) - P(t)u.$$

Consider a system comparison equations

$$(1.3.17) \quad \frac{du}{dt} = P(t)u + \Phi(t, u), \quad u(t_0) = u_0 \geq 0,$$

and its linear approximation

$$(1.3.18) \quad \frac{d\xi}{dt} = P(t)\xi, \quad \xi(t_0) = \xi_0 \geq 0.$$

DEFINITION 1.3.9. (Šiljak [167]). Matrix $P(t)$ is called a *nonautonomous M -matrix* iff

$$p_{ij}(t) \begin{cases} < 0 & \text{for all } t \in \mathcal{T}_0, i = j; \\ \geq 0 & \text{for all } t \in \mathcal{T}_0, i \neq j, i, j = 1, 2, \dots, m. \end{cases}$$

DEFINITION 1.3.10. Nonautonomous linear system (1.3.18) is called a *reducible comparison system*, provided that there exists a Liapunov transformation $\xi = Q(t)y$ by means of which it can be reduced to the system

$$\frac{dy}{dt} = By,$$

with a constant M -matrix B . Moreover

$$B = Q^{-1} \left(PQ - \frac{dQ}{dt} \right).$$

Recall that for the *Liapunov transformation*

$$(1.3.19) \quad \xi = Q(t)y$$

there exists $Q^{-1}(t)$ and $Q \in C^1(\mathcal{T}_0, R^{m \times m})$.

Besides, the values

$$k = \sup_{t \geq 0} \|Q(t)\| \quad \text{and} \quad l = \sup_{t \geq 0} \|Q^{-1}(t)\|$$

are finite.

THEOREM 1.3.21. *Let for the system (1.2.7) the following conditions hold true*

- (1) *there exists a positive definite decrescent vector function $V(t, x)$ such that*

$$(1.3.20) \quad D^+V(t, x) \leq P(t)V(t, x) + \Phi(t, V(t, x)),$$

where $P(t)$ is a nonautonomous M -matrix and $\Phi(t, u)$ is quasi-monotone in u and

$$\lim_{\|u\| \rightarrow 0} \frac{\|\Phi(t, u)\|}{\|u\|} = 0 \quad \text{uniformly in } t \geq t_0;$$

- (2) *a matrix $P(t)$ reducible in the sense of Liapunov.*

Then the following assertions are valid

- (a) *if the matrix B in the system*

$$(1.3.21) \quad \frac{dy}{dt} = By + Q^{-1}\Phi(t, Qy)$$

- has all eigenvalues with negative real parts, then the zero solution of comparison system (1.3.17) is uniformly asymptotically stable;
- (b) if the matrix B in the system (1.3.21) has all eigenvalues with negative real parts and in addition $V(t, x, \alpha) \geq \Delta \|x\|^2$ for some $\Delta > 0$, then the zero solution of comparison equation (1.3.17) is exponentially stable;
- (c) if the inequality (1.3.20) holds with a reversed sign and the matrix B in system (1.3.21) has at least one eigenvalues with positive real parts, then the zero solution of comparison system (1.3.17) is unstable.



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PROOF. We apply to system (1.3.17) the Liapunov transformation $u = Q(t)y$ and get system (1.3.21). By condition (1) of the Theorem 1.3.21

$$\|\Phi(t, Qy)\| \leq \varepsilon \|Qy\|$$

for some $\varepsilon > 0$ and hence, the fact that $\|y\| \leq \frac{\eta}{k}$ yields

$$\|Q^{-1}\Phi(t, Qy)\| \leq \varepsilon lk \|y\|.$$

So, it is clear that if all eigenvalues of the matrix B in the system (1.3.21) have negative real parts, then the solutions of the system

$$\frac{dy}{dt} = By$$

vanish and furthermore the solutions of the systems (1.3.21) and (1.3.17) respectively possess the same property.

Assertions (b) and (c) are proved in the same manner.

If in Theorem 1.3.21 inequality (1.3.20) is satisfied with a constant matrix P being an M -matrix, then all assertions of the Theorem 1.3.21 remain valid without the transformation of the system (1.3.17) to (1.3.21).

1.4 Notes

1.2. The work by Liapunov [101] was published more than 100 years ago; nevertheless its ideas still inspire many investigations today. Therefore in Sections 1.2 and 1.3 are included not to repeat the contents of this paper but to cite the basic statements of the second Liapunov method according to the original (see Liapunov [101]).

Comments 1.2.1–1.2.7, Theorem 1.2.1 and Definitions 1.2.1–1.2.3 are set out according to Grujić, Martynyuk and Ribbens-Pavella [57], where a huge bibliography on stability theory is available as well.

1.3. A short survey of main directions of the method of Liapunov functions begins with a review of its original results (see Liapunov [101]). The survey of classical and new trends of the method of scalar Liapunov functions is based on the results by Zubov [178, 179] (Theorems 1.3.3–1.3.9), Hahn [66] and Krasovskii [89] (Theorems 1.3.10 and 1.3.11). Theorem 1.3.12 is due to Rumyantzev [160] and Theorem 1.3.13 is due to Marachkov [108]. Theorem 1.3.14 is based on the results by Salvadori [162], while Theorem 1.3.16 is due to Hatvani [71]. For recent development in the method of scalar Liapunov functions see Lakshmikantham and Martynyuk [92].

The survey of the development of the method of the vector Liapunov function takes into account the results by Bellman [16], Matrosov [132], Melnikov [139], Corduneanu [20, 21], Kamke [81], etc. Theorem 1.3.17 is due to Michel and Miller [143]. Theorem 1.3.18 is a generalization of results by Corduneanu [20, 21] and is related to the results by Grujić, Martynyuk and Ribbens-Pavella [57]. Theorem 1.3.19 is a development of Theorem 1.6.1 by Matrosov, Lakshmikantham and Sivasundaram [96]. Theorem 1.3.20 is due to Martynyuk and Obolenskii [129]. Theorem 1.3.21 is new.

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2

MATRIX LIAPUNOV FUNCTION METHOD IN GENERAL

2.1 Introduction

The short survey of the direct Liapunov method development cited in Chapter 1 shows that the generalizations of this method in terms of multicomponent functions make this method more versatile in applications. On the other hand unsolved still is the problem of construction of appropriate functions or systems of functions in terms of which the further development of this fruitful technique is possible. In this regard a two indices system of functions (a matrix-valued function) is proposed in this chapter as a basis for construction of both scalar or vector Liapunov functions.

This chapter gives an account of the foundations of the method of matrix Liapunov functions that is a new method of qualitative analysis of nonlinear systems.

The Chapter is organized as follows.

In Section 2.2 all necessary notions of the direct Liapunov method based on matrix-valued function are introduced.

In Section 2.3 the theorems of direct Liapunov method on motion stability are set out where a scalar function constructed on the set of the two-indices system of functions is applied.

In Section 2.4 a scalar function constructed in terms of a matrix-valued function is incorporated together with the principle of comparison.

The basic theorems of the method of matrix Liapunov functions are presented in Section 2.5. Also the aggregation forms are developed for autonomous large scale systems in terms of matrix-valued functions and the estimates of asymptotic stability domains are discussed.

Section 2.6 deals with a new direction in stability theory referred to as a “multistability of motion”. For the analysis of multistability of large scale



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systems consisting of two, three or four subsystems the method of matrix-valued Liapunov functions is employed in combination with the method of comparison with scalar and vector Liapunov functions.

Section 2.7 presents applications of some general results in the problems of mechanics, automatics regulation and mathematical biology.

2.2 Definition of Matrix-Valued Liapunov Functions

2.2.1 The property of having a fixed sign of the matrix-valued function

Together with the system (1.2.10) we shall consider a two-indices system of functions

$$(2.2.1) \quad U(x) = [v_{ij}(x)], \quad i, j = 1, 2, \dots, m,$$

where $v_{ii} \in C(R^n, R_+)$ and $v_{ij} \in C(R^n, R)$ for all $i \neq j$. Moreover it is assumed that

- (i) $v_{ij}(x)$ are locally Lipschitzian in x ;
- (ii) $v_{ij}(0) = 0$ for all $i, j = 1, 2, \dots, m$;
- (iii) $v_{ij}(x) = v_{ji}(x)$ in any open connected neighborhood of point $x = 0$.

REMARK 2.2.1. If $v_{ij} \equiv 0$ for all $i \neq j = 1, 2, \dots, m$ then $U(x) = \text{diag}[v_{11}(x), \dots, v_{mm}(x)]$ and

$$(2.2.2) \quad V(x) = U(x)e, \quad e \in R^m$$

is a vector function.

REMARK 2.2.2. If $v_{ij} \equiv 0$ for all $i \neq j = 1, 2, \dots, m$ and there exists at least one value of $k \in [1, m]$ such that $v_{ii} \equiv 0$ for all $i = 1, 2, \dots, k-1, k+1, \dots, m$ and $v_{kk}(x) > 0$ satisfies the conditions (i)–(ii), then

$$(2.2.3) \quad U(x) = v_{kk}(x) \quad \text{for all } x \in N, \quad N \subseteq R^n,$$

is a positive definite scalar function.

Thus the two-indices system of functions (2.2.1) is a basis for construction of both scalar and vector Liapunov functions.

However, for the matrix-valued function (2.2.1) to solve the stability problem for the equilibrium state $x = 0$ of the system (1.2.10) it should possess the property of having a fixed sign in the sense of Liapunov.

It runs as follows:

- (i) the concept of positive definiteness of a matrix-valued function (2.2.1) should be compatible with the well-known concept of positive definiteness of a matrix;
- (ii) the concept of positive definiteness of a matrix function (2.2.1) should be compatible with Liapunov's original concept of positive definiteness of scalar functions;
- (iii) the concept of positive definiteness of a matrix function (2.2.1) should be directly applicable to stability analysis and adequate to Liapunov's (second) method.

For the sake of preciseness the following definition will be used through the book, which is based on the corresponding definition by Liapunov [101] and Hahn [66], Grujić [47] and Martynyuk [116].

DEFINITION 2.2.1. The matrix-valued function $U: R^n \rightarrow R^{m \times m}$ is:

- (i) *positive semi-definite* iff there is a time-invariant neighborhood \mathcal{N} of $x = 0$, $\mathcal{N} \subseteq R^n$, such that
 - (a) U is continuous on \mathcal{N} : $U(x) \in C(\mathcal{N})$,
 - (b) U vanishes at the origin: $U(0) = 0$,
 - (c) $v(x, y) = y^T U(x) y \geq 0 \quad \forall (x \neq 0, y \neq 0) \in \mathcal{N} \times R^m$;
- (ii) *positive semi-definite on a neighborhood \mathcal{S} of $x = 0$* iff (i) holds for $\mathcal{N} = \mathcal{S}$;
- (iii) *positive semi-definite in the whole* iff (i) holds for $\mathcal{N} = R^n$;
- (iv) *negative semi-definite (on a neighborhood \mathcal{S} of $x = 0$ in the whole)* iff $(-U)$ is positive semi-definite (on the neighborhood \mathcal{S} or in the whole, respectively).

REMARK 2.2.3. Stability analysis shows sufficiency of using a fixed vector $\eta \in R^m$ instead of any y in (c), that is $v = R^n \rightarrow R$ is defined by

$$v(x) = \eta^T U(x) y, \quad \eta = (\eta_1, \dots, \eta_m)^T, \quad \eta_i \neq 0, \quad i = 1, 2, \dots, m.$$

Iff all $\eta_i = 1$ in η , then $\eta = I = (1, 1, \dots, 1)^T \in R^m$ and

$$U(x) = \sum_{i,j=1}^m u_{ij}(x), \quad u_{ij}(x) = u_{ji}(x).$$

REMARK 2.2.4. In case $m = 1$, then Definition 2.2.1 reduces to Liapunov's original definition of positive definiteness concept (cf. Liapunov [101]).

REMARK 2.2.5. It is to be noted that matrix-valued function U defined by $U(x) = 0$ for all $x \in \mathbb{R}^n$ is both positive and negative semi-definite. This ambiguity can be avoided by introducing the notion of strictly positive (negative) semi-definite and there is $\tilde{x} \in \mathcal{N}$ such that $U(\tilde{x}) > 0$ ($U(\tilde{x}) < 0$), respectively.

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DEFINITION 2.2.2. The matrix-valued function $U: R^n \rightarrow R^{m \times m}$ is:

- (i) *positive definite* iff there is a time-invariant neighborhood \mathcal{N} , $\mathcal{N} \subseteq R^n$, of $x = 0$, such that it is both positive semi-definite on \mathcal{N} and $v(x, y) = y^T U(x) y > 0 \quad \forall (x \neq 0, y \neq 0) \in \mathcal{N} \times R^m$;
- (ii) *positive definite on a neighborhood \mathcal{S} of $x = 0$* , iff (i) holds for $\mathcal{N} = \mathcal{S}$;
- (iii) *positive definite in the whole*, iff (i) holds for $\mathcal{N} = R^n$;
- (iv) *negative definite (on a neighborhood \mathcal{S} of $x = 0$ in the whole)* iff $(-U)$ is positive definite (on the neighborhood \mathcal{S} or in the whole, respectively).

Together with the system (1.2.7) we shall consider a two-indices system of functions

$$(2.2.4) \quad U(t, x) = [v_{ij}(t, x)], \quad i, j = 1, 2, \dots, m,$$

where $v_{ii} \in C(\mathcal{T}_\tau \times R^n, R_+)$, $v_{ij} \in C(\mathcal{T}_\tau \times R^n, R)$ for all $i \neq j$. Moreover the next conditions are making

- (i) $v_{ij}(t, x)$ are locally Lipschitzian in x ;
- (ii) $v_{ij}(t, 0) = 0$ for all $t \in R$ ($t \in \mathcal{T}_\tau$) $i, j = 1, 2, \dots, m$;
- (iii) $v_{ij}(t, x) = v_{ji}(t, x)$ in any open connected neighborhood \mathcal{N} of point $x = 0$ for all $t \in R$ ($t \in \mathcal{T}_\tau$).

PROPOSITION 2.2.1. *The matrix-valued function $U: R \times R^n \rightarrow R^{m \times m}$ is positive definite on \mathcal{T}_τ , $\tau \in R$ iff it can be written as*

$$y^T U(t, x) y = y^T U_+(t, x) y + a(\|x\|),$$

where $U_+(t, x)$ is a positive semi-definite matrix-valued function and $a \in K$.

DEFINITION 2.2.3. Set $v_\zeta(t)$ is the largest connected neighborhood of $x = 0$ at $t \in R$ which can be associated with a function $U: R \times R^n \rightarrow R^{m \times m}$ so that $x \in v_\zeta(t)$ implies $v(t, x, y) < \zeta, y \in R^m$.

REMARK 2.2.6. In order to understand and appreciate deepness and importance of Liapunov's concept of definite functions let scalar functions v and w be considered, $v, w: R \times R^n \times R^m \rightarrow R$. Let them obey the following on $\mathcal{T}_\tau \times \mathcal{N}$, where \mathcal{N} is a connected neighborhood of $x = 0$:

- (i) v is positive definite on $\mathcal{T}_\tau \times \mathcal{N}$;
- (ii) w is positive semi-definite on $\mathcal{T}_\tau \times \mathcal{N}$ and $w(t, x, y) > 0 \forall (t, x \neq 0) \in \mathcal{T}_\tau \times \mathcal{N}$, but it is not positive on $\mathcal{T}_\tau \times \mathcal{N}$.

Let $v_\zeta(t)$ and $w_\zeta(t)$ be associated with v and w in sense of Definition 2.2.3. Then, the following is true:

- (a) there is $\xi \in (0, +\infty)$ such that $v_\zeta(t) \subseteq \mathcal{N}, \forall t \in \mathcal{T}_\tau, \forall \zeta \in (0, \xi)$;
- (b) for any $\xi \in (0, +\infty)$ for which $w_\zeta(\tau) \subseteq \mathcal{N}$ there is $t \in \mathcal{T}_\tau, t > \tau$, such that $w_\zeta(t) \setminus \mathcal{N} \neq \emptyset$.

DEFINITION 2.2.4. The matrix-valued function $U: R \times R^n \rightarrow R^{s \times s}$ is:

- (i) decreasing on $\mathcal{T}_\tau, \tau \in R$, iff there is a time-invariant neighborhood \mathcal{N} of $x = 0$ and a positive definite function w on $\mathcal{N}, w: R^n \rightarrow R$, such that $y^T U(t, x) y \leq w(x), \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;
- (ii) decreasing on $\mathcal{T}_\tau \times \mathcal{S}$ iff (i) holds for $\mathcal{N} = \mathcal{S}$;
- (iii) decreasing in the whole on \mathcal{T}_τ iff (i) holds for $\mathcal{N} = R^n$.

The expression "on \mathcal{T}_τ " is omitted iff all corresponding conditions still hold for every $\tau \in R$.

PROPOSITION 2.2.2. The matrix-valued function $U: R \times R^n \rightarrow R^{m \times m}$ is decreasing on $\mathcal{T}_\tau, \tau \in R$, iff it can be written as

$$y^T U(t, x) y = y^T U_-(t, x) y + b(\|x\|), \quad (y \neq 0) \in R^m,$$

where $U_-(t, x)$ is a negative semi-definite matrix-valued function and $b \in K$.

Barbashin and Krasovskii [12, 13] discovered the concept of radially unbounded functions. They showed necessity of it for asymptotic stability in the whole.

DEFINITION 2.2.5. The matrix-valued function $U: R \times R^n \rightarrow R^{m \times m}$ is:

- (i) radially unbounded on $\mathcal{T}_\tau, \tau \in R$, iff $\|x\| \rightarrow \infty$ implies $y^T U(t, x) y \rightarrow +\infty, \forall t \in \mathcal{T}_\tau, y \in R^m$;

- (ii) *radially unbounded* iff $\|x\| \rightarrow \infty$ implies $y^T U(t, x)y \rightarrow +\infty, \forall t \in \mathcal{T}_\tau, \forall \tau \in \mathbb{R}, y \in \mathbb{R}^m$.

PROPOSITION 2.2.3. *The matrix-valued function $U: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is radially unbounded in the whole (on \mathcal{T}_τ) iff it can be written as*

$$y^T U(t, x)y = y^T U_+(t, x)y + a(\|x\|) \quad \forall x \in \mathbb{R}^n,$$

where $U_+(t, x)$ is a positive semi-definite matrix-valued function in the whole (on \mathcal{T}_τ) and $a \in KR$.



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2.2.2 Dini derivative and Eulerian derivative

In this section the notations of upper and lower limit of a function $\psi: R \rightarrow R$ are needed (see McSchane [138]). In brief (see Demidovich [24]) they can be explained as follows.

Let t_k be a member of a sequence S_τ^- (S_τ^+) obeying

- (i) $t_k \in R$ for every integer k , $t_k < \tau$ ($t_k > \tau$)

and

- (ii) $t_k \rightarrow \tau^-$ ($t_k \rightarrow \tau^+$) as $k \rightarrow +\infty$.

DEFINITION 2.2.6.

- (i) Number $\alpha \in R$ is the *partial limit of the function ψ over the sequence S_τ^- (S_τ^+)* iff for every $\varepsilon > 0$ there is an integer N such that $k > N$ implies $|\psi(t_k) - \alpha| < \varepsilon$;
- (ii) the symbol $\alpha = +\infty$ ($\alpha = -\infty$) is the *partial limit of the function ψ over the sequence S_τ^- (S_τ^+)* iff for every $\varepsilon \in (0, \infty)$ there is an integer N such that, respectively, $k > N$ implies $\psi(t_k) > 1/\varepsilon$ ($\psi(t_k) < -1/\varepsilon$);
- (iii) the *greatest (smallest) partial limit of the function ψ over the sequence S_τ^-* is its left upper (lower) limit at $t = \tau$, respectively, which is denoted by $\limsup [\psi(t): t \rightarrow \tau^-]$, ($\liminf [\psi(t): t \rightarrow \tau^-]$);
- (iv) *right upper (lower) limit of ψ at $t = \tau$* is analogously defined when everywhere in (iii) τ^- and S_τ^- are respectively replaced by τ^+ and S_τ^+ .

DEFINITION 2.2.7. Let U be a continuous function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{m \times m}$, $U \in C(\mathcal{T}_\tau \times \mathcal{N})$ and let solutions χ of the system (1.2.7) exist and be defined on $\mathcal{T}_\tau \times \mathcal{N}$. Then, for $(t, x) \in \mathcal{T}_\tau \times \mathcal{N}$

(i)

$$D^+U(t, x) = \limsup \left\{ \frac{U[t + \theta, \chi(t + \theta(t, x))] - U(t, x)}{\theta} : \theta \rightarrow 0^+ \right\}$$

is the upper right Dini derivative of U along the motion χ at (t, x) ;

(ii)

$$D_+U(t, x) = \liminf \left\{ \frac{U[t + \theta, \chi(t + \theta(t, x))] - U(t, x)}{\theta} : \theta \rightarrow 0^+ \right\}$$

is the lower right Dini derivative of U along the motion χ at (t, x) ;

(iii)

$$D^-U(t, x) = \limsup \left\{ \frac{U[t + \theta, \chi(t + \theta(t, x))] - U(t, x)}{\theta} : \theta \rightarrow 0^- \right\}$$

is the upper left Dini derivative of U along the motion χ at (t, x) ;

(iv)

$$D_-U(t, x) = \liminf \left\{ \frac{U[t + \theta, \chi(t + \theta(t, x))] - U(t, x)}{\theta} : \theta \rightarrow 0^- \right\}$$

is the lower left Dini derivative of U along the motion χ at (t, x) .

(v) The function U has Eulerian derivative \dot{U} , $\dot{U}(t, x) = \frac{d}{dt}U(t, x)$ at (t, x) along the motion χ iff

$$D^+U(t, x) = D_+U(t, x) = D^-U(t, x) = D_-U(t, x) = DU(t, x)$$

and then

$$\dot{U}(t, x) = DU(t, x).$$

If u_{ij} is differentiable at (t, x) then (see Liapunov [101])

$$\dot{u}_{ij}(t, x) = \frac{\partial u_{ij}}{\partial t} + (\text{grad } u_{ij})^T f(t, x)$$

and

$$\text{grad } u_{ij} = \left(\frac{\partial u_{ij}}{\partial x_1}, \frac{\partial u_{ij}}{\partial x_2}, \dots, \frac{\partial u_{ij}}{\partial x_n} \right), \quad i, j = 1, 2, \dots, s.$$

Effective application of D^+U in the framework of the second Liapunov method is based on the result by Yoshizawa [174], which enables calculation of D^+U without utilizing system motions themselves.

THEOREM 2.2.1. *Let the matrix-valued function U be continuous and locally Lipschitzian in x over $\mathcal{T}_\tau \times \mathcal{S}$ and \mathcal{S} be an open set. Then,*

$$D^+U(t, x) = \limsup \{ [U(t + \theta, x + \theta f(t, x)) - U(t, x)] \theta^{-1} : \theta \rightarrow 0^+ \}$$

holds along solutions χ of the system (1.2.7) at $(t, x) \in \mathcal{T}_\tau \times \mathcal{G}$.

D^*U will mean that both D^+U and D_+U can be used.

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2.3 Direct Liapunov's Method in Terms of Matrix-Function

The following results are useful in the subsequent sections.

PROPOSITION 2.3.1. *Suppose $m(t)$ is continuous on (a, b) . Then $m(t)$ is nondecreasing (nonincreasing) on (a, b) iff*

$$D^+m(t) \geq 0 \ (\leq 0) \quad \text{for every } t \in (a, b),$$

where

$$D^+m(t) = \limsup \{[m(t + \theta) - m(t)] \theta^{-1} : \theta \rightarrow 0^+\}.$$

Following Liapunov [101], Persidskii [152], Yoshizawa [174] and Grujić, Martynyuk and Ribbens-Pavella [54], the next result is obtained.

THEOREM 2.3.1. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on \mathcal{T}_τ). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{S} \subseteq \mathcal{N}$ of point $x = 0$;*
- (2) *a positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) matrix-valued function $U(t, x)$ and vector $y \in R^m$ such that function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x and $D^+v(t, x, y) \leq 0$.*

Then

- (a) *the state $x = 0$ of system (1.2.7) is stable (on \mathcal{T}_τ), provided $U(t, x)$ is weakly decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (b) *the state $x = 0$ of system (1.2.7) is uniformly stable (on \mathcal{T}_τ), provided $U(t, x)$ is decrescent on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$).*

PROOF. We shall prove first assertion (a) of Theorem 2.3.1. The fact that function $U(t, x)$ is weakly decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) implies that for

any $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) and $x_0 \in \mathcal{G}$ there exists a constant $\delta_0 = \delta(t_0) > 0$, a vector $y \in R^m$ and a function $b \in CK$ such that

$$(2.3.1) \quad y^T U(t_0, x_0) y \leq b(t_0, \|x_0\|), \quad \|x_0\| < \delta_0.$$

Further, since $U(t, x)$ is positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) then

$$(2.3.2) \quad a(\|x\|) \leq y^T U(t, x) y \quad \forall (t, x) \in R \times \mathcal{G} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{G}).$$

Let $\varepsilon > 0$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) are arbitrary. The properties of functions $a \in K$ and $b \in CK$ yield the existence of a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ continuously dependent on t_0 and such that

$$(2.3.3) \quad b(t_0, \delta_1) < a(\varepsilon).$$

We define $\delta(t_0) = \min\{\delta_0, \delta_1\}$. It is clear that inequalities (2.3.1)–(2.3.3) are satisfied for $\|x_0\| < \delta$. Therefore,

$$(2.3.4) \quad a(\|x_0\|) < y^T U(t_0, x_0) y \leq b(t_0, \|x_0\|) < a(\varepsilon)$$

which yield $\|x_0\| < \varepsilon$.

Now we claim that for any solution $\chi(t; t_0, x_0)$ of system (1.2.7) with the initial conditions $x_0: \|x_0\| < \delta$ the inequality $\|\chi(t; t_0, x_0)\| < \varepsilon \quad \forall t \in \mathcal{T}_0$ holds. If not, there exists a $t_1 > t_0$ such that

$$(2.3.5) \quad \|\chi(t_1; t_0, x_0)\| = \varepsilon \quad \text{and} \quad \|\chi(t; t_0, x_0)\| < \varepsilon \quad \forall t \in [t_0, t_1]$$

for some solution $\chi(t; t_0, x_0)$ of system (1.2.7). Let

$$m(t) = y^T U(t, \chi(t; t_0, x_0)) y \quad \text{when } t \in [t_0, t_1].$$

Since $v(t, x, y)$ is locally Lipschitzian in x , then we get by condition (2) $D^+ v(t, x, y) = D^+ m(t) \leq 0$.

Hence, we find in view of Proposition 2.3.1 that $m(t)$ is a nonincreasing function on $[t_0, t_1]$. Thus, we have

$$\begin{aligned} a(\varepsilon) &= a(\|\chi(t_1; t_0, x_0)\|) \leq y^T U(t_1, \chi(t_1; t_0, x_0)) y \\ &\leq y^T U(t_0, x_0) y \leq a(\varepsilon). \end{aligned}$$

The contradiction obtained shows that the state $x = 0$ of system (1.2.7) is stable (on \mathcal{T}_τ).

To prove assertion (b) of Theorem 2.3.1 it is sufficient to note that by condition (b) of Theorem 2.3.1 function $U(t, x)$ is decreasing and function b in inequality (2.3.1) can be taken independent of $t_0 \in R$. This proves the theorem.

THEOREM 2.3.2. *Let the vector function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$). If there exist*

- (1) *radially unbounded positive definite in the whole matrix-valued function $U \in C(R \times R^n, R^{m \times m})$ (or $U \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$ (on \mathcal{T}_τ) and vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x)y$ is locally Lipschitzian in x and*

$$D^+v(t, x, y) \leq 0 \quad \forall (t, x) \in R \times R^n \quad (\forall (t, x) \in \mathcal{T}_\tau \times R^n).$$

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Then

- (a) the state $x = 0$ of system (1.2.7) is stable in the whole (on \mathcal{T}_τ), provided $U(t, x)$ is weakly decreasing in the whole (on \mathcal{T}_τ);
- (b) the state $x = 0$ of system (1.2.7) is uniformly stable in the whole (on \mathcal{T}_τ), provided $U(t, x)$ is decreasing in the whole (on \mathcal{T}_τ).

REMARK 2.3.1. If f is locally Lipschitzian on $R \times \mathcal{N}$ (on \mathcal{T}_τ) then U in the preceding theorems is also locally Lipschitzian on $R \times \mathcal{N}$ (on \mathcal{T}_τ) which enables effective calculation of D^+U via Theorem 2.2.1.

REMARK 2.3.2. The preceding theorems hold also when D^+U is replaced by D_+U (McShane [138] and LaSalle [97]).

Following Liapunov [101], Massera [130, 131], Yoshizawa [174], Halanay [67], Hahn [66], Grujić, Martynuk and Ribbens-Pavella [57] the next result is obtained.

THEOREM 2.3.3. Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist

- (1) open connected time-invariant neighborhood $\mathcal{G} \subseteq \mathcal{N}$ of the point $x = 0$;
- (2) positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) matrix-valued function $U(t, x)$, a vector $y \in R^m$ and positive definite on \mathcal{G} function ψ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x and

$$D^+v(t, x, y) \leq -\psi(x) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

Then

- (a) iff $U(t, x)$ is weakly decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$), the state $x = 0$ of system (1.2.7) is asymptotically stable (on \mathcal{T}_τ);
- (b) iff $U(t, x)$ is decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$), the state $x = 0$ of system (1.2.7) is uniformly asymptotically stable (on \mathcal{T}_τ).

PROOF. *Necessity.* Consider assertion (b) of Theorem 2.3.3. Let $x = 0$ of (1.2.7) be uniformly asymptotically stable (on \mathcal{T}_τ). Let $\varepsilon > 0$ be arbitrarily chosen, ζ be such that $B_\zeta \subseteq \mathcal{N}$, $\Delta \in (0, +\infty)$ and $\xi = \min\{\delta_M(\varepsilon), \Delta, \zeta\}$. Let $\mathcal{G} = B_\xi$, $\varphi \in K_{[0, \varepsilon]}$, $\alpha \in (1, +\infty)$, $y = (1, 1, \dots, 1)^T \in R^m$ and

$$y^T U(t, x)y = v(t, x) = \sup\{\varphi[\|\chi(t + \sigma; t, x)\|] (1 + \alpha\sigma)(1 + \sigma)^{-1} : \sigma \in [0, +\infty)\}, \quad \forall t \in R.$$

The function v is decreasing and positive definite on \mathcal{N} (on $\mathcal{T}_\tau \times \mathcal{G}$) because χ is continuous in all its arguments, $\varphi \in K_{[0, \varepsilon]}$, $(1 + \alpha\sigma)(1 + \sigma)^{-1}$ is also continuous, $\chi(t; t_0, 0) \equiv 0$, $\varphi(0) = 0$, and $\varphi(\|x\|) \leq v(t, x) \leq \varphi[\Pi(x)]$ $\forall t \in \mathcal{T}_0$, $\forall t_0 \in R$ ($\forall t_0 \in \mathcal{T}_\tau$), $\forall x \in \mathcal{G}$, where $\Pi \in K_{[0, \varepsilon]}$.

Let $x^* = \chi(t + \theta; t, x)$, $x = \chi(t; t_0, x_0)$, $\theta > 0$, so that

$$\begin{aligned} & v(t + \theta, x^*) \\ &= \sup\{\varphi[\|\chi(t + \theta + \sigma; t + \theta, x^*)\|] (1 + \alpha\sigma)(1 + \sigma)^{-1} : \sigma \in [0, +\infty)\} \\ &= \sup\{\varphi[\|\chi(t + \theta + \sigma; t, x)\|] (1 + \alpha\sigma)(1 + \sigma)^{-1} : \sigma \in [0, +\infty)\} \\ &= \varphi[\|\chi(t + \theta + \sigma^*; t, x)\|] (1 + \alpha\sigma^*)(1 + \sigma^*)^{-1}, \quad \forall t \in R. \end{aligned}$$

Let $\Delta = \min\left\{1, \frac{\Delta}{\alpha}\right\}$. The existence of $\sigma^* \in [0, \tau_u(\Delta, \nu)]$ obeying the last equation is guaranteed by continuity of χ , $\varphi \in K_{[0, \varepsilon]}$ continuity of $(1 + \alpha\sigma)(1 + \sigma)^{-1}$ and uniform attraction of $x = 0$.

Let $\sigma = \theta + \sigma^*$. Then (see Halanay [67]),

$$\frac{1 + \alpha\sigma^*}{1 + \sigma^*} = \frac{1 + \alpha\sigma}{1 + \sigma} \left[1 - \frac{(\alpha - 1)\theta}{(1 + \alpha\sigma^*)(1 + \sigma)}\right] > 0$$

so that

$$\begin{aligned} v(t + \theta, x^*) &= \varphi[\|\chi(t + \theta; t, x)\|] \frac{1 + \alpha\sigma}{1 + \sigma} \left[1 - \frac{(\alpha - 1)\theta}{(1 + \alpha\sigma^*)(1 + \sigma)}\right] \\ &\leq v(t, x) \left[1 - \frac{(\alpha - 1)\theta}{(1 + \alpha\sigma^*)(1 + \sigma)}\right], \\ &\quad \forall t_0 \in R \quad (\forall t_0 \in \mathcal{T}_\tau) \quad \forall t \in \mathcal{T}_0, \end{aligned}$$

or $\forall t \in \mathcal{T}_0$

$$\frac{v(t + \theta, x^*) - v(t, x)}{\theta} \leq -\frac{(\alpha - 1)v(t, x)}{(1 + \sigma^*)(1 + \alpha\sigma^* + \alpha\theta)} \quad \forall t_0 \in R \quad (\forall t_0 \in \mathcal{T}_\tau).$$

This inequality in limit as $\theta \rightarrow 0^+$ takes the form

$$D^*v(t, x) \leq -\psi(x) \quad \forall (t, x) \in R \times \mathcal{G} \quad (\forall (t, x) \in \mathcal{T}_\tau \times \mathcal{G}),$$

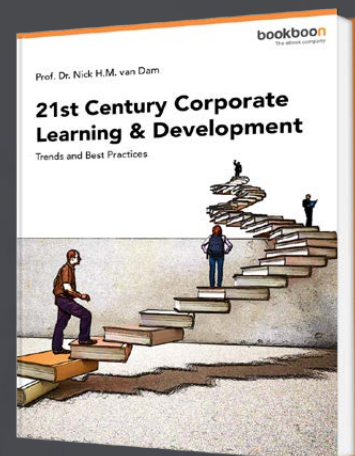
where

$$\psi(x) = \frac{(\alpha - 1)\varphi(\|x\|)}{[1 + T(\Delta)][1 + \alpha T(\Delta)]},$$

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is the continuous function, $T(\Delta) \in [\tau_u(\Delta, \nu), +\infty)$, because $x = 0$ is uniformly attractive (see Halanay [67]).

Hence, ψ is positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$).

Sufficiency. Under the conditions of Theorem 2.3.3 all conditions of Theorem 2.3.1 are fulfilled. Hence, $x = 0$ of (1.2.7) is uniformly stable (on \mathcal{T}_τ). Its uniform attraction (on \mathcal{T}_τ) is proved as follows.

Let ζ be such that $B_\zeta \subseteq \mathcal{G}$. Let φ_1 and $\varphi_2 \in K_{[0, \zeta]}$ obey

$$(2.3.6) \quad \begin{aligned} \varphi_1(\|x\|) \leq v(t, x) \leq \varphi_2(\|x\|) \\ \forall (t, x) \in R \times B_\zeta \quad (\forall (t, x) \in \mathcal{T}_\tau \times B_\zeta). \end{aligned}$$

Let

$$(2.3.7) \quad \Delta = \varphi_2^I[\varphi_1(\zeta)].$$

As shown in the proof of the sufficiency part of Theorem 2.3.1, the conditions (2.3.3) and (2.3.4) guarantee that $\|x_0\| < \Delta$ implies

$$v(t, \chi(t; t_0, x_0)) \leq v(t_0, x_0) \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in R \quad (\forall t_0 \in \mathcal{T}_\tau)$$

and that v is decreasing in t along motions χ of (1.2.7).

Let

$$(2.3.8) \quad \begin{aligned} \inf \{v(t, \chi(t; t_0, x_0)) : t \in \mathcal{T}_0\} = \nu, \\ \forall t_0 \in R \quad (\forall t_0 \in \mathcal{T}_\tau), \quad \|x_0\| < \Delta. \end{aligned}$$

Obviously $\nu \geq 0$. If $\nu > 0$ then

$$D^*v(t, \chi(t; t_0, x_0)) \leq -\gamma,$$

where

$$\gamma = \inf \{w(x) : x \in \partial B_\rho, \rho = \varphi_2^I(\nu)\}.$$

Therefore,

$$v(t, \chi(t; t_0, x_0)) \leq v(t_0, x_0) - \gamma(t - t_0),$$

so that

$$v(t, \chi(t; t_0, x_0)) < \nu \quad \text{for } t \in \left(\frac{v(t_0, x_0) - \nu}{\gamma} + t_0, \infty \right),$$

which contradicts (2.3.8). Hence, $\gamma = 0$ which together with (2.3.5), (2.3.8) and positive definiteness of v on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) prove that $\|x_0\| < \Delta$ implies

$$\lim [\|\chi(t; t_0, x_0)\| : t \rightarrow +\infty] = 0, \quad \forall t \in R \quad (\forall t_0 \in \mathcal{T}_\tau),$$

i.e. that $x = 0$ is attractive. Let now $\rho > 0$ be arbitrarily chosen,

$$\gamma = \lim \{w(x) : x \in B_\Delta \setminus B_\rho\}, \quad \gamma = \gamma(\rho),$$

and

$$\tau_u(\Delta, \rho) = \min \left\{ 0, \frac{\varphi_2(\Delta) - \varphi_1(\rho)}{\gamma(\rho)} \right\}.$$

Then

$$D^*v(t, x) \leq -\gamma,$$

$$\forall (t, x) \in R \times (B_\Delta \setminus B_\rho) \quad (\forall (t, x) \in \mathcal{T}_\tau \times (B_\Delta \setminus B_\rho))$$

and for $t = \tau_u(\Delta, \rho) + t_0$, $\tau_u(\Delta, \rho) > 0$, $\tau_u(\Delta, \rho) = 0$ implies

$$\|\chi(t; t_0, x_0)\| < \rho \quad \forall t \in \mathcal{T}_0,$$

$$v(t, \chi(t; t_0, x_0)) \leq v(t_0, x_0) - \gamma(t - t_0) \leq \varphi_2(\Delta) - \varphi_2(\Delta) + \varphi_1(\rho) = \varphi_1(\rho)$$

so that

$$\varphi_1(\|\chi(t; t_0, x_0)\|) \leq \varphi_1(\rho), \quad \forall t_0 \in R \quad (\forall t_0 \in \mathcal{T}_\tau)$$

yields

$$\|\chi(t; t_0, x_0)\| \leq \rho \quad \text{at } t = \tau_u(\Delta, \rho) + t_0, \quad \forall x_0 \in B_\Delta.$$

For $t \in (\tau_u(\Delta, \rho), +\infty)$

$$v(t, \chi(t; t_0, x_0)) < v(t_0 + \tau_u(\Delta, \rho));$$

$$\chi(t_0 + \tau_u(\Delta, \rho), t_0, x_0) \leq \varphi_1(\rho)$$

so that

$$\|\chi(t; t_0, x_0)\| < \rho,$$

$$\forall t \in (t_0 + \tau_u(\Delta, \rho), +\infty), \quad \forall t_0 \in R \quad (t_0 \in \mathcal{T}_\tau), \quad \forall x_0 \in B_\Delta$$

which proves that attraction of $x = 0$ is uniform (on \mathcal{T}_τ).

Following Barbashin and Krasovskii [12, 13] and Martynyuk [116], and the preceding proof in which we choose $\varphi \in KR$ it is easy to prove.

THEOREM 2.3.4. *Let the vector function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$). If there exist*

- (1) *radially unbounded positive definite in the whole matrix-valued function $U(t, x) \in C(R \times R^n, R^{m \times m})$ (or $U(t, x) \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$ (on \mathcal{T}_τ), a vector $y \in R^m$ and a positive definite in the whole function θ , such that the function*

$$v(t, x, y) = y^T U(t, x) y$$




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is locally Lipschitzian in x and

$$D^+v(t, x, y) \leq -\theta(x) \quad \forall (t, x, y) \in R \times R^n \times R^m \\ \forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m.$$

Then

- (a) iff $U(t, x)$ is weakly decreasing in the whole (on \mathcal{T}_τ), the state $x = 0$ of system (1.2.7) is asymptotically stable in the whole (on \mathcal{T}_τ);
- (b) iff $U(t, x)$ is decreasing in the whole (on \mathcal{T}_τ), the state $x = 0$ of system (1.2.7) is uniformly asymptotically stable in the whole (on \mathcal{T}_τ).

Following Krasovskii [89], Grujić, Martynyuk and Ribbens-Pavella [57] and He and Wang [72] and utilizing $\varphi(\zeta) = \zeta^p$ in the proof of Theorem 2.3.3, it is easy to prove the following result.

THEOREM 2.3.5. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) an open connected time-invariant neighborhood $\mathcal{G} \subseteq \mathcal{N}$ of the point $x = 0$;
- (2) a matrix-valued function $U(t, x)$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x ;
- (3) functions $\varphi_1, \varphi_2 \in K$ and a positive real number η_1 and positive integer p such that

$$\eta_1 \|x\|^p \leq v(t, x, y) \leq \varphi_1(\|x\|) \quad \forall (t, x, \eta \neq 0) \in R \times \mathcal{G} \times R^m$$

and

$$D^+v(t, x, y) \leq -\varphi_2(\|x\|) \quad \forall (t, x, \eta \neq 0) \in R \times \mathcal{G} \times R^m \\ (\forall (t, x, \eta \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

Then, iff the comparison functions φ_1 and φ_2 are of the same magnitude, the state $x = 0$ of system (1.2.7) is exponentially stable (on \mathcal{T}_τ).

REMARK 2.3.2. The statement of Theorem 2.3.6 remains valid, if $\varphi_1(\|x\|) = \eta_2\|x\|^p$ and $\varphi_2(\|x\|) = \eta_3\|x\|^p$, $\eta_2, \eta_3 = \text{const} > 0$.

THEOREM 2.3.6. Let the vector function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$). If there exist

- (1) radially unbounded positive definite in the whole matrix-valued function $U(t, x) \in C(R \times R^n, R^{m \times m})$ (or $U(t, x) \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$) (on \mathcal{T}_τ) and vector $y \in R^m$ such that the function

$$v(t, x, \eta) = y^T U(t, x) y$$

is locally Lipschitzian in x ;

- (2) functions $\psi_1, \psi_2 \in KR$ a positive real number η_1 and positive integer q such that

$$\begin{aligned} \eta_2 \|x\|^q \leq v(t, x, y) \leq \psi_1(\|x\|) \quad \forall (t, x, y \neq 0) \in R \times R^n \times R^m \\ \forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times R^n \times R^m \end{aligned}$$

and

$$\begin{aligned} D^+ v(t, x, y) \leq -\psi_2(\|x\|) \quad \forall (t, x, y \neq 0) \in R \times R^n \times R^m \\ \forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times R^n \times R^m. \end{aligned}$$

Then, if the comparison functions ψ_1, ψ_2 , are of the same magnitude, the state $x = 0$ of system (1.2.7) is exponentially stable in the whole (on \mathcal{T}_τ).

PROOF. The proof is similar to that of Theorem 2.3.5.

REMARK 2.3.3. The assertion of Theorem 2.3.6 remains valid, if $\varphi_1(\|x\|) = \eta_2\|x\|^q$ and $\varphi_2(\|x\|) = \eta_3\|x\|^q$.

PROPOSITION 2.3.2. In order that the state $x = 0$ of system (1.2.7) be exponentially stable (on \mathcal{T}_τ) in the whole, it is necessary and sufficient for it to be exponentially stable (on \mathcal{T}_τ) and uniformly asymptotically stable in the whole (on \mathcal{T}_τ).

Following Zubov [178] and taking into account the results by Martynyuk [116] we shall formulate and prove a result on instability.

THEOREM 2.3.7. *Let the vector function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{G} \subseteq \mathcal{N}$ of the point $x = 0$;*
- (2) *a matrix-valued function $U(t, x) \in C^{1,1}(R \times \mathcal{G}, R^{m \times m})$ or $U(t, x) \in C^{1,1}(\mathcal{T}_\tau \times \mathcal{G}, R^{m \times m})$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is strictly positive semi-definite (on \mathcal{T}_τ) and satisfies the relation*

$$\frac{dv}{dt} = \lambda v + \tilde{\theta}(x), \quad \lambda = \lambda(t, x),$$



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where $\tilde{\theta}(x)$ is a positive semi-definite function on \mathcal{G} ;

- (3) a number $\varepsilon > 0$ such that when $\delta > 0$ ($\delta < \varepsilon$) for continuous on $\mathcal{T}_0 \times R \times \mathcal{G}$ (on $\mathcal{T}_0 \times \mathcal{T}_\tau \times \mathcal{G}$) solution $\chi(t; t_0, x_0)$ of system (1.2.7) which satisfies the condition $\|x_0\| < \delta$, $v(t_0, x_0) > 0$ implies $\|\chi(t; t_0, x_0)\| < \varepsilon \quad \forall t \in R \quad (\forall t_0 \in \mathcal{T}_\tau)$ the inequality

$$|v(t, \chi(t; t_0, x_0), y)| \geq v(t_0, x_0, y) \exp \left(\int_{t_0}^t \lambda(s) ds \right)$$

does not hold for all $t \geq t_0$, $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$), $t \in \mathcal{T}_0$.

Then and only then the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

PROOF. *Necessity.* Let the state $x = 0$ of system (1.2.7) be unstable (on \mathcal{T}_τ). We construct two functions v and $\tilde{\theta}$ satisfying the conditions of Theorem 2.3.7. The instability (on \mathcal{T}_τ) of state $x = 0$ of system (1.2.7) yields the existence of an $\varepsilon^* > 0$ such that for any $\delta > 0$ a x_0 and a $t_0, t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) can be taken so that the inequality

$$(2.3.9) \quad \|\chi(t; t_0, x_0)\| < \varepsilon^*$$

does not hold for all $t \geq 0$ in spite of the fact that $\|x_0\| < \delta$, $t_0 \geq 0$.

Let $t = t(t_0, x_0)$ be the next time after t_0 when inequality (2.3.9) is violated. The set of points $\Pi = \{(t_0, x_0): \|x_0\| < \delta, t_0 \geq 0\}$ is divided conventionally into sets Π_1 and Π_2 such that

- (A) for $(t_0, x_0) \in \Pi_1$ the solutions $\chi(t; t_0, x_0)$ of system (1.2.7) satisfy condition (2.3.9) provided all $t \geq t_0$.
- (B) for $(t_0, x_0) \in \Pi_2$ the solutions $\chi(t; t_0, x_0)$ of system (1.2.7) intersect the surface $\|x\| = \varepsilon^*$ when the time increases.

We set in case (A) $y = (1, 1, \dots, 1) \in R^m$ and

$$(2.3.10) \quad y^T U(t, x) y = v(t, x) \equiv 0 \quad \forall t \in \mathcal{T}_0.$$

For the case (B) we set $y = (1, 1, \dots, 1) \in R^m$ and

$$(2.3.11) \quad y^T U(t_0, x_0) y = v(t_0, x_0) = \exp(t_0 - t(t_0, x_0)).$$

It is clear that $v(t, x_0) \equiv 0$ for function (2.3.10) when $t \geq t_0$, and

$$v(t, x_0) = \exp(t_0 - t(t_0, x_0))$$

for function (2.3.11).

Hence, we get $dv/dt = v$. Comparing this result with condition (2) of Theorem 2.3.7 we obtain $\lambda = 1$ and $\tilde{\theta} = 0$.

Function v is strictly positive semi-definite (on \mathcal{T}_τ) and bounded, $\int_{t_0}^t \lambda(s) ds$ diverges as $t - t_0 \rightarrow \infty$, since $\lambda \equiv 1$. Therefore, condition (3) of the Theorem 2.3.7 is also satisfied.

Sufficiency. Let all hypotheses of Theorem 2.3.7 be satisfied.

We are going to show that the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ). If not, then using $\varepsilon > 0$ a $\delta > 0$ can be taken so that

$$(2.3.12) \quad \|\chi(t; t_0, x_0)\| < \varepsilon \quad \forall t \in \mathcal{T}_0, \quad \forall t \in R \quad (\forall t_0 \in \mathcal{T}_\tau),$$

when $\|x_0\| < \delta$.

According to condition (2) of Theorem 2.3.7 we take t_0 and x_0 so that $v(t_0, x_0, y) > 0$ and consider along the solution $\chi(t; t_0, x_0)$ of system (1.2.7) the correlation

$$(2.3.13) \quad \frac{dQ}{dt} = \lambda Q(t) + P(t) \quad \forall t \in R \quad (\forall t \in \mathcal{T}_\tau),$$

where $Q(t) = v(t, \chi(t; t_0, x_0))$ and $P(t) = \tilde{\theta}(\chi(t; t_0, x_0))$.

In view of $P(t) \geq 0$ for all $t \in R$ we find from correlation (2.3.13)

$$Q(t) \geq Q(t_0) \exp\left(\int_{t_0}^t \lambda(s) ds\right) \quad \forall t \in R \quad (\forall t \in \mathcal{T}_\tau).$$

This inequality contradicts condition (3) of Theorem 2.3.7 and, therefore, inequality (2.3.9) can be satisfied for all $t \in \mathcal{T}_0$, i.e. the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

COROLLARY 2.3.1. If conditions (1) and (2) of Theorem 2.3.7 are satisfied and

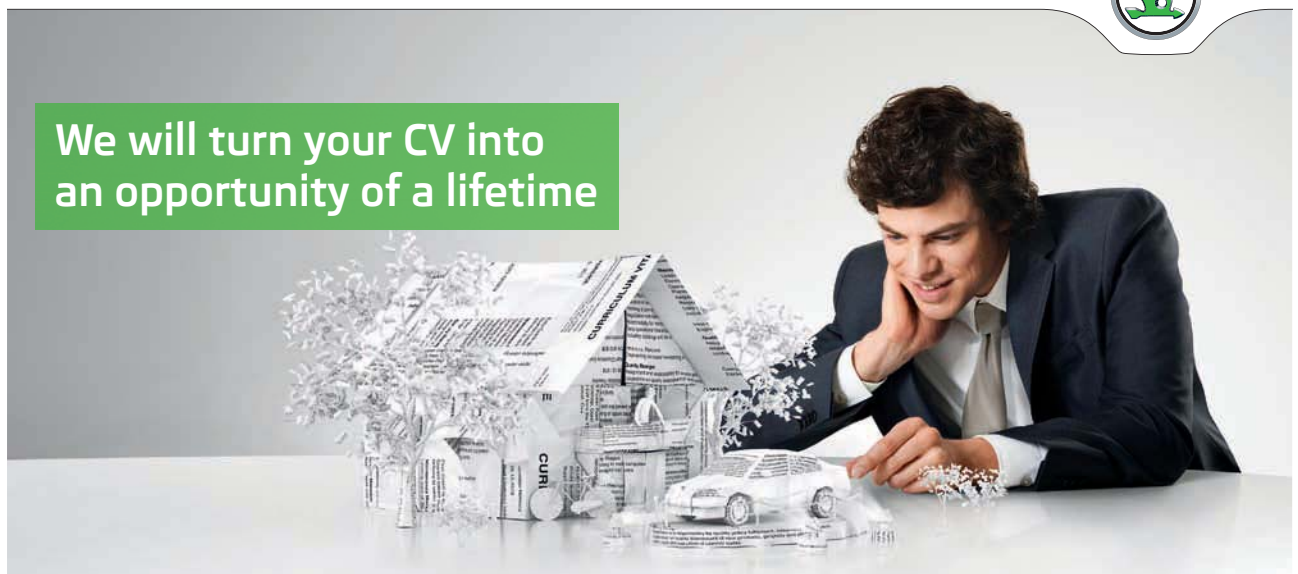
- (1) the function $v(t, x, y) = y^T U(t, x) y$ is bounded (on \mathcal{T}_τ);
- (2) $\int_{t_0}^t \lambda(s) ds \rightarrow +\infty$ as $t \rightarrow +\infty$.

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Then the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

COROLLARY 2.3.2. If conditions (1) and (2) of Theorem 2.3.7 are satisfied and

- (1) the function $v(t, x, y) = y^T U(t, x)y$ is bounded (on \mathcal{T}_τ);
- (2) the function λ is a positive constant.

Then the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

REMARK 2.3.4. Corollary 2.3.2 is a new version of Liapunov's theorem on instability (cf. Liapunov [101], Theorem III, pp. 68).

COROLLARY 2.3.3. If conditions (1) and (2) of Theorem 2.3.7 are satisfied and

- (1) $\frac{dv}{dt} = \tilde{\theta}(x) \quad \forall t \in \mathcal{T}_0 \quad (\forall t \in \mathcal{T}_\tau) \quad \forall x \in \mathcal{G};$
- (2) using number $\varepsilon > 0$ and $\delta > 0$ can be taken so that $\tilde{\theta}(x) > 0$ for $v(t, x, y) > \varepsilon$.

Then the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

REMARK 2.3.5. Corollary 2.3.3 is a new version of Chetaev's theorem on instability (cf. Chetaev [19], pp. 33).

2.4 On Comparison Method

The concept of the matrix Liapunov function together with the theory of differential inequalities provides a very general comparison principle under much less restrictive assumptions. In this set up, the matrix Liapunov function may be viewed as a transformation that reduces the study of a given complicated differential system to the study of relatively simpler scalar differential equations.

2.4.1 Differential inequalities

Let us consider the following scalar differential equation

$$(2.4.1) \quad \frac{du}{dt} = g(t, u), \quad u(t_0) = u_0 \geq 0, \quad t_0 \in R \ (t_0 \in \mathcal{T}_\tau),$$

where $g \in C(R \times R, R)$ (or $g \in C(\mathcal{T}_\tau \times R, R)$) and $g(t, 0) = 0 \ \forall t \in \mathcal{T}_0$.

DEFINITION 2.4.1. Let $\gamma(t)$ be a solution of (2.4.1) existing on some interval $J = [t_0, t_0 + \alpha)$, $0 < \alpha \leq +\infty$, $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$). Then $\gamma(t)$ is said to be *the maximal solution of (2.4.1)* if for every solution $u(t) = u(t; t_0, x_0)$ of (2.4.1) existing on J , the following inequalities hold

$$(2.4.2) \quad u(t) \leq \gamma(t), \quad t \in \mathcal{G}, \quad t_0 \in R \ (t_0 \in \mathcal{T}_\tau).$$

A minimal solution is defined similarly by reversing the inequality (2.4.2).

We need the following known results for our discussion the proof of which may be found in (see e.g. Olech and Opial [150], Yoshizawa [174], and Lakshmikantham, Leela and Martynyuk [94]).

PROPOSITION 2.4.1. Let $g \in C(R \times R, R)$ (or $g \in C(\mathcal{T}_\tau \times R, R)$) and $\gamma(t) = \gamma(t; t_0, x_0)$ be the maximal solution of (2.4.1) existing on J . Suppose that $m \in C(R, R_+)$ ($m \in C(\mathcal{T}_\tau, R_+)$) and

$$(2.4.3) \quad D^*m(t) \leq g(t, m(t)), \quad t \in J,$$

where D^* is any fixed Dini derivative.

Then $m(t_0) \leq u_0$ implies

$$(2.4.4) \quad m(t) \leq \gamma(t), \quad \forall t \in J.$$

PROPOSITION 2.4.2. Let $g \in C(R \times R, R)$ (or $g \in C(\mathcal{T}_\tau \times R, R)$) and $\rho(t) = \rho(t; t_0, x_0)$ be the minimal solution of (2.4.1) existing on J . Suppose that $m \in C(R, R_+)$ ($m \in C(\mathcal{T}_\tau, R_+)$) and

$$(2.4.5) \quad D^*m(t) \geq g(t, m(t)), \quad t \in J.$$

Then $m(t_0) \geq u_0$ implies

$$(2.4.6) \quad m(t) \geq \rho(t), \quad \forall t \in J.$$


PROPOSITION 2.4.3. *Let for system (1.2.7) there exist*

- (1) *a matrix-valued function $U \in C(R \times R^n, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$) and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x)y$ is locally Lipschitzian in x for every $t \in R$ ($t \in \mathcal{T}_\tau$):*
- (2) *a majorizing function $g \in C(R \times R_+, R)$ $g \in C(\mathcal{T}_\tau \times R_+, R)$, $g(t, 0) = 0 \forall t \in \mathcal{T}_0$ ($\forall t \in \mathcal{T}_\tau$) such that*

$$D^+v(t, x, y) \leq g(t, v(t, x, y))$$

$$\forall (t, x, y) \in R \times R^n \times R^m \quad (\mathcal{T}_\tau \times R^n \times R^m);$$

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- (3) a maximal solution $\gamma(t) = \gamma(t; t_0, u_0)$ of comparison equation (2.4.1) on J .

Then along any solution $\chi(t; t_0, x_0)$ of system (1.2.7) existing on $J_1 \subseteq J$ the estimate

$$(2.4.7) \quad v(t_0, x_0, u_0) \leq u_0, \quad t_0 \in R \ (t_0 \in \mathcal{T}_\tau)$$

implies the inequality

$$(2.4.8) \quad v(t, \chi(t; t_0, x_0), y) \leq \gamma(t) \quad \forall t \in J_1 \cap J.$$

PROOF. Let $m(t) = v(t, \chi(t; t_0, x_0), y)$ and $\chi(t; t_0, x_0)$ being a solution of (1.2.7) such that (2.4.7). Since $v(t, x, y)$ is locally Lipschitzian in x , we get, by (1.2.7) and (2.4.1), the differential inequality

$$D^+ m(t) \leq g(t, m(t)), \quad m(t_0) \leq u_0, \quad t_0 \in R \ (t_0 \in \mathcal{T}_\tau), \quad t \in J,$$

and Proposition 2.4.1 gives the desired result (2.4.8).

A comparison result analogous to Proposition 2.4.3 which yields lower bounds is the following.

PROPOSITION 2.4.4. *If in Proposition 2.4.3, assumption (2) is reversed to*

$$D^+ v(t, x, y) \geq g(t, v(t, x, y)) \\ \forall (t, x, y) \in R \times R^n \times R^m \quad (\mathcal{T}_\tau \times R^n \times R^m)$$

and $\rho(t) = \rho(t; t_0, u_0)$ is the minimal solution of (2.4.1) existing for $t \geq t_0$, then

$$v(t, \chi(t; t_0, x_0), y) \geq \rho(t) \quad \forall t \in J, \quad t_0 \in R \ (t_0 \in \mathcal{T}_\tau)$$

whenever $v(t_0, x_0, y) \geq u_0$.

PROOF is similar to proof of Proposition 2.4.3.

In some situations, estimating $D^+ v(t, x, y)$ as a function of t, x and $v(t, x, y)$ is more natural (see e.g. Matrosov [134], Hatvani [71], and Grujić, Martynyuk and Ribbens-Pavella [57]).

PROPOSITION 2.4.5. *Let for system (1.2.7) there exist*

- (1) *a matrix-valued function $U \in C(R \times R^n, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$) and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x)y$ is locally Lipschitzian in x for every $t \in R$ ($t \in \mathcal{T}_\tau$):*
- (2) *a majorizing function $g \in C(R \times R^n \times R_+, R)$ $g \in C(\mathcal{T}_\tau \times R^n \times R_+, R)$, $g(t, x, u)$ nondecreasing in u , $g(t, 0, 0) = 0 \quad \forall t \in \mathcal{T}_0$ such that*

$$D^+ v(t, x, y) \leq g(t, x, v(t, x, y)) \quad \forall (t, x, y) \in R \times R^n \times R^m \\ (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m);$$

- (3) *a maximal solution $r(t) = r(t; t_0, u_0, x_0)$ of comparison equation*

$$\frac{du}{dt} = g(t, x(t), u), \quad u(t_0) = u_0 \geq 0$$

exist for all $t \geq t_0$, $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$).

Then $v(t_0, x_0, y) \leq u_0$ implies

$$v(t, \chi(t; t_0, x_0), y) \leq r(t; t_0, u_0, x_0) \quad t \in J.$$

PROOF is similar to proof of Proposition 2.4.3.

COROLLARY 2.4.1. *If in conditions of Proposition 2.4.3*

- (i) $g(t, u) \equiv 0 \quad \forall t \in R$ ($\forall t \in \mathcal{T}_\tau$)
then

$$v(t, \chi(t; t_0, x_0), y) \leq v(t_0, x_0, y) \quad \forall t \in R \quad (\forall t \in \mathcal{T}_\tau);$$

- (ii) $g(t, u) = \lambda(t)u$
then

$$v(t, \chi(t; t_0, x_0), y) \leq u_0 \exp \left(\int_{t_0}^t \lambda(s) ds \right), \\ t \geq t_0, \quad t_0 \in R \quad (t_0 \in \mathcal{T}_\tau);$$

- (iii) $g(t, u) = a \exp [-ku] + \varphi(t) - a$, where $\varphi: R \rightarrow R$; $a, k = \text{const}$,

then

$$v(t, \chi(t; t_0, x_0), y) \leq \psi(t) + k^{-1} \ln \left\{ \exp(ku_0) + ak \int_0^t \exp(-k\psi(s)) ds \right\}, \quad t \geq t_0;$$

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then

$$v(t, \chi(t; t_0, x_0), y) \leq G^{-1} [G(u_0) - (t - t_0)], \quad t \geq t_0$$

where $G(u) = \int_0^u \frac{ds}{c(s)}$ if $\int_0^u \frac{ds}{c(s)} < +\infty$ and

$$G(u) = \int_{\delta}^u \frac{ds}{c(s)} \text{ if } \int_0^u \frac{ds}{c(s)} = \infty; \delta = \text{const} > 0,$$

G^{-1} is a function converse to the function G .

2.4.2 Theorems on stability via matrix Liapunov functions and scalar comparison equations

The estimates of function $v(t, x, y)$ found in Propositions 2.4.3–2.4.4 allow the reduction of the problem on stability of state $x = 0$ of system (1.2.7) to the stability investigation of solution $u = 0$ of equation (2.4.1). Let us formulate first stability definitions for solution $u = 0$ of equation (2.4.1).

DEFINITION 2.4.2. The state $u = 0$ of the equation (2.4.1) is:

- (i) *stable with respect to \mathcal{T}_i* iff for any $t_0 \in \mathcal{T}_i$ and any $\varepsilon \in (0, \infty)$ there exists $\delta(t_0, \varepsilon)$ such that for any u_0 , $0 \leq u_0 < \delta$ an estimation $\gamma(t; t_0 u_0) < \varepsilon$ is fulfilled for all $t \in \mathcal{T}_0$;
- (ii) *uniformly stable with respect to \mathcal{T}_i* iff conditions of the definition 2.4.2 (i) are fulfilled and for every $\varepsilon \in (0, \infty)$ the corresponding maximal δ denoted by $\delta_M(t, \varepsilon)$ obeys:

$$\inf (\delta_M(t, \varepsilon) : t \in \mathcal{T}_i) > 0;$$

- (iii) *stable in the whole with respect to \mathcal{T}_i* iff conditions of the definition 2.4.2 (i) are fulfilled and $\delta_M(t, \varepsilon) \rightarrow +\infty$, $\forall \varepsilon \rightarrow +\infty$, $\forall t \in \mathcal{T}_i$;
- (iv) *uniformly stable in the whole with respect to \mathcal{T}_i* iff conditions of the definition 2.4.2 (ii), (iii) are fulfilled.

DEFINITION 2.4.3. The state $u = 0$ for the equation (2.4.1) is:

- (i) *attractive with respect to \mathcal{T}_i* iff for any $t_0 \in \mathcal{T}_i$ there exists $\Delta(t_0) > 0$ and for any $\zeta > 0$ and $u_0, : 0 \leq u_0 \leq \Delta(t_0)$ there is $\tau(t_0, x_0, \zeta) \in [0, \infty)$ such that an estimation $\gamma(t; t_0, u_0) < \zeta$ is fulfilled for all $t \in (t_0 + \tau(t_0, x_0, \zeta) + \infty)$;
- (ii) *u_0 - attractive with respect to \mathcal{T}_i* iff conditions under (i) are fulfilled and for any $t_0 \in \mathcal{T}_i$ and any $\eta \in (0, +\infty)$ there exists $\Delta(t_0) > 0$ and $\tau_u(t_0, \Delta(t_0), \eta) \in [0, +\infty)$ such that

$$\sup (\tau_m(t, u_0, \eta) : 0 \leq u_0 \leq \Delta(t_0)) = \tau_u(t_0, \Delta(t_0), \eta);$$

- (iii) *t_0 - uniformly attractive with respect to \mathcal{T}_i* iff conditions of (i) are fulfilled and for any $\eta \in (0, +\infty)$ there exists $\Delta > 0$ and $\tau_u(u_0, \eta) \in [0, +\infty)$ such that

$$\sup (\tau_m(t_0, u_0, \eta) : (t_0, u_0 \in \mathcal{T}_i)) = \tau_u(u_0, \eta);$$

- (iv) *uniformly attractive with respect to \mathcal{T}_i* iff conditions of the definitions 2.4.3 (i) – (iii) are fulfilled and for any $\eta \in (0, +\infty)$ there exists $\Delta > 0$ and $\tau_u(\Delta, \eta) \in [0, +\infty)$ such that

$$\sup (\tau_m(t_0, u_0, \eta) : (t_0, u_0) \in \mathcal{T}_i \times [0 \leq u_0 \leq \Delta]) = \tau_u(\Delta, \eta).$$

DEFINITION 2.4.4. The state $u = 0$ of the equation (2.4.1) is:

- (i) *asymptotically stable with respect to \mathcal{T}_i* iff it is stable with respect to \mathcal{T}_i and attractive with respect to \mathcal{T}_i ;
- (ii) *equi-asymptotically stable with respect to \mathcal{T}_i* iff it is stable with respect to \mathcal{T}_i and u_0 - uniformly attractive with respect to \mathcal{T}_i ;
- (iii) *quasi-asymptotically stable with respect to \mathcal{T}_i* iff it is uniformly stable with respect to \mathcal{T}_i and t_0 - uniformly attractive with respect to \mathcal{T}_i ;
- (iv) *exponentially stable with respect to \mathcal{T}_i* iff there exists $\Delta > 0$ and real values $\alpha \geq 1$ such that for $0 \leq u_0 \leq \Delta$ the inequality

$$u(t; t_0, u_0) \leq \alpha u_0 \exp(-\beta(t - t_0)), \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i$$

is valid.

Definitions 2.4.4 (i) – (iv) become the corresponding definitions of asymptotic stability in the whole provided both the corresponding type of stability in the whole and attraction in the whole.

In Definitions 2.4.2 – 2.4.4, the expression “with respect to \mathcal{T}_i ” can be omitted iff $\mathcal{T}_i = \mathcal{R}$.

THEOREM 2.4.1. *Let vector-function f in system (1.2.7) be continuous on $R \times N$ (on $\mathcal{T}_\tau \times N$). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{G} \subset N$ of point $x = 0$;*
- (2) *a matrix-valued function $U \in C(R \times \mathcal{G}, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times \mathcal{G}, R^{m \times m})$) and vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for every $t \in R$ ($t \in \mathcal{T}_\tau$);*
- (3) *a majorizing function $g \in C(R \times R_+, R)$ ($g \in C(\mathcal{T}_\tau \times R_+, R)$) $g(t, 0) = 0 \quad \forall t \in \mathcal{T}_0$ ($\forall t \in \mathcal{T}_\tau$) such that*

$$D^+ v(t, x, y) \leq g(t, v(t, x, y)) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m$$

$$(\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

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Then properties of the function

$$(2.4.9) \quad y^T U(t, x) y = v(t, x, y)$$

and properties of the zero solution of the equation (2.4.1)

$$\frac{du}{dt} = g(t, u), \quad u(t_0) = u_0 \geq 0$$

provide the corresponding properties of the state $x = 0$ of system (1.2.7).

REMARK 2.4.1. In condition (2) of Theorem 2.4.1 alongside the function defined by (2.4.9) another suitable function, such as

$$\begin{aligned} v(t, x) &= \max \{u_{ij}(t, x) : (i, j) \in [1, m]\}, \\ v(t, x, \eta) &= \eta^T U(t, x) \eta, \quad \eta \in R_+^m, \quad \eta > 0, \\ \text{or } v(t, x) &= Q(U(t, x)) \end{aligned}$$

can be utilized, where $Q \in C(R^{m \times m}, R_+)$, $Q(u)$ is nondecreasing in u and $Q(0) = 0$.

We shall state some properties of the zero solution of equation (2.4.1) and function (2.4.9) and prove Theorem 2.4.1.

PROOF. *Case A.* Let $g(t, u) = 0$, solution $u = 0$ of equation (2.4.1) be stable with respect to \mathcal{T}_i and function (2.4.9) be positive definite on \mathcal{G} ($\mathcal{T}_\tau \times \mathcal{G}$). Then, by Theorem 2.3.1 the state $x = 0$ of system (1.2.7) is stable (on \mathcal{T}_τ).

Case B. Let solution $u = 0$ of equation (2.4.1) be stable with respect to \mathcal{T}_i , u_0 – uniformly attractive on \mathcal{T}_i and function (2.4.9) is decreasing on \mathcal{G} ($\mathcal{T}_\tau \times \mathcal{G}$). We are going to show that in this case the state $x = 0$ of system (1.2.7) is equi-asymptotically stable (on \mathcal{T}_τ). In fact, for the function $v(t, x, y)$ mentioned above there exist functions $a, b \in K$ such that

$$(2.4.10) \quad \begin{aligned} a(\|x\|) \leq v(t, x, y) \leq b(\|x\|) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \\ \forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m. \end{aligned}$$

Let $\varepsilon \in (0, H)$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) be prespecified. The fact that $u = 0$ is stable with respect to \mathcal{T}_i implies that for $a(\varepsilon) > 0$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) there exists a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that it follows from $u_0 < \delta_1$ that $u(t; t_0, u_0) < a(\varepsilon)$ for all $t \geq t_0$. We take $u_0 = v(t_0, x_0, y)$ and $\delta = \delta(t_0, \varepsilon) > 0$ so that

$$(2.4.11) \quad b(\delta) < \delta_1.$$

Let the solution $\chi(t; t_0, x_0)$ of system (1.2.7) start in domain: $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) and $\|x_0\| < \delta$. We claim that $\|\chi(t; t_0, x_0)\| < \varepsilon$ for all $t_0 \in \mathcal{T}_0$. If not, there exists other solution $\chi(t)$ with the initial conditions in the same domain and value $t_1 > t_0$ such that

$$(2.4.12) \quad \|\chi(t_1; t_0, x_0)\| = \varepsilon \quad \text{and} \quad \|\chi(t; t_0, x_0)\| \leq \varepsilon \quad \forall t \in [t_0, t_1].$$

By Proposition 2.4.3 we have the estimate

$$(2.4.13) \quad v(t, \chi(t; t_0, x_0), y) \leq \gamma(t) \quad \forall t \in [t_0, t_1],$$

where $\gamma(t)$ is the maximal solution of equation (2.4.1).

Seeing that

$$(2.4.14) \quad v(t_0, x_0, y) \leq b(\|x_0\|) \leq b(\delta) < \delta_1$$

and in view of inequalities (2.4.11) – (2.4.13) we get

$$a(\varepsilon) \leq v(t_1, \chi(t_1; t_0, x_0), y) \leq \gamma(t_1) < a(\varepsilon).$$

This proves stability with respect to \mathcal{T}_i of the state $x = 0$ of system (1.2.7).

Further it follows from the property of u_0 – uniform attraction of the solution $u = 0$ of equation (2.4.1) that, given $a(\eta) > 0$ and $t_0 \in R$

$(t_0 \in \mathcal{T}_\tau)$, $0 < \eta < H$ there exists a $\delta_1^* = \delta_1^*(t_0) > 0$ and $\tau = \tau(t_0, \eta) > 0$ such that $u_0 < \delta_1^*$ implies $u(t; t_0, u_0) < a(\eta)$ for all $t \geq t_0 + \tau$. We take $u_0 = v(t_0, x_0, y)$ the same as before and find $\delta_0^* = \delta_0^*(t_0) > 0$ such that

$$(2.4.15) \quad b(\delta_0^*) < \delta_1^*.$$



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Let $\delta_0 = \min(\delta_1^*, \delta_0^*)$ and $\|x_0\| < \delta_0$. Proceeding as in the proof of stability (on \mathcal{T}_τ) of the state $x = 0$ of system (1.2.7) we conclude that $\|\chi(t; t_0, x_0)\| < H$ for $t \geq t_0$. Assume that there exists a sequence $\{t_k\}$, $t_k \geq t_0 + \tau$, $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$ and $\eta \leq |\chi(t; t_0, x_0)|$, where $\chi(t; t_0, x_0)$ is a solution of system (1.2.7) with the initial conditions $\|x_0\| < \delta_0$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$). Taking into account estimates (2.4.8) and (2.4.15) we get

$$a(\eta) \leq v(t_k, \chi(t_k; t_0, x_0), y) \leq \gamma(t_k) < a(\eta).$$

This proves that the state $x = 0$ of system (1.2.7) is attractive (on \mathcal{T}_τ). By Definition 2.4.4 (ii) the state $x = 0$ of system (1.2.7) is equi-asymptotically stable (on \mathcal{T}_τ).

THEOREM 2.4.2. *Let vector-function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If*

- (1) *conditions (1)–(2) of Theorem 2.4.1 are satisfied and*
- (2) *there exists a majorizing function G such that $G \in C(R \times R_+, R)$ ($G \in C(\mathcal{T}_\tau \times R_+, R)$) $G(t, 0) = 0 \quad \forall t \in \mathcal{T}_0$ ($\forall t \in \mathcal{T}_\tau$) such that*

$$D^+v(t, x, y) \geq G(t, v(t, x, y)) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \\ (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

Then properties of the function

$$(2.4.16) \quad y^T U(t, x) y = v(t, x, y)$$

and instability properties of the zero solution of the equation

$$(2.4.17) \quad \frac{du}{dt} = G(t, u), \quad u(t_0) = u_0 \geq 0$$

imply instability (on \mathcal{T}_τ) of the state $x = 0$ of the system (1.2.7).

PROOF. In order to prove Theorem 2.4.2 we shall state some properties of the function (2.4.16). Namely, assume positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) function, and for any $\delta > 0$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$) an x_0 is found, $\|x_0\| < \delta$,

such that $v(t_0, x_0, y) > 0 \quad \forall y \in R^m$. Instability of the zero solution of equation (2.4.17) ensures that given $\varepsilon^* > 0$ and $t_0 \in R$ ($t_0 \in \mathcal{T}_\tau$), for any $\delta^* > 0$ a $u_0: 0 \leq u_0 < \delta^*$ can be found so that $\rho(t; t_0, u_0) \geq \varepsilon^*$. Since function (2.4.16) is positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$), a function $a \in K$ can be taken so that

$$\begin{aligned} a(\|x\|) \leq v(t, x, y) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \\ \forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m. \end{aligned}$$

We take $\varepsilon > 0$ so that

$$(2.4.18) \quad a(\varepsilon) < \varepsilon^*.$$

This is possible due to assumptions on function $v(t, x, y)$.

Now we determine $u_0 \leq \delta^*$ and $t \geq t_0$ so that $\rho(t; t_0, u_0) \geq \varepsilon^*$. If x_0 is taken in accordance with (2.4.18) and $t \notin \mathcal{T}_0$ ($t \notin \mathcal{T}_\tau$) then the theorem is proved, since the solution $\chi(t; t_0, x_0)$ cannot cease its existence without leaving the domain $\|x\| < \varepsilon$. Let $t_0 \in \mathcal{T}_0$ ($t_0 \in \mathcal{T}_\tau$). Then we get according to Proposition 2.4.4

$$a(\|\chi(t; t_0, x_0)\|) \geq v(t, \chi(t), y) \geq \rho(t) \geq \varepsilon^* > a(\varepsilon).$$

Consequently, $\|\chi(t; t_0, x_0)\| > \varepsilon$ and the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

2.5 Method of Matrix Liapunov Functions

As already mentioned in the introduction the application of matrix Liapunov functions make it possible to establish easily verified stability conditions for the state $x = 0$ of system (1.2.7) in terms of the property having a fixed sign of special matrices. The results presented in this section demonstrate the opportunities of the matrix Liapunov functions technique.

2.5.1 Nonautonomous systems

General Theorems 2.3.1–2.3.7 allows sufficient stability conditions for the state $x = 0$ of system (1.2.7) to be constructed as follows.

THEOREM 2.5.1. *Let the vector-function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
- (2) *a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m \times m})$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for all $t \in R$ ($t \in \mathcal{T}_\tau$);*
- (3) *functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K, \tilde{\psi}_{i2} \in CK, i = 1, 2, \dots, m$;*
- (4) *symmetric $m \times m$ matrices $A_j(y), j = 1, 2, 3, \tilde{A}_2(y)$ such that*

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- (a) $\psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \leq v(t, x, y) \leq \tilde{\psi}_2^T(t, \|x\|)\tilde{A}_2(y)\tilde{\psi}_2(t, \|x\|)$
 $\forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$
- (b) $\psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \leq v(t, x, y) \leq \psi_2^T(\|x\|)A_2(y)\psi_2(\|x\|)$
 $\forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$
- (c) $D^+v(t, x, y) \leq \psi_3^T(\|x\|)A_3(y)\psi_3(\|x\|)$
 $\forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$

Then, if the matrices $A_1(y)$, $A_2(y)$, $\tilde{A}_2(y)$, $(y \neq 0) \in R^m$ are positive definite and $A_3(y)$ is negative semi-definite, then

- (a) the state $x = 0$ of system (1.2.7) is stable (on \mathcal{T}_τ), provided condition (4)(a) is satisfied;
- (b) the state $x = 0$ of system (1.2.7) is uniformly stable (on \mathcal{T}_τ), provided condition (4)(b) is satisfied.

PROOF. We shall prove assertion (a). Since matrices $A_1(y)$ and $\tilde{A}_2(y)$ $\forall (y \neq 0) \in R^m$ are positive definite, then $\lambda_m(A_1) > 0$ and $\lambda_M(\tilde{A}_2) > 0$, where $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ are minimal and maximal eigenvalues of matrices $A_1(y)$ and $\tilde{A}_2(y)$ respectively.

Condition (3) of Theorem 2.5.1 provides the existence of functions $\pi \in K$ and $\rho \in CK$ such that

$$\pi(\|x\|) \leq \psi_1^T(\|x\|)\psi_1(\|x\|)$$

and

$$\rho(t, \|x\|) \geq \tilde{\psi}_2^T(t, \|x\|)\tilde{\psi}_2(t, \|x\|).$$

Consequently,

$$(2.5.1) \quad \begin{aligned} \lambda_m(A_1)\pi(\|x\|) &\leq v(t, x, y) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \\ &(\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m) \end{aligned}$$

and

$$(2.5.2) \quad \begin{aligned} v(t, x, y) &\leq \lambda_M(\tilde{A}_2)\rho(t, \|x\|) \quad \forall (t, x, y) \in R \times \mathcal{G} \times R^m \\ &(\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

Since matrix $A_3(y)$ is negative semi-definite, then

$$(2.5.3) \quad \begin{aligned} D^+v(t, x, y) &\leq 0 \quad \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m \\ &(\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

Taking into account (2.5.1)–(2.5.3) one can easily see that all conditions of Theorem 2.3.1 are satisfied and the state $x = 0$ of system (1.2.7) is stable (on \mathcal{T}_τ).

The proof of assertion (b) of the Theorem 2.5.1 is the same, seeing that $\psi_{i2} \in K$.

THEOREM 2.5.2. *Let the vector-function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$). If there exist*

- (1) *a matrix-valued function $U \in C(R \times R^n, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$) and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for all $t \in R$ ($t \in \mathcal{T}_\tau$);*
- (2) *functions $\varphi_{1i}, \varphi_{2i}, \varphi_{3i} \in KR$, $\tilde{\varphi}_{2i} \in CKR$, $i = 1, 2, \dots, m$;*
- (3) *symmetric $m \times m$ matrices $B_j(y)$, $j = 1, 2, 3$, $\tilde{B}_2(y)$ such that*

$$(a) \quad \begin{aligned} \varphi_1^T(\|x\|)B_1(y)\varphi_1(\|x\|) &\leq v(t, x, y) \leq \tilde{\varphi}_2^T(t, \|x\|)\tilde{B}_2(y)\tilde{\varphi}_2(t, \|x\|) \\ &\forall (t, x, y) \in R \times R^n \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m); \end{aligned}$$

$$(b) \quad \begin{aligned} \varphi_1^T(\|x\|)B_1(y)\varphi_1(\|x\|) &\leq v(t, x, y) \leq \varphi_2^T(\|x\|)B_2(y)\varphi_2(\|x\|) \\ &\forall (t, x, y) \in R \times R^n \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m); \end{aligned}$$

$$(c) \quad \begin{aligned} D^+v(t, x, y) &\leq \varphi_3^T(\|x\|)B_3(y)\varphi_3(\|x\|) \\ &\forall (t, x, y) \in R \times R^n \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m). \end{aligned}$$

Then, provided that matrices $B_1(y)$, $B_2(y)$ and $\tilde{B}_2(y)$, $\forall (y \neq 0) \in R^m$ are positive definite and matrix $B_3(y)$ is negative definite,

- (a) under condition (3)(a) the state $x = 0$ of system (1.2.7) is stable in the whole (on \mathcal{I}_τ);
- (b) under condition (3)(b) the state $x = 0$ of system (1.2.7) is uniformly stable in the whole (on \mathcal{I}_τ).

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PROOF. Under conditions (1)–(3)(a) of Theorem 2.5.2 the function $v(t, x, y)$ is radially unbounded positive definite in the whole (on \mathcal{T}_τ) and weakly decreasing in the whole (on \mathcal{T}_τ). Since the matrix $B_3(y)$, $\forall (y \neq 0) \in R^m$ is negative semi-definite, then we have in consequence of condition (3)(c) of Theorem 2.5.2

$$D^+v(t, x, y) \leq 0 \quad \forall (t, x, y \neq 0) \in R \times R^n \times R^m \\ (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times R^n \times R^m).$$

According to Theorem 2.3.2 the state $x = 0$ of system (1.2.7) is stable in the same manner taking into account conditions (1)–(3)(b) and (3)(c).

THEOREM 2.5.3. *Let the vector-function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*
- (2) *a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times \mathcal{N}, R^{m \times m})$) and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for all $t \in R$ ($t \in \mathcal{T}_\tau$);*
- (3) *functions $\eta_{1i}, \eta_{2i}, \eta_{3i} \in K$, $\tilde{\eta}_{2i} \in CK$, $i = 1, 2, \dots, m$;*
- (4) *symmetric $m \times m$ matrices $C_j(y)$, $j = 1, 2, 3$, $\tilde{C}_2(y)$ such that*

$$(a) \quad \eta_1^T(\|x\|)C_1(y)\eta_1(\|x\|) \leq v(t, x, y) \leq \tilde{\eta}_2^T(t, \|x\|)\tilde{C}_2(y)\tilde{\eta}_2(t, \|x\|) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$$

$$(b) \quad \eta_1^T(\|x\|)C_1(y)\eta_1(\|x\|) \leq v(t, x, y) \leq \eta_2^T(\|x\|)C_2(y)\eta_2(\|x\|) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$$

$$(c) \quad D^*v(t, x, y) \leq \eta_3^T(\|x\|)C_3(y)\eta_3(\|x\|) + m(t, \eta_3(\|x\|)) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m),$$

where function $m(t, \cdot)$ satisfies the condition

$$\lim \frac{|m(t, \eta_3(\|x\|))|}{\|\eta_3\|} = 0 \quad \text{as} \quad \|\eta_3\| \rightarrow 0$$

uniformly in $t \in R$ ($t \in \mathcal{T}_\tau$).

Then, provided the matrices $C_1(y)$, $C_2(y)$, $\tilde{C}_2(y)$ are positive definite and matrix $C_3(y)$ ($y \neq 0$) $\in R^m$ is negative definite, then

- (a) under condition (4)(a) the state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ);
- (b) under condition (4)(b) the state $x = 0$ of the system (1.2.7) is uniformly asymptotically stable (on \mathcal{T}_τ).

PROOF. Following the arguments from the proof of Theorem 2.5.1 under conditions (1)–(4)(a) the function $v(t, x, y)$ is positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$) and weakly decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$). Consider condition (4)(c). Since $\eta_{3i} \in K$, $i = 1, 2, \dots, m$ there exists a function $\omega \in K$ such that

$$(2.5.4) \quad \omega(\|x\|) \geq \eta_3^T(\|x\|)\eta_3(\|x\|).$$

Due to matrix $C_3(y)$ ($y \neq 0$) $\in R^m$ being negative definite all its eigenvalues are negative so that $\lambda_M(C_3) < 0$. Therefore, we get in view of (2.5.4)

$$(2.5.5) \quad D^*v(t, x, y) \leq \lambda_M(C_3)\omega(\|x\|) + m(t, \eta_3(\|x\|)) \\ \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

Under condition (4)(c) for the given neighborhood $\mathcal{G} \subset \mathcal{N}$ of point $x = 0$ a $0 < \mu < 1$ can be taken so that

$$(2.5.6) \quad |m(t, \eta(\|x\|))| < -\mu\lambda_M(C_3)\eta_3^T(\|x\|)\eta_3(\|x\|) \\ \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$

Together with inequalities (2.5.5) condition (2.5.6) yields the estimate

$$D^*v(t, x, y) \leq (1 - \mu)\lambda_M(C_3)\omega(\|x\|), \quad \lambda_M(C_3) < 0.$$

Thus, function $D^*v(t, x, y)$ is negative definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$). Therefore, all conditions of Theorem 2.3.3 are satisfied and the state $x = 0$ of the system (1.2.7) is asymptotically stable (on \mathcal{T}_τ).

Assertion (b) of Theorem 2.5.3 is proved in the same manner taking into account that condition (4)(b) ensures function $v(t, x, y)$ decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$).

THEOREM 2.5.4. *Let the vector-function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$) and conditions (1)–(3) of Theorem 2.5.2 are satisfied.*

Then, provided that matrices $B_1(y)$, $B_2(y)$ and $\tilde{B}_2(y)$ are positive definite and matrix $B_3(y) \forall (y \neq 0) \in R^m$ is negative definite,

- (a) *under condition (3)(a) of Theorem 2.5.2 the state $x = 0$ of system (1.2.7) is asymptotically stable in the whole (on \mathcal{T}_τ);*
- (b) *under condition (3)(b) of Theorem 2.5.2 the state $x = 0$ of system (1.2.7) is uniformly asymptotically stable in the whole (on \mathcal{T}_τ).*



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PROOF. Under conditions (1)–(3)(a) of Theorem 2.5.2 the function $v(t, x, y)$ is radially unbounded positive definite in the whole (on \mathcal{T}_τ).

Because matrix $B_3(y) \forall (y \neq 0) \in R^m$ is negative definite, proceeding as in the proof of Theorem 2.5.2 we arrive at the estimate

$$D^*v(t, x, y) \leq \lambda_M(B_3)\varphi_3^T(\|x\|)\varphi_3(\|x\|) \\ \forall (t, x, y) \in R \times R^n \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m).$$

Since $\varphi_{3i} \in CK$, $i = 1, 2, \dots, m$, there exist a function $\theta(\|x\|) \in KR$ such that

$$\theta(\|x\|) \geq \varphi_3^T(\|x\|)\varphi_3(\|x\|).$$

Therefore,

$$D^*v(t, x, y) \leq \lambda_M(B_3)\theta(\|x\|), \quad \lambda_M(B_3) < 0 \\ \forall (t, x, y \neq 0) \in R \times R^n \times R^m \quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times R^n \times R^m).$$

Thus, function $D^*v(t, x, y)$ is negative definite in the whole (on \mathcal{T}_τ).

According to Theorem 2.3.4 the state $x = 0$ of system (1.2.7) is asymptotically stable in the whole (on \mathcal{T}_τ).

The proof of assertion (b) of Theorem 2.5.4 is similar to the above and takes into account the fact that by conditions (2) and (3) of Theorem 2.5.2 the function $v(t, x, y)$ is radially unbounded positive definite and decreasing in the whole (on \mathcal{T}_τ).

THEOREM 2.5.5. *Let the vector-function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist*

- (1) *an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;*

- (2) a matrix-valued function $U \in C(R \times \mathcal{N}, R^{m \times m})$ and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x)y$ is locally Lipschitzian in x for all $t \in R$ ($t \in \mathcal{T}_\tau$);
- (3) functions $\sigma_{2i}, \sigma_{3i} \in K$, $i = 1, 2, \dots, m$, a positive real number Δ_1 and positive integer p , symmetric $m \times m$ matrices $F_2(y)$, $F_3(y)$ such that

$$(a) \quad \begin{aligned} \Delta_1 \|x\|^p \leq v(t, x, y) &\leq \sigma_2^T(\|x\|)F_2(y)\sigma_2(\|x\|) \\ \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m &\quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m); \end{aligned}$$

$$(b) \quad \begin{aligned} D^*v(t, x, y) &\leq \sigma_3^T(\|x\|)F_3(y)\sigma_3(\|x\|) \\ \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m &\quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

Then, provided that the matrices $F_2(y)$ ($y \neq 0$) $\in R^m$ are positive definite, the matrix $F_3(y)$ ($y \neq 0$) $\in R^m$ is negative definite and functions σ_{2i}, σ_{3i} are the same magnitude, then the state $x = 0$ of system (1.2.7) is exponentially stable (on \mathcal{T}_τ).

PROOF. Under conditions (1)–(4)(a) function $v(t, x, y)$ is positive definite and decreasing (on \mathcal{T}_τ). In fact, we have the estimate

$$\begin{aligned} v(t, x, y) &\leq \lambda_M(F_2)\sigma_2^T(\|x\|)\sigma_2(\|x\|), \quad \lambda(F_2) > 0 \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m &\quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

Since the functions $\sigma_{3i} \in K$, $i = 1, 2, \dots, m$, there exists a function $\varkappa \in K$ such that

$$\varkappa(\|x\|) \geq \sigma_2^T(\|x\|)\sigma_2(\|x\|).$$

Therefore

$$(2.5.7) \quad \begin{aligned} \Delta_1 \|x\|^p \leq v(t, x, y) &\leq \lambda_M(F_2)\varkappa(\|x\|), \quad \lambda_M(F_2) > 0 \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m &\quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

We reduce condition (4)(b) of Theorem 2.5.5 to the form

$$(2.5.8) \quad \begin{aligned} D^*v(t, x, y) &\leq \lambda_M(F_3)\pi(\|x\|), \quad \lambda_M(F_3) < 0 \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m &\quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m), \end{aligned}$$

where $\pi \in K$ is such that

$$\pi(\|x\|) \geq \sigma_3^T(\|x\|)\sigma_3(\|x\|).$$

Since functions α and π are of the same magnitude, there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$k_1 \alpha(\|x\|) \leq \pi(\|x\|) \leq k_2 \alpha(\|x\|).$$

We get from inequalities (2.5.7) and (2.5.8)

$$(2.5.9) \quad \begin{aligned} & D^*v(t, x, y) \leq \lambda v(t, x, y) \\ & \forall (t, x, y \neq 0) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m), \end{aligned}$$

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where $\lambda = \lambda_M(F_3)\lambda_M^{-1}(F_2)$, $\lambda < 0$.

In view of the estimate from the left in (2.5.7) we obtain from (2.5.9)

$$v(t, x, y) \leq v(t_0, x_0, y) \exp(\lambda(t - t_0))$$

and

$$(2.5.10) \quad \|\chi(t; t_0 x_0)\| \leq \Delta_1^{-\frac{1}{p}} \lambda_M^{\frac{1}{p}}(F_2) \varkappa^{\frac{1}{p}}(\|x_0\|) \exp\left(\frac{\lambda}{p}(t - t_0)\right).$$

We designate according to Definition 1.2.3 (vi)

$$\alpha = \Delta_1^{-\frac{1}{p}} \lambda_M^{\frac{1}{p}}(F_2), \quad \beta = \frac{\lambda}{p}, \quad \beta < 0.$$

From (2.5.10) we obtain

$$\|\chi(t; t_0 x_0)\| \leq \alpha \varkappa^{\frac{1}{p}}(\|x_0\|) \exp(\beta(t - t_0)) \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$$

This proves Theorem 2.5.5.

THEOREM 2.5.6. *Let the vector-function f in system (1.2.7) be continuous on $R \times R^n$ (on $\mathcal{T}_\tau \times R^n$). If there exist*

- (1) *a matrix-valued function $U \in C(R \times R^n, R^{m \times m})$ ($U \in C(\mathcal{T}_\tau \times R^n, R^{m \times m})$) and a vector $y \in R^m$ such that the function $v(t, x, y) = y^T U(t, x) y$ is locally Lipschitzian in x for all $t \in R$ ($\forall t \in \mathcal{T}_\tau$);*
- (2) *functions $\nu_{2i}, \nu_{3i} \in KR$, $i = 1, 2, \dots, m$, a positive real number $\Delta_2 > 0$ and a positive integer q ;*
- (3) *symmetric $m \times m$ matrices H_2, H_3 such that*

$$(a) \quad \Delta_2 \|x\|^q \leq v(t, x, y) \leq \nu_2^T(\|x\|) H_2(y) \nu_2(\|x\|) \\ \forall (t, x, y \neq 0) \in R \times R^n \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times R^n \times R^m);$$

$$(b) \quad D^* v(t, x, y) \leq \nu_3^T(\|x\|) H_3(y) \nu_3(\|x\|) \\ \forall (t, x, y \neq 0) \in R \times R^n \times R^m \quad (\forall (t, x, y \neq 0) \in \mathcal{T}_\tau \times R^n \times R^m).$$

Then, if the matrix $H_2(y) \forall (y \neq 0) \in R^m$ is positive definite, the matrix $H_3(y) \forall (y \neq 0) \in R^m$ is negative definite and functions ν_{2i}, ν_{3i} are of the same magnitude, the state $x = 0$ of system (1.2.7) is exponentially stable in the whole (on \mathcal{T}_τ).

PROOF of this Theorem is similar to that of Theorem 2.5.5 taking into account the fact that under conditions of Theorem 2.5.6 the function $v(t, x, y)$ is radially unbounded (on \mathcal{T}_τ). Inequality (2.5.10) is replaced by

$$\|\chi(t; t_0 x_0)\| \leq \Delta_2^{-\frac{1}{q}} \lambda_M^{\frac{1}{q}}(H_2) g^{\frac{1}{q}}(\|x_0\|) \exp\left(\frac{\lambda_1}{q}(t - t_0)\right)$$

where $g(\|x\|) \in KR$ and $g(\|x\|) \geq \nu_2^T(\|x\|)\nu_2(\|x\|)$,

$$\lambda_1 = \lambda_M(H_3)k_1\lambda_M^{-1}(H_2), \quad k_1 > 0, \quad \lambda_1 < 0.$$

We designate $\beta = \lambda_1 q^{-1}$ and define function $\Phi(\Delta) = \Delta_1^{-\frac{1}{q}} \lambda_M^{\frac{1}{q}}(H_2) g^{\frac{1}{q}}(\Delta)$ whenever $\|x_0\| < \Delta, \Delta = +\infty$. Then

$$\|\chi(t; t_0 x_0)\| \leq \Phi(\Delta) \exp(\beta(t - t_0)), \quad \beta < 0 \quad \forall t \in \mathcal{T}_0, \quad \forall t_0 \in \mathcal{T}_i.$$

This proves Theorem 2.5.6.

THEOREM 2.5.7. Let the vector-function f in system (1.2.7) be continuous on $R \times \mathcal{N}$ (on $\mathcal{T}_\tau \times \mathcal{N}$). If there exist

- (1) an open connected time-invariant neighborhood $\mathcal{G} \subset \mathcal{N}$ of the point $x = 0$;
- (2) a matrix-valued function $U \in C^1(R \times \mathcal{N}, R^{m \times m})$ ($U \in C^1(\mathcal{T}_\tau \times \mathcal{N}, R^{m \times m})$) and a vector $y \in R^m$;
- (3) functions $\psi_{1i}, \psi_{2i}, \psi_{3i} \in K, i = 1, 2, \dots, m$, symmetric $m \times m$ matrices $A_1(y), A_2(y), G(y)$ and a constant $\Delta > 0$ such that

$$(a) \quad \psi_1^T(\|x\|)A_1(y)\psi_1(\|x\|) \leq v(t, x, y) \leq \psi_2^T(\|x\|)A_2(y)\psi_2(\|x\|) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$$

$$(b) \quad Dv(t, x, y) \geq \psi_3^T(\|x\|)G(y)\psi_3(\|x\|) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m);$$

- (4) point $x = 0$ belong to $\partial\mathcal{G}$;
- (5) $v(t, x, y) = 0$ on $\mathcal{T}_0 \times (\partial\mathcal{G} \cap B_\Delta)$, where $B_\Delta = \{x: \|x\| < \Delta\}$.

Then, if matrices $A_1(y)$, $A_2(y)$ and $G(y) \forall (y \neq 0) \in R^m$ are positive definite, the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

PROOF. Under conditions (1)–(3)(a) of Theorem 2.5.7 it is easy to obtain for function $v(t, x, y)$ the estimate

$$(2.5.11) \quad \lambda_m(A_1)\gamma(\|x\|) \leq v(t, x, y) \leq \lambda_M(A_2)\zeta(\|x\|) \\ \forall (t, x, y) \in R \times \mathcal{G} \times R^m \quad (\forall (t, x, y) \in \mathcal{T}_\tau \times \mathcal{G} \times R^m).$$



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Here $\gamma \in K$ and $\gamma(\|x\|) \leq \psi_1^T(\|x\|)\psi_1(\|x\|)$, $\zeta \in K$ and $\zeta(\|x\|) \geq \psi_2^T(\|x\|)\psi_2(\|x\|)$.

Since $\lambda_M(A_1) > 0$, $\lambda_M(A_2) > 0$, then by estimate (2.5.11) function $v(t, x, y)$ is positive and bounded (on \mathcal{T}_τ). Hence, for every $\Delta > 0$ an $x_0 \in \mathcal{G} \cap B_\Delta$ and a $a > 0$ can be found such that $a \geq v(t_0, x_0, y) > 0 \forall (y \neq 0) \in R^m$.

Condition (3)(b) of Theorem 2.5.7 is reduced to the form

$$(2.5.12) \quad \begin{aligned} Dv(t, x, y) &\geq \lambda_m(G)\xi(\|x\|), & \lambda_m(G) &> 0, \\ \forall (t, x, y \neq 0) &\in R \times \mathcal{G} \times R^m & (\forall (t, x, y) &\in \mathcal{T}_\tau \times \mathcal{G} \times R^m). \end{aligned}$$

Here $\xi \in K$ and $\xi \leq \psi_3^T(\|x\|)\psi_3(\|x\|)$.

In view of (2.5.11) and (2.5.12) we have for $\chi(t; t_0, x_0 \in \mathcal{G})$

$$\begin{aligned} a &\geq v(t, \chi(t; t_0, x), y) = v(t_0, x_0, y) + \int_{t_0}^t Dv(\tau, \chi(\tau; t_0, x_0)y) d\tau \\ &\geq v(t_0, x_0, y) + \lambda_m(G)\xi(\|x_0\|)(t - t_0) \quad \forall t \in \mathcal{T}_0 \quad (\forall t \in \mathcal{T}_\tau). \end{aligned}$$

Hence, it follows that the solution $\chi(t; t_0, x_0)$ must leave neighborhood \mathcal{G} some time later. But because of condition (5) it cannot leave \mathcal{G} through $\partial\mathcal{G} \in B_\Delta$. Consequently, $\chi(t; t_0, x_0)$ leaves the domain B_Δ and the state $x = 0$ of system (1.2.7) is unstable (on \mathcal{T}_τ).

2.5.2 Autonomous systems

2.5.2.1 Definitions of stability domains and their estimates. For a while our attention will be focused on the difference between the notions “domain” and “region”.

Referring to LaSalle and Lefschetz [98] a “region” is an open connected set. However, Santalo [163] defined “domain” as an open and connected set, and “region” as the union of a domain with some, none, or all its boundary points.

We want to emphasize that, for stability analysis of nonlinear systems, only a neighborhood (either open or closed or neither open nor closed) of the origin is of interest herein. Hahn [66] used “domain” in this sense. The reason for using a neighborhood that can be closed is that the domain of asymptotic stability of an equilibrium of a nonlinear system can be closed.

We accept:

DEFINITION 2.5.1. A set D_s , $D_s \subseteq R^m$, is the domain of the equilibrium state $x = 0$ defined by

$$D_s = \bigcup \left[D_s(\varepsilon) : \varepsilon \in \overset{\circ}{R}_+ \right],$$

where $D_s(\varepsilon)$ is such a neighborhood of $x = 0$ that $\|\chi(t; 0, x_0)\| < \varepsilon$ $\forall t \in R_+$, holds provided only that $x_0 \in D_s(\varepsilon)$ for every $\varepsilon \in \overset{\circ}{R}_+$.

The next definition has been commonly used (see Krasovskii [89], Hahn [66], LaSalle and Lefschetz [98]).

DEFINITION 2.5.2. A set D_a , $D_a \subseteq R^m$, is the domain of attraction of the equilibrium state $x = 0$ of the system (1.2.10) if and only if it is such a neighborhood of $x = 0$ that

$$\lim [\|\chi(t; 0, x_0)\| : t \rightarrow +\infty] = 0$$

holds provided only that $x_0 \in D_a$.

It is now natural to accept the definition of the domain of asymptotic stability of $x = 0$ in the form.

DEFINITION 2.5.3. A set D , $D \subseteq R^m$, is the domain of asymptotic stability of $x = 0$ of the system (1.2.10) if and only if it is both a neighborhood of $x = 0$ and the intersection of its domain of stability and domain of attraction, that is, that $D = D_s \cap D_a$ is a neighborhood of $x = 0$.

The exact determination of the domain of asymptotic stability has great engineering and theoretical importance. Unfortunately, we can realize it only in special cases. For these reasons we investigate its estimate E defined as follows.

DEFINITION 2.5.4. A set E , $E \subseteq R^m$, is an *estimate set* (in brief, *estimate*) of the asymptotic stability domain D of $x = 0$ of the system (1.2.10) if and only if

- (i) E is a neighborhood of $x = 0$,
- (ii) $E \subseteq D$

and

- (iii) E is positively invariant set of the system (1.2.10), that is, that $x_0 \in E$ implies $\chi(t; 0, x_0) \in E$ for every $t \in R_+$.



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2.5.2.2 *System description and decomposition.* Suppose autonomous system (1.2.10) to be decomposed into m interconnected subsystems

$$(2.5.13) \quad \frac{dx_i}{dt} = g_i(x_i) + h_i(x)$$

with individual subsystems

$$(2.5.14) \quad \frac{dx_i}{dt} = g_i(x_i), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, m,$$

where $x_i \in R^{n_i}$, $g = (g_1^T, g_2^T, \dots, g_m^T)^T$, $x = (x_1^T, \dots, x_m^T)^T$. Besides $g_i \in C(R^{n_i}, R^{n_i})$, $h_i \in C(R^n, R^{n_i})$ and $g_i(0) = 0$, $h_i(0) = 0 \quad \forall i = 1, 2, \dots, m$.

ASSUMPTION 2.5.1. There are connected neighborhoods N_i of $x_i = 0 \quad \forall i = 1, 2, \dots, m$ such that both

- (i) motions $x_i(t, x_{i0})$ of (2.5.14) are continuous in $(t, x_{i0}) \in R_+ \times N_i$, where $x_i(0, x_{i0}) \equiv 0 \quad \forall i = 1, 2, \dots, m$;

and

- (ii) motions $x(t, x_0)$ of (1.2.10) (or (2.5.13)) are continuous in $(t, x_0) \in R_+ \times N$, where $N = N_1 \times N_2 \times \dots \times N_m$ and $x(0, x_0) \equiv 0$.

Let $U: R^n \rightarrow R^{m \times m}$ be the matrix-valued function with elements $u_{ij} \in C(R^n, R)$ for $i \neq j$ and $u_{ij} \in C(R^n, R_+)$ for $i = j$.

Let us construct the function

$$v(x, y) = y^T U(x) y$$

by means vector $y \in R^m$ which was used above. We shall use expressions of one of Dini derivatives of function U

$$D^+U(x) = \limsup \{ [U(x(t+\theta), x) - U(x)] \theta^{-1} : \theta \rightarrow 0^+ \},$$

$$D_+U(x) = \liminf \{ [U(x(t+\theta), x) - U(x)] \theta^{-1} : \theta \rightarrow 0^+ \}$$

with the function $U(x)$. We shall denote by symbol $D^*U(x)$ the possibility of utilizing any of functions $D^+U(x)$ or $D_+U(x)$.

ASSUMPTION 2.5.2. Matrix-valued function U is radially increasing on N , that is, the following inequality holds elementwise

$$U(\lambda_1 x) < U(\lambda_2 x) \quad \forall (x \neq 0) \in N, \quad \lambda_i \in (0, +\infty), \quad i = 1, 2, \quad \lambda_1 < \lambda_2.$$

Let K be the elementwise greatest $m \times m$ matrix, $K = (k_{ij})$ satisfying

$$(2.5.15) \quad U_K(x) \subseteq \text{int } N$$

for the set $U_K(x)$ defined by

$$(2.5.16) \quad U_K(x) = \{x : U(x) < K\},$$

where $\text{int } N$ is the interior of N . In case N is unbounded then $k_{ij} = +\infty$ is possible for some $(i, j) \in [1, m]$.

Let

$$(2.5.17) \quad E = \bigcup \{E_{ij} : (i, j) \in [1, m]\}, \quad E_{ij} = \{x : u_{ij}(x) < k_{ij}\}, \\ k_{ij} < +\infty, \quad \partial E_{ij} = \{x : u_{ij}(x) = k_{ij}\}.$$

2.5.2.3 *A metric aggregation form.* Metric on R^n will be introduced by the Euclidean norm $\|\cdot\|$. A metric aggregation form is determined by

ASSUMPTION 2.5.3. There are $U \in C(R^n, R^{m \times m})$, $w \in C(R^n, R^m)$ and real number α_{ij} such that

(i) $w(x) = 0$ for $x \in N$ iff $x = 0$;

(ii) $U(x) \in C(N, R^{m \times m})$;

(iii) $D^*U(x) \leq \begin{pmatrix} \alpha_{11}\|w(x)\|^2 & \dots & \alpha_{1m}\|w(x)\|^2 \\ \dots & \dots & \dots \\ \alpha_{m1}\|w(x)\|^2 & \dots & \alpha_{mm}\|w(x)\|^2 \end{pmatrix} \quad \forall x \in N.$

THEOREM 2.5.8. *Let Assumptions 2.5.1 – 2.5.3 hold. In order for the set E (2.5.17) to be an estimate of D it is sufficient that U is positive definite on N , $\alpha_{ij} < 0$ and $u_{ij}(x)$ is radially unbounded in case N is unbounded, $\forall i, j = 1, 2, \dots, m$.*

PROOF. Positive definiteness of $U(x)$ on N implies positive definiteness of $u_{ij}(x)$ on $N \quad \forall (i, j) \in [1, m]$. The conditions (iii) of Assumption 2.5.3 proves

$$(2.5.18) \quad D^*u_{ij}(x) \leq \alpha_{ij}\|w(x)\|^2 \quad \forall x \in N \quad \forall (i, j) \in [1, m].$$

Since $\alpha_{ij} < 0$ and $w(x) = 0$ iff $x = 0$ due to (i) of Assumption 2.5.3 then Assumption 2.5.1, (2.5.18) and Assumption 2.5.2 prove that E_{ij} (2.5.17) is an estimate of $D \forall (i, j) \in [1, m]$. Hence E (2.5.17) is also an estimate of D .

Let $v_i: R^n \rightarrow R^m$ be defined by

$$(2.5.19) \quad v_i = (u_{i1}, u_{i2}, \dots, u_{im})^T, \quad \forall i = 1, 2, \dots, m.$$

ASSUMPTION 2.5.4. Vector $b \in R^m$ is elementwise positive, $b^T v_i$ is positive definite on N and radially unbounded in case N is unbounded, and $k_i \in (0, +\infty)$ is such that a set

$$(2.5.20) \quad V_i = \{x: b^T v_i(x) < k_i\}, \quad k_i < +\infty \rightarrow \partial V_i = \{x: b^T v_i(x) = k_i\}, \\ \forall i \in [1, m]$$

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is the largest connected neighborhood of $x = 0$ in N determined by $b^T v_i(x)$.

THEOREM 2.5.9. *Let Assumptions 2.5.1, 2.5.3 and 2.5.4 hold. In order for the set E (2.5.21)*

$$(2.5.21) \quad E = \bigcup \{V_i : i \in [1, m]\}$$

to be estimate of D it is sufficient that the matrix $A = (\alpha_{ij})$ and the vector b obey elementwise $Ab < 0$.

PROOF. From (iii) Assumption 2.5.3 and $b > 0$ (Assumption 2.5.4) it results

$$(2.5.22) \quad D^*U(x)b \leq Ab \|w(x)\|^2 \quad \forall x \in N.$$

The condition $Ab < 0$, (i) of Assumption 2.5.3 and (2.5.22) prove $D^*U(x)b < 0$ elementwise on N , $x \neq 0$. This result, Assumption 2.5.4, and (2.5.19) prove that both $b^T v_i$ is positive definite and $D^*U(x)b$ elementwise negative ($x \neq 0$) on the closure \bar{E}_i (2.5.21), $\forall i \in [1, m]$. These facts and Assumption 2.5.1 prove that $E_i = V_i$ is an estimate of D . Since this holds for every $i \in [1, m]$, then E (2.5.20), (2.5.21) is an estimate of D .

Let k be the greatest number or the symbol $+\infty$ such that the set V_k

$$(2.5.23) \quad V_k = \{x : b^T U(x)b < k\}, \quad k < +\infty \rightarrow \partial V_k = \{x : b^T U(x)b = k\}$$

is the largest connected neighborhood of $x = 0$ in N determined by b and V .

THEOREM 2.5.10. *Let Assumptions 2.5.1 and 2.5.3 hold. In order for the set $E = V_k$ (2.5.23) to be an estimate of D of $x = 0$ of (1.2.10) it is sufficient that U is positive definite on N , $v(x) = b^T U(x)b$ is radially unbounded in case N is unbounded, the vector b is elementwise positive and the scalar $b^T A b$ is negative for $A = (\alpha_{ij})$.*

PROOF. Positive definiteness of U on N means that $v(x) = b^T U(x)b$ is positive definite on N . Condition (iii) of Assumption 2.5.3 and $b > 0$ imply

$$D^* v(x) = b^T D^* U(x)b \leq (b^T A b) \|w(x)\| \quad \forall x \in N.$$

These results, $b^T A b < 0$, the condition (i) of Assumption 2.5.3 and Assumption 2.5.1 prove that $E = V_k$ (2.5.21) is an estimate of D .

2.5.2.4 A quadratic aggregation form. A generalized quadratic aggregation form is this setting introduced by

ASSUMPTION 2.5.5. There are $U \in C(R^n, R^{m \times m})$, $w \in C(R^n, R^m)$ and matrices $A_{ij} \in R^{m \times m}$ such that

- (i) $w(x) = 0$ for $x \in N$ iff $x = 0$;
- (ii) $U(x) = 0$ for $x \in N$ iff $x = 0$;
- (iii) $D^* U(x) \leq \begin{pmatrix} w^T(x)A_{11}w(x) & \dots & w^T(x)A_{1m}w(x) \\ \dots & \dots & \dots \\ w^T(x)A_{m1}w(x) & \dots & w^T(x)A_{mm}w(x) \end{pmatrix} \quad \forall x \in N.$

THEOREM 2.5.11. *Let Assumptions 2.5.1, 2.5.2 and 2.5.5 hold. In order for the set E (2.5.17) to be an estimate of D it is sufficient that U is positive definite on N , $u_{ij}(x)$ is radially unbounded in case N is unbounded $\forall (i, j) \in [1, m]$, and the matrix $(A_{ij} + A_{ij}^T)$ is negative definite $\forall (i, j) \in [1, m]$.*

PROOF. Let $\lambda_M(A_{ij} + A_{ij}^T)$ be the maximal eigenvalue of $(A_{ij} + A_{ij}^T)$ and

$$(2.5.24) \quad \alpha_{ij} = \frac{1}{2} \lambda_m(A_{ij} + A_{ij}^T).$$

Negative definiteness of $(A_{ij} + A_{ij}^T)$ implies $\alpha_{ij} < 0 \quad \forall (i, j) \in [1, m]$. This result and the conditions of Theorem 2.5.11 satisfy all the requirements of Theorem 2.5.8, which proves the statement of Theorem 2.5.11.

THEOREM 2.5.12. *Let Assumptions 2.5.1, 2.5.4 and 2.5.5 hold. In order for the set E (2.5.21), (2.5.22) to be an estimate of D it is sufficient that the matrix $\sum_{j=1}^m [b_j(A_{ij} + A_{ij}^T)]$ is negative definite for all $i \in [1, m]$.*

PROOF. Using $b > 0$ elementwise (Assumption 2.5.4) we derive

$$(2.5.25) \quad D^+U(x)b \leq \begin{pmatrix} w^T(x) \left(\sum_{j=1}^m b_j A_{1j} \right) w(x) \\ \dots\dots\dots \\ w^T(x) \left(\sum_{j=1}^m b_j A_{mj} \right) w(x) \end{pmatrix} \quad \forall x \in N$$

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from (2.5.24). Negative definiteness of $\sum_{j=1}^m [b_j(A_{ij} + A_{ij}^T)]$, and the conditions (i) and (iii) of Assumption 2.5.5 prove negativeness of $b^T D^* v_i(x)$ for every $(x \neq 0) \in N \forall i \in [1, m]$, due to (2.5.25). This result, and Assumption 2.5.1, positive definiteness of $b^T v_i$ on $N \forall i \in [1, m]$ prove that E (2.5.21), (2.5.22) is an estimate of D .

THEOREM 2.5.13. *Let Assumptions 2.5.1 and 2.5.5 hold. In order for the set $E = V_k$ (2.5.23) to be an estimate of D it is sufficient that U is positive definite on N , $v(x) = b^T U(x)b$ is radially unbounded in case N is unbounded, the vector b is elementwise positive and the matrix $\sum_{i,j=1}^m [b_i b_j (A_{ij} + A_{ij}^T)]$ is negative definite.*

PROOF. Function $v(x)$, $v(x) = b^T U(x)b$, is positive definite on N due to positive definiteness of $U(x)$ on N . Its derivative $D^* v(x)$ is negative for every $(x \neq 0) \in N$ in view of (i) and (iii) of Assumption 2.5.5,

$$D^* v(x) \leq \frac{1}{2} w^T(x) \left[\sum_{i,j=1}^m [b_i b_j (A_{ij} + A_{ij}^T)] \right] w(x)$$

and negative definiteness of $\sum_{i,j=1}^m [b_i b_j (A_{ij} + A_{ij}^T)]$. These results and Assumption 2.5.1 prove that $E = V_k$ (2.5.23) is an estimate of D .

2.5.2.5 Generalized Michel's aggregation form. The aggregation form will be generalized by referring to Grujić, Martynyuk and Ribbens-Pavella [57] and Michel [141] as follows:

ASSUMPTION 2.5.6. There are $U \in C(R^n, R^{m \times m})$, $w \in C(R^n, R^m)$, $w(x) = [w_1(x), \dots, w_m(x)]^T$, and vector $a_{ij} \in R^s$ such that

- (i) $w_{ij}(x) = 0$ for $x \in N$ iff $x = 0$;
- (ii) the matrix-valued function $U(x)$ is continuous on N , $U \in C(N, R^{m \times m})$;
- (iii) the matrix-valued function $U(x)$, the vector function w and the vector a_{ij} obey (2.5.26)

$$(2.5.26) \quad D^*U(x) \leq \begin{pmatrix} w_1(x)a_{11}^T w(x) & \dots & w_1(x)a_{1s}^T w(x) \\ \dots & \dots & \dots \\ w_s(x)a_{s1}^T w(x) & \dots & w_s(x)a_{ss}^T w(x) \end{pmatrix} \quad \forall x \in N.$$

Let

$$A^j = (a_{1j}, a_{2j}, \dots, a_{mj})^T, \quad V^j = (u_{1j}, u_{2j}, \dots, u_{sj})^T, \\ K^j = (k_{1j}, k_{2j}, \dots, k_{mj})^T.$$

Let

$$E^j = (e_{ki}^j) \in R^{m \times m}, \quad e_{ki}^j = [r_{ki}^1 \delta_{ki} + r_{ki}^2 (1 - \delta_{ki})] a_{ki}^j,$$

where $\delta_{ki} = 1$ for $k = i$, $\delta_{ki} = 0$ for $k \neq i$ and

$$r_{ki}^1 = \inf [w_i(x) : x \in \partial E_{ki}], \quad r_{ki}^2 = \sup [w_i(x) : x \in \bar{E}_{ki}].$$

THEOREM 2.5.14. *Let Assumptions 2.5.1, 2.5.2 and 2.5.6 hold. In order for the set E (2.5.17) and its closure \bar{E} to be estimates of D it is sufficient that $U(x)$ is positive definite on N , $u_{ij}(x)$ is radially unbounded in case N is unbounded $\forall k = 1, 2, \dots, m$, of A^j is non-negative and the vector $E^j \mathbf{1}$ is negative elementwise $\forall j \in [1, m]$.*

PROOF. Since $U(x) = [v^1(x), v^2(x), \dots, v^m(x)]$ then (2.5.26) can be rewritten as

$$(2.5.27) \quad D^*U(x) \leq W(x) [A^1 w(x), A^2 w(x), \dots, A^m w(x)], \\ W(x) = \text{diag} \{w_1(x), w_2(x), \dots, w_m(x)\}.$$

Let $j \in [1, m]$ be arbitrarily chosen. Positive definiteness of $U(x)$ on N implies positive definiteness of $u_{ij}(x)$ on $N \forall (i, j) \in [1, m]$. From $E^j \mathbf{1} < 0$, the definitions of E^j and V^j , (2.5.27) and Assumption 2.5.6 it follows that $D^*u_{ij}(x) < 0 \forall x \in \partial E_{ij} \forall i \in [1, m]$. This result, Assumption 2.5.1 and

Assumption 2.5.3 prove positive invariance of \bar{E}_{ij} with respect to motions of (1.2.10). The definitions of A^j and E^j imply $A^j \leq E^j$ elementwise. Hence $E^j \mathbf{1} < 0$ implies $A^j \mathbf{1} < 0$.

Since $a_{ki}^j \geq 0$, $k \neq j$, then there is positive diagonal $D^j = \text{diag}\{d_{1j}, d_{2j}, \dots, d_{mj}\}$ such that $[(A^j)^T D^j + D^j A^j]$ is negative definite. Hence a function v^j , $v^j(x) = (d^j)^T v^j(x)$ for $d^j = (d_{1j}, d_{2j}, \dots, d_{mj})^T$, is positive definite and

$$D^* v^j(x) \leq \frac{1}{2} w^T(x) [(A^j)^T D^j + D^j A^j] w(x) < 0$$

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$\forall (x \neq 0) \in N$ due to negative definiteness of the matrix $[(A^j)^T D^j + D^j A^j]$ and (i) of Assumption 2.5.6. These results, Assumption 2.5.3 together with positive invariance of all \overline{E}_{ij} prove that E_{ij} and \overline{E}_{ij} are estimates of D . Since this holds for every $(i, j) \in [1, m]$, then E and \overline{E} are estimates of D of $x = 0$ of (1.2.10).

Let

$$A_j = (a_{j1}, a_{j2}, \dots, a_{jm})^T, \quad b = \text{diag}(b_1, b_2, \dots, b_m),$$

$$A(b) = (A_1^T b, A_2^T b, \dots, A_m^T b)^T.$$

THEOREM 2.5.15. *Let Assumptions 2.5.1, 2.5.4 and 2.5.6 hold. In order for the set E (2.5.20), (2.5.21) to be an estimate of D it is sufficient that the vector $A_i^T b$ is negative elementwise $\forall i \in [1, m]$.*

PROOF. Since $b > 0$ (Assumption 2.5.4) then (2.5.19) and (2.5.27) yield (2.5.28) due to (iii) of Assumption 2.5.6,

(2.5.28)

$$D^*U(x)b = \begin{pmatrix} b^T D^* v_1(x) \\ \dots\dots\dots \\ b^T D^* v_m(x) \end{pmatrix} \leq \begin{pmatrix} w_1(x) b^T A_1 w(x) \\ \dots\dots\dots \\ w_m(x) b^T A_m w(x) \end{pmatrix} \quad \forall x \in N.$$

Elementwise negativeness of $A_i^T b_i \quad \forall i \in [1, m]$, (i) of Assumption 2.5.6 and (2.5.28) imply $b^T D^* v_i(x) < 0 \quad \forall (x \neq 0) \in N$. Hence, Assumption 2.5.1 and Assumption 2.5.4 prove that E (2.5.20), (2.5.21) is an estimate of D .

THEOREM 2.5.16. *Let Assumptions 2.5.1 and 2.5.6 hold. In order for the set $E = V_k$ (2.5.23) to be an estimate of D it is sufficient that $U(x)$ is positive definite on N , $v(x) = b^T U(x)b$ is radially unbounded in case N is unbounded and the matrix $[A^T(b)B + BA(b)]$ is negative definite for the elementwise positive vector b .*

PROOF. Theorem 2.5.16 is proved in the same way as Theorem 2.5.13. In order to achieve this the matrix $\sum_{i,j=1}^m [b_i b_j (A_{ij} + A_{ij}^T)]$ should be replaced by the matrix $[A^T(b)B + BA(b)]$ in the proof of Theorem 2.5.13.

2.5.2.6 Grujić–Šiljak’s aggregation form. The aggregation form can be applied to matrix-valued function aggregation of (1.2.10) as follows:

ASSUMPTION 2.5.7. There are $U \in C(R^n, R^{m \times m})$, $w \in C(R^n, R^m)$, and vectors $a_{ij} \in R^m$ such that

- (i) $w(x) = 0$ for $x \in N$ iff $x = 0$;
- (ii) $U(x) \in C(R^n, R^{m \times m})$;
- (iii) $D^*U(x) \leq \begin{pmatrix} a_{11}^T w(x) & \dots & a_{1m}^T w(x) \\ \dots & \dots & \dots \\ a_{m1}^T w(x) & \dots & a_{mm}^T w(x) \end{pmatrix} \quad \forall x \in N.$

THEOREM 2.5.17. Let Assumptions 2.5.1, 2.5.3 and 2.5.7 hold. In order for the set E (2.5.17) and its closure \bar{E} to be estimates of D it is sufficient that $U(x)$ is positive definite on N , $u_{ij}(x)$ is radially unbounded in case N is unbounded $\forall (i, j) \in [1, m]$, off-diagonal element a_{ki}^j ($k \neq i, k, i = 1, 2, \dots, m$) of A^j is nonnegative and the vector $E^j \mathbf{1}$ is negative elementwise $\forall j \in [1, m]$.

PROOF. The condition (iii) of Assumption 2.5.7 can be set in the form

$$(2.5.29) \quad D^*U(x) \leq [A^1 w(x), A^2 w(x), \dots, A^m w(x)] \quad \forall x \in N.$$

We consider now $v^j(x) = \mathbf{1}^T v^j(x)$. Positive definiteness of $U(x)$ on N implies positive definiteness of all $u_{ij}(x)$, hence of all $u^j(x)$, on N . Radial unboundedness of all $u_{ij}(x)$ implies radial unboundedness of all $v^j(x)$ in case N is unbounded. Assumption 2.5.2 implies radial increasing of all $v^j(x)$. From (2.5.29) and (i) of Assumption 2.5.7 it follows that $D^*v^j(x) < 0 \forall (x \neq 0) \in N, \forall j \in [1, m]$. The definition of E^j and $E^j \mathbf{1} < 0$ prove positive invariance of $E_{ij} \forall (i, j) \in [1, m]$. These results and Assumption 2.5.1 prove that both E (2.5.17) and \bar{E} are estimates of D .

THEOREM 2.5.18. Let Assumptions 2.5.1, 2.5.4 and 2.5.6 hold. In order for the set E (2.5.20), (2.5.21) to be an estimate of D it is sufficient that the vector $A_i^T b$ is negative elementwise $\forall i \in [1, m]$.

PROOF. Since $b > 0$ (Assumption 2.5.4) then (2.5.19) and (2.5.29) in view of (iii) of Assumption 2.5.7 yield

$$(2.5.30) \quad D^*U(x)b = \begin{pmatrix} b^T D^* v_1(x) \\ \dots\dots\dots \\ b^T D^* v_m(x) \end{pmatrix} \leq \begin{pmatrix} b^T A_1 w(x) \\ \dots\dots\dots \\ b^T A_m w(x) \end{pmatrix} \quad \forall x \in N.$$

Now, $A_i^T b < 0$ elementwise, (i) of Assumption 2.5.7 and (2.5.30) imply $D^* v_i(x) < 0 \forall (x \neq 0) \in N$ for $v_i(x) = b^T V_i(x) \forall i \in [1, m]$. This result, Assumption 2.5.1 and Assumption 2.5.4 prove that E (2.5.20), (2.5.21) is an estimate of D .

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THEOREM 2.5.19. *Let Assumptions 2.5.1 and 2.5.7 hold. In order for the set $E = V_k$ (2.5.23) to be an estimate of D it is sufficient that $U(x)$ is positive definite on N , $v(x) = b^T U(x)b$ is radially unbounded in case N is unbounded and the vector $A^T(b)b$ is negative elementwise for the elementwise positive vector b .*

PROOF. Since $U(x)$ is positive definite on N for $y = b \in R^m$, then v , $v = b^T U(x)b$, is also positive definite on N . From $b > 0$ and (iii) of Assumption 2.5.7 we derive

$$D^*v(x) \leq b^T A(b)w(x) \quad \forall x \in N$$

so that

$$D^*v(x) < 0 \quad \forall (x \neq 0) \in N$$

due to (i) of Assumption 2.5.7 and $A^T(b)b < 0$. These results and Assumption 2.5.1 prove that $E = V_k$ (2.5.23) is an estimate of D .

2.5.2.7 L-aggregation form. L -aggregation form is being introduced in this framework by

ASSUMPTION 2.5.8. There are $U \in C(R^n, R^{m \times m})$, $w \in C(R^n, R^m)$, $b \in R^s$, $b = (b_1, b_2, \dots, b_m)^T$ and $A \in R^{m \times m}$ such that

- (i) $\|w(x)\| = 0$ for $x \in N$ iff $x = 0$;
- (ii) $U(x) \in C(R^n, R^{m \times m})$;
- (iii) $v(x) = b^T U(x)b$ obeys

$$D^*v(x) \leq w^T(x)(A^T B + BA)w(x) \quad \forall x \in N$$

for $B = \text{diag}(b_1, b_2, \dots, b_m)$.

THEOREM 2.5.20. *Let Assumptions 2.5.1 and 2.5.8 hold. In order for the set $E = V_k$ (2.5.23) to be an estimate of D it is sufficient that $U(x)$ is positive definite on N , $v(x)$ is radially unbounded in case N is unbounded and the matrix $(A^T B + BA)$ is negative definite.*

PROOF. The function $U(x)$ is positive definite on N due to positive definiteness of $V(x)$ on N . Negative definiteness of $(A^T B + BA)$ and conditions (i) and (iii) of Assumption 2.5.8 imply $D^*v(x) < 0 \quad \forall (x \neq 0) \in N$. These results and Assumption 2.5.1 prove that $E = V_k$ (2.5.23) is an estimate of D .

2.6 On Multistability of Motion

As is well known, stability analysis of nonlinear systems is made under the assumption of the “equality” of all solutions coordinates with respect to dynamical properties as it is accepted in classical papers by Liapunov [101] and his adherents. The exception is made for stability with respect to a part of variables. In the problem, phase vector of variables is divided into two subvectors, the norm of one of which is said to be “nonincreasing” to infinity for the finite time.

2.6.1 General problem on multistability

A large-scale system of dimension n is governed by

$$(2.6.1) \quad \frac{dx_i}{dt} = f_i(t, x_1, \dots, x_s), \quad x_i(t_0) = x_{i0},$$

where $x_i \in R^{n_i}$, $t \in \mathcal{T}_\tau$, $\mathcal{T}_\tau = [\tau, +\infty)$, $\tau \in R$, $t_0 \in \mathcal{T}_i$, $\mathcal{T}_i \subset R$, $f_i: \mathcal{T}_\tau \times R^{n_1} \times \dots \times R^{n_s} \rightarrow R^{n_i}$ and it is assumed that $f_i(t, x_1, \dots, x_s) = 0$ for all $t \in \mathcal{T}_\tau$ iff $x_1 = x_2 = \dots = x_s = 0$. Together with (2.6.1) we shall show in vector notation the system (1.2.7)

$$(2.6.2) \quad \frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where $x \in R^n$, $n = \sum_{i=1}^s n_i$; $f: \mathcal{T}_\tau \times R^n \rightarrow R^n$, $x_0 = (x_{10}^T, \dots, x_{s0}^T)^T$. It is clear that $f(t, x) = 0$ for all $t \in \mathcal{T}_\tau$ iff $x = 0$.

DEFINITION 2.6.1. System (2.6.1) is called *multistability* (on \mathcal{T}_τ) iff its zero solution $(x_1^T, \dots, x_s^T)^T = 0$ is stable in some type (on \mathcal{T}_τ) and attractive (on \mathcal{T}_τ) with respect to groups of variables $\{x_i^T\}$, $i = 1, 2, \dots, s$ (with respect to totality of groups of variables $\{x_1^T, \dots, x_l^T\}$, $l < s$).

REMARK 2.6.1. When multistability of solution $x = 0$ of (2.6.1) is discussed with respect to all groups of variables $\{x_1^T, \dots, x_s^T\}$ system (2.6.2) is defined in domain $B(\rho) = \left\{x_i: \sum_i \|x_i\| < \rho\right\}$ or in R^n as usual.

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REMARK 2.6.2. If multistability of solution $x = 0$ of (2.6.1) is discussed with respect to a group of variables $\{x_1^T, \dots, x_l^T\}$, $l < s$ then it is sufficient to define system (2.6.1) in the domain

$$B_{(\cdot)}(\rho) = \{x_i^T : \|(x_1^T, \dots, x_l^T)^T\| < \rho\}, \quad \rho = \text{const},$$

$$D_{(\cdot)}(\rho) = \{x_i^T : 0 < \|(x_{l+1}^T, \dots, x_s^T)^T\| < +\infty\},$$

here solution $x(t, \cdot) = (x_1^T(t, \cdot), \dots, x_l^T(t, \cdot))^T$ of the system (2.6.1) is assumed to be continuable along $(x_{l+1}^T, \dots, x_s^T)^T$, i.e. solution $(x_1^T, \dots, x_s^T)^T$ is definite for all $t \in \mathcal{T}_\tau$ for which $\|(x_1^T(t), \dots, x_l^T(t))^T\| \leq \rho$.

The construction of sufficient (and necessary) conditions ensuring multistability of zero solutions of (2.6.1) in terms of Definition 2.6.1 makes the general problem on multistability of motion.

2.6.2 On the relationship of the definition of multistability with the other notions of stability of motion

We shall recall the well known definition with reference to system (2.6.1).

DEFINITION 2.6.2. The zero solution $x_1 = x_2 = \dots = x_s = 0$ of system (2.6.1) is

- (i) *stable relatively* \mathcal{T}_i if for any $\varepsilon > 0$ and $t_0 \in \mathcal{T}_i$ there exist $\delta(t_0, \varepsilon) > 0$ such that $\sum_i^s \|x_i(t; t_0, x_0)\| < \varepsilon$ for all $\sum_i^s \|x_{i0}\| < \delta$ and all $t \geq t_0$;
- (ii) *asymptotically stable relatively* \mathcal{T}_i if the conditions of Definition 2.6.2 (i) are satisfied and $\sum_{i=1}^s \|x_i(t; t_0, x_0)\| \rightarrow 0$ as $t \rightarrow +\infty$.

Having compared Definition 2.6.1 with Definition 2.6.2, we see that if all subvectors x_i in system (2.6.1) are homogeneous with respect to the

dynamical properties and in Definition 2.6.1 of stability one Euclidean norm $\|x\| = \sum_{i=1}^s \|x_i\|$ is used, Definition 2.6.1 of multistability degenerates into Definition 2.6.2 of stability in the sense of Liapunov of the zero solution of the system (2.6.1) iff $\mathcal{T}_i = R$.

DEFINITION 2.6.3. The zero solution of the system (2.6.1) is

- (i) *stable relatively the subvectors* x_1, \dots, x_k ($k < s$) *and respect to* \mathcal{T}_i , if for every $\varepsilon > 0$ and $t_0 \in \mathcal{T}_i$ there exist $\delta_1(t_0, \varepsilon) > 0$ and $\delta_2(t_0, \varepsilon) > 0$ such that $\sum_{i=1}^k \|x_i(t; t_0, x_{i0})\| < \varepsilon$ for $\sum_{i=1}^k \|x_{i0}\| < \delta_1$ and $\sum_{i=k+1}^s \|x_{i0}\| < \delta_2$ for all $t \geq t_0$;
- (ii) *asymptotically stable with respect to the subvectors* x_1, \dots, x_k ($k < s$) *relatively* \mathcal{T}_i if under conditions (i) of Definition 2.6.3 the relation $\sum_{i=1}^k \|x_i(t; t_0, x_{i0})\| \rightarrow 0$ holds for all $t \rightarrow +\infty$.

The comparison of Definition 2.6.1 and 2.6.3 shows that if the subvectors x_i , $i < k$ are homogeneous relatively the dynamical properties and the solution of the system (2.6.1) is continuable relatively x_{k+1}, \dots, x_s , the Definition of multistability with respect to a part of the variables implies Definition 2.6.1.

According as Movchan [147], Lakshmikantham and Salvadori [93], Lakshmikantham, Leela and Martynyuk [94] we consider the classes of functions

$$M = \{\rho \in C(R_+ \times R^n, R_+) : \inf_{(t,x)} \rho(t, x) = 0\},$$

$$M_0 = \{\rho \in M : \inf_x \rho(t, x) = 0 \text{ for all } t \in R_+\}.$$

DEFINITION 2.6.4. System (2.6.1) is

- (i) (ρ_0, ρ) -*stable with respect to* \mathcal{T}_i , if for any $\varepsilon > 0$ and $t_0 \in \mathcal{T}_i$ there exists a positive function $\delta(t_0, \varepsilon)$, being continuous in $t_0 \in \mathcal{T}_i$ for every $\varepsilon > 0$ and such that $\rho_0(t_0, x_0) < \delta$ implies $\rho(t, x(t)) < \varepsilon$ for all $t \leq t_0$;
- (ii) *asymptotically* (ρ_0, ρ) -*stable with respect to* \mathcal{T}_i if under the conditions of Definition 2.6.4 (i) $\rho(t, x(t)) \rightarrow 0$ as $t \rightarrow +\infty$.

The comparison of Definitions 2.6.1 and 2.6.4 yields that the Definition 2.6.4 provides the general characteristics of the dynamical properties of the subvectors x_i , $i = 1, 2, \dots, s$, without distinguishing between them.

Let us consider the system (2.6.1) and introduce the measures

$$\begin{aligned}\rho_0 &= \rho_0(t, x_1, \dots, x_s) \in M_0; \\ \rho_1 &= \rho_1(t, x_1, \dots, x_{s-1}) \in M; \\ &\dots \\ \rho_{s-1} &= \rho_{s-1}(t, x_1) \in M.\end{aligned}$$

DEFINITION 2.6.5. System (2.6.1) is *multistable with respect to the measures* $(\rho_0, \dots, \rho_{s-1})$ *relatively* \mathcal{T}_i , iff it is (ρ_0, \dots, ρ_k) -stable in some type (on \mathcal{T}_τ), $k < s - 1$.

Thus, the examination of Definitions 2.6.2–2.6.5 indicates that only Definition 2.6.5 is a generalization of Definition 2.6.1, while the rest of the definitions follow from it.

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2.6.3 Multistability investigation

In order to apply the method of matrix Liapunov function to the problem in question, we introduce classes of matrix-valued function with particular properties.

Together with (2.6.1) we consider a two-indexed system of functions

$$(2.6.3) \quad U(t, x) = [v_{ij}(t, x)], \quad i, j \in [1, s]$$

with $v_{ii} \in C(\mathcal{T}_\tau \times R^n, R_+)$ and $v_{ij} \in C(\mathcal{T}_\tau \times R^n, R)$ for $i \neq j \in [1, s]$.

The notion of the definiteness of an auxiliary function (that is used in the direct Liapunov's method) is a main one, since this behaves as a scalar function having all norm properties.

DEFINITION 2.6.6. The matrix-valued function $U: \mathcal{T}_\tau \times B_{(1,l)} \times D_{(l+1,s)} \rightarrow R^{s \times s}$ is:

- (i) *positive definite on \mathcal{T}_τ , $\tau \in R$, with respect to variables (x_1^T, \dots, x_l^T)* iff there exist time-invariant connected neighborhoods \mathcal{N}^* , $\mathcal{N}^* \subset R^l$ of $x = 0$, a vector $\varphi \in R_+^s$, $\varphi > 0$ and a scalar positive definite in the sense of Liapunov function $w: \mathcal{N}^* \rightarrow R_+$ such that
- (a) $U(t, x) \in C(\mathcal{T}_\tau \times \mathcal{N}^* \times D_{(l+1,s)}, R^{s \times s})$;
 - (b) $U(t, x) = 0$ for all $t \in \mathcal{T}_\tau$ and $(x_1^T, \dots, x_l^T) = 0$;
 - (c) $\varphi^T U(t, x) \varphi \geq w(x_{1,l}^T)$ for all $(t, x \neq 0, \varphi \neq 0) \in \mathcal{T}_\tau \times \mathcal{N}^* \times D_{(l+1,s)} \times R_+^s$, $x_{1,l}^T = (x_1^T, \dots, x_l^T)$;

- (ii) *positive definite on $\mathcal{T}_\tau \times \mathcal{G}^*$ with respect to variables (x_1^T, \dots, x_l^T) if conditions of Definition 2.6.6 (i) hold for $B_{(1,l)} = \mathcal{G}^*$;*
- (iii) *positive definite in the whole (on \mathcal{T}_τ) with respect to variables $(x_{1,l}^T)$ if condition of Definition 2.6.6 (i) hold for $B_{(1,l)} = R^l$;*
- (iv) *negative definite (in the whole) on $\mathcal{T}_\tau \times B_{(1,l)}$ (on \mathcal{T}_τ) with respect to variables $(x_{1,l}^T)$, iff $(-U)$ is positive definite (in the whole) on $\mathcal{T}_\tau \times B_{(1,l)}$ (on \mathcal{T}_τ) with respect to variables $(x_{1,l}^T)$.*

PROPOSITION 2.6.1. *The matrix-valued function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ is positive definite on \mathcal{T}_τ with respect to (x_1^T, \dots, x_l^T) , iff it can be represented in the form*

$$(2.6.4) \quad \varphi^T U(t, x) \varphi = \varphi^T U_+(t, x) \varphi + w(x_1^T, \dots, x_l^T),$$

where $U_+(t, x)$ is positive semi-definite with respect to all variables (x_1^T, \dots, x_s^T) and w is a function explicitly independent of $t \in \mathcal{T}_\tau$ and positive definite with respect to variables (x_1^T, \dots, x_l^T) , $l < s$.

PROOF. *Necessity.* Let the matrix-valued function $U(t, x)$ be (x_1^T, \dots, x_l^T) positive definite on \mathcal{T}_τ . Then, by Definition 2.6.6 there exists a positive definite in the sense of Liapunov function $w(x_1^T, \dots, x_l^T)$ such that on the domain $\mathcal{T}_\tau \times B_{(1,l)} \times D_{(l+1,s)} \times R_+^s$ condition (i) of Definition 2.6.6 is satisfied. We introduce the function

$$\varphi^T U_+(t, x) \varphi = \varphi^T U(t, x) \varphi - w(x_{1,l}^T)$$

which, is non-negative by condition 2.6.6 (c). Hence the function $\varphi^T U(t, x) \varphi$ can be presented in the form (2.6.4).

Sufficiency. Let equality (2.6.4) be satisfied, where $\varphi^T U_+(t, x) \varphi \geq 0$ and $w(x_{1,l}^T)$ is a positive definite function with respect to the variables (x_1^T, \dots, x_l^T) . Then equality (2.6.4) implies

$$\varphi^T U(t, x) \varphi - w(x_{1,l}^T) = \varphi^T U_+(t, x) \varphi \geq 0.$$

Hence condition 2.6.6 (c) for the function $\varphi^T U(t, x) \varphi$ holds. This proves the Proposition 2.6.1.

PROPOSITION 2.6.2. *The matrix-valued function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ is positive definite on \mathcal{T}_τ with respect to variables (x_1^T, \dots, x_l^T) (in the whole) iff there exist function $a \in K(KR)$ such that*

$$(2.6.5) \quad \varphi^T U(t, x) \varphi \geq a(\|(x_{1,l}^T)^T\|)$$

in the domain $\mathcal{T}_\tau \times \mathcal{N}^* \times D_{(l+1,s)} \times R_+^s$.

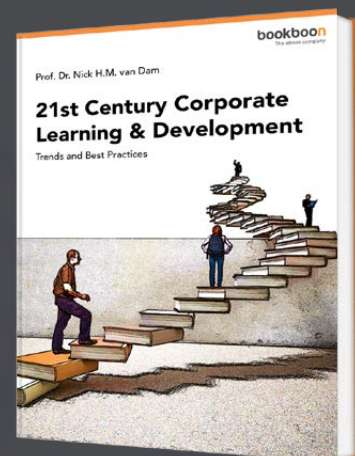
DEFINITION 2.6.7. The matrix-valued function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ is called

- (i) *decreasing on \mathcal{T}_τ with respect to variables (x_1^T, \dots, x_l^T)* iff there exists time-invariant connected neighborhood $\mathcal{N}^* \subseteq R^l$ of $x = 0$, a positive definite function $w_2: \mathcal{N}^* \rightarrow R_+$ and a vector $\varphi \in R_+^s$, $\varphi > 0$ such that
- (a) conditions (a), (b) of Definition 2.6.6 hold and
- (b) $\varphi^T U(t, x) \varphi \geq w_2(x_{1,l}^T)$ for all $(t, x \neq 0, \varphi \neq 0) \in \mathcal{T}_\tau \times \mathcal{N}^* \times D_{(l+1,s)} \times R_+^s$.

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PROPOSITION 2.6.3. *The matrix-valued function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ is decreasing on \mathcal{T}_τ with respect to variables (x_1^T, \dots, x_l^T) , iff it can be presented in the form*

$$(2.6.6) \quad \varphi^T U(t, x) \varphi = \varphi^T U_-(t, x) \varphi + w_2(x_1^T, \dots, x_l^T),$$

where $U_-(t, x)$ is negative semi-definite with respect to all of variables (x_1^T, \dots, x_s^T) , and w_2 is independent of $t \in \mathcal{T}_\tau$ positive definite function of variables (x_1^T, \dots, x_l^T) , $l < s$.

PROOF. Repeating the same argument as in Proposition 2.6.1, one can show there is a matrix-valued function $U(t, x)$ for which the condition (2.6.6) holds.

PROPOSITION 2.6.4. *The matrix-valued function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ is decreasing on $\mathcal{T}_\tau \times \mathcal{N}^*$ with respect to variables (x_1^T, \dots, x_l^T) iff there exist a function $b \in K_{[0, \alpha]}$, where $\alpha = \sup\{x_{1,l}^T \in \mathcal{N}^*\}$ and estimate*

$$(2.6.7) \quad \varphi^T U(t, x) \varphi \leq b(\|(x_{1,l}^T)^T\|)$$

holds for all $(t, x) \in \mathcal{T}_\tau \times \mathcal{N}^* \times D_{(l+1, s)} \times R_+^s$.

Let $U \in C(\mathcal{T}_\tau \times R^n, R^{s \times s})$. The right-hand upper Dini derivative of functions $U(t, x)$ along solutions of the system (2.6.1) are defined by

$$(2.6.8) \quad D^+ U(t, x) = [D^+ v_{ij}(t, x)] \quad \forall i, j \in [1, s],$$

where

$$D^+ v_{ij}(t, x) = \limsup \{ [v_{ij}(t + \theta, x + \theta f(t, x)) - v_{ij}(t, x)] \theta^{-1} : \theta \rightarrow 0^+ \}.$$

2.6.4 Principle of comparison and multistability

The investigation of multistability of the solution of systems of differential equations (2.6.1) via the comparison technique assumes the presence of the corresponding comparison theorems.

2.6.4.1 *The functions of SL-class.* All scalar functions of the type

$$(2.6.9) \quad v(t, x, a) = a^T U(t, x) a,$$

where $U \in C(\mathcal{T}_\tau \times R^n, R^{s \times s})$ are attributed to the class SL.

The vector a can be defined as

- (i) $a = y \in R^s, y \neq 0$;
- (ii) $a = \psi \in C(R^n, R_+^s), \psi(0) = 0$;
- (iii) $a = \theta \in C(\mathcal{T}_\tau \times R^n, R^s) \theta(t, 0) = 0, \forall (t, x) \in \mathcal{T}_\tau \times \mathcal{N}$;
- (iv) $a = \varphi \in R_+^s, \varphi > 0$.

Applying function (2.6.9) and quasimonotone nondecreasing in u for each t function $g \in C(R_+^2, R), g(t, 0) = 0$ we shall formulate the following comparison result.

PROPOSITION 2.6.5. *Let the function $U: \mathcal{T}_\tau \times R^n \rightarrow R^{s \times s}$ be locally Lipschitzian in x . Suppose that the function*

$$(2.6.10) \quad \varphi^T D^+ U(t, x) \varphi \triangleq D^+ v(t, x, \varphi)$$

and the function $g \in C(R_+ \times R^n \times R_+, R)$ such that

$$D^+ v(t, x, \varphi) \leq g(t, x, v(t, x, \varphi))$$

holds for $(t, x, \varphi) \in R_+ \times R^n \times R_+^s$. Let $x(t) = x(t; t_0, x_0)$ be a solution of (2.6.1) existing on $[t_0, \infty)$ and $r(t; t_0, x_0, u_0)$ be the maximal solution of

$$(2.6.11) \quad \frac{du}{dt} = g(t, x(t), u), \quad u(t_0) = u_0 \geq 0$$

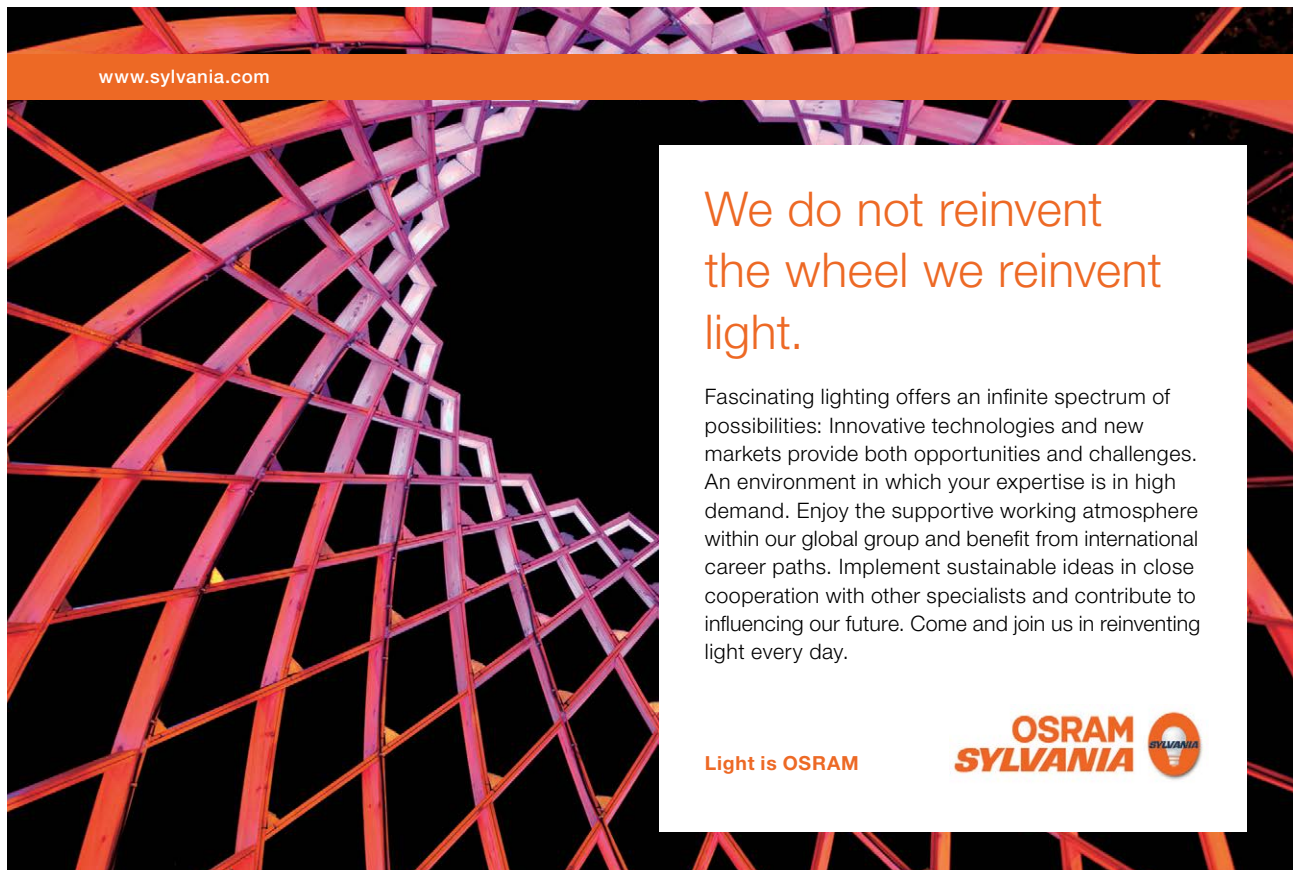
existing for $t \geq t_0$. Then $v(t_0, x_0, \varphi) \leq u_0$ implies

$$(2.6.12) \quad v(t, x(t), \varphi) \leq r(t; t_0, x_0, u_0), \quad t \geq t_0.$$

PROOF is similar to the proof of Proposition 2.4.3.

COROLLARY 2.6.1. If all conditions of Proposition 2.6.5 are satisfied and function $g(t, x, v)$ satisfies either of the conditions

- C₁. $g(t, x, v) = 0$ for all $t \geq t_0$;
- C₂. $g(t, x, v) = \psi^T A \psi$, where $\psi \in C(R^n, R_+^s)$, $\psi(0) = 0$, A is a constant matrix $s \times s$;
- C₃. $g(t, x, v) = w^T B w + r(t, w, \varphi)$, where $w \in C(R^n, R_+^s)$, B is a constant matrix $s \times s$, $r \in C(R_+ \times R_+^s \times R_+^s, R)$ is a polynomial in power higher than two;
- C₄. $g(t, x, v) = W(t, w, \varphi) + r^*(t, w, \varphi)$, where $W \in C(R_+ \times R_+^s \times R_+^s, R)$ is at least a second-power polynomial, and r^* is the same polynomial as in case C₃;
- C₅. $g(t, x, v) = w^T(x) [A^T B + B A] w(x)$, where $w \in C(R^n, R^s)$, $A \in R^{s \times s}$, $b \in R^s$, $B = \text{diag}(b_1, \dots, b_s)$, then estimate (2.6.12) is satisfied, and the investigation of comparison equation (2.6.11) is simplified.




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2.6.4.2 *The functions of VL-class.* All vector functions of the type

$$(2.6.13) \quad L(t, x, b) = AU(t, x)b,$$

where $U \in C(\mathcal{T}_\tau \times R^n, R^{s \times s})$, A is a constant matrix $s \times s$, and vector b is defined according to (i)–(iv) similarly to the definition of the vector a .

For any function $U(t, x)$, which is associated with system (2.6.1) we shall define the function

$$(2.6.14) \quad D^+L(t, x, \varphi) = AD^+U(t, x)\varphi$$

for all $(t, x, \varphi) \in \mathcal{T}_\tau \times R^n \times R_+^s$.

PROPOSITION 2.6.6. *Let there exist*

- (1) *a matrix-valued function $U \in C(\mathcal{T}_\tau \times R^n, R^{s \times s})$ such that $U(t, x)$ is locally Lipschitzian in x ;*
- (2) *a constant $s \times s$ matrix A , a vector $\varphi \in R_+^s$ and vector $y \in R^n$ such that*

$$y^T L(t, x, \varphi) \geq a(\|x\|),$$

where $a \in K$;

- (3) a vector function $G \in C(\mathcal{T}_\tau \times R^n \times R_+^s, R^s)$ such that $G(t, x, u)$ is quasimonotone nondecreasing in u for every $t \in R_+$ such that the estimate

$$(2.6.15) \quad D^+L(t, x, \varphi) \leq G(t, x, L(t, x, \varphi))$$

holds;

- (4) let $x(t; t_0, x_0)$ be any solution of (2.6.1) existing on $[t_0, \infty)$ and $w(t; t_0, w_0, x_0)$ be the maximal solution of

$$(2.6.16) \quad \frac{du}{dt} = G(t, x, u), \quad u(t_0) = w_0 \geq 0$$

existing for $t \geq t_0$. Then $L(t_0, x_0, \varphi) \leq w_0$ implies

$$(2.6.17) \quad L(t, x(t), \varphi) \leq w(t; t_0, w_0, x_0) \quad t \geq t_0.$$

PROOF. It is proved in a standard way by the comparison method (see e.g. Lakshmikantham, Leela and Martynuk [94]).

COROLLARY 2.6.2. Let conditions (1) and (2) of Proposition 2.6.6 be satisfied and in conditions (3) and (4) the function $G \in C(\mathcal{T}_\tau \times R^n, R^s)$. Then, estimate (2.6.17) is satisfied for the maximal solution $w^*(t; t_0, w_0)$ of the comparison system

$$(2.6.18) \quad \frac{du}{dt} = G(t, u), \quad u(t_0) = u_0 \geq 0.$$

COROLLARY 2.6.3. Let conditions (1) and (2) Proposition 2.6.6 be satisfied and the function $G(t, x, L)$ have the form

$$G(t, x, L(t, x, \varphi)) = PL(t, x, \varphi) + m(t, L(t, x, \varphi)),$$

where $P = [p_{ij}]$ is a $s \times s$ matrix with elements $p_{ij} \geq 0$ ($i \neq j$) and $m \in C(\mathcal{T}_\tau \times R^s, R^s)$ is quasimonotone in L and

$$\lim_{\|L\| \rightarrow 0} \frac{\|m(t, L)\|}{\|L\|} = 0$$

uniformly in $t \geq t_0$.

Then, estimate (2.6.18) is true for the maximal solution $w(t; t_0, w_0)$ of the comparison system

$$\frac{du}{dt} = Pu + m(t, u), \quad u(t_0) = w_0 \geq 0.$$



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2.6.4.3 The functions of ML-class. In order to formulate the theorem of comparison with matrix-valued Liapunov function relatively to arbitrary cone K in space R^n we shall need some auxiliary information. Following Lakshmikantham, Leela and Martynyuk [94] a proper subset $K \subset R^n$ is called a cone if the following properties hold:

$$(2.6.19) \quad \begin{aligned} \lambda K \subset K, \quad \lambda \geq 0, \quad K + K \subset K, \quad K = \overline{K} \\ K \cap \{-K\} = \{0\} \quad \text{and} \quad \text{int } K \neq \emptyset \end{aligned}$$

where \overline{K} denotes the closure of K , $\text{int } K$ is the interior of K . We shall denote by ∂K the boundary of K . The cone K induces the order relations on R^n defined by

$$(2.6.20) \quad \begin{aligned} x \stackrel{K}{\leq} y \quad \text{iff} \quad y - x \in K \quad \text{and} \\ x \stackrel{K}{<} y \quad \text{iff} \quad y - x \in \text{int } K. \end{aligned}$$

The set K^* defined by $K^* = \{\varphi \in R^n : \varphi(x) \geq 0 \text{ for all } x \in K\}$, where $\varphi(x)$ denotes the scalar product $\langle \varphi, x \rangle$, is called the adjoint cone and satisfies the properties (2.6.19).

We note that $K = (K^*)^*$, $x \in \text{int } K$ iff $\varphi(x) > 0$ for all $\varphi \in K_0^*$ and $x \in \partial K$ iff $\varphi(x) = 0$ for some $\varphi \in K_0^*$, where $K_0 = K - \{0\}$.

We can now define as quasimonotone property a function relative to the cone K .

A function $f \in C(R^n, R^n)$ is said to be quasimonotone nondecreasing relative to K if $x \stackrel{K}{\leq} y$ and $\varphi(x - y) = 0$ for some $\varphi \in K_0^*$ implies $\varphi(f(x) - f(y)) \leq 0$.

If f is linear, that is, $f(x) = Ax$ where A is an n by n matrix, the quasimonotone property of f means the following: $x \geq 0$ and $\varphi(x) = 0$ for some $\varphi \in K_0^*$ imply $\varphi(Ax) \geq 0$.

If $K = R_+^n$, the function f is said to be quasimonotone nondecreasing if $x < y$ and $x_i = y_i$ for some i , $1 \leq i \leq n$, implies $f_i(x) \leq f_i(y)$.

We consider the system

$$(2.6.21) \quad \frac{dz}{dt} = g(t, z), \quad g(t, 0) = 0,$$

where $g \in C(R_+ \times R^n, R^n)$, $g(t, z)$ is a locally Lipschitzian in z . Let $z_1(t; t_0, z_{10})$, $z_2(t; t_0, z_{20})$ be solutions of the system (2.6.21) with the initial conditions (t_0, z_{10}) and (t_0, z_{20}) respectively.

DEFINITION 2.6.8. We shall say that system (2.6.21) has monotone (strictly) solutions, if

$$z_{20} - z_{10} \in K, \quad z_{20} \neq z_{10}$$

imply the inclusions

$$z_2(t) - z_1(t) \in K \quad (z_2(t) - z_1(t) \in \text{int } K)$$

for all $t \geq t_0$ respectively.

DEFINITION 2.6.9. System (2.6.21) is said to belong the class $W_0(K)$ ($W_s(K)$) if $(z - y) \in \partial K$, $z \neq y$ implies the inequalities

$$g(t, z) - g(t, y) \geq 0 \quad (g(t, z) - g(t, y) > 0)$$

respectively.

DEFINITION 2.6.10. The operator $p(t, z)$ is *positive on* $J_1 \times D$ if $z \in D$ implies $p(t, z) \geq 0$ for all $t \in J_1$, with respect to the cone K .

We shall formulate now a basic Proposition of the principle of comparison in the space, ordered by an arbitrary cone.

PROPOSITION 2.6.7. *Let*

- (1) *there exists a function $g(t, z) \in W_0(K)$ continuous in open (t, z) set $J_1 \times D$ and satisfying the uniqueness conditions of solutions $z(t; t_0, z_0)$ of system*

$$(2.6.22) \quad \frac{dz}{dt} = g(t, z), \quad z(t_0) = z_0;$$

- (2) *there exists a function $h(t, y)$ continuous on open (t, y) set $J_2 \times D \subset J_1 \times D$, $J_2 \subseteq J_1$ such that $g(t, y) - h(t, y) = p(t, y)$, where $p(t, y)$ is a positive operator on set $J_0 \times D$, where $J_0 = J_1 \cap J_2$.*

Then satisfy the relation

$$z(t) - y(t) \in K$$

whenever $z_0 - y_0 \in K$, where $y(t)$ is an arbitrary solution of the system

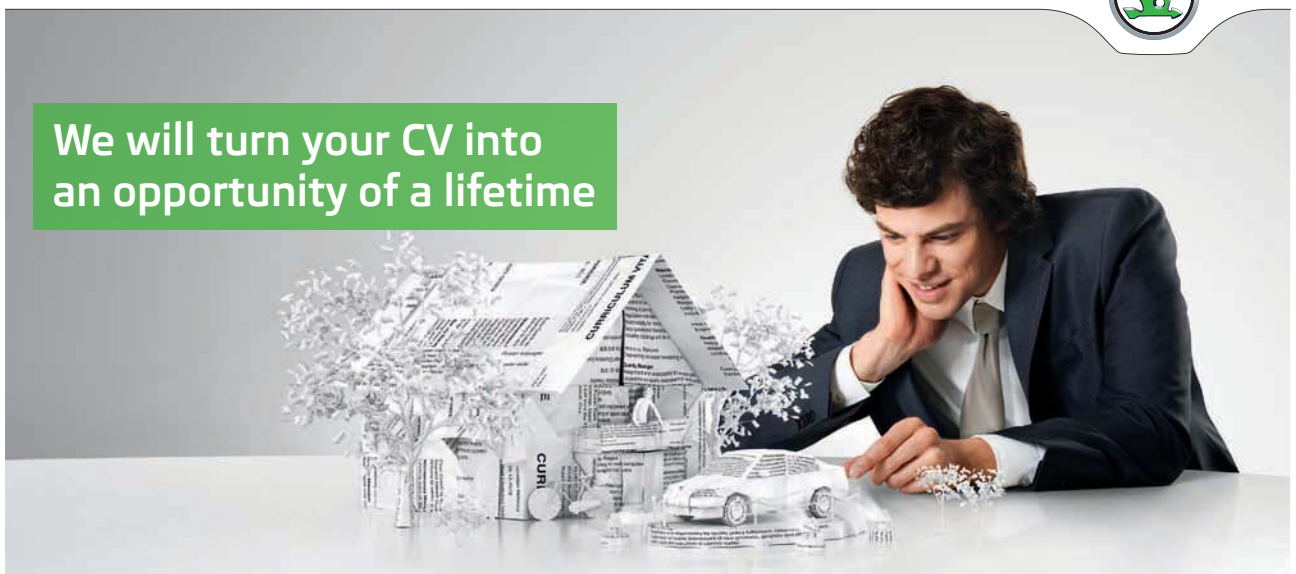
$$(2.6.23) \quad \frac{dy}{dt} = h(t, y), \quad y(t_0) = y_0.$$

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PROOF. Together with system (2.6.21) consider a weakly perturbed system

$$(2.6.24) \quad \frac{dz}{dt} = g(t, z) + \varepsilon(z, u^*)u,$$

where $u \in K$, $u^* \in K^*$ and $\varepsilon \in (0, \varepsilon^*)$, the solution $z(t, \varepsilon) = z(t; t_0, z_0, \varepsilon)$ of which exists on $[t_0, \tau]$, where $\tau \in J_0$ and $\lim_{\varepsilon \rightarrow 0} z(t, \varepsilon) = z(t)$ uniform on $[t_0, \tau]$, where $z(t)$ is a solution of system (2.6.22). Let $z(t, \varepsilon) - y(t) \notin K$ for all $t \in [t_0, \tau]$. Then there exists a $t^* \in [t_0, \tau]$ such that

$$z(t, \varepsilon) - y(t) \in K \quad \text{for all } t \in [t_0, t^*]$$

and $z(t, \varepsilon) - y(t) \notin K$ for the values $t > t^*$ arbitrarily close to t^* . For $t = t^*$ the inclusion

$$(2.6.25) \quad z(t^*, \varepsilon) - y(t^*) \in \partial K$$

and the condition

$$(2.6.26) \quad z(t^*, \varepsilon) \neq y(t^*)$$

are satisfied. For the function $m(t, \varepsilon) = z(t, \varepsilon) - y(t)$ we make the differential equation, in view of the system of equations (2.6.22) and (2.6.23). Namely

$$\begin{aligned} \frac{dm}{dt} &= g(t, z) + \varepsilon(z, u^*)u - h(t, y) \\ &= g(t, z) - g(t, y) + g(t, y) - h(t, y) + \varepsilon(z, u^*)u. \end{aligned}$$

By condition (2) of Proposition 2.6.7

$$(2.6.27) \quad \frac{dm}{dt} = g(t, z) - g(t, y) + p(t, y) + \varepsilon(z, u^*)u,$$

where $p(t, y)$ is a positive operator. By conditions (1) and (2) of Proposition 2.6.7 $p(t^*, y) \geq 0$ and $g(t^*, z) - g(t^*, y) \geq 0$ whenever $(z - y) \in \partial K$. The last condition is satisfied due to (2.6.25) and (2.6.26). The item $\varepsilon(z, u^*)u$ is also non-negative, since $u \in K$ and $u^* \in K^*$.

We confront with the set of point m from the boundary of cone K the indicatory function $\delta(\cdot | K)$, setting

$$\delta(m | K) = \begin{cases} 0, & \text{if } m \in K; \\ +\infty, & \text{if } m \notin K. \end{cases}$$

For the indicatory function $\delta(m | K)$ we compute the subgradient $\gamma(m)$ and scalar multiply the right and left side of the equation by $\gamma(m)$. We get

$$\left(\gamma, \frac{dm}{dt} \right) < -\alpha, \quad \alpha = \text{const} > 0$$

at point $t = t^*$. Therefore, $m(t, \varepsilon)$ will not leave the cone K for all $t > t^*$ as $\varepsilon \rightarrow 0$. The proof is complete.

2.6.5 The system (2.6.1) analysis for $s = 2$

For the system (2.6.1) we construct a matrix-valued function

$$(2.6.28) \quad U(t, x) = [v_{ij}(t, x)], \quad i, j = 1, 2,$$

where $x \in R^{N_0}$, $N_0 = n_1 + n_2$ and v_{ij} is locally Lipschitzian in x . With the aid of vector $y \in R^2$, $y \neq 0$ we construct a scalar function

$$(2.6.29) \quad v(t, x, y) = y^T U(t, x) y.$$

Function (2.6.29) allows us to investigate multistability of the system under definite conditions.

2.6.5.1 Direct application of matrix-valued function. Suppose that system (2.6.1) is defined in domain

$$(2.6.30) \quad \mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho), \quad \rho = \text{const} > 0$$

and the following stability definition holds true for it.

DEFINITION 2.6.11. System (2.6.1) is called *multistable* (on \mathcal{T}_τ) if its zero solution $(x_1^T, x_2^T)^T = 0$ is

- (i) uniformly (x_1^T, x_2^T) -stable with respect to \mathcal{T}_i ;
- (ii) uniformly asymptotically x_2^T -stable with respect to \mathcal{T}_i .



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THEOREM 2.6.1. *Let vector-function $f = (f_1^T, f_2^T)^T$ in (2.6.1) continuous on $R \times B_1(\rho) \times B_2(\rho)$ on $(\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho))$. If there exists*

- (1) *open connected time-invariant neighborhood \mathcal{G} of $x = 0$;*
- (2) *matrix-valued function $U(t, x)$ is*
 - (a) *positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
 - (b) *decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
- (3) *matrix-valued function $D^+U(t, x)$ is*
 - (a) *negative semi-definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}$);*
 - (b) *x_2^T -negative definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}$).*

Then system (2.6.1) is multistability (on \mathcal{T}_τ) in the sense of Definition 2.6.11.

PROOF. If conditions (1), (2), (3)(a) of the Theorem 2.6.1 hold for system (2.6.1) with function (2.6.29), then all hypotheses of Theorem 2.5.1 are fulfilled and state $(x = 0) \in R^{N_0}$ is uniformly stable (on \mathcal{T}_τ).

If conditions (1), (2), (3)(b) of Theorem 2.6.1 hold for system (2.6.1) with function (2.6.29), then all hypotheses of Theorem 2.5.3 are fulfilled and state $(x = 0) \in R^{N_0}$ is uniformly asymptotically x_2^T -stable (on \mathcal{T}_τ).

The Theorem 2.6.1 is proved.

Further we suppose that multistability of (2.6.1) for $s = 2$ is investigated in the domain

$$(2.6.31) \quad \mathcal{T}_\tau \times B_1(\rho) \times D_2, \quad D_2 = \{x_2: 0 < \|x_2\| < +\infty\}.$$

The next result can be easily verified (see e.g. Martynyuk [122]).

THEOREM 2.6.2. *Let vector function $f = (f_1^T, f_2^T)^T$ in (2.6.1) be continuous on $R \times B_1(\rho) \times D_2$ (on $\mathcal{T}_\tau \times B_1(\rho) \times D_2$). If there exists*

- (1) *an open connected time-invariant neighborhood \mathcal{G} of $(x = 0) \in R^{n_1}$;*
- (ii) *matrix-valued function $U(t, x)$ is*
 - (a) *x_1^T -positive definite on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
 - (b) *decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*
 - (c) *x_1^T -decreasing on \mathcal{G} (on $\mathcal{T}_\tau \times \mathcal{G}$);*

- (3) matrix-valued function $D^+U(t, x)$ is
- (a) negative semi-definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}$);
 - (b) x_1^T -negative definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}$);
 - (c) negative definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}$).

Then, respectively

- (a) the conditions (1), (2)(a) and (3)(a) are sufficient for stability of state $(x = 0) \in R^{N_0}$ of (2.6.1) (on \mathcal{T}_τ);
- (b) the conditions (1), (2)(a), (2)(b) and (3)(a) are sufficient for uniform x_1^T -stability of state $(x = 0) \in R^{N_0}$ of (2.6.1) (on \mathcal{T}_τ);
- (c) the conditions (1), (2)(a) and (3)(c) are sufficient for asymptotic x_1^T -stability of state $(x = 0) \in R^{N_0}$ of (2.6.1) (on \mathcal{T}_τ);

2.6.5.2 The application of matrix-valued Liapunov function via transition to vector function. Basing on matrix-valued function $U(t, x)$ and vector $y \in R^s$, $y \neq 0$, $s = 2$ we construct a vector function

$$(2.6.32) \quad L(t, x, y) = AU(t, x)y$$

where A is a constant 2 by 2 matrix. Consider a system of comparison

$$(2.6.33) \quad \frac{du}{dt} = G(t, u), \quad u(t_0) = u_0 \geq 0$$

where $u \in R_+^2$, $G(t, u) = (g_1(t, u_1), g_2(t, u_1, u_2))^T$, $g_1 \in C(\mathcal{T}_\tau \times R_+, R)$, $g_2 \in C(\mathcal{T}_\tau \times R_+ \times R_+, R)$

$$g_1(t, 0) = g_2(t, 0, 0) = 0 \quad \text{for all } t \in \mathcal{T}_\tau.$$

DEFINITION 2.6.12. A comparison system (2.6.33) is called *multistable* (on \mathcal{T}_τ), if its zero solution is

- (i) u_1 -stable with respect to \mathcal{T}_τ ;
- (ii) uniformly u_2 -stable with respect to \mathcal{T}_τ .

THEOREM 2.6.3. Let vector function $f = (f_1^T, f_2^T)^T$ in (2.6.1) be continuous on $R \times B_1(\rho) \times D_2$ (on $\mathcal{T}_\tau \times B_1(\rho) \times D_2$). If there exists

- (1) a matrix-valued function $U: \mathcal{T}_\tau \times B_1(\rho) \times D_2 \rightarrow R^{2 \times 2}$, vector $y \in R^2$, $y \neq 0$ and a constant matrix A 2 by 2 such that components $L_i(t, x, y)$, $i = 1, 2$ of vector function (2.6.32) are locally Lipschitzian in x and satisfy the conditions
- (a) $L_1(t, 0, y) = 0 \quad \forall t \in R \quad (t \in \mathcal{T}_\tau)$;
- (b) $a(\|x_1\|) \leq L_2(t, x, y) \leq b(\|x_1\|) + b_1(L_1(t, x, y))$
for all $(t, x) \in \mathcal{T}_\tau \times B_1(\rho) \times D_2 \cap B_1^c(\eta)$ when each $0 < \eta < \rho$;

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(2) a vector function $g \in C(\mathcal{T}_\tau \times \mathbb{R}_+^2, \mathbb{R}^2)$, $G(t, u)$ is quasimonotone nondecreasing with respect to u for the components of which

(a) $D^+L_1(t, x, y) \leq g_1(t, L_1(t, x, y), 0)$
hold for all $(t, x) \in \mathcal{T}_\tau \times B_1(\rho) \times D_2$, and

(b) $D^+L_2(t, x, y) \leq g_2(t, L_1(t, x, y), L_2(t, x, y))$
hold for all $(t, x) \in \mathcal{T}_\tau \times B_1(\rho) \times D_2 \cap B_1^c(\eta)$ for $0 < \eta < \rho$;

(3) zero solution of system (2.6.33) is multistable (on \mathcal{T}_τ) in the sense of Definition 2.6.12.

Then the system (2.6.1) is x_1^T -stable (on \mathcal{T}_τ).

PROOF. Let $(t_0, \varepsilon): t_0 \in \mathcal{T}_i$ and $0 < \varepsilon, \rho$ be given. It follows from condition (3) of the theorem that for given $\varepsilon_1, \varepsilon_2 > 0$ and $t_0 \in \mathcal{T}_i$ there exist $\delta_{10} = \delta_{10}(t_0, \varepsilon_1) > 0$ and $\delta_{20} = \delta_{20}(\varepsilon_2) > 0$ such that

- (a) $\eta^T u_0 < \delta_{10}$ implies that $u_1(t; t_0, u_0) < \varepsilon_1 \quad \forall t \geq t_0$ and
(b) $\eta^T u_0 < \delta_{20}$ implies that $u_2(t; t_0, u_0) < \varepsilon_2 \quad \forall t \geq t_0$.

Let $\varepsilon_2 = a(\varepsilon)$ and $\varepsilon_1 = b_1^{-1} \left(\frac{1}{2} \delta_{20} \right)$. It follows from the continuity of the function $L_1(t; x, y)$ and condition (1)(a) that there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that

$$L_1(t_0, x_0, y) < \delta_{10} \quad \text{and} \quad \|x_0\| < \delta_1.$$

Let $\delta = \min(\delta_1, \delta_2)$. It is clear that δ depends on $t_0 \in \mathcal{T}_i$ and on $0 < \varepsilon < \rho$. For δ defined in this way, we can assert that the zero solution of (2.6.1) is x_1^T -stable (on \mathcal{T}_τ) with respect to \mathcal{T}_i .

Assume the contrary, i.e., that the zero solution of (2.6.1) is not x_1^T -stable (on \mathcal{T}_τ) when all the conditions of Theorem 2.6.3 are fulfilled. Then

for the solution $x(t; t_0, x_0)$ of (2.6.1) with initial conditions $t_0 \in \mathcal{T}_i$ and $\|x_0\| < \delta$ there exists a time $t_2 > t_1 > t_0$ such that

$$(2.6.34) \quad \begin{aligned} \|x_1(t_2)\| &= \varepsilon < \rho, & \|x_1(t_1)\| &= \delta_2(\varepsilon) \\ x_2(t) &\in \mathcal{N}_{(1,2)}(\rho) \cap \mathcal{N}_{(1,2)}^c(\eta), & \eta &= \delta_2(\varepsilon) > 0 \end{aligned}$$

at the same time that $\|x_2(t)\| < +\infty$.

Let $m(t) = L(t, x(t), y)$; in view of condition (2) of the theorem we obtain

$$(2.6.35) \quad D^+ m_1(t) \leq g_1(t, m_1(t), 0), \quad t_0 \leq t \leq t_2;$$

$$(2.6.36) \quad D^+ m_2(t) \leq g_2(t, m(t)), \quad t_1 \leq t \leq t_2.$$

Let $u^*(t) = u(t; t_1, m(t_1)) \geq 0$ be the extension of $u(t)$ to the left from t_1 to t_0 , and let $u^*(t_0) = u_0^*$. We assume that $L_1(t_0, x_0, y) = u_1(t_0)$ and that $u^*(t_0) = u_0$.

From the differential inequality

$$D^+ m_1(t) \leq g_1(t, m_1(t), u_2^*(t)), \quad m_1(t_0) = u_1(t_0)$$

and the comparison theorem we have

$$(2.6.37) \quad m_1(t) \leq u_1(t; t_0, u_0), \quad t_0 \leq t \leq t_1, \quad u_0 = (u_1(t_0), u_2(t_0))^T.$$

From this it is clear that $u(t) = (u_1(t; t_0, u_0), u_2^*(t; t_1, m(t_1)))^T$ is a solution of (2.6.33) on $[t_0, t_1]$. From condition (1) of Theorem 2.6.3 and inequalities (2.6.34), (2.6.35) and (2.6.37) we obtain

$$(2.6.38) \quad a(\varepsilon) = a(\|x_1(t_2)\|) < L_2(t_2, x(t_2), y) \leq u_2(t_2; t_1, m(t_1)).$$

From the fact that

$$L_1(t_1, x(t_1), y) \leq u_1(t_1; t_0, u_0) < b_1^{-1} \left(\frac{1}{2} \delta_{20} \right)$$

as soon as $\eta^T u_0 < \delta_{10}$ and also from conditions (2.6.34), we have, by condition (1)(a)

$$(2.6.39) \quad \begin{aligned} L_2(t_1, x(t_1), y) &\leq b(\|x_1(t)\|) + b_1(L_1(t_1, x(t_1), y)) \\ &\leq b(\delta_2(\varepsilon)) + b_1 \left(b_1^{-1} \left(\frac{1}{2} \delta_{20} \right) \right) < \frac{1}{2} \delta_{20} + \frac{1}{2} \delta_{20} = \delta_{20}. \end{aligned}$$

It follows from the uniform u_2 -stability of the zero solution of (2.6.33) with respect to \mathcal{T}_i that

$$(2.6.40) \quad u_2(t_2; t_1, m(t_1)) < a(\varepsilon).$$

Inequality (2.6.40) contradicts condition (2.6.37). This completes the proof of the theorem.

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2.6.6 The system (2.6.1) analysis for $s = 3$

Suppose that for $s = 3$ the right-hand side of (2.6.1) are defined in the region

$$(2.6.41) \quad \mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3, \quad D_3 = \{x_3 : 0 < \|x_3\| < +\infty\}.$$

Consider a matrix-valued function

$$(2.6.42) \quad U(t, x) = [v_{ij}(t, x)], \quad i, j = 1, 2, 3,$$

where $v_{ij} \in C(\mathcal{T}_\tau \times R^{N_1}, R)$, $v_{ij}(t, x)$ are locally Lipschitzian in x , $N_1 = n_1 + n_2 + n_3$. With the aid of vector $\varphi \in R_+^3$, $\varphi > 0$ and matrix-valued function (2.6.42) we construct the function

$$(2.6.43) \quad v(t, x, \varphi) = \varphi^T U(t, x) \varphi,$$

where $v \in C(\mathcal{T}_\tau \times R^{N_1} \times R_+^3, R)$.

Function (2.6.42) is applied in two approaches as in Section 2.6.5.

2.6.6.1 Direct application of matrix-valued function.

DEFINITION 2.6.13. System (2.6.1) is *multistable* (on \mathcal{T}_τ) if its zero solution is

- (i) t_0 -uniformly (x_1^T, x_2^T) -stable in the whole (on \mathcal{T}_τ);
- (ii) asymptotically x_2^T -stable in the whole (on \mathcal{T}_τ).

THEOREM 2.6.4. Let vector function $f = (f_1^T, f_2^T, f_3^T)^T$ in (2.6.1) be continuous on $R \times B_1(\rho) \times B_2(\rho) \times D_3$ (on $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3$). If there exists

- (1) an open connected time-invariant neighborhood \mathcal{G}^* of point $(x = 0) \in R^{N_0}$, $N_0 = n_1 + n_2$;

- (2) a matrix-valued function $U(t, x)$
 - (a) (x_1^T, x_2^T) -positive definite on \mathcal{G}^* (on $\mathcal{T}_\tau \times \mathcal{G}^* \times D_3$);
 - (b) decreasing on $\mathcal{G}^* \times D_3$ (on $\mathcal{T}_\tau \times \mathcal{G}^* \times D_3$);
- (3) a matrix-valued function $D^+U(t, x)$ is
 - (a) negative semi-definite on $R \times \mathcal{G}^* \times D_3$ (on $\mathcal{T}_\tau \times \mathcal{G}^* \times D_3$);
 - (a) x_2^T -negative semi-definite on $R \times \mathcal{G}$ (on $\mathcal{T}_\tau \times \mathcal{G}^* \times D_3$);
- (4) a constant $m > 0$ for which

$$\|f_2(t, x_1, x_2, x_3)\| \leq m \quad \forall (t, x) \in \mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3$$

Then, respectively

- (a) hypotheses (1), (2)(a) and (3)(a) are sufficient for (x_1^T, x_2^T) -stability of $(x = 0) \in R^{N_1}$, $N_1 = n_1 + n_2 + n_3$ of the system (2.6.1) (on \mathcal{T}_τ);
- (b) hypotheses (1), (2)(a), (2)(b) and (3)(a) are sufficient for uniform (x_1^T, x_2^T) -stability of $(x = 0) \in R^{N_1}$ of the system (2.6.1) (on \mathcal{T}_τ);
- (c) hypotheses (1), (2) and (3)(b) are sufficient for asymptotical x_2^T -stability in the whole of state $(x = 0) \in R^{N_1}$ of (2.6.1) (on \mathcal{T}_τ).

PROOF. We show that if all hypotheses of Theorem 2.6.4 are satisfied, then for $\|x_0\| < \Delta$, $\Delta < +\infty$ the correlation

$$(2.6.44) \quad \lim_{t \rightarrow +\infty} \|x_2(t; t_0, x_0)\| = 0$$

is valid. Suppose on the contrary. That there exists a number $\delta^* > 0$, a point $x_0^*: \|x_0^*\| < \Delta$ and a sequence $t_k \rightarrow \infty$ such that inequality

$$(2.6.45) \quad \|x_2(t_k; t_0, x_0^*)\| \geq \delta^*, \quad k = 1, 2, \dots$$

holds true.

Let $t_k - t_{k-1} \geq \alpha > 0$, $k = 1, 2, \dots$. We present x_2 -component of solution $x(t) = (x_1^T(t), x_2^T(t), x_3^T(t))^T$ of the system (2.6.1) in the neighborhood of $t = t_k$ in the form

$$(2.6.46) \quad x_2(t; t_0, x_0^*) = x_2(t_k; t_0, x_0^*) + \int_{t_k}^t f_2(s, x(s; t_0, x_0^*)) ds.$$

In view of (2.6.45) and (2.6.46) we have

$$\|x_2(t; t_0, x_0^*)\| \geq \delta^* - m(t - t_k), \quad k = 1, 2, \dots$$

Hence, there exists a β , $0 < \beta < \frac{1}{2}\alpha$ such that

$$(2.6.47) \quad \frac{1}{2}\delta^* \leq \|x_2(t; t_0, x_0^*)\| \leq \rho \quad \forall t \in [t_k - \beta, t_k + \beta]$$

for $k = 1, 2, \dots$. By force of Proposition 2.6.3 and hypotheses (3)(b) of the theorem we have

$$(2.6.48) \quad D^+v(t, x, \varphi) \leq -c(\|x_2\|)$$



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where $c \in KR$.

From the Liapunov correlation for function $v(t, x, y)$ we have

$$0 \leq v(t_k + \beta, x(t_k + \beta; t_0, x_0^*), \varphi) \leq v(t_0, x_0^*, \varphi) - \sum_{i=1}^k \int_{t_i - \beta}^{t_i + \beta} c(\|x_2(s; t_0, x_0^*)\|) ds \leq v(t_0, x_0^*, \varphi) - 2k\beta c\left(\frac{\delta^*}{2}\right).$$

This shows that the condition $v(t_k + \beta, x(t_k + \beta; t_0, x_0^*), \varphi) \geq 0$ is violated for k being large enough. Therefore, (2.6.44) is proved.

2.6.6.2 The application of matrix-valued function via transition to vector-function. Suppose $U: \mathcal{T}_\tau \times R^n \rightarrow R^{3 \times 3}$, Q is a 3 by 3 constant matrix and $y \in R^3$. Construct a vector function

$$(2.6.49) \quad L(t, x, \varphi) = QU(t, x)y$$

where $L \in C(\mathcal{T}_\tau \times R^n \times R^3, R^3)$.

Let a, b be functions from classes K_1 and K_2 , where $K_1 = \{a \in C(0, \rho), R_+\}$ increases with u and $a(u) \rightarrow 0$ as $u \rightarrow 0$, and $K_2 = \{b \in C(0, 3\rho), R_+\}$ increases with u and $b(u) \rightarrow 0$ as $u \rightarrow 0$.

Suppose that the components $L_1(t, x, y), \dots, L_3(t, x, y)$ of the function (2.6.49) satisfy the following conditions:

(A) $L_1(t, 0, y) = 0$ for all $t \in R$ or $t \in \mathcal{T}_\tau$, and

$$L_1(t, x, y) \in C(\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times R^3, R_+).$$

(B) There is a constant $0 < \eta_1 < \rho$ such that

$$L_2(t, x, y) \in C(\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times R^3 \cap B_2^c(\eta_1), R_+)$$

and

$$a_2(\|x_2\|) \leq L_2(t, x, y) \leq b_2(\|x_1\| + \|x_2\| + \|x_3\|) \\ \forall (t, x, y) \in \mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3 \cap B_2^c(\eta_1),$$

where $a_2 \in K_1$ and $b_2 \in K_2$.

(C) For any $0 < \eta_1 < \rho$ there exist an $\eta_2 < \rho$ such that

$$L_2(t, x, y) \in C(\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3 \cap B_1^c(\eta_2), \mathbb{R}_+); \\ a_3(\|x_1\|) \leq L_3(t, x, y) \leq b_3(\|x_1\| + \|x_2\| + \|x_3\|) \\ \forall (t, x, y) \in \mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3 \cap B_1^c(\eta_2),$$

where $a_3 \in K_1$ and $b_3 \in K_2$.

DEFINITION 2.6.14. The comparison system

$$(2.6.50) \quad \frac{dz}{dt} = G(t, z), \quad z(t_0) = z_0 \geq 0,$$

where $z = (u, v, w)^T$, $G = (g_1(t, u), g_2(t, v), g_3(t, w))^T$ is *multistable* (on \mathcal{T}_τ) if its zero solution is

- (i) u -equistable (on \mathcal{T}_τ), and
- (ii) (v, w) -uniformly stable (on \mathcal{T}_τ).

Following Lakshmikantham, Leela and Martynyuk [94], Martynyuk [118] and Koksal [88] the next result is obtained.

THEOREM 2.6.5. Let vector function $f = (f_1^T, f_2^T, f_3^T)^T$ in (2.6.1) be continuous on $\mathbb{R} \times B_1(\rho) \times B_2(\rho) \times D_3$ (on $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3$). If there exists

- (1) matrix-valued function $U(t, x)$, a vector $y \in \mathbb{R}^3$ and a constant 3 by 3 matrix Q for which components L_1, L_2, L_3 of (2.6.49) the conditions (A)–(C) are satisfied;
- (2) functions $g_k \in C(\mathcal{T}_\tau \times \mathbb{R}_+, \mathbb{R})$, $g_k(t, 0) = 0 \quad \forall t \in \mathcal{T}_\tau$ such that
 - (a) the inequality

$$D^+ L_1(t, x, y) \leq g_1(t, L_1(t, x, y))$$

holds in the domain $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3$,

(b) *the inequality*

$$D^+L_1(t, x, y) + D^+L_2(t, x, y) \leq g_2(t, L_1(t, x, y), L_2(t, x, y))$$

holds in the domain $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3 \cap B_2^c(\eta_1)$,

(c) *the inequality*

$$D^+L_1(t, x, y) + D^+L_3(t, x, y) \leq g_3(t, L_1(t, x, y), L_3(t, x, y))$$

holds in the domain $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times D_3 \times \mathbb{R}^3 \cap B_1^c(\eta_2)$;

(3) *the zero solution of system (2.6.50) is multistable (on \mathcal{T}_τ) in the sense of Definition 2.6.14.*

Then the zero solution of system (2.6.1) is (x_1^T, x_2^T) -stable (on \mathcal{T}_τ).

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2.6.7 The system (2.6.1) analysis for $s = 4$

For $s = 4$ the system (2.6.1) is considered in region

$$\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times B_3(\rho) \times D_4, \quad D_4 = \{x_4: 0 < \|x_4\| < +\infty\}.$$

Let $N_1 = n_1 + \dots + n_4$.

DEFINITION 2.6.15. System (2.6.1) is *multistable* (on \mathcal{T}_τ) if its zero solution $((x_1^T, \dots, x_4^T) = 0) \in R^{N_2}$ is

- (i) t_0 -uniformly (x_1^T, x_2^T, x_3^T) -stable (on \mathcal{T}_τ);
- (ii) asymptotically (x_2^T, x_3^T) -stable (on \mathcal{T}_τ);
- (iii) practically x_3^T -stable (on \mathcal{T}_τ), i.e. if given (λ, A) with $0 < \lambda < A$, the inequality $\|x_0\| < \lambda$ implies $\|x_3(t)\| < A$ for all $t \in \mathcal{T}_\tau$.

THEOREM 2.6.6. Let vector function $f = (f_1^T, \dots, f_4^T)^T$ in (2.6.1) be continuous on $R \times B_1(\rho) \times B_2(\rho) \times B_3(\rho) \times D_4$ (on $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times B_3(\rho) \times D_4$). If there exists

- (1) a matrix-valued function $U(t, x)$ which is
 - (a) (x_1^T, x_2^T, x_3^T) -positive definite (on \mathcal{T}_τ);
 - (b) (x_1^T, x_2^T, x_3^T) -decreasing (on \mathcal{T}_τ);
- (2) the matrix-valued function $D^+U(t, x)$ which is (x_2^T, x_3^T) -negative definite (on \mathcal{T}_τ);

(3) a constant $m_1 \in R_+$, $m_1 > 0$ for which

$$\| (f_2^T(t, x), f_3^T(t, x))^T \| \leq m_1$$

in the domain $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times B_3(\rho) \times D_4$;

(4) a vector $\varphi \in R_+^4$, $\varphi > 0$ such that for all $t \in \mathcal{T}_\tau$, given (λ, A) inequality

$$\sup (\varphi^T U(t, x) \varphi \quad \text{for} \quad \|x\| < \lambda) < \inf (\varphi^T U(t, x) \varphi \quad \text{for} \quad \|x_3\| = A)$$

holds true.

Then system (2.6.1) is multistable in the sense of Definition 2.6.15.

PROOF. Properties (i) and (ii) of the zero solution of the system (2.6.1) are implied by hypotheses (1) and (2) of the Theorem 2.6.6, when function $v(t, x, \varphi) = \varphi^T U(t, x) \varphi$ and its derivative $D^+ v(t, x, \varphi)$ are considered along with solution of the system (2.6.1). To prove practical stability (on \mathcal{T}_τ) of state $(x = 0) \in R^{N_2}$ with respect to variables of vectors x_3^T it is sufficient to make sure that when hypotheses of Theorem 2.6.6 hold, the value of norm $\|x_3(t; t_0, x_0)\|$ does not reach the value of A for all $t \in \mathcal{T}_\tau$ provided $\|x_0\| < \lambda$ for any $t_0 \in \mathcal{T}_i \subseteq R$. By hypotheses (2) of the Theorem 2.6.6 we have

$$D^+ v(t, x, \varphi) \leq 0$$

in the domain $\mathcal{T}_\tau \times B_1(\rho) \times B_2(\rho) \times B_3(\rho) \times D_4$. Hence

$$(2.6.51) \quad \begin{aligned} v(t, x, \varphi) &\leq v(t_0, x_0, \varphi) \\ &\leq \sup (\varphi^T U(t, x) \varphi \quad \text{for} \quad \|x\| < \lambda). \end{aligned}$$

Let hypotheses (3) of Theorem 2.6.6 be satisfied and inequality $\|x_3(t)\| < A$ be false for some $t \in \mathcal{T}_\tau$. If the violation of the inequality takes place at $t^* \in \mathcal{T}_\tau$, then

$$(2.6.52) \quad v(t^*, x, \varphi) \geq \inf (\varphi^T U(t, x) \varphi \quad \text{for} \quad \|x_3\| = A).$$

From (2.6.51) and (2.6.52) we get

$$(2.6.53) \quad \begin{aligned} \sup (\varphi^T U(t^*, x) \varphi \quad \text{for} \quad \|x\| < \lambda) &\geq v(t^*, x, \varphi) \\ &\geq \inf (\varphi^T U(t, x) \varphi \quad \text{for} \quad \|x_3\| = A). \end{aligned}$$

The inequality (2.6.53) contradicts hypothesis (4) of the Theorem 2.6.6 and proves that $(x = 0) \in R^{N_2}$ is practically stable (on \mathcal{T}_τ) with respect to variable x_3^T .

2.7 Applications

In this section we present some applications of general theorems of matrix-valued Liapunov functions method to system of equations that model real engineering problems.

2.7.1 The problem of Lefschetz

We consider a problem on stability in a product space for a system of differential equations of the perturbed motion

$$(2.7.1) \quad \begin{aligned} \frac{dy}{dt} &= g(y) + G(y, z), \\ \frac{dz}{dt} &= h(z) + H(y, z). \end{aligned}$$

Here $y \in R^p$, $z \in R^q$, $g: R^p \rightarrow R^p$, $G: R^p \times R^q \rightarrow R^p$, $h: R^q \rightarrow R^q$, $H: R^p \times R^q \rightarrow R^q$. In addition, function g , G ; h , H are continuous on R^p , R^q , $R^p \times R^q$ and they vanish for $y = z = 0$.

The problem itself is to point out the connection between the stability properties of equilibrium state $y = z = 0$ with respect to system (2.7.1) on $R^p \times R^q$ and its nonlinear approximation

$$(2.7.2) \quad \begin{aligned} \frac{dy}{dt} &= g(y), \\ \frac{dz}{dt} &= h(z). \end{aligned}$$

ASSUMPTION 2.7.1. Let there exist the time-invariant neighborhood $\mathcal{N}_y \subseteq R^p$ and $\mathcal{N}_z \subseteq R^q$ of the equilibrium state $y = 0$ and $z = 0$, respectively and let there exist a matrix-valued function

$$(2.7.3) \quad U(y, z) = \begin{pmatrix} v_{11}(y) & v_{12}(y, z) \\ v_{21}(y, z) & v_{22}(z) \end{pmatrix}$$

the element v_{ij} of which satisfy the estimations characteristic to the quadratic forms

$$(2.7.4) \quad \begin{aligned} v_{11}(y) &\geq c_{11}\|y\|^2 & \forall (y \neq 0) \in \mathcal{N}_y; \\ v_{22}(z) &\geq c_{22}\|z\|^2 & \forall (z \neq 0) \in \mathcal{N}_z; \\ v_{12}(y, z) = v_{21}(y, z) &\geq c_{12}\|y\|\|z\| & \forall (y \neq 0, z \neq 0) \in \mathcal{N}_y \times \mathcal{N}_z. \end{aligned}$$

ASSUMPTION 2.7.2. Let there exist constants α_{ij} , $i = 1, 2$; $j = 1, 2, \dots, 8$ such that

$$(2.7.5) \quad \begin{aligned} \left(\frac{\partial v_{11}}{\partial y}, g \right) &\leq \alpha_{11} \|y\|^2; \\ \left(\frac{\partial v_{11}}{\partial y}, G \right) &\leq \alpha_{12} \|y\|^2 + \alpha_{13} \|y\| \|z\|; \\ \left(\frac{\partial v_{22}}{\partial z}, h \right) &\leq \alpha_{21} \|z\|^2; \\ \left(\frac{\partial v_{22}}{\partial z}, H \right) &\leq \alpha_{22} \|z\|^2 + \alpha_{23} \|y\| \|z\|; \\ \left(\frac{\partial v_{12}}{\partial y}, g \right) &\leq \alpha_{14} \|y\|^2 + \alpha_{15} \|y\| \|z\|; \\ \left(\frac{\partial v_{12}}{\partial y}, G \right) &\leq \alpha_{16} \|y\|^2 + \alpha_{17} \|y\| \|z\| + \alpha_{18} \|z\|^2; \\ \left(\frac{\partial v_{12}}{\partial z}, h \right) &\leq \alpha_{24} \|z\|^2 + \alpha_{25} \|y\| \|z\|; \\ \left(\frac{\partial v_{12}}{\partial z}, H \right) &\leq \alpha_{26} \|y\|^2 + \alpha_{27} \|y\| \|z\| + \alpha_{28} \|z\|^2. \end{aligned}$$

THEOREM 2.7.1. Suppose that

- (1) all conditions of Assumptions 2.7.1, 2.7.2 are fulfilled;
- (2) the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad c_{12} = c_{21}$$

be positive definite;

and

- (3) the matrix

$$S = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad \sigma_{12} = \sigma_{21}$$

be negative definite, where

$$\sigma_{11} = \eta_1^2 (\alpha_{11} + \alpha_{12}) + 2\eta_1 \eta_2 (\alpha_{14} + \alpha_{16} + \alpha_{26});$$

$$\sigma_{22} = \eta_2^2 (\alpha_{21} + \alpha_{22}) + 2\eta_1 \eta_2 (\alpha_{18} + \alpha_{24} + \alpha_{28});$$

$$\sigma_{12} = \frac{1}{2} (\eta_1^2 \alpha_{13} + \alpha_{23} \eta_2^2) + \eta_1 \eta_2 (\alpha_{15} + \alpha_{25} + \alpha_{17} + \alpha_{27}),$$

η_1, η_2 being positive numbers.

Then the state of equilibrium $y = z = 0$ of the system (2.7.1) is uniformly asymptotically stable.

If conditions of Assumptions 2.7.1, 2.7.2 are fulfilled for $\mathcal{N}_y = \mathbb{R}^p$, $\mathcal{N}_z = \mathbb{R}^q$ and conditions (2), (3) of the theorem hold, then the equilibrium state $y = z = 0$ of the system (2.7.1) is uniformly asymptotically stable in the whole.

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PROOF. On the basis of estimations (2.7.4), it is not difficult to show that the function $v = \eta^T U(y, z) \eta$ satisfies the estimate

$$(2.7.6) \quad v \geq u^T \Phi^T C \Phi u,$$

where $u^T = (\|y\|, \|z\|)$, $\Phi = \text{diag} [\eta_1, \eta_2]$.

Also, in view of Assumption 2.7.1 and the estimates (2.7.5), the derivative $Dv(y, z)$ defined by $Dv(y, z) = \eta^T D U(y, z) \eta$ satisfies

$$(2.7.7) \quad Dv(y, z) \leq u^T S u.$$

By virtue of (2) and (3) and the inequalities (2.7.6), (2.7.7), we see that all conditions of Theorem 2.5.3 are verified for the function $v(y, z)$ and its derivative. Hence the proof is complete.

If in estimate (2.7.5) we change the sign of inequality for the opposite one, then by means of the method similar to the given one we can obtain an estimate

$$Dv(y, z) \geq u^T \tilde{S} u$$

which allows us to formulate instability conditions for the equilibrium state $y = z = 0$ of system (2.7.1) on the basis of Theorem 2.5.7.

The statement of Theorem 2.7.1 shows that asymptotic stability of the equilibrium state $y = z = 0$ of system (2.7.1) can hold even if the equilibrium state $y = z = 0$ of system (2.7.2) has no properties of asymptotic quasi-stability (cf. Lefschetz [100]).

2.7.2 Autonomous large scale systems

We consider a large scale systems be decomposed into three subsystems

$$(2.7.8) \quad \begin{aligned} \frac{dx}{dt} &= Ax + f(x, y, z), \\ \frac{dy}{dt} &= By + g(x, y, z), \\ \frac{dz}{dt} &= Cz + h(x, y, z), \end{aligned}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$, $z \in R^{n_3}$, $n_1 + n_2 + n_3 = n$; A , B and C are constant matrices of the corresponding dimensions

$$f \in C(R^{n_1} \times R^{n_2} \times R^{n_3}, R^{n_1});$$

$$g \in C(R^{n_1} \times R^{n_2} \times R^{n_3}, R^{n_2});$$

$$h \in C(R^{n_1} \times R^{n_2} \times R^{n_3}, R^{n_3}).$$

Moreover, the vector-functions f , g and h vanish for $x = y = z = 0$ and contain variables x , y and z in first power, i.e. the subsystems

$$(2.7.9) \quad \frac{dx}{dt} = Ax;$$

$$(2.7.10) \quad \frac{dy}{dt} = By;$$

$$(2.7.11) \quad \frac{dz}{dt} = Cz;$$

are not complete linear approximation of the system (2.7.8). Physically speaking this corresponds to the situation when the connections between subsystems (2.7.9)–(2.7.11) are carried out by time-invariant linear blocks. For different dynamical properties of subsystems (2.7.9)–(2.7.11) sufficient total stability conditions will be established for the state $x = y = z = 0$ of the system (2.7.8).

The solution algorithm for this problem is based on actual construction of the matrix-valued function

$$(2.7.12) \quad U(x, y, z) = [v_{ij}(\cdot)], \quad v_{ij} = v_{ji} \quad \forall (i \neq j)$$

with the elements

$$(2.7.13) \quad \begin{aligned} v_{11}(x) &= x^T P_{11} x, \\ v_{22}(y) &= y^T P_{22} y, \\ v_{33}(z) &= z^T P_{33} z; \\ v_{12}(x, y) &= x^T P_{12} y, \\ v_{13}(x, z) &= x^T P_{13} z, \\ v_{23}(y, z) &= y^T P_{23} z, \end{aligned}$$

where P_{ii} , $i = 1, 2, 3$, are symmetrical and positive definite matrices, P_{12} , P_{13} and P_{23} are constant matrices. It can be easily verified that for the functions (2.7.13) there exist estimates

(2.7.14)

$$\begin{aligned}
v_{11}(x) &\geq \lambda_m(P_{11})\|x\|^2 && \forall (x \neq 0) \in \mathcal{N}_x; \\
v_{22}(y) &\geq \lambda_m(P_{22})\|y\|^2 && \forall (y \neq 0) \in \mathcal{N}_y; \\
v_{33}(z) &\geq \lambda_m(P_{33})\|z\|^2 && \forall (z \neq 0) \in \mathcal{N}_z; \\
v_{12}(x, y) &\geq -\lambda_M^{1/2}(P_{12}P_{12}^T)\|x\|\|y\| && \forall (x \neq 0, y \neq 0) \in \mathcal{N}_x \times \mathcal{N}_y; \\
v_{13}(x, z) &\geq -\lambda_M^{1/2}(P_{13}P_{13}^T)\|x\|\|z\| && \forall (x \neq 0, z \neq 0) \in \mathcal{N}_x \times \mathcal{N}_z; \\
v_{23}(y, z) &\geq -\lambda_M^{1/2}(P_{23}P_{23}^T)\|y\|\|z\| && \forall (y \neq 0, z \neq 0) \in \mathcal{N}_y \times \mathcal{N}_z,
\end{aligned}$$

where $\lambda_m(P_{ii})$ are minimal eigenvalues of matrices P_{ii} , $i = 1, 2, 3$, $\lambda_M^{1/2}(P_{12}P_{12}^T)$, $\lambda_M^{1/2}(P_{13}P_{13}^T)$, $\lambda_M^{1/2}(P_{23}P_{23}^T)$ are norms of matrices P_{12} , P_{13} and P_{23} respectively.


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By means of the function

$$U(x, y, z) = \begin{pmatrix} v_{11}(x) & v_{12}(x, y) & v_{13}(x, z) \\ v_{12}(x, y) & v_{22}(y) & v_{23}(y, z) \\ v_{13}(x, z) & v_{23}(y, z) & v_{33}(z) \end{pmatrix}$$

and the vector $\eta \in R_+^3$, $\eta_i > 0$, $i = 1, 2, 3$ we introduce the function

$$(2.7.15) \quad v(x, y, z, \eta) = \eta^T U(x, y, z) \eta.$$

PROPOSITION 2.7.1. *Let for system (2.7.8) there exists matrix-valued function (2.7.12) with elements (2.7.13) and estimates (2.7.14). Then for function (2.7.15) the estimate*

$$(2.7.16) \quad v(x, y, z, \eta) \geq u^T H^T P H u \\ \forall (x \neq 0, y \neq 0, z \neq 0) \in \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{N}_z$$

is satisfied, where $u^T = (\|x\|, \|y\|, \|z\|)$; $H = \text{diag}[\eta_1, \eta_2, \eta_3]$,

$$(2.7.17) \quad P = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^T) & -\lambda_M^{1/2}(P_{13}P_{13}^T) \\ -\lambda_M^{1/2}(P_{12}P_{12}^T) & \lambda_m(P_{22}) & -\lambda_M^{1/2}(P_{23}P_{23}^T) \\ -\lambda_M^{1/2}(P_{13}P_{13}^T) & -\lambda_M^{1/2}(P_{23}P_{23}^T) & \lambda_m(P_{33}) \end{pmatrix}.$$

Together with function (2.7.15) we shall consider its total derivative

$$(2.7.18) \quad Dv(x, y, z, \eta) = \eta^T DU(x, y, z) \eta$$

by virtue of system (2.7.8).

PROPOSITION 2.7.2. *Let for system (2.7.8) there exist matrix-valued function (2.7.12) with elements (2.7.13). For total derivatives of functions (2.7.13) by virtue of subsystems (2.7.9)–(2.7.11) the following estimates are satisfied*

- (1) $(\nabla_x v_{11})^T Ax \leq \rho_{11} \|x\|^2 \quad \forall x \in \mathcal{N}_x;$
- (2) $(\nabla_x v_{12})^T Ax \leq \rho_{12} \|x\| \|y\| \quad \forall (x, y) \in \mathcal{N}_x \times \mathcal{N}_y;$
- (3) $(\nabla_x v_{13})^T Ax \leq \rho_{13} \|x\| \|z\| \quad \forall (x, z) \in \mathcal{N}_x \times \mathcal{N}_z;$
- (4) $(\nabla_y v_{22})^T By \leq \rho_{21} \|y\|^2 \quad \forall y \in \mathcal{N}_y;$
- (5) $(\nabla_y v_{21})^T By \leq \rho_{22} \|x\| \|y\| \quad \forall (x, y) \in \mathcal{N}_x \times \mathcal{N}_y;$
- (6) $(\nabla_y v_{23})^T By \leq \rho_{23} \|y\| \|z\| \quad \forall (y, z) \in \mathcal{N}_y \times \mathcal{N}_z;$
- (7) $(\nabla_z v_{33})^T Cz \leq \rho_{31} \|z\|^2 \quad \forall z \in \mathcal{N}_z;$
- (8) $(\nabla_z v_{31})^T Cz \leq \rho_{32} \|x\| \|z\| \quad \forall (x, z) \in \mathcal{N}_x \times \mathcal{N}_z;$
- (9) $(\nabla_z v_{32})^T Cz \leq \rho_{33} \|y\| \|z\| \quad \forall (y, z) \in \mathcal{N}_y \times \mathcal{N}_z,$

where $\nabla_u = \partial/\partial u$ and

$$\begin{aligned}\rho_{11} &= \lambda_{\max} [P_{11}A + A^T P_{11}], \\ \rho_{21} &= \lambda_{\max} [P_{22}B + B^T P_{22}], \\ \rho_{31} &= \lambda_{\max} [P_{33}C + C^T P_{33}], \\ \rho_{12} &= \|A^T P_{12}\|, \\ \rho_{13} &= \|A^T P_{13}\|, \\ \rho_{22} &= \|P_{12}B\|, \\ \rho_{23} &= \|B^T P_{23}\|, \\ \rho_{32} &= \|P_{13}C\|, \\ \rho_{33} &= \|P_{23}C\|\end{aligned}$$

respectively, ρ_{12} , ρ_{13} , ρ_{22} , ρ_{23} , ρ_{32} , ρ_{33} are norms of matrices $A^T P_{12}$, $A^T P_{13}$, $P_{12}B$, $B^T P_{23}$, $P_{13}C$, $P_{23}C$.

ASSUMPTION 2.7.3. There exist constants ρ_{ij} , $i = 1, 2, 3$, $j = 4, 5, \dots, 12$, such that in open connected neighborhoods $\mathcal{N}_x \subseteq R^{n_1}$, $\mathcal{N}_y \subseteq R^{n_2}$, $\mathcal{N}_z \subseteq R^{n_3}$ or in its product there exist the estimates

- (1') $(\nabla_x v_{11})^T f \leq \rho_{14} \|x\|^2 + \rho_{15} \|x\| \|y\| + \rho_{16} \|x\| \|z\|;$
- (2') $(\nabla_x v_{12})^T f \leq \rho_{17} \|y\|^2 + \rho_{18} \|x\| \|y\| + \rho_{19} \|y\| \|z\|;$

$$(3') \quad (\nabla_x v_{13})^T f \leq \rho_{1.10} \|z\|^2 + \rho_{1.11} \|x\| \|z\| + \rho_{1.12} \|y\| \|z\|;$$

$$(4') \quad (\nabla_y v_{22})^T g \leq \rho_{2.4} \|y\|^2 + \rho_{2.5} \|x\| \|y\| + \rho_{2.6} \|y\| \|z\|;$$


$$(5') \quad (\nabla_y v_{21})^T g \leq \rho_{2.7} \|x\|^2 + \rho_{2.8} \|x\| \|y\| + \rho_{2.9} \|x\| \|z\|;$$

$$(6') \quad (\nabla_y v_{23})^T g \leq \rho_{2.10} \|z\|^2 + \rho_{2.11} \|x\| \|z\| + \rho_{2.12} \|y\| \|z\|;$$

$$(7') \quad (\nabla_z v_{33})^T h \leq \rho_{3.4} \|z\|^2 + \rho_{3.5} \|x\| \|z\| + \rho_{3.6} \|y\| \|z\|;$$

$$(8') \quad (\nabla_z v_{13})^T h \leq \rho_{3.7} \|x\|^2 + \rho_{3.8} \|x\| \|y\| + \rho_{3.9} \|x\| \|z\|;$$


$$(9') \quad (\nabla_z v_{23})^T h \leq \rho_{3.10} \|y\|^2 + \rho_{3.11} \|x\| \|y\| + \rho_{3.12} \|y\| \|z\|.$$

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PROPOSITION 2.7.3. *If estimates (1) - (9) and (1') - (9') are satisfied, then for all total derivatives of function (2.7.15) by virtue of system (2.7.8) the inequality*

$$(2.7.19) \quad Dv(x, y, z, \eta) \leq u^T S u \quad \forall (x, y, z) \in \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{N}_z$$

takes place, where

$$(2.7.20) \quad \begin{aligned} S &= [\sigma_{ij}], \quad \sigma_{ij} = \sigma_{ji} \quad \forall (i, j) \in [1, 3]; \\ \sigma_{11} &= \eta_1^2(\rho_{11} + \rho_{14}) + 2\eta_1(\eta_2\rho_{27} + \eta_3\rho_{37}); \\ \sigma_{22} &= \eta_2^2(\rho_{21} + \rho_{24}) + 2\eta_2(\eta_1\rho_{17} + \eta_3\rho_{3.10}); \\ \sigma_{33} &= \eta_3^2(\rho_{31} + \rho_{34}) + 2\eta_3(\eta_1\rho_{1.10} + \eta_2\rho_{2.10}); \\ \sigma_{12} &= \frac{1}{2}\eta_1^2\rho_{15} + \frac{1}{2}\eta_2^2\rho_{25} + \eta_1\eta_2(\rho_{12} + \rho_{22} + \rho_{18} + \rho_{28}) \\ &\quad + \eta_3(\eta_1\rho_{38} + \eta_2\rho_{3.11}); \\ \sigma_{13} &= \frac{1}{2}\eta_1^2\rho_{16} + \frac{1}{2}\eta_3^2\rho_{35} + \eta_1\eta_3(\rho_{13} + \rho_{32} + \rho_{1.11} + \rho_{39}) \\ &\quad + \eta_2(\eta_1\rho_{29} + \eta_3\rho_{2.11}); \\ \sigma_{23} &= \frac{1}{2}\eta_2^2\rho_{26} + \frac{1}{2}\eta_3^2\rho_{36} + \eta_2\eta_3(\rho_{23} + \rho_{33} + \rho_{2.12} + \rho_{3.12}) \\ &\quad + \eta_1(\eta_2\rho_{19} + \eta_3\rho_{1.12}). \end{aligned}$$

REMARK 2.7.1. The dynamical properties of subsystems (2.7.9)-(2.7.11) influence only the sign of coefficients ρ_{11} , ρ_{21} and ρ_{31} . The constants ρ_{12} , ρ_{13} , ρ_{22} , ρ_{23} , ρ_{32} , ρ_{33} can always be taken positive and the rest of the constants are independent of matrices A , B and C .

In view of the above remark we introduce the following designations

$$\begin{aligned}c_{11} &= \eta_1^2 \rho_{14} + 2\eta_1(\eta_2 \rho_{27} + \eta_3 \rho_{37}); \\c_{22} &= \eta_2^2 \rho_{24} + 2\eta_2(\eta_1 \rho_{17} + \eta_3 \rho_{3.10}); \\c_{33} &= \eta_3^2 \rho_{34} + 2\eta_3(\eta_1 \rho_{1.10} + \eta_2 \rho_{2.10}).\end{aligned}$$

Hence we have

$$\sigma_{11} = \eta_1^2 \rho_{11} + c_{11}; \quad \sigma_{22} = \eta_2^2 \rho_{21} + c_{22}; \quad \sigma_{33} = \eta_3^2 \rho_{31} + c_{33}.$$

PROPOSITION 2.7.4. *The matrix S is negative definite if and only if*

- (1) $\eta_1^2 \rho_{11} + c_{11} < 0$;
- (2) $\eta_1^2 \eta_2^2 \rho_{11} \rho_{21} + \eta_1^2 \rho_{11} c_{22} + \eta_2^2 \rho_{21} c_{11} + c_{11} c_{22} - \sigma_{12}^2 > 0$;
- (3) $\eta_1^2 \rho_{11} (\eta_2^2 \eta_3^2 \rho_{21} \rho_{31} + \eta_2^2 \rho_{21} c_{33} + \eta_3^2 \rho_{31} c_{22} + c_{22} c_{33} - \sigma_{23}^2) + \eta_2^2 \rho_{21} \times (\eta_3^2 \rho_{31} c_{11} + c_{11} c_{33} - \sigma_{13}^2) + \eta_3^2 \rho_{31} (c_{11} c_{22} - \sigma_{12}^2) + c_{11} c_{22} c_{33} + 2\sigma_{12} \times \sigma_{13} \rho_{23} - c_{11} \sigma_{23}^2 - c_{22} \sigma_{13}^2 - c_{33} \sigma_{12}^2 < 0$.

REMARK 2.7.2. If subsystems (2.7.9)–(2.7.11) are nonasymptotically stable, i.e. $\rho_{11} = \rho_{21} = \rho_{31} = 0$, the conditions of Proposition 2.7.4 become

- (1') $c_{11} < 0$;
- (2') $c_{11} c_{22} - \sigma_{12}^2 > 0$;
- (3') $c_{11} c_{22} c_{33} + 2\sigma_{12} \sigma_{13} \sigma_{23} - c_{11} \sigma_{23}^2 - c_{22} \sigma_{13}^2 - c_{33} \sigma_{12}^2 < 0$.

REMARK 2.7.3. If subsystem (2.7.9) is nonasymptotically stable, subsystem (2.7.10) is asymptotically stable and (1.7.11) is unstable, i.e. $\rho_{11} = 0$, $\rho_{21} < 0$, $\rho_{31} > 0$, the conditions of Proposition 2.7.4 become

- (1'') $c_{11} < 0$;
- (2'') $\eta_2^2 \rho_{21} c_{11} + c_{11} c_{22} - \sigma_{12}^2 > 0$;
- (3'') $\eta_2^2 \rho_{21} (\eta_3^2 \rho_{31} c_{11} + c_{11} c_{33} - \sigma_{13}^2) + \eta_3^2 \rho_{31} (c_{11} c_{22} - \sigma_{12}^2) + c_{11} c_{22} c_{33} + 2\sigma_{12} \sigma_{13} \sigma_{23} - c_{11} \sigma_{23}^2 - c_{22} \sigma_{13}^2 - c_{33} \sigma_{12}^2 < 0$.

PROPOSITION 2.7.5. *Matrix S is negative semi-definite iff the inequality signs $<$ and $>$ in Proposition 2.7.4 are replaced by \geq and \leq correspondingly.*

Function (2.7.15) and its total derivative (2.7.18) together with estimates (2.7.16) and (2.7.19) allows us to establish sufficient conditions of stability (in the whole) and asymptotic stability (in the whole) for system (2.7.8).

THEOREM 2.7.2. Suppose that the system (2.7.8) be such that

- (1) in product $\mathcal{N} = \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{N}_z$ there is the matrix-valued function $U: \mathcal{N} \rightarrow R^{3 \times 3}$;
- (2) there exist the vector $\eta \in R_+^3$, $\eta_i > 0$, $i \in [1, 3]$;
- (3) the matrix P is positive definite;
- (4) the matrix S is negative semi-definite or equals to zero.



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Then the state $x = y = z = 0$ of the system (2.7.8) is uniformly stable.

If all estimates mentioned in conditions of Theorem 2.7.2 are satisfied for $\mathcal{N}_x = R^{n_1}$, $\mathcal{N}_y = R^{n_2}$, $\mathcal{N}_z = R^{n_3}$ and function (2.7.15) is radially unbounded, the state $x = y = z = 0$ of the system (2.7.8) is uniformly stable in the whole.

PROOF. Under all conditions of Theorem 2.7.2 the conditions of well-known Barbashin-Krasovskii's theorem are satisfied, and hence, the corresponding type of stability of state $x = y = z = 0$ of the system (2.7.8) takes place (see Theorem 2.5.2).

Let there exists the domain $\Omega = \{(x, y, z) \in \mathcal{N}, 0 \leq v(x, y, z, \eta) < a, a \in \overset{\circ}{R}_+\} \subset R^n$ where $Dv(x, y, z, \eta) \leq 0$.

We designate by \mathcal{M} the largest invariant set in Ω where

$$Dv(x, y, z, \eta) = 0.$$

THEOREM 2.7.3. Suppose that the system (2.7.8) be such that

- (1) the conditions (1) - (3) of Theorem 2.7.2 be satisfied;
- (2) on the set Ω $Dv(x, y, z, \eta) \leq 0$ i.e. the matrix S is negative semi-definite.

Then the set \mathcal{M} is attractive relative to the domain Ω , i.e. all motions of system (2.7.8) starting on set Ω tend to the set \mathcal{M} as $t \rightarrow +\infty$.

Proof of this Theorem is similar to that of Theorem 26.1 by Hahn [66].

THEOREM 2.7.4. Suppose that the system (2.7.8) is such that

- (1) the conditions (1) - (3) of Theorem 2.7.2 are satisfied;
- (2) the matrix S is negative semi-definite.

Then the equilibrium state $x = y = z = 0$ of the system (2.7.8) is uniformly asymptotically stable.

If all estimates mentioned in conditions of Theorem 2.7.4 are satisfied for $\mathcal{N}_x = R^{n_1}$, $\mathcal{N}_y = R^{n_2}$, $\mathcal{N}_z = R^{n_3}$ and function (2.7.15) is radially unbounded, the state $x = y = z = 0$ of the system (2.7.8) is uniformly asymptotically stable in the whole.

The proof is similar to that of Theorem 25.2 by Hahn [66].

2.7.3 Large scale Lur'e-Postnikov system

We consider the system of equations

$$(2.7.21) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y + A_{13}z + q_1 f_1(\sigma_1) \triangleq f_1^*; \\ \frac{dy}{dt} &= A_{21}x + A_{22}y + A_{23}z + q_2 f_2(\sigma_2) \triangleq f_2^*; \\ \frac{dz}{dt} &= A_{31}x + A_{32}y + A_{33}z + q_3 f_3(\sigma_3) \triangleq f_3^*, \end{aligned}$$

where

$$\begin{aligned} \sigma_i &= c_{i1}^T x + c_{i2}^T y + c_{i3}^T z, \\ f_i(\sigma_i)/\sigma_i &\in [0, k_i], \quad i = 1, 2, 3, \quad \sigma_i \in (-\infty, +\infty). \end{aligned}$$

Assume that for system (2.7.8) matrix-valued function (2.7.12) is constructed with elements (2.7.13) for which estimates (2.7.14) are satisfied, and matrix (2.7.17) is positive definite. It is easy to verify that for the total derivative of function (2.7.15) by virtue of system (2.7.21) the following estimate

$$(2.7.22) \quad Dv(x, y, z, \eta) \leq u^T \tilde{S} u$$

is satisfied, where $\tilde{S} = [\tilde{\sigma}_{ij}]$, $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} \quad \forall (i, j) \in [1, 3]$ and

$$\begin{aligned} \sigma_{11} &= \lambda_{\max} \left[\eta_1^2 (A_{11}^T P_{11} + P_{11} A_{11} + P_{11} (q_1 k_1^* c_{11}^T) + (q_1 k_1^* c_{11}^T)^T P_{11}) \right. \\ &\quad \left. + 2\eta_1 \eta_2 (P_{12} A_{21} + P_{12} (q_2 k_2^* c_{21}^T)) + 2\eta_1 \eta_3 (P_{13} A_{31} + P_{13} (q_3 k_3^* c_{31}^T)) \right], \end{aligned}$$

$$\begin{aligned} \sigma_{22} &= \lambda_{\max} \left[\eta_2^2 (A_{22}^T P_{22} + P_{22} A_{22} + P_{22} (q_2 k_2^* c_{22}^T) + (q_2 k_2^* c_{22}^T)^T P_{22}) \right. \\ &\quad \left. + 2\eta_1 \eta_2 (A_{12}^T P_{12} + (q_1 k_1^* c_{12}^T)^T P_{12}) + 2\eta_1 \eta_3 (P_{23} A_{32} + P_{23} (q_3 k_3^* c_{23}^T)) \right], \end{aligned}$$

$$\begin{aligned} \sigma_{33} &= \lambda_{\max} \left[\eta_3^2 (A_{33}^T P_{33} + P_{33} A_{33} + P_{33} (q_3 k_3^* c_{33}^T) + (q_3 k_3^* c_{33}^T)^T P_{33}) \right. \\ &\quad \left. + 2\eta_1 \eta_3 (A_{13}^T P_{13} + (q_1 k_1^* c_{13}^T)^T P_{13}) + 2\eta_2 \eta_3 (A_{23}^T P_{23} + (q_2 k_2^* c_{23}^T)^T P_{23}) \right], \end{aligned}$$


σ_{ij} , $i \neq j$, $i, j \in [1, 3]$ are norms of matrices:

$$\begin{aligned} \sigma_{12} = & \left\| \eta_1^2 (P_{11}A_{12} + P_{11}(q_1k_1^*c_{12}^T)) + \eta_2^2 (P_{22}A_{21} + P_{22}(q_2k_2^*c_{21}^T)) \right. \\ & + \eta_1\eta_2 (A_{11}^T P_{12} + P_{12}A_{22} + (q_1k_1^*c_{11}^T)^T P_{12} + P_{12}(q_2k_2^*c_{22}^T)) \\ & \left. + \eta_1\eta_3 (P_{13}A_{32} + P_{13}(q_3k_3^*c_{32}^T)) + \eta_2\eta_3 (P_{23}A_{31} + P_{23}(q_3k_3^*c_{31}^T)) \right\|, \end{aligned}$$

$$\begin{aligned} \sigma_{13} = & \left\| \eta_1^2 (P_{11}A_{13} + P_{11}(q_1k_1^*c_{13}^T)) + \eta_3^2 (P_{33}A_{31} + P_{33}(q_3k_3^*c_{31}^T)) \right. \\ & + \eta_1\eta_2 (P_{12}A_{23} + P_{12}(q_2k_2^*c_{23}^T)) + \eta_1\eta_3 (P_{13}A_{33} + A_{11}^T P_{13} \\ & \left. + P_{13}(q_3k_3^*c_{33}^T) + (q_1k_1^*c_{11}^T)^T P_{13}) + \eta_2\eta_3 (A_{21}^T P_{23} + (q_2k_2^*c_{21}^T)^T P_{23}) \right\|, \end{aligned}$$

$$\begin{aligned} \sigma_{23} = & \left\| \eta_2^2 (P_{22}A_{23} + P_{22}(q_2k_2^*c_{23}^T)) + \eta_3^2 (P_{33}A_{32} + P_{33}(q_3k_3^*c_{32}^T)) \right. \\ & + \eta_1\eta_2 (A_{13}^T P_{12} + (q_1k_1^*c_{13}^T)^T P_{12}) + \eta_1\eta_3 (A_{12}^T P_{13} + (q_1k_1^*c_{12}^T)^T P_{13}) \\ & \left. + \eta_2\eta_3 (A_{22}^T P_{23} + P_{23}A_{22} + (q_2k_2^*c_{22}^T)^T P_{23} + P_{23}(q_3k_3^*c_{33}^T)) \right\|, \end{aligned}$$

$$k_i^* = \begin{cases} k_i & \text{for } \sigma_i q_i^T P_{ij} x > 0 \text{ (or } \sigma_i q_i^T P_{ij} y > 0 \\ & \text{or } \sigma_i q_i^T P_{ij} z > 0); \\ 0 & \text{in other cases.} \end{cases}$$



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Estimate (2.7.22) of total derivative $Dv(x, y, z, \eta)$ along solutions of system (2.7.21) makes possible the application of Theorems 2.7.2–2.7.4.

EXAMPLE 2.7.1. Let in system (2.7.21) matrices and vectors be defined as

$$\begin{aligned}A_{11} &= \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}; & A_{12} &= \begin{pmatrix} -5 & 0 \\ -1 & -5 \end{pmatrix}; & A_{13} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\A_{21} &= \begin{pmatrix} 5 & 0 \\ 1 & 5 \end{pmatrix}; & A_{22} &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}; & A_{23} &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}; \\A_{31} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; & A_{32} &= \begin{pmatrix} 2.3 & 0 \\ 0 & 2.3 \end{pmatrix}; & A_{33} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \\q_1 &= \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}; & q_2 &= \begin{pmatrix} -0.1 \\ 0 \end{pmatrix}; & q_3 &= \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}; \\c_{11} &= \begin{pmatrix} -0.1 \\ 0 \end{pmatrix}; & c_{12} &= \begin{pmatrix} -0.01 \\ 0 \end{pmatrix}; & c_{13} &= \begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix};\end{aligned}$$

$$\begin{aligned}
c_{21} &= \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}; & c_{22} &= \begin{pmatrix} 0.01 \\ 0 \end{pmatrix}; & c_{23} &= \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}; \\
c_{31} &= \begin{pmatrix} -0.1 \\ -0.1 \end{pmatrix}; & c_{32} &= \begin{pmatrix} 0 \\ -0.01 \end{pmatrix}; & c_{33} &= \begin{pmatrix} 0 \\ -0.1 \end{pmatrix};
\end{aligned}$$

$$k_i = 1, \quad i = 1, 2, 3.$$

We take matrix-valued function $U(x)$ elements in the form

$$\begin{aligned}
v_{11}(x) &= x^T \text{diag}(1, 1)x; & v_{22}(y) &= y^T \text{diag}(1, 1)y; \\
v_{33}(z) &= z^T \text{diag}(1, 1)z; & v_{12}(x, y) &= x^T \text{diag}(0.1, 0.1)y; \\
v_{13}(x, z) &= x^T \text{diag}(0.1, 0.1)z; & v_{23}(y, z) &= y^T \text{diag}(0.1, 0.1)z.
\end{aligned}$$

For elements $v_{ij}(\cdot)$ estimates

$$\begin{aligned}
v_{11}(x) &\geq \|x\|^2, & v_{22}(y) &\geq \|y\|^2, & v_{33}(z) &\geq \|z\|^2, \\
v_{12}(x, y) &\geq -0.1\|x\|\|y\|; & v_{13}(x, z) &\geq -0.1\|x\|\|z\|; \\
v_{23}(y, z) &\geq -0.1\|y\|\|z\|
\end{aligned}$$

are satisfied, and matrix \tilde{P} corresponding to matrix P in estimate (2.7.16)

$$\tilde{P} = \begin{pmatrix} 1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & 1 \end{pmatrix}$$

is positive definite.

If $\eta = (1, 1, 1)^T$ then, given choice of elements $v_{ij}(\cdot)$, $i, j \in [1, 3]$ matrix-valued function $U(x, y, z)$, the matrix \tilde{S} takes the values

$$\tilde{S} = \begin{cases} \begin{pmatrix} -5.2 & 0.16 & 0.2 \\ 0.16 & -0.34 & 0.15 \\ 0.2 & 0.15 & -0.2 \end{pmatrix} & \text{for } k_i^* = 0; \\ \begin{pmatrix} -5.202 & 0.18 & 0.03 \\ 0.18 & -0.34 & 0.012 \\ 0.03 & 0.012 & -0.202 \end{pmatrix} & \text{for } k_i^* = k_i = 1. \end{cases}$$

It is easy to verify that in both cases matrix \tilde{S} is negative definite.

By Theorem 2.7.4 we find that the state $x = y = z = 0$ of system (2.7.21) with vectors and matrices defined in Example 2.7.1 is asymptotically stable in the whole (i.e., system (2.7.21) is absolutely stable).

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2.7.4 A generalized Lotka-Volterra system

We consider a generalized Lotka-Volterra system of the form

$$(2.7.23) \quad \begin{aligned} \frac{dx_1}{dt} &= x_1 (b_1 + a_{11}(x)x_1 + a_{12}(x)x_2), \\ \frac{dx_2}{dt} &= x_2 (b_2 + a_{21}(x)x_1 + a_{22}(x)x_2), \end{aligned}$$

where $x_1, x_2 \in R_+$, $a_{ij} \in C(R_+^2, R)$, b_1, b_2 are constants, $x \in R_+^2$. The generalized Lotka-Volterra system (2.7.23) can have several equilibrium states x_e determined as solutions of

$$(2.7.24) \quad x_e = 0 \quad \text{or} \quad A(x)x_e = -b$$

when $b \neq 0$ and $\det A(x) = 0 \quad \forall x \in S(\rho)$, $S(\rho) \subseteq R_+^2$ or $b = 0$, $\det A(x) \neq 0 \quad \forall x \in S(\rho)$, in which case $x_e = 0$ is the unique equilibrium state of (2.7.23) which is a singular case.

Otherwise, the system (2.7.23) can have finitely many ($\det A(x) \neq 0 \quad \forall x \in S(\rho)$, $b \neq 0$) or infinitely many ($\det A(x) = 0 \quad \forall x \in S(\rho)$, $b = 0$) equilibrium states. If we are interested in properties of $x \neq 0$, then we use the Liapunov transformation of the state variables,

$$(2.7.25) \quad y_1 = x_1 - x_{e1}, \quad y_2 = x_2 - x_{e2}$$

and transform (2.7.23) into

$$(2.7.26) \quad \begin{aligned} \frac{dy_1}{dt} &= (a_{11}(x)y_1 + a_{12}(x)y_2)x_{e1} + (a_{11}(x)y_1 + a_{12}(x)y_2)y_1, \\ \frac{dy_2}{dt} &= (a_{21}(x)y_1 + a_{22}(x)y_2)x_{e2} + (a_{21}(x)y_1 + a_{22}(x)y_2)y_2. \end{aligned}$$

Together with equations (2.7.26) for $i = 1, 2$, we consider the real functions $v_{ij}(y_1, y_2)$ and matrix-valued function

$$(2.7.27) \quad U(y_1, y_2) = \begin{pmatrix} v_{11}(y_1) & v_{12}(y_1, y_2) \\ v_{12}(y_1, y_2) & v_{22}(y_2) \end{pmatrix}$$

with elements

$$(2.7.28) \quad \begin{aligned} v_{11}(y_1) &= \alpha y_1^2, & v_{22}(y_2) &= \beta y_2^2, \\ v_{12}(y_1, y_2) &= v_{21}(y_1, y_2) &= -\gamma y_1 y_2 \end{aligned}$$

$\alpha, \beta > 0$ and γ a constant.

By means of the vector $\eta^T = (\eta_1, \eta_2) \in R_+^2$, $\eta_i > 0$ we shall construct a scalar function

$$(2.7.29) \quad v(y) = \eta^T U(y) \eta$$

for the generalized Lotka-Volterra system (2.7.23).

For all $y \in S(\rho)$ the inequality

$$(2.7.30) \quad v(y) \geq u^T H^T P H u$$

holds, where $u^T = (|y_1|, |y_2|)$, $H = \text{diag}(\eta_1, \eta_2)$,

$$(2.7.31) \quad P = \begin{pmatrix} \alpha & -\gamma \\ -\gamma & \beta \end{pmatrix}.$$

The total derivatives of the matrix-valued function (2.7.27) along solutions of (2.7.23) are given by

$$\begin{aligned} \frac{dv_{11}}{dt} &\leq 2\alpha|a_{11}|x_{e1}|y_1|^2 + 2\alpha|a_{12}|x_{e1}|y_1||y_2| + 2\alpha|a_{11}||y_1|^3 \\ &\quad + 2\alpha|a_{12}|x_{e1}|y_1|^2|y_2|; \\ \frac{dv_{22}}{dt} &\leq 2\beta|a_{22}|x_{e2}|y_2|^2 + 2\beta|a_{21}|x_{e2}|y_1||y_2| + 2\beta|a_{22}||y_2|^3 \\ &\quad + 2\beta|a_{21}|x_{e2}|y_1||y_2|^2; \\ \frac{dv_{12}}{dt} &\leq \gamma|a_{21}|x_{e2}y_1^2 + \gamma|a_{12}|x_{e1}y_2^2 + |\gamma(a_{22}x_{e2} + a_{11}x_{e1})||y_1||y_2| \\ &\quad + |\gamma(a_{21} + a_{11})||y_1|^2|y_2| + |\gamma(a_{22} + a_{12})||y_1||y_2|^2, \end{aligned}$$

and

$$(2.7.32) \quad \frac{dv}{dt} \leq u^T(C + G(y_1, y_2))u,$$

where $u^T = (|y_1|, |y_2|)$,

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}, \quad G(y_1, y_2) = \begin{pmatrix} \sigma_{11}(y_1) & \sigma_{12}(y_1, y_2) \\ \sigma_{21}(y_1, y_2) & \sigma_{22}(y_2) \end{pmatrix}.$$

Here we have

$$c_{11} = 2\eta_1(\alpha\eta_1 a_{11}x_{e1} + \eta_2\gamma a_{21}x_{e2});$$

$$c_{22} = 2\eta_2(\beta\eta_2 a_{22}x_{e2} + \eta_1\gamma a_{12}x_{e1});$$

$$c_{12} = \alpha\eta_1^2|a_{12}|x_{e1} + \beta\eta_2^2|a_{21}|x_{e2} + \eta_1\eta_2|\gamma|(a_{11}x_{e1} + a_{21}x_{e2})$$



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and

$$\begin{aligned}\sigma_{11}(y_1) &= 2\alpha\eta_1^2|a_{11}||y_1|, \\ \sigma_{22}(y_2) &= 2\beta\eta_2^2|a_{22}||y_2|, \\ \sigma_{12}(y_1, y_2) &= (\alpha\eta_1^2|a_{12}| + \eta_1\eta_2|\gamma(a_{21} + a_{11})|) y_1 \\ &\quad + (\beta\eta_2^2|a_{21}| + \eta_1\eta_2|\gamma(a_{22} + a_{12})|) y_2.\end{aligned}$$

Inequalities (2.7.30) and (2.7.32) imply the following theorem, which is the main result of this section.

THEOREM 2.7.5. *The equilibrium x_e of the generalized Lotka-Volterra system (2.7.23) is asymptotically stable if*

- (1) *the matrix P is positive definite;*
- (2) *there exists a constant matrix \overline{G} such that*

$$G(y_1, y_2) \leq \overline{G} \quad \forall (y_1, y_2) \in S(\rho);$$

- (3) *there exists a constant matrix \overline{C} such that*

$$C(x_1, x_2) \leq \overline{C} \quad \forall (x_1, x_2) \in S(\rho);$$

- (4) *the matrix $\overline{C} + \overline{G}$ is negative definite.*

We believe that this result is the first of its kind for such generalized Lotka-Volterra systems.

2.8 Notes

2.1. The following is a summary of the formulation of the matrix Liapunov function method:

- * discovery of double-index system of functions, as a structure suitable for constructing Liapunov functions (see Martynyuk and Gutowski [123]);

- * formulation of the basic concepts of the MLMF on the basis of double-index system function (see Djordjević [27, 29], Grujić [47], Martynyuk [109, 112, 116].);
- * formulation of the principle of invariance and investigation of autonomous systems (see Djordjević [28]; Grujić [47]; Grujić, Martynyuk and Ribbens-Pavella [57]; Martynyuk [116], etc.);
- * development of methods for constructing matrix Liapunov functions (see Djordjević [30], Martynyuk and Krapivny [124], Grujić and Shaaban [61], etc.);
- * construction of sufficient condition of stability for
 - (a) systems with lumped parameters (see Djordjević [27–30], Grujić [47], Martynyuk [126], Martynyuk and Miladzhanov [125], etc.);
 - (b) systems with a small parameter multiplying a derivative (see Martynyuk [114], Martynyuk and Miladzhanov [128], etc.);
 - (c) systems with random parameters (see Azimov [7], Azimov and Martynyuk [8], Martynyuk [115], etc.);

2.2. The results in this section are due to Grujić [47], Martynyuk [116, 121]. Propositions 2.2.1–2.2.3 are new.

2.3. Theorems 2.3.1–2.3.4 uses the results of Liapunov [101], Persidskii [152], Yoshizawa [174], Zubov [178] and Grujić, Martynyuk and Ribbens-Pavella [57].

2.4. Theorems 2.4.1–2.4.2 are new. They generalize well-known theorems of comparison method in motion stability theory (see e.g. Lakshmikantham, Leela and Martynyuk [94]).

2.5. Theorems 2.5.1–2.5.7 of this section are new. The results of the investigation of autonomous system (Theorems 2.5.8–2.5.20) are presented based on those by Grujić [47] and Grujić, Martynyuk and Ribbens-Pavella [57].

2.6. The notion of multistability of motion is formulated in terms of refusal from “homogeneous” behavior of components of solutions for nonlinear system. This notion can be viewed as well as generalization of stability with respect to a part of variables (see e.g. Rumiantzev [160] and Aminov and Sirazetdinov [2]). The results of sections 2.6.1 – 2.6.4 are new. Theorem 2.6.3 is taken from Martynyuk [118]. Theorems 2.6.4 and 2.6.5 were published by Martynyuk [117] and Theorem 2.6.6 the same author [119]. In the investigation of nonlinear systems by vector Liapunov functions the notion of multistability of comparison system was used by Lakshmikantham, Leela and Rao [95].

2.7. In subsection 2.7.1 the solution of the Lefschetz [100] problem is presented according to Martynyuk [111]. Moreover, the results by Djordjević [29] are used. The results of subsections 2.7.2 – 2.7.3 are taken from Martynyuk and Miladzhanov [125]. The results of subsection 2.7.4 are taken from Freedman and Martynyuk [37].

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3

STABILITY OF SINGULARLY-PERTURBED SYSTEMS

3.1 Introduction

The physical system can consist of subsystems that react differently to the external impacts. Moreover, each of the subsystems has its own scale of natural time. In the case when the subsystems are not interconnected, the dynamical properties of each subsystem are examined in terms of the corresponding time scale. It turned out that it is reasonable to use such information when the additional conditions on the subsystems are formulated in the investigation of large scale systems. The existence of various time scales related to the separated subsystems is mathematically expressed by arbitrarily small positive parameters μ_i present at the part of the higher derivatives in differential equation. If the parameters μ_i vanish, the number of differential equations of the large scale system is diminished and, hence the appearance of algebraic equations.

This is just the singular case allowing the consideration of various peculiarities of the system with different time scales.

The chapter is arranged as follows.

Section 3.2 provides mathematical description of the system with quick and slow variables and states the problem of investigation.

Section 3.3 deals with asymptotic stability conditions for singularly perturbed system in terms of scalar Liapunov function.

Section 3.4 deals with Lur'e-Postnikov systems in terms of scalar Liapunov function.

In Section 3.5 the notion of the property of having a fixed sign is formulated for matrix-valued function for singularly perturbed system.

In Section 3.6 the matrix-valued Liapunov function is introduced and the structure of estimation of this function total derivative along solution of the system under consideration is determined.

In the Section 3.7 and 3.8 general results of the direct Liapunov method are stated for singularly perturbed system via matrix-valued function.

In Section 3.9 the method of constructing elements of the matrix-valued function is concretized and linear singularly perturbed systems are investigated using this method.

Section 3.10 contains some applications of general results to systems modeling mechanics problems such as oscillating system of solid bodies and Lur'e-Postnikov system.

The final Section 3.11 is supplied with detailed bibliography comments to the sections of the chapter.

3.2 Description of Systems

The singularly perturbed system S being considered below, is described by two systems of nonlinear differential equations

$$(3.2.1) \quad \frac{dx}{dt} = f(t, x, y, \mu),$$

$$(3.2.2) \quad \mu \frac{dy}{dt} = g(t, x, y, \mu),$$

where $(x^T, y^T)^T$ is a vector of state of the whole system, $x \in R^n$, $y \in R^m$, $f \in C(R \times R^n \times R^m \times \mathcal{M}, R^n)$, $g \in C(R \times R^n \times R^m \times \mathcal{M}, R^m)$. The parameter μ is positive and is supposed to be arbitrarily small. We set $\mu \in (0, 1] = \mathcal{M}$.

The states $x = 0$ and $y = 0$ have open connected neighborhoods $\mathcal{N}_x \subseteq R^n$ and $\mathcal{N}_y \subseteq R^m$ respectively. The vector-function f and g are such that for $(x^T, y^T)^T = 0$ system (3.2.1), (3.2.2) has the only equilibrium state in the Cartesian product $\mathcal{N}_x \times \mathcal{N}_y$ of the sets \mathcal{N}_x and \mathcal{N}_y for any $\mu \in (0, 1]$. If μ takes zero value, system (3.2.1), (3.2.2) degenerates into system S_0 , which is described by the differential and algebraic equation

$$(3.2.3) \quad \frac{dx}{dt} = f(t, x, y, 0),$$

$$(3.2.4) \quad 0 = g(t, x, y, 0).$$

It is supposed that $g(t, x, y, 0)$ vanishes for any $t \in R$ and $x \in \mathcal{N}_x$, iff $y = 0$. This requirement is motivated by an effective application of the Liapunov's

coordinates transformation by Hoppensteadt [74] in the investigation of singularly-perturbed systems. The system of lower order

$$(3.2.5) \quad \frac{dx}{dt} = f(t, x, 0, 0)$$

obtained in result, is of importance in the stability investigation of system (3.2.1), (3.2.2). If $\mu > 0$ is a sufficiently small value of the parameter, then system (3.2.1), (3.2.2) consists of the parts which accomplish slow and quick motions. The quick system S_τ (or the boundary layer) is obtained from system (3.2.1), (3.2.2) after the change of the time scale by introducing the variable

$$\tau = (t - t_0)\mu^{-1}.$$

Then, the quick system corresponding to system (3.2.2) becomes

$$(3.2.6) \quad \frac{dy}{d\tau} = g(\alpha, b, y, 0).$$

In this system α and b , $b = (\beta_1, \dots, \beta_n)$, are scalar and vector parameters, introduced instead of $t \in R$ and $x \in \mathcal{N}_x$ respectively. We suppose as earlier, that g vanishes for any $t \in R$, $x \in \mathcal{N}_x$, $\mu \in (0, 1]$ iff $y = 0$. The separation of the time-scales in the investigation of stability of system (3.2.1), (3.2.2) is essential due to the fact that the analysis of the degenerate system S_0 (3.2.5) and the quick system S_τ (3.2.6) is a more simple problem in comparison with the general problem of stability of system (3.2.1), (3.2.2). The next problem to be considered is to establish conditions for the vector-function f and g under which the property of uniform asymptotic stability in the product $\mathcal{N}_x \times \mathcal{N}_y$ of system (3.2.1), (3.2.2) can be obtained from the same property of solutions of system (3.2.5) and (3.2.6).

3.3 Asymptotic Stability Conditions

Let

$$\mathcal{N}_{x0} = \{x: x \in \mathcal{N}_x, x \neq 0\}, \quad \mathcal{N}_{y0} = \{y: y \in \mathcal{N}_y, y \neq 0\}.$$

The function $V(\alpha, b, y) \in C^{(1,1,1)}(R \times R^n \times R^m, R)$ and

$$V_\alpha = \frac{\partial V}{\partial \alpha}, \quad V_b = \left(\frac{\partial V}{\partial \beta_1}, \frac{\partial V}{\partial \beta_2}, \dots, \frac{\partial V}{\partial \beta_n} \right)^T.$$

We introduce two assumptions on systems (3.2.5) and (3.2.6) connected with positive definite functions θ and V .

ASSUMPTION 3.3.1. There exist

- (1) a decreasing positive definite on \mathcal{N}_x and radially unbounded for $\mathcal{N}_x = R^n$ function $\theta \in C^{(1,1)}(R \times \mathcal{N}_{x_0}, R_+)$;
- (2) positive definite function $\varphi \in C(R^n, R_+)$ and $\psi \in C(R^m, R_+)$ on \mathcal{N}_x and \mathcal{N}_y , respectively;
- (3) non-negative numbers ζ_1 and ζ_2 , $\zeta_1 < 1$, and the conditions are satisfied:
 - (a) $\theta_t(t, x) + \theta_x^T(t, x)f(t, x, 0) \leq -\varphi(x) \quad \forall (t, x) \in R \times \mathcal{N}_{x_0}$;
 - (b) $\theta_x^T(t, x)[f(t, x, y, \mu) - f(t, x, y, 0)] \leq \zeta_1\varphi(x) + \zeta_2\psi(y)$,
 $\forall (t, x, y, \mu) \in R \times \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{M}$.

Conditions (1)–(3)(a) of Assumption 3.2.1 ensure uniform asymptotic stability of $x = 0$ of system (3.2.5) in the whole, when $\mathcal{N}_x = R^n$. Condition (3)(b) is a requirement to the qualitative properties of the vector-function f on $\mathcal{N}_x \times \mathcal{N}_y$.

ASSUMPTION 3.3.2. There exist

- (1) a decreasing positive definite on $\mathcal{N}_x \times \mathcal{N}_y$ and radially unbounded in y uniformly relatively $x \in \mathcal{N}_x$ for $\mathcal{N}_y = R^m$ function $V(t, x, y) \in C^{(1,1,1)}(R \times \mathcal{N}_x \times \mathcal{N}_y, R_+)$ (or $V(t, y) \in C^{(1,1)}(R \times \mathcal{N}_{y_0}, R_+)$ decreasing and positive definite on \mathcal{N}_y and radially unbounded for $\mathcal{N}_y = R^m$);
- (2) non-negative numbers $\xi_1, \xi_2, \xi_3, \xi_4$ ($\xi_1 < 1, \xi_2 < 1$) and an integer $\pi > 1$;
- (3) positive definite functions $\varphi \in C(R^n, R_+)$, $\psi \in C(R^m, R_+)$ on \mathcal{N}_x and \mathcal{N}_y respectively and the following conditions are satisfied
 - (a) $V_y^T g(\alpha, b, y, 0) \leq -\psi(y) \quad \forall (\alpha, b, y) \in R \times \mathcal{N}_x \times \mathcal{N}_y$ or $(\forall (\alpha, b, y) \in R \times \mathcal{N}_x \times \mathcal{N}_{y_0})$ respectively;
 - (b) $V_y^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] \leq \xi_1 \mu^\pi \varphi(b) + \xi_2 \psi(y) \quad \forall (\alpha, b, y, \mu) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$ or $(\forall (\alpha, b, y, \mu) \in R \times \mathcal{N}_x \times \mathcal{N}_{y_0} \times \mathcal{M})$ respectively;
 - (c) $V_\alpha + V_b^T f(\alpha, b, y, \mu) \leq \xi_3 \varphi(b) + \xi_4 \psi(y) \quad \forall (\alpha, b, y, \mu) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$ or $(\forall (\alpha, b, y, \mu) \in R \times \mathcal{N}_x \times \mathcal{N}_{y_0} \times \mathcal{M})$ respectively.

The constants $\zeta_1, \zeta_2, \xi_1, \xi_2$ and ξ_3, ξ_4 mentioned in Assumption 3.3.1, 3.3.2 must be taken as small as possible. If the function V does not depend on x , then it is to be positive definite on \mathcal{N}_y only. If, in addition \mathcal{N}_y is time-invariant, then condition (c) in Assumption 3.3.2 is omitted.

Let

$$\tilde{\mu} = \frac{1 - \xi_2}{\zeta_2 + \xi_4}.$$

This value is a lower estimate of the upper boundary of the admissible change of μ .

THEOREM 3.3.1. *In order that the equilibrium state $(x^T, y^T)^T = 0$ of system (3.2.1), (3.2.2) to be uniformly asymptotically stable, it is sufficient that conditions of Assumptions 3.3.1 and 3.3.2 be satisfied for every $\mu \in (0, \tilde{\mu})$ and for $\mu \rightarrow 0$ as soon as the inequality*

$$1 > \zeta_1 + \xi_1 \tilde{\mu}^{\pi-1} + \xi_3$$

holds.



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If moreover $\mathcal{N}_y \times \mathcal{N}_y = R^{m+n}$, then the equilibrium state is uniformly asymptotically stable in the whole for every $\mu \in (0, \tilde{\mu})$ and for $\mu \rightarrow 0$.

PROOF. Let the function ν be defined by the formula $\nu = \theta + V$. Then $\nu(t, x, y) \in C^{(1,1,1)}(R \times \mathcal{N}_{x_0} \times \mathcal{N}_{y_0})$ and, since the conditions of Assumptions 3.3.1 and 3.3.2 are satisfied, it is decreasing and positive on $\mathcal{N}_x \times \mathcal{N}_y$. The Euler derivative $\frac{d\nu(t, x(t), y(t), \mu)}{dt}$ of it along the motion of system (3.2.1), (3.2.2) $z(t) = (x^T(t), y^T(t))^T \neq 0$ ($z(t) = 0$, $t \in [t_0, +\infty[$) means that the equilibrium state is reachable and therefore is not considered, due to system (3.2.1), (3.2.2) is

$$\frac{d\nu}{dt} = \theta_t + \theta_x^T f + V_t + V_x^T f + \frac{1}{\mu} V_y^T g.$$

The right-side part of this expression is transformed to the form

$$\begin{aligned} \frac{d\nu}{dt} &= \theta_t + \theta_x^T f(t, x, 0, 0) + \theta_x^T [f(t, x, y, \mu) - f(t, x, 0, 0)] + V_x^T f(t, x, y, 0) \\ &\quad + \frac{1}{\mu} V_y^T g(t, x, y, 0) + \frac{1}{\mu} V_y^T [g(t, x, y, \mu) - g(t, x, y, 0)]. \end{aligned}$$

Conditions (3)(a) and (3)(b) of Assumption 3.3.1 and (3)(a)–(3)(c) of Assumption 3.3.2 lead to the estimate

$$(3.3.1) \quad \frac{d\nu}{dt} \leq -(1 - \zeta_1 - \xi_1 \mu^{\pi-1} - \xi_3) \varphi(x) - \frac{1}{\mu} [1 - \xi_2 - \mu(\zeta_2 + \xi_4)] \psi(y),$$

$$\forall \mu \in (0, \tilde{\mu}) \quad \mu \rightarrow 0 \quad \forall (t, x, y) \in R \times \mathcal{N}_{x_0} \times \mathcal{N}_{y_0}.$$

Let

$$\begin{aligned} \mathcal{N}_{0x} &= \{z: x = 0, y \in \mathcal{N}_{y_0}\}, & \mathcal{N}_{0y} &= \{z: x \in \mathcal{N}_{x_0}, y = 0\}, \\ \mathcal{N}_0 &= \mathcal{N}_{0x} \times \mathcal{N}_{0y}. \end{aligned}$$

It is clear that

$$\mathcal{N}_x \times \mathcal{N}_y = \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{N}_0 \times \{z : z = 0\}.$$

Let ν_M be a maximal positive number, for which the largest connected neighborhood $U_{\nu_M}(t)$ of point $z = 0$ is such, that

$$\nu(t, x, y) \in [0, \nu_M), \quad \forall (x, y) \in U_{\nu_M}(t) \quad \forall t \in R,$$

is a subset of the product $\mathcal{N} = \mathcal{N}_x \times \mathcal{N}_y$ for every $t \in R$. The existence of the value $\nu_M > 0$ is implied by the positive definiteness of function ν on \mathcal{N} and the time-invariance of the neighborhood of point $z = 0$.

Let $\tau_i, \tau_i^*, t_0 \leq \tau_i < \tau_i^* \leq +\infty$ denote the times when $z(t) \in U_{\nu_M}(t) \setminus \mathcal{N}_0 \quad \forall t \in (\tau_i, \tau_i^*), \tau_i > t_0$ and $z(t) \in \mathcal{N}_0 \quad \forall t \in [\tau_{i-1}^*, \tau_i]$. If $z(t_0) \in U_{\nu_M}(t_0) \setminus \mathcal{N}_0$ then $i = 0, \tau_0 = t_0, [\tau_0, \tau^*) = [t_0, \tau^*)$ is the first interval to be considered and the next is $[\tau_0^*, \tau_1]$. If $z(t_0) \in \mathcal{N}_0$, then $i = 1, \tau_0^* = t_0$ and $[\tau_0^*, \tau_1]$ is the first interval to be considered, and the next is (τ_1, τ_1^*) . In what follows, $i \geq 0$ is an integer.

Let

$$\begin{aligned} \zeta(t; t_0, z_0, \mu) &= (\chi^T(t; t_0, z_0, \mu), \eta^T(t; t_0, z_0, \mu))^T, \\ \zeta(t_0; t_0, z_0, \mu) &\equiv z_0, \end{aligned}$$

is a motion of system (3.2.1), (3.2.2) for the initial values z_0 and $t = t_0$ when $\mu > 0$.

PROPOSITION 3.3.1. *The function ν is strictly decreasing in $t \in [\tau_{i-1}^*, \tau_i]$ along motions $\zeta(t; t_0, z_0, \mu)$ of system (3.2.1), (3.2.2) for every $\mu \in (0, \tilde{\mu})$ and for $\mu \rightarrow 0$.*

PROOF.

Part 1. Let there exist a time $\hat{t} \in [\tau_{i-1}^*, \tau_i[$ when $\nu(t, x(t), y(t)) \leq \nu(\hat{t}, x(\hat{t}), y(\hat{t}))$ for some $t \in (\tau_{i-1}, \tau_{i-1}^*)$. If $\hat{t} = \tau_{i-1}^*$, then there exist $\bar{\tau}_1, \bar{\tau}_2 \in (\tau_{i-1}, \tau_{i-1}^*), \bar{\tau}_1 < \bar{\tau}_2$ such that $\nu(\bar{\tau}_1, x(\bar{\tau}_1), y(\bar{\tau}_1)) \leq \nu(\bar{\tau}_2, x(\bar{\tau}_2), y(\bar{\tau}_2))$ due to the continuity of function V and ζ at $t \in \mathcal{T}_0, \forall t \in R$ which ensures the continuity of functions f and g . Therefore, there exists a $\tau_3 \in [\bar{\tau}_1, \bar{\tau}_2]$, when

$$\left. \frac{d\nu}{dt} \right|_{t=\tau_3} \geq 0.$$

However, this contradicts estimate (3.3.1) because of the positive definiteness of functions φ and ψ and the fact that

$$(1 - \zeta_1 - \xi_1 \mu^{\pi-1} - \xi_3) > 0, \quad \frac{1}{\mu} [1 - \xi_2 - \mu(\zeta_2 + \xi_4)] > 0 \quad \forall \mu \in (0, \tilde{\mu}).$$

Hence, the equality $\hat{t} = \tau_{i-1}^*$ is impossible and a value $\hat{t} \in (\tau_{i-1}^*, \tau_i]$ is to be considered. Let $T_1 \subseteq [\tau_{i-1}^*, \tau_i)$ be a set of all times t such that $x(t) = 0$, $T_2 \subseteq [\tau_{i-1}^*, \tau_i)$ be a set of all times t , such that $y(t) = 0$. Since $z(t) = 0$ is excluded $\forall t \in [t_0, +\infty[$, then, by virtue of the continuity of the system motion it should be $T_1 = [\tau_{i-1}^*, \tau_i)$ or $T_2 = [\tau_{i-1}^*, \tau_i)$. To be specific, we suppose that $T_1 = [\tau_{i-1}^*, \tau_i)$. Then $\theta(t, x(t)) = \theta(t, 0) \forall t \in T_1$ and $\nu(t, x(t), y(t)) = \nu(t, 0, y(t))$. Moreover,

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$$(3.3.2) \quad \begin{aligned} \frac{d}{dt}\nu(t, 0, y(t)) &= \frac{d}{dt}V(t, 0, y(t)) \leq -\frac{1}{\mu}(1 - \xi_2 - \xi_4)\psi(y(t)) \\ &\quad \forall t \in T_1, \quad \forall \mu \in (0, \tilde{\mu}), \quad \mu \rightarrow 0. \end{aligned}$$

This contradicts the assumption that $\hat{t} \in T_1$. Now let $T_2 = [\tau_{i-1}^*, \tau_i)$. Then $y(t) = 0 \quad \forall t \in T_2$. Therefore

$$\begin{aligned} \nu(t, x(t), y(t)) &= \nu(t, x(t), 0) \quad \forall t \in T_2, \\ \frac{d}{dt}\nu(t, x(t), 0) &\leq -(1 - \zeta_1 - \zeta_3)\varphi(x(t)) \quad \forall t \in T_2, \end{aligned}$$

that contradicts the assumption that $\hat{t} \in T_2$. In general, there exists no value $\hat{t} \in [\tau_{i-1}^*, \tau_i[$ mentioned above.

Part 2. Inequalities (3.3.1), (3.3.2), estimates of $\tilde{\mu}$ and conditions $1 > \zeta_1 + \xi_1\tilde{\mu}^{\pi-1} + \xi_3$, $\zeta_2 > 0$, $\xi_3 > 0$ together with the positive definiteness of functions φ and ψ prove that the function ν strictly decreases on interval $[\tau_{i-1}^*, \tau_i)$, $\tau_{i-1}^* \geq t_0$, $\forall i \geq 1$.

Part 3. Let there exist $\hat{t} \in [\tau_{i-1}^*, \tau_i]$ such that $\nu(t, x(t), y(t)) \geq \nu(\hat{t}, x(\hat{t}), y(\hat{t}))$ for some $t \in (\tau_i, \tau_i^*)$. Hence, there exist $\bar{\tau}_1, \bar{\tau}_2 \in (\tau_i, \tau_i^*)$, $\bar{\tau}_1 < \bar{\tau}_2$ such that $\nu(\bar{\tau}_1, x(\bar{\tau}_1), y(\bar{\tau}_1)) \leq \nu(\bar{\tau}_2, x(\bar{\tau}_2), y(\bar{\tau}_2))$ due to the continuity of $\nu(t, x(t), y(t))$ and ζ in t and because of description of Section 3.2.

Therefore, $\exists \bar{\tau}_3 \in [\bar{\tau}_1, \bar{\tau}_2]$ is such that $\frac{d}{dt}\nu(t, x(t), y(t))|_{t=\bar{\tau}_3} \geq 0$ and this contradicts condition (3.3.1).

The combination of assertions of Parts 1–3 proves Proposition 3.3.1.

In view of the positive definiteness of ν we establish according to the results Part 1 the uniform stability of state $z = 0$ of system (3.2.1), (3.2.2) for $\forall \mu \in (0, \tilde{\mu})$ and for $\mu \rightarrow 0$. Further on, because of the positive definiteness of functions φ and ψ and the fact that $(1 - \zeta_1 - \xi_1\mu^{\pi-1} - \xi_3) > 0$ and $(1 - \zeta_1 - \xi_1\mu^{\pi-1}) > 0 \quad \forall \mu \in (]0, \tilde{\mu})$ as $\mu \rightarrow 0$ and due to the estimate of $\tilde{\mu}$, $\frac{d}{dt}\nu$ is proved to be smaller than a negative definite function on $\mathcal{N}_{x_0} \times \mathcal{N}_{y_0}$,

on \mathcal{N}_{ox} and on \mathcal{N}_{oy} . This result together with the conditions of positive definiteness and decrease of function ν proves uniform attraction in the whole of the state $z = 0$ of system (3.2.1), (3.2.2) and completes the prove of the first assertion of the theorem. In the case when $\mathcal{N}_x \times \mathcal{N}_y = R^{n+m}$, the function ν will be radially unbounded and this together with the other conditions proves the second assertion of the theorem. This theorem is applied in the absolute stability analysis of singularly perturbed Lur'e-Postnikov systems.

3.4 Singularly Perturbed Lur'e-Postnikov Systems

Let system (3.2.1), (3.2.2) be the Lur'e-Postnikov type system (see Grujić [54])

$$(3.4.1) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y + q_1\Phi_1(\sigma_1), \\ \sigma_1 &= c_{11}^T x + c_{12}^T y; \end{aligned}$$

$$(3.4.2) \quad \begin{aligned} \mu \frac{dy}{dt} &= \mu A_{21}x + A_{22}y + q_2\Phi_2(\sigma), \\ \sigma_2 &= \mu c_{21}^T x + c_{22}^T y. \end{aligned}$$

The matrices $A_{(\cdot)}$ and vectors $c_{(\cdot)}$ and $q_{(\cdot)}$ are of the appropriate dimensions. The nonlinearities Φ_i , $i = 1, 2$ are continuous, $\Phi_i(0) = 0$, and in Lur'e sectors $[0, k_i]$, $k_i \in (0, +\infty)$ satisfy the conditions

$$\frac{\Phi_i(\sigma_i)}{\sigma_i} \in [0, k_i], \quad i = 1, 2; \quad \forall \sigma_i \in (-\infty, +\infty).$$

The nonlinearities Φ_i are considered incidentally, for which the state $x = 0$, $y = 0$ is the only equilibrium state of the degenerate system

$$(3.4.3) \quad \frac{dx}{dt} = A_{11}x + q_1\varphi_1(\sigma_1^0), \quad \sigma_1^0 = c_{11}^T x$$

and the system, describing the boundary layer respectively

$$(3.4.4) \quad \frac{dy}{dt} = A_{22}y + q_2\Phi_2(\sigma_2^0), \quad \sigma_2^0 = c_{22}^T y.$$

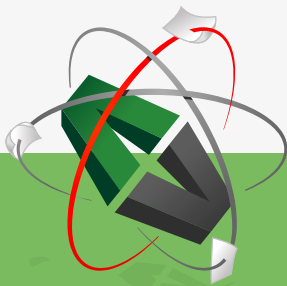
This assumption is valid if

$$c_{ii}^T A_{ii}^{-1} q_i > 0, \quad i = 1, 2.$$

We suppose the matrix A_{11} is stable, the pair (A_{11}, q_1) is controlled and there exist numbers $\psi_1 \in [0, +\infty[$ and $\varepsilon_1 \in (0, +\infty)$ such that

$$\begin{aligned} & k_1^{-1} + \operatorname{Re}(1 + j\psi_1\omega)c_{11}^T(A_{11} - j\omega I_1)^{-1}q_1 \\ & - \varepsilon_1 q_1^T (A_{11}^T + j\omega I_n)^{-1} (A_{11} - j\omega I_n)^{-1} q_1 \geq 0 \quad \forall \omega \in [0, +\infty]. \end{aligned}$$

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Then

$$\Theta(x) = \left(x^T H_1 x + \psi_1 \int_0^{\sigma_1^0} \Phi_1(\sigma_1^0) d\sigma_1^0 \right)^{1/2}$$

is the Liapunov function for degenerate system (3.4.3) for any Φ_i taking the values in $[0, K_1]$, where H_1 is a solution of the equations

$$(3.4.5) \quad A_{11}^T H_1 + H_1 A_{11} + q_1 q_1^T = -\varepsilon_1 I_1, \quad h_1 + H_1 q_1 = -\sqrt{\gamma} q_1$$

for

$$(3.4.6) \quad \gamma = k_1^{-1} - \zeta_1 c_{11}^T q_1, \quad h_1 = \frac{1}{2}(\psi_1 A_{11}^T c_{11} + c_{11}).$$

Now we shall verify the conditions of Assumptions 3.3.1 and 3.3.2.

The verification of conditions of Assumption 3.3.1: Let H_1 and $\theta(x)$ be defined as above. Hence, the function $\theta(x)$ is decreasing positive definite on R^n and radially unbounded. We shall check up the condition (3)(a) first

(a) in this case $\theta_t = 0$ and

$$\Theta_x^T(x) f(x, 0, 0) \leq -\frac{1}{2} \varepsilon_1 \eta_2^{-1} \|x\| \quad \forall (x \neq 0) \in R^n,$$

where $\eta_2 = \Lambda^{1/2}(H_1 + \frac{1}{2} \psi_1 k_1 c_{11} c_{11}^T)$ and $\Lambda(\cdot)$ is a maximal eigenvalue of matrix (\cdot) . Hence

$$\varphi(x) = \eta_3 \|x\|, \quad \eta_3 = \frac{1}{2} \varepsilon_1 \eta_2^{-1}$$

and

$$\theta_t + \theta_x^T f(x, 0, 0) \leq -\varphi(x) \quad \forall (x \neq 0) \in R^n$$

and, besides, $\mathcal{N}_x = R^n$, $\mathcal{N}_{x0} = \{x: x \neq 0, x \in R^n\}$;
 (b) for the function $\theta(x)$ we have

$$\begin{aligned} \theta_x^T [f(x, y, \mu) - f(x, 0, 0)] &= \frac{1}{2} \theta(x) x^T \left(2H_1 + \zeta \frac{\Phi_1(\sigma_1^0)}{\sigma_1^0} c_{11} c_{11}^T \right) \\ &\times \{ A_{12} y + q_1 [\Phi_1(\sigma) - \Phi_1(\sigma_1^0)] \} \leq \zeta_1 \varphi(x) + \zeta_2 \psi(y), \\ &\forall x \in \mathcal{N}_{x0}, \quad \forall y \in R^m \quad \forall \mu \in (0, 1]. \end{aligned}$$

Incidentally

$$\psi(y) = \rho_3 \|y\|, \quad \zeta_1 = k_1 (\eta_1 \eta_3)^{-1} \eta_2 \|q_1\| \|c_{11}\|$$

and

$$\zeta_2 = (\eta_1 \rho_3)^{-1} \eta_2 (k_1 \|c_{12}\| \|q_1\| + \|A_{12}\|), \quad \eta_1 = \lambda^{1/2}(H_1),$$

where $\lambda(\cdot)$ is a minimal eigenvalue of matrix (\cdot) . The value $\rho_3 > 0$ will be defined below. The numbers ζ_1 and ζ_2 and the functions θ , φ and ψ satisfy the conditions of Assumption 3.3.1.

The verification of the conditions of Assumption 3.3.2: We take the function $V(y) = \|y\|$ as the auxiliary function. This choice shows the alternative to the choice of the Liapunov functions. The function V is decreasing positive definite in R^n and radially unbounded. In order to verify condition (3)(a) of Assumption 3.3.2, we present the system of the boundary layer in the form suggested by Rosenbrok

$$\frac{dy}{d\tau} = D_{22}(\alpha_2)y,$$

where

$$D_{22}(\alpha_2) = A_{22} + \alpha_2(\sigma_2^0) q_2 c_{22}^T, \quad \alpha_2(\sigma_2^0) = \frac{\Phi_2(\sigma_2^0)}{\sigma_2^0}.$$

The matrix $\hat{D}_2(\alpha_2) = D_{22}^T(\alpha_2) + D_{22}(\alpha_2)$ is negative definite for each $(\sigma, \varphi_2) \in R \times \mathcal{N}_0([0, K_2])$ iff $D_{22}(0)$ and $D_{22}(K)$ are negative definite. In the case under consideration this assumption is fulfilled. At last $\psi(y) = \rho_3 \|y\|$ and $V_y^T g(\alpha, b, y, 0) \leq -\psi(y) \quad \forall (y \neq 0) \in R^m$ ensure the satisfaction of condition (3)(a).

For condition (3)(a) we have

$$\begin{aligned} V_y^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] &= \frac{1}{V} y^T \{ \mu A_{21} b + q_2 [\Phi_2(\sigma_2) - \Phi_2(\sigma_2^0)] \} \\ &\forall (y \neq 0) \in R^m. \end{aligned}$$

Let

$$\xi_1 = 2\varepsilon_1\eta_2 \sup_{\alpha \in [0, k_2]} \|A_{21} + \alpha q_2 c_{21}^T\|,$$

$$\xi_2 = k_2 \|q_2 c_{21}^T\| \rho_3^{-1}.$$

We assume that $\xi_2 < 1$, then

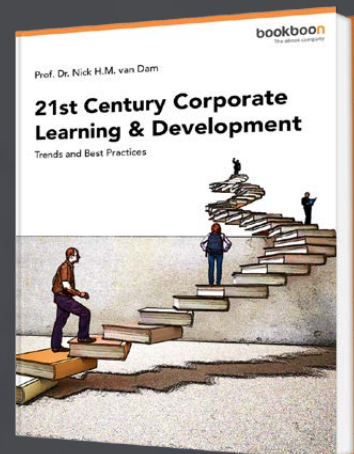
$$V_y^T [g(\alpha, b, y, \mu) - g(\alpha, b, 0, 0)] \leq \xi_1 \mu \varphi(b) + \xi_2 \psi(y)$$

$$\forall (\alpha, b, y, \mu) \in R \times R^n \times R^m \times (0, \infty).$$

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This corresponds to condition (3) in Assumption 3.3.2 for $\pi = 1$. Checking up condition (3)(c) we take into account that $V_\alpha \equiv 0$ and $V_b \equiv 0$ and, therefore, $\xi_3 = 0$ and $\xi_4 = 0$. The lower estimate of the upper bound of the parameter μ changes and has the form

$$\tilde{\mu} = \frac{1 - \xi_2}{\zeta_2}.$$

Now the inequality $1 > \zeta_1 + \xi_1$ ensures absolute stability of the state $z = (x^T, y^T)^T = 0$ of system (3.4.1), (3.4.2).

EXAMPLE 3.4.1. Let

$$A_{11} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 \\ 10^{-1} \end{pmatrix}, \quad c_{11} = \begin{pmatrix} -10^{-2} \\ 0 \end{pmatrix},$$

$$A_{12} = I, \quad c_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad k_1 = 2$$

and

$$A_{21} = 10^{-3}I_2, \quad c_{21} = \begin{pmatrix} 10^{-3} \\ 0 \end{pmatrix}, \quad k_2 = 1,$$

$$A_{22} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this example we take $\psi_1 = 1$, $\varepsilon_1 = 10^{-1}$ so that

$$\frac{1}{k_1} + \operatorname{Re}(1 + j\psi_1\omega)c_{11}^T(A_{11} - j\omega I_2)^{-1}q_1$$

$$- \varepsilon_1 q_1^T(A_{11}^T + j\omega I_2)^{-1}(A_{11} - j\omega I_2)^{-1}q_1 \equiv \frac{1}{k_1} > 2.$$

Further

$$g_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

is defined from the equation

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} + \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the form

$$H_1 = \frac{1}{20} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence $\eta_1 = 0.16$ and $\eta_2 = 0.45$. The matrix $\hat{D}_{22}(\alpha_2)$ reads

$$\hat{D}_{22}(\alpha_2) = \begin{pmatrix} -8 + 2\alpha_{22} & 2 + \alpha_{22} \\ 2 + \alpha_{22} & -8 \end{pmatrix}.$$

The matrices \hat{D}_{22} and $\hat{D}_{22}(1)$ are negative definite. Finally, $\zeta_1 = 0.05$, $\zeta_2 = 1.88$, $\xi_1 = 0.02$ and $\xi_2 = 0.002$. Therefore $\tilde{\mu} = 0.52$. Since $\zeta_1 + \xi_1 = 0.53$ is smaller than 1, the state $z = (x^T, y^T)^T = 0$ of the system defined in this example is absolutely stable for each $\mu \in (0; \tilde{\mu})$, i.e. $\mu \in (0; 0.52)$ on $\mathcal{N}_0(L)$, $L = [0, K]$, $K = \text{diag}(2, 1)$. The advantage of the separation of the time-scales in this example is that the order of the system in question is diminished. Namely, instead of the system of the fourth order one investigates two systems of the second order and verifies the inequality $1 > \zeta_1 + \xi_1$. Moreover, the lowering of the order of the systems simplifies the construction of the Liapunov functions.

However, the dimensions m and n of the reduced systems (3.2.5) and (3.4.3) and the systems of the boundary layer (3.2.6) and (3.4.4) are high enough so that one faces the problem of the lowering their order again.

3.5 The Property of Having a Fixed Sign of Matrix-Valued Function

Alongside the system (3.2.1)–(3.2.2) we shall consider first a more simple case.

3.5.1 Case A.

Let perturbed motion equations be given in the form

$$(3.5.1) \quad \frac{dx}{dt} = f(t, x, y),$$

$$(3.5.2) \quad \mu \frac{dy}{dt} = g(t, x, y),$$

where $x \in R^n$, $y \in R^m$, $f \in C(R \times R^n \times R^m, R^n)$ and $g \in C(R \times R^n \times R^m, R^m)$, $\mu \in \mathcal{M}$. For $\mu = 0$ we obtain from (3.5.1) and (3.5.2)



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$$(3.5.3) \quad \frac{dx}{dt} = f(t, x, y),$$

$$(3.5.4) \quad 0 = g(t, x, y),$$

Assume that $g(t, x, y)$ vanishes if and only if $y = 0$. Then we get from system (3.5.3)–(3.5.4) the system

$$(3.5.5) \quad \frac{dx}{dt} = f(t, x, 0),$$

which describes slow motions in system (3.5.1)–(3.5.2). The quick system (boundary layer) corresponding to system (3.5.2) has the form

$$(3.5.6) \quad \frac{dy}{dt} = g(\alpha, b, y),$$

where $\tau = (t - t_0)\mu^{-1}$, α and b are the same as in system (3.2.6).

We define the functions

$$f^*(t, x, y) = f(t, x, y) - f(t, x, 0);$$

$$g^*(t, x, y) = g(t, x, y) - g(\alpha, b, y).$$

and represent system (3.5.1)–(3.5.2) as

$$(3.5.7) \quad \begin{aligned} \frac{dx}{dt} &= f(t, x, 0) + f^*(t, x, y), \\ \mu \frac{dy}{dt} &= g(\alpha, b, y) + g^*(t, x, y). \end{aligned}$$

In order to investigate systems (3.5.1) and (3.5.2) with subsystems (3.5.5) and (3.5.6) we shall consider the matrix-valued function

$$(3.5.8) \quad U(t, x, y, \mu) = \begin{pmatrix} v_{11}(t, x) & v_{12}(t, x, y, \mu) \\ v_{21}(t, x, y, \mu) & v_{22}(t, y, \mu) \end{pmatrix}.$$

The elements v_{11} and v_{22} of matrix U corresponds to the subsystems (3.5.5) and (3.5.6) and functions $v_{12} = v_{21}$ are responsible for the interconnections of the subsystems. Using the matrix-valued function $U(t, x, y, \mu)$ we introduce the scalar function

$$(3.5.9) \quad V(t, x, y, \mu) = w^T U(t, x, y, \mu) w,$$

where $w \in R^2$.

DEFINITION 3.5.1. The matrix-valued function $U: R_+ \times R^m \times R^n \times \mathcal{M} \rightarrow R^{2 \times 2}$ is referred to as

- (i) *positive definite*, iff there exist connected neighborhoods \mathcal{N}_x and \mathcal{N}_y of points $x = 0$ and $y = 0$ $\mathcal{N}_x \subseteq R^m$, $\mathcal{N}_y \subseteq R^n$ such that
 - (a) $U \in C(R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}, R^{2 \times 2})$
 - (b) $U(t, 0, 0, \mu) = 0 \quad \forall t \in R_+, \forall \mu \in \mathcal{M};$
 - (c) $w^T U(t, x, y, \mu) w > u(x, y) \quad \forall (t, x \neq 0, y \neq 0, w \neq 0) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \times R^2;$
- (ii) *positive definite on S* iff the conditions of Definition 3.5.1, (i) are satisfied on $\mathcal{N}_x \times \mathcal{N}_y = S$;
- (iii) *positive definite in the whole*, iff all conditions of Definition 3.5.1, (i) are satisfied for $\mathcal{N}_x \times \mathcal{N}_y = R^m \times R^n$.

REMARK 3.5.1. It can be easily seen that this definition of the property of having a fixed sign of matrix-valued function U agrees with the well-known notions such as

- (i) positive definiteness of the numerical matrix;
- (ii) positive definiteness of the scalar Liapunov function;
- (iii) conceptual applicability of function (3.5.9) in the construction of the direct Liapunov's method of motion stability investigation.

In many problems of stability it is sufficient to use a fixed vector $\eta \in R^2$ (or $\eta \in R_+^2$) instead of the vector in formula (3.5.9).

Let $\eta = (\eta_1, \eta_2)^T$, $\eta_i > 0$, $i = 1, 2$ then

$$(3.5.10) \quad V(t, x, y, \mu) = \eta^T U(t, x, y, \mu) \eta.$$

DEFINITION 3.5.2. The matrix-valued function $U \in C(R_+ \times R^m \times R^n \times \mathcal{M}, R^{2 \times 2})$ is called

- (i) η -positive definite, iff there exist connected neighborhoods \mathcal{N}_x and \mathcal{N}_y of points $x = 0$ and $y = 0$, $\mathcal{N}_x \subseteq R^m$, $\mathcal{N}_y \subseteq R^n$ such that
 - (a) $U \in C(R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}, R^{2 \times 2})$;
 - (b) $U(t, 0, 0, \mu) = 0 \quad \forall t \in R_+, \quad \forall \mu \in \mathcal{M}$;
 - (c) $\eta^T U(t, x, y, \mu) \eta > u(x, y) \quad \forall (x \neq 0, y \neq 0) \in \mathcal{N}_x \times \mathcal{N}_y, \quad \forall (t, \mu) \in R_+ \times \mathcal{M}$;
- (ii) η -positive definite on S , iff all conditions of Definition 3.5.2 (i) are satisfied for $\mathcal{N}_x \times \mathcal{N}_y = S$;
- (iii) η -positive definite in the whole, iff all conditions of Definition 3.5.2 (i) are satisfied for $\mathcal{N}_x \times \mathcal{N}_y = R^m \times R^n$.



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Definition 3.5.2 agrees with points (i)–(iii) of Remark 3.5.1. In particular, the vector η can be unique, i.e. $\eta_i = 1$ and $i = 1, 2$.

REMARK 3.5.2. The definitions of positive semi-definiteness and η -positive semi-definiteness of matrix-valued function U are introduced on the basis of Definitions 3.5.1 and 3.5.2, in conditions (c) of which the $u(x, y)$ should be replaced by ≥ 0 .

REMARK 3.5.3. Functions (3.5.9) and (3.5.10) can be also constructed in the form

$$V(t, x, y, \mu, w) = w^T U^T U w, \quad w \in R^2$$

or

$$V(t, x, y, \mu) = \eta^T U^T U \eta, \quad \eta \in R_+^2.$$

In addition, the requirements to the elements of matrix-valued function U satisfying the conditions of Definitions 3.5.1 and 3.5.2 can be weakened.

The algebraic conditions of the property of having a fixed sign of function (3.5.10) are formulated in terms of the assumptions on elements $v_{ij}(t, \cdot)$ of the matrix-valued function U .

ASSUMPTION 3.5.1. There exist functions $v_{11}(t, x)$, $v_{22}(t, y, \mu)$, $v_{12}(t, x, y, \mu)$, functions φ_i and ψ_i of class $K(KR)$, $i = 1, 2$ and constants $\underline{\alpha}_{ii} > 0$, $\bar{\alpha}_{ii} > 0$, $i = 1, 2$ and $\underline{\alpha}_{12}$, $\bar{\alpha}_{12}$ such that

- (1) $\underline{\alpha}_{11}\varphi_1^2(\|x\|) \leq v_{11}(t, x) \leq \bar{\alpha}_{11}\varphi_2^2(\|x\|)$
 $\forall(t, x) \in R_+ \times \mathcal{N}_x$ ($\forall(t, x) \in R_+ \times R^m$);
- (2) $\mu\underline{\alpha}_{22}\psi_1^2(\|y\|) \leq v_{22}(t, y, \mu) \leq \mu\bar{\alpha}_{22}\psi_2^2(\|y\|)$
 $\forall(t, y, \mu) \in R_+ \times \mathcal{N}_y \times \mathcal{M}$ ($R_+ \times R^n \times \mathcal{M}$);

- (3) $\mu\underline{\alpha}_{12}\varphi_1(\|x\|)\psi_1(\|y\|) \leq v_{12}(t, x, y, \mu) \leq \mu\bar{\alpha}_{12}\varphi_2(\|x\|)\psi_2(\|y\|)$
 $\forall(t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \quad (R_+ \times R^m \times R^n \times \mathcal{M});$
- (4) $v_{12}(t, x, y, \mu) = v_{21}(t, x, y, \mu)$
 $\forall(t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \quad (R_+ \times R^m \times R^n \times \mathcal{M}).$

The following assertion is valid.

PROPOSITION 3.5.1. *If for the elements $v_{ij}(t, \cdot)$, $i, j = 1, 2$, of matrix-valued function (3.5.8) the conditions of Assumption 3.5.1 are satisfied, then function (3.5.10) satisfies the bilateral estimate*

$$(3.5.11) \quad u_1^T A(\mu)u_1 \leq V(t, x, y, \mu) \leq u_2^T B(\mu)u_2$$

$$\forall(t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \quad (R_+ \times R^m \times R^n \times \mathcal{M}),$$

where $u_1^T = (\varphi_1, \psi_1)$, $u_2^T = (\varphi_2, \psi_2)$,

$$A(\mu) = H^T A_1(\mu)H, \quad B(\mu) = H^T A_2(\mu)H, \quad H = \text{diag}(\eta_1, \eta_2); \quad \eta_1, \eta_2 > 0,$$

$$A_1(\mu) = \begin{pmatrix} \underline{\alpha}_{11} & \mu\underline{\alpha}_{12} \\ \mu\underline{\alpha}_{21} & \mu\underline{\alpha}_{22} \end{pmatrix}, \quad A_2(\mu) = \begin{pmatrix} \bar{\alpha}_{11} & \mu\bar{\alpha}_{12} \\ \mu\bar{\alpha}_{21} & \mu\bar{\alpha}_{22} \end{pmatrix},$$

$$\underline{\alpha}_{12} = \underline{\alpha}_{21}; \quad \bar{\alpha}_{12} = \bar{\alpha}_{21}.$$

PROOF. We get the estimate from above in inequality (3.5.11). In view of expression (3.5.10) and inequalities (1)–(4) of Assumption 3.5.1 we have

$$V(t, x, y, \mu) \leq \begin{pmatrix} \eta_1 & \varphi_2 \\ \eta_2 & \psi_2 \end{pmatrix}^T \begin{pmatrix} \bar{\alpha}_{11} & \mu\bar{\alpha}_{12} \\ \mu\bar{\alpha}_{21} & \mu\bar{\alpha}_{22} \end{pmatrix} \begin{pmatrix} \eta_1 & \varphi_2 \\ \eta_2 & \psi_2 \end{pmatrix}$$

or

$$V(t, x, y, \mu) \leq \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix}^T \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}^T \begin{pmatrix} \bar{\alpha}_{11} & \mu\bar{\alpha}_{12} \\ \mu\bar{\alpha}_{21} & \mu\bar{\alpha}_{22} \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix}.$$

Hence, in view of the designations adopted in Proposition 3.5.1 we get the estimate from above in inequality (3.5.11). The estimate from below is obtained in the same way.

3.6 Matrix-Valued Liapunov Function

The conception of the property of having a fixed sign of matrix-valued function admitted in Definitions 3.5.1 and 3.5.2 allows us to introduce the matrix-valued Liapunov function in the following way. We introduce the designations

$$D^*V(t, x, y, \mu, w) = w^T D^*U(t, x, y, \mu)w,$$

$$D^*U(t, x, y, \mu) = [D^*v_{ij}(t, \cdot)]; \quad i, j = 1, 2.$$

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The sign D^*U shows that both derivatives D^+U and D_+U can be used, where

$$D^+U(t, x, y, \mu) = \limsup\{[U(t + \theta, x(t + \theta, \cdot), y(t + \theta, \cdot), \mu) - U(t, x, y, \mu)]\theta^{-1} : \theta \rightarrow 0^+\};$$

$$D_+U(t, x, y, \mu) = \liminf\{[U(t + \theta, x(t + \theta, \cdot), y(t + \theta, \cdot), \mu) - U(t, x, y, \mu)]\theta^{-1} : \theta \rightarrow 0^+\}.$$

In this notation D^+U (D_+U) is the upper (lower) right-side Dini derivative of matrix-valued function U relatively (t, x, y) .

DEFINITION 3.6.1. Matrix-valued function $U: R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \rightarrow R^{2 \times 2}$ is referred to as

- (i) *matrix-valued Liapunov function of the $\mathcal{S}(w)$ type*, if
 - (a) the matrix-valued function $U(t, x, y, \mu)$ is positive definite and decreasing on $R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M} \rightarrow R^{2 \times 2}$;
 - (b) the matrix-valued function $D^*U(t, x, y, \mu)$ is nonpositive on $R_+ \times \mathcal{N}_x \times \mathcal{N}_y$ for $\mu \in (0, \mu_0)$ and as $\mu \rightarrow 0$ and $D^*U(t, 0, 0, \mu) = 0$ for all $t \in R_+$;
- (ii) *matrix-valued Liapunov function of $AS(w)$ type*, if
 - (a) the matrix-valued function $U(t, x, y, \mu)$ is positive definite and decreasing on $R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;
 - (b) the matrix-valued function $D^*U(t, x, y, \mu)$ is strictly negative on $R_+ \times \mathcal{N}_{x0} \times \mathcal{N}_{y0}$ for $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$ and $D^*(t, 0, 0, \mu) = 0$ for $t \in R_+$, $\mathcal{N}_{x0} = \{(x \neq 0) \in \mathcal{N}_x\}$, $\mathcal{N}_{y0} = \{(y \neq 0) \in \mathcal{N}_y\}$.
- (iii) *matrix-valued Liapunov-Chetayev function of $NS(w)$ type*, if there exist a $t_0 \in (\tau, \infty)$, $\tau \in R$, some value $\varepsilon > 0$ ($\overline{B}_\varepsilon \subset \mathcal{N}_x \times \mathcal{N}_y$)

and an open set $D \in B_\varepsilon$ such that on $[t_0, \infty) \times D$ the following conditions are satisfied

- (a) $0 < U(t, x, y, \mu) \leq Q < \infty$ component wise, there Q is a 2×2 matrix;
- (b) $w^T D^+ U(t, x, y, \mu) w \geq a(V(t, x, y, \mu, w))$ for $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$, where a is of class K , and moreover
- (c) $(x = 0, y = 0) \in \partial \mathcal{D}$;
- (d) $U(t, x, y, \mu) = 0$ on $[t_0, \infty) \times (\partial \mathcal{D} \cap B_\varepsilon)$.

The definitions of η – matrix-valued function are formulated in a similar manner, using the definitions of η -positive definiteness of matrix-valued function $U(t, x, y, \mu)$.

ASSUMPTION 3.6.1. There exist

- (1) functions $\varphi_i, \psi_i, i = 1, 2$ and $v_{sk}; s, k = 1, 2$ mentioned in Assumption 3.5.1 and, moreover
 - (a) function $v_{11}(t, x) \in C(R_+ \times \mathcal{N}_{x0}, R_+)$;
 - (b) function $v_{22}(t, y, \mu) \in C(R_+ \times \mathcal{N}_{y0} \times \mathcal{M}, R_+)$;
 - (c) function $v_{12}(t, x, y, \mu) \in C(R_+ \times \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}, R)$,
- (2) constants ρ_{ij} ($i = 1, 2; j = 1, \dots, 8$) and the following conditions are satisfied
 - (a) $D_t^+ v_{11} + (D_x^+ v_{11})^T f(t, x, 0) \leq \rho_{11} \varphi_2^2(\|x\|)$
 $\forall (t, x) \in R_+ \times \mathcal{N}_x$;
 - (b) $D_t^+ v_{22} + (D_y^+ v_{22})^T g(\alpha, b, y, 0) \leq \mu \rho_{21} \psi_2^2(\|y\|)$
 $\forall (\alpha, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;
 - (c) $(D_x^+ v_{11})^T [f(t, x, y) - f(t, x, 0)] \leq \rho_{12} \varphi_2^2(\|x\|)$
 $+ \rho_{13} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (t, x, y) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y$;
 - (d) $(D_y^+ v_{22})^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] \leq \rho_{22} \psi_2^2(\|y\|)$
 $+ \mu \rho_{23} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (t, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;
 - (e) $D_t^+ v_{12} + (D_x^+ v_{12})^T f(t, x, 0) \leq \mu \rho_{14} \varphi_2^2(\|x\|)$
 $+ \mu \rho_{15} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;
 - (f) $(D_x^+ v_{12})^T [f(t, x, y) - f(t, x, 0)] \leq \mu \rho_{16} \varphi_2^2(\|x\|)$
 $+ \mu \rho_{17} \varphi_2(\|x\|) \psi_2(\|y\|) + \mu \rho_{18} \psi_2^2(\|y\|)$
 $\forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;
 - (g) $(D_y^+ v_{12})^T g(\alpha, b, y, 0) \leq \mu \rho_{24} \psi_2^2(\|y\|) + \mu \rho_{25} \varphi_2(\|x\|) \psi_2(\|y\|)$
 $\forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$;

$$\begin{aligned}
 \text{(h)} \quad & (D_y^+ v_{12})^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] \leq \mu \rho_{26} \varphi_2^2(\|x\|) \\
 & + \mu \rho_{27} \varphi_2(\|x\|) \psi_2(\|y\|) + \mu \rho_{28} \psi_2^2(\|y\|) \\
 & \forall (t, b, y, \mu) \in \mathbb{R}_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}.
 \end{aligned}$$


PROPOSITION 3.6.1. *If all conditions of Assumption 3.6.1 are satisfied, then for the upper right Dini derivative of function (3.5.10) the upper estimate*



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$$(3.6.1) \quad D^+V(t, x, y, \mu) \leq u_2^T C(\mu) u_2 \quad \forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$$

is satisfied, where

$$C(\mu) = [c_{ij}(\mu)], \quad c_{12}(\mu) = c_{21}(\mu); \quad i, j = 1, 2,$$

and

$$\begin{aligned} c_{11}(\mu) &= \eta_1^2(\rho_{11} + \rho_{12}) + 2\eta_1\eta_2(\mu\rho_{14} + \mu\rho_{16} + \rho_{26}); \\ c_{22}(\mu) &= \eta_2^2(\rho_{22} + \rho_{21}) + 2\eta_1\eta_2(\mu\rho_{18} + \mu\rho_{24} + \rho_{28}); \\ c_{12}(\mu) &= \frac{1}{2}(\eta_1^2\rho_{13} + \eta_2^2\rho_{23}) + \eta_1\eta_2(\mu\rho_{15} + \rho_{25} + \mu\rho_{17} + \rho_{27}). \end{aligned}$$

PROOF. In view of the fact that

$$D^+V(t, \cdot) = \eta^T D^+[v_{ij}(t, \cdot)]\eta, \quad i, j = 1, 2$$

the estimates (a)–(h) for the elements of matrix $U(t, x, , \mu)$ lead to inequality (3.6.1).

We introduce the values μ_j , $j = 1, \dots, 4$, μ_0 , μ^* by the formulas

$$\begin{aligned} \mu_1 &= -\frac{2\eta_2\rho_{26} + \eta_1(\rho_{11} + \rho_{12})}{2\eta_2(\rho_{14} + \rho_{16})}; \\ \mu_2 &= -\frac{\eta_2(\rho_{21} + \rho_{22}) + 2\eta_1(\rho_{24} + \rho_{28})}{2\eta_1\rho_{18}}; \\ \mu_3 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \mu_4 = \frac{\alpha_{11}\alpha_{22}}{\alpha_{12}^2}, \\ \mu_0 &= \min(\mu_1, \mu_2, \mu_3), \quad \mu^* = \min(\mu_0, \mu_4). \end{aligned}$$

Besides,

$$\begin{aligned}
 a &= 4\eta_1^2\eta_2^2[(\rho_{15} + \rho_{17})^2 - \rho_{18}(\rho_{14} + \rho_{16})]; \\
 b &= \left[\frac{1}{2}\eta_1^2\rho_{13} + \frac{1}{2}\eta_2^2\rho_{23} + \eta_1\eta_2(\rho_{25} + \rho_{27}) \right] 2\eta_1\eta_2(\rho_{15} + \rho_{17}) \\
 &\quad - 2\eta_1\eta_2\rho_{18}[\eta_1^2(\rho_{11} + \rho_{12}) + 2\eta_1\eta_2\rho_{26}] \\
 &\quad - 2\eta_1\eta_2(\rho_{14} + \rho_{16})[\eta_2^2(\rho_{21} + \rho_{22}) + 2\eta_1\eta_2(\rho_{24} + \rho_{28})]; \\
 c &= \left[\frac{1}{2}\eta_1^2\rho_{13} + \frac{1}{2}\eta_2^2\rho_{23} + \eta_1\eta_2(\rho_{25} + \rho_{27}) \right]^2 \\
 &\quad - [\eta_1^2(\rho_{11} + \rho_{12}) + 2\eta_1\eta_2\rho_{26}][\eta_2^2(\rho_{21} + \rho_{22}) + 2\eta_1\eta_2(\rho_{24} + \rho_{28})].
 \end{aligned}$$

Should $\mu_0 > 1$, we shall consider $\mu \in (0, 1]$.

PROPOSITION 3.6.2. *The matrix $C(\mu)$ is negative definite for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$, provided that*

- (a) $\rho_{14} + \rho_{16} > 0$;
- (b) $2\eta_2\rho_{26} + \eta_1(\rho_{11} + \rho_{12}) < 0$;
- (c) $\eta_2(\rho_{26} + \rho_{22}) + 2\eta_1(\rho_{24} + \rho_{28}) < 0$;
- (d) $\rho_{18} > 0$;
- (e) $a > 0$;
- (f) $c < 0$.

PROOF. Conditions (a) and (b) imply that $c_{11} < 0$ for every $\mu \in (0, \mu_1)$ and $\mu \rightarrow 0$; conditions (c) and (d) imply that $c_{22} < 0$ for every $\mu \in (0, \mu_2)$ and for $\mu \rightarrow 0$; and conditions (e) and (f) imply that $c_{11}c_{22} - c_{12}^2 > 0$ for every $\mu \in (0, \mu_3)$ and for $\mu \rightarrow 0$.

All these conditions hold for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$, where $\mu_0 = \min(\mu_1, \mu_2, \mu_3)$. The conditions are sufficient for the matrix $C(\mu)$ negative definite.

REMARK 3.6.1. If for conditions (a)–(c), (e) and (f) Proposition 3.6.2 is satisfied and $\rho_{18} \leq 0$, then its assertion is true for $\mu_0 = \min(\mu_1, \mu_3)$.

REMARK 3.6.2. If for conditions (b)–(f) Proposition 3.6.2 is satisfied and $\rho_{14} + \rho_{16} \leq 0$, then its assertion is true for $\mu_0 = \min(\mu_2, \mu_3)$.

REMARK 3.6.3. If for conditions (b), (c), (e) and (f) Proposition 3.6.2 is satisfied and $\rho_{18} \leq 0$, $\rho_{14} + \rho_{16} \leq 0$, then its assertion is true for $\mu_0 = \mu_3$.

We note that the quadratic form $u_2^T C(\mu) u_2$ is given in the cone R_+^2 formed by the functions (φ_2, ψ_2) . Therefore the following result is valid.

PROPOSITION 3.6.3. *The matrix $C(\mu)$ is conditionally negative definite, i.e. $u_2^T C(\mu) u_2 < 0$ for $u_2 \in \mathbb{R}_+^2 \setminus 0$, for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$, if*

$$\max(\det[-C(\mu)], c_{12}(\mu)) > 0$$

for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.

Estimates (3.5.11) and (3.6.1) allow us to formulate the generalizations of the classical results on stability and instability of unperturbed motion of system (3.5.1), (3.5.2) as follows.

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3.7 General Theorems on Stability and Instability in Case A

The equilibrium state $(x^T, y^T)^T = 0$ of system (3.5.1), (3.5.2) is investigated by means of function (3.5.10) being a special case of function (3.5.9). Estimates (3.5.11) and (3.6.1) allows us to formulate algebraic conditions ensuring the presence of some properties of the equilibrium state $(x^T, y^T)^T = 0$.

THEOREM 3.7.1. *Let the motion $(x^T(t; t_0, x_0, \mu), y^T(t; t_0, y_0, \mu))^T$ of system (3.5.1), (3.5.2) be continuous for $(t_0, x_0, y_0) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y$ and $\mu \in \mathcal{M}^0 \subset \mathcal{M}$. In order that the equilibrium state $(x^T, y^T)^T = 0$ of system (3.5.1), (3.5.2) be uniformly stable for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$ it is necessary that all conditions of Assumptions 3.5.1 and 3.6.1 be satisfied and it is sufficient that*

- (1) *the matrices $A_1(\mu) + A_1^T(\mu)$ and $A_2(\mu) + A_2^T(\mu)$ be conditionally positive;*
- (2) *the matrix $C(\mu)$ be non-positive for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.*

If in addition, $\mathcal{N}_x \times \mathcal{N}_y = R^{m+n}$, then the equilibrium state $(x^T, y^T)^T = 0$ is uniformly stable in the whole for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.

PROOF. Estimate (3.5.11) implies that if Assumption 3.5.1 and condition (1) of Theorem 3.7.1 hold, the function $V(t, x, y, \mu)$ is definite positive and decreasing. The conditions of Assumption 3.6.1 and condition (2) of Theorem 3.7.1 ensure nonpositiveness of function $D^+V(t, x, y, \mu)$ on $R_+ \times \mathcal{N}_x \times \mathcal{N}_y$ for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$. The combination of this conditions is equivalent to the conditions of Liapunov's theorem on stability of the equilibrium state $(x^T, y^T)^T = 0$ (see Liapunov [101], and Grujić, Martynuk and Ribbens-Pavella [57]).

If $\mathcal{N}_x \times \mathcal{N}_y = R^{m+n}$, then functions (φ_i, ψ_i) belong to class KR and estimates (3.5.11) and (3.6.1) are satisfied for all $(x, y) \in R^{m+n}$. Together with conditions (1) and (2) of Theorem 3.7.1 this ensures stability in the whole of the state $(x^T, y^T)^T = 0$.

The theorem is proved.

THEOREM 3.7.2. *Let the motion $(x^T(t; t_0, x_0, \mu), y^T(t; t_0, y_0, \mu))^T$ of system (3.5.1) and (3.5.2) be continuous for $(t_0, x_0, y_0) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y$ and $\mu \in \mathcal{M}^0 \subset \mathcal{M}$. For the equilibrium state $(x^T, y^T)^T = 0$ of system (3.5.1), (3.5.2) be uniform asymptotically stable for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$ it is necessary that all conditions of Assumptions 3.3.1 and 3.3.2 be satisfied and it is sufficient that*

- (1) *the matrices $A_1(\mu) + A_1^T(\mu)$ and $A_2(\mu) + A_2^T(\mu)$ be conditionally positive;*
- (2) *the matrix $C(\mu)$ be conditionally negative for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.*

If, in addition, $\mathcal{N}_x \times \mathcal{N}_y = R^{m+n}$, then the equilibrium state $(x^T, y^T)^T$ is uniformly asymptotically stable in the whole for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.

PROOF. The proof of Theorem 3.7.2 is similar to that of Theorem 3.7.1, taking into account that its conditions are equivalent to the conditions of the theorem on uniform asymptotic stability (Grujić, Martynyuk and Ribbens-Pavella [57]).

The theorem is proved.

PROPOSITION 3.7.1. *Let in Assumption 3.6.1 in conditions (a)–(h) the inequality sign “ \leq ” be replaced by “ \geq ”, the constants ρ_{ij} ($i = 1, 2; j = 1, \dots, 8$) be replaced by $\tilde{\rho}_{ij}$ ($i = 1, 2; j = 1, \dots, 8$) and the pair of functions (φ_2, ψ_2) be replaced by the pair of function (φ_1, ψ_1) .*

Then for the upper right-side Dini derivative of function (3.5.10) the estimate from below

$$(3.7.1) \quad D^+V(t, x, y, \mu) \geq u_1^T \tilde{C}(\mu) u_1 \quad \forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$$

is satisfied, where the matrix $\tilde{C}(\mu)$ has the same structure as the matrix $C(\mu)$.

THE PROOF of Proposition 3.7.1 is similar to that of Proposition 3.6.1.

THEOREM 3.7.3. *Let the motion $(x^T(t; t_0, x_0, \mu), y^T(t; t_0, y_0, \mu))^T$ of system (3.5.1) and (3.5.2) be continuous for $(t_0, x_0, y_0) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y$. For the equilibrium state $(x^T, y^T)^T = 0$ of system (3.5.1), (3.5.2) is unstable for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$ it is necessary that the conditions of Assumption 3.5.1 and Proposition 3.7.1 be satisfied, and it is sufficient that*

- (1) *the matrices $A_1(\mu) + A_1^T(\mu)$ and $A_2(\mu) + A_2^T(\mu)$ be conditionally positive;*
- (2) *the matrix $\tilde{C}(\mu)$ be conditionally positive for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.*

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PROOF. Due to inequality (3.5.11) and condition (1) of Theorem 3.7.3 the function $V(t, x, y, \mu)$ is positive definite and bounded for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$. Inequality (3.7.1) and condition (2) of Theorem 3.7.3 together with the above condition are equivalent to the conditions of the second Liapunov's theorem on instability (see Liapunov [101]).

This completes the proof.

3.8 General Theorems on Stability and Instability in Case B

We consider the general system (3.2.1)–(3.2.2) and matrix-valued function (3.5.8). Systems of (3.2.1)–(3.2.2) type are attributed to Case B of inclusion of a small parameter. Functions

$$\begin{aligned} f^0(t, x, y, \mu) &= f(t, x, y, \mu) - f(t, x, y, 0), \\ g^0(t, x, y, \mu) &= g(t, x, y, \mu) - g(\alpha, b, y, 0). \end{aligned}$$

are considered as perturbed systems describing slow motions and as a boundary layer of systems (3.2.5) and (3.2.6) respectively.

ASSUMPTION 3.8.1. For the systems of equations (3.2.1) and (3.2.2) all conditions of Assumption 3.5.1 are satisfied, and for function (3.5.8) estimates (3.5.11) are valid.

ASSUMPTION 3.8.2. There exist

- (1) the functions $\varphi_i, \psi_i \in K, i = 1, 2, v_{sk}, s, k = 1, 2$ mentioned in Assumption 3.6.1;
- (2) a constants $\rho_{ij} (i = 1, 2, j = 1, 2, \dots, 8)$ such that

- (a) $D_t^+ v_{11} + (D_x^+ v_{11})^T f(t, x, 0, 0) \leq \rho_{11} \varphi_2^2(\|x\|)$
 $\forall (t, x) \in R_+ \times \mathcal{N}_x;$
- (b) $D_t^+ v_{22} + (D_y^+ v_{22})^T g(\alpha, b, y, 0) \leq \mu \rho_{21} \psi_2^2(\|y\|)$
 $\forall (\alpha, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M};$
- (c) $(D_x^+ v_{11})^T [f(t, x, y, 0) - f(t, x, 0, 0)] \leq \rho_{12} \varphi_2^2(\|x\|)$
 $+ \rho_{13} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (t, x, y) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y;$
- (d) $(D_y^+ v_{22})^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] \leq \rho_{22} \psi_2^2(\|y\|)$
 $+ \mu \rho_{23} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (\alpha, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M};$
- (e) $D_t^+ v_{12} + (D_x^+ v_{12})^T f(t, x, 0, 0) \leq \mu \rho_{14} \varphi_2^2(\|x\|)$
 $+ \mu \rho_{15} \varphi_2(\|x\|) \psi_2(\|y\|) \quad \forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M};$
- (f) $(D_x^+ v_{12})^T [f(t, x, y, 0) - f(t, x, 0, 0)] \leq \mu \rho_{16} \varphi_2^2(\|x\|)$
 $+ \mu \rho_{17} \varphi_2(\|x\|) \psi_2(\|y\|) + \mu \rho_{18} \psi_2^2(\|y\|)$
 $\forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M};$
- (g) $(D_y^+ v_{12})^T g(\alpha, b, y, 0) \leq \mu \rho_{24} \psi_2^2(\|y\|) + \mu \rho_{25} \varphi_2(\|x\|) \psi_2(\|y\|)$
 $\forall (\alpha, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M};$
- (h) $(D_y^+ v_{12})^T [g(\alpha, b, y, \mu) - g(\alpha, b, y, 0)] \leq \mu \rho_{26} \varphi_2^2(\|x\|)$
 $+ \mu \rho_{27} \varphi_2(\|x\|) \psi_2(\|y\|) + \mu \rho_{28} \psi_2^2(\|y\|)$
 $\forall (\alpha, b, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}.$

PROPOSITION 3.8.1. *If all conditions of Assumption 3.8.2 are satisfied, then for the upper right Dini derivative of function (3.5.8) along a solution of (3.2.1)–(3.2.2) the upper estimate*

$$D^+ V(t, x, y, \mu) \leq u_2^T C^0(\mu) u_2 \quad \forall (t, x, y, \mu) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$$

is satisfied, where

$$C^0(\mu) = [s_{ij}(\mu)], \quad s_{12}(\mu) = s_{21}(\mu), \quad i, j = 1, 2,$$

and

$$\begin{aligned} s_{11} &= \eta_1^2(\rho_{11} + \rho_{12}) + 2\eta_1\eta_2\mu(\rho_{14} + \rho_{16} + \rho_{26}); \\ s_{22} &= \eta_2^2(\rho_{21} + \rho_{22}) + 2\eta_1\eta_2(\mu\rho_{18} + \rho_{24} + \rho_{28}); \\ s_{12} = s_{21} &= \frac{1}{2}\eta_1^2\rho_{13} + \frac{1}{2}\eta_2^2\mu\rho_{23} + \eta_1\eta_2(\mu\rho_{15} + \mu\rho_{17} + \rho_{25} + \rho_{27}). \end{aligned}$$

THE PROOF is similar to the proof of Proposition 3.6.1.

We introduce the values $\tilde{\mu}_i$, $\tilde{\mu}_0$, $\tilde{\mu}^*$, $i = 1, 2, 3, 4$ by the formulas

$$\begin{aligned}\tilde{\mu}_1 &= -\frac{\eta_1(\rho_{11} + \rho_{12})}{2\eta_2(\rho_{14} + \rho_{16} + \rho_{26})}, \\ \tilde{\mu}_2 &= -\frac{\eta_2(\rho_{21} + \rho_{22}) + 2\eta_1(\rho_{24} + \rho_{28})}{2\eta_1\rho_{18}}, \\ \tilde{\mu}_3 &= \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}, \\ \tilde{\mu}_4 &= \mu_4, \quad \tilde{\mu}_0 = \min(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3), \quad \tilde{\mu}^* = \min(\tilde{\mu}_0, \tilde{\mu}_4).\end{aligned}$$



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Moreover,

$$\begin{aligned}\tilde{a} &= \left[\frac{1}{2} \eta_2^2 \rho_{23} + \eta_1 \eta_2 (\rho_{15} + \rho_{17}) \right]^2 - 4 \eta_1 \eta_2 \rho_{18} (\rho_{14} + \rho_{16} + \rho_{26}); \\ \tilde{b} &= \left[\frac{1}{2} \eta_1^2 \rho_{13} + \eta_1 \eta_2 (\rho_{25} + \rho_{27}) \right] \left[\frac{1}{2} \eta_2^2 \rho_{23} + \eta_1 \eta_2 (\rho_{15} + \rho_{17}) \right] \\ &\quad - 2 \eta_1 \eta_2 \rho_{18} (\rho_{11} + \rho_{12}) - 2 \eta_1 \eta_2 (\rho_{14} + \rho_{16} + \rho_{26}) \\ &\quad \times [\eta_2^2 (\rho_{21} + \rho_{22}) + 2 \eta_1 \eta_2 (\rho_{24} + \rho_{28})], \\ \tilde{c} &= \left[\frac{1}{2} \eta_1^2 \rho_{13} + \eta_1 \eta_2 (\rho_{25} + \rho_{27}) \right]^2 \\ &\quad - \eta_1^2 (\rho_{11} + \rho_{12}) [\eta_2^2 (\rho_{21} + \rho_{22}) + 2 \eta_1 \eta_2 (\rho_{24} + \rho_{26})].\end{aligned}$$

PROPOSITION 3.8.2. *The matrix $C^0(\mu)$ is negative definite for every $\mu \in (0, \tilde{\mu}_0)$ and for $\mu \rightarrow 0$, provided that*

- (a) $\rho_{11} + \rho_{12} < 0$;
- (b) $\rho_{14} + \rho_{16} + \rho_{26} > 0$;
- (c) $\eta_2 (\rho_{21} + \rho_{22}) + 2 \eta_1 (\rho_{24} + \rho_{28}) < 0$;
- (d) $\rho_{18} > 0$;
- (e) $\tilde{a} > 0$;
- (f) $\tilde{c} < 0$.

THE PROOF is similar to that of Proposition 3.6.2.

REMARK 3.8.1. If conditions (a), (b), (c), (e) and (f) of Proposition 3.8.2 are satisfied and $\rho_{18} \leq 0$, then its assertion is true for $\tilde{\mu}_0 = \min(\tilde{\mu}_1, \tilde{\mu}_3)$.

REMARK 3.8.2. If conditions (a), (c), (e) and (f) of Proposition 3.8.2 are satisfied and $\rho_{14} + \rho_{16} + \rho_{26} \leq 0$, then its assertion is true for $\tilde{\mu}_0 = \min(\tilde{\mu}_2, \tilde{\mu}_3)$.

REMARK 3.8.3. If conditions (a), (c), (e) and (f) of Proposition 3.8.2 are satisfied and $\rho_{18} \leq 0$, $\rho_{14} + \rho_{16} + \rho_{26} \leq 0$, then its assertion is true for $\tilde{\mu}_0 = \tilde{\mu}_3$.

THEOREM 3.8.1. *Let motion $(x^T(t; t_0, x_0, \mu); y^T(t; t_0, y_0, \mu))^T$ of the system (3.2.1)–(3.2.2) be continuous for $(t_0, x_0, y_0) \in R_+ \times \mathcal{N}_x \times \mathcal{N}_y$ and $\mu \in M^0 \subset M$. In order that the equilibrium state $(x^T, y^T)^T = 0$ of system (3.2.1)–(3.2.2) be uniformly asymptotically stable for every $\mu \in (0, \tilde{\mu}_0)$ and for $\mu \rightarrow 0$ it is sufficient that*

- (1) *conditions of Assumptions 3.8.1 and 3.8.2 be satisfied;*
- (1) *matrices $A_1(\mu) + A_1^T(\mu)$ and $A_2(\mu) + A_2^T(\mu)$ be conditionally positive definite;*
- (2) *matrix $C^0(\mu)$ be negative definite for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.*

If, moreover, $\mathcal{N}_x \times \mathcal{N}_y = R^{m+n}$, functions $\varphi_i, \psi_i \in KR$, $i = 1, 2$, then the equilibrium state $(x^T, y^T)^T = 0$ of the system (3.2.1)–(3.2.2) is uniformly asymptotically stable in the whole.

THE PROOF is similar to that of Theorem 3.7.1.

Sufficient instability conditions for state $(x^T, y^T)^T = 0$ of the system (3.2.1)–(3.2.2) are established in the same way as in Theorem 3.7.2.

3.9 Asymptotic Stability of Linear Autonomous Systems

For the mentioned class of systems two cases of singular perturbation are considered.

3.9.1 Case A

Consider the system

$$(3.9.1) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y, \\ \mu \frac{dy}{dt} &= A_{21}x + A_{22}y, \end{aligned}$$

where $x \in R^n$, $y \in R^m$, A_{11} , A_{12} , A_{21} , A_{22} are constant matrices with corresponding dimensions, $\mu \in [0, 1]$ is a small parameter.

We construct the matrix-valued function (3.5.1) of elements v_{ij} ($i, j \in [1, 2]$) in the form

$$(3.9.2) \quad \begin{aligned} v_{11}(x) &= x^T B_1 x, \\ v_{22}(y, \mu) &= \mu y^T B_2 y; \\ v_{12}(x, y, \mu) &= v_{21}(x, y, \mu) = \mu x^T B_3 y. \end{aligned}$$

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Besides, matrices B_1 and B_2 are symmetric and positive definite, and B_3 is a constant matrix.

Further we need the following estimate (see Djordjević [28])

PROPOSITION 3.9.1. *Let an $A \in R^{n \times m}$ and $B \in R^{m \times r}$, $x \in R^n$, $y \in R^r$. Then the bilinear form $x^T A B y$ satisfies the bilateral estimate*

$$-\lambda_M^{1/2}(AA^T)\lambda_M^{1/2}(B^TB)\|x\| \|y\| \leq x^T A B y \leq \lambda_M^{1/2}(AA^T)\lambda_M^{1/2}(B^TB)\|x\| \|y\|,$$

where $\lambda_M(AA^T)$ and $\lambda_M(B^TB)$ are maximal eigenvalues of the matrices AA^T and B^TB respectively.

PROOF. Let $\alpha \in R$. We construct the vector

$$w = \alpha A^T x + B y,$$

and consider the inequality

$$(3.9.3) \quad w^T w \geq 0$$

Since $w^T = \alpha x^T A + y^T B^T$, then (3.9.3) is equal to

$$(3.9.4) \quad \alpha^2 x^T A A^T x + 2\alpha x^T A B y + y^T B^T B y \geq 0.$$

In order that the polynomial (3.9.4) be non-negative it is sufficient that its discriminant be non-positive. Hence, we get

$$(x^T A B y)^2 \leq (x^T A A^T x)(y^T B^T B y)$$

and

$$(3.9.5) \quad |x^T A B y| \leq (x^T A A^T x)^{1/2} (y^T B^T B y)^{1/2}.$$

Hence, it follows the estimate from Proposition 3.9.1.

COROLLARY 3.9.1. If in Proposition 3.9.1 $B = I$ (I is an identity matrix) and $r = n$, then bilateral estimate becomes

$$(3.9.6) \quad -\lambda_M^{1/2}(AA^T)\|x\| \|y\| \leq x^T A y \leq \lambda_M^{1/2}(AA^T)\|x\| \|y\|$$

for all $A \in R^n$, $x \in R^n$, $y \in R^n$.

In view of estimates typical for the quadratic forms and with regard to Corollary 3.9.1 it is easily seen that for functions (3.9.2) the following inequalities are valid:

$$(3.9.7) \quad \begin{aligned} v_{11}(x) &\geq \lambda_m(B_1)\|x\|^2 && \forall x \in \mathcal{N}_{x0}; \\ v_{22}(y, \mu) &\geq \mu\lambda_m(B_2)\|y\|^2 && \forall (y, \mu) \in \mathcal{N}_{y0o} \times \mathcal{M}; \\ v_{12}(x, y, \mu) = v_{21}(x, y, \mu) &\geq -\mu\lambda_M^{1/2}(B_3B_3^T)\|x\| \|y\| \\ &&& \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}. \end{aligned}$$

For the function $V(x, y, \mu) = \eta^T U(x, y, \mu)\eta$, $\eta \in R_+^2$, the matrix $A_1(\mu)$ from estimate (3.5.11) has the form

$$A_1(\mu) = \begin{pmatrix} \lambda_m(B_1) & -\mu\lambda_M^{1/2}(B_3B_3^T) \\ -\mu\lambda_M^{1/2}(B_3B_3^T) & \mu\lambda_m(B_2) \end{pmatrix}.$$

Since by assumption on matrix B_1 have $\lambda_m(B_1) > 0$, and then for the function $V(x, y, \mu)$ to be positive definite it is sufficient that

$$(3.9.8) \quad \lambda_m(B_1)\lambda_m(B_2) > \mu\lambda_M(B_3B_3^T)$$

for every $\mu \in (0, \mu_0^*)$ and for $\mu \rightarrow 0$.

The fact that

$$\frac{dV(x, y, \mu)}{dt} = \eta^T \frac{dU(x, y, \mu)}{dt} \eta$$

yields

$$(3.9.9) \quad \frac{1}{2} \frac{dV(x, y, \mu)}{dt} = z^T \begin{pmatrix} c_{11} & c_{12} + \mu\sigma_{12} \\ c_{12} + \mu\sigma_{12} & c_{22} + \mu\sigma_{22} \end{pmatrix} z,$$

where $z = (x^T, y^T)^T$ and

$$\begin{aligned} c_{11} &= \frac{1}{2}\eta_1^2(B_1A_{11} + A_{11}^TB_1) + \frac{1}{2}\eta_1\eta_2(B_2A_{21} + A_{21}^TB_3^T); \\ c_{22} &= \frac{1}{2}\eta_2^2(B_2A_{22} + A_{22}^TB_2), \\ \sigma_{22} &= \frac{1}{2}\eta_1\eta_2(B_3^TA_{12} + A_{12}^TB_3), \\ c_{12} &= \frac{1}{2}\eta_1^2B_1A_{12} + \frac{1}{2}\eta_1\eta_2B_3A_{22} + \frac{1}{2}\eta_2^2A_{21}^TB_2; \\ \sigma_{12} &= \frac{1}{2}\eta_1\eta_2A_{11}^TB_3, \quad \eta_1 > 0, \quad \eta_2 > 0. \end{aligned}$$

Let $\frac{d}{dt}V_M(x, y, \mu)$ be an upper bound of the expression (3.9.6). It is easy to verify that

$$(3.9.10) \quad \frac{d}{dt}V_M(x, y, \mu) \leq 2u^TC(\mu)u,$$

where $u = (\|x\|, \|y\|)^T$ and

$$C(\mu) = \begin{pmatrix} \lambda_M(c_{11}) & \lambda_M^{1/2}(c_{12}c_{12}^T) + \mu\lambda_M^{1/2}(\sigma_{12}\sigma_{12}^T) \\ \lambda_M^{1/2}(c_{12}c_{12}^T) + \mu\lambda_M^{1/2}(\sigma_{12}\sigma_{12}^T) & \lambda_M(c_{22}) + \mu\lambda_M(\sigma_{22}) \end{pmatrix}.$$

Here $\lambda_M(c_{ii})$ and $\lambda_M(\sigma_{22})$ are maximal eigenvalues of matrices c_{ii} , $i = 1, 2$ and σ_{22} respectively; and $\lambda_M^{1/2}(c_{12}c_{12}^T)$ and $\lambda_M^{1/2}(\sigma_{12}\sigma_{12}^T)$ are norms of matrices c_{12} and σ_{12} respectively.

In this case, the values μ_2 , μ_3 and μ_0 are expressed as follows

$$\mu_2 = -\lambda_M(c_{22})/\lambda_M(\sigma_{22}), \quad \mu_3 = (-b + \sqrt{b^2 - 4ac})/2a, \quad \mu_0 = \min(\mu_2, \mu_3),$$

where

$$\begin{aligned} a &= \lambda_M(\sigma_{12}\sigma_{12}^T), \\ b &= \lambda_M^{1/2}(c_{12}c_{12}^T)\lambda_M^{1/2}(\sigma_{12}\sigma_{12}^T) - \lambda_M(c_{11})\lambda_M(\sigma_{22}), \\ c &= \lambda_M(c_{12}c_{12}^T) - \lambda_M(c_{11})\lambda_M(c_{22}). \end{aligned}$$

Sufficient conditions for uniform asymptotic stability of the state $(x^T, y^T)^T = 0$ of (3.9.1) are established in terms of Theorem 3.7.1. Namely,

the equilibrium state $(x^T, y^T)^T = 0$ of system (3.9.1) is uniformly asymptotically stable in the whole if

- (1) inequality (3.9.8) is satisfied;
- (2) the following inequalities are satisfied
 - (a) $\lambda_M(c_{11}) < 0$;
 - (b) $\lambda_M(c_{22}) < 0$;
 - (c) $\lambda_M(\sigma_{22}) > 0$;
 - (d) $\lambda_M(c_{12}c_{12}^T) - \lambda_M(c_{11})\lambda_M(c_{22}) < 0$.

This assertion follows from the fact that for functions (3.9.2) under condition (3.9.8) the function $V(x, y, \mu) = \eta^T U(x, y) \eta$ is positive definite and radially unbounded, and under condition (2) $D^+V(x, y, \mu)$ along solutions of system (3.9.1) is negative definite. Therefore, all conditions of Theorem 3.7.1 are satisfied.

3.9.2 Case B

Consider the system

$$(3.9.11) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y, \\ \mu \frac{dy}{dt} &= \mu A_{21}x + A_{22}y, \end{aligned}$$

where, $x \in R^n$, $y \in R^m$, $\mu \in (0, 1]$ and matrices A_{11}, \dots, A_{22} are the same as in system (3.9.1).

In order to establish conditions for uniform asymptotic stability of equilibrium state $(x^T, y^T)^T = 0$ of system (3.9.11) we incorporate the Theorem 3.7.3. To this end we take the elements of a matrix-valued function in the form of (3.9.2) and assume that the estimate (3.5.11) is satisfied for the function $V(x, y, \mu)$.

We have for the total derivative of function $V(x, y, \mu)$ along a solutions of system (3.9.11)

$$(3.9.12) \quad \frac{1}{2} \frac{d}{dt} V(x, y, \mu) = z^T \begin{pmatrix} c_{11}^0 + \mu \sigma_{11}^0 & c_{12}^0 + \mu \sigma_{12}^0 \\ c_{12}^0 + \mu \sigma_{12}^0 & c_{22}^0 + \mu \sigma_{22}^0 \end{pmatrix} z,$$

where

$$\begin{aligned}
c_{11}^0 &= \frac{1}{2}\eta_1^2(B_1A_{11} + A_{11}^TB_1); \\
\sigma_{11}^0 &= \frac{1}{2}\eta_1\eta_2(B_3A_{21} + A_{21}^TB_3^T); \\
c_{22}^0 &= \frac{1}{2}\eta_2^2(B_2A_{22} + A_{22}^TB_2); \\
\sigma_{22}^0 &= \frac{1}{2}\eta_1\eta_2(B_3^TA_{12} + A_{12}^TB_3); \\
c_{12}^0 &= \frac{1}{2}\eta_1^2B_1A_{12} + \frac{1}{2}\eta_1\eta_2B_3A_{22}; \\
\sigma_{12}^0 &= \frac{1}{2}\eta_2^2A_{21}^TB_2 + \frac{1}{2}\eta_1\eta_2A_{11}^TB_3, \quad \eta_1 > 0, \quad \eta_2 > 0.
\end{aligned}$$

For the upper bound $\frac{d}{dt}V_M(x, y, \mu)$ of expression (3.9.12) we have the estimate

$$(3.9.13) \quad \frac{d}{dt}V_M(x, y, \mu) \leq 2u^TC^0(\mu)u,$$

where

$$C^0(\mu) = \begin{pmatrix} \lambda_M(c_{11}^0) + \mu\lambda_M(\sigma_{11}^0) & \lambda_M^{1/2}(c_{12}^0c_{12}^{0T}) + \mu\lambda_M^{1/2}(\sigma_{12}^0\sigma_{12}^{0T}) \\ \lambda_M^{1/2}(c_{12}^0c_{12}^{0T}) + \mu\lambda_M^{1/2}(\sigma_{12}^0\sigma_{12}^{0T}) & \lambda_M(c_{22}^0) + \mu\lambda_M(\sigma_{22}^0) \end{pmatrix}.$$

In this case, the values $\tilde{\mu}_i$, $i = \overline{1, 4}$, $\tilde{\mu}_0$ and $\tilde{\mu}^*$ are defined as

$$\begin{aligned}
\tilde{\mu}_1 &= -\frac{\lambda_M(c_{11}^0)}{\lambda_M(\sigma_{11}^0)} & \tilde{\mu}_2 &= -\frac{\lambda_M(c_{22}^0)}{\lambda_M(\sigma_{22}^0)}, \\
\tilde{\mu}_3 &= \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, & \tilde{\mu}_4 &= \mu_4, \\
\tilde{\mu}_0 &= \min(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3), & \tilde{\mu}^* &= \min(\tilde{\mu}_0, \tilde{\mu}_4);
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \lambda_M(\sigma_{12}^0\sigma_{12}^{0T}) - \lambda_M(\sigma_{11}^0)\lambda_M(\sigma_{22}^0); \\
b_1 &= \lambda_M^{1/2}(c_{12}^0c_{12}^{0T})\lambda_M^{1/2}(\sigma_{12}^0\sigma_{12}^{0T}) - \lambda_M(c_{11}^0)\lambda_M(\sigma_{22}^0) - \lambda_M(\sigma_{11}^0)\lambda_M(\sigma_{22}^0); \\
c_1 &= \lambda_M(c_{12}^0c_{12}^{0T}) - \lambda_M(c_{11}^0)\lambda_M(c_{22}^0).
\end{aligned}$$

According to estimates (3.5.11) and (3.9.13) for functions $V(x, y, \mu)$ and $DV_M(x, y, \mu)$ the sufficient conditions for uniform asymptotic stability in the whole of state $(x^T, y^T)^T = 0$ of the system (3.9.11) are

- (a) $\lambda_m(B_1)\lambda_m(B_2) > \mu\lambda_M^{1/2}(B_3B_3^T)$;
- (b) $\lambda_M(c_{11}^0) < 0$;
- (c) $\lambda_M(\sigma_{11}^0) > 0$;
- (d) $\lambda_M(c_{22}^0) < 0$;
- (e) $\lambda_M(\sigma_{22}^0) > 0$;
- (f) $\lambda_M(\sigma_{12}^0\sigma_{12}^{0T}) - \lambda_M(\sigma_{11}^0)\lambda_M(\sigma_{22}^0) > 0$;
- (g) $\lambda_M(c_{12}^0c_{12}^{0T}) - \lambda_M(c_{11}^0)\lambda_M(c_{22}^0) < 0$.

3.9.3 Example

Let the system (3.9.1) be

$$(3.9.14) \quad \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} 0.5 & 0.1 \\ -0.5 & 0.6 \end{pmatrix} x + \begin{pmatrix} 3.6 & 0.3 \\ -0.2 & 5 \end{pmatrix} y, \\ \mu \frac{dy}{dt} &= \begin{pmatrix} -7 & 0.5 \\ -1 & -8 \end{pmatrix} x + \begin{pmatrix} -3 & 0.5 \\ 1 & -8 \end{pmatrix} y, \end{aligned}$$

where $x \in R^2$, $\mu \in (0, 1]$.

We take for the system (3.9.14) the matrix-valued function $U(x, y, \mu)$ with the elements

$$(3.9.15) \quad \begin{aligned} v_{11}(x) &= x^T \text{diag}[2, 2]x; \\ v_{22}(y, \mu) &= \mu y^T \text{diag}[1, 1]y; \\ v_{12}(x, y, \mu) &= v_{21}(x, y, \mu) = \mu x^T \text{diag}[0.4; 0.4]y. \end{aligned}$$

It is easy to see that $v_{ij}(\cdot)$, $i, j = 1, 2$ satisfy the estimates

$$(3.9.16) \quad \begin{aligned} v_{11}(x) &\geq 2\|x\|^2 & \forall (x) \in \mathcal{N}_{x_0}, \\ v_{22}(y, \mu) &\geq \mu\|y\|^2 & \forall (y, \mu) \in \mathcal{N}_{y_0} \times \mathcal{M}, \\ v_{12}(x, y, \mu) &\geq -0.4\mu\|x\|\|y\| & \forall (x, y, \mu) \in \mathcal{N}_{x_0} \times \mathcal{N}_{y_0} \times \mathcal{M}. \end{aligned}$$

Let $\eta = (1, 1)$. Then matrix $A_1(\mu)$ in estimate (3.5.11) for the function

$$V(x, y, \mu) = \eta^T U(x, y, \mu) \eta, \quad \eta \in R_+^2$$

with elements (3.9.15) and the estimates (3.9.16) has the form

$$A_1(\mu) = \begin{pmatrix} 2 & -0.4\mu \\ -0.4\mu & \mu \end{pmatrix}.$$

It can be easily verified that matrix $A_1(\mu)$ is positive definite for every $\mu \in (0, 1]$ and for $\mu \rightarrow 0$.

The elements of matrix $C(\mu)$ from the estimate

$$\frac{d}{dt}V_M(x, y, \mu) \leq 2u^T C(\mu)u,$$

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where $u = (\|x\|, \|y\|)^T$, have the values

$$\begin{aligned}
 \lambda_M(c_{11}) &= -1.291723; \\
 \lambda_M(c_{22}) &= -2.89; \\
 \lambda_M(\sigma_{22}) &= 2.000713; \\
 \lambda_M^{1/2}(c_{12}c_{12}^T) &= 0.784953; \\
 \lambda_M^{1/2}(\sigma_{12}\sigma_{12}^T) &= 0.165452;
 \end{aligned}
 \tag{3.9.17}$$

The values of parameters μ_2 , μ_3 and μ_0 are

$$\begin{aligned}
 \mu_2 &= 1.444485; & \mu_3 &= 1.779742; \\
 \mu_0 &= \min(\mu_2, \mu_3) = 1.444485.
 \end{aligned}$$

With regard to (3.9.17) we find that

- (a) $\lambda_M(c_{11}) < 0$;
- (b) $\lambda_M(c_{22}) < 0$;
- (c) $\lambda_M(\sigma_{22}) > 0$;
- (d) $\lambda_M(c_{12}c_{12}^T) - \lambda_M(c_{11})\lambda_M(c_{22}) = -3.117332 < 0$,

and $\mu_0 = 1.444485$.

By Theorem 3.7.3 the equilibrium state $(x^T, y^T)^T = 0$ of the system (3.9.14) is uniformly asymptotically stable in the whole for every $\mu \in (0, 1]$ and for $\mu \rightarrow 0$.

3.10 Applications

Consider some applications of general results to the problems of mechanics.

3.10.1 Plane two-component pendulum

Let two absolutely solid bodies form a pendulum as shown on Figure 3.10.1.

Body I is rotating around hinge O_1 and contains a sphere cavity. A round body II is placed into this cavity and is freely connected with body I at point O_2 . For the sake of simplicity we assume that the center of mass of body II coincides with point O_2 .

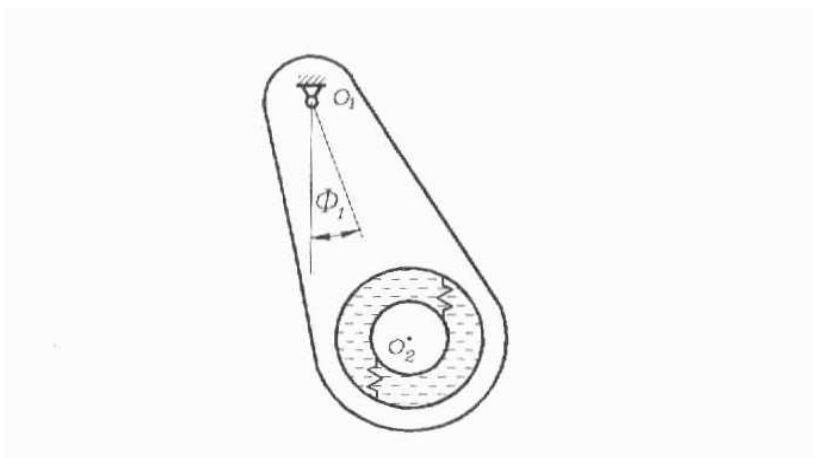


FIGURE 3.10.1 Plane two-component pendulum

The bodies forming such a pendulum are subjected to the weight force and moments of elasticity force and friction with a large coefficient of proportionality to relative rotation angulars and relative angular velocities of the links. Body I moves in the medium with viscous friction. The motion equations of this system in the form of moment of momentum equations for the total system relative to point O_1 and for body II relative to point O_2 are

$$\begin{aligned}
 & \frac{d}{d\tau}(I_1\Omega_1 + I_2\Omega_2) = -Pl \sin \Phi_1 - N_1\Omega_1, \\
 (3.10.1) \quad & \frac{d}{d\tau}I_2\Omega_2 = -K_2(\Phi_2 - \Phi_1) - N_2(\Omega_2 - \Omega_1), \\
 & \frac{d\Phi_1}{d\tau} = \Omega_1, \quad \frac{d\Phi_2}{d\tau} = \Omega_2.
 \end{aligned}$$

Here we designate by Φ_1 and Φ_2 the rotation angulars of the system elements, by Ω_1 and Ω_2 its angular velocities, by I_1 and I_2 the moments of inertia, by τ the natural time, by P the total weight of the system, by l the distance from point O_1 to the center of masses, by N_1 the coefficient of moment of friction of outer forces for the system, by K_2 and N_2 the coefficients of stiffness and friction of moments of interaction forces between the bodies.

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In system (3.10.1) we get over from variables $\Phi_1, \Phi_2, \Omega_1, \Omega_2$ to the set $\varphi_1, \Omega_1, \Delta, U$ containing the variables $\Delta = \Phi_2 - \Phi_1, U = \Omega_2 - \Omega_1$ with respect to which tight co-actions take place. Then we obtain the following equations

$$(3.10.2) \quad \begin{aligned} I_1 \frac{d}{d\tau} &= -Pl \sin \Phi_1 - N_1 \Omega_1 + K_2 \Delta + N_2 U, \\ I_2 \frac{d}{d\tau} &= \frac{I_2}{I_1} Pl \sin \Phi_1 - \left(1 + \frac{I_2}{I_1}\right) (K_2 \Delta + N_2 U), \\ \frac{d\Phi_1}{d\tau} &= \Omega_1, \\ \frac{d\Delta}{d\tau} &= U. \end{aligned}$$

In the system (3.10.2) we get over to the pure normalized values

$$(3.10.3) \quad \begin{aligned} t = \frac{\tau}{\tau_*}, \quad i_1 = \frac{I_1}{I_*}, \quad i_2 = \frac{I_2}{I_*}, \quad \varphi_1 = \frac{\Phi_1}{\Phi_*}, \\ \delta = \frac{\Delta}{\Delta_*}, \quad \omega_1 = \frac{\Omega_1}{\Omega_*}, \quad u = \frac{U}{U_*}, \end{aligned}$$

Let us consider a class of motions for which

- (a) the oscillations of body I are large ($\Phi_* = I$);
- (b) the moments of inertia are of the same order ($I_* = I_1$);
- (c) the stiffness of elastic forces is essentially larger than the coefficient of regeneration $K_1 = Pl$ due to the condition $K_1 \ll K_2$.

We estimate partial time constants of the system. Time constants τ_i of slow oscillations due to condition (c) are estimated by $\tau_1^2 = I_1/K_1$, the time constant τ_2 of quick oscillations of body II due to elasticity is estimated by the correlation $\tau_2 = I_2/k_2$. For $K_2 \geq K_1$ we have $\mu = \tau_2/\tau_1 \ll 1$. We estimate characteristic angular velocities of the system with respect to variables Ω_1, U by the correlations $\Omega_* = \Phi_*/\tau_1$ and $U_* = \Delta_*/\tau_2$.

Assume that the oscillation moments and moments of forces of elastic interaction are the values of the same order ($K_2 \Delta_* = K_1$). We take the

value of the order of slow partial oscillations ($\tau_* = \tau_1$) as the characteristic time. In result of the normalization of (3.10.3) equations (3.10.2) become

$$(3.10.4) \quad \begin{aligned} \frac{d\omega}{dt} &= -\sin \varphi_1 - 2\xi_1\omega_1 + \delta + 2\xi_2u, \\ \frac{d\varphi_1}{dt} &= \omega_1; \\ \mu \frac{du}{dt} &= i_2 \sin \varphi_1 - (1 + i_2)(\delta + 2\xi_2u), \\ \mu \frac{d\delta}{dt} &= u. \end{aligned}$$

Here all variables φ_1 , ω_1 , δ and u have the values of the order of one, $i_2 = I_2/I_1$ and ξ_1, ξ_2 are dimensionless coefficients of damping of the first and second partial oscillating links. In system (3.10.4) we make the change of variables

$$\sin \varphi_1 = x_1, \quad \omega_1 = x_2, \quad u = y_1, \quad (1 + i_2)\delta - i_2 \sin \varphi_1 = y_2$$

and linearize the system. In result we get

$$(3.10.5) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y, \\ \mu \frac{dy}{dt} &= A_{21}x + A_{22}y, \end{aligned}$$

where

$$\begin{aligned} A_{11} &= \begin{pmatrix} 0 & 1 \\ \frac{1}{1+i_2} & -2\xi_1 \end{pmatrix}, & A_{12} &= \begin{pmatrix} 0 & 0 \\ 2\xi_2 & \frac{1}{1+i_2} \end{pmatrix}, \\ A_{21} &= \begin{pmatrix} 0 & 0 \\ 0 & -i_2 \end{pmatrix}, & A_{22} &= \begin{pmatrix} -2(1+i_2)\xi_2 & -1 \\ 1+i_2 & 0 \end{pmatrix}, \end{aligned}$$

$x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$, μ is a small parameter.

For system (3.10.5) we construct matrix-valued function with elements

$$(3.10.6) \quad \begin{aligned} v_{11}(x) &= x^T \begin{pmatrix} 2(\xi_1\gamma_1 + 1) & \gamma_1 \\ \gamma_1 & 2(1+i_2) \end{pmatrix} x; \\ v_{12}(x, y, \mu) &= \mu x^T \begin{pmatrix} 0 & 0 \\ 2 & 0.01 \end{pmatrix} y, \\ v_{22}(y, \mu) &= \mu y^T \begin{pmatrix} 2(1+i_2) & \gamma_2 \\ \gamma_2 & 2(\xi_2\gamma_2 + 1) \end{pmatrix} y, \end{aligned}$$

where γ_1 and γ_2 are constants satisfying the conditions

$$(3.10.7) \quad \gamma_1 < 4\xi_1(1 + i_2), \quad \gamma_2 < 4\xi_2(1 + i_2)$$

Functions (3.10.6) satisfy the estimates

$$\begin{aligned} v_{11}(x) &\geq k_1 \|x\|^2 && \forall x \in R^2 \\ v_{12}(x, y, \mu) &\geq -2\mu \|x\| \|y\| && \forall (x, y) \in R^2 \times R^2 \\ v_{22}(y, \mu) &\geq \mu k_2 \|y\|^2 && \forall y \in R^2, \end{aligned}$$

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where

$$k_1 = \xi_1 \gamma_1 + i_2 + 2 - \sqrt{(\xi_1 \gamma_1 - i_2)^2 + \gamma_1^2};$$

$$k_2 = \xi_2 \gamma_2 + i_2 + 2 - \sqrt{(\xi_2 \gamma_2 - i_2)^2 + \gamma_2^2}.$$

It can be easily verified that when inequalities (3.10.7) are satisfied, then $k_1 > 0$ and $k_2 > 0$.

Matrix $A_1(\mu)$ in estimate (3.5.11) for matrix-valued function with elements (3.10.6) has the form

$$(3.10.8) \quad A_1(\mu) = \begin{pmatrix} k_1 & -2\mu \\ -2\mu & k_2\mu \end{pmatrix}$$

and is positive definite for any $\mu \in (0, \tilde{\mu}_4)$, where

$$\tilde{\mu}_4 = \frac{1}{4} k_1 k_2.$$

If $\eta^T = (1, 1)$, then the elements of matrix $C^0(\mu)$ are

$$\lambda_M(c_{11}^0) = \max \left(-\frac{\gamma_1}{1+i_2}; -4\xi_1(1+i_2) + \gamma_1 \right);$$

$$\lambda_M(\sigma_{11}^0) = 0;$$

$$\lambda_M(c_{22}^0) = \max \left(-4(1+i_2)^2 \xi_2 + (1+i_2)\gamma_2; -\gamma_2 \right);$$

$$\lambda_M(\sigma_{22}^0) = 2\xi_2 + \frac{1}{400(1+i_2)}$$

$$+ \left[\left(2\xi_2 + \frac{1}{400(1+i_2)} \right)^2 + \left(\frac{\xi_2}{100(1+i_2)} \right)^2 + \left(\frac{1}{1+i_2} \right)^2 \right]^{1/2}.$$

$$\lambda_M^{1/2}(c_{12}^0 c_{12}^{0T}) = \left[\frac{1}{2} P_1 + \left(\frac{1}{4} P_1^2 - \left(\frac{\gamma_1}{200} \right)^2 \right)^{1/2} \right]^{1/2};$$

$$P_1 = (\xi_1 \gamma_1)^2 + \left(\frac{\gamma_1}{2(1+i_2)} \right)^2 + \left(\frac{1+i_2}{100} \right)^2;$$

$$\lambda_M^{1/2}(\sigma_{12}^0 \sigma_{12}^{0T}) = \left[\frac{1}{2} P_2 + \left(\frac{1}{4} P_2^2 - \left(i_2(\xi_2 \gamma_2 + 1) - \frac{i_2 \gamma_2}{100} \right)^2 \right)^{1/2} \right]^{1/2};$$

$$P_2 = 1 + \left(\frac{1}{200} \right)^2 + \left(2\xi_1 + \frac{i_2 \gamma_2}{2} \right)^2 + \left(\frac{\xi_1}{100} + i_2(\xi_2 \gamma_2 + 1) \right)^2.$$

Matrix $C^0(\mu)$ is negative definite for every $\mu \in (0, \tilde{\mu}_0)$ and $\mu \rightarrow 0$, where $\tilde{\mu}_0 = \min(\tilde{\mu}_2, \tilde{\mu}_3)$ and

$$\tilde{\mu}_2 = -\frac{\lambda_M(c_{22}^0)}{\lambda_M(\sigma_{22}^0)}, \quad \tilde{\mu}_3 = \frac{-b_1 + \sqrt{b_1^2 - 4a_1 c_1}}{2a_1},$$

where

$$a_1 = \lambda_M(\sigma_{12}^0 \sigma_{12}^{0T}),$$

$$b_1 = \lambda_M^{1/2}(c_{12}^0 c_{12}^{0T}) \lambda_M^{1/2}(\sigma_{12}^0 \sigma_{12}^{0T}) - \lambda_M(c_{11}^0) \lambda_M(\sigma_{22}^0);$$

$$c_1 = \lambda_M(c_{12}^0 c_{12}^{0T}) - \lambda_M(c_{11}^0) \lambda_M(c_{22}^0)$$

if one of the following conditions (i)–(iv) is satisfied

- (i) $\frac{\gamma_1 \gamma_2}{1+i_2} > \lambda_M(c_{12}^0 c_{12}^{0T})$ for $\gamma_1 < \frac{4\xi_1(1+i_2)^2}{2+i_2}$ and $\gamma_2 < \frac{4\xi_2(1+i_2)^2}{2+i_2}$;
- (ii) $(4\xi_2(1+i_2) - \gamma_2)\gamma_1 > \lambda_M(c_{12}^0 c_{12}^{0T})$ for $\gamma_1 < \frac{4\xi_1(1+i_2)^2}{2+i_2}$ and $\frac{4\xi_2(1+i_2)^2}{2+i_2} < \gamma_2 < 4\xi_2(1+i_2)$;
- (iii) $(4\xi_1(1+i_2) - \gamma_1)\gamma_2 > \lambda_M(c_{12}^0 c_{12}^{0T})$ for $\frac{4\xi_1(1+i_2)^2}{2+i_2} < \gamma_1 < 4\xi_1(1+i_2)$ and $\gamma_2 < \frac{4\xi_2(1+i_2)^2}{2+i_2}$;
- (iv) $(1+i_2)(4\xi_1(1+i_2) - \gamma_1)(4\xi_2(1+i_2) - \gamma_2) > \lambda_M(c_{12}^0 c_{12}^{0T})$ for $\frac{4\xi_1(1+i_2)^2}{2+i_2} < \gamma_1 < 4\xi_1(1+i_2)$ and $\frac{4\xi_2(1+i_2)^2}{2+i_2} < \gamma_2 < 4\xi_2(1+i_2)$.

By Theorem 3.7.1 the equilibrium state $(x^T, y^T)^T = 0$ of system (3.10.5) is uniformly asymptotically stable for every $\mu \in (0, \tilde{\mu}^*)$ and for $\mu \rightarrow 0$, where $\tilde{\mu}_* = \min(\tilde{\mu}_4, \tilde{\mu}_0)$.

3.10.2 Singularly perturbed Lur'e systems

In this section, the stability of a singularly perturbed system of the Lur'e form is analyzed on the basis of the Liapunov matrix-valued function. We obtain sufficient conditions for the absolute stability of a system of the Lur'e form and we indicate the bounds of the variation of the small parameter.



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3.10.2.1 *Singularly Perturbed Lur'e System. Case A.* We consider the autonomous singularly perturbed system of Lur'e type

$$(3.10.9) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y + q_1 f_1(\sigma_1), & \sigma_1 &= c_{11}^T + c_{12}^T y; \\ \mu \frac{dy}{dt} &= A_{21}x + A_{22}y + q_2 f_2(\sigma_2), & \sigma_2 &= c_{21}^T + c_{22}^T y, \end{aligned}$$

where $x \in N_x \subseteq R^n$, $y \in N_y \subseteq R^m$, $\mu \in (0, 1]$ is a small parameter, the matrices $A(\cdot)$ and the vectors $c(\cdot)$, $q(\cdot)$ having appropriate dimensions. The nonlinearities f_i , $i = 1, 2$, are continuous, $f_i(0) = 0$ and in the Lur'e sectors $[0, k_i]$, $k_i \in (0, +\infty)$ satisfy the conditions $f_i(\sigma_i)/\sigma_i \in (0, k_i]$, $i = 1, 2$; $\forall \sigma_i \in (-\infty, +\infty)$.

Moreover, we consider only those nonlinearities f_i for which the state $(x^T, y^T)^T = 0$ is the unique equilibrium state of the degenerate system

$$(3.10.10) \quad \frac{dx}{dt} = A_{11}x + q_1 f_1(\sigma_1^0); \quad \sigma_1^0 = c_{11}^T x$$

and of the system, describing the boundary layer,

$$(3.10.11) \quad \mu \frac{dy}{dt} = A_{22}y + q_2 f_2(\sigma_2^0); \quad \sigma_2^0 = c_{22}^T y$$

This assumption holds if

$$c_{ii}^T A_{ii}^{-1} q_i > 0.$$

We introduce the following notations:

$$\begin{aligned} f(x, 0) &= A_{11}x + q_1 f_1(\sigma_1^0); \\ f^*(x, y) &= A_{12}y + q_1 [f_1(\sigma_1) - f_1(\sigma_1^0)]; \\ g(0, y) &= A_{22}y + q_2 f_2(\sigma_2^0); \\ g^*(x, y) &= A_{21}x + q_2 [f_2(\sigma_2) - f_2(\sigma_2^0)]. \end{aligned}$$

Then the system (3.10.9) takes the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, 0) + f^*(x, y); \\ \mu \frac{dy}{dt} &= g(0, y) + g^*(x, y).\end{aligned}$$

Together with system (3.10.9) and subsystems (3.10.10), (3.10.11) we shall consider the matrix-valued function

$$(3.10.12) \quad U(x, y, \mu) = \begin{pmatrix} v_{11}(x) & v_{12}(x, y, \mu) \\ v_{21}(x, y, \mu) & v_{22}(y, \mu) \end{pmatrix}; \quad v_{12} = v_{21},$$

where

$$v_{11} = x^T B_1 x; \quad v_{12} = \mu y^T B_2 y; \quad v_{22} = \mu x^T B_3 y;$$

where B_1 and B_2 are symmetric, positive-definite matrices; B_3 is a constant matrix. With the aid of the matrix-valued function (3.10.12) we introduce the scalar function

$$(3.10.13) \quad V(x, y, \mu) = \eta^T U(x, y, \mu) \eta,$$

where $\eta^T = (\eta_1, \eta_2)$; $\eta \in R_+^2$; $\eta_i > 0$, $i = 1, 2$.

We assume that the elements of the matrix-valued function (3.10.12) satisfy the estimates

$$(3.10.14) \quad \begin{aligned}v_{11}(x) &\geq \lambda_m(B_1) \|x\|^2 && \forall x \in \mathcal{N}_{x0} = \{x: x \in \mathcal{N}_x; x \neq 0\}; \\ v_{22}(y, \mu) &\geq \mu \lambda_m(B_2) \|y\|^2 && \forall (y, \mu) \in \mathcal{N}_{y0} \times \mathcal{M}; \\ v_{12}(x, y, \mu) &\geq -\mu \lambda_M^{1/2}(B_3 B_3^T) \|x\| \|y\| && \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M},\end{aligned}$$

where $\lambda_m(B_i)$ are the minimal eigenvalues of the matrices B_i , $i = 1, 2$; $\lambda_M^{1/2}(B_3 B_3^T)$ is the norm of the matrix $(B_3 B_3^T)$; $\lambda_M(B_3 B_3^T)$ is the maximal eigenvalue of the matrix $B_3 B_3^T$; $\mathcal{N}_{y0} = \{y: y \in \mathcal{N}_y, y \neq 0\}$; $\mathcal{M} = (0, 1]$.


Under the estimates (3.10.14), for the function (3.10.13) we have the estimate

$$v(x, y, \mu) \geq u^T H^T A H u \quad \forall (x, y, \mu) \in \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M},$$

where $u^T = (\|x\|, \|y\|)$, $H = \text{diag}(\eta_1, \eta_2)$;

$$A(\mu) = \begin{pmatrix} \lambda_m(B_1) & -\mu\lambda_M^{1/2}(B_3B_3^T) \\ -\mu\lambda_M^{1/2}(B_3B_3^T) & \mu\lambda_m(B_2) \end{pmatrix}.$$

For the derivatives of the elements of the matrix-valued function (3.10.12) along the solutions of the system (3.10.9) we have the following estimates;



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$$\begin{aligned}
(3.10.15) \quad & \text{(a)} \quad (\nabla_x v_{11})^T f(x, 0) \leq \rho_{11} \|x\|^2 \quad \forall x \in \mathcal{N}_{x0}; \\
& \text{(b)} \quad (\nabla_x v_{11})^T f^*(x, y) \leq \rho_{12} \|x\|^2 + 2\rho_{13}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0}; \\
& \text{(c)} \quad (\nabla_y v_{22})^T g(0, y) \leq \mu\rho_{21} \|y\|^2 \quad \forall (y, \mu) \in \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(d)} \quad (\nabla_y v_{22})^T g^*(x, y) \leq \mu\rho_{22} \|y\|^2 + \mu\rho_{23}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(e)} \quad (\nabla_x v_{12})^T f(x, 0) \leq \mu\rho_{15}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(f)} \quad (\nabla_x v_{12})^T f^*(x, y) \leq \mu\rho_{17}^{1/2} \|x\| \|y\| + \mu\rho_{18} \|y\|^2 \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(g)} \quad (\nabla_y v_{12})^T g(0, y) \leq \mu\rho_{25}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(h)} \quad (\nabla_y v_{12})^T g^*(x, y) \leq \mu\rho_{26} \|x\|^2 + \mu\rho_{27}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M},
\end{aligned}$$

where ρ_{11} , ρ_{12} , ρ_{21} , ρ_{22} , ρ_{18} , ρ_{26} are the maximal eigenvalues of the matrices

$$\begin{aligned}
& B_1 A_{11} + A_{11}^T B_1 + B_1 q_1 k_1^* c_{11}^T + (q_1 k_1^* c_{11}^T)^T B_1, \\
& \quad B_1 q_1 k_1^* c_{11}^T + (q_1 k_1^* c_{11}^T)^T B_1, \\
& B_2 A_{22} + A_{22}^T B_2 + B_2 q_2 k_2^* c_{22}^T + (q_2 k_2^* c_{22}^T)^T B_2, \\
& \quad B_2 q_2 k_2^* c_{22}^T + (q_2 k_2^* c_{22}^T)^T B_2, \\
& \quad A_{12}^T B_3 + (q_1 k_1^* c_{12}^T)^T B_3, \\
& \quad B_3 A_{21} + B_3 q_2 k_2^* c_{21}^T,
\end{aligned}$$

respectively; $\rho_{13}^{1/2}, \rho_{23}^{1/2}, \rho_{15}^{1/2}, \rho_{17}^{1/2}, \rho_{25}^{1/2}, \rho_{27}^{1/2}$ are the norms of the matrices

$$\begin{aligned} & B_1 A_{12} + B_1 q_1 k_1^* c_{12}^T \\ & B_2 A_{21} + B_2 q_2 k_2^* c_{21}^T, \\ & A_{11}^T B_3 + (q_1 k_1^* c_{11}^T)^T B_3, \\ & (q_1 k_1^* c_{11}^T)^T B_3, \\ & B_3 A_{22} + B_3 q_2 k_2^* c_{22}^T \\ & B_3 q_2 k_2^* c_{22}^T \end{aligned}$$

respectively,

$$k_i^* = \begin{cases} k_i & \text{for } \sigma_i q_i B_j x > 0 \text{ (or } \sigma_i q_i B_j y > 0); \\ 0 & \text{for } \sigma_i q_i B_j x \leq 0 \text{ (or } \sigma_i q_i B_j y \leq 0); \end{cases} \quad (i = 1, 2; \quad j = 1, 2, 3).$$

Denoting the upper bound of the derivative of the function (3.10.13) by $\frac{d}{dt}V_M(x, y, \mu)$, we find the estimate

$$(3.10.16) \quad \frac{d}{dt}V_M(x, y, \mu) \leq u^T C(\mu) u,$$

where

$$\begin{aligned} C(\mu) &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad \sigma_{12} = \sigma_{21}; \\ \sigma_{11} &= \eta_1^2(\rho_{11} + \rho_{12}) + 2\eta_1\eta_2\rho_{26}; \\ \sigma_{22} &= \eta_2^2(\rho_{21} + \rho_{22}) + 2\mu\eta_1\eta_2\rho_{18}; \\ \sigma_{12} &= \eta_1^2\rho_{13}^{1/2} + \eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\mu\rho_{15}^{1/2} + \mu\rho_{17}^{1/2} + \rho_{25}^{1/2} + \rho_{27}^{1/2}). \end{aligned}$$

We introduce the quantities

$$\mu_1 = -\frac{\eta_2(\rho_{21} + \rho_{22})}{2\eta_1\rho_{18}}; \quad \mu_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad \mu_0 = \min(\mu_1, \mu_2),$$

where

$$\begin{aligned} a &= \eta_1^2\eta_2^2(\rho_{15}^{1/2} + \rho_{17}^{1/2})^2; \\ b &= \eta_1\eta_2(\rho_{15}^{1/2} + \rho_{17}^{1/2})[\eta_1^2\rho_{13}^{1/2} + \eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\rho_{25}^{1/2} + \rho_{27}^{1/2})] - 2\eta_1\eta_2\rho_{18}\sigma_{11}; \\ c &= [\eta_1^2\rho_{13}^{1/2} + \eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\rho_{25}^{1/2} + \rho_{27}^{1/2})]^2 - \eta_2^2(\rho_{21} + \rho_{22})\sigma_{11}. \end{aligned}$$

If it turns out that $\mu_0 > 1$, then we consider $\mu \in (0, 1]$.

PROPOSITION 3.10.1. *The matrix $C(\mu)$ is negative-definite for every $\mu \in (0, 1]$ and for $\mu \rightarrow 0$ if the following conditions hold:*

- (a) $\sigma_{11} < 0$,
- (b) $\eta_1 \rho_{18} > 0$,
- (c) $\eta_2(\rho_{21} + \rho_{22}) < 0$,
- (d) $c < 0$.

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REMARK 3.10.1. If $\eta_1 \rho_{18} \leq 0$ and the conditions (a), (b), (d) of Proposition 3.10.1 are satisfied, then its assertion remains valid for $\mu_0 = \mu_2$.

THEOREM 3.10.1. *Assume that the singularly perturbed Lur'e system (3.10.9) is such that the matrix-valued function (3.10.12) has been constructed for it, the elements of which satisfy the estimates (3.10.14), and for the upper bound of the derivative of the function (3.10.13) the estimate (3.10.15) holds.*

In this case, if

- (a) *the matrix A is positive-definite;*
- (b) *the matrix $C(\mu)$ is negative-definite for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$,*

then the equilibrium state $(x^T, y^T) = 0$ of the system (3.10.9) is uniformly asymptotically stable for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.

If, furthermore, $\mathcal{N}_x \times \mathcal{N}_y = R^{n+m}$ then the equilibrium state of the system (3.10.9) is uniformly asymptotically stable on the whole for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$.

PROOF. On the basis of the matrix-valued function (3.10.12), with the aid of the vector $\eta \in R_+^2$, $\eta > 0$, we construct the scalar function (3.10.13). Under the estimates (3.10.14) one can show that

$$v(x, y, \mu) \geq u^T H^T A H u, \quad \forall (x, y, \mu) \in \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}.$$

Then from condition (a) of Theorem 3.10.1 there follows that the function $V(x, y, \mu)$ is positive-definite.

For the derivative $\frac{d}{dt}V(x, y, \mu)$ the estimate (3.10.15) holds. From here and from condition (b) of Theorem 3.10.1 there follows that the derivative $\frac{d}{dt}V(x, y, \mu)$ of the function (3.10.13) is negative-definite for every

$\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$. As is known (see Grujić, Martynyuk and Ribbens-Pavella [1]), these conditions are sufficient for the uniform asymptotic stability of the equilibrium state of the system (3.10.9).

In the case $\mathcal{N}_x \times \mathcal{N}_y = R^{n+m}$ the function $V(x, y, \mu)$ is radially unbounded which, together with the other conditions, proves the second assertion of this theorem. This is the absolute stability of the system (3.10.9), μ_0 being an estimate of the upper bound of the variation of the parameter μ .

3.10.2.2 Singularly Perturbed Lur'e System. Case B. Assume that the singularly perturbed system is the Lur'e-type system:

$$(3.10.17) \quad \begin{aligned} \frac{dx}{dt} &= A_{11}x + A_{12}y + q_1 f_1(\sigma_1), & \sigma_1 &= c_{11}^T x + c_{12}^T y; \\ \mu \frac{dy}{dt} &= \mu A_{21}x + A_{22}y + q_2 f_2(\sigma_2), & \sigma_2 &= c_{21}^T x + c_{22}^T y. \end{aligned}$$

Here we preserve all the assumptions made regarding the system (3.10.9), including the assumption on the equilibrium state, i.e., the conditions on the system (3.10.10), (3.10.11).

We assume that for the system (3.10.17) we have constructed the matrix-valued function (3.10.12) for the elements of which the estimated (3.10.14) are satisfied. We introduce the following notations:

$$\begin{aligned} f(x, 0) &= A_{11}x + q_1 f_1(\sigma_1^0); \\ f^*(x, y) &= A_{12}y + q_1 [f_1(\sigma_1) - f_1(\sigma_1^0)]; \\ g(0, y) &= A_{22}y + q_2 f_2(\sigma_2^0); \\ g^*(x, y, \mu) &= \mu A_{21}x + q_2 [f_2(\sigma_2) - f_2(\sigma_2^0)]. \end{aligned}$$

The system (3.10.17) takes the form

$$\begin{aligned} \frac{dx}{dt} &= f(x, 0) + f^*(x, y); \\ \mu \frac{dy}{dt} &= g(0, y) + g^*(x, y, \mu). \end{aligned}$$

By virtue of the system (3.10.17), for the derivatives of the elements v_{ij} of

the matrix-valued function (3.10.12) we have the estimates:

$$\begin{aligned}
(3.10.18) \quad & \text{(a)} \quad (\nabla_x v_{11})^T f(x, 0) \leq \rho_{11} \|x\|^2 \quad \forall x \in \mathcal{N}_{x0}; \\
& \text{(b)} \quad (\nabla_x v_{11})^T f^*(x, y) \leq \rho_{12} \|x\|^2 + 2\rho_{13}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0}; \\
& \text{(c)} \quad (\nabla_y v_{22})^T g(0, y) \leq \mu \rho_{21} \|y\|^2 \quad \forall (y, \mu) \in \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(d)} \quad (\nabla_y v_{22})^T g^*(x, y, \mu) \leq \mu \rho_{22} \|y\|^2 + 2\mu^2 \rho_{23}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(e)} \quad (\nabla_x v_{12})^T f(x, 0) \leq \mu \rho_{15}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(f)} \quad (\nabla_x v_{12})^T f^*(x, y) \leq \mu \rho_{17}^{1/2} \|x\| \|y\| + \mu \rho_{18} \|y\|^2 \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(g)} \quad (\nabla_y v_{12})^T g(0, y) \leq \mu \rho_{25}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}; \\
& \text{(h)} \quad (\nabla_y v_{12})^T g^*(x, y, \mu) \leq \mu^2 \rho_{26} \|x\|^2 + \mu \rho_{27}^{1/2} \|x\| \|y\| \\
& \quad \forall (x, y, \mu) \in \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}.
\end{aligned}$$

Here, ρ_{11} , ρ_{12} , ρ_{21} , ρ_{22} , ρ_{18} , ρ_{26} and $\rho_{13}^{1/2}$, $\rho_{23}^{1/2}$, $\rho_{15}^{1/2}$, $\rho_{17}^{1/2}$, $\rho_{25}^{1/2}$, $\rho_{27}^{1/2}$ are the same quantities as in the estimates (3.10.15). We note that the presence of the small parameter μ in the right-hand side of the system (3.10.18) leads only to the modification of the estimates (3.10.15)(d) and (3.10.15)(h) to the form (3.10.18)(d).

Denoting the upper bound of the derivative of the function (3.10.13) along the solution of the system (3.10.17) by $\frac{d}{dt}V(x, y, \mu)$, we find the estimate

$$(3.10.19) \quad \frac{d}{dt}V_M(x, y, \mu) \leq u^T \tilde{C}(\mu)u,$$

where

$$\begin{aligned}
\tilde{C}(\mu) &= \begin{pmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} \end{pmatrix}; \quad \tilde{\sigma}_{12} = \tilde{\sigma}_{21}; \\
\tilde{\sigma}_{11} &= \eta_1^2(\rho_{11} + \rho_{12}) + 2\mu\eta_1\eta_2\rho_{26}; \\
\tilde{\sigma}_{22} &= \eta_2^2(\rho_{21} + \rho_{22}) + 2\mu\eta_1\eta_2\rho_{18}; \\
\tilde{\sigma}_{12} &= \eta_1^2\rho_{13}^{1/2} + \mu\eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\mu\rho_{15}^{1/2} + \mu\rho_{17}^{1/2} + \rho_{25}^{1/2} + \rho_{27}^{1/2}).
\end{aligned}$$

We introduce the quantities

$$\begin{aligned}\tilde{\mu}_1 &= -\frac{\eta_1(\rho_{11} + \rho_{12})}{2\eta_2\rho_{26}}; & \tilde{\mu}_2 &= -\frac{\eta_2(\rho_{21} + \rho_{22})}{2\eta_1\rho_{18}}; & \tilde{\mu}_3 &= -\frac{\tilde{b} + \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}; \\ \tilde{\mu}_0 &= \min(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3); \\ \tilde{a} &= [\eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\rho_{15}^{1/2} + \rho_{17}^{1/2})]^2 - 4\eta_1^2\eta_2^2\rho_{18}\rho_{26}; \\ \tilde{b} &= [\eta_1^2\rho_{13}^{1/2} + \eta_1\eta_2(\rho_{25}^{1/2} + \rho_{27}^{1/2})][\eta_2^2\rho_{23}^{1/2} + \eta_1\eta_2(\rho_{15}^{1/2} + \rho_{17}^{1/2})] \\ &\quad - 2\eta_1\eta_2[\eta_1^2\rho_{18}(\rho_{11} + \rho_{12}) + \eta_2^2\rho_{26}(\rho_{21} + \rho_{22})]; \\ \tilde{c} &= [\eta_1^2\rho_{13}^{1/2} + \eta_1\eta_2(\rho_{25}^{1/2} + \rho_{27}^{1/2})]^2 - \eta_1^2\eta_2^2(\rho_{11} + \rho_{12})(\rho_{21} + \rho_{22}).\end{aligned}$$



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If it turns out that $\mu_0 > 1$, then we consider $\mu \in (0, 1)$.

PROPOSITION 3.10.2. *The matrix $\tilde{C}(\mu)$ is negative-definite for every $\mu \in (0, \mu_0)$ and for $\mu \rightarrow 0$ the following conditions hold:*

- (a) $\eta_1(\rho_{11} + \rho_{12}) < 0$;
- (b) $\eta_2\rho_{26} > 0$;
- (c) $\eta_2(\rho_{21} + \rho_{22}) < 0$;
- (d) $\eta_1\rho_{18} > 0$;
- (e) $\tilde{a} > 0$;
- (f) $\tilde{c} < 0$.

REMARK 3.10.2. If $\eta_2\rho_{26} \leq 0$ and conditions (a), (c)–(f) of Proposition 3.10.2 are satisfied, then its assertion remains valid for $\tilde{\mu}_0 = \min(\tilde{\mu}_2, \tilde{\mu}_3)$.

REMARK 3.10.3. If $\eta_1\rho_{18} \leq 0$ and conditions (a)–(c), (e), (f) of Proposition 3.10.2 are satisfied, then its assertion remains valid for $\tilde{\mu}_0 = \min(\tilde{\mu}_1, \tilde{\mu}_3)$.

REMARK 3.10.4. If $\eta_2\rho_{26} \leq 0$, $\eta_1\rho_{18} \leq 0$ and conditions (a), (b), (e), (f) of Proposition 3.10.2 are satisfied, then its assertion remains valid for $\tilde{\mu}_0 = \tilde{\mu}_3$.

THEOREM 3.10.2. *Assume that the singularly perturbed Lur'e system (3.10.17) is such that the matrix-valued function (3.10.12) has been constructed for it, the elements of which satisfy the estimates (3.10.14), and for the upper bound of the derivative of the function (3.10.13) the estimate (3.10.19) holds.*

In this case, if

- (a) *the matrix A is positive-definite;*

(b) the matrix $\tilde{C}(\mu)$ is negative-definite for every $\tilde{\mu} \in (0, \tilde{\mu}_0)$ and for $\mu \rightarrow 0$,

then the equilibrium state $(x^T, y^T)^T = 0$ of the system (3.10.17) is uniformly asymptotically stable for every $\tilde{\mu} \in (0, \tilde{\mu}_0)$ and for $\mu \rightarrow 0$.

If, furthermore, $\mathcal{N}_x \times \mathcal{N}_y = R^{n+m}$ then the equilibrium state of the system (3.10.17) is uniformly asymptotically stable on the whole for every $\tilde{\mu} \in (0, \tilde{\mu}_0)$ and for $\mu \rightarrow 0$.

3.10.2.3 Example. We consider a system of the form (3.10.17) in which

$$\begin{aligned} A_{11} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}; & q_1 &= \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}; & c_{11} &= \begin{pmatrix} -0.01 \\ 0 \end{pmatrix}; \\ A_{12} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; & c_{12} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} & (k_1 &= 2); \\ A_{21} &= \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}; & c_{21} &= \begin{pmatrix} 0.001 \\ 0 \end{pmatrix}; & q_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \\ A_{22} &= \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}; & c_{22} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & (k_2 &= 1). \end{aligned}$$

The matrix-valued function (3.10.12) has the elements

$$\begin{aligned} v_{11}(x) &= x^T \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{pmatrix} x; & v_{22}(y, \mu) &= \mu y^T \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} y; \\ v_{12}(x, y, \mu) &= v_{21}(x, y, \mu) = \mu x^T \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} y, \end{aligned}$$

for which we have the estimates

$$\begin{aligned} v_{11}(x) &\geq 0.2\|x\|^2; & v_{22}(y, \mu) &\geq 2\mu\|y\|^2; \\ v_{12}(x, y, \mu) &\geq -0.01\mu\|x\|\|y\|. \end{aligned}$$

If $\eta_i = 1$, $i = 1, 2$, then the matrix

$$A = \begin{pmatrix} 0.2 & -0.01\mu \\ -0.01\mu & 2\mu \end{pmatrix}$$

is positive-definite for every $\mu \in (0, 1)$.

Moreover, for the elements of the matrix $\tilde{C}(\mu)$ we have:

- (1) for $k_i^* = k_i$: $\rho_{11} = -0.15290$, $\rho_{12} = 0.00043$, $\rho_{21} = -7.67545$,
 $\rho_{22} = 4.82843$, $\rho_{18} = 0.012$, $\rho_{26} = 0.00002$, $\rho_{23}^{1/2} = 0.00490$,
 $\rho_{13}^{1/2} = 0.46165$, $\rho_{15}^{1/2} = 0.02415$, $\rho_{17}^{1/2} = 0.00002$, $\rho_{25}^{1/2} = 0.05117$,
 $\rho_{27}^{1/2} = 0.01414$, and $\tilde{a} = 0.00084$, $\tilde{b} = 0.01909$, $\tilde{c} = -0.15638$,
 $\tilde{\mu}_{11} = 7623.205$, $\tilde{\mu}_{12} = 118.6257$, $\tilde{\mu}_{13} = 6.13236$;
- (2) for $k_i^* = 0$: $\rho_{11} = -0.15279$, $\rho_{21} = -12$, $\rho_{18} = 0.01$, $\rho_{26} =$
 0.00001 , $\rho_{13}^{1/2} = 0.4$, $\rho_{23}^{1/2} = 0.002$, $\rho_{15}^{1/2} = 0.02414$, $\rho_{25}^{1/2} = 0.05$,
 $\rho_{12} = \rho_{22} = \rho_{17}^{1/2} = \rho_{27}^{1/2} = 0$, and $\tilde{a} = 0.00068$, $\tilde{b} = 0.01506$,
 $\tilde{c} = -1.6398$, $\tilde{\mu}_{21} = 7639.5$, $\tilde{\mu}_{22} = 600$, $\tilde{\mu}_{23} = 28.84152$.

It is easy to verify that in both cases the conditions of Proposition 3.10.2 are satisfied.

The quantity $\tilde{\mu}_0 = \min(\mu_{ij}, i = 1, 2; j = 1, 2, 3) = 6.13236 > 1$.

Thus, on the basis Proposition 3.10.2, in the given example the matrix $\tilde{C}(\mu)$ is negative-definite for every $\mu \in (0, 1)$ and for $\mu \rightarrow 0$. On the basis of Theorem 3.10.2, the state $(x^T, y^T) = 0$ of the system, determined in this example, is absolutely stable for every $\mu \in (0, 1)$ and for $\mu \rightarrow 0$.

We note that this example has been investigated in (Grujić, Martynyuk and Ribbens-Pavella [57]) by the vector function method. The obtained estimate has been $\tilde{\mu} = 0.52$.

The use of the Liapunov matrix-valued function in the theory of absolute stability of a singularly perturbed system may turn out to be preferable to the method of the scalar or vector function because of two circumstances: the Liapunov matrix-valued function broadens the possibilities for the dynamical properties of the degenerate system (3.10.10) and of the boundary-layer system (3.10.11), and may give a more accurate estimate of the upper value of the parameter μ .

3.11 Notes

3.1. Singularly-perturbed systems are known to be rather widely used in the engineering and technology as models of real processes (see e.g. surveys by Vasiljeva and Butuzov [170]; Kokotović, O'Malley, and Sannuti [86, 87]; Grujić [50, 51]; and some others). Stability properties of SPS were studied by Gradshtein [43]; Tikhonov [169]; Klimushev and Krasovskii [84, 85]; Hoppensteadt [73–77]; Wilde and Kokotović [172]; Šiljak [168]; Zien [176]; Porter [156–158]; Habets [64, 65]; E. Geraschenko and M. Geraschenko [41];

Grujić [48, 52, 54]; Martynyuk and Gutowsky [123]; Martynyuk [114]; Martynyuk and Miladzhanov [126–128]. Monograph by Grujić, Martynyuk and Ribbens-Pavella [57] contains rather full list of bibliography on the SPS stability.

The present chapter describes a way of the Liapunov's direct method application basing on auxiliary matrix-valued function. This approach admits a weakening of some requirements to dynamical properties of the subsystems. In the chapter the investigation of the problem on absolute stability of singularly-perturbed Lur'e-Postnikov system is made minutely.

3.2. The description of system (3.2.1), (3.2.2) follows Grujić [48] and Grujić, Martynyuk and Ribbens-Pavella [57].

3.3–3.4. The contents of Section 3.3 and 3.4 may be found in Grujić, Martynyuk and Ribbens-Pavella [57].

3.5–3.8. The presentation of these sections is based on results by Martynyuk [114] and Martynyuk and Miladzhanov [128].

3.9. The results of this section are due to Martynyuk and Miladzhanov [126, 127].

3.10. The motion equations of the plane two-component pendulum are due to Novozhilov [149]. The investigation of these equations made in this section corresponds to Miladzhanov [145].

The problem of absolute stability plays a central role in stability theory as a consequence of its theoretical and applied importance. In 1944, Lur'e and Postnikov have shown that the mathematical model of hydraulic servosystems is described by a system of differential equations of a special form. These systems have been called Lur'e–Postnikov systems or Lur'e systems. The problem of absolute stability, closely related with these systems, has become classical in control theory. Since 1944, various approaches to the solution of the stability problem of Lur'e systems have been suggested. The majority of them are directed at the determination of sufficient conditions for absolute stability. The first results have been obtained by Lur'e. The conditions obtained by him are purely algebraic and the stability problem reduces to the verification of the existence of solutions for nonlinear algebraic equations. A sufficiently complete bibliography regarding this problem can be found in Grujić, Martynyuk and Ribbens-Pavella [57] and Gelig, Leonov and Yakubovich [39].

Popov's elegant frequency criteria (see Popov [155]) have been widely used and have stimulated further investigations in various directions of the

theory of absolute stability. Grujić [53] has established necessary and sufficient conditions (Liapunov-type conditions) for absolute stability, from which follows that, in the family of functionals, one has to use more than one function, the form of which need not be affected by the form of the nonlinearities of the system. Likhtarnikov and Yakubovich [103] have presented a new approach for the analysis of the absolute stability of nonlinear systems. The essence of this approach consists in the fact that to a linear block in an automatic system one associates a linear manifold in some functional space; the non-linear blocks are described in an analogous manner. Moreover, the intersection of the sets of all possible processes on the input $\sigma(t)$ and the output $\xi(t)$ characterizes a closed system (the class of the corresponding systems). Then, on the basis of the theorem on the minimization of quadratic functionals in linear spaces under quadratic constraints, one constructs absolute stability criteria.

The investigation of the Lur'e-Postnikov system in subsection 3.10.2 is presented in accordance with Martynyuk and Miladzhanov [126].

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