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HANDBOOK *of* THE HISTORY OF LOGIC

VOLUME 3

THE RISE OF MODERN LOGIC:
FROM LEIBNIZ TO FREGE

Edited by

Dov M. Gabbay
John Woods

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PREFACE

With the publication of the present volume, the *Handbook of the History of Logic* turns its attention to the rise of modern logic. The period covered is 1685–1900, with this volume carving out the territory from Leibniz to Frege. What is striking about this period is the earliness and persistence of what could be called 'the mathematical turn in logic'. Virtually every working logician is aware that, after a centuries-long run, the logic that originated in antiquity came to be displaced by a new approach with a dominantly mathematical character. It is, however, a substantial error to suppose that the mathematization of logic was, in all essentials, Frege's accomplishment or, if not his alone, a development ensuing from the second half of the nineteenth century. The mathematical turn in logic, although given considerable torque by events of the nineteenth century, can with assurance be dated from the final quarter of the seventeenth century in the impressively prescient work of Leibniz. It is true that, in the three hundred year run-up to the *Begriffsschrift*, one does not see a smoothly continuous evolution of the mathematical turn, but the idea that logic is mathematics, albeit perhaps only the most general part of mathematics, is one that attracted some degree of support throughout the entire period in question. Still, as Alfred North Whitehead once noted, the relationship between mathematics and symbolic logic has been an "uneasy" one, as is the present-day association of mathematics with computing. Some of this unease has a philosophical texture. For example, those who equate mathematics and logic sometimes disagree about the directionality of the purported identity. Frege and Russell made themselves famous by insisting (though for different reasons) that logic was the senior partner. Indeed logicism is the view that mathematics can be re-expressed without relevant loss in a suitably framed symbolic logic. But for a number of thinkers who took an algebraic approach to logic, the dependency relation was reversed, with mathematics in some form emerging as the senior partner. This was the precursor of the modern view that, in its four main precincts (set theory, proof theory, model theory and recursion theory), logic is indeed a branch of pure mathematics. It would be a mistake to leave the impression that the mathematization of logic (or the logicization of mathematics) was the sole concern of the history of logic between 1665 and 1900. There are, in this long interval, aspects of the modern unfolding of logic that bear no stamp of the imperial designs of mathematicians, as the chapters on Kant and Hegel make clear. Of the two, Hegel's influence on logic is arguably the greater, serving as a spur to the unfolding of an idealist tradition in logic — a development that will be covered in a further volume, *British Logic in the Nineteenth Century*.

The story of logic's modernisation in the twentieth century is taken up in another companion volume *Logic from Russell to Gödel*, also in preparation.

The Editors wish to record their considerable debt to this volume's able authors. Thanks are also due, and happily rendered, to the following individuals: Professor Mohan Matthen, Head of the Philosophy Department, and Professor Nancy Gallini, Dean of the Faculty of Arts, at the University of British Columbia; Professor Bryson Brown, Chair of the Philosophy Department, and Professor Christopher Nicol, Dean of the Faculty of Arts and Science, at the University of Lethbridge; Professor Alan Gibbons, Head of the Department of Computer Science at King's College London; Jane Spurr, Publications Administrator in London; Dawn Collins and Carol Woods, Production Associates in Lethbridge and Vancouver, respectively; and our colleagues at Elsevier, Senior Publisher, Arjen Sevenster, and Production Associate, Andy Deelen.

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LEIBNIZ'S LOGIC

Wolfgang Lenzen

1 INTRODUCTION

The meaning of the word 'logic' has changed quite a lot during the development of logic from ancient to present times. Therefore any attempt to describe "the logic" of a historical author (or school) faces the problem of deciding whether one wants to concentrate on what the author himself understood by 'logic' or what is considered as a genuinely logical issue from our contemporary point of view. E.g., if someone is going to write about Aristotle's logic, does he have to take the entire *Organon* into account, or only the First (and possibly the Second) Analytics? This problem also afflicts the logic of Gottfried Wilhelm Leibniz (1646–1716).

In the late 17^{th} century, logic both as an academic discipline and as a formal science basically coincided with Aristotelian syllogistics. Leibniz's logical work, too, was to a large extent related to the theory of the syllogism, but at the same time it aimed at the construction of a much more powerful "universal calculus". This calculus would primarily serve as a general tool for determining which formal inferences (not only of syllogistic form) are *logically valid.* Moreover, Leibniz was looking for a "universal characteristic" by means of which he hoped to become able to apply the logical calculus to arbitrary (scientific) propositions so that their *factual truth* could be "calculated" in a purely mechanical way. This overoptimistic idea was expressed in the famous passage:

> If this is done, whenever controversies arise, there will be no more need for arguing among two philosophers than among two mathematicians. For it will suffice to take the pens into the hand and to sit down by the abacus, saying to each other (and if they wish also to a friend called for help): *Let us calculate.*[1]

Louis Couturat's well-known monograph *La logique de Leibniz*, published in 1901, contains, besides a series of five appendices, nine different chapters on "La Syllogistique, La Combinatoire, La Langue Universelle, La Caractéristique Universelle, L'Encyclopédie, La Science Générale, La Mathématique Universelle, Le

[1] Cf. **GP** 7, 200: "Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: *Calculemus*". The abbreviations for the editions of Leibniz's works are explained at the beginning of the bibliography.

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Calcul Logique, Le Calcul Géométrique ". This very broad range of topics may perhaps properly reflect Leibniz's own understanding of 'logic', and it certainly does justice to the close interconnections between Leibniz's ideas on logic, mathematics, and metaphysics as expressed in often quoted statements such as "My Metaphysics is entirely Mathematics"[2] or "I have come to see that the true Metaphysics is hardly different from the true Logic"[3]. In contrast to Couturat's approach (and in contrast to similar approaches in Knecht [1981] and Burkhardt [1980]), I will here confine myself to an extensive reconstruction of the *formal core of Leibniz's logic* (sections 4–7) and show how the theory of the syllogism becomes provable within the logical calculus (section 8). In addition, it will be sketched in section 9 how a part of Leibniz's "true Metaphysics" may be reconstructed in terms of his own "true logic" which had been prophetically announced in a letter to Gabriel Wagner as follows:

> It is certainly not a small thing that *Aristotle* brought these forms into unfailing laws, and thus was the first who wrote mathematically outside *Mathematics.* [...] This work of *Aristotle*, however, is only the beginning and quasi the *ABC*, since there are more composed and more difficult forms as for example *Euclid*'s forms of inference which can be used only after they have been verified by means of the first and easy forms [...] The same holds for *algebra* and many other formal proofs which are naked, though, and yet perfect. It is namely not necessary that all inferences are formulated as: *omnis*, *atqui*, *ergo*. In all unfailing sciences, if they are proven exactly, quasi higher logical forms are incorporated which partly flow from Aristotle's [forms] and partly resort to something else.
>
> [...] I hold for certain that the art of reasoning can be further developed in uncomparable ways, and I also believe to see it, to have some anticipation of it, which I would not have obtained without *Mathematicks.* And though I already discovered some foundation when I was not even in the mathematical *novitiate* [...], I eventually felt how entangled the paths are and how difficult it would have been to find a way out without the help of an inner mathematicks. Now what, in my opinion, might be achieved in this field is of such great an idea that, I am afraid, no one will believe before presenting real examples.[4]

The systematic reconstruction of Leibniz's logic to be developed in this chapter reveals five different calculi which can be arranged as follows:

[2]Cf. **GM** 2, 258: "Ma Metaphysique est toute mathematique".

[3]Cf. **GP** 4, 292: "j'ay reconnu que la vraye Metaphysique n'est guères differente de la vraye Logique".

[4]Cf. Leibniz's old-fashioned German in **GP** 7, 519–522.

$$
\begin{tikzcd}
L0.4 \arrow[d] & \\
L0.8 \arrow[d] & \\
L1 \arrow[r] \arrow[d] & PL1 \\
L2 &
\end{tikzcd}
$$

Four of these calculi form a chain of increasingly stronger logics $L0.4$, $L0.8$, $L1$, and $L2$, where the decimals are meant to indicate the respective logical strength of the system. All these systems are *concept logics* or term-logics, to use the familiar name from the historiography of logic. Only the fifth calculus, *PL1*, is a system of *propositional logic* which can be obtained from $L1$ by mapping the concepts and conceptual operators into the set of propositions and propositional operators.

The most important calculus is $L1$, the full *algebra of concepts* which Leibniz developed mainly in the *General Inquiries* (**GI**) of 1686 and which will be described in some detail in section 4 below. As was shown in Lenzen [1984b], $L1$ is deductively equivalent or isomorphic to the ordinary algebra of sets. Since Leibniz happened to provide a complete set of axioms for $L1$, he "discovered" the Boolean algebra 160 years before Boole.

Also of great interest is the subsystem $L0.8$. Instead of the conceptual operator of negation, it contains subtraction (and some other auxiliary operators). Since, furthermore, the conjunction of concepts is symbolized there by the addition sign, it is usually referred to as *Plus-Minus-Calculus*. Leibniz developed it mainly in the famous essay "A not inelegant Specimen of Abstract Proof"[5]. This system is inferior to the full algebra $L1$ in two respects. First, it is conceptually weaker than the latter; i.e. not every conceptual operator of $L1$ is present (or at least definable) in $L0.8$. Second, unlike the case of $L1$, the axioms or theorems discovered by Leibniz fail to axiomatize the Plus-Minus-Calculus in a complete way. The decimal in '$L0.8$' can be understood to express the degree of *conceptual incompleteness* – just 80 percent of the operators of $L1$ are able to be handled in the Plus-Minus-Calculus. In the same sense, the weakest calculus $L0.4$ contains only 40 percent of the conceptual operators available in $L1$. In view of the main operators of containment and converse containment, i.e. being contained, Leibniz occasionally referred to it as "Calculus of containing and being contained" [Calculus de Continentibus et Contentis]. He began to develop it as early as in 1676; and he obtained the final version in the "Specimen Calculi Universalis" (plus "Addenda") dating from around 1679. Leibniz reformulated this calculus some years later in the so-called "Study in the Calculus of Real Addition", i.e. fragment # XX of **GP** 7 [236–247; **P.**, 131–144]. In view of the fact that the mere Plus-Calculus is only a weak subsystem of the Plus-Minus-Calculus, it must appear somewhat surprising that

[5] "Non inelegans specimen demonstrandi in abstractis" – **GP** 7, 228-235; **P.**, 122-130.

many Leibniz-scholars came to regard the former as superior to the latter.[6] Both calculi will be described in some detail in section 5.

Now a characteristic feature of Leibniz's algebra $L1$ (and of its subsystems) is that it is in the first instance *based upon* the propositional calculus, but that it afterwards serves as a *basis for* propositional logic. When Leibniz states and proves the laws of concept logic, he takes the requisite rules and laws of propositional logic for granted. Once the former have been established, however, the latter can be obtained from the former by observing that there exists a strict analogy between *concepts* and *propositions* which allows one to re-interpret the conceptual connectives as propositional connectives. This seemingly circular procedure which leads from the algebra of concepts, $L1$, to an algebra of propositions, *PL1*, will be described in section 6. At the moment suffice it to say that in the 19th century George Boole, in roughly the same way, first presupposed propositional logic to develop his algebra of sets, and only afterwards derived the propositional calculus out of the set-theoretical calculus. While Boole thus arrived at the classical, two-valued propositional calculus, the Leibnizian procedure instead yields a *modal* logic of strict implication. As was shown in Lenzen [1987], *PL1* is deductively equivalent to the so-called Lewis-modal system S2°.

The final extension of Leibniz's logic is achieved by his theory of indefinite concepts which constitutes an anticipation of modern *quantification theory*. To be sure, Leibniz's theory is, in some places, defective and far from complete. But his ideas concerning quantification about concepts (and, later on, also about individuals or, more exactly, about*individual-concepts*) were clear and detailed enough to admit an unambiguous reconstruction, which will be provided in section 7. The resulting system, $L2$, differs from an orthodox second-order logic in the following respect. While normally one begins by quantifying over individuals on the first level and introduces quantification over predicates only in a second step, in the Leibnizian system quantification over *concepts* comes first, and quantifying over *individual(-concept)s* is introduced by definition only afterwards. Within calculus $L2$, there exist various ways of formally representing the categorical forms of the theory of the syllogism. They will be examined in some detail in section 8 where we investigate in particular the so-called theory of "quantification of the predicate" developed in the fragment "Mathesis rationis". Furthermore, in the concluding section 9 it will be indicated how a good portion of Leibniz's metaphysics can be reconstructed in terms of his own logic.

The entire system of Leibniz's logic, then, may be characterized as a *second-order logic of concepts based upon a sentential logic of strict implication.* This is somewhat at odds with the standard evaluation, e.g. by Kneale and Kneale [1962, p. 337], according to which Leibniz "never succeeded in producing a calculus which covered even the whole theory of the syllogism". Some of the reasons for this rather notorious underestimation of Leibniz's logic will be discussed in section 3 below.

[6]Cf., e.g., Loemker's introductory remark to his translation of the Plus-Calculus: "This paper is one of several which mark the most advanced stage reached by Leibniz in his efforts to establish the rules for a logical calculus" (**L** 371).

2 MANUSCRIPTS AND EDITIONS

Gottfried Wilhelm Leibniz was born in 1646. When he died at the age of 70, he left behind an extraordinarily extensive and widespread collection of papers, only a small part of which had been published during his lifetime. The bibliography of Leibniz's printed works [Ravier, 1937] contains 882 items, but only 325 papers had been published by Leibniz himself, and amongst these one finds many brief notes and discussions of contemporary works.

Much more impressive than this group of printed works is Leibniz's correspondence. The Bodemann catalogue (**LH**) contains more than 15,000 letters which Leibniz exchanged with more than 1,000 correspondents all over Europe, and the whole correspondence can be estimated to comprise some 50,000 pages. Furthermore, there is the collection of Leibniz's scientific, historical, and political manuscripts in the Leibniz-Archive in Hannover which was described in another catalogue (**LH**). The manuscripts are classified into fourty-one different groups ranging from Theology, Jurisprudence, Medicine, Philosophy, Philology, Geography and all kinds of historical investigations to Mathematics, the Natural Sciences and some less scientific matters such as the Military or the Foundation of Societies and Libraries. The whole manuscripts have been microfilmed on about 120 reels each of which contains approximately 400-500 pages. This makes all together about 50- to 60,000 pages which are scheduled to be published (together with the letters) in the so-called Akademie-Ausgabe ('**A**'). This edition was started in 1923, and it will probably not be finished, if ever, until a century afterwards.

Throughout his life, Leibniz published not a single line on *logic*, except perhaps for the mathematical Dissertation "De Arte Combinatoria" or the Juridical Disputation "De Conditionibus". The former incidentally deals with some issues in the traditional theory of the syllogism, while the latter contains some interesting observations about the validity of certain principles of what is nowadays called deontic logic. Leibniz's main aim in logic, however, was to extend Aristotelian syllogistics to a "Universal Calculus". And although we know of several drafts for such a logic which had been elaborated with some care and which seem to have been composed for publication, Leibniz appears to have remained unsatisfied with these attempts. Anyway he refrained from sending them to press. Thus one of his fragments bears the characteristic title "Post tot logicas nondum Logica qualem desidero scripta est"[7] which means: After so many logics the logic that I dream of has not yet been written.

So Leibniz's genuinely logical essays appeared only posthumously. The early editions of his philosophical works by Raspe (**R**), Erdmann (**OP**), and C. I. Gerhardt (**GP**) contained, however, only a very small selection. It was not until 1903 that the majority of the logical works were published in Couturat's valuable edition of the *Opuscules et fragments inédits de Leibniz* (**C**). Some years ago I borrowed from the Leibniz-Archive a copy of those five or six microfilm reels which contain group IV, i.e. the philosophical manuscripts. It took me quite some time to

[7]Cf. **A** VI 4, # 2 (pp. 8-11).

work through the 2,500 pages in search of hitherto unpublished logical material. Though I happened to find some interesting papers that had been overlooked by Couturat, the search eventually turned out less successful than I had thought. I guess that at least 80 percent of the handwritten material relevant for Leibniz's logic are already contained in **C**.

Although, then, Couturat's edition may be considered as rather complete, there is another reason why any serious student of Leibniz's logic cannot be satisfied with these texts alone. The *Opuscules* simply do not fulfil the criteria of a *text-critical* edition as set up by the Leibniz-Forschungsstelle of the University of Münster, i.e. the editors of series VI of the Akademie-Ausgabe. In particular, Couturat all too often suppressed preliminary versions of axioms, theorems, and proofs that were afterwards crossed out and improved by Leibniz. A full knowledge of the gradual ripening of ideas as revealed in a text-critical presentation of the different stages of the fragments, however, is essential for an adequate understanding both of what Leibniz was looking for and of what he eventually managed to find.

Since the recent publication of the important and impressive volume **A** VI, 4 which contains Leibniz's Philosophical Writings from ca. 1676 to 1690[8], the situation for scholars of Leibniz's logic has drastically improved. The majority of the drafts of a "Universal Calculus" now are available in an almost perfect text-critical edition. Just a few works especially on the theory of the syllogism such as "A Mathematics of Reason" [**P**. 95-104; cf. "Mathesis rationis", **C**., 193–202;] and "A paper on 'some logical difficulties"' [**P**., 115–121; cf. "Difficultates Quaedam Logicae" **GP** 7, 211–217] have not yet been included in **A** VI 4 but will hopefully be published in the next (and final?) volume of that series.

As regards English translations of Leibniz's philosophical writings in general, the basic edition still is Loemker's **L**. A much more comprehensive selection of Leibniz's *logical* papers is contained in Parkinson's edition **P**. Another translation of the important *General Inquiries about the Analysis of Concepts and of Truths* was given by W. O'Briant in [1968].

3 THE TRADITIONAL VIEW OF LEIBNIZ'S LOGIC

The rediscovery of Leibniz's logical work would not have been possible without the pioneering work Louis Couturat. On the one hand, **C** still is an important tool for all Leibniz scholars; on the other hand, Couturat is also (at least partially) responsible for the underestimation of the value of traditional logic in general and of Leibniz's logic in particular as it may be observed throughout the 20th century. In the "Résumé et conclusion" of chapter 8, Couturat compares Leibniz's logical achievements with those of modern logicians, especially with the work of George Boole:

[8]This volume appeared in 1999 and it contains 522 pieces with almost 3,000 pages distributed over three subvolumes (A, B, and C).

> Summing up, Leibniz had the idea [...] of all logical operations, not only of multiplication, addition and negation, but even of subtraction and division. He knew the fundamental relations of the two copulas [...] He found the correct algebraic translation of the four classical propositions [...] He discovered the main laws of the logic calculus, in particular the rules of composition and decomposition [...] In one word, he possessed almost all principles of the Boole–Schröder-logic, and in some points he was even more advanced than Boole himself. (Cf. Couturat [1901, pp. 385–6])

Despite this apparently very favourable evaluation, Couturat goes on to maintain that Leibniz's logic was bound to fail for the following reason:

> Finally, and most importantly, he did not have the idea of combining logical addition and multiplication and treating them together. This is due to the fact that he adopted the point of view of the comprehension [of concepts]; accordingly he considered only one way of combing concepts: by adding their comprehensions, and he neglected the other way of adding their extensions. This is what prevented him to discover the symmetry and reciprocity of these two operations as it manifests itself in the De Morgan formulas and to develop the calculus of negation which rests on these formulas. (Cf. Couturat [1901, pp. 385–6])

A similar judgement may be found in C. I. Lewis' *A Survey of Symbolic Logic* of 1918. Lewis starts by appreciating:

> The program both for symbolic logic and for logistic, in anything like a clear form, was first sketched by Leibniz [...]. Leibniz left fragmentary developments of symbolic logic, and some attempts at logistic which are prophetic. [Lewis, 1918, p. 4]

But in the subsequent passage these attempts are degraded as "otherwise without value", and as regards the comparison of Leibniz's logic and Boolean logic, Lewis says:

> Boole seems to have been ignorant of the work of his continental predecessors, *which is probably fortunate, since his own beginning has proved so much more fruitful.* Boole is, in fact, the second founder of the subject, and all later work goes back to his. (ibid., my emphasis)[9].

In the introduction of his 1930 monograph *Neue Beleuchtung einer Theorie von Leibniz,* K. Dürr describes the historical development of logic from Leibniz to modern times as follows:

> ...It is well known that Leibniz was the first who attempted to create what might be called a logic calculus or a symbolic logic [...] In the

[9]Cf. in the same vein chapter I of Lewis and Langford [1932].

> mid of the 19^{th} century the movement aiming at the creation of a logic calculus was reanimated by the work of the Englishman Boole, and it is beyond every doubt that Boole was entirely independent of Leibniz. (Cf. Dürr [1930, p. 5]).

Dürr wants to clarify the relations between Leibniz's logic and modern logic by providing a formal reconstruction of the Plus-Minus-Calculus, and he announces that his comparative studies will provide results quite different from those of Couturat. Unfortunately, however, Dürr fails to give a detailed comparison between Leibniz's logic and Boole's logic. Moreover, as was already mentioned in the preceding section, unlike Leibniz's "standard system", $L1$, developed in the *General Inquiries*, the fragments of the Plus-Minus-calculus in **GP** 7 remain fundamentally incomplete.

In a 1946 paper, "Uber die logischen Forschungen von Leibniz", H. Sauer deals with the issue of whether Leibniz or Boole should be considered as the founder of modern logic. He mentions two reasons why Leibniz's logical oeuvre was neglected or underestimated for such a long time. First, the majority of Leibniz's scattered fragments was published only posthumously – as a matter of fact almost 200 years after having been written. Second, even after the appearance of **C** the time was not yet ripe for Leibniz's logical ideas. When Sauer goes on to remark that Leibniz created a logical calculus which was a precursor of modern propositional and predicate calculus, one might expect that he wants to throw Boole from the throne and replace him by Leibniz. However, the following prejudice[10] changes his opinion:

> [Leibniz's logic calculus] is, however, imperfect in so far as Leibniz, under the spell of Aristotelian logic, fails to get rid of the old error that all concepts can be build up from simple concepts by mere conjunction and that all propositions can be put into the form 'S is P'. (Cf. Sauer [1946, p. 64]).

Thus in the end also Sauer disqualifies Leibniz's logic as inferior to "the essentially more perfect 19th century algebra of logic".

Even more negative is the verdict of W. & M. Kneale in their otherwise competent book *The Development of Logic* published in 1962. After charging Leibniz with the fault of committing "himself quite explicitly to the assumption of existential import for all universal statements [...] which prevented him from producing a really satisfactory calculus of logic", and after blaming him with the "equally fateful" mistake that he "[...] accepted the assimilation of singular to universal statements because it seemed to him there was no fundamental difference between the two sorts" [Kneale and Kneale, 1962, p. 323], they sum up Leibniz's logical achievements as follows:

[10] Sauer may have adopted this reproach from Couturat [1901], but a similar critique was already put forward by Kvet [1857].

> When he began, he intended, no doubt, to produce something wider than traditional logic. [...] But although he worked on the subject in 1679, in 168[6] and in 1690, *he never succeeded in producing a calculus which covered even the whole theory of the syllogism.* ([Kneale and Kneale, 1962, p. 337], my emphasis).

The common judgment behind all these views thus has it that Leibniz *in vain* looked for a general logical calculus like Boolean algebra but never managed to find it.

First revisions of this sceptical view were suggested by N. Rescher in a [1954] paper on "Leibniz's interpretation of his logical calculi" and by R. Kauppi's [1960] dissertation *Über die Leibnizsche Logik.* Both authors tried in particular to rehabilitate Leibniz's "intensional" approach. However, it was not until the mid-1980ies when strict proofs were provided to show that – contrary to Couturat's claim

- the "intensional" interpretation of concepts is equivalent (or isomorphic) to the modern extensional interpretation;

- Leibniz's "algebra of concepts" is equivalent (or isomorphic) to Boole's algebra of sets;

- Leibniz's theory of "indefinite concepts" constitutes an important anticipation of modern quantifier theory;

- Leibniz's "universal calculus" allows in various ways the derivation of the laws of the theory of the syllogism.[11]

This radically new evaluation of Leibniz's logic was summed up in Lenzen [1990a] which, like the majority of all books about this topic, was written in German.[12] To be sure, there exist many English works on Leibniz's *philosophy* in general. To mention only some prominent examples: Russell [1900], Parkinson [1965], Rescher [1967; 1979], Broad [1975], Mates [1986], Wilson [1989], Sleigh [1990], Kulstad [1991], Mugnai [1992], Adams [1994], and Rutherford [1995]. But these monographs as well as the important selections of papers in Frankfurt [1972], Woolhouse [1981], and Rescher [1989], only occasionally deal with logical issues. As far as I know, only two English studies are devoted to a more detailed investigation of Leibniz's logic, viz. Parkinson's [1966] introduction to his collection **P** and Ishiguro's [1972] book on *Leibniz's Philosophy of Logic and Language.*

[11] Cf. Lenzen [1983; 1984a; 1984b] and [1988].

[12] Cf. Kvet [1857] (written by a Czech author), Dürr [1930], Kauppi [1960] (written by a Finnish author), Poser [1969] and Burkhardt [1980]; in addition cf. the two monographs in French by Couturat [1901] and by the Swiss author Knecht [1981].

4 THE ALGEBRA OF CONCEPTS (L1) AND ITS EXTENSIONAL INTERPRETATION

The starting point for Leibniz' universal calculus is the traditional "Aristotelian" theory of the syllogism with its categorical forms of universal or particular, affirmative or negative propositions which express the following relations between two concepts A and B:

U.A.	Every A is B	U.N.	No A is B
P.A.	Some A is B	P.N.	Some A is not B

Within the framework of so-called "Scholastic" syllogistics[13] negative concepts Not-A are also taken into account, which shall here be symbolized as $\overline{A}$. According to the principle of so-called obversion, the U.N. 'No A is B' is equivalent to a corresponding U.A. with the negative predicate: Every A is Not-B. Thus in view of the well-known laws of opposition – according to which P.N. is the (propositional) negation of U.A. and P.A. is the negation of U.N. – the categorical forms can uniformly be represented as follows:

U.A.	Every A is B	U.N.	Every A is $\overline{B}$
P.A.	$\neg$(Every A is $\overline{B}$)	P.N.	$\neg$(Every A is B).

The algebra of concepts as developed by Leibniz in some early fragments of around 1679 and above all in the **GI** of 1686 grows out of this syllogistic framework by three achievements. First, Leibniz drops the expression 'every' ['omne'] and formulates the U.A. simply as 'A is B' ['A est B'] or also as 'A contains B' ['A continet B']. This fundamental proposition shall here be symbolized as '$A \in B$', and the negation $\neg(A \in B)$ will be abbreviated as '$A \notin B$'. Second, Leibniz introduces the new operator of conceptual *conjunction* which combines two concepts A and B by juxtaposition to AB. Third, Leibniz disregards all traditional restrictions concerning the number of premisses and concerning the number of concepts in the premisses of a syllogism. Thus arbitrary inferences between sentences of the form $A \in B$ or $A \notin B$ will be taken into account, where the concepts A and B may be arbitrarily complex, i.e. they may contain negations and conjunctions of other concepts. Let the resulting language be referred to as $L1$.

One possible axiomatization of $L1$ would take (besides the tacitly presupposed propositional functions $\neg, \wedge, \vee, \rightarrow$, and $\leftrightarrow$) only negation, conjunction and the $\in$-relation as primitive conceptual operators. As regards the relation of conceptual containment, $A \in B$, it is important to observe that Leibniz's formulation 'A contains B' pertains to the so-called *intensional* interpretation of concepts as *ideas*, while we here want to develop an *extensional* interpretation in terms of *sets of individuals*, *viz.* the sets of all individuals that fall under the concepts A and B, respectively. Leibniz explained the mutual relationship between the "intensional" and the extensional point of view in the following passage of the *New Essays on Human understanding*:

[13] Cf. Thom [1981]

> The common manner of statement concerns individuals, whereas Aristotle's refers rather to ideas or universals. For when I say *Every man is an animal* I mean that all the men are included amongst all the animals; but at the same time I mean that the idea of animal is included in the idea of man. 'Animal' comprises more individuals than 'man' does, but 'man' comprises more ideas or more attributes: one has more instances, the other more degrees of reality; one has the greater extension, the other the greater intension. (cf. **GP** 5: 469; my translation).

If 'Int(A)' and 'Ext(A)' abbreviate the "intension" and the extension of a concept A, respectively, then the so-called *law of reciprocity* can be formalized as follows:

(RECI 1) $\quad \text{Int}(A) \subseteq \text{Int}\ (B) \leftrightarrow \text{Ext}(A) \supseteq \text{Ext}(B).$

This principle immediately entails that two concepts have the same "intension" if and only if they also have the same extension:

(RECI 2) $\quad \text{Int}(A) = \text{Int}\ (B) \leftrightarrow \text{Ext}(A) = \text{Ext}(B).$

But the latter "law" appears to be patently false! On the basis of our modern understanding of intension and extension, there exist many concepts or predicates A, B which have the same extension but which nevertheless differ in intension. Consider, e.g., the famous example in Quine [1953, p. 21], A = 'creature with a heart', B = 'creature with a kidney', or the more recent observation in Swoyer [1995, p. 103] (inspired by Quine and directed against RECI 1):

> For example, it might just happen that all cyclists are mathematicians, so that the extension of the concept *being a cyclist* is a subset of the extension of the concept *being a mathematician.* But few philosophers would conclude that the concept *being a mathematician* is in any sense included in the concept *being a cyclist.*

However, these examples cannot really refute the law of reciprocity *as understood by Leibniz.* For Leibniz, the *extension* of a predicate A is not just the set of all *existing* individuals that (happen to) fall under concept A, but rather the set of all *possible* individuals that have that property. Thus Leibniz would certainly admit that the intension or "idea" of a mathematician is not included in the idea of a cyclist. But he would point out that even if in the *real world* the set of all mathematicians should by chance coincide with the set of all cyclists, there clearly are other possible individuals *in other possible worlds* who are mathematicians and not bicyclists (or bicyclists but not mathematicians). In general, whenever two concepts A and B differ in intension, then it is possible that there exists an individual which has the one property but not the other. Therefore, given Leibniz's understanding of what constitutes the extension of a concept it follows that A and B differ also in extension.[14]

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[14]As regards the ontological scruples against the assumption of merely possible individuals, cf. the famous paper "On What There Is" in Quine [1953, pp. 1–19] and the critical discussion in Lenzen [1980, p. 285 sq.].

In Lenzen [1983] precise definitions of the "intension" and the extension of concepts have been developed which satisfy the above law of reciprocity, RECI 1. Leibniz's "intensional" point of view thus becomes provably equivalent, i.e. translatable or transformable into the more common set-theoretical point of view, provided that the extensions of concepts are taken from a universe of discourse, U, to be thought of as a set of *possible individuals*. In particular, the "intensional" proposition $A \in B$, according to which concept A *contains* concept B, has to be interpreted extensionally as saying that the set of all As *is included* in the set of all Bs. The first condition for the definition of an extensional interpretation of the algebra of concepts thus runs as follows:

(DEF 1) Let U be a non-empty set (of possible individuals), and let ϕ be a function such that $\phi(A) \subseteq U$ for each concept-letter A. Then ϕ is an extensional interpretation of Leibniz's concept logic $L1$ if
(1) $\phi(A \in B) = \text{true iff } \phi(A) \subseteq \phi(B)$.

Next consider the identity or *coincidence* of two concepts which Leibniz usually symbolizes by the modern sign '=' or by the symbol '∞', but which he sometimes also refers to only informally by speaking of two concepts being the same [idem, eadem]. As stated, e.g., in §30 **GI**, identity or coincidence can be defined as mutual containment: "That A is B and B is A is the same as that A and B coincide", i.e.:

(DEF 2) $A = B \leftrightarrow_{\text{df}} A \in B \wedge B \in A$.

This definition immediately yields the following condition for an extensional interpretation ϕ:

$$(2)\ \phi(A = B) = \text{true iff } \phi(A) = \phi(B).$$

In most drafts of the "universal calculus", Leibniz symbolizes the operator of conceptual *conjunction* by mere juxtaposition in the form AB. Only in the context of the Plus-Minus-Calculus, which will be investigated in more detail in section 5 below, he favoured the mathematical '+'-sign (sometimes also '$\oplus$') to express the conjunction of A and B. The intended interpretation is straightforward. The extension of AB is the set of all (possible) individuals that fall under both concepts, i.e. which belong to the intersection of the extensions of A and of B:

$$(3)\ \phi(AB) = \phi(A) \cap \phi(B).$$

Let it be noted in passing that the crucial condition (1) which reflects the reciprocity of extension and "intension" would be derivable from conditions (2) and (3) if the relation $\in$ were defined according to §83 **GI** in terms of conjunction and identity: "Generally, 'A is B' is the same as '$A = AB$' " (**P**, 67), i.e. formally:

(DEF 3) $A \in B \leftrightarrow_{\text{df}} A = AB$.

For, clearly, a set $\phi(A)$ coincides with the intersection $\phi(A) \cap \phi(B)$ if and only if $\phi(A)$ is a subset of $\phi(B)$! Furthermore, the relation "A is in B" [A inest ipsi B] may simply be defined as the converse of $A \in B$ according to Leibniz's remark in §16 **GI**: "[...] 'A contains B' or, as Aristotle says, 'B is in A"'

(DEF 4) $\quad A \iota B \leftrightarrow_{\mathrm{df}} B \in A.$

In view of the law of reciprocity, one thus obtains the following condition:

$$(4)\ \phi(A\iota B) = \text{true iff } \phi(A) \supseteq \phi(B).$$

The next element of the algebra of concepts — and, by the way, one with which Leibniz had notorious difficulties — is *negation.* Leibniz usually expressed the negation of a concept by means of the same word he also used to express propositional negation, viz. 'not' [non]. Especially throughout the **GI**, the statement that one concept, A, contains the negation of another concept, B, is expressed as 'A is not-B' [A est non B], while the related phrase 'A isn't B' [A non est B] has to be understood as the mere negation of 'A contains B'. As was shown in Lenzen [1986], during the whole period of the development of the "universal calculus" Leibniz had to struggle hard to grasp the important difference between 'A is not-B' and 'A isn't B'. Again and again he mistakenly identified both statements, although he had noted their non-equivalence repeatedly in other places. Here the negation of concept A will be expressed as '$\overline{A}$', while propositional negation is symbolized by means of the usual sign '$\neg$'. Thus 'A is not-B' must be formulated as '$A \in \overline{B}$', while 'A isn't B' has to be rendered as '$\neg A \in B$' or '$A \notin B$'. The intended extensional interpretation of $\overline{A}$ is just the set-theoretical complement of the extension of A, because each individual which fails to fall under concept A eo ipso falls under the negative concept $\overline{A}$:

$$(5)\ \phi(\overline{A}) = \overline{\phi(A)}.$$

Closely related to the negation operator is that of ***possibility*** or self-consistency of concepts. Leibniz expresses it in various ways. He often says 'A is possible' [A est possibile] or 'A is [a] being' [A est Ens] or also 'A is a thing' [A est Res]. Sometimes the self-consistency of A is also expressed elliptically by 'A est', i.e. 'A is'. Here the capital letter '**P**' will be used to abbreviate the possibility of a concept A, while the impossibility or inconsistency of A shall be symbolized by '**I**(A)'. According to **GI**, lines 330–331, the operator **P** can be defined as follows: "A not-A is a contradiction. *Possible* is what does not contain a contradiction or A not-A":

(DEF 5) $\quad \mathbf{P}(B) \leftrightarrow_{\mathrm{df}} B \notin A\overline{A}.$[15]

It then follows from our earlier conditions (1), (3), and (4) that **P**(A) is true (under the extensional interpretation ϕ) if and only if ϕ(A) is not empty:

[15]This definition might be simplified as follows: $\mathbf{P}(B) \leftrightarrow_{\mathrm{df}} B \notin \overline{B}$.

(6) $\phi(\mathbf{P}(A)) = \text{true iff } \phi(A) \neq \varnothing$.

At first sight, this condition might appear inadequate, since there are certain concepts – such as that of a unicorn – which happen to be empty but which may nevertheless be regarded as possible, i. e. not involving a contradiction. Remember, however, that the universe of discourse underlying the extensional interpretation of $L1$ does not consist of *actually existing* objects only, but instead comprises all *possible* individuals. Therefore the non-emptiness of the extension of A is both *necessary and sufficient* for guaranteeing the self-consistency of A. Clearly, if A is possible then there must exist at least one *possible* individual that falls under concept A.

The main elements of Leibniz's algebra of concepts may thus be summarized in the following diagram.

Element of L1	*Symbolization*	*Leibniz's Notation*	*Set-theoretical Interpretation*
Identity	$A = B$	$A \infty B; A = B$; coincidunt A et B; ...	$\phi(A) = \phi(B)$
Containment	$A \in B$	A est B; A continet B	$\phi(A) \subseteq \phi(B)$
Converse Containment	$A \iota B$	A inest ipsi B	$\phi(A) \supseteq \phi(B)$
Conjunction	AB	$AB; A + B$	$\phi(A) \cap \phi(B)$
Negation	$\overline{A}$	Non-A	$\overline{\phi(A)}$
Possibility	$\mathbf{P}(A)$	A est Ens; A est res; A est possibile	$\phi(A) \neq \emptyset$

Some further elements will be discussed in the subsequent section 5 when we investigate the operators and laws of the Plus-Minus-Calculus. Before we do this, however, let us have a look at some fundamental laws of $L1$! The subsequent selection of principles, all of which (with the possible exception of the last one) were stated by Leibniz himself, is more than sufficient to derive the laws of the Boolean algebra of sets:

Laws of L1	Formal version	Leibniz's version
CONT 1	$A \in A$	"B is B" (**GI**, §37)
CONT 2	$A \in B \wedge B \in C \to A \in C$	"[...] if A is B and B is C, A will be C" (**GI**, §19)
CONT 3	$A \in B \leftrightarrow A = AB$	"Generally 'A is B' is the same as '$A = AB$' " (**GI**, §83)
CONJ 1	$A \in BC \leftrightarrow A \in B \wedge A \in C$	"That A contains B and A contains C is the same as that A contains BC" (**GI**, §35; cf. **P** 58, note 4)
CONJ 2	$AB \in A$	"AB is A" (**C**, 263)
CONJ 3	$AB \in B$	"AB is B" (**GI**, §38)
CONJ 4	$AA = A$	"$AA = A$" (**GI**, §171, Third)
CONJ 5	$AB = BA$	"$AB \infty BA$" (**C**. 235, # (7))
NEG 1	$\overline{\overline{A}} = A$	"Not-not-$A = A$" (**GI**, §96)
NEG 2	$A \neq \overline{A}$	"A proposition false in itself is 'A coincides with not-A' " (**GI**, §11)
NEG 3	$A \in B \leftrightarrow \overline{B} \in \overline{A}$	"In general, 'A is B' is the same as 'Not-B is not-A' " (**GI**, §77)
NEG 4	$\overline{A} \in \overline{AB}$	"Not-A is not-AB" (**GI**, §76a)
NEG 5	$[\mathbf{P}(A) \wedge] A \in B \to A \notin \overline{B}$	"If A is B, therefore A is not not-B" (**GI**, §91)
POSS 1	$\mathbf{I}(A\overline{B}) \leftrightarrow A \in B$	"if I say 'A not-B is not', this is the same as if I were to say [...] 'A contains B' " (**GI**, §200).[16]
POSS 2	$A \in B \wedge \mathbf{P}(A) \to \mathbf{P}(B)$	"If A contains B and A is true, B is also true" (**GI**, §55)[17]
POSS 3	$\mathbf{I}(A\overline{A})$	"A not-A is not a thing" (**GI**, §171, Eighth)
POSS 4	$A\overline{A} \in B$	

[16]Parkinson translates Leibniz's "Si dicam AB non est ..." somewhat infelicitous as "If I say, AB does not exist' ..." thus blurring the distinction between (actual) existence and mere possibility. For an alterative formulation of Poss 1 cf. C., 407/8: "[...] si A est B vera propositio est, A non-B implicare contradictionem", i.e. 'A is B' is a true proposition if A non-B includes a contradiction.

[17]At first sight this quotation might seem to express some law of propositional logic such as

CONT 1 and CONT 2 show that the relation of *containment* is reflexive and transitive: Every concept contains itself; and if A contains B which in turn contains C, then A also contains C. CONT 3 shows that the fundamental relation $A \in B$ might be defined in terms of conceptual conjunction (plus identity).

CONJ 1 is the decisive characteristic axiom for *conjunction*, and it establishes a connection between *conceptual* conjunction on the one hand and *propositional* conjunction on the other: concept A contains 'B and C' iff A contains B and A also contains C. The remaining theorems CONJ 2–CONJ 5 may be derived from CONJ 1 with the help of corresponding truth-functional tautologies.

Negation is axiomatized by means of three principles: the law of double negation NEG 1, the law of consistency NEG 2, which says that every concepts differs from its own negation, and the well known principle of contraposition, NEG 3, according to which concept A contains concept B iff $\overline{B}$ contains $\overline{A}$. The further theorem NEG 4 may be obtained from NEG 3 in virtue of CONJ 2.

The important principle POSS 1 says that concept A contains concept B iff the conjunctive concept A Not-B is impossible. This principle also characterizes negation, though only indirectly, since according to DEF 4 the operator of *self-consistency* of concepts is definable in terms of negation and conjunction. POSS 2 says that a term B which is contained in a self-consistent term A will itself be self-consistent. POSS 3 easily follows from POSS 1 in virtue of CONT 1. POSS 4 is the counterpart of what one calls "ex contradictorio quodlibet" in propositional logic: an inconsistent concept contains every other concept! This law was not explicitly stated by Leibniz but it may yet be regarded as a genuinely Leibnitian theorem because it follows from POSS 1 and POSS 3 in conjunction with the observation that, since $A\overline{A}$ is inconsistent, so is, according to POSS 2, also $A\overline{AB}$. Furthermore, in **GP** 7, 224–5 Leibniz remarks that "[...] the round square is a quadrangle with null-angles. For this proposition is true in virtue of an impossible hypothesis". As the text-critical apparatus in **A** VI, 4, 293 reveals, Leibniz had originally added: "Nimirum de impossibile concluditur impossibile". So in a certain way he was aware of the principle "ex contradictorio quodlibet" according to which not only a contradictory *proposition* logically entails any arbitrary proposition, but also a contradictory or "impossible" *concept* contains any other concept.

As was shown in Lenzen [1984b, p. 200], the set of principles {CONT 1, CONT 2, CONJ 1, NEG 1, POSS 1, POSS 2} provides a complete axiomatization of the algebra of concepts which is isomorphic to the Boolean algebra of sets.

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modus ponens: If $A \rightarrow B$ and A, then B. However, as Leibniz goes on to explain, when applied to concepts, a "true" term is to be understood as one that is self consistent: "[...] By 'a false letter' I understand either a false term (i.e. one which is impossible, or, is a non-entity) or a false proposition. In the same way, 'true' can be understood as either a possible term or a true proposition" (**P**, 60). As to the contraposited form of POSS 2, $A \in B \wedge \mathbf{I}(B) \rightarrow \mathbf{I}(A)$, cf. also the special case in **C.**, 310: "Et sanè si DB est non Ens [...] etiam CDB erit non ens".

5 THE PLUS-MINUS-CALCULUS

The so-called Plus-Minus-Calculus (together with its subsystem of the mere Plus-Calculus) was developed mainly in two essays of around 1686/7[18] which have been published in various editions and translations of widely varying quality. The first and least satisfactory edition is Erdmann's **OP** (# XIX), the last and best, indeed almost perfect one may be found in vol. VI, 4 of **A** (## 177, 178). The most popular and most easily accessible edition, however, still is Gerhardt's **GP** 7 (## XIX, XX). English translations have been provided in an appendix to Lewis [1918], in Loemker's **L** (# 41), and in Parkinson's **P** (## 15, 16).

The Plus-Minus-Calculus offers a lot of problems not only concerning interpretation, meaning and consistency of these texts, but also connected with editorial and translational issues. Since the latter have been discussed in sections 2 and 3 of Lenzen [2000], it should suffice here to point out that an adequate understanding of the Plus-Minus-Calculus can hardly be gained by the study of the two above-mentioned fragments alone. On the one hand, some additional short but very important fragments such as **C**. 250–251, **C**. 251, **C**. 251–252 and **C**. 256 (i.e., ## 173, 174, 175, 180, 181 of **A** VI, 4) have to be taken into account. Second, both the genesis and the meaning of the Plus-Minus-Calculus will become clear only if one also considers some of Leibniz's mathematical works, in particular his studies on the foundations of arithmetic.

After sketching the necessary arithmetical background in section 5.1, I will examine in 5.2 how Leibniz gradually develops his ideas of "real addition" and "real subtraction" from the ordinary theory of mathematical addition and subtraction. Strictly speaking, the resulting Plus-Minus-Calculus is not a *logical* calculus but a much more general calculus which allows of quite different applications and interpretations. In its abstract form, it is best viewed as a theory of set-theoretical containment, $\subseteq$, set-theoretical "addition", $A \cup B$, and set-theoretical subtraction, $A - B$, while it comprises neither set-theoretical "negation", $\overline{A}$, nor the elementship-relation, AεB! Furthermore, Leibniz's drafts exhibit certain inconsistencies which result from his vacillating views concerning the laws of "real" subtraction. These inconsistencies can be removed basically in three ways. The first possibility would consist in dropping the entire theory of "real subtraction", $A-B$, thus confining oneself to the mere Plus-Calculus. Second, one might restrict $A - B$ to the case where B is contained in A — a reconstruction of this conservative version of the Plus-Minus-Calculus was given by Dürr [1930]. The third and logically most rewarding alternative consists in admitting "real subtractions" $A - B$ also if $B \not\subseteq A$; in this case, however, one has to dispense with Leibniz's idea that there might exist "privative" entities which are "less than nothing" in the sense that, when $-A$ is added to A, the result will be 0.

[18]This dating by the editors of **A** VI, 4 rests basically on extrinsic factors such as the type of paper and watermarks. Other authors suspect these fragments to have been composed during a much later period. Cf., e.g., Parkinson's classification "after 1690" in the introduction to **P** (p. lv) and the references to similar datings in Couturat [1901, p. 364] and Kauppi [1960, p. 223].

In section 5.3 the application of the Plus-Minus-Calculus to the "intensions" of concepts is considered. One thus obtains two *logical calculi*, L0.4 and L0.8, which are subsystems of the full algebra of concepts, $L1$, and which can accordingly be given an extensional interpretation as developed in section 4 above.

5.1 Arithmetical Addition and Subtraction

From a modern point of view, the operators of elementary arithmetic should be characterized axiomatically by a set of general principles such as:

(ARITH 1) $a = b \vdash \tau(a) = \tau(b)$

(ARITH 2) $a = a$

(ARITH 3) $a + b = b + a$

(ARITH 4) $a + (b + c) = (a + b) + c$

(ARITH 5) $a + 0 = a$

(ARITH 6) $a - a = 0$

(ARITH 7) $a + (b - c) = (a + b) - c.$

Guided by the idea that only identical propositions are genuinely axiomatic while all other basic principles in mathematics (as well as in logic) should be derivable from the definitions of the operators involved, Leibniz tried to reduce the number of axioms to an absolute minimum. Thus in a fragment on "The First Elements of a Calculus of Magnitudes" ["Prima Calculi Magnitudinum Elementa", PCME, for short] only ARITH 2 receives the status of an "*Axiom* $a = a$" (**GM** 7, 77). The rule of substitutivity, ARITH 1, is presented as a definition: "Those are *equal* which can be substituted for one another salva magnitudine" (ibid.). The axiom of commutativity, ARITH 3, appears as a "*Theorem* $+a + b = +b + a$" (**GM** 7, 78).[19] The characteristic axiom of the neutral element 0, ARITH 5, is conceived as an "*Explication* $+0 + a = a$, i.e. 0 is the sign for *nothing,* which adds nothing" (ibid.). The subtraction axiom ARITH 6 is introduced as a logical consequence of the definition of the '-'operation: "Hence [...] $+b - b = 0$" (ibid.). And the structural axiom ARITH 7 is put forward as a "*Theorem* Those to be added are written down with their original signs, i.e. $f + (a - b) = [\ldots]f + a - b$." (**GM** 7, 80).

[19] Leibniz sometimes conceives arithmetic as a theory of positive ($+a$) and negative ($-b$) magnitudes which can be conjoined by the operation of "positing" (denoted by juxtaposition) so as to yield the sum $+a + b$ or the difference $+a - b$: cf. **GM** 7, 78. If the operation of positing itself is assumed to be commutative ("... nihil refert, quo ordine collocentur"), then not only '+' is provably commutative, but so is also '−' in the sense of: "$a - b = -b + a$" (AEAS, 19 v.); or "$-a - b = -b - a$ seu transpositio" (AEAS, 20 v.). In "Conceptus Calculi" Leibniz mistakenly claimed subtraction to be symmetric in the stronger sense: "In additione et subtractione [...] ordo nihil facit, ut $+b + a$ aequ. $+a + b, b - a$ aequ. $a - b$" (**GM** 7, 84).

The latter, unbracketed formulation of the term '$(f + a) - b$' already indicates that Leibniz never took very much care about bracketing. This is not only confirmed by the fact that he habitually "forgot" to state the law of associativity, ARITH 4, but also by various other examples. For example, the theorems:

(ARITH 8) $(a + b) - b = a$

(ARITH 9) $(a - b) + b = a$

were stated by Leibniz in an hitherto unpublished manuscript "De Aequalitate; Additione; Subtractione" (**LH** XXXV, 1, 9, 18–21 — AEAS, for short) quite ambiguously as "$a + b - b = a$" (AEAS, 21 r.) and "$+a - b + b$ will be equiv. to a".[20] This unbracketed formulation seduced him to think that ARITH 8 might be proved as follows: "for $b - b$ putting 0 gives $a + 0 = a$" (AEAS, 21 r.). Actually, however, ARITH 7 has to be presupposed to guarantee that $(a + b) - b$ equals $a + (b - b)$. That Leibniz really had ARITH 8 and 9 in mind is evidenced by the fact that he considered

(ARITH 10) "If $a + b = c$ then $c - b = a$" (AEAS, 21 r.)

(ARITH 11) "If $a - b = c$ then $a = c + b$" (AEAS, 20r)

as immediate corollaries of the former theorems. The subsequent two principles are special instances of the rule ARITH 1:

(ARITH 12) "If you add equals to equals, the results will be equal, i.e. if $a = l$ and $b = m$, then $a + b = l + m$" (**GM** 7, 78)

(ARITH 13) "If you subtract equals from equals, the rest will be equal, i.e. if $a = l$ and $b = m$, then $a - b = l - m$" (**GM** 7, 79)

By contrast, the converse inference

(ARITH 14) "Si $a = l$ et $a + b = l + m$ erit $b = m$" (AEAS, 19 v.)

(ARITH 15) "Si $a - b = l - m$ et sit $b = m$ erit $a = l$" (*ibid.*)

cannot be derived from the axioms of equality, ARITH 1 and 2, alone. Leibniz's negligent attitude towards bracketing veils that the "proof" of, e.g., ARITH 14: "For $b + a = m + l$ (by transpos. of add.) therefore (by the preced.) $b + a - a = m + l - l$. Hence $b = m$" (AEAS, 20 v.) makes use not only of ARITH 3 ("transpos. of add.") and ARITH 13 ("preced."), but also presupposes either ARITH 8 or ARITH 7 when $(b + a) - a$ is tacitly equated with $b + (a - a)$.

It may be interesting to note that in the unpublished fragment, "Fundamenta Calculi Literalis", Leibniz came to recognize the axiomatic status of ARITH 1, 2, 3, 5, and 6. After stating the usual principles of the equality relation, he listed the relevant

[20] The latter quotation is not from AEAS but from Knobloch [1976, p. 117].

Axioms in which the meaning of the characters is contained [...]

(4) $+a + b = +b + a[\ldots]$

(5) $a + 0 = a[\ldots]$

(9) $a - a = 0[\ldots]$ (**LH** XXXV, XII, 2, 72 r.)

Originally he had also included "(2) $a = c$ is equivalent to $a + b = c + b$" (*ibid.*); but later on he thought that this equivalence "can be proved [...] by the Def. of equals" (*ibid.*). Once again his negligence concerning brackets may have been due to his recognizing that only one half of the equivalence, viz. ARITH 12, follows from the above axioms while the other implication, ARITH 14, additionally presupposes the crucial axiom ARITH 7. Anyway, it is quite typical of Leibniz that he "forgot" to state just those two basic principles, ARITH 4 and 7, which involve brackets.

For the sake of the subsequent discussion it should be pointed out that (on the basis of the remaining axioms ARITH 1-6) ARITH 7 can be replaced equivalently by the conjunction of ARITH 8 and 9.[21] Furthermore the related structural laws

(ARITH 16) $a - (b + c) = (a - b) - c$

(ARITH 17) $a - (b - c) = (a - b) + c$

can be derived either from ARITH 7 or from ARITH 8 + 9.[22] ARITH 17 was formulated by Leibniz as the rule: "Those to be subtracted will be written down with signs changed, + in −, and − in +, i.e. $f - (a - b) = f - a + b$" (**GM** 7, 80). And in AEAS he presented an elliptic version of ARITH 16 in a way that indicates that here at least he became aware of the logical function of brackets: "$-(a+b) = -a-b$. This is the meaning of brackets" (o.c., 19 r.) It will turn out in the next section that it is just axiom ARITH 7 (and the theorems that depend on it) which lead into difficulties when one tries to transfer the mathematical theory of '+' and '−' to the field of "real entities".

5.2 "Real" Addition and Subtraction

Already in PCME Leibniz envisaged to apply the arithmetical calculus to "things", e.g. to "straight lines to be added or subtracted" (o.c., # (25)). In the fragments # XIX and XX of **GP** 7, he mentions two further applications: the addition or composition, i.e. conjunction, of concepts, or the addition, i.e. union, of sets. In what follows we will concentrate upon the latter interpretation where accordingly '−' represents set-theoretical subtraction and '0' stands for the empty set which shall therefore be symbolized as '$\emptyset$'. The underlying theory of '=' now, of course,

[21] According to ARITH 4 and 9 $(a + (b - c)) + c = a + ((b - c) + c) = a + b$; from this it follows by ARITH 10 which is an immediate corollary of ARITH 8 that $(a + b) - c = a + (b - c)$.

[22] According to ARITH 3, 4, 9: $((a-b)-c)+(b+c) = ((a-b)-c)+(c+b) = (((a-b)-c)+c)+b = (a - b) + b = a$; hence it follows by ARITH 10: $a - (b + c) = (a - b) - c$. Similarly, according to ARITH 16 and 9: $(a - (b - c)) - c = a - ((b - c) + c) = a - b$, from which it follows by ARITH 11 that $(a - b) + c = a - (b - c)$.

no longer refers to the relation of numerical *equality* but to the stricter relation of *identity* or coincidence. Thus, e.g., the basic rule of substitutivity, $A = B \vdash \tau(A) = \tau(B)$, has to be reformulated with 'salva veritate' replacing 'salva magnitudine' (cf. **GP** 7, 236, Def. 1). Accordingly ARITH 12 and 13 now reappear as "If coinciding [terms] are added to coinciding ones, the results coincide" (**GP** 7, 238) and "If from coinciding [terms] coinciding ones are subtracted, the rests coincide" (**GP** 7, 232). The law of reflexivity, $A = A$, can be adopted without change. The law of symmetry of set-theoretical addition now is presented as "*Axiom. 1* $B + N = N + B$, i.e. transposition here makes no difference" (**GP** 7, 237). The "real nothing", i.e. the empty set $\emptyset$, is characterized as follows "It does not matter whether Nothing [nihil] is put or not, i.e. A+Nih.= A" (**C.** 267),

(NIHIL 1) $\quad A + \emptyset = A.$

The subtraction of sets is again conceived in analogy to the arithmetical case as the converse operation of addition: "If the same is put and taken away [...] it coincides with Nothing. I.e. $A[\ldots] - A[\ldots] = N$" (**GP** 7, 230), formally:

(MINUS 1) $\quad A - A = \emptyset.$

The main difference between arithmetical addition on the one hand and "real addition" on the other is that, whereas for any number $a \neq 0, a + a$ is unequal to a, the addition of one and the same set A does not yield anything new:

(PLUS 1) $\quad$ "$A + A = A$ [...] or the repetition here makes no difference" (**GP** 7, 237).

However, this new axiom cannot simply be added to the former collection without creating inconsistencies. As Leibniz himself noticed, it would otherwise follow that there is no real entity besides $\emptyset$: "For e.g. [by PLUS 1] $A + A = A$, therefore one would obtain [by the analogue of ARITH 10] $A - A = A$. However (by [MINUS 1]) $A - A =$ Nothing, hence A would be = Nothing" (**C.** 267, # 29). Thus any non-trivial theory of real addition satisfying PLUS 1 has to reject as least the counterparts of the laws ARITH 10 (or ARITH 8) and ARITH 7.

As was suggested by Leibniz, ARITH 10 should be restricted to the special case where A and B are uncommunicating or have nothing in common: "Therefore if $A + B = C$, then $A = C - B$[...] But it is necessary that A and B have nothing in common" (**C.** 267, # 29).[23] A precise definition of this new relation presupposes that one first introduces the more familiar relation 'A contains B' or its converse 'A is contained in B', formally $A \subseteq B$, as follows:

> $A + Y = C$ means 'A is in C', or 'C contains A'. (cf. **C.** 265, ## 9, 10).

[23]Leibniz also recognized that the same restriction was necessary in the case of ARITH 14: "Si $A + B = D + C$ et $A = D$, erit $B = C$.[...] Imo non sequitur nisi in incommunicantibus" (**C.**, 268).

That is, C contains A iff there is some set Y such that the union of A and Y equals C. As Leibniz noted in Prop. 13 and Prop. 14 of fragment XX, this definition may be simplified by replacing the variable 'Y' by 'C':

(DEF 6) $A \subseteq B \leftrightarrow_{\text{df}} A + B = B.$

It is now possible to define:

> If some term, M, is in A, and the same term is in B, this term is said to be 'common' to them, and they will be said to be 'communicating'.[24]

I.e., two sets A and B have something in common iff there exists some Y such that $Y \subseteq A$ and $Y \subseteq B$. Since, trivially, the empty set is included in any set A (cf. NIHIL 1)

(NIHIL 2) $\emptyset \subseteq A,$

one has to add the qualification that Y is not empty:

(DEF 7) $\text{Com}(A, B) \leftrightarrow_{\text{df}} \quad \exists Y(Y \neq \emptyset \wedge Y \subseteq A \wedge Y \subseteq B).$

The necessary restriction of ARITH 8 can then be formalized as

(COM 1) $\neg\text{Com}(A, B) \rightarrow (A + B) - B = A.$

According to Leibniz this implication may be strengthened into a biconditional:

> Suppose you have A and B and you want to know if there exists some M which is in both of them. Solution: combine those two into one, $A+B$, which shall be called L [...] and from L one of the constituents, A, shall be subtracted [...] let the rest be N; then, if N coincides with the other constituent, B, they have nothing in common. But if they do not coincide, they have something in common which can be found by subtracting the rest N, which necessarily is in B, from B [...] and there remains M, the commune of A and B, which was looked for.[25]

What is particularly interesting here is that Leibniz not only develops a criterion for the relation $\text{Com}(A, B)$ in terms of whether $(A + B) - B$ coincides with A or not, but that he also gives a formula for "the commune" of A and B in terms of addition and subtraction. If '$A \cap B$' denotes the commune, i.e. the intersection of A and B, Leibniz's formula takes the form:

(COM 2) $A \cap B = B - ((A + B) - A).$

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[24]**P.**, 123; cf. **GP** 7, 229: "Si aliquid M insit ipsi A, itemque insit ipsi B, id dicetur ipsis commune, ipsa autem dicentur communicantia".

[25]Cf. **C.**, 250: "Sint A et B, quaeritur an sit aliquod M quod insit utrique. Solutio: fiat ex duobus unum $A + B$ quod sit L [...] et ab L auferatur unum constituentium A [...] residuum sit N, tunc si N coincidit alteri constituentium B, nihil habebunt commune. Si non coincidant, habebunt aliquid commune, quod invenitur, si residuum N quod necessario inest ipsi B detrahatur a B [...] et restabit M quaesitum commune ipsis A et B."

Closely related with COM 2 is the following theorem: "If, however, two terms, say A and B, are communicating, and A shall be constituted by B, let again be $A + B = L$ and suppose that what is common to A and B is N, one obtains $A = L - B + N$";[26] formally:

(COM 3) $\quad A = ((A + B) - B) + (A \cap B).$

The subsequent theorems also may be of interest: "What has been subtracted and the remainder are uncommunicating" (**P.**, 128; cf. **GP** 7, 234), formally:

(COM 4) $\quad \neg\mathrm{Com}(A - B, B).$

"Case 2. If $A + B - B - G = F$, and everything which both A and B and B and G have in common is M, then $F = A - G$"[27], formally:

(COM 5) $\quad A \cap B = A \cap C \rightarrow ((A + B) - B) - C = A - C.$

Furthermore one gets the following necessary restriction of ARITH 14: "In symbols: $A + B = A + N$. If A and B are uncommunicating, then $B = N$" (**P.**, 130; cf. **GP** 7, 235), formally:

(MINUS 2) $\quad \neg\mathrm{Com}(A, B) \wedge \neg\mathrm{Com}(A, C) \rightarrow (A + B = A + C \rightarrow B = C).$

Finally, when Leibniz remarks: "Let us assume meanwhile that E is everything which A and G have in common — if they have something in common, so that if they have nothing in common, A = Nothing",[28] he thereby incidentally formulates the following law which expresses the obvious connection between the relation of communication and the operator of the commune:

(COM 6) $\quad (A \cap B) = \emptyset \leftrightarrow \neg\mathrm{Com}(A, B).$

In this way Leibniz gradually transforms the theory of mathematical addition and subtraction into (a fragment of) the theory of sets. It is interesting to see how the problem of incompatibility between the arithmetical axiom ARITH 7 and the new characteristic axiom of set-theoretical union, PLUS 1, leads him to the discovery of the new operators '$\subseteq$', 'Com', and '$\cap$' which have no counterpart in elementary arithmetic.

It cannot be overlooked, however, that the theory of real addition and subtraction is incomplete in two respects. First, the axioms and theorems actually found by Leibniz are insufficient to provide a complete axiomatization of the set of operators $\{=, +, \emptyset, -, \subseteq, \mathrm{Com}, \cap\}$; second, when compared to the full algebra of sets, Leibniz's operators turn out to be conceptually weaker. In particular, it is not possible to define negation or complementation in terms of subtraction (plus the remaining operators listed above). Leibniz only pointed out that there is a *difference* between negation (i.e., set-theoretical complement) and subtraction:

[26]Cf. **C.**, 251: "Sin communicantia sint duo, ut A et B, et A constitui debeat per B, fiat rursus $A + B = L$ et posito ipsis A et B commune esse N, fiet $A = L - B + N$".

[27]**P.**, 127; cf. **GP** 7, 233: "Si $A + B - B - G = F$, et omne quod tam A et B, quam G et B commune habent, sit M, erit $F = A - G$."

[28]**P.**, 127; cf. **GP** 7, 233: "Ponamus praeterea omne quod A et G commune habent esse E [...] ita ut si nihil commune habent, E sit = Nih.".

> *Not* or the negation differs from *Minus* or the subtraction in so far as a repeated 'not' destroys itself while a repeated subtraction does not destroy itself.[29]

Furthermore he believed that just as the "negation" of a positive number a is the negative number $(-a)$, i.e. $(0-a)$, so also in the domain of real things the "negation" of a set A should be conceived of as a "privative" thing $(\emptyset - A)$:

> If from a B some C shall be subtracted which is not in B, the rest A or $B-C$ will be a semi-privative thing, and is a D is added, then $D+A = E$ means that in a way D and B have to be put in E, yet first C has to be removed from D [...] Thus let be [...] $E = L - M$ where L and M have nothing else in common; now if L and M (uncommunicating) are both positive, then E will be a *semi-privative thing.* If $M =$ Nothing, then $E = L$ and E will be a *positive thing* [...]; finally, if L is $=$ Nothing, then $E = M$ and E will be a *privative thing.*[30]

To be sure, if ARITH 7, 9, or 11 would also hold in the case of real addition and subtraction, then it might be shown that there exist privative sets which are "less than nothing" in the sense that when $(-M)$ is added to M, the result equals the empty set $\emptyset$. E.g., letting be $A = \emptyset$ in ARITH 9, one immediately obtains $(\emptyset - B) + B = \emptyset$; and ARITH 7 analogously entails that $B + (\emptyset - B) = (B + \emptyset) - B = B - B = \emptyset$. However, the existence of a privative set $-B$ which is "less than nothing" is inconsistent with the rest of Leibniz's theory of sets, in particular with the characteristic axiom PLUS 1. Since $B = B + B$, it follows that $B + (-B) = (B + B) + (-B) = B + (B + (-B))$; hence if $B + (-B)$ were equal to $\emptyset$, one would obtain that $\emptyset = B + \emptyset = B$, i.e. each set B would coincide with $\emptyset$.[31]

It is somewhat surprising to see that, although Leibniz clearly recognized that the first half of ARITH 7, viz. ARITH 8 or 10, is no longer valid in the field of real entities, he failed to recognize that the other half, i.e. ARITH 9 or 11, which involves the existence of "privative sets", also has to be abandoned. In fragment

[29] Cf. **C.**, 275: "Differunt *Non* seu negatio a [...] *Minus* seu detractione, quod 'non' repetitum tollit se ipsum, at vero detractio repetita non seipsam tollit." Leibniz goes on to explain that "non-non B est B, sed $-B$ idem est quod Nihilum. Verbi gratia [...] $A - B$ est A." This happens to be true, though, in the sense that $A - (-B) = A - (0 - B) = A - 0 = A$; but this equation is based upon the non-existence of "privative sets" which contradicts Leibniz's explicit statements some lines earlier.

[30] Cf. **C.**, 267–8: "Si ab aliquo B detrahi jubeatur C quod ipsi non inest, tunc residuum A seu $B - C$ erit res semi-privativa et si apponatur alicui D, tunc $D + A = E$ significat D quidem et B esse ponenda in E, sed tamen a D prius esse removendum C [...] sic ut sit [...] $E = L - M$ et L atque M nihil amplius habebunt commune; quodsi jam L et M (incommunicantia) ambo sint aliquid positivum, erit E *res semiprivativa.* Sin sit $M =$ Nih. erit $E = L$, seu E erit *res positiva* [...]; denique sin sit $L =$ Nih. erit $E = M$, seu E erit *res privativa.*" Cf. also **C.**, 275: "Hinc si ponatur $D - B$, et D non contineat B, non ideo putandum est notam omissivam nihil operari. Saltem enim significat provisionaliter, ut ita dicam, et in antecessum, si quando contingat augeri $D - B$ per adjectionem alicujus cui insit B, tunc saltem sublationi illi locum fore. Exempli causa si $A = B + C$ erit $A + D - B = D + C$."

[31] This proof, by the way, presupposes the axiom of associativity, ARITH 4, $A + (B + C) = (A + B) + C$.

XIX of **GP** 7, which may be considered as an attempt to give a final form of the theory of real addition and subtraction, Leibniz "solved" the problems at hand by just restricting subtractions ($A - B$) to the case where $B \subseteq A$:

> Postulate 2. Some term, e.g. A, can be subtracted from that in which it is – e.g., from $A + B$. (**P**. 124; cf. **GP** 7, 230).

Leibniz still stuck to the idea that otherwise "privative sets" would result[32], and he failed to see that ARITH 16 (which he had tacitly presupposed in several other places[33]) is set-theoretically valid and entails that

(MINUS 3) $\emptyset - B = \emptyset$.[34]

Hence real subtractions never yield "less than Nothing".

To conclude this section let me point to some modifications of Leibniz's theory of real addition which are (necessary and) sufficient for obtaining a *complete* version of the algebra of sets. First, one has to introduce a new constant, U, denoting the universal set (or the universe of discourse). This set may be characterized axiomatically by the principle that U contains any set A:

(UD 1) $A \subseteq U$.

Second, the commune of A and B will have to be characterized by the axiom

(COM 7) $C \subseteq A \cap B \leftrightarrow C \subseteq A \wedge C \subseteq B$.

Leibniz put forward this defining principle only indirectly when he referred to the commune of two sets as "that in which there is whatever is common to each"[35]. Third, instead of ARITH 7, which becomes invalid in the area of set-theory, one has to adopt former theorem ARITH 16:

(MINUS 4) $A - (B + C) = (A - B) - C$,

plus the following refinement of ARITH 17:

(MINUS 5) $A - (B - C) = (A - B) + (A \cap C)$.

It may then be shown that the resulting collection of principles[36] forms a complete axiomatization of the algebra of sets, where negation is definable by $\overline{A} =_{\text{df}} U - A$.

[32] **P**. 127, fn. 1; cf. **GP** 7, 233: "[...] hinc detractiones possunt facere nihilum [...] imo minus nihilo".

[33] Cf. his "proof" of "Theor. IX" in **GP** 7, 233.

[34] According to MINUS 1, ARITH 16, and 8: $0 - B = (B - B) - B = B - (B + B) = B - B = 0$!

[35] **P**., 128; cf. **GP** 7, 234: "id cui inest quicquid utrique commune est".

[36] I.e., the counterparts of ARITH 1–6, and the "new" principles UD 1, COM 7, MINUS 4 and 5. For details cf. Lenzen [1989a].

5.3 Application of the Plus-Minus-Calculus to Concepts

The main draft of the Plus-Minus-Calculus was aptly called by Leibniz "A not inelegant specimen of *abstract* proof". This led some commentators to attribute to him the insight:

> [...] that logics can be viewed as abstract formal systems that are amenable to alternative interpretations. [...] In Leibniz's intensional interpretations of his system, $\oplus$ is a *conjunction*-like operator on concepts, but in his extensional interpretations, it becomes a *disjunction*-like operation on extensions (in effect, it becomes set-theoretic union).[37]

This view of the dual interpretability of '+' as conjunction and as disjunction is, however, misleading. It is true, though, that *if* the Plus-Calculus is considered as an abstract structure whose operators $\langle +, \subseteq \rangle$ are only implicitly defined by the axioms, then there exist different *models* for this system. As was shown, e.g., in Dürr [1930], in a first model '$A + B$' may be interpreted as the conjunction (or intersection) of A and B, while in a second model '$A + B$' is interpreted as the disjunction (or union) of A and B. However, these models will satisfy the axioms of the Plus-Minus-Calculus only if the interpretation of the remaining operators of the abstract structure also are duly adjusted. Thus in view of the equivalence expressed in "Theorem VII" + "Converse of the preceding Theorem":

> [...] if B is in A, then $A + B = A$. [...] If $A + B = A$, then B will be in A. (**P**., 126/7; cf. **GP** 7, 232)

in the first model (with '+' taken as '$\cap$') the fundamental inesse-relation would have to be interpreted as the *super*set-relation $B \supseteq A$; while only in the second model (with '+' taken as '$\cap$') "B is in A" might be interpreted like in DEF 1 as the *subset*-relation $B \subseteq A$.

Dürr [1930, p. 42] holds that Leibniz himself had envisaged the dual interpretation of the abstract structure either as $\langle \cap, \supseteq \rangle$ or as $\langle \cup, \subseteq \rangle$ because he thought that Leibniz had used the expression "A is in B" *alternatively* in the sense of $A \subseteq B$ or in the sense of $B \subseteq A$. Dürr quotes the remark that "the concept of the genus is in the concept of the species, the individuals of the species in the individuals of the genus" (**P** 141) as evidence for Leibniz's allegedly vacillating interpretation of the phrase "A is in B" [A inest ipsi B]. But this is untenable. For Leibniz, the logical operator "A is in B" *always* means exactly what it literally says, namely that A *is contained in* B. The crucial quotation only expresses the law of reciprocity, RECI 1, according to which the intension of the concept of the *genus* is contained in the intension of the concept of the species, while at the same time the extension of the concept of the *species* is contained in the extension of the concept of the genus. In both cases one and the same logical (or set-theoretical) relation of containment, $\subseteq$, is involved.

[37]Swoyer [1995, p. 104]. Cf. also Schupp [2000, LII].

There is one further, elementary point which proves that Leibniz's addition $A + B$ always has to be interpreted as the *union* of A and B. Within the framework of the Plus-Minus-Calculus, the operators $\langle +, \subseteq \rangle$ are only part of a larger structure which contains in particular also the distinguished element '0' ("Nothing"). Thus, if $\langle \cap, \supseteq \rangle$ would constitute a model of the Plus-Minus-Calculus, then the defining axiom AX 5, $A + 0 = A$, would have to hold. But with '+' interpreted as '$\cap$', this would mean that '0' is not the empty but the *universal* set! Such an interpretation, however, is entirely incompatible with Leibniz's characterization of '0' as "Nihilum"![38]

What is at issue, then, is not a dual (or multiple) interpretation in the sense of Dürr's different *models*, but rather, as Leibniz himself stressed, different *applications* of the Plus-Minus-Calculus.[39] One particularly important application concerns the realm of:

> [...] absolute concepts, where no account is taken of order or of repetition. Thus it is the same to say 'hot and bright' as to say 'bright and hot', and [...] 'rational man' — i.e. 'rational animal which is rational' — is simply 'rational animal'. (*ibid.*).

Let us now take a closer look at this interpretation of the Plus-Minus-Calculus, where the entitites A, B are viewed as (intensions of) *concepts* and where the sum $A+B$ therefore corresponds to (the intension of) the *conjunction* AB in accordance with Leibniz's remark: "For $A + B$ one might put simply AB".[40] Hence the extensional interpretation of $A + B$ coincides with our earlier requirement:

$$(4) \quad \phi(A \oplus B) = \phi(AB) = \phi(A) \cap \phi(B).$$

Most of the basic theorems for conjunction mentioned in section 4 now reappear in the Plus-Minus-Calculus as theorems of conceptual addition. For example, one half of the equivalence CONJ 1 is put forward as "*Theorem* V [...] If A is in C and B is in C, then $A+B$ [...] is in C" (**P**, 126). CONJ 2 is formulated in passing when Leibniz notes that "N is in $A \oplus N$ (by the definition of 'inexistent')" (**P**, 136). CONJ 4 simply takes the shape of "*Axiom 2* [...] $A + A = A$" (**P**, 132); and CONJ 5 is similarly formulated as "*Axiom 1* $B \oplus N = N \oplus B$".

The law of the reflexivity of the $\in$-relation, CONT 1, reappears as "Proposition 7. A is in A" which, interestingly, is proven by Leibniz as follows: "For A is in $A \oplus A$ (by the definition of 'inexistent' [...]), and $A \oplus A = A$ (by axiom 2). Therefore [...] A is in A" (**P**, 133). The counterpart of the law of transitivity of the $\in$-relation, CONT 2, is formulated straightforwardly as "THEOREM IV [...] if A is in B and B is in C, A will also be in C" (**P**, 126). And the analogue of

[38] **C.**, 267, # 28: "*Nihilum* sive ponatur sive non, nihil refert. Seu A + *Nih.* ∞ A". Dürr [1930: 96] was well aware of this axiom and pointed out that in the second model "Nihil" corresponds to the "allumfassende Klasse".

[39] Cf. **P.**, 142: "[...] whenever these laws [$A + B = B + A$ and $A + A = A$] are observed, the present calculus can be appplied".

[40] Cf. **C.**, 256: "Pro $A + B$ posset simpliciter poni AB".

CONT 3, $A \in B \leftrightarrow A = AB$, is formulated in two parts as "THEOREM VII [...] if B is in A, then $A + B = A$" and as "*Converse of the preceding theorem* [...] If $A + B = A$, then B will be in A" (**P**, 126-7). Here, of course, 'A is in B' is taken to hold if and only if, in the terminology of $L1$, "B contains A".

The mere Plus-Calculus, $L0.4$, as developed in the "Study in the Calculus of Real Addition" is the logical theory of the operators 'ι' (or '$\in$'), '$\oplus$', and '$=$'. Although the theorems for identity (coincidence) are developed there in rather great detail, it remains a very weak and uninteresting system (at least in comparison with the full algebra of concepts, $L1$); thus it shall no longer be considered here. Much more interesting, however, is the Plus-*Minus*-Calculus, $L0.8$, which contains many challenging laws for conceptual subtraction and for the auxiliary notions of the empty concept 0, the relation of communication among concepts, $\mathrm{Com}(A, B)$, and for the commune of A and B, $A \otimes B$, which comprises all what two concepts A and B have in common.

The "empty concept"

When the Plus-Minus-Calculus is applied to (intensionally conceived) concepts, the empty *set* "Nihil" corresponds to the empty *concept*, i.e. the concept which has an (almost) empty *intension*. Leibniz tried to define or to characterize this concept as follows:

> *Nothing* is that which is capable only of purely negative determination, namely if N is not A, neither B, nor C, nor D, and so forth, then N can be called *Nothing.*[41]

The 'and so forth'-clause should be made more precise by postulating that *for no concept* Y, N contains Y. Within the framework of Leibniz's quantifier logic (to be developed systematically in section 6 below), this definition would take the form $N = 0 \leftrightarrow \neg\exists Y(N \in Y)$. However, according to CONT 1, each concept contains itself; hence the empty concept always contains at least one concept, namely 0. Therefore one has to amend Leibniz's definition by adding the restriction that 0 contains no *other* concept Y (different from 0):

(DEF 8) $\quad A = 0 \leftrightarrow_{\mathrm{df}} \neg\exists Y(A \in Y \wedge Y \neq A)$.

As we saw earlier, the "addition" of 0 to any concept A leaves A unchanged, i.e. $A + 0 = A$ or, equivalently, $A0 = A$. According to CONT 3 this means that 0 is contained in each concept A:

(NIHIL 1) $\quad A \in 0$.

[41] Cf. **A** VI, 4, 625: "*Nihil* est cui non competit nisi terminus mere negativus, nempe si N non est A, nec est B, nec C, nec D, et ita porro, tunc N dicitur esse *Nihil*". Cf. also **A** VI, 4, 551: "Si N non est A, et N non est B, et N non est C, et ita porro; N dicetur esse Nihil" or **C**., 252: "Esto N *non est* A, item N *non est* B, item N *non est* C, et ita porro, tunc dici poterit N *est Nihil*".

Furthermore it is easy to prove that the empty concept 0 coincides with the *tautological* concept:

(NIHIL 2) $\quad 0 = \overline{A\overline{A}}$

For according to POSS 4, $A\overline{A} \in Y$ for every Y. Hence by the law of contraposition, the negation of $A\overline{A}$, i.e. the tautological concept, *is contained* in every Y. Thus if there exists some Y such that $\overline{A\overline{A}}$ *contains* Y, it follows by DEF 2 that $Y = \overline{A\overline{A}}$.

If it is further observed that, according to RECI 1, a concept with minimal intension must have maximal extension, we obtain the following requirement for the extensional interpretation of the empty (or tautological) concept 0:

(7) $\phi(0) = U$ (universe of discourse).

(Un)communicating concepts and their commune

Under the present application of the Plus-Minus-Calculus, the relation of communication no longer expresses the fact that two *sets* A and B are overlapping, but $\mathrm{Com}(A, B)$ means that the *concepts* A and B "have something in common" [A et B habent aliquid commune; A et B sunt communicantia]. This relation can be defined as follows:

> If some term, M, is in A, and the same term is in B, this term will be said to be 'common' to them, and they will be said to be 'communicating'. If, however, they have nothing in common [...], they will be called 'uncommunicating'. (**P**, 123)

This explanation might be formalized straightforwardly as $\mathrm{Com}(A, B) \leftrightarrow \exists X(A \in X \wedge B \in X)$. But since the empty, tautological concept 0 is contained in each A, it has to be modified as follows:

(DEF 9) $\quad \mathrm{Com}(A, B) \leftrightarrow_{\mathrm{df}} \exists X(X \neq 0 \wedge A \in X \wedge B \in X)$.

Now, whenever A and B are communicating, Leibniz refers to what they have in common as "quod est ipsis A et B commune", and he explained the meaning of this operator quite incidentally as follows:

> In two communicating terms [A and B, M is] that in which there is whatever is common to each [iff ...] $A = P + M$ and $B = N + M$, in such a way that whatever is in A and [in] B is in M but nothing of M is in P or N. (**P**, 128).

The first equation, $A = P + M$, says that the commune of A and B, M, together with some other concept P constitutes A, i.e. M is contained in A. If we symbolize the commune of A and B, i.e. the "greatest" concept C that is contained both in A and in B, by '$A \otimes B$', this condition amounts to the law:

(COMM 1) $\quad A \in A \otimes B$.

Similarly, the second equation, $B = N + M$, entails that

(COMM 2) $\quad B \in A \otimes B.$

Moreover, "whatever is in A and [in] B is in M", i.e. whenever some concept C is contained both in A and in B, it will also be contained in the commune:

(COMM 3) $\quad A \in C \wedge B \in C \rightarrow A \otimes B \in C.$

Thus in sum the commune may be defined as that concept C which contains all and only those concepts Y that are contained both in A and in B:

(DEF 10) $\quad A \otimes B = C \leftrightarrow_{\mathrm{df}} \forall Y(C \in Y \leftrightarrow A \in Y \wedge B \in Y).$

Now it is easy to prove (although Leibniz himself never realised this) that the commune of A and B coincides with the disjunction, i.e. the 'or-connection' of both concepts:

(COMM 4) $\quad A \otimes B =_{\mathrm{df}} \overline{\overline{A}\,\overline{B}}.$

According to DEF 10, it only has to be shown that for any concept $Y : \overline{\overline{A}\,\overline{B}} \in Y$ iff $A \in Y$ and $B \in Y$. Now if (1) $A \in Y \wedge B \in Y$, then by the law of contraposition, NEG 3, $\overline{Y} \in \overline{A} \wedge \overline{Y} \in \overline{B}$, hence by CONJ 1 $\overline{Y} \in \overline{A}\,\overline{B}$, from which one obtains by another application of NEG 3 that $\overline{\overline{A}\,\overline{B}} \in Y$; (2) if conversely for any $Y \overline{\overline{A}\,\overline{B}} \in Y$, then the desired conclusion $A \in Y \wedge B \in Y$ follows immediately from the laws

(DISJ 1) $\quad A \in \overline{\overline{A}\,\overline{B}}$

(DISJ 2) $\quad B \in \overline{\overline{A}\,\overline{B}}$

in virtue of CONT 2. The validity of DISJ 1, 2 in turn follows from the corresponding laws of conjunction (CONJ 2, 3), $\overline{A}\,\overline{B} \in \overline{A}$ and $\overline{A}\,\overline{B} \in \overline{B}$ by means of contraposition, NEG 3, plus double negation, NEG 1. In view of COMM 4, then, one obtains the following condition for the extensional interpretation of the commune of A and B:

$$(8)\ \phi(A \otimes B) = \phi(A) \cup \phi(B).$$

Furthermore, as Leibniz noted in passing[42], two concepts are communicating iff the commune of A and B is not the empty concept:

(COMM 5) $\quad \mathrm{Com}(A, B) \leftrightarrow A \otimes B \neq 0.$

Hence the extensional interpretation for the relation $\mathrm{Com}(A, B)$ amounts to the condition that the extensions of A and B are non-exhaustive:

$$(9)\ \phi(\mathrm{Com}(A, B)) = \text{true iff } \phi(A) \cup \phi(B) \neq U.$$

[42]Cf. **P**, 127, Theorem IX: "Let us assume meanwhile that E is everything which A and [B] have in common — if they have something in common, so that if they have nothing in common, E = Nothing".

Conceptual subtraction

To conclude our discussion of the Plus-Minus-Calculus, we have to (re)consider the operation of real subtraction, $A - B$, as applied to (intensionally conceived) *concepts*. Leibniz tried to define this operation as follows:

> *Definition 5.* If [B] is in [A], and some other term, [C], should be produced in which there remains everything which is in [A] except what is also in [B] (of which nothing must remain in [C]), B will be said to be subtracted or removed from [A], and C will be called the 'remainder'. (**P**, 124).

Thus $(A - B)$ is said to contain all and only those (non-empty) concepts Y which are contained in A but which are not contained in B:

(DEF 11*) $A - B = C \leftrightarrow_{\text{df}} \forall Y(Y \neq 0 \rightarrow (C \in Y \leftrightarrow A \in Y \wedge B \notin Y))$.

This definition entails, firstly, that, as Leibniz postulated in an extra "*Axiom 2*: If the same term is added and subtracted, then [...] this coincides with Nothing. That is A [...] — A [...] = Nothing" (**P**, 124):

(MINUS 1) $A - A = 0$.[43]

Second, a concept Y can remain in the "remainder" $A - B$, only if Y was originally contained in A itself: $\forall Y((A - B) \in Y \rightarrow A \in Y)$. Substituting $(A - B)$ for Y, one thus obtains (in view of the trivial law CONT 1):

(MINUS 2) $A \in (A - B)$.

Third, whenever some (non-empty) concept C is contained both in A and in B, then C is no longer contained in the remainder $(A - B) : A \in C \wedge B \in C \wedge C \neq 0 \rightarrow (A - B) \notin C$. Thus in particular there does not exist a (non-empty) concept C which is contained both in B and in $(A - B)$, or, as Leibniz put it: "What has been subtracted and the remainder are uncommunicating. If $L - A = N$, I assert that A and N have nothing in common" (**P**, 128):

(MINUS 3) $\neg\text{Com}(A - B, B)$.

Fourth, the above version of DEF 11* would allow to infer that any (non-empty) concept C which is contained in A but not in B will therefore be contained in $(A - B)$:

(MINUS 4*) $A \in C \wedge B \notin C$ [$\wedge C \neq 0$][44] $\rightarrow (A - B) \in C$.

[43] For according to DEF 11* $A - A$ would contain a non-empty concept Y only if both $A \in Y$ and $A \notin Y$!

[44] Unlike in DEF 11*, this restriction now is redundant since in view of NIHIL 1 $B \notin C$ already entails that $C \neq 0$.

But this is incompatible with certain other basic principles of the Plus-Minus-Calculus. Consider, e.g., the case where A is the sum of two uncommunicating (non-empty) concepts B and $C, A = B + C$, or $A = BC$. Clearly, A contains B, but not conversely. Hence one could derive from MINUS 4* (with 'A' substituted for 'C') that $(A - B) \in A$ which, in view of MINUS 2, would mean that $(A - B) = A$! But this is absurd, since if you subtract from $A = BC$ one of the (uncommunicating) components, B, then, as Leibniz's himself noted elsewhere[45], the remainder will be just the other component, C:

(MINUS 5) $\quad A = BC \land \neg\mathrm{Com}(B,C) \rightarrow (A - B) = C.$

The problem behind MINUS 4* becomes clearer when one considers another (slightly more complicated) counterexample. Let A contain B which in turn contains some $D(\neq 0)$, and suppose that A contains another concept $E(\neq 0)$ such that $\neg\mathrm{Com}(B,E)$; let C be the "sum" of D and E. Since B and E are uncommunicating, it follows a fortiori that B does not contain E. Hence B does not contain the "larger" concept $C(= DE)$ either. According to MINUS 4*, however, the premisses $A \in C \land B \notin C$ would entitle us to conclude that C remains (entirely) in $(A - B)$ while, intuitively, only a part of C, namely E, should remain in $(A - B)$ since everything that was contained in B, in particular D, must be removed from A in order to yield $(A - B)$.

Generalizing from this example, one finds that Leibniz's requirement $B \notin Y$ (in DEF 11*) is too weak to warrant that a concept Y which was originally contained in A may remain in $(A-B)$. This inference is valid only if Y does not itself contain a component X which is also contained in B. In other words, Y must be *entirely outside* B, i.e. Y and B may have *nothing in common.* Principle MINUS 4*, and the corresponding clause of DEF 11*, therefore have to be corrected as follows:

(MINUS 4) $\quad A \in C \land \neg\mathrm{Com}(B,C) \rightarrow (A - B) \in C$

(DEF 11) $\quad A - B = C \leftrightarrow_{\mathrm{df}} \forall Y(Y \neq 0 \rightarrow (C \in Y \leftrightarrow A \in Y \land \neg\mathrm{Com}(B,Y))).$

It may then be shown that conceptual subtraction $(A - B)$ might alternatively (and much more simply) be defined as the commune of A and *Non*$-B$:

(MINUS 6) $\quad (A - B) = A \otimes \overline{B}.$

All that has to be proved, according to DEF 11, is that for each (non-empty) concept $Y: A \otimes \overline{B} \in Y$ iff $A \in Y \land \neg\mathrm{Com}(B,Y)$. Suppose (1) that $A \otimes \overline{B} \in Y$. Then COMM 1 immediately gives us $A \in Y$, while $\neg\mathrm{Com}(B,Y)$ is obtained indirectly as follows. Assume that B and Y would have something in common, i.e. there exists some $X(\neq 0)$ such that $B \in X \land Y \in X$; premiss (1) by way of COMM 1 entails that $\overline{B} \in Y$, hence because of $Y \in X$ also $\overline{B} \in X$. Together with $B \in X$ one thus obtains by COMM 3 that $\overline{B} \otimes B \in X$, hence by COMM 4 $\overline{\overline{B}\,\overline{\overline{B}}} \in X$ i.e.

[45]Cf. C. 267, # 29: : "[...] if $A + B = C$, then $A = C - B$ [...] but it is necessary that A and B have nothing in common".

$\overline{\overline{B}\,B} \in X$. But this is a contradiction since any concept contained in the empty or tautological concept must itself be tautological while it was presupposed that $X \neq 0$!

For the proof of the converse implication suppose (2) that $A \in Y \wedge \neg\mathrm{Com}(B, Y)$. In view of COMM 3 it suffices to show that $\overline{B} \notin Y$. Again, this shall be proved indirectly. So if one assumes that $\overline{B} \notin Y$, it follows by POSS 1 that $\mathbf{P}(\overline{B}\,\overline{Y})$, i.e. $\overline{B}\,\overline{Y}$ doesn't coincide with the contradictory concept $A\overline{A}$. Hence by contraposition its negation, i.e. according to COMM 4 the commune of B and Y, $B \otimes Y$, does not coincide with the negation of $A\overline{A}$, i.e. with the tautological concept $\overline{A\overline{A}}$. But according to COMM 5 this means that B and Y are communicating, which contradicts our premiss $\neg\mathrm{Com}(B, Y)$.

In the end, then, conceptual subtraction $(A - B)$ turns out to be tantamount to the disjunction of A and $\overline{B}$, and this gives rise to the subsequent condition for the extensional interpretation of $A - B$:

$$(10)\ \phi(A - B) = \phi(A) \cup \overline{\phi(B)}.$$

We are now in a position to sum up our definition of an extensional interpretation of Leibniz's algebra of concepts which at the same time serves also as an instrument for the extensional interpretation of the Plus-Minus-Calculus (as applied to intensions of concepts):

(DEF 1) Let U be a non-empty set (of possible individuals). Then the function ϕ is an *extensional interpretation* of the algebra of concepts, $L1$, and of the Plus-Minus-Calculus, $L0.8$, if and only if:

(I) $\phi(A) \subseteq U$ for each concept-letter A, and

(II) (1) $\phi(A \in B) = \text{true iff } \phi(A) \subseteq \phi(B)$

(2) $\phi(A = B) = \text{true iff } \phi(A) = \phi(B)$

(3) $\phi(A \iota B) = \text{true iff } \phi(A) \supseteq \phi(B)$

(4) $\phi(A \oplus B) = \phi(AB) = \phi(A) \cap \phi(B)$

(5) $\phi(\overline{A}) = \overline{\phi(A)}$

(6) $\phi(\mathbf{P}(A)) = \text{true iff } \phi(A) \neq \emptyset$

(7) $\phi(0) = U$

(8) $\phi(A \otimes B) = \phi(A) \cup \phi(B)$

(9) $\phi(\mathrm{Com}(A, B)) = \text{true iff } \phi(A) \cup \phi(B) \neq U$

(10) $\phi(A - B) = \phi(A) \cup \overline{\phi(B)}$.

This summary also allows me to explain why the Plus-Minus-Calculus and the mere Plus-Calculus have been dubbed '$L0.8$' and '$L0.4$', respectively. While the full algebra of concepts, $L1$, contains all of the above ten elements either as primitive or as defined operators, in $L0.4$ only 40 %, namely $\{\in, \iota, =, \oplus\}$, and in $L0.8$ only 80 %, namely $\{\in, \iota, =, \oplus, 0, \otimes, \text{Com}, -\}$, are available.[46]

To conclude this section let me add some further interesting theorems involving subtraction $(A - B)$ plus the commune of A and B:

Formalization	*Leibniz's formulation*
$A = ((A + B) - B) + (A \otimes B)$	"[...] if $A+B = L$ and it is assumed that what A and B have in common is N, then $A = L-B+N$" (**C.**, 251)
$A \otimes B = B - ((B + A) - A)$	"[...] let $A + B$ be L [...] and from L one of the constituents A, is subtracted [...] let the remainder be N [...] if the remainder is subtracted from B [...] there remains M, the common part of A and B"; (**C.**, 250)
$A \otimes B = \{(A + B) - [((A+B) - A) + ((A + B) - B]\}$	"From $A + B$ one subtracts A, remains L; from the same one subtracts B, remains M. Now the given $L + M$ is subtracted from $A + B$; remains the commune" (**C.**, 251/2).

6 ALETHIC AND DEONTIC MODAL LOGIC

Although Leibniz never spent much time for the investigation of the proper laws of (ordinary or modal) propositional logic, he may yet be credited with three important discoveries in this field:

1. By means of a simple, ingenious device Leibniz transformed the algebra of concepts into an algebra of propositions;

2. Leibniz developed the basic idea of possible-worlds-semantics for the interpretation of the modal operators;

3. Leibniz not only discovered the strict analogy between the logical laws for deontic operators ('forbidden', 'obligatory', 'allowed') on the one hand and the alethic operators ('impossible', 'necessary', 'possible') on the other hand; but he even anticipated A. R. Anderson's [1958] idea of "defining" the former in terms of the latter.

46Neither negation nor the (Im-)Possibility operator can be defined in terms of "Nihil" and/or subtraction!

6.1 Leibniz's Calculus of Strict Implication

In the fragment *Notationes Generales*, probably written between 1683 and 1685[47], Leibniz pointed out to the parallel between the containment relation among concepts and the implication relation among propositions. Just as the simple proposition 'A is B' (where A is the "subject", B the "predicate") is true, "when the predicate is contained in the subject", so a conditional proposition 'If A is B, then C is D' (where 'A is B' is designated as 'antecedent', 'C is D' as 'consequent') is true, "when the consequent is contained in the antecedent" (cf. **A** VI, 4, 551). In later works Leibniz compressed this idea into formulations such as "a proposition is true whose predicate is contained in the subject *or more generally* whose consequent is contained in the antecedent".[48] The most detailed explanation of the basic idea of deriving the laws of the algebra of *propositions* from the laws of the algebra of *concepts* was sketched in §§75, 137 and 189 **GI** as follows:

> If, as I hope, I can *conceive all propositions as terms, and hypotheticals as categoricals* [...] this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance. [**P**, 66 ...]
>
> We have, then, discovered many secrets of great importance for the analysis of all our thoughts and for the discovery and proof of truths. We have discovered [...] how *absolute and hypothetical truths have one and the same laws* and are contained in the same general theorems. [**P**, 78 ...]
>
> Our principles, therefore, will be these [...] Sixth, *whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows.* (**P**, 85, all italics are mine).

To conceive all propositions in analogy to concepts ("instar terminorum") means in particular that the hypothetical proposition 'If α then β' will be logically treated exactly like the fundamental relation of containment between concepts, 'A contains B'. Furthermore, as Leibniz explained elsewhere, negations (and conjunctions) of propositions are to be conceived just as negations (and conjunctions) of concepts:

> If A is a proposition or statement, by non-A I understand the proposition A to be false. And if I say 'A is B', and A and B are propositions, then I take this to mean that B follows from A [...] This will also be useful for the abbreviation of proofs; thus if for 'L is A' we would say 'C' and for 'L is B' we say 'D', then for this [hypothetical] 'If L is B, it follows that L is B' one could substitute 'C is D'.[49]

[47]**A** VI, 4, # 131.

[48]Cf. **C**. 401: "vera autem propositio est cujus praedicatum continetur in subjecto, *vel generalius* cujus consequens continetur in antecedente" (my emphasis); cf. also **C**. 518: "Semper igitur praedicatum seu consequens inest subjecto seu antecedenti".

[49]Cf. **C**., 260, # 16: "Si A sit propositio vel enuntiatio, per non-A intelligo propositionem A esse falsam. Et cum dico A est B, et A et B sunt propositiones, intelligo ex A sequi B. [...]

One thus obtains the following "mapping" of the primitive formulas of the algebra of concepts into primitive formulae of an algebra of propositions:

$A \in B$	$\alpha \to \beta$
$\overline{A}$	$\neg\alpha$
AB	$\alpha \wedge \beta$

As Leibniz himself mentioned, the fundamental law POSS 1 does not only hold for the containment-relation between concepts but equally for the entailment relation between propositions:

> A contains B is a *true proposition* if A non-B entails a contradiction. *This applies both to categorical and to hypothetical propositions*, e.g., 'If A contains B, C contains D' can be formulated as follows: 'That A contains B contains that C contains D'; therefore 'A containing B and at the same time C not containing D' entails a contradiction.[50]

Hence $A \in B \leftrightarrow \mathbf{I}(A\overline{B})$ may be "translated" into $(\alpha \to \beta) \leftrightarrow \neg\Diamond(\alpha \wedge \neg\beta)$. This formula shows that Leibniz's implication is not a material but rather a *strict* implication. As was already noted by Rescher [1954, p. 10], Leibniz's account provides a definition of "entailment in terms of negation, conjunction, and the notion of possibility", for α implies β iff it is *impossible* that α is true while β is false. This definition of strict implication "re-invented", e.g., by C. I. Lewis[51] was formulated also in the "Analysis Particularum":

> Thus if I say 'If L is true it follows that M is true', this means that one cannot suppose at the same time that L is true and that M is false.[52]

As regards the other, non-primitive elements of $L1$, the relation 'A is in B' represents, according to DEF 4, the converse of $A \in B$. Hence its propositional counterpart is the "inverse implication", $\alpha \leftarrow \beta$. According to DEF 2, the coincidence relation $A = B$ is tantamount to mutual containment, $A \in B \wedge B \in A$, which will thus be translated into a mutual implication between propositons, $\alpha \to \beta \wedge \beta \to \alpha$, i.e. into strict equivalence, $\alpha \leftrightarrow \beta$. Next, according to DEF 5, the possibility or self-consistency of a concept B amounts to the conditions $B \notin A\overline{A}$. In the field

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Utile etiam hoc ad compendiose demonstrandum, ut si pro L est A dixissemus C et pro L est B dixissemus D pro ista si L est A sequitur quod L est B, substitui potuisset C est D."

[50]Cf. **C.**, 407: "*Vera propositio* est A continet B, si A non-B infert contradictionem. *Comprehenduntur et categoricae et hypotheticae propositiones*, v.g. si A continet B, C continet D, potest sic formari: A continere B continet C continere D; itaque A continere B, et simul C non continere D infert contradictionem" (second emphasis is mine).

[51]Cf. e.g., [Lewis and Langford, 1932, p. 124]: "The relation of strict implication can be defined in terms of negation, possibility, and product [...] Thus "p implies q" [...] is to mean "It is false that it is possible that p should be true and q false".

[52]Cf. **A** VI, 4, 656: "Itaque si dico *Si L est vera sequitur quod M est vera*, sensus est, non simul supponi potest quod L est vera, et quod M est falsa"'.

of propositions one hence obtains that α is possible, $\Diamond\alpha$, if and only if α does not entail a contradiction: $\neg(\alpha \rightarrow \beta \wedge \neg\beta)$.

$A\iota B$	$(\alpha \leftarrow \beta)$	$[\leftrightarrow_{\mathrm{df}} (\beta \rightarrow \alpha)]$
$A = B$	$\alpha \leftrightarrow \beta$	$[\leftrightarrow_{\mathrm{df}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)]$
$\mathbf{P}(A)$	$\Diamond\alpha$	$[\leftrightarrow_{\mathrm{df}} \neg(\alpha \rightarrow (\beta \wedge \neg\beta))]$

Finally one could also map the specific elements of the Plus-Minus-Calculus into the following somewhat unorthodox propositional operators:

0	$\neg(\alpha \wedge \neg\alpha)$
$\mathrm{Com}(A, B)$	$\Diamond(\neg\alpha \wedge \neg\beta)$[53]
$A \otimes B$	$\alpha \vee \beta$
$A - B$	$\alpha \vee \neg\beta$.

Given this "translation", the basic axioms and theorems of the algebra of concepts listed at the end of section 4 may be transformed into the following set of laws of an algebra of propositions:

[53]The converse relation which obtains between α and β iff the falsity of the one proposition entails the truth of the other has been referred to by Leibniz as "incondestructibilia" or also as "inconnegabilia", Cf., e.g. **A** VI, 4, 389: "Si ex propositione *A non est* [verum] sequitur *B est* [verum] tunc vicissim ex propositione *B non est* sequitur *A est*, et *A*, *B* nondum nomen invenere cum scilicet unum saltem eorum [verum] existere debet, possis appellare, *incondestructibilia*."

	Basic Principles of PL1
IMPL 1	$(\alpha \to \alpha)$
IMPL 2	$((\alpha \to \beta) \wedge (\beta \to \gamma)) \to (\alpha \to \gamma)$
IMPL 3	$(\alpha \to \beta) \leftrightarrow (\alpha \leftrightarrow \alpha \wedge \beta)$
CONJ 1	$(\alpha \to \beta \wedge \gamma) \leftrightarrow ((\alpha \to \beta) \wedge (\alpha \to \gamma))$
CONJ 2	$\alpha \wedge \beta \to \alpha$
CONJ 3	$\alpha \wedge \beta \to \beta$
CONJ 4	$\alpha \wedge \alpha \leftrightarrow \alpha$
CONJ 5	$\alpha \wedge \beta \leftrightarrow \beta \wedge \alpha$
NEG 1	$(\neg\neg\alpha \leftrightarrow \alpha)$
NEG 2	$\neg(\alpha \leftrightarrow \neg\alpha)$
NEG 3	$(\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$
NEG 4	$\neg\alpha \to \neg(\alpha \wedge \beta)$
NEG 5	$[\Diamond\alpha\wedge](\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$
POSS 1	$(\alpha \to \beta) \leftrightarrow \neg\Diamond(\alpha \wedge \neg\beta)$
POSS 2	$(\alpha \to \beta) \wedge \Diamond\alpha \to \Diamond\beta$
POSS 3	$\neg\Diamond(\alpha \wedge \neg\alpha)$
POSS 4	$(\alpha \wedge \neg\alpha) \to \beta$

Although Leibniz didn't care very much about propositional logic, he happened to put forward at least some of these laws in scattered fragments. For instance, in the first juridical disputation *De Conditionibus* the transitivity of the inference relation, IMPL 2, is characterized as follows: "The Co[ndition] of the co[ndition] is the co[ndition] of the co[nditioned]. If by positing A B will be posited and by positing B C will be posited, then also by positing A C will be posited".[54] As regards IMPL 1 and CONJ 2, 3, Leibniz mentions in the fragment "De Calculo Analytico Generale" the "Primary Consequences: A is B, therefore A is B [...] A is B and C est D, therefore A is B, or as well [therefore] C is D",[55] and the corresponding "Axioms [...] 3) If A is B, also A is B. If A is B and B is C, also A is B". Furthermore the definition of strict implication in terms of strict equivalence (and conjunction), IMPL 2, is exemplified in another fragment as follows:

[54]Cf. **A** VI, 1, 110: "C[onditi]o C[onditio]nis est C[onditi]o C[onditiona]ti. Si posito A positur B, et posito B positur C; etiam posito A positur C." For a discussion of Leibniz's early work on juridic (or deontic) logic cf. Schepers [1975].

[55]Cf. **A** VI, 4, 149; "PRIMAE CONSEQUENTIAE *A est B ergo A est B*. [...] *A est B et C est D ergo A est B* vel *ergo C est B*".

> *A true hypothetical proposition* of first degree is 'If A is B, and from this it follows that C is D' [...] Let the state of affairs 'A is B' be called L, and the state of affairs 'C is D' be called M. Then one obtains $L = LM$; *in this way the hypothetical [proposition] is reduced to a categorical.* (cf. **C**. 408, second emphasis is mine).

Moreover in "De Varietatibus Enuntiationum" Leibniz forwards principle CONJ 1 for the special case A = 'a is b', B = 'e is d' and C = 'l is m' by maintaining that the proposition "If a is b it follows that e is d and l is m" can be resolved into the conjunction of the propositions "If a is b it follows that e is d" and "If a is b it follows that l is m" (cf. **A** VI, 4, 129). Versions of the principle of double negation, NEG 1, may be found in §4 **GI** or, for the special cases of propositions of the type '$A = B$' and '$A \in B$', more formally in **C**. 235[56]. Finally the "Analysis particularum" contains besides the above quoted paraphrase of POSS 1 also the law of (propositional) contraposition NEG 3: "If a proposition M [...] follows from a proposition L [...], then conversely the falsity of the proposition L follows from the falsity of the proposition M".[57]

The above collection of basic principles does not yet, however, constitute a genuine *calculus* of (modal) propositional logic. At least some additional *rules of deduction* are needed which allow one to derive further theorems from these "axioms". As was shown elsewhere, Leibniz was well aware at least of the validity of the rule of (strict) *modus ponens*:

$$\text{(MP)} \qquad (\alpha \rightarrow \beta), \alpha \vdash \beta$$

and of the rule of conjunction:

$$\text{(RC)} \qquad \alpha, \beta \vdash \alpha \wedge \beta.$$

Furthermore it was argued there that the mapping of $L1$ into *PL1* yields a calculus of strict implication in the vincinity of Lewis' system S2°. This does not mean, however, that Leibniz would have favoured such a weak system as the proper calculus of (alethic) modal logic. For example, Leibniz would certainly have subscribed to the validity of the truth-axiom $\Box\alpha \rightarrow \alpha$ (or, equivalently, $\alpha \rightarrow \Diamond\alpha$). But, for purely syntactical reasons, these laws can never be obtained by Leibniz's consideration of propositions "instar terminorum" from corresponding theorems of $L1$.[58] For reasons of space, this issue shall not be discussed here further — the reader is referred to the detailed exposition in [Lenzen, 1987]. Only a few more theorems for the modal operators $\Box$ and and $\Diamond$ shall be considered in the subsequent section where Leibniz's version of a possible worlds semantics is presented.

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[56] "Idem sunt $A \infty B$ [...] et A non non ∞B"; cf. also **C**. 262: "A non non est B, idem est quod A est B"

[57] Cf. **A** VI, 4, 655/6: "Si ex propositione L [...] sequitur propositio M [...] tunc contra ex falsitate propositionis M sequitur falsitas propositionis L".

[58] E.g., $\alpha \rightarrow \Diamond\alpha$, could only result from mapping the formula $A \in \mathbf{P}(A)$ or $A \rightarrow \mathbf{P}(A)$ into PL1; but none of these is syntactically well-formed!

6.2 Leibniz's Possible Worlds Semantics

The fundamental logical relations between necessity, $\Box$, possibility, $\Diamond$, and impossibility can be expressed, e.g., by:

(NEC 1) $\Box(\alpha) \leftrightarrow \neg\Diamond(\neg\alpha)$

(NEC 2) $\neg\Diamond(\alpha) \leftrightarrow \Box(\neg\alpha)$.

Of course, these laws were familar already to logicians long before Leibniz. However, Leibniz not only formulated, e.g., NEC 1 already as a youth, at the age of 25, as follows:

> Whenever the question is about necessity, the question is also about possibility, for if something is called necessary, then the possibility of its opposite is negated.[59]

But he also "proved" these relations by means of an admirably clear semantic analysis of modal operators in terms of "possible cases", i.e. possible worlds:

"Possible	is whatever can	happen	or what is true in some cases
Impossible	is whatever cannot	happen	or what is true in no [...] case
Necessary	is whatever cannot	not happen	or what is true in every [...] case
Contingent	is whatever can	not happen	or what is [not] true in some case".[60]

Hence a proposition α is *possible* iff α is true in at least one case; α is *impossible*, iff α is true in no case; α is *necessary* iff α is true in each case; and, finally, α is *contingent*, i.e. non-necessary, iff α is not true in at least one case.[61] Now this analysis of the truth-conditions for modal propositions not only entails the above mentioned laws NEC 1 and 2, but it also gives rise to the principle that whenever α is necessary, α will be possible as well, and by contraposition: "Because all that is necessary is possible, all that is impossible is contingent":[62]

(NEC 3) $\Box\alpha \rightarrow \Diamond(\alpha)$,

(NEC 4) $\neg\Diamond(\alpha) \rightarrow \neg\Box(\alpha)$.

[59]Cf. **A** VI, 1, 460: "Quoties autem de necessitate quaestio est, de possibilitate quaestio est, nam si quid necessarium dicitur, possibilitas oppositi negatur".

[60]Cf. **A** VI, 1, 466:

"Possibile	est quicquid potest	fieri seu quod verum est quodam	casu
Impossibile	est quicquid non potest	fieri seu quod verum est nullo [...]	casu
Necessarium	est quicquid non potest non	fieri seu quod verum est omni [...]	casu
Contingens	est quicquid potest non	fieri seu quod verum est quodam non	casu."

[61]As this quotation shows, Leibniz uses the notion of contingency not in the modern sense of 'neither necessary nor impossible' but as the simple negation of 'necessary'.

[62]Cf. **A** VI, 4, 2759: "Quia omne necessarium est possibile omne impossibile est contingens seu potest non fieri".

Leibniz "demonstrates" these laws by reducing them to corresponding laws for (universal and existential) quantifiers such as: "If α is true in each case, then α is true in at least one case". These quantificational principles were tacitly presupposed by Leibniz who only mentioned them in passing by maintaining (very elliptically), e.g.: "'All' is the same as 'none not"'or "'All not' is the same as 'none"'. Cf. the following "proof" of NEC 2:

> [...] 'necessarily not happen' and 'impossible' coincide. For also 'none' and 'everything not' coincide. Why so? Because 'none' is 'not something'. 'Every' is 'not something not'. Therefore 'everything not' is 'not something not not'. The two latter 'not' destroy each other, thus remains 'not something'.[63]

On the background of certain rules for the negation of the quantifier expressions 'all', 'some', and 'none', which reflect the core ideas of the traditional theory of opposition of categorical forms, Leibniz thus argues that an impossible proposition which is false in every case is the same as a proposition which is not true in any case. Let it be mentioned in passing that the analogue "proof" of NEC 3 contains a minor mistake which is quite typical of Leibniz:[64]

> [...] everything which is necessary is possible. For always, when 'everything is', also 'something is' [the case]. Thus if 'everything is', 'not something is not', or 'something is not not'. Hence 'something is'.[65]

To be sure, a necessary proposition α which is true in every case a fortiori has to be true in at least one case, hence α is possible. But this principle — or the corresponding quantificational law ($\forall x\alpha \to \exists x\alpha$) — cannot be correctly derived from the presupposed equivalence ($\forall x\alpha \leftrightarrow \neg\exists x\neg\alpha$) plus the law of double negation, ($\neg\neg\alpha \leftrightarrow \alpha$) in the way attempted by Leibniz. For 'not something is not', i.e. $\neg\exists x\neg\alpha$, is not the same as 'something is not not',i.e. $\exists x\neg\neg\alpha$!

It cannot be overlooked, however, that the truth conditions quoted from the early *De Conditionibus*, even when combined with Leibniz's later views on possible worlds, fail to come up to the standards of modern possible worlds semantics, since in Leibniz's work nothing corresponds to the accessability relation among worlds. Therefore it is almost impossible to decide which of the diverse modern systems like T, S4, S5, etc. best conforms with Leibniz's views. According to Poser [1969], Leibniz's modal logic is tantamount to S5. This means in particular that Leibniz acknowledged the characteristic axiom of S4:

[63]Cf. **A** VI, 1, 469: "[...] necessarium non fieri et impossibile, coincidunt. Nam etiam Nullus et omnis non coincidunt. Cur ita? quia nullus est non quidam. Omnis est non quidam non. Ergo omnis non, est non quidam non non. Abjiciant se mutuò duo posteriora non, superest non quidam."

[64]In so far as, again and again, Leibniz had serious problems in distinguishing 'non est' and 'est non'; cf. [Lenzen, 1986].

[65]Cf. **A** VI, 1, 469: "[...] omne necessarium est possibile. Nam semper, si omnis est, etiam quidam est. Si enim Omnis est, non quidam non est seu quidam non non est. Ergo quidam est".

(NEC 5) $\Box\alpha \rightarrow \Box\Box\alpha$.

Poser pointed out to the following passage in "De Affectibus": "For what can impossibly be actually the case, that can impossibly be possible"[66] which rather convincingly shows that, in Leibniz's view, any impossible proposition is impossibly possible:

(NEC 6) $\neg\Diamond\alpha \rightarrow \neg\Diamond\Diamond\alpha$.

However, Poser failed to give any quotation (or any other compelling reason) to show that Leibniz would also have accepted the stronger S5-principle $\Diamond\alpha \rightarrow \Box\Diamond\alpha$, according to which any possible proposition would be necessarily possible. Moreover, as was argued by Adams [1982], the latter principle appears to be incompatible with Leibniz's philosophical view of necessity as expressed, e.g., in the **GI**:

> (133) A true necessary proposition can be proved by reduction to identical propositions, or by reduction of its opposite to contradictory propositions; hence its opposite is called 'impossible'.
>
> (134) A true contingent proposition canot be reduced to identical propositions, but is proved by showing that if the analysis is continued further and further, it constantly approaches identical propositions, but never reaches them. (**P**, 77).

If a *necessary* proposition α can be reduced in finitely many steps to an "identity", this means that a proposition α is *possible* if and only if it is not refutable in finitely many steps (i.e. its negation cannot be reduced in finitely many steps to an "identity"). But on this understanding of possibility and necessity, the S5 principle $\Diamond\alpha \rightarrow \Box\Diamond\alpha$ appears to be blatantly false.

6.3 Leibniz's Deontic logic

Leibniz saw very clearly that the logical relations between the "Modalia Iuris" *obligatory*, *permitted* and *forbidden* exactly mirror the corresponding relations between the alethic modal operators *necessary*, *possible* and *impossible* and that therefore all laws and rules of alethic modal logic may be applied to deontic logic as well:

> Just like 'necessary', 'contingent', 'possible' and 'impossible' are related to each other, so also 'obligatory', 'not obligatory', 'permitted', and 'forbidden'.[67]

[66] Cf. **Grua**, 534: "Nam quod impossibile est esse actu, id impossibile est esse possibile".

[67] Cf. **A** VI, 4, 2762: "Uti se habent inter se necessarium, contingens, possibile, impossibile; ita se habent debitum, indebitum, licitum, illicitum".

This structural analogy rests on the important discovery that the deontic notions can be defined by means of the alethic notions plus the additional "logical" constant of a morally perfect man ["vir bonus"]. Such a "virtuous man", b, is characterized by the requirements that (1) b strictly obeys all laws, (2) b always acts in such a way that he does no harm to anybody, and (3) b loves or is benevolent to all other people.[68] Given this understanding of the "vir bonus", b, Leibniz explains:

Obligatory is	what is	necessary	for the virtuous man as such
not obligatory is	what is	contingent	for the virtuous man as such
permitted is	what is	possible	for the virtuous man as such
forbidden is	what is	impossible	for the virtuous man as such.[69]

If we express the restriction of the modal operators $\Box$ and $\Diamond$ to the virtuous man by means of a subscript 'b', these definitions can be formalized as follows:

(DEON 1) $O(\alpha) \leftrightarrow \Box_b(\alpha)$

(DEON 2) $E(\alpha) \leftrightarrow \Diamond_b(\alpha)$[70]

(DEON 3) $F(\alpha) \leftrightarrow \neg\Diamond_b(\alpha)$

Now, as Leibniz mentioned in passing, all that is unconditionally necessary will also be necessary for the virtuous man as such:[71]

(NEC 7) $\Box(\alpha) \rightarrow \Box_b(\alpha)$.

Hence the fundamental laws for the deontic operators can be derived from corresponding laws of the alethic modal operators in much the same way as Anderson

[68]Cf. **A** VI, 1, 466: "Vir bonus est quisquis amat omnes"; **A** VI, 4, 2851: "Vir bonus est qui benevolus est erga omnes" and **A** VI, 4, 2856: "Vir bonus censetur, qui hoc agit ut prosit omnibus noceat[que] nulli." It is interesting to note that Leibniz denotes the entire discipline of jurisprudence as the "science of the virtuous man" ("scientia viri boni") and justice as the "voluntas viri boni".

[69]Cf. **A** VI, 4, 2758:

"Debitum	est, quod viro bono qua tali	necessarium
Indebitum	est, quod viro bono qua tali	contingens
Licitum	est, quod viro bono qua tali	possibile
Illicitum	est, quod viro bono qua tali	impossibile."

In the former edition in **Grua** 605 'debitum' was mistakenly associated with 'contingens'. Cf. also **A** VI, 4, 2863: "quod Viro bono possibile, impossibile, necessarium est, si nomen suum tueri velit, id justum sive licitum, injustum, ac denique debitum esse."

[70]We here use the letter 'E' (reminding of the German 'erlaubt') instead of 'P' for ‚permitted' in order to avoid any confusions with the operator for the possibility (or self-consistency) of concepts!

[71]Cf. **A** VI, 4, 2759: "Nam omne necessarium est necessarium viro bono".

[1958] reduced deontic logic to alethic modal logic. As Leibniz pointed out, two different classes of theorems may be distinguished. First we have some "Theorems in which the juridic modalities are combined by themselves", i.e. theorems describing the logical relations among the deontic operators, e.g.:

> Everything which is obligatory is permitted [...] Everything which is forbidden is not obligatory [...] Nothing which is obligatory is forbidden [...] Nothing which is forbidden is obligatory [...] Everything that is forbidden is obligatory to omit. And everything that is obligatory to omit is forbidden. [...] Everything that is forbidden to omit is obligatory and everything which is obligatory is forbidden to omit [...] Everything which is not obligatory is permitted to omit and everything that is permitted to omit is not obligatory.[72]

(DEON 4a) $O(\alpha) \rightarrow E(\alpha)$

(DEON 4b) $\neg E(\alpha) \rightarrow \neg O(\alpha)$

(DEON 5a) $O(\alpha) \rightarrow \neg F(\alpha)$

(DEON 5b) $F(\alpha) \rightarrow \neg O(\alpha)$

(DEON 6) $F(\alpha) \leftrightarrow O(\neg\alpha)$

(DEON 7) $O(\alpha) \leftrightarrow F(\neg\alpha)$

(DEON 8) $\neg O(\alpha) \leftrightarrow E(\neg\alpha)$

As Leibniz "demonstrates" (or, at least, makes it plausible to suppose), these laws are immediate counterparts of the well-known logical relations between the alethic modalities. E.g., concerning DEON 6 he remarks:

> Everything which is forbidden is obligatory to omit. And everything that is obligatory to omit is forbidden, i.e. 'forbidden' and 'obligatory to omit' coincide. Because 'necessarily not happen' and 'impossible' coincide. For also 'none' and 'everything not' coincide.[73] (Cf. **A** VI, 1, 469).

As a second class of theorems one obtains certain "Theorems in which the juridic modalities are combined with the logical modalities" [Theoremata quibus combinantur Iuris Modalia Modalibus Logicis seu justum cum possibili]. Thus in the

[72]Cf. **A** VI, 1, 468/9: "Omne debitum est justum" [...] "Omne injustum est indebitum" [...] "Nullum debitum est injustum" [... or equivalently] "Nullum injustum est debitum" [...] "Omne injustum est debitum omitti. Et omne debitum omitti est injustum" [...] "Omne injustum omitti est debitum et Omne debitum est injustum omitti" [... and] "Omne indebitum juste omittitur et omne quod juste omittitur est indebitum".

[73]Cf. **A** VI, 1, 469: "Omne injustum est debitum omitti. Et omne debitum omitti est injustum, seu injustum et debitum non fieri coincidunt. Quia necessarium non fieri et impossibile, coincidunt. Nam etiam Nullus et omnis non coincidunt".

"Elementa Juris Naturalis" Leibniz mentions the following principles concerning the relations between the alethic concepts 'necessary', 'possible' and 'impossible' on the one hand and the deontic notions 'obligatory, 'permitted' and 'forbidden' on the other hand: "Everything which is necessary is obligatory" [Omne necessarium debitum est], or, by contraposition: "Everything that is not obligatory is not necessary but contingent" [Cf. **A** VI, 1, 470: "Omne indebitum nec necessarium est, sed contingens"]:

(DEON 9a) $\Box(\alpha) \rightarrow O(\alpha)$

(DEON 9b) $\neg O(\alpha) \rightarrow \neg\Box(\alpha)$

Furthermore: "Everything that is necessary is permitted" [Omne necessarium justum est], or, again by contraposition, "Everything that is forbidden is not necessary but contingent" ["Quicquid injustum est, id nec necessarium est, sed contingens", *ibid.*]:

(DEON 10a) $\Box(\alpha) \rightarrow E(\alpha)$

(DEON 10b) $\neg E(\alpha) \rightarrow \neg\Box(\alpha)$

Next, "Everything that is permitted is possible" [Omne justum possibile est], or "Everything that is impossible is not permitted" ["Quicquid est impossibile, id injustum est", *ibid.*]:

(DEON 11a) $E(\alpha) \rightarrow \Diamond(\alpha)$

(DEON 11b) $\neg\Diamond(\alpha) \rightarrow \neg E(\alpha)$

Finally, "Everything which is obligatory is possible" [Omne debitum possibile est], or "Everything which is impossible is not obligatory, i.e. may be omitted by the virtuous man" ["Omne impossibile indebitum seu omissibile est viro bono", *ibid.*]:

(DEON 12a) $O(\alpha) \rightarrow \Diamond(\alpha)$

(DEON 12b) $\neg\Diamond(\alpha) \rightarrow \neg O(\alpha)$

To illustrate Leibniz's way of demonstrating these laws in "Modalia et Elementa Juris Naturalis" let us consider DEON 10a which is formulated there with the word 'licitum' instead of 'justum' for 'permitted':

> Everything which is necessary is permitted, i.e. necessity has no law.
>
> For everything which is necessary is necessary for the virtuous man. If something is necessary for the virtuous man, its opposite is impossible for the virtuous man. What is impossible for the virtuous man is anyway not possible for the virtuous man as such, i.e. it is not permitted. Therefore the opposite of something necessary is not permitted.

> However, if the opposite of something is not permitted, then itself is permitted.[74]

By means of the "bridge principle", NEC 7, $\Box(\alpha)$ is first shown to entail $\Box_b(\alpha)$. Next Leibniz makes use of the following law NEC 8 which relativizes the usual equivalence NEC 1 to the "virtuous man":

(NEC 8) $\quad \Box_b(\alpha) \leftrightarrow \neg\Diamond_b(\neg\alpha)$.

According to DEON 2, the resulting formula $\neg\Diamond_b(\neg\alpha)$ is equivalent to $\neg E(\neg\alpha)$ which in turn entails the desired conclusion $E(\alpha)$ by way of the further theorem:

(DEON 13) $\quad \neg E(\neg\alpha) \rightarrow E(\alpha)$.

Note, incidentally, that in an earlier proof which was later deleted by Leibniz, the conclusion $\Diamond_b(\alpha)$ or $E(\alpha)$ had been obtained more directly by inferring $\Box_b(\alpha)$ from the premiss $\Box(\alpha)$ and then making use of the following law which relativizes NEC 3 to person b:

(NEC 9) $\quad \Box_b(\alpha) \rightarrow \Diamond_b(\alpha)$

For, as Leibniz remarks: "Everything which is necessary for the virtuous man is anyway possible for the virtuous man as such, i.e. it is permitted"[75]. Similarly Leibniz proves DEON 12b as follows:

> Nothing which is impossible is obligatory, i.e. there is no obligation for impossibles.
>
> For everything which is imposible is impossible for the virtuous man. Nothing which is impossible for the virtuous man is anyway possible for the virtuous man as such. What is not possible for the virtuous man as such is not necessary for the virtuous man as such, i.e. it is not obligatory.[76]

Here again by means of the "bridge principle" NEC 7, $\neg\Diamond_b(\alpha)$ is first shown to follow from $\Box(\neg\alpha)$ or $\neg\Diamond(\alpha)$; second, NEC 9 in its contraposited form $\neg\Diamond_b(\alpha) \rightarrow \neg\Box_b(\alpha)$ is used to derive $\neg\Box_b(\alpha)$ which, thirdly, according to DEON 1, gives the desired conclusion $\neg O(\alpha)$.

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[74] Cf. **A** VI, 4, 2759/60: "Omne necessarium est licitum, seu necessitas non habet legem. Nam omne necessarium est necessarium viro bono. Quod est necessarium viro bono, ejus oppositum est impossibile viro bono. Quod impossibile viro bono utcunque non est possibile viro bono qua tali seu licitum. Ergo necessarii oppositum non est licitum. Cujus autem oppositum non est licitum, id ipsum est licitum."

[75] Cf. **A** VI, 4, 2759: "Omne necessarium viro bono utcunque est possibile viro bono qua tali; hoc est licitum".

[76] Cf. **A** VI, 4, 2759: "Nullum impossibile est debitum, seu impossibilium nulla est obligatio. Nam omne impossibile est impossibile viro bono. Nullum impossibile viro bono utcunque est possibile viro bono qua tali. Quod non est possibile viro bono qua tali non est necessarium viro bono qua tali, seu non est debitum."

7 "INDEFINITE CONCEPTS" (QUANTIFIER LOGIC L2)

In many logical fragments Leibniz uses letters from the end of the alphabet $(x, y, \ldots, X, Y, Z, \ldots)$ and occasionally also from the mid of the alphabet $(Q, L, \ldots)$ for the representation of "indefinite concepts", while the "normal" concepts are symbolized by letters from the beginning of the alphabet $(A, B, C, \ldots, a, b, \ldots)$[77]. Below it will be shown

1. that indefinite concepts primarily function as (existential and universal) *quantifiers ranging over concepts*;

2. that Leibniz somehow "felt" the difference between an indefinite concept's functioning as an existential quantifier and as a universal quantifier, but that his elliptic formalization fails to bring out this difference with sufficient clarity and precision;

3. that Leibniz nevertheless anticipated some fundamental laws of quantifier logic and may thus be considered at least as a forerunner of modern quantification theory.

The bare essentials of his theory of indefinite concepts — as developed mainly in the **GI** — shall be outlined in this section (7), while some more details will be presented in the subsequent sections devoted to the theory of "quantification of the predicate" (8) and to Leibniz's view of possible individuals and possible worlds (9).

7.1 The Existential Quantifier

By the time around 1679 Leibniz became aware of the possibility to represent the universal affirmative (U.A.) proposition 'Every A is B' by the formula $A = BY$. The origin of this formalization appears to be due to the semantics of so-called "characteristic numbers", i.e. a numerical *model* for the theory of the syllogism which (1) assigns to the concepts $A, B, \ldots$ certain numbers a, b,...[78] where (2) the 'est'-relation among concepts is semantically interpreted by the condition of divisability of the corresponding numbers.

> *A categorical universal affirmative proposition* as 'Man is animal' will be expressed as follows: $\frac{b}{a} = y$, or $b = ya$. For it signifies that the number by which 'man' is expressed can be divided by the number by

[77]Cf. **GI**, §21: "Deinde definitas a me significari prioribus Alphabeti literis, indefinitas posterioribus, nisi aliud significetur." Similarly in **C**. 274-6: "Literae posteriores ut V, W, X, Y, Z, etc. significabunt indefinitum" or also in **C**. 264-70, ## (7,8): "A significat determinatum, Y vel Z vel alia litera posterior significat indeterminatum."

[78]In a later, more sophisticated approach Leibniz assigns a *pair* of such numbers to each concept. For details cf., e.g., Lukasiewicz [1957].

> which 'animal' is expressed, although the result of the division, namely y, is not considered here.[79]

Here y represents an "indefinite number" which is implicitly bound by an existential quantifier. In §16 **GI** the "*Affirmative Proposition* A is B" is similarly analyzed (without specific reference to characteristic numbers) as follows:

> [...] That is, if we substitute a value for A, 'A coincides with BY' will appear [...] For by the sign Y I mean something undetermined, so that BY is the same as some B [...] So 'A is B' is the same as 'A is coincident with some B', or $A = BY$.[80]

This principle, according to which $A \in B$ is equivalent to $A = BY$, has to be interpreted more exactly as the existentially quantified proposition that A contains B if and only if *there exists some* Y *such that* $A = BY$:

(CONT 4) $\quad A \in B \leftrightarrow \exists Y(A = BY).$

This explicit introduction of the existential quantifier not only accords with Leibniz's own intentions but it was also anticipated by him in some other fragments. Thus in §10 of "The Primary Bases of a Logical Calculus" (**C**. 235–7) he used the expression "there can be assumed a Y such that $A = YB$" (**P**, 90). And in fragment **C**. 259–61 Leibniz starts by putting forward the law

(NEG 6*) $\quad A \notin B \leftrightarrow \exists Y(YA \in \overline{B})$

elliptically as "A is not B is the same as QA is non B" (§9), but when he later offers a proof of this principle in §18, he uses the unambiguous and explicit formulation "there exists a Q such that QA is $\overline{B}$" [datur Q tale ut QA sit non B].

Now, there is a minor problem connected with NEG 6*. In view of CONJ 2, the concept $\overline{B}A$ contains $\overline{B}$; hence, trivially, there always exists at least one Y such that $YA \in \overline{B}$, namely $Y = \overline{B}$. Therefore one should improve NEG 6* by saying more exactly that the negation of the U.A., 'Some A is not B', is true if and only if for some Y which is *compatible with* A: YA contains $\overline{B}$:

(NEG 6) $\quad A \notin B \leftrightarrow \exists Y(\mathbf{P}(YA) \wedge YA \in \overline{B}).$

As a matter of fact, Leibniz himself hit upon the necessity of postulating that QA is self-consistent when he proved NEG 6 by means of the former principle POSS 1 as follows:

[79] Cf. **C**., 57: "*Propositio categorica universalis affirmativa*, ut homo est animal, sic exprimetur: $\frac{b}{a}$*aequ.* y, vel b aequ. ya. significat enim numerum quo exprimitur homo, divisibilem esse per numerum quo exprimitur animal, tametsi is quod dividiendo prodit nempe y hic non consideretur".

[80] **P**, 56; cf. also §§17, 158, 189 and 198 **GI** or **C**. 301. In the fragments **C**. 259-61 and **C**. 261-4, Leibniz used the letter 'L' as an "indeterminate concept": "*A est B*, sic exponitur literaliter $A \infty LB$, ubi L idem quod indefinitum quoddam" (**C**. 259); cf. also **C**. 262/3: "cum A est B dici potest $A \infty LB$ [...] per L intelligi Ens vel aliud quiddam quod jam in A continetur".

> 'A is not B' and 'QA is non B' coincide, i.e. to say 'A isn't B' is the same as to say 'there exists a Q such that QA is non B'. If 'A is B' is false, then 'A non B' is possible by [POSS 1]. 'Non B' shall be called 'Q' Therefore QA is possible.[81]

In other places, however, Leibniz often overlooked this requirement or he simply took the self-consistency of the corresponding concept for granted. Thus in §§47, 48 **GI** after stating that "'A contains B' is a universal affirmative in respect of A" he suggests the following formalization for the P.A.: " 'AY contains B' is a particular affirmative in respect of A". Since $AY \in B$, i.e. more explicitly $\exists Y(AY \in B)$, follows from the trivial law $AB \in B$, this condition cannot, however, adequately express the content of the P.A. which rather has to be formalized by $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$.

The basic inference of existential generalization,

(EXIS 1) $\phi(A) \vdash \exists Y \phi(Y)$,

according to which any proposition asserting that a certain concept A has the property ϕ entails that for some indefinite concept $\phi(Y)$, was formulated in §23 **GI** as follows:

> For any definite letter there can be substituted an indefinite letter not yet used [...] i.e. one can put $A = Y$.

Furthermore Leibniz provided several special instances or applications of this rule, e.g.:

(EXIS 1.1) $A = AA \vdash \exists Y(A = AY)$

(EXIS 1.2) $AB \in C \vdash \exists Y(AY \in C)$

(EXIS 1.3) $A = AB \vdash \exists Y(A = YB)$.

Thus in §24 **GI** he derives $\exists Y(A = AY)$ from the principle of idempotence, CONJ 4, by noting:

> To any letter a new indefinite one can be added; e.g., for A we can put AY. For $A = AA$ (by 18 [i.e. CONJ 4]), and A is Y (or, for A one can put Y, by 23 [i.e. by EXIS 1]); therefore $A = AY$. (**P**, 57).

In §49 **GI** he proves EXIS 1.2 as follows: "If AB is C, it follows that AY is C; or, it follows that some A is C. For it can be assumed by 23 [i.e. by EXIS 1] that $B = Y$" (**P**, 59). Furthermore, the validity of EXIS 1.3 (that had already been maintained in §117 **GI**)[82] was proved, e.g., in ♯10 of a fragment of August 1st, 1690 as follows:

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[81] Cf. **C.** 261: "A non est B et QA est non B coincidere seu dicere A non est B, idem esse ac dicere: datur Q tale ut QA sit non B. Si falsum est A est B, *possibile est A non B* per [POSS 1]. Non B vocetur Q. Ergo *possibile est QA*" (my emphasis).

[82] "$A = BY$ is the same as that $A = BA$". Cf. also §8 of fragment **C.**, 261–4.

> If $A = AB$, there can be assumed a Y such that $A = YB$. This is a postulate but it can also be proved, for A itself at any rate can be designated by Y. (**P**, 90).

In # 13 of the same fragment Leibniz also shows the converse implication:

> If $A = YB$, it follows that $A = AB$. I prove this as follows. $A = YB$ (by hypothesis), therefore $AB = YBB$ (by [11]) $= YB$ (by 6 [i.e. CONJ 4]) $= A$ (by hypothesis).

Note, incidentally, that the inference from $A = YB$ to $AB = YBB$ is licensed by principle # 11 of the same essay ("If $A = B$, $AC = BC$") and not, as the editions of Couturat and Parkinson have it, by # 10. It is true that the manuscript contains "per (10)", but this slip is owing to the fact that Leibniz originally numbered the quoted principle as # (10), and when he later renumbered it as # 11, he forgot to change the reference accordingly.

Anyway, these examples show that Leibniz had a fairly good understanding of the rule for *introducing* an existential quantifier, EXIS 1. Moreover, one may also ascribe to him at least a partial insight into the validity of the converse rule for *eliminating* existential quantifiers. In modern systems of natural deduction this rule says that from an existential proposition of the form $\exists Y\alpha[Y]$ one may deduce a corresponding singular proposition $\alpha[A]$ provided that the singular term A is a "new" one, which does not yet occur in the corresponding context:

(EXIS 2) $\exists Y\phi(Y) \vdash \phi(A)$, for some "new" constant A.

In this vein also Leibniz notes in **GI** §27:

> Some $B = YB$, and therefore some $A = ZA$ [...] *but a new indefinite letter, namely* Z, is to be assumed for the latter equation just as Y had been assumed a little earlier. (**P**, 57; my emphasis).

This passage may be interpreted as saying that from a proposition, e.g., of the form 'Some A is C', i.e. $\exists Y(AY \in C)$, one may deduce that $AZ[\in C]$, provided that the indefinite concept Z is "new". In Lenzen [1984a] various other examples were discussed which show that Leibniz often applied the rule of inference, EXIS 2, is just this sense.

7.2 The Universal Quantifier

Leibniz did not always recognize that the *negation* of a formula containing an indefinite concept as an *existential* quantifier gives rise to a *universally* quantified proposition. Thus in "De Formae Logicae Comprobatione" (**C**, 292–321) he tried to prove the syllogisms of the first figure within the quantifier system $L2$ as follows:

Barbara:	Every C is B	Every D is C	Therefore Every D is B.
	$C = BX$	$D = CY$	Therefore $D = BXY$.
Celarent:	No C is B	Every D is C	Therefore No D is B
	$C = X$ non-B	$D = CY$	Therefore $D = YX$ Non-B
Darii:	Every C is B	Some D is C	Therefore Some D is B
	$C = BX$	$D \neq Y$ non-C	Therefore $D \neq Y$ non-BX.

But the desired $D \neq YX$ non-B does not follow from this [representation]. Hence there is still another difficulty in this calculus. Let's take an example: Every man is an animal. Some wise [being] is a man. Therefore Some wise [being] is an animal. According to the calculus: 'Man' is the same as 'rational animal'; 'wise' is not the same as 'Y not-man'. Therefore 'wise' is not the same as 'Y not-(rational animal)'.[83]

The proof of *Barbara* rests on the formalization of the universal affirmative proposition according to CONT 4. Thus 'Every C is B' is represented by '$C = BX$', i.e. more explicitly $\exists X(C = BX)$; similarly 'Every D is C' is represented by the corresponding formula $[\exists Y](D = CY)$; now substitution of BX for C in the latter equation yields $[\exists Y \exists X](D = BXY)$ which can easily be transformed into $\exists Z(D = BZ)$, i.e. 'Every D is B'. The latter inference, though not mentioned explicitly in the above quoted passage, had been stated, e.g., in the **GI** as follows:

(19) [...] So when $A = BY$ and $B = CZ, A = CYZ$; or, A contains C.
(20) It must be noted [...] that one letter can be put for any number of letters together: e.g. $YZ = X$. (**P**, 56/7).

Next *Celarent* is proved in quite the same way as *Barbara* by making use of the traditional principle of obversion according to which the universal negative proposition (U.N.) 'No C is B' is equivalent to a U.A. with the negated predicate

[83]Cf. C., 301:

"*Barbara*:	Omne C est B.	Omne D est C.	Ergo Omne D est B.
	$C = BX$.	$D = CY$.	Ergo $D = BXY$.
Celarent:	Nullum C est B.	Omne D est C.	Ergo Null. D est B.
	$C = X$ non-B.	$D = CY$.	Ergo $D = YX$ Non-B.
Darii:	Omne C est B.	Qu. D est C.	Ergo Qu. D est B.
	$C = BX$.	D non $= Y$non$C$.	Ergo D non $= Y$non BX.

Sed hinc non sequitur: D non $= YX$non$B$ quod desideratur. Unde est alia adhuc in tali calculo difficultas. Exemplum sumamus: Omnis homo est animal. Quidam sapiens est homo. *E.* quidam sapiens est animal. Secundum calculum: Homo idem est quod animal rationale; sapiens non idem est quod Y non homo. Ergo sapiens non idem est quod Y non animal-rationale".

'Every C is not-B'. Hence $C \in \overline{B}$, i.e., according to CONT 4, $\exists X(C = \overline{B}X)$, plus the second premiss $[\exists Y](D = CY)$ yields by substitution $[\exists Y \exists X](D = \overline{B}XY)$, which may be simplified to $\exists Z(D = \overline{B}Z)$, i.e. 'Every D is $\overline{B}$' or 'No D is B'.

However, during his attempt to give a similar proof for *Darii* Leibniz faces another difficulty in his calculus [**C.** 301: "Unde est aliqua adhuc in tali calculo difficultas"] which is due, among others, to the fact that in '$D \neq Y$ not-C' the indefinite concept Y functions as a *universal* quantifier. The difficulty can be analyzed as follows. From 'Every C is B', i.e. $[\exists X](C = BX)$, plus 'Some D is C' which, as the negation of $D \in \overline{C}$, would have to be formalized explicitly as $\neg\exists Y(D = Y\overline{C})$, or $\forall Y(D \neq Y\overline{C})$, one obtains by way of substitution $\forall Y(D \neq Y\overline{BX})$. Leibniz formalizes this elliptically as $D \neq Y\overline{BX}$ and does not see how one might get from this the desired conclusion $D \neq YX\overline{B}$. As a matter of fact, the inference from $\forall Y(D \neq Y\overline{C})$ and $\exists X(C = BX)$ to $\forall Z(D \neq Z\overline{B})$ is not at all obvious, in particular for someone like Leibniz who never developed any laws that would allow him to transform a negated conjunction like $\overline{BX}$ into, say, a disjunction of $\overline{B}$ and $\overline{X}$. However, Leibniz might have solved this difficulty by observing that according to the law of contraposition, NEG 3, the premiss $C \in B$ entails $\overline{B} \in \overline{C}$, i.e. by CONT 4 $\exists X(\overline{B} = X\overline{C})$. Using *this* equation, $\forall Y(D \neq Y\overline{C})$ is easily shown to entail $\forall Z(D \neq Z\overline{B})$, because if there would exist some Z such that $D = Z\overline{B}$, the substitution $\overline{B} = X\overline{C}$ would yield $D = ZX\overline{C}$ which contradicts the premiss $\forall Y(D \neq Y\overline{C})$.

In view of the other difficulties that Leibniz encountered during his attempt to prove the syllogistic laws in "De Formae Logicae Comprobatione", it may be understandable that he did not fully realize the difference between the use of indefinite concepts functioning as existential and as universal quantifiers, respectively. In other fragments, however, he became more or less aware of this distinction. Thus in a somewhat confused passage of §112 **GI**[84] he said:

> It must be seen whether, when it is said that AY is B (i.e. that some A is B), Y is not taken in some other sense than when it is denied that any A is B, in such a way that not only is it denied that some A is B — i.e. that this indeterminate A is B — but also that any A out of a number of indeterminates is B, so that when it is said that no A is B, the sense is that it is denied that $A\hat{Y}$ is B; for $\hat{Y}$ is Y, i.e. any Y will contain this Y. So when I say that some A is B, I say that this some [hoc quoddam] A is B; if I deny that some A is B, or that this some A is B, I seem only to state a particular negative. But when I deny that any A is B, i.e. that not only this, but also this and this A is B, then I deny that $\hat{Y}$ is B. (**P**, 72).

While the P.A. shall be formalized, according to Leibniz, by '$AY \in B$' with Y functioning as an *existential* quantifier, its negation shall not be represented as

[84]In order to avoid confusion with our formalization of conceptual negation, the symbol $\overline{Y}$ which Leibniz here uses for the "universal" indeterminate concept was replaced by '$\hat{Y}$'. Cf. also §§80–82 GI where Leibniz similarly uses two different symbols for indefinite concepts.

$AY \notin B$, but rather by means of a new symbol $\hat{Y}$ as $A\hat{Y} \notin B$, where this new type of indefinite concept $\hat{Y}$ denotes "any Y" [quodcunque Y] and thus represents a *universal* quantifier. To put it less elliptically: whereas 'Some A is B' may be formalized in $L2$ as $[\exists Y](AY \in B)$[85], the negation takes the form $[\forall\hat{Y}](A\hat{Y} \notin B)$ in accordance with the well-known law

(UNIV 1) $\quad \neg\exists Y\alpha[Y] \leftrightarrow \forall Y\neg\alpha[Y]$,

or its special instance

(UNIV 1.1) $\quad \neg\exists Y(AY \in B) \leftrightarrow \forall Y(AY \notin B)$.

In view of this explanation, Leibniz's incidental remark "$\hat{Y}$ is Y, i.e. any Y will contain this Y" [$\hat{Y}$ est Y, seu quodcunque Y continebit hoc Y] expresses another important law of the logic of quantifiers, namely: Each proposition of the form $\alpha[\hat{Y}]$ entails the corresponding proposition $\alpha[Y]$, or less elliptically:

(UNIV 2) $\quad \forall Y\alpha[Y] \rightarrow \exists Y\alpha[Y]$.

This principle was anticipated also in fragment **C.** 270–3 where Leibniz had similarly used two types of indefinite concepts, Y and $\tilde{Y}$:[86]

> Let us see in which way Y and $\hat{Y}$ differ from each other, namely like 'something' and 'whatsoever' but this happens by accident, and I want it to be Y simpliciter. This must be examined more carefully.[87]

Unfortunately, Leibniz never carried out the closer examination of this topic. Nevertheless it should be clear that Y as 'something' represents the existential quantifier $\exists Y$ while $\tilde{Y}$ as 'whatsoever' corresponds to the universal quantifier $\forall Y$, and the remark that $\tilde{Y}$ should be "Y simpliciter" means that a universal proposition of the type $\forall\tilde{Y}\alpha[\tilde{Y}]$ *entails* the corresponding existential proposition $\exists Y\alpha[Y]$.

There are various other logical laws where Leibniz used indefinite concepts as universal quantifiers. Thus in **C.** 259–61 he formulates: "(15) A is B is the same as to say: If L is A, it follows L is also B" [A est B, idem est ac dicere si L est A sequitur quod et L est B]. Couturat [1901, p. 347, fn 2] thought that this principle would represent only a variant of the "principe du syllogisme", i.e. the law of transitivity of the $\in$-relation. But this interpretation is incompatible with the fact that CONT 2 has the form $A \in B \wedge L \in A \rightarrow L \in B$, or, equivalently, $A \in B \rightarrow (L \in A \rightarrow L \in B)$, where the first implication must never be strengthened into a biconditional. Furthermore Leibniz's explanation "L is to be understood as

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[85]More exactly, in view of the trivial law $AB \in B$, the P.A. should be formalized by $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$ — cf. the discussion of principles NEG 6* and NEG 6 in section 7.1; this complication can, however, be ignored here.

[86]Here for typographical reasons 'X' has been replaced by 'Y' because my word processor only generates '$\tilde{Y}$' but not Leibniz's sign composed of an 'X' and '$\sim$'.

[87]Cf. C., 271: "Videndum quomodo Y et $\hat{Y}$ differant, scilicet ut aliquod et quodcunque sed id contingit per accidens, et velim qui sit Y simpliciter. Haec melius examinanda".

any term of which 'L is A' can be said" [Intelligitur autem L quicunque terminus de quo dici potest L est A] makes clear that here L is not a definite but an indefinite concept, i.e. a *variable* functioning as a universal quantifier. Therefore the principle has to be formalized more explicitly as follows:

(UNIV 3) $\quad (A \in B) \leftrightarrow \forall L(L \in A \rightarrow L \in B).$

Leibniz's proof contains an anticipation of the contemporary rules for eliminating and introducing universal quantifiers:

> Let us assume the proposition 'A is B'. I say that it entails 'If L is A, it follows that L is B', which I prove as follows: Since A is B, hence $A = AB$[...]. But if L is A, then $L = LA$. Whereby (substituting for A the value AB) one obtains $L = LAB$. Therefore L is AB, hence L is B [...]. Now let us conversely prove that 'If L is A, it follows that L is B' entails 'A is B'. L however is to be understood as any term of which 'L is A' can be said. So assume the one [$\forall L(L \in A \rightarrow L \in B)$] to be true and yet the other [$A \in B$] to be false. [...] Therefore the following proposition will be stated: QA is non-B. [...] But QA is A. Therefore QA is B (*because QA is subsumed under L*). Hence QA is B non-B what is absurd.[88]

In the first part Leibniz derives $[\forall L](L \in A \rightarrow L \in B)$ from the premiss $A \in B$ by showing that, for any $L, L \in A$ (in conjunction with $A \in B$) entails $L \in B$. This follows the basic idea of the rule of $\forall$-introduction according to which $\forall Y \alpha[Y]$ may be established by showing that, for any arbitrary constant $A, \alpha[A]$. In the second part Leibniz proves indirectly that $A \notin B$ is incompatible with the premiss $[\forall L](L \in A \rightarrow L \in B)$, because if $A \in B$ was false, then according to NEG 6 there would exist some Q such that $QA \in \overline{B}$ (and $\mathbf{P}(QA)$); now, trivially, according to CONJ 3 $QA \in A$; thus $[\forall L](L \in A \rightarrow L \in B)$ would allow us to conclude that $QA \in B$ ("because QA is subsumed under [the variable] L"); hence (by CONJ 1) we would obtain $QA \in B\overline{B}$ which is "absurd" or, more correctly, which contradicts $\mathbf{P}(QA)$. This kind of proof follows the basic idea of $\forall$-elimination according to which $\forall Y \alpha[Y]$ entails, for any arbitrary constant $A, \alpha[A]$.

Another interesting law implicitly containing a universal quantifier may be found in a marginal note to §18 **GI**, where Leibniz first notes that $AC = ABD$ does not generally entail $C = BD$; and where he adds that the following special case of this inference is valid:

[88]Cf. **C**. 260: "Assumamus hanc propositionem A *est* B. dico hinc inferri si L est A, sequitur quod L est B. Hoc ita demonstro: Quia A est B, ergo $A \infty AB$ [...]. Jam si L est A, erit $L \infty LA$. Ubi (pro A substituendo valorem AB) fit $L \infty LAB$. Ergo L est AB. Ergo L est B [...]. Nunc inverse demonstremus, ex hac: Si L est A sequitur quod L est B, vicisssim inferri A est B. Intelligitur autem L quicunque terminus de quo dici potest L est A. Ponamus illud [$\forall L(L \in A \rightarrow L \in B)$] esse verum, et tamen hoc [$A \in B$] esse falsum. [...] Statuatur ergo haec enuntiatio: QA est non B. [...] Jam QA est A. Ergo QA est B (*quia QA comprehenditur sub L*) Ergo QA est B non B quod est abs." (my emphasis).

> For it to be inferred from $AC = ABD$ that $C = BD$, it must be presupposed that nothing which is contained in A is contained in C unless it is also contained in BD, and conversely. (**P**, 56, Note 2).

If, for the sake of simplicity, we substitute 'B' for 'C' and also 'C' for 'BD', this principle says that $AB = AC$ entails $B = C$ provided that each concept Y which is contained in A will be contained in B if and only if it is also contained in $C : \forall Y(A \in Y \rightarrow (B \in Y \leftrightarrow C \in Y)) \rightarrow (AB = AC \rightarrow B = C)$. Some further laws are discussed in [Lenzen, 1984a].

8 THE "QUANTIFICATION OF THE PREDICATE"

Leibniz's theory of "Quantification of the predicate" (TQP, for short) was developed mainly in the fragment "Mathesis rationis" which had first been edited in 1903 by Couturat (**C**, 193–206; cf. **P**, 95–104).[89] However, Couturat published not much more than the final version of the essay (sheets 1 and 2 of the manuscript **LH** IV, 6, 14),[90] while a preliminary draft and some related studies (sheets 3–5) were edited only in a very abridged form (cf. **C**, 203–206). Even the main text is far from complete since, among others, three important paragraphs that Leibniz decided to omit[91] did not find entrance into Couturat's edition. As will be shown below, the additional material of these §§ provides the key for a proper understanding of §24 which - together with the related §§ 3-6 — forms the core of the whole essay.

Perhaps due to the lack of a complete and critical text, the real meaning of this fragment seems not to have been recognized so far. Most scholars agreed to Couturat's verdict that Leibniz sketched TQP, *only in order to refute it*.[92] Couturat [1901, p. 24] maintained this view although he was aware of the fact that Leibniz had stressed at several places the importance of TQP for a "foundation of all rules of the figures and moods of syllogistic theory". Couturat thought it necessary to close an apparent gap in Leibniz's syllogistic studies by providing a "Précis of classical logic" which basically consisted in a derivation of the theory of the syllogism from TQP. However, a closer analysis of the *Mathesis* reveals that Leibniz was in no need of such help since he not only developed TQP all by himself but also used it in much the same way as Couturat as a tool for deriving the basic laws of the syllogism.

[89]The most important logical works are abbreviated as follows: *Comprobatione* = "De formae logicae comprobatione per linearum ductus" (**C**, 292–321); *Dissertatio* = *Dissertatio de Arte Combinatoria* (**A** VI, 1, 168–230).

[90]The classification of Leibniz's manuscripts (**LH**) follows the catalogue of E. Bodemann (**LH**).

[91]Cf. **LH** IV, 6, 14, 1 recto: "Omitti possunt 48, 49, 50".

[92]Cf. **C**, 194, fn.1: "Ici Leibniz conçoit nettement la quantification du prédicat, et la rejette."

8.1 Theory of the syllogism and universal calculus

Leibniz's great aim in logic was to construct a general calculus of concept logic that would enable him to strictly verify the traditional theory of the syllogism. It is not easy to chronologize this enterprise but the following can be claimed with some degree of certainty. On the one hand, Leibniz dealt with issues in the traditional theory of the syllogism practically throughout his (adult) life, namely from 1665 when he composed the *Dissertatio* until 1715 when the "Schedae de novis formis et figuris syllogisticis" (**C**, 206–210) were written. The various drafts of a general calculus, on the other hand, date from a much shorter period between 1680 and 1690, approximately. The validation of the theory of the syllogism by means of the "Calculus universalis" involves two tasks which can be referred to as 'soundness' and 'completeness', respectively. The *proof of soundness* amounts to showing that both the simple inferences of subalternation, opposition, and conversion and the 24 moods that were generally regarded as valid[93] can be derived as theorems of $L1$ or $L2$. If, as usual, **A**, **E**, **I**, and **O** symbolize the categorical forms of a universal affirmative, universal negative, particular affirmative, and particular negative proposition, the simple consequences may be formalized as follows:

(Opp 1) $\neg\mathbf{A}(B,C) \leftrightarrow \mathbf{O}(B,C)$

(Opp 2) $\neg\mathbf{E}(B,C) \leftrightarrow \mathbf{I}(B,C)$

(Sub 1) $\mathbf{A}(B,C) \rightarrow \mathbf{I}(B,C)$

(Sub 2) $\mathbf{E}(B,C) \rightarrow \mathbf{O}(B,C)$

(Conv 1) $\mathbf{E}(B,C) \leftrightarrow \mathbf{E}(C,B)$

(Conv 2) $\mathbf{E}(B,C) \rightarrow \mathbf{O}(C,B)$

(Conv 3) $\mathbf{A}(B,C) \rightarrow \mathbf{I}(C,B)$

(Conv 4) $\mathbf{I}(B,C) \leftrightarrow \mathbf{I}(C,B)$.

The perfect moods of the Ist figure accordingly take the shape:

(Barbara) $\mathbf{A}(C,D) \wedge \mathbf{A}(B,C) \rightarrow \mathbf{A}(B,D)$

(Celarent) $\mathbf{E}(C,D) \wedge \mathbf{A}(B,C) \rightarrow \mathbf{E}(B,D)$

(Darii) $\mathbf{A}(C,D) \wedge \mathbf{I}(B,C) \rightarrow \mathbf{I}(B,D)$

[93] In many places Leibniz defended the view that there are exactly 6 valid moods in each of the 4 figures. He put forward this claim already in the *Dissertatio* (**A** VI, 1, 184: "Ita ignota hactenus figurarum harmonia detegitur, singulae enim modis sunt aequales"), but one may doubt whether at that time he was entitled to do so. On the one hand the table of the valid moods contained a 25th syllogism named Frisesmo which "[...] ex regulis modorum non sit inutilis" (**A** VI, 1, 185/6). On the other hand Leibniz mistakenly listed a syllogism Colanto among the valid moods of the IVth figure while in fact it had to be replaced by Calerent.

(FERIO) $\quad \mathbf{E}(C,D) \wedge \mathbf{I}(B,C) \to \mathbf{O}(B,D)$.

Actually, the proof of soundness could be simplified to demonstrating these 4 moods only plus the laws of opposition. For Leibniz had shown in "De formis syllogismorum Mathematice definiendis" (**C**, 410–416) that:

1. the laws of subalternation, SUB 1, 2, follow from DARII and FERIO;
2. by means of SUB 1 and 2 the remaining two moods of the Ist figure, BARBARI and CELARO, can be proved;
3. the moods of figures II and III can be reduced to those of the Ist by means of a primitive inference called '*regressus*'; and
4. the laws of conversion can be derived from moods of the IInd and IIIrd figure.

Finally in *Mathesis* Leibniz also proved that

5. the moods of the IVth figure follow from the previous ones by means of the rules of conversion.[94]

Hence {BARBARA, CELARENT, DARII, FERIO, OPP 1,2} constitutes an axiomatic basis of the theory of the syllogism.

Leibniz who already in 1679 had developed a *semantical* method for validating these principles by means of characteristic numbers[95] started a series of *syntactic* derivations in *Comprobatione* which was probably written around 1686. At that time, however, the various attempts to derive the basic principles of the theory of syllogism from the "universal calculus" remained without success. As was shown in Lenzen [1988], it was not before 1690 that Leibniz found a satisfactory proof of the soundness of syllogistic theory[96]. The *proof of completeness*, on the other hand, should have

- to demonstrate the traditional canon of general rules including the so-called rules of quantity and quality;
- to derive from them some more specific rules for the single figures; and
- to show that the latter suffice to invalidate all but those syllogisms already proven to be sound.

Before investigating how Leibniz tackled this threefold task in *Mathesis*, let us take a closer look at the traditional version of this syllogistic doctrine as described, e.g., in the famous *Port-Royal Logic.*

[94]Cf. **LH** IV, 6, 14, 3 recto - 3 verso. Another proof of the IVth figure is given in **C**, 209.

[95]Cf. the series of essays of April 1679 (**C**. 42-92 + 245-247) where Leibniz maintains "Ex hoc calculo omnes modi et figurae derivari possunt per solas regulas Numerorum" (**C**. 247). For a possible extension of Leibniz's method to a language containing negation cf. [Sotirov, 1999].

[96]Cf. the marginal note: "Hic demonstrantur Modi primae figurae, et regulae oppositionum. Quarum ope (ut alibi jam ostendimus) demonstrantur deinde conversiones et modi reliquarum figurarum." (**C**, 229).

8.2 Axioms and rules of traditional syllogistics

The first axiom of Arnauld/Nicole [1683] is nothing but the above mentioned law of subalternation. Three further axioms contain the *theory of quantity and quality*, that is:

(QUAN) The subject of a universal proposition is universal. The subject of a particular proposition is particular.

(QUAL) The predicate of an affirmative proposition is particular. The predicate of a negative proposition is universal.

These axioms are said to be the basis for the subsequent *general rules of the syllogism*, although Arnauld/Nicole fail to show how the latter might be derived from the former.

(GR 1) The middle term may not be particular in both premisses.

(GR 2) If a term is universal in the conclusion then it must also be universal in the premiss.

(GR 3) At least one of the premisses must be affirmative.

(GR 4) If the conclusion is negative, one of the premisses also has to be negative.

Next: "The conclusion always follows the weaker part, i.e. if one of the two propositions is negative, the conclusion must be negative, and if one is particular, it must be particular"[97]. It will be convenient to split this rule up into

(GR 5.1) If one of the premisses is particular, then the conclusion must be particular;

(GR 5.2) If one of the premisses is negative, then the conclusion must be negative.

Finally one has:

(GR 6) At least one of the premisses must be universal.

These general rules in turn are supposed to entail the following *special rules* for the single figures, although, again, Arnauld/Nicole fail to indicate how the latter might be obtained from the former. The *first figure* is defined by the fact that the middle term, C, is the subject in the minor-premiss, i.e. the premiss containing the minor-term, B, while C is the predicate in the major-premiss (which contains the major-term D). Here the following restrictions obtain:

[97]Cf. Arnauld/Nicole [1683, p. 186]: "La conclusion suit toûjours la plus foible partie, c'est-à-dire, que s'il y a une des deux propositions negatives, elle doit être negative; & s'il y en a une particuliere, elle doit être particuliere".

(SR I.1) In the first figure the minor-premiss must be affirmative

(SR I.2) In the first figure the major-premiss must be universal.

In the *second figure*, which is defined by having the middle term both times as a predicate, the corresponding restrictions run as follows:

(SR II.1) In the second figure one of the premisses must be negative

(SR II.2) In the second figure the major-premiss must be universal.

The *third figure* is characterized by having the middle term both times as subject. Here the following conditions apply:

(SR III.1) In the third figure the minor-premiss must be affirmative

(SR III.2) In the third figure the conclusion must be particular.

Finally, with regard to the *fourth figure* where the middle term is predicate in the major-premiss and subject in the minor-premiss, [Arnauld and Nicole, 1683, p. 200] mention three conditions:

> If the major is affirmative, the minor is always universal [...] If the minor is affirmative, the conclusion is always particular [...] In all negative moods the major must be general.

In view of the general rules GR 4 and GR 5.2, a mood is negative if and only if it has a negative conclusion. Hence we can paraphrase the above conditional restrictions as follows:

(SR IV.1) In the fourth figure, if the major-premiss is affirmative, the minor-premiss must be universal

(SR IV.2) In the fourth figure, if the minor-premiss is affirmative, the conclusion must be particular

(SR IV.3) In the fourth figure, if the conclusion is negative, the major-premiss must be universal.

8.3 Leibniz's early attempts at a proof of completeness

Leibniz appears to have been acquainted with this traditional doctrine already as a youth. In the *Dissertatio* he does not state the axioms QUAN and QUAL, though, but he mentions in passing the general rules GR 2, 3, 5, 6[98], and he also

[98] "Ex puris particularibus nihil sequitur [...] Conclusio nullam ex praemissis quantitate vincit [...] Ex puris negativis nihil sequitur [...] Conclusio sequitur partem in qualitate deteriorem" (A VI, 1, 181).

formulates the special rules in a very condensed way[99]. Only Leibniz's conditions for the IVth figure differ quite considerably from the traditional restrictions: "In the IVth the conclusion is never a UA. The major never PN. And if the minor is N, the major is UA".[100] In *Comprobatione*, probably written 2 decades after the *Dissertatio*, Leibniz gives a riper version of the laws of the syllogism, and he makes some first steps towards a proof of completeness. First he mentions (although he does not prove yet) the proper rules of quantity and quality when he points out that

> A *distributed term* is the same as a total or universal one; a non-*distributed* is one which is particular or partial. The subject has the same quantity as the proposition. [...] But the predicate in each affirmative proposition is partial or non-distributed, and in each negative proposition it is total or distributed.[101]

Second he is now able to demonstrate the validity of the general rules (omitting only GR 4) as follows. As regards GR 1:

> *The middle [term] must be distributed or total in at least one of the premisses*, otherwise no coincidence can be established; if something of the minor term coincides or fails to coincide with a part of the middle term, and something of the major term in turn coincides or fails to coincide with a part of the middle term, different parts of the middle term might be concerned.[102]

Similarly, we read with respect to GR 2:

> [...] it can generally be said that a term cannot be more ample in the conclusion than it is in the premisses, otherwise that which would not enter into the logical consideration, namely that part of the term which is not concerned in the premisses, would enter into the conclusion [...]. And this is what is ordinarily stated as '*A term which is not distributed* [...] *in a premiss cannot be distributed in the conclusion.*[103]

[99]Cf. **A** VI, 1, 184: "Imae autem et 2dae figurae semper major propositio est U[niversalis ...] Imae et IIItiae semper minor A[ffirmativa ...] In IIda semper Conclusio N[egativa ...] In IIItia Conclusio semper est P[articularis]".

[100]Cf. **A** VI, 1, 184: "In IV^{ta} Conclusio nunquam est UA. Major nunquam PN. Et si Minor N, Major UA".

[101]Cf. **C.**, 312: "*Terminus distributivus* est idem qui totalis seu universalis; *non distributus*, qui particularis seu partialis. Subjectum est ejusdem quantitatis cujus propositio. [...] Sed praedicatum in omni propositione affirmativa est partiale seu non distributum, et in omni propositione negativa est totale seu distributum".

[102]Cf. C., 317: "*Medius debet esse in alterutra praemissarum distributus seu totalis*; alioqui nulla potest effici coincidentia, si minoris termini aliquid parti medii coincidit aut non coincidit, et majoris termini aliquid rursus parti medii coincidit aut non coincidit, diversae partes medii affici poterunt".

[103]Cf. C., 316: "[...] generaliter dici potest terminum non posse [esse] ampliorem in conclusione quam in praemissa, alioqui id quod non venisset in ratiocinationem, ea nempe pars termini, quae in praemissa non afficitur, veniret in conclusionem [...] Atque hoc est quod vulgo dicitur *Terminum non distributum [...] in praemissa nec posse esse distributum in conclusione*".

Concerning GR 3 Leibniz explains:

> It is also evident that nothing can be inferred from merely negative propositions. For if you only exclude that which is in an extreme [minor or major] term from that which is in the middle [term] you cannot infer any coincidence, indeed you cannot even infer the exclusion of that what is in one of the extremes from that which is in the other.[104]

The proof of the remaining rules GR 5, 6 is somewhat less satisfactory because Leibniz restricts it to the case of affirmative propositions noting that "all negative syllogisms can be transformed into affirmative ones by changing a negative [proposition] into an affirmative with an indefinite [i.e. negative predicate]".[105]

The special rules for the single figures, however, are not derived very systematically by Leibniz. He just mentions some restrictions that happen to come to his mind as immediate consequences of the general rules. Thus, as a corollary of GR 1, he notes: "Therefore in the figures [?] where the middle term is always the predicate [i.e., only in the IInd figure] the conclusion must be negative" [Hinc in figuris ubi medius terminus semper est praedicatum conclusio debet esse negativa], i.e. SR II.1, and "where [the middle term] always is the subject [i.e., in the IIIrd figure], the conclusion must be particular" [ubi semper est subjectum conclusio debet esse particularis], i.e. SR III.2. Furthermore Leibniz infers from GR 2 some *conditional* restrictions which, however, are much weaker than the traditional rules.[106] Finally, Leibniz promises to derive further rules for the Ist and IVth figure once GR 6 and GR 5 were proven, but he fails to make this announcement true.

8.4 Proving the special rules

By the time of the *Mathesis,* probably around 1705[107], Leibniz has gained a clear knowledge of the logical foundations of the general rules. In what I consider as a preliminary version of the essay, he gives the following summary of the "fundaments of all theorems of the figures and the moods":

> (1) The middle term must be universal in at least one premiss [...]

[104]Cf. C., 318: "Manifestum etiam est ex meris negativis propositionibus nil sequi. Nam sola exclusio ejus quod est in termino extremo ab eo quod est in medio non infert utique ullam coincidentiam, sed ne quidem inferre potest exclusionem ejus quod in uno extremo ab eo quod est in alio extremo."

[105]Cf. C, 319: "omnes syllogismos negativos posse mutari in affirmativos, ex negativa faciendo affirmativam indefiniti [praedicati]".

[106]Cf., e.g., C, 316: "[...] si conclusio est universalis, Minorem propositionem esse universalem in figuris ubi terminus minor est praemissae suae subjectum, scilicet prima et secunda". This condition and three similar ones reappear in *Mathesis* as §§34 - 36.

[107]According to a communication of Prof. Schepers from the Leibniz-Forschungsstelle Münster, the water-sign of the manuscript indicates that *Mathesis* was written at about that time. The present investigation also suggests that *Mathesis* is a rather late fragment, at any rate later than *Comprobatione* because the TQP-version of the categorical forms given there (cf. C, 311) is clearly inferior to the one presented in *Mathesis*.

> (2) At least one premiss must be affirmative [...]
>
> (3) A particular term in a premiss is also particular in the conclusion [...]
>
> (4) If one premiss is negative, also the conclusion is negative [...]
>
> (5) The subject of a universal proposition is universal, that of a particular is particular [...]
>
> (6) Because of the logical form, the predicate of an affirmative proposition is particular, that of a negative is universal.
>
> From these [six] fundamentals all theorems concerning the figures and moods can be proved."[108]

It is not without interest to note that Leibniz sees no need to distinguish the traditional axioms QUAL and QUAN from the theorems GR 1–6; he rather considers them all alike as fundamentals. Actually, the above list contains only a part of the traditional rules, *viz.* GR 1, 2, 3, and 5.1. Leibniz evidently forgot to state also GR 4, but in the final version of *Mathesis* he recognizes this slip when he inserts into his formulation of GR 5.2 "Nor is it less evident that if one of the premisses is negative, the conclusion also must be negative"[109] the remark "and vice versa". In contrast, the fact that also GR 5.1 and GR 6 no longer range among the fundamentals should not be taken as another slip of Leibniz but rather as the result of his insight that both principles follow from the remaining ones. Corresponding proofs are provided in §§32 and 33 of the main text.

In an admirably clear and strictly deductive way Leibniz shows in §§37, 38, 39, 42, 43 that the fundamental principles (in conjunction with the definition of the figures as stated in §22) entail the following special rules for the first 3 figures:

- SR II.1: "[...] in the second figure, the conclusion must be negative";
- SR II.2: "In the same figure, the major proposition is always universal";
- SR III.2: "[...] in the third figure, the conclusion must be particular";
- SR III.1 + SR I.1: "In the first and the third figure the minor proposition is affirmative";
- SR I.2: "In the first figure, the major proposition is universal."

[108]Cf. **LH** IV, 6, 14, 4 verso: "(1) Medius terminus debet esse universalis in alterutra praemissa [...]
(2) Alterutra praemissa debet esse affirmativa [...]
(3) Terminus particularis in praemissa est particularis in conclusione [...]
(4) Si una praemissa sit negativa, etiam conclusio est negativa [...]
(5) Subjectum propositionis universalis est universale, particularis particulare
(6) Praedicatum propositionis affirmativae vi formae est particulare, negativae universale.
Ex his [sex] fundamentis omnia Theoremata de Figuris et modis demonstrari possunt."

[109]Cf. **C.**, 196: "Nec minus manifestum est, una praemissa existente negativa, etiam conclusionem esse negativam".

Moreover, the number of special rules for the IVth figure also can be reduced to two. The former SR IV.1 is stated in §46 as follows: "In the fourth figure, the minor proposition is not particular at the same time as the major proposition is affirmative"; and instead of SR IV.2 + IV.3 Leibniz now formulates: "In the fourth figure, the major proposition is not particular at the same time as the minor proposition is negative." (§45). Hence Leibniz who in general was fond of symmetries and harmonies happily concludes: "Any figure, therefore, has two limitations" (§47).

A careful analysis of the Leibnitian proof of the special rules reveals that each of the six fundamentals (and no other principle) is used as a premiss. As will be shown in section 8.6 below, the special rules in turn are necessary and sufficient to carry out the final step in the proof of completeness by proving "[...] that there are not more [than the 24 valid moods], and this must be done, not by an enumeration of illegitimate moods, but from the laws of those which are legitimate" (**P**, 104). First, however, we will have to describe Leibniz's version of TQP which is the basis for the first step of the completeness proof, *viz.* for validating the six fundamentals.

8.5 The Quantification of the Predicate

In order to discuss Leibniz's TPQ let us consider, e.g., the universal affirmative proposition:

> (3) When I say 'Every A is B', I understand that any of those which are called A is the same as some one of those which are called B.

What kind of entitites are the informal quantifier-expressions 'any' and 'some' assumed to refer to, and how is the relation of 'being called' A (or B) to be understood? For a contemporary logician it may be most natural to interpret the quantifiers as referring to *individuals* which are elements of the *set* A (or individuals to which the *predicate* A applies). In this case one arrives at the following version of TQP. The universal affirmative proposition 'Every A is B' will be paraphrased as: 'Every individual x which is an element of A is identical with some individual y which is an element of B'. Since the symbol '$\in$' is here used to designate the containment relation between concepts, we now better chose another symbol, say ε, for expressing the set-theoretical relation between a certain object x and a set A. Furthermore, in distinction to Leibniz's quantifiers, $\forall$ and $\exists$, ranging over *concepts*, let us introduce another pair of quantifiers, Λ and V, which range over *objects*. Leibniz's extensional characterization of the U.A. then takes the following form:

(UA 1) $\quad \Lambda x(x\varepsilon A \rightarrow \mathrm{V}y(y\varepsilon B \wedge y = x))$.

The particular affirmative proposition 'Some A are B' in the sense of "(4) [...] some one of those which are called A [are] the same as some one of those which are called B" accordingly can be formalized as follows:

(PA 1) $\quad Vx(x\varepsilon A \wedge Vy(y\varepsilon B \wedge y = x)).$

The universal negative proposition, 'No A are B', in the sense of "(5) [...] any one of those which are called A is different from any one of those which are called B" amounts to:

(UN 1) $\quad \Lambda x(x\varepsilon A \rightarrow \Lambda y(y\varepsilon B \rightarrow y \neq x)).$

Finally, the particular negative proposition, 'Some A are not B', in the sense of "(6) [...] some one of those which are called A [are] different from any one of those which are called B" can be rendered as:

(PN 1) $\quad Vx(x\varepsilon A \wedge \Lambda y(y\varepsilon B \rightarrow y \neq x)).$

Under the present interpretation the additional propositions mentioned in §7 make a clear sense, although they are "superfluous" [inutile] and "not in accordance with our linguistic usage" [non est in usu in nostris linguis]. To say that "every A is every B" means that "all those which are called A are the same as all those which are called B" (**P**, 95; cf. **C**., 193: "omnes qui dicuntur A esse eosdem cum omnibus qui dicuntur B"). This can be formalized as follows:

(NC 1) $\quad \Lambda x(x\varepsilon A \rightarrow \Lambda y(y\varepsilon B \rightarrow y = x)).$

But this will never be the case unless the sets A and B are singletons which contain exactly one and the same element.

In the same way the corresponding proposition "Some As are the same as all Bs" (**P**., 95, cf. **C**., 194: "quosdam A esse eosdem cum omnibus B") has to be formalised as

(NC 2) $\quad Vx(x\varepsilon A \wedge \Lambda y(y\varepsilon B \rightarrow y = x)).$

Again this can't be true unless the set B is a singleton.[110]

The other two propositions which Leibniz obtained by negating NC 1 and NC 2: "[...] any one of those which are called A is different from some one of those which are called B" and "[...] some one of those which are called A is different from some one of those which are called B" (**P**., 95), i.e.

(NC 3) $\quad \Lambda x(x\varepsilon A \rightarrow Vy(y\varepsilon B \wedge y \neq x))$

(NC 4) $\quad Vx(x\varepsilon A \wedge Vy(y\varepsilon B \wedge y \neq x)),$

[110] Note, incidentally, that Leibniz commits a fallacy when he says that NC 2 might equivalently be expressed by saying "Omnes B esse A". According to UA 1, the latter amounts to the condition $\Lambda x(x\varepsilon B \rightarrow Vy(y\varepsilon A \wedge y = x))$. However, one may not at all interchange the two quantifiers within that formula.

will in general be tautological statements the truth of which is self-evident ["per se patet"] unless, again, "B is unique" (**P.**, 95, cf. **C.** 194: "nisi B sit unicum").

It strikes me as somewhat incomprehensible that not only Couturat but also modern commentators regarded this as a *rejection* of TQP[111]. Even if Leibniz's remarks about the artificiality ("non est in usu in nostris linguis") and the redundancy ("inutilis") of the non-categorical propositions NC 1–4 (which exhaust all possibilities of a quantification of the predicate) might be interpreted as a rejection of this particular *part* of TQP, still it could hardly be denied that Leibniz advocated the other, more relevant part of TQP which relates to the categorical forms UA 1, PA 1, UN 1, and PN 1. Furthermore, it cannot be overlooked that Leibniz took this very (semi)-formalization of the categorical forms as a conclusive proof of the traditional rules of quantity and quality:

> (9) So [...] it is evident that every affirmative proposition (and only such a proposition) has a particular predicate, by art. 3 et 4.,
>
> (10) and that every negative proposition (and only such a proposition) has a universal predicate, by art. 5 et 6.
>
> (11) Further, the proposition itself is called 'universal' or 'particular' by virtue of the universality or particularity of its subject. (**P**, 96)

As a matter of fact, these counterparts of QUAL and QUAN follow immediately from the quantification both of the subject and of the predicate as illustrated in UA 1, PA 1, UN 1, and PN 1, provided that the terms A, B are taken to be universal or particular just in case they are modified by a universal or by a particular (i.e., existential) quantifier.

Before discussing a second version of TQP presented in §§24, 48–50, let me briefly touch upon Leibniz's proofs of the remaining fundamentals. They basically follow the lines of the corresponding demonstrations in *Comprobatione*. Thus Leibniz immediately infers the fundamental principles GR 3, GR 4 + GR 5.2 from the logical laws for identity stated in §§12 and 13[112]:

> (15) It is at once inferred from this that a syllogism cannot be made out of two negative propositions; for in this way it would be stated that L is different from M, and that M is different from N. [**P**, 96 ...]
>
> (21) It is none the less evident that if one premiss is negative, the conclusion also is negative, and conversely; for the reasoning used here

[111] Parkinson remarked in the same vein as Couturat that: "[...] Leibniz conceives the idea of the quantification of the predicate, only to reject it." (**P**, liii). [Kauppi, 1960, p. 199] says that "[...] die Quantifikation des Prädikats [wird] als unnötig verworfen". Burkhardt [1980, p. 44] shares Couturat's opinion that "[Leibniz hatte] die Quantifizierung des Prädikates [...] noch im arithmetischen Kalkül von 1679 abgelehnt". He correctly recognizes, however, that in §24 "Leibniz noch ein Zeichensystem zur Darstellung der vier kategorischen Satzformen entwickelt [hat], mit dessen Hilfe es möglich ist, Subjekt und Prädikat zu quantifizieren" (o.c., 45).

[112] "[...] thus if L is the same as M and M is the same as N, L and N are the same"; "[...] Thus, if L is the same as M, and M is different from N, L and N are also different."

> is just the same as that whose principle was stated in article 13 [...] (**P**, 97).

The proof of the other fundamentals GR 1, 2 resorts in addition to the following definition of a categorical syllogism:

> (12) What are called 'simple categorical syllogisms' elicit a third proposition from two others [...]
>
> (16) It is also evident that in the simple categorical syllogism there are three terms, as we are using some third term, and while we compare this equally with the one and the other of the extremes we are seeking a method of comparing these extremes with each other. (**P**, 96)

This third term, the medius, must be universal in at least one premiss, as Leibniz argues in §:

> (19) [...] For [...] if the middle term in each premiss is particular, it is not certain that the contents of the middle term which are used in one premiss are the same as the contents of the middle term which are used in the other premiss, and therefore nothing can be inferred from this about the identity and difference of the extremes. (**P**, 97)

And in the subsequent § he shows that if a term is particular in a premiss, it will also be particular in the conclusion:

> (20) It can also be seen easily that a particular term in the premiss does not imply a universal term in the conclusion, for it is not known to be the same or different in the conclusion unless it is known that it is the same as or different from the middle term in the premiss.

8.6 *The* $\Psi B\Psi D$*-formalism*

Another version of the TQP is developed in §24 which is difficult to read in several places since the text is written in very small letters on the margin. The main differences between the text-critical edition given in [Lenzen, 1990b] and the previous edition in **C** (or in **P**) are the following. Leibniz inserted the last sentence of §24 'propositionis quaecunque [...]' on top of the sentence 'S significabit [...]'. That's why a certain word which Couturat somewhat diffidently interpreted as 'unurarem' seemed to belong to the former sentence while in fact it reads as 'terminum' and belongs to the latter sentence. Accordingly, the passage:

> S signifies the universal, P the particular, V, Y, Ψ the indetermined. [cf. **C.**, 196: "S significabit universalem, P particularem, V, Y, Ψ incertam"]

has to be corrected to "S significabit *terminum* universalem, P particularem, V, Y, Ψ *incertum*." This is quite important since it conclusively establishes that the symbols 'S' and 'P' characterize the universality and particularity of a *term* and not, as, e.g., Parkinson assumed[113], the corresponding property of a ***proposition.*** Accordingly 'Ψ' symbolizes that it is undetermined whether the subsequent term is universal or particular; it does not, however, as Burkhardt [1980, p. 47] has maintained, constitute itself an indefinite term. The resulting formalisation of the categorical forms is read by Couturat as

> Therefore the sign $SBSD$ is the universal negative proposition, $SBPD$ the universal saffirmative. $IBSD$ the particular negative. $IBID$ the particular affirmative.
>
> Signum itaque $SBSD$ est propositio universalis negativa. $SBPD$ universalis affirmativa. $IBSD$ particularis negativa. $IBID$, particularis affirmativa. (**C.**, 196)

The opening word, however, actually belongs to the preceding sentence: "The quantity of the proposition will be designated by the universal sign of the subject, the quality [of the proposition] by the sign of the predicate". [Propositionis quantitas designabitur per subjecti signum universale, qualitas per praedicati signum]. Furthermore, the text of the manuscript does not necessarily speak in favor of a letter 'I' within the formulae '$IBSD$' and '$IBID$', but allows one to read this letter instead as a very slim 'P' where what at first glance to be a point above 'I' really is a tiny crook of a 'P'. That Leibniz at any rate *meant to write* 'P' instead of 'I' is evident from the deleted §§ 48 where one can read very clearly:

> If we do not take care about what are the premisses, the terms will be F, G, and similar ones. In general the universal proposition $SF\Psi G$, the particular proposition $PF\Psi G$, the affirmative proposition ΨFPG, the negative proposition ΨFSG. In particular, the universal affirmative proposition $SFPG$, the particular affirmative $PFPG$, the universal negative $SFSG$, the particular negative $PFSG$.[114]

This unambiguous statement also confirms that the concluding sentence of § 24 ends with the words "is generally expressed by $\Psi F\Psi G$" [generaliter exprimitur $\Psi F\Psi G$] and not, as **C** has it, with "generaliter exprimitur unurarem $\Psi F.\Psi S$."[115]

Let us now consider in which way Leibniz used this symbolism to complete his proof of completeness. In §45 he proved the special rule IV.1 indirectly as follows. If one would have at the same time that "the major proposition is not particular [...and] the minor proposition is negative", one could argue:

---

[113]Cf. **P**, 98: "S will stand for a universal, P for a particular, V, Y, Ψ for an indefinite proposition".

[114]Cf. **LH** IV, 6, 14, 2v., margin: "(48) [...] Ubi nullus respectus ad praemissas, termini erunt F, G, vel tales. In genere propositio universalis SFΨG propositio particularis $PF\Psi G$ propositio Affirmativa ΨFPG propositio negativa ΨFSG. In specie Universalis Affirmativa $SFPG$, Particularis affirmativa $PFPG$, Universalis negativa $SFSG$, particularis negativa $PFSG$."

[115]Even more misleading is the interpretation of this formula by Parkinson who suggests "$\Psi P.\Psi S$" — cf. **P**, 98, fn. 1.

> [...] Let the particular major proposition in this figure (by 24) be $PD\Psi C$, and the negative minor proposition be [ΨCSB]; then the negative conclusion will be $PBSD$. But this is absurd, since (art. 20 [i.e. GR 2]) there cannot be PD in the major proposition and SD in the conclusion. (**P**, 103)[116]

In §46 it is similarly shown that:

> [...] the minor proposition is not particular at the same time as the major proposition is affirmative. For suppose that they are: then the major proposition will be ΨDPC, and the minor proposition $PC\Psi B$. But in this way the middle term, C, is particular in each, which is contrary to art. 19 [i.e. contrary to GR 1]. (**P**, 103/104).

Systematically much more important, however, is the sketch of a proof that Leibniz gives at the very end of *Mathesis* to show that there are not more valid moods than the 24 ones proven elsewhere:

> "It must be maintained that there are no more moods, and this must be done, not by an enumeration of illegitimate moods, but from the laws of those which are legitimate. For example, in the first figure the premisses $SC.\Psi D, \Psi B.PD$ give:

$$\left\{\begin{array}{ll} SCPD & \left\{\begin{array}{lll} SBPD & \mathrm{AA} & \left\{\begin{array}{lll} \mathrm{A} & \text{Barbara} & 1 \\ \mathrm{I} & \text{Barbari} & 2 \end{array}\right. \\ PBPD & \mathrm{AI} & \quad\mathrm{I} \;\; \text{Darii} \quad\;\; 3 \end{array}\right. \\ SCSD & \left\{\begin{array}{lll} SBPD & \mathrm{EA} & \left\{\begin{array}{lll} \mathrm{E} & \text{Celarent} & 4 \\ \mathrm{O} & \text{Celaro} & 5 \end{array}\right. \\ PBPD & \mathrm{EI} & \quad\mathrm{O} \;\; \text{Ferio} \quad\;\; 6 \end{array}\right. \end{array}\right.$$

In its present form, however, this schema is incomplete and incorrect. As was stated in §22, the position of the terms in the Ist figure is: "Fig.1. $CD.BC.BD$." The special rule I.1, according to which the minor-premiss is affirmative, therefore has to be formalized as 'ΨBPC', whereas Leibniz erroneously has 'ΨBPD' which would symbolize an affirmative *conclusion*. Hence only the following combination of premisses (obtained by substituting 'S' and 'P' successively in the place of 'Ψ') is legitimate:

[116] Couturat pointed out in **C**, 202, fn. 1 and 2, that the formula for the negative minor-premiss has to be ΨCSB instead of Leibniz's $SC\Psi B$, and that the "in minore" of the manuscript must be read as "in conclusione". Leibniz's third inaccuracy of symbolizing the "conclusio negativa" as $PBSD$ instead of ΨBSD is harmless, since under the given premisses the conclusion also has to be particular, hence $PBSD$.

$$\begin{array}{ll} SCPD & \begin{cases} SBPC \\ PBPC \end{cases} \\ SCSD & \begin{cases} SBPC \\ PBPC \end{cases} \end{array}$$

In the first two cases, in view of GR 4, the conclusion must itself be affirmative: ΨBPD; moreover, in the second subcase it has to be particular according to GR 3: $PBPD$. In the last two cases, in contrast, the conclusion has to be negative on account of GR 4: ΨBSD; in the second subcase, again, it also must be particular: $PBSD$. Hence Leibniz's schema for the only valid moods of the Ist figure has to be modified as follows:

$$\left\{\begin{array}{l} SCPD \left\{\begin{array}{l} SBPC \begin{cases} SBPD & \text{BARBARA} \quad 1 \\ PBPD & \text{BARBARI} \quad 2 \end{cases} \\ PBPC \quad\; PBPD \quad \text{DARII} \quad 3 \end{array}\right. \\ SCSD \left\{\begin{array}{l} SBPC \begin{cases} SBSD & \text{CELARENT} \quad 4 \\ PBSD & \text{CELARO} \quad 5 \end{cases} \\ PBPC \quad\; PBSD \quad \text{FERIO} \quad 6 \end{array}\right. \end{array}\right.$$

As was shown at length in Lenzen [1990b], this formal method of eliminating the invalid moods "ex legibus legitimorum" can be applied to the other figures as well. To round off the present discussion of the *Mathesis*, I want to delineate in the following section in which respect the $\Psi B\Psi C$-formalism may be considered as a second version of TQP.

8.7 Formalisations of the Categorical forms [117]

The most immediate way of expressing the universal affirmative proposition within the general calculus of a logic of concepts is simply to drop the informal quantifier-expression 'Every' in 'Every A is B', thus obtaining the formula 'A is B', or symbolically

(UA 2) $\quad A \in B$.

According to CONT 3 and CONT 4 this formula can be reduced to one of the following identities:

[117] Leibniz made enormous efforts to formalize the single categorical forms within his system(s) of concept logic, and he worked with enumerable "homogeneous" and inhomogeneous combinations of these formulas, not all of which turned out to be correct and useful. Here only the most important homogeneous schemata shall be considered. For more details cf. [Lenzen, 1988].

(UA 3) $\quad A = AB$

(UA 4) $\quad \exists Y(A = BY)$.

Now in "A paper on 'some logical difficulties" (**P**, 115–121) Leibniz recognized that the UA can equivalently be expressed by the generalized statement that every A is B in the sense of $\forall X(XA \in B)$. Somewhat more exactly, Leibniz first defined the following formal criterion for the universality or non-universality, i.e. particularity, of a term B (within a certain proposition):

> In general we can tell if a term [...] B is universal if [...] YB can be substituted for [...] B, where Y can be anything which is compatible with B (**P**, 119).

Next he went on to prove that the term A is in fact universal within the proposition $A \in B$ by pointing out: "In the universal affirmative, $AB = A$, therefore [for every Y] $YAB = YA$".[118] Hence $A \in B$ entails $\forall Y(AY \in B)$. On the other hand, $\forall Y(AY \in B)$ entails, for arbitrary concepts Y, that $AY \in B$, especially for $Y = A : AA \in B$, i.e., because of the trival law CONJ 4, $A \in B$. Hence one obtains the further formalisation

(UA 5) $\quad \forall X(XA \in B)$.

The remaining '$\in$' can either be eliminated, as Leibniz did in the quoted passage, by means of CONT 3, or by means of CONT 4. In the latter case one obtains the following representation with *two* quantifiers:

(UA 6) $\quad \forall X \exists Y(XA = YB)$.

The *PA* 'Some A is B', on the other hand, was formalized by Leibniz among others as 'XA est B' where the indefinite concept X now plays the rôle of an *existential* quantifier:

(PA 2) $\quad \exists X(XA \in B)$.[119]

Eliminating, again, the '$\in$' by means of CONT 4, one obtains the doubly-quantified version

(PA 3) $\quad \exists X \exists Y(XA = YB)$,

[118] **P** 119. In the same passage Leibniz also proves all the remaining theorems of quantity and quality.

[119] Cf., e.g., **GI**, §48: "AY contains B is [the] particular affirmative". However, in view of the trivial law CONJ 2 there always exists at least one Y such that $AY \in B$. Therefore Leibniz's formalisation of the P.A. should be modified by requiring that Y is compatible with A. Corresponding remarks apply to the subsequent formulas PA 3 and UN 2.

which Leibniz expressed somewhat elliptically as: "the *particular affirmative* Some C is B can be expressed thus: $XB = YC$" [cf. **C.**, 302: "*particularis affirmativa* Qu. C est B sic exprimetur: $XB = YC$"]

In view of the laws of opposition, the universal negative proposition can accordingly be formalized as: "No C is B, i.e. $XC \neq YB$" [cf. **C.**, 303: "Nullum C est B id est XC non $= YB$"], where both indefinite concepts X, Y now function as *universal* quantifiers:

$$\text{(UN 2)} \qquad \forall X \forall Y (XA \neq YB).$$

Finally, for the particular negative proposition one obtains as the negation of UA 6:

$$\text{(PN 2)} \qquad \exists X \forall Y (XA \neq YB).$$

Putting these formal representations together into the schema:

$$\begin{array}{llll} \text{(UA)} & \forall X \exists Y (XA = YB) & \forall X \forall Y (XA \neq YB) & \text{(UN)} \\ \text{(PA)} & \exists X \exists Y (XA = YB) & \exists X \forall Y (XA \neq YB) & \text{(PN)} \end{array}$$

one obtains *the real meaning of the* $\Psi B \Psi C$*-formalism*. All that has to be observed is that the original version of § 24:

$$\begin{array}{llll} \text{(UA)} & \text{SA PB} & \text{SA SB} & \text{(UN)} \\ \text{(PA)} & \text{PA PB} & \text{PA SB} & \text{(PN)} \end{array}$$

implicitly contained corresponding '$=$' and '$\neq$'-symbols as Leibniz explained in the deleted §49:

> We can also reduce everything by means of the calculus to identities and non-identitites. [...] thus if I want to express a negative proposition [...] ΨFSG, it will be ΨF non $= SG$.[120]

Hence the intended meaning of the above schema is better formalized as follows:

$$\begin{array}{llll} \text{(UA)} & \text{SA} = \text{PB} & \text{SA} \neq \text{SB} & \text{(UN)} \\ \text{(PA)} & \text{PA} = \text{PB} & \text{PA} \neq \text{SB} & \text{(PN)} \end{array}$$

Here the "sign" [signum] S has to be interpreted as an indefinite concept governed by a universal quantifer while P accordingly represents an indefinite concept governed by a particular (or existential) quantifier.

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[120]Cf. **LH** IV, 6, 14, 2v.: "Possumus etiam reducere omnia ad principium identitatis et diversitatis per calculum. [...] ut si velim exprimere propositionem negativam fiet ΨFSG, erit ΨF non $= SG$".

8.8 Conclusion

To conclude, I want to show that the first, "extensional" version of TQP discussed in section 8.6 is provably equivalent to the second, "intensional" version elaborated in the preceding section, where this equivalence can be established by means of principles of a genuinely Leibnizian logic. For reasons of space, however, I can here only sketch how the two version of, e.g., the UA can be derived from each other. A more detailed account may be found in [Lenzen, 1990b].

In section 8.7 several laws of $L2$ were quoted to show that the "intensional" UA with quantified subject and quantified predicate, $\forall X \exists Y(XA = YB)$, is equivalent to the simple formalization of the "*Affirmative Proposition* A is B or A contains B" (**GI**, §16). Now, as Leibniz observed in **C**, 260, the UA can also be expressed as a *universal conditional*: "A is B, is the same as to say If L is A, it follows that L is B" [A est B, idem est ac dicere si L est A, sequitur quod et L est B]. Hence another formalisation of the UA is:

(UA 7) $\quad \forall X(X \in A \rightarrow X \in B).$

Next observe that Leibniz developed several logical criteria for a concept A being a *complete* concept (of an individual substance) or, for short, *an individual concept*, e.g.:

> [..] if two propositions *with exactly the singular subject* are presented such that one of them has one of two contradictory terms as predicate while the other proposition has the other term as predicate, then necessarily one proposition is true and the other false" [121]

This can be formalized as follows:

(DEF 12) $\quad \mathrm{Ind}(A) \leftrightarrow_{\mathrm{df}} \forall X(A \in \overline{X} \leftrightarrow A \notin X).$

With the help of this definition, one can introduce new quantifiers ranging over individual(concept)s:

(DEF 13) $\quad \Lambda X\alpha \leftrightarrow_{\mathrm{df}} \forall X(\mathrm{Ind}(X) \rightarrow \alpha)$

(DEF 14) $\quad VX\alpha \leftrightarrow_{\mathrm{df}} \exists X(\mathrm{Ind}(X) \wedge \alpha).$

These quantifiers allow us to represent the UA, alternatively to UA 7, also as

(UA 8) $\quad \Lambda X(X \in A \rightarrow X \in B).$

This formula captures the meaning of Leibniz's example:

[121] **LH** IV, 5, 8d, 17 verso; cf. **C**, 67: "[...] si duae exhibeantur propositiones *ejusdem praecise subjecti singularis* quarum unius unus terminorum contradictoriorum, alterius alter sit praedicatum, tunc necessario unam propositionem esse veram et alteram falsam". A discussion of this important passage may be found in [Lenzen, 1986], esp. pp. 23–24.

> The universal affirmative proposition *Every b is c* can be reduced to this hypothetical proposition *If a is b, a will be c*, e.g.: *Every man is an animal*, i.e. *If someone is a man (b), he (a, or Titus) is c (animal).*[122]

The last but one step in the proof of the equivalence between the "extensional" and the "intensional" approach consists in the trivial law according to which the condition $Vy(y = x \wedge \alpha)$ is only a complicated version of $\alpha[x]$. Hence UA 1 may be simplified to

$$\text{(UA 9)} \qquad \Lambda x(x\varepsilon A \to x\varepsilon B).$$

Now, the intension and the extension of a concept A in general are linked together by the so-called law of reciprocity which also applies to *individual*-concepts. As captured in DEF.1, their intension is maximal. The extension of an individual-concept, therefore, will be minimal, which means that it consists of exactly *one* (possible) individual only. In this sense individuals may properly be called the lowest species "whose name cannot be restricted to fewer"[123] , or in other words: "The absolutely lowest species is the individuum" [Cf. **A** VI, 4, 32: "Species absoluta infima est *individuum*"].

To sum up: the individual concept X contains the concept A: '$X \in A$', iff X's extension, i.e. the unit-set $\{x\}$ containing exactly the individual x, is contained in the extension of A, i.e. iff x itself has the property A or is a member of the set of all As: $x\varepsilon A$![124] In this sense the "extensional" formalisation UA 9 coincides with the "intensional" version UA 8.

9 POSSIBLE INDIVIDUALS AND POSSIBLE WORLDS

Since the publication of Russell [1900], a lot of books and articles have been written about Leibniz's logic on the one hand and about his metaphysics on the other. Most Leibniz scholars followed Russell in recognizing the intimate relationship between these two areas of Leibniz's philosophy. After all, Leibniz himself had repeatedly pointed out the close connection between his metaphysical and his logical ideas. Thus in a famous letter to Duchess Sophie he declared that "[...]the true Metaphysics is hardly different from the true Logic" (**GP** 4, 292). However, modern commentators consider this statement as an absolutely unfounded exaggeration. They are confident that Leibniz's logic of concepts is much too weak to serve as a basis either for defining the central notions of his ontology or even

[122] Cf. **A** VI, 4, 126: "Propositio Universalis affirmativa *Omne b est c* reduci potest ad hanc hypotheticam *Si a est b, a erit c*, verbi gratia: *Omnis homo est animal* id est, *Si quis est homo (b) is (a vel Titius) est c (animal)*".

[123] Cf. **A** VI, 4, 31:"[...] cuius nomen ad pauciora restringi non potest"

[124] As the formalizations UA 8 and UA 9 make clear, there is always a logical relation between the individual(-concept) x (or X) and the general concept A whether the latter is taken extensionally as a set or intensionally as an idea. Modern predicate logic, however, misleadingly veils this relation behind the functional brackets of '$A(x)$'. For a more detailed discussion of this point cf. [Lenzen, 1989b].

for deriving certain metaphysical propositions which Leibniz had referred to as "logical" propositions. Thus in their standard exposition of *The Development of Logic*, W. and M. Kneale [1962, p. 337] summarize their evaluation of Leibniz's logical achievements as follows:

> When he began, he intended, no doubt, to produce something wider than traditional logic. [...] But although he worked on the subject in 1679, in 168[6], and in 1690, he never succeeded in producing a calculus which covered even the whole theory of the syllogism.

If this were correct, then it would be absurd to expect that any interesting element of Leibniz's "true metaphysics" might be derived from his "true logic". In particular, it would be silly to believe that the core of Leibniz's proof of the existence of God, namely the statement "If the necessary being is possible, then it exists" might turn out as a *logical* truth. But this is at any rate what Leibniz himself claimed to be the case when he characterized this statement as "a modal proposition, perhaps one of the best fruits of the entire logic".[125]

Hopefully the present exposition has convincingly shown that 20th century scepticism concerning the strength of traditional logic in general and concerning Leibniz's achievements in particular is rather unfounded. Anyway, in Lenzen [1990a] a self-consistent reconstruction of the "Universal Calculus" has been provided which actually allows one to derive the quoted thesis about the existence of the necessary being as a logical theorem! For reasons of space I will here confine myself to giving a logical reconstruction of the main elements of Leibnitian *ontology*, to wit the notions of a possible individual and of a possible world. Let us begin by considering §§71–72 **GI** where Leibniz presents his views on *existence* and on *individuals*:

> (71) What is to be said about the proposition 'A is an existent' or 'A exists'? Thus, if I say about an existing thing, 'A is B', it is the same as if I were to say 'AB is an existent'; e.g. 'Peter is a denier', i.e. 'Peter denying is an existent'. The question here is how one is to proceed in analysing this; i.e. whether the term 'Peter denying' involves existence, or whether 'Peter existent' involves denial — or whether 'Peter' involves both existence and denial, as if you were to say 'Peter is an actual denier', i.e. is an existent denier; which is certainly true. Undoubtedly, one must speak in this way; and this is the difference between an individual or complete term and another. For if I say 'Some man is a denier', 'man' does not contain 'denial', as it is an incomplete term, nor does 'man' contain all that can be said of that of which it can itself be said.
>
> (72) So if we have BY, and the indefinite term Y is superfluous (i.e., in the way that 'a certain Alexander the Great' and 'Alexander the Great' are the same), then B is an individual. If there is a term BA

[125] Cf. **GP** 4, 406: "On pourrait encore faire à ce sujet une proposition modale qui seroit un des meilleurs fruits de toute la logique, scavoir que *si L'Estre necessaire est possible, il existe.*"

and B is an individual, A will be superfluous; or if $BA = C$, then $B = C$.

First we have to clarify the central notions 'existing', 'individual', and 'individual-term'. Leibniz has often been blamed for not carefully distinguishing between terms and their denotations. The quoted passage certainly justifies such a criticism, but Leibniz's rather careless use of the word 'individual' to refer alternatively either to individual-terms or to individuals does not give rise to serious misunderstandings. One may assume that there is a 1-to-1-correspondence between individuals and individual-terms, and the context makes perfectly clear what Leibniz is talking about. What has to be kept in mind, however, is that an individual-term for Leibniz nevertheless is a concept, i.e. an "intensional" entity which may contain (or be contained in) other concepts. Hence its extension must be conceived of as a *subset* — and not as an *element* — of the universe of discourse. E.g., the extension of the individual-concept 'Peter' is not the individual Peter but the *unit-set* containing exactly that individual.

As regards the notion of existence, Leibniz is treating it on a par with the other concepts by forming corresponding conjunctions 'Petrus existens', 'abnegans existens' which enter into the fundamental relation of containment, '$\in$'. Therefore 'existens' may be abbreviated by a distinguished concept letter, say E^*, which has to be interpreted extensionally, like any other concept letter, as a certain subset of the universe of discourse.[126]

Now, generalizing from the above examples, Leibniz is maintaining that whenever A is the complete term of an existing individual, then the statement 'A is B' is equivalent both to i) 'AB is Existing' and to ii) 'A Existing is B', and also to iii) 'A is Existing B'. These principles may easily be shown to be theorems of the algebra of concepts regardless of whether the subject-term A is a "normal" concept or an individual-concept. What, then, had Leibniz in mind when he went on to explain: "Undoubtedly, one must speak in this way; and this is the difference between an individual or complete term and another."

At first sight the answer may be surprising. The difference between an individual concept and an ordinary one is that the proposition 'A exists' or 'A is existing' may only in the former but not in the latter case be regarded as a relation of conceptual containment and hence be formalized as '$A \in E^*$'. Why this is the case will be explained below in connection with §§144–150 **GI**. First, however, I want to deal with some other criteria for distinguishing individual-concepts from ordinary concepts.

A first difference is vaguely outlined by Leibniz's remark that from the truth of the particular proposition 'Some man is a denier' it does not follow that the universal proposition 'Every man is a denier' or, for short 'Man is denier' be true as well: "'man' does not contain 'denial"'. Here one evidently has to add the

[126] According to Leibniz, the extreme cases that this set is either empty or universal should be excluded. For he not only believed it to be "impossible that nothing exists" (**A** VI 4, 17), but he also held the view that not all of the possible individuals are compossible and that therefore some individuals will not be created by God but will remain mere possibles.

unspoken claim that the corresponding inference from a particular to a universal proposition does hold if the subject term is an individual-concept. This stands in close connection with the parenthetical remark of §72: "'a certain Alexander the Great' and 'Alexander the Great' are the same", and also with the following passage from "A paper on 'some logical difficulties'":

> How is it that opposition is valid in the case of singular propositions - e.g. 'The Apostle Peter is a soldier' and 'The Apostle Peter is not a soldier' - since elsewhere a universal affirmative and a particular negative are opposed? Should we say that a singular proposition is equivalent to a particular and to a universal proposition? Yes, we should. [...] For 'some Apostle Peter' and 'every Apostle Peter' coincide, since the term is singular. (**P** 115; cf. **GP** 7, 214)

Let us see how this claim, which has been dubbed by Englebretsen [1988] the "Wild Quantity Thesis", can be verified within Leibniz's calculus. Observe, first, that the UA 'Every A is B', i.e. $A \in B$, can in general (for arbitrary subject-terms A) be represented, in $L2$, in the form of $\forall Y(YA \in B)$.[127] In the case of a singular proposition - i.e. a proposition with an individual term such as 'Apostle Peter' as subject - this means that, e.g., Apostle Peter is a denier if and only if every Apostle Peter is a denier, or, in short, that the subject term 'Apostle Peter' is equivalent to the universally quantified term 'every Apostle Peter'. Thus the first part the "Wild Quantity Thesis" is already verified.

As regards the second part, observe that according to NEG 6 the particular affirmative proposition 'Some A is B', i.e. the negation of the UN $A \in \overline{B}$, can in general be formalized as $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$. Now if the subject-term A is an individual concept — formally Ind(A) — then the predication 'A is B' turns out to be *equivalent* to the formula $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$:

(IND 1) $\qquad \mathrm{Ind}(A) \rightarrow (A \in B \leftrightarrow \exists Y(\mathbf{P}(AY) \wedge AY \in B)).$

In other words: the singular predication 'A is B' is tantamount to the particular proposition 'Some A are B', or — in our previous example — Apostle Peter is a denier iff some Apostle Peter is a denier.

The validity of IND 1 is based on the completeness-condition for individual concepts which Leibniz mentions in the concluding sentence of §72 **GI**. There he calls a concept A "superfluous" (with respect to concept B) iff (for every C) $BA = C$ entails that $B = C$. This condition may be simplified by just requiring that A is already contained in B.[128] Now, when Leibniz goes on to maintain "If there is a term BA and B is an individual, A will be superfluous" (**P**., 65,

[127] On the one hand, $\forall Y(YA \in B)$ immediately entails $AA \in B$ and thus, because of the trivial law $AA = A$ also $A \in B$; conversely $YA \in B$ follows, for arbitrary Y, from the premiss $A \in B$ and from the trivial conjunction law $YA \in A$ by means of the transitivity of '$\in$'.

[128] For, on the one hand, substituting 'BA' for 'C' yields that $BA = BA$ entails $B = BA$; conversely, if $BA = B$ then (for any C) $BA = C$ entails that $B = C$. Hence A is superfluous with respect to B just in case that $B = BA$, i.e. $B \in A$.

fn. 1), he seems to maintain that any term A is superfluous with respect to any individual term B. But this is absurd since otherwise an individual-concept B would be "completely complete" in the sense of containing every concept A, in particular besides A also Non-A, and hence B would be inconsistent.

To resolve this difficulty, observe that Leibniz begins the sentence in question by saying "Si sit terminus BA" which Parkinson translated as "If there is a term BA". In other contexts, this translation surely would be appropriate to express the sense of a mere stipulation: "Let there be a term BA [...]". In the present context, however, Leibniz meant to say: "Let the term BA *be*", i.e. let BA *be a consistent term*, or, let us suppose that $\mathbf{P}(BA)$! There are several passages within and without the **GI** where Leibniz paraphrases the condition of self-consistency of a concept A just by saying 'A is'. Therefore the interpretation of "Si sit terminus BA" as meaning 'Let BA be a possible term' is very plausible, and it entails the necessary condition: B is an individual-concept only if — unlike other concepts — B is complete in the precise sense of already containing any concept A with which it is compatible (i.e. for which $\mathbf{P}(BA)$ holds). Since A here stands for any arbitrary concept, it may be replaced by an indefinite concept Y and then be bound by a universal quantifier:

(IND 2) $\qquad \mathrm{Ind}(B) \to \forall Y(\mathbf{P}(BY) \to B \in Y).$

That this is what Leibniz had in mind is evidenced by the fact that the converse implication

(IND 3) $\qquad \forall Y(\mathbf{P}(BY) \to B \in Y) \to \mathrm{Ind}(B)$

is recognized by him as a sufficient condition for B to be an individual-concept when he says: "So if BY is [possible], and the arbitrary indefinite term Y is superfluous, then B is an individual". We thus obtain the following Leibnizian definition of individual-concepts:

(IND 4) $\qquad \mathrm{Ind}(A) \leftrightarrow \mathbf{P}(A) \wedge \forall Y(\mathbf{P}(AY) \to A \in Y),$

where the trivial condition $\mathbf{P}(A)$ not mentioned by Leibniz has been added. This definition is semantically adequate and it enables us to prove the open part of the "Wild Quantity Thesis", IND 1, as follows: If $\mathrm{Ind}(A)$ and $A \in B$, then, trivially, $AA \in B$ and $\mathbf{P}(AA)$, from which $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$ follows by existential generalization; conversely, let there be some Y such that $\mathbf{P}(AY) \wedge AY \in B$; since A is presupposed to be an individual-concept, $\mathbf{P}(AY)$ according to IND 4 implies that $A \in Y$, i.e. $A = AY$, so that $AY \in B$ yields the desired $A \in B$.

So far I have been concerned with the truth-conditions for attributing existence to individuals. Let us now consider §§144–150 **GI** where Leibniz investigates the truth-conditions for corresponding non-singular categorical propositions.

> (144) Propositions are either essential or existential, and both are either *secundi adjecti* or *tertii adjecti*. [...] An existential proposition

tertii adjecti is 'Every man exists liable to sin'. [...] An existential proposition *secundi adjecti* is 'A man liable to sin exists, i.e. is actually an entity' ["existit seu est ens actu"].

(145) From every proposition *tertii adjecti* a proposition *secundi adjecti* can be made, if the predicate is compounded with the subject into one term and this is said to [be or to] exist ["esse vel existere"], i.e. is said to be a thing, whether in any way whatsoever, or actually existing ["esse res sive utcunque, sive actu existens"].

(146) The particular affirmative proposition, 'Some A is B', transformed into a proposition *secundi adjecti*, will be 'AB exists' ["AB est"], i.e. 'AB is a thing' - either possible or actual ["AB est res, nempe vel possibilis vel actualis"], depending on whether the proposition is essential or existential. [...]

(148) The particular negative proposition, 'Some A is not B', will be transformed into a proposition *secundi adjecti* as follows: 'A, not-B exists'. That is, A which is not B is a certain thing - possible or actual, depending on whether the proposition is essential or existential.

(149) The universal negative is transformed into a proposition *secundi adjecti* by the negation of the particular affirmative. So, for example, 'No A is B', i.e. 'AB does not exist' ["AB non est"], i.e. 'AB is not a thing' [...]

(150) The universal affirmative is transformed into a proposition *secundi adjecti* by the negation of the particular negative, so that 'Every A is B' is the same as 'A not-B does not exist, i.e. is not a thing' ["A non B non est, seu non est res"] (**P**, 80-81; cf. **C.**, 392).

These ideas may be summarized and formalized in the following diagram:

Categorical form	*Formalization "secundi adjecti"*	*Formalization "tertii adjecti"*
U.A. "Every A is B"	$\neg P(A\overline{B})$	$A \in B$
U.N. "No A is B"	$\neg P(AB)$	$A \in \overline{B}$
P.A. " Some A is B"	$P(AB)$	$A \notin \overline{B}$
P.N. "Some A is not B"	$P(A, \overline{B})$	$A \notin B$

Figure 1. "Essential" propositions

Leibniz's thesis of the reducibility of the categorical forms *tertii adjecti* to propositions *secundi adjecti* amounts to the claim that the corresponding formulae are provably equivalent. This, however, easily follows from our former axiom Poss 1.

Let us now turn to the "existential" interpretation of the categorical forms. Just as the truth of the "essential" P.A. "AB is a possible [...] thing" according to our semantics requires that there is at least one possible individual $x \in U$ such that x is an AB, i.e. x is both an A and a B, so the stronger "existential" P.A. "AB is an actual [...] thing" should be considered as true if and only if there is an actually existing individual x which is both an A and a B. How can this be expressed, however, within the logic of concepts? The answer to this question may be found in an untitled fragment where Leibniz is wondering whether:

> [...] the way of transforming logical propositions into terms by adding just 'ens' or 'non ens' also works in the case of existential propositions. [...] For example: 'Some pious is poor', i.e. 'Pious poor is existing'. [...] Let us see whether 'existing' can also be moved into the term so that only 'ens' or 'non Ens' remains. Such that 'Pious poor is existing' yields 'Pious poor existing is Ens'. (cf. **C**, 271).

Generalizing from this example, an "existential" P.A. "AB is existing" shall be reduced to a proposition *secundi adjecti* by maintaining that the conjunction ABE(xistens), or $ABE*$, is "Ens", i.e. is self-consistent: $\mathbf{P}(ABE^*)$! Similarly, an "existential" P.N. "A Not-B is existing" will have to be represented by $P(A\overline{B}E^*)$, as Leibniz illustrates when he transforms "Some pious [man] is not poor, i.e. 'Pious not poor' is existing" ["quidam pius non est pauper, seu Pius non pauper est existens"] into "'Pious not poor existing' is Ens or possible" ["Pius existens non pauper est Ens seu possible", *ibid.*]. Since "existential" versions of the universal propositions can be obtained by negating P.A. and P.N., respectively, one arrives at the following schema:

Categorical form	*Formalization "secundi adjecti"*	*Formalization "tertii adjecti"*
U.A.* "Every existing A is B"	$\neg P(A\overline{B}E^*)$	$A \in^* B$
U.N.* "No existing A is B"	$\neg P(ABE^*)$	$A \in^* \overline{B}$
P.A.* " Some existing A is B"	$P(ABE^*)$	$A \notin^* \overline{B}$
P.N.* "Some existing A isn't B"	$P(A,\overline{B}E^*)$	$A \notin^* B$

Figure 2. "Existential" propositions

Here, of course, the new operator of existential containment, $\in^*$, must be interpreted extensionally as saying that each actually existing individual which falls under concept A also falls under concept B. This operator might be defined in terms of ordinary containment plus the concept of existence as follows:

(DEF 15) $\quad A \in^* B \leftrightarrow AE^* \in B.$

To conclude the discussion of §§144–150 GI, let me explain and prove the former claim that only in the case of individual concepts A, the statement 'A exists' may be represented by the formula '$A \in E^*$'. If A is an ordinary concept, say that of a horse, then a statement of the form '$A \in B$' always has to be understood as a *universal* affirmative proposition saying that every individual which is an A also is a B, say, every horse is an animal. Hence substituting the concept 'E^*' in the place of the predicate 'B' one obtains that 'Horse $\in$ E(xistence)*' is true if and only if *every* horse actually exists. Existential propositions of the type 'Horses exist', however, only maintain that *some* horses exist. Hence, where A is a normal concept, 'As exist' will have to be represented in $L2$ by the formula '$A \notin \overline{E^*}$ which expresses an particular affirmative proposition.

Now, as was shown in connection with the "Wild quantity thesis", the completeness of an individual-concept A entails that the particular proposition 'Some A are B' becomes equivalent to the universal proposition 'Every A is B'. Therefore the existence of an *individual* may well be expressed also in the form of the simple attribution 'A$\in E^*$'.

So far, only a very small portion of Leibnitian ontology has been dealt with. Let me conclude by sketching in bare outlines how a more complete logical reconstruction of Leibniz's metaphysics would have to proceed[129]. First, quantification over individuals should be modelled by restricting the quantifiers to *individual concepts* as in DEF 13, 14. With the help of these quantifiers, the following axiom can be formulated which reflects the basic idea of the set-theoretical semantics underlying concept-logic, namely the idea that a concept is possible if and only if, within the realm of all possible individuals, it has a non-empty extension:

(POSS 5) $\mathbf{P}(A) \leftrightarrow VX(X \in A)$.

The second step towards a logical reconstruction of Leibnizian ontology requires the introduction of the modal *propositional operators* of possibility and necessity. This involves a generalization of our former extensional semantics in the usual way: i.e. one has to take into account of a non-empty set W of possible worlds; relativize the truth of each propositions to the elements of W; and let the modalized propositions $\Diamond\alpha$ and $\Box\alpha$ be true if and only if the unmodalized proposition α is true in at least one/or in every world w.

Third, the former concept of actual existence, E^*, has to be generalized or relativized in such a way that in every possible world w it refers to the set of all individuals which belong to w and in this sense "exist in w". Then the crucial relation of compossibility among individuals can be defined to obtain if and only if X and Y will co-exist in some possible world, i.e. if they possibly coexist:

(DEF 16) $\Lambda X \Lambda Y(\mathrm{Comp}(X,Y) \leftrightarrow_{df} \Diamond(X \in E \wedge Y \in E))$.

Fourth, *possible worlds* will be constructed as maximal sets of compossible individuals in roughly the following way:

[129] Cf. Lenzen [1991; 1992].

(Def 17) $W(A) \leftrightarrow_{\mathrm{df}} \Lambda X(X \in A \leftrightarrow \Lambda Y(Y \in A \rightarrow \mathrm{Comp}(Y, X)))$.

Finally the *actual world* w^* may be singled out from the set of all possible worlds by the fact that it contains the greatest number of elements. Then our former notion of (actual) existence, E^*, may be regarded as the extension of the world-bound concept of existence, E, in the real world w^*. This chain of logical moves seems to stand behind Leibniz's insight that 'existens'

> can be defined as 'that which is compatible with more things than anything else which is incompatible with it'. (**P** 51).

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KANT: FROM GENERAL TO TRANSCENDENTAL LOGIC

Mary Tiles

Let us first acknowledge that when it comes to logic, Kant has something of a public relations problem. He has not figured prominently in standard histories of logic now in use and rated only a few passing, hardly laudatory, comments from the Kneales. Even the translator of the relatively recently published collection of Kant's lectures on logic [Kant, 1992a] seems to find Kant decidedly wanting as a logician.

> Kant is not a major contributor to the development of formal logic. He fails, too, in his most conspicuous efforts to build his transcendental logic on clues provided by formal logic. [Young, 1992, p. xvi]

By disputing both these claims I hope to justify the inclusion of a substantial entry devoted to Kant in this revised history of logic.

To be sure, Kant contributes nothing directly to the development of logic in the way of formal or symbolic techniques. He is, nonetheless, the architect who provides conceptual design sketches for the new edifice that was to be built on the site once occupied by Aristotelian, syllogistic logic but which in the eighteenth century was covered by rubble left by Ramist and Cartesian demolition gangs. Frege, Hilbert, Russell, Gödel and others would do the actual technical engineering work necessary to put up the new building.

Kant lays the groundwork for three important structural features of modern logic: the distinction between concept and object, the primacy of the proposition (or sentence) as the unit of logical analysis, and the conception of logic as investigating the structure of logical systems, and not merely the validity of individual inferences. Furthermore, Kant's work is pivotal in that its critical perspective reveals avenues other than those taken by the logical positivists and followed later within analytic philosophy. Cavaillès for example suggests that

> Two possibilities are opened for the doctrine of science after the Kantian analysis, depending on whether the accent is put on the notion of a demonstrative system or on that of a mathematical organon. The first is picked up in the conception of logic inaugurated by Bolzano and continued simultaneously, but in different ways, by the formalists and by Husserl. The second is taken up by what one might call the epistemological philosophies of immanence of Leon Brunschvicg and Brouwer. [Cavaillès, 1976, pp. 14–15]

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Those resistant to the claims of the new formal logic who drew on Kant include not only Brouwer, Poincaré, intuitionists and constructivists, but also Husserl who, in his opposition to Frege, inspired alternatives to analytic philosophy developed in the phenomenological tradition broadly conceived.

To substantiate these remarks I offer three related readings of Kant's *Critique of Pure Reason:* (1) as a critique of the conception of reason developed in what Stillingfleet [1697] called 'the new way of ideas' — the style of philosophizing made popular by Descartes and adopted by both rationalists and empiricists in the seventeenth and early eighteenth centuries. This was incorporated into logic through the widely disseminated *Port Royal Logic*, written by Arnauld and Nicole in 1662 [Arnauld, 1964]; (2) as a critique of the claims made on behalf of logic by rationalists such as Leibniz and Wolff; and (3) as an innovative account of the structures of reasoning.[1] The need to move from general to transcendental logic is central to both critiques and to the account of reason developed to support them. It is in the arguments supporting this move that links between the categories and the functions of judgement are to be found. In addition, and this was the lesson ignored by logical positivists and other enthusiasts for the new post-Fregean symbolic logic, the perspective of transcendental logic opens up a conception of rational structures which are essentially incomplete, not simply in the sense most frequently associated with Kant, of a restriction of theoretical reason to the realm of possible experience, but in the sense that the identity and individuation of objects requires two mutually irreducible rational frameworks. Moreover, by distinguishing between understanding and reason and by reflecting on the way in which reason will always overreach understanding, Kant in some respects anticipates the kind of incompleteness results associated with the names of Gödel and Tarski.

In what follows sections 1 and 2 identify Kant's targets. Section 3 discusses Kant's distinctive conception of reason and reasoning. Because this way of thinking is not shared by analytic philosophers, there has been some reluctance to recognize any of what Kant calls transcendental logic as properly part of logic. Sections 4 and 5 concern the move from general to transcendental logic. Sections 6 and 7 get to the heart of the relationships between judgement, object, concept, form of judgement and category. Section 8 picks up themes from the transcendental dialectic which

[1] It might be expected that the natural place to start when writing on Kant's contributions of logic would be with his lectures on logic, as recovered from notes taken by some of his students. However, while these provide useful insights into the development of Kant's thoughts about logic and about its centrality in the *Critique of Pure Reason*, there are several reasons for not focusing on this material. Foremost is that Kant's really innovative thought is only to be found in the *Critique*. Second, these lectures which Kant was obliged to give year in and year out, using a text belonging to the Wolffian school targeted for criticism in the *Critique*, were themselves a perpetuation of the university tradition of teaching general logic for which Kant can have had little real enthusiasm. His remarks about lecturing, quoted by Cassirer, confirm that this was how he earned his living but not viewed as a particularly rewarding experience. 'I sit daily at the anvil of my lectern and keep the heavy hammer of repetitious lectures going in some sort of rhythm. Now and then an impulse of a nobler sort, from out of nowhere, tempts me to break out of this cramping sphere, but ever-present need leaps on me with its blustering voice and perpetually drives me back forthwith to hard labor by its threats.' [Cassirer, 1981, p. 42].

reveal Kant's focus on the structure of systems and arguably constitute a move into what would now be called metatheory. Section 9 picks up again the theme of concept and object and the inevitable incompleteness of any formal (general) logic.

1 REASON IN THE WAY OF IDEAS

Kant lived at a time when, at least amongst philosophers, formal logic had been by turns over-rated (by those infected with enthusiasm for the Lullian and later Leibnizian art) and disdained (by Ramists and Cartesians). Philosophers espoused the way of ideas in part as an attempt to escape from entanglement in the grammatical complexities of linguistic expression in which Scholastic discussions had tended to get bogged down and in part as a route to a nominalistic rejection of many aspects of Aristotelian metaphysics, thus paving the way for the physics of mechanism.

As in many other instances, Kant grasped and was able to characterize, in a way that others could not, the shifts of alignment between logical forms, reasoning and knowledge that had occurred as a result of the changes which, in the wake of the rejection of Aristotelian physics and metaphysics, now linked knowledge and reason to representation. The promise of the adequation of word to world afforded by the goal of syllogistic demonstration in an earlier age, had been disrupted by the successes of science, and of Newtonian science in particular, in particularizing and mathematising physical reality. Rules for syllogistic reasoning which related universals (part of the furniture of the world) now become laws of thought relating ideas or concepts; they no longer have the status of laws of truth revealing natures or essences when the object of knowledge is an empirical world which has come to be thought of as a realm of particulars. As the logic of tradition was displaced from philosophy by the new mathematical, and experimental physics, the theory of demonstration lost its hold on explanations and causes. Logical categories and syllogistic pathways no longer 'cut nature at its joints' nor could they 'limn the ultimate traits of reality' (to borrow phrases from Quine). The result was a 'flattening' of logical space; it had lost the dimension which gave it depth. This occurred in the works of both empiricist and rationalist philosophers, hence Kant's complaint, that

> In a word, Leibniz *intellectualized* appearances, just as Locke ..... had *sensualized* all of the concepts of understanding, ... These great men did not seek in understanding and sensibility two quite different sources of presentations which could, however, only in *connection* make objectively valid judgement about things. Instead each man kept to only one of the two sources viz., the one source that in his opinion referred directly to things in themselves, while the other source did nothing but confuse or order the presentations of the first source. A271 (B327)[2]

[2] All references to the *Critique of Pure Reason*, will be abbreviated to giving the line numbers

In other words, there were confusions, particularly in the works of empiricist philosophers, between the particularity of an idea, and its being the idea of a particular, as well as between an idea's being general and its being the idea of a universal (a general thing). This is in part a result of not making a sharp distinction (a distinction in kind) between sensations and conceptions, and equally importantly, of failing to recognize the regulative and motivating role of ideas (in Plato's sense) as ideals. Kant makes a number of distinctions which are mapped in Figure 1, below.

After making these distinctions he remarks

> Once someone has become accustomed to these distinctions, he must find it unbearable to hear the presentation of some red color to be called an idea; it must not even be called a notion (concept of understanding) A320 (B377)

What is important, for present purposes, is to note that Kant sees both Locke and Leibniz as having limited theoretical reasoning to the framework of general logic, which in turn is equated with the logic appropriate for expressing relations of ideas. Thus arguments for the move from general to transcendental logic are at the same time arguments for rejecting both empiricist and rationalist versions of the way of ideas. (Leibniz' *Nouveaux Essais,* in which he provides a detailed critical response to Locke's *Essay on Human Understanding,* was not published in Prussia until 1765, at which time Kant (according to Cassirer [1981, p. 98]) gave it considerable attention.)

In addition there is another, more specific agenda here; it is to discredit the sweeping claims made on behalf of logic and mathematical method by Wolff and his followers. It is in pursuit of this agenda that Kant emphasizes distinctions between logic, philosophy (which includes natural philosophy, or science) and mathematics. He argues not just for the synthetic a priori status of mathematical knowledge but also that mathematics is an inappropriate model for philosophy; philosophy cannot hope to attain the kind of a priori guarantees afforded to mathematical knowledge, even though natural philosophy can only become worthy of the name 'science' once it uses mathematical methods and works within a mathematical framework. Here again a crucial role is played by deflationary remarks which seem to trivialize general logic by severely limiting its cognitive scope and which point to the need for a transcendental logic.

2 GENERAL LOGIC

In the university setting within which late medieval scholastic logic traditions were formed and flourished, the subject of logic appeared both as an *art*, taught in the

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of the Akademie A and B editions. The translations are throughout by Pluhar, from Kant [1996].

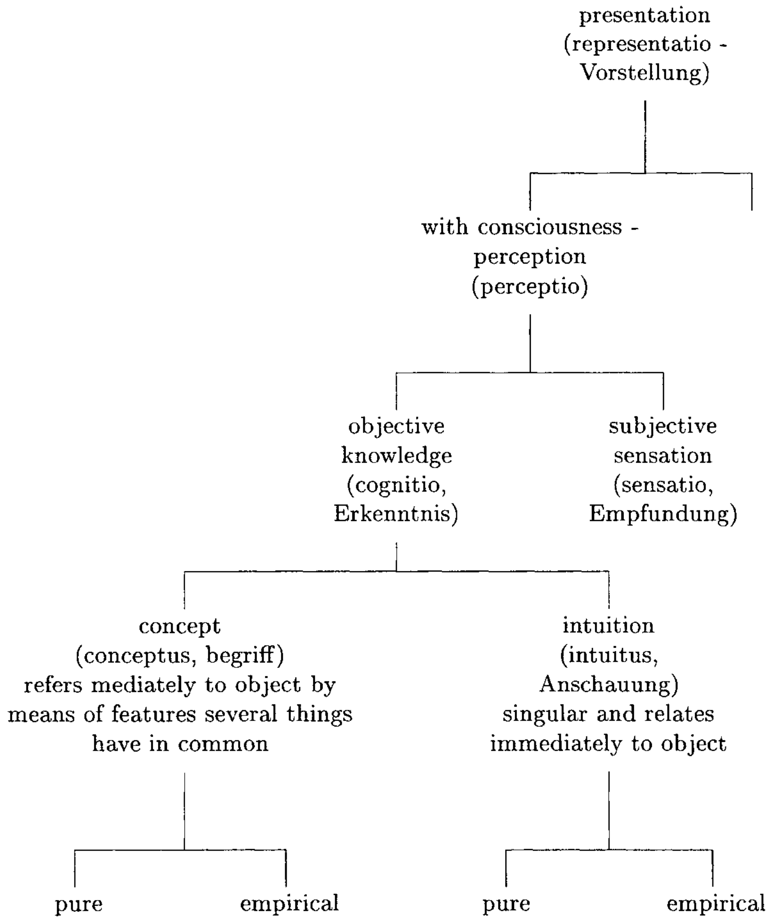

Figure 1.

Trivium to young boys and as a *science* taught in association with theology and philosophy in the more advanced *Quadrivium*, the latter being given very different treatments in the nominalist and in the more orthodoxly Aristotelian traditions [Moody, 1953, pp. 5–6]. The logic of the *Trivium* and its theory, presented as the elements of logic reached by analysis, was commonly referred to as the formal part of logic.

Although the art and the science of logic were connected, the former tended to be tied closely to persuasion, to the techniques of good argument and disputation, whereas the latter was tied to the pursuit of knowledge. This bifurcation of function is already present in Aristotle – the same syllogistic forms and immediate inferences (systematically explored in the *Prior Analytics*) form the basis for discussions of scientific knowledge in the theory of demonstration presented in the *Posterior Analytics* and of rhetorical strategies in the *Topics*. But just as the use of logic in rhetoric takes it beyond questions of formal validity, the Aristotelian theory of demonstration leading to scientific knowledge (knowledge of causes) also goes beyond the bounds of purely formal logic. Although all demonstrations must be formally valid, not all formally valid inferences constitute demonstrations. It is here that logic becomes entangled with epistemology and metaphysics.

Kant, who lectured on logic for most of his career as a university teacher, was almost as scathing in his assessment of the value of traditional presentations of syllogistic logic as Descartes. He dismissively acknowledges the role of syllogistic as part of the art of logic—the art of learned disputation—which 'belongs only to the athletics of the learned.... an art which does not contribute much to the advancement of truth' [Kant, 1992b, p. 91]. Even so in his lectures he was not wholly able to eliminate running over the syllogistic figures. As he says, in his lectures he cannot arrange everything according to his own view but 'must do much in conformity with the prevailing fashion' [Kant, 1992b, p. 91]. That fashion was set by Christian Wolff who took much of his inspiration from Leibniz, but, unlike Leibniz he published prolifically and with systematizing zeal. Kant himself, derived from Wolff the sense that philosophy needed to be conducted systematically in order to build a unified and structured whole. But Kant rejected key components of Wolff's dogmatic approach to philosophy. He rarely mentions Wolff by name, choosing instead to offer a sustained critique of Leibniz, but there are many points at which Kant makes remarks which are clearly targeted directly at Wolff's position.

Kant was obliged to continue the university tradition of teaching logic as the art, or skill, of reasoning. and using an approved text book — Georg Friedrich Meier's *Excerpts from the Doctrine of Reason (Auszug aus der Vernunftlehre)*. In one set of notes taken on Kant's logic lectures (the Jäsche Logic) the brief history of logic says that Wolff's general logic is the best there is, that Baumgarten condensed Wolffian logic and that Meier in turn commented on Baumgarten. [Kant, 1992a, pp. 534–5]. (Ak.IX 21) This being the case, Kant was not confident of his ability to overthrow 'the colossus which hides its head in the clouds of antiquity, and whose feet are of clay.' [Kant, 1992b, p. 91] He characterizes scholastic philosophy as instruction aimed at skill, i.e. it is classified *in toto* as an art, whereas true philosophy is doctrine aimed at wisdom [Kant, 1992a, p. 259]. Although Kant acknowledges Aristotle to be the father of logic, he dismisses his work; it is

> too scholastic, full of subtleties, and fundamentally has not been of much value to the human understanding. It is a dialectic and an organon for the art of disputation. ... All our logical terminology is

> from him. Otherwise it tends to mythology and subtlety and is banned from the schools. Still, the principal ideas from it have been preserved, and this is because logic is not occupied with any object and hence it can be quickly exhausted. [Kant, 1992a, p. 257]

Logic as the art of reasoning (general logic)

> abstracts from all content of cognition, i.e. from all reference of cognition to its object. It examines only the logical form in the relation that cognitions have to one another; i.e. only the form of thought as such. A55 (B80)

The completeness of this logic is a function of its limited scope and utility. Logic is the science of the rules of the understanding in general, but general logic in abstracting from all content of knowledge deals with nothing but the mere form of thought. ([Kant, 1996], A54 (B78)) It can consider only the relation of one cognition to another; it can thus arrive only at laws of thought, not at laws of truth, to use Frege's expression. It is barred from considering any formal principles of truth determination, because these are based on the relation of knowledge to its object. Wolff on the other hand claimed:

> If a predicate, either affirmative or negative, fits the subject [*subjecto convenit*] either absolutely or under a given condition, the proposition is said to be true. Truth is therefore the agreement of our judgement with the object or the represented thing [*Est itaque veritas consensus judicii nostri cum objecto, seu re representata*]. *Philosophia rationalis,* §505
>
> *Truth is the determinability of the predicate by the notion of the subject [Veritas est determinabilitas praedicati per notionem subjecti]* ......You have the real definition of truth if you conceive it as the determinability of the predicate by the notion of the subject. (Ibid. §513)
>
> The *criterion of truth* is an element which is intrinsic to the proposition [*Criterium veritatis est propositioni intrinsecum*], whereby one knows that it is true; consequently *it presents sufficient marks [notas] in any circumstance to know the truth and therefore to tell a true proposition from a false one. (Ibid. §523)*
> Passages quoted in Longeneusse 1998, from Wolff 1740

Note that what Wolff gives as a general account of truth is what Kant gives as truth for an analytic judgement (A7 (B11)) when he also insists that analytic judgements do not expand our knowledge. Wolff still believes that it is possible to use Euclid's geometry as a model for acquisition of all knowledge. Euclid is his paradigm of mathematical method, and this he claims to be identical with philosophical method. First principles will be definitions and everything must be logically demonstrated from first principles or from previously established results.

The consequence is that all philosophical knowledge, knowledge from principles, ends up being analytically true. Philosophy itself is 'the science of possibles insofar as they can be'.

> §139 The rules of philosophical method are the same as the rules of mathematical method. For according to philosophical method one must use only terms which have been accurately defined. (§116) And only that which has been sufficiently demonstrated can be admitted as true (§§117, 118). Both the subject and the predicate of every proposition must be accurately determined (§§121, 130). And everything should be ordered those things come first through which later things are understood and established (§§123, 124, 133). [Wolff, 1963, p. 76]

Moreover Wolff insists

> §88 If you wish to study philosophy fruitfully, then logic must be given the very first place. Logic treats the rules which direct the cognitive faculty to the knowledge of truth. [Wolff, 1963, p. 45]

But Kant is equally insistent that general logic cannot yield any knowledge (of objects), or any rules of the understanding pertaining to acquisition of knowledge of objects. A conceptual relation established on the basis of logical conceptual analysis may be empty because there are no objects falling under the concepts concerned. General logic can only establish logical possibilities and impossibilities. It provides only a negative touchstone of truth, avoidance of formal contradiction. Moreover, Kant argues there could be no formal criterion of truth, if truth is the agreement of cognition with its object, nothing in the form or logical structure of a concept could mark it out as guaranteeing knowledge of its object or even of the existence of an object falling under it. (Not even the concept of God carries this guarantee.) Kant sees no route from 'essence' to existence, or from the logical perfections of a concept to any assurance that it affords accurate knowledge of objects.

General logic then is merely a logic of the relation of ideas, of the comparison of concepts. And the possible relations can be quickly exhausted: they are, inclusion, exclusion and overlap, corresponding to the four kinds of categorical judgements (overlap giving both particular affirmative and particular negative. The theory of syllogisms can be grounded in consideration of these relations (whether the interpretation of concepts is extensional or intensional). Diagrams (see Figure 2) that Kant sketched in his copy of Meier [Kant, 1924, p. 715] indicate that this was the way in which he thought about syllogisms.

In the *New Essays* Leibniz too says that the whole theory of the syllogism could be demonstrated from the theory of the container and the contained [Leibniz, 1981, p. 486]

3215. $\nu?\pi - \varrho?(x - \lambda?)\eta??$ L 101′. *Zu L* §. 363:

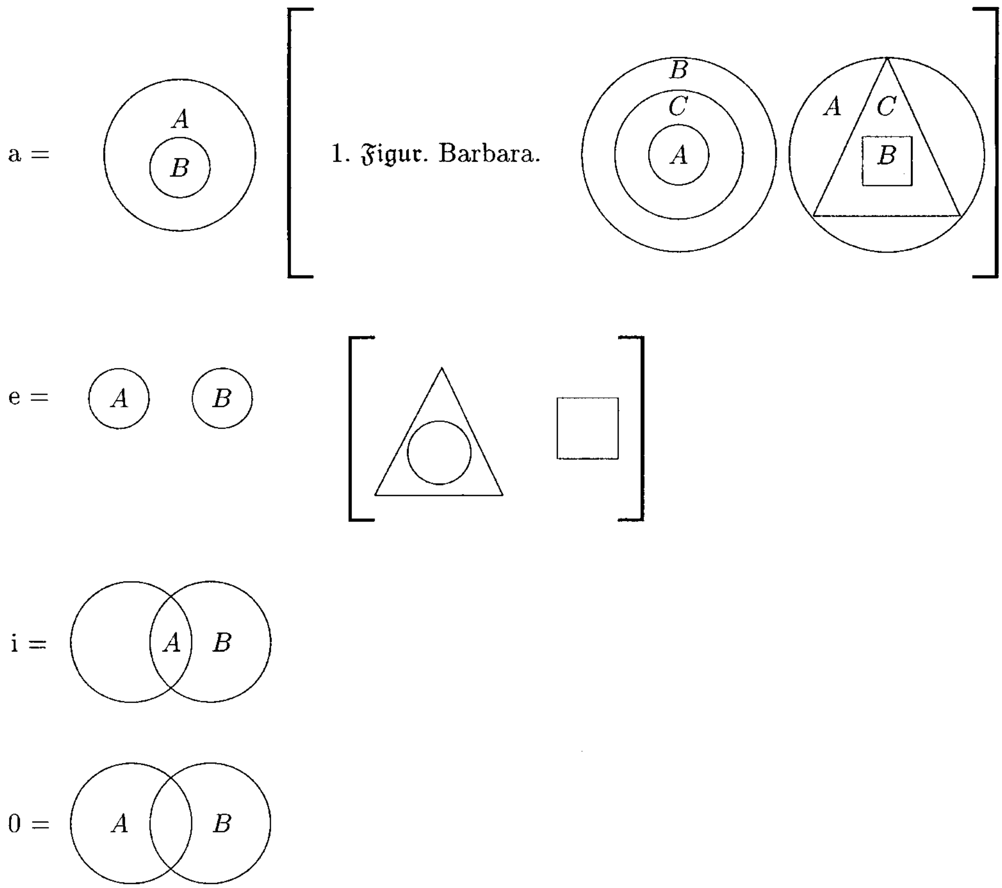

Figure 2.

3 LOGIC, REASON AND REASONING

Kant's critique of general logic and the philosophic role assigned to it by Wolff and his followers falls in two parts (coming in the sections of *Critique of Pure Reason* labeled Transcendental Analytic and Transcendental Dialectic). The nuts and bolts basis of Kant position is contained in the theory of judgement laid out in the Transcendental Analytic. But since that position itself builds on assumptions about general logic is is perhaps advisable first to look ahead to what Kant says about logic, reason and reasoning in the Transcendental Dialectic. We will find that to progress far with this we have to return to the account of understanding and judgement before being able to appreciate the force of Kant's critique of reason — his argument that reason can be a source of illusion.

There is an important respect in which Kant's approach to logic differs both from that of those who subscribe to the 'way of ideas' and from the majority of philosophers working within contemporary analytic philosophy. There are two, intimately interconnected, ways of stating this difference. The first concerns Kant's conception of reason and the second concerns grounding the distinction between logic as a normative discipline, having something to say about the way rational people *should* think, and psychology as a descriptive discipline, having something to say about the ways people actually *do* think.

Kant's conception of reason is distinctive in that (a) he draws a firm distinction between understanding and reason, and (b) he characterizes both as *active* faculties. Understanding and reason, as forms of activity directed toward knowledge, are each concerned to unify disparate presentations.

> The understanding may be considered a power of providing unity of appearances by means of rules; reason is then the power of providing unity of the rules of understanding under principles. Hence reason initially never deals with experience or any object, but deals with the understanding in order to provide the understanding's manifold cognitions with a priori unity through concepts. This unity may be called the unity of reason, and is quite different in kind from what unity the understanding can achieve. A302 (B359)
>
> Reason, in making inferences seeks to reduce the great manifoldness of understanding's cognition to the smallest number of principles (universal conditions) and thereby to bring about the highest unity of this cognition. A305 (B361)

The significance of the distinction between reason and understanding is that on Kant's view reason never deals directly with experience of any object, but only with unifying thought on the basis of previously made judgements. The significance of seeing reason as an active faculty having unity as a goal is that this will be the basis for viewing reason as a source of concepts, and it is misuse of these concepts which leads reason astray. In assigning unity as a goal of reason, Kant is reflecting the emphasis placed on order, method and system in the Port Royal logic and in Wolff's philosophy. Reason demands the construction of theories hierarchically organized under first principles, not isolated inferences. But the difference is that Kant does not see reason as a faculty of intellectual intuition or as striving for intellectual intuition; humans he thinks are incapable of intellectual intuition. The force of his critique is to say that it is an illusion to think of the finite limitation of human cognitive powers as a mere contingent limitation of an intuitive faculty, which merely needs to be brought to 'see' with greater clarity and distinctness. The view of reason and reasoning which unfolds leads Kant to distinguish two kinds of reasoning — reasoning from concepts (treated in general logic) and reasoning from the construction of concepts, on which the possibility of constructing general logic as a theory of forms of reasoning itself depends. The latter is the product of reason's capacity for reflection on its own principled (rule governed) activity and thus is a consequence of portraying reason and understanding as active faculties.

By way of contrast, Arnauld and Nicole, in the *Port Royal Logic*, capture the view of logic, associated with the more passive conception of reason employed in the 'way of ideas'.

> To reason is to form one judgement from several others.... [Arnauld, 1964, p. 29]
>
> Logic does not teach us *how* to conceive, to judge, to reason or to order; for nature in giving us reason gave us the means to perform these operations. Logic consists rather in reflecting on these natural operations. [Arnauld, 1964, pp. 29–30]
>
> If any man is unable to detect by the light of reason alone the invalidity of an argument, then he is probably incapable of understanding the rules by which we judge whether an argument is valid – and still less able to apply these rules. [Arnauld, 1964, p. 176]

Many of the features of this view were already outlined by Descartes (and Arnauld and Nicole acknowledge that they were using an unpublished manuscript of Descartes' *Rules for the Direction of Mind*). Descartes talks of 'the Natural Light of reason', which knows by intellectual intuition (understanding); it is a quasi-perceptual and passive faculty. Intellectual truths revealed to the Natural Light of reason are 'clearly and distinctly perceived' and are such that try as one might one cannot, whilst holding them in mind, doubt their truth. The clarity of vision allows objectively compelling truth to impose itself. This is the mode in which, for rationalists like Leibniz and Wolff, God is supposed timelessly to know all things and it is the link to God which supplies the imperative force; humans should strive for perfection in knowledge and perfection in knowledge is knowing in the manner of God (attaining the God's-eye view). This model incorporates not just the goal of making correct judgements, but of incorporating all knowledge into a comprehensive vision.

Of the empiricists it is Hume who most definitively shows how powerless this conception of reason as understanding by the 'Natural Light of reason' is when stripped of it theological underpinnings. It leads to knowledge of nothing more than the relations between our own ideas and is incapable of generating or justifying beliefs about material objects or the external, world. It is, moreover, completely lacking in motive force. Hence 'reason is, and ought only to be, the slave of the passions' [Hume, 1985, II.iii.3]. But one should also remember that both Descartes and Hume do make reference to the human capacity to learn from experience. Descartes calls this 'the Light of Nature' and Hume talks about the the interconnected roles of custom, habit and imagination in the formation of concepts (such as those of a material object and cause) and of beliefs about the natural world. In discussing 'the understanding' Kant is giving an alternative account of our ability to learn from experience (gain empirical knowledge) and it is one which links this process much more closely with reason and reasoning, than the accounts offered by Descartes and Hume.

On the 'Natural Light' conception of reason, reasoning and logic are relegated to the status of methods for leading the mind to clear and distinct perceptions. The need for such methods is a reflection of the limitations of embodied, human reason. As Arnauld and Nicole say, 'It is the human mind's limitations which force man to reason [Arnauld, 1964, p. 175]. They focus on knowledge as the product of judgement and logic as including everything that is useful for training human judgement to overcome its limitations and this includes a stress on orderly method as the vehicle for achieving unified knowledge. The objective is the formation of clear and distinct ideas on the basis of which all comparison of ideas, judgement of their relations, will be intuitively immediate. In the eighteenth century, Leibniz and others, expended considerable effort on the construction of an ideal, universal language which would, by symbolically encoding the complexity of concepts, enable their relations to be made transparent.[3] Logic is then described as an art (not a science) and 'This art consists in man's reflecting on the mind's four principle operations – conceiving, judging, reasoning, and ordering.' [Arnauld, 1964, p. 29]

Crucial to Kant's divergence from both the *Port Royal* and the Leibnizian conceptions of logic is his denial that humans have a capacity for intellectual intuition, and his consequent reconception of understanding as a discursive faculty for making judgements. All knowledge is thus mediated by concepts; there is no immediate intellectual cognition. Understanding gives rise to both judgements and concepts, which are inseparably linked. Similarly reasoning, the making of inferences, is inseparable from imposition of order. But even more crucially Kant latches on to the capacity for reflection, Leibniz' apperception. Without this capacity, logic would clearly be impossible; in fact this whole Cartesian style of doing philosophy rests on the capacity for reflection, for self-knowledge. It had conceived this as a kind of inner perception, but this at best leads only to an empirically descriptive self-knowledge unless our affiliation with God can somehow be brought in to the picture. It could not otherwise provide any grounding for the normative claims made on behalf of logic.

[3] For discussion see, for example, [Rider, 1990].

Analytic philosophy has sought to finesse this problem by taking the linguistic turn; returning to the more scholastic practice of linking logic to the study of thought as expressed in language, thus seeing it as concerned with objective relations of implication, or consequence between assertoric sentences, or the propositions they express. On this view logic is a science, not an art, since it is concerned with characterizing these relations, not with processes of reasoning. In other words it laws are laws of truth, not of thought.[4] Kant, in conjunction with insisting on the discursive nature of all knowledge, takes a practical turn, which is a small, but significant step from thinking of logic as the *art* of reasoning, for it recognizes a sense in which logic can also be a science, but not however a science descriptive of a static, independently given structure. Rather logic can be a science by characterizing the structures generated by operating according to the rules and principles demanded by understanding and reason respectively, in their quest for unity. This will be a form of reasoning from the construction of concepts.

What is it that is distinctive of rational beings? It is the capacity for not only acting in accordance with a rule, but acting in accordance with a conception of a rule (Kant Ak. IV 412). It is the conception of a rule which is a prerequisite for knowing what would and what would not be in accordance with it. It is only this which can ground recognition of a rule as requiring action of a certain kind, and hence providing a foundation for both moral and logical imperatives — this is what ought to be done, should be concluded or believed. It is also for making judgements since all judgement, for Kant involves the application of a rule. Automatic rule following does not require judgement. Thus Kant says [Kant, 1992b, p. 93] 'it is a quite different thing to *distinguish* things from one another, and to *cognize the distinction of things.* The latter is only possible by means of judgements, and cannot be accomplished by an irrational animal.' And a little later (p. 95) 'The following distinction may be of great use: *Logical distinction* is the cognition that a thing A is not B, and is always a negative judgement. To distinguish physically is to be impelled by different ideas to different actions.' What distinguishes rational beings is their ability to make their own thoughts and actions objects of their own thought.

If understanding and reasoning are rule governed activities whose goal is, by ordering, to unify our cognitions, they are also sources of conceptions of what these rules require, and of the nature of the activities of judging and reasoning. There can be no science of reason without such a reflexive exercise of reason for there is no external standpoint from which the human mind can assess its own capacities; any limits discerned must be discerned from within the sphere of its own operation. This means that once rules of inference are recognized and formulated (whatever they are), there can be no stopping here, since the formulation of rules leads to conceptualization of the way they operate and of the kinds of structures that can be produced by allowing their repeated application, i.e. the step to metatheory is unavoidable. Yet it is also a trap if it is thought to be a way of stepping outside, or of reaching beyond the sphere of reason, for this theoretical activity it itself

[4] See [Frege, 1959, Introduction].

carried out by human reason. This is why there needs to be a distinction between this mode of conceiving and reasoning, and that which is concerned with objective knowledge (knowledge of external objects).

So long as the goal of knowledge is portrayed as imitation of God's knowledge, the structure aimed at is conceived as a timeless structure of relations of ideas in the mind of God. For God does not arrive at knowledge through any process of reasoning, but apprehends everything once and for all in an act of intellectual intuition. Kant is quite insistent that this is an inappropriate model for human knowledge (B 72) and hence for logic as a study of the formal structures of human knowledge.

In this respect his departure from the 'way of ideas' takes him into new territory. He is assisted in this task by the radical changes that had occurred in mathematics during the course of the seventeenth century and which continued into the eighteenth. In particular this period saw the development of algebra, calculus and analytic geometry. Kant draws on mathematical analogies for his thinking about logic (just as Leibniz had earlier drawn on algebra and arithmetic in his work on logic and the development of a universal characteristic, and as Frege was to draw on the concept of a function for his reform of the notion of a concept.). But in order to see the nature of Kant's extended view of reason and logic it is necessary to see how he views reasoning from concepts, the topic of general logic.

We have already said that as is clear from his lecture notes, Kant is quite aware that the standard theory of the syllogism reduces it to a matter of the comparison of ideas and their relations of containment. This is exactly how Arnauld and Nicole, in their section devoted to reasoning, treat syllogistic reasoning; it is reduced to the comparison of ideas - in this case three ideas corresponding to the subject, middle and predicate terms. In other words reasoning and making judgements are alike a matter of the comparison of ideas.

In the Transcendental Dialectic Kant starts with the logical use of reason, covered in general logic, and derives from it an account of reason as a 'transcendental' power. His section on the logical use of reason is very short. He distinguishes, as was customary, between immediate and mediate inferences, assigning the former to understanding and the latter to reason. Mediate inferences are discussed simply under the heading of syllogism, although included here are hypothetical and disjunctive syllogisms along with the Aristotelian categorical syllogisms. Hypothetical syllogism, for Kant, covers both *modus ponens* and *modus tollens,* and disjunctive syllogism goes either from the truth of one member of the disjunct to the falsity of all the others, or from the falsity of all but one of the members of the disjunct to the truth of this one [Kant, 1992a, pp. 622–3]. Thus disjunction is clearly conceived as an exhaustive and mutually exclusive listing of cases.

In some respects this is standard fare, but it would be unwise to jump immediately to the conclusion that there is nothing innovative going on. What is crucial in this brief account is his general characterization of inference for it is not based on comparison of ideas, but on the application of a rule.

> In every syllogism I first think a *rule* (*maior*) by the understanding. Second, I *subsume a* a cognition (*minor*) under the condition of the rule by means of the *power of judgement*. Finally, I *determine* my cognition (*conclusio*) by the predicate of the rule and hence a priori by *reason*. Therefore, the various kinds of syllogism consist in the relation that the major premise, as the rule, presents between a cognition and its condition. A304 (B361)

There are a several points to note here. First, this account of the exercise of reason accords with the way Kant characterizes reason as the faculty of principles. A principle is a universal rule. In reasoning from concepts the particular is thought through the universal, by subsuming a particular case under a universal rule (in reasoning from the construction of concepts the universal is thought through the particular). Reason seeks unity by seeking to organize knowledge under principles which are as universal as possible; it seeks ever greater generality as a way of encompassing more, but it also seeks to make as many distinctions as possible within this range, for otherwise it would have no representation of the diverse extent of its range. The drive to unify is the drive to encompass as much diversity as possible in a single system. We should also note that Kant is here reading hypothetical and disjunctive premises as universal (distinctly non-truth functional); the hypothetical, used as a rule is saying that whenever its antecedent is given, its consequent can be affirmed. The disjunctive premise is claiming coverage of *all* of a range of mutually exclusive possibilities.[5]

But Kant also characterizes judgement as the faculty of rules, and talks of concepts as rules, so we should expect to look here for connections between logic as concerned with reasoning, the logical forms of judgement, and the functions of unity in judgement. Finally, a major premise, viewed as a rule presents a *relation* between a cognition and its condition, which means that each kind of major premise (categorical, hypothetical, disjunctive) must express a distinctive *logical relation* between a cognition and its condition. But Kant gives us very little explanation of what exactly he understands by a condition.

This terminology is used by Wolff and is explicitly involved in his understanding of the task of philosophy. He says that philosophy must give a reason why those things which can occur actually do occur [Wolff, 1963, §31, p. 18]. This he equates with saying why something should be affirmed or denied of a thing. The philosopher should make it clear whether the predicate is attributed to the subject because of a definition (which would make it a straightforward true categorical judgement) or because of some condition (which limits the range in which the proposition can be used to those cases in which the condition holds). Once this condition is included in the proposition, as Wolff insists it should be, the result will be a hypothetical proposition [Wolff, 1963, §121, p. 63]. The whole point is that the propositions of philosophy should be demonstrated from first principles

[5]In modern terminology it is thus fair to say that Kant is much closer to a natural deduction approach to logic than to either an algebraic or axiomatic approach.

and one needs to know whether the demonstration should start with a definition or with the condition by which the subject is determined [Wolff, 1963, §121, p. 64]. And if we recall Wolff's definition of truth we see that what has to be filled out is the *concept* of the subject so that it determines the predicate. When definition is the basis for this determination, it is an inner determination, when it is a condition which provides the basis, it is an external determination.

Now an external determination is what Kant suggests we should be looking for when considering synthetic truths. And when we are thinking of empirical, a posteriori truths it is experience which provides this external determination.

> ...(1) that analytic judgements do not at all expand our cognition, but spell out and make understandable to myself the concept that I already have; (2) that in synthetic judgements, where the predicate does not lie within the concept of the subject, I must have besides this concept something else (X) on which the understanding relies in order to cognize nonetheless that the predicate belongs to the subject.
>
> In empirical judgements, or in judgements of experience, it is not difficult to find this X. For here this X is the complete experience of the object that I think by means of the subject concept, the concept amounting only to part of the experience. A8

Kant then adds that if there is to be any synthetic a priori knowledge this X, the condition of the synthetic a priori judgement cannot be experience; it must be a concept and this will be the concept of a possible object (of experience). We can now see why Kant says that all the judgements of the understanding are conditioned; actual or possible experience of objects provide their (truth) conditions.

In both the contexts from which I have been quoting Kant mentions the principle that every event has a cause. One of his aims, is to undercut the idea that the demonstrations provided by general logic, starting from definitions of concepts, can be demonstrations of causes. Thus he hopes to refute the idea that the Principle of Sufficient Reason can suffice for moving one from the realm of the logically possible to determination of what is actually the case. Thus it is in discussion of the pure use of reason that Kant notes that

> The principle that everything that occurs has a cause is not at all a principle cognized and prescribed by reason. It makes possible the unity of experience and borrows nothing from reason: reason could not, without this reference to possible experience and hence from mere concepts, have commanded such synthetic unity. A307 (B364)

Here already, in our discussion of general logic we have been pushed back into the account of judgement.

4 GENERAL LOGIC AND THE FORMS OF JUDGEMENT

If general logic is thought simply as concerned with judgements based on the comparison of concepts (ideas), then the forms of judgement will be determined by the various ways in which concepts can be compared. Each comparison would then yield a specific way of bringing the concepts together in thought (function of unity in judgement). Kant (A261) lists the relations in which concepts can stand to one another as those of identity/difference, agreement/opposition, inner/outer, and determinable/ determination (matter/form).

> Before all objective judgements we compare the concepts in order to hit upon the *sameness* (of many representations under one concept) for the sake of *universal* judgements, *difference* for producing *particular* judgements; and upon the *agreement* from which *affirmative* judgements and the *opposition* from which *negative* judgements can come to be, etc A262 (B318)

The list of relations accords with the terminology we have seen used by Wolff and would suggest the following table of corresponding forms of judgement

Quantity
Universal—sameness
Particular—difference

Quality
Affirmative—agreement
Negative—opposition

Relation
Categorical—inner
Hypothetical—outer

Modality
Problematic—determinable
Assertoric—determination

Figure 3.

The completeness of this list depends upon having exhausted the logical relations in which a pair of concepts can stand to one another and this in turn depends on whether there are dimensions of comparison other than those of quantity, quality, relation and modality. Kant is following tradition in thinking this list to be exhaustive, in the context of general logic. A very similar list can be found in Leibniz' *New Essays*. Here Philalethes (Locke) suggests that the agreement and disagreement of ideas and propositions, relevant to knowledge of truth can be reduced to four sorts: (1) identity or diversity, (2) relation, (3) coexistence or necessary connection, and (4) real existence. And from his comment it is clear that these would map onto Kant's quantity, quality, relation and modality. Leibniz (Theophilus) then suggests that all relation involves either comparison or concurrence, with relations of comparison yielding identity and diversity, in all respects or in some only, thus giving both sameness and difference as well as like or unlike. Concurrence includes Locke's coexistence, which Leibniz interprets as connectedness of existence and he notes that when something is said to exist, either this is connecting the notion of existence with the idea in question, or the existence of the object of an idea may be conceived as the concurrence of that object with myself [Leibniz, 1981, pp. 357–58]. Leibniz' way of reducing the relations is of interest because it matches the way in which Kant will distinguish between mathematical and dynamical categories.

But there is an important caveat to Kant's remark about the comparison of concepts, it is 'before all objective judgements'. He adds that correct determination of this relation depends on distinguishing whether the objects about which judgement is made are objects for understanding or sensibility. He further observes that these relations of comparison between concepts 'differ from categories in as much as they do not exhibit the object according to what makes up its concept (magnitude, reality) but exhibit in all its manifoldness only the comparison of presentations that precedes the concept of things.' (A269 (B325)) General logic deals only in concepts, not in things, and deals with relations of concepts only as objects of understanding. So general logic makes nothing of this distinction between objects of understanding and objects of sensibility, because it doesn't talk about objects. Thus Kant's caveat indicates two things, first the need to move toward transcendental logic and second, that once we do so we should anticipate that the judgements of the various forms will be subject to a double reading.

5 TOWARD TRANSCENDENTAL OF LOGIC

The problem with general logic is that its deals only in forms (universals) and not with the intersection of form and matter necessary for thought of concrete individuals. So general logic is formal not just in the sense that it abstracts from all content of concepts, but in the sense that in dealing only with conceptual relations it abstracts from everything pertaining to the existence of individuals objects. Transcendental logic, on the other hand, does not abstract from the entire content of knowledge but contains the rules of pure thought of an object. It is a science of reason whereby we think objects entirely a priori and which concerns itself with the laws of understanding and of reason solely in so far as they relate a priori to objects. It is also concerned with delimiting the scope and objective validity of such knowledge. (B82) This is a logic of existence as well as of essence, of individuals and their relations as well as of universals, and contains the laws of understanding and reason in so far as they relate a priori to objects. Thus Kant gives the following definition

> Logic is a science of reason not only as to mere form but *as to matter*; a science a priori of the necessary laws of thought, not in regard to particular objects, however, but to all objects in general — hence a science of the correct use of the understanding and of reason in general, not

subjectively, however, i.e. not according to empirical (psychological) principles for how the understanding does think, but objectively, i.e. according to a priori principles for how it ought to think. [Kant, 1992a, p. 531]

The philosopher whom Kant most specifically and systematically criticizes for being misled into thinking that general logic could be a framework for all thought of objects is Leibniz, because his account of the true objects of knowledge (monads, metaphysical atoms) treats knowledge of objects as knowledge of their infinitely complex concepts.

We already noted that logic, as defined by the Port Royal logicians, consists in reflecting on the operations of conceiving, judging, reasoning and ordering. Their account of conceiving and judging is as follows:

> To conceive a thing is simply to view that thing as it present itself to the mind.... The form by which we represent a thing to ourselves is called an idea .... To judge is to join two ideas, affirming or denying the one idea of the other. [Arnauld, 1964, p. 2]
>
> Once we have formed ideas of things, we compare the ideas. We unite those which belong together by affirming one idea of another; we separate those which do not belong together by denying one idea of another. To judge is to affirm or deny. [Arnauld, 1964, p. 109]

Port Royal logicians assume that knowledge of things is the overall objective of reason and that this is achieved by getting progressively better, more distinct and more adequate ideas of things. Thinking is an activity with this as its objective, the objective is not to think *true thoughts*, but to gain *improved ideas* (conceptions) of things. Very similar statements are to be found in the voices of both Locke (Philalethes) and Leibniz (Theophilus) in Leibniz' *New Essays*.[6] But the problem posed by Philalethes (Locke), who is open in his disregard for the utility of syllogistic logic in the acquisition of knowledge, is that

> It is said 'that no syllogistic reasoning can be ... conclusive, but what has, at least, one [universal] proposition in it. [But is seems that] the immediate object of all our reasoning and knowledge, is nothing but particulars.' Knowledge rests wholly on the agreement and disagreement of our ideas, each of which is only a particular existence which represents only an individual thing. [Leibniz, 1981, p. 485]

Theophilus (Leibniz) seeks to address this concern by talking about the way in which singular premises are treated as universals in syllogistic logic, but this

[6]This framework is, incidentally, one in which truth functional logic finds no natural place; concepts can be conjoined, disjoined or negated, but not arbitrarily. Disjunction, for example would not correctly occur between arbitrary concepts but only between those known to divide things falling under a given concept into mutually exclusive but jointly exhaustive classes. 'All natural numbers are even or odd', for example, but not 'Some natural numbers are even or prime.'

hardly constitutes a satisfactory response to Locke's concern. In addition, the question to which Philalethes repeatedly returns is that of how we get knowledge of the existence of empirical objects and of how we get knowledge of their material natures, i.e. his concern is with empirical scientific knowledge, knowledge that would be expressed in synthetic (not analytic) judgements. Locke is aware that mere comparison of ideas will not yield this knowledge and that logic, in spite of Leibniz' best efforts to argue to the contrary, does not have the resources to give an account of the synthetic truths. In fact even Leibniz acknowledged that the principle of sufficient reason was required to distinguish between the actual world and the many logically possible worlds. Kant, in seeking to formulate a logic appropriate to synthetic truths also seeks to undermine the idea that the principle of sufficient reason is an apriori truth of reason.

Kant addresses these issues by distinguishing three cognitive capacities and giving to each capacity a distinctive mode of presenting content for thought — **sensibility**, by means of which objects are *given* to us through *intuitions;* **understanding**, by means of which objects are *thought*, giving rise to *concepts*; (A20 (B34)), also characterized as the faculty of rules (A299 (B356)) and **reason**, by means of which inferences are made, which gives rise to *ideas*, and characterized as the faculty of principles. Morerover the overarching theme in Kant's characterization of rational cognition, that which constitutes knowledge at all levels, is bringing unity to the manifold — recognizing both the plurality and the unity. Each capacity is a capacity for a distinctive kind of unification. The forms of sensibility (space and time) make possible a synthesis of the manifold in intuition; judgements brings unity to the manifold through cognition under a concept in judgement; reason brings unity to concepts and judgements in the construction of ordered systems of thought.

The way in which Kant uses the distinction between sensibility and understanding, intuition and concept, in his account of judgement, marks a complete break with the account of knowledge as comparison of ideas, and of the improvement of knowledge as the improvement of the quality of ideas — making them clear and distinct. On this latter account sensation was supposed to yield confused ideas, ideas needing to be made distinct, but not as yielding presentations different in kind from those of the understanding. Thus Kant says that for Leibniz sensibility was only a confused way of presenting, and not a separate source of presentations (A270 (B326)). And

> Hence the philosophy of Leibniz and Wolff, by considering the distinction between what is sensible and what is intellectual as a merely logical one, has imposed an entirely wrong point of view on all investigations about the nature and origin of our cognitions. For plainly the distinction is transcendental, and does not concern merely the form of these cognitions, i.e. their distinctness or indistinctness, but concerns their origin and content. Hence sensibility does not merely fail to provide us with a distinct cognition of the character of things in themselves; it provides us with none whatsoever. A 44 (B62)

By recognizing intuitions as distinct in kind from concepts and as the product of exercise of a different human capacity, Kant opens the way for being able to treat of the forms of intuition. In other words there is something to be said in general about the way in which thoughts are referred to objects in the attempt to say things about them, i.e. to express objective truths. Kant goes on to insist that there is no knowledge without both concepts and intuitions.

> Without sensibility no object would be given to us; and without understanding no object would be thought. Thoughts without content are empty; intuitions without concepts are blind. (A51 B75)

This restores the missing dimension to logic, now called transcendental logic, and distinguishes it from the logic which deals only in the comparison of concepts. If truth is the agreement of an idea with its object, then sensibility must be part of the picture, for it is only in intuition that objects are presented. Thought is knowledge of objects by means of concepts, and not just the comparison of ideas. Moreover, Kant insists that a concept is a concept solely in virtue of its comprehending other representations, by means of which it can relate to objects; concepts are 'predicates of possible judgements and as such refer to some presentation of an as yet undetermined object' (B94). Concept, object and judgement are thus terms which cannot be independently defined but have to be understood in interconnection, as Frege was also to insist

> We may in say in brief, taking 'subject' and 'predicate' in the linguistic sense: A concept is the reference of a predicate; an object is something that can never be the whole reference of a predicate, but can be the reference of a subject. [Frege, 1960, pp. 47–8, see also p.32]

Nonetheless understanding, in making the logical distinction between concept and object must be capable of conceptualizing, if not defining them. These concepts would, in Wittgenstein's terms, have to be formal concepts (a phrase which he uses to replace Russell's language of logical types).[7] Moreover, while in some judgements, concepts are related to other concepts, there must be exercises of judgement in which intuitions are unified under a concept, and these will not be judgements recognized in the standard lists of general logic.

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[7]In the *Tractatus* [Wittgenstein, 1961] he says

> 4.126 ...When something falls under a formal concept as one of its objects, this cannot be expressed by means of a proposition. Instead it is shewn by the very sign for this object....
> Formal concepts cannot, in fact, be represented by means of a function, as concepts proper can. For their characteristics, formal properties, are not expressed by means of functions.
> 4.1271 Every variable is the sign for a formal concept...
> 4.12721 ...It is not possible, therefore, to introduce as primitive ideas objects belonging to a formal concept *and* the formal concept itself.

6 JUDGEMENT

Writers on Kant's logic have most frequently started by addressing his list of the logical forms of judgement and its relation to his table of categories. Kant claims completeness for his list of forms of judgement and it is perhaps this claim, more than any other that has led modern logicians to dismiss Kant's significance as a logician. Kant omits some of the forms recognized by modern logic while including more forms than would be required if he were taking a minimalist approach, such as that adopted by Russell, using only negation, material conditional, and the universal quantifier.

The relationship said to exist between the categories and the logical forms of judgement is crucial to the project of Kant's first critique, for it is on the basis of the categories (as a priori concepts) that the possibility of synthetic a priori knowledge is made to rest. The apparently simple, steps by which Kant makes this move have caused much bafflement. He makes it sound very easy, just list the functions of unity in judgement and then notice that

> The same function that gives unity to the various presentations *in a judgement* also gives unity to the mere synthesis of various presentations *in an intuition.* This unity — speaking generally — is called pure concept of understanding. Hence the same understanding — and indeed through the same acts whereby it brought about in concepts, the logical form of a judgement by means of analytic unity — also brings into its presentation a transcendental content, by means of the synthetic unity of the manifold in intuition as such; and because of this, these presentations are called pure concepts of understanding applying a priori to objects. Bringing such a transcendental content into these presentations is something that general logic cannot accomplish. (A79 (B105))

The problem is that it is difficult to discern what exactly Kant means by the functions of unity in judgement.

He does say that by 'function' he means the unity of the act of arranging various presentations under one common presentation, and that all judgements are functions of unity among our presentations. And also

> Bringing various presentations *under* a concept (a task dealt with by general logic) is done analytically. But bringing, not presentations but the *pure synthesis* of presentations, *to* concepts is what transcendental logic teaches. (A 79 (B104))

> By *synthesis, in the most general sense of the term,* I mean the act of putting various presentations with one another and of comprising their manifoldness in one cognition.... Before any analysis can take place, these presentations must first be given, and hence in terms of *content* no concepts can originate analytically. (A77 (B103))

And there is one further example which gives perhaps a better clue as to what is intended here.

> By pure synthesis I mean the synthesis that rests on a basis of synthetic a priori unity. E.g. our act of counting (as is more noticeable primarily with large numbers) is a *synthesis according to concepts*, because it is performed according to a common basis of unity (such as the decimal system). Hence under this concept the unity of the manifold's synthesis becomes necessary. (A78 (B104))

If we bear this example in mind it gives some clue as to the way in which object and concept are coordinated but not independently given 'functions' in judgement; both arise in the context of judgement in which the units (constituting objects) are given by the rule which unites them under a concept. Concepts presuppose objects as the things of which they can be predicated, or to use older, more Platonist language, a concept is a universal, is a one over many. But for such unity in diversity to be possible, concepts and objects must have different identity criteria and either the concept as basis of the unity must carry with it identity criteria for the sort of object which can fall under it (be a sortal concept), or the domain of possible objects of predication must be presupposed as given, along with criteria of identity and individuation, independently of the concept. Now transcendental logic, as logic, is concerned with objects in general, not with specific kinds of objects. So we need to ask what are the logical functions of an object?

As indicated above, judgement aims at knowledge of objects.

> Judgement — the indirect cognition of an object..... In every judgement there is a concept [that comprises and thus] holds for many [presentations], and among them comprises also a given representation that is referred directly to the object. A68 (B 93)

This given representation may or may not be a concept.

The logic of judgements aimed at knowledge of objects, cannot then be adequately covered by thinking only of the comparison of ideas. When a judgement is made and we hold it to be an objective truth (a truth about an independently existing object), we also assume that the judgements of other people should be the same, because the object is that in virtue of which the judgement is correct or otherwise (agrees of fails to agree with its object). Kant expresses this by saying that we assume the judgement has necessary universal validity. So another way of describing what Kant is doing would be to say that he is beginning to think through what a logic giving principle of objective (inter-subjective) truth should take into account.

In judgement aimed at objective truth we cannot think that we are just subjectively comparing our ideas. Or, to put it another way, intensional relations between concepts can no longer ground the truth of judgements. This may serve for (analytic) logical truth, but not for objective truth. One might then suppose that this shift is simply that from an intensional to an extensional logic, treating concepts via their extensions[8] . However, this is not the case. Extension is certainly brought in to the picture, but for Kant, as still for Frege, interpreted concepts determine extensions. The ground of truth here is still the intension of the concept but now in application to objects. This means that subject and predicate terms become strongly asymmetric in their role, which they are not in mere comparison of ideas and in general logic, where conversion and contraposition are allowed. ('No A is B' is converted into 'No B is A', and 'All A are B' is contraposed to yield 'All non-B are non-A', for example.) Such interchanges between subject and predicate terms treat them as being of the same logical type. If, on the other hand, a judgement is to be read as aimed at knowledge of an object, then a part of the judgement must be interpreted as referring to that object (which introduces an indeterminacy, an indistinctness, into that concept) and the other as saying something about it. The subject term has to be thought to be referring to some object, in which the content of the judgement is united as a conception of that object. But in making this reference we simultaneously form a schematic conception of that about which judgement is made. It is this that the categories provide; they are formal concepts in the sense that they do not describe an object but merely indicate the logical contours of that about which judgement is made; contours which indicate what would constitute truth or falsity for a judgement of this logical kind.

> The logical functions of judgements as such — unity and plurality, affirmation and negation, subject and predicate — cannot be defined without committing a circle; for the definition itself would, after all, have to be a judgement, and hence would already have to contain these functions. The pure categories, however, are nothing other than presentations of things as such insofar as the manifold of their intuition must be thought through one or another of these logical functions. A 245 (B 303)

So, reality is simply what is attributed in an affirmative judgement; inherence of a quality in a thing is the relation assessed in a categorical judgement, consequence is the relation assessed in a hypothetical judgement. Logic alone does not give any content to these concepts, does not give any criteria of application, so that as pure categories they are in a sense empty, but by their relation to one another via the logical relations between forms of judgement they are formally constrained.

Thus, in transcendental logic judgements are not viewed as statements of relations between ideas. The pure concepts of understanding are the rules for uniting presentations treated as presentations of the conditions which determine empirical

[8]And indeed this is what it became at the hands of Frege as picked up by Russell and the logical positivists, and later by those using an extensional semantics. The move to formal extensional semantics is arguably a move to what Kant called transcendental logic, although equally arguably it is not made in a way which he would have approved.

judgements as synthetic truths (i.e. the specific form of synthesis is projected back into the object so that it can be conceived to require this rather than that form of judgement). This is said to require that the given intuition 'must be subsumed under a concept which determines the form of judgement in general with respect to the intuition in a consciousness in general, and thereby provides the empirical judgement with objective reality' (A110). In other words in the act of judgement the intuition is thought as presentation of an object about which a particular kind of judgement is appropriate. The object is that in virtue of which the judgement is true or false. It thus requires use of a formal concept (or category) which determines the role intuition is to play in assessing the correctness of the judgement (A 111). The categories are the concepts which frame objects in this way. Frege would make the same move, in a somewhat different manner (because of his different account of judgement) under the guise of specifying truth conditions. Formal semantics requires that, to each logical form a sentence can have, one should have a corresponding account of what would have to be the case for it to be true. Indeed, one of the reasons why there seems to be so much obscurity surrounding Kant's move to the categories is that it occurs in the context of his transcendental deduction, aiming to establish the categories as a priori concepts with guaranteed application in the realm of experience, but not beyond. But the parallel with formal semantics may enable us to see that, at least to an extent, modern logic moved in the transcendental direction, counting delimitation of domains of possible interpretation as being with in its scope. And with this hindsight we may also see the logically innovative moves that Kant is making, whether we do or do not want to buy in to the argument of the transcendental deduction, whatever we think that argument is or how we think it is supposed to work.

7 OBJECTS, CONCEPTS AND CATEGORIES

So lets start again simply looking at the logical situation. General logic dealing only in relations between concepts can at most determine what is logically possible, which concepts can be combined without contradiction. But the whole framework presumes that concepts as universals don't simply contain, fail to contain, or exclude other concepts, but also that concepts as universals stand over against particulars which they aggregate into classes. Particulars are real existents; determining what is actually the case, as opposed to what is logically possible, requires reference to the domain of particulars. What do we know logically about particulars? In other words, is there anything than can be said about particulars as possible existents simply from the role they have to play in logic as the counterpoints to concepts?

Kant has already to hand an account of what from the point of view of general logic would be required of particulars if they were to be given through concepts. This is provided by Leibniz in his *Monadology.* Kant can use this account to leverage his way to an account of possible objects given intuition. First and foremost they are the ultimate subjects of predication; that of which predication is made but which can never themselves be predicates. In other words the first understanding of their role comes from singular categorical judgements. To borrow an image from Searle, they are like pegs on which to hang concepts [Searle, 1967, p. 95].

In the Amphiboly of Reflection, where Kant gives an extended criticism of the Leibnizian way of moving to a transcendental logic we see him going through just these moves. Leibniz treats individual logical subjects, logical and metaphysical atoms, as intelligible objects whose nature is expressed via infinitely complex concepts. In the process we gain insight into the contrasting Kantian way of looking at the requirements of a transcendental logic.

Let us start from the framework of general logic where judgements express the relation of ideas (representations) without any fundamental asymmetry between subject and predicate terms. As we have seen, the move to transcendental logic requires recognition of the asymmetry between subject and predicate in judgements. Logic can ground such an asymmetry in the notion of an individual object as ultimate subject of predication. But from the point of view of general logic the singular is treated as universal.

> As far as the universality of a proposition is concerned, it makes no difference whether the extension of its subject ideas be great or small, provided only that the whole extension be referred to. Consequently, in an argument singular propositions function as do universal propositions. [Arnauld, 1964, p. 110]

From the point of view of transcendental logic, just as universals unify by grouping together their particular instances as being the same, individual objects, as logical subjects, unify the multiple concepts they instantiate. In this respect singular judgements should be recognized as introducing a distinctive function of unity in judgement, and hence as different in logical form from universal judgements, and so in turn as reflecting a distinct relation – the relation of object to concept in a singular affirmative categorical judgement is distinct from that of concept to concept (containment) in a universal affirmative categorical judgement. Now what distinguishes individuals as logical atoms[9] , ultimate subjects of predication, is that they cannot be further divided by concepts, they are like logical points . As Kant puts it

> because singular judgements have no range at all, any predicate of them cannot be referred to some part of what is contained under the concept of the subject and be excluded from some other part of it.... if a singular judgement is compared in terms of quantity with a generally valid one merely as two kinds of cognition then the singular judgement relates to the generally valid one as unity relates to infinity, and hence is in itself essentially distinct from it. A71(B97)

[9] Note 'individual' is etymologically linked to 'indivisible'.

Another way of saying this is that individual objects are fully determinate, i.e. every predication made of such an object should be either true or false. In this respect individual objects are distinct from concepts. As Kant notes

> Every *concept* is, as regards what is not contained in this concept itself, indeterminate, and subject to the principle of *determinability*: viz., that of *every two* predicates contradictorily opposed to each other only one can belong to the concept. This principle rests on the principle of contradiction, and hence is merely a logical principle that abstracts from all content of cognition and has in view nothing but its cognition's logical form.
>
> But every *thing* is, with regard to its possibility, subject also to the principle of *thoroughgoing determination*, whereby of *all possible* predicates of *things*, insofar are these predicates are compared with their opposites, one must belong to the thing. This principle rests not merely on the principle of contradiction. For besides considering the relation of two predicates that conflict with each other, the principle considers every thing also in relation to *possibility in its entirety*, i.e. to the sum of all predicates of things as such. A571–2 (B599–600)

In a footnote Kant says that the *determinability* of every *concept* is subordinated to the **universality** (*universalitas*) of the principle of the excluded middle between two opposite predicates; but the *determination* of every *thing* is subordinated to the **allness** (*universitas*) or the sum of all possible predicates. Now it follows from this that in order to know a thing completely (from its concept) one would have to run through all possible predications; but since this thing is posited as an independent existent and ground of truth it is also thought as already determinate in all respects. But as Kant points out, this is a totally unusable concept, it is purely formal (an idea of reason), it will never serve as the basis for identifying an object. Nonetheless it does point up one of the logical functions of the concept of an object. Kant is insistent that because this determinateness of things brings with it a presupposition of the entire field of possible predication, it belongs to transcendental, not general logic, because it arises not from comparing predicates, but the thing in itself with the sum of all possible predicates. It means that if an intuition is treated as presentation of an object, it must be thought as presentation of something which *is* determinate with respect to a range of possible predications, whether or not we know which of a pair of contradictory predicates is to apply (bi-valence). Effectively one could say that Kant is distinguishing between the principle of excluded middle and that of bivalence, and noting that *universal* assertion of excluded middle requires justification from bi-valence. Kant, however, does not allow excluded middle as a universal logical principle because he believes there will always be uses of reason beyond the realm of objective truth and because the totality of all possible predications is itself, an indeterminate and unusable totalisation of reason.[10] (More will be said about the illusions of rea-

[10]We could also note that this is the point picked up by Brouwer, and also made by Michael

son in section 8.) It also entails that one way of specifying a range of objects (things to be counted as particulars) is to delimit in some way a range of possible predications (properties, qualities) in respect of which they are determinate, an idea which underlies the concept of a state space, used in the physics of complex systems [Auyang, 1998, Ch. 3].[11]

We now see why the category paired with singular judgements is not unity, as one might have supposed, but totality. However, once the singular categorical judgement is recognized as a distinct form, giving the founding logical character of an object (given in intuition, not through a concept) it has to be interpreted not as asserting concept containment but in a way which treats predication as attributing to an object some characteristic which may, or may be found in it. It then follows that (a) all the already recognized forms of judgement must be reinterpreted so that their truth is regarded as grounded in the objects to which their subject terms refer (rather than in intensional relations between concepts) and (b) other forms of judgement will also have to be recognized, for it is also necessary to think through how quality, relation and modality may be manifest in the case of singular judgements.

We already see why disjunction must be added to the list of forms of judgement. The very characterization of thing in relation to possibility in its entirety involves recognizing disjunction as a form of judgement. A thing falls under exactly one of a group of concepts which are mutually opposed and which between them exhaust the sphere of possibility. The disjuncts have to be mutually exclusive but together must divide without remainder an antecedently given domain of possibility; the components form a relational system where what is true of one part has consequences for what is true of the other parts.

Let us first go back to categorical judgements to rethink the interpretation of affirmative universal and particular judgements. The (transcendental) reading Kant gives to universal judgements is of interest in that it comes quite close to the reading Frege was to use. Universal judgements in relation to the domain of objects are treated via the way they license rules of inference, as rules saying under what condition predication (judgement) can be made about something already given. 'All men are mortal' tells us that any object which can be subsumed under the condition of being a man, can be judged to be mortal. In other words, that for any given object x, judgement that x is a man is a sufficient condition for the judgement x is mortal.[12] (This is very close to the intuitionist reading of the universal quantifier proposed by Dummett [1977, Ch. 1].) The 'given' here is important, for there is no license for thinking of all possible objects; indeed it is

Dummett in his characterization of realism [Dummett, 1977].

[11]Kant himself goes a considerable way toward working out this kind of conception in his *Metaphycial Foundations of Natural Science*, when he tries to think how to work with objects given solely as a system of objects given by their relations to one another.

[12]This reading is suggested also by a note in the Jäsche Logic. 'An example of a *synthetic* proposition is, To everything x, to which the concept of body $(a + b)$ belongs, belongs also *attraction* (c). Synthetic propositions increase cognitions *materialiter*, analytic ones merely *formaliter*. [Kant, 1992a, p. 607]. Ak. IX §36, 111.

precisely the logical contours of the concept of an object that we are trying to explore.

> Thus in the conclusion of a syllogism we restrict a predicate to a certain object, after having previously, in the major premise though it in its entire range under a certain condition. This complete magnitude of range in reference to such conditions is called *universality* (*universalitas*). To it corresponds in the synthesis of intuitions *allness* (*universitas*) or totality of conditions. A 322 (B 379).

Here we see explicitly the switch required in moving from intensions to extensions. From the point of view intensions the function of the universal is to unify, from the point of view of its application to objects, it is to totalise (just as the individual object has the function of totalising the realm of concepts). So although this does not amount to fully extensional reading of the universal affirmative categorical judgement, it is one which places the universality with the totality of objects, the rule is valid for every object, rather that limiting it to the totality of the subject class, and it introduces conditionality into the connection between subject concept and predicate concept. However, to express an objective truth, the subject term must not be empty.

The particular affirmative categorical judgement, 'Some men are bald' once thought as grounded in the domain of independently existing objects rather than concept relations is not just a matter of the logical possibility of adding 'bald' to the concept 'man', but of there actually being men who are bald.

In the case of the disjunctive judgement. Leibniz again gives Kant the model for what would be required of a realm of particulars, with complete (infinite) concepts. These concepts represent, as it were, the limits of division of all being. Each concept of an individual is maximal so that if there were to be any change in the concept of one, it would entail change in all the others (this is the preestablished harmony between the windowless monads). Such individuals are positions in a coordinated system and so although all their determinations are internal, they nonetheless form an organic whole. If we take away the complete concepts and retain merely what this says about objects, what we have is the concept of objects forming a reciprocally coordinated whole (B 112–13). The hypothetical judgement thought as a relation between singular judgements, asserts a relation of consequence (determination) between either states of an individual object or of the state of one object and that of another.

Once an asymmetry is recognized between subject and predicate terms in a categorical judgement, with the subject term referring to the objects about which the predication is made, negated terms need to be distinguished from unnegated ones (as Aristotle, Boethius and others in the despised scholastic tradition had recognized). Affirmation of a negated predicate does not classify an object; it merely serves to exclude it from a class. For example, any non-living thing could truly be said to be non-warmblooded, so to assert of some living thing that it is non-warmblooded is merely to relate it to an undelimited range of non-living things; it gives no characterization of its being as a living thing. In Kant's words such judgements are infinite (in the sense of indefinite) in their range, their function is merely to limit the intension of the subject concept. But here we also see that attributing a characteristic to an object is interpreted not as expressing agreement between ideas, but as saying something about the objects mode of being, its reality. Further, at the conceptual level the opposition between affirmation and negation is what, through contradiction, limits conceptual possibility. Contradictory concepts are logically opposed to one another, and to bring them together is to create a necessarily empty concept. Kant suggests that if opposition is to have a grounding in the reality of objects then there must similarly be a real opposition between features of reality such that their coming together results in an empty intuition (his example is the action of equal and opposite forces on a single object which produce no effect).

Finally, general logic as concerned only with relations between concepts knows only two modalities; that of possibility/impossibility associated with problematic judgements, and the logical actuality of the determination of one concept by another, the limits of the actuality being set by contradiction. But when the domain of existing objects becomes the ground of truth, there must be a distinction between conceptual possibility and real existence (of objects). And from the points of view of objects, conceptually determined relations (things capable of proof) appear as necessities. The complete table is now:

Quantity
Universal —sameness —unity
Particular—difference—plurality
Singular totality

Quality
Affirmative—agreement—reality
Negative —opposition—negation
Infinitive limitation

Relation
Categorical —intrinsic —inherence
Hypothetical—extrinsic—causality
Disjunctive community

Modality
Problematic—determinable —possible/impossible
Assertoric —determination—existent/non-existent
Apodeictic necessary/contingent

Figure 4.

The categories are purely formal concepts. Their role is somewhat like that of a specification a possible model for a theory written in first order predicate logic. The first thing that needs specification is a domain of quantification, a universe of objects. We are not told how to recognize an object in any such domain nor how to recognize any other component of the model. The difference is that Kant is not talking about just a limited formal system, but potentially about all objects of thought. However, his arguments of the transcendental analytic and dialectic are to the effect that any interpretation must be within a limited domain (he limits it to the domain of experience, objects are possible object of experience). This particular limitation has its source in his insistence that experience is the only source of content for concepts. But the idea that there has to be some limitation has a more structural, logical, source in argument that objects need to be given in intuition, that is in some reference frame which is independent of concepts. This has to to with his arguments rejecting Leibniz' metaphysics of monads, and his conception of reason as always able to conceptualize beyond the bounds of understanding (judgement), and thus setting goals or tasks for knowledge, which it would not merely be a mistake to believe already determined in some reality, but would lead reason into conflict with itself.

8 ILLUSIONS OF REASON

The distinctive features of Kant's conception of reason were outlined in section 3. Reason in its logical use, makes inferences. Kant's account of inference sees all reasoning as a matter of making an application of a universal rule, by subsuming a cognition under the condition of the rule. Each possible major premise (universal categorical, hypothetical, disjunctive) thus is seen as expressing a relation between a condition and what it is a condition of (the conditioned). From this Kant draws his account of the pure use of reason (A306 (B363)), noting first that in its logical use it does not deal with intuitions, but only relates judgements, and second that it seeks the universal condition of a judgement. Thus the principle of pure reason, in its logical use 'is to find for understanding's conditioned, the unconditioned whereby the cognition's unity is completed.' (A307 (B364))

Kant points out that if this were taken as a principle, saying that for every series of conditions there is an unconditioned limit, it would be a synthetic one. For while it follows (is an analytic truth) that whatever is conditioned has a condition, it does not follow that there is an unconditioned (that would serve as the condition for everything). ('For all x there is a y such that Rxy' does not entail 'There is a y such that for all x, Rxy'.) Kant calls the concept of the totality of conditions for a given conditioned the transcendental concept of reason, and naturally he suggests that there must be as many kinds of pure concepts of reason as there are kinds of relations of condition to conditioned. But in addition the concept of the totality of conditions is an empty one; there is and can be no object which falls under it. It is a *focus imaginarius* which draws us on to find more conditions, to further unify our knowledge; but it does not draw us to an already sealed fate; we have to make our own way. We are deluded when we mistake the ideal for the real.

The unity of reason, Kant says, is the unity of a system (A 680 (B708)). Its goal is systematic cognition and this consists in coherence based on an ordering principle. This in turn presupposes an idea, the form of a whole of cognition, a whole that precedes any determinate cognition of its parts and contains conditions for determining a priori for each part its position in relation to remaining parts (A 644 (B645)). It is a mistake, however, to think of this idea as the idea of an actual existing object (such as knowledge in the mind of God), an independent reality in which the system exists as already completed. The idea of a completed system is one which should only be posited as a heuristic to guide the search for knowledge. What Kant wants to block is the move from grasp of logical structure, expressed in concepts, to a dogmatic metaphysics postulating objects of which there can be no empirical knowledge (are not possible objects of experience).

What is of interest however is that in the process he has a lot to say about the logic of infinite series, ordering relations and ordered structures, and of the inconsistencies to be encountered if one attempts to treat the domain of all objects as itself an object. (This was the problem that bedeviled, Frege, Russell and others working on the logicist and set theoretic foundations of mathematics in the late nineteenth and early twentieth centuries.) Kant's thought here is guided by work in eighteenth century mathematics with its development of techniques for use in association with calculus, many of which concerned infinite series, questions of infinite divisibility, and so on. For example he says 'of every series the exponent of which is given.... can be continued' (A331 (B388)) And as Reich [1992, p. 76] has noted, in a mathematical context the exponent is the *ratio* of a series, i.e. the coefficient of proportionality between one term and the next (e.g. the exponent of the relation of 3 and 12 is 4 or 1/4), so given this information one could continue the series 3, 12, 48, 172... or the series .... 3/16, 3/4, 3, 12). By analogy Kant is treating the relation of condition to conditioned as the exponent of a series so that once one such judgement is given, it should be clear how that series should be continued. And it would be continued by constructing chains of syllogisms of the relevant kind, taking one back from condition to condition of the condition, etc. In this Kant anticipates the idea that a logical system can be treated as a generalized arithmetic — a structure which can be recursively generated.

What Kant recognizes is that once a term as starting point is given together with a specification of how one term in a series is to be related to, or found from, another, it is known that the series can be continued indefinitely and, although never completed, is nonetheless in a sense given in that we can form a concept of the series. An analogy that he uses is the series of progenitors of a person. If we take it that every person P has two biological parents and that we can judge the relation of parent to offspring, we can think of forming the series which starts from P, goes to P's parents, then to their parents, and so on. We imagine that if we keep going in this way we would get the total collection of P's progenitors. This series could only come to an end if there were some non-biological being or beings which produce biological human offspring without having itself (themselves) to have parents. Each would be an unconditioned condition of every member of the series, but as such could not, on pain of contradiction, be thought to be a person, and could not itself be a term in the series. If the series has no such end, we still feel we have a conception of the series in its totality, on the basis of knowing the relation which links one term to the next. Yet the concept of this totality does not serve us as ordinary concepts would, for it does not give any indication of how, given an arbitrary person X, we could, by inspection of X, determine whether he/she was or was not a progenitor of P. The only way to do this would be by tracing P's family tree and locating X on it, but having failed to find X on it as far as we have gone back, cannot tell us that X isn't there. Of course this example is not a perfect analogy for the kind of abstract system Kant is talking about, because with generations we can always use time as a way of ruling a lot of people out (if we know P's parents, we can rule out all their contemporaries as not being amongst P's progenitors). Additionally if there were some feature of P's DNA that had to be carried by each of P's progenitors, but could not be shared by anyone who was not, then we would have a test applicable to anyone, for membership of this series, which would work independently of determining their location in it.

This example is formally analogous to the problem of determining whether a given sentence in a formal system is or is not a theorem of the system. Knowing the system's axioms and rules of inference, we know how to generate sequences of theorems, but this doesn't in and of itself yield a decision procedure for theoremhood. Sometimes, with a very simple system (such as Hofstadter's MIU system [1979, Ch. 1]) there may be a morphological trait (number of occurrences of a given symbol, perhaps) which is characteristic of axioms and is transmitted through use of the rules of inference, and thus to all theorems. This then allows theorems to be detected independently of their derivations. This was the idea behind Hilbert's formalist approach to tackling problems in the foundations of mathematics. But, as we now know, even first order predicate calculus lacks such a decision procedure.

It is interesting to note that Frege also encountered exactly these issues in his attempt to reduce arithmetic to logic by defining numbers as extensions of concepts. He identifies the need to be able to have numbers defined and identifiable as objects independently of their location in the series generated from 0 by repeated addition of a unit as the crux of his attempt to render arithmetic truths analytic. Since he has not defined numbers via their generation in a series, Frege finds that he needs to prove that every number has a successor. To do this he defines the relation 'y follows in the ϕ series after x' and he is deliberately trying to replace the idea that this is specified by saying that it will be the case if starting from x and running through the ϕ-series we finally reach y. What he says he wants is

> a criterion which decides in every case the question 'Does it follow after?' wherever it can be put; and however much in particular cases we may be prevented by extraneous difficulties from actually reaching a decision, that is irrelevant to the fact itself.
>
> We have no need always to run through all the members of a series intervening between the first member and some given object, in order to ascertain that the latter does follow after the former. [Frege, 1959, p. 93]

What is the definition which is supposed to do the trick? In modern notation it is

y follows in the R series after x iff
$\forall F(\forall z(Rxz \rightarrow Fz)\&\forall z(Fz \rightarrow \forall w(Rzw \rightarrow Fw))) \rightarrow Fy)$

As has frequently been observed, one problem with this definition is that it is (to use Russell's terminology) impredicative, i.e. it is defined by reference to a totality (that of all hereditary properties) to which it will itself belong, since the property of following in the R series after a will be an R-hereditary property possessed by every R-relative of a. How then is this to provide, even in principle, a criterion which 'decides in every case' whether something does or does not follow in the R-series after a? By trying to use broad totalising definitions to avoid recursive definitions of series, and by treating relations simply as predicates of pairs (or n-tuples) of objects, Frege ignores all of Kant's caveats. He freely treats all concepts, even those defined using quantifiers over objects and over concepts, as if they defined corresponding objects (classes), and ends, foreseeably from Kant's perspective, in contradiction.

From one perspective one could say that Frege was justified; he was following what Kant acknowledges as a dictate of reason

> Reason's principle is that if the conditioned is given, then the entire sum of conditions and hence the absolutely unconditioned (through which alone the conditioned was possible) is also given. A498 (B526)

Kant allows that if one were to take a transcendental realist attitude, treating conditioned and conditions as things in themselves then the condition would have to be already given with the conditioned. In which case the unconditioned, whether as the totality of the series, or as a limiting term (and unconditioned condition) has been presupposed.

He divides the Antinomies into two groups according to whether they relate to illegitimate extension of the use of a mathematical or a dynamical category. The mathematical categories are those pertaining to quality and quantity and the first two Antinomies concern the infinite extent and infinite divisibility of space and time. The dynamical categories are those pertaining to relation and modality and the third and fourth Antinomies concern causality. The logical point relating to this division is that because Kant does not take a transcendental realist position on space and time, his resolution of the first two Antinomies is to say that space and time are not completed (infinite) wholes, so it is inappropriate to try to assign any magnitude to their extent (A522(B550)). Similarly he denies that they can be thought to be actually infinitely divided (composed of either infinitesimals or points) (A526(B554)). This allows him to deny the applicability of excluded middle (because bivalence fails); space is *neither* finite *nor* infinite in extent, time *neither* has *nor* fails to have a beginning. Space and time exist as we know them solely through relations (involve a synthesis of the homogeneous), thus the times series extends back indefinitely and space is unbounded. Similarly, regions of space and time are given as continuous wholes capable of indefinite division, but given before their parts and thus not presuming the independent preexistence of those parts. Since they are not *composed* of indivisibles, it is *neither* true *nor* false that the limits of division of space and time are extended. In these cases Kant thinks this to be the only way to avoid contradiction.

In the case of the two Antinomies concerned with causality Kant allows that a transcendental realist can consistently postulate the existence of causal agents lying outside the realm of empirical causation, and thus as acting as uncaused causes (a creator of the world, or an agent with free will). Kant will not, admit that any such things can be known to exist. What makes the logical difference on his account is that in this case the series of conditioned to condition is not a series of homogeneous, but of heterogeneous items. A cause does not in all respects resemble it effects and can be identified independently of them. The concept generating the series thus already allows within it the possibility of difference in kind. This resolution allows that it is at least logically possible to assert *both* that the series has no limit (there is no beginning in the chain of empirical causation) *and* that the whole series had a cause (itself uncaused), by disambiguating the meaning of 'cause'. This is the structure of resolution that has been used by mathematicians when constructing transfinite numbers. There is no largest natural number, but if we change what we mean by number (in particular drop the idea that all numbers are generated from 0 by repeated applications of the successor operation), we can supposed there to be a number which comes next after all the natural numbers.

The fact that in these cases what many had taken for contradictorily opposed propositions have to be shown not to stand in that relation, leads Kant to insist, in the Transcendental Doctrine of Method, that all proofs in transcendental logic should be direct, and should not employ *reductio ad absurdum*. (A790 (B818)) This is because transcendental logic is concerned with grounding truth in objects. If there can be no object corresponding to a given concept Fx, then generating a contradiction from an attempt to employ the concept (say Fa) can be no license for concluding that the negation of the concept can be applied (not-Fa). There is no truth about objects to be expressed using this concept.

Whether one looks to realist or non-realist resolutions of the Antinomies, Kant's arguments in the Antinomy of Pure Reason suggest, one will have real problems (like running in to contradictions) if one tries to treat the totality of all objects as just another object. And this is exactly what the set-theoretic paradoxes demonstrated. Set theory no longer attempts this feat.[13]

Whether or not one accepts Kant's way of limiting understanding to experience, his dialectic of reason is instructive.

> Such a dialectical doctrine will refer not to the unity of understanding in experiental concepts, but to the unity of reason in mere ideas. But

[13]One way of resolving this issue, and the method adopted in Gödel–Bernays set theory is to distinguish between sets and classes. Sets are objects which can belong to other collections; classes however are not.

> as synthesis according to rules this unity is still congruent, first with the understanding, and yet as absolute unity of this synthesis it is to be concurrent simultaneously with reason. Hence if this unity is adequate to reason then its conditions will be too great for understanding, and if the unity is commensurate with the understanding then its conditions will be too small for reason. And from this there must arise a conflict that cannot be avoided, no matter how one goes about doing so. A422 (B 450)

He points out that the problem here is that there is a potentially ambiguous concept; there is understanding's always finite but potentially infinite series — a generative concept of series (analogous to a generative definition in mathematics). Then reason comes along and treats this concept as the concept of a completed totality. From within the standpoint of understanding the series is never complete. The problem is that reason's concepts are 'too large' because the means available (on understanding's definition of them) for determining whether something does or does not fall under it are inadequate to the task. Reason tempts us to frame the idea of a limit to a recursive process—a largest number, a first cause, a limit of division and to think of it both as limit and as being the same kind of thing as the members of the series. But if the limit is the same in kind as the members of the series, then we know how to go beyond it (hence it is not a limit). The alternative is to treat the limit as something lying outside the series, as different in kind and hence something which can never be reached by working through the series.[14] Thus a point is a limit of division of a line but it is not a part of the line, and no division of a line, no matter how prolonged, will arrive at a point. In this way Kant points to the capacity of logic, once treated as dealing with forms of judgement about things, or objects, to frame concepts which we could never actually apply. These are ideas of reason, 'the power which prescribes to understanding the rule of its complete use' [Kant, 1996, A574]. But of course, his point it that we will never be able to complete the use of our understanding.

[14]There is a striking analogy between the structure of this situation and the strategy which underlies proofs of two of Gödel's major results — his incompleteness theorems for arithmetic and his demonstration of the consistency of the continuum hypothesis and the axiom of choice with the axioms of Zermelo-Fraenkel set theory. 'In each case a fundamental property of predicates is introduced — in the former instance, that of being primitive recursive, in the latter, that of being 'absolute for the constructible sub-model' — ....in each case one crucial notion – that of being provable in formal number theory, or of being a cardinal number within a model of set theory — conspicuously fails to have the specified property, and the rest of the proof hinges on that failure. In particular, central to both proofs is a distinction between internal and external points of view: in the incompleteness paper, between mathematical and metamathematical notions; in the set theoretic consistency proofs, between those functions that exist within a given sub model and those that exist outside it' [Dawson, 1997, p. 61]. It is known that Gödel was fully familiar with Kant's philosophy as a young student and that although dissenting from many of Kant's views, acknowledged his influence. Kant had, of course, no knowledge of the particular problems created by later mathematicians, but might still be credited with having outlined the global structure of the framework in which they would be played out, a structure on which Gödel was to draw.

Another example, which we have in a sense already seen, is the question of the existence of noumena (logically complete objects). From the framework of general logic, working only with concepts, individuals, as ultimate subjects of predication, are not definable. A genus may be divided into species, and then into subspecies (this is the series generated by successive disjunctive syllogisms) but there is never a place to stop. As Leibniz knew any finite conjunction of concepts is still a complex concept which may well be satisfied by more than one object. Kant takes it to be a logical law that since a species is always a concept, containing only what is common to different things, it can never be completely determined.

> Hence the concept also cannot be referred proximately to an individual, but must consequently always contain under itself other concepts, i.e. sub-species, must always be contained under it. A656 (B684)

In this context Kant makes an assertion which is characteristic of his position that potential infinity does not presuppose the antecedent existence of an actual infinity (as Leibniz and Descartes would have supposed, and as Cantor was also to insist.)

> One can readily see, however, that this *logical* law would likewise be without meaning and application if there did not lie at is basis a *transcendental law of specification.* This law does not indeed demand of the things that can become our objects an actual *infinity* of differences, for the logical principle, which asserts merely the *indeterminateness* of the logical sphere in regard to possible division, gives no occasion for such a demand. But the transcendental law nonetheless imposes on the understanding the demand that under every species we encounter it must seek subspecies, and for every difference must seek smaller differences. A565 (B 684)

The complete concept of an individual (e.g. Kant) is also a (humanly) unattainable limit concept. It is the limit of a series which can be conceptualized on the basis of the series, but which cannot belong to the series, and in this sense its concept (were it to be admitted) has to be a concept of a different order from those used in its construction. (The analogy here is with a point in a line, which can be approximated by taking ever smaller intervals around it, but which is itself not an interval but the limit of the sequence of approximations.) Nevertheless, both Leibniz and Spinoza had suggested that there are logical counterparts of concrete individuals having infinitely complex concepts. Kant rejects this, suggesting that these rationalists have been taken in by a characteristic illusion generated by reason. They have taken the idea that reason gives of the goal toward which we should strive in our cognitive efforts for a depiction of some already existing reality, when in fact the function of such ideas is merely regulative.

Thus Kant insists that it is an important practical principle of reason in the construction of scientific knowledge that we both try to unify theories by seeking principles of maximal generality, and at the same time seek ever more precise descriptions, making ever finer discriminations between things. At no stage however, should we think the process complete nor should we suppose that there is a preexistent reality in which, as it were, the process is already completed. What we are committed to by adopting these scientific goals is a belief that the processes can be continued indefinitely. There are two logical principles at work pushing to the systematic completeness of all knowledge. One says that starting from the genus we should descend to the manifold that may be contained thereunder. The other that starting from the manifold of species we should ascend to the genus to endeavor to secure unity for the system.

Our empirical scientific knowledge will therefore always be incomplete and capable of further improvement. The logical framework envisioned is not that of Leibniz or of the logical positivists for there are no absolutely simple concepts. The system is open ended. The point is that regulative principles and regulative ideas guide scientific enquiries, but they are the product of requirements of reason in its logical ordering, not required or validated in advance by the way the world is. More to the immediate point, there remains a complete gulf between the concepts organized within such a logical structure and individual objects presented in intuition. How is then do concepts, on Kant's account ever get applied to objects?

9 THE POSSIBILITY OF OBJECTS OF EXPERIENCE

Determining sameness and difference is to determine the kind of object we are referring to as well as how to count or measure objects of that kind. As Kant points out the identity criteria for universals are different from those for individuals. Thus the presentation of individuals in thought requires something other than concepts; these presentations are what Kant, following received usage, calls intuitions. Once intuition is recognized as a capacity distinct from understanding, Kant argues, the comparison of 'ideas', or presentations, must be systematically duplicated depending on whether the presentation is thought as presentation of an object of understanding or of intuition. On this basis he rejects Leibniz' form of the identity of indiscernibles according to which no *two* objects, such as two drops of water, can be qualitatively and quantitatively identical — there must be, according to Leibniz, some internal distinction between them, that is some concept which applies to the one but not to the other and in virtue of which their distinction is logically reflected in the complete analysis of each of their concepts. Kant's response is to say that Leibniz is confusing objects of the understanding with objects of intuition by applying an inappropriate identity criterion.

Objects presented in empirical intuition are individuated without reference to any noumenal realm. Qualitatively identical presentations may be presentations of the same object of understanding (and thus united under a concept) but distinct in virtue simply of their spatio-temporal relations to other objects. Objects presented in pure intuition are individuated solely on the basis of their extrinsic relations to one another.

Sensation yields ideas of qualities. To gain an idea of something which can serve as a logical subject (a subject of predication, an individual substance, a concrete particular) requires putting together a complex of qualities as qualities of that object — which is itself thought simply as the x which has those qualities. It is to unify the manifold provided by sensory experience in a particular way. On the other hand to form the idea of a quality (a universal, a predicate) as something which can be common to many particulars, requires a different kind of unification of the manifold of experience — grouping by similarity and difference. In fact these two different unifications are not logically independent since qualities are qualities of objects and objects as potential subjects of multiple predications cannot be thought without thinking of qualities as possessed by objects. It is for this reason that judgement takes precedence over its components. Subjects and predicates, objects and concepts, arise out of judgement and are explained only in the context of the components of a judgement. This is another version of the argument that recognition of universals requires a coordinate recognition of particulars which have their own, and different identity criteria.

How then do we succeed in thinking of and making reference to individuals, if not by listing individuating characteristics? Basically what Kant does is to reintroduce some elements of the structure of the Aristotelian framework (albeit in a completely new guise). For Aristotle the individual substance is a composite of matter and form. Forms are universal (common to things of a kind) and matter individuates amongst things of the same kind. In the Aristotelian framework prime matter is another limit concept. It is the pure potential for being enformed and thus in itself has no determinations. As Kant comments 'matter is not at all an object for pure understanding' (A277 (B333)) Kant does not invoke prime matter; instead the pure intuitions of space and time take over some of its logical functions (although somewhat confusingly these are now forms of intuition, for which sensibility provides the content) and the material substance (or just matter) of eighteenth century physics takes over others.

Space and time provide the frameworks for individuating between qualitatively identical objects; space and time are homogeneous continua and as such have the potential for indefinite division and the characterization of indefinitely many complex structures. Space and time are not objects of experience, but are the forms of sensibility within which all experience takes place, is ordered and organized. (They are *ens imaginaria*, not objects, but forms without substance, forms for intuiting but not objects intuited (A291 (B347)). Spaces and times, the parts of space and time respectively, have, in themselves no intrinsic distinguishing features; their distinctness and identity rest solely on their relation to each other. Space and time are thus whole given before their parts, and the parts form an organic whole in the sense that they have no identity beyond their location in the system, for which each depends on the others. Furthermore, no identification of parts is possible without some determination of them based on qualitative differences between the objects they contain. Qualities can serve as the marks for noting the presence of an object of a given kind but qualitative sameness and difference will not provide its identity criteria, for these one must refer back to spatial and or temporal relations as revealed through the object's relations to other objects of its kind.

Logically we have two types of cognitions to consider and to think about how they relate. As Kant argues, the whole system of concepts, the apparatus of general logic, or the functioning of understanding as they are portrayed, presupposes the existence of a manifold whose manyness, whose structure as a manifold, is not a product of conceptual discriminations marking inherent differences between its parts. It is a domain whose objects and the complexes they can form is of a logically different order from that captured in general logic. But as a manifold, it has to have a structure and for us to recognize it as manifold we have to be able to represent it to ourselves as a many comprising one system. Kant makes this point as follows:

> A presentation which is to be thought as common to different presentations is regarded as belonging to presentations that, besides having it, also have something *different about them.* Consequently it must beforehand be thought in synthetic unity with other presentations (even if only possible ones). Only then can I think in it the analytic unity of consciousness, which makes the presentation a *conceptus communis.* And thus the synthetic unity of apperception is thus the highest point, to which we must attach all use of the understanding, even the whole of logic.... indeed this power is the understanding itself. fn B 134

Just adding a manifold of intuition to a field of concepts is not enough to provide a basis for knowledge of individuals. We have to be capable of recognizing the multiplicity of the manifold, i.e. to recognize it *as* manifold. It thus has to be unified in consciousness under some concept. Thus the same synthetic unity of apperception underlies both giving unity to a combination of representations in a judgement and giving unity to a combination of representations in intuition. If we can recognize and give labels to (form concepts of) distinct forms of combination in judgement we should be able to do the same thing for combination in intuition.

Kant, aware of the directions taken in natural science, knew that the mechanistic approach to understanding natural objects and events is to discern their parts and to seek to understand the whole on the basis of the way it is made out of parts. In other words, in order to reflect the reasoning of the natural sciences it would seem that one needs to be able to represent objects as complex and to be able to reason about objects on the basis of that complexity. In his *Metaphysical Foundations of Natural Science* Kant works out in some detail the 'logic' of a system of objects that are relationally constituted; something which few others have done.[15] The sense in which an object can be composed of parts is quite different from that in which a concept can have parts, and the operations for constructing complex objects are not directly mappable by corresponding operations for putting together

[15]So it is not perhaps surprising that when Sunny Auyang [1998]came to write her book on the foundations of complex systems theory, it was to Kant that she turned for conceptual assistance.

concepts of the parts into a complex concept. By failing to make this distinction Leibniz was led to insist that material objects are not real because they don't have the simplicity required of ultimate logical subjects, as anything spatially extended is divisible.

Thus Kant recognized that operations of construction and unification go on both at the level of concepts and at the level of objects and that they have different structures. Operations within the imagination lead to the structuring of complex objects – intuitions of complex objects. There may be a concept of this complexity but the corresponding concept will not have been put together by operations of conceptual compounding. Thus besides distinguishing between intuition and concept, Kant also distinguishes between reasoning from concepts and reasoning from the construction of concepts as concepts of complex objects, i.e. as the product of constructive operations productive of intuitions of those objects.

> With analytic cognition I make a given concept distinct. Synthetic cognition gives me the concept simultaneously with distinctness. The *mathematicus* makes concepts with distinctness. The philosopher makes concepts distinct......In the synthetic [act] I add new marks to the concept of an object through experience. It arises in such a way that the parts of the cognition precede the whole cognition. In the analytic [act] the whole precedes the parts. [Kant, 1992a, p. 299]

Here Kant has moved beyond traditional logic in that he recognizes ways of defining concepts other than by conjunction (in definition by genus and differentia). What he does at the level of understanding as well as at the level of reason is to recognize that generative rules produce structure and that structures themselves can become (abstract) objects of theoretical knowledge. In so doing rules can, give, from another point of view, a priori knowledge of anything generated in accordance with them. This holds both at the level of generation in space or time (spatial or temporal structure mathematically characterized) and at the level of reasoning to generate cognitive structures. (Thus the ability to engage in the study of formal logic as a logic which abstracts from all content, looking only at forms of combination, itself requires recognition of particulars which are not qualitatively marked, but are identified only by their mutual relationships in a structure. So this knowledge is itself knowledge through the construction of concepts. Kant acknowledges that the principles of pure understanding (such as, that every event has a cause) derived from the necessary applicability of the categories to possible objects of experience, are 'anything but cognitions from concepts'. They rely on recognition of the constitutive role of the categories in relation to experience, they generate not actual objects, but the 'space' of possible objects.

In this way Kant moves beyond the conception of logic as concerned with rules or principles of correct inference to a concern for logically structured systems of knowledge, and thereby rendering it closely akin to mathematics and its concern for knowledge of the structures of systems. So, for example, if it is a principle of reason to seek to explain by reference to previously accepted general laws, one can anticipate that theoretical sciences will exhibit a structure in which there is an explanatory hierarchy of laws ordered by their degree of generality. In fact Kant is more concerned with deductive systems as systems than with developing rules of deduction.

What Frege and the logical positivists tried to do was to incorporate the distinction between object and concept, as well as the logic of (external) relations into a single, expanded system of logic. The price for this unification is loss of any recognition of the need for domains whose parts are individuated solely by their mutual relationships, and whose relations are thus internal, constitutive relations.[16] Logical atomists, in particular tried to accomplish the kind of flattening of logical space that Kant was resisting. They wanted to reduce everything to states of affairs, facts, or propositions, and their logical relations. A principle of this reduction was to reduce any statement about an apparently complex object (such as a class of objects) into a logically complex statement about its simple components. (See [Whitehead and Russell, 1962, p. xxxv] and [Wittgenstein, 1961, p. 3.144–3.24].) The problems encountered by these programs suggest that Kant was on target in his diagnosis of what went wrong when Leibniz attempted to conduct a similar exercise on the basis of a different logic.

Objects and concepts each have their own 'formal frameworks', which in many respects work as duals to each other. How are they made to articulate over one another, in such a way that concepts can be used to form judgements about objects? This is perhaps the most innovative of Kant's moves, and one which was only partially picked up when Frege taught us to think of concepts, by analogy with mathematical functions. Kant repeatedly says that concepts are rules, that application of concepts is a matter of judgement, and that judgement is what it takes to bring a case under a rule. Moreover, he emphasizes the fact that general logic contains, and can contain, no rules for the exercise of such judgement, for that would presuppose knowing how to apply a rule. In fact 'A person may understand a universal *in abstracto* but not be able to distinguish whether a case *in concreto* comes under it.' (A134 (B173)) And correcting this is a matter of training to develop the judgemental skills, not of giving more rules. So at the center of the articulation there lies the rational being with its capacity for judgement and for reflection on its exercise of that judgement.

What are concepts rules for doing? They are rules for the representation of objects in intuition.

> Thus we think a triangle as an object, in that we are conscious of the combination of three straight lines according to a rule by which such an intuition can always be represented. This unity of rule determines the manifold. (A106)

[16]The battle over external relations was a battle fought out at the beginning of the twentieth century between Russell [1924, p. 159] and Bradley [1914, p. 280]; Russell won Bradley lost and the dominance of the language of predicate calculus makes it difficult to reopen the question.

> It is schemata, not images, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. For it would never reach to a concept's universality that makes the concept hold for all triangles (whether right angled, oblique angled, etc.) but would always be limited to part of this sphere. The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space. Even less is an object of experience or an image thereof ever adequate to the empirical concept; rather the concept always refers directly to the schema of the imagination, this schema being a rule for determining our intuition in accordance with such a general concept. The concept 'dog' signifies a rule according to which my imagination can delineate the figure of a four-footed animal in a general manner, without limitation to any single determinate figure such as experience, or any possible image that I can represent *in concreto*, actually presents. (A142 B181)

Kant also talks of a schema as a method of presenting something in accordance with a certain concept. Here he is again taking his cue from mathematics, where, especially in the eighteenth century, concepts were defined using generative definitions, saying how an object of the kind in question is produced, e.g. by the motion of a point according to a given rule (expressed an algebraic equation). There is an important difference between the case of mathematical and empirical concepts however. In the mathematical case, since mathematical objects are not material, their construction according to a schema is the construction of the object, the method of construction is what defines the object, not merely the concept. But for empirical concepts, the schema generates a schematic, incomplete image of an object. Experience is needed to assure us that there are objects falling under the concept, and any empirical object will have more specificity than is contained in the schematic image; i.e. the empirical objects are not (and cannot be) defined by the rule which gives the concept of an empirical object. There are indefinitely many modes of access to empirical objects for these are objects which we will never know completely (never capture fully in concepts). This is why the method of philosophy (science) cannot be that of mathematics, and why scientific knowledge will never attain the same kind of certainty that is available in mathematics.

What Kant sees is that the concept of a rule can bridge the gap between universal and particular. A rule is general; can be repeatedly applied, yet each application has a particular outcome. A rule for producing something will confer certain common characteristics on the items produced (the more mechanized the more uniform as mass production has shown). Imagination then is the crucial faculty which links sensibility to understanding. It is the reproduction of a schematic image, to be overlaid on present experience, or on another reproduction to yield recognition in a concept. The account of application of a concept 'in concreto' is then very close to that given for inference using a universal categorical major premise. A concept is after all a universal. It provides the rule. Its schema, as it were provides the condition. Judgement that the case in hand satisfies the condition licenses application of the concept, results in judgement that one is perceiving an object of such and such a kind. This is as it should be on an account on which objects are never directly presented in experience, but are always as it were inferred, and referred to the space-time framework within which they are individuated.

The objects of mathematics are, however, immaterial. Repeated application of the same constructive rule gives exactly the same result for there is not question of material instantiation (this is left for applied mathematics). Nonetheless mathematics is about operations on these non material objects and the structures they form; it is not (syllogistic) reasoning from concepts. This is why it is, on Kant's view, incorrect to classify mathematics as consisting of analytic judgements. Grasp of definitions in mathematics already requires grasp of a synthesis (method of construction) for an object, and it is this which grounds mathematical truth.[17]

In so far as the space-time framework and the framework of categories and concepts, required by the logical forms of judgement contribute to (are constitutive of) the identity of possible objects of experience, each gives rise to synthetic a priori principles. Mathematical principles true of the structure of space and time (at least those required for the individuation of objects and events) play a constitutive role *vis à vis* experience because they supply identity and individuation criteria for things of a given kind. But they aren't in themselves sufficient for knowledge of empirical objects. Space and times, or spatio-temporal regions or structures are not objects of any possible experience unless determined and delimited by qualities which can be made observable in some way. Having two sets of forms which work as duals of each other, and which have to be combined, forces an indeterminacy at their nexus (the possible object of experience); neither framework can be reduced to the other, hence the objects of experience similarly remain stubbornly multidimensional, they cannot be reduced to something fully knowable within the terms available from either framework. In this sense Kant could be read as insisting that any single, purely formal framework for reasoning will be incomplete.

The irony is that Kant has been criticized for his claim that we can have a complete knowledge of logic, or of the forms of judgement, when in fact the whole underlying theme of his critique of pure reason is to map its structures in such as way that reason's incompleteness can be internally acknowledged. Reason must leave a space for synthetic knowledge even as it holds before us ideals of what complete knowledge might be like. Taking these ideals as goals, we have a conception of what we ought to do to improve our state of knowledge.

[17] Whether Kant would still take this position after the burgeoning development of infinitistic methods in mathematics is doubtful. Certainly if intuitionists and constructivists are those whose faithfully represent the continuation of a Kantian position on mathematics in the changed setting of currently mathematical practice, one who have to say that Kant would not take those infinitistic methods as yielding certainty and would restrict mathematics too to the use of constructive proofs.

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HEGEL'S LOGIC

John W. Burbidge

Although two of Hegel's works are called Logics, he is seldom considered a major figure in the history of logic. For the *Science of Logic* and the *Encyclopedia Logic* do not focus on the terms, propositions and syllogisms that make up the bulk of logic textbooks, but range through a number of concepts that sound more like traditional metaphysics, such as being, quality, quantity, essence, actuality, teleology and life. In addition, the discussion is cloaked in a dense and obscure language that appears to abandon the traditional conventions of argument that start from accepted premises and move on to justified conclusions.

Nonetheless Hegel has much to contribute to an understanding of logic as the science of reasoning. For Hegel's professed aim was to use thought to examine its own processes of thinking — the logic, he says, is thought thinking itself. He announced that he intended to examine the assumptions and immediate inferences that all too often are simply adopted without examination as the arsenal of reflective thinking, and thus assumed to be so self-evident that any reasonable person could be expected to agree. Hegel's objective was to think back over these implicit presuppositions and trace how they develop one from another. His project, then, was to provide a systematic study of the processes that characterize all rational thought.

From 1812 to 1816, while serving as headmaster of the classical gymnasium in Nürnberg, he published the *Science of Logic*, a three-volume detailed analysis of these basic rational moves. While he interspersed the dense analytic description with remarks, which traced connections between such fundamental rational concepts and more conventional disciplines — mathematics and science, the history of philosophy and religion — or even ordinary experience, the crux of his argument is found in the basic text: abstract, compact, and using few illustrations to ease the way for the reader. Then, when he moved into a secure university post at Heidelberg in 1817, he needed a more accessible handbook for his students; so he prepared what he called the *Encyclopedia of the Philosophical Sciences*, which was a series of short succinct paragraphs that would serve as theses upon which he could expand in his lectures. The first third of this recapitulated the earlier *Science of Logic*, but the outline form of the theses was not able to reproduce the careful development of thought as it moved from concept to concept in the earlier work.

For the purposes of this chapter I shall use the longer text, and endeavour to show how Hegel sought to uncover the fundamental presuppositions of all thinking. I shall proceed in the following manner: First I shall suggest both how Hegel

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arrived at his conception of what a speculative logic ought to be from a careful study of the first edition of Immanuel Kant's *Critique of Pure Reason*, and from the way Johann Gottlieb Fichte tried to complete Kant's project; and how this project developed during his early years as a university lecturer in Jena, up to the point of his *Phenomenology of Spirit.* I shall then turn to the beginning of the logic, both tracing the most primitive logical moves and using them to illustrate his systematic method. In less detail I shall suggest how this initiative continued through the first two volumes, called the Doctrine of Being and the Doctrine of Essence, pausing where appropriate to explore some of his analyses further.

Of particular interest for the history and philosophy of logic is what Hegel does in his third volume, on the Doctrine of the Concept. So I shall concentrate on how he organizes and characterizes concepts, judgements and syllogisms, suggesting how the more complicated forms are introduced to address the limitations of earlier forms. Of equal importance is Hegel's claim that thought conceives objectivity in mechanical, chemical and teleological ways; and his further claim that fully comprehensive thought (what he calls the idea) unites conceiving and objectivity into a single conception. This discussion culminates in his treatment of method.

I shall then turn to three supplementary questions: the relation between the longer and the shorter logic; the changes Hegel made when he revised both these texts for second (and in the case of the *Encyclopedia*, third) editions; and how the logic relates to other philosophical disciplines that reflect on nature and human society. As a conclusion I shall discuss how Hegelian logic developed in the Anglo-Saxon world during the nineteenth century, and some of the ways in which it is currently being interpreted.

1 THE BACKGROUND TO HEGEL'S *LOGIC*

The understanding of logic that underlies Hegel's philosophy can be traced back to a critical move in the first edition of Immanuel Kant's *Critique of Pure Reason.* When he turns from sensible intuition to the work of understanding, Kant uses the traditional forms of logical judgement as the guiding thread or clue to the discovery of all pure concepts of the understanding. Whereas sense and introspection present intuitions, the understanding is discursive and uses concepts. Concepts, Kant says, are *functions* which serve to *unite* diverse representations under something common. He then suggests that one and the same function not only unites subject and predicate in the various types of logical judgement but also unites the multiple offerings of intuition, for the function as function operates irrespective of the material it operates on.[1] Kant goes on to distinguish the act of synthesis — of holding diverse representations together in a single cognition (which he attributes to imagination) — from the act of uniting, which collapses the synthetic diversity into a single, isolatable thought or concept (and is the work of understanding).[2]

[1] This argument is in both the first and the second edition of the *Critique*: A67–9, B92–4.

[2] While this can be found in both editions at A76–9 and B 102–4, it is continued into the chapter on the transcendental deduction only in the first edition. By the second edition, Kant

While the twelve types of logical judgement provide the fundamental categories of the understanding, Kant notes that a number of other categories could be derived from those twelve - his examples are 'force', 'action', and 'passion' from causality; 'presence' and 'resistance' from community; 'coming to be', 'ceasing to be' and 'change' from the categories of modality. (B108) And much later, at the end of the *Critique* he defines the metaphysics of nature as "all the principles of pure reason that are derived from mere concepts and are employed in the theoretical knowledge of all things" and the metaphysics of morals as "the principles which in an a priori fashion determine and make necessary all our actions." (B869)

On one point Kant is insistent. The uniting function of conceiving cannot be derived analytically from the content united. The spontaneous activity of understanding brings it to the content imagination collects.

There are several important points to notice in this discussion. In the first place, the uniting concepts of the understanding provide the foundation for the standard types of logical judgements as much as for the organization of intuitions. They are, so to speak, the explanation of the structures of formal logic. In the second place, the unity of conceiving is to be distinguished from imagination's synthesis, which only collects diversity into a single perspective. In the third place, Kant himself undertook the further elaboration of a priori concepts into a metaphysics in his own works: the *Metaphysical Foundations of Natural Science*, and the *Metaphysics of Morals.*

Kant's successors, although impressed and excited by this probing into the ground of rational thinking, found his overall theory unsatisfying. There seemed to be something contingent and positivistic about working from a simple table of judgements that had emerged from the history of logic to determine what are the basic categories of the understanding. For all that Kant claimed that these exhausted the full range of fundamental principles, he had not shown *why* these twelve and just these twelve should be given pride of place. Further, if the understanding is spontaneous, in contrast to the receptivity of sensible intuition, what is it that makes thought determine itself to unite content according to these specific functions.[3]

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argues that the conceptual unity is the condition for synthesis. It is the first version of the transcendental deduction, however, which inspired the romantic interest in the role of imagination, and I am suggesting here that its distinction between the synthesis of imagination and the union of conceiving was also of significance in Hegel's thinking as he began to investigate what a science of logic would be like. He changes it (and thereby separates himself from the romantics) by indicating how thinking itself (and not imagination) generates synthesis.

[3]See here Hegel's Introduction to the *Science of Logic*: "What has here been called objective logic would correspond in part to what with [Kant] is transcendental logic. He distinguishes it from what he calls general logic in this way, (α) that it treats of the concepts which refer *a priori* to *objects*, and consequently does not abstract from the whole *content* of objective cognition, or, in other words, it contains the rules of the pure thinking of an *object*, and (β) at the same time it treats of the origin of our cognition so far as this cognition cannot be ascribed to the objects." (*Hegel's Science of Logic*, [SL] tr. A.V. Miller, 62; Hegel, *Gesammelte Werke,* [GW] 21, 47) He went on to write: "But if philosophy was to make any real progress, it was necessary that the interest of thought should be drawn to a consideration of the formal side, to a consideration of the ego, of consciousness as such, i.e. of the abstract relation of a subjective knowing to an

It was Johann Gottlieb Fichte who, using Kant's transcendental method, suggested that the spontaneous self-positing activity of the I or ego — Kant's transcendental unity of apperception — not only underlies but itself constitutes the category of reality; that this act conditions a second, oppositing activity that is nothing but the category of negativity; and that these two condition a third, limiting activity, which is the category of limitation. From these three he proceeded to derive the other nine categories; thus extending back into the basic twelve that kind of derivation which Kant had proposed for metaphysics in the *Critique.*[4]

It is against this background that we can turn to Hegel's logic. He is interested in deriving pure concepts, the functions of thought that unite diverse representations under a concept; but unlike Kant, he is interested not simply in the uniting of sensible intuitions and imaginative syntheses but how conceptual functions integrate representations and conceptions generated by thought itself.

In working out how this might happen, Hegel found inspiration in Plato's dialogue, *Parmenides.* In that setting the aged Parmenides takes the young Socrates through a series of dilemmas, showing that, of contrary descriptions, neither of them, once isolated by understanding, can legitimately be predicated of "the one". Such negative *reductios* anticipated Kant's antinomies of pure reason, in that they also reveal how presupposing one contrary leads perforce to its opposite, while starting with the other reverses the shift. "This skepticism," writes Hegel, "does not constitute a particular part of a system, but it is itself the negative side of the cognition of the absolute, and immediately presupposes reason as the positive side."[5]

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object, so that in this way the cognition of the *infinite form*, that is, of the concept, would be introduced. But in order that this cognition may be reached, that form has still to be relieved of the finite determinateness in which it is ego, or consciousness. The form, when thus thought out into its purity, will have within itself the capacity to *determine* itself, that is, to give itself a content, and that a *necessarily* explicated content — in the form of a system of determinations of thought." (SL 63; GW 21, 48) Thus the objective logic which completes the transcendental logic needs to be supplemented by a logic of subjectivity or of conceiving,

[4]See the "Fundamental Principles of the Entire Science of Knowledge," in *Fichte: Science of Knowledge*, ed. & tr. P Heath & J. Lachs, New York: Appleton, 1970, 93–119; *Fichtes Werke*, hg. I.H. Fichte, Berlin: Gruyter, 1971, I, 91–123. Once again it is instructive to note what Hegel writes in his Introduction: "If other disciples of Kant have expressed themselves concerning the determining of the *object* by the ego in this way, that the objectifying of the ego is to be regarded as an original and necessary act of consciousness, so that in this original act there is not yet the idea of the ego itself — which would be a consciousness of that consciousness or even an objectifying of it — then this objectifying act, in its freedom from the opposition of consciousness, is nearer to what may be taken simply for *thought* as such. But this act should no longer be called consciousness; consciousness embraces within itself the opposition of the ego and its object which is not present in that original act. The name consciousness gives it a semblance of subjectivity even more than does the term *thought*, which here, however, is to be taken simply in the absolute sense as *infinite* thought untainted by the finitude of consciousness, in short, *thought as such.*" SL 62–3; GW 21, 47–8.

[5]"On the Relationship of Skepticism to Philosophy," in *Between Kant and Hegel*, translated by G. di Giovanni and H.S. Harris, Albany: State University of New York Press, 1985, 323; GW 4, 207.23–5.

Kant had argued that one cannot derive the categorial concepts from the content provided by intuition. However, where one has two contrary intellectual transitions in which opposites, once analysed reciprocally lead into their antitheses, one has a synthesis that is not the product of imagination, but a function of reflective thought itself. A leads to not-A; and not-A leads back to A; the two are parts of a single pattern. New categories can then emerge where these negative syntheses are positively united in a new concept. A discursive function of unity is thus applied, not to the alien content of sensation and introspection, but to the products of thought's own thinking. A logic which derives concepts in this way requires a careful, reflective isolation of the different intellectual operations, noting their distinctive features and identifying how they are derived one from another.

To write about the processes of thought must use language that captures conceptual meaning. Illustrations and examples quite often draw attention to incidental details rather than to the intellectual functions and shifts of meaning that are Hegel's primary concern. So Hegel develops his argument in the abstract language of concepts. This results in a density and obscurity in Hegel's texts which inevitably pose problems for his interpreters. In addition, because his focus is on the discursive dynamic of reasoning, he endeavours to show how terms shift in meaning as reflection turns its attention to their various components. Rather than relying on conventional syllogisms, then, he offers instead an analytical description of how meanings shift and coalesce.

On the basis of these observations, the following discussion uses as its guiding thread the principle that the language Hegel is using names and describes intellectual functions — not only those that unite, but also those that discriminate and those that shift (or pass over) from one thought to its counterpart or completion. By thought he means at the very least that "common store of thoughts" which humans have "transmitted from one generation to another."[6] (So in the following exposition, at times I shall talk as if the meaning of a concept itself requires the

[6]The phrase is from G. Frege's "Sense and Reference," in *Translations from the Philosophical Writings of Gottlob Frege*, ed. P. Geach & M. Black (Oxford: Blackwell, 1966) 59. To talk about the logic as thought thinking through its own processes might appear to fall afoul of the charge of psychologism as originally made by Frege. But Frege, in making this charge, draws a distinction rather similar to the one made by Hegel between pure concepts and representations in general. "In order to be able to compare one man's mental images with another's we should have to have united them into one and the same state of consciousness, and to be sure that they had not altered in the process of transference. It is quite otherwise for thoughts; one and the same thought can be grasped by many men. The constituents of the thought and *a fortiori* things themselves, must be distinguished from the images that accompany in some minds the act of grasping the thought — images that each man forms of things." [Review of Husserl's *Philosophie der Arithmetik* in *Zeitschrift für Philosophie und phil. Kritik*, 103 (1894), 317–8; translation from *Translations from the Philosophical Writings of Gottlob Frege*, ed. P. Geach & M. Black, Oxford: Blackwell, 1966, 79.] In "Sense and Reference" he again draws a distinction between what he calls an idea or internal image and "the sign's sense, which may be a common property of many and therefore is not a part or a mode of the individual mind. For one cannot deny that mankind has a common store of thoughts which is transmitted from one generation to another." [Ibid. 59] It is the "act of grasping the thought" or the "sign's sense", the common store "transmitted from one generation to the next", which is the focus of Hegel's attention, not the transient and idiosyncratic images that accompany them.

move to some other thought; at times, however, I shall suggest that it is the way "we" think, reflect or conceive. Both forms are meant to refer to the same kind of process.)

It took some time for Hegel to develop his analysis of self-thinking thought to the point of publication. We have a manuscript he was preparing in 1804–5 on Logic, Metaphysics and the Philosophy of Nature, but which was never completed. In it he began to distance himself from a method, adopted by Kant in his *Metaphysical Foundations of Natural Science* and Schelling in his *Ideas for a Philosophy of Nature*, in which thought constructs new terms by breaking them up into their constituent parts and rearranging them in appropriate ways. Construction according to this method is balanced by a proof, which involves using empirical evidence to show that the construction legitimately applies in the world of experience. Although Hegel still used a version of this method in his Philosophy of Nature of 1804–5, he abandons it for the logic itself, where he works with the more abstract approach of describing the conceptual functions of thought.

Before turning to his full *Science of Logic*, however, Hegel wrote a *Phenomenology of Spirit*. This work, which defies traditional categories of philosophical literature, shows how confident claims to knowledge break down when they are consistently put into practice, requiring new approaches to knowing that have learned from that failure. As experience accumulates through a series of such moves, human thinking incorporates into its own functions what it has learned from putting its ideas into practice in the real world. The result of this experiential accretion, Hegel suggests, is that, when we do reflect on pure thought thinking through its own operations, we are not simply working in a realm independent of the concrete realm of particulars (a charge often made of Plato's ideas), but rather with the distilled essence of human experience as it has developed over the millennia. Into this thinking or conceiving are resolved all the experienced forms of consciousness.[7]

This identification of thinking with the essence of human cumulative experience adds a further dimension to Hegel's logic. For it means that the concepts pure thought thinks, as well as the transitions it makes, capture the structures and processes of reality. Traditional metaphysics had used reason to determine the nature of being per se. By showing in the *Phenomenology* that human reason has been educated over the ages by its experience of the world and society to embody the patterns and structures of reality, Hegel can claim that what pure thought discovers as it works through its own thoughts are not only the logical principles underlying all thought, but also the metaphysical principles of whatever is.

[7]Again from Hegel's Introduction: "In the *Phenomenology of Spirit* I have exhibited consciousness in its movement onwards from the first immediate opposition of itself and the object to absolute knowing. The path of this movement goes through every form of the *relation of consciousness to the object* and has the concept of science for its result. The concept therefore (apart from the fact that it emerges within logic itself) needs no justification here because it has received it in that work, and it cannot be justified in any other way than by this emergence in consciousness, all the forms of which are resolved into this concept as their truth." SL 48 (Throughout I have altered Miller's "Notion" to "concept" to make clear the relationship to Kant); GW 21, 32.

It is thus possible to read Hegel's logic as a reworking of traditional metaphysics. Since this chapter is part of a *Handbook on the History of Logic*, however, it will develop an interpretation based solely on the pure operations of thought: thinking, reflecting and conceiving.

2 BEING AND IMMEDIATE INFERENCE

Since Hegel wants to show how the various concepts are systematically related, he must start out his logic with the conceptual function that is most indeterminate, and so can be used to unite the widest range of terms. This, he suggests, is 'being'. The verb 'to be' can be predicated of anything whatever without discrimination and on its own cannot serve to distinguish irrational or imaginary concepts from real ones. Pure thought finds, therefore, it is really contemplating nothing. With this realization, thought has moved from the simple concept 'being' to what one would think of as its opposite: 'nothing'. 'Nothing' as well is completely indeterminate; yet in being thought it is also a conceptual function. So it has the same defining characteristics as 'being'. Conceiving has thus moved back to the term with which it began.

Since conceiving is a spontaneous function of the intellect, it can reflect on the total dynamic that has thus emerged. 'Being' and 'nothing' are radical opposites; yet they appear to have identical characteristics. To be identical yet opposed is contradictory, so thought must find an explanation or ground which will resolve the paradox.[8] This emerges when we realize that with 'nothing' thought has *pass over* from 'being', and then with 'being' thought has *passed over* from 'nothing'. This move of passing over is a conceptual function and needs to be identified. The appropriate term in ordinary language to name it is 'becoming'.

There is no simple becoming, however. The movement from 'being' to 'nothing' should be called 'passing away', while the move from 'nothing' to 'being' involves more specifically 'coming to be'.

At this point Hegel makes his most distinctive move. Recall that, for Kant in the first edition of the *Critique*, the *unity* of the concept is to be distinguished from the *synthesis* of the imagination, even though the latter sets the conditions for the former. When in this instance thought (not imagination) synthetically combines 'passing away' with 'coming to be', we find ourselves with a circle: 'being' passes away into 'nothing', then comes to be from 'nothing', only to pass away again, and again come to be. The simple transitions of thought, by reciprocally leading into each other, have become not only a synthesis but also a self-perpetuating circle. Conceiving that self-contained dynamic now as a unity and separating it from the thinking that led up to it so that it can be thought immediately produces a new concept: of a kind of being that involves coming to be and passing away.

[8] As we shall see again, contradiction is not the end of reasoning, but rather the clue that some conceptual function is operative that must be isolated and identified.

For this concept Hegel adopts the German term '*Dasein*', sometimes translated 'determinate being', but at this point much more indeterminate.[9]

The episode so far described has traced the first chapter of Hegel's *Science of Logic.* In it we can notice several important features that are characteristic of his method.

First, of central importance to his project is the way thought reflectively focuses on its own functions. In the first of these moves, reflection looks back over an intellectual transition that has taken place and brings its various components together in a synthesis. This happens in its most primitive form with the simple moves from 'being' to 'nothing' and vice versa; but it happens again with the synthesis of the two kinds of becoming: 'passing away' and 'coming to be'. This move of synthesis, however, leads on to a second reflective manoeuvre, which can take two forms. On the one hand, reflection can isolate the simple movements of thought when it passes over [in German *übergehen*] from one concept to its contrary. What starts out as a contradiction is resolved by recognizing that thought is not static but dynamic — in Kant's terms, not receptive but discursive. Such functions of passing over can then be identified as concepts in their own right. On the other hand, where the moves are reciprocal and lead into a recurring cycle, conceiving can collapse the synthesis into a new functional unity and (because the process that led up to this unity can be ignored) arrive at the definition of a new *immediate* concept. For this final discursive act of unification, in which a new conceptual unity leaves behind the mediation that led up to it, Hegel adopts the German term '*Aufheben*', which in its conventional uses may have several meanings: to retain, to cancel and to go beyond. By uniting the recurrent cycle into a unified concept, he suggests, thought thus *cancels* the earlier mediating interplay, even though much of its determinate meaning is *retained* and *raised up* into a more sophisticated meaning. Since English does not have an obvious counterpart for *Aufheben*, some translators have adopted from Latin the verb 'sublate' which could contain similar senses, and which has on occasion become a technical term in Hegelian discussions. Whatever it may be called, however, this uniting and collapsing function is critical to Hegel's systematic project.

The second feature worth noting is how the movement of this chapter benefits from some comments at the beginning of the shorter, or *Encyclopedia Logic.* There Hegel provides a preliminary outline of his logical method, and distinguishes three sides. In the first, understanding abstracts and remains with fixed determinations and distinctions. In the second, dialectical, moment a particular fixed determination dissolves itself and passes over to its opposite. The third, or speculative, moment grasps the unity of the determinations in their opposition, the affirmative that is contained in their dissolution.

When we apply this threefold schema to the first chapter, where Hegel takes us from pure being to *Dasein*, we can notice how understanding seeks to define, first 'being", then 'nothing' and finally 'becoming'. At each point it endeavours

[9] One could adopt Paul Tillich's suggestion and distinguish pure being (in German *Sein*) from a being or beings (*Dasein*), although this does not capture the generality of the latter term.

to fix the distinctive characteristics. The moves of passing over (or *Übergehen*) from 'being' to 'nothing' and vice versa would be the work of dialectical reason, in which the act of thinking through what is meant in the initial definitions has led willy-nilly over to something that looks like its direct opposite. Finally, the reflective move in which the various moments of a process are held together in a synthesis is the work of speculative reason. Collapsing that synthesis into a unity, and divorcing it from the thinking that led up to it so that it becomes a new immediate concept, is once again the work of understanding.

By making explicit the conceptual functions of passing over (or *Übergehen*), of reflective synthesis and of unifying (or *Aufheben*), Hegel sets the stage for a systematic treatise in which a new concept surfaces as the name for an intellectual function that has itself emerged from previous concepts. He recognizes that the content synthesized need not simply be presented through sensible intuition, as Kant had claimed,[10] but rather arises within the processes of thinking itself. The choice of the appropriate term to match any new definition involves looking for a suitable word in conventional language, for it is in human language where "the forms of thought are, in the first instance, displayed and stored".[11] But the term taken over does not bring with it all its conventional associations; it needs rather to be carefully defined, and distinguished from apparent synonyms. Only then will the development be rigorous and scientific.

The first chapter, because of the indeterminacy of its beginning, is rudimentary and basic. As the *Logic* proceeds, terms become more sophisticated, but they do so only gradually and as the argument requires it. The concept which ended the first set of moves, *Dasein*, is still rather imprecise. But when it is compared to the original concept, pure being, it is seen to contain an element of nothingness or non-being within its meaning, so that it is qualified. A qualified being could be called something; but the thought of something inevitably evokes the thought of something else, or other. And this leads into considering the determinations that are to distinguish them from each other. These defining features become limits, where something would become qualitatively something else whenever the limit is passed. Such a limited being is in fact finite, and the limit is a barrier beyond which it would cease to be. Whatever is beyond the determinate limit is to that extent qualitatively indeterminate or infinite. But once one starts to think about any such a beyond, it becomes itself determinate, and so limited and finite. Thought has thus moved from the thought of the finite, to the thought of an infinite beyond, only to have the latter become something finite in its turn, for which there is another beyond ... and so it proceeds in an infinite qualitative progress. In other words, 'infinite', which started as simply meaning an indefinite beyond now acquires a second meaning: of an ongoing, repetitive process, in which the finite in general does not disappear but persists (and is to that extent infinite)

[10] Although some argue that, in the *Philosophy of Nature* and *Philosophy of Spirit*, new concepts must integrate not only logical meanings but also the results of empirical observation.

[11] SL, Preface to the Second Edition, 31; GW 21, 10.

and the infinite can only be defined as the opposite of some finite (and becomes to that extent limited and finite).

Hegel has thus reached a cyclical pattern that resembles, yet is more complicated than, the reciprocal moves from being to nothing and back again. The concept 'finite" leads into 'infinite', and 'infinite' leads back to 'finite'. The two moments of finite and infinite are reciprocally interconnected in a conceptual synthesis where each constitutes and requires the other. In fact this whole synthesis could be called 'infinite' in a third sense: a self constituting process which both generates finite moments and at the same time goes beyond them. But if we do not simply hold the two moments in a synthesis but conceive the synthesis as a unity, we have reached a new immediate concept: the thought of a self-contained being, or a being on its own account. For this Hegel uses another German term *Fürsichsein*, which is translated literally as 'being-for-itself'.

In the second phase of his *Logic*, then, Hegel has been able to derive such terms as 'quality', 'something' and 'other', 'determination', 'limit', 'finite' and three senses of 'infinite'. He then moves on, under *Fürsichsein* to introduce units and atoms, the void, many, repulsion and attraction, showing the categorical underpinning of both Lucretius' atomism and Kant's construction of matter from attractive and repulsive forces.[12]

By considering the concepts of attraction and repulsion on their own, abstracted from the qualities of the entities involved, Hegel moves beyond considerations of quality into the concepts of continuous and discrete magnitude, which then become refined as intensity of degree and quantity of number. Since any number is a discrete moment of a continuous series, Hegel is able to return to the concept of infinity, now in its strictly mathematical sense as a series in which any determinate member has a successor. Having been a teacher of mathematics both at the University of Jena and in the Nürnberg secondary school, he then adds some remarks on the theoretical foundation of the infinitesimal calculus.

In any such infinite series, any two specific quanta are related to each other as a ratio or proportion, but the limit which distinguishes them from each other (which mathematicians call the exponent) functions not as part of the continuous or discrete magnitude, but as a qualitative determination. Thus, while the analysis of quality through *Dasein*, infinity and being-for-itself led to the concept of magnitude, reflection on the implications of quantitative concepts leads back to the concept of quality. Once again we have a reciprocal pattern which brings together contrary concepts. The synthesis of quantity and quality can then be united in the concept of measure, under which quantities are used to determine and define qualities.

The concept of measure as well undergoes development, from using a simple ruler with conventionally defined units, to using ratios between quantities. At first these may be between measurements of abstract features (in the way units of space and time are used to measure velocity or acceleration), but they can become more sophisticated as ratios between two qualities of a single thing (as in specific

[12]See Kant's *Metaphysical Foundations of Natural Science.*

weight or the elements of a chemical compound). Since, as we have seen, any number is a discrete member of a continuous series, one can think about altering the quantitative ratios in any real thing to see how this transforms its quality and change it into something else. But such a move has a surprising implication, for it means that one can no longer simply appeal to the surface quality or being of something to determine it exactly. One must look beneath the surface to discern its underlying reality or essence.

With this move Hegel completes the first book of his *Logic*, which he has titled "Being". In it he has looked at the way thought moves forward in simple transitions (or immediate inferences) from something to other, from quality to quantity, from mere magnitude to number, and from simple measurement to using ratios. Now he has introduced a second mode of thinking — one which distinguishes between what simply happens and its underlying essence or ground. The second book of the *Logic* explores the logical categories involved in this kind of thinking.

3 ESSENCE AND THE LAWS OF THOUGHT

As we might expect, the logic that explores the meaning of 'essence' turns out to be more complicated than what has gone on before. There thought simply moves over from one thought to its complement or contrary. An essence, however, is thought of as distinguished from what is immediately present. At first it might appear as if the immediate being and the essence are simply two independent terms. But when we look more closely we see that the immediate shows itself to be transient and inessential, requiring the move to something more substantial underlying it.

Hegel turns to this immediate show. It is rather like the Hindu concept of *maya*, under which the world of experience is thought of as an illusion, hiding yet suggesting what is ultimate and essential. So we have a reciprocal relationship between the essence and this show. The essence is understood to be the otherness of the surface being; yet this being is *maya*, not to be considered in its own right, but rather as both veiling and revealing what is genuinely important. In a sense, then, the essence is nothing other than *maya* understood as *maya*.

Here we have been introduced to a distinctive pattern of thought. Earlier we moved from being to nothing, and then moved back again. But now we remain with essence as we distinguish it from *maya*, and we remain with the *maya* as it both shows and veils the essence. Each is reflected in the other. This mode of thinking we call reflection.[13] The next move in Hegel's logical progress explores what is involved in reflection.

Reflection takes its immediate starting point to be something inessential, and its role is to dissolve that surface show so that thought can reach what it hides, which is the essence. The whole process is defined by negatives: the starting point is a nonentity; the act of thought dissolves itself; and the result is other

[13] In Hegel's German, the word used for show or *maya* is *Schein*, a cognate of the English "shine", so that the logical move to reflection is prefigured in a word play of ordinary language.

than the original given. What holds these three negative moments together is the presupposition that there is a reality, lying beneath the surface, which is posited as genuinely immediate in contrast to what is presented directly in the process of thought. So, to grasp what is really unmediated, reflection must not only dissolve the immediate surface show, but also cancel the effects of its own mediating activity. It thinks of itself as standing outside of the material it is thinking about.

The fact that reflection, to be effective, must cancel the mediating effects of its own activity has an impact on the way it functions. For the changes introduced by its activity cannot be allowed to mediate or influence the characteristics of the essence that is to emerge from the process of reflecting. That essence has been presupposed as genuinely immediate. In other words, the task of reflection is to remove from the surface being that initiates its activity those inessentials which are in fact not genuinely immediate. To do this it must presuppose some determination that essentially characterizes the real immediate, and which can be discerned as both shown and veiled in the surface *maya*. These fixed determinations are required to ensure its success. So reflection is not only external to the material it is thinking about; it relies as well on reflective determinations that must be both constitutive of the essential to be discerned and regulative for its own operations. So Hegel moves on to explore what he calls the determinations of reflection.

The determinations that both define the essence and constrain the activity of thinking are unmediated, and so to be distinguished from the mediating activity of reflection. They persist and are presupposed by thought. Indeed, precisely because they are not to be the function of any synthesizing intellectual activity, there may be a number of such determinations, compatible, yet unrelated to each other.

The result is a paradox. For reflection has now two conflicting functions. On the one hand it is actively to reflect on the surface *maya* to derive the underlying essence. On the other hand, in this process it is to use fixed and persistent determinations that are unaffected by the dynamic of thought, but are nonetheless its critical component. Each function requires the other. For reflection as such cannot discern the essence unless it uses the determinations; and the determinations can only be effective if they become the norms that regulate the reflective activity.

At this point, rather than reaching a reciprocal cycle that can be united into a single new concept (as happened in the earlier part of the *Logic*), thought has come up with a fundamental paradox: two features which are explicit opposites, both of which are yet constitutive of a single intellectual operation. The way forward involves looking at the determinations of reflection — at the laws which determine how thought should function — to see what they explicitly involve.

Before moving on to review what Hegel has to say about these laws of thought, let us reflect on what he has been doing so far in this discussion of essence. He has been investigating what Kant would call the categories that underlie a common, and fundamental, type of thinking. Whenever we look for an explanation we are showing that we are not satisfied with the surface show of things, but are interested in determining what is their essential ground. Such is the procedure, not only of all religion, but also of all natural science and all social theory The reflection that this quest involves follows the pattern that Hegel has exposed: It is presumed that the given is not the real picture — indeed that it may be fundamentally misleading. It is presumed that the act of thinking about this puzzle will not affect or influence the real explanations that are to be the results of the endeavour. And it hopes to be successful because it accepts as given some fundamental principles that not only govern all thought, but are understood to be constitutive of what is essentially the case.

For there is far more to thinking than simply uniting subject and predicate. Thinking may also involve reflecting: searching for explanations, looking for underlying principles, questing for unchanging truth. If it is true that the modern world has stopped talking about essences in any ontological sense, it nonetheless continues to look for what is really going on underneath natural and social phenomena. And this quest presupposes what Hegel calls the categories of essence, show (or *maya*) and reflection.

But there is something else to notice. Earlier we suggested that the simple transitions of thought — from being to nothing, from something to other, from finite to infinite — were parallel to the work of what Hegel calls dialectical reason. Speculative reason in contrast brings together into a synthesis the contrary features that have emerged through dialectic in their opposition. It looks for the affirmative that is contained in their mutual dissolution and transition into each other. (Cf. *Enz*, §82) Although we have not yet followed through its logic to the end, reflection would seem to have parallels with this description. It takes up opposition — between essence and show, or between persisting determinations and its own dissolving activity — and holds them together in a synthesis, if not yet a unity. It looks for the affirmative essence that is contained in the dissolution and transience of mere surface show. That parallel between reflection and speculative reason will only become more explicit as we explore further the determinations of reflection.

Since reflection presupposes that the essence to be discussed remains consistent despite surface variations, the most basic determination it relies on is the law of identity. What is essential are the patterns or characteristics that remain the same. Yet identity only makes sense when contrasted with difference. Similarly pure difference requires reference to some kind of identity. So thought must consider how identity and difference may be brought together in a synthesis.

The way things are identified may have nothing at all to do with the way they are differentiated. We then have a diversity, in which one can discern through comparison how some things are alike and others are unlike. When we turn to this relationship between likeness and unlikeness (or equality and inequality) we find an opposition, for like and unlike are the positive and negative sides of a single reflective act of comparison. Sometimes when we distinguish opposites as positive and negative it is a matter of indifference which we call the positive and which the negative as long as they are opposed to each other in a single perspective (a perspective which Hegel calls the "positive-and-negative"). But if we are going to get at what makes some things essentially positive and others essentially negative we find that we must go further. For the positive is positive because it excludes (negatively) the negative; and the negative positively constitutes itself when it excludes its positive opposite. So each of them are both positive and negative at the same time.

The reflection that draws these inferences about the genuinely positive and negative is affected by the same paradox. For as a single act it must exclude from itself these contradictory entities, yet it must at the same time positively affirm them as contradictory. So the result is no longer simply an opposition, but a contradiction, in which what is denied is affirmed, and what is affirmed is denied. Such a kind of reflection in effect destroys that upon which it stands, and so it must "fall to the ground". With this metaphor Hegel suggests that when contradictions emerge they cannot be maintained; rather reflection must investigate their underlying ground or reason. So the ultimate law of thought is the principle of sufficient reason, in which reflection determines the ground of explicit paradoxes.

Hegel has, then, outlined the categorical basis for the laws of thought and how they can be derived one from another. The law of identity leads into the principle of diversity, contained in Leibniz' assertion that no two things are fully identical with each other. This then passes over into the basis for arithmetic's distinction between positive and negative numbers. The fact that contradiction can be shown to emerge from reflection on opposition does not mean, as Popper claims, that "Hegel's intention is to operate freely with all contradictions."[14] Rather our philosopher claims that any effort to analyze the surface features of anything to get at its essence will lead to contradictory results. And it is this very contradiction which generates the need for an explanation or ground. Not only does the law of non-contradiction hold. Because it holds, one must take seriously the nature of such contradictions when they emerge if one is to have any success in discerning the underlying essence which they veil.[15]

Hegel then moves on to explore the various ways in which reflection thinks of a grounding relationship. One starts from the simple distinction between the essence as ground and its form as grounded; but this leads on to the relation between matter as indeterminate potential and form as its determination. When reflection recognizes that the distinction between matter and form is really its own doing, then it thinks of their union as content in contrast to the form which it introduces by abstraction.

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[14]See *The Open Society and its Enemies*, (London, 1966) II, 40.

[15]Because both traditional and modern logics work with fixed logical constants and variables, they set aside the transitions of thought involved in analysis, implication and reflection. For these are dynamic, rather than static. Hegel's dialectic attempts to surmount this shortcoming by retaining the process in the understanding of its result. Thus it can see how one thought has led over to its opposite, creating a contradiction; and it recognizes that this contradiction, because it has arisen in some kind of inevitable way, must be thoroughly explored to determine its ground.

One now thinks of ground and grounded as in content the same, with only their form distinguishing them. But since the two are not really the same, thought must be able to think what happens when the grounding relationship introduces real changes. In that case, then, ground and grounded will have identical content in some respects, although in others they are different. For a complete explanation, we must not only explain the way the two are identical, but what it is in addition that introduces the difference.

One now has several factors combining to ground a single thing, each one functioning as a condition for the differentiated result. A condition, however, is not a complete ground. It is rather an immediate being which serves as only part of the material for a full process of grounding. And this process of grounding in turn is an immediate way of assembling an appropriate set of conditions to produce a grounded result. A complete ground, then, requires both the conditions and the relationship they have to each other. At first it might appear as if these various constituents function as immediate beings that are simply combined externally by some reflective act. But further reflection reveals that more is involved. For thought has shown that beings come to be and pass away; they are inherently transitory. Therefore immediate conditions never persist unaltered; and the grounding relationship can be understood as simply whatever dynamic of becoming occurs when a set of changing entities are brought together. When all the conditions are present, in other words, the grounded just comes to be. This whole pattern simply describes the nature of things.[16] As this nature of things works out its inherent process it will emerge into existence.

In Hegel's analysis, two things have happened here. In the first place, the transitions of simple beings are now considered as a totality. This ability to consider all the moments together is the work of reflection. But in the second place reflective thought reaches this thought of the nature of things by showing that its own thinking activity in distinguishing ground from grounded is not an essential constituent of the dynamic. It has dissolved its own mediating role.

The new content being thought is captured by the concept 'existence'. For what exists in this sense is the immediate being of what is essential. In this way Hegel takes up the traditional relationship between essence and existence. For the analysis of essence has led to the conclusion that, when fully worked out, it must come to be — it exists.

With this shift Hegel turns to examine the concept of existence. What exists is a world of mutually dependent, existing things. At first things are distinguished from their properties, but when reflective thought works through this distinction, it realizes that all it is left with are a set of interpenetrating properties which are the way the thing appears.

[16]For this sense of 'nature of things' Hegel adopts a German term, *Sache*, for which my dictionary offers as translations: 'thing', 'object', 'matter', 'legal case', 'task', 'business', 'affair', question', 'subject', and 'cause'. I have at times been tempted to translate it as 'heart of the matter', or 'real thing'. For more on *Sache* see below in Section VI.

With the concept of appearance the distinction between surface show and essence has re-emerged, this time with reference to the world of existence. The way the world is in itself (as distinct from the way it appears) is at first thought of as persisting laws; then as a world in which what is really happening is the inverted counterpart of what appears on the surface, so that appearance and reality are essentially connected. From this perspective, for example, what appears as the south pole of a magnet is really its north pole; and what appears as punishment is really a way of reintegrating the offender into the community.

The tools reflection uses to capture this relationship between appearance and reality are, first, the difference between whole and its parts; when this fails to capture the inherent dynamic of existence it appeals to force and its expression; and when even this distin ction cannot be maintained satisfactorily thought simply contrasts what is internal to what is external. However, the external, fully understood, reveals its inner essence, so the distinction between existence and appearance dissolves into a single, unified concept: actuality.

The basic sense of the term 'actuality' involves the simple identity of inner and outer, so that one need not think of it being formed from anything else. This is nothing other than Spinoza's definition of substance: what causes itself; but Hegel here appropriates another term that comes from his colleague Friedrich Schelling: 'the absolute'. In the double sense of this term, taken over from Kant, the absolute is considered to be in itself, and as valid in all respects without limitation.[17] Because it starts out from a simple identity of inner and outer, however, this concept cannot explain how differences emerge, so external reflection has to introduce the language of attributes and modes. Nonetheless, modes can only function as the essential determinations of the absolute — as the realization of its essence or possibility.

With this move, Hegel is able to turn to an examination of the modal categories: actual, possible and necessary, moving from the strictly formal sense of what cannot be otherwise to the way conditions are sufficient to make an actual possible and finally to the sense of necessity as a totality in which contingencies must emerge to become the conditions for other parts of the whole.

The concepts that comprehend how the whole necessarily determines its constituent moments are, first, substance and attribute; then, (since substance mediates the way accidents interact) cause and effect; and finally, (since the causal relationship requires an interaction between any specific cause and its attendant conditions) reciprocal interaction.

Hegel stresses the importance of this final concept, for it captures an essential moment of the whole preceding analysis. Being and nothing reciprocally interact in the concept of becoming, as do finite and the beyond in the concept of infinity, quantity and quality in the concept of measure, essential and inessential in the

[17]See *Critique of Pure Reason* B381: "The word 'absolute' is now often used merely to indicate that something is true of a thing considered in itself, and therefore of its inward nature.... On the other hand the word is also sometimes used to indicate that something is valid in all respects, without limitation."

concept of reflection, identity and difference in contradiction and ground, existence and appearance in the concept of actuality. As we have seen it is a key feature of the critical move Hegel calls sublation, or *Aufheben.*

What we have in every such pattern of mutual interaction is a totality which incorporates its constituent moments while cancelling their independence. By encompassing this variety into a single whole, it functions as a universal. And when this complex synthesis collapses into a discursive unity and is isolated from the thinking that led up to it, we have what Kant had called a concept. For a concept is a single thought which, though universal, determines itself in the very act of conceiving, defining its particular moments and features. With this shift, Hegel is able to conclude the second book of his *Science of Logic* on Essence and prepare to consider the central element of conceptual thought as such — Kant's discursive functions of unity or concepts. And, following Kant, he will investigate their critical role in determining the structure of logical judgements and arguments.

4 CONCEPT AND TRADITIONAL LOGIC

Concepts are functions, equivalent to the act of conceiving. They unite the syntheses reflection has brought together into an immediate thought. As we have seen, Hegel has reached this term by way of the thought of reciprocity: where each moment leads over to, and presupposes, its counterpart, generating a synthesis. Conceiving is this mediating, integrating dynamic, now considered as a unity; because it incorporates a number of moments into its own simplicity it is universal. The universality of conceiving at this point is not an abstraction, common to a number of particulars but unrelated to their mutual differences. Rather it unites the various moments of its own dynamic, and to this extent is both determinate (and so in a sense particular) and concrete (and so in a sense singular).

As dynamic, thinking activity, conceiving can focus on this element of determination, think of alternative possible determinations, and then unite the original universal with its contrary species into a more comprehensive, generic universal. At this point in the analysis, that higher universal is still not an abstract genus, but rather remains a self-determining dynamic that incorporates the various subordinate moments into its own integrity.

Understood in this way, the intellectual activity called conceiving is simply the way self-conscious life (what Hegel calls "spirit") thinks through its own self-constituting dynamic. Since that dynamic has united the reciprocal interaction of its determinate moments, a full comprehension of its nature requires that we look at that determinacy. In the context of conceiving, a determination is called particularity.

With this move we have two distinct moments: the concept's determination — its particularity — and its comprehensive universality. Once they are distinguished in this way the two terms function as particulars vis-à-vis each other, even though one of the particulars is called the universal. What marks them out as particulars is that they are distinguished from each other within a more general context.

Focusing on this moment of difference has significant implications. For now the universal moment is no longer comprehensive, but separated from its determinate content; in this way it becomes abstract, since it now excludes this other moment and has become indifferent to it.

Usually when people talk of determinate concepts they are referring to such abstract universals. The kind of thinking that fixes the meanings of such abstractions and proceeds to work with them as unalterable is called understanding. Frequently that is as far as conventional thinking goes. According to Hegel's analysis, however, such abstract terms are not original but the results of conceiving when, working through its own dynamic, thought focuses on its internal distinctions and separates them into independent terms. If abstractions are not the first word, though, they are also not the last. For Hegel says that understanding's 'infinite power' of separating concrete terms into their abs tract determinations and then grasping the depth of their differences, when pushed to its limits, leads over dialectically to contrary concepts. Nonetheless, understanding's process of abstraction is the way reason works when it fully determines the structure of of its own thoughts. In other words, while the immediate transitions of becoming are parallel to dialectical reason, and reflection is the counterpart to speculative reason, it is in the particularizing, or determining, activity of conceptual thought where understanding finds its place.

Particularizing, then, is the process of abstraction — of understanding. But when thought particularizes a determination and thinks of it simply on its own terms, it no longer has the thought of one particular over against another, nor does it retain any universal under which this term can be subsumed. It is thinking something singular — no longer strictly conceiving it as a universal, but referring to it as in some sense unique.[18]

Once we move on to talk about the singular we find that it plays a number of functions. In the first place the abstract universal, although universal in form, is singular in content. In the second place, the two earlier particulars which were distinguished as universal and particular concepts can be read as distinct individuals, to which set singularity can be added as a third, equivalent member. In other words, when conceiving fully determines the constituent moments of its content it distinguishes universal, particular and singular as three objects of reference, all equally singular.

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[18]It is useful to note that while Hegel has sections titled "Universal Concept" and "Particular Concept", the third section is simply called "The Singular." The unity of the concept is collapsed into something that is no longer universal but simply itself. It needs to be noted, however, that in the later *Encyclopedia Logic* Hegel claims that a totality considered as a unity can also be called the singular. The argument he sketches there thus varies from the one advanced here, which follows the larger *Science of Logic*.

It is the ability of conceiving to abstract from universal connections and focus strictly on a singular which has been operative throughout Hegel's logic in the move which he calls *Aufhebung*, when understanding takes a reciprocal movement and "collapses" it into a unity. The universality involved in the movement back and forth from 'being' to 'nothing' or from 'cause' to 'effect' is set aside and the intellect directs its attention to the resulting self-contained thought as an unmediated singular.

At the same time, singularity results in the break up of the concept. For the singular is referred to as a unit that excludes anything universal — as a 'this' that is simply related to itself. It stands radically opposed to any abstract concept, even though reference to it is the result of conceptual thought annulling its own comprehensive inclusiveness. As such a result it nonetheless contains within itself an implicit reference to the abstract universal from which it has been separated. Thought articulates this implicit relationship between a singular and its abstract universal by means of a proposition or judgement.[19]

Hegel has thus set the stage for his discussion of the formal logic of his day — the various types of judgement and forms of inference. He has done so by showing that comprehensive thought, in defining its own determinations, has to separate, and thus abstract, the universal from the determinate; and whatever is so determinate that no universal remains becomes the singular object of reference. Since the particular, although determinate, continues to be conceptual, it will be able to serve within the syllogism as a bridge between the two extremes. But all of the three moments of universal, particular and singular have lost that dynamic completeness with which conceptual thinking began, and they have become fixed individuals, each with its own determinate content; as a result they can serve as constituent terms in the judgements and syllogisms that Hegel now goes on to explore.

The most basic kind of proposition involves the simple coupling of a singular with its abstract universal: the affirmative judgement 'Socrates is human' or S is P. But this judgement does not capture the fact that the singular as singular is quite different from the universal as abstract. To express that difference we need a negative judgement: 'the singular is not a universal' or S is not P.

Negations are the way we determine a thought and distinguish it from its contraries. In negative judgements the 'not' applies to the predicate, implying that it is one particular among several and that S is in fact related to one of its mates.[20] The radical difference between the irreducible singularity of the subject, and the conceptuality of the universal predicate, then, has not been captured by this form of negation. One needs instead to apply the 'not' to the copula, and not to the predicate. The result is what traditional logicians have called the infinite judgement: It is not the case that S is P. For examples Hegel provides: "Spirit is not alkaline," and "the rose is not an elephant."

Even when it is true, an infinite judgement imparts little if any information, so reflection looks back at the connection between subject and predicate in these basic sentences to see how it can be made more instructive. In the first place, the predicate as a bare abstract universal is too empty. If we were to incorporate the implications of the negative judgement and use as predicate a universal that was determinate enough to include contrary particulars — a category or class — it

[19]Hegel here plays on words, for the German word for judgement — *Urteil* — could be read as "primordial division".

[20]This type of negation was used by Carnap and Ryle when they defined category words: when we say "this A is not blue" we are nonetheless placing A under the category of coloured.

would contribute more to the connection asserted in the judgement. In a similar way the copula, or connection between subject and predicate, need not simply assert an identity or a difference, but rather suggest that the subject is included under, or subsumed by, the predicate. The subject, then, becomes a singular existent. The resulting judgement in traditional logic would still read S is P — "the singular is universal", but its meaning is more precisely defined as a statement of membership in a class: "This is a member of the class C," or more simply the singular judgement; "this is P."

A singular 'this' does not fully capture the universality of the class that functions as predicate, so one needs to indicate this openness by including others in a particular judgement: Some S are P. But just as the thought of 'something' led over to the thought of 'something else', the particular judgement suggests its counterpart: Some S are not P. S as such includes both sets — those that are P and those that are not P. As a result thought finds itself thinking not only of all the subjects, but also of the universal that includes both P and not P. So one is led to a universal judgement: All S are U.

Just as, at the end of the discussion of the judgements of quality, the abstract predicate has come to be defined more precisely as a category with particular species, so in the judgements of quantity the subject has expanded from being an indicated singular to being all of a conceptually defined set. This means that one can now think of a judgement in which two universals — two concepts — are coupled, not as a simple association, nor as membership in a class, but as necessarily related. This is what traditional logic has called a categorical judgement. Although it may have the same form as an affirmative judgement — S is P — it presupposes a different content. In an affirmative judgement an abstract universal is simply said to inhere in a singular subject: "this rose is red"; in a categorical judgement a class is incorporated into its broader genus: "the rose is a plant."

Because of the ambiguity that remains in the form of the categorical judgement, it does not adequately express its essential conditions. For it is supposed to affirm the necessary connection between the two concepts. To capture this necessity one needs a different judgement form, which would explicitly state such a necessary link: the hypothetical judgement "if S then P". The copula or connection has now shifted from being a simple verb to showing how a condition grounds the conditioned by means of an 'if-then'.

Such a hypothetical judgement does capture the way the subject necessarily leads over to the predicate, but it leaves undefined what it is about the antecedent which makes the move necessary. Anything could be slotted into this form as long as the appropriate sequence is maintained.[21] What thought really requires is a judgement form that expresses the way one specific concept requires others. The appropriate form for this is the disjunctive judgement: A is B or C or D, which carries the attendant sense: A includes B and C and D.

[21] As more recent logic has shown, valid formal hypotheticals may include not only contrary-to-fact conditionals such as "if the moon is made of green cheese, then I'm a monkey's uncle" but also such strange statements as "if day regularly follows night, then $2 + 2 = 4$".

If the qualitative judgements defined the predicate more precisely and the quantitative judgements focused on the subject, the judgements of relation — categorical, hypothetical and disjunctive — have refined the way they are connected to each other. In a disjunction both subject and predicate are universal — the subject as a comprehensive concept, the predicate as an exhaustive listing of its particular species. Yet their identity is simply given as a categorical assertion. It does not capture the necessity of their relationship. So Hegel moves on to the kinds of judgement that define the mode of the coupling relation — the judgements of modality.

A simple assertion does not provide any warrant for why it should be said. It is a singular intellectual act, and cannot do justice to the universality of the subject that is supposed to determine the appropriate predicate. Since, as Kant argued, universals are possibles, not actuals, that universality could better be captured by a problematic judgement: S is possibly P.

No sooner is that expedient tried than thought recognizes its inadequacy. For what is only possibly P could equally well be not P. Pure possibility does not discriminate between them. So what is needed is a judgement form that does justice to the necessity involved in the universal: S must be P.

With this Hegel has completed his analysis of the traditional forms of judgement. He has shown that conceptual thought, in order to articulate its own assumptions, must construct ever more inclusive judgement forms. But the process does not stop here, for the simple assertion of a necessary coupling does not show the implicit reason for that necessity. To make that rationale explicit one must move beyond judgement to inference. So Hegel now turns to a discussion of syllogisms. Once again he sets it within a systematic, developmental framework.

In the apodictic judgement 'S must be P' we refer not only to the subject and the predicate but to the necessity of their relationship. If we want to justify this necessity we require a middle term that is related both to the subject and to the predicate. The subject is referred to as the topic of the sentence — a singular; the predicate is a general universal; the middle term, then, must share both the concrete determinacy of the subject and the universality of the predicate, so it functions as a particular. The first form of syllogism, then, is one where a particular mediates between something considered as a singular and something considered as a universal.

In the very early stages of the *Logic* a transition or becoming is identified as the most basic form of connection between two concepts, so the most obvious kind of connection here involves two transitions, first from the singular to the particular, and then from the particular to the universal. The conclusion simply collapses these two transitions into a single one — from singular to universal. Its form is the classical Barbara syllogism: "John is tall; tall things are sublime; so John is sublime."

The problem with this kind of reasoning is that singular subjects can have any number of particular determinations; and universals include any number of particulars, so it is entirely contingent which syllogism one constructs. By choosing different determinations of the singular subject, and then selecting an appropriate universal for the particular middle term it is possible to reach explicitly contradictory conclusions: "John is petulant; petulant things are abhorrent; so John is abhorrent.".

The reason for this paradox is that the immediate transitions which form the connections in the premises simply occur as singular events. In other words what is really mediating the inference is not the particular concept but something singular. To capture this sort of inference we need a different kind of syllogistic form — one in which a singular mediates between a particular and a universal. In this second syllogism the major premise is formally the same as the conclusion of the first: a singular is conjoined to a universal through a simple transition. The minor premise, however, poses a problem. For one cannot simply pass over from a particular to a singular, and so the inference cannot function transitively. Instead one must simply affirm that the two are in fact associated — that the particular happens to describe the singular. The contingency in that association allows the syllogistic form to retain the convention that the singular is the natural subject, for one can equally say that the singular happens to be particular: Some S is P. This then means that the conclusion as well can only be a particular judgement: Some S are U. The result is what Aristotle called the third figure of the syllogism, Datisi: Dogs are animals, some dogs are noisy; so some noisy things are animals.[22]

The novelty in this syllogism is the connection between the singular and the particular. This does not involve an immediate transition, but rather a reflective synthesis or association. The two descriptions are simply conjoined. But a conjunction which contains two particulars in this way is a kind of universal, although a very bare, abstract universal. Nonetheless since it has been identified as the key to the second kind of inference, its mediating role needs to be recognized by an appropriate syllogistic form: one in which a universal mediates between a singular and a particular. In the first premise, now, the particular is associated with the universal (which has the form of the conclusion of the second syllogism); in the second premise the singular is connected by an intellectual transition with the universal (which replicates the conclusion of the first syllogism). Because both the singular and the particular are associated with the same universal, the inference is that they can be associated with each other. However, as we have seen, the universal makes the connection not because of any intrinsic relationship with the particular and the singular but because of an abstract association. So the universal is left quite indeterminate. Lacking any intrinsic conceptual relationship, the two cannot be positively connected together. They can only be negatively distinguished; so the first premise must be negative, and since this exclusion defines the universal's mediating role the negative feature must be carried through to the conclusion. So we have Aristotle's second figure, Cesare: "Flowers are not dogs; spaniels are dogs; so spaniels are not flowers."[23]

[22] By converting the minor premise to "some noisy things are dogs", this becomes Darii of the first figure.

[23] By converting the major premise to "No dogs are flowers" we have Celarent of the first figure.

With this third kind of syllogism, the abstract formality of the traditional syllogism becomes explicit. For the universal that makes it work is a bare abstraction that excludes all determinate qualities. The fullest expression of how its empty abstractness functions can be found in the mathematical formula, "If two things are equal to the same thing, they are equal to each other," for here the three terms are abstracted from all qualitative characteristics and considered only in terms of their magnitude. No conceptual reasoning is involved at all.

On the one hand, this analysis of the traditional syllogism results in a bare formalism, which can tell us nothing about the determinate nature of conceptual thinking. But thought is not limited to considering the final stage of its series of inferential moves. It can reflect on them as a whole to see if there is some kind of basic principle that is implicit and essential. From this perspective one notices that the three syllogisms make a set, for each of the three conceptual determinations — particular, singular, and universal — mediates the coupling of the other two, with the result that that, in a strictly formal sense, the syllogisms justify each other's premises.

On the other hand, since the reasoning is so formal, the mediated connection asserted in the conclusion must in fact be grounded in some immediate association — an immediacy that is expressed when that type of judgement is used as a premise in the other syllogisms. Reflecting on this, thought recognizes that any effective inferential mediation cannot simply abstract from all qualitative determination but must incorporate immediate concrete content. And such content can only be provided by referring to singulars, which are both qualitatively determined, and can be counted among the members of some universal class. One needs this concrete content to establish more than a formal link between the two terms of the conclusion.

Hegel now returns to the earlier forms of the syllogism and reconstructs them, taking into account this new requirement. In the form where particularity mediates, the particular quality must be such that it specifies its individual instantiations. In addition to being predicated of a singular subject in the minor premise, then, it explicitly collects all of the individuals it qualifies in the major: All M is P; S is M; so S is P.

This inference, which Hegel calls the syllogism of allness is, however, redundant. For if we say that all plants receive nutrition continuously, and that this holly bush is a plant, then we do not introduce anything new when we conclude that this holly bush receives its nutrition continuously. That fact has already been incorporated into the major premise.

The real question posed by this form of syllogism is how we are justified in making the claim that all plants receive their nutrition continuously. For this we need a different kind of inference — induction — which (like Hegel's second figure) uses singulars, or a limited set of them, to establish the universality of the connection between a particular class and its abstract quality. Because a finite set

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Hegel notes that the only thing necessary to generate what traditional logic has called the fourth figure is the conversion of the negative and particular conclusions in the second and third figure.

of individuals are both plants and receive nutrition continuously we infer that all plants are characterized in this way.

This inference builds on the distinction between the finite set of plants that serve as its middle term and the complementary set of plants not so enumerated, but included with the former in the "all" of the conclusion. Implicit in the reasoning is the assumption that what applies to some applies to all. But much earlier in the logic of 'something', thought found that its natural inference is not from some to "all" but to something other with different qualities that becomes its contrary. A similar shift posed a problem for the particular judgement. There is, then, a basic contingency in induction, because there is no necessity when we extend a specific quality from the finite set selected for the premise to the complementary set that needs to be included in the conclusion. Yet induction assumes that the two sets are for all practical purposes identical.

Another inference is required to make this assumption explicit: an argument from analogy. In this form of inference what is known to apply to one set of individual plants, for example, is extended to other plants. Because these trees and bushes and grasses are green, then we assume that other plants, like mushrooms, will also be green. As this example shows, however, although analogical inferences are essential to the inductive extension of a predicate from a finite set to the whole class, they are themselves radically contingent. For one has no way of knowing whether the predicate one selects is essential to the class, or only contingently related to the items enumerated.

What underlies the inferences of allness, induction and analogy — which Hegel calls syllogisms of reflection — is their reliance on singular instances as their mediating term, either gathered together in a set, or taken as an individual. Since singulars, as objects of reference, are explicitly distinguished from the abstract universals used to characterize them, there can be no necessity that would legitimate any inference to new conclusions. These singulars provide only an external unity for the two extremes they are meant to mediate. What thoughts needs to effect a genuine mediation to something new is a universal concept — the kind of thought that both defines its particular determinations, and indicates the singulars to which it applies. This requires a different kind of inference — a kind Hegel calls syllogisms of necessity.

The first kind of inference that uses a fully determinate concept as the middle term is a categorical syllogism. This has the same form as both the Barbara syllogism and the inference of allness, but it is given a different content. The middle term is a substantial genus — "human" for example; the predicate is an abstract characterization of this genus, and the subject is one of its singular instances: Socrates is human; humans are mortal; so Socrates is mortal.

While the content of this inference grounds its necessity, its form still retains an element of contingency. For the singular subject, as well as the abstract universal predicate, are named independently, as if they had an unmediated being. As immediate they may have determinations other than the one specified in the middle term, which would allow them to be incorporated into other such syllogisms. In other words, the syllogistic form does not capture the necessity of the inference.

We need an argument form which explicitly expresses the necessary connection and the way it relates the independent terms. This would be modus ponens (Hegel calls it a hypothetical syllogism) in which the major premise is a hypothetical judgement: If A then B. This form of proposition asserts a necessary relationship without requiring the independent existence of its terms. The minor premise asserts that the condition, A, also has an independent existence, so that it is both mediating and immediate. B in the conclusion has been mediated by the inference, but is also ascribed immediate existence. The syllogism states that a necessary relationship has been given determinate being. But the counterpart of this claim is that the inference as a whole is simply a different way of expressing the necessary relationship that was given in the hypothetical premise. The unity of that statement is in the syllogism simply broken apart into its independent constituents, a kind of activity that Hegel calls a negative unity.

Not only does the categorical syllogism have a form that does not express the necessity of its inferential content; the same applies to modus ponens. For affirmed as immediate existents, the minor premise and the conclusion fail to indicate the mediating relationship which they nonetheless presuppose and require. A fully adequate syllogistic form will show how each term is implicitly involved in every other one. This is captured in the disjunctive syllogism: A is B or C or D; A is neither C nor D; so A is B.[24]

In this syllogism the major premise says that the universal is exhaustively defined in its constituent species. The minor says that the universal is an exclusive singular. The conclusion affirms that it is a member of a particular species. In other words it expresses explicitly the way a universal concept, as a discursive function of unity, determines itself into its constituent particulars, but then goes further to individuate itself in an exclusive singular. Rather than being an inference to something new, it simply articulates the nature of conceptual thought in and of itself.

By collapsing all these considerations together into a unity, Hegel can generate a new immediate concept. Thinking has now moved beyond the subjectivity of a form distinguished from its content. Having fully expressed its own nature, it has become objective. In other words thought is now conceiving something immediate that is fully in and of itself. With this inference, Hegel has shifted to the concept of objectivity — of that which is the complementary counterpart of the subjective dynamic of conceiving.

Before tracing the final sections of Hegel's *Logic* it is worth our while to reflect on what he has been doing in this discussion of concept, judgement and syllogism. The various forms that he explores are not his creation. Rather he has incorporated the traditional table of judgements, as well as the full range of standard inferences into a systematic pattern. In other words, Hegel is not proposing a new logic. He is rather offering a philosophy of logic — showing first of all how conceiving sets

[24] Hegel also uses a form of exclusive disjunction: A is B or C or D; A is B; so A is neither C nor D.

the conditions for a discussion of reasoning by distinguishing singular objects of reference from abstract universals — and then showing how each judgement form and type of syllogism must be introduced to articulate a relation or an inference that is only implicit in the previous form. The key transitions — from judgements of quality to judgements of quantity, or from formal syllogisms to induction and analogy — exploit the peculiar function of speculative reason — looking back on the previous moves as a whole, and then conceptually uniting this synthesis into a new form.

In his analysis of any form he shows how that form reveals its limitations, limitations that are to be resolved in the subsequent form. That analysis, while starting from its understood immediacy, leads into contrary and complementary considerations through the process Hegel calls dialectical reason. And the reflective synthesis of the different descriptions is the work of speculative reason, setting the stage for the conceptual act of unifying into a new unity, which can be considered apart from all mediation.[25]

Subjective conceiving, then, is the process by which self-reflective thought sets out in detail the form of its constituent moments, and in so doing establishes the reasons for the structures of traditional logic. By articulating that structure, thought has given to its own content an appropriate form and has thus become fully self-contained. So not only does it think the concept of objectivity. It has itself become something objective — immediately in and of itself.

5 OBJECTIVITY, LIFE, COGNITION AND METHOD

In Hegel's analysis, the disjunctive syllogism, by fully articulating its constituent conditions, is complete in and of itself. That result, removed by understanding from the mediating process that has led to it, can be called "objectivity". While Hegel admits that this term is frequently used as a synonym for "actuality", "reality" and even "existence", it can nonetheless have its sense restricted to that which, in contrast to the self-determining dynamic of conceiving, is unmediated and self-contained. In this sense it is used not only for the world beyond thought, but also for intellectual material, such as the categorical imperative, Plato's ideas, or Kant's causal necessity, which are thought of as fully self-contained. It serves, then, as a dialectical complement for the subjective dynamic of conceiving.

To conceive of something as immediately in and of itself is to think of a mechanical object.[26] As immediate it is indeterminate; but to be in and of itself, it

[25]In *On Hegel's Logic* (Atlantic Highlands, N.J.: Humanities Press, 1981) I made a first attempt at reconstructing Hegel's analysis to show how the various forms of symbolic logic can be derived, moving from the conjunction of a function with a singular, to membership in a class to propositional forms, and then how this can become expanded into a) a formal variation of the Aristotelian syllogisms, b) modern induction and analogy, and c) the basic inferences of propositional calculus. On rereading it is apparent that this needs to be reworked in detail, although I suspect the project is still feasible.

[26]Not only physical things can be conceived mechanically. Hobbes constructs his social theory using mechanical categories, with humans functioning as self-contained irreducible units; asso-

must be a totality, incorporating all its determinations. Mechanical objectivity, then, involves a universe of objects, each one immediate and indeterminate on its own, but together incorporating all determinations. To resolve the contradiction involved in thinking indeterminate objects that yet constitute a determinate totality, thought must introduce a process, or becoming, by which each object acts on, and reacts to, the others without thereby losing its distinctive immediacy. This means, however, that this object is no longer completely indeterminate, but its immediate objectivity is the result of an interaction with other objects.

Understood in this way, mechanical objects are no longer indifferent to each other; they particularize themselves in terms of their relationships to each other. The stronger exert an influence over the weaker without thereby destroying their immediate objectivity. They become centres to which the weaker objects respond according to regular patterns or laws. These laws determine how the independent objects relate to each other.

Conceptual thought thinks of this pattern as a unity, and the result is the thought of objects which are independent, yet oriented towards each other. This kind of objectivity is no longer strictly mechanical, but chemical.

But with the chemical object as well we have an implicit paradox — between the independence of the objects, and their mutual orientation. This contradiction can be resolved only by positing a process by which their independence is overcome and they become one. Any such process cannot be immediate or direct, else there would be no initial independence. So one must postulate the presence of some kind of mediating object, or catalyst, that, once introduced, will bring about the union.

The product of any such combination can be thought of as a chemical object, however, only if it is not self-contained and inert but, through some other catalyst, can be broken up into independent constituents or elements. And these elements can be considered chemical objects only if they have characteristics that, in appropriate circumstances, enable objects to be chemically oriented towards each other: one as an animating principle, the other as susceptible to animation.

When conceptual thought brings this cycle of processes — chemical union, chemical separation, and the distinctive functions of the elements — together into a unity it realizes that the whole dynamic has a peculiar structure. The processes do not take place immediately, but require mediation; and that mediation or catalyst does not just happen, but is introduced by conceptual thought to satisfy the requirements of the concept of chemical object. Taking this whole structure on its own, divorced from the reasoning that led up to it, provides us with a new category: a framework in which a concept governs or determines which objective processes to use. This is the concept of teleology — of objects being manipulated or organized according to a purpose or goal.

When one now turns to thinking of teleology, or final causes, one distinguishes between a subjective purpose and the objective world in which it is to be realized.

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ciation psychology uses mechanical concepts to explain the process of thinking, and arithmetic uses mechanical operations to add and subtract.

Since the objective world is not immediately what it is supposed to become, the subjective purpose must find something that it can immediately use which will trigger objective processes — mechanical and chemical — that lead to the desired result. The use of these means bridges the gap between purpose and realized end, but they do so not on their own, but because they satisfy the requirements of the original purpose. In other words a single content persists throughout the whole process. At first it is a concept, pure and simple; then it becomes the governing principle that manipulates instruments and means; and in the end it is objectively realized.

While this common content integrates the moments of a teleological process into a basic unity, it nonetheless leads into a paradoxical consideration. For the objective world is not static. Each end achieved is itself transient, turning into something else; and this means that each particular teleological sequence can be understood as simply a means within a larger conceptual framework.[27] We are thus led into an infinite regress in which each end becomes in its turn a means towards a further end.

As we have already seen, conceiving is not impotent before such ongoing repetitions; it can unify syntheses. And so thought is able to look at this cycle which moves from intention through means to an objective end as a whole, and divorce it from the reasoning that led up to it. What it is now thinking is a whole that is not only objective but also fundamentally determined by the dynamic of subjective conceiving. This objectivity is permeated with subjectivity; and such a relationship can well serve as a definition of life.

With the category of life thought has not just moved on to another form of objectivity. Instead it is now thinking of a union of subjectivity and objectivity. Just as the analysis of conceiving led to what is immediately in and of itself (and so objective), so the analysis of immediate objectivity has led to an objectivity permeated by conceptual subjectivity. The categories which express this unity of subjectivity and objectivity can no longer be called simply concepts, however, for they incorporate more than conceiving as such. So Hegel adopts from the tradition another word — idea — to name them. For Plato an idea or form was objective, in the sense that it is immediately in and of itself, but it is at the same time a subjective concept. And Hegel prefers this sense of the term to that adopted by Kant. For his ideas do not simply regulate the function of reason, but also constitute objectivity.

The category "idea", however, will also undergo revision as it is more precisely defined. It starts out as life — the simple identity of the subjective and the objective. But this is not an indeterminate identity. For the objective is integrated by the subjective so that its various components are not independent mechanical objects, but become members of a single body — a living individual. Governed by the subjective "soul", these members interact, each one both determining, and being determined by, the others. They do this by being open, or sensitive, to each

[27] This, Hegel will suggest elsewhere, is the rational basis for the development of teleological arguments for the existence of God.

other; by reacting to the others in a process the contemporary biologists called irritability; and by combining both openness and resistance reconstituting their own unique functions — what was called reproduction.

Since this structure of mutual interaction characterizes life, it must apply not only to this reciprocal dynamic among members of a single body, but also to the way that individual interacts with its environment. As sensitive, the individual is open to, and able to overreach, this alien other, even though this latter has its own agenda. At the same time the living individual resists the intrusions of that environment when its own integrity is threatened. And it combines both moments by appropriating that other and transforming its independent prickliness into a component of its own life. In this way it renews itself as an individual — no longer strictly identical with that it was before, but an individual of the same general sort or genus.

With the thought of a genus we are thinking of a structure of life that is common to several individuals. But these individuals need not be reconstituted versions of a single living process over time, different simply because different kinds of objectivity have been appropriated. They could equally well be independent individuals that share a common structure. As something simply shared, however, the genus is in danger of becoming simply an abstract universal, not a living union of objectivity with subjectivity. Though thought, it does not have an objective existence of its own. Since it is to be the universal expression of life, however, the genus should acquire actuality. This can be achieved when what is common (and so generic) to two individuals becomes objective as a seed. The achievement is, however, flawed; for the seed can only develop into another individual instance of the genus. The genus thus continues to be abstract, even though it may provide the impetus for another union of two individuals in a new seed — a cycle that can continue in another infinite regress.

When thought conceives this cycle as a unity, however, it realizes that something has been achieved. For the genus has found a corresponding objectivity — not in any single living individual, but rather in the ongoing cycle of reproduction. What we have is not a simple identity of subjective genus and objective cycle that would mimic the integrity of a living individual, but rather a correspondence between a conceptual universal, or genus, and an objective reality. If one focuses on that feature of correspondence between two universals, one subjective and the other objective, and leaves aside the reasoning that led to it, we have a new category: the correspondence of subjectivity with objectivity, which serves as a definition of what we call cognition or knowledge.

In this relationship the concept first distinguishes itself from its object, which now stands over against it as a distinctive other. Yet at the same time it has the drive to establish a correspondence with that other — to give it a fully conceptual form. This, then, defines the cognitive quest: to establish under the idea of the true a correspondence between object and concept. This quest progresses through several stages. Conceptual thought starts by trying to grasp the object as it is, without introducing any mediating activity of its own. Since there are to be no conceptual discriminations, the object is thought in the form of a simple identity — as an abstract universal. Not only is the object as such conceived in this way, but the concrete characteristics by which it is distinguished from other objects receive the same treatment. Thought thereby *analyses* an object into its elements — a collection of diverse universals which, though originating from the same object, yet have no conceptual relationship one to another. For any such synthesis would require that mediating activity of thought which was to be held in check to ensure complete correspondence. If thought considers the whole object as distinct from its parts, or as a cause that has certain events as its effects, this is not to be the work of the implicit connections of thought but rather relationships found within the object itself. This presupposes, however, that the object itself is not simply a collection of analysed components, but a universal that determines itself into particulars, a dynamic that mirrors the activity of thought. To achieve a full correspondence between concept and object, then, the concept must itself reproduce that synthetic dynamic, and so introduce mediation into its own cognitive activity.

The first synthetic move the concept makes is to construct a definition of the object. Even though the latter is considered to be a singular, thought subsumes it under a universal genus, and then distinguishes it by means of its specific differences. While this process can be quite successful when the object is already the work of abstract reason (as in mathematics), it becomes much less effective when thought tries to establish the defining characteristics of real objects in the natural and social world. For what specifically differentiates humans — the lobe of the ear, for example — does not necessarily capture anything very essential about them. And the wider genus could be defined by any common features, whether essential or not.[28] In addition, malformed individuals are born into any species, requiring exceptions to any full definition. So the work of definition on its own is prone to error, and cannot provide the desired correspondence.

The strategy adopted to avoid this fallibility is the method of division. Whereas definition starts from the singular object and tries to derive both its universal genus and its specific difference, division starts from a universal and then distinguishes within this universal its particular species. Any such division should be exhaustive, capturing all the various components of the universal. While this, again, can be successful with the abstract objects of conceptual systems, however, it is woefully inept when applied to concrete objects in nature and society. For the universal by itself lacks any principle that determines how it is to be broken up into particulars, since it expresses only what is common to a number of objects.

Thought can resolve this dilemma by constructing theorems that articulate the rationale by which the universal and its components are mutually related to each other. While at this point Hegel spends his time talking about the theorems of Euclid (which illustrate only conceptual systems), the approach he defines could equally apply to the formation of hypotheses and theories in natural science, in which thought articulates a network of internal relations to explain how a whole is

[28]For example, although the extinct Tasmanian thylacine filled the biological space for wolves in that environment, and has been classified as such, it is a marsupial rather than a mammal.

articulated into its parts or a set of conditions combines to produce a determinate effect.

While this type of cognition shows how the various elements of the object may be related to each other, it nonetheless fails to attain full correspondence. For it remains the subjective act of conceiving that has constructed this network of interrelations, and there is no guarantee that the object itself has been in fact determined and formed in this way. A carefully constructed hypothesis, even if it explains all the known facts, need not be true.

This means that a completely new approach must be taken to achieve the desired correspondence. Instead of working with the idea of the true, thought starts from the idea of the good — a conceptual structure that interconnects determinate moments — and then wills to realize this idea in objective reality. It thereby hopes to reach its goal of a full correspondence between concept and actuality. But its hopes also remain unrealized, because the objective world has its own structure and its own agenda, and the practical idea (or will) finds itself confronted by unconquerable limits which frustrate its intentions. To the extent that the idea of the good — with its aim of realizing its conceptual construction in reality — is separated from cognition, or the idea of the true, it will be as unsuccessful as the latter is when *it* is divorced from willed action. Each, then, requires the other if there is any hope of achieving a full correspondence between concept and object — of realizing what Hegel has called the idea.

In other words, the idea in and of itself involves the integration of the theoretical drive for truth with the practical drive to achieve the good. Not only does each supplement the other, but each on its own shows up the limitations of the other. The characteristics of the object identified by analysis show to practical reason the determinations which it must build into its idea of the good; and the success or failure of the attempts to realize the good confirms or belies the theoretical constructions of pure cognition.

With this we have a reciprocal relationship which is complete in itself, and can be collapsed into a new unified concept to which Hegel gives the name "absolute idea"; it is to be the integration of concept and actuality that is valid in all respects.[29] The final task of Hegel's *Logic* is to explore this last functional unity, and in so doing to articulate the method that not only combines theory and practice but also underlies the whole proceeding discussion.

The method of pure reason starts from something immediate. The very fact that it is the beginning means that no mediation that might have led up to it is to be considered. As immediate it is simple and universal. Though universal, however, this beginning is determinate — it is in itself something. And the task of thought is to develop this determination.[30]

Any such determination is inevitably different from the original simplicity. So

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[29] As we have seen, this definition of "absolute" as "valid in all respects" is derived from Kant's *Critique of Pure Reason* B381.

[30] One could also say that the further development is to discover the mediation that led to this beginning.

thought moves on to the moment of difference: of determination over against indeterminacy, of defining limit over against inherent nature. And when it focuses on this new moment of difference, thought finds that it is again led back to *its* opposite, which is the concept with which it began. On the one hand these two reciprocal processes involve an analysis of what is already there; on the other hand they introduce synthetically new material. This is, says Hegel, the moment of dialectic.

As we have already suggested, the working out of such differences that have emerged will lead from an initial diversity into opposition and — when each moment is recognized as fully defining the original beginning — contradiction. Since these contradictory moments developed from a single process of thought, they are related not simply negatively as opposites, but each is also the result of the other; each is required because it is conditioned by the other.

We have a relationship which is both positive and negative, a reciprocity (which speculative reason can consider in a synthetic whole) by which the various moments are incorporated into a more comprehensive perspective even as their independence is cancelled.

Both the immediate dialectical inference that leads to an opposing, distinctive term and the reflective speculative synthesis which considers the opposites in both their negative and positive relationship are moves familiar in ordinary discussions of reasoning. What is distinctively Hegelian is the final move, in which the speculative synthesis is then united into a single concept, complete in itself, which can be considered on its own apart from all the mediation that led up to it. This is what Kant called the work of understanding — focusing on concepts as discursive functions of unity. As Hegel has shown, such conceiving involves particularizing a thought until one has identified it as a singular. As a unity it is universal; as now given a new immediacy through abstraction, it is indeterminate in form, even though its content incorporates the results of the earlier development. Conceiving thus introduces a new beginning ready for further dialectical analysis. This move, Hegel says, is the third, after dialectical and speculative reason; but he admits it could also be called the fourth: beginning, transition to another and back to the first, synthesis, new beginning.

It is this final move, by which conceptual thought creates a new immediate beginning out of a complex development, that enables Hegel to set his discussion of the various moves of thought into a system. He claims, in addition, that the moves he is describing have already been made in the historical development of human culture, and that they are enshrined, though not with full precision, in our language. This is why, whenever he reaches a new conceptual unity he looks to the treasury of our ordinary vocabulary to find the term that most closely matches all the nuances of the new universal term. Leaving aside conventional associations not germane to the immediate context, he then explores what this new concept implies.

At times in logical thought these various moments of the absolute method fall apart. The dialectical moment leads to ever more detailed analysis without synthesis. The speculative moment ascends to ever more encompassing, and abstract, universals. However only when the two contrary moments are integrated can rational thought hope to be fully comprehensive.[31]

With this Hegel has competed his *Logic*. The discussion of method captures the full nature of thought that thinks only its pure concepts, for it describes the dynamic that is common both to the subjectivity of the concept and the objectivity of whatever is conceived. For all that continued analysis and reflection may reveal new details that need to be incorporated into the logical discussion, its method, and so its distinctive essence, is fully transparent.[32]

6 THE VERSIONS OF HEGEL'S *LOGIC*

That Hegel did not hold the *Science of Logic* to be complete in every detail can be seen in what he himself did with it. As we have seen, the original three-volume work was written while he was headmaster of the classical secondary school in Nürnberg, before he finally attained tenured positions, first at Heidelberg and then at Berlin. On arrival in Heidelberg he needed a textbook for his courses which would be more manageable than the highly theoretical volumes of the *Phenomenology of Spirit* or the *Logic*. And so he wrote an *Encyclopedia of the Philosophical Sciences* (1816) as a series of compact theses that would encapsulate in dense prose the essential theses of his logic, philosophy of nature and philosophy of spirit. As well, he added to many of these succinct paragraphs remarks that applied the philosophical substance to the history of philosophy, current debates, or contemporary work in other disciplines.

The *Encyclopedia* received an extensive revision ten years later, and further amendments for a third edition in 1830. On Hegel's death in 1831 it became the skeleton on which his disciples prepared what they called his "System of Philosophy". Working from notes taken by students over the years, and on occasion from manuscripts Hegel himself had used for lectures, they prepared what are called "additions", which indicated how Hegel had expanded orally on the written text. Unfortunately, by collating many different lecture sequences, they were not able to show the way Hegel modified his organization or the details of his discussion from year to year.[33] At the same time they provided a text that was more accessible to

[31] It is worth noting that this final achievement of the absolute idea, while valid in all respects, does not necessarily reach closure. For the result of the method is to reach a new immediate which, as immediate, simply provides a new beginning. Similarly, the reciprocal interaction of theoretical and practical reason enables each to correct and modify the other, but since each functions on its own, neither can fully resolve the incompleteness of the other. Each can, so to speak, show up the inadequacies of the other, but never fully confirm or realize what the other proposes, either as the true or the good. Paradoxically, Hegel could be espousing a fallibilism that anticipates both Pierce and Popper.

[32] "I could not pretend," writes Hegel, "that the method which I follow in this system of Logic — or rather which this system in its own self follows — is not capable of greater completeness, of much elaboration in detail; but at the same time I know that it is the only true method." SL 54; GW 21: 38.17–20.

[33] In Berlin Hegel lectured on the *Logic* during the summer semester every year from 1819 to

the average philosophical reader than either the original *Science of Logic* or the dense telegraphic theses that initiate each paragraph of the *Encyclopedia.* The editors of the posthumous *Works* were also able to replace the first volume of the *Science of Logic*, originally published in 1812,[34] with a second edition, which was in press when Hegel died in November, 1831.

Although sections of the larger *Logic* had been included in J.H.Stirling's *The Secret of Hegel*, in W.H. Harris' *The Doctrine of Reflection* and in H.S. Macran's *Hegel's Doctrine of Formal Logic*, it was the first part of the posthumous "system" that introduced the English-speaking world to Hegel's logic as a whole. W. Wallace's *The Logic of Hegel* appeared in 1874 and has been reprinted many times. (In 1991, T.F. Geraets, W.A. Suchting and H.S. Harris prepared a new translation of the *Encyclopedia Logic.*) The larger *Logic* was then translated in full by W.H. Johnston and L.G. Struthers in 1929 and by A.V. Miller in 1969.

Before considering the changes Hegel made as he moved from edition to edition, it is worth looking first at the differences between the larger and the shorter *Logics* in general. The dense prose of the earlier work uses the abstract language of conceptual thought to indicate the moves reason makes as it reflects on concepts. Not only does the language have little illustrative content, but Hegel assumes that the pure conceptual meaning will be quickly grasped by his reader, as well as those implications that connect it to preceding and subsequent terms. Even so, the moves he makes are detailed and painstakingly articulated. He offers a rarefied and sophisticated discourse that few can navigate with ease. At times he may interrupt this strictly logical development with remarks that apply what he has written to mathematics, the history of philosophy, contemporary science, or theology, but these offer little help in grasping the systematic nature of his basic thought.

In the *Encyclopedia* an analysis and development that might take several pages in the longer work is telegraphed into a single paragraph of two or three sentences. Because he expects to expand the conceptual connections in his lectures, Hegel introduced his terms by drawing parallels to earlier concepts, and specifying the differences between them. Invariably he spends more space on the crucial transitional paragraphs, where conceptual thought functionally integrates a synthesis into a new immediate unity. Even so, this shorter *Logic* lacks the elaboration of all the transitions, reflections and interrelationships that provide the systematic substance of the full *Science of Logic.*

To many of the *Encyclopedia* paragraphs Hegel added remarks, comparable to those he wrote in the larger *Logic*, in which he discusses the application of the logic to philosophy in particular and intellectual disciplines in general. Since Wallace's translation did not distinguish these remarks typographically, they have frequently been read as of equal significance to the key paragraphs. Fortunately, the more

1831.

[34]Not yet translated into English. The analysis of the first book of the *Logic*, on Being, in this article has followed the more readily available second edition of 1831, and not the original first edition.

recent translation by Geraets and others indents them from the left-hand margin.

The lecture material provided by Hegel's editors (in both English translations set in smaller type) contribute even more to the accessibility of the *Encyclopedia Logic*. While one thus has a better general sense of Hegel's logical theory in this version, however, the fact that the strictly logical material is not distinguished from its applications makes it more difficult to discern how Hegel's method works its way through the various concepts and categories. This difficulty is exacerbated by the fact that, as we have seen, the basic paragraphs are themselves of telegraphic density.

Nonetheless the three editions of the *Encyclopedia* allow us to see Hegel reworking his thoughts. In addition, we can look not only at the move from the larger *Logic* to the *Encyclopedia* and the revisions of the second and third editions of the latter, but also at the differences between the first edition of the first part of the larger work — on Being — and the second, posthumously published version. Sometimes, as in his discussions of measuring, Hegel confesses that the material is too difficult, and he omits it from the compendium for his lectures (although he reworked it in some detail when he revised the larger work). But in other areas he makes quite significant shifts.

Some examples may illustrate this. In the first edition of the *Science of Logic*, the basic discussion of *Dasein* makes the following moves: *Dasein*, being other, being for another and being in itself, reality, and then something. Hegel then went from limit through determination, quality, alteration and negation to infinity. By the second edition of 1831 the shift from *Dasein* to something moves quite directly by way of quality. Then Hegel discusses other, being in itself and for another, determination, limit and finitude before reaching a much expanded discussion of qualitative infinity. The earlier categories of reality and negation (which he had taken over from Kant) are reduced to elements within the discussion of quality.

When one turns to the three editions of the *Encyclopedia* (which span the whole interim period between the two versions of the larger work) we find that Hegel changed the order of the categories several times. Already in 1817, quality introduces the discussion of being other, being for other, being in itself, reality and something, rather than appearing later. By 1827 quality alone remains in this first set of moves, anticipating Hegel's last position, but negation and reality continue to play a role in the transition from something to limitation and finitude.

Some new material added in the 1831 second edition of the larger *Logic* suggests that, in the time since 1812, he had become more aware of the need to differentiate the simple transitions of being, when one term simply proceeds from its predecessor, from the work of reflection, which simultaneously thinks contraries as directly referring to each other. In other words he has become more careful in distinguishing the way the different operations of thought — immediate transitions, reflective syntheses, and conceptual unities — function so that they are not introduced prematurely nor become caught in confusion.

A second example of Hegel's reworking of material is his discussion of the German term, *Sache*. This word, difficult to translate, is conventionally used for a number of purposes: a legal case, a task, a question under discussion, the business at hand, the subject or theme. Earlier in this exposition I used "nature of things", when it emerged within the discussion of ground in its transition to the concept of existence. That exposition reflected the text of 1813.

That particular text, as part of the Doctrine of Essence, never was revised. However, *Sache* disappeared from the 1817 version to re-emerge in 1827 in the discussion of real possibility. Though this appears later in the Doctrine of Essence — in the material on the modal categories — it nonetheless retains associations with the 1813 text. For in the digested version of the *Logic* for his students, Hegel discusses the structure of conditioning only at this point — a set of conditions constitutes a real possibility — rather than earlier as the way a complete ground functions. To this extent there remains some parallelism to the 1813 text.

The most significant change, however, occurs in 1831. For Hegel now introduces *Sache* into the final stages of his discussion of measure, toward the end of his first book on the Doctrine of Being. He uses it for the reality that underlies the surface change of qualities whenever quantitative ratios are varied. It now anticipates the distinction between an essence and its superficial show, rather than being one part of the logic developed from that distinction. It appears that, in reworking the way changes of quantitive ratios produce qualitative shifts even though the underlying components remain the same, Hegel has discovered that he is preparing the way for a primitive sense of "substratum", for which "substance" would be too specific, and "essence" would pre-empt the shift to the second part of the *Logic.* So he draws on *Sache* to satisy his needs. Because the second book was never revised, we shall never know what he intended to do once he got to the discussion of conditioning, ground and existence.

These examples show that Hegel was constantly rethinking his material: drawing important new distinctions, allowing earlier ones to recede into the background. Although we do not have as yet more than one or two of the transcripts of single lecture series in print (one has been published from 1817 and one from 1831), we can expect that there as well he would have regularly reworked the order of his presentation.[35]

The *Science of Logic*, then, was not a closed study, complete on the death of Hegel. Indeed he was revising and reworking the material up to the moment of his death. One could say that the method he described in the final chapter on the absolute idea was continuing to function. Any ordered analysis of the structure of pure thought, once complete, became a new beginning which could then become subject to revision and alteration — changes that would then have to be reintegrated systematically into a coherent discussion. Whether this was simply Hegel attempting to correct his earlier thought on the way to a complete and closed system, or whether the process of revision is endemic to the logical method itself is a question that must remain open.

Although the logic underwent a number of revisions, however, Hegel nonetheless

[35]The way the early editors collated the lecture material to follow the order of the 1830 *Encyclopedia* means that this important information has been effectively veiled from view.

claimed that the method articulated in the chapter on the absolute idea is the only true method. Because it involves the move into something other, a reciprocal move back again, the synthesis of the two, and finally the collapse into a new immediate unity, it is not simply one moment in an infinite regress. For the new beginning is, in a sense, simply the earlier starting point now enriched with new determinations and new content. So what from one point of view could be read as a movement onto new determinations could also be seen as an exploration of the presuppositions underlying that indeterminate beginning. And since the pattern is self-correcting, in that the move to difference shows the inadequacies of what had gone on before, it is able to reaffirm its own validity even as it is subject to revision. It is, therefore, valid in all respects, or absolute.

Hegel suggests that, when the logic has finally reached the comprehension of its own method, it can think of this pattern as the simple expression of everything that has gone on before, incorporating all the logical determinations and categories into a new conceptual immediacy. Any other logical move would simply reconfirm that method. So, in a sense, the method is only related to itself. Were thought to look for a term to name a simple immediacy that is related only to itself, and divorce it from all the reasoning that led up to it, it would find that such a term had already been considered. For self-related immediacy is the definition of pure being, with which the whole operation began, even though that category as now understood contains implicitly a much richer content.

The one fact that the method, in moving on to new ideas, is simply fleshing out the content that was implicit in its starting point, and the second fact that the method, as the culmination of the whole logical project, returns to its beginnings, provide the justification for Hegel's claim that the logic is systematic and that it is a system of totality. For all the elaboration in detail that it might yet have to explore, pure thought is in principle fully transparent.

Nonetheless the result remains only a logical result, enclosed within the subjectivity of pure thinking. As an idea it is to be the absolute union of concept and reality. But as pure thinking it is bedevilled by the drive of cognition to overcome its subjectivity. So thought posits itself enmeshed in the immediacy of pure being which is in some way other than itself. Once pure being is thought of as a totality, we find ourselves thinking of nature.

This move, however, cannot be a simple transition of thought — from one concept to another — for then thought would continue to be confined to the subjectivity of pure thinking, and it would recreate the category of cognition already investigated. Rather, pure thought, as the idea, must freely make itself redundant and let itself go in absolute self-confidence. Abandoning its own network of internal relations, it moves to an externality that lacks all subjectivity — space and time. Ultimately, once thought has fully explored this realm of external nature, it will be led back to an externality that is yet integrated by thought — a realm Hegel calls spirit. And thought will complete its liberation through a disciplined investigation of this realm.

The move from the science of logic to the philosophy of nature has been one of the most difficult aspects of Hegel's whole philosophy. His early colleague, Friedrich Schelling, was to call it an illegitimate leap into another genus. And contemporary interpreters disagree on what is involved. Does Hegel simply extend the developmental logical analysis, but vary it from that of the logic by explicitly including the element of otherness and externality? Or does thought freely declare its own redundancy and, while anticipating in general what it will find, only discover the detail from disciplined observation — details which must themselves be incorporated into subsequent investigations? The answer to those questions, mercifully, lies outside the domain of this article.

SUMMARY

This exposition of the argument in Hegel's *Science of Logic* has been designed to show that, although not a logic in the conventional sense, it nonetheless investigates the processes of thought that underlie logical reasoning and which find expression in formal syllogisms. Thought moves from one concept to another in immediate transitions; it reflects on the totality of such moves; and, says Hegel following Kant, it unites these transitions together with their starting and concluding terms into new concepts, which can be isolated and considered on their own apart from any mediation.

To fully comprehend what is going on, however, Hegel says that we cannot remain content with simply a schema of intellectual moves. Despite the common structure, each transition, each reflective synthesis, each conceptual unity is conditioned and determined by the specific content being thought.

Based on these assumptions Hegel endeavours to show that the concepts we think are not a simple collection of diverse terms. They can be understood as developing one from another — from the most simple and indeterminate to the most complex and comprehensive. This developmental framework, he claims, underlies the assumptions, inferences and judgements of reasonability that permeate all intellectual discourse. And understanding this development will affect the way we conceive these terms and processes.

AFTER HEGEL

Hegel's claim that all rational discourse could ultimately be integrated into a systematic framework caught the imagination of British and American philosophers during the 19th century. In 1844 the young Benjamin Jowett travelled from Oxford to Germany where he was introduced to Hegel's thought by Johann Eduard Erdmann; from then on he not only encouraged the study of Hegel among his students at Oxford, but even began a translation of the *Encyclopedia* Logic. Around 1858 in St. Louis, Missouri, the German emigré, Henry Conrad Brokmeyer converted William Torrey Harris and with others they inaugurated a systematic study of Hegel based on Brokmeyer's translation of the larger Logic. The result was a flourishing of neo-Hegelian logics.

For Harris (1835–1909), the "objective dialectic" is a process which results when a thought is assumed to be universally valid and true. Whether it is taken simply on its own, as negatively related to its contraries, or as the positive identity of such contraries, logic will lay bare its imperfections and in so doing pass over to a more profound thought, which contains explicitly what had been only implicit. This dialectic, when developed into a system, will contain the solutions to all the problems posed by experience.

For the Cambridge philosopher, J.M.E. McTaggart (1866–1925) on the other hand, the logic analyses what happens when categories are predicated of a subject. Whenever we predicate any concept except the last with any consistency we are forced to apply its contrary to the same subject. The resulting contradiction between the thesis and its antithesis requires a synthesis, which reconciles them in a higher category.

This pattern, however, is gradually modified in the course of the logic: in the discussion of essence, the contraries mutually imply each other, and in the logic of concept each category expresses the "truest significance" of its predecessor, so that at this stage the logic requires no opposition or contradiction, and so no reconciliation. McTaggart draws the implication that reality, the only subject that can truly be characterized by these logical predicates, is itself continuous and developmental. The early moves through thesis and antithesis to synthesis then do not describe reality as it actually is, but rather reflect the way finite and incomplete thought corrects its subjective and limited predications on the way to completeness. Reality, however, or the absolute, is not affected by such negativity.

The study of logic as the way thoughts are predicated of reality is taken up by F.H. Bradley (1846–1924) and Bernard Bosanquet (1848–1923), both educated at Oxford. In his *Principles of Logic*, Bradley argues that one must distinguish grammatical distinctions from the logical relation, in which a judgement predicates a single complex thought (expressed in the sentence) to reality as the ultimate subject. Thus all judgments are both categorical and hypothetical: hypothetical in that they express the connections between thoughts; categorical in that they affirm reality itself to be connected in this way. The role of inference is to elaborate the systematic interconnections that are true of the self-existent, self-contained and complete absolute. It can do so, however, because the mediating connections it brings forward depend on "the unbroken individuality of a single subject." Logic, then, both analyses reality into its elements and at the same time shows how those elements are synthetically interconnected.

Bosanquet's *Logic or the Morphology of Knowledge* builds on Bradley's theory of judgement, but shows in a more Hegelian way how the various types of judgement and inference can be orderd in a systematic way so that they present a continuous development of forms from simple qualification through comparison, measurement, singular, universal, negative and disjunctive judgements into those inferences which make explicit the necessary ground of any judgement. These, too, have what Bosanquet calls a morphology, in that one can trace how each one grows logically out of what precedes it. Starting from enumerative induction and mathematical reasoning, he proceeds to analogy, perceptive analysis, hypothesis and finally concrete systematic inference. Since all such reasoning is predicated of reality, it provides the foundation for knowledge. "The comparative value of these forms of knowledge," he writes, "and the affinities between them, are the object-matter of Logical Science."

In British Idealism, then, the systematic interrelation of thought which is the focus of Hegel's logic is explicitly metaphysical. Working from several remarks in Hegel's *Encyclopedia Logic*, they understand logical categories to be predicates of a singular reality (or the absolute), and the role of logic is to educate the mind to the point where it can grasp this reality as a whole.

The American, Charles S. Peirce (1839–1914) initially had contempt for the Hegelianism of his contemporaries, but eventually allowed that his philosophy resuscitated Hegel in a strange costume. Peirce's three categories were anticipated in Hegel's being, essence and concept. Indeed, "it appears to me that Hegel is so nearly right that my own doctrine might very well be taken for a variety of Hegelianism." [CP 5.38 (1903)] Hegel, however, erred in incorporating fact, and to a lesser extent quality, into the interconnections of thought, so that "the element of Secondness, of *hard fact*, is not accorded its true place in his system." [CP 1.524 (1903)] In a sense, what Peirce missed was a critical aspect of that negativity and finitude which McTaggart, as well as the other idealists who equated reality with a single individual absolute, had wanted to discount because of its partiality.

The interpretation of Hegel's logic offered in this article is based on the conviction that Hegel does in fact take "secondness" seriously. His *Phenomenology of Spirit* traces the development of human experience which has learned from the failure of its confident claims to knowledge when they have been put into practice until it reaches the point where it can show that the self-determining life of spirit — that is, of humans both individually and as cultural communities — knows by putting its convictions into practice and learning from both the successes and the failures of those ventures. Human reasoning is nothing else but this dynamic of lived experience as it has been distilled into the essence of thought. Thought, then, is the way humans conceive not only the world they experience but also their own intellectual activity. The simple dialectical transitions, the reflective syntheses and the conceptual unities of Hegel's logic are just the way intelligent life determines itself as thought.

This reading, however, is only one among many. A selection from English-speaking Hegel interperters during the twentieth century will suggest some of the alternatives.

For many years the only available commentary after McTaggart's was that of G.R.G. Mure (*A Study of Hegel's Logic*), an heir of the nineteenth century British idealists. For Mure the absolute is active spirit, and the philosopher as a participant in its life thinks the categories of that life as they apply in perception (the logic of being), in empirical explanation (the logic of essence) and in philosophical thinking itself. The moves from thought to thought are not explained by the inherent character of each category *per se*, but by the fact that all are grounded in the persisting presence of spirit, yet separated out as singular thoughts.

For Errol E. Harris (*An Interpretation of the Logic of Hegel*) the absolute is simply to be understood as the whole, and speculative thought is that which thinks this reality as a whole. The teleological drive to completeness thus "sublates" or overcomes the finitude of each limited concept by forcing its implicit antithesis to become explicit so that both contraries can be incorporated into their synthesis.

Charles Taylor (*Hegel*), avoiding the language of the absolute, suggests that for Hegel the categories apply to reality in general. Each one is indispensable as a way spirit posits the things to which concepts apply. But because such limited terms do not satisfy the standard of coherence which any conception of reality must meet, they show themselves to be incoherent, requiring more adequate terms.

Mure, Harris and Taylor all assume that the categories of Hegel's logic function as predicates of a subject which is none other than reality as such — the absolute or the whole. For them, the logic is essentially a metaphysics. Stephen Houlgate (*Hegel, Nietzsche and the Criticism of Metaphysics*) maintains a metaphysical reading, but of a different order. For Houlgate, the logic works without foundations, and so cannot presuppose reality as the subject of which the categories are predicated. Rather, thinking is itself being. As a result, the logic conducts an immanent critique of the traditional concepts of metaphysics and, by showing their true character, yields a determinate philosophical knowledge of reality. The truth of this systematic construction becomes manifest when its conclusions accord with common experience.[36]

Robert Pippin (*Hegel's Idealism*), however, rejects any metaphysical reading of the *Logic*. For Pippin Hegel is following a Kantian project — of articulating those concepts that are required to think in a determinate way any possible object whatsoever and then exploring their categorical commitments. This requires an exposition not only of those terms that refer to objects, but also of those that apply to the self-conscious judge of objects. Since Hegel, following Fichte, rejects Kant's claim that concepts on their own are empty and must be completed with reference to sensible intuitions by way of the schematism, he has to justify their transcendental use through the internal connections that are shown to hold between categories and so reflect the self-determining power of pure thought. While the British idealists had argued that the coherence of thought reflected the coherence of reality, Pippin makes a more modest claim: that it is the systematic, or holistic "dialectical" interrelatedness of these determinations of possible objects which legitimates the transcendental use of the categories.

Terry Pinkard, in *Hegel's Dialectic: The Explanation of Possibility*, adopts a similar, non-metaphysical reading. Concepts for Hegel are defined by the rules according to which they can be used, and the role of logic is to reconstruct and reorder the categories so that they become compatible. This systematic redescrip-

[36] If such a correspondence fails to appear, says Houlgate, it reflects either a flaw in the logical explanation or an improper application to experience.

tion starts with simple determinations, moves on to the relation of substructure to superstructure and concludes with an examination of the principles used in the very act of conceiving.

In his later *Hegel: A Biography*, however, Pinkard reverts to a metaphysical reading, claiming that Hegel had taken over from his friend, the poet Hölderlin, the conviction that thought and being are one, so that the logic starts with judgements we make about finite entitites that come to be and pass away, moves on to the relation between appearance and reality, and concludes with the "normative structure" of social space. The *Logic* thus equates being to the rationality of modern humanity.

Some (for example Clark Butler, Yvon Gauthier and Dominique Dubarle) have tried to formalize Hegel's logic, but none of the proposals have won a general acceptance, since they require the introduction of non-formal conditions to generate Hegel's logical transitions.

Ultimately, the significance of Hegel's logic depends on whether he is read as developing a systematic philosophy peculiar to himself, or as articulating the structure of all human thinking to the extent that it can be divorced from its applications in experience. If, as is argued here, he is performing the latter task, then a study of Hegel's logic may well provide a way for us humans to satisfy the Delphic injunction to know ourselves. We shall do so by exploring the immediate inferences of thought, the synthetic strategies of reflection and the way we conceive immediate "self-evident" starting points, all of which constitute the domain of rational discourse.

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BOLZANO AS LOGICIAN*

Paul Rusnock and Rolf George

1 INTRODUCTION

Bernard Bolzano (1781–1848) stands out with Frege as one of the great logicians of the nineteenth century. His approach to logic, set out in the *Theory of Science* [WL] of 1837, marks a fundamental reorientation of the subject on many fronts, one which is as radical as any in the history of the field. In sharp contrast to many of his contemporaries, Bolzano insisted upon a rigorous separation of logic from psychology. It should be possible, he thought, to characterize propositions, ideas, inferences, and the axiomatic organization of sciences without reference to a thinking subject. Consistently pursuing this approach to logic and methodology, Bolzano developed important accounts of formal semantics and formal axiomatics. A talented mathematician, Bolzano developed his logic in conjunction with his mathematical research. Among the first to work on the foundations of mathematics in the modern sense of the term, he made a number of key discoveries in analysis, topology, and set theory, and had a significant influence on the development of mathematics in the nineteenth century. In logic, Bolzano is best remembered for his variation logic (section 4.2 below), a surprisingly subtle and rigorous development of formal semantics. In this article, we discuss Bolzano's logic along with some of his work in the foundations of mathematics which has some bearing on logic.

2 BIOGRAPHY

2.1 Life and Works

Bernard Placidus Johann Nepomuk Bolzano was born in Prague on October 5, 1781. His early education, conducted in the spirit of the Josephinian Enlightenment (named after the second Austrian Emperor of that name), stressed utility, practical morality, and a concern for the common good.

*We use the following abbreviations in this article: **BBGA**= B. Bolzano, *Bernard Bolzano Gesamtausgabe* ed. E. Winter, J. Berg, F. Kambartel, J. Louzil, B. van Rootselaar (Stuttgart–Bad Cannstatt: Fromann-Holzboog, 1969–); *Contributions* = B. Bolzano, *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague, 1810); WL= B. Bolzano, *Wissenschaftslehre* (Sulzbach, 1837). *Method* = B. Bolzano, *Von der mathematischen Lehrart* ed. J. Berg (Stuttgart–Bad Cannstatt: Fromann-Holzboog, 1981); also in **BBGA** II.A.7; FL = B. Bolzano, *Functionenlehre* ed. K. Rychlik (Prague, 1930).

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In 1804 he competed unsuccessfully for a chair in mathematics in Prague, and then accepted an appointment for a newly established position as professor of religious instruction. His decision to accept this position and become a priest had not come easily. He filled reams of paper with deductions, starting with the supreme moral law, determining the utility of each profession. A chance remark of one of his professors, that a doctrine may be justified if belief in it leads to moral improvement, helped to convince him that this was a justifiable choice.[1]

The foundation of Bolzano's faith was the principle of utility: "I am of the opinion that the supreme moral law demands nothing but the advancement of the common good."[2] He rigorously measured all activities, including religious pursuits, against the standard of public utility. Religion was "the sum of doctrines and opinions that have an either detrimental or beneficial influence upon the virtue and happiness of a person."[3] Virtue is "the persistent striving to make the sum of pain in this world as small as possible, and to enlarge the sum of well being as much as possible."[4]

Inevitably, in the course of the anti-enlightenment "restauration", charges of heterodoxy and political unreliability were laid against Bolzano. On December 24, 1819 he was dismissed from his university post in a purge of freethinkers, nationalists and progressives. Objectionable statements were cited in evidence, the most offensive from a volume of sermons of 1813: "There will be a time when the thousandfold distinctions of rank among men that cause so much harm will be reduced to their proper level, when each will treat the other as a brother. There will be a time when constitutions will be introduced that are not subject to the same abuse as the present one."[5] The Imperial Chancellor Saurau determined that Bolzano's "innovations" could not be justified. In German universities, Saurau pointed out, professors must live on students' fees, thus new doctrines are needed to attract them. In Austria, however, they are paid by the state "so that they teach propositions that are approved by the church and the civil administration. It is a dangerous error for a professor to think that he can instruct the youth entrusted to his care according to the drift of his individual convictions or according to his own views."[6] Bolzano was therefore forbidden to teach, preach or publish, his mail was opened and censored. Later the ban on publication was relaxed to allow the publication of works without political or religious content. He found comfort in the thought that the small pension of 300 Gulden was no more than would be his share if all goods were equally divided.

After 1823 Bolzano spent most of his time in the Bohemian countryside at the estate of his friends Anna and Joseph Hoffmann, who could give him the support his poor health demanded (he suffered from a lung ailment).

[1] B. Bolzano, *Lebensbeschreibung* (Sulzbach, 1936), p. 27.

[2] *Wissenschaftslehre*, 4.27.

[3] *Lebensbeschreibung*, p. 199.

[4] Homily on the first Sunday of Advent, 1810. *Erbauungsreden* (Prague/Vienna, 1852), Vol. IV, p. 19.

[5] Eduard Winter, *Der Bolzanoprozess* (Brno, 1944), p. 29-30.

[6] *Ibid.*, p. 35 f.

During that period he wrote a monadological essay, *Athanasia*, on the immortality of the soul, to console Anna Hoffmann when her fifth and last child died. But most of the decade 1820-1830 was spent writing the logic he had planned, the *Wissenschaftslehre*, completed about 1830, but not published until 1837.[7]

Efforts were then made to publicize Bolzano's views, especially in Germany. A discussion of his philosophy was to be stimulated in the leading journals, important philosophers should be asked to review his books, a prize was to be awarded for the best critical discussion of the *Wissenschaftslehre*. He himself wrote a book of responses to reviews of that work,[8] and another summarizing his views.[9] On the whole, these efforts, which were supported by students and friends, were not successful. No philosophical school or tradition formed.

In the 1830s, Bolzano began to assemble materials for a planned mathematical treatise, the *Theory of Magnitudes* (*Größenlehre*). The early parts of this work are of particular interest to the historian of logic, containing a brief exposition of Bolzano's logic, called "On the Mathematical Method", a theory of collections (including sets) and theories of the natural, rational, and real numbers. Later parts, especially the *Theory of Functions*, continue earlier studies in the foundations of mathematics, pursuing them to greater depth and in a more systematic way. The *Theory of Magnitudes* was not printed in Bolzano's lifetime, but some manuscripts did circulate within Austria.[10] Little historical research has been done to determine whether Bolzano's manuscripts on the foundations of mathematics had any influence. They are in any case of considerable intrinsic interest. Towards the end of his life, Bolzano, seeing that he would not have the time to finish the work he had begun, decided to write a shorter book presenting some of his mathematical discoveries. It was published after his death by his friend Prihonský as The *Paradoxes of the Infinite*.

In 1841, Bolzano moved back to Prague. Official hostility had abated. He became Director of the Royal Bohemian Academy of Sciences in 1842/43, and was chair of the division for pure mathematics and philosophy until his death. On December 18, 1848 he died after a life "of physical and mental suffering", and was buried in Prague.

2.2 Reception

Bolzano's mathematical work was carried forward by Weierstrass and others, and his name became attached to some important theorems in analysis. Dedekind and

[7] Both *Athanasia, oder Gründe für die Unsterblichkeit der Seele* (1827) and *Wissenschaftslehre* (1837) had to be published outside Austria, due to Bolzano's troubles with the Austrian authorities.

[8] *Dr. Bernard Bolzano und seine Gegner* (Sulzbach, 1839).

[9] *Bolzano's Wissenschaftslehre und Religionswissenschaft in einer beurteilenden Uebersicht* (Sulzbach, 1841).

[10] The logical essay "On the Mathematical Method", for example, formed the basis of a lengthy exchange between Bolzano and Franz Exner, who was professor of philosophy at Prague from 1832 to 1848.

Cantor knew the *Paradoxes of the Infinite*. Bolzano's logical and philosophical work, by contrast, dropped out of public view until the end of the 19th century. At that time, it was mostly Franz Brentano's students E. Husserl, A. Meinong, K. Twardowski and B. Kerry who returned to the study of Bolzano. This was not due, however, to Brentano's mediation, who rather thought of ideas and propositions that "exist from eternity" as "astonishing aberrations." "They would have done better to learn from me than from him."

> I must be allowed totally to refuse all responsibility for so much that is bizarre and absurd in Husserl as well as Meinong, produced under the influence of Bolzano. As I ... have never accepted even a single proposition of Bolzano's, I have never suggested to my students that they could find there true enrichment of their philosophical knowledge.[11]

Brentano notes that Robert Zimmermann (1824-98), his colleague in Vienna and the only student of Bolzano to hold a chair in philosophy, was responsible for these deviations from his teachings.[12]

Bolzano fortified, if he did not engender, Husserl's anti-psychologism, the chief topic of *Logical Investigations* I. His enthusiastic endorsement of Bolzano deserves to be cited:

> [Bolzano's] theory of the elements of logic far surpasses all else that world literature has to offer as a systematic exposition of logic .... It contains such an abundance of original, scientifically secured and fruitful thought that he must be considered one of the greatest logicians of all time.... Logic as a science must be based upon Bolzano's work, and from him logic must learn what is needed: mathematical precision of distinctions, mathematical exactness of theories. It will then acquire a standpoint for evaluating the "mathematical" theories of logic that mathematicians, unconcerned about philosophical disdain, have so successfully constructed.[13]

With the rise of modern logic, interest in Bolzano continued to grow, due largely to the influence of Jean Cavaillès, Heinrich Scholz, Eduard Winter, later Jan Berg, Edgar Morscher, Jan Sebestik and others (see the Bibliography).

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[11] From a letter Brentano's to Samuel Hugo Bergman, of June 1, 1909. *Archiv für Geschichte der Philosophie* **48** (1966) 306-311, p. 308. Bergman was a student of Anton Marty, himself a student of Brentano.

[12] Ibid.

[13] Edmund Husserl, *Logische Untersuchungen* Vol. 1. *Husserliana* Vol. 17, (Den Haag: Nijhoff, 1975), pp. 227 f.

3 BOLZANO'S CONCEPTION OF LOGIC

3.1 Mathematics and Logic in the Contributions

Bolzano's primary scientific interest was the foundations of mathematics, and it was this interest that led him into a deeper study of logic. As a young man, struck by the imperfections of contemporary mathematics, he undertook to reconstruct the entirety of the science upon more secure foundations. In 1810, he published the first installment of his *Contributions to a Better Founded Presentation of Mathematics,*[14] presenting the grand lines of this project. The second part of this book is devoted to the mathematical method, which, for Bolzano, is just logic (II, §1).

In the *Contributions*, mathematics is defined as a general science of forms, (I, §8), a proposal made more precise in the second (unpublished) installment on "universal mathematics".[15] There it becomes clear that mathematics is the general science of what Bolzano calls *systems*,[16] that is, sets of elements possessing various properties and relations.

With this general conception of mathematics, the work cut out for the foundations of mathematics (which Bolzano called the philosophy of mathematics) is clear: to characterize a structure, one must identify its elements, their properties and relations, and the important truths concerning these. Since many properties and relations can be defined in terms of others, and many propositions proved from others, the fundamental problem of characterizing structures becomes one of determining primitive concepts and rules for forming complex concepts, along with primitive truths (axioms) and rules of inference, and then providing definitions of important concepts and proofs of important theorems. Bolzano's approach is, in short, axiomatic in the modern sense of the term.

Bolzano followed this approach in all his work in the foundations of mathematics. Geometry, for example, is developed as a theory of collections of points possessing metrical and topological relations, analysis as a theory of quantities possessing relations of order, functional connections, etc. Of more interest to the present purpose, however, is the use Bolzano makes of this conception in logic. His insight was that not just the *objects* of mathematics were structures, but that mathematics *itself* (and in general any science) was a structure, namely, a collection of *truths* (themselves structured entities) ordered by the relation of ground-consequence. The same sort of foundational analysis, therefore, was appropriate in logic. The formal core of logic, that is, is mathematical, the theory of the particular kind of structures called sciences.

[14] *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague, 1810); hereafter *Contributions.*

[15] *Allgemeine Mathesis.* In **BBGA**, II.A.5.

[16] He also calls them *wholes* or *sums*; in his later work Bolzano adopts the term *collections* (*Inbegriffe*). For discussion of collections in Bolzano's work, see J. Sebestik, *Logique et mathématique chez Bernard Bolzano*, Part 3, Ch. 2; P. Simons, "Bolzano on Collections," *Grazer phil. St.* **53** (1997) 87-108.

Already at this early date, Bolzano indicates that he understands sciences to have a certain independence from human thought. "This much," he writes, "seems to me to be certain: in the realm of truths, i.e. in the collection of all true judgments, there is an objective connection, independent of our subjective recognition of it; and that, as a consequence, some of these judgments are the grounds for others, their consequences"(II, §2). Sciences, that is, are intrinsically ordered. The goal of scientific exposition is to discover and display this order (*Ibid.*). Logic—itself a science—tells us how to do this.

Bolzano takes pains to distance himself from the epistemological approach to axiomatics, as put forward, for instance by Pascal in his essay on the *esprit géométrique*. For Bolzano primitive concepts are merely *semantically* primitive, i.e. indefinable, axioms are truths which are not consequences of other truths, and proofs are indications of what he calls the "objective dependence" of a given truth on others. He explicitly rejects the notions that primitive concepts are those clearly understood in themselves, that axioms are self-evident truths, and proofs devices that engender conviction. We see here an early expression of Bolzano's non-psychologistic approach to logic, an approach very much at odds with that of many of his contemporaries.

Overall, the logic of the *Contributions* is immature, more remarkable for its general viewpoint than for the details of the treatment. By contrast, Bolzano's early mathematical writings (especially the *Purely Analytic Proof* of 1817) contain achievements of lasting value. In them, Bolzano develops central points of the classical foundation of the calculus, including precise definitions of the continuity of a function, the convergence of an infinite series, and clear statements of the completeness of the real numbers.

3.2 From the Contributions to the Theory of science

Bolzano seems to have recognised the deficiencies of the logic developed in the *Contributions*. Already in 1812 he noted his intention to write a new treatise on logic. This project, which was pursued especially vigorously in the 1820s, gave rise to Bolzano's logical masterwork, the *Wissenschaftslehre* (*Theory of Science*; hereafter WL), a book that was substantially complete by 1830, but not published until 1837 due primarily to Bolzano's troubles with the Austrian establishment.

The WL is broader in scope than the discussion of mathematical method in the *Contributions*. In this later work, Logic, or the theory of science, is the science concerned with the organization and presentation of scientific information. A science, according to Bolzano, is a collection of truths worth expounding systematically in a treatise. The theory of science teaches us how to "divide the total domain of truths into individual sciences" as well as how to "represent the sciences in treatises" (WL §1).

Of the four volumes of the WL only the last deals with the "theory of science proper", that is, with the actual composition of scientific treatises, their division into chapters, choice of audience, etc. It is, however, more than a manual of style and contains, among other matters, important sections on definitions, criteria and indirect proofs. The other three volumes are foundational studies. It is consistent with Bolzano's conception of a science that supporting truths can be more important and interesting than conclusions drawn from them. In the theory of science the most important parts occur early in the development. The first two volumes of WL contain Bolzano's treatment of deductive logic, probability theory and related matters, while the third is devoted to epistemology and heuristics, the "Art of Discovery", where Bolzano anticipates Mill's Canons, and describes what was later known as the hypothetico-deductive method. Our discussion will focus on the first two volumes.

In the subtitle of WL, Bolzano promises "assiduous attention to earlier authors." And, indeed, WL is an invaluable source for the history of logic, the attention to earlier authors taking up as much as a third of the first two volumes. For instance, after introducing the concept of a proposition, Bolzano spends three chapters outlining the views of more than forty philosophers on the subject. Among them are Aristotle, Condillac, Euler, Fries, Hegel, Herbart, Hobbes, Kant, Lambert, Leibniz, Locke, Malebranche, Wolff and several of his contemporaries. Some had a vague, not properly explained, notion, most of them tried to define propositions in terms of judgements, others committed various definitional improprieties, too complex to discuss here. Bolzano's careful, and always generous (except for Hegel) assessment of earlier views accompanies every subject he discusses, but we shall pay no further attention to this part of his project.

Psychologism

In contrast to Frege and Husserl, Bolzano does not engage in a lengthy polemic against those who had thought to found logic upon psychology. Early in the *Wissenschaftslehre* the question is raised if logic is an independent science, specifically if, as many have claimed, psychological theses (*Lehrsätze*) can have a place in it. With typical thoroughness he first asks what it means for a science to be independent of all other sciences. A science A is independent of another B, if no proposition from B needs to occur in A. Given this, even geometry, the very paradigm of a rigorous stand-alone science, is not independent, since it includes propositions from arithmetic and analysis. Just so, certain non-logical theses will be found in logic (§13), specifically those of psychology, the only science he mentions. This has to do with Bolzano's broad conception of his subject. After the development of the abstract foundational part in the first two volumes of WL, he turns in the third volume to epistemological matters, discussing the problems of how truths may be discovered, how causes for given effects, and effects for given causes, may be found, and so forth.

> Logic is to teach us rules by which our knowledge can be organized into a scientific whole. To do this, it must also teach us how truth may be found, error discovered, etc. It cannot do the latter without attending to the way in which the mind acquires its ideas and knowledge. The

> proofs of its rules and theories must therefore make reference to the faculty of representation, to memory, the association of ideas, imagination, etc. But the human mind and its faculties are the subject of an already existing science, namely empirical psychology. From this it follows that logic is dependent at least on psychology, and that it must forego the reputation of being a completely independent science (WL §13).

He follows, but argues anew for, the old custom of making discovery one of the tasks of logic. This is no more than a terminological issue; in the properly logical part, the theory of concepts, propositions and arguments, psychology plays no role. He did not, as we shall see, make judgements and inferences, mental occurrences, the foundation of logic, but propositions in themselves, ideas in themselves and the relations among them. As well, he denies the central role usually allotted to the so-called "laws of thought", the laws of identity and contradiction. While they are certainly true,

> one can by no means say that they contain "the ground of all truth in thinking, and are sufficient to determine the truth and correctness of all our thoughts." For they contain, as far as I can see, neither the objective, nor even some subjective, ground for the deduction of any truth worth talking about, let alone of all truths (WL §45).

This was obvious to him because he understood, and had advanced, the first clear and viable definition of, logical consequence. He also denied the common conceit that a collection of contradictory thoughts "cannot be united in a unity of consciousness, is not thinkable" (§45.4). How could one discover that there is no dodecahedron with hexagonal sides if one could not form that concept?

After developing the abstract logical theory, Bolzano did not hesitate to advance bold speculations about the working of the human mind. Here is an example. He held that "mediated" judgements could arise in the mind in only three ways, as consequences of previously recognised grounds, as conclusions that follow deductively, or are made probable by previously accepted propositions. In all three cases, the judgements that are the conclusions are *caused* by the premisses, occur after the premisses, which have not wholly disappeared from the mind when the conclusion forms. In these cases the premisses are the complete cause of, and not only the occasion for forming, the judgement that is the conclusion. Sometimes fallacious rules are employed, but the human mind is so structured that it never concludes M directly from A, B, C, D unless it follows in one of the three ways. The mind does not contain a defective routine that generates false conclusions from true premisses. Such a destructive routine could be shown to exist only if carefully established conclusions could contradict each other, "but no one has shown that this can happen" (WL §300).

4 THE LOGIC OF THE *THEORY OF SCIENCE*

4.1 Propositions and Ideas

Propositions

The weight of logical tradition spoke in favor of beginning with terms or ideas, and defining propositions or judgments in terms of the former. Bolzano reverses this order. He begins with the concept of an *objective proposition* or *proposition in itself*, and later defines ideas in terms of propositions. Not satisfied with the definitions others had given of the concept "proposition" and unable to find a suitable one himself, he did not attempt to *define* this concept, that is, lay out its constituents, but instead tries to convey the meaning of the word as he uses it by employing it in a variety of contexts (WL, §23). He calls this a *Verständigung*, an "arriving at an understanding" or "explication". Indeed, he introduced the now common expressions "contextual definition" and "definition in use" to describe this device (see section 4.1 below). In a letter to Franz Exner, he writes that the following two sentences should suffice for this purpose:[17]

1. A proposition in itself is either true or false (but not both).
2. A proposition does not have actual existence.

Elsewhere he says that a proposition is "any assertion that something is or is not the case, regardless of whether somebody has put it into words, and regardless even of whether it has been thought" (WL §19; Berg 45 f. gives a more comprehensive list of explicatory sentences). A proposition, that is, is a sort of abstract or mathematical object, more specifically a truth bearer. Propositions may also be thought of as the *contents* of judgments, and as the *meanings* of sentences.[18]

Propositions in themselves must be rigorously distinguished, Bolzano thinks, from both thought and expressed propositions. For logical relations do not depend upon whether or not a proposition has been thought or expressed. When, for example, one judges that all of the propositions of the form

If $\sqrt{n}$ is rational, then n is a perfect square.

are true, for $n = 1, 2, 3, \ldots$, one is most decidedly not making a judgment concerning actual human thoughts or existing expressions. Otherwise it might, for example, have been true to say that all propositions of the form '$2n + 1$ is a prime' were true until someone actually formed the thought that, e.g. 9 is prime. One might be said to be making a judgment about *possible* thoughts or expressions in such cases, and indeed Leibniz had spoken of possible thoughts as objects similar to Bolzano's propositions in themselves.[19] But "possible thought" won't do as a

[17] B. Bolzano and F. Exner, *Der Briefwechsel B. Bolzano's mit F. Exner* ed. E. Winter (Prague, 1935), p. 63.

[18] B. Bolzano, *Von der mathematischen Lehrart* ed. J. Berg (Stuttgart–Bad Cannstatt: Fromann-Holzboog, 1981; also in **BBGA** II.A.7.), §2. Hereafter *Method.*

[19] *Phil Schriften* ed. Gerhardt, 7.190-93.

definition of proposition, Bolzano explains, for *ideas* (i.e. *parts* of propositions) may also be the matter of thoughts:

> ...it would therefore be completely wrong to say that every possible thought is a proposition. If we want to correct this mistake, we shall have to narrow down the concept of a thought. But how? I can see no other way but to declare a thought to be a mental proposition. But then we can obviously not define the concept of a proposition via that of a thought (WL, §23).

The introduction of propositions and ideas in themselves has earned Bolzano the name of a "logical Plato", and commentators have been moved to say that he postulated a *supersensible world* of propositions and ideas. But Bolzano merely claimed that *there are* propositions, and denied emphatically that they exist, or have reality. In the course of his argument he claims that none of the ordinary nouns indicating existence are applicable: propositions have neither *Sein*, nor *Dasein*, nor *Existenz*, nor *Wirklichkeit* (WL. §19). He thus distinguishes between two sorts of being: being something, a status shared by abstract and real objects, and really existing, i.e. in time, having causal relations with other real objects. Logically this difference is marked as follows. Real, or actual existence is an *attribute* of objects, and a proposition stating that something exists has the canonical form 'A has actuality,' where 'A' refers to the thing in question. The weaker sort of being, on the other hand, is not an attribute of objects. To say that there are As, e.g. there are propositions, is a statement not about propositions but rather about the *concept* of a proposition, namely, that this concept has an object (or, in Bolzano's technical terms, 'The concept A has reference [*Gegenständlichkeit*]').

It is quite possible to worry about exactly what may truly be said to be in this weaker sense, but Bolzano shows little interest in this question, reflecting fairly closely the relaxed attitude towards abstract objects later expressed in Carnap's "Empiricism, Semantics and Ontology".[20] As we would now say, he quantifies over logical objects in certain well-defined contexts. He did this to develop an adequate semantics and logic, not because he claimed transcendental insight into things that subsist without actually existing. The main point of interest is "the relations among propositions", and to speak of these, Bolzano thinks, we must accept that there are propositions. This, we think, can be read as saying that the business at hand is to investigate the relations among propositions, and perhaps their internal structure, without asking for the nature of the relata. The various relations investigated will tell as all we can say with assurance, or at all, about propositions. The objects themselves, the propositions, are identified and so to speak defined by the relations in which they are embedded. It is informative to compare this approach with Carnap's structuralist work *The Logical Structure of*

[20] In L. Linsky ed., *Semantics and the Philosophy of Language* (University of Illinois Press, 1952), p. 208–40.

the World and his claim that "each scientific statement can in principle be so transformed that it is nothing but a structure statement."[21]

Ideas

Propositions are *structured* abstract objects, composed of parts combined in a certain way. The parts of propositions which are not themselves complete propositions are called *Vorstellungen*, a term which has been variously translated as *ideas*, *presentations* and *representations*. It should be noted that this was a significant departure from the then current uses of the term. The etymology of the word suggests a relation to an object as an intrinsic feature of a presentation—presenting being a matter of putting something before the mind—and indeed the word was customarily understood in just this way. For Bolzano, on the contrary, an idea is just an element of a proposition, a constituent of meaning; and ideas like those designated by the words 'and', 'or', 'not' are called ideas by the same right as ideas like those designated by words such as 'philosopher' 'wise' and 'Aristotle'.

Extension, Content and Structure of Ideas

Logicians before Bolzano characterized ideas in terms of their content and extension. Bolzano adheres fairly closely to the traditional definitions of these terms, but adds a further refinement in the case of content, namely, he recognizes the way the parts of an idea are ordered as a third factor responsible for the individuation of ideas. The concepts of extension and content continue to play an important role in Bolzano's logic, for he attempts to define a number of central concepts in terms of them, including the distinction between intuitions and concepts, the distinction between purely conceptual and empirical propositions and sciences, and even modal notions like necessity and possibility (see below, section 4.1).

Let us begin with extension. Many ideas have, or refer to, objects, for example, 'Aristotle', 'city', 'prime number'. Others, for various reasons, do not, e.g. 'and', 'nothing', 'rational square root of 2', 'golden mountain'. When an idea has one or more objects, the collection of all its objects is called its *extension*. Ideas may have extensions of all sizes. The ideas 'golden mountain' 'and', for example, have no objects; 'even prime' has one; 'natural number less than n' has n; 'natural number' infinitely many.

Some ideas are complex, built of parts combined in a certain way. Since the same parts can be combined in different ways to produce different ideas (for example, 'a man bitten by a dog' and 'a dog bitten by a man'), we can distinguish two important features of complex ideas: the set of their parts (which Bolzano calls their *content*), and the way they are put together (Bolzano doesn't provide a term for this second feature; we will call this the *arrangement* of the parts of an idea) (WL, §§56, 58). By contrast, some ideas are simple, having no parts. Ultimately, Bolzano thinks, all ideas are either themselves simple or else composed of simple

[21](Berkeley, 1967), p. 29

parts. Simple ideas, having neither parts nor arrangement, are individuated only by their extensions (WL, §96.3). The core of Bolzano's theory of representation is thus the notion of reference, or having an object, a notion which he thinks is indefinable.[22]

Two ideas may have the same extension and yet different content, witness: 'icosahedron', 'regular solid which has the greatest volume among all those of a given surface area'. Two ideas may have the same content, and yet have different extensions, as for instance: '2^3' and '3^2'. Finally, two ideas may differ only in the arrangement of their parts, having the same extension and content, e.g. 2^4 and '4^2'.

According to many theories of representation current at that time, extension and content vary inversely: the more parts an idea contains, the fewer objects it represents. This doctrine was defensible under the assumption that ideas were merely sets of characteristics. Adding more characteristics to an idea, provided they are not redundant, makes the condition it expresses more exigent, and thus narrows the extension; removing characteristics—abstracting—makes the extension wider. Bolzano saw that this thesis could not be maintained for a sufficiently rich theory of ideas, however. Counterexamples abound: the idea 'man who speaks French or German', for example, has a greater content *and* a greater extension than the idea 'man who speaks German.'[23]

To our knowledge, Bolzano says nothing concerning the individuation of syncategorematic ideas such as 'and' or 'not'. In light of the logical importance of such ideas, and his careful treatment of the features that individuate denoting ideas, this seems a serious oversight.

Relations between Extensions of Ideas

Bolzano defines the usual relations between extensions of ideas, thus providing the concepts for an elementary logic of classes (WL, §94 f.). Ideas $A, B, C, D, \ldots$ are said to be *compatible* iff they have an object in common. An idea A is said to *include* an idea B iff all the objects of B are also objects of A. When A includes B and B includes A, the two ideas are said to be *equivalent* or *interchangeable* (*Wechselvorstellungen*). When the relation of inclusion is strict, i.e. A includes B but the two are not equivalent, it is said that B is *subordinate* to A, or that A is *higher* than B.

Bolzano proves a few elementary theorems concerning these relations (WL, §105), but there is little noteworthy or original in this part of his treatment.[24] In §108 (cf. *Method*, §5.7), however, he does take a new step when he seeks to

[22] "Aufsatz, worin eine von Hrn. Exner in seiner Abhandlung: 'Über den Nominalismus und Realismus' angeregte logische Frage beantwortet wird," (Prague, 1843; reprinted in **BBGA**, I.A18.71-78), p. 74.

[23] Bolzano-Exner correspondence (ed. Winter) p. 81 ; cf. WL, §120.

[24] J. G. E. Maass (*Grundriss der Logik* [Leipzig 1793; there were numerous later editions]), as Bolzano remarks, had defined a similar set of relations. Cf. J. Sebestik, *Logique et mathématique chez Bernard Bolzano*, p. 171 f.

extend these relations to ideas with no objects. Consider the ideas 'regular solid with more than 20 faces' and 'regular solid with 21 faces'. Bolzano remarks that many logicians would find it acceptable to say that the latter idea is subordinate to the former even though neither has any objects. In order to make sense of this usage, he suggests the following method of extending the concept of subordination to such ideas:

> we consider certain of their components $i, j, \ldots$ *variable*, and pay attention to the behaviour of the infinitely many *new* ideas that are produced when we replace $i, j, \ldots$ with different ideas, noting what occurs whenever one or the other of them becomes a referring idea. .... We say that A is *higher* than B if on every occasion when a certain determination of the variable parts $i, j, \ldots$ for which A or B becomes a referring idea, A represents all the objects of B as well as some others. (Method, §5.7)

In the case under consideration, we can vary the part 'regular'. In every case where a substitution (e.g. 'three-dimensional') produces new ideas that have objects, we find that the latter is subordinate to the former. This technique is especially useful, Bolzano suggests, in dealing with non-referring concepts in mathematics such as $\sqrt{-1}$. (For Bolzano, the expression $\sqrt{-1}$ designated an idea of a *magnitude* that, multiplied by itself, yielded -1, and there was no such magnitude). Thus, for example, he thinks we should consider equations (which he construes as expressions of the equivalence of ideas) such as $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{-2 \cdot -3}$ as being verified in this extended sense (i.e. since $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ whenever a and b are chosen so as to render these expressions referring ideas). It should be noted that the extended logical relations spoken of in such cases are relative to a choice of components in the ideas in question, a constant feature of his use of this technique of variation (cf. below, section 4.2).

Intuitions and concepts

Bolzano defines *intuitions* as ideas that are simple and have exactly one object (WL, §72; Method, §6; intuitions are discussed at length in the correspondence with Exner). Concepts, by contrast, are ideas that are not intuitions and contain no intuitions as parts, while ideas that contain both concepts and intuitions as parts are called *mixed*. Bolzano's intuitions closely resemble the *logically proper names* of Russell's "Logical atomism"—they are best expressed by the demonstrative 'this', refer exclusively to a present mental state, and are necessarily involved in any reference to existing particulars. They are relevant to a discussion of Bolzano's logic because he uses the distinction between concept and intuition to define that between purely conceptual and empirical propositions and sciences (which, for him, supplants the traditional *a priori-a posteriori* distinction) as well as the modal notions of necessity and possibility. A purely conceptual proposition is one that contains no intuition, while a purely conceptual science contains only purely conceptual propositions. Other propositions and sciences, those containing intuitions, are called *empirical*. Purely conceptual sciences, that is, make no reference to existing particulars, and can be expressed without indexicals or demonstratives. Finally, to say that a proposition is *necessarily* true/false means, according to Bolzano, that it is a purely conceptual truth/falsehood; while to say that a proposition is possible is just to say that it is not the case that the negation of the proposition is a purely conceptual truth.

Definitions and Explications

Bolzano recognizes a variety of means for conveying the meaning of an expression or symbol. His general term for such means is *Verständigung*, which may be translated as "arriving at an understanding" or more familiarly as "explication". One prominent variety of explication is a *definition*, a sentence indicating the parts of an idea and their arrangement (WL, §555, §668.8; *Method*, §9). Bolzano recognizes that this means is not always available: not, for instance, when the idea in question is simple, and thus has no parts; nor in cases where we are not sure what the parts are; nor in cases where the expressions we need to use to define an idea are unknown to the reader. For such cases, he singles out an especially important form of explication, which is a sort of contextual definition or definition in use.

> One of the most general [means of explication] is the following: we set out various sentences in which the concept which is designated by our sign appears in such combinations that no other concept could be thought in its place if these sentences were to express something reasonable. By considering and comparing these sentences the reader will gather by himself the meaning of our sign (*Method*, §9[25]).

Bolzano recognizes that this is not a fine-grained means of communicating content: the best one may hope to achieve by it is that the reader will connect an idea with the sign which is *equivalent* to, i.e. co-extensive with, the idea we connect with it.[26]

Form(s) of Propositions (1): The Subject-Predicate Form

Based in large part upon a grammatical analysis, Bolzano contends in §127 of the WL that all (?true) propositions have the proximal form 'A has b', where 'A' is a concrete idea (i.e. an idea which refers to objects), and 'b' an idea which refers to

[25] Cf. *Contributions*, II §8; WL, §668.9: "If we find a so far unknown symbol in connection with others whose meaning we already know, then the mere assumption that the author did not intend to say something obviously absurd often allows us to determine more or less exactly what he meant to express by it. In such cases one says that the meaning of the symbol was recognized through its use or context."

[26] *Method*, §9, also WL, §668.9

attributes of objects.[27] He thought that expressing propositions in this way was a good way of clearly understanding their content, and spent not a little ingenuity in attempting to find expressions of this form for a variety of sentences. 'There are As', for example, has the canonical form 'The idea A has reference'; 'Some A are B' is rendered as 'The idea of an A which is B has reference'; 'α or β or γ' as 'the idea of a true proposition among α, β, γ has reference.'; as mentioned above, 'Necessarily A' = 'A is a purely conceptual truth.' 'Possibly A' = 'Not-A is not a purely conceptual truth' (WL, §182.4).

Bolzano has fifteen chapters in which he analyses the forms of sentences, and ten treating the history of the subject (WL §§169–194). Significantly, this occurs after the discussion of consistency, consequence and probabilification. It is, in other words, not a chapter that sets out the grammar of a formal structure as a preliminary to the discussion of logical properties and relations. It is meant, rather, as an aid to the exegesis of texts (*Auslegungskunde*, WL §169, 2.212). He has selected for analysis propositions that are important because they occur "in several sciences" (WL 2.211), in a separate and independent chapter in his broad treatment of logical topics.

Bolzano's attempt to reduce all sentences to the subject-predicate form seems to have been a poor decision in certain respects. It has been suggested, for instance, that the development of argument patterns, the calculational aspect of logic, was not notably advanced by Bolzano, due to his rigidity in matters of logical form.[28] At the same time, it is worth pointing out that his logic is not thereby limited to the power of a first-order monadic predicate calculus (traditional syllogistic is sometimes interpreted in this way). For Bolzano allows, among others, the predicate 'reference (*Gegenständlichkeit*)', i. e. the property of having objects. Thus, as noted above, he renders 'There are *A*s' as 'The idea *A* has reference'; and 'some *A* are *B*' as 'the idea of an *A* which is also a *B* has reference;' one recognizes the similarity of these renderings with the standard formulations $(\exists x)Ax$ and $(\exists x)(Ax \cdot Bx)$. Since he also has negation, recognizes relations, and allows the formation of concepts of indefinite complexity, Bolzano's subject-predicate form seems to possess at least the expressive power of first order logic.

4.2 Variation Logic, Ground and Consequence

A science, for Bolzano, is a collection of propositions ordered by what he called the relation of *ground and consequence* (*Abfolge*). After the analysis of propositions, the second order of business in the theory of elements was, accordingly, to produce an account of this relation. Bolzano, as we have mentioned, thought of sciences as possessing an intrinsic order in this respect. Just as ideas had components structured in a certain way and permitted of only one definitive analysis into parts, so too, he thought, there must be a single, intrinsic, deductive order for the propositions of a given science.

[27] As discussed below [section 4.2], Bolzano also speaks of another notion of form in the context of his variation logic.

[28] W. and M. Kneale, *The Development of Logic* (Oxford, 1964), p. 370 f.

Bolzano's attempts to characterize the notion of ground and consequence involve the more general notion of *deducibility* (*Ableitbarkeit*), which is itself integrated into a general theory of variation logic. We will now give an account of this latter theory before returning to his attempts to pin down the relation of ground and consequence.

Consequence and the Logic of Variation

It is generally agreed that Bolzano was the first to give a viable definition of logical consequence. But he deemed this concept "too obvious and important ... to have escaped entirely the attention of logicians." Yet it seemed to him that "the nature of this relation has not always been correctly grasped, or, if comprehended, discussed with insufficient generality, or without a precise definition..." (WL §155.11). A proper account must begin by exploring Bolzano's claim that logical relations hold between abstract entities, i.e. propositions and ideas in themselves. We have seen that, according to Bolzano, pure logic is not concerned with judgments, mental manifestations, but with their contents. Less appreciated is the equally important point that the objects of logical inquiry are not linguistic entities. In a logic so conceived, problems of synonymy can be addressed only by considering ideas of linguistic expressions. Synonymy and equivocation can occur only in the latter, not in ideas in themselves. The same holds for well-formation. There are no ill formed formulas, no formal nonsense, in the realm of propositions.

Analytic and Synthetic Propositions

From a given proposition other propositions can be generated by replacing certain of its ideas. For example, let 'The man Caius is mortal' be given. There is then a set of propositions differing from this only in the element 'Caius'. As it is replaced by others, e.g. 'Sempronius', 'Titus', 'rose', 'triangle' a set of propositions is generated containing 'The man Titus is mortal,' 'The man triangle is mortal,' and so on. We use the word *variand* to designate constants tagged for substitution, like 'Caius' in this example. Properly speaking, there are no variables and no propositional functions in Bolzano, only propositions, from which sets of propositions can be generated by replacing some constants by others.

In a still unpublished paper "On Substitution", of 1904 (in the McMaster Archives) Russell espoused a view not unlike that of Bolzano, and indeed of a definitely Bolzanesque flavour. In Monk's synopsis:

> Instead of working with propositional functions like 'x is mortal', one works with straightforward propositions such as 'Socrates is mortal' and 'Plato is mortal'; and instead of having the notion of a variable 'x' which can be determined by individuals like Socrates and Plato, one has merely a technique of substituting one individual [name] for another in any given proposition. The advantage of this is that it does

away with both the notions of propositional functions and classes in favour of simply propositions.[29]

To return to our example: the substitution of 'triangle' results in a proposition whose subject idea has no referent, is "empty". Such propositions Bolzano deemed to be false. He now defines as follows:

> If there is merely a single idea in a proposition which can be arbitrarily replaced without changing its truth or falsity (provided reference [*Gegenständlichkeit*] is preserved), that is, if the propositions that turn up are either all true, or all false..., then I allow myself to call such propositions, with an expression borrowed from Kant, *analytic* propositions... and all others *synthetic* propositions (WL §148).

Thus, 'The man Caius is mortal' is analytically true, 'The man Caius is omniscient' analytically false with respect to 'Caius'. 'Caius is mortal,' on the other hand, is synthetic with respect to the same idea.

Bolzano notes that to determine the analyticity of propositions more than just "logical knowledge" is often required, as in our example. There is, however, a notable class of propositions whose analyticity can be determined with only logical knowledge, e.g. 'an A that is B is A', 'an A that is B is B'. In these cases we would now say that the set of variands is the set of extralogical terms. This distinction, he notes, is not definite "since the domain of concepts that belong to logic is not clearly delineated" (WL §148.4). Such propositions may be called *logically analytic*. The concept of analyticity is extended to sets of propositions, which are said to be analytic if all their elements are.

Bolzano's definition shows its mettle in mathematical contexts. For instance (his examples) 'If A is larger than B, then B is smaller than A', 'If $P = MN$, then $M = \frac{P}{N}$.' Assuming, as we must, that A, B, etc. are not variables, but constants whose value we don't know, these propositions are analytic with respect to them. The advantage over Kant's definition, that a judgment is analytic if the predicate "is contained" in the subject, is obvious. First, Bolzano allows for analytically false propositions, and second, his definition applies to propositions of other than subject-predicate form. It must be borne in mind that analyticity is a binary relation: a proposition is analytic with respect to certain of its component ideas.

Form(s) of Propositions (2): Form in Variation Logic

The concept of variation gives rise to another notion of the forms of logical objects. In §§12 and 81 of the WL, he distinguishes two senses of "form" which are important for logic. In the first place, a form is a class of propositions generated by variation. The form of the equation

$$(1)\quad 2 \cdot \frac{1}{2} = 1$$

[29] R. Monk, *Bertrand Russell: The Spirit of Solitude* (London: Jonathan Cape, 1996), p. 185).

relative to the designation of both occurrences of 2 as variable, for example, is the class of all propositions which may be generated from it by uniform substitutions of appropriate ideas for the two occurrences of 2. Thus, among infinitely many others, the equations:

$$1 \cdot \tfrac{1}{1} = 1 \tag{2}$$

$$\pi \cdot \tfrac{1}{\pi} = 1 \tag{3}$$

$$-17 \cdot \tfrac{1}{-17} = 1 \tag{4}$$

belong to the form in question.

Bolzano explains that the theorems of logic do not deal with individual propositions and consequences, but with whole classes or species of them, and that it is a formal science only in this sense (WL §12.1). Hence,

> whenever one calls logic a formal science ... and speaks of the forms of representations, propositions and arguments, the word *form* is used in the sense of *kind* (WL §81.2).

A second sense of form refers to expressions in which some meaningful terms are replaced by symbols serving as place-holders. Thus one could speak of

$$(5) \quad a \cdot \frac{1}{a} = 1$$

as the form of (1), relative to ... etc. (WL, §12.2).

It will be noted that because form (in both senses) is relative to a specification of variable parts, logical objects may be said to have a number of distinct forms. Thus it rarely if ever makes sense in Bolzano's logic to speak of *the* form of a proposition or argument.

Relations among Propositions in Variation Logic

Compatibility The method of variation establishes certain relations among propositions. A collection of propositions $A, B, C, \ldots$ is said to be *compatible* (*verträglich*) relative to the specification of certain parts $i, j, k, \ldots$ as variable iff there exist $i', j', k', \ldots$ which, when substituted for $i, j, k, \ldots$ in $A, B, C, \ldots$ result in propositions $A', B', C', \ldots$, all of which are true. Note that, as usual, propositions are not compatible *simpliciter* but only with respect to the designation of certain parts as variable. To illustrate this point, consider:

$$3 < 5 \tag{6}$$

$$5 < 3 \tag{7}$$

The propositions expressed by these equations are incompatible if all occurrences of the parts '5' and '3' are considered variable, but compatible when the two occurrences of '$<$' are taken as the variable parts (since, for example, substituting '$\neq$' for '$<$' results in two true propositions).

The example suggests that compatible proposition must share an idea, but this is not so. Bolzano maintains that all true propositions are compatible, no matter which ideas are thought variable, "for the ideas originally occurring in them ... make them all true" (§154.6). Since not all true propositions share ideas, the ideas with respect to which $A, B, \ldots$ are compatible need not occur in them.

If there is no substitution on $i, j, \ldots$ that makes all of $A, B, C, \ldots$ true, then they are *incompatible* with respect to these ideas. Thus 'No finite being has omniscience,' 'Man is a finite being,' and 'Some man has omniscience' are incompatible with respect to 'finite being', 'man' and 'omniscience' (§154.2). Note that none of these three propositions contains all three of the variable terms. It follows, although Bolzano does not pursue this line, that inconsistent propositions can be compatible. '4 is a prime number' and '4 is divisible by 2' are inconsistent, but compatible with respect to 'prime number'.

Several theorems are established: any set of incompatible propositions contains at least one false one (7); all propositions in a compatible set may be false, but in this case each must contain at least one of the ideas that are marked for variation (but they need not share them) (8). Incompatible sets may contain subsets that are compatible with respect to the same variands. But if a given set is incompatible with respect to certain variands, then so is any superset.

Deducibility and equivalence *Deducibility* (*Ableitbarkeit*) is Bolzano's general notion of logical consequence. He defines it as a special case of compatibility (as a result of this decision, nothing follows from incompatible premises in Bolzano's variation logic). A set of propositions $M, N, O, \ldots$ is said to be deducible from the propositions $A, B, C, \ldots$ with respect to the variants $i, j, k, \ldots$ iff

1. The propositions $A, B, C, \ldots$ are compatible with respect to the variants $i, j, k, \ldots$; and
2. Every substitution of ideas $i', j', k', \ldots$ for $i, j, k \ldots$ which makes all of $A, B, C, \ldots$ true also makes all of $M, N, O, \ldots$ true (WL, §155).

Thus, for example, $e < \pi$ is deducible from the propositions $e < 3$ and $3 < \pi$ with respect to the variable ideas 'e', '3', 'π'. *Equivalence* is defined as mutual deducibility (WL, §156).

A number of theorems about consequence are established. Since consequence is a triadic relation, several of these cannot be rendered if consequence is, as usual, construed as diadic, and deserve special comment. An argument will be valid with respect to a set of variands if its form, i.e. the class of propositions generated from it, contains no argument whose premisses are true, and conclusion false. It follows that every subclass of that form contains only valid arguments, from which we obtain this theorem:

> If a conclusion follows from a set of premisses with respect to a certain set of variands, then it also follows with respect to any subset thereof (WL §155.19).

'No fishes are mammals' follows from 'No mammals are fishes' with respect to 'mammal' and 'fish'. Now, if only 'fish' is a variand and 'mammal' a constant, the class generated is a subclass of the previous, hence also valid, and so for all other cases.

By contrast, it does not hold that if a relation of consequence holds with respect to certain variands, then it will also hold if further ideas are varied. For example, 'Socrates is mortal' follows from 'All men are mortal' by variation of 'mortal' but not by variation of all extralogical terms (WL §155.20). However, variands that do not occur in the argument at all may be added without impairing validity (WL §155.23).

Further, if a variand does not occur in a premiss, then deletion of that premiss still leaves an argument valid with respect to the given variands. For example, 'All men are mortal, Socrates is a man, therefore Socrates is mortal' is valid with respect to 'mortal', but so is the enthymeme 'All men are mortal, therefore Socrates is mortal,' resulting if the minor premiss (which does not contain 'mortal') is deleted.

In contemporary logic, a form of an argument is a schema from which the argument results by introducing constants for variables or schematic letters. Validity and other logical concepts are defined in the first instance for schemata, and hold derivatively for the arguments generated from them. The argument $A \to A, A \therefore A$, in a logic of schemata, is an instance of the valid schemata *modus ponens* and *iteration*, but also of the invalid schema *asserting the consequent*, and some others. In the logic of variation, by contrast, the form is a class of arguments generated *from the argument itself* by varying its components. If A is the only variand, then this set consists of all arguments in which any proposition replaces the A. It is a subset of the more inclusive set modus ponens, but the latter cannot be obtained by variation from the argument at hand.

There are always more schemata than there are Bolzano-style forms. His manner of delineating these forms has the advantage that it is always clear *which argument is meant.* In his conception, an argument of valid form is valid, of invalid form invalid. In a logic of schemata, by contrast, every argument instantiates invalid schemata. To show that it is invalid, one must show that it instantiates no valid schema.

We have seen that some of the consequences envisaged are logical, others enthymematic. Bolzano does not use this term, simply distinguishing logical consequences from those that require knowledge that goes beyond logic. Consider again 'All men are mortal, therefore Socrates is mortal.' This is valid with respect to 'mortal', but to see this, one needs to know that Socrates was a man. It turns out that this additional premiss is strictly equivalent to the claim that every substitution on 'mortal' that makes the premiss true also makes the conclusion true. The same can be said for all enthymemes resulting from the deletion of a premiss from a classical categorical syllogism. Bolzano's logic thus embraces a treatment of an important class of enthymematic, or material consequences.[30] This removes an awkwardness inherent in the classical treatment of the subject, where an enthymeme is described as an argument with a "missing premiss", requiring a cluster of *ad hoc* admonitions, principles of charity, that restrict the additional premiss that is to be supplied.

Enthymematic consequence, unfortunately, is not transitive. 'Some vertebrates have hearts' implies 'Some vertebrates have kidneys' with respect to 'vertebrate', and the second sentence implies 'Some creatures that have kidneys have bones' with respect to 'has a kidney'. But the last sentence does not follow from the first. Bolzano's ingenious solution to this problem is a theorem, an image of Gentzen's cut rule for triadic consequence:

> If, with respect to certain ideas $i, j, \ldots$ propositions $M, N, O, \ldots$ follow from $A, B, C, D, \ldots$, and $X, Y, Z, \ldots$ follow from $M, N, O, \ldots$, $R, S, T, \ldots$ *with respect to the same ideas*, then $X, Y, Z, \ldots$ follow from $A, B, C, D, \ldots, R, S, T$ also with respect to the same ideas (WL §155.24).

The rider "with respect to the same ideas" is crucial: if two consequences do not share the same variands, then they cannot be chained. Bolzano concluded that every consequence, logical or enthymematic, satisfies a necessary condition of *relevance*, namely, that premisses and conclusion share an element (§155.21). He did not give a strict proof of this, but one can be given for all arguments whose premisses (as he required) are compatible with respect to variands of the deduction, and whose conclusion is not analytic with respect to them. For if they did not share an element, then some substitutions on the variands occurring only in the premisses will make them true, and substitution on the variands occurring only in the conclusion will make it false. Hence in his system the classically valid $A \& \neg A \therefore B$ (*ex absurdo quodlibet*) does not hold.

Bolzano's relation of consequence, being triadic, cannot simply be described as transitive, non-symmetrical, etc. But it can be said to be in a sense transitive, *viz.* if the variands are the same.

The requirement that the premisses be compatible has certain undesirable consequences. First, not every sentence follows from itself. Second, the relation is not monotonic, that is, one cannot add arbitrary further premisses to an argument and preserve validity. Third, it is not the case that if A follows from B, the denial of A follows from the denial of B (though this can be resolved by disallowing analytic conclusions).

Exact deducibility Bolzano distinguishes a special case of consequence, which he calls *exact* or *adequate*. M is said to be an *exact consequence* of $A, B, C, \ldots$ relative to variables $i, j, k, \ldots$, according to the essay on method, when M is deducible from $A, B, C \ldots$ but not from any proper subset of $A, B, C, \ldots$. (*Method,*

[30] See R. George, "Enthymematic Consequence", *American Philosophical Quarterly*, January 1972.

§8.2, there called "deducibility in the strict sense"; WL, §155.26 has a slightly different version).

A notable theorem of this relation is that in exact consequence, no element of the premiss set, nor the conclusion may be analytic. For if one of the premisses had that character, it could be deleted without invalidating the consequence, while an analytic conclusion, for its part, "does not need for its truth the condition that the premisses are true" (WL §155.27). Hence in neither case would the consequence be exact. A further interesting theorem is this: If M follows exactly from $A, B, C, \ldots$ with respect to some variands, than its negation Neg.M is compatible with any proper subset of the premisses. For if Neg.M were incompatible with any such subset, then Neg.Neg.M would follow from it, hence M would follow, and the consequence would not be exact. It follows further that in such a relation no premiss can be a consequence of the rest, and the negation of any of them must be compatible with the rest.

Probability What is now called absolute probability is discussed under the heading of "degrees of satisfiability" (*Gültigkeitsgrad*, WL §147). A proposition A containing the variand j is said to be satisfiable if there is some appropriate substitution on j that converts A into a true proposition A'. In certain well defined circumstances, one can determine the proportion of substitutions, out of all of a given class of substitutions, that generate true propositions. If a lottery has 90 outcomes, with five winners, then the degree of satisfiability of the proposition 'Ticket no. 8 wins' is 5/90, or 1/18. If every ticket wins, then this sentence will be said to have degree of satisfiability 1, and will be reckoned among analytic sentences. Since Bolzano uses *concept* variation, he recognizes that the substitutions must be constrained in order for this approach to work. In the above example, for instance, ticket number 8 could equally well be represented by 'the ticket with the number 32/4', 'the ticket with the number 2^3', etc. Bolzano deals with such cases, in effect, by asking us to substitute only a single idea from each equivalence class and thus attains the same result as with object variation (WL, §147).

Bolzano uses the expression 'probability' (*Wahrscheinlichkeit*, WL §161) only for conditional probability, and sometimes uses the expression 'comparative satisfiability' for it. If $A, B, C, D, \ldots$ are compatible with respect to $i, j, \ldots$ it is often "enormously important" to ascertain the proportion of substitutions that make these propositions true to those that make some other proposition M true. This is "the probability that accrues to the proposition M on the assumption of $A, B, C, D, \ldots$" (§161.1). This proportion can be represented by a fraction, always ≤ 1, since M cannot be true more often than $A, B, C, D, \ldots$. Indeed, if it is true whenever they are, then it is deducible from them. Bolzano then proves a substantial number of theorems of the calculus of probability. His conception is not unlike that of Wittgenstein in *Tractatus* 5.15,[31] and that of Carnap.

[31] "If **Tr** is the number of the truth-grounds of a proposition 'r', and if **Trs** is the number of the truth-grounds of a proposition 's' that are at the same time truth-grounds of 'r', then we call the ratio **Trs**:**Tr** the degree of ***probability*** that the proposition 'r' gives to the proposition 's'."

It is important, Bolzano notes, to distinguish probability, the measure of a relation between propositions, from the confidence in a proposition, though the confidence should be, and often is, a function of ascertained probabilities.

Deducibility and Ground-consequence

We return, finally, to the notion of ground and consequence (*Abfolge*), discussed in WL, §§162, 198–222 and *Method* §§13–14. Recall that Bolzano assumed that the propositions of a science had an intrinsic order, some being the grounds of others, their consequences. An important task for logic was to characterize this intrinsic order so that we could put the truths of a science "in the order that they themselves prescribe" (WL, §2). Bolzano found support for his view that there was such an order in Aristotle's distinction between demonstrations that merely show *that* a proposition is true and those that also show *why* it is true,[32] and in Leibniz's remarks in the *New Essays*:

> A reason is a known truth whose connection with some less well-known truth leads us to give our assent to the latter. But it is called a "reason", especially and *par excellence*, if it is the cause not only of our judgment but of the truth itself.... A *cause* in the realm of things corresponds to a *reason* in the realm of truths, which is why causes themselves—and especially final ones—are often called "reasons".[33]

Just as ideas permit of definitive analysis, Bolzano thought, so too the order of the propositions of a science must somehow be uniquely determined.[34] In line with his anti-psychologism, it is clear that this order is not the order in which we come to recognize the truths of a science, but rather an *objective* order holding between truths in themselves (*Method*, §§13–14). Bolzano seems to have thought that the ground-consequence relation was in some respects like deducibility, in some like causality, and in others like "providing an explanation for". Not surprisingly, these apparently discordant intentions are not unified into a single coherent theory. Bolzano tells us that a certain uncertainty plagues everything he says on the subject, and warns us not to expect anything by way of a complete account, his goal being to draw the reader's attention to an important unsolved problem. (WL, §195) We will give a sketch of his remarks.[35]

[32] *Posterior analytics*, $78^a 22$ f.

[33] IV.xvii.§3, tr. Remnant and Bennett.

[34] One might contrast with this view Quine's statement ("Two dogmas of empiricism"): "Relative to a given set of postulates, it is easy to say what a postulate is: it is a member of the set. ... But given simply a notation, mathematical or otherwise, and indeed as thoroughly understood a notation as you please in point of the translations or truth conditions of its statements, who can say which of its true statements rank as postulates? Obviously the question is meaningless—as meaningless as asking which points in Ohio are starting points."

[35] More thorough accounts may be found in Buhl *Ableitbarkeit und Abfolge in der Wissenschaftslehre Bolzanos*, *Kantstudien* Ergänzungshefte 83 (1961), P. Mancosu, "Bolzano and Cournot on Mathematical Explanation," Revue d'Histoire des Sciences **52**(1999)429-455, and Sebestik, *Logique et mathématique chez Bernard Bolzano*, Part 2, Ch. 4.

The order observed in axiomatic presentations of mathematics is an order of proof, and so it might be thought that the relation of ground-consequence is just the relation of deducibility. However, Bolzano thought it clear that deducibility alone would not do the job, not even the narrower notion of exact deducibility. For there are cases of mutually deducible, or equivalent propositions where it seemed clear to Bolzano that the ground-consequence relation could only go one way. Consider, for example, the following two propositions:

1. The barometer reads higher today than yesterday.

2. The air pressure is higher today than yesterday.

These propositions are mutually deducible when 'today' and 'yesterday' are considered variable. What is more, knowing (1) allows us to recognize the truth of (2); (1) is what Bolzano calls a subjective ground or "ground of knowledge". On the other hand, he thinks it absurd to say that (1) is an *objective* ground for (2); rather, (2) should be thought of as a (partial) *objective* ground for (1):

> ...we will never believe that the truth that the barometer stands lower today than yesterday is the *objective ground* of the truth that the atmospheric pressure is lower today than yesterday. Rather, it is obvious that between these two truths it is rather the converse relation that holds; the latter truth is one of the partial grounds from whose connection the former is produced as a consequence: *because* the air pressure has decreased, the mercury has sunk. (*Method*, §13)

The causal language is no accident in this case. Indeed, Bolzano thinks that causal statements are at bottom just statements of ground/consequence relations:

> "X is the cause of Y" actually means "the truth that X is the case is related to the truth that Y is the case" as ground (partial ground) to consequence (partial consequence) (WL, §168).

But the relation of ground-consequence is not limited to, nor is it reducible to, cause-effect relations, for Bolzano thinks that it also applies to propositions dealing with things that have no causal relations, e.g. propositions in themselves or mathematical objects. He claims, for example, that although the following two propositions are mutually deducible relative to the variables 'A' and 'B', the former should be considered the consequence of the latter but not vice versa (we shall see why he thought this in a moment).

1. A pair of circles in the same plane, one with centre A and radius AB, the other with centre B and radius AB must intersect

2. A and B being distinct points, there exists a third point C at distance AB from both A and B.

In general, while deducibility may be mutual, Bolzano thinks that ground-consequence is anti-symmetric. Other structural features also separate deducibility from the ground-consequence relation: any proposition not analytically false is deducible from itself, but no proposition is its own ground. Finally, deducibility is transitive, while ground-consequence is not.

We have seen that in some cases causal relations may determine the ground-consequence relation. But this cannot be the case in. e.g. mathematics. How is the relation determined there? Bolzano frankly admits that he has no completed theory. However, he does have several suggestions. In *Method*, §17, he points to two features, simplicity and generality, which place some conditions on ground-consequence relations in mathematics. A proposition A is called *simpler* than B if the content (section 4.1 above) of A is a proper subset of the content of B. A is called *more general* "when either its subject or its predicate idea, or both, are of greater extension." He sets out the following rules:

1. The simpler truth cannot be the consequence of the more complex.

2. Assuming two truths to be equally complex, the more general truth cannot be the consequence of the more special.

To illustrate the use of the first of these criteria, let us return to the two propositions mentioned above:

1. A pair of circles in the same plane, one with centre A and radius AB, the other with centre B and radius AB, must intersect

2. A and B being distinct points, there exists a third point C at distance AB from both A and B.

(1) should be considered the consequence of (2), Bolzano argues, but not vice versa (*Method*, §13). Having settled on the view that geometrical objects should be defined as point-sets, Bolzano saw that to say that two circles intersect is just to say that two point-sets have an element in common. But to say that the set of points at distance AB from A and the set of points at distance AB from B have a common member we first have to know that there is a point such as described in (2), and we can know this without even considering the sets in question. (2) is thus *simpler* than (1), since it does not involve the notion of set, and thus cannot be a consequence of (1).

Finally, Bolzano offers what he considers a sufficient condition for truths $A, B, C,$ to be the grounds of another truth M (WL, §221.7). M is a consequence of $A, B, C, \ldots$, he states, if M is *exactly deducible* from $A, B, C, \ldots$ relative to variable parts $i, j, k, \ldots$, and if $A, B, C, \ldots$ are the *simplest* of all the propositions equivalent to $A, B, C, \ldots$ relative to $i, j, k, \ldots$ We can see the general thrust of Bolzano's inquiry: exact deducibility involves deducibility as well as the independence and indispensability of the premises. The criterion of simplicity provides the means to choose between rival axiomatizations meeting the previous criteria.

As Bolzano himself conceded, there are large gaps in this account of ground-consequence. One might even wonder if Bolzano's problem is sufficiently well-defined to admit of a solution. He ends his discussion with the following remark:

> I occasionally doubt whether the concept of ground and consequence, which I have above claimed to be simple, is not complex after all; it may turn out to be none other than the concept of an ordering of truths which allows us to deduce from the smallest number of simple premises the largest possible number of the remaining truths as consequences (WL, §221).

5 MATHEMATICS

In the nineteenth century, Bolzano was known as a mathematician primarily on the basis of two works, the *Purely Analytic Proof*... (1817) and the posthumously published *Paradoxes of the Infinite* (1851). The publication of Bolzano's mathematical manuscripts in the twentieth century showed that these works were just small parts of Bolzano's wide-ranging and systematic mathematical studies.[36] The *Purely Analytic Proof* was both an important and influential work. Published four years before Cauchy's *Cours d'analyse*, it was among the earliest works in the "new analysis" of the nineteenth century. Although brief, this paper contains a wealth of conceptual distinctions and fruitful proof techniques, providing a remarkably detailed model of how to "arithmetize" analysis. We find here, for example, a rigorous definition of continuity, the "Cauchy" criterion for the convergence of an infinite series, a precise statement of the least upper bound property of the reals (every bounded set of real numbers has a least upper bound), and a nice proof of this result using the technique of iterated bisection. (A slightly earlier work, the *binomische Lehrsatz*... (1816), contains one of the first rigorous developments of the theory of power series).

The *Theory of Functions*[37], which Bolzano worked on beginning in the 1830s, extends this early work and provides a strikingly modern development of real analysis. There he gives a still sharper definition of pointwise continuity,[38] distinguishing left-, right- and two-sided continuity at a point, and apparently also between

[36] Accounts of Bolzano's work in the foundations of mathematics may be found in A. Behboud, *Bolzanos Beiträge zur Mathematik und ihrer Philosophie* (Bern: Bern Studies in the History and Philosophy of Science, 2000); P. Rusnock, *Bolzano's philosophy and the emergence of modern mathematics* (Amsterdam: Rodopi, 2000); J. Sebestik, *Logique et mathématique chez Bernard Bolzano* (Paris: Vrin, 1992).

[37] *Functionenlehre* ed. Rychlik (Prague, 1931).

[38] "If a function Fx ... is so constituted that the variation it undergoes when ... its variable passes from a determinate value x to the different value $x + \Delta x$ diminishes *ad infinitum* as Δx diminishes *ad infinitum*—if, that is, Fx and $F(x+\Delta x)$ (the latter of these at least from a certain value of the increment Δx and all smaller values) are measurable [i.e., roughly speaking, real and finite], and the absolute value of the difference $F(x + \Delta x) - Fx$ becomes and remains less than any given fraction $\frac{1}{N}$ if one takes Δx small enough (and however smaller one may let it become): then I say that *the function Fx is continuous for the value x*, and this *for a positive increment* or *in the positive direction*, when that which has just been said occurs for a positive

pointwise and uniform continuity (FL, I, §13).[39] Some of the most important theorems about continuous functions are given elegant proofs, many employing the "Bolzano-Weierstrass" theorem (i.e. an infinite set of real numbers contained within a closed interval has a limit point in the interval). Finally, Bolzano has a solid grasp of the relations between continuity and differentiability. Among other things, he constructs a function which is everywhere continuous and nowhere differentiable (FL I, §75; II,§19).

Nor did his work in the foundations of analysis end there. In the *Größenlehre*, Bolzano develops an arithmetical theory of real numbers (Bolzano calls his numbers "measurable", and his theory is actually slightly more general than a theory of real numbers).[40] This is developed around the notion of an infinite number concept (and associated expressions), e.g.

$$1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\cdots$$

An infinite number expression A is said to be measurable iff for every integer q there is exactly one positive integer p and two infinite number expressions P_1, P_2 (where P_1 is either zero or "purely positive" and P_2 purely positive), such that:

$$A=\frac{p}{q}+P_1=\frac{p+1}{q}-P_2$$

A measurable number, in effect, induces a "cut" in the rationals. The fractions $\frac{p}{q}$ are called the measuring fractions of A. Two number concepts are said to be equivalent if they have the same measuring fractions. Bolzano proves a selection of important results concerning measurable numbers, among them the Bolzano-Cauchy theorem (every "Cauchy" sequence of measurable numbers has a measurable limit), the least upper bound theorem, and, apparently, the Bolzano-Weierstrass theorem (In the FL, Bolzano refers to a proof of this result in the theory of measurable numbers, but none has been found in his papers).

In the *Größenlehre*, as also to a limited extent in the WL and the *Paradoxes of the Infinite*, Bolzano develops a very general theory of collections (*Inbegriffe*). An interesting variety of collection is the *set* (*Menge*). This is a collection of elements "whose basic conception renders the arrangement of its members a matter of indifference, and whose permutation therefore produces no essential change

value of Δx; *for a negative increment* or *in the negative direction*, on the other hand, when that which has been said holds for a negative value of Δx; if, finally, the stated condition holds for a positive as well as a negative increment of x, I say, simply, that Fx is *continuous* at the value x." (FL , I, §2).

[39] So at least it has been argued. See P. Rusnock and A. Kerr-Lawson, "Bolzano and uniform continuity," *Historia Mathematica* (forthcoming).

[40] Classic references are K. Rychlik, notes to B. Bolzano, *Theorie der reellen Zahlen im Bolzanos handschriftlichen Nachlasse* (Prague, 1962); B. van Rootselaar, "Bolzano's Theory of Real Numbers," *Archive for History of Exact Sciences* **2** (1964) 168-80; D. Laugwitz, "Bemerkungen zu Großenlehre Bolzanos," *Arch. Hist. Ex. Sci.* **2** (1965) 398-409; also discussed in Behboud, Rusnock, Sebestik.

from the current point of view" (*Paradoxes* §4). Bolzano notes as a characteristic property of infinite sets that they can be mapped 1 : 1 onto proper parts of themselves, a result up to then thought inconsistent. A theory of natural numbers is also developed in the theory of collections.

Bolzano's contributions to the foundations of mathematics, including his pioneering work in set theory, along with the similar work of Gauss, Cauchy, Abel, Dirichlet, Weierstrass, Heine, Schwartz, et al., made the centrality of quantificational concepts in mathematics quite obvious. This may have had some influence on later logicians like Frege and Peirce, who gave a prominent place to quantificational concepts in their logical theories.

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HUSSERL'S LOGIC

Richard Tieszen

Edmund Husserl (1859–1938) wrote extensively on logic and mathematics. Most of the books that he published were on these subjects, as were many of his published articles. His books that were not devoted exclusively to logic or mathematics usually contain at least some remarks on these topics. In addition, Husserl left thousands of pages of research manuscripts, lecture notes, correspondence, and other writings on logic and mathematics. Much of this material has now been published in the Collected Works of Husserl, the *Husserliana*, but there is more to come. My approach in this article will be to follow the historical progression of Husserl's thought on logic, primarily as it is represented in his publications. In order to minimize controversies about different interpretations of and emphases in Husserl's works, I will follow the texts rather closely. This will allow us to see how his views changed over the years, how new concepts entered into his thinking about logic, how some views were rejected or set aside, and so on. I will also comment on the materials on logic and mathematics that are part of the Husserl *Nachlass*. Not only do Husserl's views change and mature over time but there are also various obscurities in these views. I will indicate some of these as we proceed.

Husserl began his career as a mathematician. He received a doctorate in 1883 for a thesis on the calculus of variations. He soon turned to 'psychological' and philosophical issues concerning logic and mathematics. He continued to work on the philosophy of logic and mathematics for the rest of his life, although his philosophical vision widened to include many other topics. Husserl was personally acquainted with or corresponded with many of the leading logicians and mathematicians of his day. These included Weierstrass, Cantor, Hilbert, Zermelo, Frege, Weyl, Schröder, and others. His work on logic up to roughly 1904 shows a familiarity with a remarkably wide literature on the subject that includes the writings of both philosophers and mathematicians. Husserl's own contributions are primarily philosophical or phenomenological. There is little in his work by way of technical, formal logic as we now think of it. He did not create a particular formal system, did not prove new theorems, did not devise new technical methods, and so on. Some of his ideas lend themselves to such development but Husserl typically did not pursue the work in this direction. In some of his writings he describes the division of labor between technical and philosophical work in logic and mathematics. The philosophical and the technical work are said to have different ends but they are both valuable and they can be viewed as complementing one another. Husserl saw himself as investigating the epistemology, ontology and methodology of logic and mathematics. Instead of creating a particular formal system, for example, he wished to reflect on the nature of formal systems, their place in logic and

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mathematics, and on how or in what sense they could supply knowledge. We find extended analyses of the nature of logic, the nature of arithmetic and geometry, formal systems and algorithms, meaning theory, the nature of reference, essences, the intuition of ideal objects, the origin of logical and mathematical concepts, and so on.

Husserl's views on logic were shaped by a variety of forces. There were different influences at different points in his career (see, e.g., [Mohanty, 1995]). His own teacher Brentano certainly had an impact on Husserl but Husserl also rejected some of Brentano's ideas about logic. Bolzano was an important influence, as was Lotze. Husserl often cites Leibniz with approval when he speaks of his conception of logic as *mathesis universalis*. Frege may have had an influence but its exact nature has been a subject of some dispute in the literature (see, e.g., [Føllesdal, 1994], [Mohanty, 1982], [Brown, 1991], [Drummond, 1985], [Hill and Rosado-Haddock, 2000], [McIntyre, 1987], [Willard, 1989]). Husserl in turn had an impact on a number of people. His assistant Oskar Becker, for example, took up some of Husserl's ideas on mathematics and discussed them with people like Hermann Weyl and Arend Heyting (see Becker [1923; 1927], [Mancosu and Ryckman, 2002]). That there was some influence on Weyl can be seen in a number of Weyl's writings (see Weyl [1994; 1985; 1967]; also [da Silva, 1997], [Tieszen, 2000], [van Dalen, 1984], [Tonietti, 1988]). There was also some impact on the Polish school of logic, especially in connection with Twardowski, Lesniewski and Adjukiewicz. More recently, Kurt Gödel has displayed a strong interest in some of Husserl's ideas about meaning clarification and our knowledge of abstract (or 'ideal') objects (see [Gödel, 1961], [Tragesser, 1977], Tieszen [1992; 1998], [Føllesdal, 1995]). For whatever reasons, some of Husserl's ideas have resonated with philosophers of logic and mathematics at different points in history and some have not. It is ironic that there has been almost no consideration of Husserl's view on logic and mathematics by those philosophers in so-called Continental philosophy who are viewed as Husserl's successors. Many of Husserl's contributions to logic and mathematics are only now being investigated in earnest. This is probably due to the changing philosophical climate but in the English-speaking world it is no doubt also due to the presence of more translations of Husserl's work. Husserl's work is richly suggestive and it speaks to many issues of contemporary relevance in the philosophy of logic and mathematics. This should become apparent as we proceed.

1 PHILOSOPHY OF ARITHMETIC (1891) AND EARLY WRITINGS ON LOGIC AND MATHEMATICS (1890–1908)

Husserl's first book is called *Philosophie der Arithmetik: Psychologische und Logische Untersuchungen (PA)*. Although it is usually regarded as a pre-phenomenological work it is an important book in the development of Husserl's thought about logic. The major influences on Husserl at this time seem to have been Stumpf and Brentano. A number of themes are present or prefigured in the book, however, that persisted throughout Husserl's later thinking about logic and mathematics. In particular, we already see in this book the emphasis on the analysis of the origins in everyday experience of concepts of mathematics and logic. As in later writings, Husserl discusses some of the cognitive acts and processes (e.g., attention, abstraction, collective combination, and reflection) required to obtain the concepts of logic and mathematics from everyday perceptual experience. Husserl also rejected or considerably revised some of the views he had expressed in *PA*. In particular, he went to great pains in later writing to avoid the 'psychologism' that critics like Frege found in the book. Husserl would be careful in later work, for example, to distinguish characteristics of numbers from characteristics of presentations of numbers. There seem to be places in *Philosophy of Arithmetic* where this distinction is at work but there are also places where the distinction should be made or at least be made more clearly. Husserl does not yet distinguish numbers as 'ideal' objects from the 'real' mental acts and processes in which we come to know about numbers. Ideal objects are not yet part of his ontology. Husserl went to great pains in later work to distinguish the phenomenological investigation of numbers and other objects of mathematics and logic from the (empirical) psychological investigation of these phenomena. Some other ideas that are very important in later works are not present in *PA* at all. The later view of intentionality, for example, is not here nor is the distinction between empty and fulfilled intentions that plays such an important role in Husserl's later epistemology. In Part II of *PA* 'logic' is conceived largely as a theory of calculation involving concrete, sensible signs. The conception of logic already broadens by the time of the *Logical Investigations*. I will outline some of the main ideas of *PA*, without attempting comprehensive treatment.

1.1 Philosophy of Arithmetic

In the Part I of *PA* Husserl treats central psychological questions about the concepts of plurality (Vielheit), unity (Einheit) and number (Anzahl) in the case where these phenomena are directly given to us in concrete experience. This is distinguished from the case where they are represented only through indirect symbolization, as in the case of large numbers. At this point of his career Husserl saw himself as concerned with the descriptive psychology (in Brentano's sense) of the concepts of plurality, unity and number. Part II of the book takes up the 'symbolic' methods of arithmetic and the symbolic ideas of plurality and number. In this latter part Husserl argues that most of arithmetic must be understood in terms of symbolic concepts and methods since we are so limited in our ability to experience pluralities or numbers in an 'authentic' or 'genuine' way. Here the investigation centers on the 'logical' or symbolic constructions that lead to the extension of arithmetical knowledge beyond what can be given in the limited, authentic manner. Thus, what we have in *PA* may be viewed as largely a combination of descriptive psychology and a kind of formalism.

Husserl begins by saying that he will restrict himself to the notion of cardinal number. In later research he saw that an account of the ordinal conception of number would also be important. In *PA* he wants to consider the origin and content of the genuine concepts of plurality and number. He provides a psychological analysis of the type of abstraction that leads to the concepts of plurality and number. Many terms are used more or less interchangeably in this text with the notion of plurality: multitudes, aggregates, sets (Menge), totalities (Inbegriffe). Husserl thinks the notion of number presupposes the concept of plurality and, thus, he turns his attention to the latter notion. The basis of these concepts lies in totalities or pluralities of definite objects that are given to us in concrete sensory experience. Suppose we perceive as a group (1) a cat and a dog and a house, (2) a pencil and a chair and a car. What is common to all of the different types of specific multiplicities that we might experience in this way is the manner in which their elements are combined or connected. The nature of the particular 'contents' that are compounded does not enter into consideration. Now if we are after the *general* concepts of plurality and number it is clear that the contents need not be restricted to physical things. The definite objects that make up a plurality can in principle be of any kind whatever, e.g., a few persons, several houses, an angel, a feeling, and so on. The concepts of number and plurality only arise, however, if the connection of the individual elements into a whole is present. The concepts of plurality and number do not arise merely from the presence of the particular contents. The presentation of a totality of given objects is a *unity* in which the presentations of the individual objects are contained as partial presentations. What is the nature of the unification that must be present in order for the concept of plurality to arise? Husserl says that there must be a 'collective connection'. Collective connection is a said to be a type of synthesis. The concept of plurality arises through reflection on the union of contents of a concrete totality. Husserl says a psychical act of second order gives us awareness of a plurality.

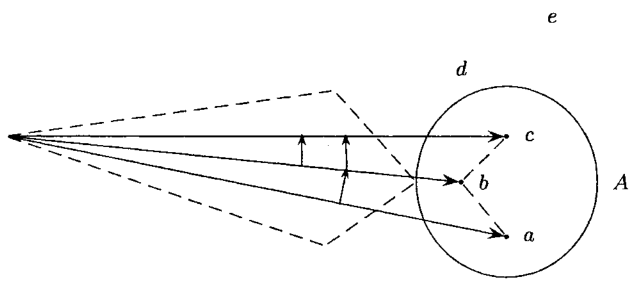
For purposes of comparison it might be of some use here to think of Cantor's definition of a set (Menge) as a many that allows itself to be thought as a One. Husserl is concerned with a psychological account of how it is possible to be aware of a cardinal number as a One based on a 'many' that is given to us. What Husserl is after here, however, is an analysis of the authentic presentation of numbers (as opposed to the merely symbolic representation of numbers) and his account (at least at this stage of his work) applies only to very small finite cardinals.

Husserl investigates the psychological nature of collective connection. The problem is to obtain the *general* concepts of plurality and number from the concrete pluralities or totalities that are given to us in sense experience. We know that the nature of the individual objects of a concrete plurality cannot contribute to the content of the general concept of plurality or number. Thus, one must consider the connection of the objects in the unified presentation of their totality. Collective connection, Husserl argues, cannot be reduced to any other kind of relation, such as a relation of time, of difference or of equality. Following Brentano, Husserl argues that there are different kinds of relations: relations of 'primary contents' or 'physical phenomena' but also relations that are psychical. In the latter case the contents are united by a cognitive act. Collective connection, Husserl says, is a psychical relation. The syncategorematic term 'and' expresses in ordinary usage the elementary nature of this collective connection. The presentation of collective connection is obtained by reflection on the psychical act that connects contents into a totality. We have, for example, some content *and* another content *and* another content. One could simply have these contents one by one in experience without reflecting on the whole. It is through reflection on the whole, however, that the concept of plurality is formed. The concept of collective connection, by the way, is not identical to the concept of plurality but it is said to be the most important constituent of the latter concept.

Husserl holds at this point that no concept can be thought without being founded on a concrete intuition. The general concept of plurality depends on the intuition of a concrete plurality from which the general concept is abstracted. We abstract from the particular nature of the compounded contents while their connection is retained. It is possible to abstract from the particular contents while attending to the collective connection because abstracting just amounts to not observing something. Husserl defines ***plurality in general*** as "something and something ..." or as "one and one and..." The general concept of plurality has an indeterminateness that arises from the "..." or the "and so forth". This is not supposed to mean that there is no limit to the collection of 'somethings' or 'ones' but only that no limit is found or that a given limit does not matter. If this indeterminateness is removed the concept of plurality breaks up into a multiplicity of distinct concepts of numbers. Concepts such and "one and one", "one and one and one", and "one and one and one and one" arise, which have been named 'two', 'three', and 'four', respectively. Number concepts, however, need not be derived from the general concept of plurality. They can be obtained directly from concrete pluralities. Numbers can be abstracted by regarding the compounded single contents as 'somethings' or 'ones'. The concepts of number and plurality agree in their essential contents except that the concept of number involves the distinction of the abstract forms of plurality from one another whereas in the concept of plurality this is not involved. The concept of number arises out of the comparison of distinct forms of plurality (or numbers). Husserl also discusses in more detail the concepts 'one' or 'something' since these play an important role in his view of number. Indeed, the concept of number is said to be constituted by (i) the concept of collective connection and (ii) the concept of 'something'. It is in this manner that Husserl thinks he has provided a basic account of the origin and content of the concept of number.

We might indicate basic ideas of Husserl's account in the following diagram (see [Willard, 1984, p. 43], [Tieszen, 1989, p. 151]):

Suppose a–e are objects in a given field of consciousness. Now we could be aware of different groups of these objects. Husserl thinks we can be aware of something like 3 or perhaps 5 objects at a glance (in one act). For very small totalities we would not need a succession of acts to determine their number. We know that we



are intuiting a totality all at once and we can determine the number at a glance. Otherwise, 'authentic' counting is required and even here we soon reach our limits. Thus, suppose that the unbroken lines to a, b and c are distinct acts in which these objects are noticed or considered. The arrows crossing between the unbroken lines are awarenesses of earlier such acts built into subsequent acts, ordering them into a progression. There is a retention of the awareness of one thing in the awareness of the next thing. This taking together of objects, however, does not yet constitute one object. Indeed, in ordinary perception we often perceive objects in a sequence like this without thinking of a totality of which they are parts. We must direct our awareness toward the connection of the objects itself. The diffuse arrow formed from the broken lines is the founded (reflective) consciousness of the higher-order object, the totality consisting of a, b, and c, which we label as A. The dotted lines connecting a to b to c are the 'collective connections', excluding the other objects in the field of consciousness. If we think of this as underlying the cardinal conception of number then we leave out or abstract from the ordering. Consideration of the ordering would, however, be involved in the origins of the ordinal conception of number (although, as indicated above, this is not explicitly considered by Husserl in *PA*). Notice that we have also in effect already abstracted from what the particular objects of consciousness are. A formal abstraction is at work in which we have something a and something b and something c.

Do these remarks imply that the collective connection is mental or that pluralities or numbers are therefore mental entities? It was Frege in particular who raised these kinds of questions about Husserl's work in *PA*. Arithmetic, it appears, is concerned with formal properties of pluralities or multiplicities formed by the mind. Husserl does say things in parts of *PA* that would lead us to believe that numbers themselves are purely formal and objective. They are not subjective. On the other hand, there is language in *PA* that suggests that these formal properties of pluralities are formed by the mind, and Husserl's ontology at this point seems to include only the physical and the psychical but not ideal objects. One could presumably go back and rewrite *PA* to make the matter clear but Husserl himself moved on to other things (but see p. 226 below and Section 8 below on the origin of sets).

Husserl goes on to provide a psychological analysis of the relations '$<$' and '$>$'. It is psychologically less easy to distinguish 24 from 25 than to distinguish 5 from 6. Intuition either begins to fail or to become more prone to error as the size of the numbers increase. Hence, we resort to mechanical operations of counting or reckoning. Psychologically speaking, however, the foundations of the relations '$<$' and '$>$' must come together in one act of consciousness. Thus, the totalities that are to be compared must be present in one act, in which the two totalities are united. The limits of actual presentation are thereby reach very quickly and so everything else is a matter of figurative or symbolic presentation.

Husserl discusses the definition of the equality of number through the concept of 1-to-1 correspondence. He refers to the work of Grassmann, Leibniz, Frege, and some others. He again approaches this by way of psychological considerations. Psychologically speaking, one can *compare* two aggregates or pluralities with respect to their number. They have the same number if their units can be placed in 1-to-1 correspondence in thought. In order to determine equality of number the two pluralities can simply be counted in a symbolic sense. In this manner their equality or inequality will be determined and the numbers themselves will be provided. One of his claims in this part of the book, however, is that numbers are not themselves relational concepts based on extensional equivalence, for every number statement would then refer to its relations to other classes or aggregates instead of referring to a concrete given aggregate. To assign a number to a class would be to classify it under a group of equivalent classes. Husserl thinks this is not really the meaning of a statement about numbers. A given class of trees is not said to be four because it belongs to a certain class of infinitely many classes that can be placed into 1-to-1 correspondence with one another. Indeed, Husserl says in *PA* that the extensions of the concepts of plurality and number are precise and are given, even if we do not grasp the essence and origins of these concepts. Husserl expresses his opposition to what he considers remote and artificial constructions that reinterpret basic concepts of arithmetic in terms of "strange or unhelpful concepts". He remarks at one point that mathematicians have sometimes been too zealous in their desire to attain rigor in all definitions even though some elementary ideas are neither in need of definition nor capable of it. One can, however, try to clarify such concepts. At this point in his career he thinks such clarification can be obtained through descriptive psychology. Later he will say that this kind of clarification is to take place through phenomenology.

Frege's account of number, in particular, is considered unacceptable by Husserl. Husserl claims that one can only define that which is a logically complex. Simple, elementary concepts like equality, similarity, whole, part, plurality, and unity are incapable of formal, logical definition. All we can do is exhibit the concrete phenomena from which they are abstracted and make clear the nature of the abstraction process. Husserl, unlike Frege, does not find it objectionable for mathematicians to describe the way in which one comes to awareness of the concept of number, instead of beginning with a logical definition of number. The concept of number, like the concepts of plurality and unity to which it is so closely related, also cannot be defined. This kind of view is similar in some respects to the views of mathematicians like Poincaé, Weyl and Brouwer. Frege's goal of defining number is said to be chimerical. Husserl points out that Frege gives us a defintion of the extension of the concept of number but not its content or meaning. The extension of a concept consists of the totality of all the objects falling under it. For Frege, 'the number that attaches to the concept F' is defined as the extension of the concept 'equal in number to the concept F'. The concept of this number is the sum of the concepts with the same number as F. This is a sum of infinitely many 'equivalent' classes. Frege does not arrive at what we mean or intend in thinking of the number belonging to F. Husserl, for reasons mentioned above, thinks it is pointless to comment further. Many of Husserl's strong remarks about Frege's work are of course considered by Frege himself in his famous review of *PA* [Frege, 1977].

Husserl's own analysis of number and his views on unity and plurality do not seem to allow for the numbers 0 and 1. Frege is quoted to the effect that what does not apply to 0 and 1 cannot be essential to the concept of number. Husserl is prepared here to deny that 0 and 1 are concepts of number. Numbers are regarded by Husserl as all of the conceivable determinations of the indeterminate concept of plurality. Now 1 is not a collection of objects, nor is 0. Husserl suggests that the terms 'one' and 'zero' function linguistically as number determinations, as this would be understood by grammarians and there may be some scientific reasons for such extended usage but, logically speaking, they are not numbers. Husserl also responds to the following problem that is posed by Frege: if we attempt to account for the origin of number as arising from compounding of different objects we simply get a heap of different objects and not a number, but if we try to form a number by compounding equals then by virtue of their equality we never get a plurality. Suppose we start from a concrete aggregate. We say, for example, that Jupiter, a house and a tree are three. Now in Husserl's account of number Jupiter, a house and a tree are alike insofar as each is a unit or a mere 'something' after formal abstraction. Their 'equality' in this sense, however, is a result of the abstraction of numbers and not their basis. We have "something and something and something", not just "something". The concept of number is constituted by the concepts of something and collective connection. Now the unification that gives us a number is due to the relation of collective connection and not to equality. Recall that the presentation of collective connection is obtained by reflection on the psychical act that connects the contents. In this manner we have the concept of a whole that connects parts in a collective manner. Frege continued to be unimpressed with this analysis, as can be seen in his review of *PA*.

In his review of *PA* Frege charged Husserl with psychologism and objected to Husserl's views on abstraction, definition, the numbers 0 and 1, the idea that number does not attach to a concept, and other points. Various writers have discussed the views of Frege and Husserl on number, including their different views about numbers as logical objects, definition, extensions and intensions, the origins of concepts, the grounds of arithmetic knowledge, and so on (see, e.g., Tieszen [1990; 1994], [Resnik, 1980, Chp. 1], [Hill and Rosado-Haddock, 2000]).

In discussing the meaning of statements of number Husserl thinks that it follows from his discussion that a number cannot be regarded as a determination of a concept. Number does not, for example, refer or attach to a concept in Frege's sense. Instead, number is the general form of plurality under which the totality of some objects $a, b, c, \ldots$ falls. This plurality or totality is the subject of a number statement. Husserl says that the extension of a concept has a number but not the concept itself. We can say, at most, that a concept has the property that its extension has the number three. The typical statement of number, however, does not have this complex meaning. It should be noted that part of the disagreement between Frege and Husserl here results from their different views of what 'concepts' are and from their different ideas about the role of intensions and extensions in mathematics (see, e.g., [Tieszen, 1994]).

Husserl rounds out the first part of *PA* with a supplement in which he presents and criticizes the nominalistic treatments of number in the work of Herbart and Kronecker.

Part II of *PA* is on the symbolic concepts of number and the *logical* sources of arithmetic. Husserl provides a psychological and 'logical' account of our operations with numbers. He holds, for example, that we are able to retain a number of presentations of totalities at the same time and to unite them collectively to form totalities of totalities. As Husserl has been saying, however, our abilities to do this in an authentic way are rather restricted. Numbers arise directly through the counting of pluralities and indirectly through operations of calculation. Husserl discusses in an interesting way some aspects of the psychology of counting.

The fundamental operations by which new numbers can be formed from given numbers are addition and partition. With the exception of 0, 1 and 2, all numbers permit of further partitions into numbers. Addition amounts to the formation of a new number by the collective connection of the units of two or more numbers. Multiplication appears basically to be a new kind of notation that abbreviates a type of addition. It is a symbolization that expresses the way in which a number is to be found. In order to obtain the number intended the underlying additions must actually be carried out. Addition connects numbers but partition separates a given number into a plurality of number parts. Subtraction and division are two special cases of partition.

If the term 'operation' refers to an activity that is actually carried out with the numbers themselves then there are no other operations than connection and partition. Generally speaking, however, the notion of operation in arithmetic is not taken to be so limited. Arithmetic, in other words, is not concerned with actual numbers. All presentations beyond the first few numbers in the number series are only symbolic. Husserl thinks this is important for understanding the nature of arithmetic. If we had genuine presentations of all numbers there would be no arithmetic. The most complicated relations among numbers that are presently discovered by complicated calculations would be just as evident intuitively as $4 +$

$2 = 6$. As a matter of fact, we are very limited in our capacity for presentation. The actual presentation of all numbers could only be expected in the case of an infinite understanding. Such a form of understanding would have the ability to unite a real infinity of elements into an explicit presentation. It is also conceivable that there are finite beings who could attain actual presentations of numbers in the trillions but for them there would also be no practical reason for developing an arithmetic. Arithmetic is just the sum of technical means used to overcome the imperfections of our intellect. It is only under the most favorable circumstances, as indicated by psychological research, that we can actually present concrete pluralities of around twelve elements, i.e., that we can determine at a glance, in one act, the number. Husserl thus says that the upper limit of our genuine (or 'actual') number concepts is reached at around twelve. The fact that we do not feel inhibited by this limit in arithmetic shows that other factors must be involved in arithmetical knowledge.

It is worth noting that on this early view of Husserl's almost none of our arithmetic knowledge counts as intuitive, actual or genuine. The 'symbolic' takes precedence over the intuitive almost immediately, and at the very lowest levels of arithmetic knowledge. Later on, Husserl's conception of intuition and of what can be given in intuition broadens quite significantly. In *LI*, for example, he speaks of the intuition of numbers as ideal objects. He also distinguishes empty from fulfilled intentions and the static from the (rule-governed) dynamic fulfillment of an empty intention. In PA, however, the language of empty and fulfilled intentions is not present nor is there any talk of intuiting ideal objects, essences, concepts or meanings. I discuss this in more detail below (see especially Section 2.8).

In *PA* Husserl is led to ask how we can speak of concepts (or objects) that are not actually given. The most certain of all sciences, arithmetic, is based on such concepts. The answer is that the concepts (or objects) are given in a symbolic manner. A content can be presented to us indirectly only through signs that characterize it uniquely. In such a case it is presented in a symbolic but not in a genuine manner. (It appears that Husserl was led to these ideas on the basis of attending some lectures of Stumpf.) We have, for example, a genuine presentation of the external appearance of a house when we actually view it but only a symbolic presentation if someone characterizes it indirectly for us by saying that there is a house in such and such a location. Symbolic presentations serve as provisional substitutes for genuine presentations. In cases where the real object is inaccesssible, they are lasting substitutes for genuine presentations. There can also be symbolic presentations of abstract or general concepts. A certain species of red, for example, can be actually presented but it can also be symbolically or figuratively presented.

What is the origin and meaning of symbolic presentations in the sphere of number? Husserl investigates this in some detail. Among other things, we see that we can only speak of larger sensuous pluralities in a symbolic sense. The successive apprehension of each element of the plurality may be possible but not their comprehensive collection. A genuine presentation of a plurality requires as many psychical acts as there are contents present and these must then be united by a psychical act of second order. If we look at the sky, however, we immediately judge that there are 'many' stars. How do we account for cases in which in which hundreds or perhaps thousands of elements are noted immediately if we can only genuinely apprehend about a dozen elements? The presentation in these cases must presumably be symbolic and not genuine. Husserl answers this question by appealing to a kind of 'fusion' of elements that is analogous to the fusion that Stumpf discovered in the case of simultaneous sense-qualities. 'Quasi-qualitative' factors (Momente), or what von Ehrenfels called 'Gestalt qualities', are involved in these kinds of cases. Husserl goes into the psychology of these 'figural factors' in some detail. They allow us to grasp a large concrete plurality as a plurality without grasping all or even many of its elements individually. Here there are some interesting early applications of Gestalt theory to the psychology of number awareness.

Husserl also considers infinite pluralities. It is impossible to actually form such pluralities or to symbolize them through the succesive enumeration of all the individuals concerned. In every case of an infinite plurality there is a symbolic presentation of a process of conceptual formation that can be continued without limit. Husserl says that what such a continually extending plurality can include or not include is determined a priori by means of precise concepts. It can be decided concerning every given object of thought whether it can or cannot be an element of this process. These comments are made in the context of Husserl's 'formalism' (i.e., his early views on merely symbolic presentation) and they might appear rather naïve or at least unclear in light of later developments in logic and mathematics. Is Husserl speaking here merely of some 'ideal' situation? In any event, this view does prefigure Husserl's later ideas, to be discussed below, about 'definite manifolds' and definite axiom systems.

In the case of an infinite plurality the concept of a last step or a last element is meaningless, whereas the same is not true in the case of a finite plurality. Husserl thinks that the intention of forming an actual infinite plurality is absurd and is ruled out logically. The presentation of a determinate unlimited process, however, and the concept of all that is included in such a domain on account of its conceptual unity, are regarded as logically acceptable. Husserl says that the concept of a plurality in this case is contained in the notion of the process involved. It is not the concept of plurality in the true sense of the term.

The symbolic presentations of pluralities form the foundation for the symbolic presentations of numbers. Numbers, as we saw, are the discriminated species of the general concept of plurality. Husserl says that a rigorous and systematic principle is needed for the formation of symbolic number-forms (numerals) that supplement the narrow domain of numbers that can be actually presented. The process providing the number-forms must be determined uniquely so that for every genuine number not more than one symbolic number-form will result. The idea is to find a principle by which a system of signs can be constructed out of a few basic signs. The system should attach an easily distinguishable sign to every number and indicate its place in the number series. Since we cannot actually construct large numbers we are led in this way to the use of 'logical' postulates and concepts. Husserl says that we defer to systems of signs that are set up according to rules.

Toward the end of *PA* Husserl defines arithmetic as the science of the relations of numbers. One derives certain numbers from given numbers by means of relations that obtain among the numbers. The method of derivation is either essentially conceptual, with the notation playing a subordinate role, or it is an essentially sensuous operation that derives signs from signs by means of rules. The latter method, Husserl says, is more advantageous in arithmetic. It is concrete, sensuous, comprehensive and easy to use. There are some interesting points of comparison with Hilbert's formalism here (see, e.g., [Mahnke, 1977; Heelan, 1988; Majer, 1997; Hill, 2000]). The conceptual method, on the other hand, is highly abstract, restricted and difficult to practice. Husserl suggests that there is no problem that the method of sensible signs could not solve. He says that it is accordingly the logical method of arithmetic that he was seeking. If we attend only to the technical methods permitted by such a system we have the pure mechanics of calculation that is basic to arithmetic and that makes up the technical side of its method. In various writings from this period that he did not publish Husserl discusses the role of calculation and algorithms in mathematical knowledge (see [Husserl, 1983]).

Husserl believes that in the second part of *PA* he has provided an important investigation of arithmetic as symbolic knowledge of a certain type. He has indicated the central types of symbolic representation involved in it and has shown how they depend on a purely formal system of signs for their construction. Arithmetic, as such a system of symbolic representations, results from the 'logical' demands made upon our epistemic limitations concerning number. The theory of symbolic representations will provide a 'logic' (or an algebra) for arithmetic. In light of these views on the role of computation on concrete, sensuous signs in our arithmetical knowledge it is interesting that Husserl was to take a position at Göttingen in 1900. It is known that Hilbert had some role in helping Husserl to obtain this position. Husserl's later views, however, were destined to diverge rather significantly in a number of respects from Hilbertian formalism.

It is worth mentioning that Husserl had planned a second volume of *PA* which would consider the general logic (or algebra) of symbolic methods ('semiotic'). In this latter volume the arithmetic of numbers would appear as a member of a whole class of arithmetics. Husserl said he would provide more detailed investigations concerning symbolic representation and the methods of knowledge grounded on such representation. A philosophy of Euclidean geometry was also to be included in the volume. The projected second volume never appeared.

1.2 Some Other Early Writings on Logic and Mathematics

There were a great many other writings on logic and mathematics in the period following *PA* and preceding the publication of *LI*. Much of this material was not published by Husserl but it is now available in the *Husserliana* (see [Husserl, 1970]

Ergänzende Texte, and [Husserl, 1979; Husserl, 1983; Husserl, 1984; Husserl, 1994] and [Husserl, 1996]). I will briefly describe some of this work.

It is known that Husserl worked on geometry from 1886 to 1903. In the writings on space from this period Husserl distinguishes four different concepts of space: the prescientific space of everyday life, the space of pure geometry, the space of applied geometry, and the space of metaphysics. The first three concepts of space lead, genetically speaking, to the last. He also distinguishes the logical from psychological treatment of space. The psychological treatment would include a descriptive account of the content of the idea of space and a genetic account of the origin of the idea of space. The concept of prescientific space already involves some idealization since this space, as a whole, is not perceived. The space of pure geometry is a logical or conceptual structure, based on an idealization of perceived space. This work also contains various reflections on the privileged status of 3-dimensional geometry (at least in a genetic account of geometry), on the formalization of geometry, and on how one arrives at the idea of the 'Euclidean manifold'. While Husserl did not publish the material on geometry from this period some of the ideas made their way into his later writings (e.g., see Section 7.2 on the "Origin of Geometry" below).

There is a good deal of work from this period that continues the line of thinking in Part II of *PA* concerning calculation with concrete symbol systems, algorithms, and related themes. Husserl's 1901 double lecture in Göttingen is of interest in this regard. If authentic presentations of the natural numbers already break off around the number 12 then what do we say about negative, rational and irrational numbers? How can we justify our reasoning in the case of these 'imaginary' concepts? Apart from the fact that it will need to be justified in some way by recourse to symbolic presentations the answer in the manuscript of these lectures is not as clear as it could be. Years later, in *Formal and Transcendental Logic (FTL)* (see Section 6 below), the answer is given as follows: if the formal, axiomatic symbol systems we are using are 'definite' then calculating with imaginary concepts can never lead to contradictions. I will return to the notion of 'definite' formal systems below but at the moment it can be noted that a 'definite' axiom system is evidently a consistent and complete axiom system. Husserl's answer to the questions, it seems, would be like Hilbert's insofar as we need to shift to properties of a purely formal symbol system to justify our reasoning about concepts or objects that cannot be given to us in intuition or in an authentic manner. Following out the line of thinking that starts in Part II of *PA*, this is how one would justify real analysis. One would turn to the idea of computation in a consistent purely 'symbolic' system. As mentioned above, the comparison with Hilbert is quite interesting here. In Hilbert's work the idea of justifying parts of mathematics on a purely symbolic basis was eventually formulated in a very precise program.

It is interesting to note how the distinction between the intuitive and the symbolic in mathematics is worked out by different writers (especially Husserl, Hilbert and Weyl) during this period (see, e.g., [Tieszen, 2000]). Husserl's early view seems more like Hilbert's but later on Husserl distinguishes the meaning-fulfillment of expressions, which requires intuition, from the meaning-intention of expressions, which is associated with mere concepts or meanings, and he also distinguishes these meaning-intentions from what he calls the 'games meaning' of signs. In other words, Husserl's later view is more complex.

One of the more revealing documents from this period is Husserl's long letter to Stumpf ([Husserl, 1994]) in which he describes some of the changes that are taking place in his thinking about logic and mathematics. There is also a long review of Schröder's *Vorlesungen über die Algebra der Logic* and a paper entitled "The Deductive Calculus and the Logic of Contents". Among other things, Husserl shows in these writings that he is concerned with the debate between the extensional (Umfangs) and the 'content' (Inhalt) logicians. While Husserl had not yet clearly distinguished himself from psychologistic logicians, he spared no effort in criticizing the formal, extensional logic of Schröder and others. In "The Deductive Calculus and the Logic of Contents" he tries to show how a merely formal calculus like Schröder's can be interpreted intensionally (as a calculus of contents) just as well as extensionally (as a calculus of classes). He notes that even under the intensional interpretation, however, the calculus amounts to a technique which is not to be confused with genuine thinking.

During this period Husserl prepared reports on German writings in logic from 1894 and 1895-99 as well as notes on and reviews of the work of Marty, Twardowski and others. Among the most interesting items from this period are those that show how Husserl was pursuing various avenues in his thinking, retaining some ideas and abandoning others in an effort to arrive at a satisfactory position. We see some of the ideas involved in the transition from *PA* to *LI*, for example, in "On the Logic of Signs (Semiotic)". Here we find the idea that signs can refer by virtue of their meaning, and that in this way an intention can be present even if it cannot be carried out to completion with respect to its subject matter. In "Psychological Studies in the Elements of Logic" Husserl says we must examine the mind itself and its acts and contents to see how it achieves knowledge, whether through algorithms or otherwise. In the first part of this essay there is discussion of how cognitive contents can be either independent or dependent, and either abstract or concrete. The much longer second part is concerned with clarifying the notions of intuition and representation and with developing an analysis of the intention-fulfillment relation. Here Husserl argues that there can be intuitions of abstract as well as concrete contents. It is interesting to compare this work with the VIth *Logical Investigation.* The psychology of the intention-fulfillment relation is further explored in the very interesting "Intuition and Representation, Intention and Fulfillment" (see [Husserl, 1994]).

In "Intentional Objects" Husserl starts with what he calls the 'paradox of objectless representations' which results from juxtaposing theses of Brentano and Bolzano. On the one hand, it seems that to each representation there corresponds an object (Brentano), but on the other hand, it appears that this is not true (Bolzano). This is a fine essay in which Husserl points out some problems in the position of Brentano (and in work of some members of the Brentano school), and sides with Bolzano. The essay concludes with an important discussion of truth and the experience of truth in fulfillment. The views on mathematics that are expressed in this piece are quite interesting but they are revised in various ways in Husserl's later work.

For some of the secondary literature on Husserl's views during this period see [Willard, 1984], and Hill in [Hill and Rosado-Haddock, 2000]. In [Tieszen, 1989] I attempted to indicate what some of Husserl's views on number would look like in light of ideas in his later philosophy on intentionality, ideal objects, intention and fulfillment, dynamic fulfillment, ordinals, and so on. The idea was to develop Husserlian views on mathematical intuition in connection with natural numbers and finite sets.

2 LOGICAL INVESTIGATIONS (FIRST EDITION, 1900-1901; REVISED SECOND EDITION 1913-1921)

The conception of logic in the *Logical Investigations* is broadened considerably. Early in the work Husserl considers the ideas of logic as algorithmic, as normative, as technical practice and as purely theoretical. We are now presented with the idea that purely theoretical logic is concerned with 'ideal' meanings. Moreover, it is also concerned with the ontological correlates of ideal meanings, e.g., ideal objects and ideal states of affairs. In *LI* pure mathematics is taken to be part of pure logic in this broad sense.

The *Logical Investigations* are comprised of the "Prolegomena to Pure Logic" and six investigations centered around issues about expression and meaning, 'ideal' objects and theories of abstraction, the theory of wholes and parts, the idea of pure grammar, intentionality, the phenomenology of knowledge, and related topics. I will highlight basic ideas of the "Prolegomena" and each of the six investigations. In the first edition of the *Logical Investigations* Husserl characterized his investigation as a form of descriptive psychology, in the manner of his teacher Brentano. In the second edition, he strives to make it clear that phenomenology is in no sense to be understood as an empirical science. Rather, it seeks *a priori* conditions and essences in its epistemological criticism and clarification of pure logic. Where there is a difference between the two editions I will tend to focus on the more mature views of the second edition.

2.1 "The Prolegomena to Pure Logic"

In the "Prolegomena to Pure Logic" Husserl sets out to demarcate the domain of pure logic as an independent science. Logic is endangered, he says, through confusions with other sciences, especially psychology and some other empirical sciences. Husserl seeks in these Prolegomena to sharpen the conception of pure logic as an *a priori*, theoretical discipline that is formal and demonstrative in nature.

Husserl now describes logic in a broad sense as the "science of all possible sciences" or "theory of science". He says (in §12) that excellent thoughts towards the circumscription of logic are to be found in Bolzano's *Wissenschaftslehre*. The essence of science consists of the unity of knowledge in a whole system of grounded validations. Husserl examines at some length the nature of the unity of science, i.e., the interconnections of things and the interconnections of truths in science (§§62-64). The realm of truth is not a disordered chaos but is dominated and unified by law. The investigation and presentation of truths must therefore also be systematic. Connections of validation are governed by reason and order, by regulative laws, not by caprice or chance. In mathematics, for example, we can find many examples of reasoning about different kinds of objects (e.g., triangles, numbers) that have the form "Every A is B, a is an A, so a is a B". Here we are carrying out a validation that is precisely not a function of chance or caprice. Arguments that take us validly from given pieces of knowledge to new knowledge always have a form that applies to countlessly many examples. Validating procedures do not stand in isolation. With them a definite *type* is always brought out. Forms of reasoning of the type just cited are free of all essential relation to some limited field of knowledge. Logic is what makes possible the existence of sciences. All testing, invention and discovery rests on regularities of form. It is the wide degree of independence of form from a field of knowledge that makes possible a 'theory of science'. Were there no such independence there would be only coordinated logics corresponding separately to the different sciences. There would not be a general logic. Husserl says that in fact both are needed. There should be investigations into the theory of science as it concerns all sciences and, supplementary to these, particular investigations concerning the theory and method of the separate specific sciences.

Husserl says that some of the methods used in science are validation procedures but some are simply auxiliary devices for validation procedures. The theory of science needs to take both of these into account. Some scientific procedures are abbreviations or substitutions for validating arguments and are used to economize thought. These are methods or procedures that originally received their sense and value from such validations but are now used without cognitive insight. Among these auxiliary devices Husserl includes algorithmic methods. Their function is "to save as as much genuine deductive mental work as is possible by artificially arranged mechanical operations on sensible signs." (§9) They may be executed blindly. Husserl says that whatever marvels these methods may achieve, their sense and justification depends on validitory thought. All 'mechanical methods', including those of calculating machines, are included here.

Logic, as theory of science, is in a sense a normative discipline. It seeks that which pertains to genuine, valid science as such, so as to use this Idea of science to measure whether the empirically given sciences are in agreement with this Idea, to what degree they approach it, and where they offend against it. With this norm as its end, logic readily gives rise to a technology. Logic will have many practical uses in this role. Husserl asks whether the definition of logic as a technology, as an applied or practical logic, really captures the essential character of logic. His answer is that it does not.

Furthermore, even if logic has a normative function the logical laws are not themselves normative prescriptions. They do not, as part of their content, tell us how one *should* judge. They can be employed for normative purposes but they are not therefore norms. Anyone who judges both that every A is a B and every B is a C ought to also judge that every A is a C. What we are told in logic, however, is only that if every A is a B and every B is a C then it is also true that every A is a C. There are no normative terms in the latter case. It has a different thought content. Purely theoretical statements admit of normative transformations but that does not make them normative statements.

Logic has a normative function but every normative discipline presupposes one or more theoretical disciplines as its foundation, in the sense that it must have a theoretical content that is not itself normative. Every normative discipline demands that we know certain non-normative truths and the latter are taken from certain theoretical sciences. We are therefore led to the question: which theoretical sciences provide the essential foundations of the theory of science? In particular, does logic have its place in sciences that have already been marked off and independently developed?

One of the central answers to these questions in Husserl's time was that it is psychology that provides the foundations of logic. Philosophers like Mill, Lipps, Sigwart and Wundt had made such claims. In thinking of logic as normative, psychologistic thinkers point out that what is regulated by logic is the mental activities or products of those who reason. What logic talks about are concepts, judgments, deductions, and so on. These are all taken to be psychological activities or products. Husserl claims that psychologistic arguments show only that normative logic may be helped by psychology, not that psychology provides the essential foundation of normative logic. The possibility remains open that 'pure logic' is the foundation of normative logic. Earlier thinkers may not have succeeded in making clear what pure logic is but this should give us all the more reason to attempt such a thing.

Husserl proceeds to critically examine the view that psychology is the foundation of logic. He starts by pointing out that psychology is supposed to be a factual or empirical science (§21). It has so far lacked genuine and *exact* laws. The propositions of psychology are merely vague generalizations from experience. They are propositions about approximate regularities. If this is the case then there are serious consequences for the psychologistic logicians. If psychological laws lack exactness then the same must be true for the laws of logic. Laws of logic and mathematics, however, are exact. Even if one had exact natural laws in psychology it is still true that natural laws are not *a priori*. They are instead established by induction from singular facts of sense experience. They are probabilities. Thus, laws of logic would have to be probabilities. But this seems patently false. Laws of logic have an *a priori* validity. Husserl says that we know about basic *a priori* laws on the basis of direct insight. Laws of logic are not causal laws. Psychologistic logicians confuse the contents of judgment with judgments as psychological processes or entities. The latter are real events having causes and effects. A law, however, ought not to be confused with the judging or with knowledge of the law. The ideal ought not to be confused with the real. There is a fundamental and essential gulf between ideal and real laws, between normative and causal regulation, logical and real necessity, logical and real grounds. There is no conceivable gradation that could mediate between the ideal and the real.

Another consequence of psychologism is that logical laws must themselves be psychological in content. This, according to Husserl, is palpably false. No logical law imples a matter of fact. Laws of logic presuppose nothing mental. They presuppose no facts of psychic life. They do so no more than the laws of pure mathematics do. One should not confuse the psychological 'presuppositions' and 'bases' of the *knowledge* of a law with the *logical* presuppositions, the grounds and premisses, of that law. Psychological dependence, or dependence of origin, is distinct from logical demonstration and justification. Following Kant, Husserl says that all knowledge 'begins' with sense experience but it does not therefore 'arise' from sense experience. The truth of laws of logic, Husserl says here, is raised above time. One cannot attach temoral being to it. It does not arise or perish (§24).

Husserl considers Mill's psychologistic account of some particular laws of logic, e.g., the law of contradiction. For Mill, this law states nothing more than the *real* incompatibility of two acts of judgment. Mill's interpretation, Husserl argues, yields a wholly vague, scientifically unproven empirical proposition, not a law of logic. What the law of contradiction is about is the *ideal* impossibility that the two propositions could *both be true.*

Husserl is led to a broader consideration of basic errors of empiricism by his examination of psychologism (§26). He says that extreme empiricism is as absurd as extreme skepticism. It destroys the possibility of the rational justification of mediate knowledge (in the form of deductive inference) and so destroys its own possibility as a scientifically proven theory. Since it puts full trust only in the singular judgments of sense experience it abandons all hope of rationally justifying mediate knowledge. It will not acknowledge as immediate insights and as given truths the ultimate principles upon which the justification of mediate knowledge depends. Instead, it tries to derive them from sense experience and induction. Empiricism appeals to a naive, uncritical everyday experience to found logical laws, instead of to immediately evident universal principles. Husserl considers Hume to be a moderate empiricist since he distinguished matters of fact from relations of ideas and surrendered only the former to sense experience. Nonetheless, Hume's position is also untenable. For Hume, mediate judgments of fact never permit of rational justification but only of psychological explanation. This must apply to Hume's theory itself. Husserl says that our capacity to ideate universals in singulars, to 'see' a concept in an empirical presentation and to be assured of the identity of our conceptual intentions in repeated presentations, is presupposed by the possibility of knowledge. Just as we can intuit *one* concept in an act of ideation as the single species whose unity against various instances is given with insight, so can we apprehend with evidence the logical laws relating to concepts. These concepts are ideal unities. Wherever we can carry out conceptual presentations in this sense we can also *apply* logical laws. The validity of these laws, however, is absolutely unrestricted. It does not depend on our power nor on anyone's power to achieve acts of conceptual presentation, nor to sustain or repeat such acts.

Husserl argues that psychologism and, more generally, empiricism, is a skeptical relativism. Empiricism undermines itself. One can distinguish individual from specific relativism. In the former case one makes truth relative to each person, while in the latter case one makes it relative to humans as a species. Husserl's term for species relativism is 'anthropologism'. Husserl of course argues that both forms of relativism are absurd. Sigwart in particular is singled out for his anthropologism. Sigwart resolves truth into conscious experiences. Experiences are real particulars, temporally determinate, that come into being and pass away. According to Husserl, however, truth is a 'Idea' that is beyond time. It makes no sense to give truth a date in time nor a duration that extends through time. Truth can of course be apprehended or grasped but this is not like apprehending some empirical content that comes into being and then vanishes at some later stage. The experience of truth is experience of a universal, an Idea. Husserl's rationalism emerges very clearly here. If we think of truth as ideal in this sense then the empirical sciences as a whole only approximate truth, just as real objects only approximate ideal objects.

Husserl says that we are conscious of truth as we are conscious of a species, e.g., the color red. A red object may stand before us but this object is not the species red. The concrete object here does not contain the species as a psychological or metaphysical part. The non-independent moment of red (as opposed to a piece –see Investigation III below) that is given to us is itself something individual, something here and now that arises and vanishes with the concrete whole object. It is *similar* but is not identical in different red objects. Redness, however, is an ideal unity that does not come into being or pass away. The part (moment) red is not Redness itself but is an instance of redness. Just as universal objects differ from singular ones, so too do our acts in apprehending them. Reference to an individual in consciousness is different from reference to its species, or its Idea (see also Section 2.4 below). In several acts of ideation we come to be aware of the identity of the ideal unities that are meant in our single acts. This is a strict identity. There is awareness of an identical species. Truth is likewise an Idea. We are aware of the unity and identity of truth over against the dispersed multitude of concrete compared cases. Husserl says that the statements "It is the truth that P" and "There could have been thinking beings having insight into judgments to the effect that P" are equivalent. If there are not intelligent beings or if they are in a *real* sense impossible then the *ideal* possibilities remain without actual fulfillment. The apprehension of truth is simply nowhere realized. Each truth, however, retains its ideal being and remains what it is. It is a case of validity in the timeless realm of Ideas (§39). This idea of truth could not be merely relative to the human species. That would be to miss its *sense*. The relativization of truth presupposes the objective being of the fixed point with respect to which things are relative. This is the contradiction in relativism.

Other beings could not have logical principles different from ours. To think that this is a possibility is to confuse psychological or anthropological possibility with logical possibility. All change affects sensory individuals. It makes no sense in regard to concepts. Real possibilities involve sensory individuals but ideal possibilities do not.

Logic might seem to be about mental phenomena and processes since it speaks of judgments, proofs, and so on. That it could not be about mental phenomena, however, is shown by the comparison with mathematics. Psychologism would also make mathematics a branch of psychology. There is, however, a theoretical unity of logic and mathematics. Like pure mathematics, the territory to be investigated by pure logic is an ideal territory (§46). Mathematics no longer needs to fight for its independent existence, Husserl says. We would have no numbers without counting, no sums without addition, and so on, and yet no one regards the theories of pure mathematics as parts of psychology. Counting and arithmetical operations as *facts*, as mental acts proceeding in time, are certainly of concern to psychology since psychology just is the empirical science of mental facts. Arithmetic, however, is different. Numbers, sums, products and so on are not causal acts of counting, adding and multiplying that are carried out here and there. They are not the same as the presentations in which they are given. The number 5 is not my own or any other person's counting of 5. In the counting of 5 we have 5 as a possible object of acts of presentation whereas the number 5 itself is the ideal species of a form whose concrete instances are found in what becomes objective in certain acts of counting, in the collective whole constituted thereby. The number cannot be regarded as a part or side of a mental experience. Therefore, it is not something real. If we are to conceive of 5 correctly we will first have an articulate, collective presentation of this or that set of five objects. In this act a collection is intuitively given in a certain formal articulation as an instance of the number species in question. On the basis of this intuited individual we perform an abstraction. That is, we not only isolate the non-independent moment of collective form in what is before us but we apprehend the Idea in it. The number 5, as the species of the form, is the reference of this conscious act. (This should be compared with the account in *PA*. Among other things, Husserl is now grafting his ontology of ideal objects onto his earlier *PA* account of number.) What we are now meaning is not this individual instance, not the intuited object as a whole, not the form immanent in it but still inseparable from it. What we mean is rather the ideal form-species that is identical in whatever mental act it may be individuated in as an intuitively constituted collection. It is a species that is untouched by the contingency, temporality and transience of our mental acts. Acts of counting arise and pass away. Arithmetical propositions tell us nothing about what is real, neither about the real things counted nor the real acts in which they are counted. The propositions of arithmetic are laws rooted in the ideal essence of the genus Number. The singulars that come within the range of these laws are *ideal singulars*, the determinate numbers that are the lowest specific differences of the genus number. What has been said here about pure arithmetic likewise carries over at all points to pure logic. Terms like 'judgment', 'concept', 'proof', and so on are equivocal. On the one hand they stand for mental states that belong to psychology but, on the other hand, they stand for ideal entities. One must always be careful to separate these two meanings.

It is yet another prejudice of psychologism to hold that the recognition of the truth of a judgment can be adequately understood in terms of the psychology of inner evidence. Pure laws of logic, however, say nothing about the psychology of inner evidence or its conditions. Psychological possibilities about inner evidence are real possibilities but what is psychologically impossible may very well be ideally possible. There are, for example, decimal numbers with trillions of places and there are truths relating to them. No one can actually imagine such numbers nor do the additions, multiplications and so on relating to them. Inner evidence here is psychologically impossible yet, ideally speaking, it represents a possible state of mind. Moreover, inner evidence is often taken to be a special feeling that attends some judgments. This view must be categorically rejected. The correct view of 'inner evidence' is that inner evidence is the experience of truth as ideal. Truth is an Idea whose particular case is an actual experience in an inwardly evident judgment. A judgment that is not self-evident stands to a self-evident judgment much as an arbitrary positing of an object in imagination stands to its adequate perception. A thing adequately perceived is not a thing merely meant in some matter or other. It is a thing given in our act as what we mean. As in the realm of perception, the unseen does not coincide with the non-existent. Lack of inner evidence does not amount to untruth. Evidence is the experience of agreement between meaning and what is itself present, the meant. It is the experience of agreement between the sense of an assertion and the self-given state of affairs. (These ideas are discussed in more detail in Investigations I and VI.) The Idea of this agreement is truth, whose ideality is also its objectivity. To have evidence in this sense is also to be aware that no other person could have evidence that is at variance with our own. If evidence were merely a matter of feeling then one cannot escape skepticism about claims to evidence.

The problems of psychologism, generally speaking, result from failing to clearly grasp the distinction between the real and the ideal. Ideal objects of thought are not mere pointers to 'thought-economies' or verbal abbreviations whose true content reduces to merely individual, singular experiences.

Yet another empiricist attempt to find a basis for logic and epistemology can be found in the efforts to provide a biological basis for these disciplines. Here Husserl mentions the work of Avenarius and Mach. Avenarius' doctrine of least action and Mach's doctrine of an economy of thought are examined. Here one thinks of science in terms of evolution or adaptation. One conceives of science as the most purposive (economical, power-saving) adaptation of thought to the varied fields of phenomena. In particular, a creature will be better adapted the more rapidly or efficiently it can perform the acts needed for its survival and success. This leads to the notion of an 'economy of thought'. Our intellectual powers are severely limited

There is a fairly narrow sphere within which complex, abstract notions can be fully understood. A significant effort has to be made to understand complexities of this sort. When these fact are considered it is all the more amazing that the more comprehensive rational theories and sciences should have been developed at all. How could sciences like mathematics, with their towering structures of thought, be possible? Art and method make it possible to overcome the defects of our cognitive constitution. They permit an indirect achievement by way of symbolic processes from which intuition, all true understanding and inner evidence are absent. One has some sense of security, however, in using such arts and methods to economize thought. There are certain natural processes of thought economy that are then perfected and developed. Once the methods have been developed and justified they can be used without insight. They can be used mechanically. The reduction of insight to mechanism in our thought processes leads to an indirect mastery over the complexities of thought that admit of no direct mastery. Here Husserl gives many examples from mathematics (§54). The surrogative, operational concepts that are developed on this basis turn signs into 'counters' and make possible extensive fields of mathematical thought and research. They take these areas down from the exhausting heights of abstraction to comfortable intuitive ways where imagination, guided by insight, can move within the limits of rules, as in regulated games.

A vast thought economy is present in recent purely mathematical disciplines. Genuine thought is replaced by surrogative, signitive thinking. This economy leads to formal generalizations of our original trains of thought. In this manner, the horizon of deductive disciplines is greatly enlarged. Out of elementary arithmetic, for example, arises a more generalized form of arithmetic in which numbers and magnitudes no longer count as basic concepts but merely as chance objects of application. Fully conscious reflection now takes place and the pure theory of manifolds emerges as a further extension. In its form it covers all possible deductive systems. The form-system of formal arithmetic is merely one of its special instances.

Husserl thinks all of this contains important insights and ought to be investigated in great detail. Its relation to logic as a practical technology is immediately understandable. It yields an important foundation for such a technology. Here Husserl especially praises the work of Mach. We need to keep in mind, however, that Avenarius and Mach relate their ideas on an economy of thought to certain biological facts. Ultimately, we are dealing with a branch of the theory of evolution. For just this reason, these thinkers are able to throw light on practical epistemology and the methodology of scientific research but not at all on pure epistemology and the ideal laws of pure logic. Husserl says that all of the arguments against psychologism and relativism can be brought to bear on the effort to found logic and pure epistemology in this manner on a biological economy of thought.

Husserl says that the principle of economy can be thought of either as something factually given or as logically ideal. People like Mach and Avenarius tacitly substitute the former for the latter. We see that it is a supreme goal of science to arrange facts under laws that are as general as possible and in this manner bring them together with the maximum possible rationality. This maximization is an ideal of a pervasive, all-embracing rationality. The 'basic laws' would be laws of supreme coverage and efficacy, whose knowledge yields the maximum of insight and explanation in some field. The axioms of elementary geometry are examples of this. If we idealize this, we have the notion that there are no limits to our power to deduce and subsume.

The goal or principle of maximal rationality in this sense is the supreme goal of the rational sciences. It is self-evident that it would be better for us to know laws more general than those that we already possess at a given time. Such laws would lead us back to grounds that are deeper and more embracing. This principle, however, is clearly no mere biological principle or principle of thought economy. It is a purely *ideal* principle, and an eminently normative one. To identify the movement toward maximum possible rationality with a drift towards biological adaptation, or to derive the former from the latter, amounts to confusion. It parallels the psychologistic misreadings of laws of logic and their misconception as laws of nature. The ideal movement of logical thinking is towards rationality. The thought economist turns this into a *real* drift of human thought, bases it on a *vague* principle of power-saving, and ultimately on adaptation. One is certainly justified in speaking of an economy of thought but only in that one compares one's actual thought with an ideal norm. The ideal validity of this norm is presupposed by all talk of an economy of thinking. It is therefore not a possible explanatory outcome of a theory of such economy. We measure our empirical thinking against our ideal thinking and we then say that the former to some extent runs as if guided by the latter. Before all economizing of thought we must already know our ideal. We must know what science ideally aims at. Pure logic is prior to all thought economics. It is absurd to base the former on the latter.

Husserl now begins to turn toward his own view of pure logic. He discusses how his view of pure logic is linked to the views of Kant, Herbart, Lotze, Leibniz, Lange and Bolzano. Leibniz and Bolzano are held in high regard and it is Bolzano and Lotze who are especially praised as the anti-psychologistic logicians.

Husserl's positive suggestions about the nature of 'pure logic' begin with reflections on what constitutes the unity of science. Given Husserl's very broad conception of logic, this becomes a question about the conditions of the possibility of theory in general. He is of course concerned with *ideal* conditions, not *real* conditions of knowledge. Truths of science are what they are whether we have insight into them or not. Since they do not hold insofar as we have insight into them, but we can only have insight into them insofar as they hold, they must be regarded as objective or ideal conditions for the possibility of our knowledge of them. Logical justification of a given theory, i.e., justification in virtue of its pure form, demands that we go back to the essence of its form, to the concepts and laws that are ideal constituents of theory in general that regulate in an *a priori*, deductive fashion all specializations of the Idea of theory in its possible kinds. Thus, we are dealing with an *a priori*, theoretical, nomological science that concerns the ideal essence of science as such. We are, Husserl says, dealing with the theory of theory, or the science of the sciences. Husserl says that pure logic in this broad sense will even include the pure theory of probability.

Husserl describes three sets of tasks that should be assigned to pure logic in this sense. First, we must lay down and clarify the primitive concepts that make possible the interconnected web of theory. Here we are concerned with the concepts that constitute the Idea of a unified theory. A given theory is a certain deductive combination of given propositions which are themselves certain combinations of given concepts. The 'form' of the theory arises if we substitute variables for these given elements. The relevant concepts here, among others, are concept, proposition, and truth. The elementary connective forms of logic also play a role here: conjunction, disjunction, conditional linkage of propositions, and so on. Husserl says these concepts involve categories of *meaning*. There are correlative concepts such as object, state of affairs, unity, number, relation, connection, and so on that are pure formal *objective categories*. The phenomenological origin of all of these notions must be investigated. That is, we must seek insight into the essence of these concepts. This can only be done through the intuitive representation of the essence in each case, which is to be fixed in unambiguous, sharply distinct verbal meanings. Husserl admits that it may be difficult to make headway with this first set of problems but he thinks that they are also among the most important.

The second set of problems lies in the search for the laws that are grounded in the two classes of concepts just mentioned. On the one hand, there are laws involving the truth or falsity of meanings as such, purely on the basis of their categorial formal structure. On the other hand, there are laws concerning their objective correlates, concerning the being and not being of objects as such and states of affairs as such, on the basis of their pure categorial form. The laws in the one case concern meanings and in the other case concern objects as such. (This should be compared with what Husserl says in *FTL*, Section 6.1 below). On the side of meaning we have, for example, theories of inference, while on the side of the objective correlates we have, for example, the pure theory of pluralities or the pure theory of numbers. We should try to find the laws, which in their formal universality, span all possible meanings and objects, under which every particular theory or science is ranged, and which it must obey if it is to be valid. Husserl says that not every such theory presupposes every such law as the ground of its possibility and validity. Rather, the ideal completeness of the theories and laws in question will yield a comprehensive fund from which each particular valid theory derives the ideal grounds of essential being appropriate to its form.

With these tasks completed we will have done justice to the Idea of a science of the conditions of the possibility of a theory in general. Such a science, however, points beyond itself to a completing science that investigates possible theories in an *a priori* fashion, rather than the possibility of a theory in general. What is needed, according to Husserl's third task, is a theory of the possible forms of theories or a 'pure theory of manifolds'. Husserl says that the objective correlate of the concept of a possible theory, definite only in respect of its form, is the concept of a possible field of knowledge over which a theory of this form will preside. Such a field is known in mathematical circles, Husserl says, as a *manifold.* The objects in a manifold are not determined directly as individual or specific singulars, nor indirectly by way of their material species or genera, but solely by the *form* of the connections attributed to them. These connections are as little determined in content as are their objects. Only their form is determined through the forms of the laws that are assumed to hold of them. As an example, Husserl says that in the theory of manifolds '+' is not the sign for numerical addition but for any connection for which laws of the form $a + b = b + a$ hold. The manifold is determined by the fact that its 'thought-objects' permit of these 'operations'.

The most general idea of a theory of manifolds is that it is to be a science that works out the form of the essential types of possible theories. All actual theories are then specializations or singularizations of corresponding forms of theory, just as all fields of knowledge are individual manifolds. As an examples of what he has in mind, Husserl points to the theories of manifolds that arose from generalizations of geometric theory and its forms and to extensions of the formal theory of real numbers into the formal two-dimensional field of ordinary complex numbers. Regarding geometry, Husserl is referring to the theory of n-dimensional manifolds, whether Euclidean or non-Euclidean. He mentions especially the work of Riemann. He also mentions Grassman's theory of extensions, Lie's theory of transformation groups, some work of Helmholtz, and Cantor's investigations into numbers and manifolds. Husserl says it is senseless to speak of different geometries if 'geometry' names the science of the space of everyday phenomena. If we mean by 'space' the categorial form of world-space, however, and by 'geometry' the categorial theoretic form of geometry, then we can extend our conceptions of these fields. From this point of view, the theory of a Euclidean manifold of three dimensions is an ultimate ideal singular in a connected range of *a priori*, purely categorial theoretic forms. Several other examples are given (see §70). Husserl returns to these ideas in later writings to be discussed below.

One of the last sections of the Prolegomena concerns the division of labor between mathematicians and philosophers. Husserl says that the construction of theories and the solution of formal problems is the work of the mathematician. Logic, once it was taken over by mathematicians, has made great progress. The philosopher oversteps his bounds if he attacks the mathematization of logic, for this is really the only scientifically rigorous treatment of the subject. What then is left over for the philosophers? Mathematicians are not really pure theoreticians, Husserl says, but only ingenious technicians. They are the constructors who build up theories like works of art. As a mechanic constructs machines without needing to have insight into the essence of nature and its laws, so the mathematician constructs theories of numbers, quantities, manifolds, and so on, without insight into the essence of theory in general, and the concepts and laws that are its conditions. The various special sciences are more concerned with practical results and mastery than essential insight. For just this reason, they are in need of continuous epistemological reflection that only the philosopher can provide. Philosophical investigation, Husserl says here, does not seek to meddle in the work of the specialist but seeks insight into the sense and essence of his achievements as regards method and manner. The work of the mathematician and the philosopher are mutually complementary scientific activities through which complete theoretical insight, comprehending all relations of essence, can first come into being. The individual logical investigations that follow the Prolegomena are, Husserl says, preparatory to the philosophical side of our subject. They will further elucidate what the mathematician will not and cannot do but what must be done.

2.2 Introduction to the Six Logical Investigations

The six logical investigations that follow the Prolegomena are said by Husserl (in the second edition of *LI*) to be phenomenological investigations that contribute to the epistemological criticism and clarification of pure logic. In the "Introduction to Volume II of the *Logical Investigations*" Husserl says that phenomenology is supposed to lay bare the sources from which the basic concepts and ideal laws of pure logic flow, back to which they once more must be traced in order to give them all the clarity and distinctness needed for the understanding of and for an epistemological critique of pure logic. It will be important to begin with considerations of language and expression since the objects of logic are always given to us through these means. To be somewhat more precise, the objects of logic come before us in concrete mental states that function either as the meaning-intention or meaning-fulfillment of certain verbal expressions (see Section 2.3 below). The pure logician is interested in the logical judgment itself, not primarily in the concrete mental phenomena. The logician is interested in the identical asserted meaning which is a unity over against manifold descriptively different judgment experiences. There is in the singular experiences that correspond to this ideal unity a certain pervasive common feature but since the concern of the logician is not with the concrete instance but with the corresponding Idea (the abstractly apprehended universal) it seems he would have no reason to leave the field of abstraction. He would have no reason to make the concrete experiences the theme of his interest instead of Ideas. Husserl says that even if phenomenological analysis of concrete thought-experiences does not fall within the proper sphere of pure logic, it is nonetheless indispensable to purely logical research. For all that is logical must be given in fully concrete fashion if it is to be made our own and if we are to bring to self-evidence the *a priori* laws that have their roots in concrete experience.

That which pertains to logic is first given to us in an imperfect manner: a concept is given to us as more or less wavering in meaning, and the laws built out of concepts are therefore more or less wavering in meaning. We have logical insights but these depend on the verbal meanings that come alive in actually passing judgments regarding the law. Unnoticed equivocations may permit the substitution of other concepts beneath our words. Equivocation may distort the sense of the propositions of pure logic (as happens in the case of psychologism or other forms of empiricism). This is where phenomenology, with its emphasis on epistemological clarity and distinctness, must begin. Concepts of logic must have their origin in intuition. They must arise out of an ideational intuition founded on certain experiences and must admit of reconfirmation and of recognition of their self-identity on the reperformance of such abstraction. We cannot rest content with 'mere words', i.e., with a purely symbolic understanding of words. Meanings that result from remote, confused, inauthentic intuitions, if by any intuitions at all, are not enough. We must 'go back to the things themselves' to render self-evident what is given in actually performed abstractions. We must strive to arouse dispositions in ourselves that will keep our meanings unshakably the same, and that will measure them often against the marks set by reproducible intuitions or by an intuitive carrying out of our abstraction. In this manner we can achieve the desired clearness and distinctness. In the analysis of meanings we must keep in mind that matters of expression and matters of meaning must both be considered. Grammatical distinctions are sometimes essential to questions of meaning and are sometimes contingent. It is therefore important to make clear the relationships between expression and meaning.

Husserl says that the phenomenology of logical experiences aims at giving a sufficiently wide descriptive (but not empirical/psychological) understanding of these mental states and their indwelling sense to enable us to give fixed meanings to all of the fundamental concepts of logic. Meanings will be clarified both by going back to the analytically explored connections between meaning-intentions and meaning-fulfillments and also by making their possible function in cognition intelligible and certain. We especially need to avoid confusing the objective with the psychological attitude when it comes to logic. Phenomenology should make this possible since it calls for a theory of the essence of our experience.

We are constantly faced with questions of the following sort: "How we are to understand the fact that the intrinsic being of objectivity becomes 'presented', 'apprehended' in knowledge, and so ends up becoming subjective? What does it mean to say that the object has 'intrinsic being', and is 'given' in knowledge? How can the ideality of the universal *qua* concept or law enter the flux of real mental states and become an epistemic possession of the thinking person?" (§2) These questions cannot be separated from questions regarding the clarification of pure logic and, hence, they must be subjected to phenomenological analysis.

There will no doubt be difficulties in such pure phenomenological analysis due to the seemingly unnatural direction of intuition and thought required by phenomenology. Instead of becoming lost in the performance of acts built intrinsically on one another and instead of naively positing the existence of objects, we must practice reflection. We must, that is, make these acts themselves and their meaning-content our objects. This is a direction of thought that runs counter to deeply ingrained habits. It has also been held that when we pass over from naively performed acts to an attitude of reflection, or when we perform acts proper to such reflection, our former acts necessarily undergo change. Further, there are the difficulties of stating results of reflection and communicating them to others. Husserl says that while these difficulties should be noted there is no reason to believe that they cannot be overcome with time and effort.

In this Introduction Husserl also mentions that his subsequent investigations will aspire to freedom from metaphysical, scientific and psychological presuppositions. The phenomenology of logic should take 'freedom from presuppositions' as a basic principle in its investigations. This theme is frequently repeated in Husserl's work. Even if it cannot be actually achieved it should be regarded as an ideal that we should not abandon.

2.3 Investigation I: Expression and Meaning

Husserl starts this Investigation by noting that the terms 'expression' and 'sign' are often taken as synonyms but that they do not always coincide in their application. Every sign is a sign for something but not every sign expresses a meaning or sense. A flag, for example, is the sign of a nation in the sense of an *indication*, just as fossil vertebrae are signs of prediluvian animals. A flag is not, however, an expression that has a meaning. Fossil vertebrae are not expressions that have a meaning. Signs in the sense of indications do not express anything unless they happen to fulfill a signifying as well as as indicative function. Indicative signs are thus to be distinguished from *meaningful* signs, i.e., *expressions*. Each part of speech counts as an expression but Husserl wishes to exclude facial expressions and gestures that involuntarily accompany speech without communicative intent. Expressions, as Husserl understands them, fulfill a communicative function. Expressions in communicative speech function as indications. They serve the hearer as signs of the thoughts of the speaker. They intimate inner experiences. The meaning of an expression does not coincide with its intimating function, however, since expressions continue to have meaning in uncommunicated, interior mental life.

Now whether expressions occur in dialogue or soliloquy they can be analyzed on the one hand into a physical phenomenon and, on the other hand, into the acts that give this physical sign or utterance meaning and possibly intuitive fullness (in which its relation to an expressed object is constituted). The difference between the physical sign configuration and the meaning-intention that makes it into an expression becomes most clear when we turn our attention to the sign *qua* sign, e.g., to the printed word. If we do this we have merely an external percept that loses its verbal character. If it again functions as a word then the character of its presentation is wholly altered. The sign configuration remains present but we no longer intend it. It is no longer properly the object of our cognitive activity. Instead, we are directed toward the thing meant in the sense-giving act. The intentional character of the experience is altered. In virtue of the acts that confer meaning, the expression is more than merely a sounded word. The acts attaching to a string of signs or sounds make it an expression but these acts are not outside of, beside or merely simultaneous with it in consciousness. They are one with it. They make up a unitary total act. One can say they are parts of a whole.

Insofar as an expression means something, it relates to what is objective. The object may either be actually present through accompanying intuitions or it may appear in representation, e.g., in a mental image. But this need not occur. An expression can function significantly and remain more than a mere sound of words but still lack an intuition that would give its object. In this case the relation of expression to object is unrealized. What we have is a mere *'meaning-intention'.* A name, for example, might mean an object without the object being present. If the empty meaning-intention is now fulfilled then the relation to an object is realized. Thus, we need to distinguish meaning-intentions that are void of intuition from those that are intuitively fulfilled. There are *meaning-conferring acts* and *meaning-fulfilling acts.* The former are essential to an expression if it is to be an expression at all. The latter are not essential to the expression as such but stand to it in the relation of fulfilling it more or less adequately, thus actualizing its relation to its object. These latter acts are 'fused' with the meaning-conferring acts in cases where we have knowledge.

Thus far we have considered the expression and the sense-conferring or sense-fulfilling *experiences* but we need to also consider what is objectively given in these experiences: the expression itself, its sense and its objective correlate. The expression itself, as distinct from the experience of the expression, is an ideality. Here we are not referring to a sound pattern that is uttered and then vanishes, never again to recur in an identical manner. We are instead referring to the expression *in specie.* Similarly, the meaning or sense of the expression is an ideality. It is distinct from the meaning-conferring experience. In this selfsame meaning nothing about the judging or the one who judges is discoverable. The act of judging arises and passes away. The content of the judgment, however, neither arises nor passes away. It is an identity, e.g., one and the same geometrical truth uttered by many different people at different times. The meaning is in this sense a unity in plurality. We do not arbitrarily attribute it to our assertions but we discover it in them.

Husserl turns to the 'objective correlate' meant by a meaning. Each expression not only has a meaning (content) but also refers to certain objects. There is what is meant or said and there is also that which is spoken of. The object never coincides with the meaning (§12). This distinction is required if we realize that several expressions may have the same meaning but different objects, or again that they may have different meanings but the same object. In the case of names, the latter situation occurs with expressions like 'the victor at Jena' and 'the vanquished at Waterloo', or 'the equilateral triangle' and the 'equiangular' triangle'. The former situation occurs with expressions like 'a horse'. This has the same meaning across different contexts but can pick up different objects in those contexts. Similarly, 'one' is a name whose meaning never differs but the various 'ones' that occur in a sum should not be identified for that reason. They all have the same meaning but differ in objective reference. Later in this Investigation Husserl considers indexical expressions in more detail.

Expressions might also differ with respect to both meaning and reference or they might agree with respect to both meaning and reference. The latter situation occurs with synonymous expressions. The parallels here with Frege's views on sense and reference have been noted by many commentators. Husserl's views imply that an intensional logic should be developed alongside any purely extensional logic. This will apparent at various points below.

Proper names function differently depending on whether they name individual or general objects. A word like 'Socrates' can only name different things by meaning different things, that is, by becoming equivocal. Wherever a word has one meaning it also names one object. Thus, a distinction is required between equivocal names that have many meanings and general or class names that have many values. It is one thing to have multiple senses of equivocal names and another to have multiple values of general names. When collective meanings are fulfilled, for example, we intuit a plurality of items. Fulfillment is articulated into a plurality of individual intuitions.

An expression only refers to an objective correlate because it means something and it can be correctly said to signify or name the object through its meaning (§13). An act of meaning is the determinate manner in which we refer to our object of the moment, although this mode of reference and the meaning itself can change while the objective reference remains fixed. Husserl says that the same intuition can offer fulfillment of different expressions. This matter is taken up in much more detail in Investigation VI (see Section 2.8 below).

Husserl notes that in speaking of meaning the term 'content' is often used in the literature. It is used, however, in an ambiguous manner. The 'content' might be (i) the intending sense or meaning simpliciter, (ii) the fulfilling sense, or (iii) the object. One must always heed these differences. In particular, the act that confers meaning yields the idea of the intending meaning, while we can also speak of the fulfilling meaning associated with an expression in the case in which the mere intention is fulfilled.

In connection with these distinctions, we need to be aware of equivocations in talk of 'meaningless' or 'senseless' expressions. It is part of the notion of an expression to have a meaning. There are sound patterns or strings of signs, however, that are meaningless in the sense that they do not express anything (e.g., "Green is or"). In another sense, people sometimes hold that an expression has a meaning when the object corresponding to it exists and is meaningless when no such object exists. Husserl says that this usage cannot be consistently maintained since there are situations in which an expression is perfectly meaningful but lacks an object (§15). Some thinkers have also held that expressions like "round square" or other expressions infected with contradictions are senseless. In response, Husserl says he agrees with Marty that we could not understand the question whether such things exist if these words were senseless. In particular, mathematics is full of examples where one demonstrates, sometimes by lengthy arguments, that expressions are objectless *a priori*. Thus, these thinkers are confusing true meaninglessness of the sort first mentioned in this paragraph with another quite different meaninglessness, i.e., the *a priori* impossibility of a fulfilling sense. In Investigation IV Husserl returns to these points and distinguishes nonsense (Unsinn) from countersense (Widersinn). A string of words like 'Green is or' is nonsense while an expression like 'round square' is countersensical. The former should be ruled out by logical grammar alone.

Husserl notes that one might form a conception of meaning according to which an expression has meaning if and only if its meaning-intention is fulfilled. This, however, confuses meaning with fulfilling intention (see also Investigation VI, §25). Husserl points out that meanings of expressions can be clarified by appealing to corresponding intuitions (i.e., meaning-fulfillments) where this is possible, but it does not follow from this that an expression may not have meaning apart from intuition (fulfillment) (§21).

It is worth noting that Husserl argues at some length against the view according to which an expression functions by arousing certain mental images or pictures that regularly accompany it (§17-18). An expression may remain meaningful even in the absence of any accompanying mental picture. One can show how vastly the imaginative accompaniments may vary while the meaning or words remain constant.

Husserl is also careful to separate the meaning that an expression has when it lacks a fulfilling intuition from what he calls the 'games-meaning' of signs (§20). As an example, one can note that arithmetical signs have, in addition to their original meaning, a 'games-meaning' that is determined by the game of calculation and its rules. One can treat arithmetical signs as mere counters in accordance with these rules to arrive at solutions of problems, just as one treats chess pieces as counters in a chess game, through whose fixed rules the chess pieces acquire their meaning. One operates with the games-meaning of the arithmetical signs in place of the original meaning. Thus, in purely symbolic arithmetical thought and calculation we do not operate with meaningless signs. Mere signs in the sense of physical marks bereft of all meaning do not do duty for the same signs alive with arithmetical meaning. Rather, the signs taken in an operational or games-sense do duty for the signs in full arithmetical meaningfulness. Non-intuitive meaningful thought and the kind of symbolic thought that employs operational or games-meaning are quite different things.

We have seen that there is a distinction between acts of meaning and the meaning itself, where the latter is viewed as an ideal unity against the multiplicity of possible acts. There are some special cases, however, that require further attention: cases of expressions whose meaning shifts due to vagueness or indexicality (starting at §24). In particular, Husserl offers what must be one of the first accounts of indexical expressions in recent times. Consider the words 'I wish you luck'. While these words can express my wish they can also serve countless other persons to express wishes having 'the same' content. Not only do the wishes themselves differ from case to case but the meanings of the wish utterance do too. Cases of this sort might shake our faith in the supposed ideality and objectivity of meanings. Husserl distinguishes essentially occasional and subjective expressions (i.e., indexical or demonstrative expressions) from objective expressions. An expression is 'objective' if its meaning can be understood without necessarily directing one's attention to the person uttering it or the circumstances of the utterance. In the case of occasional expressions one must consider the occasion, the speaker and the situation. Only then can one identify a definite meaning from a unified group of possible meanings. Expressions in theories, expressions that make up the principles, proofs and theorems of the abstract sciences are, for example, objective. What a mathematical expression means, for example, is not in the least affected by the circumstances of our actual use of it.

Many expressions, however, that serve the practical needs of everyday life or that prepare the way for the sciences may involve occasional expressions. All expressions including a personal pronoun are of this type. The word 'I' names a different person from case to case and does so by way of a changing meaning. We should not, however, suppose that the immediate presentation of the speaker gives us the entire meaning of the word 'I'. The word is not to be regarded as an equivocal expression, as though its meaning were to be identified with all possible proper names of persons. The idea of self-reference as well as an implied pointing to the individual idea of the speaker both belong to the meaning of this word. Husserl says that in this kind of case we will need to say that two kinds of meaning are built one upon the other. There is the *indicating* meaning of the expression. This pertains to the general function of the word in such a way that its indicative function can be exercised once something is actually presented. Once something is actually presented we have the *indicated* meaning of the expression. This same distinction holds for demonstratives (e.g., expressions like 'this'). When someone utters 'this' she does not arouse in the hearer the idea of what she means but in the first place the idea or belief that she means something lying within her intuitive or thought-horizon, something she wishes to point out to the hearer. 'This' in isolation has only an indicating meaning. The meaning indicated only appears in the context of its use. Other such expressions are 'here', 'there', 'above', 'below', 'now', 'yesterday', 'tomorrow', 'later', and so on. For some of the secondary literature on occasional expressions see D. Smith [1981; 1982]; [Philipse, 1982]; [Mulligan and Smith, 1986]; and [Schuhmann, 1993].

Husserl discusses other sorts of 'fluctuating' expressions: complete and incomplete expressions, expressions that are functioning normally from those functioning abnormally, and exact and vague expressions. The remarks on vague and exact expressions are mirrored in Husserl's later distinction between expressions for 'morphological' as opposed to exact essences (see Section 5.1 on *Ideas I* below).

Husserl asks whether the recognition of such cases in which meaning seems to fluctuate should dissuade us of the idea of meanings as ideal unities, or at least lead us to restrict the generality of this view. Do meanings themselves divide into objective and subjective, into meanings that are fixed and meanings that change with occasion? The answer is 'no'. The content meant by the subjective expression, with its sense oriented to the occasion, is an ideal unit of meaning in precisely the same sense as the fixed expression. This is shown by the fact that, ideally speaking, each subjective expression can be replaced with an objective expression that will preserve the identity of each momentary meaning-intention. Of course in practice we cannot actually eliminate essentially occasional expressions from language. Meanings as such, however, do not differ among themselves. What may seem to be changes in meaning are actually changes in the act of meaning.

Pure logic is exclusively concerned with meanings as ideal unities. It is the science of meanings as such, of their essential types and differences, and of the ideal laws that rest on these types and differences. The scientific investigator knows that expressions are contingent and that what is essential is the ideal, selfsame meaning. She does not create the objective validity of thoughts or the ideal connections of thoughts but rather she discovers or sees them. Their ideal being is not taken as a psychological 'being in mind'. All theoretical science consists, in its objective content, of one homogeneous stuff: an ideal fabric of meanings.

Husserl thus contrasts the meaning-conferring experience with the single, self-identical meaning that is its 'content' and is set over against the dispersed multiplicity of actual and possible experiences of speakers and thinkers. This content or ideal sense is not at all what psychology means by 'content'. Husserl argues, like Frege, that one and the same meaning can be associated with many different subjective ideas, images, and so on. The strict identity of what is meant, as distinct from the mental character of meaning it, is forced on us if we are to do justice to logic. In speaking of the proposition or truth that π is a transcendental number, for example, I do not have in mind the individual experience of any particular person. What is meant in this sentence is the same thing whether I think it or not, and whether there are any thinking persons or acts. The identity here, Husserl says, is an identity of species. Only in this way can it embrace a multiplicity of individuals. The manifold singulars for the meaning as an ideal unity are the corresponding act-moments of meaning, the meaning-intentions. Meaning is related to varied acts of meaning just as Redness *in specie* is to the slips of paper lying here that all have the same redness.

In drawing this Investigation to a close, Husserl collects together some additional observations on the concept of meaning that has emerged. He says that meanings are 'universal objects'. There are indeed universals and we can be aware of them as such. This theme is discussed in more detail in later Investigations, especially II and VI. Further, the ideality of meanings must be distinguished from normative ideality. The latter concerns an ideal of perfection, over against particular cases that realize it more or less approximately. Meanings in themselves, however, are specific unities and are not themselves ideals in this sense. They are not goals toward which we strive in our creations. Their 'ideality' lies in being unities in multiplicity that are independent of the 'real'.

Husserl distinguishes meaning from concept (in the sense of species). Each species presupposes a meaning in which it is presented. It has already been said that a meaning is a species. The meaning in which an object is thought, however, and its object, the species itself, are not the same thing (§33). What Husserl is saying is that we are directed by way of a meaning toward an object. The object can itself be a species and so we must keep the two (meaning and species) distinct. Just as in the field of individuals we can distinguish between Bismarck himself and the meaning by virtue of which we are directed toward Bismarck, so in the field of species we distinguish between the number 4 itself and the meanings that have 4 as their object. The universal that we think of does not resolve itself into the universality of the meanings in which we think of it.

Another point is that in the act of meaning we are not conscious of meaning as an object. If we perform an act and live in it we naturally refer to its object and not to its meaning. The latter first becomes an object for us in an act of reflection. Such reflection is quite common in logical research, which takes place in the context of a theoretical interest.

Finally, this investigation as a whole has considered meanings as meanings of expressions. Husserl says, however, that there is no intrinsic connection between meanings as ideal unities and the signs to which they are tied, through which they become real in human cognitive life. We cannot say that all ideal unities of this sort are expressed meanings. When a new concept is formed we see how a meaning becomes realized that was previously unrealized. Numbers, for example, neither spring forth nor vanish with the act of enumeration. The endless number series represents an objectively fixed set of objects delimited by ideal law. It is similar to the ideal unities of pure logic. They are an ideally closed set of objects to which being thought of or being expressed are both contingent. There are therefore countless meanings that, relative to us, are merely possible since, owing to our limits, they never are or never can be expressed.

2.4 Investigation II: "The Ideal Unity of the Species and Modern Theories of Abstraction"

Investigation II is concerned with the problem of abstraction. It is filled with objections to nominalism and conceptualism. Husserl says that he wants to consider this problem early in his Investigations in order to secure the basic foundations of pure logic and epistemology by defending the right of specific (or ideal and universal) objects to be granted objective status alongside individual (or real) objects. Empiricist psychology and epistemology have consistently misconstrued the nature of abstraction. A good deal of this investigation consists of a critique of the accounts of abstraction in Locke, Berkeley, Hume and Mill.

Husserl opens the Investigation with the claim that there are acts in which we intend or mean an individual and that, just as clearly, there are acts in which we intend or mean an ideal species. What we need to do is to compare these two types of acts. In the first case a whole concrete thing or an individual piece or property attaching to it might appear. Now the same concrete thing that makes an appearance can sustain different acts in the two cases. In the first case it provides a basis for an act of individual reference while in the second case it provides a basis for an act of conception and reference directed toward a species. In the second case the thing or a feature of a thing appears but it is not this feature here and now that we *mean*. Rather, we mean its 'Idea'. We do not mean, for example, this aspect of red in the house but Red as such. This act of meaning is 'founded' on underlying apprehensions in a manner that Husserl will discuss in Investigation VI. A new mode of consciousness is built on the intuition of the individual house or its red aspect, a mode of consciousness constitutive of the intuitive presence of the Idea of red. This mode of consciousness sets the species before us as a universal object. The primitive relation between species and instance thereby emerges. It becomes possible to survey and compare a range of instances. The individual aspects differ but in each the same species is realized.

Husserl says he is aware of the excesses of realism about universals, species, and so on, but he thinks that the question of whether species ought to be treated as objects can be answered only by going back to the meaning of the names standing for species and to the meaning of assertions claiming to hold for species. If these names and assertions can be interpreted as making the true objects of our intentions individual then we must yield to the view of the opponents. Otherwise, their view is evidently false. Now Husserl says we cannot help but distinguish between *individual singulars*, like the things of sense experience, and *specific singulars*, like the numbers and manifolds of pure mathematics. Number is a concept that has $1, 2, 3, \ldots$ as its subordinate singulars. The number 2, for example, is not a group of two individual, 'real' objects. The latter is not what we *mean* by the number two. Similarly, we must distinguish *individual* from *specific universals*. There are individually singular judgments, like "Socrates is a man", and specifically singular judgments, like "Two is an even number". Universal judgments are either individually universal, like "All men are mortal", or are specifically universal, like "All analytic functions can be differentiated". Species really do become objects of knowledge. Judgments about them have the same logical force as judgments about individual objects.

Meanings, as ideal objects, count as units in our thought and as such we can pass judgment on them as units. They can be compared with other meanings and distinguished from other meanings. They can be an identical subject for numerous predicates and an identical term in numerous relations. This is true just as much for meanings as it is for other objects that are not meanings, e.g., horses, stones, etc. A meaning can be treated as self-identical because it is self-identical.

We sometimes speak of things being the same when in fact they are only like one another or only resemble one another. There are improper uses of the expression 'the same'. The improper use of the expression for things that are alike refers us back, through its impropriety, to a proper use of the expression. Husserl argues that whenever things are alike, an identity in the strict sense is also present. We cannot predicate likeness of two things without stating the respect in which they are alike. Each likeness relates to a species under which the objects compared are subsumed. This species cannot itself be merely alike in the two cases. Otherwise an infinite regress threatens. If we specify the respect in which they are alike then we point by way of a more general class term to the range of specific differences among which the one that appears in our compared members is to be found. If two things are alike as regards form, then the form-species in question is the identical element. If they are alike as regards color, then the color-species is the identical element. Since not every species has an unambiguous verbal expression a suitable expression for a 'respect' may be lacking. Even though stating it clearly might be difficult, we nonetheless keep it in mind. One cannot define identity as a limiting case of 'likeness'. Identity is indefinable, whereas 'likeness' is definable. 'Likeness' is the relation of objects falling under one and the same species. Talk of likeness loses its meaning if one is not allowed to speak of the identity of the species.

Husserl raises several objections to efforts to reduce ideal unities to dispersed multiplicities. The objections are based on his theory of intentionality. We compare two types of intentions: (1) our intention when we grasp intuitively like objects, or when we recognize their likeness in a single glance, or when in single acts of comparison we recognize the likeness of one definite object to certain others, and (2) our intention when, possibly based on the same foundations, we grasp as an ideal unity the attribute that constitutes the respect in which the things are alike or are compared. The object of our intention that is meant and named in the two cases is different. What we *mean* in the second case is the universal, the ideal unity, and not the units or pluralities meant in the first case. The two intentional situations are quite different from one another. In the second case no intuition of likeness, not even a comparison, is needed. For example, I recognize this paper as paper and as white and thereby make clear to myself the general sense of the expressions 'paper' and 'white' but in order to do this I need not carry out any intuitions of likeness nor any comparisons.

We cannot account for an intention to a species by presenting singular things belonging to groups of similars because the presented singular things comprise only a few members of such groups and can never exhaust their total range. What would give unity to this range, what would make it a possible object of awareness and knowledge if the unity of the species lapsed? How can anything unify if it must first be unified? Every attempt to transform the being of what is ideal into the possible being of what is real is faced with the problem that possibilities, in the sense relevant to logic, are themselves ideal objects. They can as little be found in the real world as can numbers and triangles (in the mathematical sense).

The empiricist attempt to dispense with species as objects by having recourse to their extensions can therefore not be carried out. It cannot tell us what gives unity to such extensions. This can be made clear by considering yet another argument. The view Husserl is attacking operates with the notion of 'circles of similars'. Each object, however, belongs to a plurality of 'circles of similars' and we must be in a position to say what distinguishes these circles of similars from one another. Suppose that an object a is similar to other objects: to one object in respect A, to another in respect B, and so on. Now what unifies the circle of similars determined by, for example, redness, as opposed to triangularity? The empiricist or nominalist view says that these are differing similarities. If a and b are similar in respect of red, and a and c in respect of triangularity, these similarities must differ *in kind*.

Here again we come up against kinds. These 'kinds' cannot in turn be understood as just similarities or we start down the path of an infinite regress.

Husserl also addresses fictionalism in this part of Investigation II. He says that he does not wish to put the being of what is ideal on a level with the being-thought-of that characterizes the fictitious or the nonsensical. The fictitious and the nonsensical do not exist at all. There can be certain necessary and valid connections among objectless ideas, but that is all. Ideal objects, however, genuinely exist. These objects sustain predicates and we have insight into certain categorial truths relating to such ideal objects. If these truths hold then everything presupposed as an object by their holding must have being. The sense of this being and the sense of this predication need not coincide with their sense in cases where a real predicate is asserted or denied of a real object. There is a fundamental categorial split in our unified conception of being, or in our conception of object. This difference does not, however, do away with the concept of an object as a unity. In both cases something pertains or does not pertain to an object, there can be structures of evidence, and so on.

Mill's view, according to which abstraction is a function of attention, is considered at some length by Husserl. On this view, there are no ideal or general objects. Rather, we simply devote exclusive attention to certain parts or sides of an object. An attribute arrived at in this manner cannot exist by itself but it can be regarded by itself and in this sense become an 'object' of our attention. One then appeals to associations of general names with attributes attended to, and so on. Mill and others who hold such a theory do not start with what is *meant* in our cognitive acts in the two kinds of cases and so they overlook what is given. Meaning what is general is distinct from meaning what is individual. Empirical psychology is not in a position to recognize and clarify this *meaning*, the kind of meaning relevant to logic and epistemology. The consciousness in which we mean the universal is what it is whether or not we know anything about psychology, or mental antecedents and consequences, or causes and effects, associative dispositions, etc. The empiricist can try to give a account of consciousness as a fact of our human nature by considering unconscious dispositions, causal factors, appeals to previous experience, and so on. Husserl says that this is legitimate but that such empirical psychological facts would simply not be of interest for pure logic and epistemology. Only sense and essence matter.

To help make his point, Husserl contrasts the expressions 'an A', 'all A', and 'the A' with one another. These are logical forms that are irreducible to one another. They differ in meaning. They are different as intentions and we are therefore directed in different ways by these different intentions. Nominalism ignores these different forms of consciousness or confuses them with one another.

Husserl argues that abstraction as exclusive concern or attention does not by itself produce generalization. It does not remove the sensory individuality of the aspect or attribute to which we attend. Suppose, for example, that we concentrate our attention on the green of a tree that stands before us. Suppose, as Mill suggests, that we increase our concentration until we are completely unaware of associated aspects of the green of the tree. The result, it is claimed, is generality. If another object with exactly the same coloring were substituted we would see no difference. The green would be for us one and the same. Husserl argues, however, that *this* green would not really be the same as the other green. This kind of deliberate forgetfulness or blindness does not alter the fact that the aspect we are now heeding is just *this* aspect here and now and not some other. Comparison of two concrete separated phenomena of the same quality, e.g., green, shows that each has its own green. The green of the one is as much separated from the green of the other as are the concrete wholes in which these greens are given. Thus, we do not obtain the ideal species as a unity in multiplicity. The species is different from the instance of the species that occurs in the sensory phenomenon. Assertions that are true for the instance are false and even nonsensical for the ideal species. The coloring has its place and time, it is spread out and has an intensity, and it arises and vanishes. These predicates yield nonsense when applied to the color as a species. When a house burns down all of its parts burn down. Its individual forms and qualities, its constituent parts and aspects are all gone. The relevant geometrical, qualitative and other species, however, have not burned. With the attention theory of abstraction we never get beyond sensory, 'real' individuals. There are countless cases, however, where what we mean and name is not an individual but rather its Idea. Mill's theory is therefore unable to clarify our consciousness of universals. It does not inquire into the relation of attention to meaning and reference.

Using another example, Husserl says that no geometrical proposition holds for a drawn figure as a physical object. In the latter we can find no ideal geometrical properties. A mathematician might look at drawings that appear on blackboards but in none of his acts of geometrical thought does he *refer* to these drawings or any individual features of them. What we attend to is not this concrete object nor an abstract partial content (i.e., a non-independent aspect) in it. Rather, we attend to the ideal species.

Husserl also raises objections to accounts, especially Locke's, that psychologically hypostatize universals. He argues that Locke's account of abstract ideas is rife with confusions. In particular, Locke equivocates in many different ways on the word 'idea' (§10). Once this is cleared up, based on a deeper descriptive account of the components and structures of cognition, we see that Locke incorrectly takes universals to be real (instead of ideal) data in consciousness. Husserl is sharply critical of Locke's example of the 'universal triangle' (see §11). He also reminds us that inner pictures that may accompany names are not the meanings of those names. He objects to such a picture theory of meaning.

When we have a presentation or judgment about, e.g., a horse, it is a horse, not our sensations of the moment, that is presented and judged about. Our sensations are only presented and judged about in psychological reflection. The Lockean view does not inquire into the meaning and reference of our acts but instead takes the objects to which consciousness is directed, and the objects of attention in particular, to be mental contents as real occurrences in consciousness. It does not have the proper view of the relation of meaning and reference to attention. Husserl proposes a theory that recognizes 'sensuous abstraction' but also non-sensuous (and partially sensuous) abstraction. In non-sensuous abstraction we would have thought-forms or categories that do not permit of sensory fulfillment. These types of abstraction are discusssed in greater detail in Investigation VI (see Section 2.8 below). Meanwhile, Husserl suggests that the phenomenology of attention can be explored in much more detail.

Husserl also considers the idea that general concepts and names are mere devices for economizing thought. They are devices that spare us the individual consideration and naming of all individual things. Locke mentions this kind of view but one already finds it in medieval nominalism. This is rejected on several grounds, including the grounds already covered in the discussion of thought economy in the Prolegomena.

Husserl presents a critique of Berkeley's views on abstraction, and an entire chapter is devoted to Hume's theory of abstraction and several later variations on it. Berkeley and Hume suppose that talk of general ideas requires Locke's absurd interpretation, as in his absurd general triangle. They are reacting to Locke's views. Neither Berkeley nor Hume considers what is *meant* in different kinds of acts. They are blind to the fact that different intentional act-characters make the difference between the awareness of individuals and the awareness of ideal species. Of course they do not consider act-characters to be palpable features in our experience but this only shows a blindness or prejudice about our experience. They are already bringing certain principles to bear in their analysis of experience and cognition and whatever is not compatible with these principles is neglected.

In the later sections of this Investigation Husserl also begins to distinguish and apply many different concepts of what is 'abstract'. He distinguishes, for example, the non-independent parts ('moments') of an object from the independent parts ('pieces') of an object, and indicates the manner in which the former count as 'abstract'. He points out, however, that while the moments of an object are abstract they are so in a sense that is different from the sense in which ideal species are abstract (§40). Here he is already depending on some of the ideas about parts and wholes that are developed in the next Investigation.

2.5 Investigation III: "On the Theory of Wholes and Parts"

Husserl says that the distinction between 'abstract' and 'concrete' contents must be submitted to a thorough analysis. The distinction was already made by his teacher Stumpf, using the terms 'dependent' (non-independent) and 'independent' contents, respectively. Stumpf's work lies in the background of this investigation. There is also some influence from Brentano. (A theory of parts and wholes can of course already be found in the work of Aristotle.) Husserl says the distinction between dependent and independent contents extends beyond the sphere of conscious contents, however, and plays an extremely important role in the field of objects as such. The investigation of the distinction should play a role in the pure, *a priori* theory of objects. The pure theory of parts and wholes should be viewed as part of formal ontology. Such a pure theory could be applied to many different kinds of phenomena. This Investigation is perhaps the most extensive and interesting discussion of the logic of parts and wholes in the literature of the period and it certainly remains of interest today.

Objects can be related to one another as wholes to parts or also as coordinated parts of a whole. Every object is either actually or possibly a part. That is, there are actual or possible wholes that include it. Not every object, however, need have parts. Thus, we can in principle distinguish objects into the *simple* and the *complex*. This distinction is explored in more detail by Husserl but we omit the details here.

Some wholes can be divided into parts that Husserl calls *pieces*. Pieces are *independent* parts. If we are speaking of wholes and parts with respect to objects (i.e., ontology) then pieces are parts that can exist independently of the objects of which they are parts. We can also speak of parts and wholes with respect to what Husserl calls 'presentations' or 'contents'. (Husserl is at times thinking of part/whole structures with respect to our presentations and at other times of part/whole structures with respect to objects.) In this case, a piece is a part that permits of a separate presentation. Parts of wholes that cannot exist or be presented independently of wholes are called 'moments' of those wholes. Husserl also calls them *non-independent* parts. They are inseparable from the wholes of which they are parts. The effort to separate them would lead to the modification or elimination of other parts or to the whole to which they are related. In our sensory experience, for example, visual quality and extension are inseparable if vision is present. They cannot have an isolated and mutually independent existence in our experience. Husserl says that the inability of non-independent parts to exist by themselves points to laws of essence. Non-independent objects are objects belonging to such pure species (e.g., the species color) as are governed by a law of essence to the effect that they only exist (if at all) as parts of more inclusive wholes of a certain appropriate species.

Examples of pieces might be the parts of a machine, the players on a team, or the hairs on your head. A hair detached from your head is of course no longer a living thing but it can still exist and be perceived as an independent thing. Pieces, when separated from the wholes of which they are part, become wholes in themselves and are no longer parts. Pieces are parts that can become wholes.

Examples of moments are colors, which cannot occur apart from some surface or spatial extension, or musical pitch, which cannot exist except as bound up with a sound. Using an example from physics (see [Sokolowski, 2000, Chp. 3]), we can say that a body in motion possesses the moments of mass, velocity, momentum and acceleration. These moments are all interdependent: there cannot be momentum without mass and velocity, or acceleration without mass and force. Moments cannot exist except as bound up with other moments. They are the kinds of parts that cannot become wholes.

Something that is a piece in one respect can be a moment in another. A door, for example, can be separated from a house but as an object of perception it cannot be separated from a background. It has to be perceived against a background of some sort. It is important to keep this in mind in applications of the theory of parts and wholes.

As we noted, there is a kind of necessity in (or essence about) the way that moments are fused or blended together in their wholes. In particular, some moments will be *founded* on others. One can distinguish *founded* from *founding* parts. For example, hue is founded on color. Conversely, color founds or is the substrate for hue. There can be layers of founding. For example, shade is founded on hue, which is in turn founded on color. Husserl would thus say that shade is *mediately founded* on color, while hue is *immediately founded* on color. If we consider pitch and timbre, however, we find they are both immediately founded on sound.

It is clear that not every part is included in its whole in the same manner and that not every part is woven together with every other (in the unity of a whole) in the same way. A hand, for example, forms part of a person in a quite different way from the color of her hand, or from her body's total extent, or from her mental acts. Thus, Husserl considers additional types of part/whole and part/part relationships. Any pair of parts of a whole, for example, can be such that there is a relation of foundedness between both parts or not. If there is such a relation it can be either reciprocal or one-sided. Color and extension are mutually founded in a single intuition since no color is thinkable without a certain extension and no extension without a certain color. The character of being a judgment, however, is one-sidedly founded on underlying presentations since the latter need not function as foundations of judgments.

A whole can be termed a *concretum* (see also Section 5.1 below). It is something that can exist or present itself to experience as a concrete individual. A piece is thus a part that can itself become a concretum. A moment, however, cannot become a concretum. Whenever moments exist and are experienced they are inseparable from their other moments. They exist only as fused or bound up with other moments. It is clearly possible to speak and think about moments by themselves. We can speak about pitch without mentioning sound or about hue without mentioning color. When we consider moments by themselves they are *abstracta*. They are being thought of abstractly. We can speak about abstracta as a function of our use of language. Language permits us to deal with a moment apart from its necessary complement of other moments and its whole. This can be a problem, however, if we begin to think that a moment can exist by itself and can become a concretum. In applications one might mistakenly take what is actually a moment to be a piece.

We can also speak of *relative independence* and *relative non-independence* (§13). The absolute distinction between the two then becomes the limiting case of the relative. One might have a moment (with its own parts) of some whole. The moment is by definition non-independent. Within this moment, conceived in abstracto, there might be pieces. These would then count as relatively independent, while any moments at this level would count as relatively non-independent.

Abstract (non-independent) parts can thus have pieces and pieces can have abstract parts. Pieces that have no piece in common are *exclusive* (disjoined) pieces. The division of a whole into a plurality of mutually exclusive pieces is a piecing. Two such pieces may still have a common identicial moment: their common boundary, for example, is an identical moment of the adjoining pieces of a divided continuum (see also, e.g., B. Smith [1997]). Pieces are isolated when they are strictly disjoined, i.e., when they have no such identical moments. Thus, parts might interpenetrate, or they might be mutually external. The same whole can be interpentrative in relation to certain parts and combinatory in relation to others.

These ideas about inseparability or about parts existing in isolation lead Husserl to some reflections on the continuity/discontinuity of parts. 'Fusion' is involved in continuity, and separation in discontinuity. Some ideas on the continuity of space and the continuity of the stream of consciousness in time are discussed. Husserl also makes some interesting comments about sharper as opposed to more confused separation. This leads him to point out that in mathematics we have exact or ideal notions whereas such notions are not at work in sensory perception. The spatial shape of a tree as perceived, for example, is not an exact or ideal shape in the sense of exact geometry. The essences elicited from sensory intuition are 'inexact essences' and are not to be confused with the 'exact essences' that are, as Husserl says here, Ideas in the Kantian sense and which arise through a different kind of ideation. These remarks parallel what Husserl says later in the *Ideas I* about exact and morphological essences (Section 5.1 below).

Extensive wholes are those in which the pieces belong to the same lowest genus as is determined by the undivided whole. The pieces of an extended whole are extended pieces. As examples one could consider the division of a spatial stretch into spatial stretches or of a temporal stretch into temporal stretches.

We can speak of mediate and immediate parts, or nearer and remoter parts, in relation to the whole to which they belong but we can also use these notions in connection with the relations of the parts to one another. Some parts may be closer to one another, some further from one another.

Non-independent parts, as was said above, must be governed by various *a priori* laws. The laws that serve to define given types of non-independent contents rest on the specific essence associated with the contents, on their particular nature. All possible individual objects (existing things) will have their correlated essences. To these essences correspond concepts or propositions that have not only a 'content' or 'matter' but also a 'form'. Purely formal concepts and propositions lack all matter or content. The form that results is quite indifferent to its matter. It can persist through arbitrary variations of its comprised contents. The categories of formal logic and formal ontology are concerned solely with such purely formal concepts and propositions. As in the Prolegomena, Husserl lists some of these categories or concepts. They include concepts like 'something', 'one', 'object', 'quality', 'relation', 'plurality', 'number', 'order', 'whole', 'part' and so on. These have a different character from concepts like 'house', 'tree', 'color', 'tone', 'sensation', and so on. The former group themselves around the empty notion of 'something' or 'object as such', and are associated with this through formal axioms. The latter, however, are grouped around various highest material genera or categories, in which material ontologies have their basis. Thus, there is a basic distinction between formal and material spheres of essence, or between *formal* and *material ontologies.* Husserl says that on this basis we are given the true distinction between the *analytic a priori* and the *synthetic a priori* disciplines. All the laws or necessities governing different sorts of non-independent items fall into spheres of the synthetic *a priori.* Purely formal lawfulness is clearly different from lawfulness that depends on matter. Analytic (or formal) generalization is different from laws concerning material essences (see also Section 5.1 below). Analytic laws, Husserl says, are unconditionally universal propositions that are free from all explicit or implicit assertions of individual existence. They contain nothing but purely formal concepts. They stand opposed to their specifications, which arise when we introduce concepts with content. We cannot obtain from them any empirical assertions of existence. Analytically necessary propositions are said to permit of complete formalization.

It is such forms that will make up, in particular, the purely formal theory of parts and wholes. Only what is formally universal in the foundation relation is then relevant, together with the *a priori* combinations that it permits. We rise in the case of any type of whole to its pure form, its categorial type, by abstracting from the specificity of the contents in question. This is a formalizing abstraction. It is quite different from the performance of an 'abstraction' that gives us the universal redness in a concrete visual datum, or the generic moment of color in the redness previously abstracted. In formalization we replace the names standing for the sort of content in question by indefinite expressions of a particular type.

Husserl includes in this Investigation some thoughts towards a formal theory of wholes and parts (§14). Such a theory would include, among other things, the following definitions and propositions:

DEFINITIONS. If a law of essence means that an A cannot as such exist except in a more comprehensive unity that associates it with an M then the A requires *foundation* by the M. Equivalently, A needs to be *supplemented* by the M. A is *exclusively founded* upon M if A's need for supplementation is satisfied by M alone. To say that A requires supplementation or is founded is to say it is *non-independent.*

PROPOSITION 1. If an A as such requires foundation on an M then every whole having an A but not an M as a part requires a similar foundation.

As a corollary, using the previous definition,

PROPOSITION 2. A whole that includes a non-independent moment without including as its part the supplement which that moment demands, is likewise non-independent, and is so relatively to every subordinate independent whole in which that non-independent moment is contained.

PROPOSITION 3. If W is an independent part of (and so also relatively to) F, then every independent part w of W is also an independent part of F.

This can also be expressed as: If A is an independent part of B, and B is an independent part of C, then A is also an independent part of C.

PROPOSITION 4. If C is a non-independent part of a whole W then it is also a non-independent part of every other whole of which W is a part. Otherwise expressed: if A is a non-independent part of B and B is a non-independent part of C then A is a non-independent part of C.

PROPOSITION 5. A relatively non-independent object is also absolutely non-independent, whereas a relatively independent object may be non-independent in an absolute sense.

PROPOSITION 6. If A and B are independent parts of some whole W, they are also independent relatively to one another. For if A required supplementation by B, or any part of B, there would be, in the range of parts determined by W, certain parts (those of B) in which A would be founded. A would therefore not be independent relatively to its whole W.

An interesting secondary literature has grown around these ideas. See, e.g., [Sokolowski, 1967/68]; B. Smith (ed.) [1992]. Formalizations of some of Husserl's ideas in this Investigation can be found in Simons [1982; 1987]; [Null, 1989]; and [Fine, 1995].

2.6 Investigation IV: "The Distinction Between Independent and Non-Independent Meanings and the Idea of Pure Grammar".

In this Investigation Husserl applies his general distinction between independent and non-independent objects to ideal meanings. The distinction between independent and non-independent meanings is at the foundation of essential categories of meaning on which many *a priori* laws of meaning rest. These laws abstract from the objective validity or truth of propositions. They precede such matters. They provide pure logic with possible meaning-forms. These are the *a priori* forms of complex meanings that are significant as wholes. Such laws guard against *non-sense* (Unsinn) and are to be distinguished from laws that guard against formal or analytic *countersense* (Widersinn) or formal absurdity (compare with the account in *FTL*, discussed below in Section 6.1). The former merely tell us what is required in the case of complex meanings if we are to have a significant semantic unity. As such, they are *a priori* patterns in which meanings belonging to different semantic categories can be united to form one meaning instead of producing chaos. Within pure logic, we need to distinguish the pure theory of semantic forms from the pure theory of validity which presupposes it. Building on these ideas, Husserl wishes to promote the old idea, conceived by the rationalists in the seventeenth and eighteenth centuries, of an *a priori* or *universal grammar*. As is to be expected, he contrasts the idea of universal grammar with the idea of founding grammar on psychology or other empirical sciences.

This Investigation is focused on meanings but Husserl again wishes to track what can be said about expressions as well as what can be said about meanings. He starts by distinguishing simple from complex expressions and simple from complex meanings. Husserl thinks there are simple, elementary meanings and he probes the distinction between simple and complex meanings in some detail. He asks, for example, whether complexity or simplicity of meanings merely reflects complexity or simplicity of the objects that such meanings present. The answer is 'no'. Complex meanings can present simple objects and simple meanings can present complex objects. There are also many questions about whether a given meaning should count as simple or complex. For example, is the meaning of a proper name simple or complex? Consider the case of the proper name of a person, 'William', who is known to us. What Husserl says here is that the proper name has, as it were, a 'proper meaning' that is simple. The consciousness of meaning in this case, however, already involves a range of possibilities, of explicative meanings, associated with or implied by the proper meaning. These possibilities of further determination cannot proceed in just any direction whatever, but only in connection with the person William whom we mean when we use the expression 'William'. There are, therefore, two directions in which we can speak of the simplicity or complexity of the meaning of this expression. On the one hand there is the simplicity of the 'proper meaning'. This presupposes, however, a wider intentional background of content with a variable set of determining marks, meanings that would be involved in explicating the meaning of 'William'. The 'presentative content' by virtue of which William is presented can change in many ways while the proper name goes on performing the same significant role naming William directly. The variation and complexity of this set does not, however, touch the meaning itself, the 'proper meaning' that remains invariant through these possible variations.

Once we consider complex meanings we can ask whether each word in a complex of words has its own correlated meaning. A grammatical distinction has been drawn between *categorematic* and *syncategorematic* expressions. Words like 'house', 'alive', and 'red' are categorematic and words like 'but', 'and', 'with' are syncategorematic. Husserl says we must also distinguish categorematic from syncategorematic meanings. This is to be spelled out in terms of the distinction between independent and non-independent meanings, respectively, and Husserl explores some of the relationships of meaning to expression in this context. Meanings are conceived of as ideal unities but Husserl thinks the distinction between independent and non-independent parts can hold in either the real or the ideal realm.

In Investigation I Husserl had already discussed the concrete acts in which we mean something. These acts can themselves be simple or complex. A concrete act of meaning can involve several acts of meaning as parts. Such partial acts can be parts of a whole, whether as independent or non-independent parts. A total meaning then belongs to the whole act, and to each partial act belongs a partial meaning. A meaning itself is independent when it can constitute the full, entire meaning of a concrete act of meaning, and non-independent when this is not the case. When it is non-independent it can be realized only in a non-independent part-act in the concrete act of meaning. It can only achieve concreteness in relation to certain other complementary meanings. In other words, it can exist only as part of a meaningful whole. The non-independence of meaning thus characterized determines the essence of the syncategorematica.

Husserl considers several issues concerning this view. First, is the distinction between independence and non-independence of meanings reducible to the distinction between independence and non-independence of objects meant? Is it the case that categorematic expressions are directed to independent objects and syncategorematic expressions to non-independent objects? This could not be correct. The expression 'non-independent moment', for example, is categorematic but it presents a non-independent object. Every non-independent object whatever can be made the object of an independent meaning (e.g., redness, figure, likeness, size, unity, being).

Another issue is this: given this account of syncategorematica, how is it possible to understand syncategorematic expressions in isolation? It seems that the non-independent elements of categorematically closed speech cannot be isolated. One thing that Husserl says here is that isolated syncategorematica like 'equals', 'and', 'or' and 'together with' can be considered as merely intending meanings or as fulfilling meanings. He says they can achieve no fulfillment of meaning, no intuitive understanding, except in the context of a wider meaning-whole. In order to be clear what the word 'equals' means we must turn to an intuitive equation, we must actually (genuinely) perform a comparison to bring understanding and fulfillment to a sentence of the form $a = b$. The same holds in the other cases. The non-independent status of the fulfilling meaning, which forms part of a fulfilling meaning of a wider context, serves to base derived talk about the non-independent status of the intending meaning. Thus, Husserl says that no syncategorematic meaning, no act of non-independent meaning-intention, can function in knowledge outside of the context of a categorematic meaning. But can we still speak of intuitively unfulfilled meanings as non-independent? We must be able to hold that even empty meaning-intentions reveal a difference between independence and non-independence. Words like 'and' are understood even as isolated. They are not just hollow noises. Husserl's answer to this question is not entirely clear. His view seems to be that in such cases we form a thought of some familiar conjunction, or a thought of the type 'A and B', and that this supplies the meaning-intention even though there is no meaning-fulfillment in this case.

It follows from the fact that the distinction between independence and non-independence applies to meanings that meanings are subject to *a priori* laws regulating their combination into new meanings. To each case of a non-independent meaning a law of essence applies, as is the case with all non-independent objects. The law regulates how the meaning needs to be completed by further meanings and so points to the forms and kinds of context into which it can be fitted. How are meaningful wholes to be obtained from meaningful parts? Meanings cannot be combined to form new meanings without the aid of connective forms (which are themselves non-independent) and so there must be *a priori* laws of essence governing all meaning combinations. We are not free to combine meanings in any way we like. Meanings only fit together in antecedently definite ways to compose other unified meanings while other possibilities of combination are excluded by laws. Violating these laws yields only a heap of meanings, not a single unified meaning. The impossibility of certain combinations is not merely subjective. It rests on a law of essence. It is not merely a factual incapacity on our part. It is an objective, ideal impossibility, rooted in the pure essence of the realm of meanings. It is an impossibility concerning essential kinds, concerning the *semantic categories* that meanings fall under.

The following example can be given. The expression 'This tree is green' has a unified meaning. We can formalize this expression to obtain the corresponding pure form of meaning. Partially formalizing it, we obtain 'This S is P'. The 'materialization' of this form, its specification in definite propositions, is possible in infinitely many ways but we are not completely free in such specification. We work within definite limits. We cannot substitute any meanings we like for the variables 'S' and 'P'. We can substitute any nominal material for 'S' and 'P' and also any adjectival material for 'P'. Thus, we could even obtain expressions like 'This blue raven is green'. The point is that we still have an expression with a unified meaning. (We are not yet worried about the truth-values of the expressions. In this case the expression is false.) If we depart from the categories of our meaning-material, however, the unitary sense vanishes. Nominal material can be replaced freely by nominal material but not by adjectival, relational or propositional material. Material from a given category can be replaced with other material from the same category. In this free exchange of materials within each category we may find that false, foolish or ridiculous meanings result but we will nonetheless have unified meanings. When we transgress the boundaries of the categories this is no longer true. We no longer obtain meaningful wholes from meaningful parts. We can string together words like 'This careless is green', 'More intense is round', and so on, but now we achieve only a word series in which each word is significant but in which the whole formed from the words is not. One of the consequences of these reflections, according to Husserl, is that the task lies before us, fundamental for logic and for grammar, of setting forth the *a priori* constitution of the realm of meanings, of investigating the *a priori* system of formal structures in a 'form-theory of meanings' that leaves open all material specificity of meaning.

One must distinguish the law-governed incompatibilities introduced by this study of syncategorematica from the kind of incompatibility found in an expression like 'the round square'. As Husserl said in §15 of Investigation I, one must not confuse the senseless or nonsensical (Unsinn) with the absurd or countersensical (Widersinn). The latter is a subspecies of the meaningful. The combination 'a round square' yields a unified meaning. There is a meaning here but no object exists that corresponds to this meaning. By contrast, combinations of words like 'a round or' and 'a man and is' do not form meaningful wholes. In the one case, certain partial meanings fail to combine in a unity of meaning that could have objective validity. The meaning itself exists, however, in cases like 'wooden iron', 'round square', and 'all squares have five angles'. In the case of nonsense, however, the combination of words does not even form a meaningful whole.

In the pure logic of meanings it is pure grammar that provides a foundation for everything else. As Husserl will say in *FTL*, this is the lowest or first level of logic. The higher aim of logic is to find the laws of objective validity for meanings but this presupposes the logico-grammatical theory of forms. To illustrate his point, Husserl describes what we now think of as clauses in an inductive definition of sentences (or propositions). This is part of what it means to provide a grammar of logic. Such an account of the forms in the realm of meaning would allow us to distinguish sense from nonsense and would immediately reveal nonsense (Unsinn).

The formal laws that have the mere function of separating sense from nonsense leave it open whether meanings built on such forms have objects or not, or whether they yield possible truths or not. The laws of grammar are entirely distinct from laws that distinguish a formally consistent from a formally inconsistent (absurd) meaning. The consistency or absurdity of meanings expresses either objective, *a priori* possibility (consistency or compatibility) or objective impossibility (incompatibility). It expresses the possibility or impossibility of the being of the objects meant to the extent that this depends on the meanings.

In addition to drawing a line between nonsense and countersense, we also need to distinguish *material (synthetic) absurdity* from *formal (analytic) absurdity*. In the former case concepts with content must be given, e.g., 'A square is round'. All false propositions of geometry would be examples. The latter case covers every purely formal incompatibility without regard to any material content of knowledge. The laws of contradiction, double negation and modus ponens are laws for the avoidance of formal absurdity. They allow us to avoid formal fallacies. They show us what holds for objects in general in virtue of their pure 'thought-form', in advance of all objective matters signified. These laws are, in the sense of Investigation III, §11, 'analytic' laws, as opposed to the synthetic *a priori* laws that contain non-formal concepts and depend upon such concepts for their validity.

2.7 Investigation V: "On Intentional Experiences and their 'Contents'"

In his introduction to this Investigation Husserl reminds us that Investigation II was devoted to clarifying the general sense of the ideality of the species and, in particular, the ideality of meanings with which pure logic is concerned. As with all ideal unities, there are real possibilities and actualities that correspond to ideal meanings. To meanings *in specie* correspond acts of meaning. Meanings *in specie* are said to be ideally apprehended aspects of acts of meaning. This leads us naturally to questions about the kinds of experiences in which the supreme genus 'meaning' has its origin. Husserl says he now wants to inquire into the originative source of the concept of meaning and its essential specifications. Experiences of meaning are classifiable as 'acts'. The meaningful element in each single act must be sought in the act-experience and not in the fulfillment of the act or in the object of the act. We can consider the relation of acts to objects through intuition (i.e., fulfillment of the act) and Husserl will write about this at length in Investigation VI. In the present context he says he will do some preparatory work toward that end by investigating the element that makes the act an intentional experience, one that is directed toward objects. This involves clarifying the notion of act, in particular in the sense of an intentional experience. Husserl says that to put meaning-experiences into the genus of acts that exhibit intentionality will certainly be worthwhile. In this Investigation he sets about distinguishing and clarifying the ideas of act, the 'character' and 'content' of acts, and the notion of 'presentation'.

Husserl begins with an examination of different conceptions of consciousness. This leads him in particular to Brentano's concept of consciousness as intentional experience. He says that only conscious acts that exhibit intentionality are relevant to the highest ranks of the normative sciences. They alone, of all the different modes of being conscious, furnish the concrete bases for abstracting the fundamental notions that function systematically in logic, ethics and aesthetics, and that enter into the ideal laws of these sciences. To be conscious is to be conscious of something, to be directed toward some object or state of affairs. Intentionality is just this directedness. Husserl works through the details of Brentano's theory of intentionality, clarifying various points and making various adjustments in it. On Husserl's view one might, for example, have the idea or meaning-intention of the god Jupiter but it does not follow that the god Jupiter is an immanent or mental object nor does it follow that the god Jupiter exists extramentally. It simply does not exist at all. There are many circumstances in which an intention may exist, in which we may be directed toward an object, even though the object does not exist or is incapable of existence. In Husserl's view, the directedness toward or reference to an object belongs to an act-experience and can exist even if there is no object. Brentano's language, however, can promote misunderstandings about the 'intentional object' in such cases.

Husserl now distinguishes between the quality and matter of an act (§20). The *quality* of an act is the act-character, i.e., the type of act. The act may be a judgment, an assertion, a desire, a perception, a memory, and so on. The *matter* of the act is the 'content' that determines what it is about. For example, we might have two assertions, '$7 + 5 = 12$' and 'Ibsen is the principal founder of modern dramatic realism'. As *assertions* they are of the same kind. Their common feature is their judgment-quality. Their 'contents', however, differ. Husserl wishes to use the term 'matter' for this kind of content to distinguish it from other notions of content. Matter in this sense is a component of the concrete act-experience that it may share with acts of quite different quality. We can easily find many examples in which the act-qualities change while the matter remains identical. The matter fixes the objective direction of the act but of course there need not actually be an object in order for the act to be directed. One also has many examples in which the quality remains the same while the matter varies.

Every act has its quality and its matter. Husserl calls the union of the quality and matter of an act the 'intentional essence' of the act. In an attempt to pin down the notion of matter he notes that certain variations remain possible even if the quality and the objective reference of the act are both fixed at the same time. Two identically qualified acts may appear directed toward the same object without full agreement in 'intentional essence'. The intentional essence forms only one part of the complete act. (To the extent that we are dealing with acts functioning in expressions in a sense-bestowing fashion, Husserl wishes to speak more specifically of the 'semantic essence' of the act. He says that "the ideational abstraction from this essence yields a 'meaning' in our ideal sense" (§21).)

The ideas 'equilateral triangle' and 'equiangular triangle', for example, differ in content although both are evidently directed to the same object. They present the same object but in a different fashion. The same is true of such presentations as 'a length of $a + b$ units' and 'a length of $b + a$ units'. This is also true of statements that are synonymous and differ only in respect of 'equivalent' concepts. Husserl therefore says that matter must be that element in an act that first gives it reference to an object that is so definite that it not merely fixes the object meant in a general way but also the precise way in which it is meant (§20). (See also §25 of Investigation VI.) This suggests a fine-grained notion of matter or 'intension'. This includes the properties, relations, and categorial forms that it attributes to the object. The matter is said to be the objective, interpretative sense (Sinn der gegenständlichen Auffassung, Auffassungssinn) that serves as the basis for the act's quality. Identical matters can never yield distinct objective references, as is shown by the examples above. Husserl says that differences of equivalent (but not tautologically equivalent) expressions certainly affect matter but he does not go as far as one would like in addressing issues about how narrow the notion of matter is. It appears that 'equilateral triangle' and 'equiangular triangle' express different matters but there are other comments in this section (§20) that complicate the picture.

The quality and matter together do not make up the concrete, complete act. Acts with the same intentional essence can still differ from one another descriptively. The same person at different times, or different people at the same time or at different times, may have the same beliefs, memories, judgments, and so on, even though there is no individual sameness of their acts. It is not as if my individual presentations are identical with those of another person. The intentional essence or the meaning can, however, be the same. This is like Frege's idea that different people can share the same sense even if they associate different subjective ideas with that sense. The identity of a judgment or a statement consists in an identity of meaning repeated as the same in many individual acts, and represented in them by their semantic essence.

A considerable portion of Investigation V is devoted to analyses of the notion of 'presentation' and its relation to the matter of acts. Since judgments are of particular interest to the logician, Husserl spends a lot of time investigating the notion of presentation in relation to judgments. Let us forego the ins and outs of the arguments here and simply note a few points that Husserl makes in these sections about judgments and names. First, he says that judgments are not to be confused with perceptions, memories, and so on. In a judgment a *state of affairs* appears before us, or becomes intentionally objective to us. A state of affairs, even one concerning what is sensibly perceived, is not itself an object that could be sensibly perceived (§28, and also Investigation VI, discussed below). In perception an object is given to our senses and we may then judge that it is or is thus and so. In the judgment it is not the existent sensible object that appears but rather the fact *that this is.* The state of affairs judged is of course to be distinguished from the judging itself. It is different from the act in which the state of affairs appears. Now we can investigate various questions about the 'matter' of judgments in particular. It is possible for states of affairs to be our object but it is also possible for judgments to be our object. Judging about judgments differs from judging about states of affairs. One can name either a judgment or a state of affairs as a logical subject.

Husserl examines some of the difficulties attending the concept of a name. He says that we should not understand 'names' as mere nouns. To see what names are we should look at statements in which names function with their normal meaning (§34). We will then be able to note that words and word groupings that are to count as names only express complete acts when they either stand for some complete simple subject of a statement or at least could perform such a simple subject function in a statement without change in their intentional essence. Thus, a mere noun, even when coupled with an attributive or relative clause, does not make a full name. We must also add the definite or indefinite article. The following examples of names can be given: 'the horse', 'a bunch of flowers', 'a house built of sandstone', 'the opening of the Reichstag', and also expressions like 'that the Reichstag has been opened'. A distinction is drawn between 'positing' and 'non-positing' names. Positing names are those that give what they name the status of an existent and non-positing names do not do so. If a name is merely understood then such a mere understanding is not part of the positing use of a name. Nominal acts that are positing refer to an object as existent but without claiming to seize the object itself in intuition.

In the final chapter of this Investigation Husserl sums up the most important ambiguities that have emerged earlier in the Investigation concerning the terms 'presentation' and 'content'. This should allow us to disambiguate these terms when they are used in talk of the 'matter' of acts, the fulfillment of meaning-intentions, imagination, picturing, and so on. Husserl says that by 'content', for example, we can have in mind the meaning, as an ideal entity, of a nominal presentation. This is a 'content' or 'presentation' in the sense of pure logic. To this corresponds a real moment in the real 'content' of an act, the intentional essence with its quality and matter. We can further distinguish in this real content the separable contents not belonging to the intentional essence, i.e., the 'contents' (sensations and images) that receive their interpretation in the act. Additional uses of the term 'content' may also be discerned. Thus, we need to be sure that we are not glossing over all of these differences or equivocating on them when we use the term 'content'.

2.8 Investigation VI: "Elements of a Phenomenological Elucidation of Knowledge"

Investigation VI is by far the longest in the book. It is one of the most interesting Investigations and parts of it are also the most difficult. Husserl says that all thought and all knowledge is a function of certain acts that occur in the context of expressive discourse. In these acts lies the source of all unities of validity that are objects of thought and knowledge for the thinker. Thus, the source of the pure, universal Ideas connected with such objects, whose ideally governed combinations pure logic attempts to set forth, must also lie here. By seeing that our logical experiences belong in this class we will have taken an important step toward an analysis that will make sense of the sphere of logic. Now one of the central distinctions in a phenomenological theory of knowledge is the distinction between *meaning-intention* and *meaning-fulfillment* that was already mentioned in Investigation I. This is, roughly speaking, like the distinction between concept or thought on the one hand, understood as mere meaning without intuitive fulfillment, and a corresponding intuition on the other. Knowledge requires more than mere meaning-intention. It requires that the meaning-intention be fulfilled. This distinction is explored in great detail in Investigation VI.

Husserl opens the Investigation by revisiting some themes about meaning and expression in Investigation I. He considers expressions of perceptions ('judgments of perception') in order to point out that the meaning of such an expression cannot lie in the perception itself. Quite different statements could be based on the same percept, thus leading us to unfold quite different senses. Conversely, the sound of my words and their sense might remain the same even though my percept varies in different ways. Percepts may in fact not only vary but may also vanish altogether without causing a judgment of perception to lose its meaning. Thus, perception is not the act in which the sense of a perceptual statement is achieved. Nonetheless, perception does seem to make some kind of contribution to the meaning of a statement grounded on perception. If, for example, I use the occasional expression 'this' to say 'This blackbird is perched' as I observe a particular blackbird, the expression 'this' only becomes fully significant in the actual circumstances of the utterance. The perceived object is what the word 'this' signifies. Husserl reminds us of his earlier discussion of essentially occasional expressions in Investigation I, and of his distinction between indicating and indicated meaning, and he now develops his views in more detail (§5). Intuition of the object may contribute to the meaning of a perceptual statement but only in the sense that the meaning could not acquire a determinate relation to the object it means without some intuitive aid. This does not imply that the intuitive act is itself a carrier of meaning.

Occasional expressions have a meaning (indicated meaning) that varies from case to case but in all such cases a common element (indicating meaning) is left over. This shows that their ambiguity is different from typical forms of equivocation. Husserl says that perception accordingly realizes the possibility of an unfolding of my act of this-meaning, with its definite relation to the object, but it does not itself constitute the meaning or even part of it. In these cases perception is an act that may determine meaning but it does not embody meaning. Once the intention to an object has been formed on a suitable intuitive basis it can be revived and exactly reproduced without the help of a suitable act of perception or imagination. Occasional expressions are like proper names insofar as the latter name objects directly. The meaning of both proper names and occasional expressions has an intuitive origin from which their naming intentions first orient themselves toward an individual object. They are different from one another, however, in some other respects. No ostensive character, for example, is associated with a proper name as it is in the case of an occasional expression. Proper names belong as fixed appellations to their objects. The upshot is that no part of the meaning itself should be located in the percept. The percept, which presents the object, and the statement that thinks and expresses it must be distinguished even though they stand to one another in a relation of mutual coincidence or unity of fulfillment.

Husserl says that there can be a *static* coincidence of meaning and intuition, as is the case when the object referred to by a meaning is indeed intuitively present to us. This happens, for example, if I speak of my computer and my computer is indeed present to me in intuition. There is also the matter of *dynamic* coincidence or unity where an expression first functions in a merely symbolic fashion and is then accompanied by an intuition that more or less corresponds to it. In this case we experience a consciousness of fulfillment. The act of pure meaning, like a goal-seeking intention, finds its fulfillment in the act that renders the matter intuitive. The same object that was merely thought of symbolically is now presented in intuition as precisely that which it was at first merely thought or meant to be. It is through fulfillment that we can first come to speak of knowing or recognizing something. In the dynamic relationship the intention and the fulfillment are disjoined in time. The fulfillment unfolds in a temporal pattern. In the static relationship, which can be viewed as the outcome of this temporal process, they occur in temporal and material coincidence. It is the difference between coming into coincidence and being coincident. There is an experience of identity between the object meant or thought and the object intuited.

Husserl notes that the difference between intention and fulfillment is seen in many kinds of experiences, not only those concerning signification. We have, for example, wishful intention and wish-fulfillment, the fulfillment of hopes and fears, expectation and fulfillment of expectation, and so on. Intention should not be identified straight away with expectation. If we consider the example of ordinary perception we see that intentions in stasis lack the character of expectancy but they acquire this character when the perception is in flux. In this later case there will be a perceptual manifold pertaining to one and the same object if the perception is veridical. The object shows itself from a variety of sides. What was merely adumbrated or anticipated from one perspective may be confirmed from another perspective. All perceiving and imagining consists of a web of partial intentions fused together in the unity of a single total intention. The correlate of this last intention is the thing while the correlate of its partial intentions are the things parts and aspects. Husserl says that it is only in this way that we can understand how consciousness reaches out beyond what it actually experiences. It can mean beyond itself and it is possible in some cases that this meaning can be fulfilled. Not only does every nuance in intention correspond to some nuance of the correlated fulfillment but to different classes of intentions there will correspond different classes of fulfillment. Husserl also says that there can be fulfillments of meaning-intentions even if those meaning-intentions are not expressed. Here there is a kind of 'wordless' fulfillment that is divorced from the signitive contents that would otherwise pertain to a meaning-intention (§15).

Just as there can be agreement between intention and what is given in intuition so there can be disagreement or conflict. Intuition may not accord with intention, in which case Husserl speaks of the 'frustration' of the intention. In the case of agreement there is a kind of synthesis, a synthesis of identity. The experience of conflict is also a kind of synthesis, for it also puts things into relation and unity. It is a synthesis of distinction. The two kinds of synthesis are not completely parallel: conflict presupposes a certain basis of agreement. If I think an object $\underline{a}$ to be red and it shows itself in intuition to be green then the intention 'red' is incompatible with the intention 'green'. This can only be the case because the same object $\underline{a}$ is intended in both acts. In *Experience and Judgment* Husserl describes the origin of negation in these terms (see Section 8 below). There can also be partial agreement or disagreement between an intention and the acts that fulfill or frustrate it.

Husserl also speaks of how acts of imagination can be fulfilled through what he calls a synthesis of 'image resemblance'. In the fulfillment of perceptual acts, however, we have the presence of the perceptual object that was merely intended. Now in the case of either perception or imagination the object is not given wholly and in its entirety (§14). It is only given in a perspectival manner. Other parts or 'sides' of an object are foreshadowed but are not themselves intuitively given. There could be indefinitely many percepts of the same object, all differing in content. This does not mean that only the percepts are given and not the object. The object is realized imperfectly in the percepts but it is the object itself that is given in the total act, even if only by way of an aspect. It is the same object that is there through the manifold of percepts. Each individual percept is a mixture of fulfilled and unfulfilled intentions. Some parts of the object are given and some parts are not yet given although they might be given in new percepts. This suggests the ideal, limiting case of an adequate perception in which the object is not given imperfectly. This 'ideal' of adequation, Husserl says, enters into the sense of all perception. Husserl thus says that the relation of fulfillment is one that admits of degrees in which epistemic value steadily increases. Each such ascending series points to absolute knowledge or absolute adequation as an ideal limit.

Once again Husserl discusses the notion of 'clarity' in connection with the idea of fulfillment of a meaning-intention. Fulfillment, even if only partial, helps to give clarity to one's thought but Husserl says that it is also possible to achieve a kind of clarity with a signitive intention alone (§17). We should not run these two things together.

In §18 Husserl gives us an example of some of these ideas in connection with mathematical concepts. He says that the formation of every mathematical concept that unfolds itself in a chain of definitions reveals the possibility of fullment chains built member upon member out of signitive intentions. We clarify the concept $(5^3)^4$, for example, by having recourse to the 'definition' according to which it is the number that arises when one forms the product $5^3 \bullet 5^3 \bullet 5^3 \bullet 5^3$. To clarify this later concept we must go back to the sense of 5^3, i.e., the formation $5 \bullet 5$ 5. We clarify 5, in turn, through the chain of definitions $5 = 4 + 1$, $4 + 3 + 1$, $3 + 2 + 1$, $2 = 1 + 1$. In this manner we would at last come to the completely explicated sum of ones of which we could say 'This is the number $(5^3)^4$ itself'. Husserl says that we could not actually carry out this entire process but it is something that can be done 'in itself'. Now an act of fulfillment not only corresponds to this final result but to each individual step along the way that clarifies and enriches the content of the original number expression. In this manner each ordinary decimal number points to a possible chain of fulfillments so that chains of indefinitely many numbers are possible *a priori*. Husserl says that we usually speak in mathematics as if the meaning of a word were identical with the content of its complex defining expression. In this case, however, there could be no talk of fulfillment chains. We would be moving among pure identities wholly tautological in character. If, however, we consider the complexity of the thought formations that arise through substitutions like those given above we can hardly think that all of the complication we uncover was present in the signitive intention first experienced. It is plain, Husserl says, that there are real differences in intention in these cases.

Husserl also says that cases of this kind have the remarkable property that the 'matter' of our intentions dictates a determinate order of fulfillment *a priori*. To each signitive intention of this class there is a definite, proximate fulfillment or group of fulfillments which in its turn has a definite, proximate fulfillment, etc. Thus, Husserl says we may speak in cases of this sort of mediate intentions or fulfillments. Every mediate intention requires a mediate intention which, after a finite number of steps, terminates in an immediate intuition.

Mediate presentations in this sense must be distinguished from presentations of presentations, i.e., presentations directed upon other presentations as their objects. Such presentations of presentations are, generally speaking, themselves intentions and are thus capable of fulfillment. The presenting presentations in question require no mediate fulfillment through the fulfillment of the presented presentation. The intention of $P_1(P_2)$ is directed to P_2. It is fulfilled and completely fulfilled when P_2 itself is present. It is not enriched when the intention of P_2 is fulfilled in its turn. Husserl says, for example, that as the thought of a color has its fulfillment in the act of intuiting this color, so the thought of a thought has its fulfillment in an act of intuiting this thought and has its final intuitive fulfillment in an adequate intuition of the same. Note that intuition is not always outward intuition of external physical objects. 'Inner' perception or imagination can also function as a fulfilling intuition.

Husserl says that if we take the 'weight' of the intuitive (i) or signitive (s) content as the sum total of the intuitively or signitively presented moments of the object then both weights i and s will add up to a single total weight, i.e., the sum total of the object's properties. So we always have $i + s = 1$, where the weights i and s can vary in many respects. That is, the same object, intentionally speaking, can be given intuitively with more or less numerous, ever varying properties. The signitive content is then altered correspondingly, either increasing or diminishing. Ideally, the two limiting cases are $i = 0$ and $s = 1$, and $i = 1$ and $s = 0$. In the former case we have only signitive content. There would be no property of its intentional object that would be brought to intuition. These are the pure meaning-intentions. In the second case there is no signitive content whatever. All is fullness. No part, no side, no property of its object fails to be intuitively given and none is indirectly or subsidiarily meant.

There may also be a mixture of perceptual and imaginative components in intuition. Thus, there could be purely perceptual content or purely imaginative content and we could also assign weights to these components. Husserl mentions that we could also consider the 'extent' or 'richness' of the fullness, the 'liveliness' of the fullness, or the 'reality level' of the fullness.

Husserl also discusses meaning-intentions that are 'possible', in the sense of being internally consistent, and impossible, in the sense of being internally inconsistent. He refers to the possible here as the 'real' and the impossible as the 'imaginary'. Two intuitive acts are said to have the same essence (Essenz) if their pure intuitions have the same matter. All objectively complete intuitions with one and the same matter have the same essentia. The 'possibility' of a meaning may be defined by saying that there is an adequate essence that corresponds to it in specie in the sphere of objectifying acts, as an essence whose matter is identical with its own. Alternatively put, it must have a fulfilling sense. There exists in specie a complete intuition whose matter is identical with its own. The 'exists' here has the same ideal sense as in mathematics. This definition provides the *ideal* necessary and sufficient conditions of possibility. Husserl goes on to discuss the nature of compatibility and incompatibility (conflict) of contents in general. He specifies a few axioms of unity and conflict (compatibility and incompatibility) (§34).

It is in the context of these discussions of the intention-fulfillment relation that Husserl begins to spell out his ideas on self-evidence (Evidenz), truth, and being. The ideal of ultimate fulfillment has already been mentioned. Husserl now says that self-evidence can be thought of as this most perfect synthesis of fulfillment. We have an identification of what is meant with what is given. There is full agreement between meaning-intention and the corresponding state-of-affairs, the object meant. This agreement is experienced in self-evidence. The concepts of truth and being, Husserl suggests, should be so distinguished that our concepts of truth are applied from the side of our *acts* while our concepts of being are applied to the corresponding *objective correlates* of our acts (§39). Truth could thus be defined as the Idea of adequation or of 'rightness' of objectifying assertion and meaning, whereas being would be understood as the identity of the object meant and given in adequation or as the adequately perceivable thing as such, which stands in an indefinite relation to an intention it is to make true or to fulfill adequately.

Husserl again makes a point about his notion of self-evidence that he already made in the Prolegomena: It is not possible for two people to disagree about what is self-evident. If someone experiences the self-evidence of A then it is self-evident that no second person can experience the absurdity of this same A, for to be self-evident is for A not merely to be meant but also genuinely given and given precisely as it was thought to be. In the strict sense A is itself present. It would be a different matter if self-evidence were interpreted in terms of our feelings, but it is exactly such an interpretation that must be rejected.

One of the sections of Investigation VI that has been of great interest to generations of readers is on the topic of 'categorial intuition' (kategoriale Anschauung). If knowledge and understanding are to be thought of in terms of the meaning-intention/meaning-fulfillment relation then how are we to think of fulfillment or intuition in the case of intentions directed toward categorial forms? This is an especially important question given Husserl's views on logic, mathematics and science. Husserl opens this discussion by noting that parts of very different kinds are to be found in our meanings. Among these will be meanings expressed by formal words such as 'the', 'a', 'some', 'many', 'two', 'is', 'not', 'and', 'or', and so on. Are there parts and forms of perception corresponding to all parts and forms of meaning? Husserl puts forward a suggestion toward the solution of his problem at the outset of his discussion: it is in 'founded' acts that we find fulfillment in such cases. Perception, for example, may serve to base certain connective, relational or otherwise formative acts and it is in these later kinds of acts that our categorial meaning-intentions find fulfillment. The original perceptual acts are 'founding' acts. They are directed toward individuals given to us in sensory intuition. In the 'founded' acts, however, it is not the perceived individual that is meant or given. Rather, we are directed toward what is universal. Husserl devotes a large portion of Chapters 6 and 7 of Investigation VI to the development of this kind of account. In his Foreword to the second edition of Investigation VI he tells us that he is not entirely happy with all of the aspects of his account (e.g., his doctrine of categorial representation) but that the work still deserves study in any case (see [Lohmar, 1990]).

Husserl says that in perceptual statements of the form 'All S are P', 'An S is P', and so on, it is only at places indicated by letters or variables that meanings can be placed that are themselves fulfillable in sensory perception. The supplementary forms here cannot find fulfillment in perception. One could speak of this as a form/matter distinction but Husserl says here that it will be better to use the term 'stuff' instead of 'matter' since 'matter' has already been used for the intentional or interpretative sense associated with our acts. Husserl says that 'being', in its attributive or predicative function, is not fulfilled in any percept. Being, as Kant said, is no *real* predicate. We can see color but not being-colored and we can feel smoothness but not being-smooth. Being is nothing in the object nor is it anything attaching to the object. It is no *real* feature of an object. The expressions 'a', 'the', 'all', 'if...then', 'and', 'something', 'nothing', and so on, are meaningful propositional elements but they do not have their objective correlates in the sphere of *real* objects, i.e., the sphere of objects of possible sense perception. They do not have their objective correlates in either 'inner sense' or 'outer sense'. In particular, they do not arise through reflection on mental acts in inner sense, as philosophers like Locke might hold.

Where, then, do categorial forms of our meanings find their fulfillment? It cannot be in sense perception, either as inner or outer perception. Husserl holds that there is an essential homogeneity of the function of fulfillment across different domains of knowledge. There must be an act that renders identical services to the categorial elements of meaning that sensory perception renders to the material elements. Just as there are intuitions (fulfillments) and objects in the one case so there must be intuitions and objects in the other case. To say that categorially structured meanings find fulfillment in intuition or 'perception' means that they relate to the object in its categorial structure. The object with these categorial forms is not merely referred to, as is the case where meanings are functioning only symbolically, but it is intuited. Hence, we must widen our concepts of intuition (or perception) and object so that we may speak of both sensory and ('supersensory') categorial intuition. This allows the possibilitity of speaking of aggregates, indefinite pluralities, totalities, numbers, disjunctions, predicates and states of affairs as objects and to speak of the acts through which they might be given as 'percepts' or intuitions.

Husserl offers at least a provisional phenomenological analysis of the distinction between sensuous and categorial perception. Sensory (real) objects are are said to be objects of the lowest level of possible intuition, while categorial or ideal objects are objects of higher levels. In sensory perception an object is directly apprehended in a 'straightforward' (schlichter) manner. The object is immediately given, in the sense that it is not constituted in relational, connective or otherwise articulated acts. It is not constituted in acts founded on other acts. In the later case we would have a different kind of object. Sensory objects are present in perception at a single-act level. They do not need to be constituted in 'many-rayed' fashion in acts of higher level whose objects are set up for them by way of other objects already constituted in other acts. Husserl says that each straightforward act of perception, by itself or together with other acts, can serve as a basic act for new acts which include it or merely presuppose it. The new acts bring a new awareness of objects that presupposes the old. When new acts involving conjunction, disjunction, generalization, and relational knowledge arise we have acts that set up new objects, acts in which something appears as actual and self-given that was not self-given and could not have been given in the foundational acts alone. The new objects, however, are based on the older ones (§46). These objects can only appear in such founded acts. In the first instance the founding will be on straightforward perceptual acts but there can in principle be whole series of foundings upon foundings.

In sense perception the external object appears at once, as soon as our glance falls upon it. No apparatus of founding and founded acts is required. The unity of perception in this case does not arise through our own synthetic activity but is rather an immediate fusion of part-intentions without the addition of new act-intentions. The continuous percept is built out of individual percepts and we can say it is founded on them in the sense in which a whole is founded on its parts. This is not the kind of founding involved in categorial intuition since in the latter case we have a new act-character that is grounded in the underlying act-characters and is unthinkable apart from them. In sense perception the perception is merely extended. The unification of percepts is not the performance of some new act through which there is consciousness of a new object. The same object is meant in the extended act that was meant in the part-percepts taken singly. There is as it were a passive unity of identification through these acts but this is not the same thing as the unity of an *act* of identification. Identity itself is not made objective in ordinary perception. The perceptual series can be used to found a new act when we articulate our individual percepts and relate their objects to one another. In this latter case the unity of continuity holding among the individual percepts provides a basis for a consciousness of identity itself. Husserl says that the moment of coincidence linking our act-characters with one another serves as representative content for the new percept, founded upon our articulated individual percepts.

In clarifying what a straightforward percept is we have also clarified what a sensible or real object is. A real object is just the possible object of a straightforward perception. Sensible objects are, in general, the possible objects of sensible intuition and sensible imagination. We can then also define real part, real piece, real moment and real form. Each part of a real object is a real part. In straightforward perception the whole object is explicitly given while each of its parts is implicitly given. Every concrete sensory object and every piece of such an object can be perceived in explicit fashion. Abstract moments of such objects, however, are incapable of separate being. Their representative content cannot be experienced alone but only in a more comprehensive concrete setting. Husserl says that the apprehension of a moment, and of a part generally, *as* part of a whole already points to a founded act since this is already a relational kind of act. The sphere of 'sensibility' has been left behind and that of 'understanding' entered. 'Understanding', as opposed to sensibility, can be defined as the capacity for categorial acts. Husserl says we can consider acts in which concretely determinate states or affairs, collections and disjunctions are given as complex thought-objects (objects of a higher order) that include their foundational objects as *real parts* in themselves. We can also consider acts whose objects are of a higher level but which do not include their foundational objects in themselves.

When a sensible object is apprehended in a straightforward manner it simply stands before us. The parts that constitute it are in it but are not made our explicit objects. We can also, however, grasp the same object in an explicating fashion in articulating acts in which we put certain parts into relief. Relational acts can then bring the parts into relation to one another or to the whole. This is a new kind of (active) 'synthesis' that we bring to the situation. Only through such new modes of interpretation will the connected and related members assume the character of 'parts' or of 'wholes'. Now the articulating acts and the act we call 'straightforward' are experienced together in such a way that new objects, the relationships of the parts, are constituted. These are, therefore, already categorial acts and objects. All such relationships of wholes to parts or parts to parts are already of a categorial, ideal nature. They are not part of the straightforward intuition. In a similar manner, external relations like 'A is to the right of B' or 'A is larger than B', are given as states of affairs in founded acts. Neither the straightforward percept of the complex whole nor the specific percepts pertaining to its members are in themselves the relational percepts (§48). Husserl gives two additional examples of this type: collectiva and disjunctiva (§51). In all these cases we see that in straightforward perception we are completely and passively engaged or immersed and that the kind of articulation or 'reflection' involved in categorial shaping is founded on straightforward perception but is not itself straightforward perception.

In the examples considered thus far the synthetic acts are so founded on straightforward percepts that the synthetic intention is subsidiarily directed to the objects of the founding percepts insofar as it brings them into a relational unity. There are, however, other kinds of categorial acts in which the objects of the founding acts do not enter into the intention of the founded one. This is what we have in the field of what Husserl calls 'universal intuition' (§52). In this case there is a kind of 'abstraction' but it is not an abstraction that amounts to setting some non-independent moment into relief. Instead, Husserl calls it an 'ideational abstraction'. Here it is an 'Idea' or universal that is brought to consciousness and achieves actual givenness. We become aware of the identity of the universal on the basis of different individual intuitions. It is the unity through this multiplicity. Husserl says that the universal is itself given to us in this manner. We are not merely thinking of it in a signitive manner. Husserl had already commented on this to some extent in Investigation II. He says here, as he does elsewhere, that our consciousness of a universal has as satisfactory a basis in perception as it does in imagination. The 'red' or the 'triangle' exemplified in mere imagination is specifically the same 'red' or 'triangle' as that exemplified in perception.

Acts of categorial intuition should be similar in their basic structure to acts of straightforward intuition. For any intuition, no matter how near or far it stands from sensory intuition, there is a corresponding expressive meaning as its possible ideal counterpart. Sensory components will of course not play the same role in the founded acts that they play in the ultimate founding acts but Husserl says that categorial acts should still have a quality, a matter (or interpretative sense), and a 'representing content'. These distinctions in the case of acts of categorial intuition do not reduce to distinctions among the underlying acts. It is fairly easy to see that acts of categorial intuition will have a quality and matter like other acts, but what could their 'representing contents' be? This is the next problem that Husserl takes up. In the case of sensory intuition there is a 'representing sensum' that could stay the same while the interpretative sense varied, and could vary while the latter remained constant. The distinction in sensory intuition between matter and representing content can be pointed out and easily acknowledged. In the case of categorial acts the situation is not as simple. There must be such a thing, Husserl thinks, because signitive acts lack representing contents but intuitive acts do not. Representing contents constitute the difference between 'empty' signification and 'full' intuition. Since it is not the objects of the underlying acts that are presented in categorial intuition it cannot be the representing content of such acts that exclusively makes up the representing content of categorial intuition.

In a merely signitive intention the identity of the meant object, for example, is not experientially *lived through* but is merely *thought of*, whereas in the case of intuited objects the identity is indeed given. This leads Husserl to say that the mental bond that establishes the synthesis in the case where we 'live through' the identification is a bond of thought or meaning and as such is more or less fulfilled. It is this constituent that must therefore exercise a representative function. The mental bond experienced in actual (i.e., intuitive or authentic) identification, collection, etc., must be the *universal common feature* that we can distinguish both from the quality and matter, and that constitutes the representing content in the case of categorial intuition. There will be a common element that confronts us, for example, in all forms of collective synthesis. The representing content in this particular case thus appears to be what Husserl called the 'collective connection' in *PA*. In *PA* it is a 'second-order' act directed toward the collecting that gives us awareness of *one* object having many elements. Husserl now says that we can merely think of or intend a collection but we can also 'live through' or carry out the collecting and it is in the latter case that we have representing content appropriate to the collection as such. The sensory or imaginative content of the underlying acts is not relevant to such a founded synthesis. It is inessential.

Husserl says that it is essential to categorial acts, in which all that is 'intellectual' is constituted, that they should be achieved in stages (§57). Objectivations arise on the basis of objectivations and constitute objects that, as objects of a higher order or objects in a wider intellectual sense, can only be given in such founded acts. Such synthetic acts exclude the immediate unity of representation that unites representative contents of straightforward intuitions. Categorial intuition arises, if this account is correct, insofar as the mental content that binds the underlying acts itself sustains interpretation as the objective unity of the founded objects, as their relation of identity, of part to whole, etc. Husserl thinks we can say that the same mental moments that are sensuously given in inner perception may, in a founded act of the character of a categorial intuition, set up a categorial form and so sustain a totally different categorial representation. Husserl also distinguishes here what he calls 'primary contents' from 'contents of reflection' and says that only reflective contents can serve as purely categorial representing contents. Such is Husserl's account of representing content in the case of categorial acts. Unfortunately, the account is somewhat obscure and it raises a number of questions.

The varied forms of founded acts, Husserl goes on to argue, permit many complications into new forms. As a result of certain *a priori* laws, categorial unities may again and again become the objects of new connecting, relating or ideating acts. Universal objects, for example, can be collectively connected, and the collections thus formed can in their turn be collectively connected with other collections of similar or different type, and so on. There are or must be certain law-governed limits within which one can unify states of affairs in new states of affairs, investigate internal and external relations among such unities, etc. The 'complication' here is achieved in founded acts of ever higher level. The governing laws in this field are intuitive counterparts of the laws of grammar in pure logic. In addition to the pure theory of the forms of meaning we would have a corresponding pure theory of the forms of intuition. This would all have to be investigated.

Since underlying objects may themselves be categorial in type it is worthwhile to distinguish between purely categorial acts (acts of 'pure understanding') and mixed acts of understanding that are blended with sensory elements. Husserl says that everything categorial ultimately rests upon sensory intuition. When it comes to knowledge, a categorial intuition (i.e., an intellectual insight or a case of thought in the highest sense) without any foundation in sensory intuition is a piece of nonsense (§60). Eidetic abstraction rests on what is individual but does not for that reason mean what is individual. The intuition of universality excludes individuality and sensibility. We must distinguish between *sensory abstraction*, which yields sensory concepts (either purely sensory or mixed with categorial forms), and purely *categorial abstraction*, which yields purely categorial concepts. Husserl says that 'color', 'house', 'judgment', and 'wish' are purely sensory concepts, 'coloredness', 'virtue', 'the axiom of parallels' have a categorial admixture, and 'unity', 'plurality', 'relation', and 'concept' are purely categorial. Sensory concepts have their immediate basis in the data of sensory intuition, categorial concepts in the data of categorial intuition. In particular, Husserl says that all logical forms and formulae, such as 'All S are P', 'No S is P', and so on, are purely categorial. Pure logic, pure arithmetic, pure theory of manifolds, and so on, contain no sensory concepts in their theoretical fabric.

Categorial structuring does not involve any 'real' reshaping of 'real' objects. Categorial forms leave the 'primary objects' untouched. There are many ways to categorially form real contents. For example, one can divide up a sensorily unified group into part-groups in many ways. Many possibilities of categorial shaping arise on the foundation of the same sensory stuff. Although there is great freedom here there are still law-governed limits. We cannot intuit sensory stuff in any categorial form we like. For example, we cannot just intuit a part, with unchanged real content, as the whole, or the whole as a part. Since there are law-governed limits there must be ideal laws that determine what variations in any given categorial forms there can be in relation to the same definite but arbitrarily chosen matter. Indeed, Husserl says there must be pure laws of authentic thinking, that is, of categorial intuitions in virtue of their purely categorial forms.

There is, running parallel to all possible founding and founded intuitions, a system of founding and founded meanings that could possibly express these intuitions. The realm of meaning, however, is much wider than that of intuition, i.e., than the total realm of possible fulfillment. There are countless meanings that lack 'reality' or 'possibility'. No possible fulfillment could correspond to them. Thus, there cannot be a complete parallel between types of categorial intuition and types of meaning.

Husserl now distinguishes *authentic* from *inauthentic* thinking. Inauthentic acts of thinking would be the signitive intentions behind statements where these intentions are not fulfilled, while authentic acts of thinking would lie in the corresponding fulfillments. In inauthentic thinking we are free to form meanings in any way we like so long as these meanings are not nonsensically combined. This was already covered in Investigation IV where the pure logico-grammatical laws would distinguish sense from nonsense. If, however, we are concerned with the objective possibility of complex meanings in which we would have fulfillment, then the sphere in narrowed considerably. Intermediate cases of partial fulfillment would of course also need to be considered.

Husserl reminds us that we are to consider ideal (not real) laws here. None of this is merely empirical psychology. A course of experience of the world that violated the laws of logic, i.e., the laws of authentic thinking and the norms of inauthentic thinking, is a piece of nonsense. No metaphysical theory is needed to explain the agreement of the course of nature with the 'regularities' of the understanding. We only need a phenomenological clarification of meaning, thinking and knowing, and of the ideal laws that spring from these. The world is set up for us as a sensuous unity. Its meaning is to be the unity of actual and possible straightforward percepts. Its true being, however, precludes its being adequately given in any closed process of perception. It is always an inadequately meant unity for theoretical research, in part intended through straightforward and categorial intuition and in part through mere signification. The further our knowledge progresses, the better and more richly will the idea of the world be determined. The more too will inconsistencies be excluded from it. The true structure of the world could never conflict with the forms of thinking.

Toward the end of this Investigation Husserl sums up his efforts at clarifying the most important differences involved in the distinction between 'thinking' and 'intuiting' (§66). First, intuition as perception or imagination (whether categorial or sensory, adequate or inadequate) is opposed to mere thinking as merely signitive reference. Sensory intuition is to be distinguished from categorial intuition. The latter is a founded form of intuition. There is adequate and inadequate intuition.

There is also individual intuition (not necessarily confined to sensory intuition) and universal intuition.

Husserl suggests how Kant's epistemology could be criticized from this point of view, for Kant fails to draw these kinds of distinctions. Categorial (logical) functions play a great role in Kant's thought but he does not have a place for categorial intuition. He fails to appreciate the deep difference between intuition and signification, with their possible separation and their usual commixture. Thus, he does not complete his analysis of the difference between the inadequate and the adequate adaptation of meaning to intuition. He therefore also fails to distinguish between concepts as the meaning of words and concepts as species of authentic presentation. He does not recognize the possibility of concepts as correlates of intentional acts of particular types. Kant never made clear to himself the character of pure ideation, the survey of conceptual essences and the laws of universal validity that have their origins in those essences. He lacked the phenomenologically correct concept of the *a priori*. A scientific critique of reason should involve the investigation of the pure essential laws that govern acts as *intentional* experiences, in all their modes of sense-giving and in their fulfilling constitution of 'true being'. This was not carried out by Kant but it is just what is required for an account of the 'possibility of knowledge'.

One of the issues raised by the account of categorial intuition presented in this Investigation, and discussed to some extent in the secondary literature, is just how far the notion of intuition extends in mathematics and logic. What do we intuit in mathematics and logic? One account, according to which we do not intuit every object that we may possibly intend in mathematics, can be found in [Tieszen, 1989].

3 THE PHENOMENOLOGY OF THE CONSCIOUSNESS OF INTERNAL TIME (1893-1917)

These lectures and manuscripts are devoted to the phenomenological analysis of internal time but they contain a number of ideas that relevant to Husserl's views on logic and mathematics. Internal time is the form of all of our experiences. Internal time is to be distinguished from 'external' or 'objective' time. Internal time is the form of the stream of consciousness itself. Anything that can be an object for us is constituted in the stream of consciousness, including all of those objects that are 'external' to us, like the objects of the physical world that are given to us as spatial and temporal. Since all of our experiences are constituted in the flow of internal time this must also be true of our experiences of ideal objects. In several sections of these lectures (see especially §45 and Appendix XIII) Husserl therefore considers the constitution of non-temporal, transcendent objects, like the objects of mathematics and logic. If we judge that $2 \times 2 = 4$ then what is *meant* as such in this judgment is a non-temporal idea. Furthermore, the numbers here are not things with temporal extension nor are $\times$ and $=$. On the other hand, there are the various *acts* of judgment. There is that which is judged and then there is also the judging. The same thing, $2 \times 2 = 4$, can be meant in countless acts of judgment. The judgings are processes that take place in time and can then be stored in (secondary) memory. They have a beginning in time, take place as a process, and immediately sink back into the past. Judging that $2 \times 2 = 4$ begins with forming the subject thought '2×2' and on this basis 'is equal to 4' is formed. Earlier phases of the judging must be retained in an appropriately modified manner during later phases of the judging or else the judging would not be possible at all. The underlying structure of retention/primal impression/protention will be involved in judging, as it is in all types of consciousness. Similarly, secondary memory and acts of anticipation will be involved in the judging, and these are also conditions for the possibility of awareness of any objects. The meaning itself or the objects posited by the judgment do not have temporal extension. They are ideal objects that are non-temporal invariants through the flow of subjective time.

The flow of consciousness in internal time is continuous. Husserl's reflections on the continuity involved here are quite interesting. The phases of consciousness, the 'parts' of the stream of consciousness, are evidently non-independent. Furthermore, in the language of Investigation III of *LI*, they interpenetrate one another. One cannot find or exhibit a fixed, durationless time-point in the stream of consciousness. To speak of such a point is already an abstraction or idealization. If we take internal time to be the most fundamental continuum, as Hermann Weyl has suggested ([1994, Chp. 2, §6]), then it will not do to think of the continuum as composed of a set of durationless points. To resolve the continuum into such a set is precisely to break up the flow. It gives us an atomistic, static view of the continuum. Could there possibly be a mathematics or logic of the continuum of internal time? Weyl turned to Brouwer's mathematics of the intuitive continuum on this issue, with its idea that a 'point' is something that is itself becoming as it is generated through time. Such 'points' are given by free choice sequences (see [Tieszen, 2000]; [van Atten *et al.*, 2002]). Some principles of classical logic (e.g., excluded middle), with their built-in assumptions about the exactness or discreteness of objects, would need to be abandoned if we take such an intuitive continuum seriously.

Might all of this lead us to recognize mathematical or logical objects that are themselves dynamic, in the sense that they have a beginning in time and a duration (though, like choice sequences, not necessarily an end in time)? There are no indications in his work that Husserl was prepared to recognize such objects as part of mathematics but one could certainly investigate the matter in more detail (see [van Atten, 1999]).

One additional point that is worth mentioning is that in the lectures on internal time Husserl describes ideal objects as non-temporal or atemporal (unzeitlich). Around 1917, however, he began to speak of ideal objects as 'omnitemporal'. It is not that they are somehow outside of time altogether but, rather, that they have being at all times. Husserl discusses this explicitly in *Experience and Judgment* and I will return to the matter below in Section 8.

4 "PHILOSOPHY AS RIGOROUS SCIENCE" (1911)

In "Philosophy as Rigorous Science" Husserl argues that "the highest interests of human culture demand the development of a rigorously scientific philosophy". The impulse to develop philosophy as rigorous science is not at all foreign to the present age. It is fully alive in the naturalism that dominates our age. Husserl sees in naturalism, however, a "growing danger to our culture" and in this essay he attacks naturalism, historicism, and the *Weltanschaung* philosophy of Dilthey. Many of the themes are familiar from his earlier criticisms of empiricist skepticism, relativism, psychologism, etc. He deplores the "prejudices" that attend naturalism and the naturalizing of consciousness. The critique is not aimed at natural science itself but at the effort to naturalize philosophy. His idea is to set out an alternative vision of philosophy as rigorous science. Natural science is itself 'naive'. There are a host of questions that natural science is not itself in a position to answer. These are questions, for example, about the *meanings* of expressions like 'nature', 'theory', 'method', 'evidence', 'validity', 'objectivity', and so on. They are the most general kinds of questions about the relationship of subjects to objects, about how objects of different types are constituted in consciousness, and so on. Natural science at any given stage of its development makes a host of presuppositions about these things. The experimental method itself presupposes what no experiment can accomplish: the analysis of various aspects of the meanings found in consciousness and through which we think of things. Certain riddles are inherent in natural science but the solution of these riddles on the basis of rational inquiry in principle transcends natural science. To expect from natural science itself the solution to these problems is to be involved in a vicious circle.

Husserl says that "the spell of the naturalistic point of view... has blocked the road to a great science unparalleled in its fecundity...". This spell makes it difficult for us to see essences or ideas or, rather, since we in fact do constantly see them, it makes it difficult for us to let them them have the peculiar value which is theirs instead of absurdly naturalizing them. Husserl says that intuiting essences conceals no more difficulties or mystical secrets than does perception.

Philosophy must not give up its will be to rigorous science. One must in no instance abandon one's radical lack of prejudice, prematurely identifying ideals or essences given in immediate intuition with empirical facts. To do this is to stand like a blind man before ideas that are given in intuition.

5 IDEAS I (1913) AND IDEAS III

In *Ideas I* Husserl introduces many of the ideas that characterize his later view of phenomenology as a transcendental, eidetic discipline. The phenomenological and eidetic 'reductions' are introduced, phenomenology is portrayed as a kind of transcendental 'idealism', the concept of the noema is introduced, and the theory of intentionality is now developed in terms of noetic-noematic correlation. After *LI*

Husserl rejected his account of meaning as a species whose instantiations are the particular acts intending that meaning. The idea of the 'meant object precisely as it is meant' is now incorporated into his noematic concept of meaning (see below). Since phenomenology is now portrayed as a science of essence, the concept of essence also comes to the fore in this work.

From this point on Husserl says that phenomenology is a transcendental idealism. Transcendental phenomenological idealism is the view that it is only in our own experience that things are 'there' for us, given as what they are, with the whole content and mode of being that experience attributes to these things. In short, "nothing exists for me otherwise than by virtue of the actual and potential performance of my own consciousness" (*FTL* §94). Whatever I encounter as an existing object is something that has received its whole sense of being from my intentionality. Illusion also receives its sense from me. Experience teaches me that the 'object' could be an illusion. Objects can be thought of as intentional 'poles of identity' through the manifold activities of consciousness. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent object that had any other sense than that of an intentional unity making its appearance in the subjectivity of consciousness. Thus, if what is experienced has the sense of 'transcendent being' then it is experience itself that constitutes this sense. If an experience is 'imperfect' in the sense that an object is given only partially, then it is only experience that tells me this. And so on. What this means is that the sense of objects as ideal (and hence, non-mental), transcendent, partially given, acausal, etc., is constituted in the experience of the human subject. The characteristics just mentioned, however, are just those that have traditionally been emphasized by realists or platonists, except that now they are said to be constituted in the subjectivity of consciousness. So is Husserl a realist or an idealist about logic and mathematics? This is has been a constant source of debate in the secondary literature on Husserl. In the opinion of the present author, Husserl was trying to show how one could accommodate both a kind of idealism and a kind of realism. In somewhat the same way that Kant claimed to be both a transcendental idealist and a realist about sensory objects, Husserl claimed to be both in the case of ideal objects.

5.1 Ideas I (1913)

In *Ideas I* Husserl says that phenomenology is not a science of matters of fact but is a science of essences. Here he speaks of the 'eidetic reduction' as the procedure that leads from psychological phenomena to the pure essence of consciousness and experience. It is not concerned with the 'real' but rather with the 'irreal' or 'ideal'. He wishes to make it perfectly clear, in response to readers of the first edition of the *LI*, that phenomenology does not consist of descriptions of mental processes of the sort one would find in (introspective) empirical psychology. Chapter 1 of Part I of the *Ideas* consists of 'logical considerations' in terms of which the book is to be read. Part I is entitled "Essence and Eidetic Cognition". Its first chapter is on "Matter of Fact and Essence" and its second chapter is on "Naturalistic Misinterpretations" of the phenomenological view of essence.

In Chapter 1, empirical sciences are said to be sciences of matters of fact. They posit real individuals as existing in space-time. Individual existence of this sort is contingent. This contingency, however, is correlative to a necessity, an eidetic universality. Husserl says that any matter of fact, by its very essence, could be otherwise. Thus, it belongs to the sense of anything contingent to have an essence and, furthermore, that this essence can be grasped. Each individual object has its own specific character, its own stock of essential predicables that must belong to it if other, secondary, relative determinations can belong to it. Any material thing has its own essential species. The highest eidetic universalities concerning material things delimit what Husserl calls 'regions' or 'categories' of individuals. The essence of an individual is the 'what' of the individual. Husserl argues that the intuition of something individual can be transformed into eidetic intuition. Here he also calls eidetic intuition 'ideation'. In this case, it is the pure essence that is seen, whether it be the highest category or a particularization thereof, down to full concretion. The intuition of an essence (Wesenschau) can be adequate or inadequate. If it is inadequate it is an imperfect 'seeing', and not only with respect to greater or lesser clarity and distinctness. It is characteristic of essences that they can be given only 'onesidely'. One can intuit many 'sides' of an essence in a sequence but one can never intuit it completely.

The essence, or *Eidos*, is a new object. Just as the datum of individual sensory intuition is an individual object, so the datum of eidetic intuition is a pure essence. Husserl points to various analogies between the intuition of sensory individuals and the intuition of essences. Essences, for example, can be intuited vaguely or distinctly, they can be made the subject of true or false predications, just like any other object in the broadened sense proper to formal logic.

Husserl says that there is a relationship between the two types of intuition of the following sort: thre can be no intuition of essence without the possibility of turning one's regard to a corresponding individual and forming a consciousness of an example, but no intuition of an individual is possible without the possibility of bringing about an ideation in which one directs one's regard to the essence exemplified in the intuition of the individual. Husserl thinks that since we can grasp with evidence the distinctions between individual existence and essence, or *matter of fact* and *Eidos*, we can set aside various mystical views that have been associated with essences.

Essences can be exemplified in different ways: in perception, in memory, and in phantasy (i.e., imagination). Ideation can operate in the sphere of imagination just as well as in the sphere of actual perception. Grasping or positing essences, therefore, does not imply positing of any individual factual existence. Pure eidetic truths contain not the slightest assertion about matters of fact. No truth in the sphere of matters of fact can be deduced from pure eidetic truths alone.

Eidetic cognition does not, however, always have essences as its objects (§5). There are other forms of consciousness that involve essences while excluding every positing of factual existence. In geometry, for example, we do not as a rule make judgments about essences, e.g., the essence "straight line" or the essence "triangle". Rather, the judgments are about *any* straight line or *any* triangle. These kinds of judgments have the characteristic of eidetic universality. Consider as an example the judgment "Any color whatever is distinct from any sound whatever". This judgment has eidetic universality but it does not make essences its object. Husserl argues that we can always shift from such a judgment to the corresponding 'objectivating attitude', in which the judgment becomes "The essence (the genus) color is other than the essence (the genus) sound". The former judgment posits no factual existents any more than the latter does. The former is about any individuals that might be subsumed under essences, and imagined individuals would do just as well as perceived individuals. Husserl argues that, conversely, any judgment about essences can be converted into an equivalent universal judgment about individuals subsumed under essences. The judgments have a different logical form but nonetheless belong together as purely eidetic judgments. What they have in common is that they posit no individual existence even when they judge about something individual. Eidetic judgment about an individual can be combined with positing the factual existence of something individual. In this case, eidetic universality is transferred to the factual individual or to an indeterminately universal sphere of individuals. This is what happens, for example, in the case of applications of geometric truths to nature. The universality of natural laws is, of course, not eidetic universality. In the former kind of universality there is a different meaning, a positing of factual existence, even if there is no positing of some definite physical situation.

There are pure eidetic sciences and sciences of matters of fact. Among the former Husserl counts pure logic, pure mathematics, and what he calls "pure theories of time, space, motion, and so forth". In these sciences sensory experience plays no role in grounding judgments. The use of figures drawn on a board by a geometer does not ground her seeing of essences or her eidetic thinking. The figures might just as well be imagined. They could even be hallucinated. The geometer explores ideal possibilities, not actualities. The geometer's work is grounded in the seeing of essences. In purely eidetic science one proceeds with eidetic intuition and eidetically valid inferences. Husserl says that the *sense* of eidetic science necessarily precludes the incorporation of results yielded by the empirical sciences. Nothing ever follows from matters of fact but matters of fact. Eidetic science is necessarily independent of sciences of matters of fact, but there could be no sciences of fact that were independent of formal or material eidetic sciences. The empirical sciences depend on formal logic. Formal logic is concerned with the essence of 'anything objective whatever'. It concerns the broadest concept of object. Empirical science indeed depends on other disciplines that make up *mathesis universalis* (e.g., arithmetic, pure analysis, pure theory of manifolds). Furthermore, any matter of fact includes a *material* essential composition. Given factual particulars will be bound by 'material' essences.

Any concrete empirical objectivity has a place within a highest material genus, a 'region', of empirical objects. To the pure region there corresponds a regional eidetic science (also called a 'regional ontology'). Any science of matters of fact has essential theoretical foundations in eidetic ontologies. Husserl says we might fashion the idea of a perfectly rationalized experiential science of nature. This would be a science so far advanced in theorization that every particular in it has been traced back to that particular's most universal and essential grounds. The realization of such an idea obviously requires the elaboration of the corresponding eidetic sciences, i.e., formal *mathesis* and the disciplines of material ontology that explicate the essence 'nature'.

Corresponding to regional ontologies are the various material essences. These material essences, Husserl says, are in a sense the 'essences proper'. There are also the mere 'essence-forms'. These, Husserl says, are completely empty forms. All material universalities fall under such formal essences. Formal essences have a purely formal universality. There is therefore a purely formal domain that is not just another region but is the empty form of any region whatever. All the material regions fall under it, though only formally. Formal ontology contains the forms of all possible ontologies. Formal ontology (as pure logic in its full extent as *mathesis universalis*) is the eidetic science of any object whatever. Husserl says that *mathesis universalis* will have many disciplines but that they all lead back to fundamental truths that function as the axioms of the disciplines of pure logic. The fundamental concepts of pure logic Husserl calls 'categories' (§10). Logical form-essences, in other words, are called 'categories'. These categories make up the 'analytic' (as opposed to 'synthetic') region (compare with Investigation III of *LI*). Husserl gives the following as examples of 'logical' categories in this sense: property, relative determination, predicatively formed affair-complex, relationship, identity, equality, aggregate (collection), cardinal number, whole and part, genus and species. Here Husserl also speaks of the 'signification categories' concerning judgments. These involve the notion of "any signification whatever", as distinct from "any object whatever". These are the fundamental concepts belonging to the essence of the proposition (apophansis): concepts of different kinds of propositions, proposition members, and proposition forms. As we will see, Husserl distinguishes the stratum of "apophantic logic" from other strata of logic and, in particular, from the level concerning the *objects* that propositions signify. One can distinguish categorial concepts (as significations) from categorial essences (as the signified objectivities).

Husserl says (§11) that any object that can be logically determined (i.e., explicated, related to other objects) takes on various 'syntactical' forms. Husserl is using the term 'syntactical' here for the kind of structuring due to the understanding. As correlates of a determining thinking, objectivities of a 'higher level' are thereby constituted. Among such objectivities Husserl lists conditions, qualities, relationships between objects, pluralities of units, members of ordered sets, objects as bearers of ordinal numerical determinations, etc. Where our thinking is 'predicative' (i.e., involves making judgments) there accrue expressions and relevant apophantic signification- formations that mirror the 'syntactical objectivities'.

Husserl calls the categories corresponding to these syntactical objectivities 'syntactical categories'. He also calls them 'categorial objectivities', recalling terminology from the *LI.* These categorial objectivities can then function as substrates for further categorial formations, which can then do the same, and so on. Every such formation, however, refers back to ultimate substrates, objects of the first or lowest level that are no longer syntactical-categorial formations, that no longer contain any of the ontological forms that are mere correlates of the thinking functions. The formal region is accordingly divided into ultimate substrates and syntactical objectivities. The ultimate substrates would be syntactically formless.

Husserl says that each essence, whether material or formal, has its place in a hierarchy of essences, a hierarchy of *generality* and *specificity.* Descending through the hierarchy we arrive at the *infimae species*, which Husserl also calls 'eidetic singularities'. Ascending through specific and generic essences, we arrive at the highest genus. The highest genus involving signification in pure formal logic, for example, is "any signification whatever". Each determinate proposition form and each determinate form of a proposition-member is an eidetic singularity. Similarly, "any cardinal number whatever" is a highest genus, while two, three, etc., are its *infimae species.* Husserl also gives examples involving material essences. The genus/species relationships are not relationships among sets. Rather, he says, in the particular essence the more universal essence is "immediately or mediately *contained*" (§12). The relationship of eidetic genus or species to its eidetic particularization is included among the relationships of *part* to *whole.* Whole and part then express the broadest concept of "that which contains and that which is contained". The eidetically singular essence thus implies collectively the universals lying above it which, for their part, lie one inside another, the higher always lying inside the lower.

Generalization is to be distinguished from *formalization.* The relation of generalization to specialization is different from the universalization of something materially filled to the formal (in the sense of formal logic) or the materialization of something logically formal. In the latter case, one 'fills out' an empty logico-mathematical form or a formal truth. Formalization of course plays a large role in mathematics. Thus, subordinating an essence to the formal universality of a *pure logical* essence must not be mistaken for the subordinating of an essence to its higher essential *genera.* A determinate inference in physics, for example, is a singularization of a determinate purely logical form of inference. The pure form is not a genus relative to the materially filled inference but is an infima species of the purely logical genus "inference". *Mathesis universalis* includes nothing but purely empty forms. Filling out an empty logical form is an operation entirely different from genuine specialization down to the infimae species. The transition from 'space' to "Euclidean manifold", for example, is not a generalization but a formal universalization. Logical form-essences are not inherent in materially filled singularizations in the same manner in which the universal "red" is inherent in the different nuances of red. It is not a part-relationship that would justify speaking of containment.

Any essence that is not an infima species has an *eidetic extension* made up of specificities and always ultimately of eidetic singularities. Any formal essence, on the other hand, has a formal or *mathematical extension*. One can also speak of an *empirical extension*, which involves restriction to a sphere of factual being by virtue of a positing of factual being that annuls the pure universality.

There is a distinction between (§14) materially filled substrates and empty formal substrates. The latter class consists of the totality of predicatively-formed affair-complexes belonging to the realm of pure logic as *mathesis universalis*, with all of the categorial objectivities out of which they are constructed. Every predicatively formed affair-complex expressed by some logical or arithmetical axiom or theorem, every form of inference, every number, every numerical formation, every function in pure analysis, and every Euclidean or non-Euclidean manifold belongs in this class. If we focus on the class of materially filled objectivities, we arrive at ultimate materially filled substrates as the cores of all syntactical formations. At this lowest level we find the pure, syntactically formless, individual, singular particular, the 'this here'.

As in the *LI*, Husserl says we need to distinguish *selfsufficient* (independent) from *non-selfsufficient* (non-independent) objects. A categorial form, for example, is non-selfsufficient insofar as it refers back to a substrate whose form it is. Substrate and form are referred to one another and are unthinkable without one another. A purely logical form, therefore, is non-selfsufficient. A non-selfsufficient essence is called an *abstractum*, while an absolutely selfsufficient essence is a *concretum*. A 'this-here', the material essence of which is a concretum, is called an *individuum*. The individuum is the primal object required by pure logic. A concretum is an eidetic singularity because species and genera (excluding infima species) are non-selfsufficient in principle. On the basis of these distinctions eidetic singularities can be either abstract or concrete.

A 'region' may now be defined as the total highest generic unity belonging to a concretum. It is, Husserl says, the unitary nexus of the summa genera pertaining to the infimae species within the concretum. The 'eidetic extension' of the region "comprises the ideal totality of concretely unified complexes of infimae species belonging to these genera". The 'individual extension' comprises the ideal totality of possible individua having such concrete essences. Each regional essence determines "synthetic" eidetic truths (compare with Investigation III of *LI*). These are truths grounded in the regional essence that are not mere particularizations of truths included in formal ontology. Neither the regional concept nor any of its specifications is freely variable in these synthetic truths. The substitution of indeterminate terms for the related determinate terms does not yield a law of formal ontology as it does in the case of any "analytic" necessity. The set of synthetic truths grounded in the regional essence makes up the content of the regional ontology. The total set of fundamental truths among them — the regional axioms — defines the set of regional categories. One can make certain comparisons here with Kant's views: one would have to understand by synthetic cognitions *a priori* the regional axioms and we would have as many irreducible classes of such cognitions as we have regions. We would have as many different groups of categories (in a Kantian sense) as there are regions to differentiate. Formal ontology takes its place outside the regional ontologies. Its concept of "object" determines the formal axioms and the set of formal ("analytical") categories.

Husserl says that these logical considerations set a task for us: to determine, within the circle of our intuitions of individuals, the summa genera of concretions and in this manner to effect a distribution of all intuited individual being according to regions of being, each of which marks off an eidetic and an empirical science that is necessarily distinct from other sciences, even if some interweaving and partial overlapping may occur. This will require both material and formal ontology.

Since empiricism denies the existence of 'ideas' or essences and denies the cognition or intuition of essences, Husserl launches (in Chp. 2) an attack once again on empiricistic naturalism. Empiricism identifies or confuses a demand for a return to the "things themselves" with the demand for basing all cognition on sense experience. It requires that judgments be legitimated by sense experience. This seems to Husserl to be a prejudice, given that one has not first made a study of judgments to determine whether there are different kinds of judgments. It amounts to a kind of speculative construction *a priori*. Genuine science requires freedom from prejudice.

Husserl, however, also wants to avoid the obscurities and errors that have been made in defending essences and the intuition of essences. The anti-empiricists have not recognized that there is a kind of intuition of essences that parallels sensory intuition in crucial respects, especially in providing evidence. Instead, they have depended on appeals to mere feelings of evidence in a way that makes their view difficult to distinguish from mysticism. Husserl portrays his notion of intuition of essence, on the contrary, as the very stuff of reason, science, and evidence. Thus, in so many of his writings he tries to show how the notions of science, intuition and evidence must be broadened if we are to do justice to the things themselves and to avoid prejudice. Husserl also responds here to some objections to his alleged platonism. He says that if "object" and "something real", or "actuality" and "real actuality", have the same sense then he would indeed be guilty of a perverse "platonic hypostatization". As in the *LI*, however, he distinguishes the two and argues that the notion of 'object' needs to be used in a very wide sense, e.g., as anything that can be the subject of a true statement. It is this universal concept of object that is required by pure logic and that determines a universal scientific language.

At a later point in the book Husserl discusses some issues about clarifying our grasp of essences (§§66-70). He says the intuition of essences has degrees of clarity. The two extremes are complete clarity and complete unclarity. In the later case, consciousness is blind. It is no longer in the least intuitive. Nothing at all has been given. Yet another way of saying this, in Husserl's terminology, is that we do not have a 'presentive' consciousness in this case. There are degrees of clarity, intuitiveness, and giveness. In the normal situation, objects are intuited with respect to certain sides or moments and are intended only emptily with respect to other sides or moments. Husserl describes the "method of perfectly clear seizing upon essences" at this point. This is the method of free variation in imagination that is described in many of Husserl's writings (see, e.g., Husserl [1977; 1973; 1969]). I discuss it below in Section 6.2. Husserl says that what is given at any particular time is usually surrounded by a 'halo of undetermined determinability' which may be separated in explication into a number of intendings. Moreover, it is not the case that all seizing upon essences requires a complete clarity of the underlying singular particulars in their concreteness. Husserl says here that phantasy (imagination) has a primacy over sensory perception in the method of clarification of essences. A geometer, for example, operates on a figure or model far more in imagination than in sensory perception. Sensory perception is far more restricted. In phantasy, however, one can run through many possible shapings, and so on. A kind of freedom thereby opens up for the geometer that is not possible in perception. Phantasy, however, can be fertilized by the study of history, art, poetry, literature, and so on. Husserl says that, in this sense, "feigning is the source from which the cognition of 'eternal truths' is fed" (§70).

Husserl contrasts the eidetics of mental processes called for by phenomenology with the eidetics of mathematics and logic (§72). First, there is the central division of essences and eidetic sciences into the material and the formal. Phenomenology, unlike mathematics, is a material eidetic science. It is descriptive. It is concerned with the essences of mental processes, i.e., with concreta, not abstracta. In geometry, for example, one does not seize upon the lowest eidetic species (the countless spatial shapes that can be drawn in space) in intuitions of singular particulars. Rather, one fixes some fundamental structures in axioms and then deduces consequences of the axioms. The multiplicity comprising all of the spatial formations thereby determined can be called a 'manifold'. More specifically, it will be a 'definite manifold'. Husserl tells us more about definite manifolds here than he did in the *LI*. A "definite manifold" is characterized by the fact that a finite number of concepts and propositions derivable from the essence of the province in question completely and unambiguously determines the totality of all possible formations belonging to the province in the manner of purely analytic necessity so that nothing in the province is left open. A definite manifold can be exhaustively defined. The 'definition' consists of the axiomatic system of concepts and axioms. 'Mathematically exhaustible' means that the defining assertions are such that nothing remains undetermined. Husserl says that this is equivalent to the following: any proposition that can be constructed out of the axioms is either a pure formal-logical consequence of the axioms or a pure formal-logical anti-consequence (i.e., a proposition that formally contradicts the axioms, so that its contradictory opposite would be a consequence of the axioms). He says that in the case of a mathematically definite manifold the concepts "true" and "formal-logical consequence of the axioms" are equivalent, as are the concepts "false" and "formal-logical anti-consequence of the axioms". He says that a system of axioms that exhaustively defines a manifold in this manner is a "definite set of axioms". A deductive discipline based on such a set of axioms is a 'definite discipline'. In a footnote in this section Husserl says that the close relationship of his concept of definiteness to Hilbert's "axiom of completeness" will be obvious to every mathematician.

Husserl's notion of a definite formal axiom system may seem fairly clear, especially in light of his references to Hilbert. Although the sharp distinction between syntax and semantics that one finds in modern logic is not so clear in Husserl, a definite formal system is evidently a syntactically consistent and complete axiom system. One can raise additional questions about his notion of a definite manifold. What exactly is a manifold? In some ways it seems to be the forerunner of the modern conception of a model, except that the concept of set may not always play a central role in manifold theory (e.g., in the ontological correlate of a theory of parts and wholes). There has been some discussion of what manifolds are in the secondary literature (see, e.g., D. Smith [2002]; [Hill, 2000]) but the answer is not so easy as it may seem.

Husserl says that the concept of a mathematical manifold presupposes that the essences in the domain under consideration are *exact*. The concepts of descriptive natural science, by contrast, are inexact. A natural scientist might use *morphological* concepts of vague configurational types, like "notches", "scalloped", "umbelliform", and so on. The concepts used by a geometer, on the other hand would not be of this type. Geometrical concepts are 'ideal' concepts. They express something that cannot be seen in sensory perception. Their origin and their content is different from *descriptive* concepts. The latter do not express ideals, but they express essences drawn immediately from sensory intuition. Exact concepts, however, have as their correlates essences that are 'ideas' in the Kantian sense. Husserl calls them 'ideal essences'. The correlates of descriptive concepts are called morphological essences. The ideation that yields ideal essences, as ideal limits, is different from the simple abstraction in which a salient moment of a perceptual object is raised into the region of essences as something essentially vague or 'typical'. It is impossible to find ideal limits in sensory intuition. Morphological essences may only approach such ideal limits more or less closely without ever reaching them. Exact sciences and purely descriptive sciences can be combined but one cannot take the place of the other.

Husserl had asked in §73 whether the stream of consciousness could itself be a genuine mathematical manifold. It is interesting to ask what the implications of such a view would be. Since it is now claimed that exact sciences cannot replace descriptive sciences it appears that the phenomenology of the stream of consciousness must be something different from the mathematical manifold of the stream of consciousness (if there were such a thing). Husserl leaves open the question whether there could be a *mathesis* of mental processes that could be developed alongside (but not as a replacement for) descriptive phenomenology (§75).

Husserl says that the theorizing logician isolates the formal theory of signification by virtue of his one-sided direction of interest and this then becomes something dealt with by itself. The formal logician does not heed or comprehend the noematic and noetic context in which formal logic is phenomenologically interwoven (§134). The phenomenologist should, however, take into account its full interrelationship. She should trace out the phenomenological complex of essences from all sides. Here one thinks of the roles that forms and essences play in our experience. Our experience is to be investigated in terms of the intentionality of consciousness. To say that consciousness exhibits intentionality is to say that it is always directed toward objects or states-of-affairs. As in *LI*, Husserl holds that it is referred to or directed toward objects by way of meanings or *Sinne*. In particular, Husserl singled out the 'matter' of an act for this function in the *LI*. In *Ideas I*, however, he introduces his concept of the noema and the 'noematic nucleus'. There is a substantial literature on the nature and structure of noemata. A number of philosophers, but especially Dagfinn Føllesdal, have drawn attention to the similarities between Frege's meaning/reference distinction and Husserl's related distinction in his theory of the intentionality of consciousness (see, e.g., Føllesdal [1969; 1990]; Mohanty [1982; 1977a]; [Bell, 1994]; [Rosado-Haddock, 2000]; B. Smith [1994]; and [Smith and McIntyre, 1982]). Just as expressions can have a meaning but lack a reference, for example, so acts of consciousness can be directed by noemata but lack an object. Just as expressions with different senses can refer to the same object, so can acts with different noemata refer to the same object. Noemata are expressed in language, they are ideal objects, and so on. There are of course various differences between the views of Frege and Husserl. To mention only one difference, the referents of propositions for Frege are truth values while for Husserl they are states of affairs. For Husserl a proposition is made true by its relation to a state of affairs. On the basis of Husserl's views one could say that there is a 'logic' of consciousness and it would certainly be an intensional logic. Husserl's ideas on noemata, intensions, etc., are related to many interesting issues in philosophical logic.

There has also been some investigation of possible world semantics in connection with Husserl's views on intentionality (see, e.g., [Hintikka, 1975]; [Mohanty, 1981]; [Harvey, 1986]; [Harvey and Hintikka, 1991]; [Hutcheson, 1987]; [Smith and McIntyre, 1982]).

Finally, let us mention that Husserl's hylomorphism is clearly apparent in *Ideas I*, as it is in his other works. Here it takes the form of holding that the 'hyletic data (sensory components) of experience are shaped or informed by noemata. There is a noetic-noematic-hyletic structure in our experience.

5.2 *Ideas III.*

Ideas III was not composed as a book. It is comprised of manuscripts that are now included in *Husserliana* V. In *Ideas III* Husserl returns to many of the themes just discussed. One thing that is included in this text is his elaboration on the relationship of noema to essence (§16). The noema, as the meaning by virtue of which acts are directed toward objects, must not be confused with essence (see also, e.g., *LI*, Investigation I, §33; *EJ*, §64d). The intuitive grasping of essences is different from the intuitive grasping of noemata. To say that a noema exists is not to say that the object corresponding to the noema exists. What is thought by way of a noema can be 'countersensical'. One might, for example, think of a 'round rectangle'. This is a meaningful expression. There is, however, no essence 'round rectangle'. In order to judge that there is no round rectangle it is presupposed that the expression 'round rectangle' has a meaning. Husserl holds that there is 'dissonant unity' of the intended essences here. The intended essences are incompatible. Given the incompatibility, we do not grasp an essence. One might also put this by saying that contradictory concepts are essenceless. To intuit a noema is to simply grasp a meaning, more or less clearly. In this case one has evidence for the existence of the noema. This is different from the intuition that the corresponding essence exists, or the intuition of the incompatibility that rejects the existence of an essence. This is just a special case of the general distinction according to which positing significations and positing objects are two different things.

6 FORMAL AND TRANSCENDENTAL LOGIC (1929)

In *FTL* Husserl returns to his investigation of logic in the broad sense of theory of science. He notes how logic itself has become a special, technical science and says that he wants to keep alive the broader sense of logic as it is related to a pure and universal theory of reason. He wishes to undertake an intentional explication of the proper sense of formal logic. This will start with the traditional objective contents of formal logic as they are given to us but will put these theoretical formations back into the living intentions of logicians from which they originated as sense-formations (meaning-formations). We will need to turn back to the intentionality of the scientists. *FTL* is composed of two parts. Part I of *FTL* is entitled "The Structures and the Sphere of Objective Formal Logic" while Part II is entitled "From Formal to Transcendental Logic". In Part I Husserl arrives at the view that there is a three-fold stratification of formal logic that he had not yet completely recognized in the *Logical Investigations*. In connection with this three-fold stratification he attempts to clarify the relationship between formal logic and formal mathematics. Here he returns again to the Leibnizian idea of a formal *mathesis universalis*. He arrives at the view that the sense of pure formal mathematics is to be a 'pure analytics of non-contradiction' (or a 'logic of consistency') that can be considered independently of the concept of truth. Formal ontology and the theory of manifolds also comes in for further discussion and development.

In Part II Husserl says that the 'subjective-logical' is the chief theme, and that here he marks out the path from formal to transcendental logic. Of course Kant already introduced the idea of transcendental logic as a logic of possible experience or, as Kant puts it, a logic that (unlike formal logic) does not abstract from our knowledge or experience. Husserl is clearly working in this tradition but his own vision of transcendental logic will be much more detailed and will involve many concepts (e.g., that of intentionality) that were not part of Kant's conception. First, Husserl returns to the theme of psychologism to set aside psychologistic prejudices about logic. He then begins to uncover a series of presuppositions involved in logical cognition. None of these problems of sense (Sinn) relating to the subjective are problems of natural human subjectivity (i.e., of empirical psychology). Already in the Introduction to *FTL* he says they are all problems of transcendental subjectivity and hence of transcendental phenomenology. A philosophical logic, he says, can be developed only in the nexus of a transcendental phenomenology. A science of logic developed independently of transcendental logic is just as unphilosophical as any other science. Such a science does not understand its own productions as those of a productive intentionality, and is unable to clarify either the concepts that underlie its province or the genuine meaning of being of its province. It is unable to say what sense belongs to the existents of which it speaks or what sense-horizons those existents presuppose. It will be as dogmatic and naive as any other positive science. The existing positive sciences, however, are clues to guide transcendental research and it is only the later kind of research that will allow us to create genuine science for the first time. We must rise above the forgetfulness of the theoreticians in the objective, positive science of logic and consider the cognitive production of the theories and methods of logic. The logician or mathematician lives in this producing but does not have this productive living as itself a theme within her view. Husserl therefore says that only a science clarified and justified in terms of transcendental phenomenology can be an ultimate science. Only a transcendental logic can be an ultimate, deepest and most universal theory of science. What the modern sciences lack is a true logic, i.e., a transcendental logic, that investigates the cognition behind science and thereby makes science understandable in all its activities. This logic does not intend to be a mere pure and formal logic, a *mathesis universalis*, for while *mathesis* may be a science of logical idealities it is still only a 'positive' science. A transcendental logic should bring to light the system of transcendental principles that gives to the sciences the possible sense of genuine science. The positive sciences are completely in the dark about the true sense of their fundamental principles. Transcendental logic makes it understandable how the positive sciences can bring about only a relative, one-sided rationality (see also [Husserl, 1966] for more on transcendental logic).

The "Preparatory Considerations" in *FTL* open with some comments on language, thinking, and the fact that thinking is directed toward objects. As in the earlier works, verbal and written expressions are said to be expressions of our thinking and our thinking is directed toward objects of different types. Our words carry signitive intentions, i.e., meanings, and a sharp distinction must be made between the mental process of meaning and the meaning intended. There are meaning-bestowing acts and there is the meaning itself. Scientific thinking in particular consists of judicative thinking and so we will be focusing on the role of judgments in logic. Logic, Husserl tells us, is 'two-sided'. It has a 'subjective' side and an 'objective' side. On the one hand there are the *productive activities and habitualities* and, on the other hand, the persisting *results* produced by these activities and habitualities (§8). On the side of the results we have the various forms of judgment and various theoretical objects and truths. Here we have objective validity. We have identities of which we can be aware over and over again, that are available intersubjectively, that are understood in the same sense by everyone, and that are existent even when no one is thinking of them. In the opposite direction we have the subjective side of logic. It concerns the subjective structures and processes in which theoretical reason brings about its productions. In particular, it concerns the ongoing intentional cognition in which the objective formations have their origin. This may at first remain hidden but it can be uncovered in reflection and made the theme of research directed to the subjective side of logic. It is through these subjective processes and structures that ideal objectivities are constituted as existing 'in themselves'. In all objective or positive sciences we consider the objects that the science is about but we do not typically consider the thinking itself that is involved in the science. The thinking requires uncovering and this would take place in a new thinking. The thematic field of the science is overstepped by reflections turned toward the subjective. The geometer, for example, will not think of exploring, besides geometrical shapes, geometrical thinking. The objects of logic and mathematics are given, in particular, in categorial experience. In logic and mathematics, as in other sciences, we have data of experience. We have identifiable and intuitable objects, e.g., judgments, proofs, and so on. It's just that here the objects and truths are 'ideal', not 'real'.

6.1 Part I: The Structures and the Sphere of Objective Formal Logic

In Part I of *FTL* Husserl begins, as we said, with objective formal logic. In particular, he explores the structures and the sphere of formal logic as 'apophantic analytics'. ('Apophantic' is the Greek term for judgment or proposition. Husserl is not using the term only for what we now think of as propositional logic. Elementary predication is basic but apophantic logic is generally concerned with any information that can in a broad sense be expressed in judgments or propositions.) It was Aristotle who first brought out the sense of the 'form' of propositions that would determine what we now think of as formal logic. Leibniz extended this with his idea of *mathesis universalis*, and Vieta's work on formalizing algebra helped in distinguishing formal mathematics from 'material' mathematical disciplines like traditional Euclidean geometry. These developments, and others, made possible the idea of a theory of forms. The theory of the 'pure forms of significations' is in fact just what Husserl has already characterized in the *LI* as the grammar of pure logic. It concerns the mere possibility of judgments as judgments, without inquiring into whether they are true of false or even whether they are merely consistent or contradictory. This is now described as the first level of formal logic. At the next, higher level there is the science of the possible forms of true judgments. Husserl calls this *'consequence-logic'* (Konsequenzlogik) or the *'logic of non-contradiction'*. (See also Appendix III of *FTL*.) This concerns the question of whether a given form is included in or excluded by the forms of judgments in a premise set. In the former case we have an analytic consequence relation while in the later case we have an analytic anti-consequence, or an analytic contradiction. Husserl says that this second level of logic concerns only the (non)-contradictoriness of judgments.

It is not yet concerned with the *truth* of judgments. Judgments may be formally consistent with one another or not. If not, then they have no possibility of all being true, and this is purely a matter of form. Next, Husserl distinguishes a third level of logic, what he call *truth-logic* (Warheitslogik) . This is an inquiry into the formal laws of possible truth. Non-contradiction is a condition for possible truth. Each level, starting from the lowest, is a condition for the possibility of the next level.

Husserl comes back to the notion of truth-logic at a later point in *FTL*. First, he discusses different types of evidence associated with judgments. Here there is some discussion of both the modes of judging and of judgments themselves. Evidence may, for example, be clear or distinct and there may be clarity in the awareness of something itself or in the anticipation of what will be given. The general point here, however, is to claim that what the purely analytic logician is concerned with in particular is distinct judgment. The cognitional striving involved in obtaining evidence is beside the point in the sphere of pure apophantic analytics. The logician abstracts from it. The fundamental question of pure analytics, Husserl says, is this: when and in what relations are any judgments possible within the unity of one judgment, insofar as their mere form is considered? 'Non-contradiction' thus means that it is possible that the judger can judge distinct judgments within the unity of a judgment performable with distinctness.

Now every 'analytic' or purely formal countersense would be excluded by the rules of pure analytics but it does not follow that every material countersense would be thereby excluded, nor that other untruths would be excluded. When we bring the concept of the truth of judgments into consideration we are thinking of judgments as pervaded by a cognitional striving, as meanings that have to be fulfilled (§19). Judgments that are formally incompatible cannot possibly be fulfilled. The possibility of adequation is excluded. Truth and falsity are predicates that belong only to a judgment that is distinct or can be made distinct. In a mediated fashion, therefore, a pure analytics is a fundamental part of a formal logic of truth. Any consequence relationship of judgments, if it can be effected with intuition, becomes a consequence relationship of truths or of material possibilities. Any contradiction, however, excludes from the start all questions of adequation.

The separation of consequence-logic from truth-logic results in a separation that extends to the principles of logic. Husserl thus states two versions of some of these principles. For example, the 'double principle of contradiction and excluded middle', as a principle explicating the concepts of truth and falsity is stated as follows: "if a judgment is true then its contradictory opposite is false"; and "of two contradictory judgments, one is necessarily true". If we combine these we have "any judgment is exclusively one or the other, true or false". The analog in consequence-logic is stated as follows: "a judgment in which two mutually contradictory judgments are conjoined is not possible as a judgment proper; it cannot be given as a possible judgment in distinct evidence. It does not have ideal 'mathematical existence'. But at least one of them can become given as a possible judgment in distinct evidence" (§20). Husserl also gives two readings of *modus ponens* and *modus tollens.* In consequence-logic we can say that 'N' follows analytically from two judgements of the forms 'If M then N' and 'M'; and that 'not M' follows from two judgments of the forms 'If M then N' and 'not N'. The corresponding truth-principles are: if 'If M then N' and 'M' are true then 'N' is true; and if 'If M then N' are true then 'not M' is true.

Husserl goes on to consider the relationship of formal apophantics to formal mathematics and he elaborates somewhat on his notion of formal ontology. Traditional syllogistic logic is certainly not the whole of logic. Leibniz already enlarged significantly on the idea of a formal theory of science with his conception of *mathesis universalis* and his idea of unifying traditional syllogistics and formal mathematics. Once one becomes acquainted with deductive techniques one can see, as Leibniz saw, that one can calculate with propositions just as one can calculate with numbers. Indeed, a universal deductive theory of propositions can be built up in this manner. The idea of calculating with propositions is part of 'apophantics'. There is also 'non-apophantic mathematics', i.e., the traditional mathematics of sets, combinations and permutations, numbers, and so on. Here it is not the case that judgments (or propositions) become thematic in one's work. The idea will be to unify the study of judgments and their relations with the study of objects and states of affairs and their relations. One can see that in their broadest universality the theory of sets and numbers relate to anything whatever. They have a formal universality that leaves out of consideration all material determinations of objects. The various formal mathematical disciplines have as their fundamental concepts various derivative formations of 'anything whatever'. This gives rise to the idea of an all-embracing science, a comprehensive formal mathematics. Husserl says that it is natural to view this whole mathematics as a formal ontology of the pure modes of 'anything whatever'. One could then determine the separate provinces of this ontology by *a priori* structural considerations.

Formal apophantics and formal ontology (or formal mathematics) must be closely related to one another even though they have in effect developed as separate sciences. There must, however, be some kind of correlation because judging is judging about objects, predicating properties of them, and so on. All forms of objects, all the derivative formations of 'anything whatever', make their appearance in formal apophantics since they have being for us only insofar as they are given to us through judgments. In formal distinctions pertaining to judgments there are formal differences pertaining to objects although admittedly one has to be careful about the nature of the correlations.

Husserl offers some reasons why the problem of the unity of formal apophantics and formal mathematics was masked at earlier points in history. One reason is that the concept of pure empty form was lacking. Another reason is that judgments, unlike judgings, are ideal formations. People were less inclined in mathematics to subjectivize their objects but in logic there was a recurring tendency to confuse judging as a psychological act with judgments. Husserl also thinks that some of the problems stemmed from reinterpreting syllogistic logic as extensional. Admirable as their achievements were, Aristotle, Leibniz and Bolzano all come in for some criticism here. Leibniz is judged to have had the deepest insights into the relationship of the existing logic to mathematics.

Husserl notes how he introduced the notion of formal ontology, although not under that name, in his earlier "Prolegomena to Pure Logic". He did not, however, consider the question there of the relationship of formal ontology to apophantic logic. Nonetheless, he points out how even in *PA* he was, in effect, already investigating the constitution of categorial objectivities (§27). In *PA* he sought to make categorial objectivities (e.g., cardinal numbers and sets) understandable on the basis of the constituting intentional activities in which they originally make their appearance. No part of the material content of the collected elements of sets or the counted units of numbers enters into the formal and universal concepts of set and number. Here we already have examples of syntactic forms of anything whatever, or of the 'empty something'. In the "Prolegomena" the categories of signification (associated with judgments) were already contrasted with the formal object-categories. In effect, formal logic was characterized there as both an apophantics and an *a priori* formal theory of objects. In *FTL*, Husserl says, all of this is pursued in more detail and with more clarity.

In *LI* Husserl had already spoken of the task of investigating the theory of possible forms of theories and, correlatively, the theory of manifolds (Mannigfaltigkeiten). He reminds us of his remarks in §§69-70 of the "Prolegomena". Now suppose we make thematic the idea of judgment systems in their entirety, where we think of each of these as a possible deductive theory. We can think of a manifold as "the form-concept of the province belonging to a deductive science."(§28) It is the correlate of a formal deductive theory. For example, we can consider Euclidean geometry as Euclid himself understood it, i.e., as a theory of the intuited space of the world. This can be reduced to a theory-form, i.e., we can formalize it. In doing this the materially determinate contents are converted into indeterminate modes of the empty 'anything whatever'. Materially determinate geometry is thereby converted into a system form. To each geometrical truth a truth-form corresponds, and to each proof a proof-form. The determinate object-province becomes the form of a province, a manifold. It is of course not just any manifold or the form of just anything. It is the form derived from Euclidean geometry by formalization.

Husserl says that the great advance in mathematics occasioned by the work of Riemann and his successors consists in having viewed the system-forms themselves obtained in this manner as mathematical objects. Once one has such system-forms one can alter them freely, universalize them mathematically and particularize the universalities in different ways. This is not done by following the rules for differentiating the species of a genus according to the Aristotelian tradition. It's something altogether different that has its roots in the sphere of the formal.

In this manner mathematicians freely constructed different manifolds, using geometry and the Euclidean ideal evinced in geometry as their guides. Taking the Euclidean ideal as a guide, Husserl says that one can think of a manifold as the ontological correlate of a purely formal axiom system. Husserl says that a 'definite' manifold is a 'manifold in the pregnant sense'. It would be the "form-idea of an infinite object-province for which there exists the unity of a theoretical [deductive] explanation." (§31). There would be no truth about such a province that would not be deducible from the fundamental laws. Such a manifold is defined by a consistent and 'complete' formal axiom system. The axiom system that defines such a manifold is distinguished by the circumstance that any proposition that can be constructed in accordance with the grammar of pure logic out of the concept-forms occurring in the system is either 'true' (i.e., an analytic, deducible consequence of the axioms) or 'false' (i.e., an analytic contradiction) (§31). This immediately raises questions of the following sort: how can one know that a system of axioms is definite (i.e., complete)? Since any science consists of a manifold or multiplicity of truths, when does a set of axioms for the science (or when does its objective correlate) count as 'mathematical' and 'definite'? It seems that the provinces of some sciences are not definite manifolds (e.g., psychology or phenomenology itself). In his remarks on definiteness Husserl says that he borrows the expression "complete set of axioms" from Hilbert although he and Hilbert arrived at the notion of completeness (or definiteness) independently. (As mentioned above in Section 1, Husserl explains how in his double lecture to the Göttingen Mathematical Society he originally used the notion of a definite manifold in his studies of computation in a formally defined deductive system of imaginary concepts. His idea was to indicate how one can never be led to contradictions when calculating with such concepts if one is working in a system that is definite.)

Husserl says that in our efforts to build up a comprehensive theory of manifolds, or theory of possible theories, we need to guard against the idea that what we have is merely a deductive game with symbols. We need to keep in mind the relationship of the symbols to actual objects of thinking, e.g., numbers, sets, etc. There is an inclination, out of the desire for greater exactness, to put in place of the actual theory of manifolds its purely symbolic analog, to define manifolds in terms of mere 'rules of the game'. It is not merely computational manipulation of signs that makes up logic. We must also say that there are *objects* of a certain manifold for which certain laws hold, objects that have certain properties, stand in certain relations, and so on.

In formal logic we can focus on judgments or on objects (as distinct from judgments). Formal logic is two-sided in this sense: there is formal apophantics and there is formal ontology. Formal ontology, at its highest level, is the theory of manifolds. Formal ontology, as Husserl has said in earlier works, is distinct from material ontology. In *FTL* he is especially interested in further clarifying the relationship of formal apophantics to formal ontology. What is involved in shifting the focus from propositions to objects? Since formal objects are given to us through formal judgments how have we gone beyond formal judgment theory? Why is it that the theme of formal object theory is not objects after all but judgments?

Husserl argues that being focused on judgments as one's theme, on the one hand, and being focused on objects and their 'syntactical' or structured forms, on the other hand, are two different things. In judging we are directed toward objects of different types. The judgment we make is not itself made thematic in this process. It is not itself our object. We can, however, change our focus and make the judgment itself our theme. This takes place in a judging at a second level, a judging about judgments. Here the original judgment becomes our object. If we consider, for example, our judgments about nature we are not apt to conflate the objects about which we judge with the judgments themselves. Now in formal logic we also have objects, that is, categorially structured or formed objects. These categorial objects are not, as it were, given to us before any thinking or judging but we can say that in making judgments about nature we are also already engaged in categorially structuring or shaping. Of course there is also the experience of nature that underlies the judging but Husserl will say that all scientific thinking and scientific objects have an 'origin' in experiential intuition. Science, logic included, is built up or constituted from this basis. In formal logic we are directed toward categorial objects in respect of their pure forms. There is abstraction from the 'material cores' with which we may have started. Thus, in formal logic the ontology is purely formal.

In making judgments our intentions may be fulfilled. We may have an intuition of the object or state of affairs itself toward which we are directed by the judgment. It may also happen that our intention is disappointed. This shows that we can distinguish between what is merely supposed, as when our judgments are disappointed, and what is actual. In the one case we have the mere judgment but in the other case we can speak of a sphere of objects. In science in particular we exercise a critical attitude in which what lies before us is both our objects and our suppositions. In the 'supposed as such' we lay hold of what is called judgment in logic. The final result of such a critical process, ideally speaking, is either truth or falsity. In this sense, truth amounts to a correct critically verified judgment. There is verification by means of adequation to the corresponding categorial objectivities themselves. There is fulfillment. As in Investigation VI of *LI*, Husserl also distinguishes a second sense of truth: the true as the actually or truly *existent*, as the correlate of the evidence that gives the actuality itself. This is truth in the sense of true *being*.

Judgments, as mere suppositions, belong to the region of 'senses' (Sinne) or 'meanings'. To ask about the signification or sense of a statement, and to makes its sense distinct to ourselves, is to shift from that straightforward stating and judging attitude to the reflective attitude in which meanings are seized upon or posited. Senses, Husserl says, are "ideal poles of unity, 'transcending' the acts and subjects related to them, in quite the same fashion in which objects that are not senses do." (§48) (This notion of sense, Husserl says, can be broadened to cover any form of consciousness that is object-directed.) Now pure formal apophantics is just a theory of senses. To be more precise, the first two strata of logic, the grammar or pure logic and the pure analytics of non-contradiction, are theories exclusively of the region of senses. Consequence-logic excludes all questions of truth. In using the predicate 'true' we go beyond the sphere of senses. Truth requires adequation.

Husserl notes that the pure mathematician need not be concerned at all with whether the manifolds or systems of judgments that she investigates has or may have concrete applications. She needn't be concerned with whether her constructions are true of the world or even with the question of truth at all. Formal ontology is the science of possible objects purely as possible. It is possible to take judgments purely as senses that, in themselves, say nothing about actual being. There can be a purely formal mathematics in which we worry about nothing other than non-contradiction. This kind of mathematics, Husserl says, has its own legitimacy.

6.2 Part II: From Formal to Transcendental Logic

Part II of *FTL* is concerned with laying a transcendental foundation for logic. In the 'Preparatory Considerations' of *FTL*, as we saw above, Husserl already argued that logic is two-sided, in the sense that it has both a subjective side and an objective side. The idea is to now turn to the subjective side of logic and to the cognitive structures and processes through which theoretical reason brings about its 'productions'. The idea is to turn back to the living intentionality of the scientists who work in logic and mathematics. Only a transcendental logic, Husserl says, can be an ultimate, deepest and most universal theory of science. As soon as we mention this turn to the subjective foundations of logic, however, the bogy of psychologism appears. In particular, Husserl was criticized by some for falling back into psychologism in the sections of the *LI* that followed the "Prolegomena to Pure Logic". The investigations that were aimed at clarifying the fundamental concepts of pure logic were considered psychologistic. Husserl wants to show that a logic directed only to its own thematic sphere remains naive and cut off from the radical self-understanding and self-justification required by philosophy, and that this situation can be remedied not by psychological investigations but only by transcendental phenomenological investigations.

Psychologism amounts to equating the formations 'produced' by judging with the phenomena of internal experience, i.e., with mental processes, acts, or entities. The judging mental process and the formations produced by judging are, however, quite different from one another. Husserl says that there is evidence that in repeated acts the produced judgments, proofs, and so on are not merely alike or similar but are *numerically the same* judgments, proofs, etc. Their 'making an appearance' in the domain of consciousness is multiple. The processes or acts of thinking are temporally distinct from one another. They are individually different and separated. The same is not true of the thoughts that are thought in the thinking. The thoughts are, to be sure, not 'real' objects, not spatial, and so on. They are 'irreal' and their essence excludes spatial extension, location and mobility. They do, however, admit of a physical embodiment in the form of sensible verbal or written signs. Every ideality or irreality has manners of possible participation in reality but this in no way alters the separation of the real from the irreal.

Husserl proposes to show how the evidence of the real and of the irreal parallel one another in certain ways. By studying this we can gain a better understanding of the universal homogeneity of objects as objects. The evidence of irreal objects is quite analogous to the evidence of ordinary experience. The identity and therefore the objectivity of something ideal can be directly experienced with the same originality as the identity of an object of experience in the usual sense. In both cases there is an identity through a manifold of possible acts, a synthesis in which the same thing appears. The experience in both cases is repeatable at will, giving the same object. One difference is that an ideal object, unlike a real object, is not itself temporally individuated. Another similarity, however, is that there can be deception in both cases. Husserl says that even an ostensibly apodictic evidence (i.e. evidence of necessity) can be disclosed as deception. There could be further evidence by which it is overturned. One must consider evidence, apodictic evidence included, in the whole context of acts and processes in which it occurs. Our experience is imperfect but it is still experience of objects. It is because it is imperfect that experience can adjust itself to experience and correct itself by experience. It will not do in particular to ascribe apodicticity to some single mental process torn from the concrete, unitary context of our mental life. (Føllesdal [1988] links these ideas to recent holistic views about claims to 'necessity'.) As in the *LI*, Husserl also says that we should not confuse evidence in his sense with feelings of evidence. When we have evidence for an object we are presented with the object itself. We have adequation of an intention to the affairs themselves.

It is a law of intentionality that consciousness of any object belongs to an openly endless multiplicity of possible modes of consciousness which can be connected synthetically in a consciousness of one and the same thing. There can be evidence in this process for this same thing or for something other that annuls it. Thanks to evidence, Husserl says, the life of consciousness has an all-pervasive teleological structure, a directedness toward 'reason', i.e., toward the discovery of correctness (and the lasting acquisition of correctness) and the cancellation of incorrectness. Now evidence is always the giving of an object itself but it does not follow that the structure of evidence is identical in all domains. Husserl says that "category of objectivity and category of evidence are perfect correlates" (§60). To every fundamental species of objectivity a fundamental species of experience, or evidence, corresponds. It is the business of phenomenology to explore all of this.

If we substituted for ideal objects the temporal occurrences in the life of consciousness in which they make their appearance then, to be consistent, one would also have to do this in the case of everyday sensory objects. In neither case, however, will this do. For in both cases the object is *transcendent*. That is, it is an identical pole in the individual mental processes and yet it transcends them by virtue of having an identity that surpasses them (§61). 'It' can be given in a more and more determinate manner through time. This transcendence lies in the essence of the experience itself. In a sense then, an object draws the ontic sense peculiar to it from the mental processes of experience alone, from the way it shows itself in experience. This is true for both real and irreal objects. The evidential giving of something itself must be characterized as a process of constitution. Of course the object is not itself the actual and openly possible experiential process constituting it, nor is it the possibility of repetitive synthesis. It follows that a certain ideality lies in the sense of every experienceable object. It is an ideality through the manifold cognitive processes that are separated from one another by individuation in immanent time. In this ideality consists the transcendence belonging to all species of objectivities over against the multiplicities constituting them.

Husserl is aware that caution must be exercised in speaking of the 'production' of logical formations in consciousness. In many cases where we speak of producing we are referring to a real sphere and we are thinking about the production of real physical things or processes. In the case of logic, however, it is irreal objects we are talking about. They are given in real psychic processes but our activity in logic and mathematics is not directed toward such real psychic processes. Otherwise, logic and mathematics would not be possible at all. There is nonetheless a productive activity in logic and mathematics. As in other types of activity, there are ends of our actions that are intended by us beforehand in empty modes of anticipation. We are conscious of these ends as things toward which we are striving and, should we have success, then the process of actualizing these things themselves just is an action that is accomplished step by step. Our end, for example, might be some judgment that we strive to produce. Judging is acting and in judging something irreal is intentionally constituted. What is constituted in this case is also a producible end obtained by some means. This is not to say, however, that irreal objects are what they are only *in* and *during* the original production. They are 'in' the original production only in the sense of being intended to in it and their being transcends the original production in the sense that the same object can be arrived at in different times and places by different persons.

Husserl says that "in respect of its being, reality has precedence to every irreality whatsoever", since all irrealities relate back essentially to an actual or possible reality (§64). Husserl makes this claim in various places in his writings. We will see more of it in the *FTL* but we have already seen it in his comments about the 'founding' of categorial objects in the *LI*. The analysis of origins or of founding and founded structures is itself a taken to be a major part of the phenomenology of logic. In *FTL* Husserl thinks that by reminding us of the ontic precedence of real to irreal objects he can steer clear of the 'vehement opposition' to putting ideal objects on a par with real objects. What he is doing, he tells us, is simply broadening the concepts of 'object' and 'evidence' to conform with the intentionality of consciousness in the case of logic and mathematics. For in these fields too there are data of experience. We can broaden the concepts of object and evidence and yet still claim that the meaning of being of irreal objects is founded on or has its origins in the meaning of being of real objects.

Psychologism, in its general form, denies irreal objects their *sense* as a species of object in favor of subjective mental occurrences. This is a characteristic of every 'bad idealism', like that of Berkeley or Hume. Phenomenological idealism is quite different from this. It mounts a 'radical criticism' of psychologism. This 'good idealism' and the critique of psychologism is of course presented in many of Husserl's later works. One thing that distinguishes phenomenology from other forms of philosophical criticism is its critique of the performances of cognition that remain hidden in theorizing directed straightforwardly to a province. In this sense, it offers a 'critique of reason' or a 'transcendental criticism of cognition'. At the next step of the *FTL* Husserl wishes to clarify the nature and the necessity of the investigation of the subjective side of logic.

The formations of logic are at first given to us in straightforward evidence in our activities in logic but a thematizing reflection on this evidence is required (§§69-70). As in *LI*, he says that these formations must now be clarified in reflection so that we can correctly apprehend and delimit their objective sense. In this manner we can secure the identity of this sense against the various shiftings and disguises that may occur when this sense is produced naively. Thus, we look in reflection to the activity that is hidden throughout the naive practice of logic and mathematics. We consider the activity involved in constituting the themes of logic. In this manner we can identify and fix or trace the possible variations concerning these themes that had previously gone unnoticed. We can distinguish the corresponding intentions and actualizations. In short, we can become aware of the shifting processes of forming concepts that pertain to logic. These are shiftings of intentionality that can then in turn lead to verbal equivocations. We can only formulate these equivocations and remove them through reflective examination of the original constitution of the formations and the intentional directedness involved. Husserl thinks that this kind of clarification is already illustrated through the investigations of Part I of *FTL*. If we consider how logic functions in scientific cognition then we see how distinguishing the three strata of logic constitutes a clarification of logic. We have succeeded in distinguishing important features of logic that were previously hidden or unclear. There might have been equivocation for example, between how judgments are understood in consequence-logic and how they are understood in truth-logic. Unsinn might not be distinguished from Widersinn, and so on. Here we have only a few examples. Much more can be accomplished and Husserl thinks he is carrying out just this kind of work in the later sections of *FTL*. If it is true that to the objective *a priori* structures of logic there correspond subjective *a priori* structures, as Husserl believes, then much more work remains to be done.

Husserl says that we now require a constitutive criticism of analytic logic that can make us conscious of some of the 'idealizing presuppositions' with which logic operates as if they were truisms. These are presuppositions that Husserl says we have thus far in our investigations taken over unconsciously. This new level of criticism continues the earlier work, especially the three-fold stratification of logic, and presupposes it.

First, Husserl says that logic, already at the level of consequence-logic, presupposes the ideal identity of judgments. What assures us of this identity? These judgments are supposed to be accepted as identical objects that are fixed and that exist for us at all times, through all the pauses in and vicissitudes of mental activity. Logicians and mathematicians do not trouble themselves much about this. They simply presuppose the identity of their objects and the identity of their judgments. This idealizing presupposition raises a problem of constitution that can be considered by transcendental phenomenology.

Another idealization concerns 'reiterational infinity', as when the logician or mathematician uses the expression "and so forth" to indicate something infinite. The subjective correlate of this is that "one always can again". This is clearly an idealization since in fact one cannot always again take another step. This form, however, plays a sense-determining role everywhere in logic. For example, one can always return to the same judgment, or obtain another set not included in a presently given set, or obtain another natural number, and so on. Mathematics is the realm of infinite constructions, a realm of ideal existents, including both senses or judgments and mathematical objects. Now if infinities as categorial formations are considered evident then this evidence must itself become a theme in the investigations of transcendental logic.

Husserl considers some laws of logic along with the 'subjective versions' of these laws. In consequence-logic (as distinct from truth-logic), for example, we have the law of contradiction which says that every contradictory judgment is 'excluded' by the judgment that it contradicts (§75). On the subjective side, we have the following: whoever has a judgment-meaning and in explicating it to himself sees any analytic consequence, not only judges the consequence in fact but cannot do otherwise than judge the consequence. In this process he who judges becomes conscious with a formal universality of the necessity involved, of the inability to do otherwise. This can be seen on the basis of eidetic variation. On the objective side we have the general impossibility of the unity of the two contradictory judgments, while on the subjective side we have the impossibility of the judicative believing, an impossibility for anyone at all who might be judging (with distinct evidence). Of two judgments that contradict one another only one can be accepted by anyone who judges in a proper or distinct unitary judging. Here we are speaking merely of judging in a distinct mode (as in the logic of non-contradiction), not of truth (as in truth-logic). Objectively speaking, the law of analytic contradiction is a proposition about ideal mathematical 'existence' and coexistence, i.e., about the compossibility of judgments as distinct. Subjectively speaking, there is the *a priori* structure of evidence and the subjective performances pertaining to it. It is just this latter material that must be exposed and understood in a legitimately grounded analytics, an analytics in which there would be no paradoxes or questions about legitimate application.

It is easy to see that there are also problems pertaining to the subjective side of the logic of truth. When we introduce the concept of truth into apophantic logic we obtain a theory of science proper, a formal ontology. There are also various presuppositions and difficulties at this level. When we consider the notion of truth we must also refer to the idea of adequation and to the giving of objects themselves. Husserl now considers the idealizing presuppositions contained in the laws of contradiction and excluded middle, where we view these from the perspective of truth-logic (§77). The law of contradiction expresses the general impossibility of contradictory judgments being true (or false) together. In what evidence is this grounded? We can say that if a judgment can be brought to adequation in a positive material evidence then, *a priori*, its contradictory opposite is not only excluded as a judgment but also cannot be brought to such an adequation. This is not yet to say that every judgment, without exception, can be brought to adequation. This latter situation, however, is just what is involved in the law of the excluded middle when it is viewed in terms of evidence, i.e., viewed from its subjective side. Every judgment can be confronted with 'its affairs themselves' and adjusted to them in either a positive or negative adequation. In one case the judgment is evidently true. It is in fulfilling and verifying coincidence with the categorial objectivity meant in the relevant judging. In the other case it is evidently false because a categorial objectivity is given that conflicts with the judging. Husserl says that in its subjective aspect the law of the excluded middle has two parts. It decrees not only that if a judgment can be brought to adequation then it can be brought to either a positive or negative adequation but it also decrees that every judgment necessarily admits of being brought to an adequation. Husserl says that the word 'necessarily' here is "understood with an *ideality* for which no responsible evidence has ever been sought" (§77). We all know very well how few judgments anyone can in fact legitimate intuitively, even with the best of efforts, and yet it is supposed to be a matter of *a priori* insight that there can be no judgments that do not in themselves admit of being made evident in either a positive or negative evidence.

Do these remarks on the principle of the excluded middle or on other idealizing presuppositions of logic have any implications for revisionism in mathematics and logic? How much revisionism could be tolerated? The other side of this question is this: how much idealization about knowledge of objects can be tolerated? Husserl's ideas should lead to interesting perspectives on these questions.

Husserl also mentions idealizations involved the principle of identity (A = A). Yet another idealization has to do with intersubjectivity. The same judgment is not merely an ideal unity pertaining to *my* cognitive life. The identity of the judgment is taken as a universal intersubjective evidence. The adequation effected by one subject that yields the truth for her as an ideality is such that this ideality must relate to everyone. In this respect, the idealization is that 'everyone is in harmony with everyone else'. These are remarkable sense-determinations of the concept of truth that logic takes as fundamental. They determine the concept of 'objective truth'. The principles of logic claim to be valid once and for all for everyone.

The laws of *modus ponens* and *modus tollens* are also considered in relation to their subjective correlates. The principle of analytic consequence in consequence-logic can be transmuted into a law pertaining to subjective evidence as follows: the possibility of distinct evidence of the analytic antecedent judgment necessarily entails the possibility of such evidence of the consequent judgment. Husserl says that the novelty in the transmutation of the corresponding law in truth-logic is that when the categorial actions involved in judging the antecedent are performed on the basis of having 'the affairs themselves', the same possibility of *material evidence* must exist also for the actions involved in judging the consequent.

In truth-logic we include in our considerations the predicates "is true" and "is false" as these apply to judgments. Judgments in themselves have no claim to truth or falsity. A claim to truth is not included in the proper essence of judgments but any judgment can take up into itself the practical intention aimed at verification. Now when this occurs it is, Husserl says, a fundamental conviction of the logician that there is truth-in-itself and falsity-in-itself (§79). While many judgments in fact remain undecided for us and perhaps most judgments will never be decided in fact it is believed that, *in themselves*, they can be decided. The presupposition, in short, is this: In itself every judgment is decided. Husserl says that this is very remarkable. Since they are supposed to be 'decided in themselves' this signifies that there is a method, a course of cognition, that leads to an adequation that makes evident either the truth or the falsity of any judgment. "All of this imputes an astonishing *a priori* to every subject of possible judging and therefore to every actual or conceivable human being — astonishing: for how can we know *a priori* that courses of thinking with certain final results "exist in themselves"; paths that can be, but never have been trod; actions of thinking that have unknown subjective forms that that can be, though they never have been, carried out?" (§79)

As a matter of fact, we have experience with some judgments that were undecided but that have been decided. Husserl says that there are indeed truths in themselves that one can seek and also find by avenues already predelineated in themselves. This is one of life's truisms (§80). One does not ask whether there is a truth but only how it can be reached or, at worst, whether it is not utterly unattainable by our factually limited powers of cognition or else unattainable only because of our temporarily insufficient previous knowledge and methodology. In this manner we have not only the domains of truths that make practical living possible but the infinite fields of cognition that belong to the sciences. The possibility of sciences depends on the view that their provinces exist in truth and that, concerning these provinces, theoretical truths-in-themselves exist, as truths that can be actualized by following explorable and gradually realizable paths of cognition. Husserl says that he does not intend to give up any of these truisms. They surely rank as evidences. This must not, however, keep us from subjecting them to criticism and asking about their peculiar sense and range. Such evidences can have presuppositions. Husserl mentions as another example the presuppositions involved in occasional judgments. He notes that these presuppositions are frequently overlooked by logicians who simply proceed in a naive fashion and thus fail to grasp the range of what is involved in such judgments. On the whole, there is a need to consider critically and reflectively the evidence pertaining to the presupposition of truth in logic. Logic itself does not do this. Its formal universalities stand in need of intentional criticism that prescribes the sense and limits of their fruitful application. Many problems related to the notions of 'objective truth', 'absolute truth' and so on, are to be taken up by transcendental phenomenology.

Husserl in fact pursues some of this in later sections of *FTL*.

The criticism of the evidence associated with principles of logic, Husserl argues, is to be carried back to the criticism of the evidence associated with the experience of individuals and, ultimately, to real individuals. This reflects themes we have already seen in Husserl's earlier work about how logic and mathematics are founded ultimately on or have their origins in ordinary, everyday experience. We have already seen this theme clearly, for example, in the *Philosophy of Arithmetic* and in the *Logical Investigations*. Husserl says that without an analysis of the genesis of the senses involved in the principles of logic we do not know what hidden presuppositions may lie within these principles.

Husserl argues that we can trace judgments back to 'ultimate judgments', that is, to judgments in which subjects are not themselves nominalized predicates, relations or the like, predicates are not predicates of predicates, and so on. These primitive or 'ultimate' judgments are of no particular interest for *mathesis universalis* as formal mathematics but they are important for truth-logic because here we are speaking about the primitive individuals back to which all truth can be traced. In this case one can intuit the primitive individuals. One must bring these 'ultimate cores' to adequation. On the one hand there is the reduction of judgments and, on the other hand, the correlative reduction of truths. There are truths at lower and higher levels. In truths at the lowest level the matter and material spheres concern individual objects. These objects contain in themselves no 'judgment-syntaxes' or categorial shaping. They are experienceable prior to all judging (§83). Now purely formal-analytic universalities can have different possible applications to arbitrarily selected objects with material content. Formal logic is indeed intended to serve the ends of sciences with material content. It is not merely to consist of play with empty thoughts. It is part of its purpose to relate ultimately to different spheres of individuals.

This means that there must be a hierarchy of evidences that parallels the levels of senses and truths. The truths and types of evidence that are first in themselves are the individual ones. They are the most original. 'Experience', in the pregnant sense, involves directedness to something individual. Now genetic deliberations of the type being considered here are supposed to uncover hidden intentional implications included in our judging and in the judgment itself. Uncovering the genesis of the sense of our judgments signifies an unravelling of the components of sense implicit in the judgment. Judgments, taken as the finished products of a constitution or genesis, can and must be asked about their genesis. In them the sense points back, level by level, to an original sense. Each sense can be asked about its sense-history. In particular, one can ask about the eidetically apprehensible essential form of this genesis. It is this feature, not the actual empirical history, that the phenomenologist wishes to uncover. In this manner one can uncover various intentional implications. Not only the overt but also the implied sense must have its say. This is important for the process of making logical principles evident.

The 'tracing back' of the sort involved in the genesis of sense brings us to judgments about individuals. These are the pure and simple experiential judgments,

judgments about the data of possible perception and memory. It is a general proposition of the theory of consciousness that for objectivities of every sort the consciousness that gives the objects themselves precedes all other modes of consciousness. All other modes, genetically speaking, are secondary. Thus, from the point of view of genetic phenomenology the basis of the theory of judgment is the theory of evident judgments concerning real individuals. One should trace 'predicative' evidence back to 'pre-predicative' or 'non-predicative' evidence. The basic intentional structures are already present at this level. The experiential judgment is the original judgment. Logic thus needs a theory of experience since all judgments and truths are ultimately related back to this primitive basis. Husserl says that his investigations into these basic judgments and their relation to experience will be carried out elsewhere (in what ultimately becomes *Experience and Judgment*, discussed below). This founding experience itself has a kind of shape even if the conceptual and grammatical formings that characterize the categorial (and thus, predicative judgments) are not present.

From this basis one must ascend to the possible universalizations built on experience and ask how the evidence of the lower levels is related to the evidence at higher levels. There are two basic types of universalization: universalizations brought about in connection with (i) the material *a priori* and, (ii) the formal *a priori*. In the former case we acquire materially filled genera and species and eidetic laws that have material content. With a 'formalizing universalization', however, each individual is emptied of material or content to become 'anything whatever'. Every *a priori* with material content demands a return to intuition of individual examples, a possible experience, if criticism is to bring about genuine evidence. The evidence of laws pertaining to the analytic *a priori* requires no such intuitions of determinate individuals. It only needs some examples of categories and even categories with indeterminate universal cores will do (as when propositions about numbers serve as examples). These may point back intentionally to something individual but they need not be further examined in this respect. One does not have to go into some materially filled sense in this case. Nevertheless, the sense-relations of all categorial meanings to something individual cannot be insignificant for the sense and the possible evidence of the laws of analytics. Otherwise these laws could not claim to be valid for everything that conceivably exists. The sense of these principles must have a source in more basic forms of experience.

From the point of view of the genesis of the sense of logical principles we come upon a fundamental presupposition included in these principles, at least in the principle of the excluded middle. It is that any judgment can be converted into a possible 'distinct' and 'proper' judgment. A sentence like "This color plus one equals three" makes no proper sense. It does not contain a contradiction of the sort pertaining to pure analytics. Rather, its parts each have a sense but the whole formed from them does not. There is no unity of sense in the whole. This, as we have seen before, is different from logically contradictory judgments. Logically contradictory judgments already presuppose a unity of sense. This unity of sense concerns the ideal existence of the judgment-content of an act. The unity of sense at this level already presupposes a unity of possible experience at the lowest levels. It is tacitly presupposed that the symbols that we use in purely formal logic have something to do with one another materially. There is a presumed coherence that has its basis in the coherence or unity of possible experience, and there is such a coherence whether the experience is harmonious or discordant. For even in the latter case there must be meaningful relations among contents in order for the frustration to appear. In the evidence that we have at the most basic, universal experiential level there is always such coherence. We do not have complete freedom to vary the components of judgments even in the case where intuition supporting the judgment is lacking. There is a relation to the unity of possible experience even at this level. The principles of logic hold for all judgments that have such a unity of sense. They do not apply in the case where the conditions of grammar, conditions for forming judgments as meaningful wholes from meaningful parts, have not been met.

Husserl says that the subjective grounding of logic is a problem that belongs to transcendental philosophy. 'Objective logic' is a positive, naive science. One sees even more clearly how it was related to the 'real' world in its early developments, beginning with Aristotle. Even later philosophers who saw the need to investigate the subjective foundations of science as part of their rationalism (e.g, Descartes), did not really understand the idea of exploring the subjective foundations of logic. Without investigations directed to the subjective side of logic the principles of logic "are left hanging in the air, scientifically unsupported" (§93b). Husserl now takes the opportunity to tell us in more detail about the proper way to explore the subjective foundations of logic. It is, of course, through transcendental phenomenological idealism.

Transcendental phenomenological idealism, as we said above, holds that there is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent object that had any other sense than that of an intentional unity making its appearance in the subjectivity of consciousness. Every existent is constituted in the subjectivity of consciousness. This means that each person must start from his or her own subjectivity. It is from me that everything receives its sense of being. Of course the world is a world for all of us but there are philosophical problems of intersubjectivity that must be explored by transcendental phenomenology. There is a level of pure, individual subjectivity that can be explored but there is also a level of intersubjectivity, of a 'community of monads'. Transcendental solipsism is an illusion but this point itself is to be explicated through phenomenological analyses and descriptions. Husserl in fact wrote extensively on the problems of intersubjectivity.

There are specific problems about the constitution of the objective world of everyday experience but also about the higher-level objectivity of the 'theoretical world' that we find in the sciences. In particular, it is necessary to clarify the *idealizations* involved in the intentional sense of the sciences. The currently constituted intentional unities all have a sedimented history, a history that one can uncover. Every object thereby becomes a 'transcendental clue' to guide constitutional analyses, that is, uncoverings of intentional implications. These intentional implications include advancing from given experiences to those experiences that are predelineated as possible. Moreover, this is not a question of empirical or inductive 'history'. Rather, it is a matter of inquiry into essences. The fact that there are essential structures here is what makes inquiry into empirical history possible in the first place. Any straightforwardly constituted objectivity points back to its essential type, to a correlative essential form of intentionality that is constitutive for that objectivity. It points back to an actual or possible manifold of experiences. The whole life of consciousness is governed by a universal constitutional *a priori*, embracing all intentionalities. Exploration of this extensive *a priori* is the task of transcendental phenomenology.

What is the method of this inquiry into essences? The 'method', as mentioned before, is that of free variation in imagination (§98). One starts with an example and then varies it freely in imagination and observes the manner in which the object takes shape as a synthetic unity through the variations. There is some systematic universe of possible harmonious experiences that pertains to the object. Speaking ideally, there is a totality of determinations that belong to the object in which it would be given 'from all sides'. The variation of the example is a performance in which the 'eidos' should emerge, in which the eidetic correlation between constitution and constituted should emerge. This is not an empirical variation. Rather, it is carried on with the freedom of pure imagination. What persists through this free and always repeatable variation is some invariant, the essence common to all, by which all imaginable variants are restricted. This invariant is the eidos. Even in the case of the broadest, formal analytic universalities one sees that any object thought of as a completely optional 'anything whatever' is thinkable only as the correlate of an intentional constitution inseparable from it. The constitution is indeterminately empty and yet it is not variable without restriction. With each particularization of the 'something', and with each ontic category thus substituted, the constitution must become correlatively particularized.

One can inquire into the 'static' constitution of objects, the already developed subjectivity and objectivity, but one can also inquire into the *a priori* 'genetic' constitution, based on the static constitution. In the analysis of genetic constitution one uncovers what lies in the sedimented history of the constitution of sense.

Husserl makes a number of interesting remarks about the development of a transcendental philosophy of formal logic (§100), especially in relation to Hume, Locke, Descartes and Kant. He says, for example, that Kant does not ask transcendental questions about formal logic itself. Instead, Kant ascribes to it a kind of priority that exalts it above such questions. He fails to see that logic itself is concerned with ideal formations produced by thinking. No one ventured to take the ideality of the formations with which logic is concerned as characteristic of a separate self-contained 'world' of ideal objects. Thus, they did not come face-to-face with the 'painful question' of how subjectivity can in itself bring forth formations that can be considered ideal objects in an ideal world. There was no unravelling of the intentionalities involved in this, of the sense-constitution involved. Logic, however, must be more than a merely positive, naive science of logico-mathematical idealities. Rather, there must be a two-sided inquiry that goes back from the ideal formations to the consciousness that consitutes them. It is especially Leibniz, Bolzano and Lotze who help us to see that logic is concerned with a world of idealities. Kant and the empiricists did not clearly see this and, thus, they did not extend their investigations to this sphere.

In the final chapter of *FTL* Husserl makes a number of additional comments about the nature of transcendental phenomenology and its investigation of evidence and truth. The subjective foundation of logic just is the transcendental phenomenology of reason. The evidential or constitutional problems of logic are to be treated from this perspective. Logic is at first naive, and involved in a 'natural positivity'. This positive, naive logic is to be overcome through successive and more penetrating investigations of its origins. One would aim toward an 'ultimate' logic that furnishes the norms for its own transcendental clarification. The ideal is to ground scientific cognition with "an absolute freedom from presuppositions" (§103). Every existent that ever had or can have sense for us stands in a hierarchy of intentional functions and existents already constituted intentionally. Contrary to the false ideal of an absolute existent and absolute truth, every existent is ultimately relative. It is relative, that is, to transcendental subjectivity. Transcendental subjectivity alone exists 'in itself and for itself', but as part of a hierarchical order corresponding to the constitution of the different levels of intersubjectivity. Transcendental phenomenology is scientific self-explication or self-examination on the part of transcendental subjectivity. It is, as Husserl portrays it here, a radical striving to uproot all prejudice (§104). "Every existent given beforehand, with straightforward evidence thereof, is taken by it to be a 'prejudice'." This does not necessarily mean 'prejudice' in the usual disparaging sense, but rather that transcendental criticism and grounding is required. This is the notion of prejudice taken up in hermeneutics by Gadamer and others, according to which some interpretation or other is always involved in our thinking and perceiving. For Husserl, we are to continuously subject these interpretations to examination and criticism in order to arrive at truth. Logic, in particular, is continually hampered by prejudices (in the usual bad sense) and the worst of all of these prejudices are those concerning evidence.

The usual theories of evidence are misguided by the presupposition of absolute truth. According to this view, there must be an evidence that is an absolute grasping of the truth since otherwise we could neither have nor strive for truth and science. Husserl asks a series of questions about this view that are meant to indicate his own view (§105). What if truth is an *idea*, lying at infinity? What if every truth about reality remains involved in relativities by its very essence, and refers to regulative ideas as its norms? Consider the relativity of truth and of evidence of truth, on the one hand, and, on the other hand, the infinitely distant, ideal, absolute truth beyond all relativity. What if each of these has its legitimacy and each demands the other? It is high time, Husserl says, that people got over being dazzled by the ideal and regulative ideas and methods of the exact sciences, as though the 'in itself' of such sciences were actually an absolute norm for objective being and for truth. Such people overlook the infinitudes of life and cognition, the infinitudes of relative being with its relative truths. To rush ahead and philosophize on high about such matters is wrong — it creates a wrong skeptical relativism and a no less wrong logical absolutism. When we follow the reflective, critical procedure of transcendental, eidetic phenomenology, however, "we have continuously anew the *living truth from the living source, which is our absolute life*, and from the self-examination turned toward that life, *in the constant spirit of self-responsibility.* We have the *truth* then, not as falsely absolutized, but rather, in each case, as within its *horizons* – which do not remain overlooked or veiled from sight, but are systematically explicated. We have it, that is to say, in a living intentionality (called "evidence of it") whose own content enables us to distinguish between "actually given in itself" and "anticipated", or "still in our grip" retentionally, or "appresented as alien to the ego's own", and the like – a content that, with the uncovering of the attendant intentional implications, leads to all those relativities in which being and validity are involved" (§105). Husserl concludes with some remarks on a transcendental theory of evidence in order to show how this view differs from some of the incorrect views. On the Husserlian view, evidence is said to be an effective intentional performance.

Thus does Husserl attempt in *FTL* to map the way from traditional to transcendental logic. Transcendental logic, he says, is not a second logic but only radical and concrete logic itself, which accrues by the phenomenological method. We have the traditionally limited, analytic logic, but also a preliminary understanding of those 'logics' (in another sense) that should be established: the material theories of science, among which the highest and most inclusive would be the logic of absolute science, the logic of transcendental-phenomenological philosophy itself.

7 THEMES FROM *THE CRISIS OF THE EUROPEAN SCIENCES AND TRANSCENDENTAL PHENOMENOLOGY* (1936) AND "THE ORIGIN OF GEOMETRY" (1936)

7.1 The Crisis

The *Crisis* is not devoted specifically to logic but there are many themes in it that are related to Husserl's views on logic and mathematics. The book begins with an investigation of the sciences in general, and of natural science in particular, and it leads the reader in this way to transcendental phenomenology. Husserl argues that there is a kind of 'crisis' of the modern sciences that has resulted from their attempt to be 'merely factual' and to place increasing value on technization, formalization, specialization and some other developments. He contrasts this 'positivistic' conception of science with a broader conception of science of the sort that is built into transcendental phenomenology. There has been, he argues, a positivistic reduction of science to mere factual science. The crisis of science, in this sense, is its loss of meaning for life. What is the meaning of science for human existence? The modern sciences seem to exclude precisely these kinds of questions. The physical sciences abstract from everything subjective and have nothing to say in response to such questions. The 'human sciences' on the other hand are busy trying to exclude all valuative positions and are attempting to be merely factual. They also fail to speak to such questions. There is even hostility in some quarters toward the modern sciences for reasons of this sort. Such human questions, however, were not always banned from the realm of science. There is a broader notion of science that deals with the notion of 'reason' and the role it can play in finding meaning in the world in which we live. Philosophy itself is involved in this task. Positivism, however, 'decapitatates philosophy' (§3). Various forms of skepticism about reason have set in. Skepticism insists on the validity of the factually experienced world and finds in the world nothing of reason or its ideas. Reason itself becomes more and more enigmatic under these conditions. Now in attempting to combat such trends one need not resort to naive and even absurd forms of rationalism. Rather, one needs to find the genuine sense of rationalism.

The modern sciences are made possible by mathematics and, in particular, by formal mathematics of the type that we see in algebra, analytic geometry, and so on. Husserl focuses on Galileo's mathematization of nature and the role of 'pure geometry' in making modern science possible. With Galileo nature becomes a mathematical manifold. What is involved in the mathematization of nature? Husserl discusses this at some length. It depends on the rise of pure geometry with its idealizations and its exactness. In pure geometry we do not have the shapes and the objects of everyday perceptual experience. We cannot obtain awareness of the objects of pure geometry by simply subjecting the bodies given to us in space and time to free variation in imagination. To obtain the 'pure' straight lines, planes, figures, and so on, of Euclidean geometry certain idealizations are required. This can be understood in terms of a gradual increase toward perfection. Our technical abilities to create a perfectly straight line, a perfectly flat surface, and so on, reach certain limits. The ideal of perfection, however, can be pushed beyond that, further and further. Out of the praxis of perfecting we can understand that in pressing towards the horizons of *conceivable* perfecting certain 'limit shapes' emerge toward which the series of perfectings tends, as toward invariant and never attainable poles. It is these *ideal* shapes that make up the subject matter of pure geometry. Here we obtain an exactness that is denied to us in the intuitively given surrounding lifeworld. It is the measuring of shapes in the prescientific lifeworld that underlies these idealizations. The idealizations are a natural outcome of refining and perfecting measurement. Every measurement acquires the sense of an approximation to an unattainable but ideally identical pole, i.e., to one of the definite mathematical idealities or to one of the numerical constructions belonging to them.

Galileo inherits pure geometry and with it he begins to mathematize nature. In the process, nature itself come to be viewed as more idealized. In interpreting nature through pure geometry we begin to view nature in a different manner. It is no longer the 'nature' of prescientific, lifeworld experience. Various idealizations of nature are involved in seeing nature in terms of pure geometry. In the process we leave out or abstract from some of the aspects of the lifeworld experience of nature. The 'plenum' of original intuitive experience, however, is not fully mathematizable. The mathematization of nature is an achievement, Husserl says, that is 'decisive for life' (§9f). With mathematics one has at hand the various formulae that are used in scientific method. It is understandable, Husserl says, that some people were misled into taking these formulae and their 'formula-meaning' for the true being of nature itself. This 'formula-meaning', however, constitutes a kind of superficialization of meaning that unavoidably accompanies the technical development and practice of method. With the arithmetization of geometry there is a kind of emptying of its meaning. The spatiotemporal idealities of geometric thinking are transformed into numerical configurations or algebraic structures. In algebraic calculation one lets the geometric signification recede into the background. Indeed, one drops it altogether. One calculates, remembering only at the end of the calculation that the numerals signify multitudes. This process of method transformation eventually leads to completely universal 'formalization'. This happens through the improvement and broadening of the algebraic theory of numbers and magnitudes into a universal and purely formal 'analysis', theory of manifolds, or logistic. Leibniz first caught hold of the idea of a highest form of algebraic thinking, a *mathesis universalis*. Husserl says that in its full and complete sense it is nothing other than a formal logic carried out universally, a science of the forms of meaning of the 'something in general' that can be constructed in pure thought and in empty, formal generality. On this basis, it is a science of the 'manifolds' that, according to formal elementary laws of the noncontradiction of these constructions, can be built up as in themselves free of contradiction. At the highest level, it is a science of the universe of manifolds. Manifolds are themselves compossible totalities of objects in general, which are thought of as distinct only in empty, formal generality and are conceived of as defined by determinate modalities of the something-in-general. Among these totalities the 'definite' manifolds are distinctive. They are defined through complete axiomatic systems. With such a totality one can say that the formal logical idea of a 'world-in-general' is constructed.

Husserl speaks of emptying the meaning of mathematical natural science through 'technization'. Through calculating techniques we can become involved in the mere art of achieving results the genuine sense and truth of which can be attained only by concrete intuitive thinking actually directed at the subject matter itself (§9g). Only the modes of thinking and the type of clarity that are indispensable for technique as such are in action in calculation. One operates with symbols according to the rules of the game (as in the 'games meaning' Husserl had discussed in *LI*). Here the original thinking that genuinely gives meaning to this technical process and gives truth to the correct results is excluded. In this manner it is also excluded in the formal theory of manifolds itself. The process whereby material mathematics is put into such formal logical form is perfectly legitimate. Indeed, it is necessary. Technization is also necessary, even though it sometimes completely loses itself in merely technical thinking. All of this must, however, be a method that is practiced in a fully conscious way. Care must be taken to avoid dangerous shifts of meaning by keeping in mind always the original bestowal of meaning upon the method, through which it has the sense of achieving knowledge about the world. Even more, it must be freed of the character of an unquestioned tradition whose meaning has been obscured in certain ways. This technization in which one operates with symbolic concepts often admits of mechanization. All of this leads to a transformation of our experience and thought. Natural science undergoes a far-reaching transformation and there is a covering over of its meaning.

There is, Husserl says, a surreptitious substitution of the mathematically structured world of idealities for the only real world, the one that is actually given through perception: the everyday lifeworld. This substitution already occurred as early as Galileo and it was subsequently passed down through the generations. This is a substitution of idealized nature for prescientifically intuited nature (§9h). What has happened is that the lifeworld, which is the foundation of the meaning of natural science, has been forgotten. A type of naiveté has developed. Galileo is at once both a discovering and a concealing genius. Various misunderstandings arise from the lack of clarity about the meaning of mathematization (§9i). For example, one holds to the merely subjective character of specific sense qualities. All concrete phenomena of sensibly intuited nature come to be viewed as merely subjective. If the intuited world of our life is merely subjective then all the truths of pre- and extrascientifc life that have nothing to do with its factual being are deprived of value. Nature, in its 'true' being, is taken to be mathematical. The obscurity is strengthened and transformed even more with the development and constant application of pure formal mathematics. 'Space' and the purely formally defined 'Euclidean manifold' are confused. The true axiom (in the early sense of the term) is confused with the 'inauthentic' axiom (of manifold theory). In the theory of manifolds, however, the term 'axiom' does not signify judgments or propositions but forms of propositions, where these forms are to be combined without contradiction.

Now these techniques and methods of the sciences are handed down through the generations but their true meanings are not necessarily handed down with them. It is the business of the philosopher and phenomenologist to inquire back into the original meanings through an eidetic analysis of the sedimentation involved. There is a 'historical meaning' associated with the formations of the sciences but, as we have seen before, Husserl is not interested primarily in empirical history. He is interested in finding the *a priori* unity that runs through all of the different phases of the historical becoming and the teleology of philosophy and the sciences (§15). The things taken for granted should be viewed as prejudices. These are the obscurities arising out of a sedimentation of tradition. The genetic investigation is thus meant to allow us to become aware of such prejudices and to enable us to free ourselves of various presuppositions. It is therefore the deepest kind of self-reflection aimed at self-understanding.

The lifeworld is "the intuitive surrounding world, pregiven as existing for all in common" (§§33-34). All of our activities, included our loftiest sciences, presuppose the everyday practical and situational truths of the lifeworld. Our praxis and our prescientifc knowledge in the lifeworld play a constant role in all of our activities. Husserl wants to subject the lifeworld to investigation in its own right. The lifeworld was always there for us, even before science. Human beings even now do not always have scientific interests. We have the intuited, everyday world that is prior to theory and then the various theories that are built up from this basis. One can investigate the meaning and the manner of being of the lifeworld and it is even possible to do this in a 'scientific' manner. This will involve a broader notion of 'science' that does not systematically exclude our subjectivity as human beings situated in history. There is a difference between 'objective' science and science in general (§34). Husserl is suggesting a broader conception of science, and indeed of reason, that is friendly to human consciousness. It will be the business of transcendental phenomenology to investigate the lifeworld and its structures (§36). We can say, for example, that the world is, prescientifically, already a spatiotemporal world. Here there is no question of ideal mathematical points, or 'pure' straight lines, of the exactness belonging to geometry, and the like. It is also already a world in which there is causality. Husserl thus begins to plumb the formal and most general structures of the lifeworld. At this point we begin to enter into transcendental phenomenology. We find, for example, that there is a basic distinction between the world and consciousness of the world. This leads, naturally, to the notion of intentionality. The phenomenological 'reduction' is motivated and brought in to the investigation, and so on.

While Husserl does not focus on logic in this book he does say that the supposedly self-sufficient logic that modern mathematical logicians think they are able to develop is "nothing but naiveté" (§36). Its 'self-evidence' lacks scientific grounding *a priori* in the universal lifeworld, which it always presupposes in the forms of things taken for granted which are never scientifically, universally formulated, never put in the form proper to a science of essence. Only when this radical, fundamental science exists can such a logic itself become a science. Prior to this it hangs in mid-air without support and is so naive that it is not even aware of the task that attaches to every objective logic: that of discovering how this logic is to be grounded by being traced back to the universal prelogical *a priori* through which everything logical achieves its legitimate sense and from which all logic must receive its norms.

7.2 "The Origin of Geometry"

"The Origin of Geometry", included as a supplement to the *Crisis*, elaborates on geometry instead of on natural science as a whole. It contains a number of points that are useful for understanding Husserl's ideas on genetic analysis, the lifeworld and idealization. Husserl says that we can inquire into the meaning of the geometry that has been handed down to us. This geometry continues to be valid with the very same meaning as the earlier geometry, although there have of course been many developments. There is a continuous synthesis through time in which the various acquisitions in geometry maintain their validity. The implicit knowledge that is part of this genesis can be made explicit. Geometric existence is 'supertemporal'. Geometry has an ideal objectivity. The Pythagorean theorem, for example, exists only once, no matter how many times or in what language it may be expressed. It is not like tools or other kinds of products that have a repeatability in many like exemplars. The sensible utterances of the theorem have spatiotemporal individuation in the world like all corporeal occurrences but geometry itself is ideal. Such ideal objects do exist in the world in a certain way, but only in virtue of repeated sensory embodiments. Language itself, Husserl says, is made up of ideal objects. The word 'Löwe' occurs only once in the German language. It is identical throughout its innumerable utterances by any given persons. The idealities of geometrical words and sentences, considered purely as linguistic structures, are not the idealities that make up what is expressed and brought to validity as truth in geometry. The latter are ideal geometric objects, states of affairs, and so on.

How does this latter geometric ideality proceed from its primary intrapersonal origin, where it is a structure within the conscious space of the first inventor's soul, to its ideal objectivity? We see that it occurs by means of language, through which it receives its linguistic living body. How does linguistic embodiment make out of the merely intrasubjective structure the objective structure that is valid as a geometric proposition for all of the future? Here Husserl speaks about how a common language belongs to the 'horizon of civilization'. It is within this common language with its possibilities for communication that the original geometer can express his internal structure. But how does the intrapsychically constituted structure arrive at an intersubjective being of its own as an ideal object, as something that is precisely not a real psychic object? Husserl indicates how it becomes an object for an individual subject, how it becomes something identical over against a manifold of processes and acts. As usual, he emphasizes the repeatability of the processes that give the object and he adds that this repeatability extends beyond one particular subject by way of communication. A repeatably produced structure can become an object of consciousness for several people.

This does not yet fully constitute the objectivity of the ideal structure. What is lacking is the persisting existence of the ideal objects even during periods in which the inventor and his fellow geometers are no longer consciously related to them or are even no longer alive. Husserl argues that it is the function of written, documenting expression that effects a transformation of the original mode of being of the meaning structure. The geometric structure becomes sedimented in this manner. The reader of the written expressions can then make the meaning self-evident again. There is a distinction between passively understanding an expression and making it self-evident by reactivating its meaning. Now in the proliferation of a science like geometry the scientist cannot run through the whole immense chain of groundings and reactivate everything. Meaning is built upon meaning and the earlier meanings must give something of themselves to the later meanings. No building block in this structure is self-sufficient and none can be immediately reactivated by itself. Geometry, as a deductive science, depends on chains of logical inference. One can formulate a law about reactivation here: If the premisses can actually be reactivated back to the most original self-evidence then their self evident consequences can be also. The original genuineness must propagate itself through the chain of inferences, no matter how long it is. Since we are finite beings, however, and since these chains have proceeded down through the centuries, we must say that the law just formulated contains an idealization: the removal of limits of our capacity. In a sense, it involves an 'infinitization' of our capacity.

As in the *Crisis*, Husserl is concerned about losing the deeper meaning of geometry in favor of purely formal logical activities in which there is no reactivation. One can render the concepts involved sensibly intuitable by drawn figures and then substitute this for the actual production of idealities. This is coupled with finding that the method is successful, that it has practical success. None of this, however, amounts to the success of actual insight into the meanings involved.

Husserl responds to two objections to his project of investigating the origin of geometry. First, the presently available concepts and propositions of geometry have their own meanings. Why try to trace geometry back to some undiscoverable Thales of geometry? What does this add? What is its use? In response, Husserl says that it is a ruling dogma that there is a separation in principle between epistemological elucidation and historical explanation, or between epistemological and genetic origin. Epistemology cannot, however, be separated in this way from genetic analysis. To know something is to be aware of its historicity, if only implicitly. Every effort at explication and clarification is nothing other than a kind of historical disclosure. The whole of the cultural present implies the whole of the cultural past in an undetermined but structurally determined generality. It implies a continuity of pasts that imply one another, each in itself being a past cultural present. This whole continuity is a unity of traditionalization up to the present. Here Husserl speaks about unity across difference on a global scale, not just in the case of the unities through difference that arise for us in our own personal cognitive lives. Of course in the latter case too there is a temporality and a 'history'. Anything historical has an inner structure of meaning. There is an immense structural *a priori* to history. It is not merely factual history that we are considering. Husserl in fact argues against the relativistic historicism that he sees around him.

The second objection concerns just this historicism. Historicism claims there is or could be no such historical *a priori*, no supertemporal validity. Every people has its world. Every people has its logic. Husserl responds by pointing out some of the background assumptions that are necessary for factual historical investigation to occur at all. These are things that we must know or assume before we can even get started with any factual historical investigation. In spite of all the indeterminacy in the horizon of 'history', it is through this concept or intention that we make our historical investigations. This is a presupposition of all determinability. But what kind of method can we use to make apparent to ourselves the universal and *a priori* features? We need to use the method of free variation in which we run through the conceivable possibilities for the lifeworld. In this way we remove all bonds to the factually valid historical world. We determine what is invariant through all of the variations. In connection with the origin of geometry, we can start by noting the following invariant essential structure of the surrounding world of the first geometers: that it was a world of things with a bodily character. These bodies had spatiotemporal shapes and material qualities (e.g., color, warmth, hardness, etc.). Certain particularizations of shape must have stood out in connection with practical needs. A technical praxis must have aimed at the production of certain preferred shapes and at the gradual improvement of them. Surfaces of things would have been singled out, more or less smooth, with edges more or less even, and so on. In other words, more or less pure lines, angles, and so on. The perfecting of some shapes would have been desirable for certain practical purposes. Estimates of magnitudes would also have been perfected gradually. Techniques for measurement would have improved. This is not yet pure geometry. Rather, it provides materials for the idealizing cognitive acts that would lead to pure geometry.

Husserl says that only if the necessary, general content (invariant through all conceivable variation) of the spatiotemporal sphere of shapes is taken into account can an ideal construction arise that can be understood for all future time and thus be capable of being handed down and reproduced with an identical intersubjective meaning. Were the thinking scientist to introduce something time-bound into her thinking, something bound to what is merely factual about her present, her construction would likewise have a merely time-bound validity or meaning. This meaning would be understandable only to those who shared the same merely factual presuppositions of understanding. This, however, is not the nature of geometry.

8 EXPERIENCE AND JUDGMENT (1939)

The full title of this work is *Experience and Judgment: Investigations in a Genealogy of Logic*. The book was prepared by Husserl's assistant Ludwig Landgrebe from a number of Husserl's manuscripts and was published soon after Husserl's death. It continues the line of thought concerning the origins of logic that Husserl had discussed in *Formal and Transcendental Logic*, although we have seen that in its general outlines this approach was present in Husserl's earliest work on logic and mathematics. Apart from the fact that it carries out the investigations of the origins of logic (e.g., for different forms of judgment) more thoroughly than his other books, *EJ* also includes some shifts in emphasis along with a few new and distinctive ideas. I will indicate some of the new or distinctive ideas and mention a couple of noteworthy examples of Husserl's genetic analysis.

EJ is divided into three parts. Part I is entitled "Prepredicative (Receptive) Experience". It begins with the data of everyday practices and 'passive' perceptual experience. Husserl wishes to exhibit at this level the prepredicative or 'prejudgmental' conditions for predication or predicative judgment. It is in the structures uncovered at this level that the forms of judgment that are part of logic are founded. Husserl begins with the general structures of receptivity, including the passive synthesis of data of sensory experience and the nature of association. Retention and protention, memory, anticipation, the horizon of experience, and the unity of conscious experience are already involved at this level. The structure of intentionality, with its distinction between mere intention and fulfillment is of course also present, as are many other structures. Already at this level he discusses the origin of negation (§21a). Let us pause over this as one example of his genetic analysis. Husserl discusses a situation in ordinary perception in which we have observed a ball as uniformly red over a course of perceptions. At a particular stage, however, a part of the back of the ball is revealed as green. Thereby a consciousness of 'otherness' emerges that nullifies the original intention "uniformly red". The anticipation associated with "uniformly red" is disappointed or frustrated. A conflict arises among some of our intentions. This is the original phenomenon of negation, of the "other". Thus, negation is not in the first case the business of an act of predicative judgment but it already appears in the prepredicative sphere of receptive experience. It then occurs in any other kind of intending, object-positing consciousness, including all of those forms that are founded on basic sense experience. Negation, Husserl says, presupposes normal, original object constitution, i.e., normal perception. Normal perception must be present in order to be modified. Husserl therefore says that negation is, in the first instance, a modification of consciousness. There must also be an overarching unity in the phases of consciousness in order for the 'not' to emerge. There is a unity in which intentions are directed against one another. When we encounter $\neg P$ in logic we must evidently view it as expressing a modification of consciousness, at least so far as its origins are concerned. (Under the influence of Husserl, Oskar Becker and Arend Heyting have discussed negation in a vein that is similar to these comments. See, e.g., [Heyting, 1983].) Immediately after this section on negation Husserl discusses the origins of doubt, possibility, and probability.

The apprehension of pieces and dependent parts of wholes is already covered at this level and this leads Husserl to discuss more generally the apprehension of relations and the foundation of this apprehension in passivity. Relations of connection as well as relations of comparison are discussed.

Part II of *EJ* is entitled "Predicative Thought and the Objectivities of the Understanding". In this part Husserl investigates the beginnings of predicative judgment in the pregiven elements of experience. At this level we are dealing with the ego in its more 'active', goal-directed mode of acquiring knowledge. The distinction between the passivity or receptivity of sensory experience and the activity or spontaneity of understanding is of course reminiscent of Kant. The 'objectivities of understanding' arise from acts of categorial judgment and form the structures that are at the center of logic. Husserl starts with the general structures of predication and the genesis of the most important categorial forms. Predication in its simplest form is a two-membered process and from there Husserl works up to more complicated forms. One of the first such forms is the form in which we use the expression "and so on". We have seen some of Husserl's comments on this already. The origin of judgments of identity is also discussed.

One of the distinctive features in this part of the book is Husserl's discussion of the distinction between states of affairs (Sachverhalten) and situations (Sachlagen). States of affairs are the objective correlates of judgments. They are not apprehended in simple receptivity. Rather, they are objects of a new kind that can only occur at the higher level of understanding at which we first have judgments. Hence, they are to be referred to as 'syntactical' or 'categorial' objectivities. States of affairs, as we have seen, are also not to be identified with the senses or propositions expressed by sentences. Husserl now says that what corresponds in receptivity to states of affairs are 'relations' or 'situations' (§59). Situations constitute something identical that is explicated in different ways in such a manner that equivalent predicative judgments refer to one and the same situation as an intuitively given fact. Every situation involves or gives rise to several states of affairs. Identical situations can be explicated in different states of affairs. Suppose, for example, that we consider the state of affairs "a is larger than b" and the state of affairs "b is smaller than a". Here we have the same underlying relation or situation correlated with two states of affairs. Similarly, 'a is part of b' and 'b contains a as a part' can evidently be viewed as two states of affairs founded on the same underlying situation or relation. (See also §48 of Investigation VI of *LI* for an earlier intimation of these 'relations' or 'situations'.) Situations are passively preconstituted foundations for states of affairs. In receptivity we do not yet have situations as objects. To the same situation there can correspond two or more states of affairs and to the same state of affairs there can correspond two or more senses or propositions. Equivalent propositions correspond to the same situation. In such cases the states of affairs that correspond to equivalent propositions are at least equivalent if not intensionally identical. We must also distinguish equivalence with respect to truth value from equivalence based on sameness of situation. See, e.g., [Rosado-Haddock, 2000].

The constitution of sets in productive spontaneity is also discussed in this part of *EJ* (§61). It is interesting to compare this with Husserl's early views in *PA*. Husserl says here that in the domain of receptivity there is already an apprehension of plurality in the act of collectively taking some things together (§24d). This is a retaining in awareness of one thing in the apprehension of the next. This taking together, however, does not yet have *one* object, e.g., the pair as an object. We can, however, direct our awareness toward the pair itself. Husserl says that in order for the plural explication of two things a and b to become a set a turning of regard is required. As long as we carry out a merely collective assembly we have only a preconstitutued object, a plurality. Only in a retrospective apprehension (Rückgreifen) do we have as an object a plurality as a unity, as a set. A syntactical objectivity is preconstituted in spontaneity but only after it is completed can it become a theme, can it become an object in retrospective apprehension. It is possible at any time to make what is thus preconstituted into an object, the subject of a judgment. The set is thereby itself constituted as an identifiable object. It can be identified in many modes of givenesss, it can be given in ever renewed identifications, it can enter as subject into new connections of judgment, and so on. Sets can stand in various relations to other sets, e.g., subsets, intersections, and so on. Sets can then be collected together with other sets and we can thereby obtain sets of higher order. Husserl says that every set, however, finally leads to ultimate constituents that are no longer sets. It belongs to the idea of such a set that in its first giveneness there is already a pregiven multiplicity of particulars. Every set, Husserl says, must be conceived *a priori* as capable of being reduced to ultimate constituents, to constituents that are themselves no longer sets. Thus, a set is an objectivity, preconstituted by an activity of collecting that links distinct objects to one another. The active apprehension of this objectivity consists in a simple reapprehension or grasp of that which has just been preconstituted. There are no original passively preconstituted sets. Passivity can only create preconditions. Sets are totalities of a higher level that should not be confused with the sensuous wholes given to us in receptivity. The members of a set are not related to it as the parts of a sensuous whole are to the whole itself. Here we do not have a synthesis of partial coincidence that we have between sensuous wholes and their parts. The members of a set remain in a certain way external to one another. Their form of connection is not sensuous but is 'syntactical' or categorial. It is the form of 'being collected'. We can collect anything and everything we please. There need not be the kinds of conditions of homogeneity that we find in sensuous wholes and parts.

After spelling out yet again some of the differences between the constitution of objects of the understanding, like sets or states of affairs, and the constitution of objects of receptivity, Husserl considers the difference in the temporality of both types of objects (§64). Indeed, the essential difference in the modes of being of these objects is a difference in their temporality. First, immanent (or internal) time is taken to be the form of givenness of all objects. Husserl, as usual, distinguishes internal time from 'objective' time and in these sections he briefly discusses the constitution of objective time. All lived experiences, however, are constituted in immanent time and it is the form of givenness of all objects intended in our experience. It is not as if there is an 'in itself' for objects given to us in our experience that is without relation to time. The necessary relation to time is always present. The real objects given to us in receptivity, however, are always given as themselves temporal objects. They have a beginning and an end in time, and a duration through time. Temporal predicates attach to them. Irreal or ideal objects, however, are omnitemporal. A proposition or sense, for example, is given in a temporal act that has a determined temporal position or it is given in several such acts. The judging in which it is given has a temporal duration as a cognitive process. The proposition itself, unlike a real object, has no duration in time. It is not individuated in an objective point or stage in time. A proposition is the same at all times. It sustains no temporal differentiation. It has no extension or expansion in time. It is contingently in time insofar as it can 'be' the same at any time. Irreal objects make their spatiotemporal appearance in the world but they can appear simultaneously in many spatiotemporal positions and yet be numerically identical. They can be given to the same subject at different times or to many different subjects at the same time or at different times.

Such objectivities of the understanding, upon making their appearance or upon being 'discovered' in the spatiotemporal world, can be returned to again and again. Afterwards, we say that even before they were discovered they were already 'valid'. Or we say that they can be assumed to be producible at any time. Similarly, we say 'there are' mathematical and other irreal objects that no one has yet constructed. Their existence is revealed only by their construction (the experience of them) but the construction of those already known opens in advance a horizon of objects capable of being further discovered, although still unknown. Husserl says that as long as they are not discovered by anyone they are not actually in spatiotemporality. As long as it is possible that they never will be discovered it may be that they will have no reality in the world. Once they have been actualized or realized, however, they are also localized spatiotemporally, albeit in a way that does not individualize them.

The timelessness of objects of the understanding is thus a privileged form of temporality, a supertemporality. Husserl now says that such objects are omnitemporal. Omintemporality just is a mode of temporality. In his earlier work Husserl had not characterized irreal objects as omnitemporal. Instead, they were usually described as atemporal or non-temporal (unzeitlich). Now, however, objects or truths of mathematics and logic are not taken to be somehow outside of time. Objects are given to us as synthetic identities. The sense 'identical object' is constituted on the basis of different partial experiences. An identity synthesis presupposes a temporal structure. Thus, ideal objects have temporal being, but in the sense of 'being at all times'. This is compatible with the idea that, in principle, we must have experiential access to such objects. By placing them completely outside of time it is not clear how we could have access to them.

Husserl goes on to distinguish 'bound' from 'free' idealities. What is irreal is founded with regard to its spatiotemporal appearance in a real thing but it can appear in different realities as identical. Now 'cultural objectivities' satisfy this condition. For example, Goethe's *Faust* is a cultural objectivity that is irreal. It is found in any number of real books that are exemplars of *Faust*. Another example of such an irreal objectivity is a civil constitution. A civil constitution has an ideality since it is an expression of a national will that is repeatable at different times and places, is capable of being reactivated, and can be understood and identified by different people. Cultural idealities, however, are not free idealities. Free idealities are bound to no specific territory. They have their territory in the totality of the universe and in every possible universe. They are valid once and for all. Systems of mathematics and logic, along with all pure essential structures, consist of free idealities. Bound idealities do not have these characteristics. They carry the meaning of being of the real world with them. They are bound to some specific territory, to Earth, to Mars, etc.

Part II of *EJ* concludes with some considerations about the origin of the modalities of judgment: predications of existence and predications of actuality, doubt, conjecture, modes of certainty, and questions and answers.

Part III of *EJ* is called "The Constitution of General Objectivities and the Forms of Judging 'In General'". It continues the analysis of forms of judgment, moving up to judgments that are concerned with general, conceptual thought. At this level we are dealing with universals as the types under which objects are known and classified. It is here that we attend more fully to scientific knowledge as knowledge that can be separated from given situations or experiences, communicated and made available to everyone. Husserl starts with the constitution of empirical generalities. It is in this context that he first discusses the original constitution of universals. We then consider levels of generality and arrive at the distinction between material and formal generalities. At this point there is a discussion of "the acquisition of pure generalities by the method of essential seeing" (Wesenserschauung). Husserl again discusses the method of free variation in imagination (§§87 - 93). This part of the book concludes with a discussion of universal judgments, judgments in the mode 'in general', as acts of judgment at the highest level of spontaneity.

EJ thus provides something that was projected in Husserl's earlier books on logic: a full-scale treatment of the origins of logic in receptive or prepredicative experience.

9 CONCLUSION

Many of the ideas we have discussed in this chapter call for clarification, further analysis or development. The secondary literature provides some of this but there are also various controversies about Husserl's views on logic and there are a number of problems of interpretation and emphasis. Some of the disagreements center around the different positions that Husserl already took at different stages of his career. One can see, in any case, that it would be very much in the spirit of phenomenology to subject Husserl's ideas to the kind of criticism that might lead us closer to the ideal of truth that governs all scientific investigations.

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ALGEBRAICAL LOGIC 1685–1900

Theodore Hailperin

INTRODUCTION

The phrase 'algebraical logic', as applied in this chapter, does not refer to a *kind* of logic but to a *style* of doing logic, a style in which concepts and relations are expressed by mathematical symbols—for example rendering 'All A is B' by the equation $A = AB$— so that mathematical techniques can be applied. Here mathematics shall mean mostly algebra, i.e., the part of mathematics concerned with finitary operations on some set. It was unquestionably significant for the development of logic that operations on logical terms turned out to have a high degree of analogy with algebraic operations on numbers. On an even more fundamental level was the analogy with algebra as a formal language with symbols subject to definite rules.

1 EARLY EFFORTS[1]

An association of algebraic symbols and their properties with those of logic would hardly have been conceivable before the 16th century. Systematic use of letters for numbers in general, and symbols for operations on them, were a development of the 16th and 17th centuries. Although Aristotle's theory of the syllogism used letters of the alphabet for arbitrary general terms, i.e., used variables, there was no notion of the composition of terms, hence no notion of an operation on terms. An early, perhaps the earliest, mention of such an algebraic-like operation occurs in Jakob Bernoulli's *Parallelismus ratiocinii logici et algebraici...* (1685).

In this 'open disputation' Bernoulli alludes to a parallel between compounding general terms (e.g., *Virtus & eruditio*) and an indicated algebraic literal sum $a+b$; likewise that of the logical operation of removing a term from a compound and the algebraic difference. He also remarks on the parallel between algebraic and logical inferences, citing the examples:

$$\begin{array}{rlrlrl} & a=b & & b=a & & a=b \\ & c=a & & c=a & & a=c \\ \text{therefore} & c=b & \text{therefore} & c=b & \text{therefore} & c=b \end{array}$$

[1]Portions of this section, and of §2 and §6, were presented in a talk at a conference held at Lausanne 26–27 September 1997. The talk, "Algebraical Logic: Leibniz and Boole", was printed in the collection *A Boole Anthology*, ed. James Gasser, (2000), 129–138. The author wishes to thank its publisher, Kluwer Academic Publishers, for kind permission to incorporate this material in the present volume.

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as well as (using Bernoulli's symbols '$\sqsubset$' and '$\sqsupset$' for 'less than' and 'greater than')

$$\begin{array}{rlcrlcrl} & a \sqsubset b & \qquad & & b \sqsupset a & \qquad & & a \sqsubset b \\ & c = a & & & c = a & & & a = c \\ \text{therefore} & c \sqsubset b & & \text{therefore} & c \sqsubset b & & \text{therefore} & c \sqsubset b \end{array}$$

which, he points out, correspond respectively to the first, second and third syllogistic figures.

These few brief remarks of Bernoulli's are in marked contrast to what Leibniz had accomplished by that time. Unfortunately Leibniz's achievements were effectively unknown, residing in manuscripts that were not published until the 19th and early 20th centuries. By then symbolic logic had developed independently of him. A proper appreciation of what Leibniz had achieved can only be obtained by entering to some extent into the details of his work.[2]

LEIBNIZ

To begin with we shall present material based on Leibniz's lengthy manuscript of 1686, *Generales Inquisitiones de Analysi Notionem et Veritatum (Couturat 1903*, 356–399). Although "analysis of concepts and truths" is featured in Leibniz's title we shall be mostly ignoring these topics as not being germane to our theme of logic in algebraic dress; similarly for the extensive discussions pertaining to grammatical , philosophical and metaphysical matters. Also we shall not reproduce, interesting though it may be, the gradual transition as Leibniz progresses with the writing of his paper, from an essentially verbal form of logic to one quite algebraic in character. Emphasizing this latter form we shall present it as a list of logical principles, much as Leibniz himself did in two short notes (*Couturat 1903*, 235–237, 421–423) written some four years later. We preserve historical ambience by stating these principles as Leibniz did, i.e., using his terminology and language (translated). [3] We depart inessentially from Leibniz in using '$\neq$' for his 'non =', 'not-A' for his 'non A' and in using single quotes about expressions being mentioned. We precede our list with some explanatory remarks.

In his paper Leibniz's efforts were, along with much else, directed to the construction of a logical calculus. His concept of a calculus was quite up to present day standards, but his conception of what constitutes logic was, from a modern perspective, quite limited—at least in its formal aspects—hardly more than the Aristotelian-medieval scholastic syllogistic theory of general terms. He believed

[2]For a wide ranging, in-depth account of Leibniz's *mathesis universalis* and its influence on the development of formal logic, see *Peckhaus 1997*.

[3]For our English translations we rely on *Parkinson 1966*. Our citations, however, will be to the original Latin (mainly *Couturat 1903*). The corresponding English translations are readily found in *Parkinson*, which includes in the margin the page numbers of the Latin original. In its Introduction the Parkinson book has a detailed summary and evaluation of Leibniz's work in logic.

that hypotheticals could be reduced to categoricals. Aware of the difficulties syllogistic theory had with relations, he thought these could be avoided by use of relative terms; moreover that one could treat all types of propositions in terms of universals, i.e., sentences of the form '[all] A is B'. He knew, and gave examples to that effect, that terms could be interpreted intensionally as a composition of properties (represented by terms), as well as extensionally as a collection of individuals. Habitually, though, he chose to express himself intensionally. For example, his 'A is B' was often alternatively written 'A contains B', meaning that when A and B are analyzed into a composition of terms, those of B would be among those of A.

In the logic that preceded Leibniz there was no systematic meaningful use of symbols other than that—originating with Aristotle—of letters for arbitrary terms. One of Leibniz's innovations was the introduction of the relation 'same as', or 'coincides with' between terms which, after awhile, gets replaced without comment by the symbol '$=$'. Another is that of representing the composition of two literal terms A and B by the juxtaposition 'AB'. He gives no name to it, though on occasion it is referred to as multiplication. If 'A' is 'rational' and 'B' is 'animal' then 'AB' is 'rational animal'. A third innovation is the notion of the ***privative*** of a term A, symbolized 'not-A'. There isn't much explanation. He says not-(not-A) is the same as A, and that A is ***positive*** if it is not a not-Y of any kind, assuming that Y is not not-Z, and so on. "Every term is understood to be positive unless advised that it is privative." Further he says *non-entity* is the privative of everything but, significantly, although he takes 'A not-A' to coincide with 'non-entity', he has no expression for 'everything'. A fourth novelty is the introduction of indefinite or undetermined terms for which Leibniz uses letters from the end of the alphabet:

> For by the sign Y I mean something undetermined, so that BY is the same as 'some B', ... Thus 'A is B' is the same as 'A is coincident with some B', or $A = BY$.

The following is our compilation of logical principles abstracted from Leibniz's *Generales Inquisitiones* and the two later notes relating to it. Although the order of arrangement is ours, the language (Englished) is Leibniz's. The title is taken from the first of his two notes.

Primary Bases of a Logical Calculus

(*1) $A = B$ means that one can be substituted for the other, B for A or A for B, i.e., that they are equivalent.

(*2) $A = A$. [Add: If $A = B$, then $B = A$. If $A = B$ and $B = C$, then $A = C$.]

(*3) $A = AA$.

(*4) $A =$ not-(not-A).

(*5) $AB = BA$. [Add: $(AB)C = A(BC)$]

(*6) $A = AB$ and not-B = not-B not-A coincide [are equivalent].

(*7) That in which there is [has as a component] B not-B is a non-entity or false term; for example the terms AB not-B would be a non-entity. [Since entities A, AB, etc., are not non-entities, Leibniz uses this principle to justify having $A \neq AB$ not-B and $AB \neq ACB$ not-B.]

(8) If $A = B$, $AC = BC$.

(9) $B \neq$ not-B; more generally, $AB \neq C$ not-(EB). [Erroneous. Delete the 'E' and use $AB \neq C$ not-B.]

(10) $A \neq B$ and $B \neq A$ are equivalent.

(11) $A = B$ and not-A = not-B are equivalent.

(12) not-B = not-B not-(AB).

(13) $A \neq B$ not-A.

(14) If $A = A$ not-B, then $A \neq AB$.

(15) If $A = B$, it follows that $A \neq$ not-B.

(16) If $A = AB$, one can assume a Y such that $A = YB$.

(17) If $A = YB$, it follows that $A = AB$.

Commentary on Leibniz's Primary Bases

Re (*1). In his paper Leibniz explains:

> A coincides with [i.e., $=$]B if one can be substituted in place of the other with truth being preserved, or if, on resolving each of the two by substitution of values (or definitions) in place of terms, the same terms appear on both sides [of the equation]—the same I mean formally .... For truth is preserved by changes made by substituting a definition in place of a defined term, or conversely. Hence if A coincides with B, B also coincides with A.

In this passage Leibniz is, apparently, describing a definition of $A = B$. However as employed in his formal calculus, it is a rule of inference to the effect that if $A = B$ then $S[A]$ is equivalent to $S[B]$, where $S[A]$ is a sentence and $S[B]$ one obtained from it by replacing occurrences (some or all) of A by B.

Re (*2). This expresses the reflexive property of $=$. In his paper Leibniz states what we now refer to as the symmetric and transitive properties of $=$, but using the verbal "coincides with". In a later paper (*Gerhardt 1890*, 236) they are formally stated with $=$

Re (*3). Here we have, more than a century and a half before Boole, a statement of the idempotency property of logical product.

Re (*4). The comparable statement to this in Boole's class calculus would be

$$x = 1 - (1 - x).$$

Surprisingly, despite its logical significance there is no explicit statement of it in Boole's writings.

Re (*5). This expresses commutivity of Leibniz's product. The associativity, though implicitly used by Leibniz, is not mentioned. The absence is understandable- not until the 19th century (W. R. Hamilton in 1844) did recognition of its need as a formal postulate in (numerical) algebra appear.

Re (*6). Viewed extensionally, this expresses the equivalence of A's being contained in B with the contrapositive, not-B's being contained in not-A.

Re (*7). Although Leibniz uses 'term' to refer to either an entity (possible term) or a non-entity (i.e., an impossible or contradictory term), letters A, B, $C, \ldots$ at the beginning of the alphabet, or concatenations of such letters, always refer to possible terms. No possible term is the same as an impossible term. Thus he has $A \neq CB\,$not-B or, more generally, $AB \neq ABC\,$not-B.

Re (8). This is a simple consequence of (*2) and (*1). With regard to its converse Leibniz remarks:

> But it does not follow that, because $AC = BC$ therefore $A = B$. For if [we take] $A = BC$ the result [conclusion] by (*3) would be $AC = BC$ [correct this to $BC = B$].

This apparent slip-of-the-tongue of Leibniz's occurs in both of his notes and is uncorrected by Couturat. Even with this correction the proof is not cogent. Leibniz wishes to establish the invalidity of

$$\text{If } AC = BC, \quad \text{then } A = B.$$

He takes A to be BC, resulting in the antecedent becoming

$$(BC)C = BC,$$

which by [associativity and] (*3) is valid. But the consequent becomes

$$BC = B,$$

which has not been shown to be invalid. Leibniz must have later realized this and, as we shall presently see, filled the lacuna, not by a model-theoretic counter-example, but by adding a postulate to the effect that to each term there is a term not in it.[4]

[4]Postulate 1 in XX below.

Re (9). Originally Leibniz's manuscript had

$$AB \neq C\,\text{not-}B \tag{1.1}$$

instead of $AB \neq C\,\text{not-}(EB)$. Couturat (*1903*, 422) remarks in a footnote that the insertion of the 'E' in the formula and in the proof was an after-thought of Leibniz's, but says nothing further. The insertion renders the formula invalid, and Leibniz's proof fallacious. The fallacious proof concludes with "Therefore, ..., $AB = ABC\,\text{not-}(EB)$; which is absurd by article 9 [our (*7)] for AB would be a false term, i.e., implying a contradiction." However, contrary to what Leibniz says, $ABC\,\text{not-}(EB)$ is not a contradictory term, but would be if the 'E' were deleted; the proof then does become a correct proof of $AB \neq ABC\,\text{not-}B$. The other statement in this principle,

$$B \neq \text{not-}B,$$

is a consequence of (1.1) obtained by replacing A by B and C by not-B and using (*3).

Note that replacing, in (1.1), A by not-B and C by B produces

$$B\,\text{not-}B \neq B\,\text{not-}B.$$

This appears to be in contradiction with (*2). However in (*2) Leibniz is tacitly requiring that A be a possible term; hence substitution of $B\,\text{not-}B$ for it is inadmissible.

Re (10). Immediate from the equivalence of $A = B$ and $B = A$.

Re (11). Easily established by (*2), (*1) and (*4).

Re (12). Immediate from (*6) on replacing A by AB.

Re (13). Replace in (9) B by A and C by B.

Re (14). We reproduce Leibniz's proof:

> For by (*7), $A \neq AB\,\text{not-}B$. Therefore (substituting, from hypothesis, $A\,\text{not-}B$ for A)
>
> $$A\,\text{not-}B \neq AB\,\text{not-}B.$$
>
> Therefore [by the contrapositive of (8)] $A \neq AB$.

Re (15). Immediate from (*1) and (9).

Re (16). A is such a Y.

Re (17). Leibniz's proof:
For if $A = YB$, then

$$AB = YBB[(YB)B] = YB = A.$$

Note the implicit use of associativity in the omitted step $(YB)B$ to $Y(BB)$.

Unquestionably this "logical calculus" of Leibniz's is a remarkable accomplishment. Although developed with a specific interpretation in mind, it is an abstract deductive system in which, from (*1)–(*7) taken as axioms, consequences are formally derived without appeal to meanings. Some explanation is needed to make this clear. In addition to the equality relation '$\alpha = \beta$', there are two operations, the unary 'not-α' and the binary '$\alpha\beta$'. Leibniz has no notation for a variable ranging over terms in general (such as the α and β just used). His variables A, B, $C, \ldots$ are restricted to range over possible terms. If A is a possible term, so is not-A. But if A and B are possible terms it need not be that AB is—e.g., A not-A is not. In (*1) the variables are general, but in (*2)–(*7) unless explicitly indicated as not being so (e.g. B not-B) products are restricted to being possible. Thus in (*5) there is the tacit hypothesis that A, B, AB and BA are possible terms; although one could replace in it B by not-A, the conclusion

$$A\,\text{not-}A = \text{not-}A\,A$$

is 'vacuous', since 'A not-A' is not a possible term. Leibniz didn't formalize his use of restricted variables, i.e., by explicitly introducing the condition, but for that matter, neither do present day mathematicians. Since modern logicians are unpracticed in intensional ways of thinking we present an interpretation of this logical calculus of Leibniz's in more familiar extensional terms.

Let 'term' denote subset of some non-empty set U, let 'not-' denote complementation with respect to U, and let product denote set intersection. Let A, B, $C, \ldots$ range over $\mathcal{P}(U) \setminus \{\emptyset, U\}$, i.e., all subsets of U excluding the empty set and also U. Axioms (*1)–(*6) hold for this interpretation since they are true of sets in general; and (*7) holds, since if AB is a possible set it can't equal ACB not-B, which is the empty set.

Our commentary on (16) and (17) is in need of amplification. In essence Leibniz is using a free variable (i.e., Y) with an understood existential quantifier governing it, corresponding to the common use of a free variable governed by a universal quantifier. Only here the scope of the quantifier is to be taken as narrow as possible instead of as wide as possible (and ranging over possible terms). When so viewed (16) and (17) are then

(16′) If $A = AB$, then $(\exists Y)(A = YB)$

(17′) If $(\exists Y)(A = YB)$, then $A = AB$.

We turn now to Leibniz's treatment of the syllogistic inference forms. He says (*Couturat 1903*, 236) that the universal affirmative (All A is B) can be expressed by $A = AB$, or by $A = YB$ (i.e., $(\exists Y)(A = YB)$); the particular affirmative by AB is an entity, or $A \neq A$ not-B; the universal negative (No A is B) by $A = A$ not-B, or AB is a non-entity; and the particular negative (Some A is not B) by $A \neq AB$, or by A not-B is an entity. Choosing the representations

$$\begin{aligned}
\text{Universal affirmative}: &\quad A = AB\\
\text{Universal negative}: &\quad A = A\,\text{not-}B\\
\text{Particular affirmative}: &\quad A \neq A\,\text{not-}B\\
\text{Particular negative}: &\quad A \neq AB
\end{aligned}$$

Leibniz proceedes to formally prove the syllogistic inference rules:

(i) from a universal affirmative a particular affirmative follows (subalternation)

(ii) from a universal negative a particular negative follows, and

(iii) $A \neq A\,\text{not-}B$ and $B \neq B\,\text{not-}A$ are equivalent (conversion *simpliciter*).

That (i) and (ii) hold for Leibniz shows that he is committed to subject terms of universals having existential import, i.e., are not non-entities. Two proofs of (i) are given, the first by *reductio ad absurdum*, the second the following direct one:

> Or, more briefly: $A\,\text{not-}B \neq AB$ (by (4) [our (9) with A in place of C]); in this I substitute A for AB (for they are equivalent, by hypothesis), and the result will be $A\,\text{not-}B \neq A$. Q.E.D.

Leibniz gives three proofs of (iii). They are not easy to follow, especially since the first one has incorrect references (uncorrected in *Couturat 1903*, and in Parkinson's translation). Here is a polished up version:

PROOF of (iii). By (12) [corrected from the '(9)' of *Couturat 1903*, 236], i.e., our (*6), with not-B replacing B,

$$\begin{aligned}
A = A\,\text{not-}B &\Leftrightarrow \text{not-(not-}B) = \text{not-(not-}B)\text{not-}A\\
&\Leftrightarrow B = B\,\text{not-}A \qquad \text{(by (*4) and (*1)}\\
\text{so that}&\\
A \neq A\,\text{not-}B &\Leftrightarrow B \neq B\,\text{not-}A.
\end{aligned}$$

Especially noteworthy here is the successful treatment of existential import, subalternation (i) being valid. As we shall see in §6, Boole's attempt to justify this inference via his calculus was in error since, unlike Leibniz's, his terms do not carry existential import.

The modern logician immediately notices the absence of disjunctive terms and their relationship to conjunctive terms, and wonders if Leibniz's calculus would really be adequate for all inferences—for Leibniz this would essentially be the syllogisms. If enough axioms are assumed there should be, in principle, no difficulty since we know that negation and conjunction are an adequate set of connectives for the logic of simple terms. Let us look at a couple of examples, something which for some reason Leibniz doesn't do in his papers.

Barbara—if A is B and B is C, then A is C—is simply

$$\begin{array}{rcl} A & = & AB \\ B & = & BC \\ \text{hence} & & \\ A & = & A(BC) = (AB)C = AC. \end{array}$$

Cesare—if no C is B and every A is B, then no A is C—is:

$$\begin{array}{rcl} C & = & C\,\text{not-}B \\ A & = & AB \\ \text{hence} & & \\ AC & = & ABC\,\text{not-}B \\ \text{that is} & & \\ AC & & \text{is a non-entity, or } A = A\,\text{not-}C. \end{array}$$

Hidden in the proof of this last syllogism is what to a modern logician, but not to Leibniz, would be a gap. In his representation of the universal negative Leibniz took it for granted that '$A = A\,\text{not-}B$' and 'AB is a non-entity' were equivalent. In one direction the proof is easy: if $A = A\,\text{not-}B$ then $AB = AB\,\text{not-}B$, so that AB is a non-entity. The converse is not so easy. There is a proof (of its contrapositive) which starts out (*Couturat 1903*, 237):

> Let us see if it can be shown conversely that $A \neq A\,\text{not-}B$, therefore AB is an entity—assuming that A and B are entities. Now if, assuming that A and B are entities, AB were not an entity, then one of them—A or B—must involve the contradictory of that which the other involves. ... Let $A = EC$ and $B = F\,\text{not-}C$.

Thus Leibniz assumes that if AB is a non-entity then there are E and F such that $A = EC$ and $B = F\,\text{not-}C$. Presumably he thinks that this follows from his definition of non-entity. We think it should be postulated. It suffices to assume:

(*18). If AB is a non-entity, then

$$(\exists C)(A = AC \text{ and } B = B\,\text{not-}C).$$

That is, if AB is a non-entity then there is some C which 'separates' A and B. Leibniz's result could then be established as follows:

(19). If AB is a non-entity, then $A = A\,\text{not-}B$.

Proof. Assuming the antecedent of (19), and using (*18), we have a C such that

$$A = AC \tag{1.2}$$

$$B = B\ \text{not-}C \tag{1.3}$$

From (12) above, by replacing in it B by not-C and A by B, we obtain

$$C = C\,\text{not-}(B\,\text{not-}C),$$

so that, on multiplying by A,

$$AC = AC\,\text{not-}(B\,\text{not-}C).$$

Whence from (1.2) and (1.3)

$$A = A\,\text{not-}B.$$

It is easy to show that, conversely, (19) implies (*18).

Leibniz was quite aware of and knowledgeable on the limitations of syllogistic doctrine, e.g., its inability to validate inferences involving relations, or propositions. But instead of considering extensions to accomodate such inferences he tried forcing them into the syllogistic framework. Thus (*Couturat 1903*, 377):

> If, as I hope, I can conceive all propositions as terms, and hypotheticals as categoricals, and if I can treat all propositions universally [i.e., as universals], this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance.

We devote a few paragraphs to this.

By Leibniz's time it had become common to absorb singular terms to general terms, i.e., treat them (more or less) as though they were general terms. Leibniz introduces a classification of terms into 'integral' and 'partial'. A term is integral if without further addition it can be the subject or predicate of a proposition. It is partial if something further has to be added ('obliquely') for an integral term to arise; a 'direct' addition is that of compounding of two integral terms to form another. (The terms 'direct' and 'oblique' apparently come from *Jungius 1638*.) Leibniz says (*Couturat 1903*, 357):

> If we wish to always use in our symbolism just integral terms, we must say, not 'Caesar is like Alexander' but 'Caesar is like the A which is Alexander' or 'like the thing which is Alexander'. So our term will be, not 'like' [which is partial], but 'like the A'.

The *Generales Inquisitiones* does not have any examples of inferences with relative terms, but a year later in a letter to Vagetius, Jungius' editor, Leibniz gives a proof of the proposition (*Leibniz 1768*, 39):[5]

> Painting is an art; therefore he who learns painting learns an art.

[5] Jungius' non-syllogistic example was: Circulus est figura; ergo qui circulum describit, is figuram describit. In the 19th century De Morgan's example was: A horse is an animal; therefore the head of a horse is the head of an animal.

We reproduce Leibniz proof, omitting the accompanying justifications and, for comparison, parallel it with a version in modern terms using the relation 'learns'. The underlining is our insertion.

1. *For he who learns painting learns a thing which <u>is</u> painting.*

$$[x \text{ learns } p \rightarrow (x \text{ learns } t) \wedge (t = p)] \tag{a}$$

2. *But painting <u>is</u> an art.*

$$[p \in A]$$

3. *Therefore he who learns a thing which is painting learns a thing which is an art.*

$$\begin{gathered}[\text{Therefore (since } (t = p) \wedge (p \in A) \rightarrow t \in A), \\ (x \text{ learns } t) \wedge (t = p) \rightarrow (x \text{ learns } t) \wedge (t \in A)]\end{gathered} \tag{b}$$

4. *Further: he who learns a thing which is an art, learns an art.*

$$[(x \text{ learns } t) \wedge (t \in A) \rightarrow (\exists t)((x \text{ learns } t) \wedge (t \in A))] \tag{c}$$

5. *Therefore he who learns painting, [<u>is</u> one who] learns an art.*

[Hence combining (a)–(c), x learns $p \rightarrow (\exists t)((x \text{ learns } t) \wedge (t \in A))$, or

$$\{\, x \mid x \text{ learns } p \,\} \subset \{\, x \mid (\exists t)(x \text{ learns } t \wedge t \in A) \,\}$$

i.e., Learners of painting <u>are</u> learners of an art.]

It may be, as Leibniz believed, that all propositions in his demonstration are of the universal affirmative, subject-predicate form with all terms integral. But, if so, this is obtained at the cost of stretching the meaning of 'universal affirmative'—as indicated by our underlinings, the word 'is' is used with three different meanings: $=$, $\in$, and $\subset$.

With regard to Leibniz's hoped for ideal of being able to conceive of propositions as terms, there are some scattered remarks in *Generales Inquisitiones* but no recognizable formal inferences of a propositional calculus nature. (See our §9 below for Peirce's treatment of this conception.)

Sometime after 1690 Leibniz wrote a pair of studies which should certainly be included in an account of his logical achievements. The studies are titled simply XIX and XX in *Gerhardt 1890*. Originally for the first one Leibniz had as its title "Non inelegans specimen demonstrandi in abstractis", but then crossed it out. Both studies introduce the relation 'same' and 'coincide' as in *Generales Inquisitiones*, with now a symbol for it being consistently used. (Leibniz changes over to '∞' in these papers, but we shall continue to use '$=$'.) Our earlier description and discussion of his '$=$' carries over unchanged to these studies.

As noted above, Leibniz represented the universal affirmative 'A is B' by $A = AB$, or by $A = YB$, Y an indefinite term. He also introduces the reading 'B is in

A' in accordance with the intensional view that the concept B is in the concept A. (E.g., man = rational animal; hence the concept *animal* is in the concept *man*.) In each of the present two papers Leibniz undertakes a formal characterization of the 'B is in A' relation by assuming axioms and postulates governing the binary operation used in its definition, namely when defined as $A = YB$. (As we have noted, the modern logician sees the defining formula as $(\exists Y)(A = YB)$.) But now Leibniz considers the operation to be additive in character, for instead of the multiplication sign he uses that for addition. In the first paper it is '+' and in the second it is '$\oplus$'. The two treatments, moreover, are quite different.

In XIX Leibniz introduces '$A + B = L$' to mean that A *is in* L or *is contained in* it. This is followed by a note of explanation which, however, embodies a confusion. "Even if A and B have something in common, so that both taken together are greater than L, what we have said or shall say here will still hold." He gives the example of a line segment (representing L) divided into two overlapping segments (representing A and B) whose sum $A + B$ is "greater than" L (hence not "equal" to L) yet "it can truly be said that A and B together 'coincide' with L." Apparently Leibniz doesn't realize that in his example it isn't the relation ("equals" versus "coincides") which is different but the interpretation of $+$. Incidently throughout both papers Leibniz's illustrations employ line segments with $+$, or $\oplus$, taken (closely enough) as class union. They are not Venn diagrams as the notion of complement (and universe) is absent. Next Leibniz introduces what we would now refer to as an additive inverse, i.e., subtraction. The definition of $L - A = N$ is a conditional one, requiring that A be contained in L. His two axioms and postulates (with some explanation omitted) are:

Axiom 1. $A + A = A$

Axiom 2. If the same term is added and subtracted, then whatever is constituted in another as a result of this coincides with Nothing [so that $A - A$, $(A + A) - A$, etc. = Nothing].

Postulate 1. Several terms, whatever they may be, can be taken together to constitute one; thus if there are A and B there can be formed from these $A+B$, which can be called L. [i.e., If A and B are terms then $A + B$ is a term.]

Postulate 2. Some term, e.g., A, can be subtracted from that in which it is—e.g., from $A + B$, i.e., L—if there are given the remaining terms, such as B, which together with A constitute the container L. Or, the same being assumed, it is possible to find the remainder $L - A$. [i.e., If L contains A, then $L - A$ is defined and is its additive inverse.]

Leibniz has no formal symbol for his 'Nothing' and merely deletes expressions of the form '$A - A$'. He also has no axioms or postulates justifying algebraic manipulations with '+' and '−'—presumably because he has in mind the analogy with numerical algebra. We don't know whether he had pursued the matter, but if he had he would have run into trouble; for example, the expression $A + A - A$

could be $(A + A) - A = A - A =$ Nothing, or just as well, $A + (A - A) = A$. (In a later section we shall see how Boole handled a logical calculus which had a subtraction.) Rather than continuing with a discussion of this paper we shall switch over to the somewhat more sophisticated XX.

The paper opens with the same definition of '$=$' as in XIX though here Leibniz doesn't neglect to establish its fundamental properties, including its symmetric and transitive properties. And, again, instead of a meta-theorem, the substitutivity of A for B when $A = B$ is established in individual cases. His definition of 'is in' is the same as before, though now the symbol '$\oplus$' is used for the binary operation:

> $B \oplus N = L$ means that B is in L, or that L contains B, and that B and N together constitute or compose L. The same holds for a larger number of terms.

We can only guess at the reason for the change in notation from '$+$' to '$\oplus$'. Possibly Leibniz realized that he had to distinguish it from numerical or geometric addition. A more significant change from XIX is the omission of subtraction, and Nothing is no longer mentioned. In XIX he calls terms A and B 'communicating' if there is a common part in A and B, otherwise 'non-communicating'. In the present paper the term used for this latter notion is 'disparate'. Again there are two axioms and two postulates, but one of the axioms and one of the postulates is different from those of XIX:

Axiom 1. $B \oplus N = N \oplus B$.

Axiom 2. $A \oplus A = A$.

Postulate 1. Given any term, some term can be assumed which is different from it and, if one pleases, disparate, i.e., such that one is not in the other.

Postulate 2. Any plurality of terms, such as A and B, can be taken together to compose one term, $A \oplus B$, or L.

The two axioms and Postulate 2 require no comment. We have some remarks on Postulate 1.

It is mildly surprising that Leibniz didn't think of using a symbol for his formal 'is in' since the numerical inequality, e.g., $3 \leq 3 + 4$, is highly suggestive. On the other hand the use of a less-than-or equal symbol in mathematics was not common until much later. In order to make it easier for the modern reader we shall use '$\prec$' for Leibniz's verbal 'is in'. We do not think this to be a serious distortion of Leibniz's conceptions. Then Postulate 1 can be rephrased as saying:

Given any term A, there is a term B such that

(1) $A \not\prec B$ and $B \not\prec A$.

Incidentally, while $A \prec B$ changes its meaning when $\oplus$ is changed from extensional to intensional reading, that for disparateness is invariant—(1) becomes either

$$\begin{array}{rl} & A \not\subset B \text{ and } B \not\subset A \\ \text{or} & B \not\subset A \text{ and } A \not\subset B, \end{array}$$

which are equivalent.

We state a few "propositions" which are consequences of Leibniz's axioms and postulates.

Prop. 5. If $A \prec B$, and $A = C$, then $C \prec B$.

Prop. 7. $A \prec A$.

Prop. 8. If $A = B$, then $A \prec B$ [so that $A \not\prec B$ implies $A \neq B$].

Prop. 9. If $A = B$, then $A \oplus C = B \oplus C$.

Leibniz illustrates this proposition with an accompanying drawing using line segments (hence an extensional interpretation). The picture depicts a special case with A and C being non-overlapping segments. He also gives the (intensional) example:

$$\left.\begin{array}{l} A \text{ 'triangle'} \\ B \text{ 'trilateral'} \end{array}\right\} \text{coincide}$$

[therefore]

$$\left.\begin{array}{l} A \oplus C \text{ 'equilateral triangle'} \\ B \oplus C \text{ 'equilateral trilateral'} \end{array}\right\} \text{coincide}$$

He notes that this proposition cannot be converted and that "a method for finding an instance of this will be shown below, in the problem which constitutes proposition 23." We shall discuss this presently.

After some routine propositions we come to the pair

Prop. 13. If $L \oplus B = L$, then $B \prec L$.

Prop. 14. If $B \prec L$, then $L \oplus B = L$.

these being simple consequences of $B \prec L$ defined as $[(\exists N)](B \oplus N = L)$.

We then have

Prop. 15. If $A \prec B$ and $B \prec C$, then $A \prec C$.

Prop. 17. If $A \prec B$ and $B \prec A$, then $A = B$.

Prop. 18. If $A \prec L$ and $B \prec L$, then $A \oplus B \prec L$.

Prop. 20. If $A \prec M$ and $B \prec N$, then $A \oplus B \prec M \oplus N$.

and others of a similar nature.

We turn to Leibniz's way of showing the invalidity of

$$\text{If } A \oplus C = B \oplus C, \text{ then } A = B. \tag{1.5}$$

This is established on the basis of

Prop. 23. Given two disparate terms A and B, to find a third term C, different from them and such that $A \oplus B = A \oplus C$.

He defines $C = A \oplus B$ (Post. 2). Then

$$A \oplus C = A \oplus (A \oplus B) = A \oplus B.$$

Moreover, C can't be either A or B since, for example, if $C = A$ then

$$A \oplus B = A,$$

which, by Prop. 13, yields

$$B \prec A,$$

contrary to assumption that A and B are disparate.

But note that Leibniz hasn't quite proved what he set out to do, namely to find an instance where (1.5) fails. What his Proposition 23 shows is that *if* there are disparate A and B, then there is a C (different from A and B) such that (1.5) fails. To complete the disproof of [the general validity of] (1.5) requires an A and B which are disparate. His Postulate 1 does supply a disparate B for any A. Thus the existence of one term would suffice to make the proof cogent. But neither of the two axioms nor the two postulates provide this. The modern logician would, of course, use a model-theoretic counter-example to show the invalidity of (1.5), i.e., a model satisfying the axioms (and postulates) but in which the proposition in question is false.[6]

How is one to evaluate Leibniz's contributions to logic? Directly there were none—his manuscripts were unknown until too late to have any influence. Yet no history of logic would be complete without a description of his accomplishments. Noteworthy among these are his examples of calculi showing that (parts of) logic could be mathematically treated. While the calculi were unknown, his idea of such a treatment had been promulgated by him in his writings and many letters. It is this idea which could be considered to be his one tangible contribution to the development of logic. It was picked up by a number of logicians, e.g., Segner, Ploucquet, Lambert and Holland, who tried their hand at it. Descriptive accounts of their work may be found in the historical portions of *Venn 1894* and *Lewis 1918* (or, *1960*), and in the histories of logic *Jørgensen 1931* and *Styazhkin 1969.* We here list a few isolated items to illustrate the role that the algebraic analogy played in their thinking.

[6]We take this opportunity to point out that the English translation of XX in *Lewis 1918*, and in the Dover version *Lewis 1960*, omits Postulate 1—only Postulate 2 (so labelled) appears. We assume this was an inadvertent omission. *Kneale and Kneale 1962* reproduce Lewis' translation of XX including the omission but change his 'Postulate 2' to 'POSTULATE', apparently not realizing that the Leibniz original had two postulates. The version in *Parkinson 1966* (his Chapter 16.) is complete.

Leibniz's successors

Whereas in his manuscripts Leibniz used the verbal prefix *non-* to form the privative of a term, Segner employed the mathematical minus symbol, stressing—as 100 years later De Morgan, independently, did—that it was indifferent as to whether $-X$ represented a term or its contradictory. He used the symbol '$<$' where Leibniz had used the verbal *est in*; and he also recognized the idempotency of composition of terms.

Criticizing Ploucquet's use of the symbol '$>$' Lambert thought it "ganz natürlich" that $A > B$ should indicate that the concept A had more characteristic attributes (*Merkmale*) than B. This occurs in a letter to Holland of 18 March 1765, which also includes remarks about symbolizing the composition of terms (*Lambert 1968*, 10):

> Die Zeichnung $A > B$ scheint ganz natürlich zu bedeuten, der Begrif A enthalte ausser den Merkmalen des B noch mehrer, so daß man sagen kann: alle A sind B, etliche B sind A. Die Zeichnung ab will sagen: a welches b ist, und da wird a oder b oder auch beydes *adjective* genommen. Die Zeichnung $A+B$ will sagen: A und B zusammengenommen, zusammengesetzt. In so ferne bezieht sie sich auf körperliche Dinge. Es fehlen aber dabey noch die Bestimmungen der Art (*modus*) des Zusammensetzens.

In a responding letter to Lambert, Holland (letter of 9 April 1765) suggests the general formula

$$\frac{S}{p} = \frac{P}{\pi}$$

as representative of any [subject–predicate] judgement. Here S is the subject, P the predicate and p and π are undetermined variable numbers in the range 1 to ∞ (inclusive). Thus

$$\frac{S}{1} = \frac{P}{1}$$

renders 'all S are all P',

$$\frac{S}{1} = \frac{P}{f} \qquad [f \in (1, \infty)]$$

renders 'all S are some P', and so on. Note the symmetrical role which S and P play in this scheme so that, e.g., 'all' and 'some' could apply equally well to either. Lambert thought this idea of Holland's was quite neat ("ordentlich") and pointed out, in a letter dated 21 April 1765, that it was based on considering the subject and predicate taken, not as properties, but as [classes of] individuals. Some 80 years after Holland's letter (the correspondence appeared in *Lambert 1781*) an acrimonious dispute erupted when Sir William Hamilton (the philosopher in Edinburgh) implied that De Morgan had plagiarized *his* 'quantification of the predicate' idea.

As with Leibniz, the logical vision of his 18th century followers hardly extended beyond the subject–predicate form of sentences and the syllogistic doctrine. Yet even in this circumscribed ambit their results fell short of the logical clarity and deductive rigor displayed in the Leibniz manuscripts. Lambert, referred to by one of his admirers as “zweyter Leibniz” came the closest. His principle logical efforts of an algebraic nature are in his *Sechs Versuche einer Zeichenkunst in der Vernunftlehre*, written in the years 1753–56 (*Lambert 1782*, 3–180). A few excerpts will suffice to indicate the tenor of these investigations. The first is from his I. Versuch, the second from the II. Versuch, and the third from the IV. Versuch.

Lambert expresses himself semantically rather than syntactically—he refers, for example, to ‘concepts’ rather than ‘terms’. A concept (*Begrif[f]*) consists of a set (*Menge*) of [characteristic] attributes (*Merkmale*), part *common* (with other concepts) and part *proper* (to itself). Contradictory attributes in a concept are disallowed. Changes in a concept (as a set) can be made by addition (*Zusetzung*) or subtraction (*Absonderung*) of attributes. His symbols include $=$, $+$, $-$; but, unlike Leibniz, he has no listing of their formal properties. Letters $a, b, c, \ldots$ are used for given, $m, n, l, \ldots$ for undetermined, and $x, y, z, \ldots$ for unknown, concepts. The common part of two concepts a and b is denoted by ‘ab’. (Why didn’t Lambert list juxtaposition, i.e. multiplication, among his symbols?) He writes $a - ab$ for the (set of) proper attributes of a, and $b - ab$ for those of b, so that

$$a + b - ab - ab = \text{ the combined proper attributes of } a \text{ and } b.$$

Introducing, respectively, $a|b$ and $b|a$ for these proper attributes he has

$$a|b + b|a + ab + ab = a + b. \tag{1.6}$$

We note here that Lambert’s set addition ‘+’ is multi-set addition, that is, counts the common part twice. In general

$$2xy + x|y + y|x = x + y. \tag{1.7}$$

If y is among [is a set contained in] the attributes of x then xy becomes y, $y|x$ becomes $y - xy$, i.e., $y - y$. Then (1.7) is

$$2y + x|y = x + y,$$

so that

$$y + x|y = x. \tag{1.8}$$

All these algebraic operations are carried out informally—based on the assigned meanings—rather than appealing to axioms or rules. Using (1.8) he shows that the universal affirmative can be written as an equation: For if all a is b then the concept b is contained in the concept a so that

$$b + a|b = a.$$

On replacing $a|b$ by $a - ab$ this becomes

$$b + a - ab = a,$$

and hence

$$ab = b. \tag{1.9}$$

(Note the tacit use of algebraic operations.) That conversely '$ab = b$' implies 'all a is b' is apparent to Lambert since, ab being the common part of a and b, the equation shows that the attributes of b are among those of a.

As we shall see in §6, this informally introduced primitive algebra of $+$, $-$, $\times$ has resemblances to that of Boole's in his *Laws of Thought.* As with Boole no use is made of $\neq$. There are, however, differences:

(i) Lambert's sets are not abstract but are sets of attributes

(ii) there is no listing of algebraic properties justifying the algebraic operations used, and

(iii) there is no universal set, complementation being defined only relatively.

Our next excerpt shows Lambert making a tiny step towards a theory of relations. A relation is considered to be an attribute involving two concepts such that if one is given the other is determined (i.e., is a function). Relations can be formal ("logisch") or material ("metaphysisch"). Formal relations are specified by the internal (logical) structure of the sentence; material relations by external matters. In this latter case he symbolizes 'a is the ϕ of b' by

$$a = \phi :: b, \tag{1.10}$$

corresponding to our present-day functional expression $a = \phi(b)$. He gives the example

$$a = \text{Fire}, \qquad b = \text{Warmth}, \qquad \phi = \text{Cause}.$$

As another example we could cite

$$a = \text{Eldest child}, \qquad b = \text{Parents}, \qquad \phi = \text{First born}.$$

The converse of a relation is written as its reciprocal, so that he has

$$\frac{\phi :: b}{\phi} = b,$$

and hence (1.10) can be 'solved' for b:

$$b = \frac{a}{\phi}.$$

(Note the incongruity of having a special symbol, ::, for applications of a relation to a concept, but the standard algebraic reciprocal for [application of] the converse.) Lambert also introduces powers of a relations:

$$a = \phi :: \phi :: b = \phi^2 :: b,$$

and similarly for ϕ^3, ϕ^4, etc. But other than these two (converse, powers of a relation) no other operation or combination of relations is mentioned, not even the obvious (?) $\phi :: \psi :: b$, corresponding to what is now called the relative product of ϕ and ψ.

Although Lambert's conception of a mathematical treatment of logic is broader than numerical algebra—he refers to it as "Zeichenkunst"—it is still equationally oriented. Much is made of posing problems in the form of equations and solving for an unknown. Our final excerpt shows Lambert converting the categorical sentence forms to equations. The universalaffirmative (All A is B) is written either as

$$(i)\ A = B, \quad \text{or} \quad (ii)\ A > B,$$

depending on whether the converse is also universal or not. He uses the inequality symbol in (ii) since "in diesem Fall ist das Subject weitläufiger oder grösser als das Prädicat".[7] The particular affirmative is rendered by

$$\text{(i)} \quad A < B, \quad \text{or} \quad \text{(ii) the pair} \left\{ \begin{array}{ccc} mA & > & B \\ A & < & nB \end{array} \right.$$

depending on whether the converse is universal or not. In the case of (ii) the (set of) proper attributes of A and B are overlapping so that additional attributes (m, respectively, n) can be adjoined to form the inequalities. The form $A > B$ is converted to the equation $A = mB$ by the same device that Leibniz employed, that is, by introducing an (indefinite) concept m supplementing the attributes of the concept B to make it the same as those of A. Similarly $A < B$ is rendered by $mA = B$; and both $mA > B$ and $A < nB$ by $mA = nB$. To a modern logician these mean respectively $\exists m(A = mB)$, $\exists m(mA = B)$ and $\exists m \exists n(mA = nB)$. Recall that for Lambert no concept has a contradictory (or a vacuous) set of attributes.

For the universal negative (No A is B) Lambert notes that the proper attributes of subject and predicate are mutually exclusive so that removal of the proper attributes of the predicate results in the "symbolization"

$$A > \frac{B}{m}. \tag{1.11}$$

[7]Recall that Lambert is thinking intensionally, e.g., in 'English are European' it takes additional attributes to separate the concept English from European. Apparently the inequality symbol was rarely used at the time—the typesetter for Lambert's book used a capital Roman V turned sideways for the symbol!

(The m as a denominator operates to remove the attributes m from B.) Equally well one could remove the proper attributes of A and have

$$\frac{A}{n} < B \tag{1.12}$$

As equational equivalents of (1.11) and (1.12) Lambert has for both (neither having proper attributes)

$$\frac{A}{n} = \frac{B}{m}. \tag{1.13}$$

He says that (1.13) can be transformed (apparently using customary algebraic operations) to

$$\begin{aligned} A &= \frac{nB}{m}, \\ B &= \frac{mA}{n}, \\ \text{and} \quad mA &= nB. \end{aligned}$$

But this last equation—supposedly an equivalent to a universal negative—is the same as that for the particular affirmative. Lambert's (informal) explanation for this inconsistency is that if

$$\frac{A}{n} = \frac{B}{m}$$

represents the universal negative then mA and nB are impossible concepts.[8] However he provides no formal mechanism to prevent the incorrect inference. It is difficult seeing how this could be accomplished since Lambert's formalism has no negation or negative concepts.

Using his equational forms for the categorical sentences and the schema

$$\begin{array}{ccc} \dfrac{mA}{p} & = & \dfrac{nB}{q} \\ \dfrac{\mu C}{\pi} & = & \dfrac{\nu B}{\rho} \\ \hline \dfrac{\mu n}{\pi q} C & = & \dfrac{m\nu}{p\rho} A \end{array}$$

Lambert presents a general treatment of the valid and invalid syllogistic inferences. It is clear that the treatment, without negation, negative terms or something like Leibniz's non-entity, can't be cogent in all cases.

[8] e.g., if $A =$ Male $=$ XY-chromosomed Human and $B =$ Female $=$ XX-chromosomed Human, then an XX-chromosomed Male and an XY-chromosomed Female are impossible concepts.

2 REVIVAL OF FORMAL LOGIC IN ENGLAND

At about the time Leibniz was writing the manuscripts referred to above as XIX and XX, Aldrich's *Artis Logicae Compendium (1691)* was published in Oxford. It contains a concise account of the standard topics of the time but, except for an original treatment of the valid and invalid syllogistic forms on the basis of six "canons", is otherwise unremarkable. What *is* remarkable is that it (in its various editions) was the last book emphasizing the *formal* aspects of logic that appeared in England for over a hundred years. This was not because of an unchallengeable eminence—highly thought of as it was—but a result of the changed perception of what logic was supposed to be.

What passed for logic among 18th and early 19th century British writers may be described as a mixture of psychology ("ideas", "intellectual faculties"), epistemology ("forming a true judgement of things") and rhetoric—hardly a nuturing environment for producing an algebra of logic.

Whately's *Elements of Logic (1826, 1827)* marked an abrupt change. In it one finds a vigorous defense of logic as an abstract *science* relating to linguistic structure, as opposed to its being an *art* of reasoning (as with Aldrich) or an instrument useful in the search for truth (as with the British empiricists). Even as late as 1843 J. S. Mill's *A System of Logic*, for example, had in its subtitle the description "... being a connected View of the Principles of Evidence and the Methods of Scientific Investigations". The following excerpt will serve to bring out the flavor of Whately's *Elements of Logic*. It comes immediately after his discussion of the syllogism

$$\text{Every } X \text{ is } Y;\ Z \text{ is } X;\ \text{therefore } Z \text{ is } Y,$$

obtained from a verbal form by replacing words by "unmeaning" letters (*Whately 1852*, 35):

> It appears then, that valid Reasoning, when regularly expressed, has its validity (or conclusiveness) made evident from the mere *form of the expression* itself, independently of any regard to the sense of the words.
>
> In examining this form, in such an example as that just given [i.e., the syllogism AAA] you will observe that in the first Premiss ("X is Y,") it is assumed universally of the *Class* of things (whatever it may be) which "X" denotes, that "Y" may be affirmed of them; and in the other Premiss, ("Z is X") that "Z (whatever it may stand for) is *referred* to that Class, as *comprehended* in it. Now it is evident that whatever is said of the *whole* of a Class, may be said of any thing that is comprehended (or "included," or "contained,") in that Class: so that we are thus authorized to say (in the conclusion) that "Z" is "Y".

This quotation illustrates a significant feature of Whately's *Elements*: categorical sentences are interpreted in purely extensional terms. His X, Y, Z denote *classes* of things—no mention being made of ideas, concepts, or attributes. We see Whately validating his syllogistic inference by an appeal to the (intuitively understood) class calculus property

$$\text{If } X \subset Y, \text{ and } Y \subset Z, \text{ then } X \subset Z. \tag{$*$}$$

Actual statement of (*) using a symbol for class inclusion, doesn't occur until much later in the century. Boole, for example, expresses 'All X's are Y's' by an equation '$x = xy$', or '$x = vy$', where 'x' and 'y' are class symbols and 'v' is an indefinite class symbol.

Viewing term-logic extensionally was the norm among British logicians and was taken for granted.[9] Whately's book—one of two mentioned by Boole in the Preface of his *Laws of Thought* as a reference source—is indicative of the logical *milieu* within which Boole's ideas for an algebra of logic were developed. Our next section portrays the algebraic environment.

3 SYMBOLICAL ALGEBRA. OPERATOR ALGEBRA

As with logic, a change in the perception of the nature of algebra—a change likewise of significance for our history—took place in the early part of the 19th century. Although these algebraic ideas are no longer current they were involved in guiding Boole to a workable algebra of logic.

To provide a set of "scientific" principles for algebra, and in particular to provide a rational basis for operating with negative and imaginary numbers, Peacock (*1833*) proposed the idea of a *symbolical* algebra extending that of *arithmetical* algebra.[10] The algebraic form $a - b$, for example, has meaning in arithmetical algebra only of b is not larger than a. In symbolical algebra such restrictions are removed: "Symbolical algebra arises from arithmetical by supposing that the symbols are perfectly general and unlimited both in value and representation, and that the operations to which they are subject are equally general likewise." The "laws" of symbolical algebra are obtained from those of arithmetical algebra by principles such as that

> (i) whatever forms in general symbols are equivalent in arithmetical algebra, are also equivalent in symbolical algebra, that
>
> (ii) whatever forms are equivalent in arithmetical algebra where the symbols are general in form, though specific in their value, will continue to be equivalent when the symbols are general in their nature as well as in their form, and that

[9] A comparison of intension versus extension, and an arguement for the superiority of extension, occurs in *De Morgan 1847*, 234-235.

[10] According to *Dubbey 1978*, Chapter 5, Peacock was anticipated by his friend Charles Babbage who, in unpublished essays, had earlier expressed very similar ideas.

(iii) the symbols are unlimited in value and representation and the operations, whatever they may be, are possible in all cases.

Illustrative of (i) would be the forms $a(b-c)$ and $ab-ac$; of (ii) would be $a^m \times a^n$ and a^{m+n}, m and n being specific (i.e., positive integers). As for (iii) Peacock gives "... the operation of extracting the square root of $a - b$ is *impossible*, unless a is greater than b. To remove the limitation in such cases, (an essential condition in symbolical algebra) we assume the existence of such a sign as $\sqrt{-1}$; ..." On this basis Peacock 'shows' that $\sqrt{a-b} = \sqrt{-1}\sqrt{b-a}$.

Although "suggested" by arithmetical algebra, symbolical algebra is a "science of symbols and their combination, constructed on its own rules, which may be applied to arithmetic and all other sciences by interpretation: by this means interpretation will *follow*, and not *precede*, the operations of algebra and their results, ...". Symbolical and arithmetical algebra, however, are not independent. The same operation symbols occurring in both, Peacock has an additional principle, namely that the laws of combination of symbols in symbolical algebra should reduce to those of arithmetical algebra "when the [argument] symbols are [sic] arithmetical quantities and the operation symbols are taken to be those of arithmetical algebra."

In *Gregory 1840* we come closer to our present-day idea of an abstract algebra. Peacock's reductive restriction is dropped, and there is no longer a confusion of symbol with what is symbolized. For Gregory (*1840*, 208–209) symbolical algebra is

> ... [a] science which treats of the combination of operations defined not by their nature, that is by what they are or what they do, but by the laws of combinations to which they are subject. ... ... It is true that these laws have been in many cases suggested (as Mr. Peacock has aptly termed it) by the laws of the known operations of number; but the step which is taken from the arithmetical to symbolic algebra is, that, leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws. We are thus able to prove certain relations between different classes of operations, which, when expressed between the symbols, are called algebraical theorems.

Examples of "algebraical theorems" which differ from those of ordinary numerical algebra were not long in appearing. W. R. Hamilton's quaternions, introduced in 1843, do not satisfy the commutative law for multiplication; likewise the outer product of H. Grassmann's 1844 *Ausdehnungslehre*. These two examples are well-known in the history of mathematics. It is not well-known that Boole, in his essay of 1844 for which he received a Royal Society gold medal,[11] introduced in connection with the solution of linear differential equations non-commuting differential operators. (Linear differential operators with *constant* coefficients do commute.)

[11] On a general method in analysis. *Philosophical transactions of the Royal Society*, vol. 134 (1844), 225–282.

4 THE LAST GREAT TRADITIONAL LOGICIAN

Begun in 1839, De Morgan's writings on logic spanned a period of a quarter of a century. In this section we confine our attention to selected items from his 1847 book *Formal Logic* and related publications that preceded it. Discussion of subsequent publications is deferred to a later section (§7).

Despite many insightful and forward-looking innovations the basic framework for De Morgan's logic was still the Aristotelian syllogism with its four categorical sentence forms, each with a copula connecting subject and predicate terms. The notions of *universal* and *particular*, originally applied to such sentences, were assimilated to the terms and referred to as their 'quantity'. One of De Morgan's innovations was an enlargement of the notion of syllogism in which 'quantity' was generalized to have a numerical character—either exact (e.g., m of the A's) or approximate (e.g., *most* of the A's). At about this time Sir William Hamilton of Edinburgh thought of 'quantifying the predicate', a scheme in which not only the subject but also the predicate term could be modified by *all* or *some*. Based on a misunderstanding, Hamilton accused De Morgan of plagiarizing this idea of his.[12] The acrimonious dispute that arose between them, and which became public, had two positive effects: it led to De Morgan's thoroughgoing analysis and clarification of the use of *all (each, every, any)* and *some*, and it stimulated Boole to the writing up of his "almost-forgotten thread of former ideas" in a small book (to be discussed in our next section). We continue with others of De Morgan's innovations.

Particularly noteworthy is his introduction of a *universe* [of discourse], arbitrarily specifiable, together with the removal of any distinction between a *name* and its *contrary*, either of which could be taken as the positive term (*De Morgan 1966*[= *1849b*], 2–3):

> By not dwelling upon this power of making what we may properly (inventing a new technical name) call the *universe* of a proposition, or of a name, matter of express definition, all rules remaining the same writers on logic deprive themselves of much useful illustration. And, more than this, they give an indefinite negative character to the *contrary*, as Aristotle did when he said that not-man was not the name of anything. ...
>
> I hold that the system of formal logic is not well fitted to our mode of using language, until the rules of direct and contrary terms are associated: the words direct and contrary being merely correlative. Those who teach Algebra know how difficult it is to make the student fully aware that a may be the negative quantity, and $-a$ the positive one.

[12] Even with his reputed vast learning in the history of logic, Hamilton could easily not have known of Holland's mention of a similar idea in a letter to Lambert described in our §1. However, neither Hamilton nor De Morgan mentions George Bentham's *Outline of a System of Logic* of 1827 in which the scheme is explicitly stated.

> There is a want of the similar perception in regard to direct and contrary terms.
>
> Throughout this paper, I shall use the small letters x, y, z, &c. for names contrary to those represented by the capitals X, Y, Z, &c. Thus 'every thing *in the universe* is either X or x,' 'No X is x,' &c. are identical propositions.

(The paper from which this is excerpted was published in 1849, though 'read' 9 November 1846.) De Morgan makes it quite clear that the universe can be arbitrarily specified: "… the universe being *man, Britain* and *alien* are contraries; the universe being *property, real* and *personal* are contraries." (*1847*, 55).

The adoption of 'x' in place of 'not-X', while an abbreviative convenience, deprives one of the ability to formally express the involutionary nature of the contrary-forming operator—a prime example of such an involution being the one from Algebra which De Morgan himself mentions. With the small and capital letters De Morgan has to describe in words what is evident when stated as an equation using a symbol for the operator (e.g., Leibniz's $A =$ non-(non-A).

The following display introduces De Morgan's symbolization of the four categorical forms, and also shows equivalences and interrelations expressible with use of capital and small letters and four special symbols:

A (Every X is Y)	$X)Y = X.y = y)x$
O (Some X is not Y)	$X\,Y = Xy = y\,x$
E (No X is Y)	$X.Y = X)y = Y)x$
I (Some X is Y)	$XY = X\,y = Y\,x.$

The four symbols), :, ., (juxtaposition) are borrowed from algebra but are not given any algebraic characteristics. The symbol '=' is used informally to state equivalence of meaning. Surprisingly inappropriate algebraic notation is used to represent inferences. For example the syllogism in Barbara is written[13]

$$X)Y + Y)Z = X)Z.$$

Although he introduces a symbol U for "everything in the universe spoken of" and u for its contrary, denoting "nonexistence", De Morgan declines to use them in syllogistic inferences, considering them to be extreme cases which would only be of interest to mathematicians "on account of their analogy with the extreme cases which the entrance of zero and infinite magnitude oblige him to consider" He says (*1847*, 110–111):

> On looking into any writer on logic, we shall see that *existence* is claimed for the significance of all names. Never, in the statement of

[13] In a paper written in 1858 he admits (*1966*, 87): If A and B be the premises of a syllogism, and C the conclusion, the representation $A + B = C$ is faulty in two points. The premises are compounded, not aggregated; and AB should have been written: the relation of joint premises to conclusion is that (speaking in extension) of contained and containing, and $AB < C$ should have been the symbol."

> a proposition, do we find room left for the alternative, *suppose there should be no such things*. Existence as objects, or existence as ideas, is tacitly claimed for the terms of every syllogism.

This statement, true about writers on logic when De Morgan wrote it, was no longer the case when, in 1854, Boole's *Laws of Thought* appeared. In it the symbols 1 and 0 (Universe and Nothing) function on an equal footing with other class symbols in term-logic inferences. More about this below in §6.

Complex terms are introduced—the conjunctive 'P and Q' being denoted by 'PQ' and the disjunctive 'P or Q' (*or* taken in the non-exclusive sense) by 'P,Q'. Formulation of the so-called De Morgan laws is as a pair of informal rules (*1847*, 118):

> The contrary of PQ is p,q; that of P,Q is pq.

This symmetry of *and* and *or* with respect to *contrary of*—a prominent feature of modern Boolean Algebra—is lost if *or* is thought of, as Boole did in his algebra of logic, in the exclusive sense. (See §6.)

Complex terms are used to form complex propositions such as

$$XY)PQ, \quad X,Y)PQ, \quad XY)P,Q, \quad X,Y)P,Q,$$

but mostly informal development of inferences involving such propositions. In this regard typical is De Morgan's remark (*1847*, 117):

> According $X)P+X)Q=X)PQ$ is not a syllogism or even inference, but only the assertion of our right to use at our pleasure either one of two ways of saying the same thing instead of the other [i.e., 'Every X is P and every X is Q' and 'Every X is P and Q'].

From such simple forms he shows how others, such as

$$X)P \quad \text{and} \quad Y)Q \quad \text{give} \quad XY)PQ,$$

can be deduced. This is essentially Leibniz's *Prop. 20*, listed by us in §1.

We conclude this section with a mention of another of De Morgan's innovative ideas, one which foreshadows his later initiation of a logical theory of relations.

Although recognizing that the copula *is* has a variety of meanings he argues for the primacy of identity, symbolized by =.[14] For him 'A is B', when A and B are singular terms, means that A is identical with B; and 'Every X is Y' means 'Every X is identical with a Y'. He cites three properties of *is* which make its meaning "satisfy the requirements of logicians when they lay down the properties 'A is B'":

> (i) [symmetry] indifference as to conversion, that 'A is B' and 'B is A' must have the same meaning

[14]Yet in a letter to Boole (see §5 below) he replaces Boole's equational form by a universal categorical and its converse so as to carry out deductions.

(ii) [transitivity] 'A is B' and 'B is C' must give 'A is C'

(iii) [exclusivity] *is* and *is not* are contradictory alternatives.

(The modern logician doesn't view (iii) as a property of *is* but rather of *not*.)
To De Morgan (*1847*, 50):

> Every connexion which can be invented and signified by the terms *is* and *is not*, so as to satisfy these three conditions, makes all the rules of logic true.

We can be charitable with De Morgan's claim that properties (i)–(iii) are sufficient to make *all* rules of logic true. Nevertheless, what is noteworthy is the attempt at abstracting those relational properties of the usual copula (identity, for De Morgan) sufficient to produce validity of syllogistic inferences. The notion of a syllogism is enlarged to include forms in which the copula is a relation which has symmetry and transitivity. Some of De Morgan's contemporaries considered this to be importing into logic non-formal or material matters when the copula was not that of identity. De Morgan's retort was that the standard copula *is* is just as much material. This difference of opinion, much refined, is still with us. For example, W. V. Quine (*1970*, 61–62) discussed, and presents reasons for including, identity as part of logic, whereas Kneale and Kneale (*1962*, 742) argue against it.

5 BOOLE'S FIRST ESSAY ON LOGIC

In the history of science and mathematics there are instances of theories, originating in a form either unclear, overly complicated, in part mistaken or even incorrect, but which then eventually become clarified and corrected. Boole's algebra of logic is an example of one of these. As we know—though apparently Boole didn't—he was not the first to attempt a mathematical treatment of logic. In each of these earlier attempts, however, there was no subsequent concatenated development. After Boole's, there was. We devote this section to an abbreviated description of *The Mathematical Analysis of Logic* of 1847, the first of his two major works on logic. It was "written within a few weeks after its idea had been conceived", and Boole wished it to be superseded by his *Laws of Thought (1854)*. Nevertheless the historian finds interest in seeing how ideas develop.

Two features of Boole's first work are quite apparent: (i) a commitment, as in Whately's *Elements*, to interpreting logical terms as representing classes of objects and (ii) a reliance on operator algebra, as formulated by Gregory in the context of Symbolical Algebra. Noteworthy in this connection is Boole's introduction of a novel type of operator, a non-quantitative one: to each class X he associates a "symbol" x which, operating on any class as subject [operand], selects out those elements which are Xs. The universe of all (existing or non-existing) objects is denoted by '1'. When no subject is written, then 1 is understood. Boole writes

$$x = x(1)\ [= x1],$$

with an unmathematical use of '=' since what is on the left side is an operator while on the right is a class (i.e., X).

His first "law" is stated verbally,

> 1st. The result of an act of election is independent of the grouping or classification of the subject.

and is then expressed "mathematically" by the equation

$$x(u+v) = xu + xv,$$

"$u + v$ the undivided subject, and u and v the component parts of it", and x is any selection operator. The symbol '+' makes its first appearance here—to make sense of the equation u and v have to be viewed as mutually exclusive classes. Note also the conflict with his convention that small letters refer to operators. Boole's second law is

> 2nd. It is indifferent in what order two successive acts of election are performed.

having as its "symbolic expression"

$$xy = yx,$$

either side of the equation denoting those elements common to the classes X and Y. His third law is

> 3rd. The result of a given act of election performed twice, or any number of times in succession, is the result of the same act performed once.

Hence

$$xx = x$$

or,

$$x^2 = x;$$

and if performed n times in succession, then

$$x^n = x.$$

To Boole these three laws "are sufficient for the basis of a Calculus". Furthermore, he contends (*1847*, 18):

> ... From the first of these, it appears that the elective symbols are *distributive*, from the second that they are *commutative*, properties which they possess in common with symbols of *quantity*, and in virtue of which [and the principles of Symbolical Algebra], all the processes of common algebra are applicable to the present system. The one and sufficient axiom involved in this application is that the equivalent operations performed on equivalent subjects produce equivalent results*. [Footnote omitted].

The modern reader is surprised to find a mathematician as competent as Boole being unaware that more than what he has mentioned is needed to formally justify the processes of "common" algebra. As we shall see in our next section, he replaces this inadequate justification by a much more plausible one in his *1854*.

Noticeably absent is an elective symbol for the operation of selecting everything from a class, i.e., an elective symbol corresponding to the universe. Algebraically, a natural symbol for it would be '1'; for on letting U denote the universe, then

$$x1U = 1xU = xU,$$

so that, on "understanding" the U,

$$x1 = 1x = x.$$

Possibly this is what Boole was aiming for, and which prompted his using '1' for the Universe, but didn't get it quite right. Absence of a universal selector leads him to a non-cogent reason why the "symbol $1 - x$" determines the class not-X (*1847*, 20):

> The class X and the class not-X together make up the Universe. But the Universe is 1, and the class X is determined by the symbol x, therefore [?] the class not-X will be determined by the symbol $1 - x$.

The categorical sentence forms are expressed by means of algebraic equations whose variables are elective symbols. Thus the universal affirmative, All Xs are Ys, is expressed by

$$xy = x,$$

which is similar to what Leibniz had. But Boole goes further than Leibniz and immediately converts this equation (by "common" algebra) to

$$x(1 - y) = 0.$$

the symbol [elective ?] 0 here making its first appearance without any prior mention. Even more questionable is his treatment of the particular affirmative (*1847*, 21):

> 4. To express the Proposition, Some Xs are Ys.
>
> If some Xs are Ys, there are some terms [rather, objects] common to the classes X and Y. Let those terms constitute a separate class V, to which there shall correspond a separate elective symbol v, then
>
> $$v = xy, \quad (6).$$
>
> And as v [rather, V] includes all terms common to the classes X and Y, we can indifferently interpret it, as Some Xs, or some Ys.

In this passage the definition of V is a conditional one, the condition being that 'Some Xs are Ys' is the case, nothing being said about V when it is not the case. Thus v is not completely defined and hence (6) doesn't provide an equivalent to 'Some Xs are Ys'. Matters are not improved by taking V to be, in either case, the class of objects common to X and Y. For then (6) would have no material content, being tautologically true.

Having convinced himself that categorical sentences (of any of the four kinds) are representable as equations in elective symbols, Boole goes on to show how inferences—immediate and syllogistic—can be carried out by algebraic manipulations. More general types of sentences, also equationally represented, are constructed and inferences relating to them are investigated. We shall be looking into these matters in our next section. But before doing so we bring up for discussion one of Boole's illustrative examples which De Morgan, in a letter to Boole (*Smith 1982*, letter 12), had made the subject of a comparison between their respective methods.

Boole's example (*1847*, 75):

> Ex. Given $x = y(1-z) + z(1-y)$. The class X consists of all Ys which are not-Zs, and all Zs which are not-Ys: required the class Z.

Boole's general procedure for solving elective equations, e.g., $\phi(x, y, z) = 0$, is to *assume* that z is a function of x and y of the form

$$z = vxy + v'x(1-y) + v''(1-x)y + v'''(1-x)(1-y)$$

and then finding for any such ϕ that the coefficients v, v', v'', v''' are given by

$$v = \frac{\phi(1,1,0)}{\phi(1,1,0) - \phi(1,1,1,)} \qquad v' = \frac{\phi(1,0,0)}{\phi(1,0,0) - \phi(1,0,1)}$$

$$v'' = \frac{\phi(0,1,0)}{\phi(0,1,0) - \phi(0,1,1)} \qquad v''' = \frac{\phi(0,0,0)}{\phi(0,0,0) - \phi(0,0,1)}$$

Taking $\phi(x, y, z) = x - y(1-z) - z(1-y)$ and using this general result he obtains

$$z = x(1-y) + y(1-x),$$

that is "the class Z consists of all Xs which are not Ys, and all Ys, which are not Xs; an inference strictly logical". (For comparison, in our next section we obtain this result by the method Boole would have used with his *1854* formulation.)

To make for easier reading we shall describe De Morgan's solution using modern notation. This we can do since his $X)Y$ corresponds exactly to class inclusion $X \subset Y$, his x to class complementation X', his X, Y to class union, and his XY to class intersection XY. (We couldn't do this for Boole's solution since his + and −, let alone division, do not correspond to anything in standard class calculus.)

De Morgan's letter opens with

> I am much obliged to you for your tract [*Boole 1847*] which I have read with great admiration. I have told my publisher to send you a copy of my logic [*De Morgan 1847*] which was published on Wednesday.
>
> There are some remarkable similarities between us. Not that I have used the connexion of algebraical laws with those of thought, but that I have employed mechanical modes of making transitions, with a notation which represents our head work.

De Morgan then goes on to his solution of the Boole example. The "data", written as an equation by Boole, is represented by two propositions (class inclusions)

$$X \subset (ZY' \cup YZ') \tag{1}$$

$$(ZY' \cup YZ') \subset X. \tag{2}$$

From (1) [by contraposition, "De Morgan", and distributivity]

$$\begin{gathered}(Z' \cup Y)(Y' \cup Z) \subset X' \\ (Z'Y' \cup Z'Z \cup YY' \cup YZ) \subset X'.\end{gathered}$$

On deleting $Z'Z$ and YY' which are "non-existent" he has

$$Z'Y' \cup YZ \subset X'$$

or

$$YZ \subset X'$$

and thus

$$YZ \subset X'Y. \tag{3}$$

From (2) he obtains $Y'Z \subset X$, or $Y'Z \subset XY'$. Combining this with (3) produces

$$(YZ \cup Y'Z) \subset (XY' \cup X'Y)$$

that is

$$Z \subset (XY' \cup X'Y).$$

By comparable steps he derives the converse inclusion and hence has the two propositions equivalent to Boole's solution. De Morgan remarks at the conclusion of his derivation: "This is far from having the elegance of yours; but your system is adapted to identities, in mine an identity is two propositions."

The modern logician recognizes the right-hand side of $x = y(1-z) + z(1-y)$ as the symmetric difference $y\Delta z$ of y and z. Using well-known simple properties of symmetric difference he easily has that if

$$x = y\Delta z,$$

then, taking the symmetric difference of each side with y,

$$\begin{aligned} x\Delta y &= (y\Delta z)\Delta y \\ &= (z\Delta y)\Delta y \\ &= z\Delta(y\Delta y) \\ &= z\Delta 0 \\ &= z. \end{aligned}$$

6 BOOLE'S *LAWS OF THOUGHT (1854)*

The title "Laws of Thought" which Boole used for his major work seems nowadays odd for a book on logic. We no longer believe, as Boole did, that principles of logic are revealed by the workings of the mind. Accordingly, in keeping with the theme of this Chapter, when discussing his book we shall confine our attention mainly to its algebraic aspects.

In a significant change from his *1847* the symbols $x, y, z, \ldots$ no longer are operators selecting out classes but stand directly for the classes themselves. The common understanding of the term *class* is extended so as to include singular terms as well as "universe" and "nothing". Not only are the classes which xy and yx represent identified, via an algebraic equation

$$xy = yx, \tag{1}$$

but also the ternary forms zxy, zyx, xyz, etc. Boole is apparently unaware that in identifying these ternary forms a (tacit) use of associativity is involved. Again, for a class x he has the law

$$x^2 = x. \tag{2}$$

The symbols '+' and '−' are introduced as "signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts." His linguistic illustrations make '+' correspond to 'and' and 'or', while '−' corresponds to 'except'. Unlike xy, defined for any two classes, the combinations $x + y$ and $x - y$ are only partially defined: $x + y$ has meaning only if x and y have nothing in common, and $x - y$ has meaning only if y is part of x. On the basis of linguistic examples Boole cites for these notions the properties

$$\begin{gathered} x + y = y + x \\ z(x + y) = zx + zy \\ z(x - y) = zx - zy \\ x - y = -y + x. \end{gathered} \tag{3}$$

He contends that for purposes of logical deduction the only verb needed to form propositions is *is*, or *are*, symbolicaly expressed by '='. Logical deductions are accomplished by algebraic operations on such propositions. Thus from

$$x = y + z \tag{4}$$

("The stars are the suns and the planets") he has by the algebraic rule of transposition,

$$x - z = y \tag{5}$$

("The stars, except for the planets, are the suns"). Validation of the rule comes from affirmation of the "general axiom": if equals are added to, or taken from equals, the results are equal.

In our preceding section we have seen that Boole considered it permissable to employ algebraic techniques (of numerical algebra) on elective symbols, merely on the basis that as operators they obeyed the commutative and distributive [over sums of operands] laws. Here, in his *Laws of Thought*, we have a more sophisticated justification, one based on the idea of a variant algebra. Commenting on the analogy of the formal laws of class symbols with those of Number, all but $x^2 = x$ being true of Number, and that also if x is restricted to being either 0 or 1, he says (*1854*, 37–38):

> Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values 0 and 1.* Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

At the time Boole wrote this there were in existence two variant algebras, that of W. R. Hamilton's quaternions and H. Grassmann's *Ausdehnungsgrösse.* (Three, if a now forgotten one suggested by Hamilton's quaternions, the triple algebra of De Morgan's *1849*, is included.) It is unlikely that Boole was aware of Grassmann's work, but he certainly was aware of Hamilton's, as a brief note by Boole on quaternions appeared in an 1848 issue of the *Philosophical Magazine.*

Boole's bold and innovative conception of an algebra of logic is in need of clarification. For example, is the statement

$$\text{If } xy = 0, \text{ then } x = 0 \text{ or } y = 0, \tag{6}$$

which is true if x and y "admit" the values 0 and 1, a law of Boole's envisioned algebra? If so then from the law

$$x(1 - x) = 0, \tag{7}$$

(derived by Boole from $x - x^2 = 0$) one has

$$x = 0 \quad \text{or} \quad 1 - x = 0,$$

that is,

$$x = 0 \quad \text{or} \quad x = 1.$$

Thus 0 and 1 would be the only values admitted for any x whatever. Clearly (6), allowing as it does for only two classes (*Nothing* and *Universe*), would not be desirable as a law of Boole's algebra. Exactly what his algebra amounts to was investigated in detail in *Hailperin 1976*, 2nd edition *1986*. Here we present an abbreviated summary.

In the conception Boole has in mind no meaning is attributed to some algebraic constructs: 1+1, for example, or in general $x + y$ if x and y represent overlapping classes. Nevertheless he freely uses all algebraic expressions constructable by use of the binary operation symbols $+$, $-$, $\times$, from class variables and the constants 0 and 1. One way to handle such an 'algebra' is to consider incompletely defined operations as partial functions, i.e., functions defined only under certain circumstances, and then have hypotheses, or prove theorems, to rule out undefined situations—as one does with division in real algebra where division by zero is undefined. But this was not Boole's way.

He argues that if the initial algebraic formulas are meaningful ("interpretable") in terms of class notions, and if the end result is also so interpretable then the end result is a valid conclusion even if some of the intermediary steps are not interpretable. He likens this to the use of $\sqrt{-1}$ in trigonometry. Boole provides no explicit listing of algebraic laws or principles but from examples and Propositions which he proves, one can abstract the following. In stating these we have introduced the use of metavariables $A, B, C, \ldots$, replaceable by any formulas constructable by use of $+, -, \times$ from class variables $x, y, z, \ldots$ and the constants 0 and 1.

BOOLE'S ALGEBRA OF $+$, $-$, $\times$, 0, 1

B1. $A + B = B + A$

B2. $A + (B + C) = (A + B) + C$

B3. $A + 0 = A$

B4. $A + X = 0$ has a unique solution for X

B5. $AB = BA$

B6. $A(BC) = (AB)C$

B7. $A1 = A$

B8. $A(B + C) = AB + AC$

B9. $1 \neq 0$

B10. $A^2 = 0$ only if $A = 0$

B11$_n$. $nA = 0$ only if $A = 0$ $(n = 1, 2, \ldots)$

where 'n' in the last formula is an abreviation for the n-fold sum '$1 + 1 + \ldots + 1$' so that $nA = A + A + \ldots + A$, to n terms. As usual $B - A$ is defined as $B + (-A)$, where $-A$ is the unique solution mentioned in B4, so that $A - A = 0$. It is an easy exercise (take C in B8 to be 0 and simplify) to show that $A0 = 0$.

In modern algebra formulas B1–B9 constitute a set of axioms for commutative rings with unit. The additional formulas B10, $\mathrm{B11}_n$ serve to exclude nonzero nilpotents, either multiplicative or additive. (Further details on the nature of this set of axioms can be found in *Hailperin 1986*, §2.2). These are the properties of "common" algebra which Boole uses, some explicitly, some implicitly. For an algebra of logic he adds the idempotency (to give it its modern name) condition:

B12. $A^2 = A$, for A being any one of $x, y, z, \ldots$

Boole expresses this by (2), taking for granted the replacement of x by any other class symbol.

Expressions satisfying the idempotency condition were referred to by Boole as being "interpretable" (in logic). These include the letters $x, y, z, \ldots$, their complements $1 - x, 1 - y, 1 - z, \ldots$, products of these (letters and/or complements), and sums such as $x + (1 - x)y$ and $x(1 - y) + y(1 - x)$. When translating premises into algebraic form he insures that $+$ occurs only between exclusive terms (i.e., terms whose product equals 0). This is accomplished by using either $x + (1 - x)y$ or $x(1 - y) + y(1 - x)$ as rendering 'x or y' depending on whether the 'or' is understood in the inclusive or exclusive sense. His algebraically expressed premises then contain nothing but interpretable (idempotent) terms. It is straightforward to show that the idempotents of Boole's algebra form a Boolean algebra, using for its addition $+_B$, where $x +_B y$ is $x + (1 - x)y$, for its complementation subtraction from 1, and for its product the product of the larger algebra. (Alternatively, they form a Boolean ring with unit if the addition is $+_\Delta$ where $x +_\Delta y = x(1 - y) + y(1 - x)$.) Variables for this algebra would, naturally, be $x, y, z, \ldots$.

However Boole was not aware of Boolean algebra nor, of course, that it would equally as well suffice for the logical inferences of the kind he investigated—excluding, however, those requiring terms with existential commitment. (See the penultimate paragraph of this section.) Instead, to effectuate an algebraic solution to questions of logical inference he employed the "processes of numerical algebra" supplemented with the idempotency condition on class symbols $x, y, z, \ldots$. These processes led, in general, to equations not directly expressing logical content. A number of devices are then introduced to produce an interpretable result. Here it is entire equations, not just terms, that are given an interpretation. We outline the main features of this technique.

A key notion is that of the *development* or *expansion* of a function. For Boole a function is a polynomial, or a fraction whose numerator and denominator are polynomials, in which the logical symbols $x, y, z, \ldots$ appear at most to the first power. In addition to 0 and 1 such functions may contain other constants, e.g,. $2 (= 1 + 1)$. While Boole took it for granted that 2 was neither 0 nor 1, it is readily proved from our $B11_2$, B9, and B3, B4, B9, that, respectively, $2 \neq 0$ and $2 \neq 1$.[15] A function $f(x)$ is said to be developed if it can be expressed in the form

$$f(x) = ax + b(1-x) \tag{8}$$

with a and b free of occurrences of x. Boole shows that

$$f(x) = f(1)x + f(0)(1-x). \tag{9}$$

His proof assumes that $f(x)$ can be written in the form depicted in (8). In this form substituting first 1 for x and then 0 for x produces (9). The assumption, that an $f(x)$ can be expressed as in (8), is readily justified if $f(x)$ is a polynomial—one uses ring properties and linearity to obtain $f(x) = \alpha x + \beta = (\alpha+\beta)x + \beta(1-x)$. The result (9) is readily extended to functions of two or more variables; for example for an $f(x,y)$,

$$\begin{aligned} f(x,y) = f(1,1)xy + f(1,0)x(1-y) + f(0,1)(1-x)y \\ +f(0,0)(1-x)(1-y). \end{aligned} \tag{10}$$

In this form the $f(1,1)$, $f(1,0)$, $f(0,1)$, $f(0,0)$ are called the *coefficients*, and the parts independent of f the *constituents*. Boole believed that expansions of functions fractional in form were, just as for those which were polynomial, equal to their expansions. For the modern mathematician this doesn't make sense since coefficients with zero denominator can arise. However we shall presently see that Boole's interpreted end result can be obtained without recourse to fractional forms.

The interpretation for a polynomial equation, $f(x,y,z,\ldots) = 0$, is obtained in two steps: (i) replace $f(x,y,z,\ldots)$ by its expansion to have

$$a_1t_1 + \ldots + a_nt_n = 0, \tag{11}$$

where the t_i are the constituents on $x, y, z, \ldots$ and the a_i are the non-zero coefficients; then (ii) replace each a_i by 1 obtaining

$$t_1 + \ldots + t_n = 0. \tag{12}$$

This equation, which is meaningful in purely class terms (namely, that each t_i is an empty class), is the logical interpretation Boole assigns to $f(x,y,z,\ldots) = 0$. For more than one premise equation, e.g.,

$$\begin{cases} f = 0 \\ g = 0 \end{cases}$$

the interpretation is to be that for $f^2 + g^2 = 0$, rather than for $f + g = 0$. This is done to prevent unintended cancellations. For example if the premises were

$$\begin{cases} xy = 0 \\ -xy = 0 \end{cases}$$

[15]Interestingly, neither '$\neq$' nor any (greater or less than) inequality symbol appears in the logical portion of *Laws of Thought*. Apparently logical inclusion as a symbolizable notion did not enter into Boole's conceptions.

addition would produce $0 = 0$, whereas squaring and adding produces $xy + xy = 0$, having the interpretation $xy = 0$. In the axiom system B1–B12 it can be shown that (11) and (12) are equivalent and also that $f^2 + g^2 = 0$ and the conjunction of $f = 0$ and $g = 0$ are equivalent.

As a sample inference to illustrate Boole's technique we take the one discussed by De Morgan whose solution we presented in §5. From

$$x = y(1 - z) + z(1 - y)$$

Boole would obtain (by "common" algebra)

$$z = \frac{y - x}{2y - 1},$$

and, on replacing the right-hand side by its expansion, have (using $\overline{x}$ for $1 - x$ and $\overline{y}$ for $1 - y$)

$$\begin{aligned} z &= \tfrac{0}{1}xy + \tfrac{-1}{-1}x\overline{y} + \tfrac{1}{1}\overline{x}y + \tfrac{0}{-1}\overline{xy} \\ &= x\overline{y} + \overline{x}y. \end{aligned}$$

As another example (with two premises) we take the syllogism in Barbara. Representing 'All Xs are Ys and all Ys are Zs' by

$$\begin{cases} xy = x, & \text{or } x(1 - y) = 0 \\ yz = y, & \text{or } y(1 - z) = 0, \end{cases}$$

the combined equation is (using $\overline{y}$ for $1 - y$, $\overline{z}$ for $1 - z$)

$$x\overline{y} + y\overline{z} = 0.$$

Then on solving for x,

$$x = \frac{-y\overline{z}}{\overline{y}},$$

and replacing the right-hand side by its expansion we have

$$\begin{aligned} x &= \tfrac{0}{0}yz + \tfrac{-1}{0}y\overline{z} + \tfrac{0}{0}\overline{y}z + \tfrac{0}{1}\overline{yz} \\ &= \tfrac{0}{0}z + \tfrac{-1}{0}y\overline{z}. \end{aligned} \tag{13}$$

In general, from an equation of the form

$$Ew = F, \tag{14}$$

Boole solves for w, expands the quotient F/E and obtains

$$w = \frac{F}{E} = 1A + 0B + \frac{0}{0}C + \frac{1}{0}D. \tag{15}$$

Here A stands for the sum of the constituents with coefficient 1, B those with coefficient 0, C those with coefficient $\frac{0}{0}$, and D those with coefficients which are neither 1, 0, or $\frac{0}{0}$ or their numerical equivalents, e.g., 0 for $\frac{0}{-1}$. Boole gives arguments why the constituents of A represent classes included in w, B those which are not included, C those which may or may not be, and $D = 0$ is the condition implied by (14) which holds independently of w. It is clear that for any expansion

$$A + B + C + D = 1$$

and that any two of the sums A, B, C, D are mutually exclusive, since any two distinct constituents (on the same set of variables) are mutually exclusive. Moreover, for any two constituents t_1, t_2,

$$t_1 + t_2 = t_1 +_B t_2 = t_1 +_\Delta t_2,$$

the same holding for any two of A, B, C, D in place of t_1 and t_2, so that $+$ can have any one of the three meanings. The interpreted solution for w is written

$$\begin{cases} w = A + vC, & v \text{ arbitrary} \\ D = 0. \end{cases} \tag{16}$$

For the second of the examples the interpreted solution for x would be

$$\begin{cases} x = vz, & v \text{ arbitrary} \\ y\bar{z} = 0, \end{cases}$$

which says that x is included in z, and that independently of what x is, $y\bar{z} = 0$ (obvious from the premises). In modern notation (16) would be written

$$\begin{cases} A \subseteq w \subseteq A + C \\ D = 0. \end{cases} \tag{17}$$

Elsewhere (*Hailperin 1986*, §2.7) we have shown how to make mathematical sense of Boole's introduction of quotients. But to obtain the result that (14) is equivalent to (17)—and hence to verify (for the modern logician) the correctness of Boole's method of doing class calculus—does not require that much complication. Briefly, one first shows that $Ew = F$ is equivalent to

$$(A + B)w = A + D, \tag{18}$$

where A, B, D are as defined above; the definitions can be given in terms of E and F independently of the idea of a quotient F/E. Then multiplying the equation (18) successively by A, B, and D produces

$$Aw = A, \quad \text{i.e. } A \subseteq w, \tag{19}$$

$$Bw = 0 \tag{20}$$

$$0 = D. \tag{21}$$

Now

$$w = 1w = (A + B + C + D)w,$$

so that, using (20) and (21),

$$w = (A + C)w, \quad \text{i.e., } w \subseteq A + C.$$

Thus

$$\begin{cases} A \subseteq w \subseteq A + C \\ D = 0. \end{cases} \tag{22}$$

Conversely, that (22) implies (18) can be readily shown. Finally one needs a metatheorem to the effect that if a quantifier-free formula in Boolean algebra symbols is provable in Boole's algebra (B1–B12) then it is a theorem of Boolean algebra. (See *Hailperin 1986*, 148.)

A comparison of the principal types of mathematical structures which satisfy the two axiom systems is illuminating. For Boolean algebra they are the algebras of subsets of a given fixed nonempty set with the customary operations of set union, intersection and complementation. For Boole's algebra they are the algebras of signed (sub)multisets of a fixed nonempty set. A multiset is like a set but in which repeated occurrences of an element are allowed. In a signed multiset elements may occur negatively (like the dollars in an account that is in debt). Here the operation of addition, Boole's $+$, is (metaphorically) that of dumping the contents of two multisets together into a container. Thus when Boole expands $x+y$, x and y sets, so as to have

$$x + y = 2xy + x\overline{y} + \overline{x}y$$

we read this as saying that the multiset $x + y$ has as its elements those of xy, counted doubly, and those of $x\overline{y}$ and $\overline{x}y$, counted singly. (For more on this topic see *Hailperin 1986*, §§2.1, 2.2.)

In our account of Boole's work we have made no mention of his version of syllogistic doctrine. Since Boole restricts himself to equations it is clear that he would have trouble with nonemptyness (which requires $\neq 0$ or > 0). Thus he renders 'All x's are y's' by '$y = vx$' with v arbitrary, including the possibility of its being empty, and then by multiplication with v obtains '$vy = vx$', read as 'Some x's are y's', but now v has to be non-empty. The erroneousness of $vy = vx$ as a version of the particular affirmative was pointed out in *Peirce 1870*.

Boole's treatment of propositional inferences will be discussed in §9.

7 HOW THE LOGIC OF RELATIONS BEGAN

De Morgan and Peirce were the originators of this branch of logic. As it was developed by them in the context of the 'algebraical' logic we are presenting, a brief sketch is in order.[16]

[16]There is a full-scale monograph, *Merrill 1990*, just on De Morgan and the logic of relations. When Schröder wished to present Peirce's work on this topic he ended up writing an entire volume, of which only its first part, *Schröder 1895*, was published before his death.

In §4 above we noted De Morgan's generalization of syllogistic inference wherein the copula (=, to De Morgan) was replaceable by any symmetric and transitive relation. In his *1856* (read 25 February 1850) we find a further generalization. Here the relation in the major premise and that in the minor need not be the same, the inference (generalizing, e.g., *Barbara*) being good when the relation in the conclusion is the composition (relative product, in modern terminology) of the two relations. He cites an analogous mathematical situation (*1966*, 56):

> The algebraic equation $y = \phi x$ has the copula $=$, relative to y and ϕx; but relative to y and x the copula is $= \phi$. ... The deduction of $y = \phi\psi z$ from $y = \phi x$ and $x = \psi z$ is the formation of the composite copula $= \phi\psi$. And thus may be seen the analogy by which the instrumental part of inference may be described as the *elimination of a term by composition*[1] *of relations*. [Footnote omitted] For though in ordinary inference the concluding copula is usually identical with those premised, yet it is no less true that the composition must have taken place: X is Y, Y is Z, therefore X is that which (= is) Z.

On the basis of this idea De Morgan elaborates a generalized syllogistic doctrine, which we shall forego describing. It required the introduction of the correlative (converse, in modern terminology) copula.

De Morgan's next paper (read 23 April 1860), initiates a new direction in logic (*1966*, 208):

> In my second and third papers on logic I insisted on the ordinary syllogism being one case, and one case only, of the *composition of relations*. In this fourth paper I enter further on the subject of *relations*, as a branch of logic.

In keeping with standards of the time De Morgan's presentation is, for the most part, informal and intuitive. The notions are explained using relational sentence forms with [arbitrary] singular terms, these being denoted by the letters X, Y, Z. He introduces the notation:[17]

$$X..\mathcal{L}Y \tag{1}$$

which he reads as "X is some one of the objects of thought which stand in the relation $\mathcal{L}$ to Y". In addition to this psycho-semantic reading he also gives "X is one of the $\mathcal{L}$s of Y", i.e., using the relative term '$\mathcal{L}$s of Y'. If we use $\mathcal{L}$'Y to denote this term then the second reading has the (modern) symbolic rendering $X \in \mathcal{L}'Y$. But as it doesn't seem as though De Morgan had the idea of a relative term as a distinct notion in mind, we shall uniformly use his two-individual relational form (1).

For the relationship contrary to that of (1) De Morgan writes

$$X.\mathcal{L}Y,$$

[17]To improve readability we are replacing De Morgan's capital letters L, M, ... for relations by calligraphic versions $\mathcal{L}, \mathcal{M}, \ldots$

read as "X is not any of the $\mathcal{L}$s of Y". Here we again have an example of De Morgan's inept choice of notation. Carrying over from an earlier paper the use of a period as a sign of negation he then uses two periods in (1), as if every affirmative sentence had 'not not' as a normal part of the verb! Lower case letters are used for the contrary of a relations, so that

$$X..lY \text{ is the same as } X.\mathcal{L}Y.$$

The *converse* of $\mathcal{L}$ is denoted $\mathcal{L}^{-1}$. Its properties, and its properties in combination with the contrary operation, are listed. For example: the contrary of a converse is the converse of the contrary; and if $\mathcal{L}$ is contained in $\mathcal{M}$, then $\mathcal{L}^{-1}$ is contained in $\mathcal{M}^{-1}$ though, inversely, m is contained in l. In some of these derivations De Morgan resorts to 'not-$\mathcal{L}$' for the contrary of $\mathcal{L}$, thus tacitly acknowledging the inadequacy of his capital and small letter convention.

In addition to the composition (relative product) of two relations two other combinations are introduced in which, he says, "quantity is inherent in the relation". The notions were needed in his construction of a generalized syllogistic for relations. In relational sentence form they are

$$X\,\mathcal{L}\mathcal{M}'\,Y, \quad \text{meaning: } X \text{ is an } \mathcal{L} \text{ of every } \mathcal{M} \text{ of } Y$$

and

$$X\,\mathcal{L}{,}\mathcal{M}\,Y, \quad \text{meaning: } X \text{ is an } \mathcal{L} \text{ of none but } \mathcal{M}\text{s of } Y.$$

Again we find inept notation—juxtaposition of $\mathcal{L}$ with $\mathcal{M}'$ (which has no independent meaning) looks like juxtaposition in the composition relation $\mathcal{L}\mathcal{M}$. The difference becomes evident from the modern quantifier forms for these notions:

$$X\,\mathcal{L}\mathcal{M}\,Y \quad \text{means: } \exists Z(X\,\mathcal{L}\,Z \wedge Z\,\mathcal{M}\,Y),$$

whereas

$$X\,\mathcal{L}\mathcal{M}'\,Y \quad \text{means: } \forall Z(Z\,\mathcal{M}\,Y \to X\,\mathcal{L}\,Z).$$

However the two notions, $\mathcal{L}\mathcal{M}$ and $\mathcal{L}\mathcal{M}'$, are related: $l\mathcal{M}'$ is the contrary of $\mathcal{L}\mathcal{M}$, as De Morgan notes. An extensive list of such equivalent relations is provided but, significantly, nowhere does he write an '=' between them. An incidental use of '))', his current sign for containment of (class) terms, occurs when he expresses the property of $\mathcal{L}$ being a transitive relation by the condition $\mathcal{L}\mathcal{L}))\mathcal{L}$, but his subsequent listing of eight containment results which hold for a transitive relation (such as that $l\mathcal{L}^{-1}$ is contained in l) are all written with the verbal "is contained in". It is clear that De Morgan had no conception of (interest in?) an algebraic-like treatment of relations, anymore than he had for class terms. C. S. Peirce did.

In his *1870* entitled "Description of a Notation for the Logic of Relations ..." C. S. Peirce introduces into the study which De Morgan had initiated, a format with algebraic symbols extending that which Boole had impressed into service for class terms. In the spirit of the Symbolical Algebra school of thought, he enumerates all general properties which an "addition", "multiplication", etc. should have. These are fuller and more broadly done than in Boole. New, i.e., beyond Boole, is $-\!\!<$ ("claw") for inclusion; also two additions and two multiplications, non-invertible and invertible in both cases, and involution (i.e., exponentiation) and evolution. Multiplication is not required to be commutative. In accordance with the tenets of Symbolical Algebra he provides for these an interpretation in logic conformable with the general conditions on algebraic symbols.

Terms on which the algebraic symbols operate are three kinds: absolute terms (horse, tree, man), relative terms (father of, lover of),[18] and conjugative terms (giver of — to —). Absolute terms are denoted by lower case Roman letters, relative terms by lower case *italic*. (Conjugative terms will be omitted from our discussion.) The same algebraic operation and relation symbols are used with both absolute and relative terms, exact meaning then being determined by whether the letters present are Roman or italic. Thus '$\mathrm{f} -\!\!< \mathrm{m}$' is read as 'Every frenchman is a man' while '$m -\!\!< l$' is read as 'Every mother of anything is a lover of the same thing'. Although using both kinds of addition he argues for the superiority of the non-invertible (i.e., non-exclusive *or*, class union) which he symbolizes with a comma subjacent to $+$. His logical interpretation for one of the multiplications (indicated by juxtaposition) mixes both absolute and relative terms, the first component being a relation symbol while the second could be either. When the second component is a class term the product is then a relative term. Thus for Peirce $s(\mathrm{m} +, \mathrm{w})$ denotes whatever is servant of anything of the class of men together with women, so that

$$s(\mathrm{m} +, \mathrm{w}) = s\mathrm{m} +, s\mathrm{w}.$$

He says that this multiplication is the same as De Morgan's composition (of relations). But this can't be right. For De Morgan the composition results in a relation, whereas Peirce's $s\mathrm{m}$, for example, is not a relation but a relative term, i.e., a class (servants of men). Peirce also cites

$$(l +, s)\mathrm{w} = l\mathrm{w} +, s\mathrm{w}.$$

(Note the occurrence on the left of '$+,$' between relation symbols but, on the right, between class terms.) Additionally, for this multiplication he has the associative rule

$$(sl)\mathrm{w} = s(l\mathrm{w}). \tag{2}$$

Here on the left, in 'sl', both components are relations ('sl' is read as 'servants of lovers of'). *This* product can be identified with De Morgan's relative product.

As we have noted in §4, De Morgan shied away from empty and universal terms. For Peirce the equation

$$x1 = x,$$

is a general property which, when the indicated multiplication is taken to be relative product, requires that 1 be the identity relation if the equation is to

[18] We shall often say 'relation' in place of 'relative term' where the context demands it. Peirce doesn't seem to make a clear distinction.

be valid in logic (of relations).[19] Peirce also uses non-invertible multiplication, denoted by 'x, y', which, when x and y are class terms, is interpreted as logical multiplication of terms (class intersection). It is, of course, commutative, unlike the relative product.

The general algebraic properties for exponentiation which Peirce gives are:

$$\begin{aligned} (x^y)^z &= x^{(yz)} \\ x^{y +\!\!\!, \, z} &= x^y, x^z. \end{aligned}$$

Ingeniously, he shows that the De Morgan 'combination of relations with inherent quantity' is representable by exponentiation of relations. However, as with his multiplication, his second component (here, the exponent) can be either a class term or a relation. He interprets 'l^{w}' as denoting the class term 'lover of every woman', and hence '$s^{(l\mathrm{w})}$' as 'servant of everything that is a lover of a woman'. It is then stated that

$$(s^l)^{\mathrm{w}} = s^{(l\mathrm{w})}$$

without any explanation of what s^l (both base and exponent relations) means. Taking it to be the relation 'servant of every lover of', i.e., De Morgan's combination of inherent quantity, makes the equation come out true. In terms of quantifier logic this amounts to the equivalence of

$$\forall Y(Y \in w \to \forall Z(Z\, l\, Y \to X\, s\, Z))$$

with

$$\forall Z(\exists Y(Y \in w \wedge Z\, l\, Y) \to X\, s\, Z).$$

Demonstrating this is a simple exercise in quantifier logic. He also gives

$$\begin{aligned} s^{\mathrm{m} +\!\!\!, \, \mathrm{w}} &= s^{\mathrm{m}}, s^{\mathrm{w}} \\ (s, l)^{\mathrm{w}} &= s^{\mathrm{w}}, l^{\mathrm{w}}, \end{aligned} \tag{2}$$

both of which are readily verified. (Note the non-invertible multiplication between *relations* in the second of these equations.)

Complementation is introduced by having a special relation, n, read by Peirce as 'not' and, also, as 'other than'. As to its properties he gives

$$\begin{aligned} x, n^x &= 0 \\ x +, n^x &= 1. \end{aligned} \tag{3}$$

These properties result when n is taken to be the relation of non-identity ($\neq$). Hence when x is a class, a, then n^a is the class of non-identicals of members of

[19] In terms of quantifier logic,

$$\exists Z(X\, x\, Z \wedge Z\, 1\, Y)$$

is necessarily equivalent to $X\, x\, Y$ only if '1' is '='.

a. To verify that $a, n^a = 0$, consider an individual X. To express that X comes under the term a, n^a we write $X \in a, n^a$. Then

$$\begin{array}{rcl} X \in a, n^a & \leftrightarrow & X \in a \wedge X \in n^a \\ & \leftrightarrow & X \in a \wedge \forall Y (Y \in a \rightarrow X \neq Y) \\ & \leftrightarrow & X \in a \wedge \forall Y (X = Y \rightarrow Y \notin a) \\ & \leftrightarrow & X \in a \wedge X \notin a, \end{array}$$

thus reproducing the sense of the first equation in (3). A similar explanation can be given for the second.

There being no ready algebraic analogue for it, Peirce has difficulty introducing the converse of a relation. To handle this notion he resorts to a special conjugative term.

After stating over 80 general formulas of his algebra, and working a simple example using "Jevons' modification of Boole's algebra" (which uses non-invertible addition and commutative multiplication) he comments:

> It is obvious that any algebra for the logic of relatives must be far more complicated. In that which I propose, we labor under the disadvantages that the multiplication is not generally commutative, that the inverse operations are usually indeterminative, and that transcendental equations, and even equations like
>
> $$a^{b^x} = c^{d^{e^x}} + f^x + x,$$
>
> where exponents are three or four deep, are exceedingly common. It is obvious, therefore, that this algebra is much less manageable than ordinary arithmetical algebra.

This partial summary of Peirce's *1870*, centering on his logic of relations, omits some items of historical importance. These will be mentioned presently. Entirely omitted by us are some fanciful ones such as a binomial expansion of $(x+,\ y)^z$, a Taylor series expansion of a logical function, and also applications of relation algebra to quaternions and geometry.

8 BOOLE'S ALGEBRA SIMPLIFIED

In addition to the idempotency law $x^2 = x$ for class terms x, characteristic of Boole's algebra was the use of '+' as a binary invertible operator, interpreted as *or* when its components were mutually exclusive class terms, and otherwise uninterpreted. Although admitting to the informal use of *or* in the non-exclusive sense, Boole nevertheless insisted that "in strictness" it is implied that the components refer to distinct classes of objects. Apparently he was unaware that differences of opinion on the question go back to ancient times. According to Kneale and Kneale (*1962*, 160) the Stoic logicians recognized both kinds of disjunctive propositions, though Chrysippus, for example, held that the exclusive sense was the proper one. Coming to medieval times we find that William of Shyreswood took *or* in the non-exclusive sense (*Kneale and Kneale 1962*, 261). With De Morgan, as we have noted in §4, his 'P, Q' meant 'P or Q' with *or* in the non-exclusive sense; indeed his rule

The contrary of PQ is p, q; that of P, Q is pq

would fail if P, Q were to be taken as 'P or Q' in the exclusive sense.

Jevons

The first use of '+' to stand for non-exclusive *or* in a mathematical-like setting, with the property $A + A = A$ occurs in *Jevons 1864*. Jevons, a former student of De Morgan's, considered his version of (term) logic to be a revision and simplification of Boole's. Though, being intensionally conceived, in its fundamentals it rather resembles one of Leibniz's described in our §1. It differs from Leibniz's by having disjunctive as well as conjunctive terms, by the use of the copula '=' instead of 'is in' (i.e., inclusion), and by admitting a (vaguely described) '0'.

Departing from the long-standing English tradition of viewing logic extensionally, Jevons maintained that the "meaning of a name or term is a certain set of qualities, attributes, properties, or circumstances of a thing unknown or partly known [i.e., the qualities]". However by, in effect, specifying composition of terms using set-theoretic operations on these sets of qualities, he arrives at an essentially extensional calculus of classes, the calculus operating mostly by rules rather than as an equational algebra. We present a brief description.

A combination, AB, of two terms "must have as its meaning the sum of the meanings of the separate terms." From this intensional point of view the combination 'rational animal', for example, has the qualities constituting rationality together with those constituting animality. Presumably the two sets of qualities could be overlapping, so that "sum" refers to the set which is the union of the two sets of qualities.

The notion of a "plural term B or C", symbolized '$B + C$', is introduced as a "term of many meanings, for its meaning is either that of B or that of C, but it is not known which". A notion so described, with an ambiguous meaning, is not accomodable in a formal discipline. Examining the formal properties which Jevons attributes to his plural term (e.g., $A + B = B + A$, $A + A = A$, $B + BC = B$) we find that they are compatible with the notion having the meaning which is the set of meanings common to the terms, i.e., the notion dual to his combination of terms. This is what we think Jevons should have for his "plural" term.

Also unsatisfactory is Jevons' introduction, on the basis of his intensional view, of contrary (or negative) terms. He says, "not-A is the negative term signifying the absence of the quality or set of qualities A", but nothing is said about what qualities it does have, so that the combinations such as B not-A, have no clearly specified meaning. Nevertheless he still claims: "if $A = B.C$ [then]

$$\text{not-}A = B\,\text{not-}C + \text{not-}B.C + \text{not-}B\,\text{not-}C."$$

His introduction of the symbol '0' is obscure (*1864*, 31):[20]

> **92**. Let us denote by the term or mark 0, combined with any term, that this is contradictory, and thus excluded from thought. Then $Aa = Aa.0$, $Bb = Bb.0$, and so on. For brevity we may write $Aa = 0$, $Bb = 0$. Such propositions are tacit premises of all reasoning.

By thus treating 0 as if it were, and yet were not, a term, Jevons fudges over the need for explaining what qualities it does have and what is meant by a combination of 0 with a genuine term. Although a special symbol is thus introduced in connection with Aa, Bb, etc., there is none for $A + a$, $B + b$, etc.; that is, no meaning is assigned to isolated occurrences of these expressions, though he does claim as logical principles the equations (*1864*, 39):

$$\begin{aligned} A &= A(B+b) = AB + Ab \\ A &= A(B+b)(C+c) \\ &= ABC + ABc + AbC + Abc, \text{ and so on.} \end{aligned}$$

His intensional viewpoint leads him to conclude (p. 5) that "No term can be proposed wide enough to cover its [the universe of logic's] whole sphere;" hence "every proposition of the form $A = B + b$ must be regarded as contradictory of a law of thought."

Inference is carried out by expanding logical expressions into sums of constituents on the (atomic) logical terms present, then deleting those constituents implied to be contradictory by the premises. This is not essentially different from Boole and is justifiable without the need, as Jevons thought it did, for an intensional interpretation. What is different is that Jevons uses only his +, with $A + B = AB + Ab + aB$ and everything interpretable, whereas with Boole's +, $x+y = x\overline{y} + \overline{x}y + 0\overline{x}\,\overline{y} + \frac{1}{0}xy$. Moreover, since Jevons' + has no inverse he foregoes equational solution for additive terms. Finally, although having a mathematical dress and being more or less algorithmic, Jevons' system is noticeably lacking in formal rigor.

C. S. Peirce

A few years after Jevons' *1864*, but independently of it, C. S. Peirce's first paper in logic, *1867*, likewise introduced a non-exclusive sum of terms in an algebraic context. However, unlike Jevons, Peirce adheres to the extensional point-of-view of Boole's calculus. Moreover, he retains its problematic features, i.e., undetermined and uninterpretable terms, which Jevons had eliminated. It is clear from Peirce's writings that he wanted his later work—in which among other changes these problematic features no longer appear—to supersede that of this fledgling paper. Nevertheless, as marking a contribution to the origins of Boolean algebra and, in particular, its foreshadowing of Boolean duality, it merits description.

[20] Jevons adopts De Morgan's practice of using a lower case letter to symbolize the negative of an upper case one.

Peirce introduces logical and also "arithmetical" operations on classes and distinguishes (logical) identity from equality. The logical notions are symbolized with a comma under the symbol. Thus, for logical addition: "Let $a \underset{,}{+} b$ denote all the individuals contained under a and b together." For the "arithmetical" $a + b$ he has the conditional definition:

$$\text{If No } a \text{ is } b \qquad a \underset{,}{+} b \underset{,}{=} a + b$$

For logical multiplication (the common part of a and b) he has a, b—but no mention of 'ab'. He points out that logical addition and multiplication are commutative and associative [operations] and that

$$a \underset{,}{+} a \underset{,}{=} a \quad \text{and} \quad a, a \underset{,}{=} a.$$

He states the significant properties:

> Logical addition and multiplication are doubly distributive, so that
>
> (8) $$(a \underset{,}{+} b), c \underset{,}{=} a, c \underset{,}{+} b, c$$
>
> and
>
> (9) $$a, b \underset{,}{+} c \underset{,}{=} (a \underset{,}{+} c), (b \underset{,}{+} c)$$

and presents a proof in which the mutually exclusive parts into which a, b and c are divided are joined with $+$ (without the comma). This reciprocal distributivity result, foreshadowing the duality theorem of Boolean algebra, is not mentioned by Jevons nor, of course, by Boole who had no symbol for logical addition.

An attempt is made to define logical subtraction via a conditional definition:

(10) $$\text{If} \quad b \underset{,}{+} x \underset{,}{=} a \qquad x \underset{,}{=} a \underset{,}{-} b$$

but, as Peirce notes, "x is not completely determinate. It may vary from a to a with b taken away." The zero is defined by the identities

$$0 \underset{,}{=} x \underset{,}{-} x = x - x,$$

though no proof is offered that it is independent of x, i.e., that for any a and b, $a \underset{,}{-} a \underset{,}{=} b \underset{,}{-} b$. Equally unsatisfactory is Peirce's introduction of logical division and the unit 1. Concerning these notions he finds

> The rules for the transformation of expressions involving logical subtraction and division would be very complicated. The following method is, therefore resorted to.

The method which is described is essentially Boole's idea of development, as applied to Peirce's notions and involves "uninterpretables."

Finally, we note the conspicuous absence from Peirce's paper of the De Morgan rules, which he could have stated as

$$\overline{a +_{,} b} =_{,} \bar{a}, \bar{b} \quad \text{and} \quad \overline{a, b} =_{,} \bar{a} +_{,} \bar{b}.$$

However they do get stated in his *1870*, with negation being the relational one described at the end of our preceding section.

Boole had no symbol for class inclusion. He expresses 'All Xs are Ys', for example, by an equation: either $x = xy$ or $x = vy$, v an indefinite class. As noted in our §1, both of these forms (in intensional versions) occur in Leibniz though, not having a subtraction or a 0, he had nothing corresponding to $x(1 - y) = 0$, a form which Boole derived algebraically from $x = xy$. In our §1 we have described Leibniz's development of the containment relation, for which he used the verbal "*in est*". Its properties are derived formally from axioms, without appeal to meaning. Peirce's treatment of inclusion in his *1870*—employing his claw symbol—is the first substantial one since Leibniz's. It differs from Leibniz's in having a more general, multiple, meaning for *term*; and it goes beyond Leibniz's in having compound terms with other logical notions additional to Leibniz's $\oplus$. Peirce takes his $-\!\!<$ to be the primary logical relation in terms of which $=$ can be expressed: "To say that $x = y$ is to say that $x -\!\!< y$ and $y -\!\!< x$." The topic is resumed and further developed in his *1880* (Chapter [Part] I, §4. On the Algebra of the Copula) where a propositional sense is also attributed to the notion of *term*. The claw symbol then acquires an additional meaning of "illation" (*inference* and, derivatively, *implication*). Between these two papers Peirce had seen *Grassmann 1872, Schröder 1877*, and *MacColl 1877, 1878*. We discuss the first two in the remaining portion of this section. MacColl's, devoted to propositional logic, will be included in the next one.

Robert Grassmann

Apparently Grassmann was unaware of any contemporary work in logic as he mentions only Lambert's *Neues Organon* of 1764 and Twesten's *Logik* of 1825. As Boole and Peirce did, Grassmann bases logic on a generalized form of mathematics, called by him *Grösenlehre*. It is akin to the Symbolical Algebra of Peacock, Gregory and Boole, or to the Universal Algebra of Whitehead. The following excerpt from *Die Begriffslehre oder Logik, Zweites Buch der Formenlehre oder Mathematik*, serves to illustrate Grassmann's point of view (*1872*, 4, translated):

> In order to provide a foundation for Concept Theory, i.e., Logic, we have to take a new path, that of pure formulas, giving all proofs in equations, which can then be transformed in accordance with the laws of symbolic algebra [*Grösenlehre*]. For only this form of proof presupposes no logic or grammar, only it alone can result in a rigorous form of thought; only in it is there a unique value for each entity [*Gröse*] and

relation, only it alone is valid in general for all thought since each entity that is, or could be, an object of thought can be represented in it.

The *Grösenlehre* laws that Grassmann is referring to are the commutative and associative laws for addition and multiplication, distributivity of multiplication over addition, and existence of an additive zero and a multiplicative unit. In showing how addition and multiplication apply to concepts he makes no distinction between individual (i.e., Karl, Hans) and general (i.e., old, brave) concepts. But, in addition to these general laws, logic has its own special laws: both addition and multiplication of concepts are (to use our modern term) idempotent. Logic is then a particular kind of *Formenlehre* (mathematics) and is "indeed the first and most profound".

Grassmann leaves the impression (see the above quotation) that he will be using only algebraic transformations on equations to obtain logical results. Yet central to his development is the assumption that any concept is equal to a (finite) sum of distinct elementary concepts, these concepts being (algebraically) indecomposable and, of any two distinct ones, their product is zero. Thus the algebraic 0 comes into logic; however '1' is ignored and not used. Instead he introduces a special symbol 'T' (*Totalität*) which is the sum of all concepts. It has the standard properties of a Boolean unit (universe) such as

$$a + T = T \qquad \text{and} \qquad aT = a.$$

Negation is introduced by specifying that two disjoint (i.e., having product 0) concepts whose sum is T are *negations* of each other. The negation of a is symbolized $\bar{a}$—coincidentally the same symbol as Boole's—and negation of compounds by $(a + b)^-$ and $(ab)^-$. Inclusion of concepts, symbolized by $a \leq b$ is introduced via the equation

$$(a = ab) = (a \leq b).$$

(Apparently the use of the algebraic symbol '$\leq$' in logic came so naturally to Grassmann that there is no comment about it.)

On the basis of this material Grassmann derives just about all the standard elementary results of Boolean algebra—not surprisingly since his algebra of concepts is just a Boolean algebra of (finitely many) atoms. Absent only is the recognition of the duality principle (but no one else had at this time), and the idea of the development of a Boolean function as a sum of constituents. If *Die Begriffslehre oder Logik* had appeared 25 years earlier conceivably we might all be referring to Robert-Grassmannian algebra instead of Boolean algebra. Indeed, it is a closer fit to Boolean algebra than is Boole's algebraic system.

Ernst Schröder

Schröder's *1877, Der Operationskreis des Logikkalkuls*, opens with the expression of surprise at the lack of attention given to Boole's remarkable achievment, that of realizing the ideal of a calculus of logic which Leibniz had propounded.

Schröder was unaware of *Jevons 1864* and *Peirce 1867* since he cites as the only works subsequent to Boole's, two short notes (Cayley, A. J. Ellis) and the independently arrived at treatment of Grassmann's, just described. The neglect of Boole's work is attributed to its imperfections. Of these the most serious is the incorporation into logic of material entirely foreign to the subject, e.g., the "entire ballast" of number, of which only 0 and 1 rightfully belong to logic. As a concomitant to having this foreign material Boole has to relinquish interpretability of intermediate formulas. A similar criticism of Boole's methods had, as we have noted, been made by Jevons. But Jevons' replacement for Boole's system suffers in comparison with the rigorously clear and elegant axiomatic formulation present in Schröder's *Operationskreis.*

Unlike Jevons with his "qualities" and Grassmann with his "*Begriffe*", Schröder is forthrightly extensional—*classes* are what logic calculus is about. And, unlike Peirce, the subject is not grounded on general algebraic notions with multiple interpretations, but on clearly defined operations on classes—class union symbolized by '+', intersection by '×' and complementation by a subscript '1' (which in his *1890* becomes a short vertical line, presumable so as not to confuse it with the numeral).

Of historical interest is Schröder's calling attention to and establishing the duality principle for logic—that to each general valid formula another is obtained on interchange of '+' with '×' and '1' with '0'. Intimation of this principle occurs in *Peirce 1867* (which Schröder had not yet seen) which called attention to the double distributivity—of multiplication over addition and addition over multiplication.

We present Schröder's axiomatization of his *Logikkalkul* in terms of the three operations, the two constants 0 and 1 and equality, axioms for the last being separately stated in advance. The numberings here correspond to Schröder's. (1^0 and $1'$ being definitions, are omitted.)

Schöder's Axioms for the Class Calculus

2^0)	$ab = ba$	$2'$)	$a+b=b+a$
3^0)	$a(bc) = (ab)c = abc$	$3'$)	$a + (b + c) = (a + b) + c = a + b + c$
4^0)	$a = b$ implies $ac = bc$	$4'$)	$a = b$ implies $a + c = b + c$
5^0)	$aa = a$	$5'$)	$a + a = a$
6^0)	$a(b + c) = ab + ac$	$6'$)*	$a + bc = (a + b)(a + c)$
	$(b + c)a = ba + ca$		$bc + a = (b + a)(c + a)$

$7^{0\prime}$) To each class symbol a there is at least one other, a_1, having the properties

$$aa_1 = 0 \quad \text{and} \quad a + a_1 = 1$$

9^0)	$a \cdot 1 = a$	$9'$)*	$a + 0 = a$

The duals marked with * are derivable from the others. A contemporary improvement on this formulation would be one in which all the axioms are just equations. (See, e.g., *Chang and Keisler 1973*, 38, for such a formulation.)

9 PROPOSITIONAL LOGIC

Although hypothetical syllogisms had been considered by Aristotle's pupils, and propositional inference schemata by the Stoic philosophers, not much of this ancient logical literature—particularly that of the Stoics—has survived. From the 17th century on (e.g., John Wallis, Leibniz) the conditional was treated as a universal, the topic of hypothetical syllogisms being then submerged in that of the categorical. Not until the second half of the 19th century do we have systematic treatments as an independent part of logic. The first of these was initiated by Boole in his *Mathematical Analysis of Logic* and, with a changed viewpoint, further developed in his *Laws of Thought.* Even here the subject was not clearly separated from term (class) logic—he considered it to be another interpretation of his algebra of 0 and 1 applied to portions of time. The earliest presentations of propositional calculus not based on the class calculus were those of MacColl and Frege, both conceived independently of Boole's work and of each other's. We describe each of these in turn.

Boole

Boole begins the topic in his *1847* with the definition, quoting Whately, of a hypothetical proposition as "one or more categoricals united by a copula" (e.g., 'If A is B, then C is D' with the two categoricals 'A is B' and 'C is D'). Note the more general use of the term 'copula' to include 'if, then'. In a significant departure from tradition Boole now ignores the categorical form of the component propositions: he points out that the correctness of an inference (or syllogism) does not depend on considerations relative to the categorical terms (the A, B, C, D of the example) but only on the truth or falsity of the propositions [and also on the kind of copula]. In modern treatments the copula is replaced by a propositional connective, the subject then being based on its truth-functional properties. But this was not Boole's way. Rather, he employs his algebra of elective symbols. Only the symbols are now taken to be operators selecting entities from a Universe, not of objects, but one of "cases or conjunctures of circumstances" (*1847*, 49):

> The hypothetical Universe, 1, shall comprehend all conceivable cases and conjunctures of circumstances.
>
> The elective symbol x attached to any subject expressive of such cases shall select those cases in which the Proposition X is true, and similarly for Y and Z.
>
> If we confine ourselves to the contemplation of a given proposition X, and hold in abeyance every other consideration, then two cases only are conceivable, viz. first that the given Proposition is true, and secondly that it is false.* [footnote omitted] As these cases together make up the Universe of the Proposition, and as the former is determined by the elective symbol x, the latter is determined by the symbol $1 - x$.

For two propositions he has the association of cases with expressions:

Cases	*Elective expressions*
X true, Y true	xy
X true, Y false	$x(1-y)$
X false, Y true	$(1-x)y$
X false, Y false	$(1-x)(1-y)$

An association is then established between propositions and elective *equations*: if X is true then (Boole avers), there are no cases selected by $1-x$, hence $1-x=0$ or $x=1$. Similarly, $xy=1$ expresses the "simultaneous truth of X and Y", while $x(1-y)=0$ expresses the conditional "If X is true, then Y is true". (Note the unneccessary use of the semantic predicate *is true*.) Using this association together with algebraic rules Boole shows by examples how one can justify hypothetical syllogisms—for example:

If X is true, Y is true	$x(1-y)=0$
But X is true.	$x=1$
Therefore Y is true.	$\therefore 1-y=0$ or $y=1$.

Seven years later, in *Laws of Thought*, Boole abandons the idea of a Universe of cases on which to base propositional logic, and instead uses that of *time for which a proposition is true* (*1854*, 163):[21]

> Let us take, as an instance for examination, the conditional proposition, "If the proposition X is true, the proposition Y is true." An undoubted meaning of this proposition is, that the *time* in which the proposition X is true, is *time* in which the proposition Y is true. This indeed is only a relation of coexistence, and may or may not exhaust the meaning of the proposition, but it is a relation really involved in the statement of the proposition, and further, it suffices for all the purposes of logical inference.

Now Boole's letters x, y, ... stand for "portions" of time for which the propositions X, Y, ... are true. Algebraic combinations of these x, y, ... stand for the corresponding (class) combinations of portions of time—e.g., xy for the portion of time for which X and Y are simultaneously true. The universe 1 now designates the (class of) instants of time under consideration. He represents 'X is true' by '$x=1$' and 'X is false' by '$x=0$', thus tacitly assuming that a proposition is either true for all times or else for none. Despite this change in semantics the resulting association of algebraic *equations* with compound propositions is the same as that based on his earlier "cases" idea.

[21] Compare the quotation with the following illustration occurring in Leibniz's 1669 *Specimen juris* (*1971*, 380): V.g. quando navis ex Asia venerit, Titius 100 accipiet. Quia resolvi potest in propositionem categoricam necessariam (v.g. omni Tempus adventus navis, est tempus á Titio accceptorum) ...

From our present-day viewpoint we can see that Boole's algebraic equational formulation is a workable method of doing propositional logic. For if Boole's $+$ is used only between mutually exclusive terms—where it coincides with logical sum defined by non-exclusive *or*—and if only equations with terms constructed from 0, 1, and idempotents x, y, ... are used, then Boole's algebra is Boolean algebra. Moreover, it is known that two-valued propositional logic is mathematically equivalent to a Boolean algebra (structure) of two elements. Hence, since Boole in effect assumes as an algebraic principle that (for any idempotent x)

$$x = 0 \quad \text{or} \quad x = 1, \tag{1}$$

he then has a two-element Boolean algebra.[22]

Boole thought it quite remarkable that the same algebra could be used both for the logic of primary (uncompounded) propositions and for that of secondary (compounded) propositions. *We* can understand his misunderstanding: it happens that a Boolean polynomial equation $\phi(x, y, \ldots) = \psi(x, y, \ldots)$ with no constants other than 0 and 1, is valid in all Boolean algebras (i.e., is an identity) if and only if valid in a two-element one. An equation then can be verified to be an identity by assuming that the variables x, y, ... take on only the values 0 and 1. Boole ignored (1) when doing class logic but used it for his logic of secondary propositions.

MacColl

In a series of three papers appearing in 1877–78, Hugh MacColl produced a version of propositional logic based on the (intuitively understood) meanings of sentence connectives, rather than on the (or an) algebra of classes. In his *1877*, the first of the three, written apparently with no knowledge of Boole's work, it is given in an equational form. In addition to not being based on classes, it differs from Boole's in a number of aspects. Absent is the baggage of Time: letters (and their compounds with $\times, +, '$) stand directly for statements, and are simply true or false. Gone also is Boole's special addition connected with exclusive *or*. Conjunction, represented by multiplication, and (non-exclusive) disjunction, represented by addition, are truth-functionally conceived, though not explicitly so stated. Denial (negation) is indicated by a prime which, unlike Boole's overline or the De Morgan–Jevons change of type case, is applicable to compounds as well as single letters. Unspoken of, however, is a reliance on the analogy with numerical algebra: distributivity of multiplication over addition is explicitly stated, but that addition and multiplication are commutative and associative is not. Likewise, taking the prime as analogous to the negative sign, double primes are tacitly dropped, as well as 0 as an additive term.

Equations are referred to as assertions: $A = 1$ asserts that A is true, $A = 0$ that A is false, $A = B$ that A and B are equivalent, and $A = AB$ that A implies B. Although the constants 1 and 0 are used syntactically as statements, MacColl never explicitly says what they stand for.

[22] It was taken for granted by Boole that $0 \neq 1$.

MacColl presents no formal or organized development. Occasionally some formulas (equations) are derived from others, but mostly there is just a listing based on the appeal to "self-evident" meaning. Most of the well-known logical identities are included. The interest in this first paper is not logic and he seems not to appreciate its potential. Here is his estimate of it (*1877*, 9):

> ... The chief use of the method, as far as I have carried it, is to determine the new limits of integration when we change the order of integration or variables in a multiple integral, and also to determine the limits of integration in questions relating to probability.

Some eight months later in his second paper, there is a new emphasis (*1878a*, 177):

> The following additions to my former article on this subject (see Vol. IX, pp. 9–20), though perhaps belonging more strictly to the province of logic than to that of mathematics, will, I hope, be found interesting.
>
> DEF. 12.—The symbol $A : B$ (which may be called an *implication*) asserts that the statement A implies B; or that whenever A is true B is also true.
>
> NOTE.—It is evident that the implication and the equation $A = AB$ are equivalent statements. (See Rule 2.)

Since MacColl has a truth-functional meaning for conjunction and is taking $A : B$ as equivalent to $A = AB$, his $A : B$ then has the properties of the truth-functional conditional, i.e., is false if and only if A is true and B is false. Not surprisingly no clear distinction is made between language and metalanguage. He has, for example,

> RULE 12.—If $A : B$, then $AC : BC$, whatever the statement C may be.

as well as

> RULE 15.—If A implies B and B implies C, then A implies C.

and then later on refers to

$$(\alpha : \beta)(\beta : \gamma) : (\alpha : \gamma)$$

as its "symbolical expression". The occurrence here of a colon within the scope of another colon seems casual and does not occur elsewhere. Not until we come to Frege, for whom the conditional was a fundamental sentential connective, and Peirce, who, however, identified it with the inference relation, do we find conscious use of conditionals within conditionals.

In this second paper MacColl displays the capabilities of his calculus in two applications. After the writing of his first paper he had become acquainted with

Boole's work from a description of it in Bain's *Deductive Logic.* To show the superiority of his calculus and techniques based on it, he solves one of the "more difficult" examples that Bain had mentioned in his description. The second application is to the Syllogism. Here he does not fare too well. He says: "*All X is Y* may be denoted by the implication $x : y$ in which x denotes the statement that a certain representative individual belongs to the class X, and y denotes the statement that he belongs to the class Y." Lacking the use of individual variables MacColl here deserts the formal nature of the calculus and has to understand the connection between x and y, i.e., that they are referring to a common ("representative") individual. And, lacking quantifiers, his symbolic version of the particular affirmative *Some X is Y*, namely $(x : y')'$, fails to provide the needed existential content, being equivalent to $i \in X \wedge i \in Y$, rather than $\exists i(i \in X \wedge i \in Y)$.

MacColl's third paper on his "calculus of equivalent statements" opens with a long list of equivalences and implications—recognizable (now) as valid in propositional logic—such as

$$(a : b) : a' + b \tag{6}$$

$$(a = b) = (a : b)(b : a) \tag{7}$$

The converse of (6) does not appear. Adjoining it to the list one can readily prove

$$(a = 1) = a \quad \text{and} \quad (a = 0) = a',$$

something which apparently did not occur to MacColl.

This third paper of MacColl introduces a theme which, in the 1950's, came to occupy the attention of designers of combinational circuits (and some logicians), namely, how to simplify the expression of logical functions (truth-functional formulas). The results here were forgotten or, more likely, not known, as his papers didn't enter the mainstream of logical literature.

Under consideration are formulas ("statements") which are alternations of conjunctions of letters, plain or negated, i.e., formulas in alternational normal form. The object is to reduce a statement to its "primitive" form, a form in which "it and all of its parts are free from redundant terms and redundant factors". A simple ingenious method is described for obtaining this primitive form—yielding at first a sum of what are now called its prime implicants. (A proof that it does is not given.) He notes that a term is redundant if the product of it with the denial of the sum of the remaining terms is equal to 0. Finding redundant terms is expeditiously carried out by a truth-value like computation, based on a general theorem

$$abc \ldots f(a, b, c, \ldots) = abc \ldots f(1, 1, 1, \ldots),$$

where $abc\ldots$ is taken as the term being tested, $f(a, b, c, \ldots)$ the denial of the remaining sum, and a 1 being replaced by 0 if the corresponding letter in front is a negated instance. By an example MacColl shows that the end removal of redundant terms (i.e., from the prime implicants) can be done in more than one way, resulting in more than one primitive form. Though he says nothing about thus being able to find a shortest one—something of interest to 20th century combinational circuit designers. In general, these three papers show him to be a talented and skillful user of the calculus he had developed.

Frege

The propositional calculus contained in Frege's *Begriffsschrift* (*1879*), appearing a historical hair's-breadth later than MacColl's, does not fit easily into the algebraic analogy being used as a *leitmotiv* for this chapter. Nevertheless, its considerable importance requires its presentation. Our exposition will ignore some of Frege's special distinctions, such as that between a "thought" (considered proposition) and a "judgement" (aserted proposition). Also we shall convert his 2-dimensional quasi-diagrammatic notation to the more usual linear form.

In keeping with his intended project of tracing out the fundamentals of arithmetic by seeing "how far one could proceed in arithmetic by means of inferences alone", Frege reduces his logical apparatus to a minimum and deliberately avoids using standard algebraic symbols for logical notions. As sentential connectives he uses only the conditional, truth-functionally defined, i.e., to be true except if the antecedent is true and the consequent false, and negation. He shows that *or* (both senses) and *and* can be expressed in terms of the two connectives he has chosen and, in turn, that the conditional is expressible by *and* and *not.* The conditional is chosen in that "it enables us to express inferences more simply". All the sentential formulas he will be using are derived from a set of six formulas by use of *modus ponens* and an (unstated) rule of substituting formulas for letters in those already derived. Frege's starting formulas (or axioms), stated in modern notation and labelled with the numbers used for them in *Begriffsschrift* are:

$$
\begin{array}{ll}
(1) & a \to (b \to a) \\
(2) & (c \to (b \to a)) \to ((c \to b) \to (c \to a)) \\
(8) & (d \to (b \to a)) \to (b \to (d \to a)) \\
(28) & (b \to a) \to (\neg a \to \neg b) \\
(31) & \neg\neg a \to a \\
(41) & a \to \neg\neg a.
\end{array}
$$

Unlike MacColl and Peirce (see our next item), Frege here maintains a clear distinction between inference and the conditional.

Frege's great contribution, quantifier theory, will be described elsewhere in this volume. But since we have had occasion to mention Aristotelian syllogistic doctrine, it would not be out of place to say a word about Frege's version of the categorical forms. Translating his notation into modern quantifer notation they are:

$$\begin{array}{rl} \text{All } S \text{ are } P & \forall x(S(x) \to P(x)) \\ \text{Some } S \text{ are } P & \neg\forall x(S(x) \to \neg P(x)) \\ \text{No } S \text{ are } P & \forall x(S(x) \to \neg P(x)) \\ \text{Some } S \text{ are not } P & \neg\forall x(S(x) \to P(x)). \end{array}$$

Frege arranged these in the traditional square of opposition. But he doesn't mention that the inference from 'All S are P' to its subaltern 'Some S are P' is not valid without a supplementary premise that there are S.

C. S. Peirce

It is not likely that Peirce had seen Leibniz's manuscript *Generales Inquisitiones* which contains the statement (quoted earlier by us in §1):

> If, as I hope, I can conceive all propositions as terms, and hypotheticals as categoricals, and if I can treat all propositions universally [i.e., as universals], this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance.

This is a pretty good description of what Peirce, with similar high expectations, attempts—along with much else—in his *On the Algebra of Logic* of 1880. While written with awareness of *Grassmann 1872, Schröder 1877* and *MacColl 1877–78* (but not of *Frege 1879*), the presentation shows it to be a development mainly of Peirce's own ideas.

We have earlier remarked on Peirce's use of the claw symbol for the copula relating what could, indifferently, be either absolute or relative terms. To treat the syllogistic categorical forms the symbolism is extended to include '$A \overline{-\!\!<} B$', "*a dash over any symbol signifying in our notation the negation of that symbol* [and not only over class symbols, as with Boole]". This later form is then made to have a copular sense by writing it as '$\breve{\mathrm{A}} -\!\!< \overline{\mathrm{B}}$', where $\breve{\mathrm{A}}$ is *some*-A and $\overline{\mathrm{B}}$ is *not*-B: "The short curve mark over the letter in the subject shows that some part of the term denoted by that letter is the subject, and that that is asserted to be in possible existence." Thus Peirce has a symbol for a non-empty indeterminate part of a term, though no formal rules or criteria for its proper use is provided. In the case, then, of particular categoricals Peirce's subject terms have "possible existence", but for subject terms of universals it is not required. This differs from Leibniz whose theory of the syllogism (described in §1 above) assumed that all terms designated by letters, negations of letters, and certain specific combinations thereof, are possible terms.

Peirce also introduces a third meaning for his claw symbol, that of "illation" or inference. The sentiment expressed by Leibniz in the above quotation is echoed by Peirce (*1986*, 170):

> The forms $A -\!\!< B$, or A implies B, and $A \overline{-\!\!<} B$, or A does not imply B, embrace both hypothetical and categorical propositions ... To say, 'If A, then B' is obviously the same as to say that from A, B follows,

> logically or extralogically. By thus identifying the relation expressed by the copula with that of illation, we identify the proposition with the inference, and the term with the proposition. This identification, by means of which all that is found true of term, proposition, or inference is at once known to be true of all three, is a most important engine of reasoning, which we have gained by beginning with a consideration of the genesis of logic.[1]
>
> [1] In consequence of the identification, in $S \multimap P$, I speak of S indifferently as *subject, antecedent,* or *premiss*, and of P as *predicate, consequent,* or *conclusion.*

Based on the identification of $P \mathrel{.\!\cdot\,.} Q$ with $P -\!\!< Q$, and assuming that properties of illation are known, Peirce develops "an algebra", which, when his letters are taken to stand for propositional variables, can be recognized as a formulation of propositional logic. Thus, since $P \mathrel{.\!\cdot\,.} P$ is a valid inference, he has

$$x -\!\!< x. \tag{1}$$

As another example, recognizing the equivalent validity of the two inference schemes

$$\begin{array}{lcl} x & & \\ y & \text{and} & \quad x \\ \therefore z & & \therefore y -\!\!< z \end{array} \tag{2}$$

he then has the algebraic equation

$$\{x -\!\!< (y -\!\!< z)\} = \{y -\!\!< (x -\!\!< z)\}. \tag{3}$$

Symbols for the *possible* ('∞' instead of Boole's '1') and the *impossible* ('0') are introduced and said to have the properties

$$x -\!\!< \infty \quad \text{and} \quad 0 -\!\!< x \tag{4}$$

"whatever x may be". The formulas in (4) are referred to as "definitions". The modern reader is troubled by the unclear status of the quantifier "whatever x may be". The problem arises also with Peirce's negation—he concludes from certain of his inferences that

$$S \overline{-\!\!<} P = (S -\!\!< P) -\!\!< x$$

and hence

$$\overline{A} = A -\!\!< x.$$

Here the absence of an indication of the scope of the (understood) quantifier is acutely felt by the modern reader. Formulas expressing properties of logical + and $\times$ are introduced and referred to as definitions. Today we would refer to them as introduction and elimination rules. Thus Peirce's two-part "definition"

$$\text{If } a -\!\!< x \text{ and } b -\!\!< x, \text{ then } a + b -\!\!< x$$

and, conversely,

$$\text{If } a + b \prec x, \text{ then } a \prec x \text{ and } b \prec x,$$

would be written as the inferential rules

$$\text{(i)}\quad \frac{a \prec x,\ b \prec x}{a + b \prec x} \qquad \text{(ii)}\quad \frac{a + b \prec x}{a \prec x},\quad \frac{a + b \prec x}{b \prec x}$$

providing then for the introduction and elimination of '+' in the antecedent of a conditional.

On this conceptual basis Peirce produces all the well-known standard propositional logic formulas involving $\prec$, $^{-}$, $+$, $\times$, ∞ and 0, with '$a = b$' taken as '$a \prec b$ and $b \prec a$'. As Peirce notes, various of these formulas had been previously given by Boole, Jevons, Grassmann, Schröder (allowing for his blurring of the distinction between class terms and propositions), and by MacColl. Not being aware of *Frege 1879* he considered his systematic use of formulas with multiple and nested occurrences of $\prec$, as well as such formulas combined with $+$ and $\times$, to be new.

A few years later Peirce adds to this inference-based quasi-algebraic formulation ("quasi" because of the essential use of quantifiers) a distinctively different one based on truth-functional analysis. This is described in our next section.

10 ODDS AND ENDS

The introduction in the early part of the 20th century of a single Boolean operator or a propositional connective, in terms of which the others could be defined, created considerable interest. Yet the idea was anticipated decades earlier by Peirce.[23] In a manuscript, *Peirce 1986*, 218–221, dated by the editors as "Winter 1880–81", Peirce shows that negation and the conditional are definable by one connective, which he writes juxtapositionally as AB, with the meaning that A and B are both false. Then AA renders not-A, and

$$((AA)B)((AA)B)$$

renders 'If A then B'. Also in this note Peirce describes, for formulas in this notation, an ingenious method for doing (Boolean) development and elimination with respect to a chosen letter.

Approximately at the same time, also in unpublished manuscripts, Peirce began a development of the idea of truth-values and the association of connectives with truth-functions. The progression is interesting. At first propositional logic is considered as embedded in ordinary arithmetical algebra (*1986*, 242; dated Fall-Winter 1881):

[23] For detailed historical data see the lengthy footnote 207 in *Church 1956*.

> It will be advantageous from the very outset, to introduce certain arithmetical conceptions into logic. These notions are really somewhat extraneous to the subject; but nevertheless they will be found very valuable in giving our thoughts a definite and concrete form. Let us call the state of a proposition as being true or false (or whatever else it may be) its value. We may choose any two numbers at pleasure, to represent the values of truth and falsity, and denote these by the letters $\mathbf{v}$ and $\mathbf{f}$ respectively, so that the value of every true proposition is said to be $\mathbf{v}$, and that of every false proposition is said to be $\mathbf{f}$, while propositions neither true nor false, if there be any, take other values. ... Accordingly, using the letters of the alphabet to designate different propositions, if we write $x = y$ it will not at all signify that the propositions x and y are equivalent in meaning, but only that both are either true or false at once.
>
> The equation $x = \mathbf{v}$ or $x - \mathbf{v} = 0$ will mean that the proposition x is true,—precisely what x written alone would mean. The equation $x = \mathbf{f}$ or $x - \mathbf{f} = 0$ will signify that the proposition x is false; and the same thing would be signified by writing any algebraical expression which becomes equal to $\mathbf{v}$ when $x = \mathbf{f}$ and equal to $\mathbf{f}$ when $x = \mathbf{v}$. The simplest such expression is $\mathbf{v}+\mathbf{f}-x$. This then represents the negative of the proposition x.

Thus Peirce is mapping propositions onto the set $\{\mathbf{f}, \mathbf{v}\}$ of two arbitrarily chosen numbers. However by not introducing a symbol for the mapping he confuses a proposition with its value—his $x = \mathbf{v}$ should be $V(x) = \mathbf{v}$, where V is the mapping of propositions into $\{\mathbf{v}, \mathbf{f}\}$. (The possibility of propositions neither true nor false which he mentions is not developed in any subsequent writings.) In addition to the algebraic expression here given for the negative of x, a subsequent note (*1986*, 264–266; dated Fall 1881–Spring 1882) gives algebraic expressions for logical aggregation (non-exclusive *or*) and logical compound (*and*) in terms of $\mathbf{f}$ and $\mathbf{v}$. He remarks that Boole chose $\mathbf{f} = 0$ and $\mathbf{v} = 1$ but he prefers $\mathbf{f} = 0$ and $\mathbf{v} = \log\infty$. "It must be remembered that $\log\infty$ is, although infinite, of the zero order of infinity. Consequently

$$\begin{aligned}&\log\infty + \log\infty = \log\infty\\ &\log\infty - \log\infty = 0\end{aligned}$$

although the values of these expressions would be indeterminate with ordinary algebraical infinity $\frac{1}{0}$." Peirce's use of '$\log\infty$' as a number seems to be a passing attempt to produce $x + x = x$; it doesn't appear in subsequent writings.

A brief manuscript note of 1884 (*1993*, 111–113) contains a number of interesting items. Propositional logic is considered to be the term logic of a single individual and in this case, the copula is given truth-functional meaning:

> ... Beginning with the consideration of a single individual, let us write a to signify that the individual in question is a. In order to say "If it

> is a it is b," let us write $a \prec b$. The formulae relating to this symbol $\prec$ constitute what I have called the algebra of the copula. Speaking always of but a single individual, the predication of any mark a can only be wholly true or wholly false. The proposition $a \prec b$ is to be understood as true if either a is false or b is true, and is only false if a is true while b is false.

A truth-functional analysis, now commonplace, is then stated as a means of determining the truth [validity] of a general formula.

> .... Let $\mathbf{v}$ be a mark which is true of the individual considered, and let $\mathbf{f}$ be a mark which is false of it. Then of any mark whatever a two propositions are both true, namely $\mathbf{f} \prec a$ and $a \prec \mathbf{v}$; and two other propositions are one or other of them true, namely either $\mathbf{v} \prec a$ or $a \prec \mathbf{f}$. This is so, because it holds whether for a we put $\mathbf{v}$ or $\mathbf{f}$; and it is clear that the truth of a general formula may be tested by trying whether it will always hold when either $\mathbf{v}$ or $\mathbf{f}$ is substituted throughout for each letter.

This analysis would hold also for formulas containing other connectives besides '$\prec$' since, as the note shows, they are defineable in terms of '$\prec$' and '$\mathbf{f}$', namely by having

$$\begin{aligned}(a \times b) &= (a \prec (b \prec \mathbf{f})) \prec \mathbf{f}\\ (a + b) &= (a \prec b) \prec b\\ \bar{a} &= a \prec \mathbf{f}.\end{aligned}$$

A major advance in the idea of truth-values is made in his 1885 paper (*1993*, 167):

> But this notation [i.e., $x = \mathbf{v}$, $x - \mathbf{v} = 0$] shows a blemish in that it expresses propositions in two distinct ways, in the form of quantities, and in the form of equations; and the quantities are of two kinds, namely those which must be either equal to $\mathbf{f}$ or $\mathbf{v}$, and those which are equated to *zero*. To remedy this, let us discard the use of equations, and perform no operations which can give rise to any values other than $\mathbf{f}$ and $\mathbf{v}$.

But rather than conceiving of a new kind of algebra on a two-element set (i.e., a two-element Boolean algebra), Peirce continues to take $\mathbf{f}$ and $\mathbf{v}$ as arbitrarily chosen *numbers* and to define operations:

$$\begin{aligned}\bar{x} &\quad \text{as} \quad \mathbf{v} + \mathbf{f} - x\\ x \prec y &\quad \text{as} \quad \mathbf{v} - \frac{(x - \mathbf{f})(\mathbf{v} - y)}{(\mathbf{v} - \mathbf{f})}.\end{aligned}$$

When the number variables x and y are restricted to take on only the values $\mathbf{f}$ and $\mathbf{v}$ then the possible values for these operations are given by the tables:

x	$\bar{x}$
v	**f**
f	**v**

x	y	$x \mathrel{-\!\!\!<} y$
v	**v**	**v**
v	**f**	**f**
f	**v**	**v**
f	**f**	**v**

Although Peirce didn't write these tables, he could easily have done so. Had he chosen the numbers 0 and 1 for **f** and **v**, the arithmetic operations defining $\bar{x}$ and $x \mathrel{-\!\!\!<} y$ would be the simpler $1-x$ and $\max(1-x,\ y)$. Apparently the significance of a truth-functional analysis for providing a *semantic basis* for propositional logic was not appreciated by Peirce—or anyone else until about a half century later.

After presenting once again, as in his *1880*, a list of general formulas involving $\mathrel{-\!\!\!<}$ and $\bar{\ }$ based on the intuitive understanding of a valid inference, he then presents, as "more convenient", the now popular truth-functional analysis method of determining validity of such formulas, namely attempting to make the formula have the value **f** by assigning values of **f** and **v** to the letters.

We contrast these truth-value ideas of Peirce's with those of Frege. In his *Begriffsschrift (1879)* Frege had attributed truth-functional meaning to negation and the conditional using the language of "affirming" or "denying" of components. Actual truth-values were only introduced in his *Funktion und Begriff* (1891). These are the values associated with sentences containing a variable replaceable by the name of an object. Just as '$x^2 + 2x$' has numbers as its values, so '$x^2 = 1$' has truth-values as its values—True when x is replaced by either -1 or 1, and False otherwise. To Frege, although formulas may have different *senses* depending on their construction, the *reference* or *denotation* of a sentence can only be the True or the False. Thus '$(-1)^2 = 1$' and '$1^2 = 1$', while differing in sense, denote the same truth-value.

The algebra of logic approach, initiated by Boole and extensively pursued by Peirce, was systematized in Schröder's compendious *Vorlesungen über die Algebra der Logik, 1890, 1891.* (Only its Volumes I, II part 1, are here under discussion. Also, we confine ourselves to foundational matters.)

Schröder's view—following Peirce in this regard—was that only one primitive, a binary relation, was needed as a basis for logic. It is called *subsumption* and symbolized by '$\mathrel{\in\!\!\!\!\mid}$', which he prefers to Peirce's claw symbol. Since its primary interpretation is to be that of inclusion (of regions or domains) he finds it convenient to refer to the values taken on by his variables $a, b, c, \ldots$ as *domains*, though not neccessarily having this as the only possible meaning. Indeed, he will bring in other interpretations for $\mathrel{\in\!\!\!\!\mid}$ so that values of the variables could, at will, be either classes, concepts, propositions or components of an inference. The relation is to have the two general properties:

I. $a \mathrel{\in\!\!\!\!\mid} a$

II. If $a \mathrel{\in\!\!\!\!\mid} b$ and $b \mathrel{\in\!\!\!\!\mid} c$, then $a \mathrel{\in\!\!\!\!\mid} c$.

The expression '$a = b$' is taken as an abbreviation for the combined assertion of $a \mathrel{\in\!\!\!\!\mid} b$ and $b \mathrel{\in\!\!\!\!\mid} a$, and the usual properties of an equality are established. (Hence

Schröder has for $\neq\!\!\!\in$ the properties of a partial order.) Next "symbols" 0 and 1 are introduced so that by "definition"

$$0 \Subset a \quad \text{and} \quad a \Subset 1$$

hold for any domain a. Similary, the operations of multiplication and addition (of domains) are introduced by "definition". Currently we would view these not as definitions but, rather, as (postulated) rules for the *introduction* and *elimination* of these notions in a subsumption (expression), i.e., as the rules

$$\frac{c \Subset a \quad \text{and} \quad c \Subset b}{c \Subset ab} \qquad\qquad \frac{a \Subset c \quad \text{and} \quad b \Subset c}{a + b \Subset c}$$

and their converses.

In referring to these as definitions Schröder is following Peirce's example (see our §9) rather than the equational axioms of his *1877* (see our §8). Going beyond Peirce, Schröder also considers the consistency of the principles and definitions thus far enumerated. He claims that there is an actual realization, namely, that of the (sub-)domains of a tabletop with '$\Subset$' meaning inclusion of domains. Schröder admits that there is no actual (tabletop) domain corresponding to the symbol 0, since it would have to be a domain included in the common part of any two domains, including two possibly disjoint ones. Nevertheless one is introduced by fiat: the "nulldomain" corresponding to the concept of "nothing". It is, he says, the only improper, feigned, imaginary or pretended domain.

Schröder believed that introducing a name by a definition guarantees values for it. For example, for the product ab he says (*1890*, 211):

> Das ab in der That eines Wertes nie ermangeln kann, wenn schon den Namen ab selber als „Wert" gelten lässt, erscheint selbverständlich: eine solche Definition verbürgt zugleich die Existenz des Definirten.

Deficiencies (or unclarities) infect another of Schröder's attempts, one using classes of actual objects, to provide a realization of the axioms. He identifies, for example, an individual with the class of which it is the only member, making then no distinction between membership in a class and class inclusion. His treatment of propositional logic is patterned after Boole's—propositions are replaced by classes of instants of time for which the proposition is true.

The enterprise of conceiving logic as grounded on regions or classes was subjected to pointed criticism by Frege in a discussion of Schröder's Volume I. We quote the first two of five items in his summary conclusion (*Frege 1952*, 106):

> 1. The domain-calculus, in which the fundamental relation is that of part to whole, must be wholly separated from logic. For logic Euler's diagrams are only a lame analogy.
>
> 2. The extension of a concept [i.e., a class] does not consist of objects falling under the concept, in the way, e.g., that a wood consists of trees;

it attaches to the concept and to this alone. The concept thus takes logical precedence of its extension.

Other than Schröder's *Vorlesungen* only one other extended treatment of the algebra of logic appeared: Whitehead's *Treatise on Universal Algebra (1898)* devotes five chapters to the "Algebra of Symbolic Logic" as part of its investigation of "various systems of Symbolic Reasoning allied to ordinary Algebra". And it was the last such. With the development of quantifier logic the inadequacies of the 'algebra of logic' as a foundation for logic became apparent. Nevertheless, employing algebraic concepts in logical investigations lives on in the present-day topic of Algebraic Logic.

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THE ALGEBRA OF LOGIC

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INTRODUCTION

In this essay I am principally concerned with the nineteenth-century mathematization of logic initiated in Britain by A. De Morgan (1806–1871) and G. Boole (1815–1864). The logical framework in which this novel process resulted came to be known, presumably since [MacFarlane, 1879], as the *algebra of logic.* But the phrases *logical algebra* and *algebraic logic* have also been attached to it. For the sake of convenience I shall myself indulge in the occasional use of these alternative names.

A rough imperative description of Boole's methodology in dealing with a logical problem will help to explain these labels:

1. Translate the logical data into suitable equations.

2. Apply algebraic techniques to solve these equations.

3. Translate this solution, if possible, back into the original language.

In other words, symbolic formulation of logical problems and solution of logical equations constitute defining features of Boole's method.

De Morgan initiated the systematic logical study of binary relations. A subject Boole did not touch on. His was chiefly a logic of monadic terms. The algebra of logic was later cultivated in America by C. S. S. Peirce (1839–1914) and in Germany by E. Schröder (1841–1902). Peirce unified in his work features of the unary and binary systems of Boole and De Morgan. By common consent the work of Schröder may be regarded as the final instalment of the Boolean-De Morgan development [Couturat, 1905, 3]. These four writers are the main subjects in our narrative. A minor one is W. S. Jevons (1835–1882). He took the first steps towards the mechanization of reasoning. The focus of my concern, however, is the work of Boole and De Morgan. I consider the logical contributions of Jevons, Peirce and Schröder only inasmuch as they set forth the tradition inaugurated by those two Victorian friends.

At the beginning of the twenty-century the algebra of logic was superseded by the mathematical logic of G. Frege (1848–1925) and G. Peano (1858–1932). In defense of his logic, Schröder exchanged views with both of them. As the preface to *Principia Mathematica* shows, it was to no avail. A. N. Whitehead (1861-1947)

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Dov M. Gabbay and John Woods (Editors)

and B. Russell (1872–1970) singled Peano out as the one who made of received logic a useful mathematical instrument. And as far as logical analysis was concerned, they recognized that their main debt was to Frege. In their hands, the logical tradition that started with Boole was going to be discontinued. But not before it had achieved its most immediate goal, the mathematization of logic.

The fact that the algebra of logic became to be regarded as not suitable for the new research goal, the logical foundations of the whole of mathematics, does not mean that it disappeared from mathematics. To begin with, [Huntington, 1904] regarded one of the products of the algebra of logic, the so-called *Boole algebras*, as structures satisfying a set of equations. This is the first application outside geometry of Hilbert's notion of a set of statements defining a structure class. A product of the algebra of logic had been transformed into a mathematical subject. In the second place, according to [Goldfarb, 1979], the modern view of logic was codified in the nineteen twenties by combining the algebraic and mathematical logical traditions. Finally, as the following description of the modern algebraic logic by [Andréka *et al.*, 2001] shows, there is a conceptual continuity between Boole's initial steps and contemporary logic algebraic efforts.

> Algebraic logic can be divided into two main parts. Part I studies algebras which are relevant to logic(s), e.g. algebras which were obtained from logics (one way or another). Since Part I studies algebras, its methods are, basically, algebraic. One could say, that Part I belongs to "Algebra Country". Continuing this metaphor, Part II deals with studying and building the bridge between Algebra Country and Logic Country. Part II deals with the methodology of solving logic problems by (i) translating them to algebra (the process of algebraization), (ii) solving the alge braic problem (this really belongs to Part I), and (iii) translating the result back to logic.

The algebra of logic was, then, driven out the central stage by the logic of Frege and Peano. Obliterated it was not.

What the algebra of logic had accomplished before the arrival of mathematical logic was a cultural change of profound significance. It is important to pause briefly at this stage in order to stress this fact. Around 1850 logic was still regarded as a part of philosophy and not of mathematics. At the end of the century logic had become the shared concern of these disciplines. The pre-eminent role played by Boole in this cultural shift can never be in doubt. It is true that the range of his logical systems strikes us as rather restricted *vis-à-vis* the Fregean system [Dummett, 1959]. It is also true that the rigor of his presentation did not match the rigor displayed by Aristotle in *Analytica Priora* [Corcoran and Wood, 1980]. But such truths should not cloud our understanding of other truths that demand our assent. The revival of logic in Britain that took place in the first half of the 19th century had restored the pre-eminent position of the traditional syllogism as the paradigm of deductive reasoning. Against this background we must evaluate Boole's achievements. His algebraic representation of Aristotelian sentence forms, combined with his algebraic account of syllogistic inferences made of this paradigmatic logical system a mere part of mathematics. In other words, the whole of the then known deductive logic was shown to be open to mathematical treatment. Seen from this perspective, Boole's feat was not a small one. To the contrary.

Symptomatic of the cultural change that Boole's work brought about are the occasional papers on logic written by mathematicians and the mathematical periodicals that carried a pure logical paper. [Cayley, 1871], a historically important contribution to the algebra of logic, exemplifies the first remark. The *Proceedings of the London Mathematical Society* that published a series of logical papers by H. MacColl (1837–1909) exemplifies the second one. Equally symptomatic of the new cultural situation is a brief passage in [Peano, 1894]. When Peano listed some of his logical predecessors he mentioned, next to Boole and De Morgan, some other British mathematicians: A. Cayley, W. K. Clifford, A. J. Ellis, and A. MacFarlane. This list is a fair proof that logic had become an object of interest within the mathematical community. A melancholic fact is that Boole did not live long enough to see in print any logical publication of these mathematicians.

De Morgan's contribution to the process of mathematization of logic is varied and, on this account, difficult to evaluate. He occupied himself with expanding the expressive power of traditional logic. The famous argument about heads, animals and men (or tails, animals and horses) was used by him to expose the limitations of traditional logic and to motivate the construction of new, more enlighten syllogistic systems. Ironically, his own example and his own account of its validity were anticipated by Abelardus (1079–1142). In more than one sense we can say that De Morgan was truly the last of the English medieval logicians. His English at times more tortured and torturing than Ockham's Latin. But there is more that he contributed to logic.

His mathematical interests gave him familiarity with relational notions current in the algebra of functions. Rather earlier in his career he expressed the view that to reason is to combine given relations into new ones. At the same time he expressed the view that more than one binary predicate (copula) satisfies the Aristotelian schemata. His functional background motivated also another question he posed: which binary predicates satisfy which syllogistic schemata. This change in logical perspective, explored with regard not to the copula but to the quantifiers in [Van Benthem, 1986], is perhaps the most important consequence of De Morgan's functional background. These early preoccupations culminate in his [De Morgan, 1864b], a paper that contains his mature logic of relations. There is no arguing about the fact that this paper is difficult and obscure to the point of exasperation. Thus, [Panteki, 1992, 486] thinks that De Morgan wrote it in haste, for [Grattan-Guinness, 2000, 33] it is a ramble even by De Morgan's standards and according to [Merrill, 1990, 196] it is untidy and incomplete. Still, this paper influenced Peirce's work on relations and, as we shall see, his theory of quantification.

Besides all this, De Morgan was a devoted publicist. He used the pages of the literary journal *The Athenaeum* to review Boole's first logical publication, [Sánchez Valencia, 2001], and to advance his views on mathematics and logic, decrying the lack of mutual understanding:

> We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic, the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two. [De Morgan, 1868, 71]

Our own view of Boole's and De Morgan's work has been shaped by the conviction that their work in mathematics determined the shape of their logic. This view was advanced vigorously and convincingly with respect to Boole in [Bryan, 1888] and has relentlessly been put forward by Grattan-Guinness in several publications.[1] Grattan-Guinness expanded Bryan's insight at to include De Morgan. Important contributions within this tradition are [Laita, 1976] and [Panteki, 1992]. A fair reproduction of Laita and Panteki's views on Boole are [Laita, 1977] and [Panteki, 2000].

As can be expected, there are several expositions of Boole and De Morgan's logic. Substantial parts of [Lewis, 1918] and [Jørgensen, 1931] are devoted to them. The two standard histories of logic, [Stayazhkin, 1969] and [Kneale and Kneale, [1962] 1984], contain capable accounts of their theories. A modern and outstanding reconstruction of Boole based on the notion of multisets is [Hailperin, 1976]. Insightful contributions to our understanding of Boole of historical nature are to be found in [Laita, 1979], [Van Evra, 1977] and [Corcoran and Wood, 1980]. The study of De Morgan has benefited from [Merrill, 1990] and [Panteki, 1992].

The general plan of this essay is the following. The first of its 7 parts is devoted to the description of the mathematical and logical background of Boole's and De Morgan's work. In the subsequent 4 parts I describe the algebraic logic of absolute (monadic) terms and propositions as developed by Boole, Jevons, Peirce and Schröder respectively. The final three parts are devoted to the algebraic logic of relations developed by De Morgan, Peirce and Schröder respectively. ¿From this description it is clear that I shall not give a unified account of Peirce's and Schröder's work. I have also decided not to give a unified account of De Morgan for the following reasons. He played a role in the revival of logic that led to the publication of Boole's first logical tract. He was also a prominent mathematical figure who helped to create a receptive audience for Boole's new logical proposals. He also made substantial contributions to logic. We shall, therefore, meet him both as part of the logical and mathematical background, Part 1, and as a prime player, Part 5. Such a discontinuous account is in keeping with his multifarious contribution to the Victorian upheaval that changed logic for ever.

Now, before starting we need to agree on some conventions. The first time that I mention a publication I shall refer to its first publication date. But in referring the reader to it I shall point to an available publication. In referring to places in [Peirce, 1931] I give first the volume and then the paragraph. This is the

[1] See for instance [Grattan-Guinness, 1988], the introductory essay to [Boole, 1997] and [Grattan-Guinness, 2000].

convention in the Peirce's literature. Occasionally, I shall refer to a bibliographical item by using only a page number. The item to which this page number refers to will invariably be the last item which has been mentioned.

Part 1
The Mathematical and Logical Background

This part consists of two sections. In the first one I describe the two Lagrangian algebras that influenced the mathematical work of Boole and De Morgan: the algebra of operators and the algebra of functions. After that I describe the basic features of symbolic algebra, a theory of mathematics to whose development De Morgan contributed and in whose framework Boole placed his logical research. The second section is devoted to the description of the revival of the logical studies in Britain. I pay attention to a central figure in this revival, R. Whately (1787–1863), and to the British discussion on the nature and function of logic that he provoked. I also attend to the theory of the quantification of the predicate.

1 THE LAGRANGIAN ALGEBRAS

Introduction

In the late 18th century France dominated mathematics and among the French L. P. Lagrange (1736-1813) was the towering figure. His algebraic cast of mind fostered a novel approach to foundational aspects of the calculus that was to have a great impact both on British mathematics as on British logic. Lagrange's attempt to algebraize the calculus released the definition of the derivative from any reference to Leibniz's infinitesimals, Newton's fluxions, Euler's evanescents and Delambert's limits. His program was to reduce the calculus to algebraic analysis, thus dispensing of geometry as the source of acceptable proof methods. A central role in Lagrange's algebraization was his formalization of the idea that any function f can normally be represented as a (convergent) power Taylor series in the increment i of x:

$$f(x+i) = f(x) + if'(x) + \frac{i^2 f''(x)}{2} + \cdots + \frac{i^n f^n(x)}{n!} + \frac{i^{n+1} f^{n+1}(x+j)}{(n+1)!}$$

for some j between 0 and i. Lagrange's view was, then, to define the nth-derivative of $f(x)$ with respect to x as the coefficient of i^n in the Taylor expansion of $f(x+i)$. He was convinced that this form of the Taylor series allows the derivation by algebraic means alone of the basic results of the calculus.[2] For instance, to find

[2]The counterexample to Lagrange's claim was not soon coming. Cauchy proved in the eighteen twenties that not every function was expandable into a convergent Taylor series.

the derivatives of $f(x) = x^2$, we consider

$$f(x+i) = x^2 + 2xi + \frac{2i^2}{2!}$$

From this equation we conclude that $f'(x) = 2x$ and $f''(x) = 2$.

It is important to notice that in Lagrange's view the calculus becomes a theory of functions and operations on them: derivation is an operation, symbolized by the prime symbol, that applied to the function $f(x)$ yields its derivative $f'(x)$. For his work led to the development of two algebras:

1. The algebra of differential operators
2. The algebra of functions

These two Lagrangian algebras are in fact the first instances of algebras whose elements are non-numerical objects. Their existence refutes, incidentally, the claim that in 1897 Boole's algebra of logic was "the only known member of the non-numerical genus of Universal Algebra" [Whitehead, 1898, 36]. It is worthwhile to notice at this stage that Boole himself was a keen reader and admirer of Lagrange. While Lagrange strived to reduce the calculus (and mechanics) to algebra, Boole would strive to reduce logic to algebra. Boole's program seems, therefore, to be a bold extension of the original Lagrangian one.

The algebra of differential operators

The starting point of this algebra is the representation and treatment of the operations of the calculus as objects. To the operation of differentiation corresponds the derivative operator represented by D, so that the derivative of $f(x)$ of order n is made to correspond to D^n. This operational approach was, of course, already present in the Leibnizian tradition. Lagrange's representation of Taylor series in operator form established a formal relationship between the derivative operator and the difference operator, Δ:

$$\Delta f(x) = f(x+i) - f(x) = (e^{iD} - 1)f(x)$$

In this equation one uses the identity operator 1 and the Taylor series for e^{ih} on replacing x by iD:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

L. F. Ant. Arbogast (1759–1803) went even further in developing what he called the *method of separation of symbols.* In his approach symbols of operations are treated as symbols of quantity.[3] He separated the operator symbol from its operand so that our last equation was transformed into

[3]In his textbook Boole says that the operators D and Δ "combine each with itself, with constant quantities, and with each other, as if they were individually symbols of quantity" [Boole, [1860] 1960, 18].

$$\Delta = e^{iD} - 1$$

Two features of the use of this algebra of operators are significative for us. In the first place, mathematicians used operator methods to solve differential equations. In this process it was allowed for some formulas to lack any interpretation. The original equation and its solution(s) were required to be interpretable with respect to an intended interpretation. The formulas that codify the line of reasoning that lead to the solution, however, didn't. This methodological separation between a syntactical and a semantic phase will be stressed by De Morgan in his own theory of algebra. It will also be a salient feature of the theory of the algebra of logic that Boole set forth. I shall run the risk of forfeiting the reader's sympathy by stressing time and again this feature of Boole's logical practice.

In the second place, F.J. Servois (1768–1847) in his effort to establish an adequate theoretical basis for the calculus developed a theory of commutative and distributive operators. In other words, operators that obey the rules

$$\begin{aligned} \varphi F(x) &= F\varphi(x) \\ \varphi(x+y+\ldots) &= \varphi(x)+\varphi(y)+\ldots \end{aligned}$$

thus joining calculus and algebra. In his view, this theory gives a proper explanation of Arbogast's method of separation of symbols. The fact, for instance, that the difference operator Δ defined by

$$\Delta\varphi(x) = \varphi(x+\Delta(x)) - \varphi(x)$$

turns out to be distributive and commutative, justifies that it is manipulated as it were a magnitude.[4] In his mathematical work, Boole took the view that theorems involving the difference operator Δ can be seen to follow from Servoi's principles [Boole, [1860] 1960, 16–17]. Arbogast's approach was also adopted by De Morgan. In [De Morgan, 1842, 767] we are told that he "made free use of what used to be called the separation of the symbols of operations and quantity under the name of the calculus of operations".

The algebra of functions

The name *algebra of functions* refers to the theory of functional equations. In this case we are looking for functions, suitable determined, that are solutions of equations. Examples of such equations are the classical Cauchy equations:

$$\begin{aligned} f(x+y) &= f(x)+f(y) \\ f(x=y) &= f(x)f(y) \\ f(xy) &= f(x)+f(y)(x,y>0) \end{aligned}$$

[4]Incidentally, Servois is the first who used these names to characterize the combinatorial behavior of algebraic objects. The terms *commutative* and *distributive* themselves were technical ethical terms. Aristotle distinguished namely between commutative and distributive justice.

Using these equations Newton's binomial theorem

$$(1 = x^{\alpha}) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

was proved by Cauchy "by his clever handling of functional equations" [Dhombres, 1984, 19–20].

According to [Panteki, 1992, 21] the first systematic treatment of functional equations is due to G. Monge (1746–1818). Monge's approach was born out his dissatisfaction with the approach to functional equations of Lagrange. Panteki gives the following example of Monge's approach. Under the assumption that each function f' has an inverse f^{-1}, the following question is addressed:

ILLUSTRATION 1. Find a function φ for which holds that whenever $\Delta(x) = y$, $\varphi(v(x, y)) = \psi(x)$, for known Δ, v and ψ.

The argument then goes as follows. Given $\Delta(x) = y$, $v(x, y)$ becomes a function f on x defined by $v(x, \Delta(x) = f(x)$. The functional equation to be solved is, therefore,

$$\varphi(f(x)) = \psi(x)$$

By assumption, the inverse of f exists. Hence, if $f(x) = z$, then $f^{-1}(z) = x$. Hence our previous equation leads as follows to the solution

$$\begin{aligned} \varphi(f(f^{-1}(z))) &= \psi(f^{-1}(z)) \\ \varphi(z) &= \psi f^{-1}(z) \\ \varphi &= \psi f^{-1} \end{aligned}$$

The study of functional equations was stimulated by the preference given to functional solutions of differential and difference equations. For us the important fact is that functional equations were studied by De Morgan long before he started to write about the logical properties of relations. The direct fruit of these studies is [De Morgan, 1836]. The indirect fruit is scattered throughout all his publications on logic.

2 MATHEMATICS IN BRITAIN

Introduction

19th century saw in Britain the replacement of the Newtonian fluxional calculus by the Continental methods and the revival of British mathematics. This section describes the basic features of the adoption of those methods and the basic features of the so-called *symbolical algebra.* The relevance of these two topics for the development of the algebra of logic will become apparent in the course of the next pages.

We shall concentrate our attention to Cambridge related aspects of this process for they are the most relevant for our present concerns. But other factor need to be at least mentioned. Modern research on the history of mathematics has showed that this process of change did not originate in Cambridge alone. For instrumental in this change were the Scottish mathematician and geologist John Playfair (1748–1819) and the Cambridge graduate John Toplis (1786–1858), the reviewer and the translator of Laplace, respectively, who decried the state of mathematics in Britain. Their published dissatisfaction was the onset of a process that would lead to the British embracing Leibniz's calculus in the first place and to the reform of the mathematical education in the second place. Scotland was graced with James Ivory (1765–1842), whose mathematical work was recognized by Continental mathematicians, William Wallace (1768–1805), who introduced the differential notation into Britain, and William Spence (1777–1815) whose mathematical work was to captivate posthumously younger researchers [Panteki, 1987].

Analysis

In England the change took place at Cambridge. At this university mathematics was an obligatory part of the curriculum. Moreover, the greatest academic achievement in England could be obtained at the University Senate House Examination by finishing first at the mathematical honors examination, the so-called *mathematical tripos*. The mathematical education, however, was mainly designed for this particular event. In fact, mathematics was regarded as mere mental training void of any intrinsic value [MacFarlane, 1879].

The Scottish reception of Continental methods has its initial counterpoint at Cambridge in the work of Robert Woodhouse (1773–1827) whose dissatisfaction with the state of mathematics brought him to compose his [Woodhouse, 1803].[5] In his book Woodhouse embraced the idea that the calculus is a branch of algebra. The most important effect of his reformist zeal was the influence it exerted on the younger generation. At Trinity College, Cambridge, three undergraduates George Peacock (1791–1858), John Herschel (1792–1871) and Charles Babbage (1792–1871) became dissatisfied with the state of mathematical education. They founded in 1812 the *Analytical Society*. This short-lived society seriously aimed at promoting the notation and methods of the calculus as used on the continent. [Baggage, [1864] 1969, 26] remarks that from Woodhouse's book on the calculus he "learned the notation of Leibnitz". [Peacock, 1834, 295], speaking of another book by Woodhouse in which he used the continental notation, rates it even higher in influence than the Cambridge Lacroix by saying that it "more than any other book contributed to revolutionize the mathematical studies in England".

The historical significance of the *Society* lies in its success in provoking influential reformers to bring about a revival of science in Britain [Becher, 1980]. [Peacock, 1834, 296] would point out that the continental notation was first introduced in the Senate House examinations in 1817 and that in less than two

[5] He is also the term ad quem in [Cajori, 1919b].

years the fluxional notation has altogether disappeared. He did not tell, however, that it was he who was responsible for the change in the notation used in the examinations. Interestingly, one of the founders of the society, Edward Bromhead (1789–1855), would later stimulate Boole's interest in mathematics. [MacLane, 1979, 47] considers probable that he was the one who introduced Boole to the works of Lagrange.

The change of notation sought by the members of the *Analytic Society* was achieved thanks to the influence of the Cambridge Lacroix and to the university work in which Peacock manifested himself. According to [De Morgan, 1842, 34], Newton's notation was discarded by all writers within the universities and "by most out of them". Important as this can be, and there are writers who doubt the intrinsic value of a mere change of notation, there is an additional aspect to their contribution that is even more important.[6] The real problem for mathematics in Britain was not the use of the fluxion notation. The neglect of anything not written in this notation was the real problem. And here lies one of the most remarkable contribution of the members of the Society to the development of mathematics in Britain. Because they introduced not only the Continental notation but also a research program: the development of the calculus of operations. Both Herschel and Babbage carried out research and published within the framework of the Lagrangian algebras.[7] As a result of their activities, the Lagrangian algebras came to occupy a central position in the mathematical research carried out in Britain.

The publication date of [Lacroix, [1892] 1816] is regarded by [Boyer, [1949] 1959, 266] as significant because "it witnessed the triumph in England of the methods used on the Continent." Babbage, Herschell and Peacock were responsible for the translation of this treatise on the calculus. Lacroix exposed in his book an approach to the calculus in which he made a mild use of limits. His translators lamented that "he has substituted the methods of limits of DAlembert, in the place of the most correct and natural method of Lagrange" [Lacroix, [1892] 1816, iii]. To compensate, they augmented the original text with Lagrangian footnotes. The method of Lagrange would then become the dominant in Britain.

As is well known, the notion of limit would be central in Cauchy's analysis, his definition of this notion being widely regarded as the first rigorous one. The *Analytical Society* rejection of limits must not, however, be regarded as a rejection of Cauchy's approach. When they prepared their translation, Cauchy's *Course d'analyse* was still five years away. Their clinging to Lagrange's approach can therefore not be regarded as a reactionary choice, the product of the excessive cautiones of timid newcomers.[8] At any rate, the *Analytical Society* preferred Lagrange's approach for the foundations of the calculus, thus siding, as historians put it, in their rejection of the fluxional method with the first mathematician after Newton who assigned to power series a central role.

[6] A case in point is [Cajori, 1919a].

[7] A thorough study of the mathematical work of Herschel and Baggage is given in [Panteki, 1992]. See also [Dubbey, 1978] and [Grattan-Guinness, 1979].

[8] Resistance to Cauchy's approach was, however, expressed in [Peacock, 1834, 247–248]

The British, then, found contact with Continental mathematics at a moment in which change was going to take place there. The effect was that in a couple of years, the British were again out of pace. Twenty years after the publication of the Cambridge Lacroix, [De Morgan, 1842, iv] would defend a limit based approach to the calculus. When he first exposed his purpose he noted that the method of Lagrange "had taken deep roots in elementary works" although it represented "the sacrifice of the clear and indubitable principle of limits to a phantom, the idea that an algebra without limits was purer than one in which that notion was introduced."

Symbolical Algebra

[Nový, 1973] singles Peacock, Gregory and De Morgan as the most important exponents of a new approach to algebra that arose in Britain. I consider them separately and close my remarks with a brief comment on Boole and the movement of symbolical algebra.

Peacock

George Peacock was involved in the British effort to put the teaching of algebra on a sound basis. His symbolic algebra is regarded as directed to the determination of the status of negative and imaginary numbers [Nagel, 1935]. To this end [Peacock, 1834, 206] proposed the distinction between the arithmetical algebra and symbolical algebra. Symbolical algebra is in fact a conservative expansion of arithmetical algebra inasmuch as its laws are "of such a kind as to coincide *universally* with those in arithmetical algebra when the symbols are arithmetical quantities". In fact, symbolical algebra in Peacock's hands is a kind of generalized arithmetic. The generalization is effected thanks to the *principle of the permanence of equivalent forms* which states that algebraic equivalence must be invariant with respect to interpretation: valid equations expressed in uninterpreted symbols, must remain true for all the interpretations of the symbols involved. The thesis of the freedom of algebra (from arithmetic) is expressed in [Peacock, 1834] by the idea that in symbolical algebra the rules determine the meaning of the operations. The important feature of Peacock's algebra, therefore, is the emphasis he put on the laws that express the algebraic operations. For the meaning of the objects of this algebra are determined by the rules that operate on them.

Even though in his use of symbolical algebra Peacock limited himself to the matter of the negative and the imaginary numbers, the declarative part of his writings are more bold and suggestive of things to come:

> it ought to be remarked that ... there is nothing in the nature of the symbols of algebra which can essentially confine or limit their signification of value (p. 194).[9]

[9]See also [Pycior, 1981, 36–40] for an evaluation of Peacock's strive to algebraic freedom and his practice.

Gregory

It is important for us to single out another mathematician, Wallace's Scottish student Duncan F Gregory (1813–1844). In Cambridge he would launch with R. J. Ellis (1817–1859) in 1839 *The Cambridge Mathematical Journal.* Some years later it became *The Cambridge and Dublin mathematical Journal.* It lives forth as *The Quarterly Journal of Pure and Applied Mathematics.* This journal would publish in 1841 Boole's first mathematical article.

In 1838 Gregory presented a paper that would be published as [Gregory, 1840]. It contains a symbolic approach of the two Servois properties we mentioned before, distributivity and commutativity along with what he called the *index* property.[10]

$$\begin{aligned} f(a) + f(b) &= f(a+b) \\ f' f(a) &= f f'(a) \\ f_m f_n(a) &= f_{m+n}(a) \end{aligned}$$

One of the salient features of this paper is that Gregory establishes therein a conceptual link between Peacock's symbolical algebra and Servois's laws of operations. He agrees, for instance, with Peacock that algebra treats the combination of operations as determined by the laws they obey.

Interestingly enough, Gregory proposes to consider the numerical symbols a, b as numerical operations "indicating that any other operation to which these symbols are prefixed is taken a times, b times, &c." Moreover, the symbol a by itself can be regarded as the operation (a) performed on unity [Koppelman, 1971, 192]. We shall find echoes of this view in Boole's implementation of algebraic logic.

Gregory and Associativity

Absent from Gregory's list of properties is associativity. In fact, the first British algebraist that would use this term in print would be the Scottish William Hamilton [Crowe, 1967, 15-16]. But it is only fair to add that already in 1843 Gregory was making use of the notion. He put first

$$A_x(a) = x + a$$

And then he used the following law to characterize addition as an operation that obeys

$$A_x A_y(a) = A_{A_y(x)}(a)$$

The last equation codifies, of course, the equation

$$x + (y + a) = (y + x) + a$$

But this equation expresses the associativity of addition but by name [Koppelman, 1971, 217].

[10]Gregory must be the first mathematician to render Servois's terms into English.

The situation around associativity seems then to be this. Hamilton identified and named associativity of multiplication while Gregory used this property to characterize addition. He identified it inasmuch as he used associativity as a distinct property but he failed to name it. Intriguingly, Boole's list of logical principles would completely miss associativity.

De Morgan

De Morgan came, wrestling with the influence of Peacock, to distinguish between technical and logical algebra [Pycior, 1983]. The former concerns the manipulation of symbols "under regulations". The later concerns the semantic phase of explanation and interpretation that consists in "giving meaning to the primary symbols, and of interpreting all subsequent symbolic results" [De Morgan, 1841, 174]. As far as the technical algebra is concerned, Peacock's symbolical algebra, De Morgan establishes three principles that govern the use of uninterpreted algebraic symbols (p. 176):

1. To a given symbol corresponds exactly one operation (*process*).
2. Operations whose combined use produce a result may be seen as one process and, therefore, may be denoted by one symbol.
3. To every operation corresponds an inverse operation.

Note that a generalization of these directives into the realm of relations would yield at once the result that relations may be combined into new relations and that every relation has a converse relation.

The program of his expanded symbolic algebra was resumed by De Morgan as follows:

> We ask, firstly, what symbols shall be used (without any reference to meaning); next, what shall be the laws under which such symbols are to be operated upon; the deduction of all subsequent consequences is again an application of common logic. Lastly, we explain the meaning which must be attached to the symbols, in order that they may have prototypes of which the assigned laws of operations are true [Richards, 1987, 15–16].

It is important to keep in mind the distinction between these two aspects of symbolical algebra: the syntactic one, manipulation of symbols, and the semantic one, the interpretation of symbols. As we shall see, heeding this distinction will result in a proper understanding of Boole's logic.

The freedom of algebra which in [De Morgan, 1841, 173] "amounted to the use of an algebra in which the symbols represent something more than simple magnitude", was also to find its way into logic. Because in advocating novel interpretations for the copula he motivated his position by referring to the algebraic practice he adhered to. Here are a few schemata of combination, the syllogisms. How can they be made significant? Aristotle explored a way focusing as he did on the copula *be*. But there are others ....

> No doubt absolute identity was the suggesting connexion from which all the others arose: just as arithmetic was the medium in which the form and laws of algebra were suggested. But, as now we *invent algebras* by abstracting the forms and laws of operation, and fitting new meanings to them, so we have the power to invent new meanings for all the forms of inference ... [De Morgan, 1847, 51].

Boole and Symbolical Algebra

Boole shared some of the tenets of the symbolical movement. Be it that in his view there is nothing arbitrary about it. The symbolical methods are "the visible manifestation of truths relating to the intimate and vital connexion of language with thought" [Boole, 1859, v-vi]. The important lesson he drew from the movement of symbolic algebra was this. The use of the symbolical methods presupposes the predisposition to separate the deductive process from the interpretation of the expressions involved. He points out that this principle is by no means restricted to mathematics. According to him "it claims a place among the general relations of Thought and Language" (p. 399).

3 THE REVIVAL OF LOGIC IN BRITAIN

Introduction

The mathematization of logic was preceded by a discussion on the function and nature of logic that took place in Britain in the first quarter of the 19th century. The influential Scottish philosopher Thomas Reid (1710–1796) had remarked that Aristotelian logic was considered by some as "unworthy of a place in liberal education" [Reid, 1822, 349]. In his view, logic was an art that teaches us "to think, to judge, and to reason, with precision and accuracy". But it is not an indispensable aid. Even without logic we can acquire the habit of reasoning "in mechanics, jurisprudence, in politics, or in many other sciences" (p. 351–352). Those sentiments were aired again at the beginning of the century when Oxford decided to strengthen the position of logic within the university curriculum. The supporters of logic felt at that point obliged to confront the long standing tradition of British criticism of Aristotelian logic. As a result they followed, unwittingly, the steps of G. W. Leibniz (1646–1716) who had written about one of the founders of this tradition, J. Locke (1632–1704).[11]

Leibniz is often associated with the algebra of logic on account of the 'prophetic insight' that his logical writings show. When R. L. Ellis made Boole aware of a

[11]The cartesian roots of Locke's position are discussed in [Gaukroger, 1989].

passage in Leibniz's work that anticipated his own, he is reported to have "felt as if Leibnitz had come and shaken hands with him across the centuries" [Laita, 1976, 273]. There is, as I shall now explain, another point of contact.

In 1765, Leibniz's commentary on [Locke, [1690] 1959] had seen the light of day. It is in the course of his commentary of this book that he proclaims that logic is not a "game for schoolboys" but rather "a kind of universal mathematics" [Leibniz, [1981] 1982, 487].

Three of Locke's views on logic are particularly significant to understand Leibniz's position.

1. Locke lays great stress on the fact that the logic of Aristotle is a rather poor descriptive theory. It gives, namely, a wrong account of the ways people actually reason since people do not need to learn the syllogism to reason correctly. The position adopted by Locke to prove the uselessness of logic is encapsulated in his witty remark that God did not leave it to Aristotle to make men rational [Locke, [1690] 1959, Book IV Chapter XVII].

2. The other objection lays the stress on the insufficiency of the syllogism as the only road to knowledge. He admits, irenically, that "all right reasoning" may be reduced to the Aristotelian syllogisms" (Book IV Chapter XVII). His objection is rather that the syllogism is epistemologically vacuous.

3. In his schema of the division of sciences he defines logic (*semiotics*), as the science that studies the nature of the signs, specially words, that we use in our epistemological dealings with nature and other persons (Boob IV Chapter XXI).

Leibiniz's answer to the first objection is that the argument is not conclusive. The mere fact that people can argue without syllogistic training is not sufficient to establish the uselessness of logic. [Leibniz, [1981] 1982, 482] points out that one does not show the "uselessness of arithmetic as an art by the fact that some people who have never learned to read or write, ..., are seen to count satisfactorily in everyday situations." With regard to the second one, he retorts that instead of being superfluous, logic is a crucial component of our cognitive faculties (p. 483). His position is resumed in the remark "that any knowledge which is not self-evident is acquired by inferences, and the later are not sound unless they have the proper form". And proper form is in his view not necessarily syllogistic form. For in his response Leibniz rejects Locke's conciliatory remark. There are, namely, "valid non-syllogistic" inferences that cannot be reduced to the syllogism. Be it that they are "demonstrable through truths on which ordinary syllogisms themselves depend" (p. 479). The crucial notion in his view is not that of syllogism but that of formal argument. This is defined as any argument in which the conclusion is reached in virtue of the form alone. He counts as a formal argument "a sorites, some other sequence in which repetition is avoided, even a well drawn-up statement of accounts, an algebraic calculation, an infinitesimal account" (p. 478). Finally, Locke's redefinition of logic as semiotics is, according to Leibniz neither new nor advisable. The science of reasoning, of judgement and discovery is "quite different from the knowledge", he says, "of etymologies and language use" (p. 522).

Richard Whately

When the onset of the mathematization of logic set in, a long period of contemptuous neglect in the study of deductive logic had already been brought to an end. The turning point came in 1826 when Richard Whately (1787–1863) published a new logical text, [Whately, [1827] 1975], based on an essay on logic published in 1823 in the *Encyclopedia Metropolitana.* The book attracted considerable and sustained interest from his contemporaries. Whately did not produce any remarkable result in logic and has been, on this score, described as a minor figure in the 19th century upheaval that transformed logic. In spite of this fact, it is generally agreed that he was a figure of great historical importance, the proper bearer of the often quoted description 'the restorer of logical study in England' that De Morgan bestowed on him.

As we mentioned before, the educational reform in Oxford ignited the discussion about the proper understanding of the nature and scope of logic. Whately responded in his text to old objections against logic that could be used in this dispute to deride its presence in the university curriculum. In so doing, he was obliged to defend Aristotelian logic as it has been handed down by the post-medieval tradition. He accomplished his task shifting the ground in a remarkable way. Three features set Whateley's book apart from the logical texts of most of his British forbears. First, he regarded logic as a theoretical science and not, as Reile demanded, only as an art that prepares us in the proper use of our intellectual faculties. Second, he considers logic to be primarily concerned with language and thus only secondarily with thought. In this regard he sides with Locke. Logic is then, in his view, primarily concerned with arguments and not with mental operations. In all fairness, we have to remark that on this point he seems to waver between the primacy of language position and a position defended by Reid. For the Scottish philosopher, the analysis of language and the analysis of thought are one and the same. Finally, Whately points out to a 'striking analogy' between logic and arithmetic. And in this aspect he seems to be echoing Leibniz's words.

Whately's response to Locke's criticism that stresses the poor descriptive content of logic, was to deny that logic was concerned with providing such a description. The syllogism is not a peculiar method of reasoning that people must learn to achieve argumentative skills. It is a theoretical tool useful in the analysis of reasoning. Therefore, the explicit syllogistic knowledge people command, has no bearing on the proper task of logic. For it is the task of logic to lay down the formal conditions which valid arguments must observe. The syllogism is, therefore, not a poor descriptive theory of the way people actually use their mental faculties. In reality, it is not a descriptive theory of such actual processes, it is not a theory of performance at all. In his answer to this objection we see him shaking hands with Leibniz. He remarks, namely, that bringing forward the argumentative skills of those who have never learned logic to prove its uselessness is as to deny the utility of grammar on "the grounds that many speak correctly who never studied the principles of grammar" (p. 33).

In his response to Locke's second objection, Whateley put great stress on the formal nature of logic. Logic is not primarily concerned with the acquisition of knowledge. Its primary concern is with the validity and invalidity of arguments. In stressing this thesis Whately put forward the syllogistic fullness thesis that we saw Locke advancing. The task of logic is to elaborate the normal form to which all argument may be reduced. The syllogism is that form. It is the form to which all correct reasoning may be ultimately reduced. The proper understanding of this thesis demands to take into account the criticism he wants to nullify, i.e., the context in which it was produced. For in presenting this thesis Whately is primarily rejecting the use of the term *logic* to describe substantive concerns. Logic is about validity, validity is about form. In other words, logic is not epistemology.

The development of logic followed Whately in severing it from epistemology. Never is this more pronounced than in the fact that the noun *System of Logic* in the title of J. S. Mill's book strikes modern readers as a misnomer. Awareness of this situation led Ernest Nagel to the publication of an abridged version of Mill's book under the more faithful title of *Philosophy of Scientific Method.* The Locke-tradition did clearly not succeed in keeping the name of logic to singularize their epistemological concerns. A sad side effect of this development is that Whately's wording of his position gained a new significance. In defending the formal nature of logic, Whately singled out the syllogism as the normal form to which any argument may be reduced. With this identification of the logical underlying form with syllogistic form, he went where even Leibniz did not dare to go.

It must be said that there are passages in his book in which Whately adopts a broad interpretation of the syllogism: a valid argument form so stated that its conclusion is evident from the mere form of the expression. This interpretation allows, as Leibniz' s position shows, for the existence of valid argument forms that do not need to be reduced to the syllogism. But this is a rather excessively generous interpretation. There is a fair amount of passages in which Whately's reduction is connected explicitly to the regular syllogism.

Witness Locke's remark on this point, Whately seems to be adopting a safe argumentative position. Even the most renowned critic of deductive logic accepts the strength of the syllogism. True, Reid had pointed out that the traditional conversion rules cannot explain the validity of the inference of the conclusion *Philip is the father of Alexander* from the premise *Alexander is the son of Philip.* But he did not remark, as Leibniz incidentally did, that no syllogism can be used to capture this validity. The mistaken identification of logical form with the regular syllogism was, however, not the most important feature of Whately's position. The important feature was his defence of the formal nature of logic. When this point became uncontroversial, it was inevitable that the syllogistic fullness thesis would gain in prominence and notoriety. Whately became then to be seen as a conservative logician, unduly attached to the syllogism.

The Quantification of the Predicate I

Bentham and Solly

A review of the third edition of Whately's *Elements* published in 1833 in *The Edinburgh Review*, listed several publications on logic related to this book. Among these the reviewer mentioned [Bentham, [1827] 19901], a book on logic by the botanist George Bentham (1800-1884). The most remarkable features of this book are these. In the first place, the author introduces novel propositional forms in which both the subject as the predicate term are quantificationally modified. In the second place he represents affirmative sentences as equations and negative ones as inequalities:

> In order to abstract every idea not connected with the substance of each species, I have expressed the two terms by the letters X and Y, their identity by the mathematical sign =, diversity by the sign ⊩, universality by the words in toto, and partially by the words ex parte, or for the sake of still farther brevity, by prefixing the letters t and p, as signs of universality and brevity (p. 133).

Thus, the sentence *Every horse is a quadruped* may be represented by the equation

$$\text{t horse} = \text{p quadruped}$$

This equation, written in one of the two alternative notations Bentham uses, is interpreted as the assertion that the class of horses is equal to a part of the class of quadrupeds. The letter t is, evidently, a universal quantifier while the letter p stands for an existential one. More in general, the equations he offers in this notation are the following three. We add the intended interpretation

t X = t Y	The class X is equal to the class Y
t X = p Y	The class X is equal to only a part of the class Y
p X = p Y	Only a part of the class X is equal to only a part of the class Y

The sentence *Some quadrupeds are not flying animals*, on the other hand, is represented by the inequality

$$\text{p quadruped} \mid\mid \text{p flying animal.}$$

The interpretation of this expression is that a part of the class of quadrupeds differs from a part of the class of flying animals. The use of a special symbol to express logical inequality seems to be Bentham's rendering of a feature proper to Whately's logic. The last uses namely two copulas. A positive one, *is*, and a negative one, *isn't*.[12] Again, we list below the corresponding schematic forms.

[12] Cayley, MacColl and Peirce will also work with a positive and a negative copula. We discuss this matter later on.

t X || t Y	The class of X differs from the class of Y
t X || p Y	The class of X differs from any part of the class of Y
p X || p Y	A part of the class of X differs from any part of the class of Y

Each of these six representations illustrates the doctrine of the quantification of the predicate. According to this doctrine any categorial sentence expresses a relation between two quantified terms, and not, as the received view demanded, as a relation between two terms whose nature is determined by the copula and the quantifier. The copula indicates whether the sentence is an affirmative or a negative one. The quantifier indicates whether the sentence is a universal or a particular one. These are, then, constructions in which the quantifiers are either universal or indefinites.

Bentham's doctrine of double quantification was a gleam in Leibniz's eye [Leibniz, [1903] 1966, 59]. There are also medieval discussions on this subject inspired by Aristotle's rejection of it in *De Interpretatione* [Prior, [1955] 1962, 146–147]. Some continental writers on this topic that preceded Bentham are listed in one appendix of [De Morgan, 1847]. Still, Bentham's is reputed to be the first treatment of the subject of double quantification in English. He was not going to be the last one.

In 1839 the subject reappeared in *A Syllabus of Logic*, a book by the mathematically trained Thomas Solly (1816–1875). In [Panteki, 1993], the paper that rescued Solly from oblivion, we find the following eight forms used by him:

1. All A is Some B
2. All A is All B
3. Some A is Some B
4. Some A is All B
5. All A is not Some B
6. All A is not All B
7. Some A is not Some B
8. Some A is not All B

For Bentham and Solly alike, the quantification of the predicate is not a central concern. Bentham's claim to a place in the history of logic, however, is exhausted by his contribution to this doctrine. The case of Solly is different. His is the first English tract where symbolical representation and mathematical methods are adopted in the study of the syllogism. The resulting scheme of representing syllogisms was considered superior to that of some of his successors [Lewis, 1918, 36].

De Morgan.

[De Morgan, 1846] presents a consistent theory of the syllogism that could be considered as an alternative to the traditional theory. The paper consists of fours sections. In the first one he defines the meaning of a ground term: a term is an expression that can be predicatively applied to any object of thought. Among the ground terms De Morgan was eager to include negative terms. He was aware that Aristotles was troubled with qualms in this regard, because negative terms lack the definiteness of positive ground terms. To cure these qualms, De Morgan introduced the notion of universe of discourse. In order to give a definite character to a pair of contrary terms it suffices to determine first their universe. The universe of a term, as he initially put it, is then the notion needed to accept negative terms along the positive ones. A guiding concern of De Morgan in this paper is to effectuate a closer link between language and logic.

In the second section of this paper, de Morgan introduces a formal language in which there are two types of ground terms: capital and lower case letters. The pair of ground terms (X, x) exhausts the universe. Given a pair of contraries, it does not matter which one is considered the negative and which one is considered the positive. The important thing to know about them is that they exhaust the universe because "everything in the universe is either X or x" [De Morgan, 1966, 3]. No other information is relevant. Nevertheless, for practical purposes De Morgan adopts the convention of reading the lower case letter x as *not-X*. The traditional Aristotelian sentences are represented in the following way:

P)Q	signifies	Every P is Q
P.Q	signifies	No P is Q
PQ	signifies	Some Ps are Qs
P:Q	signifies	Some Ps are not Qs

The notation is motivated by him with the remark that the right bracket and the colon are used to represent non convertible operations. However, he took "for the convertible propositions, the symbols P.Q and PQ, which the algebraist is accustomed to consider as identical with Q.P and QP" (p. 4).

In the third section, De Morgan comment on the validity of two arguments that involve the generalized quantifier *most.*

$$\frac{\text{Most of the Ys are Xs} \quad \text{Most of the Ys are Z}}{\text{Some of the Xs are Zs}}$$

This argument illustrates the fact that to a syllogism:

> all that is necessary is that more Ys in number that there exist separate Ys shall be spoken of in both premises together (p. 9).

By numerically quantifying the predicates, de Morgan argues, the validity of the schema becomes evident. We can convince ourself of the validity of this argument by using a reductio argument. De Morgan did not express himself in this manner. There is in our reconstruction, however, hardly a notion which was not at his disposal. Let (1) and (2) represent the information contained in the premises while (3) is the negation of the conclusion.

1. $|Y \cap X| > 1/2|Y|$
2. $|Y \cap Z| > 1/2|Y|$
3. $X \cap Z = \emptyset$

To establish De Morgan's result, remember first the following fact concerning the cardinality of two arbitrary sets

4. $|A| + |B| = |A \cap B| + |A \cup B|$, or equivalently
5. $|A \cup B| = |A| + |B| - |A \cap B|$

Therefore, inasmuch as because of (3), $X \cap Z \cap Y = \emptyset$, we see that (5) yields

6. $|(Y \cap X) \cup (Y \cap Z)| = |(Y \cap X)| + |(Y \cap Z)|$

Therefore, given both that

- $(Y \cap X) \cup (Y \cap Z) = Y \cap (X \cup Z)$, and
- $|Y| \geq |Y \cap (X \cup Z)|$

we obtain the following contradictory consequence based on the numerical information provided by 1 and 2

7. $|Y| \geq |(Y \cap X)| + |(Y \cap Z)| > 1/2 \mid Y \mid + 1/2 \, |Y| = |Y|$

According to De Morgan his non-standard syllogism shows that, given suitable numerical restrictions, it turns out to be possible to deduce a conclusion from two non-universal premises.

In an addition to his submitted paper, but before its publication he generalized his result. Instead of deriving the conclusion *Some Xx are Ys* he argues that it is possible to obtain a conclusion with specific numerical information, *Every one of a specified Xs is one or other of b specified Ys.* Letting s be the number of Ys in the universe, the general form of a syllogism of *definite quantity* is the following:

$$\frac{\text{n of the Ys are Xs} \quad \text{m of the Ys are Z}}{\text{n + m - s Xs are Zs}}$$

The validity of this argument form can be established provided that n + m > s. An additional condition, is that we should read the conclusion as stating that at least n + m - s X are Zs [De Morgan, 1966, 9] Assume then

8. $|(Y \cap X)| + |(Y \cap Z)| > |Y|$

9. $|X \cap Z| < |(Y \cap X)| + |(Y \cap Z)| - |Y|$

Note first that a consequence of (8) is that the right side of (9) denotes a positive number. This is in fact the only information we extract from the combined premises. Now, the following sequence of inequalities follows from (9) and (5)

10. $|X \cap Z| + |Y| < |(Y \cap X)| + |(Y \cap Z)|$

11. $|(X \cap Z) \cup Y| + |(X \cap Z \cap Y| < |(X \cup Z) \cap Y| + |(X \cap Z \cap Y|$

12. $|(X \cap Z) \cup Y| < |(X \cup Z) \cap Y|$

Therefore, inasmuch as $|Y| \leq |(X \cap Z) \cup Y|$ and $|(X \cup Z) \cap Y| \leq |Y|$, (12) yields the contradiction we sought.

After he had generalized his initial result into the syllogism of definite quantity, where specific cardinality information "is given both to the subject and the predicate of a proposition", De Morgan learned that the Scottish philosopher William Hamilton (1788–1856) had written about the need to quantify predicates (p. 17). Hamilton had urged his students to inquiry after

> the reasons why common language makes an ellipsis of the expressed quantity, frequently of the subject, and more frequently of the predicate, though both are always expressed in thought? [Hamilton, 1860-1862, xi]

Rashly, De Morgan concluded, and stated, that his numerical arguments had been anticipated by Hamilton. Little did he imagine the consequences that his mistake would have.

The Quantification of the Predicate II

The author of the review of *Elements* we mentioned in the previous section was the already mentioned philosopher William Hamilton [Hamilton, 1853]. The great reputation he enjoyed during his lifetime as a philosopher and logician of great format has not withstood the test of time. He is mostly remembered for his dispute with A. De Morgan but it has been said that "without Hamilton, we might not have had Boole" [Lewis, 1918, 37].

The dispute with De Morgan concerned the following two points:

- the paternity of the idea of the quantification of the predicate and
- the logical merits of the version of this doctrine that Hamilton defended.

No attempt will be made in these pages to consider the first of these two points. The banal paternity discussion started when Hamilton accused De Morgan in private correspondence of stealing from him the idea of the quantification of the predicate. De Morgan then demanded a public apology for this private offence and brought the affair into light. As the facts stand it is clear that De Morgan did not steal Hamilton's idea. He defended the idea of double definite partitive quantification while Hamilton defended the idea of double non partitive quantification. The doctrine of the quantification of the predicate should not be interpreted as covering double definite partitive constructions as in *Each one of 5 men is one of 7 students.* The partitive constructions in this sentence express a quantitative modification of the terms. Something that the sentences that fall within the range of the doctrine of the double quantification do not do.

The set of double quantified forms of [Hamilton, 1860-1862, 277] is given below. At the right side we add the self explanatory, Bentham's like, abbreviation used by him.

(1)	All A is all B	toto–total
(2)	All A is some B	toto–partial
(3)	Some A is some B	parti-partial
(4)	Some A is all B	parti-total
(5)	Any A is not any B	toto-total
(6)	Any A is not some B	toto-partial
(7)	Some A is not any B	parti-total
(8)	Some A is not some B	parti-partial

Hamilton and Solly's set of forms are similar but, certainly, not the same. Hamilton uses the distributive quantifier *any* while Solly, consistently, avoids this form. Let us assume now the following facts partially supported by linguistic evidence concerning the distribution of negation and quantifiers in English [Ladusaw, 1979].

- *any* is a universal quantifier with wide scope with regard to negation
- *some* is an existential quantifier with wide scope with regard to negation

[Hamilton, 1860-1862, 615], in fact, defended explicitly this linguistic assumption with regard to *any*, listing negation among the contexts that demand its use instead of is positive counterpart *some.* De Morgan rejected the grammatical judgements of Hamilton in this regard, but he adopted them as a guiding principle in the interpretation of the double quantified forms [De Morgan, 1966, 275].[13] A direct consequence of these assumptions is the following first-order rendering of the last four quantified forms from Hamilton's set:[14]

1. $\forall\, xy((A(x) \wedge B(y)) \rightarrow x \neq y)$
2. $\exists\, x\forall y((A(x) \wedge (B(y)) \rightarrow x \neq y))$

[13][Horn, 2000] comments on the conflicting views on *any*-exchange between Hamilton and De Morgan.

[14]The representation of the last one follows De Morgan's interpretation. [Parry, 1965, 352] offers a justification of it based on Hamilton's own verbal explanations and those of his disciples.

3. $\exists y \forall x((B(y) \wedge A(x)) \rightarrow x \neq y))$
4. $\exists xy(A(y) \wedge B(x) \wedge x \neq y))$

Given these two assumptions, we can understand Hamilton's contradictory pairs listed below:

1. (1) and (2)
2. (3) and (4)
3. (5) and (6)

De Morgan noticed that under Hamilton's interpretation this set is not closed under negation. Moreover, the set is redundant. These two matters hang together. The point is that the sentence form in (2) is equivalent to the conjunction of two of the original forms:

1. All A is all B
2. All A is some B and Some A is all B

This equivalence establishes the redundancy of the Hamilton set. The closure problem follows from the fact that there are in this set no simple forms that contradict the double universal form. In particular, the forms below do not constitute a contradictory pair:

1. All A is all B
2. Some A is not some B

The double universal is taken as stating the identity of the quantified terms. The second sentence is taken as stating that *some A* is not identical to *every B*. Its negation becomes the assertion that every A is identical to every B. This last is, however, the same as the assertion that the sets involved are identical.[15]

It is with regard to these facts that [Prior, [1955] 1962, 148] came to the conclusion that Hamilton defended the idea of the quantification of the predicate with "a quite fantastic incompetence". It is, nevertheless worthwhile to observe that Hamilton's flawed elaboration of the quantification of the predicate does not exhaust his historical contribution. The dispute between him and De Morgan differs from the dispute between Whately and the critics of logic. In the former case the dispute is one among defenders of logic. Moreover, both De Morgan and Hamilton agree on the need to extend the traditional syllogistic framework. This view, exposed as it was by such prominent figures are not only an indication of an intellectual atmosphere in the making. They helped to create it. Moreover, in his writings, Hamilton defended also both an extensional approach, and an equational view of propositions. Two features that were to figure prominently in the next future. [Hamilton, 1860-1862, 273] came also to the conclusion that:

[15] A defense of Hamilton is found in [Fogelin, 1976] where the form (8) is interpreted as the disjunction that contradicts (1).

> ... a proposition is simply an equation, an identification, a bringing into congruence, of two notions with respect to their intension, for it is this quantity alone which admits of ampliation or restriction, the comprehension of a notion remaining the same being always at ifs full amount.

The heyday of this equational view of propositions was about to come.

FINAL WORDS

In this part we have described background against which we want the reader to see the change in logical culture that was about to take place. De Morgan and Boole are to be regarded as active members of the mathematical tradition that started with the algebraization of analysis proposed by Lagrange. Both of them contributed to the development of the algebras that grew up from Lagrange's concerns. And both of them shared the abstract view of algebra advanced by Peacock. The interest for logic has been revived by Whately's work and by the dispute between De Morgan and Hamilton. The new form of logic left Hamilton and his followers well behind. Still, flawed as Hamilton's doctrine of the quantification of the predicate may have been, he was the center of a new movement in logic. According to Lewis it was crucial for the development of logic that Hamilton emphasized an extensional approach. His final judgement is that without the contributions of Hamilton and his disciples, small as they may appear to us, " symbolic logic might never have been revived" [Lewis, 1918, 36–37].

Part 2

The Logic of Monadic Predicates: George Boole

Section 1 gives a general description of the subsequent sections. Section 2 is devoted to the language of [Boole, 1847]. Section 3 deals with the deductive part of this book. Section 4 is reserved for matters concerning the interpretations of Boole's logical expressions. Section 1 is also devoted to interpretation. Section 6 focusses on Boole's varied account of the symbol v. Section 7 deals with a metalogic result Boole argues for. It also considers considers Boole's notion of general solution. The last section describes the differences between [Boole, 1847] and [Boole, 1854].

1 INTRODUCTION

Boole's equational logic, particularly as expressed in [Boole, 1952b], is the subject of the following pages. In the construction of this logic Boole was guided by the basic principles of Symbolical Algebra. His lasting loyalty to these principles is apparent from his statement that "equations involving the symbols of logic may be treated in all respects as if those symbols were symbols of quantity ... until in the final stage of solution they assume a form interpretable in that system of thought with which Logic is conversant" in [Boole, 1952a, 127]. In the treatment of Boole's symbolical system I try to isolate the logical structure of his arguments and not only to reproduce them. Hereby, the reader will see his "logic resting like Geometry upon axiomatic truths, and its theorems constructed upon that general theory of signs, which constitutes the foundation of the recognized Analysis" [Boole, 1952b, 58]. One of our aims is to bring home the lack of arbitrariness that characterizes much of Boole's symbolical procedures. For instance, derivation by squaring, a typical algebraic operation of which he made use, may be regarded as the result of recursive replacement starting from a trivially valid equation.

$$\dfrac{\dfrac{\alpha = \beta \quad \alpha\alpha = \alpha\alpha}{\alpha\alpha = \beta\alpha} \qquad \alpha = \beta}{\alpha\alpha = \beta\beta}$$

In the course of the exposition, I pay special attention to Boole's indefinite symbols whose prototype is v. Their role have always appeared puzzling to historians of logic. And indeed, there is no denial of the obscurity that surrounds them in their first appearance. Boole's more mature work on logic only compounded this obscurity. For there we find two conflicting accounts. Still, I shall argue that his use of indefinite symbols is no an unmitigated evil. Furthermore, these conflicting accounts can be reconciled in an unenforced way. With specific reference to the proper understanding of indefinites, I shall stress time and again the difference between algebraic and interpretive phases in the Boolean system. For in his system the algebraic form of equations undetermines logical meaning in two senses. First, in deductions we abstract from meaning. This much is the import of the symbolical method. As a consequence of this initial step not all the algebraic conclusions need be regarded as logical conclusions. Second, equations containing an occurrence of elective symbols are undetermined even when this symbol is logically interpreted. Because an equation containing such a symbol gives rise to several equations when this symbol is fully determined. It behaves, states Boole, as the arbitrary constants in linear differential equations. Interpretable equations, thus, may be determined or undetermined. The signal of undeterminacy is the presence of an indefinite symbol. In practice, Boole always indicates which symbol is to count as the arbitrary constant. In working with Boole's logic, therefore, we have to be constantly aware of the existence of this second kind of undeterminacy. This is a clear distinction that in practice turns out to be extremely elusive. Boole himself in his thinking about indefinites was arguably foiled at least once by it.

Before starting our exposition there are two points I want to stress. In the first place, we must be aware of the fact that Boole developed an equational logic in which equations are derived from other equations, (the assumptions). Some of these assumptions may be logically true equations. In applications, however, logically true equations, like $\alpha\alpha = \alpha\alpha$ in our previous example, do not need to be visible. This silent use had Boole in mind when he wrote that often there exists "no need of more than one premiss or equation, in order to render possible the elimination of a term, the necessary law of thought virtually supplying the other premiss or equation" [Boole, 1952a, 110]. Boole's use of logical truths in his demonstrations is regarded as one of the lasting improvements he has to be credited for [Corcoran and Wood, 1980].

In the second place, we must be aware of the fact that Boole was not only concerned with the development of an equational logic. He engaged in the development of a theory of it as well. For instance, one of the crucial results of Boole's logic is the proof that any expression of the language can be put in a certain normal form — every *function* of α, as Boole terms any expression in which the symbol α enters in any way. In other words, Boole strives to prove a property of his equational logic. Modest as the result may be, it is historically important to be aware of the fact that already in his first logical tract, Boole engaged in a complex enterprize. The development of an equational logic at the one hand and of a mathematical theory of this logic at the other hand. In his view, "logic not only constructs a science, but also inquires into the origin and nature of its own principles" [Boole, 1952b, 58]. In pursuing these two themes Boole was not consolidating gains already obtained. Both in presenting logic in symbolical form as in regarding logic as a mathematical subject he was breaking truly new ground.

2 THE MATHEMATICAL ANALYSIS OF LOGIC: ITS LANGUAGE

In his deductive system Boole distinguishes between expressions that have a logical interpretation and expression than don't. The last syntactic class contains expressions that play an auxiliary role in deductive processes. Among these auxiliary expressions we find quotients denoting rational numbers and two "non existent" numbers. Unless otherwise stated, I use lower case Greek letters for interpretable expressions. Capital Latin letters, on the other hand, I use for both type of expressions. Latin lower case letters will be used to denote numbers. I reserve the name *elective symbol* for the ground terms of the system.

The vocabulary

The logical vocabulary of Boole's logic contains elective symbols, usually the last letters of the alphabet, v, w, x, y, z; the symbols 1, 0, the identity symbol $=$ and representations of arithmetical operations. The nonlogical vocabulary contains brackets and numerals.[16] These numerals correspond to numbers with the following forms, for $n \neq 0$:

[16]Single symbols and their sequences are regarded here not as the formulae of the logic but as names that refer to them. Hence, the sentence $\alpha = v\beta$ *is a particular equation* does not lack a grammatical subject. The sequence of symbols '$\alpha = v\beta$' is that subject. Note in particular that the symbols 0 and 1 do not denote zero and unity. They denote themselves.

$$\frac{n}{n} \quad \frac{0}{n}$$

$$\frac{0}{0} \quad \frac{n}{m}$$

Quotients with zero as denominator were not avoided by Boole. The way in which Boole dealt with them will be considered in due course. At this stage I shall only note two things. Firstly, Boole treated $\frac{0}{0}$ as an elective symbol. In fact, as an indefinite elective symbol typically represented by v. Secondly, the quotient with the form $\frac{n}{0}$, for non zero numeral n, is treated as any numeral other than 0 and 1. This quotient arises as a particular form of $\frac{n}{m}$.

Construction rules

Assuming that all elective symbols are ground terms, the following first 4 expressions containing the terms α and β are also elective terms and the last one is a elective equation. The one but last term is an auxiliary expression.

1. $(\alpha\beta)(product)$; $(\alpha + \beta)(sum)$; $(\alpha - \beta)$ (difference)
2. if n denotes a rational number different from zero and unit, then $n\alpha$ is a pseudo elective term
3. $\alpha = \beta$

Brackets and Associativity

As a matter of fact, Boole's mathematical practice with regard to brackets can be simulated by the following syntactical policy.

- Omit external brackets
- If A and B belong to the same syntactical category, omit brackets around them in $C(\cdots AB \cdots)$, $(\cdots A + B \cdots)$, $(\cdots A - B \cdots)$.
- If A and B belong to the same syntactical category, regard their product, sum and difference as a syntactical unit in $C(\cdots A \cdots D \cdots B \cdots)$, $(\cdots A \pm \cdots \pm D \cdots \pm B \cdots)$.

Of course, it is not demonstrable that Boole was aware of these conventions. This is a case in which our exposition reflects his practice not his doctrine. A policy of introducing brackets to signal the construction of a new term would not be a good simulation for the simple reason that he did not follow such a course of action. In a modern setting, the last two conventions are justified by an appeal to the associativity of the relevant operators. Nothing of this sort can be sought here. True, the associative algebraic structure in which Boole operates guarantees the soundness of his practice. This may suggest, perhaps, that he used associativity for elective expressions. There is, however, no reason to assume that Boole contemplated a symbolical account of it. It seems more reasonable to assume that he missed associativity completely.[17]

Abbreviations

I now introduce three useful abbreviations that rest essentially on the brackets conventions:

1. $\alpha_1 + \ldots + \ldots \alpha_n =: n\alpha$
2. $-\alpha_1 - \ldots - \ldots \alpha^n =: -n\alpha$
3. $\alpha_1\alpha_2 \cdots \alpha_n =: \alpha^n$

In these abbreviations I treat elective symbols, in accordance with the symbolical doctrine, as if they were numerical expressions.

Kinds of uninterpretatibility

I close this section with a comment on a peculiar feature of Boole's logical language. This language contains, namely, two types of non-interpretable expressions.

1. not interpretable expressions consisting of elective terms
2. not interpretable expressions consisting of numerical coefficients and elective terms

The first category of non interpretable expressions are syntactically well formed expressions that violate Boole's interpretation constraint to be explained in due course. The second category of non interpretable expressions are syntactically well-formed expressions that have no logical interpretation. They arise because the only numerals to which Boole gives a non numerical interpretation are 0 and 1. It is important, therefore, to keep in mind that the presence of an expression of the form $-\alpha$, for instance, does not demand a logical reading.

3 THE MATHEMATICAL ANALYSIS OF LOGIC: THE DEDUCTIVE SYSTEM

Characterizing Laws

[Boole, 1952b, 62] turned to the calculus of operations for the foundation of his logic. He notes, namely, that elective and pseudo elective terms are commutative and distributive. In other words, they obey the Servois rules

[17] We have already seen that awareness of associativity was slow in coming. Servois did not recognize it, Gregory came close to its formulation but he, finally, did not take the step Hamilton took.

$$\begin{aligned} AB &= BA \\ A(B+C) &= AB + AC \end{aligned}$$

It is in virtue of these two rules that the symbolical strategy becomes available for elective expressions, i.e., all the processes of common algebra are applicable to them (p. 63).

A distinctive feature of the elective symbols is that they are subjected to a special version of Gregory's index law. Thus, all the elective symbols, but not only they, satisfy the equation:

$$\alpha^n = \alpha$$

I shall refer to this rule indifferently as the *index* or the *Gregory*'s law.

These three rules will appear in Boole's demonstrations as the often unstated logical truths that do the duty of premises. They are, of course, independent. To give two examples. First, expressions that denote other numbers than zero and unity satisfy the Servois's rules but not Gregory's. Second, the syntactic operation of uniform substitution, denoted by φ^{α}_{β}, satisfies the index law but isn't commutative. Note also that the wording of these rules have a distinctive content. While the Servois rules hold for all terms indiscriminately, Gregory's law is stated only with regard to ground terms. Whether compound terms satisfy this rule cannot be settled by referring to the wording of the rule. It is a matter of demonstrable interpretability, as we shall see.

Derivations

The proof method that Boole uses in making deductions of equations from equations consists, apparently, in the unrestricted use of common algebraic manipulations: factoring, multiplication, division, sum and substraction yield new equations from given ones. They are available for Boole since elective expressions obey Servois rules. Results of algebraic manipulations on given elective equations are thus admitted as algebraic consequences. Whether they are also admitted as *logical consequences* is a matter of interpretability. The importance of this distinction between those kinds of consequences cannot be exaggerated. Without recourse to interpretation we cannot evaluate properly the strength or weakness of the system. For instance, Boole putatively derives a particular sentence from a universal one, thus proving that his system simulates the traditional passage rule of subalternation. Likewise, Peirce derives the sentence *Some A are not B* from *Some B are not A*, thus proving the unsoundness of Boole's proof method. On inspection, however, both Boole and Peirce turn out to be wrong. Certainly, they obtain an algebraic conclusion by legal means. Yet, the resulting equations do not support the interpretation they impose upon them.

Replacement

As a matter of fact, Boole's algebraic manipulations can be regarded as derived rules arising from the following replacement rule, a nonuniform version of substitution.

$$\frac{A = B \quad D = C[\cdots A \cdots]}{D = C[\cdots B \cdots]}$$

This rule may be seen to codify Boole's practice and its doctrine. For he captures the essence of the algebraic method by saying that it amounts to the stricture that "equivalent operations performed upon equivalent subjects produce equivalent results" [Boole, 1952a, 63].

Derived Rules

On this basic replacement rule rests the validity of the algebraic manipulations. For instance, the replacement nature of the algebraic derivation of $AC = BC$ by multiplication from the initial equation $A = B$ is unmistakeably. One only needs to add to the assumption the validity $AC = AC$ which than can be used as the context of replacement.

To give another example, Boole uses the following inference rule in which a common factor to both premises does not occur in the conclusion.[18]

$$\frac{\mathrm{AC = BD} \quad \mathrm{A'D = B'E}}{\mathrm{ACA' = BB'E}}$$

That this rule rests on suitable applications of the replacement rule can be seen from the following argument. Starting from the assumption

$$A'D = B'E$$

we obtain by multiplication an equation that will do duty as context of replacement:

$$BA'D = BB'E$$

By replacing now the term AC for the equivalent BD we derive the desired conclusion. Note that in this case there is a silent rearrangement inside the term $BA'D$. In applications I call this rule *Boole's Cut Rule* and I shall refer to the eliminated factor as the *Cut Expression.*

I now close this section by looking at the replacement explanation of a second rule by which Boole derives a conclusion by eliminating a common member to two premises.

[18]He writes, "A convenient mode of effecting the elimination, is to write the equation of the premises, so that y shall appear only as a factor of one member in the first equation, and only as a factor of the opposite member in the second equation, and then to multiply the equations, omitting the y" [Boole, 1952b, 76].

$$\frac{\text{AC + B= 0} \quad \text{A'C + B' = 0}}{\text{A'B - AB' = 0}} \tag{1}$$

The account for this rule is essentially the same I gave before. Multiplication and distribution yield the equations

$$AA'C + AB' = 0$$
$$AA'C + A'B = 0$$

Hence, by replacement in the validity

$$0 - 0 = 0$$

we obtain an equation that leads to the desired conclusion:

$$\begin{aligned} AA'C + AB' - (AA'C + A'B) &= 0 \\ AA'C + AB' - AA'C - A'B &= 0 \\ AB' - A'B &= 0 \end{aligned}$$

In applications of this rule we are sometimes compelled to rearrange the terms involved. The result of this rearrangement may have only symbolical significance. For instance, we may need to bring the equation $\alpha\beta - \alpha$ into the form $(-\alpha)\beta + \alpha$. This transformation does not commit us to interpret $-\alpha$ logically, let alone to interpret it as the supplement of α, i.e., as an expression that satisfies

$$-\alpha \cdot \alpha = 0$$

If we did that we would reduce Boole's system to a theory of one object structures. Because then the equation $\alpha = 0$ would become provable for any arbitrary elective symbol α.

$$\begin{aligned} \alpha(1 - \alpha) &= 0 \\ (-\alpha)\alpha + \alpha &= 0 \\ 0 + \alpha &= 0 \\ \alpha &= 0 \end{aligned}$$

Furthermore, when [Boole, 1952a, 347] considers the meaning of $-y + x$ he avoids to commit himself to the interpretation of $-y$ as the supplementary class of y. The moral we have to drawn is, accordingly, that the expression $-\alpha$ used by Boole has not always logical meaning. The symbolical framework in which he works allows him to move $-\alpha$ around without forcing him to interpret it logically.[19]

[19]This is the conclusion I draw from the comment on footnote 14 in [Corcoran and Wood, 1980, 637]. The authors of this article take another position.

Some Boolean Applications of the Replacement Rule

I shall now give two substantial demonstrations that use the replacement rule. The first example establishes that the set of terms that obeys Gregory's law is closed under product.

$$(\alpha\beta)^2 = \alpha\beta$$

This example shows also the way in which the brackets conventions eclipsed associativity. The derivation can be represented in tree form. The conclusion is the root of the tree, the assumptions are the top leaves. Any level other that the top leaves arises by replacement applied to the two equations above of it or by use of a definition.

$$\dfrac{\dfrac{\dfrac{(\alpha\beta)^2 = \alpha\beta\alpha\beta \qquad \beta\alpha = \alpha\beta}{(\alpha\beta)^2 = \alpha\alpha\beta\beta} \qquad \alpha\alpha = \alpha}{(\alpha\beta)^2 = \alpha\beta\beta} \qquad \beta\beta = \beta}{(\alpha\beta)^2 = \alpha\beta}$$

By Boole's standards, this tree contains unnecessary steps. The subtree

$$\dfrac{(\alpha\beta)^2 = \alpha\beta\alpha\beta \qquad \beta\alpha = \alpha\beta}{(\alpha\beta)^2 = \alpha\alpha\beta\beta}$$

can be dispensed with. Its content is captured symbolically by $(\alpha\beta)^2 = \alpha^2\beta^2$. Numerals satisfy this equation, hence elective expressions satisfy it as well. There is, therefore, no internal reason to prove it for elective symbols. There are, nevertheless external reasons that recommend this course of action. The fact is that, given the choice, Boole preferred arguments that might not "have startled those who are unaccustomed to the processes of Symbolical Algebra" [Boole, 1952b, 110].

A Contrast

Let us contrast a symbolical and a direct demonstration of a result obtained by Boole. We prove from $\alpha + \alpha = 0$ that $\alpha = 0$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha + \alpha = 0}{2\alpha = 0} \qquad 2^{-1}2\alpha = 2^{-1}2\alpha}{2^{-1}2\alpha = 2^{-1}0} \qquad 2^{-1}2 = 1}{1\alpha = 2^{-1}0} \qquad 1\alpha = \alpha}{\alpha = 2^{-1}0} \qquad 02^{-1} = 0}{\alpha = 0}$$

Boole would have derived from the equation $2\alpha = 0$, dividing by 2 , $\alpha = 0$. The advantage of conciseness of this argument must not make us forget the advantages of the previous derivation. Therein it was clear that the deduction makes essential use of three pieces of algebraic information

- The multiplicative inverse of 2 exists: $22^{-1} = 1$.
- 1 is the unit element with regard to elective symbols: $1\alpha = \alpha$.
- A numerical product is zero when one of its factor is that.

Now I am ready to make a couple of general observation about Boole's proof strategy. In general, Boole avoids the use of symbolical arguments when an alternative is available. The unrestricted use of algebraic manipulations is limited to numerical expressions, including 0 and 1. This amount in his case to taking all their commutative ring properties for granted. He will, for instance, divide by 2 but not by α. Moreover, he will not feel compel to say that $0 \neq 1$ even though he considers the occurrence of the equation $1 = 0$ as the indication that one has tried to "unite contradictory Propositions in a single equation" (p. 104). Likewise, he does not care to remind us that 1 denotes the unit element as far as the numerals are concerned but he points out that $\alpha \cdot 1 = \alpha$ (p. 60).

It must be admitted that within the framework of symbolical algebra he was not obligated to show such a reticence in the use of symbolical arguments. And he could not have been restrained by it, for not all the equations he uses are derivable from the Servois rules. For instance, three equations that are vital in his system and for which there is no direct Servois explanation are these

$$\begin{aligned} A+B &= B+A \\ A(B-C) &= AB-AC \\ \alpha(1-\alpha) &= 0 \end{aligned}$$

A symbolical explanation is, however, easy to find. The first two need no justification since they hold in ordinary algebra. For the last one, the supplementary law, Boole's symbolic argument would take this form. Since $\alpha = \alpha^2$, transposing, $\alpha - \alpha^2 = 0$. Hence, by factoring, $\alpha(1-\alpha) = 0$.[20]

Reconstructed Boole's logic

As we have seen in the previous section the components of Boole's deductive logic in which symbolical arguments are reduced at minimum are the following.

- AB = BA
- A + B = B + A
- A(B + C) = AB + AC
- A(B - C) = AB - AC
- $\alpha(1-\alpha) = 0$

[20]He later introduced explicitly these three principles [Boole, 1952a].

- $\alpha^2 = \alpha$
- $1\alpha - \alpha$
- The Replacement Rule
- Brackets conventions
- Algebraic manipulations restricted to numerals

In the introductory section I indicated that derivation by squaring can be seen as a the recursive application of replacement. The same holds for the result of derivation by addition. I develop Boole's logic a little further by showing that derivation by substraction is also an operation that rests on replacement. To achieve this end we derive within Boole's systems a couple of equations that codify the behavior of 0 with respect to elective symbols. Boole did, of course, not forget to list these equations. They were available to him as symbolical resources.

PROPOSITION 2.

1. The Law of Identity: $\alpha = \alpha$
2. The product of an elective symbol and zero is zero: $\alpha 0 = 0$.
3. Zero is the neutral element with regard to sum: $\alpha + 0 = \alpha$
4. The expression $\alpha - \alpha$ denotes zero: $\alpha - \alpha = 0$
5. Transposition is an elective operation: If $\alpha + \beta = \delta + \gamma$ then $\alpha + \beta - \gamma = \delta$

In the proofs of these equations I shall initially use the tree notation. Afterwards I display demonstrations in abbreviated form inasmuch as the identity that sanctions the replacement are not necessarily mentioned.

1.

$$\frac{\alpha = \alpha^2 \quad \alpha = \alpha^2}{\alpha = \alpha}$$

2.

$$\cfrac{\cfrac{\cfrac{\alpha(1-\alpha) = 0 \quad \alpha 0 = \alpha 0}{\alpha 0 = \alpha\alpha(1-\alpha)} \quad \alpha\alpha = \alpha}{\alpha 0 = \alpha(1-\alpha)} \quad \alpha(1-\alpha) = 0}{\alpha 0 = 0}$$

3.

$$\begin{aligned} \alpha + 0 &= \alpha + 0 \\ &= \alpha + \alpha 0 \\ &= \alpha(1 + 0) \\ &= \alpha 1 \\ &= \alpha \end{aligned}$$

4.

$$\begin{aligned}(\alpha - \alpha) &= \alpha(1-1)\\ &= \alpha 0\\ &= 0\end{aligned}$$

Essential in the last two derivation are the equations $1+0=0$ and $1-1=0$ that are here taken for granted. These equations constitute the symbolical part of the demonstrations. Boole regards, namely, these two elective symbols as behaving exactly as the corresponding numerical expressions.

5. Assume now $\alpha+\beta=\delta+\gamma$. Then,

$$\begin{aligned}\alpha+\beta-\gamma &= \alpha+\beta-\gamma\\ &= \delta+\gamma-\gamma\\ &= \delta+0\\ &= \delta\end{aligned}$$

4 INTERPRETATION

In this section I approach the question of the interpretation of elective expressions. I shall distinguish formal from semantic interpretation. The characteristic law of the system is Gregory's law in the sense that by definition the elective symbols obey it. Numerals other than 0 and 1 don't. Generalizing from this information about ground terms into complex ones, Boole determines that only those expressions that obey the index law are logically interpretable. These terms I shall call *formally interpretable.* Elective symbols, on the other hand, may be used to represent classes or, still better, operations on classes. These classes may be classes of ordinary objects or classes of situations. Complex expressions may then be regarded as complex names of classes. The linking of elective symbols with classes I call *semantic interpretation.*

Formal Interpretation

Boole embraced the idea that the processes of symbolical reasoning are independent of interpretation. The starting point of the process, however, may consist of interpretable forms. Furthermore, at the end we must reach a *final* interpretable form. This is, of course, one of the principles of symbolical algebra. Not interpretable forms are allowed in road to a conclusion. The result itself must consist of interpretable forms.

A formal condition on interpretability that we can use states that an expression is formally interpretable if it obeys the index law. All combinations of elective symbols that satisfy the law are logically interpretable and only those combinations are interpretable [Boole, 1952b, 128]. In other words, Gregory's law "will be found of great importance in enabling us to reduce our results to forms meet for interpretation" [Boole, 1952b, 63]. In fact, as Boole later noticed, we need only to pay attention to those constructions that satisfy the following restricted version of the index law.

$$A^2 = A$$

By definition all the non numerical elective symbols satisfy this law. Of the numerals only 0 and 1 do. Notice that the term $-\alpha$ used by Boole and to which I have already made reference has no logical interpretation given the inequality $(-\alpha)^2 \neq -\alpha$. Moreover, we have already seen that the set of Gregory terms, i.e., the set of terms that obey Gregory's law is closed under product. It is also close under restricted substraction as the following symbolical argument shows:

$$\begin{aligned}(1-\alpha)^2 &= 1-2\alpha+\alpha^2\\ &= 1-2\alpha+\alpha\\ &= 1-\alpha\end{aligned}$$

For arbitrary α and β, on the other hand, it is not the case that $(\alpha+\beta)$ and $(\alpha-\beta)$ are always interpretable. This can be seen by pointing to the symbolical facts that

$$(\alpha+\beta)^2 = \alpha+\beta+2\alpha\beta$$
$$(\alpha-\beta)^2 = \alpha+\beta-2\alpha\beta$$

Two things I must stress here. In the first place, these expressions are syntactically correct and it is allowed to use them in the course of a demonstration. In the second place, there are special cases of expressions in these forms that turn out to be interpretable. Thus, a sufficient and necessary condition for $(\alpha+\beta)$ to be interpretable is expressed by the equation $\alpha\beta = 0$. And such a condition for $(\alpha-\beta)$ is expressed by $\beta = \alpha\beta$.

Semantic Interpretation

Boole assigns meaning to the elective symbols taking as their denotations operations that satisfy Servois's and Gregory's laws of combination. Operations on classes of semantic entities do that. In other words, the elective symbols can be interpreted as denoting operations that select a class of semantic entities from a given initial class. We can, if necessary, regard these operations as the characteristic functions unequivocally associated with classes. Let 1 be the name of the universe. Then the elective symbol α will be the name of the operation that selects from 1 the class E_α of semantic entities. The product $\alpha\beta$ will be the name of the class that results from selecting first the class E_β of semantic entities and then, from this class the class, E_α of semantic entities. The expression $(1-\alpha)$ has as denotation the operation that selects from the universe the class of sematic entities that are not in E_α. The expressions $\alpha - \beta$ and $\alpha + \beta$, if formally interpretable, stand for the operation that selects the class of semantic entities that are in E_α but not in E_β and the class of sematic entities that are either in E_α or in E_β but not in both.

The semantic entities may be, according to Boole, objects or situations ("cases and conjectures of circumstances"). In the first case, elective expressions may be used to represent operations on classes of objects and, on this account, they may be used to simulate monadic reasoning. In the second case, those expressions may be used to represent operations on situations and, on this account, they may be used to simulate propositional reasoning. We look in the next paragraphs at the use of Boole's logic as a simulation mechanism for propositional inferences. The monadic case, that includes syllogistic reasoning, will be dealt with afterwards.

The Propositional Case

In his analysis of propositional reasoning, Boole takes propositional truth as the basic notion. The symbol 1 stands for the "hypothetical Universe", the totality of "conceivable cases and conjectures of circumstances" [Boole, 1952b, 89]. It is important to realize that this totality must not be read as denoting Wittgenstein's *Welt*. Boole's hypothetical universe contains all that is the case and the negation of all that is the case. His totality is the totality of logical possibilities not of facts. The elective symbol α will be the name of the operation that selects from this logical universe those situations in which the proposition P_α is true. The expression $1-\alpha$, on the other hand, selects from the universe of possible situations, those situations in which P_α is not true. The product $\alpha\beta$ selects those situations in which both P_α and P_β are true. Note that elective symbols do not stand for propositions. They stand for classes. In Boole's propositional interpretation they are linked to propositions in an indirect way.

The Universe of Propositions

For a given proposition P_α it holds that only two cases are relevant: "first that the given proposition is true, and secondly that it is false" (p. 89). Moreover, the truth value "cases together make up the Universe of the proposition" (p. 89). The turn of phrase *the Universe of the proposition* originates, as we know, with De Morgan.

That the hypothetical universe is determined by both propositional possibilities is grounded in the symbolical validity

$$\alpha + (1 - \alpha) = 1 \tag{2}$$

When two propositions are involved, the four relevant cases that make up the universe are exhausted by listing their combinatorial possibilities in the nowadays familiar way:

CASES	EXPRESSIONS
P_α true P_β true	$\alpha\beta$
P_α true P_β false	$\alpha(1-\beta)$
P_α false P_β true	$(1-\alpha)\beta)$
P_α false P_β false	$(1-\alpha)(1-\beta)$

Note again that a symbolical argument establishes that the universe is exhausted by these possibilities:

$$\alpha\beta + \alpha(1-\beta) + (1-\alpha)\beta) + (1-\alpha)(1-\beta) = 1$$

In the development of his propositional logic, Boole takes as logical units equations that express the truth value of propositional representations. To say that an elective expression α is true, Boole uses the equation $\alpha = 1$. This equation restricts the logical space. Given this equation, 1 stands not longer for the totality of possible stand of affairs. Its denotation has now been restricted inasmuch as it excludes the cases in which the proposition P_α is false. In other words, the universe coincides with the class E_α in which the proposition P_α is true. In choosing for this interpretation, Boole is not culprit of forgetting that the whole of the logical space contains situations that may falsify P_α. He is proposing a restriction of that space. There is, nothing fallacious in this move.[21]

The Fate of Negation

Boole's propositional system does not lexicalize negation, in the following sense: there is no lexical item in the vocabulary that expresses the operation of negation. Negation, in this setting, corresponds to a grammatical construction. To say that an elective expression is false Boole uses the equations $\alpha = 0$. Given the fundamental equation (2), it is easy to see that in this case we restrict the universe. Since this equation implies

$$\alpha = 0 \text{ iff } (1-\alpha) = 1$$

Boole labored under the "obvious principle that a proposition is either true or false," but he did not express the principle of bivalence for elective expressions propositionally used (p. 89). The form this principle may take in his equational setting is the following:

$$\alpha = 1 \text{ or } \alpha = 0$$

The fundamental equation (2) is, of course, too weak to establish this principle. It holds, namely, regardless of what specific interpretation we have in mind, classes or propositions. Bivalence, on the other hand, is restricted to propositions only.

[21]For a different view see, for instance, [Brody, 1983, 167].

Note now that the equation $\alpha = 1$ allows us to treat α as the unit under product and that the equation $\alpha = 0$ allows us to treat it as the unit under sum. As a consequence of that it holds for any arbitrary expression A,

$$\text{If } \alpha = 1, \text{ then} \begin{cases} A = 1 \text{ if } \alpha A = 1 \\ A = 0 \text{ if } \alpha A = 0 \end{cases}$$

$$\text{If } \alpha = 0, \text{ then} \begin{cases} A = 1 \text{ if } \alpha + A = 1 \\ A = 0 \text{ if } \alpha + A = 0 \end{cases}$$

I have, we hope, achieved here the limited objective of depicting the basic ingredients of the propositional interpretation of ground elective symbols and their negation. We shall now look at the way in which compound expressions can be propositionally interpreted.

The Booleans

Note first that elective expressions correspond to propositions while elective equations correspond to assertions. The table below codifies Boole's intended propositional interpretation of his basic equations.

Expression	Interpretation
$\alpha = 1$	P_α *is true*
$(1 - \alpha) = 1$	P_α *is false*
$(\alpha\beta) = 1$	P_α *and* P_β *are simultaneously true*
$\alpha + \beta = 1$	*Either* P_α *or* P_β *are true but not both*
$\alpha + \beta - \alpha\beta = 1$	*Either* P_α *and/or* P_β *are true*
$\alpha(1 - \beta) = 0$	If P_α then P_β

De Morgan Laws as Boole's Discovery

Let us now look at the way in which in this equational system we could express some standard propositional relations.

Modus Ponens If $\alpha(1-\beta) = 0$ and $\alpha = 1$, then $\beta = 1$

Modus Tollens If $\alpha(1-\beta) = 0$ and $\beta = 0$, then $\alpha = 0$

The following two equivalences are particularly interesting because they correspond to one of the so-called *De Morgan's Laws*. To express that two propositions are simultaneously false we can use, says, Boole any of these two equations (p. 91). The first time that De Morgan himself used the closely related identities did not precede Boole's first logical treatise.

$$\alpha + \beta - \alpha\beta = 0 \quad (1-\alpha)(1-\beta) = 1$$

Similarly, to express that it is not true that two propositions are false we can use either of these two equations:

$$\alpha + \beta - \alpha\beta = 1 \quad \text{iff} \quad (1-\alpha)(1-\beta) = 0$$

It appear then that not only the medieval writers anticipated De Morgan with respect to the laws that bear his name, as we are customarily tell. Boole partially did that as well. Moreover, Boole and the Medievals related the equivalences to propositional logic while [De Morgan, 1966, 182] restricted his version to aggregates (union) and compounds (intersections):

- The contrary of an aggregate is the compound of the contraries of the aggregants
- The contrary of a compound is the aggregate of the contraries of the components.

Illustrations

The account of propositional reasoning that Boole proposes consists of the use of the derived replacement rules I previously discussed. I limit here ourselves to two examples of abbreviated derivations. In the first one we use twice Boole's Cut Rule to simulate the *Complex constructive dilemma.* In the first application, the cut expression is α. In the second one it is β. Further, note the use of algebraic manipulations used to generate equations that meet Boole's structural conditions for the use of the Cut Rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha+\beta-\alpha\beta=1}{1-\beta=\alpha(1-\beta)} \quad \alpha(1-\gamma)=0}{(1-\beta)(1-\gamma)=0}}{(1-\gamma)=\beta(1-\gamma)} \qquad \beta(1-\gamma)=0}{(1-\gamma)(1-\gamma)=0}}{(1-\gamma)=0}}{\gamma=1}$$

The second example shows that a simulation in Boole's system of the *complex destructive dilemma.* In this derivation we use the rule given in (1).

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha(1-\beta)=0}{(-\alpha)\beta+\alpha=0} \quad \dfrac{\beta\gamma=0}{\gamma\beta+0=0}}{\alpha\gamma=0}}{\alpha\gamma+0=0} \qquad \dfrac{\delta(1-\gamma)+0=0}{(-\delta)\gamma+\delta=0}}{\alpha\delta=0}$$

Observe that in this demonstration we made use of the uninterpretable forms $(-\alpha)\beta$ and $(-\delta)\gamma$. They help us to meet the structural conditions attached to the rule. Since they do not appear in the conclusion we do not need to be concerned about their lack of logical interpretation. They are mere auxiliary expressions that keep things going.

These two derivations conclude our examen of the (indirect) propositional interpretation of elective symbols. Even though they do not stand directly for propositions but for classes of situations, Boole is able to simulate propositional reasoning. In the course of his development, the importance of propositional reasoning became even greater. To the mature Boole propositional reasoning exhibits "the reasonings of ordinary life. The discourse, too, of the moralist and the metaphysician are perhaps less often concerning things and their qualities, than concerning principles and hypotheses, concerning truths and the mutual connexion and relation of truths" [Boole, 1952a, 170].

The Monadic Case

In this section I consider the other kind of interpretation that Boole proposes. Elective symbols can be regarded as operations on 1, the universal class of ordinary objects. The term $\alpha\beta$ stands for the operation that yields the class of objects that are both in E_α as in E_β. The equation $\alpha = 0$ is interpreted as the assertion that the class E_α is empty. Provided that $(\alpha - \beta)$ is an elective expression, it stands for the operation that yields the class obtained by removing from E_α any member of the class E_β. The expression $(\alpha + \beta)$, on the other hand, denotes the operation that corresponds to the union of two disjoint classes. For the sake of conciseness , in the rest of my exposition I speak of classes instead of operations. The elective expressions interpreted in this way can be used to represent monadic predicates. The elective equations, on the other hand, can be used to represent propositions containing those predicates. Chief among these propositions are the categorial propositions of traditional logic.

Categorial Propositions

Four fundamental equations with their intended categorial interpretation are given here below:

Expression	Interpretation
$\alpha\beta = 0$	No E_α is E_β
$\alpha(1-\beta) = 0$	Every E_α is E_β
$\alpha\beta = v$	Some E_αs are E_βs
$\alpha(1-\beta) = v$	Some E_αs are not E_βs

These representations form the basis for Boole's account of the inferences that make up traditional logic. In the explanation of Boole's analysis we can take over automatically the replacement inference rules previously discussed. It can be argued that Boole's attempt to account for the traditional patterns of inference met mixed success.

Successful Derivations

As I pointed out earlier on, he was under the impression, for instance, that in his logic it was possible to simulate the traditional transition from a universal proposition to a particular one. This so-called *patter of subalternation* is, however, not derivable in his logic. The point is, as we shall see in the next section, that he misread the equations $v\alpha(1-\beta)=0$ and $v\alpha = v\beta$ as entailing that there are αs which are βs, i.e., $\alpha\beta = v$. His fault was not related to the deductive engine of his logic but one related to the proper way of interpreting equations in which the notorious elective symbol v occurs. There is, for instance, nothing to fault in the following derivation

$$\dfrac{\dfrac{\dfrac{\alpha\beta = v}{\alpha\alpha\beta = v\alpha}}{\alpha\beta = v\alpha} \qquad \dfrac{\dfrac{\alpha\beta = v}{\alpha\beta\beta = v\beta}}{\alpha\beta = v\beta}}{v\alpha = v\beta}$$

His mistake, for it is a mistake, is to take the conclusion of this inference as having the same interpretation as the premise. I devote the next section to this matter. For the moment let us attend to his account of traditional inferences. I shall give one example, and examine one matter of historical relevance with regard to his use of one of the replacement rules we have discussed before. The demonstration that I present next depicts the train of reasoning he follows in establishing a syllogisms of the first figure, *barbara*.

$$\dfrac{\dfrac{\alpha(1-\beta)=0}{(1-\beta)\alpha + 0 = 0} \qquad \dfrac{\gamma(1-\alpha)=0}{\gamma\alpha - \gamma = 0}}{\gamma(1-\beta)=0}$$

In this derivation I follow Boole in making use of the elimination rule (1). An interesting feature of this derivation is the transition

$$\dfrac{\gamma(1-\alpha)=0}{\gamma\alpha - \gamma = 0}$$

In Boolean algebra the second equation is a logical truth. Since it originates in this derivation as a transformation from a contingent initial equation it seems that we have discovered a serious gap in Boole's deductive system. This situation is, however, something of a logical mirage. We have not discovered that Boole's system is flawed but rather that $\gamma\alpha - \gamma = 0$ is not true for all the interpretations of its elective symbols. And indeed, use $\gamma = 1$ and $\alpha = 0$. The equation turns out to be transformable into $-1 = 0$. Since Boole's logic gives the standard interpretation to all numerals but 0 and 1 it follows that this last equation is not true.[22] The

[22] Corcoran and Wood were the first to stress the need of further clarification of Boole's proof of Barbara. They draw, however, the conclusion that he was guilty of fallacious reasoning [Corcoran

conclusion must be, therefore, not that Boole's logic is inconsistent but that it cannot become a Boolean Algebra.

5 THE INTERPRETATION OF v

Arguably, the weakest part of Boole's treatise is his treatment of particular categorial propositions. Still, a little consideration will show that it is possible to make sense of it. I shall focus the attention to affirmative particular sentences but my remarks can be applied to the negative case as well. Let us first recapitulate the basic facts. If the intersection of two classes is empty, Boole expresses this fact by equating to 0 the product of their elective names. If two classes have at least one element in common, Boole expresses this fact by equating to v the product of their elective names. This elective symbol is the abbreviation of the quotient $\frac{0}{0}$ I mentioned in the exposition of Boole's syntax. Now, the categorial proposition *Some S is P* is true whenever the sets denoted by the subject and the predicate have members in common. Furthermore, if two sets have any members in common and they can be named their corresponding particular proposition will be true.

Truth Conditions

Boole's truth conditions for particular sentences is, therefore, the following:

> *Some S is P* is true iff the set of Ss and the set of Ps have any element in common.

The truth condition for particular propositions is projected into the symbolical language in the following condition [Boole, 1952b, 65]:

> If E_α and E_β have members in common, then $(\alpha\beta = v)$ for an arbitrary indefinite symbol v.

To understand the import of this definition we must be aware of some fundamental facts concerning indefinite symbols.

Terminology

Before attending to them, I fix some terminology. I have been following Boole in calling v an *indefinite* symbol. Products with an indefinite symbol I call *indefinite products*. I call an equation of the form $\delta = v\gamma$ with $v\gamma \neq 0$ a *particular equation*. The equation Boole uses in his characterization of particular propositions has, of course, this form. Put $\delta = \alpha\beta$ and $\gamma = 1$. Indefinite equations in regard to which $v \neq 0$ will be called *particular equations*.

---

and Wood, 1980, 627]. Incidentally, in [Boole, 1952a, 55] Boole explicitly rejects the suggestion that -1 could be a logical symbol.

Indefinite equations as disjunctions

Note now that indefinite equations are silent disjunctions. To effect this disjunction is the crucial semantic contribution of the indefinite symbol. Any indefinite product $v\alpha$ "may vary from 0 up to the entire class of" α (p. 111). On this single characterization rests our interpretation of Boole's indefinite equations. Because [Boole, 1952b, 107] interprets v as an arbitrary elective symbol much in the same way as the arbitrary constants are dealt with in the theory of linear differential equations. This means that equations containing the indefinite symbol yield particular equations when their meaning is determined. In other words, the indefinite product $v\alpha$ allows the following determinations:[23]

1. 0, with v interpreted as $(1 - \alpha)$
2. α, with v interpreted as α
3. v, with $0 \neq v$ and $v = v\alpha$

Hence, an equation of the form $v\alpha = \beta$ is equivalent to the disjunction

$$\beta = \alpha \text{ or } \beta = 0 \text{ or } \beta = \beta\alpha \text{ (with } \beta \neq 0)$$

This disjunction does not belong to the algebraic language in which Boole carries out his equational demonstrations. It belongs to the metalanguage in which the interpretation takes place. The next step in our attempt to clarify the truth condition for particular propositions is to make sense of v occurring without visible companion term. It turns out that in this case v has the same range of interpretation as any ordinary elective symbol. Remembering the equation $v = v1$, we shall say that v stands for a part of the universe in that extended sense in which even zero is considered a part of it. Thus, the trivial following disjunction holds

$$v = 0 \text{ or } v = 1 \text{ or } v = \alpha, \text{ with } 1 \neq \alpha \neq 0$$

Indefinite symbols and emptiness

This trivial conclusion conflicts with a widespread view of indefinite symbols. It has, namely, been held that v must invariably denote a non zero class.[24] In our reading of Boole, v itself is clearly allowed to vanish, i.e., to be equalized to zero. The situation is rather this. If the equation $\alpha\beta = v$ arises as representation of the fact that the classes E_α and E_β have members in common then, obviously, one the possible interpretations of the symbol v is not available. It cannot be zero. It means then that $v \neq 0$ must be true. Thus, even though in principle zero is a possible interpretation of the arbitrary symbol v, in the context induced by particular sentences this interpretation is not an option. Boole's introduction of indefinite symbols can consequently be modified into

[23] These solutions are not unique. For instance $v = 0$, and $v = 1$ would also yield 0 and α.

[24] A modern and passionate defence of this view is to be found in [Wood, 1976].

> If E_α and E_β have members in common, then $\alpha\beta = v$ is a particular equation
> for an arbitrary elective symbol v

It holds, of course, for any arbitrary elective symbol v,

If $(\alpha\beta = v)$ is a particular equation, then E_α and E_β have members in common

Boole's Interpretation Strategy

The question is not whether this condition is correct or not. It obviously is. But whether it reflects Boole's doctrine or practice. This matter can be resolved by noticing the following directive that guides Boole's interpretation strategy :

> An indefinite equation considered as representing a particular proposition is a particular equation.

The point is that the indefinite product that arises as representation of a particular equation, takes the place of a noun phase of the form *some* N and as the representative of this existential construction "though it may include in its meaning *all*, does not include *none*" [Boole, 1952a, 133]. Of course, this directive is a conventional one and could not be derived by logical argumentation from other Boolean principles. There is, however, a Griscean argument that brings us closer to that result. Because $\alpha\beta = 0$ is a stronger statement than the disjunctive $\alpha\beta = v$. Hence, according to Grice's maxims, from $\alpha\beta = v$ we may conclude $\alpha\beta \neq 0$. In other words, an indefinite equation implies conversationally that it is a particular equation. Resuming, Boole's analysis may be expressed in the following way,

> *Some S is P* iff there is an elective symbol v; such that $v = \alpha\beta$ is a particular equation.

Elective symbols and inequalities

Before continuing let us establish a fact that effortlessly follows from the preceding considerations. This fact indicates that the inequality entailed by particular equations in the metalanguage makes the need for explicit inequalities in the algebraic language less pressing.

$\alpha\beta \neq 0$ iff there is a v such that $v = \alpha\beta$ is a particular equation.

For, if the corresponding classes to α and β have no objects in common, then $\alpha\beta = 0$. Hence, $\alpha\beta \neq 0$ meant that they have some objects in common. Hence, by definition, there is a v with $v = \alpha\beta$ and $v \neq 0$. If, on the other hand, $v' = \alpha\beta$ is a particular equation, then $v' \neq 0$ holds again by definition. Therefore, $\alpha\beta \neq 0$.

Before proceeding, I must stress again a crucial distinction. The inequality $v \neq 0$ is not expressible in Boole's logical language. It belongs rather to the metalanguage in which the interpretations of equations are expressed. The incorporation of inequalities to a Boolean algebraic language took place in [Cayley, 1871, 65]. In this paper, devoted to a simplification of Boole's logic, the algebraist Cayley proposes the forms *Some X's are Y's* and *Some Y's are X's* as readings of the symbolical form

$$XYnot = 0$$

Facing the Facts

A few remarks would be useful to close this part of our exposition. An indefinite equation by itself does not carry existential content. The indefinite symbols may, for all we know, denote zero. Only when the indefinite equation is proposed as standing for a particular proposition does its form gives away its existential nature. Otherwise we shall be in need of additional information. Moreover, even if Boole's logical language does not express inequalities, the metalanguage in which the interpretation takes place does. Hence, whenever Boole establishes that an indefinite equation represent a particular proposition he is establishing at the interpretation level an inequality. The indefinite product does not denote zero under these circumstances. Let me repeat the by now familiar point. The distinction between conclusions obtained by algebraic manipulations and conclusions that are obtained as a result of interpretation is easily overlooked. Much of the proclaimed weakness of Boole's approach to logic rests ultimately in the blurring of it. Boole, it is said, cannot give a satisfactory account of particular sentences because his language does not countenance inequalities. But this, as I argued here, is only partially true. The interpretation process is also part of Boole's logic. Moreover, it has to be part of it because the symbolical method offers nothing but uninterpreted forms. Thus, the valid derivation below does not establish that its conclusion is a particular equation.

$$\frac{\alpha = \beta}{v\alpha = v\beta}$$

To determine whether it is or not, is the task of the interpretation process. And this process is not feed by algebraic form alone. Previous interpretive decisions play a major role in it.

The Problem of Subalternation

How is this account of particular equations to be reconciled with Boole's claim that in his system there is a passage from a universal proposition to a particular one? In our attempt to find an answer to this question I focus on a more general situation. In the course of his exposition Boole offers, basically, three equations that he associates with (affirmative) particular categorial propositions:

$$\text{Some } E_\alpha\text{s are } E_\beta\text{s} = \begin{cases} \alpha\beta = v \\ v\alpha = v\beta \\ v\alpha(1-\beta) = 0 \end{cases}$$

The first equation is the primitive particular equation. The other two equations are derived by using the algebraic manipulations we have become familiar with. A minimal internal condition for the adequacy of the existential reading of the derived equations would be that they are cognitively equivalent to the primitive one. [Boole, 1952b, 68] points out that "although three different forms are given for the expression of each of the *particular* propositions, everything is really included in the first form." Hence, we may lay down the following adequacy condition.

> Any expression $\varphi(v', \alpha, \beta)$ has existential import only if it entails the particular equation $\alpha\beta = v$ for an arbitrary indefinite symbol v.[25]

Not surprisingly, the derived equations listed above do not meet this cognitive criterium. Put $\alpha = 0$ and $\beta = 0$. We can then see that $v\alpha(1-\beta) = 0$ and $v\alpha = v\beta$ are both satisfied. But for no indefinite symbol v' will $v' = \alpha\beta$ be a particular equation. This means that while $v\alpha(1-\beta) = 0$ is legally derived from $\alpha(1-\beta) = 0$, it is not suitable for the existential burden it was expected to carry. As I just mentioned, to establish the desired deductive link more information is needed. Boole's treatment of subalternation suffers chiefly from lacking of an interpretive support for the existential reading of the derived equations. They look like particular equations but aren't. Of course, he could have defined the derived forms directly as particular equations, thus obtaining by decree what eluded him by toil. Later he did. In other words, Boole failed in his attempt to account within his system for all traditional inferences. He devised a representation of particular propositions that outsmarted him by blocking the generation of some invalid derivations - i.e., invalid by modern standards. Still, the failure is noticeable only at the level of interpretation. The algebraic part is beyond reproach.

Boole's Practice

In fact, in most of the applications Boole's interpretation of indefinite equations as particular ones rests in the following fact

PROPOSITION 3. Let α, β and γ be any elective symbol while v is any indefinite elective symbol. If $v = \alpha\gamma$ is a particular equation, then both $v\alpha = v\beta$ and $v\alpha(1-\beta) = 0$ have existential import.

[25] Note that our notation suggests that the indefinite symbols v' and v may be different. [Boole, 1952b, 65-66] shows that Boole was aware that $v = \alpha\beta$ and $v\alpha = v\beta$ are not equivalent.

In this proof of this Boolean assumption, essential use is made of the equation $v = v\alpha$ that follows from $v = \alpha\gamma$ for arbitrary α. Now we reason, in one of the two cases, as follows

$$\dfrac{\dfrac{v = \alpha\gamma}{v = v\alpha} \qquad \dfrac{v\alpha(1-\beta) = 0}{v\alpha = v\alpha\beta}}{v = v\alpha\beta}$$

We shall now obtain a conclusion of the same form in the other case.

$$\dfrac{v\alpha = v\beta \qquad \dfrac{\dfrac{v = \alpha\gamma}{v = v\alpha} \quad v\alpha = v\beta}{v = v\beta}}{v = v\alpha\beta}$$

The conclusions of these derivations go a long way to establish the result we want. But there is still a small extra argument to make. We see, namely, that in both cases we reach the conclusion that there is at lest one common element to the denotation of the product $v\alpha\beta$, since by assumption $v \neq 0$. A fortiori the denotation of the factors in $\alpha\beta$ do share elements. Hence, in both cases we can find an indefinite elective symbol v' with $v' = \alpha\beta$. This last equation is a particular one since the relevant sets have elements in common, i.e., $\alpha\beta \neq 0$ and, a fortiori, v' $\neq 0$.

An Example

It may be instructive to look at Boole's account of the syllogism darii because it exemplifies the manner in which apparently weak conclusions can be seen to have the strong interpretation Boole attaches to them. The task is, given the premises *Every G is B*, *Some As are G's*, to derive a particular equation that guarantees the truth of *Some A's are B's*. The first stage of the derivation is depicted below.

$$\dfrac{\dfrac{\gamma\alpha = v}{v\alpha = v\gamma} \qquad \gamma(1-\beta) = 0}{v\alpha(1-\beta) = 0}$$

The conclusion drew is $v\alpha(1-\beta) = 0$. And now begin the second stage of the interpretation. For itself this conclusion has no existential import as we have seen before. However, as already shown, the premise $\alpha\gamma = v$ fix the interpretation of v in such a way that it warrants the desired existential interpretation. Because it now follows that there is an elective symbol v' such that $v' = \alpha\beta$ is a particular equation. An additional important observation to make is the following one. Note that we can derive $v(1-\beta)\alpha = 0$. Still, no conclusion concerning the set of not B's can be obtained. This fact has bewildered some critics. It seems to them that Boole works with the *ad hoc* condition that when v is prefixed to a product it is understood that it has members in common with only the first factor. There is, however, not need of such an explanation. The *interpretation* of $v(1-\beta)\alpha = 0$ does not support the reading *Some not-B are not-A* for the simple reason that the information contained in the premises is silent about the existential properties of the set that correspond to B. That is the only reason. Again, this is an instance of a misunderstanding that arises because the distinction between algebraic derivation and interpretation process has not been sufficiently heeded. There is nothing intrinsically existential to $v(1-\beta)\alpha = 0$. Moreover, $v(1-\beta)\alpha$ and $v\alpha(1-\beta)$ are fully equivalent. Consider, for instance, each of the possibilities dictated by the range of $v\alpha$:

- $v = (1 - \alpha)$
- $v = \alpha$
- $v = valpha$, $(v \neq 0)$

Of course, if we take $v(1-\beta)$ as starting point the result will be the same.

The demonstration of darii is typical of the whole situation. When Boole uses the weak equations to express the conclusion of valid syllogisms there is always a particular premise that warrants his interpretation. In the absence of such a piece of additional information, the legally obtained conclusion would not support the existential reading.

6 A MATTER OF INTERPRETATION

Boolean scholars have shown disagreement with respect to the interpretation of the indefinite product $v\alpha$. Some of them admit the contingent nature of the equation $v\alpha = 0$ (C. I. Lewis, T. Hailperin). Others reject this possibility (Kneale, Dippert). For them the product $v\alpha$ never vanishes. If this is the case, then, of course, the equation $\alpha = v\beta$ will not be the most general solution of $\alpha(1-\beta) = 0$. Such a disagreement, involving perceptive and thorough researches suggests an obscurity in the sources. And, indeed, in his discussion of the proper interpretation of the indefinite symbol v there in a tension that Boole introduced in [Boole, 1952a]. Because in this work Boole gave two apparently conflicting accounts of the indefinite elective symbol. According to one account the product $v\alpha$ never vanishes, according to the other account it sometimes does. I give here these passages in length.

Vanishing Product

I begin with the passage that favors the vanishing interpretation. Commenting on the sentence, the premiss, *Men not mortal do no exist* and its representation

$$x = y + \frac{0}{0}(1-y)$$

he writes in [Boole, 1952a, 97–98]:

> This implies that mortals (x) consist of all men (y), together with such a remainder of beings which are not men $(1 - y)$, as will be indicated by the coefficient $\frac{0}{0}$. Now let us inquire what reminder of "not men" is implied by the premiss. It might happen that the remainder included all the beings who are not men, or it might include some of them, and no others, or it might include none, and any one of these assumptions would be in perfect accordance with our premiss. In other words, whether those beings which are not men are *all*, or *some*, or *none*, of them mortal, the truth of the premiss which virtually asserts that all men are mortal, will be equally unaffected, and therefore the expression $\frac{0}{0}$ here indicates that *all*, *some*, or *none* of the class to whose expression it is affixed must be taken.
>
> Although the above determination of the significance of the symbol $\frac{0}{0}$ is found only upon the examination of a particular case, yet the principle involved in the demonstration is general and there are not circumstances under which the symbol can present itself to which the same mode of analysis is inapplicable. We may properly term $\frac{0}{0}$ an *indefinite class symbol*, and may, if convenience should require it, replace it by an uncompounded symbol v, subject to the fundamental law $v(1 - v) = 0$.

This lengthy passage inclines the balance strongly in favor of the disjunctive, vanishing interpretation of the indefinite symbol v.

The non-vanishing Product

The conflicting interpretation, on the other hand, may be partially related to this passage in [Boole, 1952a, 69]:

> Consider, lastly, the case in which the subject of the proposition is particular, e. g. "Some men are not wise." ... The requisite form of the given proposition is, therefore, "Some men are not-wise." Putting, then, y for "men," x for "wise," i.e., "wise beings," and introducing v as the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed, we have
>
> $$vy = v(1 - x)$$

The content of this quote is no a surprise for the reader of [Boole, 1952b]. In fact, it points to the inalienable existential character of particular sentences. The carrier of the existential import is, undoubtedly, the side condition that the indefinite products induced by such propositions cannot vanish. We can express the content of this stricture by the following truth condition

$$v\alpha = v\beta \text{ iff } v,\ \alpha \text{ and } \beta \text{ have elements in common}$$

It is now clear that the equation is a particular equation inasmuch as it entails $v \neq 0$. Boole's analysis of particular sentences thus removes a nagging problem that affected his first account of the symbolic representation of particular propositions. For now we can see that the following assertion holds

$$v\alpha = v\beta \text{ has existential import}$$

But this account of particular propositions and the interpretation of the indefinite product therein does not topple the balance in favor of the non vanishing camp. The crucial case concerns the representation of universal sentences. The relevant passage in Boole's work is this one:

> Let us consider next the case in which the predicate of the proposition is particular e.g. "All men are mortal."
>
> In this case it is clear that our meaning is, "All men are some mortal beings," and we must seek the expression of the predicate, "some mortal beings." Represent then by v, a class indefinite in every respect but this, viz., that some of its members are mortal beings, and let x stand for "mortal beings," then will vx represent "some mortal beings." Hence if y represent men, the equation sought will be
>
> $$y = vx$$

To Boole, that much is clear, a predicate is particular if it is modified by the particle *some.* This modification is not given syntactically by the universal categorial propositions. It results as consequence of Boole' analysis. The next step Boole takes is to establish that the predicate of an affirmative universal proposition has to be represented by a non empty product. Therefore, the symbol that corresponds to the subject cannot denote zero. It is worthwhile to notice that this account is not restricted to affirmative sentences. Boole defended the semantic doctrine that "the true meaning of the proposition 'No Y's are X's,' is 'All Y's are not-Xs.' The subject of that proposition is, therefore, universal-affirmative, the predicate particular-negative" [Boole, 1952a, 241-242]. By giving this account, then, Boole built into its interpretation of equations the existential import of universal propositions. Note, incidentally, that the new strictures governing the interpretation of the symbolical counterparts of categorial sentences proves the validity of subalternation. Because if the universal sentence is true, the companion equation expresses already that the denotation of the subject is not empty. And since it is a part of the predicate's denotation, the truth of the particular sentence is enforced upon us.

Facing the Facts

The situation seems, therefore, to be the following. Theoretically, the indefinite product may vanish. That is the way in which it is defined in the first place.

In practice, it never vanishes. At least, it will not vanish if it is interpreted as standing for a term making up a categorial proposition. When these terms are modified explicitly of implicitly by *some*, they carry existential import and this fact must be reflected by the interpretation of the symbolical representation.

Indefinite Product and Elimination

The situation that I have just sketched is clear. Still, therein lies the source of confusion. Because Boole is not consequent in his approach. The serpent in this paradise is his theory of elimination. The elective symbol v is not eliminable from $v\alpha = v\beta$ but it is eliminable from $\alpha = v\beta$. In the first case, he notes that "v is not quite arbitrary and therefore must not be eliminated. For v is the representative of *some*, which though it may include in its meaning *all*, does not include *none*" [Boole, 1952a, 133]. Since v can be eliminated from $\alpha = v\beta$, we must conclude that in this context v may be quite arbitrary in the sense that it may including in its meaning *all* and *none*. For to eliminate a symbol v from an equation amounts to the consideration of what follows from two circumstances

1. $v = 0$
2. $v = 1$

Since in the representation of particular sentences the interpretation of $v\alpha$ excludes zero as possible meaning, it excludes $v = 0$ as well. On the other hand, since this symbol is eliminable in the representation of universal sentences if follows that $v = 0$ is available and thus $v\alpha$ cannot exclude zero. Boole's theory of elimination throws us back to the original theory. The indefinite product can vanish and it vanish in universal contexts. It seems as if Boole could not make his mind about the proper interpretation of indefinite products. Moreover, the dissention among Boolean scholars reflects the fractured account that Boole himself gave in [Boole, 1952a]. The confusion that this dissention exposes is due to Boole's own obscurity. A reasonable alternative account is the following. The indefinite that originates as representative of natural language sentences is not allowed to vanish, while the indefinite that arises from expansion is allowed to do so. Unfortunately this account is not covered by the evidence. Because [Boole, 1952b, 68] referred to the expansion generated indefinite in his account of the proper interpretation of indefinite equations induced by natural language .

Evaluation

Still, we can question Boole's account, we do not need to question his logical sanity. At first glance there is no difference between Boole's account of the indefinite product in particular and universal propositions. Both of them favor the non vanishing interpretation . But a difference there is and one that, as elimination considerations shows, has logical consequences. In particular propositions the existential import of the subject is not inferred by interpretation. It is already there in the modification of the subject by *some.* And for this reason it cannot be suspended. For universal sentences the situation is different. For the particular reading of the predicate has been inferred by interpretation.

The most sympathetic account of Boole's position is, therefore, this. If we read the predicate particularly, then the corresponding product will never vanish and zero will never be the denotation of the subject. If we choose, for whatever reason, not to read the predicate in this fashion, then the indefinite product may be allowed to vanish, and zero becomes a denotation option for the subject. The option of suspending the particular interpretation is only open for indefinite products that arise in connection to universal propositions. As far as universal propositions are concerned, existential import can be suspended. Universal propositions can be expressed "either hypothetically, *All men (if men exist) are fallible*, or absolutely, (experience having assured us of the existence of the race), *All men are fallible*" [Boole, 1952a, 92]. It is when experience has assured us that a predicate is not empty that we can adopt the non vanishing view. Before that happens we can better rely on the vanishing view.

7 A METALOGICAL RESULT

The previous section was taken up with two fundamental uses of Boole's logic: the explanation of propositional and of monadic reasoning. In this section I pay attention to a metalogic argument that Boole developed. He tried to show that any expression of the language can be brought to a "disjunctive" form. To express this result I need to fix some extra notation. Let now $\phi(\alpha)$ be an expression containing some tokens of the elective ground symbol α. By $\varphi(0)$ ($\varphi(1)$) I refer to the result of replacing all occurrences of α by 0 (1).

PROPOSITION 4. The elective expression $\varphi(\alpha)$ is equivalent to an expression of the form $\varphi(1)\alpha + \varphi(0)(1 - \alpha)$.

To prove this proposition Boole develops a symbolic argument. His first step is a Lagrangian one. Given that the elective symbols obey the rules of the calculus of operations, he expands the elective expression $\varphi(x)$ in Lagrangian mood as

$$\varphi(\alpha) = \varphi(0) + \varphi'(0)\alpha + \frac{\varphi(0)\alpha^2}{2!} + \frac{\varphi'(0)\alpha^3}{3!} \dots$$

This power series he regarded as an infinite polynomial to be handled exactly like finite ones. Thus, applications of the index law, distributivity and substraction precede the application of the replacement rule which closes the following tree derivation. The conclusion is called the *expansion* of the original expression. Each terms in a full expansion consists of an elective part (the constituent) and of a numerical part (the coefficients).

$$\dfrac{\dfrac{\varphi(\alpha) = \varphi(0) + \varphi'(0)\alpha + \frac{\varphi''(0)\alpha}{2!} + \frac{\varphi'''(0)\alpha}{3!} \dots}{\varphi(\alpha) = \varphi(0) + (\varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} \dots)\alpha} \quad \dfrac{\varphi(1) = \varphi(0) + \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} \dots}{\varphi'(1) - \varphi(0) = \varphi'(0) + \frac{\varphi(0)}{2!} + \frac{\varphi'''(0)}{3!} \dots}}{\varphi(\alpha) = \varphi(0) + (\varphi(1) - \varphi(0))\alpha}$$

To conclude Boole's argument, we observe that the expansion just obtained is used in the equivalent form

$$\varphi(\alpha) = \varphi(1)\alpha + \varphi(0)(1 - \alpha) \tag{3}$$

In this way, assuming principles of the calculus of operations Boole establishes to his own satisfaction a derived property of elective expressions. Furthermore, he generalizes this result to expressions of the form $\varphi(\alpha, \beta)$ in the following way.[26] By expanding this expression first on α and than on β we obtain the following three equations:

$$\begin{aligned} \varphi(\alpha, \beta) &= \varphi(1, \beta)\alpha + \varphi(0, \beta)(1 - \alpha) \\ \varphi(1, \beta) &= \varphi(1, 1)\beta + \varphi(1, 0)(1 - \beta) \\ \varphi(0, \beta) &= \varphi(0, 1)\beta + \varphi(0, 0)(1 - \beta) \end{aligned}$$

Combining these equations by substitution, Boole derives

$$\varphi(\alpha, \beta) = \varphi(1, 1)\beta\alpha + \varphi(1, 0)(1 - \beta)\alpha + \varphi(0, 1)\beta(1 - \alpha) + \varphi(0, 0)(1 - \beta)(1 - \alpha) \tag{4}$$

The generalization given below, can be proven by induction in the way Boole proves the initial cases:

PROPOSITION 5.

$$\begin{aligned} \varphi(x_1, x_2, \dots, x_n) = {} & \varphi(1_1, 1_2, \dots, 1_n)x_1x_2 \dots x_n \\ & + \varphi(1_1, 1_2, \dots 0_n)x_1x_2 \dots (1 - x_n) \\ & \vdots \\ & + \varphi(0_1, 0_2, \dots, 0_n)(1 - x_1)(1 - x_2) \dots (1 - x_n) \end{aligned}$$

Consequences

Boole points out several consequences attached to the expansions. In the first place, for any pair of constituents of an expansion holds that their product vanishes, i.e., $\tau_1\tau_2 = 0$. The point is of course that any pair of constituents differs

[26] Actually, this expression is closer to the notation he uses in [Boole, 1952a]. [Boole, 1952b] writes $\varphi(\alpha\beta)$ for our $\varphi(\alpha, \beta)$.

in at least one place. There must be, namely, at least one ground term α that appears in one term while (1 - α) appears in the other. In the second place, Boole argues the following special case of a more general situation:

PROPOSITION 6. The sum of the constituents in any expansion is 1.

As an informal proof of this proposition Boole establishes the following equation:

$$1 = \alpha\beta + \alpha(1-\beta) + (1-\alpha)\beta + (1-\alpha)(1-\beta)$$

In words: the sum of the constituents is 1. To establish this result Boole seems to think in the following way. Take an arbitrary pair of symbols α and β. Consider now an arbitrary expression in these symbols that is put equal to 1: $\varphi(\alpha, \beta) = 1$. According to (4) we have

$$1 = \varphi(1,1)\beta\alpha + \varphi(1,0)(1-\beta)\alpha + \varphi(0,1)\beta(1-\alpha) + \varphi(0,0)(1-\beta)(1-\alpha)$$

The argument then goes as follow. Multiply both sides of the last equation by $\beta\alpha$. This gives, on the assumption that the constituents are elective symbols,

$$\beta\alpha = \varphi(1,1)\beta\alpha$$

These two equations combine by replacement into

$$1 = \beta\alpha + \varphi(1,0)(1-\beta)\alpha + \varphi(0,1)\beta(1-\alpha) + \varphi(0,0)(1-\beta)(1-\alpha)$$

Reiteration of this procedure, but now starting with $(1-\beta)\alpha$ leads to the equations

$$\begin{array}{rcl}(1-\beta)\alpha & = & \varphi(1,0)(1-\beta)\alpha \\ 1 & = & \beta\alpha + (1-\beta)\alpha + \varphi(0,1)\beta(1-\alpha) + \varphi(0,0)(1-\beta)(1-\alpha)\end{array}$$

Two repetitions of these steps lead finally to the result Boole wanted.

Functional Elective Equations

One of the features of algebraic logic is the preoccupation with finding the solution of the functional equation $\alpha = \psi(\vec{\beta}, \gamma)$, starting from the equation in the symbols $\gamma = \varphi(\alpha, \vec{\beta})$. Boole approaches this matter in the following way. Consider the equation in α and β, $\varphi(\alpha, \beta) = 0$. We use the first expansion form obtained to obtain

$$0 = \varphi(\alpha, 0) + (\varphi(\alpha, 1) - \varphi(\alpha, 0))\beta$$

Ordinary algebra procedures then yield

$$\beta = \frac{\varphi(\alpha, 0)}{\varphi(\alpha, 0) + \varphi(\alpha, 1)}$$

Finally, expansion on α but now using the form in (3) yields

$$\beta = \frac{\varphi(1,0)}{\varphi(1,0) - \varphi(1,1)}\alpha + \frac{\varphi(0,0)}{\varphi(0,0) - \varphi(0,1)}(1-\alpha) \tag{5}$$

As I pointed out before, Boole's quotients can have four forms.

- Numerator and denominator are the same but different from 0. The quotient then reduces to 1.
- Numerator is 0 but the denominator isn't. The quotient then reduces to 0.
- Numerator is not 0 but the denominator is. The quotient is then left as it is.
- Numerator and denominator are both 0. The quotient is then replaced by an arbitrarily chosen indefinite elective symbol.

Some comments are here in order. In the first place, Boole regards the quotient $\frac{n}{0}$, that fits in the third possibility, as any other number denoting expression. Its presence thus inviting an application of Proposition (7) below. In the second place, the quotient $\frac{0}{0}$ is conventionally interpreted as an indefinite elective symbol v. It is of the great importance to remember that the product $v\alpha$ has a disjunctive interpretation that does not exclude the possibility $v\alpha = 0$. To forget it effect misunderstanding Boole as I show in the next section. Before entering into these matters let us regard the question of the numerical coefficients.

PROPOSITION 7. If in an expansion there is a member $n\alpha$ such that the coefficient n does not obey Gregory's index law, then $\alpha = 0$.

Essentially, Boole's argument goes as follows. Let $n\alpha + m\beta$ be an expansion of a given expression. Under this assumption n and m are numerical expressions and $\alpha\beta = 0$. Suppose further

$$\gamma = n\alpha + m\beta \tag{6}$$

$$n^2 \neq n \tag{7}$$

Then, multiplying by α we have from (6),

$$\gamma\alpha = n\alpha \tag{8}$$

since $\alpha\beta = 0$. Squaring yields now

$$\gamma\alpha = n^2\alpha$$

since only n does not obey the index law. Substracting (8) from this last equation yields

$$0 = (n^2 - n)\alpha$$

But by assumption (7),

$$(n^2 - n) \neq 0$$

Hence, division by $(n^2 - n)$ is allowed, yielding

$$\alpha = 0$$

THE MISSING SOLUTION

Boole's approach to the solution of logical functional equations can be illustrated at the hand of a controversial example. Given a function $\varphi(\alpha, \beta) = 0$ and the task of resolving the equation $\beta = \psi(\alpha)$, we can resort to (5) by putting

$$\psi(\alpha) = \frac{\varphi(1,0)}{\varphi(1,0) - \varphi(1,1)}\alpha + \frac{\varphi(0,0)}{\varphi(0,0) - \varphi(0,1)}(1 - \alpha)$$

Consider, for instance, the equation $\alpha(1-\beta) = 0$. The solution to the functional equation $\alpha = \psi(\beta)$ is (9) below

$$\psi(\beta) = \frac{0}{0}\beta + \frac{0}{1}(1 - \beta)$$
$$\psi(\beta) = v\beta \tag{9}$$

The Most General Solution

Equation (9), i.e. $\alpha = \mathrm{v}\beta$ is hailed in [Boole, 1952b, 68] as "the most general solution" of the equation $\alpha(1 - \beta) = 0$. And at a later stage [Boole, 1952b, 108] calls it "the complete solution of the equation". This "arbitrary elective symbol" that occurs in this solution allows, as I have already pointed out, a disjunctive interpretation. Hence, this most general solution contains the following particular solutions obtained by letting the interpretation of $v\beta$ to " vary from 0 up to the entire class of" β.

1. $\alpha = \beta$ when $v = \beta$
2. $\alpha = 0$, when $v = (1 - \beta)$
3. $\alpha = v$, when $v \neq 0$ and $v = v\beta$

The disjunctive nature of the product $v\beta$ accounts for the generality of the solution. Unfortunately, Boole describes this product in some passages of [Boole, 1952a] as if were less general than it is, effectible suggesting that it can never become zero. If this were the case then, of course, $\alpha = v\beta$ would not be the most general solution to the equation. This *not vanishing* view of the product $v\alpha$ is in harmony with those Boole scholars who decried Boole's assertion that $\alpha = v\beta$ is the most general solution because he would have missed the obvious solution $\alpha = 0$.[27]

[27]This is the view defended in [Corcoran and Wood, 1980], [Grattan-Guinness, 2000] and [Panteki, 2000].

The Complete Solution

The completeness of Boole's functional solution is due to other logical fact. The solution to the original equation that is not included in $v\alpha$ is, of course, $\beta = 1$. But it is the case that $\beta = 1$ entails $\alpha = v\beta$. Hence, the indefinite equation $\alpha = v\beta$ and the disjunction $\alpha = v\beta$ or $\beta = 1$ are logically equivalent.

8 THE LAWS OF THOUGHT

Boole's most mature work on logic shows several differences when compared with his first essay. I shall list here the most salient changes. In [Boole, 1952a] logical laws are related to the laws of the operations of the mind inasmuch as they are mediated by language. His aim is, namely, to derive the logical laws "from a consideration of those operations of the mind which are implied in the strict usage of language as an instrument of reasoning" [Boole, 1952b, 46]. The elective symbols are regarded in this work as class symbols that guide the selection of individuals from the universe of discourse. They become effectively class symbols. The universe, on the other hand, is not longer the static notion of his previous work. It may vary according to the limits we want to impose upon the discourse. Still, "whatever may be extent of the field within which all the objects or our discourse are found, that field may properly be termed the universe of discourse" [Boole, 1952a, 46]. The laws $\alpha^2 = \alpha$ $\alpha\beta = \beta\alpha$ have mental counterparts in the sense that selection of objects from the universe of discourse shows the same structure. The logical interpretation of 0 and 1 is explicitly addressed and the laws that govern their behavior as arithmetical factors are logically interpreted. In this work Boole derives the supplementary law $\alpha(1-\alpha) = 0$ from the fundamental equation $\alpha^2 = \alpha$.

Kinds of Propositions

Boole elaborated a distinction among natural language propositions. There are, according to him, primary and secondary propositions. Primary propositions express relations among things while the secondary one express relations among propositions. Primary propositions consist of a subject and a predicate. They are divided into three types to which different representations:

- Subject and predicate are both universal: $\alpha = \beta$
- Subject is universal but the predicate is particular: $\alpha = v\beta$
- Subject and predicate are both particular: $v\alpha = v\beta$

Secondary propositions, on the other hand, are propositions that express properties of relations of propositions. Every primary proposition gives rise to a secondary one, namely "to that secondary proposition which asserts its truth, or declares its falsehood" [Boole, 1952a, 58]. Particles that often, but certainly not always, signal the presence of a secondary proposition are the connectives *or*, *and*, *if*, etc. Secondary propositions, then, include metalanguage propositions as well as object level propositions.

Symbolical Methods

The principles of symbolical algebra are endorsed again. The class symbols obey the laws of thought identical to the arithmetical laws that govern 0 and 1. Given that the "formal processes of reasoning" do not depend on the interpretation of the expressions involved, we are allowed to treat the class symbols as if they were 0 and 1 (p. 70).

> We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values 0 and 1; perform upon them as such all the requisite processes of solution; and finally restore to them their logical interpretation.

A Basic Principle

A principle that appears for the first time is expressed by [Boole, 1952a, 41] in the following words:

> Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them.

This principle yields the basis for a rather powerful proof method [Boole, 1854, 76]. Boole uses it in his proof of the theorem of development for which he had previously given a Lagrangian proof.

Development

One of the processes of solution Boole uses involves the development of symbolic expressions. Boole satisfies himself of the validity of this process by an argument different from the Lagrangian argument used in his previous work. Now he reasons as follows. He assumes

$$\varphi(\alpha) = a\alpha + b(1-\alpha)$$

By putting $\alpha = 1$ and $\alpha = 0$ he determines

$$\begin{aligned} \varphi(1) &= a \\ \varphi(0) &= b \end{aligned}$$

Substituting these values of a and b in the initial equation we obtain

$$\varphi(\alpha) = \varphi(1)\alpha + \varphi(0)(1-\alpha)$$

Boole is satisfied that the expansion and the original expression are equivalent. Regarding α as a variable ranging over 0 and 1, he notes that the expansion and the original expression assume the same value.

Expressions with more than one variable are expanded in similar fashion, leading to equations such as

$$\varphi(\alpha,\beta) = \varphi(1,1)\alpha\beta + \varphi(1,0)\alpha(1-\beta) + \varphi(0,1)(1-\alpha)\beta + \varphi(0,0)(1-\alpha)(1-\beta)$$

Elimination

The expansion of an expression plays an important role in the process of eliminating a class symbol from it. Boole provides three proofs of the fact that the result of eliminating α from $\varphi(\alpha) = 0$ is given by

$$\varphi(\alpha)\varphi(1-\alpha) = 0$$

This equation expresses "that what is equally true, whether a given class of objects embraces the whole universe of disappears from existence, is independent of that class altogether" [Boole, 1952a, 122]. I sketch here one of these proofs. From the expansion of $\varphi(\alpha) = 0$, Boole derives by a symbolical argument

$$\begin{aligned} \alpha &= \frac{\varphi(0)}{\varphi(0)-\varphi(1)} \\ 1-\alpha &= \frac{\varphi(1)}{\varphi(0)-\varphi(1)} \end{aligned}$$

Substitution in the fundamental equation $\alpha(1-\alpha) = 0$ yields

$$\begin{aligned} -\frac{\varphi(0)\varphi(1)}{(\varphi(0)-\varphi(1))^2} &= 0 \\ \varphi(0)\varphi(1) &= 0 \end{aligned}$$

To express, for instance, what is true if the indefinite symbol v takes the value 0 or 1 in the equation

$$\alpha = v\beta$$

We first bring this equation into the form

$$\alpha - v\beta = 0$$

Then proceed as follows.

$$\begin{aligned} \alpha - 0\beta &= \alpha \\ \alpha - 1\beta &= \alpha - \beta \end{aligned}$$

Hence, the result of the elimination is

$$\alpha(\alpha - \beta) = \alpha(1 - \beta) = 0$$

The Cut Rule

The process of elimination of a variable from a given equation is used now to render idle the cut rule. Because Boole now reduces any set of equations into a single equation that becomes a suitable input for the process of elimination. He discusses two ways of affecting this reduction. The first one uses an idea of Lagrange for the solution of systems of equations and was already mentioned by him in [Boole, 1952b, 112–114]. He shows that the set of equations

$$\varphi_1 = 0, \varphi_2 = 0, \ldots, \varphi_n = 0$$

may be replaced by the equation

$$\varphi_1 + c_1\varphi_2 + \ldots + c_m\varphi_n = 0$$

where c_i is an arbitrary constant that does not satisfy Gregory's law. To this new equation the process of elimination can be applied.

The second method is explained as follows. First, if all the equations to be reduced contain only positive coefficients, then the equation

$$\varphi_1 + \varphi_2 + \ldots \varphi_n = 0$$

has the same logical import as the original set. The point is that on expansion they yield the same set of elective expressions.

When a member of the set has negative coefficients then the equation that would be used for elimination is this one

$$\varphi_1^2 + \varphi_2^2 + \ldots \varphi_n^2 = 0$$

Squaring does, of course, not affect class symbols. Moreover, all the numerical coefficients will be positive. The resulting equation falls then under the strictures of the previous case.

Resuming the Differences

Let us now close this survey by noting the most important point in which Boole's first logical essay differs from his most mature work.

1. there is a special notation for supplement: $\overline{\alpha} = 1 - \alpha$
2. elective symbols represent classes not operations on classes.
3. the index law is given only in the form $\alpha^2 = \alpha$.
4. division is robustly rejected as a logical operation.
5. $\alpha(1 - \alpha) = 0$ is deduced as a theorem
6. α^n is regarded as uninterpretable
7. the following laws are explicitly introduced
   (a) $\alpha + \beta = \beta + \alpha$
   (b) $\alpha(\beta - \gamma) = \alpha\beta - \alpha\gamma$
   (c) $\alpha - \beta = -\beta + \alpha$
   (d) $0\alpha = 0$
   (e) $1\alpha = \alpha$
8. There is a Taylor-free account of the expansion theorem
9. There are three proofs of an algorithm for the elimination of elective symbols.
10. Methods for the reduction of systems of equations are further explored

Boole paid more attention in this work to propositional reasoning than before. In his treatment of secondary propositions, however, he abandoned the idea that the symbols of the system correspond to class of situations in which propositions are true. Now they correspond to portions of "the time for which the corresponding proposition is true" [Boole, 1952a, 175]. He is not the first mathematician to try to use temporal structures to model mathematical theories. Hamilton went along this path before him. But we must not exaggerate the role of time in his theory of propositional reasoning. Because we may "pass from the forms of common language to the closely analogous forms of the symbolical instrument of thought here developed, and use its processes, and interpret its results, without any conscious recognition of the idea of time whatever" (p. 174).

LAST WORDS

I have now described Boole's equational logic. I have stressed the lack of arbitrariness in Boole's symbolical procedures. The apparent arithmetical manipulations are, in last instance, recursive replacements. We have also paid special attention to the proper interpretation of the indefinite symbols. I hope that the exposition will reduce their puzzling nature. Crucial to the treatment of these symbols are two things. First, recognition of the mathematical analogy Boole was elaborating:

they are to logic what arbitrary constants are to analysis. Second, the distinction between the algebraic and the interpretative phase in a Boolean deduction. Finally, I have emphasized the novel nature of having an attempted proof of a metalogic proposition. The algebra of logic and its metalogic were born from the same man.

Part 3

The Logic of Absolute Terms: Jevons

This part consists of three sections. In (1) I describe in general terms the contents of the next sections. In (2) I describe Jevons's first logical treatise. In (3) I deal with the notion of substitution of similars. The last section is devoted to Jevons's logical machines.

1 INTRODUCTION

Jevons's logical publications started with a book intended to supersede Boole's logic [Jevons, 1864]. He considered the equational logic of Boole as a product of the doctrine of the quantification of the predicate. It was, in his eyes, "perhaps, one of the most marvellous and admirable pieces of reasoning ever put together" [Jevons, [1890] 1991, 66]. Still, his admiration was mixed with objections. His standpoint was that the logic of Boole revealed a turn of mind that did not favor the construction of a deductive system in close harmony with the logic of common thought. Two features of Boole's approach did Jevons single out for criticism. First of all, the presence in Boole's system of uninterpretable expressions. Second, the arithmetically motivated rules of inference. Born out this criticism, Jevons's own equational logic got closer than Boole's logic to Boolean algebra. Furthermore, Jevons developed a proof method, the method of indirect inference, that led to an attempt to mechanize reasoning. For his efforts to expedite his proof method culminated in the construction of a machine aimed at replacing for the most part the action of thought required in logical deduction. Even though the development of modern computers was not influenced by Jevons's computational work, his device "illustrates in a clear and direct way the general principles upon which logical machines, electrical or otherwise, are based" [Mays and Henry, 1953, 485].[28]

2 PURE LOGIC

In the following pages I give a broad description of Jevons's first logical work in which he developed an equational logic. His logical system is supposed to

[28]About the lack of influence of Jevons's logical machine on the development of modern computers, see [Burks and Burks, 1988].

be Boole's divested of its "mathematical dress" (p. 5). In the exposition I try to expose his logical ideas after the manner of modern logical discourse without distorting the historical facts. In particular, when principles, definitions or laws clearly refer to arbitrary terms I shall use metalinguistic expressions that help to express the wanted generality. I shall also occasionally use brackets to parse Jevons's formulae more easily

Language and Interpretation

The logical vocabulary of Jevons's equational logic consists of lower and upper case letters standing for positive and negative ground terms: A, B, C, ..., a, b, c, ..., the symbol 0, and the enclitic negation *not.* I do not follow Jevons practice in the formulation of his logic whereby logical laws are formulated for ground terms only. In this account I shall use Greek letters as meta-variables, and I assume, as before, that expressions refer to the proper objects of the logic, thus sparing quotation marks. The compound expressions are these:

1. $\alpha\beta$
2. $\alpha + \beta$
3. not-α

Extensionality

An extensional interpretation of these constructions yields the following readings. $\alpha\beta$ represents the intersection of two classes, $\alpha + \beta$ their union, and not-α complement. A lower case letters such as a can be regarded as an abbreviation of not-A. The symbol 0 stands for the empty class.

Intensionality

This is not the reading that Jevons had in mind. Jevons distinguishes between the extent and the intent of the meaning of an expression. The number of individuals denoted make up the extent. The qualities connoted make up its intent. A proposition expresses "the result of a comparison and judgement of the sameness or difference of meaning of terms, either in intent or extent of meaning" (p. 4). The task that Jevons took upon himself was to develop a logic "founded on comparison of quality only, without reference to logical quantity" (p. 4). In the intentional interpretation the product $\alpha\beta$ stands for the sum of the meanings of the factors. The sum $\alpha + \beta$ stands either for the meaning of α or the meaning of β but we do not know which (p. 24). The complement term, not-α, on the other hand, has as its meaning the absence of the qualities that constitutes the meaning of its positive counterpart (p. 30). In the next paragraph I touch on an argument that tips the balance against this intensional account of meaning within the general framework of Jevons's logical preferences. This argument shows that Jevons overstated his non-extensional position [Grattan-Guinness, 1991, 19], [Mosselmans, 1998, 83–90].

Laws of Logic

The following laws are considered by Jevons to be primary laws.

Law of Simplicity $\alpha\alpha = \alpha$

Law of Unity $\alpha + \alpha = \alpha$

Law of Contradiction $\alpha\,\text{not-}\alpha = 0$

Law of Duality $\alpha = \alpha\beta + \alpha\,\text{not-}\beta$

The Law of Unity is one of the distinguishing features of Jevons's logical language *vis-á-vis* the Boolean one. In this setting + is interpreted as the counterpart of the natural language inclusive connective *either*.

Note that the inclusive disjunction that Jevons clearly and famously prefers fits uneasily with his intensional account of meaning. Because it is plainly false that the meaning of $\alpha + \alpha$ is captured by saying that this expression stands for the meaning of α or α but we do not know which. Inclusive disjunction is, of course, expressible in Boole's language, but he never felt the need to lexicalize it.

Duality and Expansion

It is worthwhile to notice that in applications, Jevon's Law of Duality and Boole's expansion theorem yield the same result. The point is that any two products π_i and π_j in Boole's or in Jevon's expansions are mutually exclusive in the sense that $\pi_i\pi_j$ reduces to 0. For Boole's logical concerns, then, there is no pressing need to adopt Jevons's interpretation of +.

Next to these laws that are singularized by name. Jevons isolated some more that are not.

$\alpha\beta = \beta\alpha$	$0 \cdot 0 = 0$	$\text{not-}(\alpha\beta) = \alpha\ \text{not-}\beta +$ $\text{not-}\alpha\beta\ \text{not-}\alpha\text{not-}\beta$
$\alpha + \beta = \beta + \alpha$	$0 + 0 = 0$	$\text{not-}(\alpha + \beta + \beta) =$ $(\text{not-}\alpha)(\text{not-}\beta)(\text{not-}\gamma)$
$\alpha + (\alpha\gamma) = \alpha$ $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	$\alpha\,\text{not-}\alpha = \alpha\,\text{not-}\alpha \cdot 0$	$\text{not-not-}\alpha = \alpha$

Note that the associative law is missing from Jevons's list. On the other hand, it is worthwhile to notice the first appearance of an absorption law, $\alpha + (\alpha\gamma) = \alpha$.

Properties of =

Jevons recognizes the three fundamental properties of =, namely, reflexivity, symmetry and transitivity. Equations that express the reflexivity of = are considered as fundamental in the sense that they "state the condition of all reasoning" (p 11).

Cognitively, however, they are useless because they do not offer new information. Symmetry, is captured by the assertion that the proposition $\alpha = \beta$ and $\beta = \alpha$ are interchangeable (p. 10). Transitivity, considered by him the fundamental principle of all the sciences, is captured by the assertion that out the propositions $\alpha = \beta$ and $\beta = \gamma$, the proposition $\alpha = \gamma$ can be formed (p. 12).

Replacement

Jevons formulates three replacement rules. The first one establishes that "same parts make same wholes" (p. 15). The second one is the schema of combination by multiplication (p. 17). The third one says that the two sides of an equation "may be used indifferently one in place of the other, whenever either occurs" (p 18).

$$\frac{\alpha = \alpha \quad \beta = \beta}{\alpha\beta = \alpha\beta}$$

$$\frac{\alpha = \beta}{\alpha\gamma = \beta\gamma}$$

$$\frac{\alpha = \beta \quad \delta = \gamma[\cdots\alpha\cdots]}{\delta = \gamma[\cdots\beta\cdots]}$$

Direct and Indirect Inference

Any conclusion that is obtained by making use of transitivity is called a direct inference. Indirect inferences, on the other hand, are obtained by the following fifth steps method. Starting point is the equation $\varphi = \psi$ in the ground terms A, B.

(1) The first step in achieving a logical conclusion is to list the set of all the products, π, in these symbols and their complements:

$$\begin{array}{ll} AB & Ab \\ aB & ab \end{array}$$

(2) The second step consists of the combination by multiplication of each π with the original equation thus yielding the following 4 derived equations

$$\begin{aligned} AB\varphi &= AB\psi \\ Ab\varphi &= Ab\psi \\ aB\varphi &= aB\psi \\ ab\varphi &= ab\psi \end{aligned}$$

Each of the sides of these equations simplifies to π or 0.

(3) Hence, we have the following simplified results:

$$\pi = \pi$$
$$\pi = 0$$
$$0 = \pi$$
$$0 = 0$$

The first equation is called *included subject*, the last one is called *excluded subject.* The remaining two are called *contradictions.*

(4) Every π that yields a contradiction is eliminated from the list of products. Consider now a product α. Let $\pi_1, \ldots, \pi_n$ be the set of products including the symbols in α. (5) Then a consequence of the original premise $\varphi = \psi$ is

$$\alpha = \pi_1, \ldots, \pi_n$$

In short, Jevons's proof method consists of the production of a complete list of possibilities followed by the elimination of each possibility that is incompatible with the premises of the problem. The remaining expressions contain the answer to the original problem.

Examples

Before proceeding to generalize this account let us look at one example. Consider the equation A = B. I want to show that from this equation a = b follows.

(1) We list first the products:

$$AB \quad Ab \quad aB \quad ab$$

(2) Consequently, we consider the derived equations

$$ABA = ABB \quad AbA = AbB \quad aBA = aBB \quad abA = abB$$

(3) The simplification of the obtained equations yields

$$AB = AB \quad Ab = 0 \quad 0 = aB \quad 0 = 0$$

(4) To obtain a reduced list of products we eliminate those which yield contradictions:

$$AB \quad ab$$

(5) The conclusions can now be listed

$$A = AB \quad B = AB \quad a = ab \quad b = ab$$

Note that at this stage we can make use of a direct inference to obtain the desired conclusion

$$\frac{\mathrm{a} = \mathrm{ab} \quad \mathrm{ab} = \mathrm{b}}{\mathrm{a} = \mathrm{b}}$$

I have discussed the method with regard to one equation in two ground terms. In general, one will have to look at systems of n equations in m positive ground terms. The list of products will be m^2 and the number of derived equations will be $\mathrm{m}^2\mathrm{n}$. The rest of the procedure will remain unchanged.

A Brief Comparison

In Boole's system we can pass from the equation $A = B$ to $1 = \frac{A}{B}$. The second equation contains a quotient that cannot be logically interpreted. Through expansion, though, this quotient gives raise to an interpretable expression that we, for comparison sake, write down in a mixed Boole-Jevons notation:

$$\frac{1}{1}AB + \frac{1}{0}Ab + \frac{0}{1}aB + \frac{0}{0}ab$$

The strictures concerning the meaning of the numerical coefficients reduces this expression to

$$AB + vab$$

The constituents of this expression correspond to the terms obtained at step (4) of Jevon's procedure, provided, that is, that we put $v = 1$.

In general, the relation between the expansions of Jevons and Boole is this. Jevons's included subject corresponds to Boole's expressions with 1 as numerical coefficient, while the excluded subject corresponds to expressions with $\frac{0}{0}$ as coefficient. The contradictions correspond to those coefficients that reduce the product to 0 (p. 76). The main difference is, then, that Jevons does not discriminate between Boole's 1 and v [Lewis, 1918, 78]. I shall close the treatment of Jevons by mentioning two aspects of his later work related to the system developed in his first logical essay. The first one concerns Jevons's principle of replacement of identicals—*substitution of similars* is what he called it. The second one concerns the mechanization of inference.

3 REPLACEMENT

Remember that the second step in Jevon's indirect proof method consists of the simplification of terms. Essential in this process are both the laws that establish the identity of terms as the replacement laws that guide the simplification. In [Jevons, 1869] replacement is extensively discussed. According to Jevons, traditional logic was based on an asymmetric view of propositions. The Medieval *dictum de omni* enables us to assert of the subject whatever we know of the predicate but not the other way around for "the proposition states the inclusion of the subject in the predicate, and not of the predicate in the subject" [Jevons, [1890] 1991, 87]. The equational view of categorical propositions makes necessary to transform the dictum, making it suitable for the new symmetric forms. Because under this circumstances we can say that "whatever is known of either term of the proposition is known and may be asserted of the other" (p. 87). The replacement of equivalents becomes in Jevon's view the prototype of all reasoning. In fact, all forms of reasoning are seen by him to be instances of replacement. There must be a proposition, the active one, that establishes the equality of the terms. This active proposition shall always be an equation. But the context of substitution doesn't need to be that.

A valid schema of reasoning is the following one in which § is used "to denote any possible or conceivable kind of relation" (p. 97):

$$\frac{\mathrm{A} = \mathrm{B} \quad \mathrm{B}\ \S\ \mathrm{C}}{\mathrm{A}\ \S\ \mathrm{C}}$$

Among the examples Jevons's provides of this generalization of contexts of substitution beyond the syntactic constrains of the equational view are these:

$$\frac{\mathrm{A} = \mathrm{B} \quad \text{C is the father of B}}{\text{C is the father of A}}$$

$$\frac{\mathrm{A} = \mathrm{B} \quad \text{C is a compound of B}}{\text{C is a compound A}}$$

As we shall later see, in this passage one perceives some echoes of De Morgan's early views on relations. [Peirce, 1984, 446] registers Peirce's agreement with the idea that "all reasoning is by substitution" but objects to the suggestion that "all substitutions when algebraically denoted appear as the substitution of equals for equals".

4 MECHANIZATION OF REASONING

Jevons recognized that his indirect proof method was a long and tedious one. This tediousness can be removed, he thought, by using laborsaving mechanical aids. One of these devices is explained in [Jevons, 1869], the *Logical Abacus.* The other one is the subject of [Jevons, 1870] and it may be considered a machine capable of reasoning. This device has been called *The Logical Piano.*

The Logical Abacus

The logical abacus was used primarily for didactic purposes. It consisted of an inclined black board furnished with four legs and 4 sets of rectangular slips of wood. The first set consisted of 4 slips each of which represented one possible combination of the expressions in the set { A, B, a, b}. The second set contained 8 slips each of which represented one combination of the elements of {A, B, C, a, b, c}. The third set contained 16 slips and corresponded to the combinations of the elements of {A, B, C, D, a, b, c, d, }. Similarly, the last set of slips contained 32 members and they exhausted the possible combinations among the members of the set {A, B, C, D, E, a, b, c, d, e}. Each slip contained holes above the positive terms and below the negative ones. On these holes pins could be placed. These pins were used to remove quickly all the slips containing a determined term. A straight-edged was placed below the row of pins and by lifting it all the slips containing a determined term would be removed. Note that the slips that contain terms without pin would remain in situ. Thus, removing slips with the term A would let all the slips with its negative counterpart in place.

Consider, for instance, the derivation of the equation A = C from A = B and B = C. We take the second set of slips:

A	A	A	A	a	a	a	a
B	B	b	b	B	B	b	b
C	c	C	c	C	c	C	c

The next step is to remove all the slips with A and b. The result is

A	A	a	a	a	a
B	B	B	B	b	b
C	c	C	c	C	c

Finally, we remove the slips that contain B and c, obtaining

A	a	a	a
B	B	b	b
C	C	C	c

Take now "the only combination containing A, and observe that it is joined with C". Hence, A = C [Jevons, [1890] 1991, 116–117].

Logical Diagrams

The logical diagrams developed by John Venn (1834–1923) may be regarded as geometrical representations of Jevons's slips. The four compartments determined by two overlapping closed curves are used to represent the four combinations allowed by two terms and their complements. Three of such curves may be used to represent determine eight compartments, etc. But beyond five terms there was no hope for this representational strategy [Venn, 1881]. Soon Jevons became dissatisfied with this "primitive form of an IBM punch-card machine" because it left much room for error [Gardner, 1958, 97].

The Logical Piano

In a letter written in 1868, Jevons comments on a new design: "the machine works in a few moments any logical problem involving no more than four distinct terms or things. It will be in appearance like a large accordion or a very small piano" [Jevons, 1977, 185]. Jevons's logical piano consisted of an input and an output device. The input device of this machine was a keyboard with 21 keys.

FINIS	·|·	d	D	c	C	b	B	a	A	COPULA	A	a	B	b	C	c	D	d	·|·	FULL STOP

To initialize the display one has to press the FINIS key so that the output displayed the following logical space:

A	A	A	A	A	A	A	A	a	a	a	a	a	a	a	a
B	B	B	B	b	b	b	b	B	B	B	B	b	b	b	b
C	C	c	c	C	C	c	c	C	C	c	c	C	C	c	c
D	d	D	d	D	d	D	d	D	d	D	d	D	d	D	d

Consider now the effect of feeding this machine with the information that *every A is B* and *every B is C.* First we express this information in equational form: A = AB and B = BC. To enter A = AB, one moves from left to right pressing the keys

$$A \Rightarrow \text{COPULA} \Rightarrow A \Rightarrow B \Rightarrow \text{FULL STOP}.$$

This action refreshes the display that now shows the result of eliminating all the Ab combinations:

A	A	A	A					a	a	a	a	a	a	a	a
B	B	B	B					B	B	B	B	b	b	b	b
C	C	c	c					C	C	c	c	C	C	c	c
D	d	D	d					D	d	D	d	D	d	D	d

Finally, entering B = BC leads to the final output below:

A	A							a	a			a	a	a	a
B	B							B	B			b	b	b	b
C	C							C	C			C	C	c	c
D	d							D	d			D	d	D	d

The display shows no combination of Ac so that we can conclude that every A must be C. Note that this last step, the evaluation of the final output is not yet mechanized.

Judging the Logical Piano

Jevons's logical piano counts was the first successful mechanical digital computer [Buck and Hunka, 1999, 21]. But he did not think that his machine had any practical use beyond a didactic one. Its chief importance is theoretical:

> It demonstrates in a convincing manner the existence of an all-embracing system of Indirect Inference, the very existence of which was hardly suspected before the appearance of Boole's logical works [Jevons, [1890] 1991, 171].

His skepticism was shared by John Venn who listed the following objections against logical machines:

1. Intricate logical problems are artificial constructs devised to illustrate the use of rules and methods
2. No machine deals with the four components of the reasoning process, namely
   (a) The translation into a logical language
   (b) The translation into the machine language
   (c) The deduction of conclusions from premises
   (d) The interpretation of the conclusions

In Venn's view, logical machines can be useful only with regard to the third component. He despaired of the possibility of mechanizing the translation and interpretation steps [Venn, 1881, 119–121].

FINAL WORDS

In this section I have attended to the first long study of Boole's logic. Some of the changes Jevons proposed were independently discovered by other researches. The non exclusive interpretation of disjunction, for instance. I have stressed here the topic of the mechanization of reasoning because it is a topic we usually associate with Boole. Boole himself had no interest in such matters [Boole, 1952b, 9-10]. It is Jevons who deserve all the credit for actually putting the algebra of logic at the service of the mechanization of reasoning.

Part 4
The Logic of Monadic Predicates: C. S. Peirce

This part consists of 5 sections. In (5) we give a more specific description of the aims of the rest of the section. Section (6) describes Peirce's first paper on the algebra of logic. Section (7) deals with the paper in which Peirce abandons the equational perspective in favor of the inequality approach. Section (8) considers the topic of the negative copula. The last section is devoted to two topics: the polyvalent nature of Peirce's inequality operator and to Peirce's notion of monotone replacement.

5 INTRODUCTION

Peirce, overlooking the history of logical studies in England saw De Morgan, Hamilton and Boole as belonging to a tradition founded by William of Sherwood, Duns Scotus and William of Ockham. His first published work on logic, [Peirce, 1867], extended this tradition inasmuch as his first logical contribution was proposed as a modification of the system handed down by Boole. Peirce's article counts as the second major critical analysis of Boole's work [Grattan-Guinness, 1991]. However, Peirce wrote this article unaware of the modifications proposed by Jevons [Peirce, 1931, 3.199]. In this section I shall comment on the version of algebraic logic defined by Peirce. I shall limit myself here to the non relational part of his theory. The handling of relations and relatives within algebraic logic will be the subject of another part.

Peirce's theory of quantification is often associated with his handling of relational inferences. I shall argue here that his treatment of absolute terms in [Peirce, 1870] is essentially quantificational. True enough, in his initial papers Peirce lacks a notation for quantifiers. My contention is, however, that he had very early the notion of quantification. To be slightly more specific, the idea of implicit quantification appears perhaps for the first time in the wake of his replacement of the notion identity by the notion of implication as the key logical notion. A clear separation between quantifications and propositional logic is established when Peirce adopted the notion of explicit quantification. Peirce's implication symbol ceased hereby to do the compounded duty of being the means to express universal quantification and material implication next to being a binding device. Binding will be the work of the quantifier and the variables. The Boolean would be the realm of implication.

Some observations will also be made with respect to Peirce's view of Jevons's substitution of similars and Peirce's use of replacement as a global logical mechanism. In doing so I shall refer to Peirce's subsequent system of explicit quantification and to his final return to implicit quantification in the system of existential graphs. We shall see that one of the motivations for Peirce's study of issues of quantification was his attempt to find an adequate symbolic representation of particular propositions. He came up with more than five proposals ending with the existential graphs proposal in which the dash that marks the open place in a monadic predicate *is a man* was extended to become an existential ligature, thus expressing the same meaning as the expression $\exists x Man(x)$. In this section we shall point out to the early seeds of this logical theory.

6 PEIRCE IN 1867

Peirce's first contribution to algebraic logic is [Peirce, 1867]. In the next pages we shall describe its main features. In general, Peirce adopts in this paper a catholic attitude with regard to operators. Standard arithmetical ones are used along their logical variants — a comma beneath arithmetical symbols turns them into symbols denoting logical operations. The vocabulary of the modified logic contains the following symbols.

Symbol	Name
$\underset{,}{=}$	Numerical identity
$\underset{,}{+}$	Logical addition
$,$	Logical multiplication
$\underset{,}{-}$	Logical substraction
$;$	Logical division
$+$	Arithmetical addition
$-$	Arithmetical substraction
0	Zero
1	Unity

Interpretation

As far as the interpretation is concerned, Peirce points out that the expression $\underset{,}{=}$ stands for the relation of identity among classes, i.e., it expresses numerical identity. The numerals 0 and 1 have the standard Boolean interpretation. From [Boole, 1952a] Peirce took an overline as the sign of an operation that transform a class symbol into a complement name.

The comma denotes logical multiplication, i.e., the operator that combines the objects that belong to two classes into one class. It is, therefore, an alternative for Boole's juxtaposition of terms as representation of intersection. Logical multiplication has the following algebraic properties:

$$\alpha, \alpha \underset{,}{=} \alpha \quad \alpha, \beta \underset{,}{=} \beta, \alpha \quad (\alpha, \beta)\gamma \underset{,}{=} \alpha, (\beta\gamma)$$

Remarkable is that Peirce makes here room for associativity along the properties already singled out by Boole. In [Peirce, 1870], with generous inaccuracy, Peirce would attribute this principle to Boole [Peirce, 1931, 3.81].

The symbol $+\!\!,$ denotes inclusive disjunction among classes while $+$ has the exclusive reading. Hence, like Jevons, Peirce introduces a logical symbol to be interpreted as inclusive disjunction. Unlike him, however, Peirce does not refuse to lexicalize Boolean disjunction, or *arithmetical* disjunction as he calls it. He lists three fundamental algebraic properties of logical addition,

$$\alpha +\!\!, \alpha =\!\!, \alpha \quad \alpha +\!\!, \beta =\!\!, \beta + \alpha \quad (\alpha +\!\!, \beta) +\!\!, \gamma =\!\!, \alpha +\!\!, (\beta +\!\!, \gamma)$$

He also remarks that logical and arithmetical addition coincide whenever the relevant classes have no members in common, i.e.,

$$\text{If no } \alpha \text{ is } \beta, \text{ then } \alpha +\!\!, \beta =\!\!, \alpha + \beta$$

This is, of course, the reason why the expansions of Boole and Jevons coincide.

Distributivity

Peirce argues that both logical addition and multiplication obey the distribution laws:

$$(\alpha +\!\!, \beta),\gamma =\!\!, \alpha,\gamma +\!\!, \beta,\gamma \quad (\alpha,\beta) +\!\!, \gamma =\!\!, (\alpha +\!\!, \gamma),(\beta +\!\!, \gamma)$$

The right equation, the distribution of multiplication over addition, was already used by Boole. The left one, distribution of addition over multiplication, appears here, according to [Schröder, 1880, 85], for the very first time. Peirce, on the other hand, attributes it to Jevons [Peirce, 1931, 3.81].

Remarkable is that Peirce offers a proof for these principles. In fact what he proves is that if multiplication distributes over Boolean disjunction, then it also distributes over logical disjunction.[29] Because his proof strategy is this. He expands the three terms as follows:

$$\begin{aligned}
\alpha &= \alpha,\overline{\beta},\overline{\gamma} + \alpha,\beta + \alpha,\gamma + \alpha,\beta,\gamma \\
\beta &= \beta,\overline{\alpha},\overline{\gamma} + \alpha,\beta + \beta,\gamma + \alpha,\beta,\gamma \\
\gamma &= \gamma,\overline{\alpha},\overline{\beta} + \alpha,\gamma + \beta,\gamma + \alpha,\beta,\gamma
\end{aligned}$$

Then he shows that both sides of the distributivity equations reduce to the same term. For instance, given that

$$\alpha +\!\!, \beta = \alpha,\overline{\beta},\overline{\gamma} + \beta,\overline{\alpha},\overline{\gamma} + \alpha,\beta + \alpha,\gamma + \alpha,\beta,\gamma + \beta,\gamma$$

he concludes

$$(\alpha +\!\!, \beta),\gamma = \alpha,\gamma + \alpha,\beta,\gamma + \beta,\gamma$$

[29] [Houser, 1991, 13] offers a different assessment of Peirce's argument.

Now, since $\alpha, \gamma = \alpha\gamma + \alpha\beta\gamma$ and $\beta, \gamma = \beta\gamma + \alpha\beta\gamma$, it follows

$$(\alpha, \gamma) +\!\!\!\!, \, \beta\gamma = \alpha, \gamma + \alpha, \beta, \gamma + \beta, \gamma$$

Hence, it seems that Peirce establishes the desi There are, however, three steps that make his argument non cogent. ı is the reduction of $(\alpha +\!\!\!\!, \, \beta), \gamma$. Because it assumes the equation belov

$$(\alpha, \overline{\beta}, \overline{\gamma} + \beta, \overline{\alpha}, \overline{\gamma} + \alpha, \beta + \alpha, \gamma + \alpha, \beta, \gamma + \beta, \qquad + \alpha, \beta, \gamma + \beta, \gamma$$

This equation assumes the validity of the distribu .l multiplication over arithmetical addition. A principle in need of as much proof as the principle Peirce wanted to prove.

In [Peirce, 1880], Peirce remarks that his own proof from implicational principles is easy but tedious. Schröder proved that the distributivity law is independent from the principles that he identifies as Peirce's "axiomatic" basis. He then augmented the basis with an extra principle and managed to prove the law. Peirce's proof of a crucial lemma appears "almost verbatim" in Huntington's own proof of distributivity [Huntington, 1904, 300-302].[30]

Uninterpretable and Indefinite Expressions

The symbols $-\!\!\!,$ and $F_,$ denote, respectively, logical substraction and logical division. Peirce remarks that Boole did "not make use of the operations here termed logical addition and subtraction" [Peirce, 1931, 3.18]. Still, the treatment of these operations shows Boolean traces. In the first place, with regard to the interpretation of substraction Peirce accepted well formed expressions that lack any logical interpretation. Because for logical substraction holds that the expression

$$\alpha -\!\!\!, \, \beta$$

is uninterpretable if the class denoted by β is not contained in the class denoted by α.

In the second place, logical substraction opens the door for indeterminate expressions. For even when substractions are interpretable they do not denote a fixed object. In other words, the equation below does not have a unique solution

$$x = \alpha -\!\!\!, \, \beta$$

The range of x will vary from the α class to the class denoted by the Boolean expression $\alpha - \beta$. From Peirce's point of view, therefore, Boolean substraction is a special case of logical substraction. The disjunctive nature of logical substraction is further stressed when Peirce adds a new characterization. Consider first $0 -\!\!\!, \, 1$ as the prototype of an uninterpretable expression and let v be, as in Boole's

[30] For more details see [Dipert, 1978], [Houser, 1991] and [Crapo and Roberts, 1969].

system, the prototype of a "wholly indeterminate" expression. Then the following equations holds:[31]

$$\alpha \mathbin{-\!\!\!,} \beta = v, \alpha, \beta + \alpha, \overline{\beta} + (0 \mathbin{-\!\!\!,} 1), \beta, \overline{\alpha}$$

Now, if every β is α, then $\beta\overline{\alpha} = 0$. Hence, the uninterpretable expression $(0 \mathbin{-\!\!\!,} 1), \beta, \overline{\alpha}$ vanishes, thus leaving only interpretable expressions behind. Now consider

$$\alpha \mathbin{-\!\!\!,} \beta = v, \alpha, \beta + \alpha, \overline{\beta}$$

Under these circumstances let us consider the two extreme values that, according to Boole. the indeterminate symbol can take, namely, $v = 0$ and $v = \alpha$. First, if $v = \alpha$, then

$$\begin{aligned} \alpha \mathbin{-\!\!\!,} \beta &= \alpha, \alpha, \beta + \alpha, \overline{\beta} \\ &= \alpha, \beta + \alpha, \overline{\beta} \\ &= \alpha, (\beta + \overline{\beta}) \\ &= \alpha \end{aligned}$$

Secondly, if $v = 0$, then

$$\begin{aligned} \alpha \mathbin{-\!\!\!,} \beta &= \alpha, \overline{\beta} \\ &= \alpha - \beta \end{aligned}$$

These two alternatives fix the boundaries within which logical abstraction, by the definition Peirce gives, takes its denotation.

Division and Interpretation

Until now we have only considered substraction as a source of uninterpreted expressions. But logical division has the same effect. The equation below, for instance, does have an interpretation provided every α is β.

$$x = \alpha \mathbin{,} \beta$$

Furthermore, the range of x will vary from the α class to the class denoted by $\alpha + \overline{\beta}$. Let now $1 \mathbin{,} 0$ be an uninterpretable expression, then the following equation holds

$$\alpha \mathbin{-\!\!\!,\!.} \beta = \alpha, \beta + v\overline{\alpha}\overline{\beta} + (0 \mathbin{-\!\!\!,} 1), \alpha, \overline{\beta}$$

If every α is β, then this equation reduces to

$$\alpha \mathbin{-\!\!\!,\!.} \beta = \alpha, \beta + v\overline{\alpha}\overline{\beta}$$

[31] It is worthwhile to notice that to avoid circularity this equation cannot be regarded as an alternative definition of logical substraction.

Venn, Boole and Peirce's Logical Operations

Peirce's characterization of logical division apparently found his way into [Venn, 1881]. Speaking of the class denoted by the logical quotient $\frac{x}{y}$, Venn writes:

> The full description therefore of the desired class is given by saying that it comprises the whole of x, and a quite uncertain part of $\overline{x}\,\overline{y}$ namely of what is neither x nor y. If we like to put a peculiar symbol (v) to represent perfect uncertainty of range of application, we should write down the class symbolically as $x + v\overline{x}\,\overline{y}$ [Venn, 1881, p. 71-72].

Given that the meaningfulness of the quotients demands that the x-class be a part of the y-class, we can rewrite Venn's expression as $xy + v\overline{x}\,\overline{y}$. It is then evident that in these circumstances Venn's characterization and Peirce's are actually the same.

Boole and Peirce on Division

Venn did not mention Peirce in his discussion of logical quotients. The reason is, probably, that the roots of Peirce's analysis can be traced back to Boole's treatment of logical quotients. Because the quotient $\frac{\alpha}{\beta}$ gives under Boolean expansion

$$\frac{1}{1}\alpha\beta + \frac{1}{0}\alpha\overline{\beta} + \frac{0}{1}\overline{\alpha}\beta + \frac{0}{0}\overline{\alpha}\overline{\beta}$$

Hence,

$$\frac{\alpha}{\beta} = \alpha\beta + v\overline{\alpha}\overline{\beta}$$

Peirce's analysis of logical division can, therefore, be regarded as strictly Boolean. Furthermore, the analysis of logical substraction we just saw can be easily embedded within the Boolean framework. Suppose that for x with $= x = \alpha -_{,} \beta$ holds

$$\beta +_{,} x = \alpha$$

In Boole's logic this equation amounts to

$$\beta + x - \beta x = \alpha$$

The task is now to express x in terms of α and β:

$$\begin{aligned} \beta + x - \beta x = \alpha \quad & \text{iff} \\ x - \beta x = \alpha - \beta \quad & \text{iff} \\ x(1-\beta) = \alpha - \beta \quad & \text{iff} \\ x = \frac{\alpha - \beta}{1 - \beta} \quad & \end{aligned}$$

The Boolean expansion of the quotient yields

$$\frac{1-1}{1-1}\alpha\beta + \frac{1-0}{1-0}\alpha\overline{\beta} + \frac{0-1}{1-1}\overline{\alpha}\beta + \frac{0-0}{1-0}\overline{\alpha}\overline{\beta}$$

Hence,

$$x = \alpha \mathbin{-\!\!\!,} \beta = v\alpha\beta + \alpha\overline{\beta}$$

Again, a result that basically mirrors Peirce's analysis. Peirce resumed the situation we have been discussing by pointing out that to free himself from logical division and logical substraction, the operations that do not yield an unambiguous result, Boole has "only to expand expressions involving division ... in order to be able to use the ordinary methods of algebra" [Peirce, 1931, 3.89].

Logical Substraction and Existential Propositions

Peirce argues that one of the advantages of logical substraction is that it allows us to represent the meaning of noun phrases headed by *some*. In his view, "Boole cannot properly express some a". Peirce seems to assume that *some a* is indeterminate in denotation. It can denote the whole set denoted by a or to a set consisting of some but not all members of the a-class. Let now i be an expression denoting "a class only determined to be such that only some one individual of the class" a comes under it. The expression $\alpha -, i$ is used by Peirce to represent *some* α. Therefore,

$$\alpha -, i = v, \alpha, i \mathbin{+\!\!\!,} \alpha, \overline{i} \mathbin{+\!\!\!,} (0 \mathbin{-\!\!\!,} 1), i, \overline{\alpha}$$

If i is an α unit class, then the meaning of Peirce's expression reduces to

$$v, \alpha, i \mathbin{+\!\!\!,} \alpha\overline{i}$$

This means that in Peirce's view, the meaning of *some* α varies between all of α and some but not all of α. Likewise, to say that the classes denoted by α and β have members in common Peirce would presumingly use this equation

$$\alpha -, i = \beta$$

thus signifying

$$v, \alpha, i \mathbin{+\!\!\!,} \alpha\overline{i} = \beta$$

i.e., either the classes coincide or the β-class is a proper part of the α one. Of course, Boole achieves the same result but he must first exclude $v = 0$ as an available option.

7 PEIRCE IN 1870

A New Logical Operation

[Peirce, 1870]represents a mayor change within the Boolean tradition. For here Peirce puts identity aside as the fundamental logical relation. Its place is taken over by the notion of inclusion, represented by the symbols $\prec$, $<$ and $>$. The reason Peirce gives for preferring inclusion is that it is " a wider concept than equality and therefore logically a simpler one" [Peirce, 1931, 3. 47n]. In a later paper, [Peirce, 1880], he remarks that, in general, "if one conception implies another, but not the reverse, then the later is said to be the simpler" (3.173).

The symbol for implication, $\prec$, has been called a *quantified copula* because the expressions in which it occurs as the main symbol are interpreted as restricted universal propositions [Ladd, [1883] 1983, 25]. Thus, the expression

$$\text{Frenchman} \prec \text{man}$$

is taken by Peirce as meaning that every Frenchman is a man "without saying whether there are other men or not". Similarly, the expression

$$\text{Frenchman} < \text{man}$$

is given a quantificational reading inasmuch as it is interpreted as meaning that every Frenchman is man "but that there are men besides Frenchmen" [Peirce, 1931, 66].

The symbol $>$ is crucially used in the characterization of two important functions, 0^α and 1α:[32]

$$\begin{aligned} 0^\alpha = 0 \quad &\text{iff} \quad \alpha > 0 \\ 1\alpha = 1 \quad &\text{iff} \quad \alpha > 0 \end{aligned}$$

Inequalities and Existential Propositions

Now, the inequality $\alpha > 0$ is interpreted by Peirce as the assertion that α does not vanish. Therefore, he is able to propose three equivalent representations of a particular proposition. He proposes, for instance, the following representations of the proposition that some animals are horses [Peirce, 1931, 3.141–3]:

1. $0^{\mathrm{a,h}} = 0$

2. $1(\mathrm{a, h}) = 1$

3. a,h > 0

[32]Note that we write 1α and not $1,\alpha$. The product $\alpha\beta$ behaves differently from α,β. It is, as we shall later see, the product of two relations. In other words, it is generated by a non commutative multiplication operator.

The advantages of the inequality symbol for the representation of particular propositions are obvious. Still, it is not clear from its symbolic form alone that a particular proposition is the contradictory of a universal negative expression

$$\mathrm{a} \prec \bar{h}$$

These three representations convey but do not show this information. For instance, "as h, b = 0 means that no horse is black, so $0^{h,b} = 0$ means that some horse is black". In the closing pages of this paper Peirce comes up with an alternative representation of particular propositions. Let the denotation of an absolute term m be exhausted by the infinite disjunction

$$M' \,'\!+\; M'' \,'\!+ M''' \,'\!+$$

abbreviated by $'m'$. The particular proposition *Some M is N* can then be represented by

$$'m' \prec n$$

We witness in this paper, then, Peirce avowed preoccupation with finding the proper representation of existential propositions [Peirce, 1931, 3.138]. His preoccupation would later influence Schröder's evaluation of Frege's notation for quantifiers. It was Schröder's belief that Frege succeeded were Boole had failed, namely in finding an adequate representation for particular propositions. Pointing out to the proposals by Cayley and Peirce, he states that the task Frege commended to his binding mechanism and sentential operators could be carried out without abandoning the Boolean framework [Schröder, 1880, 91–91]. And indeed, Schröder prematurely dismissive attitude has found informed accent. For modern research in natural language semantics has reached the same conclusion. If all the purpose of the binding approach of quantification were to represent the quantificational properties of natural language, then Frege's proposal cannot be the best option. The theory of generalized quantifiers offers a decent alternative [Van Benthem, 1986].

Boole and Peirce

Peirce's representations based on the inequality symbol have obvious advantages *vis-á-vis* Boole's symbolic representation. Remember that in Boole's logic algebraic form undetermines meaning. Hence, Boole's representations

1. v = ah

2. va = vh

cannot, without further ado, be interpreted existentially. One needs the additional information that in this context the equations are derived from particular propositions. Indeed, it is an interpretive mistake and incompatible with what Boole says to give v an unconditional existential reading. Peirce's inequalities enhance the autonomy of the logical system by eliminating this undeterminacy.

Peirce himself had two reservations with regard Boole's strategy. First, by making the reading of v dependent of extra logical information it becomes inevitable to assume "the truth of a proposition, which, being itself particular, presents the original difficulty in regard to its symbolical expression" [Peirce, 1931, 3.139]. In the second place, if seems that Boole's logic makes false predictions. For the equation

$$v\alpha = v(1 - \beta)$$

can be algebraically transformed into

$$v\beta = v(1 - \alpha)$$

For

$$\begin{aligned} v\alpha = v(1-\beta) &\quad \text{iff} \\ v\alpha = v - v\beta &\quad \text{iff} \\ v\alpha - v = -v\beta &\quad \text{iff} \\ v\beta = v - v\alpha &\quad \text{iff} \\ v\beta = v(1-\alpha) & \end{aligned}$$

Hence, it appears "that the inference from Some Y's are not X's to Some X's are not Y's, is good; but it is not so, in fact" [Peirce, 1931, 3.139]. Peirce is wrong in his interpretation of the conclusion. For the premise, at best, warrants only the existential reading of $v\alpha$ and $(v1 - \beta)$. There is, therefore, no Boolean legitimate way of reading the conclusion existentially. Still, Peirce's twofold argument can be used in a slightly different manner. If Boole's analysis is intrinsically existential then it is unsound. That is established by the derivation just given. If, on the other hand, v is not intrinsically existential, then the analysis is incomplete because the algebraic form alone does not conclusively determine the existential import.

Peirce's inequality stroke Venn as a natural alternative. Initially, Venn relied on Boole's v to represent particular propositions. But at the end he conceded that expression of existential content "is better effected by the formula xy > 0" [Venn, [1894] 1971, 196]. Therefore, "if we want to lay stress upon the existential aspect of the particular proposition we naturally adopt the inequality symbol" [Venn, [1894] 1971, 185].

The Negative Copula

The abandonment of equivalence as the basic logical notion was also proposed by Hugh MacColl (1837–1909) whose series of articles on logic published around the early 1880s were considered by Peirce to be congenial to his own work. MacColl's contributions to the development of algebraic logic were also recognized by Schröder who occasionally made use of the double barred name *MacColl-Peirce* as a distinguishable attribute. MacColl's starting point is the definition of a symbolic language containing logical and non logical symbols. In [MacColl, 1880, 49] they are called *temporary* and *permanent* symbols. The non logical symbols admit only of a propositional interpretation [MacColl, 1877a, 9-10]. The equations

$$\alpha = 1 \quad \alpha = 1$$

assert that the statement α is true, respectively false. The negation of α is α'. The language makes use of + to denote inclusive disjunction. The restriction to a mere propositional interpretation is one distinctive feature of MaColl's logic. But there are more. [MacColl, 1877b, 177] introduced the implication $\alpha : \beta$ meaning that whenever α is true β is also true. Consequently he introduces a sign of non-implication in the following way (p. 180). This symbol is termed *negative* copula in [Ladd, [1883] 1983, 24].

$$(\alpha : \beta) = \alpha \div \beta$$

The positive and the negative copulas are used to represent the standard categorial propositions. Since they combine statements and not terms, it is clear that the representation of a categorial proposition as *Some men are mortal* requires to interpret the terms *men* and *mortal* in a propositional manner. The terms in a categorial proposition "denote *classifying statements* referring to some one originally unclassed individual as their common subject [MacColl, 1880, 60].

In Peirce's view, the advantages of MacColl's logical language with respect to Boole's is that it can express existence. This is one of the uses of the negative copula. To represent particular propositions MacColl makes use of the negative copula [MacColl, 1880, 55–56], [MacColl, 1877b, 180-181].

> The statement *Some X is Y* will be expressed by the non-implication $x \div y$, which ... is equivalent to the implication $y \div x'$. *Some* means *some individual at least.*

This proposal found, as we shall presently see, resonances in Peirce's never ending struggle with the proper representation of existential propositions.

8 PEIRCE IN 1880

In the first part of [Peirce, 1880] Peirce's comes up with another representation of particular propositions using for this purpose the symbol $\overline{-\!\!<}$, remarking that "a dash over any symbol signifying in our notation the negative of that symbol" [Peirce, 1931, 3.165]. This negative copula has the intended quantificational meaning *Not every α is β*. Now, it is important to keep in mind that the negative copula is intended as the negation of the positive one. One should, therefore, not give in to the temptation of interpreting it as a universally quantified copula. Hence, the expressions below are intended to be each others negation:

$$\alpha -\!\!< \beta$$
$$\alpha \overline{-\!\!<} \beta$$

Thus, the sentence *Some Frenchmen are men* would be represented by Peirce in the following way:

$$\text{Frenchman} \mathrel{\overline{\prec}} \overline{man}$$

Literally, these expressions says that not every Frenchman is a non-man. Hence, some Frenchman is a man. The iconicity value of the negative copula is undeniable and in lieu of propositional negation it does what would be done by inequalities otherwise.

Frege on Negative copulas

Commenting on Boolean followers, Frege remarks that some of them "have a further sign for inequality, which also includes denial. What strikes one in all this is the superfluity of signs" [Frege, 1969, 48]. Frege had probably MacColl and Cayley in mind. [Schröder, 1880, 92,94] made him aware of their negative copulas. We have seen that the negative copula is for MacColl only an abbreviation. The superfluity charge is therefore unwarranted in his case. Cayle, on the other hand, has not other means of expressing negation than this very copula. No superfluity on this score either. We can also argue that this charge of superfluity cannot in fairness be directed against Peirce.

In the first place, we can distinguish two subsystems. A positive implicational algebra and its negative expansion. The first one contains only term negation and the negative copula. This copula is not superfluous given the absence of sentence negation. The negative copula must therefore be regarded in this case as a primitive symbol. In other words, this is the same situation in which we find Cayley. The second system gives a "constructive definition" of negation [Peirce, 1931, 3.191]:

> On the other hand, if in the minor direct syllogism (8), we put "what does not occur" for x we have by definition
>
> $$\{(S \prec P) \prec x)\} = (S \mathrel{\overline{\prec}} P)$$

The natural generalization then follows (3.192):

> We have seen that S $\overline{\prec}$ P is of the form (S $\prec$ P)$\prec$ x. Put A for (S $\prec$ P), and we find that $\overline{\text{A}}$ is of the form A $\prec$ x.

In this case, then, the negative copula is superfluous but only in the mild sense that it is an abbreviation. In other words, this is the same situation in which we find MacColl. Note that if we interpret "what does not occur" as expressing the meaning of the constant $\perp$, Peirce's definition of negation anticipates the constructive one. The crucial distinction is that the copula is for Peirce intrinsically quantificational and not merely propositional.

It is interesting to note that later Peirce reversed the roles. [Peirce, 1885] introduced first the means to express negation. According to Prior he does that in a "little obscure" manner that ranks in explicitness below his former treatment of the subject [Prior, 1957, 135]. Consequently, Peirce noted that one could achieve the same results with a negative copula as with negation. This being a more natural approach to the matter.

9 PEIRCE IN THE EARLY 90S

The Interpretations of the Copula

It is often remarked that Peirce's copula has three interpretations, logical consequence, material implication and class-inclusion:

$$\begin{tikzcd} & -\!\!< \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \vdash & \rightarrow & \subseteq \end{tikzcd}$$

Therefore, between terms $-\!\!<$ means *every*, between propositions it means either *if* or *therefore*.

This view of the matter does not take into account the quantificational character of the copula. Peirce's copula can indeed be used to express inclusion but it does so via implicit restricted universal quantification. According to Peirce the direct inclusion interpretation is not his but Schröder's:

> Thus, when in the ordinary Boolean algebra we write $m -\!\!< l$, meaning "every man is a liar", according to me this means "if x (which is any individual object you may choose) is a man, then x is a liar. Schröder, on the other hand, would say that m 'denotes' the entire collection of men (although I do not know what definite idea can be attached to the word 'denotes'), that 'l' denotes the entire collection of liars, and that the formula states that the former collection is included in the latter.

Therefore, as Peirce saw these matters, he gave his copula only two interpretations. It can be seen as the representation of logical consequence or as the representation of restricted universal quantification. Logical consequence, on the other hand, may be given a quantificational reading. Speaking of the expression $P_i -\!\!< S_i$ Peirce notes that his copula signifies primarily that "every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true". In short, then, Peirce's copula represents only one logical relation, restricted implicit universal quantification. This is the moral one has to extract from his claim that "the conditional and categorical propositions are expressed in precisely the same form ... The *form* of relationship is the same" [Peirce, 1931, 3.445]. The different readings that arise are due to variations in the domain of the implicit quantifier, ordinary objects or state of things.[33] It is worthwhile to note that the identification of logical con-

[33] This is essentially the point of view defended in [Dipert, 1981, 586–87] be it that in this paper the homogeneous reading of the copula is associated with Peirce's posterior theory of explicit quantification.

sequence and material implication was rejected by MacColl. He noted that the relation of logical consequence is stronger than material implication. The former relation asserts that the consequent is true *because* the antecedent is true. The latter relation asserts that the consequent is true *provided* the antecedent be true [MacColl, 1880, 55].

Terms, Open Formulas and Binding

In [Peirce, 1904] Peirce sets himself apart from other Boolean logicians. He remarks that the majority of Boole's followers abandoned the idea that "every logical term has one or other of two values. For my part, I have always retained that conception, as far as non-relative terms go ... " [Peirce, 1931, 4.327]. This bivalence view precludes the class interpretation of Peirce's language for obviously it is not the case that any given class is the empty set or its complement. The alternative is not to attribute classes unusual properties but to interpret the terms as propositions like constructs. Now, Peirce explicitly states that the term *man* must be interpreted as the open formula, the rhema, *x is a man.* Time and again he would state that "every term is for me nothing but a proposition" (3.440).

In fact, the formula interpretation of terms is forced upon us the moment Peirce introduced the implicational definition of negation. Because if α were a term, then so would be $\overline{\alpha}$. But this last expression being defined as the assertion $\alpha \to \perp$ cannot be a term. It seems then that α cannot have been a term in the first place. It must, therefore, all along have been a formula. This open formula is , according to Peirce, susceptible of two values. Either it gets the value truth of the value false. This is the proper interpretation of Peirce's claim concerning the bivalent nature of logical terms. Of course, this interpretation is disallowed if we were to admit mixed implications in which terms and propositions freely imply each other. For instance, following the directives in [Prior, 1964] we could interpret $\overline{man}$ as a complex term denoting the set of objects whose being a man implies $\perp$, i.e., the complement set of the set of men. The main objection against this ingenuous interpretation is that it not Peirce's. He saw implications as propositions not as complex terms. In his view, $man \prec \perp$ would rather correspond to the assertion that there are no man.

At this point modern logic is, apparently, at odds with Peirce. For within the modern framework it is a sentence what is unconditionally true or false. Open sentences are assigned a truth values only indirectly. Still, we must no conclude from this that Peirce was confused about the notions of closed and open formulas. Who writes that "A rhema is, of course, not a proposition", must not be charged with confusion on this score [Peirce, 1931, 4.439]. The facts are more complex, and interesting, that it seems. In general, a case can be made for the standpoint that no formula of Peirce's language ends as an open formula. Every term is a proposition and given that Peirce's theory "makes every proposition a conditional proposition", it follows that every term is a conditional proposition (p. 3.440). And since conditionals are binding devices, it follows that no term corresponds to an open formula. Terms get bounded as soon they are made part of an asserted proposition. The question for Peirce was this. When a monadic rhema is asserted, "so that we have to adopt a meaning for it as a proposition, what can it most reasonably to be taken to mean? Now, a case can be made for regarding the formula x *is a man* as meaning that there are men. Remember that this term is for Peirce equivalent to $(man \prec \perp) \prec \perp$. If the embedded implication can be regarded as asserting that there are no men, the whole conditional has the reading that it is no the case that there are no men. In other words, there are men. Actually, Peirce followed another line of reasoning. Given the asserted rhema $---$is beautiful Peirce wonders whether we can interpret it as meaning that something is beautiful or that anything is beautiful. At the end he decides that the existential interpretation is "decidedly the more appropriate" [Peirce, 1931, 440]. It is worthwhile to note that in the system of implicit quantification developed in [Zeman, 1967] this is the interpretation given to ground open formulas.

Replacement and Monotonicity

Peirce agreed with Jevons that inference is in essence replacement. However, no of similars [Peirce, 1984, 446]. He came to the idea that the rules of inference must "enable us to dispense with all reasoning in our proofs except the mere substitution of particular expressions in general formulae " [Peirce, 1976, 108]. To achieve the desired effect, Peirce defined two global rules of replacement based on a particular representation of implication. Instead of $\alpha \prec \beta$, he writes a "cross placed between antecedent and consequent with a sort of streamer extending over the former" [Peirce, 1976, 108]. It is clear that the streamer is intended as a negation sign. With this notation at his disposal he defines two global replacement rulesm more specifically, two monotonicity rules, with the following words:

> The general rule of substitution is that if $\widetilde{m}+n$, then n may be substituted for m under an even number of streamers (or under none), while under an odd number m may be substituted for n [Peirce, 1976, 108].

Monotonicity and negation

Let us look at Peirce's constructive definition of negation from the perspective of the monotonicity rules. Note that if an expression occurs positively in a given context, i.e., within the scope of an even number of negations, then it occurs negatively when this context is placed under the scope of an extra negation. Moreover, the streamer notation gives an iconical representation of the fact that the antecedent position of a conditional is a negative position. Since obviously any formula α is positive in itself, it is negative in $\alpha \prec \beta$, for any arbitrary formula β. Hence, the polarity reversal effected by negation can be effected by conditionalization. This is the monotonicity basis of Peirce's constructive definition of negation. The use of a falsity constant $\perp$ or "what is not possible" is needed because one expects that from a the assertion of a negation and of the negated formula a contradiction follows. Thus, the definition has to be supplemented with the *ex falso* rule.

Monotonicity in History

These monotonicity rules belong to the folklore of modern logic be it that Peirce is never credited with being the first that gave them a formal representation. They are, for instance, the *Dictum de Omni* in [Sommers, 1982, 184], the *Semisubstitutivity of Conditional Rules* in [Zeman, 1967, 484], and theorem 24 in [Kleene, 1967, 124]. Even some years earlier they were mentioned in [Kleene, 1952, 154]. In this last book we are referred to [Curry, 1939, 290-1] for another version of the rules. Curry, in turn, refers us to [Herbrand, 1930] and [MacLane, 1979]. Still, the first modern occurrence of the monotonicity rules which we have found is [Behmann, 1922, 172-174].

LAST WORDS

In this part we have described Peirce's contributions to the monadic and propositional part of the algebra of logic. Important is the use of the notion of inequality and its symbolization. We have also stressed the fact that implication and (universal) binding go together in Peirce's early systems of logic. We have also pointed out that the search for a satisfactory representation of existential propositions was a central feature of Peirce's logical works. His formalization of monotone replacement has also been stressed and with it the logical pedigree of this notion.

Part 5

The Logic of Absolute Terms: E. Schröder

This part consists of three sections. The first one sets the stage for the next two. The second one is devoted to Schröder's first logical work. The last section deals with some aspects from the first volume of his last work on logic.

1 INTRODUCTION

The next two sections contain a brief exposition of Schröder's work. We shall pay attention to the theory of logic he develops in [Schröder, [1877] 1966]. This elegant booklet is the third equational logic written after Boole. The resulting system is "the algebra of logic as we know it today " [Lewis, 1918, 111]. As we shall see, Schröder defines a structure with two binary operations, multiplication and addition, a unary one, negation, and two constants, 0 an 1 that satisfy all the axioms of a Boolean algebra. However, when preparing this work Schröder was not aware of Jevon's nor of Peirce's contributions to algebraic logic. It is significative that already in [Schröder, 1873] the notion of logical subordination (*logische Unterordnung*) is introduced and lexicalized by means of the subordination symbol $=\!\!\!\!)$. He notices in this book that the lexicalization of negation would provide him the terminological means needed to formalize all the extensional relations between terms (p. 29).

Indeed, it has been claimed that the main influence on his work stems not even from Boole. The roots of his equational logic, is argued in [Peckhaus, 1997], lie in the symbolic logic of Robert Grassmann and the doctrine of forms of Herman Günther Grassmann and Herman Hankel. The same thesis is defended in [Peckhaus, 1996]. We shall not explore in this essay the German roots of Schröder's work, but shall concentrate our attention on the Boolean aspects of his work. We shall also pay attention to the treatment of absolute terms in Schröder's second book on logic. In this context we shall describe three aspects of this theory of classes: the view of individuals as classes, the hierarchy of universes and his proof that there is no universal class.

2 DER OPERATIONKREIS

This book opens with a diagnose of what Schröder sees as Boole's neglect. For this state of affairs he holds Boole himself partially responsible. For Boole's method of resolving logical problems made use of numerical constructions that have no logical interpretation. Even though his method at the end achieves the desired results, those meaningless steps made it intellectually unappealing. He, consequently, sets forth a logical system free of this unattractive feature. There are, in fact, in his logical calculus but two numerals to whom logical citizenship is granted, the constants: 0, denoting a class without members and 1, denoting what Schröder terms Boole's *universe of discourse* [Schröder, [1877] 1966, 7]. Other numerals are in his system logically meaningless symbols.

Two enduring Boolen features

A central feature of his equational logic is, according to Schröder, the schematic representation of its objects (terms and propositions) and the representation of inferences as computational manipulations of those representations. These representations admit two interpretations. They can stand for terms or for propositions. In the first case logic is regarded as the computational manipulation of propositions of the first kind (*Urtheile der ersten Klasse*). In the second case as the computational manipulation of propositions of the second kind (*Urtheile der zweiten Klasse*). But in both cases the computations obey the same laws. In this respect, Schröder's views echoes the Boolean division between primary and secondary propositions. The Boolean influence is more pronounced when he advances the view that the propositional interpretation can be regarded as a special case of the term interpretation. For the representation of a term denotes a class of objects and the representation of a proposition can be regarded as a class of moments (*Klassen von Zeittheilen*) in which propositions are true. These two Boolean features, the double interpretation of representations and the temporal view of propositions are enduring features in Schröder's logical writings. It is interesting to note that, in his view, classes can be regarded as the sum of its elements, thus clearly taking singletons as the basic units from which classes are constructed [Schröder, [1877] 1966, 12]. A vague anticipation of this idea is found in De Morgan's remark that "classes are made up of classes [De Morgan, 1966, 180].

Schröder's Logical Language

Schröder sets up a language that has the means to represent five operations on classes: a unary operation, two unrestricted binary operations and two restricted ones. Furthermore, the language expresses by means of the identity symbol that two terms are equal. It is also assumed that identity has satisfies the familiar equational conditions of reflexivity, transitivity and symmetry.

Let now α and β be terms,. Then the first five expressions below are terms and the last one is a statement:

1. α_1 (the negation of α)

2. $\alpha\beta$ (the multiplication of α and β).

3. $\alpha + \beta$ (the addition of α and β).

4. $\alpha :: \beta$ (the quotient of α and β).

5. $\alpha \div \beta$ (the difference of α and β).

6. $\alpha = \beta$

If α and β denote classes then their negation, multiplication and addition denote a class as well. They always yield a result [Schröder, [1877] 1966, 7, 10–11]. Addition by Schröder corresponds to Jevons's inclusive operation and to Peirce's logical addition. This operation is not, as by Boole, a partial operation. It is thus important to bear in mind that for Schröder addition, multiplication and negation are always defined. The definitions establish the unrestricted existence of their denotation. That this denotation is unique becomes demonstrable. For suppose, for instance, that $\alpha\beta = \gamma$ and $\alpha\beta = \gamma'$. Then by transitivity of identity we conclude that $\gamma = \gamma'$. These operations are called by him unambiguous (*eindeutige*) and are contrasted to senseless (*undeutig*) or ambiguous (*mehrdeutig*) operations. Of course, a system that does not assume the basic equational properties must incorporate unicity along existence when characterizing products, sums and complements. Schröder broad logical basis allows him to deal with unicity as a derived property. Still, as we presently shall see, there are partial operations in his system.

Partial Operations

Division and substraction denote partial operations defined only in restricted circumstances. Thus, $\alpha \div \beta$ exists only when β is a part of α and $\alpha :: \beta$ exists only when α is a part of β. This is, of course, nothing new. The same feature is shown by the logical systems of Boole and Peirce.

Ambiguous Operations

Moreover, even when these two last operations are defined, they are not unambiguous operations. This is a feature that Schröder's logic shares with Peirce but not with Boole. For the following Peircean equations hold also within Schröder's framework

$$\alpha \div \beta = \alpha\beta_1 + u\alpha\beta \quad \alpha :: \beta = \alpha\beta + u\alpha_1\beta_1$$

The expression u denotes an completely arbitrary class. Like Boole's indeterminate symbols, the range of this symbol goes from 1 to 0. Hence, even when we know the denotations of both α and β, and that $\alpha \div \beta$ denotes a set we still may not know which set it is. Its way of composition undetermines its meaning. Therefore, even though in Schröder's and Peirce's hands algebra and logic share all their operations, none of them assume that the fit is perfect. Logic has undetermined operations.[34]

Maximal Quotient and Minimal Difference

Still, from these two indeterminate equations arise two special cases in which the indetermination has been eliminated. If $u = 0$, then

$$\alpha \div \beta = \alpha\beta_1$$

This expression is called the *minimal difference* (*Minimaldifferenz*) and it denotes, as by Boole, the class of α objects without its β elements. Moreover, if $u = 1$, then

$$\alpha :: \beta = \alpha + \beta_1$$

This expression is called the *maximal quotient* (*Maximalquotien*) and it denotes the class that results from the product $\alpha\beta$ when we abstract from the β objects.

Boole, Schröder

In spite of the view that Schröder showed that "the two operations which Boole termed division and substraction are superfluous, as they can be defined by means of negation", his approach is closer to Boole's than to modern sensibilities [Jørgensen, 1931, 136]. Because in a modern setting $\alpha - \beta$ is not only unambiguously determined, it is also unconditionally determined being only an abbreviation for $\alpha\beta_1$.

[34] For another view on this matter see [Van Evra, 1977, 364].

For Schröder, as for Boole, the situation is different. If α is smaller than β then their difference does not exist. Of course, the product $\alpha\beta_1$ is still regarded as existent. The point is that it is not considered to be a difference. Moreover, two algebraically equivalent expressions do not always determine the same class. For instance $\alpha\alpha - \alpha = 0$ is a valid equation. But the equation $\alpha(\alpha - 1) = 0$, on the other hand, isn't. It contains $\alpha - 1$, an expression that is not always considered to be meaningful. Schröder went to repeat these Boolean views in his next book. There he points out that in such cases there is nothing that corresponds to the putative difference. Even zero is excluded as meaning [Schröder, 1890, 480].

The Discovery of Duality

The core of Schröder's booklet is the second chapter in which he presents a rather rigorous exposition of algebraic logic. In this chapter he makes a sharp distinction between definitions, axioms and theorems that contrast favorably with the expository style of Boole, Jevons and Peirce.

Axioms

He list axioms that regulate the logical behavior of the equality symbol and provides the definitions of the logical operations. He then list the following axioms:

For any α and β	$\alpha\beta$ exists	$\alpha + \beta$ exists
	$\alpha\beta = \beta\alpha$	$\alpha + \beta = \beta + \alpha$
	$\alpha(\beta\gamma) = (\alpha\beta)\gamma$	$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
If $\alpha = \beta$, then	$\alpha\gamma = \beta\gamma$	$\alpha + \gamma = \beta + \gamma$
	$\alpha\alpha = \alpha$	$\alpha + \alpha = \alpha$
For every symbol α	$\alpha\beta = 0$	$\alpha + \beta = 1$
there is a β such that	$\alpha \cdot 1 = \alpha$	
For any α, β and γ	$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	

General Results

From these axioms Schöder proves, among other things, the distribution of addition over multiplication, the absorption laws, the logical counterparts of the standard arithmetical properties of 0 and 1, the double negation law. It is interesting to note that in several passages he offers a geometric argument alongside an algebraic one. His use of shaded diagrams to this effect anticipates in a mild sense Venn's more famous use of them. The last notes that Schröder uses the diagrams "simply to direct attention to the compartments under consideration" but not "with the view of expressing propositions" [Venn, 1881, 112].

The Main Theorem

The main results obtained by Schröder are these. In the first place, he shows that the following equation holds for any α and β and arbitrary x and y independent of α

$$\beta = x\alpha + y\alpha_1$$

Then he establishes his main theorem, the crown jewel of his logic, for x and y as above, the following equations are equivalent:

$$x\alpha + y\alpha_1 \text{ and } xy = 0$$

Elimination Procedure

The combined effect of these two results is that from any given equation any given symbol can be eliminated. The procedure is this. To eliminate α from the equation $\varphi(\alpha)$, write this equation as $0 = x\alpha + y\alpha_1$. According to the main theorem, this equation is equivalent to $xy = 0$ in which α does not occur.

The Principle of Duality

It is in the second chapter of this book that Schröder expresses the principle of duality [Schröder, [1877] 1966, 3]:

> ¿From any in Logic valid general formula shall again arise a valid formula whenever one replaces throughout it the plus and minus signs by multiplication and division signs and also when the symbols 0 and 1 replace each other.

Schröder emphasized this property by presenting definitions and theorems in double columns.[35] The left one corresponding to multiplication, the right one to addition.

3 VORLESUNGEN ÜBER DIE ALGEBRA DER LOGIK

We pointed out that when preparing his booklet Schröder was not aware of the work done by Peirce and Jevons. His next book, more than 2000 pages long, bears witness of the huge amount of information that he assembled in its preparation. It consists of three volumes. In this section we shall be concerned only with the two first ones that treat roughly the same subjects as his previous book. The logic he develops here is not longer an equational one. He had discovered in the meantime the non equational logic of [Peirce, 1870] and adopted the approach that coincided with the logical program he had hinted at in 1873.

[35] According to [Nagel, 1939, 224] this is a practice initiated by Gergonne who used it to express the duality for points and lines in a plane and points and planes in space.

The Calculus of domains

The basic notions are those of domains *Gebiete*, subsets of a given manifold (*Manigfaltigkeit*), and subsumption, (*Einordnung*). Subsumption, $\mathrel{=\!\!\!\!(}$, is a binary relation that establishes a partial ordering among domains, i.e., $\mathrel{=\!\!\!\!(}$ is

1. reflexive,
2. transitive
3. antisymmetric

There are two special domains. The domain that is subordinate to every domain, 0, and the domain that contains every domain, 1. Schröder clearly assumes that there are indeed *two* objects that satisfy the conditions

$$0 \mathrel{=\!\!\!\!(} \alpha \qquad \alpha \mathrel{=\!\!\!\!(} 1$$
$$\text{for all domains } \alpha$$

These two objects are also unique. Their unicity, however, is a demonstrable property. He also defines the product $\alpha\beta$ and the sum $\alpha + \beta$ on basis of his partial order as the greatest lower bound and the least upper bound, respectively, of α and β. With regard to the unicity of products and sums he proves that the operations of multiplications are "never senseless, never ambiguous, they are rather unconditionally executable and unambiguous" [Schröder, 1890, 11]. But in name, therefore, Schröder's theory of domains is a lattice theory.

Distributivity

An important subject discussed in this volume is the question whether distributivity holds for the theory of domains just described. He proved that it is not the case. In particular, he established that not every model for his theory of domains is a model for the subsumptions

$$\alpha(\beta + \gamma) \mathrel{=\!\!\!\!(} \alpha\beta + \alpha\gamma \quad (\alpha + \beta)(\alpha + \gamma) \mathrel{=\!\!\!\!(} \alpha + \beta\gamma$$

In his commentary on this stand of affairs, Schröder notes that geometry knows such negative proofs (*negative Beweise*) and he refers to Klein's and Cayley's disproof of the Euclides's parallel axiom as examples of difficulties surrounding the task he faces. We have already mentioned that there is certain uncertainty with regard to this part of Schröder's work. They refer, however, not to the of value of his claim or the solidity of his result. The question has been whether he was justified in asserting that Peirce thought distribution to be demonstrable from what amounts to his own theory of domains. Reading his owns remarks on this subject under the guidance of Schröder's views, Peirce admitted that much. Years latter, reading his own remarks on this subject as he had intended them, Peirce came to a different conclusion. He had never entertained the idea that a basis similar to Schröder's theory of domains would establish distributivity [Houser, 1991]. At any rate, there are at least two calculi of domains. One in which distribution holds and one in which it doesn't.

The Identity Calculus

The first one is called the *identity* calculus. This identity calculus is finally extended inasmuch as Schröder determines the existence of the negation, the complement, of any ground term. As he had done before, unicity of negation is not assumed but demonstrated. Historians agree that with the establishment of this unicity "he finally has the modern abstract definition of a Boolean algebra as a distributive lattice with 0, 1 such that every element has a unique complement" [Brady, 1997].

Schröder's Theory of Classes: individuals as classes

Schröder pointed out that classes share the basic abstract structure described in his theory of domains. The manifold can be regarded as a class of individuals and the domains as subclasses of this manifold. Any element of the manifold, on the other hand, may be regarded as a unit class [Schröder, 1890, 158].[36] Thus, the sentence *Caesar was killed* is interpreted by Schröder as the assertion that the unit class consisting out the one individual Caesar is contained in the class of those persons who were killed. Clearly, Schröder's theory of classes simulates singular propositions as ordinary inclusions.

Objects versus Unit Classes

Let us now pay attention to two aspects of Schröder's theory of individuals. In the first place, objects and their unit classes are two radically different entities. For Schröder the difference is such that they cannot live in the same universe. Schröder himself notes with respect to a manifold that among their elements there is no class that contains as individuals other elements of the same manifold [Schröder, 1890, 248]. In other words, the objects that make up a Schröder universe, 1, satisfy the following atomicity condition:

$$i \in 1 \rightarrow \neg\exists x(x \in 1 \land x \in i)$$

If an object i and its unit class {i} were to be in the same universe, they would obviously violate this condition. Moreover, as we shall presently see, there cannot be any meaningful expression of Schröder's language referring both to an object and to its unit class.

[36] We have already had the opportunity pointed to the partial anticipation of this idea in [De Morgan, 1966, 188].

Unnameable Objects

In the second place, if the ground terms of the system denote unit classes, then there are unnameable objects. We assume that there is a pre-theoretical set of individuals. The subsets of this set are the denotations of the ground terms of Schröder's language. Thus, the pre-theoretical individuals turn out to be technically dispensable, allowing Schröder to avoid the use of the notion of membership. Inclusion among classes is the basic relation. This move helps him in achieving a uniform representation of natural language propositions. These can, namely, be uniformly regarded as denials or assertions of inclusion among classes. Note that here arises a situation familiar to students of formal semantics. Because this way of setting up the matter has a Montegovian flavor about it. There are in Schöder's system, as in Montague's ontology, objects that cannot be named. For a term denotes a class not its members. In particular, a term that denotes a unit class does not denote its single element.

Schröder's Theory of Classes: the hierarchy of universes

For manifolds, and thus for classes, the following description holds. The process of generating classes (*Hervorhebung von Klasssen*), starting from an initial manifold (*ursprünglich Mannigfaltigkeik*), leads to a new manifold (*neue Mannigfaltigkeit*), the manifold of the classes of the previous one. This process can continue indefinitely, each time producing a new, derived manifold (*abgeleite Mannigfaltigkeit*). The initial class consists of objects whose addition, or union, does not vanish. They are, in other words, compatible (*vereinbare*). Schröder, then, envisions the following hierarchy of universes, for all natural numbers n:

1. I^1 is a class of compatible individuals, called the *initial universe.*

2. I^{n+1} is the power set of the class I^n. This universe is called the *n + 1th derived universe.*

This hierarchy is obviously not cumulative. For instance, the initial class and its first derived class have no element in common. Each universe in this hierarchy is called pure (*rein*) to emphasize the fact that it contains no individuals from a lower or a higher universe.

A crucial feature of Schröder's system is his decision to limit the use of the calculus of domains to manifolds that are pure (and compatible). Note that while in a standard language $\alpha \in I$ and $\{\alpha\} \mathrel{=\!\!(} I$ are equivalent, for Schröder the situation is different. The first formula does not make sense, involving expressions that denote objects of different universes. For the same reason there is no way in which we can make sense in this language of the identity $\alpha = \{\alpha\}$. The decision to identify the individuals of the initial universe with unit classes is a decision about the structure of the system, a decision Schröder took outside the system and not expressible within the language.

Frege was convinced that Schröder based his theory of classes on the fateful system internal identification of unit class and its element. Actually, a sizeable part of [Frege, 1895] spells out the undesirable consequences attached to such an identification. Frege's interpretation is clearly unsustainable but it is not completely unreasonable. There are, indeed, passages in which Schröder seems to invite such an interpretation. This is due, partially, to the difficulty of seeing at glance the role that he attributes to the equational symbol. There is, for instance, a context in which the equation $A = a$ is regarded as the assertion that the left class consists of the right one and of nothing else [Schröder, 1890, 248]. In modern notation, then, this equation invites the fateful identification $\{a\} = a$. The context in which this equation occurs, however, makes it clear that it is not an expression of the logical language but rather about it. The general point Schröder is advancing is just that within his language there is no contradiction-free way of allowing arbitrary formulae that mix objects and classes of objects [Schröder, 1890, 249]. Instead of refuting Schröder's prophylactic views, Frege unwittingly supports them.

Schröder's Theory of Classes: the universal class

Consider now the universe 1 of everything: $\bigcup I^n$, for natural number n. Suppose such universe exists. Then it contains an arbitrary non zero individual b and the singular class B = {b}. Schröder then, fallaciously, reasons in this way. Since $0 \neq\!\!\!\in B$, it follows that B contains 0. Hence 0 = b. But this is a contradiction given that we assume that b was a non zero individual. There can, therefore, not be a class of everything. Husserl took exception to the argument that establishes this conclusion [Husserl, 1891, 36]. He pointed out that any manifold (and thus any class) contains 0 as a subordinate manifold but not as element.

This fallacious proof is the passage that justifies the widespread conviction that Schröder confused occasionally membership with class inclusion. It is true that the identification concerns the empty class, but still it remains a mistaken view. In preparing the reader for this interpretation of his subordination symbol he offers a rather lame additional justification of the validity $0 \neq\!\!\!\in \alpha$. That this formula is true, he says, can be seen from the following consideration. A class is said to contains all its members and beyond that nothing. Hence, the empty class is one of its members [Schröder, 1890, 241].

Frege, like Husserl, rejected the argument Schröder gave for his non existence result. He also rejects the conclusion. In rigorous scientific enquiries we need to have one universe, one zero and not the multiplicity of hierarchically organized universes Schröder proposes [Frege, 1895]. In his rejection of a restricted universe, Frege behaves likes those logicians of whom [De Morgan, 1966, 2] commented that they "do not find elbow-room enough in anything less than the whole universe of possible conceptions". Zermelo's response was different. He took notice of Schröder's argument and on 16 April 1902 he offered his view of the matter to Husserl. Zermelo agreed with Husserl that Schröder's proof is defective. But he stresses the fact that the point Schröder tried to make is valid: there cannot be a class of everything. In support of this conclusion he offered a by now familiar argument. He established, namely, that the non existence of a universal class can be established by proving the contradictory nature of the set of all sets that are not members of themselves [Husserl, 1979, 399].

LAST WORDS

In this part we have described, briefly, the treatment of absolute terms due to Schröder. In his hands the monadic part of the algebra of logic has resulted in what was to become the theory of latices and the Boolean algebra. It is important to remember that the identification of individuals with classes by itself is not a mistake. The point in which Schröder went wrong was not when he identified individuals with classes. It was rather when he failed to distinguish between membership and inclusion.

Part 6

The Logic of Relations: De Morgan

This part consists of the usual introductory section followed by a section devoted to De Morgan, traditional logics and relational arguments. This section is largely based on [Sánchez Valencia, 1997]. The third section describes De Morgan's analysis of the different kinds of copulas that there are. The fourth one deals with his relational view of the syllogism. The last one exposes his theory of relations as a branch of logic.

1 INTRODUCTION

The systematic treatment of the logic of relations started with De Morgan. In his lifelong interest in this subject I distinguish two main strands. On the one hand the study of the logical properties of relations and on the other hand the study of the limits of traditional logic. Of course, these strands are interwoven but for expository purposes it is advisable to attend to them separately.

Explanation of De Morgan's ideas suffers unnecessarily on account of the unfortunate fact that his notation and terminology seldom catched up. Therefore, I shall in these pages describe his ideas with as little recourse to his notation as possible. Among those who pursued further De Morgan's relational researches I count the Boolean followers Peirce and Schröder. Modern times have also seen a revival of De Morgan's strategy in accounting for some non syllogistic inferences.[37] And the interest of Alfred Tarski for the logic of relations did lead to a revival of the algebra of relations.[38]

[37] For instance [Suppes, 1979] and [Sommers, 1982].

[38] An important paper in this regard is [Tarski, 1941].

2 RELATIONS AND TRADITIONAL LOGIC

De Morgan, Geometry and Syllogisms

Reputedly, some logicians viewed the categorical syllogism as the ultimate touchstone by which we may try any inference. In the initial phase of his academic life, De Morgan seems to come close to share this conviction. In 1836, in the essay "On Geometrical reasoning", he published his belief that the theorem of Pythagoras was reducible to detailed syllogistic form. He regarded the proposed transformation as "a specimen of a geometrical proposition reduced nearly to a syllogistic form". On the syllogisms needed to complete the proof he noted that "we have omitted some few which the student can easily supply" [De Morgan, [1836] 1898, 221].[39]

Against this position it can be argued, however, that Euclid's proofs made use of the transitivity of equality and that the following schema is not reducibly to syllogistic form

$$\frac{\text{A is equal to B} \quad \text{B is equal to C}}{\text{A is equal to C}}$$

Nevertheless, it is not clear that De Morgan intended his reduction to be a standard one.

Relations and Inference

In the first place, in the closing remarks to his essay he noted that the validity of an inference depends both on the truth of the relations expressed in the premises and on the manner in which "these facts are combined as to produce new relations". In his early view, reasoning consists precisely in the construction of new relations. The mature De Morgan would answer to the mentioned criticism by noting that

> the composition of relation, 'equal of equals is equal, expressing the transitivity of the copula *equals*, which, with its convertibility, renders it of equal validity with *is* as the copula to be employed in a syllogism [De Morgan, 1966, 67].

Recollection

Moreover, [De Morgan, 1868, 71], long after he had published his views on syllogisms and relations, mentions his early syllogistic reduction and notes, without any hint of disapproval, that it was reprinted in various editions of the books of Euclid published by Dionysius Lardner(1793-1859).

[39] I cite from the fourth 1943 American reprint. The editor of this edition points out that the original bears the date of 1831.

Their Syllogism

There is, still, something else. More than once he used the above inference as an example of an inference recognized by logicians not to be reducible to *their* syllogism [De Morgan, 1966, 67], [De Morgan, 1966, 173], [De Morgan, 1966, 254]. The suggestion, conveyed by the dissociating pronoun, is that this inference is either reducible to or already an inference in his own enlighten syllogistic form.

De Morgan's argument

Still, in years to come he was to voice his conviction that some inferences cannot be accounted for within the rigid framework of the standard theory of logic and challenged logicians to prove him wrong:

> Observing that every inference was frequently declared to be reducible to syllogism, ..., I gave a challenge in my work on formal logic to deduce syllogistically from 'Every man is an animal' that 'every head of a man is the head of an animal' [De Morgan, 1966, 29].

The relational nature of this inference, henceforth to be denoted by *DMA*, would be highlighted in an indirect way in his most important work on relations. For there he points out that "the proposition *of* is the only word of which we can say that it is, or may be made, a part of the expression of every relation" [De Morgan, 1966, 218].

Historical Relevance

This inference deserves historical attention for several reasons. In the first place, DMA or a variant thereof is routinely used to set the bounds to traditional logic. This fact makes it mildly surprising that hardly anyone ever asks:

- How did De Morgan himself account for the validity of DMA?
- Is De Morgan's explanation right?

The reason for this curious oversight is, perhaps, the assumption that De Morgan used his logic of relations to explain the validity of DMA. An illustrative example of this attitude appears in the opening paragraphs of [Tarski, 1941]. However, when De Morgan issued his challenge he did not yet have such a logic. Once we realize that this is the case, the foregoing questions become even more important. For it is an interesting point that De Morgan claimed to have an explanation for the validity of DMA well before the inception of his mature logic of relations. Still, we should not be surprised by the appearance of relational explanations in De Morgan's early work. It is important to borne in mind that before De Morgan started publishing on logic he had studied the theory of functional equations and he published on this subject a major surveying paper.[40] It is, therefore, hardly surprising that relational topics kept appearing in almost all his logical publications.

[40][Panteki, 1992] analyzes the functional background of De Morgan's logic of relations.

Relational Explanation

A relational explanation of De Morgan's example sees it as an instance of the following validity

$$\text{If } A \subseteq B \text{ then } R\text{''}A \subseteq R\text{''}B$$

in which R"X denotes the set of objects that have the R relation to some member of the set X, the so-called *left projection of R* with regard to X. De Morgan never gave such an explanation. The first attempts along these lines I am aware of are due to the mathematician R. L. Ellis and to C. S. Peirce. In a postum note presented to the British Association for the Advancement of Science in 1870 by R. Harley, Ellis comments on the fact that Boole's logic cannot handle relational arguments. As one of the inferences Boole cannot account for he quotes an example given by Whately:

He who kills a negro kills a man

Consequently Ellis uses relational notions to argue for its validity. Hereby he makes use of the notion of projection. The account is not satisfactory but this is not the point. The interesting historical fact is that Ellis's notice is the first genuine relational account of the class of inferences that De Morgan's example epitomizes [Ellis, 1870, 13]. Remarkable is also that Ellis avoids mentioning De Morgan's concern with this kind of inferences and his logic of relations.

Peirce, on the other hand, published a relational account of De Morgan's example in his [Peirce, 1869]. His point is that the following schema is syllogistically valid:

> Every relation of a subject to its predicate is a relation of the relative "not x'd, except by the X of some", to its correlate, where X is any relative I please" [Peirce, 1931, 2.322].

De Morgan's example is represented by Peirce as an instance of the previous schema:

> Every relation of "man" to "animal" is a relation of the relative "not headed, except by the head of some", to its correlate.

In a footnote Peirce refers to the passage in [De Morgan, 1864b] that contains De Morgan's example. This reference does not need to be interpreted as Peirce's claim that there is a pure syllogistic explanation of the validity of De Morgan's example. For he concedes, in the spirit of De Morgan, that the validity of his argumentation "depends upon the assumptions of the truth of certain general statements concerning relatives".[41]

[41] For another interpretation of this passage, see [Merrill, 1978, 263].

De Morgan's Challenge

In this part I review different arguments that could, rightly, be called *De Morgan's argument.* After that I shall comment on De Morgan's course of reasoning according to which DMA lies beyond the reach of traditional logic. The argument used in De Morgan's challenge to traditional logicians is often represented as below:

$$\frac{\text{Every horse is an animal}}{\text{Every tail of a horse is a tail of an animal}}$$

Although this argument does not appear in De Morgan's work, we shall see that it is intimately related to an argument that he used. The curious fact is that out of his work arise different arguments which he obviously saw as variants of the above pattern. Unfortunately, as we shall presently see, not all these arguments are valid.

Versions of DMA

It is instructive to quote at length the place De Morgan must have had in mind when writing the recollection cited in the introduction. In the passage that follows De Morgan is discussing inferences that involve replacement in complex terms. It is interesting to note that the polemic tone of voice that colours the cited reminiscence is absent from the actual text. Instead of a challenge we read a probe into the strength of the syllogistic:

> There is another process which is often necessary, in the formation of the premises of a syllogism, involving a transformation which is neither done by syllogism, nor immediately reducible to it. It is the substitution, in a compound phrase, of the name of the genus for that of the species, when the use of the name is particular. For example, 'man is animal, therefore the head of a man is the head of an animal' is inference, but not syllogism. And it is not mere substitution of identity, as would be 'the head of a man is the head of a *rational animal*' but a substitution of a larger term in a particular sense.
>
> Perhaps some readers may think they can reduce the above to a syllogism. If *man* and *head* were connected in a manner which could be made subject and predicate, something of the sort might be done, but in appearance only. For example, 'Every man is animal, therefore he who kills a man kills an animal'. It may be said that this is equivalent to a statement that in 'Every man is an animal; some one kills a man; therefore some one kills an animal,' the first premise and the second premise conditionally involve the *conclusion* as *conditionally.* This I admit: but the last is not a syllogism: and involves the very difficulty in question. 'Every man *is* an animal; Some one *is* the killer of a man': here is no middle term. To bring the first premise into 'Every

> killer of a man is the killer of an animal' is just the thing wanted [De Morgan, 1847, 114].

Let us now turn to the three arguments introduced in this passage that I shall represent as follows:

$$\frac{\text{Man is animal}}{\text{The head of a man is the head of an animal}} \tag{1}$$

$$\frac{\text{Every man is an animal}}{\text{He who kills a man kills an animal}} \tag{2}$$

$$\frac{\text{Every man is an animal} \quad \text{Some one kills a man}}{\text{Some one kills an animal}} \tag{3}$$

(2) is interesting for several reasons. A first point to observe is that it is on the analysis of this argument that De Morgan exposes the weakness of traditional logic. In the second place, it is a little bit surprising that to make the point that there is no syllogistic account for the validity of (1) De Morgan felt compelled to introduce this new argument. The explanation for this stylistic misdemeanor is perhaps a historical one. Richard Whately closed his *Elements of Logic* with a list of exercises. The task for the reader is described in the following terms:

> In such of the following examples as are not in syllogistic form, it is intended that the student should practice the reduction of them into that form; those of them, that is, in which the reasoning is in itself sound: *viz.* where it is impossible to admit the Premises and deny the Conclusion. Of such as are apparent syllogisms, the validity must be tried by logical rules ... [Whately, [1827] 1975, 324].

Surprisingly, Whately's list contains the inference to which I have already referred to in the introduction to this part (p. 335):

$$\frac{\text{A negro is a man}}{\text{He who murders a negro murders a man}}$$

It is possible, then, that by using (2) De Morgan intended a (perhaps too) subtle teasing reference to a book he admired.

We may now consider other versions of DMA. It will be remembered that De Morgan uses another argument in the passage I quoted in the introduction, namely:

$$\frac{\text{Every man is an animal}}{\text{Every head of a man is the head of an animal}} \tag{4}$$

Now, (4) is arguably invalid. Notice that even if *every man is a man* is granted it does not follow *every friend of a man is THE friend of a man.* Equally invalid is the version of DMA that Jevons attributed to De Morgan in [Jevons, 1874, 22]:

$$\frac{\text{A horse is an animal}}{\text{The head of a horse is the head of an animal}} \tag{5}$$

Jevons's argument corresponds rather to an argument proposed by Mansel while criticising De Morgan's view of logic, and later discussed by De Morgan [De Morgan, 1966, pp. 252-253]:

$$\frac{\text{A guinea-pig is an animal}}{\text{The tail of a guinea-pig is the tail of an animal}} \tag{6}$$

It is of this version of DMA that Russell and Whitehead remarked that it was a merit of Aristotle's logic that it did not raise to De Morgan's challenge [Russell and Whitehead, [1910] 1925, p. 291].

Finally, I want to draw attention here to the valid form of DMA that we encounter in De Morgan's work [De Morgan, 1966, pp. 216]:

$$\frac{\text{Every man is an animal}}{\text{Every head of a man is a head of an animal}} \tag{7}$$

It cannot be denied that DMA took different forms in De Morgan's hands. But it would be, of course, ungracious to hold this against him. It is important to keep in mind that (7) actually occurs in De Morgan's work. Because it shows that it is justified generosity of history that he is associated with the correct version of his challenge and not with the incorrect ones.

The Syllogistic Fullness Thesis and DMA

As can be gathered from the passage cited above, De Morgan came to reject the syllogistic fullness thesis, i.e., the thesis that valid inferences are (reducible to) immediate inferences or syllogisms. It is perhaps not so difficult to spot the relative weakness in De Morgan's rejection of the fullness thesis. For his strategy consists in trying and rejecting particular Aristotelian transcriptions of the original argument. This alone, however, does not prove that there is no successful categorical representation. Interestingly enough, the weakness of his position did not escape De Morgan's attention, although it failed to discomfort him. Thus, he writes

> ¿From the total absence of attempt to answer this challenge, I conclude that not one has succeeded in whose way it has fallen. ... This would be a very unsafe conclusion from the absence of printed answer. But any one who writes on a controverted subject gains a number of private correspondents, with and without names (p. 29).

In the second place, the expanded inference 3 does not count as a syllogism by established standards, but this is due to the restricted notion of syllogism that became prevalent in post-medieval times. For 3 is an instance of the so-called syllogisms *ex obliquis* studied in medieval logic. It is noteworthy that [William of Ockham, 1974, p. 387] gives the following example of a valid syllogism:

$$\frac{\text{Every man is an animal} \quad \text{The donkey belongs to a man}}{\text{The donkey belongs to an animal}}$$

We can see from this example that a more liberal notion of syllogism could be used by traditional logicians to stall De Morgan's attack.

Of course, the rejection of the syllogistic fullness thesis does not offer any explanation for the validity of arguments. In the next section I consider how De Morgan applied himself to the task of justifying DMA.

De Morgan Rules

As I pointed out previously, when De Morgan launched his challenge he did not say that DMA was valid because it has the form of a relational valid schema. There were not such schemas available. Therefore, to explain DMA he had to resort perforce to a direct approach. It appears, accordingly, that he defined replacement rules which must be applied directly to natural language sentences, thus bypassing the necessity of translating into a formal language. These rules are variously expressed. His first formulation of the rules is this:

> For every term used universally less may be substituted, and for every term used particularly, more. The species may take the place of the genus, when all the genus is spoken of; the genus may take the place of the species when some of the species is mentioned or the genus, used particularly, may take the place of the species used universally [De Morgan, 1847, p. 115].

A sufficiently like set of rules is this one:

> A little consideration suggests as a necessary rule of inference, the right to substitute a larger term used particularly for a smaller one, however used, and a smaller, used in either way, for a larger used universally [De Morgan, 1966, p. 28-9].

It may not be amiss to encode the parts of these passages that are relevant for us as follows:

- DeMorgan$\downarrow$: The species (smaller term) may take the place of the genus (larger term) when all the genus (larger term) is spoken of.
- DeMorgan$\uparrow$: The genus (larger term) may take the place of the species (minor term) when some of the species (smaller term) is mentioned.

De Morgan claimed originality with regard to DeMorgan↑. As far as DeMorgan↓ is concerned, he thinks that this is equivalent to the medieval *dictum de omni et nullo.* In the next two subsections I discuss the historical justification for these claims.

Antecedents to De Morgan's Rules

Clearly, De Morgan's rules are intended to regulate inferences from (names of) sets to (names of) subsets and vice versa. But it cannot be much of a surprise that some of the devices of the medieval theory of consequences are aimed at regulating precisely this kind of inferences. In fact, these rules are strikingly similar to the vertical inference rules common in medieval logic [Sánchez Valencia, 1994]. For instance Ockham defines the next two rules:

- A consequence from a distributed superior [term] to a distributed inferior [term] holds good [William of Ockham, 1974, p. 591].
- An absolute consequence from an inferior [term] to a superior [term] without distribution and without negation holds good [William of Ockham, 1974, p. 600].

The similarity between DeMorgan↑ and Ockham's *ab inferiori* rule on the one hand and DeMorgan↓ and Ockham's *a superiori* rule on the other hand, cannot be denied. Each pair of rules makes the same predications as to validity. Of course, there is no evidence that De Morgan obtained his rules by reading Ockham's work.

It is interesting to observe that De Morgan↑ shows close resemblance to some of the topics. For instance [Peter of Spain, 1972, p. 64] mentions the topic *a specie* and its associated maxims:

- Whatever is predicated of the species, is also predicated of the genus
- Whatever the species is predicated of, the genus is also predicated of

This topic can be seen as the instruction to replace the name of the species by the name of the genus. Thus, in fact, it sanctions inferences from (names of) subsets into (names of) sets. The main difference between this topic and Ockham's rules is that the last are subjected to an extra restriction. De Morgan's own rules also point to a restriction in the use of the rules. For the terms involved in the replacement must to be used in a specific way. Once more, there is no evidence that De Morgan was aware of the existence of this topic -though [De Morgan, 1966] contains a few references to Peter of Spain.

It is in this respect noteworthy that in his treatment of the topics Abelard refers to this topic the validity of the inference [Peter Abelard, 1956, 327].

> Whatever is a head of a man is a head on an animal (*si est caput hominis, et animalis*

De Morgan's example and his account for it have, clearly, medieval antecedents.

De Morgan's Rules and Dictum de Omni

De Morgan called DeMorgan↓ above, a version of the *Dictum de omni et nullo.* However, the suggestion he wants to convey hereby does not appear to be well-founded. Undoubtedly, there can be some dispute about the proper formulation of the medieval dicta. But there is no reason to dispute the aims for which the dicta were devised. They may be briefly described as the ground for the direct justification for the syllogisms in the first figure. Thus the traditional *dictum de omni* is used to justify the inference:

$$\frac{\text{Every A is B} \quad \text{Some C is A}}{\text{Some C is B}}$$

It is not our concern here to indicate the way in which this justification is provided. All I want is for the reader to note that a justification using DeMorgan↓ seems to be not viable. Because in the particular sentence *Some C is A* no expression is used universally. Of course, *A* is used universally in the other premise. However, we cannot use this sentence as the context of replacement because we have no information about the genus-species relation of *A* and *C*.

As a matter of fact, it is DeMorgan↑ that has been identified with the *dictum de omni.* The idea that the traditional dicta correspond to the process by which a term is replaced by a larger one has been advanced by Venn. After saying that the dicta of Aristotle may be seen as elimination recipe, he then says:

> Now, the characteristic of this Elimination to which I wish prominently to direct the reader's attention, as containing the main clue to its significance in Logic, is this: –that we have substituted a *broader or less exact determination* in the place of the one which was first given to us. That is, we have had to let slip a part of the meaning of the data in performing this process [Venn, 1881, 286-87],

Let us now turn our attention to the purpose for which De Morgan defined the replacement rules.

A Schematic Formulation of De Morgan's Rules

De Morgan's replacement rules bring us beyond the ordinary syllogistic. For instance, with the aid of DeMorgan↓ we can explain the acceptability of this inference:

$$\frac{\text{A man sees every animal}}{\text{A man sees every horse}}$$

To achieve that goal we use semantic facts. We know that the denotation of *animal* is the genus of the denotation of *horse.* We therefore use DeMorgan↓ in order to substitute *horse* for *animal* in the given premise; in doing this we reach the desired sentence as a conclusion.

De Morgan's wording of the rules seems to imply that we need truths in order to apply them in that the terms involved must be actually related as genus and species. But there is no need to accept this. Because universal sentences like *Every A is B* are employed to assert that A is a species of B, and that B is a genus of A. Alternatively, such sentences can be used to say that A is the lesser term of the pair (A, B). Thus universal sentences entail one of the conditions needed to use De Morgan's rules. This interpretation of universal positive sentences is De Morgan's own interpretation [De Morgan, 1847, p. 75].

The preceding discussion will be resumed in the following formulation of two derived replacement rules based on DeMorgan↑ and DeMorgan↓:

- D↑

  $$\frac{\text{Every X is Y} \quad \text{F(X)}}{\text{F(Y)}}$$

  provided X is used particularly in F(X).

- D↓

  $$\frac{\text{Every Y is X} \quad \text{F(X)}}{\text{F(Y)}}$$

  provided X is used universally in F(X).

We still need to determine in which way these replacement rules can be used to justify DMA in the versions (7) and (2).

De Morgan Rules and DMA

Notice first that the application of D↑ to (3) is quite direct. All that has to be assumed is that in *Some one kills a man* the term *man* is used particularly. Then, in accordance with D↑, the conclusion follows. It is rather disappointing that we ignore how De Morgan coped with (7) or (2). We have isolated the contribution of the universal sentence given as a premise in those arguments. We know for sure that replacements have to occur. But what we do not know is in which sentences the substitutions are to be carried out. De Morgan does not indicate this explicitly.

Consider first (2). Choose as an extra premise the tautological sentence *He who kills a man kills a man.* Assume that in the second occurrence of *man* this term is used particularly. Then the following inference is generated by using D↑ with regard to the underlined occurrence of *man*:

$$\frac{\text{Every man is an animal} \quad \text{He who kills a man kills a } \underline{\text{man}}}{\text{He who kills a man kills an animal}}$$

We can also see why De Morgan could have considered (7) a valid inference. The first explanation of De Morgan's example along the lines we follow is to be found in [Merrill, 1977], although Merrill directs his explanation on the faulty (4). Assume that in the underlined occurrence of *man* in the tautological premise below *man* is used particularly, then one application of $D\!\uparrow$ will yield the desired conclusion:

$$\frac{\text{Every man is an animal} \quad \text{Every head of a man is a head of a } \underline{\text{man}}}{\text{Every head of a man is a head of an animal}}$$

Shortcomings in De Morgan's Approach

As illustrated above, De Morgan's logic seems stronger than syllogistic logic. There are, however, a few problems. It is not sufficient that the denotation of the relevant expressions be given as genus and species. It is just as important that the expressions themselves be used in a particular way. Before substituting one expression for another, we have to be certain that the genus is being used universally in one case and that the species is used particularly in the other.

Conditions on De Morgan's Rules

Up till now, I have assumed that the expressions of our examples meet those restrictions. This is a simplification, since I have not yet given any criterion to determine whether this is the case. Once more, we are in the dark about De Morgan's real choice. Our hypothesis is that he took expressions of generality as a guide-line. When dealing with about categorical sentences, he said that the words of the sentences indicate whether the subject is used universally or particularly:

> In such propositions as 'Every X is Y', 'Some Xs are not Ys', $ c., X is called the *subject* and Y the *predicate*, while the verb 'is' or 'is not' is called the *copula*. It is obvious that the words of the proposition point out whether the subject is spoken of universally or partially, but not so of the predicate [De Morgan, 1847, 6].

The generalization of this remark results in the following criteria

- C_1: In the contexts *F(an A)*, *F(some A)* the term *A* is used particularly.
- C_2 : In the context *F(every A)*, *F(all A)* the term *A* is used universally.

Counterexamples

However, a little reflection shows these criteria to be inadequate. It is true that (7) and (1) can be generated by using C_1 and $D\!\uparrow$. But, the same holds for the following invalid inference:

$$\frac{\textit{Every man is an animal} \quad \textit{Every head of a } \underline{\textit{man}} \textit{ is a head of a man}}{\textit{Every head of an animal is a head of a man}}$$

This inference shows conclusively that the combination of C_1 and $D\uparrow$ is unsound. The major difficulty is that C_1 does not differentiate between the two occurrences of *man* in the tautological premise; therefore *animal* can in both cases take the place of *man*.

Now, [Merrill, 1990, 86] stresses that the invalid argument

$$\frac{\textit{Every American is human} \quad \textit{Every person who loves all humans is happy}}{\textit{Every person who loves all Americans is happy}}$$

must be recognized as valid by De Morgan's standards. Notice that according to C_2 in the second premise *humans* must be universal. Therefore by using $D\downarrow$ we infer the conclusion. Merrill then proceeds to dismiss De Morgan's rules as intrinsically inadequate. This is a matter to which I shall return shortly after I have dealt with a natural solution to De Morgan's predicament.

The Conditions in Terms of Distribution

De Morgan's treatment of his non-monadic arguments thus fails, but it is worth emphasizing that this is not due to the format of the rules. It is rather the criteria C_1 and C_2 which have been proved wanting. The use of C_1 and C_2, which seems implicit in De Morgan's strategy, makes the rules applicable. But these criteria are clearly not adequate. At this point we may consider abandoning the literal reading of De Morgan's rules and, instead, try to interpret them in terms of the traditional doctrine of distribution. This doctrine can be seen as providing the means needed for the description of the contexts in which substitution is allowed. In fact, the description Prior gave of distribution suggests a connection between the traditional doctrine of distribution and De Morgan's original rules. In Prior's view, the notion of distribution is used to indicate sensibility regarding replacement:

> What the traditional writers were trying to express seems to be something of the following sort: a term I is distributed in a proposition f(I) if and only if it is replaceable in f(I), without loss of truth, by any term 'falling under it' in the way that a species falls under a genus [Prior, [1967] 1972, 39].

Moreover, De Morgan recognizes that what he calls *universal spoken of*, is called by others *distributed* [De Morgan, 1966, 6]. Now we can re-word the provisos on the use of the replacement rules as follows:

- Provided that X is distributed in F[X].
- Provided that X is undistributed in F[X].

Remember now that according to the tradition *A* is distributed in sentences of the form *Every A B* and undistributed in sentences of the form *Every B A*. Therefore, the distribution doctrine says that in the sentence

Every head of a man is a head of a man.

head of a man has two different distribution values. The distribution doctrine says nothing whatsoever about the distribution value of *man* therein. It is true now that we cannot longer use our distribution criteria to generate the invalid arguments mentioned before. Unfortunately, neither can we generate the valid ones. It should be apparent that we have sinned at the wrong side of the caution line.

The logical moral is this. If we want to complement De Morgan's rules with the distribution doctrine, then this doctrine itself will have to be extended so as to include the elements of compound expressions. What we need is a procedure for computing distribution values, starting from basic expressions and using distribution values induced by the expressions of generality. De Morgan himself does not, however, appear to have recognized the need for such a systematic procedure. Still, though De Morgan failed to hit on such a procedure it does exist.

De Morgan Rules and Monotonicity

In this section I argue that it is possible to give a recursive definition of distribution values. We shall, however, abandon this historically loaded terminology and I shall use, instead, the less colorful language of formal semantics. We shall first introduce the notions of increasing and decreasing monotonicity. Subsequently I shall introduce some functions with these properties. By using the standard result that the set of monotone functions is closed under composition, we shall be able to give a monotonic explanation of DMA. We shall be slightly cavalier with the precise relationship between syntax and semantics.

Let D_1, D_2 be sets partially ordered by the relations $\sqsubseteq_1$ and $\sqsubseteq_2$ respectively. Moreover, assume that f is a function from D_1 into D_2. Then we say

- f is monotone increasing if for all x, y in D_1 holds

$$\frac{x \sqsubseteq_1 y}{f(x) \sqsubseteq_2 f(y)}$$

- f is monotone decreasing if for all x, y in D_1 holds

$$\frac{y \sqsubseteq_1 x}{f(x) \sqsubseteq_2 f(y)}$$

It can be easily shown that the set of monotone functions is closed under composition. Here is an illustration of the argument. Let f and g be monotone increasing. Assume that the composition $f \circ g$ is defined. Consider x and y in the domain of g. If $x \leq y$, then, by assumption, $g(x) \sqsubseteq g(y)$. Since f is monotone increasing and $g(x)$ and $g(y)$ belong to the domain of f, we have $f(g(x)) \sqsubseteq f(g(y))$, i.e., $f \circ g(x) \leq f \circ g(y)$. Thus $f \circ g$ is increasing monotone. In a similar way one can show that

- the composition of two decreasing functions is increasing.
- the composition of a decreasing and an increasing function is decreasing.

This argument indicates that the following definition is adequate: Let f be a composition of monotone functions, i.e., $f = g_1 \circ g_2 \cdots \circ g_n$ where g_i $(1 \leq i \leq n)$ is either monotone increasing or monotone decreasing. Then we shall say that f is monotone.
We now state without proof the following proposition. Let f be monotone function according to the previous definition. Then

1. if the number of decreasing functions at the right side of the equation is even, f is increasing monotone;
2. if the number of decreasing functions at the right side of the equation is odd, f is decreasing monotone [Curry, [1963] 1977, 103].

Let X be the denotation in an ordered domain of the natural language expression x and let B be a fixed semantic object. Then we define the functions f_1-f_4 as follows:

- $f_1(X) \equiv$ B OF X.
- $f_2(X) \equiv$ A(N) X.
- $f_3(X) \equiv$ EVERY B IS X.
- $f_4(X) \equiv$ EVERY X IS B.

Assume that f_1-f_3 are monotone increasing, while f_4 is monotone decreasing. Define g_1, g_2 as follows: $g_1 = f_4 \circ f_1 \circ f_2$. $g_2 = f_3 \circ f_2 \circ f_1 \circ f_2$.
Observe that by definition g_2 is monotone increasing while g_1 is monotone decreasing. Now $g_2(X)$ corresponds to EVERY B IS B' OF A X since

1.	$g_2(X)$	=	$f_3 \circ f_2 \circ f_1 \circ f_2(X)$
2.		=	EVERY B IS $f_2 \circ f_1 \circ f_2(X)$
3.		=	EVERY B IS A $f_1 \circ f_2(X)$
4.		=	EVERY B IS A B' OF $f_2(X)$
5.		=	EVERY B IS A B' OF A X

A similar unpacking shows that $g_1(X)$ corresponds to EVERY B' OF A X IS B.
What kind of functions are g_1 and g_2? If we look at the *syntactical* side we see that X has to correspond to a noun while the whole corresponds to a sentence. In a standard way we can assume that the denotations of nouns are sets (of individuals) and the denotations of sentences are one of the two values in $\{0, 1\}$. So, we can say that they are functions from sets of individuals into truth values. Of course, these denotations have a natural ordering. Inclusion in the first case and the as-small-as relation in the second one.

A Monotonic Explanation of DMA

The discussion may be seem to have strayed away from the question of the soundness of De Morgan's rules, but, of course, this is not so in reality. Let us show that DMA 7 is valid. Put first

- $f_1(X)$ = HEAD OF X.
- $f_3(X)$ = EVERY HEAD OF A MAN IS X.

Then

- g_2(MAN) $\equiv$ EVERY HEAD OF A MAN IS A HEAD OF A MAN.
- g_2(ANIMAL) $\equiv$ EVERY HEAD OF A MAN IS A HEAD OF AN ANIMAL.

Assume now that *Every man is animal* is true. Then in a standard way we conclude that MAN $\subseteq$ ANIMAL. Since g_2 is monotone increasing we have g_2(MAN) $\leq$ g_2(ANIMAL). But, trivially, g_2(MAN) $= 1$. Therefore g_2(ANIMAL) $= 1$. But this means that *Every head of a man is a head of an animal* is true.

Let us return to the counterexamples displayed in section 2. Put first

- $f_1(X)$ = HEAD OF X.
- $f_4(X)$ = EVERY X IS A HEAD OF A MAN.

Observe that we shall have

- g_1(MAN) = EVERY HEAD OF A MAN IS A HEAD OF A MAN
- g_1(ANIMAL) = EVERY HEAD OF AN ANIMAL IS A HEAD OF A MAN.

But now our invalid inference cannot be sanctioned. We can still infer that MAN $\subseteq$ ANIMAL. Nevertheless g_1(MAN) $\leq$ g_1(ANIMAL) will not follow. For g_1 is a decreasing not an increasing function.

The natural consequence of the previous paragraphs should be the introduction of adequate provisos to De Morgan's replacement rules. Because, in fact, we have seen that the use of D$\uparrow$ is sound whenever the term that is replaced is the last link in a continuous chain of monotonic functions in which the number of decreasing ones is even. Thus the proviso attached to this rule can be replaced by

> provided X occurs in the scope of an even number of decreasing functors.

The reader will accept that in the sentence *Every head of a man is a head of a man* the first occurrence of *man* happens to fall within the scope of an odd number of decreasing functors namely the universal determiner. Therefore the use of D$\uparrow$ is no longer justified.

Similarly, D$\downarrow$ is sound whenever the designed term is the last link in a chain of monotonic functions in which the number of decreasing functions is odd. Thus the proviso then may take the form provided X occurs in the scope of an odd number of decreasing functors.

It is proper to observe that in the sentence *Every person who loves all humans is happy* the expression *humans* occurs in the scope of an even number of decreasing functors, namely the two universal determiners. Therefore we cannot longer indulge in the use of D$\downarrow$: the premise does not satisfy the monotonicity constrain.

3 FROM COPULA TO RELATIONS

Types of Copulas

Already in his essay on geometrical reasoning De Morgan noticed that the copula *be* has two logical uses. It can be used to express equality. But it can also be used to signal predication (or applicability). This awareness is seen by some historians as the seed of his more mature theory of the copula that would led him to a theory of relations [De Morgan, [1836] 1898, 203],[Panteki, 1992, 431]. In 1847 he distinguished three senses of the copula. Next to the identificational and the predicative role he discerns a modal use. The task he then addresses is that of finding the common characteristics of these three senses. The result of his findings can be expressed by the following list:

1. $\alpha C\beta = \beta C\alpha$
2. If $\alpha C\beta$ and $\beta C\gamma$, then $\alpha C\gamma$
3. For any α and β, either $\alpha C\beta$ or αnot-Cβ

Any predicate that fulfills these conditions makes, in De Morgan's view, "all the rules of logic true".

The same topic is considered again in [De Morgan, 1851]. De Morgan makes in this paper free use of notions that concern binary relations. Copulas, may be both transitive and symmetric (*convertible*). A relation may be the converse (*correlative*) of other relation or of itself. He also hints at demonstrable properties of relations: the converse of a transitive relation is also transitive. The main focus of interest is, however, the expansion of the range of logic. By appealing to the relational properties of the copulas involved, De Morgan's logic can account for more inferences than his traditional rivals. He accounts, for instance, for

- inferences involving a merely transitive relation,
- inferences involving a transitive relation and its converse,
- inferences involving two arbitrary relations and their composition.

Before attending to this aspect of this paper, I want to stress a novel aspect of De Morgan's approach that bears traces of his functional background.

An Inverse Problem

Given a syllogistic schema, say,

$$\frac{\text{Every P C a M} \quad \text{No S C a M}}{\text{No S C a P}}$$

De Morgan asks which relations C satisfy it. The traditional solution is, of course,

$$C = \text{be}$$

To this solution, De Morgan adds, among others,

$$\begin{array}{lcl} C & = & \text{be equal to} \\ C & = & \text{be superior to} \\ C & = & \text{be less than} \end{array}$$

More in general, he asks for the relations C that satisfy the set of syllogistic forms

barbara, darii, camestres, baroco, felapton, ferison, bocardo.

Instead of listing particular solutions, De Morgan points out that any transitive relation satisfies all these schemata.

For instance, a transitive predicate C validates our previous schema, camestres. For suppose that there are two objects, s and p that make *Some S C a P* true. Then C(s, p) will hold. According to the first premise p will C a M. Call this M by the name *m*. Thus C(p, m) will be the case. But, by the assumption concerning the transitive nature of C C(s, p) follows. This information contradicts the remaining premise.

The next question is, now, which pair of relations (C, C') satisfies the schema

$$\frac{\text{No M C a P} \quad \text{Some S C' a M}}{\text{No every S C a P}}$$

The answer is that any pair of transitive converse relations will do that. For suppose now that every S is C to a P. Given that there is at least one S, say s, such that for a M, m, C'(s, m) holds, we may conclude that C(m, s) is the case. Remember that C and C are converse relations. Moreover, s is C to some p, i.e., C(s, p). By the assumption of transitivity we now conclude C(m, p). But this contradicts the first premise.

From this general point of view, the remarkable property of the Aristotelian copula is that it is its own converse [De Morgan, 1966, 55]. The result to which De Morgan is steering is that any transitive and symmetric relation satisfies all the Aristotelian syllogisms. In the meantime, De Morgan stresses the fact that there are valid inferences independent of transitivity. Suppose that for a pair of non transitive relations R and S, the composition R $\circ$S exists. Then the following schema would be valid

$$\frac{\text{Every M is S to some P} \quad \text{Every S is R to some M}}{\text{Every S is R} \circ\text{S to some P}}$$

He also points out that the ordinary syllogism is a special case of such a schema. For even "though in ordinary inference the concluding copula is usually identical with those premised, yet it is not less true that the composition must have taken place" (p. 56). These words elaborate the closing remarks he made in [De Morgan, [1836] 1898]: reasoning consists in the construction of new relations. This is again a topic that betrays De Morgan's familiarity with the properties of functions.

The Notion of an Abstract Copula

The copula is regarded by De Morgan as abstract in the sense that its logical behavior is fully determined by its relational properties. Instances of the copula, actual predicates of the language, would always carry extra meaning next and above their relational import. This extra meaning of a copula does not determine, however, how it is used in inferences. This feature of De Morgan's theory neutralizes the objection that the study of the abstract copula does not belong to logic inasmuch as it introduce into it extra-logical concerns. The extra-logical part is brought in when the abstract copula is instantiated but it has no logical role to play. In this sense, even the traditional copula may be regarded as bringing into logic non formal aspects. Still the extra-logical information does not need to influence the course of reasoning. At least not in such a way as to make a distinction between the traditional copula and the new ones advocated by De Morgan. The information that *be* is transitive is no more and no less part of a particular inference than the information that *kill* is not. There is, in De Morgan's views, not much difference between *be* and other copulas. With polemical stubbornness he remarked that "the copula has been material to these days" [De Morgan, 1966, 68]. In other words, *be* regarded as a lexical component of the language has more semantic features than its two syllogistically relevant properties. One can claim against De Morgan that in categorial propositions the copula is formal just because only those two relevant properties are ever made use of. To this De Morgan would reply that that is just the point. Any transitive and converse relation can play the role of the copula precisely because these two properties are sufficient. In reasoning we abstract from the semantic surplus of the copulas and this process leads to the abstract copula.

There is still another point of view according to which the copula can be considered as part of the material component in a schematic categorial proposition. In his next study on logic, [De Morgan, 1864a], would put the matters in the following way. The most general schema of a categorial proposition takes the form

$$\text{Q X R Y}$$

where Q ranges over the classical determiners, X and Y over absolute terms and R over binary relations. Next in generality comes

$$\text{Q X TR Y}$$

where TR ranges over transitive relations. The form

$$\text{Q X be Y}$$

is less general inasmuch as it is a special case of the general structure. This form carries more information than the previous one because the copula *be* "has transitiveness and more" [De Morgan, 1966, 80]. In a standard categorial schema as

$$\text{Every X is Y}$$

in which Q and R are instantiated, X and Y "the objects of inference, being terms expressed in general symbols, are void of matter; the relations between them, and the modes of inference, are material" (p. 81).

4 RELATIONAL VIEW OF SYLLOGISMS

Syllogisms as Relations

As the last reference to [De Morgan, 1864a] indicates, in this paper De Morgan returned to the topic of binary relations. A relation, says De Morgan, between two entities (individual objects, qualities, classes, attributes) is a connection that is seen to join them. A proposition is the presentation of two objects as a relation pair. The subject of a proposition is that which is in relation. The predicate is that with which it has the relation. Categorial propositions are relational propositions.[42] They express a whole-part relation between classes (aggregates of individuals) or attributes (aggregates of qualities of individuals). For instance, the standard categorial propositions have the following relational analysis:

- A(S, P) iff S is included in P. S is called *species* of P.
- E(S, P) iff S and P are disjoint. S is called *external* of P.
- I(S, P) iff S and P overlap. S is called *partient* of P.
- O(S, P) iff S is not included in P. S is called *exient* of P.

[42]The view of categorial propositions as relational ones is one of the cornerstones of the generalized quantifier view of natural language quantification advocated by modern linguists and logicians. See in this regard, for instance, [Zwarts, 1983].

Syllogisms as composition

A syllogism, on the other hand, is defined as the "inference of the relation that exists between two terms, as a necessary consequence of their relations to the same third, or *middle term*" [De Morgan, 1966, 131]. Some years earlier, De Morgan had remarked that the algebraic deduction of "$y = \varphi\psi z$ from $y = \varphi x$ and ψz is the formation of the composite copula $= \varphi\psi$ (p. 56). This early remark is brought to bear on the theory of the standard syllogism via the following existence assumption.

> For a pair of relations R and S such that R(y, x) and S(x, z) there is a composite relation, RS, that holds between y and z.

We shall use RS to represent the relative product of the relations RS. This is not the notation De Morgan used. Be it that he entertained the idea of using it. In one of the logical passages that links his work on relations to his previous work on functions, he remarks (p. 107):

> Here the compound relation, or combined relation, may be represented by AB, but by no one except a mathematician who is used to the *functional* symbol, and to the idea of $\varphi\psi(xy)$ and its distinction between the mode of composition of x, y and that of φ, ψ.

Now, the modi of the first figure can be expressed as follows

$$\frac{\mathrm{A(M, P)} \quad \mathrm{A(S, M)}}{\mathrm{AA(S, P)}} \qquad \frac{\mathrm{E(M, P)} \quad \mathrm{A(S, M)}}{\mathrm{AE(S, P)}}$$

$$\frac{\mathrm{A(M, P)} \quad \mathrm{I(S, M)}}{\mathrm{IA(S,P)}} \qquad \frac{\mathrm{E(M, P)} \quad \mathrm{I(S, M)}}{\mathrm{IE(S, P)}}$$

Syllogism as contraposition

Following a suggestion of William Rowan Hamilton, De Morgan organized the syllogisms in cycles [De Morgan, 1966, 132-33]. 'In each cycle each syllogism is what I have called an *opponent* of each of the others", writes De Morgan. The general idea of the cycles is that if the combination of R and S is sufficient to generate T, then the presence of R together with the absence of T signals the absence of S. The point is therefore that each syllogism has "the form in which one premise admitted, and the conclusion denied, denies the other premise" (p. 133).

Barbara	Baroko	Bokardo
Celarent	Festino	Disamis
Darii	Camestres	Ferison
Ferio	Cesare	Datisi

In fact, this organization of the syllogisms was inspired by the strategy followed by old logicians to "reduce two obstinate malcontents to the first figure" (p. 132). This reference is the first hint of what [Prior, [1955] 1962, 153] considered the most intriguing thesis contained in De Morgan's essay on the logic of relations, De Morgan's Theorem K:

> If two relations combine into what is contained in a third relation, then the *converse* of either of the two combined with the *contrary* of the third, in the same order, is contained in the *contrary* of the other of the two [De Morgan, 1966, 186-87]

In modern notation, De Morgan's theorem has the form

$$\text{If } R \subseteq T \text{ then both } R^{-1}\overline{T} \subseteq \overline{S} \text{ and } \overline{T}S^{-1} \subseteq \overline{R}$$

Syllogisms and converses

The relational properties of composition and negation are not sufficient for the organization of the syllogistic material along fully relational lines. For instance, the conclusion of the syllogism below, baroko,

$$\frac{\text{A(P, M)} \quad \text{O(S M)}}{\text{O(S, P)}}$$

cannot be represented by the composition OA(S, P). For this relation holds of the objects S and P only if there is an object M such that S is not contained in M and M is contained in P. But we had rather had that P is contained in M. De Morgan saw and solved this difficulty. He took as starting point the existential assumption

Every relation R has a converse.

Later De Morgan would borrow from the notation of the theory of functional equations the expression R^{-1} to denote converse relations. Making use of this notation we can represent the conclusion of Baroko by the composition OA^{-1}. Alternatively, we could retain the conclusion OA(S, P), and modify the first premise into A^{-1}(P, M).

This last alternative allows us to take a fresh look at the cycles. Because, for instance, the cycle of barbara-baroko-bokardo can be represented as follows

If	A(M, P)	and	A(S, M)	then	AA(S, P)
If	A^{-1}(M, P)	and	O(S, P)	then	OA(S, M)
If	O(S, P)	and	A^{-1}(S, M)	then	AO(M, P)

This representation of the cycle combines the three operations on relations De Morgan discussed: converse relations, complementary relations and composition of relations. De Morgan's Theorem K is a generalization of the cycle inasmuch as it is formulated for arbitrary relations and not only for the Aristotelian ones.

In De Morgan's words

De Morgan uses the following technical terminology. If x is a species of y, then y is a genus of y and if x is an exient of y then y is a deficient of x. The external and the partient relation are their own converses. There is a further relation between the syllogistic relations. The relations of species and exient are complementary in the sense that x is a species of y iff y is not an exient of x. Similarly, x is an external of y iff y is not a partient of x. using this terminology he codifies the cycles as follows (I add a little bit of modern notation to ease the reading).

species $\circ$ species $\subseteq$ species	genus $\circ$ exient $\subseteq$ exient	exient $\circ$ genus $\subseteq$ exient
species $\circ$ external $\subseteq$ external	partient $\circ$ external $\subseteq$ exient	species $\circ$ partient $\subseteq$ partient
partient $\circ$ external $\subseteq$ exient	species $\circ$ external $\subseteq$ external	partient $\circ$ species $\subseteq$ partient

5 RELATIONS AS A BRANCH OF LOGIC

Relations as a branch of Logic: The Syllabus

We have already seen that De Morgan occasionally alluded to properties of relations. The three existence postulates mentioned in the previous pages are a case in point. These examples can be expanded. [De Morgan, 1860] characterizes a transitive relation as a relation that combined with itself yields itself. Furthermore, he points out that the converse of a transitive relation is also transitive. Identity is conceived as the relation in which any object stands to itself: everything is itself. He sketches also the proof of his theorem K.

Relations as a branch of Logic: On the Logic of Relations

The most mature treatment of the logic of relations that De Morgan produced was [De Morgan, 1864b]. In this paper he developed further the technical vocabulary and some of the ideas of previous papers. From the technical language of the calculus of functions he took over the notation for converse functions to denote the converse relation and from his own work he took over the notation for contrary terms to denote contrary relations.

Definitions

The relevant definitions are these:

1. x...Ly signify that x is an object that stands to y in the relation R. We also say that x is one of the Ls of y. We shall represent x...Ly by L(x, y).

2. L^{-1}(x, y) holds iff L(y, x) holds. L^{-1} is called the *converse* relation of L.

3. $\overline{L}$(x, y) holds iff L(x, y) does not hold. $\overline{L}$ is called the *contrary* of L.

4. LM(x, y) holds iff for some z, L(x, z) and M(z, y).

5. LM'(x, y) holds iff for every z such that M(z, y), L(x, z) holds.

6. L,M(x, y) holds iff for every z such that L(x, z), M(z, y).

Universal Quantification

The two last definitions have not been anticipated in previous papers. While in ordinary composition we can speak of implicit existential quantification, in De Morgan's new forms the quantification is universal. Note that we can not speak here of quantification of the predicate position. The position that is quantified both in existential as in universal composition is the subject position of the second predicate and the object position of the first one. It is also interesting to note that the quantified position is always implicit. De Morgan speaks of *quantified relation* but the quantification is not second order quantification [De Morgan, 1966, 221]. To most historians the use of the superior and inferior accents to express universal quantification is rather clumsy. Still, it remains remarkable that he handles his quantified relational propositions "so skilfully, lacking the quantifiers, as he does, and given the clumsiness of his notation" [Martin, 1980, 53]. In the course of his exposition De Morgan occasionally uses undefined expressions. For instance, he introduces the expression Lx with the obvious intention of referring to a class. The context make it clear that the class referred to is the left projection of L, the set of Ls of x. De Morgan also uses the expression *included in* ambiguously. It is taken to denote class inclusion, but also the membership of an object to a class. An alternative account consists in taking class inclusion as the only sense. In this case we have to regard individuals as unit sets.

Properties of Relations

To these basic definitions that have already been anticipated in his previous papers, De Morgan adds a list of provable properties of relations.

1. $\overline{\mathrm{LM}} = \overline{\mathrm{L}}\mathrm{M}' = \mathrm{L}, \overline{\mathrm{M}}$

2. Contraries of converses are converses.

3. Converses of contraries are contraries.

4. The contrary of a converse is the converse of a contrary: $\overline{(\mathrm{L}^{-1})} = (\overline{\mathrm{L}})^{-1}$

5. If a relation L is contained in a relation M, then L^{-1} is contained in M^{-1} and $\overline{\mathrm{M}}$ is contained in $\overline{\mathrm{L}}$.

6. The conversion of a compound relation converts both components and inverts their order: $(\mathrm{LM})^{-1} = (\mathrm{M})^{-1}(\mathrm{L})^{-1}$

The Proof of K

De Morgan offered a proof of his Theorem K. The proof strategy he follows consists in the use of some relational principles and in the extrapolation of monadic principles into the field of relations. The relational principles are these three:

1. If L $\subseteq$ R then SL $\subseteq$ SR
2. (LM')M^{-1} $\subseteq$ L
3. M^{-1}(ML)$\subseteq$ L

The Principles

Before describing the proof itself it is convenient to say some words about these principles. The first one can be accounted for along the familiar monotonicity lines. Because both members of a composition occur in monotone position. Hence, we can take SL $\subseteq$ SL as tautological premise and replace the superior R for its inferior L, thus obtaining SL $\subseteq$ SR. The similitude of this explanation to the one given for De Morgan's example is not unmotivated. In fact, this principle is the relational version of the challenge he once launched and that was the subject of the first pages of this part. The first of the other two has the following binding structure

$$(\forall v(M(v,y) \to L(x,v)) \wedge M(v,y)) \to L(x,y)$$

The universal closure of this version of modus ponens, a monotonic inference pattern, is clearly valid. This principle will be used by De Morgan as an additional premise. This use shows De Morgan using, in Boolean fashion, logical truths in the course of a logical argumentation. The third principle is in fact a variant of the previous one. Let us now turn out attention to the proof.

Proof

Assume RS $\subseteq$ T. Contraposition yields $\overline{\mathrm{T}} \subseteq \overline{\mathrm{RS}}$. By definition, the last inclusion may be regarded in its equivalent form $\overline{\mathrm{T}} \subseteq \overline{\mathrm{R}}\mathrm{S'}$. By using now the relational version of De Morgan's example we deduce $\overline{\mathrm{T}}\mathrm{S}^{-1} \subseteq \overline{\mathrm{R}}\mathrm{S'S}^{-1}$. But the truism $\overline{\mathrm{R}}\mathrm{S'S}^{-1} \subseteq \overline{\mathrm{R}}$ allows us to conclude $\overline{\mathrm{T}}\mathrm{S}^{-1} \subseteq \overline{\mathrm{R}}$. The proof of the second part of this theorem takes, essentially, the same form.

A Difficult Passage

Connected with his proof of Theorem K De Morgan argues for the following proposition.

$$\text{If LM} = \text{N, then both L} \subseteq \text{NM}^{-1} \text{ and M} \subseteq \text{L}^{-1}\text{N}$$

The proof of this proposition may take the following form. Let x be an arbitrary individual. Consider now the set $MM^{-1}x = \{y : \exists z(M(y, z) \wedge M(x, z))\}$. De Morgan makes the crucial assumption that for any x holds $x \in MM^{-1}x$. This means, of course, that MM^{-1} is reflexive. That is, for any x $MM^{-1}(x, x)$. We want to establish now that Lx is included in $NM^{-1}x$. Suppose then $y \in Lx$. We have both L(y, x) as $MM^{-1}(x, x)$. We can therefore conclude that $LMM^{-1}(y, x)$. Hence, $y \in LMM^{-1}x$. Thus, $Lx \subseteq LMM^{-1}x$. Since x was arbitrarily chosen, we have established $L \subseteq LMM^{-1}$. De Morgan's result follows by using the identity $LMM^{-1} = NM^{-1}$.

The proof of the other part of this proposition follows the same lines. The starting point is the set $L^{-1}Lx = \{y : \exists z(L(z, y) \wedge M(z, x))\}$. The crucial assumption yields the reflexivity of $L^{-1}L$. We then argue that for any x holds that Mx is included in $L^{-1}LMx$ and via the identity $L^{-1}LM = NM$ we reach the final conclusion.

Alternative Proof

An alternative account of the first part of this proof is given in [Merrill, 1990]. Starting from the inclusion $LM \subseteq N$, Merrill derives $LMM^{-1} \subseteq NM^{-1}$. Consequently he surmises that De Morgan used the principle $L \subseteq LMM^{-1}$ to infer the desired conclusion, namely, $L \subseteq NM^{-1}$. Even though there is no indication that De Morgan used the principle Merrill attributes him, this assumption is harmless. Merrill's principle is a consequence of the reflexivity of MM^{-1}. Of course, to apply the same strategy to the second part of the proposition we have to assume $M \subseteq L^{-1}LM$. But this principle follows from the assumption that $L^{-1}L$ is reflexive.

Total Relations

It is clear that De Morgan's proposition does not hold for all relations. The reflexivity constraints set the limits to its range. As I have presented the matters De Morgan restricted his attention to total relations, i.e., relations whose domain is the whole universe of interpretation. There is, however, another view of this issue. De Morgan, a critic say, was simply confused. He clearly believed that *all* relations are total. There is, namely, at least another passage, outside the context of proof just discussed in which he appeals to the reflexivity of the compound LL^{-1} for arbitrary L [De Morgan, 1966, 226]. But this view is obviously wrong. Take a newborn baby, Mary. Consider the relation mother-of. De Morgan must believe, in the critics's view, that there is an object x such that mother-of(Mary, x) will hold.

Explanation: Existential Import

How came De Morgan to find himself in such an outlandish position? In the absence of explicit arguments for the adoption of a wrong view, the historian should not refuse to rationalize error. True, in the eyes of the lord, sundry and uncountable are the roads away from heaven. But something about them can be said. Merrill advanced two explanations. The analysis in [Merrill, 1990, 124] takes this form

> One explanation may be in De Morgan's doctrine of existential import, which assumes that all terms in categorial propositions are nonempty. If this is true in general, it might also be assumed to be true of a term defined relationally, such as "child of Mary".

Emptiness of Compound Terms

This account is not quite satisfactory. The doctrine of existential import applies to ground terms not to compound ones. No primitive term may be empty, is the doctrine. So, even if one accepts that *mother-of* and *Mary* cannot be empty, the doctrine says nothing about *mother-of(Mary, x).* This is the common doctrine. But there are some evidence that De Morgan himself adhered to it. For according [De Morgan, 1966, 97], the set of predicates {animal, human, Roman, ancient, general, conqueror of Gaul's, writer of his own campaigns, composer for the pianoforte} specifies no individual. In this case, he says, "we have an empty box".

Explanation: Functional Bias

[Merrill, 1990, 212] points out that De Morgan's "notation for functional inverse ... may also have been a factor" in his embracing the total view of relations. As observed in [Panteki, 1992], De Morgan's proposition was anticipated in [De Morgan, 1864a]. In a footnote he refers to two properties of functions. The first one corresponds to the first part of his proposition, the second one to the second part [De Morgan, 1966, 134].

> If $\varphi\psi x < \chi x$, for all the values of x, which is the proper analogy for the composition of relations in the syllogism, then $\varphi x < \chi\psi^{-1}x$, but we must not say $\psi x < \varphi^{-1}\chi x$.

It is interesting to notice that the first part of De Morgan's proposition generalizes to relations a property he knew hold of functions and, at the same time, established to his own satisfaction that a property that fails for functions holds of relations. The reason why De Morgan's first part of his proposition can be proven is due to the assumption that MM^{-1} is reflexive. Because on account of it the proposition is restricted to those relations whose domain exhausts the universe of discourse. This property they have in common with total functions. Similarly, the second part can be proven because of the assumption that $\mathrm{L}^{-1}\mathrm{L}$ is reflexive. On account of this assumption the proposition is restricted to those relations whose range exhausts the universe. This is a property that they share with some total functions. This explains why the non provable functional inclusion mentioned by De Morgan turns out to be provable for relations. It also shows which functions satisfy the second inclusion. Remember that the two assumptions De Morgan makes are mutually independent. Some total functions are not range exhausting, i.e., surjective, some range exhausting relations are not functions. It is, therefore, not only the notation for functional inverse that might have motivated De Morgan's views. The reflexivity conditions single out those relations that share enough properties of surjective functions to help the proof go through.

Extensional Bias

Merrill's accounts and our endorsing of the second one may, however, be seen to miss the point. They are motivated by an extensional view of relations. If the objects that stand in a relation must be part of the furniture of the world, then, indeed there is something wrong with a theory that demands an object that verifies the sentence $\exists$x (mother-of(mary, x) $\wedge$ Newborn(x)). The question is whether we can attribute to De Morgan the same extensional position. For him "any two objects of thought brought together by the mind, ... are *in relation*" [De Morgan, 1966, 218]. Relations are, therefore, in the mind and relata are in first instance ideas. Moreover, in his view "a symbol is not the representation of an external object absolutely, but of a state of the mind in regard to that object [De Morgan, 1841, 174]. It seems, therefore, that De Morgan does not necessarily share the conceptual background that makes of his suggestion such a conspicuous mistake.

Transitive and symmetric relations

Symmetric Relations

About symmetric relations, De Morgan makes the following interesting observations. In the first place, for any relation L he states that LL^{-1} is symmetric. Then he expresses a conjecture: for any symmetric relation M there is a relation L such that $M = LL^{-1}$. It is with regard to this conjecture that De Morgan makes use of the reflexivity of LL^{-1}. Given M(x, y), we obtain M(y, y). This is necessary, say De Morgan, if $M = LL^{-1}$. In other words, if De Morgan's conjecture is right, then all symmetric predicates are weakly reflexive. Since this consequence is false, any symmetric relation which is not reflexive shows that much, his conjecture has to be rejected.

Transitive Relations

We close our presentation of the De Morgan's logic of relations mentioning his successful treatment of transitive relations. A transitive relation L as any relation that satisfies the condition $LL \subseteq L$. From this definition and theorem K he deduces several properties of transitive relations. For instance, any transitive relation has a transitive converse, that is $L^{-1}L^{-1} \subseteq L^{-1}$.

6 FINAL WORDS

His analysis of the inadequacies of traditional logic, the study of the abstract copula, the relational analysis of the syllogism and, above all, his study of the logical properties of relations, make of De Morgan one of the founders of the algebra of logic. Still, his most mature paper on relations failed to make an impact. For this failure he is held responsible. As I pointed out, this paper is consider an untidy and incomplete ramble written in haste [Merrill, 1990, 196], [Panteki, 1992], [Grattan-Guinness, 2000, 33]. 20 years later Venn voiced his skepticism about the whole enterprize. The fact that the properties of relations are not shared by all of them caused him to remark that "the attempt to construct a Logic of Relatives seems to me altogether hopeless owing to the extreme vagueness and generality of this conception of a Relation. Almost anything may be regarded as a relation, and when we attempt to group them into manageable portions we find that several codes of law are required [Venn, 1881, 402-403].[43] Peirce thought differently. He developed De Morgan's theory further be it that he seems initially to be more interested in class terms involving relations than in relations as such. This line of development is suggested in [De Morgan, 1966, 82]. Here is remarked that "it is possible to reduce relation to class, by throwing 'X has A-relation to Y into the form 'X is in the class of objects having A-relation to Y' . De Morgan pondered, clearly, about the transition from relations to domain of relations. Peirce's turn.

Part 7

The Logic of Monadic Relations: Peirce

This part consists of 5 sections. The introductory one is followed by 3 section devoted to the theory of relatives developed by Peirce in 1870. I consider first the theory of simple relatives. Special attention is paid to the proper understanding of the notion of relative and the theory of implicit quantification used to define some of the operations on relatives. Then I focus on the theory of elementary relatives and their properties. This part concludes with a section in which I consider the definitive form that Peirce gave to his doctrine of relatives. At this moment we see him using explicit quantification to define operations on relatives, thus preparing the ground for his abandonment of the algebra of logic in favor of the theory of quantification.

1 INTRODUCTION

Boole studied the logical behavior of absolute terms, De Morgan studied the logical behavior of relational terms. Peirce studied the logical behavior of relations applied to absolute terms. Schröder systematized Peirce's contributions. This is

[43]This sentence, in fact the whole subject, is dropped from the second edition of this book.

the standard view of the development of algebraic logic. We close this essay with a brief description of the contribution of Peirce and Schröder to the logic and algebra of relations. As we shall see, they were engaged not only in the set up of this theory. Among the tools they used for this task we can discern elements that later were incorporated into the theory of quantification. In fact, both Peirce and Schröder were involved in the development of two theories: the algebra of relations and the theory of quantification. The eclipse of the algebra of logic was brought about when the theory of quantification became to be regarded as the most important part of the whole enterprize. Ironically, it was Peirce who stressed the superiority of the theory of quantification when commenting on the last 600 pages of Schröder's book. Note that this introduction could have been sounded differently by stressing the emergence of quantification: Boole studied the logical behavior of quantifier-free expressions. De Morgan introduced the notion of implicit universal and existential quantification. Peirce studied the logical behavior of explicit quantifiers attached to quantier-free matrices. Schröder systematized Peirce's contributions.

2 NOTATION FOR THE LOGIC OF RELATIVES

In the next sections I shall be occupied with [Peirce, 1870], a paper that Peirce describes as extending Boole's logic as to apply to De Morgan's theory of binary relations [Peirce, 1931, 3.643]. Peirce, who could not always afford the luxury of appearing humble, referred to this paper as second only to Boole's logical writings. But his appreciation of [De Morgan, 1864b] was high. He recalled that, after receiving from De Morgan a copy of it, he "at once fell to upon it; and before many weeks had come to see it, as De Morgan had already seen, a brilliant and astonishing illumination of every corner and every vista of logic" [Peirce, 1931, I.562]. Writers that occupy themselves with the relationship between Peirce and De Morgan do not accept Peirce's recollection. The editors of [Peirce, 1984, xliv], for instance, remark that "there is no direct evidence" that this memoir "was ever sent" by De Morgan. [Michael, 1974] held a more sanguine position about this matter. In her view, Peirce was at most mistaken about the year in which he received the memoir. Still, Peirce remarks that at the time De Morgan's memoir reached him, he had already developed his theory of relatives. This view of his own development is substantiated to a great extent by the findings reported in [Michael, 1974].

I have already paid some attention to [Peirce, 1870]. In our previous discussion of this paper I purposely passed over its treatment of logic relations because I wanted to concentrate on its purely Boolean subjects. Peirce intended this paper, however, as a synthesis of Boole's logic of classes with De Morgan's logic of relations. Now that we have became acquainted with these two theories we can proceed to look at this paper from the perspective Peirce indicates. Of course, this section should not be regarded as a full-fledged analysis of Peirce's logic of relations. We shall follow him till his concern with the binding properties of the quantifiers turns him from the high note of the algebra of logic into the prelude of first order logic. Moreover, I shall limit the account primarily to expressions that involve only binary relations.

In the course of this exposition we shall use the language of formal semantics, essentially first order logic with some lambdas thrown in, as *lingua franca*. We think it is more insightful, even though less charming, to use this language than the language of servants, lovers, betrayers, enemies, women and men. Perspicuousness aside, first order logic has the advantage of coming with a proof theory and a fairly well understood semantics. This last fact allows us to identify, for expository reasons, the meaning of an expression with its first order translation. This translation is in strict sense no meaning but rather a recipe to determine it.

Peirce's paper may be regarded as dealing with two theories of relations. There is in the first place a theory of quantification that uses different mechanisms for quantifying the argument positions of binary predicates. This theory is organized around the notion of simple relatives. Within this theory Peirce develops further the devices of implicit universal and existential quantification that De Morgan introduced. In the second place, there is description of a calculus of relations. Within this theory, organized around the notion of elementary relatives, Peirce incorporates De Morgan's formal characterization of kinds of relations. Even though these two theories are closely connected it is important to be aware of the fact that Peirce develops as separate topics what De Morgan treated as a whole.

3 SIMPLE RELATIVES

Relatives and Relations

In Peirce's paper the notion of relative embraces "terms whose logical form involves the conception of relation, and which require the addition of another term to complete the denotation. ... They are *simple relative terms*" [Peirce, 1931, 3.63]. Quite often relative terms are considered in combination with absolute terms.

We shall assume here the standard view of relations. A simple relative R denotes a class of ordered pairs, $\lambda x \lambda y R(x,y)$. In Peirce's view the application of a relative to an absolute term denotes a class. For instance, if R is a relative and w is a term, then both

$$\mathrm{Rw} \text{ and } \mathrm{R}^{w}$$

denote a complete object, the set of Rs of at least one w in the first case and the set of Rs of every w. Two things are important to note here. In the first place, at the level of the syntax, a relative is not combined with an individual term as the ordered pair representation may lead us to expect. In the second place, the interpretation reveals a quantificational structure underlying the absolute term: it turns out that they can be interpreted as the restriction in the denotations of quantified noun phrases (generalized quantifiers). We do not need to enter into many details here. It is sufficient to know that the noun phrase *a woman* has as its meaning $\lambda P\exists x(woman(x) \wedge P(x))$, while the noun phrase *every woman* has as its meaning $\lambda P\exists x(woman(x) \rightarrow P(x))$. In such constructions we call the predicate *woman* the restriction of the generalized quantifier. For convenience alone we shall not distinguish between an expression appearing in a relative and in its meaning appearing in the lambda construction.

Semantically, then, we view the behavior of a relative as follows. It takes as input the denotation of a quantified noun phrase and yields as output a class. The meaning recipe of a relative is given by using a variable Q ranging over generalized quantifiers. Because the denotation of a relative can be represented by $\lambda Q\lambda xQ(\lambda yR(x,y))$. Moreover, the class denoting term in the structure Rw and R^w yields the restriction of the quantifier to which the relative is applied. We shall from here on speak of R either as a relation or as a relative. In the first case we think of R as denoting a relation between individual objects. In the second case we think of R as denoting a relation between an individual and a generalized quantifier. In both cases the denotation of R is, in Frege's terminology, an unsaturated object.[44] We shall not enter here into details about this shift in denotation beyond remarking that there is systematic relation between the two denotations of R. [45] Some scholars, most prominently [Martin, 1980], reserve the term *relative* to the saturated complex expression Rw, i.e., to the application of R to w. [Merrill, 1997] argues against this interpretation. His strongest argument is that for Peirce a relative is an unsaturated expression. He concedes, though, that Martin has made an important point in elaborating his position. For in [Peirce, 1870] relative terms occur almost always saturated.

Our position is achieved by carving out a middle way between Martin and Merrill. We side with Martin in regarding a relative as a term that involves both a relation as a quantified expression in its make up. But we side with Merrill in regarding a relative as an unsaturated expression. R seen as a relative is an unsaturated class denoting term than involves a relation. This fact is reflected in the lambda representation. The quantified expression that occurs in a relative turns out, at the semantic level, to be lambda bounded, thus signaling an unsaturated position. Completed with a quantified noun phrase it will denote a class. And it will be regarded as saturated in such case because in this framework terms that denote classes are considered saturated.

Computing a Denotation

Let us now informally compute, $[\![Rw]\!]$ and $[\![R^w]\!]$, the denotations of Rw and of R^w. In the first case, $[\![w]\!]$, the denotation of w corresponds to $\lambda P\exists z(w(z) \wedge P(z))$. The denotation of the application of $[\![R]\!]$ to $[\![w]\!]$ reduces to a familiar form in the

[44]There is another interpretation of relatives. They can, namely, be regarded as absolute terms. We shall come to it in a moment.

[45]Such matters are explored systematically in [Van Benthem, 1986] and [Van Benthem, 1991].

following way.

$$\begin{aligned}\lambda Q\lambda x Q(\lambda y R(x,y))(\lambda P\exists z(w(z)\wedge P(z))) &= \lambda x\lambda P\exists z(w(z)\wedge P(z))(\lambda y R(x,y))\\ &= \lambda x\exists z(w(z)\wedge(\lambda y R(x,y))(z))\\ &= \lambda x\exists z(w(z)\wedge R(x,z))\end{aligned}$$

Now, in the second case $[\![\mathrm{w}]\!]$ corresponds to $\lambda P\forall z(w(z)\rightarrow P(z))$. The same lambda steps as before yield $\lambda x\forall z(w(z)\rightarrow R(x,z))$

Implicit quantification

Remember from the exposition of the Boolean part of this system that implicit binding is a general feature of Peirce's system of logic. We already observed that $-\!\!<$ introduces implicitly universal quantification. For $v-\!\!<w$ can be read as $\forall x(x\in v\rightarrow x\in w)$. But there are other places in which the interpretation reveals implicit quantification that we shall presently consider.

De Morgan's Composition

Peirce's relatives contain the complex relations De Morgan considered in his logic of relations, namely existential and universal composition. If S and L are relation terms, then both SL as S^L are relations. The denotations of these relations are given below:

$$\begin{aligned}[\![\mathrm{SL}]\!] &= \lambda y\lambda x\exists z(S(x,z)\wedge L(z,y))\\ [\![\mathrm{S}^L]\!] &= \lambda y\lambda x\forall z(L(z,y)\rightarrow S(x,z))\end{aligned}$$

A mechanic procedure allows us to compute the meaning of the expressions (SL)w and (S^L)w. Peirce also points out that $\mathrm{S}^{Lw}=(\mathrm{R}^L)^w$ is a valid equation. We can satisfy ourselves of the correctness of his views by noting that, since the underlying quantified expressions are equivalent, these expressions determine the same class:

$$\begin{aligned}[\![\mathrm{S}^{Lw}]\!] &= \lambda z\forall y(\exists x(w(x)\wedge L(x,y))\rightarrow S(z,y))\\ [\![(\mathrm{S}^L)^w]\!] &= \lambda z\forall y\forall x((w(x)\wedge L(x,y))\rightarrow S(z,y))\end{aligned}$$

The fact that for us it is easy and rather trivial to establish the equivalence expressed by Peirce should not make us lose sight of the historical situation. We are using first order logic augmented with lambda techniques to simulate Peirce's logic of relations. But these resources were not his. He has to rely on the semantic understanding of his readers. Let us first establish the following table:

S	to serve
L	to love
w	woman

Now we can understand Peirce when he says that $(S^L)^w$ will denote whatever stands to every woman in the relation of a servant of every lover of hers; and S^{Lw} will denote whatever is a servant of everything that is a lover of a woman" [Peirce, 1931, 3.77]. From this interpretation we have to convince ourselves that the equivalence holds.

Associativity and relatives as classes

Another interesting equivalence that Peirce notes is

$$(LS)w = L(Sw)$$

To determine the denotation of the left relative we determine first the meaning of the composition, shift it into a relative and then apply it to the denotation of w. The result is the meaning

$$\lambda z \exists x(w(x) \wedge \exists y(L(z,y) \wedge S(y,x)))$$

To compute the second denotation we determine first the class that corresponds to Sw. Then apply the relative L to the generalized quantifier that corresponds to this class. The result is the meaning

$$\lambda z \exists y \exists x(w(x) \wedge S(y,x) \wedge R(z,y))$$

And these two meanings are obviously equivalent. Peirce regards this equivalence as an instance of associativity. This usage suggests that, at least occasionally, Peirce fostered the view of relatives as classes. For only of expressions of the same type can we say that they obey associativity. Relatives, then, can be regarded as denoting the domain of a relation. The relative, L, for instance, can be regarded as denoting the set of lovers. [Lewis, 1918] and [Jørgensen, 1931] adopt this view.

Nevertheless, the view of relatives as denoting sets of individuals meets at least three difficulties.

Invalidities

First, as argued in [Merrill, 1997, 162], some laws that Peirce consider valid turn out to be invalid under this interpretation. An example is the following law.

$$(M \prec L) \leq M^w \prec L^w$$

In the set interpretation this expression is invalid. Even if every lover is a servant, a lover of every woman does not need to be a servant of every woman. For instance, a lover of every woman can be a servant because he is a servant of a man. In the relational interpretation, on the other hand, this expression is valid. If to love is to serve, then who loves every woman serves every woman.

Ambiguity

In the second place, the expression ML would be ambiguous. It could be regarded as denoting composition or as denoting the result of applying the relative M to the class L. There are, however, no traces in Peirce's work of ambiguous readings of the concatenation of two binary symbols.

Implication

Finally, the expression $M \prec L$ is not interpreted by Peirce as expressing inclusion between sets of individuals. Taking M for mother-of and L for lover-of he takes the inclusion to "mean that every mother of anything is a lover of the same thing" [Peirce, 1931, 3.67]. This can be regarded as inclusion between relatives. So, even if Peirce regarded the equation LS(w) = L(Sw) as an instance of associativity, and in doing so he bent towards a class interpretation, there are compelling reasons that militate against this view.

De Morgan's Quantified Relation

Additional expressivity is achieved by the introduction of De Morgan's second kind of universal composition that Peirce calls *backward involution.* In Peirce's system the expression ${}^{L}w$ denotes the set

$$\lambda x \forall y (Lxy \rightarrow w(y))$$

This expression can be used to denote the set of those who love only women or, in the diction Peirce prefers, the set of those who love none but women.

The construction in Peirce's system that corresponds to De Morgan's quantified relation is the expression ${}^{L}S$. The meaning recipe of it is given by

$$\lambda x \forall y \exists z (L(x,y) \rightarrow S(y,z)$$

As the meaning recipe indicates, our example of quantified relation can be used to denote the set of all those who love only servants. Laws for this operation mimics those established for forward involution. We mention only two of them with Peirce's natural language semantic explanation

$$\begin{aligned} {}^{L}({}^{S}w) &= {}^{(LS)}w \\ {}^{L+S}w &= {}^{L}w, {}^{S}w \end{aligned}$$

The first one can be used to express the fact that "things which are lovers to nothing but things that are servants to nothing but women are the things which are lovers of servants to nothing but women". The second one, on the other hand, can express the fact that "things which are lovers of servants of nothing but women are the things which are lovers to noting but women and servants to nothing but women" [Peirce, 1931, 3.114]. The validity of these equivalences can be established by looking at their first order translations.

Booleans and Quantification

Disjunction and Conjunction

As the last paragraph shows, Peirce uses the syntactic means at his disposal to form sums and products of relatives denoting terms. The logical sum of the terms L and S is given by L + S and the logical product of ^{L}w and ^{L}w by $^{L}w, ^{S}w$ [46] Clearly, the disjunction and conjunction signs join together expressions in different semantic types: absolute terms or relatives. The polyvalent nature of these connectives seems to constitute a semantic problem but [Partee and Rooth, 1983] showed how to derive in a systematic way the denotation of disjunctive and conjunctive expressions that live in different semantic types. This is again a matter that we shall not pursue here. We shall simply assume the following semantic correspondences:

$$\begin{aligned}
[\![\mathrm{m} + \mathrm{w}]\!] &= \lambda x(m(x) \vee w(x)) \\
[\![\mathrm{L} + \mathrm{S}]\!] &= \lambda Q \lambda x Q(\lambda y(L(x,y) \vee S(x,y)) \\
[\![\mathrm{m,w}]\!] &= \lambda x(m(x) \wedge w(x)) \\
[\![\mathrm{L,S}]\!] &= \lambda Q \lambda x Q(\lambda y(L(x,y) \wedge S(x,y))
\end{aligned}$$

The interpretation of the relatives S(m + w) and $\mathrm{S}^{(m+w)}$ goes along these lines. We determine first the disjunctive term that corresponds to m + w and then take this term as the restriction of the relevant generalized quantifiers to which the relative S is applied. The result is in the first case

$$\lambda y \exists x((m(x) \vee w(x)) \wedge S(y,x))$$

and in the second

$$\lambda y \forall x((m(x) \vee w(x)) \rightarrow S(y,x))$$

On the other hand, the interpretation of the relatives S(m,w) and $\mathrm{S}^{(m,w)}$ is in the first case

$$\lambda y \exists x((m(x) \wedge w(x) \wedge S(y,x))$$

and in the second

$$\lambda y \forall x((m(x) \wedge w(x)) \rightarrow S(y,x))$$

Boole and Peirce on + "and"

An interesting equation Peirce mentions is $S^{m+w} = S^m, S^w$. The validity of this expression rest on the equivalence of the meanings involved

$\lambda x \forall y((m(y) \vee w(y)) \rightarrow S(x,y))$ and $\lambda x(\forall y(m(y) \rightarrow S(x,y)) \wedge \forall y(w(y) \rightarrow S(x,y)))$

[46] He uses another sign for logical sum in order to distinguish it from the Boolean sum. This distinction is not relevant here.

As we shall presently see, this equivalence allows Peirce to express within his language the medieval insight that universal propositions correspond to conjunctions of singular ones. In Peirce's natural language account the relational equivalence means that a servant of every man and woman will be a servant of every man that is a servant of every woman [Peirce, 1931, 3.77]. The editors of [Peirce, 1931] correct Peirce's paraphrase. According to them the left expression should be "more accurately read as a servant of all those who are either men or women". It is interesting to note that Peirce is following here the spirit of English and the strictures of Boole. Because [Boole, 1952a, 61] noted that

> Speaking generally, he symbol + is the equivalent of the conjunctions "and", "or" .... Of the conjunctions "and" and "or", the former is usually employed when the collection to be described forms the subject, the later when it forms the predicate.

Negation and Interdefinability

In a Boolean framework, if α is an expression, then $1 - \alpha$ denotes the negation of α. Boolean negation is used by Peirce in the following equivalences that link De Morgan's involution notions with negation and existential composition [Peirce, 1931, 3.112]:

$$\begin{aligned} L^S &= 1-(1-l)s \\ {}^LS &= 1-l(1-s) \end{aligned}$$

Peirce presents these equalities for De Morgan quantified relations but it is clear that analogue equations can be given for quantified noun phrases. In a more easily read notation we give them here below:

$$\begin{aligned} L^w &= \overline{\overline{L}w} \\ {}^wS &= \overline{w\overline{S}} \end{aligned}$$

Finally, the fact that backward involution can be defined in terms of negation and forward involution was duly noted by Peirce [Peirce, 1931, 3.116]. Adapting his general definition to two particular cases we obtain the equivalences

$$\begin{aligned} {}^wS &= \overline{w}^{\overline{S}} \\ {}^LS &= \overline{L}^{\overline{S}} \end{aligned}$$

Individual terms

Individuals are identified with unit sets and they are denoted by capital letters. Thus, if w denotes a set of individuals, one of these individuals may be denoted by W. The expression, on the other hand, LW will denote a relative. Syntactically there is no need to change anything in Peirce's language. Individuals are denoted by class terms and LW is a legitimate expression of the language. Our view of relatives does not need to change either. The generalized quantifier view we took allows us to shift the denotation of individual terms from individuals into generalized quantifiers.

To distinguish between different individuals Peirce uses a prime notation: W' and W″ are different expressions suitable to denote different individuals. Peirce regarded an absolute term as equivalent to the sum of all their individuals. Suppose that there are two objects in the set w, say d and d', then $[\![w]\!] = \{d\} \cup \{d'\}$. Therefore, the following equivalence is valid:

$$w = W + W'$$

He also notes that as far as individual terms are concerned, universal and existential composition are equivalent:[47]

$$L^W = LW$$

The usefulness of these equivalences, supplemented with the equivalence of the previous paragraph, is the following. Let w = W + W'. Then the following result illustrates the reduction of a universal proposition into a conjunction of singular propositions:

$$\begin{aligned} L^w &= L^{W+W'} \\ &= L^W, L^{W'} \\ &= LW, LW' \end{aligned}$$

Thus, who loves every woman loves every and each woman. The reduction of simple relatives to relatives involving unit classes allows Peirce to give proofs of inclusions he wants to establish. For instance, assuming as above that w = W + W', the inequality $L^w \prec Lw$ can be established as follows:

$$\begin{aligned} L^w &= LW, LW' \\ &\prec LW + LW' \\ &= L(W + W') \\ &= Lw \end{aligned}$$

The Danger of Paraphrases

Let us close this section by listing some of the validities that Peirce mentioned supplemented with a Peircian paraphrase and an instructive difficulty.

[47]This equivalence is one of the cornerstones of Fred Sommers's term logic, developed in [Sommers, 1982] as an alternative to Fregean logic.

$(LS)^w \prec LSw$	A lover of a servant of every woman is a lover of a servant of a woman
$LS^w \prec (LS)^w$	Every lover of a servant of all women stands to every woman in the relation of lover of a servant of hers
$L^{S^W} \prec LS^W$	A lover of every servant of all women is a lover of a servant of every woman
$L^{Sw} \prec L^S w$	A lover of every servant of a woman is to a woman a lover of all her servants
$L^S w = L^{S^W}$	Every lover of every servant of a particular woman is a lover of every servant of all women

The last formula illustrates the danger of the paraphrase strategy as the only source of semantic information. The choice of the expression *particular woman* suggests a definite referent. But in this case the relative inclusion is not valid. A lover of every servant of Mary does not need to be a lover of every servant of any other woman. The formula is, however valid, as its meaning recipe shows:

$$\forall x \forall y (L(x,y) \wedge \exists z(w(z) \wedge S(y,z))) \rightarrow \forall x \forall y \forall z (L(x,y) \wedge z(w(z) \wedge S(y,z)))$$

[Martin, 1980, 34] consider Peirce's relative formula itself invalid. The reason must undoubtedly be that he interprets the antecedent as $L^S W'$: lover of every servant of a specific woman. His judgement illustrates in vivid form the dangers of the paraphrase strategy.

4 ELEMENTARY RELATIVES

If A and B are class denoting terms then (A:B) denotes the cartesian product of A and B. This product is called an *elementary relative* by Peirce. If we regard individuals as unit sets, then this definition covers pairs of individuals as well.

A Multiplication Table

Let now A and B denote two disjoint sets. For instance, A denotes the body of teachers in school and B denotes the body of pupils in a school. The cartesian product with the set of teacher as domain and the set of pupils as range, (A:B), will be denoted by t. t denotes, then, the relation teacher-of. The assumption is that in every school every teacher is a teacher of every pupil. From this elementary relative, we determine three other elementary relatives:

- c that denotes the relation colleague-of
- p that denotes the relation pupil-of

– s that denotes the relation

In this way we obtain a system of four generated by A and B.

$$c = (A{:}A) \quad t = (A{:}B) \quad p = (B{:}A) \quad s = (B{:}B)$$

These relatives, augmented by 0, form a system closed under composition. This result is codified by Peirce in the following multiplication table

	c	t	p	s
c	c	t	0	0
t	0	0	c	t
p	p	s	0	0
s	0	0	p	s

In natural language, says Peirce, the first equation cc =c, expresses the proposition that the colleagues of any person are that persons colleagues. The equation sp = p, on the other hand, expresses the proposition that the schoolmates of the pupils of any person are that person's pupils.

Composition of Elementary Relatives

Let us now pay some attention to the way in which Peirce computes the composition of two elementary relatives. The product (A:B), says Peirce, can be "taken to denote the elementary relative which multiplied by B gives A" [Peirce, 1931, 3.123]. More in general, Peirce uses the following definition of the result of applying an elementary relative to a non relative term:

$$(A : B)C = \begin{cases} A & \text{if } B = C \\ 0 & B \cap C = \emptyset \end{cases}$$

The relative product of two elementary relatives is determined on basis of the previous definition in accordance with the following equation:

$$(A : B)(C : D) = ((A : B)C) : D$$

Consider, for instance, sp. We reason in the following way:

$$\begin{aligned} sp &= (B : B)(B : A) \\ &= ((B : B)B) : A \\ &= (B : A) \\ &= p \end{aligned}$$

Properties of Elementary Relatives

Every relative, states Peirce, may be regarded as a logical sum of couples of individuals. For "if a relation is sufficiently determined it can exist only between individuals" [Peirce, 1931, 3.121]. Let then R be a relative. We shall associate R with, lets say, the set (A:B) + (B:A). Note that we need to regard the couples as unit sets to make sense of the sum notation. Now, Peirce's identification of relatives with classes of couples of individuals allows him to classify simple relatives along the lines opened by De Morgan, identifying the following properties of relations:

- Reflexivity (concurrence)
- Non reflexivity (opposition)
- Symmetry (equiparance)
- Asymmetry (disquiparance)
- Irreflexive (aliorelative)
- Non irreflexive (self-relative)
- (Non) transitivity

Other properties identified by Peirce make use of the notion of multiplication of two elementary relatives. For instance, the relative R = (A:B) + (B:A) is called cyclic on account of the fact that R^2 is reflexive. This can be show as follows:

$$\begin{aligned} R^2 &= ((A:B)+(B:A))((A:B)+(B:A)) \\ &= (A:B)(A:B)+(A:B)(B:A)+(B:A)(A:B)+(B:A)(B:A) \\ &= 0+(A:A)+(B:B)+0 \\ &= (A:A)+(B:B) \end{aligned}$$

In general, a relative R is called cyclic if there is a natural number n such that R^n is reflexive. Peirce notes that all symmetric relatives are cyclic. A relative R such that for no natural number n holds $R^n = 0$ is called *inexhaustible.* All cyclic relatives are inexhaustible. A fortiori, the same holds for all symmetric predicates. Peirce's natural language example is in the line of lovers and servants: *spouse of* denotes an exhaustible predicate *husband of* doesn't.

Before closing this section, let us point out some topics of this article that we did not discussed. In the first place, Peirce also considers ternary relations, what he calls *conjugative* terms. In the second place, in this paper Peirce develops the beginnings of a notation for a theory of unrestricted quantification in introducing into the language the means to express *something* and *anything* phrases. Peirce also considers the notion of converse of a relation. This he defines by using a ternary predicate.

5 THE LOGIC OF RELATIVES

The final form of Peirce's theory of relatives is contained in [Peirce, [1883] 1983]. The simple relatives are called *dual relatives*, regarded by Peirce as "a common name signifying an ordered pair of objects". The name *individual relative* is also used by Peirce to refer to these ordered pairs. A general relative is defined as an aggregate of individual relatives to which a numerical coefficient, 0 or 1, has been attached. For instance, the general relative L can be given by the equation

$$\begin{aligned} L = \quad & 0(A:A) + 1(A:B) + 1(A:C) + \\ & 0(B:A) + 1(B:B) + 1(B:C) + \\ & 0(C:A) + 0(C:B) + 0(C:C) + \end{aligned}$$

This equation is simplified by using the usual Boolean definitions to obtain

$$L = (A:B) + (A:C) + (B:B) + B(:C)$$

A relation, then, is defined by indicating which pairs of individuals it contains. The generalization of this extensional definition of a relation is given by Peirce in the initial section of his papers [Peirce, 1931, 3.329].

> Let l denote "lover"' then we may write
>
> $$l = \Sigma_i \Sigma_j (l)_{ij} (I:J)$$
>
> where $(l)_{ij}$ is a numerical coefficient, whose value is 1 in case I is a lover of J, and O in the opposite case, and where the sums are taken for all individuals in the universe.

Peirce's characterization of $(l)_{ij}$ as a numerical coefficient must not mislead us. Since it is not the case that this coefficient can denote other number but 0 and 1, we can regard it as a characteristic function. It is important to note that, though lacking the notation for membership, Peirce has now at his disposal the means of mimic membership assertions. Because I $\in$ R and I $\notin$ R correspond to the equations $R_i = 0$ and $R_i = 1$.

To every relative R corresponds a negative one, $\overline{R}$ and a converse, $\breve{R}$. Boolean sums and products of relations are defined in the expected way. Because suppose that the aggregated L + S is given. Then we want to say that L + S holds of the individual relative (A:B) if the A individual is L related or S related to the B individual. In other words, if at least one of the coefficients $(L)_{AB}$ of $(S)_A B$ is 1. Similarly, we want to say that L,S holds of this individual relative if A is both L as S related to B. In other words, if both coefficients are 1. These expectations are borne out by the equations given by Peirce:

$$\begin{aligned} (l+b)_{ij} &= (l)_{ij} + (b)_{ij} \\ (l,b)_{ij} &= (l)_{ij} X (b)_{ij} \end{aligned}$$

Peirce recognizes also the existence of the universal relation, "the aggregate of all pairs". It is denoted by ∞. By definition, $\overline{\infty} = 0$. The following predictable properties are collected by Peirce. Let L be an arbitrary relative. Then

$$L + \overline{L} = \infty \qquad L, \overline{L}$$
$$0 \prec L \quad L \prec \infty$$

Composition

As before, existential composition is denoted by the concatenation, LB, of two relative expressions. The notation for universal composition, however, is changed. Instead of involution a new operation symbol is added. Moreover, the universal reading is not plain universal inclusion nor an exclusive one but an exception phrase. If L and S are relation denoting terms then $L\ddagger B$ denotes the set of those who have the relation L to everything but objects which are S. As the meaning recipe will show, Peirce choice of a new kind of universal composition may have had a duality motivation inasmuch as in the new version the meanings of existential and universal composition turn out to be duals.[48] The quantified import of these expressions is made plain by Peirce. He remarks that LB is called a "particular combination, because it implies the existence of something loved by its relate and a benefactor of its correlate". $L\ddagger B$, on the other hand, "is said to be universal, because it implies the non-existence of anything except what is either loved by its relate or a benefactor of its correlate".

Interpretation of Coefficients

The semantic value of the coefficient $(LB)_{ij}$ is determined by computing the value of sum of coefficients, namely $\Sigma_x(L)_{ix}(B)_{xj}$. Consider the couple (I:J). Suppose further that there is an object, denote it temporarily by X, such that L holds of (I:X) and B of (X:J). In this case $(L)_{ix} = (B)_{xj} = 1$. Therefore the Boolean sum $\Sigma_x(L)_{ix}(B)_{xj}$ reduces to 1. Hence, $(LB)_{ij} = 1$. In other words, LB holds of (I:J) iff $\Sigma_x(L)_{ix}(B)_{xj}(\mathrm{I{:}J}) = (\mathrm{I{:}J})$. The value of the universal composition $(\mathrm{L}\ddagger\mathrm{B})_{ij}$, on the other hand, is determined by computing the value of the product of coefficients $\Pi_x((L)_{ix} + (B)_{xj})$.

Composition and monotony

Peirce points out that in these compositions both relations occur undistributed *in monotone increasing position* [Peirce, 1931, 3.332]. Because the following schemata are valid:

$$\frac{\mathrm{L} \prec \mathrm{S}}{\mathrm{LB} \prec \mathrm{SB}} \quad \frac{\mathrm{L} \prec \mathrm{S}}{\mathrm{L}\ddagger\mathrm{B} \prec \mathrm{S}\ddagger\mathrm{B}} \quad \frac{\mathrm{L} \prec \mathrm{B}}{\mathrm{LB} \prec \mathrm{LS}} \quad \frac{\mathrm{B} \prec \mathrm{S}}{\mathrm{L}\ddagger\mathrm{B} \prec \mathrm{L}\ddagger\mathrm{S}}$$

48 [Russell, 1903, 26] concludes that Peirce's universal composition "is a complicated notion ... which is introduced only in order to preserve the duality of addition and multiplication".

Properties of Composition

Among the properties of these two operations that Peirce collects figure the following

$$
\begin{array}{lll}
L\ddagger(B\ddagger S) = (L\ddagger B)\ddagger S) & L(BS) = (LB)S & \overline{L\ddagger B} = \overline{LB} \\
\overline{LB} = \overline{L}\ddagger\overline{B} & (L \breve{+} B) = \breve{L} + \breve{B} & (L \breve{,} B) = \breve{L}, \breve{B} \\
(L \breve{\ddagger} B) = \breve{B}\ddagger\breve{L} & (\breve{LB}) = \breve{B}\breve{L} & (L, B)\ddagger S = (L\ddagger S), (B\ddagger S) \\
L(B\ddagger S) \prec LB\ddagger S & &
\end{array}
$$

The lovers and servant are not completely dismissed. The last inequality, for instance, is explicated as asserting that "whatever is a lover of an object that is benefactor of everything but a servant, stands to everything but servants in the relation of lover of a benefactor" [Peirce, 1931, 3.334].

From Coefficient to Proposition

Any proposition is according to Peirce equivalent to saying that either a sum or a product of coefficients is greater than zero. In a two valued system this is, of course, to say that it is equal to unit. Therefore, to say that something is a lover of something is as much as saying $\Sigma_i \Sigma_j L_{ij} > 0$. Similarly, to say that everything is a lover of something is as much as saying $\Pi_i \Sigma_j L_{ij} > 0$. Peirce's general point is that the product $\Pi_i (w)_i$ corresponds to the value 1 iff the characteristic function (w) applied to the all the individuals, say a, b and c yields the same result, namely 1. In other words, ours, $[\![\Pi_i (T)_i]\!] = 1$ iff $(w)_a \cdot (w)_b = 1 \cdot (w)_c = 1$. But from this moment on, Peirce drops the inequalities. In his view, natural language propositions can better expressed by simply writing down

$$\Sigma_i \Sigma_j L_{ij} \quad \text{and} \quad \Pi_i \Sigma_j L_{ij}$$

This transition is an important step in the development of Peirce's theory of quantification. The importance, however, does not lay exclusively in his dropping of the inequality symbol. As important is the idea that natural language propositions may be regarded as corresponding to closed coefficients, i.e., to sums or products of coefficients in which all the variables are bounded. Closed coefficients differ fundamentally from relatives inasmuch as they have truth values and do not denote classes. This fact is obscured by the overworked used of 1 and 0. Innocent when the class and the propositional interpretation are not combined, risky and confusing when both interpretations are invoked at once.

To reckon with premises expressed as products and sums of coefficients Peirce describes a useful algorithm: bring the premises in prenex form and apply the Boolean calculus to the matrix. To achieve the first objective he gives a quantifier switching rule and two variable identification ones:

$$
\begin{array}{l}
\Sigma_i \Pi_j \varphi \prec \Pi_j \Sigma_i \\
(\Pi_i \varphi(i)) \cdot (\Pi_i \psi(i)) = \Pi_i (\varphi(i) \cdot \chi(i)) \\
(\Pi_i \varphi(i)) \cdot (\Sigma_i \psi(i)) \prec \Sigma_i (\varphi(i) \cdot \chi(i))
\end{array}
$$

The application of Boolean techniques to the quantifier free matrix is possible because the coefficients are all either *zero* or *unit.*

LAST WORDS

In this part we have exposed the development of Peirce's theory of relatives. In the definition of the composition and the involution of simple relatives we witnessed the presence of De Morgan's implicit quantification notions. Implicit quantification gave way to the explicit use of the quantifiers, thus opening the road to first order logic. We shall see at the end of the next part how the theory of quantification influenced Peirce's evaluation of his own theory of relatives. It is not part of our narrative but it is sobering to remember that at the end Peirce reverted to a theory of implicit quantification, dropping along the way the quantifiers. As we mentioned before, he went to develop the theory of existential graphs.

Part 8

The Logic of Relations: E. Schröder

This final part consists of two sections apart from the introduction. The second section deals with the theory of quantification that Schöder used as part of his metalanguage. The third, and last, section describes the logic of relatives as Schröder defined it.

1 INTRODUCTION

Schröder reckons De Morgan and Peirce as the founders (*die Urheber*) of the theory of relations. His main own work on this area, published as [Schröder, 1895], systematizes Peirce's theory of dual relatives as the algebra and logic of binary relatives. To show the value of the algebra of relatives Schröder developed Dedekind's theory of chains in terms of relations. Schröder boast that his formal language is more expressive than Dedekind's. His own presentation, he says, is second to none with respect to clarity. In these closing pages we restrict our attention to the way in which Schröder developed Peirce's theory of quantification and to his formalization of the algebra of relations.

QUANTIFICATION

Theory of Quantification: Syntaxis

The symbols Π and Σ were used by Schröder for the first time in [Schröder, 1891, 26]. He duly register his debt to Peirce and his student Mitchell in the midst of the development of his own version of the theory of quantification. First he notes that in this part of his work the quantification is over sets taken from the original manifold. Then he goes over to discuss matters of scope, binding and terminology.

In the constructions Π_uA and Σ_vB, u is called the product variable (*Produktvariable*) and v the sum variable (*Summationvariable*). The scope of Π is called the *Produktfaktor* and that of Σ the sum member (*allgemeine Glied*). Schröder introduces the convention that in ΠAΣB the scope of Π does not contain ΣB. The construction in which Σ occurs in the scope of Π must be written differently: Π(AΣB). In other words, the scope of a quantifier reaches as far as the next quantifier occurrence with the same quantifier depth [Schröder, 1891, 27].

[Schröder, 1895] makes a typographical distinction between first order and higher order quantification. While, for instance, Π_u indicates that the quantification ranges over entities in the domain of individuals, $\underset{u}{\Pi}$, indicates that the quantification ranges over individuals or classes or relations [Schröder, 1895, 41]. For convenience sake, we shall not comply in our exposition with this convention.

Theory of Quantification: Semantics

Now, the proposition (*Aussage*) $\Pi_u A_u$ is explicated by Schröder along the following lines.[49] We assume first that the variable (*Produkationsvariable*) is interpretable with regard to a given range. Consequently, we assume that A_u represents a proposition over the objects in this domain. The meaning of our proposition is that this proposition apply to every object. Similarly, $\Sigma_u A_u$ means that there is at least one object in the domain to which the proposition applies [Schröder, 1895, 27]. The complete propositional nature of the sum and products is evident from the following truth definitions offered by Schröder [Schröder, 1895, 37]:

1. The proposition $\Pi_u A_u$ has the truth value (*Wahrheitswert*) 1 iff for every object u $A_u = 1$.
2. The proposition $\Sigma_u A_u$ has the truth value (*Wahrheitswert*) 1 iff for at least one object u $A_u = 1$.

Theory of Quantification: validities

Schröder lists several quantificational validities that, according to him were present in the previous volume — explicitly or in a nutshell (*ausdrücklich oder in nuce*). He lists, for instance, the reduction of universal quantification to existential quantification and negation and the other way around (p.37), the principle of alphabetical variance (p. 36) and (p. 24), the principle that handles vacuous quantification (p. 39.), universal instantiation and existential generalization (p. 37), the distribution of the quantifiers over implications (p. 40). For a term A in which u does not

[49] In the second volume this proposition is called the general factor (*allgemeine Faktor*).

occur, Peirce's shifting rules are noted (p. 39). To these Peircian laws, Schröder adds the by now familiar companions which he proudly calls *meine Schemata.*

$$\Sigma_u(A \ni B_u) = (A \ni \Sigma_u B_u) \quad \Sigma_u(B_u \ni A) = \Pi_u B_u \ni A)$$

Interesting are also the laws that govern shifting over conjunction and disjunction:

$$A\Sigma_u B_u = \Sigma_u AB_u \quad A + \Pi_u B_u = \Pi_u(A + B_u)$$

Schöder closes his review of quantificational theory by mentioning Peirce's quantifiers switch rule. All these laws and principles are considered so important by Schöder that he urges the reader to study them so that they may be turned in *succum et sanguinem.* Still, quantification is exposed by him as a means to the development of the algebra of Peirce's dual relatives.

2 THE ALGEBRA OF RELATIVES

Individual Relatives

The primitive notion of this algebra is Peirce's notion of individual relative (*individuellen binäre relative*). In the exposition of this algebra Schröder distinguishes between capital letters A, B, C that denote exactly one individual and the variables i, j, h, k, m, n, p and q that may be interpreted as referring to the same individual. Hence, the individual relative (A:B) entails A $\neq$ B but $(i : j)$ is consistent both with $i = j$ as with $(j \neq i)$. The properties the individual relative are given by noting that for every i and j

$$\begin{array}{rcl} (i = j) & \text{iff} & (i : j) = (j : i) \\ (i \neq j) & \text{iff} & (i : j) \neq (j : i) \\ (i : j) \neq 0 & & \end{array}$$

The last inequality expresses that no individual relative is equal to 0. It is important to realize that it does not say that there an individual relative. An individual relative is not an assertion.

Domains

Now, the domain of individuals, 1^1, (*Denkbereich der ersten Ordnung*), the domain of the individual relatives (*Denkbereich der zweiten Ordnung*), 1^2 and the domain of conjugatives (*Denkbereich der dritten Ordnung*), 1^3, can represented as the sums

$$\sum_i i \quad \sum_i \sum_j i : j \quad \sum_i \sum_j \sum_h i : j : h$$

[Schröder, 1895, 5] declares that the non empty domain 1^1 must contain more than one element. Still, he does not altogether exclude domains with only one individual. But they are considered the exceptional case (*Ausnahmefall*).

Relative Coefficients

A simple relative (*binäres Relative*) R, on the other can be regarded as a subset of 1^2:

$$R = \Sigma_i \Sigma_j R_{ij}(i:j)$$

We can regard the left occurrence of R as denoting a set and the right occurrence as denoting the characteristic function of this set. Schröder refers to the numerical coefficient R_{ij} as the relative coefficient (*Relativkoeffizient*) and he points out that they are subject to the logical laws of the propositional calculus. Peirce's view of coefficients characteristic functions is expressed by Schröder by means of the formula

$$(R_{ij} = 1) + (R_{ij} = 0) = 1$$

By attaching the indices to the relative expressions Schröder establishes the connection between predication and the calculus of relatives.

An Ambiguity

It is advisable to note an ambiguity in Schröder use of quantifiers that arises from the predicative interpretation of the coefficient. In the context $R_i = \Sigma\phi$, the quantifier is used to form a proposition. In the context $R = \Sigma\phi$, however, the quantifier is used to form a term —a class denoting one.

Relative Constants

Special relatives are 1, the universal relation, 0, the empty relation, 1', the diagonal relation, 0', the complement of the diagonal relation.

$$\begin{array}{rlll} 1 = \Sigma_{ij} 1_{ij}(i:j) & \text{and} & 1_{ij} =: 1 \\ 0 = \Sigma_{ij} 0_{ij}(i:j) & \text{and} & 0_{ij} =: 0 \\ 1' = \Sigma_{ij} 1'_{ij}(i:j) & \text{and} & 1'_{ij} =: (i = j) \\ 0' = \Sigma_{ij} 0'_{ij}(i:j) & \text{and} & 0'_{ij} =: (i \neq j) \end{array}$$

Operations and Ordering

In Peirce's footsteps, Schröder defines the following 6 operations for dual relatives [Schröder, 1895, 29]. For any i and j :

1. $\overline{R}_{ij} = \overline{R_{ij}}$
2. $\breve{R}_{ij} = R_{ji}$

3. $(RS)_{ij} = R_{ij}S_{ij}$

4. $(R+S)_{ij} = R_{ij} + S_{ij}$

5. $(R;S) = \Sigma_h(R_{ih}S_{hj})$

6. $(R\dagger S) = Pi_h(R_{ih} + S_{hj})$

An important definition takes care of the ordering among relations [Schröder, 1895, 32]:

$$\begin{aligned}(R \mathrel{\not\ni} S) &= \Pi_i\Pi j(R_{ij} \mathrel{\not\ni} S_{ij})\\ (R = S) &= \Pi_i\Pi j(R_{ij} = S_{ij})\end{aligned}$$

Schröder's Proof Strategy

Let us look now at the way in which Schröder sought to establish some properties of the relatives. As we shall see, principles of quantification are freely used. Consider the following theorem proven by Schröder:

$$R;(B+C) = (R;B) + (R;C)$$

The first thing proven is the equivalence $R;(B+C)_{ij} = R;B + R;C)_{ij}$

$$\begin{aligned}R;(B+C)_{ij} &= \Sigma_h R_{ih}(B+C)_{hj}\\ &= \Sigma_h R_{ih}(B_{hj} + C_{hj})\\ &= \Sigma_h(R_{ih}B_{hj} + R_{ih}C_{hj})\\ &= \Sigma_h R_{ih}B_{hj} + \Sigma_h R_{ih}C_{hj}\\ &= (R;B)_{ij} + (R;C)_{ij}\\ &= (R;B+R;C)_{ij}\end{aligned}$$

This derivation establishes the universal proposition $\Pi_i\Pi_j(R;(B+C)_{ij} = R;B+R;C)_{ij})$ from which the desired result follows on account of the ordering among relations previously defined.

Quantification and the Algebra of Relatives

This proof is illustrative in two respects. In the first place, the intuitive use of universal generalization. In the second place, the roundabout way in which the theorem is proven. Relational expressions are replaced by their quantificational definitions and the other way around, quantifiers are manipulated according to recognized laws and Boolean operations are freely used on the quantifier free matrix. Of such a strategy [Tarski, 1941, 77] remarked that it "will probably seem quite natural to any one who is familiar with modern mathematical logic". Of course, to Schröder contemporaries this way of constructing the theory was all but familiar.

Let us give another example of the quantificational bent of Schröder's work. Consider the following proof of the equation R;1' = R in which we essentially follow [Schröder, 1895, 121]:

$$\begin{aligned}(R;1')_{ij} &= \Sigma_h R_{ih} 1'_{hj} \\ &= \Sigma_h R_{ih}(h = j) \\ &= \Sigma_h R_{ij} \\ &= R_{ij}\end{aligned}$$

Next to the expected remark that the diagonal is the multiplicative unit in the dual algebra (*der Modul der relativen Multiplikation*) Schröder focusses on the quantifier side and makes an important general remark. He stresses, namely, the validity of the schema

$$\Sigma_h (1')_{kh} f(h) = f(k)$$

Against the received wisdom, illustrated for instance in [Goldfarb, 1979, 252], we have to conclude that to arrive at a calculus of relations in which the entire algebra is put "into a form involving just set-theoretic operations on relations (relative products and the like), and no use of quantifiers" was not the only aim of Schródder's third volume. He engaged as well in the development of the theory of quantification as the proof theoretical framework in which to carry proofs, as the semantic framework in which meaning could be captured. A pure relational calculus was the fruit of Tarski's labor. It was he who rejected the strategy in which one proves theorems of the calculus of relations by the usual methods of first order logic.

Numerical Expressions

The diagonal and its complement can be used to make numerical assertions. For instance, 0' = 0 expresses the fact that the non empty domain 1^1 contains only one individual [Schröder, 1895, 125]. For suppose that 0' = 0 is the case and that there are two individuals called i and j. Then, 0' applied to the individual pair (i:j) yields q, i.e., $0'_{ij} = 1$. But this contradicts the assumption about 0'. Hence, 1^1 must contains only one individual. If, on the other hand, we assume that there is only one individual, call it i, then 0' applied to the only pair around, (i:i,) yields 0. In other words 0' = 0. Similarly, the equation 0';0'= 1 is equivalent to the assertion that there are more that 2 elements in the domain (p. 124). This property of Schröder's language was highlighted in [Lowenheim, 1915, 448-450]. The first theorem of this paper, communicated to Löwenheim by Korset, states that without the support of the quantifiers and the variables, the six algebraic operators and the four constants are not suitable to express all numerical assertions. The argument of Löwenheim is that no quantifier free relative expression can discriminate between a domain 1^1 of three and one of four individuals. Therefore, the quantified expressions sentence ϕ that expresses that a domain has at most three elements and the quantified sentence ψ that expresses that a domain has at most four elements cannot be adequately represented in Schröder's quantifier fragment. Their translation would turn out to be equivalent.

Let us close this section by mentioning some of the topics in Schröder's last volume that we did not even mention before. He studied, for instance, the question of inverses to the relative operations and mentioned the question of the completeness (*Vollständigkeit*) of his system with regard to the purposes of applied and pure theories. He also formulates here the notion of general solution to a relative equation and studied algebraic rules governing the behavior of Booleans over couples. In this context he worked with infinite products, even uncountable ones. This, he notes, occurs for the first time in mathematics. Moreover, he gives an example of quantifier elimination, his method of condensation (*Verdichtung*).

LAST WORDS

Schröder's three volumes, as we have already said, are generally considered the culmination point of the line of development that started with Boole. The further development of logic did not run along this book, however. It is true that it formed the immediate source of inspiration for [Lowenheim, 1915] but the topic of this paper was quantification theory and not the calculus of relatives. Historians occasionally ask for the reason why the Boolean tradition was eclipsed by the mathematical logic of Peano and Russell. It is perhaps an idle exercise to try to resolve this question. A factor must, though, be mentioned. Peirce's reception of Schröder's third volume. In his comment of this volume, Peirce distinguishes between the theory of quantification (*general algebra of logic*) and the algebra of dual relations. He values the former above the later and criticizes Schröder's for not sharing this appreciation [Peirce, 1931, 3.498]:

> Professor Schröder attaches, as it seems to me, too high a values to this algebra. That which is in his eyes the greatest recommendation of it is to be scarcely a merit, namely that it enables us to express in the outward guise of an equation propositions whose real meaning is much simpler than of an equation.

This rejection of the algebraic approach by one of its central figures was accompanied by the description of the new paradigm [Peirce, 1931, 3.499]:

> Besides the algebra just described, I have invented another which seems to me much more valuable. ... The method of using it in the solution of special problems has also been fully developed by me. ... In this algebra every proposition consists of two parts, its quantifiers and its Boolean. ... This algebra, which has but two operations, and those easily manageable, is in my opinion, the most convenient apparatus for the study of difficult logical problems.

To young researchers the message must have been clear. One of the founders of the algebra of logic had jumped ship. Algebraic logic had run its course. The discipline became a dormant body and Schröders 2000 pages its monumental tomb. It was to be Tarski's deed to kiss it back to life.

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THE MATHEMATICAL TURNS IN LOGIC

Ivor Grattan-Guinness

1 INTRODUCTION

> Symbolic logic [...] has been dismissed by many logicians on the plea that its interest is mathematical, and by many mathematicians on the plea that its interest is logical. [Whitehead, 1898, p. vi]

This essay is a short rumination around the theme of Whitehead's remark, which still applies today to a considerable extent: namely, the uneasy relationship between mathematics and symbolic logics, and now with computing as well. Many large topics are involved, and some are little studied; for example the history of education in mathematics, logics, computing and education. Thus it is no cliché to say that the essay is intended as a foray into large areas of infer-disciplinary developments. Among the general histories of logic I mention especially [Borga and Palladino, 1997; Church, 1956; Mangione and Bozzi, 1993; Styazhkin, 1969], and the source books [Ewald, 1996; Mancosu, 1998; van Heijenoort, 1967]. Many more works are cited below, and the bibliography contains still further items.

2 VARIETIES OF SYMBOLIC LOGIC

The term 'symbolic logic' was introduced by the British logician John Venn (1834–1923), to characterise the kind of logic which gave prominence not only to symbols but also to mathematical theories to which they belonged [Venn, 1881]. He had in mind the two principal manifestations of the time: 1) algebraic representations of the modes in syllogistic logic, which had increased in number from the 1830s by the quantification of the predicate and from the 1860s by some attention to the logic of relations, launched by Augustus De Morgan (1806–1871); and 2) extensions or modifications of the algebra of logic of George Boole (1815–1864) which, as he had shown in his second book *The laws of thought* (1854), surpassed syllogistic in expressive and deductive power. This whole tradition was deeply influenced by algebras, partly the common tradition in algebraising syllogistic modes, but more closely by newer ones developed in the early 19th century: differential operators, which was a key source for Boole; and functional equations, which bore a strong analogy to the logic of relations, both of which were studied by De Morgan. From the 1870s these two strands were being brought together by the American polymath C.S. Peirce (1839–1914), and from the 1890s they were treated

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more systematically by a German semi-follower, Ernst Schröder (1841–1902) in his mammoth *Vorlesungen über die Algebra der Logik* ($2\frac{1}{2}$ volumes, 1890–1905) [Peckhaus, 1997].

Venn's term remains fairly durable, but over the decades its reference has become less clear. For from the late 1870s an alternative kind of symbolic logic was introduced, initially by Gottlob Frege (1848–1925) and about a decade later, and with far more publicity by Giuseppe Peano (1858–1932), who assembled a remarkable cohort of disciples at the University of Turin. The inspiration here came not from algebras but from mathematical analysis, especially the emphasis on rigour and the detailed exhibition of proofs in terms of a developed theory of limits. This approach had been initiated from the 1820s by Augustin-Louis Cauchy (1789–1857), together with the systematic indication of necessary and/or sufficient conditions for the truth of theorems, and the need to formulate definitions carefully and (where appropriate) with generality. However, he never *explicitly* presented logical principles, either within syllogistic logic or any other tradition of his time.

From the time of Cauchy's death this approach and its attendant practices were being promoted and refined in teaching at the University of Berlin by Karl Weierstrass (1815–1897). Peano's insight was that the refinements to rigour, definition and proof suggested that words were not precise enough, and he brought symbolisation not only to basic notions of the mathematical theories involved (both mathematical analysis and also some of these algebras and geometries, for example) but also of logic itself. There the main notions included logical connectives, which Boole and others had already given symbols, and also the predicate calculus and quantification, which both Frege and Peirce had already pioneered but which Peano popularised in his writings and those of his disciples. From the early 1890s their principal organs were a journal initially called *Rivista di matematica,* and editions of a compendium of presentations under the title *Formulario matematico* [Borga, Freguglia and Palladino, 1985].

Peano called this subject 'mathematical logic'. The name had already been proposed in 1858 by De Morgan, but as a contrast to 'philosophical logic', where verbal expression alone was pursued; Venn had proposed 'symbolic logic' as a substitute. Peano's sense became widely known, and used by principal followers, especially in Britain by Bertrand Russell (1872–1970) and A.N. Whitehead (1861–1947). However, from 1904 a rather unnecessary alternative name came in, initially as 'logistique' in French, to refer both to Peano's and Russell's approaches. But both 'logistic' and 'mathematical logic' were ambiguous in an important respect, since the Peanists (as they were sometimes known) used their logic to express a mathematical theory in rigorous and axiomatised form; a presentation began with two columns of symbols, one for logical notions and the other for the mathematical ones. By contrast, during 1901 Russell decided that *only one column was required* : Peano's mathematical logic, together with a logic of relations which algebra had recently and importantly adjoined to it, was sufficient to deliver not only the needed methods of deduction but also the objects of mathematics (Rodriguez-Consuegra 1991). This stance was presented in much (though not complete) detail in their

Principia mathematica (1910–1913) [Grattan-Guinness, 2000a, Chs. 6–7]. The position has become known as 'logicism'; I shall use it for convenience, although it was introduced in the late 1920s by Rudolf Carnap (1891–1970), seemingly to avoid the ambiguity surrounding 'logistic'.

One reason for Russell's modification of Peano's position was the place in the latter's presentations of the set theory of Georg Cantor (1845–1918); for it was prominent enough to appear in both columns of a Peanist list — hence, of course, what and where is the difference between the columns?. Cantor had developed his theory from the early 1870s, initially concerned with sets of points and other objects of mathematical analysis and conceived within Weierstrass's programme [Dauben, 1979]. He was supported in certain respects, not all of them acknowledged by Cantor, by Richard Dedekind (1831–1916) [Ferreirós, 1999]. But gradually the theory extended: the doctrine of the actual infinite took on a life of its own beyond its role in analysing sets of points, for sets themselves could be formed as collections of objects of *any* kind; in the 1880s Cantor even envisaged a way of defining integers from sets, and claimed set theory as the basis for "all" mathematics. However, his reliance on the process of mental abstraction was found philosophically wanting, especially by Russell who replaced it with definitions formed in terms of sets of equipollent sets, with sets themselves determined by the appropriate propositional functions (or predicates). The construction imagined by Weierstrass, Cantor and others was largely imitated: from integers to rational numbers, and then to irrational numbers. This latter stage was tricky; but several solutions were found, and Dedekind's 'cut' theory of 1872 became the most popular. The importance of such definitions lay also in specifying the real line and thereby providing a better grasp of the notion of continuity.

3 THE ALGEBRAIC VERSUS THE MATHEMATICAL

This discussion of the role of set theory in mathematical logic makes a nice *entrée* to a survey of the differences between the two traditions of symbolic logic. They are very great, but often poorly recognised by logicians and even historians of logic; indeed, being symbolic is about the only common factor. They are best illustrated under four headings.

3.1 Theories of collections

In the algebraic tradition the customary method from Greek antiquity of handling collections of objects was used: namely, part-whole theory, where a class of objects contain only parts (such as the class of European men as a part of the class of men), and membership was not distinguished from inclusion. Relative to set theory parthood corresponds to improper inclusion, but philosophically links are not so simple; in particular, the empty Cantorian set is not to be identified with an empty class, and paradoxes involving the set of all sets cannot be formulated in part-whole theory, although Schröder found a different one there.

This situation obtained even after Cantor's theory came into prominence from the mid 1890s. Both Peirce and Schröder wrote on aspects of it, but *not* within the framework of their algebraic logic: for example, in 1897 Schröder discussed Cantor's definition of cardinal integers in two papers, not within his *Vorlesungen*. Thus, for example, algebraic logicians from Boole onwards took note on occasion of the number associated with (members of a) class, whereas Frege and Russell *defined* cardinal numbers as sets of equipollent sets.

3.2 Principles and properties

Algebraists were accustomed to specify 'laws' which a particular algebra satisfied, and the logicians followed the habit. In some contrast Peano and Russell heeded the growing practise of the time [Cavaillès, 1938] of axiomatising mathematical theories. The difference between laws and axioms is fine but not artificial; for example, algebraic logicians noted properties such as duality in the algebra of Boole which however did not catch the interest of the mathematical logicians even when present (such as between conjunction and disjunction in the propositional calculus). Again at that time, axiomatisation, especially as then practised by David Hilbert (1862–1943) led to a heightened interest in model theory [Scanlan, 1991]; while Boole had interpreted his algebra in terms both of the mental process of forming and dissecting classes and of the (sub-)classes so formed, the model-theoretic side was not strong among the algebraists. Abstract algebras were already around, mostly in group theory, but not yet much present in place within symbolic logic, though Schröder's system had included a lattice-like structure.

3.3 Relationship with (some) mathematics

As we saw, Peano framed mathematical theories in logical dress, and Russell decided that (large parts of) mathematics was part of mathematical logic, with set theory forming a part; thus logic was being applied to mathematics. By contrast, Boole took the reverse position in applying mathematics to logic, as he made clear in his first book *A mathematical analysis of logic* (1847), imitating a common form of title of books elsewhere in applied mathematics ('A mathematical analysis of fluids', and so on). Peirce and Schröder took a more ambiguous stance. Peirce wrote of 'mathematics applied to logic and logic applied to mathematics', though he did not sort out the issues involved. In particular, no algebraic logician seemed to notice the vicious circle lurking around logic as applied mathematics; for the mathematics that it involved should be consistent, but that is a logical notion in the first place. To us, it is actually meta-logical; but neither tradition recognised the basic significance of the distinction between it and logic.

Another difference concerns logicism. Schröder put forward a version, though of an extensional kind in checking that every mathematical theory possessed five basic notions (identity, intersection, negation, conversion of a relation, and relation in general) rather than the internal organic kind of logicism of Frege and Russell, building out from basic concepts such as set and predicate. A similar difference attended the predicate calculus itself: for mathematical logicians the logic was always finitary, both with regard to the finitude of formulae and of length of proofs. Algebraic logicians also accepted the latter finitude, but they allowed for infinitely long formulae in reading quantifiers. The universal and existential modes were understood as generalisations from conjunction and disjunction respectively, including for an infinite range of significance of the variable; Peirce and Schröder even used the respective signs '$\sum$' and '$\prod$' to denote the qunatifiers.

Another difference related to mathematics, especially in logicism, concerns quantity (not be confused with quantification). In embracing mathematics within their logic Russell and Frege (the latter only for arithmetic and some mathematical analysis) included quantities such as integers and especially here real numbers and lengths of lines, and so gave their logic a quantitative as well as qualitative aspects. Although algebraic logic involved integers in various supporting roles, it made no quantitative claims in the logicistic sense.

3.4 Relationship to language

Both traditions worked with a rather simple or ideal languages; but their specific foci were very different. Algebraic logic started out as a supplement, in the end a replacement, for syllogistic logic, and therefore focused attention upon adjectives and nouns. By contrast, mathematical logic focused upon the 'six little words', as Russell put it: '*all, every, any, a, some* and *the* '. In particular, the needs in mathematics for mathematical functions to be single-valued led him to concentrate upon 'the': Russell was converted to the merits of Peano in 1900 by noting his emphasis on the word, and five years later he made a notable contribution to philosophy with his theory of definite descriptions, giving precise criteria for the referentiability of propositions containing clauses commencing with 'the'. (These criteria were similar to those proposed by Peano in 1897 for mathematical functions.) Adverbs seemed to fall between the two foci, and have gained proper attention only in later logics, such as fuzzy set theory.

A related difference concerns connectives. In tune with their linguistic focus, the algebraist formed equations and so gave emphasis to logical equivalence, together with conjunction and disjunction; Peirce modified it when he gave primacy to implication. In mathematical logic the close connection with proof of theorems gave implication prime place from the start, though Russell seriously muddled it with inference and entailment.

4 CHANGES, ESPECIALLY WITH GÖDEL AND TARSKI

These differences are exhibited with special clarity by Russell and Peirce in the early 1900s. Russell made virtually no use of the considerable literature on the logic of relations available from the algebraic logicians when extending Peano's mathematical logic in 1900; and when he outlined his logicism in prosodic but detailed form in his *The principles of mathematics* (1903) Peirce reviewed it in six lines, three of them sarcastic (but accurately predictive of the wallpaper character of *Principia mathematica*). But he had lost: algebraic logic was considerably eclipsed during the 1900s, especially after the publication of *Principia mathematica,* perhaps for want of an obvious line of further research. The only distinctive influence that it exercised during the 1910s and 1920s was the adoption of certain normal forms from Schröder, including a prominent role for duality, by Thoralf Skolem (1887–1962) and Leopold Löwenheim (1878–1957).

The revival of algebraic logic dates only from the 1940s, initially thanks to Alfred Tarski (1902-1983) and some followers. By then, of course, the range of algebras visible in mathematics was much wider, and logic just an (interesting) case to study. In some ways the tradition of the previous century was revived — including not considering the apparent vicious circle noted above of deploying a consistent algebra to examine logic [Halmos, 1972]. Its modern forms did not gain sufficient individuality to gain a chapter in the handbook [Barwise, 1977], or indeed an article of its own in this handbook. Influence of a different kind affected Saunders Mac Lane (born 1909); a doctoral dissertation of 1934 under Hermann Weyl (1885–1955) on formally shortening proofs of theorems was to lead him to a structuralist philosophy of mathematics [Mac Lane, 1986], of which there have been various proponents [Vercelloni, 1988].

Several of the most significant changes in symbolic logics in the 1930s were inspired by Gödel's two famous theorems [Gödel, 1931]. The first one, on the incompletability of arithmetic with first-order quantification, sunk logicism and badly affected Hilbert's programme of metamathematics (though see [Detlefsen, 1986]); the second (or corollary) on the unprovability of the (ω-consistency of the system sunk metamathematics and badly affected logicism. Substantially revised versions of each tradition had to be devised, for the former especially with W.V. Quine and followers [Quine, 1969] and of the latter initially with Hilbert's ex-students Paul Bernays (1888–1977) and Gerhard Gentzen (1909–1945) [Webb, 1980]. Gödel's proof method of arithemeticising syntax led to new insights on the scope of recursion and (due to the theorem) the limitation of computability, with Alonzo Church (1903–1995), S.C. Kleene (1909–1994) and Alan Turing (1912–1954) among the leading figures [Davis, 1965]. Metamathematics and its related topics came to dominate foundational work from the later 1920s as far as interested mathematicians were concerned; logicism went into some decline, and the intuitionism of L.E.J. Brouwer (1881–1966) gained few followers although it exercised some influence on various schools of constructivism.

A less well known impact of Gödel's paper was the recognition of the *central* importance of distinguishing logic from metalogic. Struggles galore had ensued in this area, especially during the 1920s as Hilbert's programme began to flower and also as the incoherence of Russell's logicism began to dawn [Grattan-Guinness, 2000a, Ch. 8 passim]. Gödel's paper brought home the distinction in all its main manifestations; not only metalogic as such but also distinguishing a sign from its referent — and remembering to observe these differences all the time.

The other principal founder was Tarski, who is said (by Tarski) to have been on the track of Gödel's theorem himself by 1931, and stressed the meta-theoretic status of truth theories in a famous long paper of the mid decade [Wolenski, 1989]. He also helped to father an allied change. Up to then semantics was usually treated as a younger brother of syntax; sort out the former in a (bivalent) logic, and the latter will arrive as a bonus. This view was more or less built in to metamathematics, and also was implicit in logicism, where Russell's concerns over truth lay largely in supporting the correspondence theory within his positivistic epistemology. Of figures of the early 20th century, only C.I. Lewis (1883-1964) emphasised semantics in his advocacy of modal logics in *A survey of symbolic logic* (1918) (a book otherwise notable for its detailed presentation of algebraic logic) and other writings, thereby rather isolating himself from other logicians. But after Tarski also stressed the importance of semantics, sensitivity to it increased substantially.

The most striking case of change was Carnap [Grattan-Guinness, 1997]; strongly influenced by Russell from the 1920s (far more so than by Frege) on both logical and epistemological fronts, he emphasised the syntactical character of (bivalent) logic in his writings. But around 1937 he thought of defining truth in terms of provability, and realised that both an axiom in an (assumedly consistent) system and its negation took falsehood, so that the law of excluded middle was lost. Hence we find his wartime books *Introduction to semantics* (1942) and the much undervalued *preceding* volume *Formalization of logic* (1943), followed in 1947 by *Introduction to semantics,* when some modal logics were given detailed study and metastudy [Grattan-Guinness, 1997]. Thereafter, and also other sources such as Lewis, the range of such logics expanded massively [Rescher, 1969]. The situation was reinforced from the late 1940s onwards by the emergence of infinitary logics [Karp, 1964], after an speculative and uninfluential anticipation by Zermelo just before the publication of Gödel's incompletability theorem for finitary logics [Grattan-Guinness, 1979]. Despite sustained objections by Quine, logical monism had become logical pluralism, and of various different kinds.

5 LIVING TOGETHER AND LIVING APART

While most of this work was highly technical and thereby mathematics-looking, the mathematical community largely continued to take little interest. France is a particularly interesting case, for its eminence in mathematics; the Bourbaki clique excised symbolic logics from the realm of permitted areas of mathematical research, although in their own presentations of mathematical theories they did use the deduction theorem, presumably because their compatriot Jacques Herbrand (1908–1931) had been one of its first provers. Elsewhere, Solomon Lefschetz (1884–1972) is recalled to have cultivated the habit of entering Church's classroom at Princeton University and telling the students: 'you are wasting your time!'. In the mid 1930s Church was to play a major role as a founder editor of the *Journal of symbolic logic,* the organ of the new Association for Symbolic Logic and publication venue for this refugee academic subject.

The contrast between the common attitudes of mathematicians to symbolic logics and to set theory is very stark. The technical and topological aspects of the latter subject became well received and used from the mid 1890s, with applications made to measure theory, functional analysis, and so on; it spread as a part of the basic language for mathematics to the extent of being taught extensively at undergraduate level from the 1950s. The theory of transfinite numbers and order-types also gained much attention around the turn of the centuries, at first with followers of Cantor such as Felix Bernstein (1878–1956) and Felix Hausdorff (1868–1942) [Fraenkel, 1953]. As far as the paradoxes were concerned, Zermelo's axiom system, laid out without adumbration by logical concepts and indeed defective in the statement of the axiom of separation [Zermelo, 1908], gave mathematicians a working basis for set theory in case they felt worried. Rather more philosophical anxiety attended his axiom of choice of 1904, for its non-constructive character and the places in set theory and mathematics in general where it seemed to be needed but might be avoided [Moore, 1982]. Yet symbolic logics did not gain much interest among mathematicians then, even though there is a natural link (reduced in generality by Russell's paradox) between predicates and sets. Gödel's theorem strengthened the difference, in that it refuted Russell's claimed reduction of mathematics to (set theory and) logic.

Thus symbolic logics and mathematics continued to go their separate ways, though with set theory playing roles in each. All the mathematician needs to know about logic is the law of *modus ponens* and *modus tollens,* attention to necessary and sufficient conditions and care with definitions (part of the heritage from Cauchy, as we saw earlier), and appreciation of the five logical connectives and maybe of quantification. But nothing more seems to be required: to mathematicians the rest of logic is marginal; as a result, even 'rigorous' mathematics appears rather sloppy to logicians [Corcoran, 1973]. A striking example, pointed out to me by Graham Priest, is in the presentation of proof by contradiction: mathematicians ancient (for instance Euclid), middle-aged or modern, normally prove theorem T this way by assuming not-T, obtaining a contradiction C of some kind, and *immediately* concluding T; omitted is the required deduction of any proposition from C, and its rejection. To a logician, such rapidity is evidence of sloppiness; to a mathematician execution of such detail is evidence of pedantry. In such ways opens up the gulf between the two subjects their practitioners.

Philosophers would take the logicians' side here. Their own interest in symbolic logics has main naturally in the implication for (formalish) languages and theories of deduction, referentiabilty and meaning in general; in the case of Frege and Russell, therefore, they are more concerned with their logics than with their logicisms. Indeed, philosophers' knowledge of mathematics is often so scanty that they appear not to recognise the difference between arithmetic and mathematics when mis-representing Frege's stance (for example among many, [Dummett, 1991]).

There is a notable similarity between mathematicians' attitude to logics and to mathematical statistics. Mathematicians understand the form and manipulation of statistical parameters, the use of matrices in multi-linear regression, and so on; but they ignore the aspects which can tax statisticians the most, such as the manner of collecting data, and sampling techniques. Mathematicians' (and also statisticans') attitude to history is much the same; if they bother with history at all, then the purpose is to render the old work in some modern dress so that we can understand what the historical figure really meant to say. But of course such study deals with the *heritage* from that old work (a perfectly legitimate enterprise, of course), not its own historical context.

Perhaps the most enduring legacy from symbolic logics for mathematics (and, in my view, for philosophy also) was the gradually more conscious and widespread recognition of the distinction between a (mathematical) theory and its metatheory, though with the latter usually left in an informal state. Partly in this connection, aspects of model theory also lie in the overlap — spectacularly so in the case of the 'non standard analysis' of Abraham Robinson (1917–1974) [Dauben, 1995].

However, the intersection between mathematics and symbolic logics is still rather modest. One of the few mathematicians to consider symbolic logics seriously (he propounded a user-friendly version of intuitionism for a time) was Weyl: in a reflective essay on the development of the foundations of mathematics he made the correct and revealing remark that the distinction between a predicate and its associated set, so important in logic and its philosophy, 'leaves the mathematician cool' [Weyl, 1946, p. 268].

The converse alienation also holds, as is well illustrated by the 'lectures on the development of mathematical logic and the study of the foundations of mathematics' given by Andrej Mostowski (1913–1965) in the mid 1960s [Mostowski, 1966]. His 16 lectures came closest to the interests of mathematicians with three on model theory and the foundations of set theory; a few others, such as one on intuitionistic logic, have a general bearing on mathematics.

Mostowski also lectured on recursion and computability, topics in which once again the contention to logic has been slighter than might be expected. They relate in very significant ways to computing, which flowered mightily after the Second World War, in particular from the 1950s. Although the pioneers had included mathematicians with a strong interest in logic, such as Turing and John von Neumann (1903–1957), the later practitioners did not plunge very much into logical details as such [Goldstine, 1972], beyond, for example, deploying Church's lambda calculus on the development of programming languages. Logic courses within first degrees or in computing seem still to be very rare. The current series 'Handbook of logic in computing science' being published by Clarendon Press exhibit a very impressive range of involvements of mathematics; yet I wonder of the extent to which the mathematical community is aware of it. If but little, then history would be repeating itself from the time of the Whitehead/Russell *Principia mathematica* : I have surprised mathematicians on several occasions by pointing out interesting mathematical content, especially in the second volume.

The split between mathematics and symbolic logics has now gone three ways, with the computing community not proving greatly more sympathetic to logic than their mathematical colleagues. An informal census of the addresses of about 200 academics active in logic or at least with a serious involvement in some part of it, suggested to me that they seem to be employed in roughly equal proportions in departments of philosophy, mathematics and computing, with very few in departments of their own.

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SCHRÖDER'S LOGIC

Volker Peckhaus

1 INTRODUCTION

1.1 The Significance of Ernst Schröder's Approach to Logic

The German mathematician Ernst Schröder (1841–1902) was one of the most important representatives of the algebra of logic. His work set standards in mathematical logic as a means for the foundation of mathematics at the beginning of the 20th century, at least until Alfred North Whitehead and Bertrand Russell took the lead with their *Principia Mathematica* (*1910–1913*). In his first pamphlet on logic, *Der Operationskreis des Logikkalkuls* (*1877a*), Schröder presented a critical revision of George Boole's logic of classes, stressing the idea of the duality between logical addition and logical multiplication introduced by William Stanley Jevons (Jevons *1864*). In 1890 Schröder started his monumental *Vorlesungen über die Algebra der Logik* which remained unfinished, although it achieved three volumes with four parts, of which one appeared only posthumously (*1890a*, *1891*, *1895a*, *1905*). Contemporaries regarded the first volume alone as having completed the algebra of logic (cf. Wernicke *1891*, 196). Among the topics treated were the calculi of classes, propositions, including a full-fledged theory of quantification, and the logic of relatives (relations) in which ideas of Charles S. Peirce were elaborated.

Schröder considered himself an algebraist. It is only by chance that his life's work is usually connected to logic. No doubt, most of his life he was concerned with logic, always regarding it, however, as means to an end, the vision of a scientific universal language. Both, logic and a universal language, are based on algebra as general theory of connecting operations. His contributions to set theory (e.g. Schröder-Bernstein theorem, cf. Schröder *1898c*) are results of his research on models of algebraic logical structures. Besides this, his name is recognized in eponyms like "Schröder Domain", "Schröder Equation", "Schröder Functional Equation", i. e. in the context of functional theory and complex dynamics (cf. Steinmetz *1993*). These contributions will not concern us here.

1.2 Re-evaluation in the Historiography of Logic

The older historiography of logic has widely ignored the work of Schröder. In his periodization of mathematical logic I. M. Bocheński has the Boolean period in logic end with Schröder. This period differs, according to Bocheński, from later periods

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in that the methods of mathematics were not made the topic of logical research, but simply applied to logic (Bocheński *1956*, 314). With their *The Development of Logic*, William and Martha Kneale dominated the historiography of logic for years. But the only representative of the algebra of logic treated at some length is George Boole.[1] This may be due to their primary interest "to record the first appearances of these ideas which seem to us most important in the logic of our own day" (Kneale/Kneale *1962*, v).

For Jan van Heijenoort (*1967a*, vi) the great epoch in the history of logic opens in 1879 when Gottlob Frege published his *Begriffsschrift* (Frege *1879*). According to van Heijenoort, the epoch was preceded by a first, algebraic phase, in which Schröder, although contemporary of Frege, has undoubtedly to be counted. This algebra of logic, however, suffered from a number of limitations. Van Heijenoort concludes (*1967a*, vi): "Considered by itself, the period would, no doubt, leave its mark upon the history of logic, but it would not count as a great epoch."

It is highly problematic, however, to speak of a certain phase or period as that of the algebra of logic, because this direction runs parallel to other variations of logic, and it kept running when Alfred North Whitehead and Bertrand Russell changed the logical scene after having published *Principia Mathematica* (Whitehead/Russell *1910–13*). And even today it has its successors. Admittedly, van Heijenoort's assessment is too vague for easy rebuttal, but one should emphasize that the mark left by the algebra of logic, especially in the form given to it by Schröder, was considerable. It was therefore ill-advised of van Heijenoort to ignore Schröder's contributions to logic , given, e. g., that logical parts of Alfred North Whitehead's *Universal Algebra* (Whitehead *1898*) were taken from Schröder's *Operationskreis* (*1877a*), that Clarence Irving Lewis's *Symbolic Logic* was based on "The Classic, or Boole-Schröder Algebra of Logic",[2] that Leopold Löwenheim never gave up Schröder's language of the calculus of relatives (cf. Löwenheim *1915*), and that also Alfred Tarski worked within the algebraic paradigm of the Peirce-Schröder tradition when axiomatizing the logic of relations (Tarski *1941*) already preceded therein by Norbert Wiener (Wiener *1913*, cf. Grattan-Guinness *1975*).

Times are changing! The algebra of logic from Boole to Schröder is, e. g., properly represented in the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* (cf. Houser *1994*). Ivor Grattan-Guinness' *The Search for Mathematical Roots*, today's standard in the history of mathematical logic, clearly sees Schröder's algebra of logic as part of a parallel process in the initial period of the development of set theory, logics and axiomatics between 1870 and 1900 (cf. Grattan-Guinness *2000*, ch. 4). The algebra of logic, and in particular Schröder's contributions, are also the subjects of a number of historical studies, starting with Randall R. Dipert's *Development and Crisis in Late Boolean Logic*

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[1] Kneale/Kneale *1962*, 404–420, with short remarks on "Later Developments of Boolean Algebra" (420–427), and the theory of relations of Augustus De Morgan and Charles S. Peirce (427–434).

[2] This is the title of ch. II of Lewis *1918*.

(*1978*), and followed by Volker Peckhaus' *Logik, Mathesis universalis und allgemeine Wissenschaft* (*1997*), Geraldine Brady's *From Peirce to Skolem* (*2000*), and Risto Vilkko's *A Hundred Years of Logical Investigations* (*2002*).

2 ERNST SCHRÖDER'S LIFE

Schröder's *Nachlaß* is lost, burned in the bombing of Münster during the Second World War, together with the papers of Gottlob Frege.[3] Biographers of Ernst Schröder can, however, use to advantage an autobiographical sketch, published in 1901, the year before Schröder's death, in the folio edition *Geistiges Deutschland* (Intellectual Germany), containing portraits and short biographies of eminent German intellectuals (Schröder *1901a*). This text is almost unknown today, but it became the basis for the widely spread obituary, written by Schröder's friend Jakob Lüroth, and published in 1903 in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Lüroth *1903*). It was taken over in the posthumous second part of the second volume of Schröder's *Vorlesungen über die Algebra der Logik*. Lüroth's obituary has become the main source for everything since written on Schöder's life.[4]

Schröder begins his sketch as follows (throughout writing in third person):

> FRIEDRICH WILHELM KARL ERNST SCHRÖDER, born on 25 November 1841 in Mannheim, descended from a family of scholars. His father was the teacher Professor G. F. Heinrich Schröder, later headmaster of the *Realgymnasium* of that town which emerged from the *Höhere Bürgerschule,* which stood at that time under his authority. He was known for his numerous mineralogical and chemical papers and his papers on physics, most perhaps for his research on the filtration of air, thus being a precursor of Pasteur. His maternal grandfather, the parish priest and senior Gottfried Walther in Haunsheim, was also active as a writer for young people. There [at Haunsheim] he guided the education of the boy for two years, and managed him, who was gifted with a good memory for words, to speak Latin rather fluently in his eighth year.

Ernst Schröder's father Georg Friedrich Heinrich Schröder (∗ 28 September 1810 in Mannheim, † 12 May 1885 in Karlsruhe), was from 1833–36 professor of mathematics and physics at the *Polytechnische Centralschule* in Munich, before he went to Switzerland to teach at the *Cantonschule* in Solothurn in 1836. In 1840 he was appointed as headmaster to the *Höhere Bürgerschule*, later *Realgymnasium* in Mannheim, where he was active until 1873.[5] Ernst Schröder's grandfather Johann Gottfried Ludwig Walther was 1812–1851 priest and head of the parish (*Senior*)

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[3] On the fate of Schröder's papers cf. Dipert *1991*, 17–21; Peckhaus *1988*.

[4] Dipert *1980*, *1991*; Baldus *1935*; cf. Peckhaus *1997*, 234–238.

[5] Cf. Poggendorff (ed.) *1863*, cols. 844–845; *1898*, 1212.

in Haunsheim. Walther (∗ 15 February 1785 in Mühlheim, Upper Palatinat, † 7 December 1852 in Haunsheim) made himself a name as an author with numerous novels, fairy tales and fables.[6] He was married to Luise Auguste Friederike Cannabich, daughter of a Frankfurt merchant. Ernst Schröder's mother, Caroline, was their daughter.

Schröder continues:

> The precociousness had, however, its shady side, by associating the boy with comrades often very much older, thereby depriving him of playmates of the same age and laying the foundations of strangeness and an inclination towards seclusion.
>
> These troubles were counteracted [...] while attending subsequently the three upper classes in his father's school, where ERNST SCHRÖDER was especially attracted by newer languages, chemistry and natural history, and were he also enjoyed the instruction of the noted mathematician, Professor August Weiler.

Schröder mentioned here Johann August Weiler (∗ 31 May 1827 in Mainz, † 22 July 1911 in Mannheim)[7] who between 1851 and 1880 was Professor at the *Realgymnasium*in Mannheim. He published numerous papers on mathematical astronomy, especially on disturbance theory.

Schröder writes:

> On the other hand, it [i. e. the inclination of seclusion] was moderated by cultivating sports devoted to all kinds of physical exercise. Finally, before changing to the gymnasium, ERNST SCHRÖDER was sent to the countryside for four months to the family of a head forester, being friends [of the Schröders.]

Schröder kept faithful to sports all his life. His friend Jakob Lüroth reported that, besides swimming and skating, also he was a serious rider for some years. Later he became an enthusiastic cyclist, "because by this he was able to work out heavily without a chance of reasoning." As late as winter 1901/02, i. e. when more than 60 years of age, he took up skiing (Lüroth *1905*, XVI). Richard Baldus reported that Schröder was known all over the town of Karlsruhe as the "cycling professor", riding on a bicycle saddle of his own construction (Baldus *1935*, cf. Ahrens *1925*, 26–27). Schröder continues his note:

> Subsequently, in the next four years, he run through the four upper classes of the Mannheim *Lyceum*. Early on, enthusiasm for a knowledge of nature and a vivid interest in philosophical speculations appeared, so that the choice of a profession was not difficult; and in his tenth year the plan was fixed for SCHRÖDER to devote himself to studies

[6] Cf. parish priest's files, *Landeskirchliches Archiv*, Nuremberg.

[7] Poggendorff (ed.) *1863*, col. 1284; *1898*, 1425; *1926*, 1346.

> in mathematics and physics, therefore with the teaching profession. Soon after having graduated for university studies, SCHRÖDER turned to Heidelberg, where he received his doctoral degree *summa cum laude* after two years of studies with Hesse, Kirchhoff and Bunsen. A grant led the young student to Königsberg in Prussia, where he, besides attending lecture courses in mathematics and physics, actively took part in seminar exercises in these disciplines, winning a couple of the awards offered for them. In the autumn of 1864 he finished his student years.

It should be noted that Schröder received his Ph. D. at the age of 21, after only seven regular years at school and two years of university studies in Heidelberg. He did not write a dissertation. He was awarded the Ph. D. solely on the basis of the doctoral exam on 2 August 1862. According to the examination protocol, Schröder was examined by the classical philologist Johann Christian Felix on Horaz's *Ars Poetica*, by Ludwig Otto Hesse on higher analysis, by Robert Wilhelm Eberhard Bunsen on chemistry, and by Gustav Robert Kirchhoff on the theory of electricity. All examiners expressed their highest satisfaction, and the "doctorate with the highest grade" was awarded.[8] With the help of a grant from the foundation for art and science of the grand duchy of Baden, Schröder was able to continue his studies in Königsberg in Prussia, where he attended Friedrich Julius Richelot's lectures on mathematics and the geophysicist Franz Neumann's lectures on physics.

On his further academic development Schröder writes:

> SCHRÖDER passed the exam for teaching trainees in Baden with the grade "good", but he immediately asked for time off in order to habilitate as a *Privatdozent* for mathematics at the *Eidgenössische Polytechnikum* in Zurich.
>
> His test lecture dealt with: Die Differenziation zu allgemeinem Index.
>
> While reading at that place on determinants, the theory of electricity etc., the young mathematician was active at the same time between autumn 1864 and Whitsun 1868 as "vicar" at the *Kantonschule* in Zurich, consisting of an *Industrieschule* and a gymnasium. He taught, especially at the first place, algebra, trigonometry, geometry and mechanics, because the regular teacher, Gräffe, was unable to do so owing to serious illness.

Presenting his curriculum vitae and an offprint of his paper, "Über die Vielecke von gebrochener Seitenzahl, oder die Bedeutung der Stern-Polygone in der Geometrie" (Schröder *1862*), Schröder applied to be "incorporated in the number of privatdocents for mathematics at the confederate polytechnic."[9] This application was

[8] *Promotionsakte Ernst Schröder*, University Archive Heidelberg, H-IV-102/60.

[9] Schröder's application, dated Zurich 21 January 1865, *ETH Bibliothek*, Zurich, *Schulrat 1865*, no. 34.

evaluated positively, and a test lecture, "Über eine Erweiterung der Leibnitz'schen Theorie der Differentialrechnung," was scheduled on the 25 February 1865. The following day, Robert Clausius, the head of the department, reported to the Swiss school inspector: [10]

> The test lecture of Dr Schröder, ordered by you, took place yesterday. According to the unanimous assessment of the conference it was sufficient in form and content. The conference therefore believes that by the lecture together with the written submissions of Dr Schröder, his qualification for the habilitation as privatdocent has adequately been proved.

It was possible only with difficulty to live from the remuneration granted for lecture courses at the polytechnic. Schröder therefore also taught at the cantonal school in Zurich substituting for the mathematician Gräffe who remained ill. This situation was not satisfactory, as Schröder wrote in his autobiographical note:

> Although Dr SCHRÖDER was able to earn his living, the chances for a career in both directions did not appear to be particularly good because of this double position. He therefore decided to return to Civil Service in Baden. After a short substitution at the *Höhere Bürgerschule* in Karlsruhe he got a teaching position at the *Pädagogium* zu Pforzheim, which carried, however, a heavy teaching load of some 26 hours.

Schröder reports that he passed a second teaching exam in 1869.[11] He continues:

> Then the war of 1870 came. Although Dr SCHRÖDER was declared unfit for service in those days, he volunteered for military service. He was now declared to be in perfect health. He was involved in the campaign as a voluntary operating gunner in the Fourth Heavy Battery of the Baden Field Artillery Regiment from 20 July to 1 November 1870. While in the field, he received an appointment as a professor of the grand duchy of Baden, and soon afterwards the news that he was detached home, on account of an application of the head school inspector. There he started his teaching activities at the *Pro- und Realgymnasium* in Baden-Baden as a teacher of mathematics and science. He then accepted a call as full professor of mathematics at the Technical University in Darmstadt in 1874, and two years later he accepted a call to the Technical University Karlsruhe, then called polytechnic.
>
> Following his interests he taught there the subjects of arithmetic, trigonometry and higher analysis.

[10] *Ibid.*, no. 157.

[11] On Schröder's career at school cf. also the "Standesliste", *Generallandesarchiv Karlsruhe*, 76/10053.

It was during this time as a school teacher that Schröder published his textbook *Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende* (*1873*) and, a year later, *Über die formalen Elemente der absoluten Algebra* as a "Programmschrift" (*1874b*), i.e., a pamphlet attached to the annual report and catalog of a 19th century German secondary school.

In the beginning of the academic year 1874/75 Schröder succeeded the mathematician Nikolaus Heinrich Dölp, who had died in summer 1874, at the *Großherzoglich Hessische Polytechnische Schule* in Darmstadt. He delivered in the summer semester of 1876 a one hour lecture course on "Logic on a mathematical basis," in the framework of courses for general education organized by the School for Mathematics and Science. It was the first lecture course on the new mathematical logic delivered in Germany. Gottlob Frege, e. g., did not begin his courses on the concept script, in Jena, until the winter semester of 1879. Unlike Frege, Schröder used his lecture courses to *prepare* his logical writings. So, his first logical book, *Der Operationskreis des Logikkalkuls*, appeared in 1877 (Schröder *1877a*), in the year after the Darmstadt lecture course.

In 1876, Schröder accepted an invitation from the *Großherzoglich Badische Polytechnische Schule* in Karlsruhe, where he started teaching in the winter semester. He taught there for 26 years until his death. In 1890/91 he was the director of this university. In Karlsruhe, he also taught logic in summer semester 1878 (e.g., about "Logic as a mathematical discipline"). From 1883/84 until 1888/89 he taught in each winter semester the two hour course on "Algebra of logic". The first volume of Schröder's main work on logic, the *Vorlesungen über die Algebra der Logik*, did not appear, however, until 1890, i. e. after Schröder had ended his lectures on logic.[12]

Schröder passed away on the 16 June 1902 in Karlsruhe. Jakob Lüroth reports that Schröder died of brain fever (*Gehirnentzündung*) after an illness of only a few days (Lüroth *1905*, III) caused, as he assumed, by a cycling tour. Lüroth remarked that Schröder preferred covering as many kilometers as possible as quickly as possible to admiring the beauty of the landscape. It was said that Schröder had made a long tour just a few days before his death, and caught a cold, which caused the fatal illness (*ibid.*, XVI).

Lüroth also mentions that this picture of an exceptionally fit man was disturbed by some contradicting observations (Lüroth *1905*, XVII):

> In the last years of his life, it appeared strange to me that frictions inevitably connected with life and the office formed a burden of growing heaviness for him, obviously hampering his efficiency to such an extent that he was not able to bring himself to complete his great life's work, his lectures on logic.

Lüroth concludes: "If in this depression the beginnings of a deeper disease had turned up, one could be grateful to kindhearted fate that Schröder had been preserved from a longer infirmity by a quick death after short illness" (*ibid.*).

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[12] On teaching mathematical logic at German universities cf. Peckhaus *1992*.

Even this was obviously only half of the truth. In the beginning of the 1930s, Andrew D. Osborn, who was at that time writing the first scientific biography of the phenomenologist Edmund Husserl (cf. Osborn *1934*), asked Schröder's former assistant Andreas Heinrich Voigt about his struggle with Husserl over an algebraic logic of intensions (*Inhaltslogik*) and possible tensions between him, Husserl and Schröder. Voigt had written his Ph. D. dissertation on the algebra of logic in 1890 under the supervision of Jakob Lüroth (cf. Voigt *1890*), and subsequently became an assistant for elementary mathematics in Karlsruhe. Voigt replied that, after having changed his profession to economics, he became alienated from Schröder. "In addition, he [i. e. Schröder] became completely inapproachable for other reasons and finally he came to a sorrowful end by committing suicide (by poisoning). He suffered from an incurable illness and had very early thoughts on committing suicide."[13] It is obvious that while the illness gave the occasion for suicide, the deeper reason has to be seen in his depression.

3 OVERVIEW OF SCHRÖDER'S WRITINGS ON LOGIC

In his autobiographical note, Schröder gives the following characterization of his scientific writings (Schröder *1901a*):

> *Schröder*'s scientific papers can be divided into three groups.
>
> First we mention a number of papers published in different journals and [school] programmes on such current problems of his scientific discipline as "Mac-Laurinsche Summenformel" [*1867*], "Algorithmen zur Auflösung der Gleichungen" [*1870a*], "Iterierte Funktionen" [*1871*], "Vier Combinatorische Probleme" [*1870b*], "v. Staudts Rechnung mit Würfen" [*1876*], "Trinomische Gleichungen" [*1880a*], "Theorem der Funktionslehre" [*1877c*] etc. Following *Schröder*'s popular reworking of "arithmetic and algebra", the research of the second group starts with the first volume of his relevant textbook [*1873*].
>
> These papers provide a broader foundation for this discipline, leading to an "absolute algebra", i. e., a general theory of connections going even beyond the associative law. From these papers, representing Schröder's very own field of research, still only little has been published.
>
> Among these we mention "Die formalen Elemente der absoluten Algebra" [*1874b*], a paper "Ueber Algorithmen und Calculn" [*1887a*], and furthermore some contributions to the reports of the British Association [*1884*, *1888*].

[13]Letter of Andreas Heinrich Voigt to Andrew D. Osborn, Frankfurt, 23 October 1932, papers of A. H. Voigt, Volker Voigt, Frankfurt a. M. On the biography of Voigt (* 18 April 1860 in Flensburg, † 6 November 1940 in Frankfurt a. M.) cf. Hamacher-Hermes *1994*, 138–150. Hamacher-Hermes's book is devoted to the debate between Husserl and Voigt.

> The third field concerns Professor *Schröder*'s work on a reform and further development of logic. Here he builds on efforts of numerous predecessors and contemporary researchers such as: Leibniz, Ploucquet, Boole, De Morgan, Ch. S. Peirce, etc. The relevant papers try to design logic as a calculating discipline, especially making possible an exact handling of relative concepts, and, from then on, an emancipation from the routine claims of spoken language, and also, from then on, to remove any breeding ground for 'cliché' in the field of philosophy by emancipation from the routine claims of spoken language. This should prepare the ground for a scientific universal language that, widely differing from linguistic efforts like Volapük [a universal language like Esperanto, very popular in Germany at that time], looks more like a sign language than like a sound language.
>
> Some little of the work of [Schröder on this topic dates from 1877: "Der Operationskreis des Logikkalkuls" [*1877a*]; a more comprehensive treatise is approaching completion since 1890: "Vorlesungen über die Algebra der Logik", of which the first volume treats the calculus of classes [*1890a*], the second the calculus of propositions [*1891*, *1905*], while volume three deals with the relatives [*1895a*].
>
> Related thoughts are also expressed in [Schröder's] articles: e. g., in his headmaster's speech "Ueber das Zeichen" [*1890b*], as well as those published in the "Monist" under the heading "On Pasigraphy" [*1898b*] and in the Bibliothèque du Congrès International de Philosophie (Paris 1900) under the title "Sur une extension de l'idée d'ordre" [*1901b*].
>
> As can be seen from the dates presented, *Schröder* is one of the few lecturers in mathematics at universities who, like quondam Weierstrass and Paul Du Bois-Reymond, has served from scratch.
>
> The disposition for schematizing, and the aspiration to condense practice to theory, led Schröder to approach physics by perfecting mathematics. This required deepening of mechanics and geometry, but, above all, of arithmetic and subsequently he became in time aware of the necessity to reform the source of all these disciplines, logic.

In his own survey of his scientific aims and results, Schröder thus divides his scientific production into the following three fields:

(1) A number of papers dealing with some current problems of his science.

(2) Studies concerned with creating an "absolute algebra," i. e., a general theory of connections.

(3) Work on the reform and development of logic.

From the quoted passages one could assume that (2) and (3) are separate from each other. This is not the case, since both are constituents of Schröder's heuristic idea of a universal science mediated by a scientific universal language. His scientific efforts served for providing the requirements to found physics as the science of material nature by "deepening the foundations," to quote a famous metaphor later used by David Hilbert (*1918*, 407) to illustrate the objectives of his axiomatic programme. Schröder thought that the formal part of logic, using a symbolical notation, could be formed as a "calculating logic," as a *model* of formal algebra that is called "absolute" in its last state of development. If doing mathematics is some sort of reasoning, logic is evidently competent for mathematics. The philosopher Rudolf Hermann Lotze, from the University of Göttingen, whose logic book was attentively studied by Schröder, sees therefore in mathematics an "autonomously developing branch of general logic" (Lotze *1880*, § 18), a formulation that can be found nearly word for word in some of Schröder's writings (Schröder *1898a*, 4). These considerations indicate that in Schröder's conception mathematics presupposes logic, and both mathematics and logic presuppose formal algebra. Formal algebra is therefore not regarded as a proper part of mathematics, but as a condition which makes mathematics possible.

From this it becomes evident that Schröder's usual characterization as an algebraist of logic is correct only in the one respect that his most influential publications concerned the algebra of logic, but it does not coincide with his own opinion of his central working field: His "very own field of research" is "absolute algebra", which is in regard to its basic problems and fundamental assumptions similar to modern abstract or universal algebra. Furthermore, when Eugen Lüroth writes in his obituary about the psychological problems that hindered Schröder from completing "his big life-work of his logic lectures," this statement is correct only by accident. Schröder regarded the completion of his *Vorlesungen* only as an interplay that had to be mastered before he could return to his original algebraic tasks.

4 THE PROGRAMME OF THE "ABSOLUTE ALGEBRA"

Schröder formulated this algebraic programme in his *Lehrbuch der Arithmetik und Algebra* published in 1873 (Schröder *1873*, outline *1874a*). The *Lehrbuch* is the first and only published part of a number theory that was originally projected to cover four volumes. It has the subtitle, "Die sieben algebraischen Operationen," alluding to the three direct algebraic operations addition, multiplication, and power, and their inverses subtraction, division, roots, and logarithms. In the *Lehrbuch* the programme of the "absolute algebra" is formulated. A first step to its development is taken in the programme pamphlet, *Über die formalen Elemente der absoluten Algebra* (*1874b*). Schröder developed this programme in some smaller notices, but also long papers such as "Ueber eine eigenthümliche Bestimmung einer Funktion durch formale Anforderungen" (*1881*), "Über Algorithmen und Calculn" (*1887a*) and "Tafeln der eindeutig umkehrbaren Functionen zweier Variablen auf den einfachsten Zahlengebieten" (*1887b*), and also in the appendices 4 to 6 of the first volume of the *Vorlesungen über die Algebra der Logik* (*1890a*, 616–697).

In the first chapter of his *Lehrbuch*, Schröder defines (pure) mathematics as the "science of number." This definition differs from the traditional doctrine of mathematics as the science of quantity. Schröder leaves the notion of number open, because it goes through "a progressive and not yet ended expansion or development" (*1873*, 2). He hints at the discovery of hypercomplex number systems, and remarks that many more further kinds of numbers could be imagined. In any case, he says, the number is a sign that is created arbitrarily for the attainment of quite different aims. Schröder uses these ideas for the later quite general definition of a "domain of numbers" that is not restricted to mathematics. In the *Programmschrift* Schröder points out that the basis of "absolute algebra" is the assumption that there is an

> unlimited manifold of objects (of any kind) that are conceptually distinguished from one another—by a feature or a boundary. Each of the elements of this assumed manifold is designated with letters $a, b, c \ldots$ [...] The given manifold can be called a *domain of numbers* in the widest sense of the word.

Examples of objects or numbers constituting such a manifold are "proper names, concepts, propositions, algorithms, numbers [of pure mathematics], symbols for dimensions and operations, points and systems of points, or any geometrical objects, quantities of substances, etc." (*1874b*, 3).

But what is "formal algebra"? To the theory of formal algebra "in the most narrow sense of the word" belong "those investigations on the laws of algebraic operations [...] that refer to nothing but general numbers in an unlimited number field without making any presuppositions concerning its nature." Formal algebra, therefore, prepares "studies on the most different number systems and calculating operations that might be invented for particular purposes" (*1873*, 233).

In the *Lehrbuch*, Schröder formulates a four-step programme of formal algebra (*1873*, 293–294):

(1) Formal algebra compiles all assumptions that can serve in defining connectives for numbers in a domain of numbers.

(2) Formal algebra compiles for every premise or combination of premises the complete set of inferences, a task that Schröder calls "separation."

(3) Formal algebra investigates which finished domains of numbers can be constructed by the operations defined.

(4) Formal algebra decides "what geometrical, physical, or generally reasonable meaning these numbers and operations can have, what real substratum they can be given" (294).

Only after having finished with the semantical steps (3) and (4), formal algebra becomes an "absolute algebra." Absolute algebra is therefore a formal algebra including all the possible models, and logic is only one of them.

5 SCHRÖDER'S SOURCES

It should be stressed that Schröder wrote his early considerations on formal algebra and logic without any knowledge of the results of his British predecessors, whether the Cambridge symbolical algebra or George Boole's algebra of logic. He rather stood in the tradition of German combinatorial algebra and algebraic analysis (cf. Peckhaus *1997*, ch. 6).

5.1 Combinatorial Analysis

Among the few sources mentioned in the textbook and the school programme pamphlet, Martin Ohm's (1792–1872) *Versuch eines vollkommen consequenten Systems der Mathematik* (*1822*) occurs, a book that stood in the German tradition of algebraical and combinatorial analysis that originated in the work of Carl Friedrich Hindenburg (1741–1808) and his school (cf. Jahnke *1990*, 161–322, *1993*).

Martin Ohm (cf. Bekemeier *1987*) aimed at applying Euclid's axiomatic programme for geometry to all of mathematics (Ohm *1853*, V). He distinguished between number (or "undesignated number") and quantity (or "designated number"), regarding the first of the two as the higher concept. Properties of the calculi of arithmetic, algebra, analysis, etc. are not seen as features of quantities, but of operations, i. e. mental activities (*1853*, VI–VII). This operational view can also be found in the work of Hermann Günther Graßmann, who stood in the Hindenburg tradition, as well.

5.2 General Theory of Forms

Hermann Günther Graßmann's *Lineale Ausdehnungslehre* (*1844*) was a decisive influence on Schröder, especially the "general theory of forms" ("allgemeine Formenlehre") of this pioneering study in vector algebra and vector analysis.[14] The general theory of forms was popularized by Hermann Hankel in his *Theorie der complexen Zahlensysteme* (*1867*).

Graßmann defined the general theory of forms as "the series of truths that is related to all branches of mathematics in the same way, and that therefore only presupposes the general concepts of equality and difference, connection and division" (*1844*, 1). Equality is taken as substitutivity in every context. Graßmann chooses $\frown$ as a general connecting sign. The result of the connection of two elements a and b is expressed by the term $(a \frown b)$. Using the common rules for brackets we get for three elements $((a \frown b) \frown c) = a \frown b \frown c$ (§ 2). Graßmann restricts his considerations to "simple connections", i. e. associative and commutative connections (§ 4). These connecting operations are synthetic. The reverse operations are called "resolving" or "analytic" connections. $a \smile b$ stands for the form which

[14]On H. G. Graßmann cf. the collection Schubring (ed.) *1996*. On Schröder's relation to the the brothers Graßmann cf. Peckhaus *1996*.

results in a if it is synthetically connected with b: $a \smile b \frown b = a$ (§ 5). Graßmann also introduces forms in which more than one synthetic operation occurs. If the second connection is symbolized as $\stackrel{\frown}{\frown}$ and if distributivity holds between the synthetic operations, then the equation $(a \frown b) \stackrel{\frown}{\frown} c = (a \stackrel{\frown}{\frown} c) \frown (b \stackrel{\frown}{\frown} c)$ is valid. Graßmann called the second connection a higher level connection (§ 9), a terminology which might have influenced Schröder's later "Operationsstufen", i. e. "levels of operations".

Whereas Graßmann applied the general theory of forms in the domain of extensive quantities, especially directed lines, i. e. vectors, Hermann Hankel later used it to construct his system of hypercomplex numbers (Hankel *1867*). If $\lambda(a, b)$ is a general connection of objects a, b leading to a new object c, i. e. $\lambda(a, b) = c$, there is a connection Θ which, applied to c and b leads again to a, i. e., $\Theta(c, b) = a$ or $\Theta\{\lambda(a, b), b\} = a$. Hankel called the operation Θ "thetic" and its reverse λ "lytic". The commutativity of these operations is not presupposed.

5.3 "Wissenschaftslehre" and Logic

Hermann Günther Graßmann had already announced that his *Lineale Ausdehnungslehre* would be part of a comprehensive reorganization of the system of sciences. His brother Robert Graßmann (1815–1901) attempted to realize this programme in a couple of writings published under the series title *Wissenschaftslehre oder Philosophie.* In its parts on logic and mathematics he anticipates modern lattice theory. He also formulated a logical calculus similar in part to that of Boole. His logical theory was obviously independent of the contemporary German philosophical discussion on logic, nor was he aware of his British precursors.[15] Graßmann wrote about the aims of his logic or theory of reasoning ("Denklehre") that logic (*1875*, 121)

> should teach us strictly scientific reasoning which is equally valid for all men of any people, any language, equally proving and rigorous. It has therefore to relieve itself from the barriers of a certain language and to treat the forms of reasoning, becoming, thus, a *theory of forms* or *mathematics.*

Graßmann tried to realize this programme in his *Formenlehre oder Mathematik*, published in six brochures consisting of an introduction (*1872a*), a general part on "Grösenlehre" (*1872b*) understood as "science of tying quantities" and the special parts, "Begriffslehre oder Logik" (theory of concepts or logic, *1872c*), "Bindelehre oder Combinationslehre" (theory of binding or combinatorics, *1872d*), "Zahlenlehre oder Arithmetik" (theory of numbers or arithmetic, *1872e*) and "Ausenlehre oder Ausdehnungslehre" (theory of the exterior or theory of extensions, *1872f*).[16]

In the general theory of quantities Graßmann introduces the letters a, b, c, ... as syntactical signs for arbitrary quantities. The letter e represents special quantities:

[15] On Graßmann's logic and his anticipations of lattice theory cf. Mehrtens *1979*.

[16] Cf. also the revision of his logic in Grassmann *1890*.

elements, or in Graßmann's strange terminology "Stifte" (pins), i. e. quantities which cannot be derived from other quantities by tying. Besides brackets which indicate the order of the tying operation he introduces the equality sign $=$, the inequality sign $\gtrless$ and a general sign for a tie $\circ$. Among special ties he investigates joining or addition ("Fügung oder Addition") ("+") and weaving or multiplication ("Webung oder Multiplikation") ("·"). These ties can occur either as interior ties, if $e \circ e = e$, or as exterior ties, if $e \circ e \gtrless e$.

The special parts of the theory of quantities are distinguished with the help of the combinatorially possible results of tying a pin to itself. The first part, "the most simple and, at the same time, the most interior", as Graßmann called it, is the theory of concepts or logic in which interior joining $e+e = e$ and inner weaving $ee = e$ hold. In the theory of binding or combinatorics, interior joining $e + e = e$ and exterior weaving $ee \gtrless e$ hold; in the theory of numbers or arithmetic, exterior joining $e + e \gtrless e$ and interior weaving $ee = e$ hold, or $1 \times 1 = 1$ and $1 \times e = e$. Finally, in the theory of the exterior or Ausdehnungslehre, the "most complicated and most exterior" part of the theory of forms, exterior joining $e + e \gtrless e$ and exterior weaving $ee \gtrless e$ hold (*1872a*, 12–13).

Graßmann thus formulates Boole's "Law of Duality", using his interior weaving $ee = e$, but he goes beyond Boole in allowing interior joining $e + e = e$, which approximates to Jevons' proposal of 1864.

In the theory of concepts, or logic, Graßmann starts by interpreting the syntactical elements, which had already been introduced in a general way. Now, everything that can be a definite object of reasoning is called "quantity". In this new interpretation, pins are initially taken as quantities underived from other quantities by tying. Equality is interpreted as substitutivity without value change, inequality as the impossibility of such a substitution. Joining is read as "and", standing for adjunction or the logical "or". Weaving is read as "times", i. e. conjunction or the logical "and". Graßmann introduces the signs $<$ and $>$ to express the sub- and superordination of concepts. The sign $\leqq$ expresses, that a concept equals or that it is subordinated another concept. This is exactly the sense of Schröder's later basic connecting relation of subsumption or inclusion. In the theory of concepts Graßmann expressed this relation more briefly with the help of the angle sign $\angle$. The sign T stands for the All or the totality, the sum of all pins. The following laws hold: $a + \mathrm{T} = \mathrm{T}$ and $a\mathrm{T} = a$. 0 is interpreted as "the lowest concept, which is subordinate to all concepts." Its laws are $a + 0 = a$ and $a \cdot 0 = 0$. Finally Graßmann introduces the "not" ("Nicht"), or negation, as complement with the laws $a + \bar{a} = \mathrm{T}$ and $a \cdot \bar{a} = 0$.

6 SCHRÖDER'S ALGEBRA OF LOGIC

6.1 *Schröder's Way to Logic*

Logical considerations first occur in Schröder's investigations on domains of numbers in his *Lehrbuch* (*1873*). Logic is a possible interpretation of the structure of general numbers dealt with in absolute algebra. Schröder assumes that there are operations with the help of which two objects from a given manifold can be connected to yield a third that also belongs to that manifold (Schröder *1873*, 4). He chooses from the set of possible operations the non-commutative "symbolic multiplication"

$$c = a \,.\, b = ab$$

with two inverse operations

$$\begin{array}{ll} \text{measuring ("Messung")} & b \,.\, (a : b) = a \ , \\ \text{and division ("Teilung")} & \frac{a}{b} \,.\, b = a \ . \end{array}$$

Schröder calls a direct operation together with its inverses, "level of operations" ("Operationsstufe"). He realizes that "the logical addition of concepts (or individuals)" follows the laws of multiplication of real numbers.

But there is still another association with logic. In his *Lehrbuch*, Schröder speculates about the relation between an "ambiguous expression", such as $\sqrt{a}$ and its possible values. He determines five logical relations, among them the subsumption relation which became essential in his mature logic. Let A be an expression that can have different values $a, a', a'', \ldots$. Then the following relations hold (Schröder *1873*, 27–29):

$$\textit{Superordination} \quad A \succcurlyeq \left\{ \begin{array}{l} a \\ a' \\ a'' \\ \vdots \end{array} \right. \quad ,$$

Examples: metal $\succcurlyeq$ silver; $\sqrt{9} \succcurlyeq -3$.

$$\textit{Subordination} \quad \left. \begin{array}{l} a \\ a' \\ a'' \\ \vdots \end{array} \right\} \preccurlyeq A \quad ,$$

Examples: gold $\preccurlyeq$ metal; $3 \preccurlyeq \sqrt{9}$.

$$\textit{Coordination} \quad a \divideontimes a' \divideontimes a'' \divideontimes \ldots \ ,$$

Examples: gold $\divideontimes$ silver (as regards the general concept "metal") or $3 \divideontimes -3$ (as regards the general concept $\sqrt{9}$).

$$\textit{Equality} \quad A = B \ ,$$

which means that the concepts A and B are identical in intension and extension.

$$\textit{Correlation} \quad A(=)B \ ,$$

which means that the concepts A and B agree in at least one value.

Schröder recognizes that if he now introduced *negation*, he would have created a complete terminology to express all relations between concepts (in respect of their extension) with short formulas that can harmonically be embedded into the schema of the mathematical sign language (*ibid.*, 29).

Schröder wrote his logical considerations in the introduction of the *Lehrbuch* without having seen any work of logic in which symbolical methods were applied. It was while completing a later sheet of his book that he came across Robert Graßmann's *Formenlehre oder Mathematik* (*1872a*). He felt it necessary to insert a comprehensive footnote running over three pages, hinting at this book (Schröder *1873*, footnote, pp. 145–147). There he reported that Graßmann used the sign + for the "collective comprehension", "really regarding it as an *addition*—one could say a 'logical' addition—that has, in addition to the features of common (numerical) addition, the basic property $a + a = a$." He wrote of his interest in the rôle the author had assigned to multiplication, regarded as the product of two concepts which unite the marks common to both concepts.

In the *Programmschrift* of 1874 Schröder again credits Robert Graßmann, but mentions that he had recently found out that the laws of the logical operations had already been developed before Graßmann "in a classical work" by George Boole (Schröder *1874b*, 7). Still in the *Operationskreis* his knowledge of the relevant literature was still quite rudimentary. Besides the works of Boole and Grassmann Schröder knew only those writings that had been reviewed in the *Jahrbücher über die Fortschritte der Mathematik*. This changed radically. Schröder's *Vorlesungen über die Algebra der Logik* (1890–1905) can be read as a very learned synopsis of the literature on logic published up to that time.

6.2 Algebra of Logic

Operationskreis des Logikkalkuls (1877)

In 1877 Schröder published his *Operationskreis des Logikkalkuls*,[17] in which he takes up and modifies the logic of Boole's *Laws of Thought*. An "Operationskreis" (circle of operations) is constituted by more than one direct operation together with their inverses. A "logical calculus" is the set of formulas which can be produced in a circle of operations with logical connecting operations. Schröder calls it a characteristic mark of "*mathematical logic* or the *logical calculus*" that its derivations and inferences can be done in the form of calculations, namely—in the first part of the logic—as calculations with concepts leading to statements about the objects themselves, i. e., categorical judgement, or, in Boole's terminology, "primary propositions" (*1877a*, 1). In its second part, the logical calculus deals

[17] Schröder *1877a*, see also his advertising note *1877b*.

with statements about judgements. Examples are conditional sentences, hypothetical or disjunctive judgements, and Boole's secondary propositions. In both parts, calculations follow the same laws; only their interpretation is different. The object of logical calculations are symbolized by letters, in the first part by "class symbols". a denotes a class or genus of objects of thought. It represents the extension of the concept denoted by the expression (*1877a*, 2). Schröder stresses the complete dualism of the connecting operations of multiplication ("determination") and addition ("collection") based on the "(empirical) principle": "By substitution of plus and minus signs with signs for multiplication and division each general formula being valid in logic has to become a valid formula again" (*1877a*, 3). Duality is made obvious by opposing the respective equations in two columns facing each other. Schröder introduces the class symbols 0 for "nothing", the class to which no element belongs, and 1 for "something", "a category comprehending everything reasonable, the totality of all that can be talked of (Boole's 'universe of discourse')." In the *Operationskreis* Schröder doesn't follow his considerations on the subsumption relation. The basic relation is equality. Besides addition and multiplication Schröder introduces negation (symbolized by a postponed stroke on the line: $a_{|}$ is the contradictory opposite of a. It is governed by the "axiom"

(7): For every class symbol a there has to be an $a_{|}$, with

$$7^{\circ})\ aa_{|} = 0 \qquad \Big| \qquad 7')\ a + a_{|} = 1$$

Calculi of Domains and Classes

Schöder developed his logic in a systematic way in the *Vorlesungen über die Algebra der Logik* (*1890–1905*). Nevertheless, he still presented it as ongoing research, or, as Paul Bernays put it, "the material is exhibited, so to speak, *in statu nascendi*, with all reflections and all difficulties made explicit" (Bernays *1975*, 614). Again he separates logic from its structure. The structures are developed and interpreted in several fields, beginning with the most general field of "domains" ("Gebiete"), manifolds of arbitrary distinct elements, then classes, i. e., kinds of individuals, especially concepts in respect to their extension (*1890a*, 160), and finally propositions (vol. 2, *1891*). The basic operation in the calculi of domains and classes is now subsumption, i. e., identity or inclusion. Equality is defined in terms of the subsumption relation: $(a \Subset b)(b \Subset a) = (a = b)$. Schröder presupposes two principles, *Reflexivity* $a \Subset a$, and *Transitivity* "If $a \Subset b$ and at the same time $b \Subset c$, then $a \Subset c$". He then defines "identical zero" ("nothing") and "identical one" ("all"), "identical multiplication" and "identical addition", and finally negation.

The universal class 1 is not an absolute universal class, but contains all elements of a domain fixed in advance. It is furthermore restricted to a pure manifold (*reine Mannigfaltigkeit*). It thus meets the condition "that there are among its elements given as individuals no classes which contain themselves elements of the same manifold as individuals" (*1890a*, 248). A second manifold could be derived, containing subsets of the first, to be individuals of the second, but "it is not allowed to mix considerations in the first with these in the second" (*ibid.*, 249). This derivation process may be extended to infinity (248). With these considerations Schröder anticipates the simple theory of types (cf. Church *1939/40*, *1976*).

In the sections dealing with statements without negation Schröder proves one direction of the distributivity law for logical addition and logical multiplication, but shows that the other direction cannot be proved. He shows rather its independence by formulating a model in which it does not hold. Schröder introduces the distributivity laws for conjunction and adjunction in the context of sentences "not dealing with negation,"[18] i. e., before having defined negation. They stand, thus, in the same position as in Charles S. Peirce's paper, "On the Algebra of Logic" (*1880*), which was one of Schröder's main sources. Peirce claims there (*1880*, 33) that the distributivity principles can easily be proved, "but the proof is too tedious to give."[19] Schröder now proves (*1890a*, 280) the theorems

$$25_{\times})\ ab + ac \mathrel{\preccurlyeq} a(b+c) \ \mid\ 25_{+})\ a + bc \mathrel{\preccurlyeq} (a+b)(a+c)$$

and cannot abstain from the remark that these proofs "can indeed be done easily, but by no means tediously in the way hinted at [already by Peirce]" (291). It is, however, impossible to prove the so-called "second subsumption"

$$26_{\times})\ a(b+c) \mathrel{\preccurlyeq} ab + ac \ \mid\ 26_{+})\ (a+b)(a+c) \mathrel{\preccurlyeq} a + bc$$

on the basis of the calculus developed up to that point. Schröder was even able to show the independence of these propositions by formulating a model in which all theorems apart from $26_{\times,+}$ are valid. This model is the "*logical calculus with groups*, e. g., of functional equations, algorithms or calculi,"[20] the first example of a non-distributive lattice.[21] From this result Schröder infers (291),

> that there exist instead of one, *two kinds* of calculi, in such a way that in one both, in the other only one of the two parts of the law of distributivity are unconditionally valid. With this insight the necessity suggests itself of naming different calculi differently. It seemed

[18] This is the heading of § 10.

[19] For the "distributivity scandal" of having claimed to have a proof without having realized it, cf. Crapo/Roberts *1969*, Curry *1977* and Houser *1991b* (on the basis of Houser *1985*). For the relation between Schröder and Peirce cf. Barone *1965*, 159–202; *1966*; Houser *1991a*.

[20] On Schröder's proof cf. Peckhaus *1994*, 359–374; Mehrtens *1979*, 51–56; a sketch in Dipert *1978*, 123–131, note pp. 146–148. For a second proof by Schröder and further proofs of other authors cf. Thiel *1994*.

[21] The big success of this result is also indicated by the fact that subsequently further independence proofs ("proofs of the unprovability") were found: Jakob Lüroth published a number theoretic independence proof in his review of the first volume of Schröder's *Vorlesungen* (Lüroth *1891*, 165–166.). Andreas Heinrich Voigt sketched a proof in his "calculus of ideal intensions" (*Kalkül idealer Inhalte*) with a geometrical interpretation in his rejoinder of Edmund Husserl's criticism of Schröder's Algebra of Logic (Voigt *1892*, 303–304). Another geometrical proof was published as "extract from a letter to the editors" by Alwin Reinhold Korselt in the *Mathematische Annalen* (Korselt *1894*). There is furthermore a "proof of the unprovability" of the second subsumption by Georg Wernick (*1929*) in an axiomatic system without negation, without reference to Schröder, however.

> appropriate to me to call the first, hitherto simply named "logical calculus", "*identical*" calculus, contrary to the other, the calculus with "groups"—maybe as the real "*logical*", still to refer both calculi, however, to the domain of the "*algebra of logic*".

Schröder presupposes a special form of the distributivity principle to the identical calculus as principle $\mathrm{III}_\times$:[22]

$$\text{If } bc = 0, \text{ then } a(b+c) \Subset ab + ac.$$

Schröder finally remarks that attempts to proof the problematic distributivity laws using negation would also fail. This claim is later qualified a bit. In definition (6) he namely introduces negation in the following way (302):

> We call "negation" of a domain a such a domain a_l which stands in a relation to it in a way that at the same time:
>
> $$aa_\mathsf{l} \Subset 0 \quad \text{and} \quad 1 \Subset a + a_\mathsf{l}$$
>
> holds.

He postulates additionally that for every domain a there is a negation of this domain a_l (303) and then remarks that he has to leave it open as to whether the relevant form of the distributivity law without principle $\mathrm{III}_\times$ can be proved using negation and theorems following from it (310). Edward Vermilye Huntington, in his methodological criticism of Schröder's algebra of logic, therefore justly accuses Schröder of having shown only the independence of just one of his principles, and this not even completely, since he didn't regard the possible independence of $\mathrm{III}_\times$ from the definition of negation.[23]

Calculus of Propositions

Schröder devotes the second volume of the *Vorlesungen* to the calculus of propositions. The step from the calculus of classes to the calculus of propositions is taken by changing the basic interpretation of the formulas used. Whereas the calculus of classes is bound to a spatial interpretation, especially in terms of the part–whole relation, Schröder employs a temporal interpretation in the calculus

[22] Schröder *1890a*, 293. In the second part of the second volume of the *Vorlesungen*, published posthumously by Karl Eugen Müller, Schröder uses the proof of full distributivity communicated by Alwin Reinhold Korselt in a letter of 1895. This proof uses a modified principle $\mathrm{III}^\circ_\times$, in which the symmetry in respect to the duality of logical multiplication and addition can be kept (Schröder *1905*, 421); cf. also Müller's comment and the further proof in the notes, which Korselt communicated to Schröder in a letter of 1899 (Schröder *1905*, 596–597). Korselt's considerations were used in Schröder's *Abriß der Algebra der Logik*, which was elaborated by Müller (Schröder *1909*, § 66, pp. 43–44; Schröder *1966*, III, 703–704).

[23] Huntington *1904*, 291, note †. Christine Ladd-Franklin stresses the same in her review of volume 1 of the *Vorlesungen* (Ladd-Franklin *1892*, 132).

of propositions, taking up an idea from Boole's *Laws of Thought* (*1854*, 164–165). This may be illustrated regarding subsumption as the basic connecting relation. In the calculus of classes, $a \Subset b$ means that the class a is part of or equal to the class b. In the calculus of propositions, this formula may be interpreted in the following way (Schröder *1891*, § 28, p. 13):

> The time during which a is true is completely contained in the time during which b is true, i. e., *whenever* [...] *a is valid b is valid* as well. In short, we will often say: "*If* a is valid, then b is valid," "a *entails* b" [...], "from a follows b."

Schröder then introduces two new logical symbols, the "sign of products" $\prod$, and the "sign of sums" $\sum$. He uses $\prod_x$ to express that propositions referring to a domain x are valid for any domain x in the basic manifold 1, and $\sum_x$ to say that the proposition is not necessarily valid for all, but for a certain domain x, or for several certain domains x of our manifold 1, i. e., for at least one x (Schröder *1891*, § 29, 26–27).

For Schröder the use of $\sum$ and $\prod$ in logic is perfectly analogous to arithmetic. The existential quantifier and the universal quantifier are therefore interpreted as possibly indefinite logical addition or disjunction, and logical multiplication or conjunction, respectively. This is expressed by the following definition, which also shows the duality of $\sum$ and $\prod$ (Schröder *1891*, § 30, 35).

$$\sum_{\lambda=1}^{\lambda=n} a_\lambda = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \quad \Big| \quad \prod_{\lambda=1}^{\lambda=n} a_\lambda = a_1 a_2 a_3 \cdot \ \ldots \ \cdot a_{n-1} a_n \ .$$

With this conception, Schröder becomes a precursor of infinitary logic, later taken up by Leopold Löwenheim and Thoralf Skolem which influenced the logical efforts of David Hilbert and his school (cf. Moore *1997*).

Geraldine Brady writes concerning the semantics of Schröder's quantifiers (*2000*, 149) that they

> have been identified with operations on truth functions on the domain; infinite products and sums (greatest lower bounds and least upper bounds of infinite sets of truth functions) are used to define the truth function that results from the truth function for a formula when a variable of that formula is quantified.

Schröder had nevertheless all the requirements at hand for modern quantification theory, which he took, however, not from Frege, but from conceptions developed by Charles S. Peirce (1836–1914) and his school, especially by Oscar Howard Mitchell (1851–1889).[24] This was a rather late insight, because when Frege first published his quantification theory in his *Begriffsschrift*, Schröder didn't get the point. In

[24] Cf. Mitchell *1883*, Peirce *1885*. On the development of modern quantification theory in the algebra of logic cf. Brady *2000*.

his review of the *Begriffsschrift* Schröder discusses Frege's quantification theory, admitting some shortcomings in Boole's treatment of particular judgements which had been given "only an inadequate, or, in a rigorous reading, no expression" (*Schröder 1880b*, 91). This was harsh, but it was harsh against Boole, not against Frege. Schröder justified his assessment by the following argument (*ibid.*, 90*f.*):

> The indefinite factor v, which is used by Boole to express in the first part of the logical calculus the statement "some a are b" in the form of $va = vb$, does not serve its purpose because this equation is always identically fulfilled by the assumption $v = ab$, even in the case that no a is b. In the section on "universality" *Frege* justly gives such stipulations which allow him to express these judgements indubitably. I will not follow him slavishly in this respect, but rather show that this does not justify his further deviations from *Boole*'s notation, and also that the latter can be modified and extended by analogy. The author reaches this essentially in the way that he introduced Gothic letters in the meaning of general signs and stipulated a notation to negate this universality [...].

This is simply not true! Frege's syntactical sign for universality is concavity. The Gothic letter signifies the scope of the quantifier, i. e., the range of arguments which can be used in the quantified formula. This quotation shows that Schröder (at least in 1880) simply did not grasp the concept of the scope of a quantifier. His easy modification of the Boolean notation consisted in plainly restricting the formula $va = vb$ by introducing the sign $\neq$ for 'not equal' and stipulating that $va \neq 0$ or $ab \neq 0$, which together would also express that some a are b (*ibid.*).

Jean van Heijenoort in his famous (and notorious) paper "Logic as Calculus and Logic as Language" (*1967b*) saw in quantification theory an essential mark of modern, Frege-style logic, thereby implying that there was no quantification theory in the algebra of logic. Warren Goldfarb acknowledged the attempts in the Peircean tradition (Goldfarb *1979*, 354). He criticized, however, that Schröder had no notion of formal proof as the authors in the logicist tradition (*ibid.*):

> Rather, the following sort of questions is investigated: given an equation between two expressions of the calculus, can that equation be satisfied in various domains—that is, are there relations on the domain that make the equation true?

This resembles the modern notion of the satisfiability of logical formulas, although the full form of this notion cannot be found in Schröder's work (*ibid.*). The treatment of quantification theory with the algebraic framework of a logic of classes and propositions may have "barred or at least impeded a clear insight into the intricate matter," as Christian Thiel remarked, and led to confusions and mistakes.[25]

[25]Thiel's evaluation (Thiel *1990/91*, 13) concerned errors in Schröder's theory of the distribution of quantifiers. Cf. Schröder *1905*, § 3, cf. also Quine *1940*, § 20, 105–109.

Calculus of Relatives

Schröder devotes the third volume of the *Vorlesungen* to the "Algebra and Logic of Relatives", of which only a first part dealing with the *algebra* of relatives could be published (Schröder *1895a*). Algebra and logic of relatives serve as an organon for absolute algebra in the sense of pasigraphy, or general script, that could be used to describe most different objects as models of algebraic structures. It will be treated in detail in section 9 below.

Schröder never claimed any priority for this part of his logic, but always conceded that it was an elaboration of Peirce's work on relatives (cf. Schröder *1905*, XXIV).

7 SOLVING LOGICAL PROBLEMS

Schröder treats the solution of logical problems in the last two paragraphs of the first volume of his *Vorlesungen über die Algebra der Logik* (Schröder *1890a*, §§ 25, 26). He characterizes the type of problems discussed in the beginning of § 26 as follows (559):

> The preceding discussion did only concern problems, whose data can be expressed by subsumptions (or equations [...]) between such classes or functions of such in the identical [i. e. Boolean] calculus, and whose solution can also be expressed by propositions of this form. It was important to eliminate certain classes from the data of the problem, to calculate others from these data [...], i. e. to find their subjects and predicates which can be described with the help of the remaining classes.

In this quotation Schröder refers to the thirty problems which he solved in the preceding paragraph with the help of his class calculus as far as it was developed at that stage. He then goes on to compare his results with solutions provided by alternative calculi. He mentions Peirce who had listed in his paper "On the Algebra of Logic" five different methods in chronological order, the ones by George Boole, William Stanley Jevons, Ernst Schröder, Hugh MacColl and his own (Peirce *1880*, 37). Schröder expresses the opinion that these five methods can be reduced to three, because his own is a modified version of Boole's which therefore has become obsolete (Schröder *1890a*, 559). Furthermore, the methods of MacColl and Peirce could be combined, because MacColl had paved the way for Peirce (589).

As an object of testing the performances of the calculi, Schröder chose a problem first published by Boole which held some prominence among the mathematical and philosophical logicians of the time because of its complexity.[26] Boole's formulation of the problem is quoted in full, but the different ways of solution are only sketched:[27]

[26]"Example 5" in (Boole *1854*, 146–149).

[27]Boole *1854*, 146, cited by Schröder in translation with some revisions (Schröder *1890a*, 522). This problem was also treated by Hermann Lotze in his "Anmerkung über logischen Calcül"

> Ex. 5. Let the observation of a class of natural productions be supposed to have led to the following general results.
>
> 1st, That in whichsoever of these productions the properties A and C are absent, the property E is present or found, together with one of the properties B and D, but not with both.
>
> 2nd, That wherever the properties A and D are found while E is absent, the properties B and C will either both be found or both be missing.
>
> 3rd, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both.
>
> Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D.

In his translation Schröder numbered the data with α, β, and γ, and he split the two questions into four (Schröder *1890a*, 522):

> Let it be required to ascertain,
>
> *first*, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D,
>
> *secondly*, also to decide whether any relations exist independently from the presence or absence of the other properties among the presence or absence of the properties B, C, and D (and, if yes, which?),
>
> *thirdly*, to determine what may be concluded in like manner from the existence of the property B in respect of the properties A, C, and D

(Lotze *1880*, 265–267). Lotze criticized Boole's claim that his solution of the problem shows the advantage of his calculus over syllogistics. Lotze agreed with Boole that it was senseless to try to solve this problem syllogistically, but didn't regard the calculatory procedure as obvious. He preferred a combinatorial way which he obviously adopted from Jevons. This combinatorial way "presents itself automatically as the more appropriate" (*ibid.*, 266). Jevons's combinatorial procedure was a subject of correspondence between Lotze and Schröder. Schröder reported on this correspondence, criticizing Lotze's devaluation of the calculatory method (cf. Schröder *1890a*, 566–568). Gottlob Frege criticized, like Lotze, the artificiality of this problem in his comparison of the *Begriffsschrift* with the Boolean calculus (Frege *1880/81*, *1983*, 52). Nevertheless he also tried to solve the problem. Frege's pathbreaking solution is thoroughly discussed by Peter Schroeder-Heister (Schröder-Heister *1997*). A favorable treatment of this problem can be found in Wilhelm Wundt's logic (Wundt *1880*, 357). Gottfried Gabriel suggested in 1989 the examination proposed of the different solutions to obtain comparison criteria for different systems of logic, i.e., traditional logic, algebra of logic, and Frege's *Begriffsschrift* (Gabriel *1989*).

(and vice versa, when the existence or absence of the property B can be inferred from that of the properties of the latter group),

fourthly, to state what follows for the properties A, C, D as such.

Schröder's notation will be used to sketch out his solution. Besides the logical apparatus already introduced, Schröder uses the function symbol $f(x)$ for representing in the identical calculus a complex expression containing x (or $x_{\shortmid}$) and other symbols connected using basic logical operations, identical multiplication, addition and negation (cf. Schröder *1890a*, 401). Schröder's solution will be outlined in as far as it is necessary to compare it with the alternative solutions discussed.

In an initial step Schröder presents the data, i. e. the conditions α–γ, as subsumptions or equations. In Schröder's calculus, equality is derived from subsumption: $a = b$ stands for a subsumption relation between the terms a ("subject") and b ("predicate") which is valid in both directions at the same time. $a = b$ is thus defined as $(a \preccurlyeq b)(b \preccurlyeq a)$. In Schröder's symbolism the data α–γ can be formalized as follows:

$$
\begin{array}{rrcl}
\alpha: & a_{\shortmid}c_{\shortmid} & \preccurlyeq & (bd_{\shortmid} + b_{\shortmid}d)e \\
\beta: & ade_{\shortmid} & \preccurlyeq & bc + b_{\shortmid}c_{\shortmid} \\
\gamma: & a(b+e) & = & cd_{\shortmid} + c_{\shortmid}d\ .
\end{array}
\tag{1}
$$

These formulas contain the class symbol e as related to the property E, which must then be eliminated because it does not affect the solutions of the questions. To eliminate this class symbol Schröder put the equations (1) to 0 on the right hand side, and finally combined these three equations by conjunction into one. For this purpose he could use two theorems proven earlier:[28]

$$38_\times \qquad (a \preccurlyeq b) = (ab_{\shortmid} = 0)$$

and

$$39_\times \qquad (a = b) = (ab_{\shortmid} + a_{\shortmid}b = 0)\ .$$

After having combined the modified premises the following equation results

$$
\begin{aligned}
&a_{\shortmid}c_{\shortmid}(bd + b_{\shortmid}d_{\shortmid} + e_{\shortmid}) + ade_{\shortmid}(bc_{\shortmid} + b_{\shortmid}c) + \\
&\quad + a(b+e)(cd + c_{\shortmid}d_{\shortmid}) + (a_{\shortmid} + b_{\shortmid}e_{\shortmid})(cd_{\shortmid} + c_{\shortmid}d) = 0\ .
\end{aligned}
$$

Several steps are required for the elimination of e. They result in the formula

$$a(cd + bc_{\shortmid}d_{\shortmid}) + a_{\shortmid}(cd_{\shortmid} + c_{\shortmid}d + b_{\shortmid}c_{\shortmid}d_{\shortmid}) = 0\ .$$

This formula is the starting point for further eliminations and resolutions of certain class symbols, finally leading to an answer to our questions.

Schröder stresses the similarity of his method with that of Boole, which latter he considers as "definitely settled" through his own modifications. For Schröder

[28] Schröder erroneously mentions Theorem 39_+ instead of $39_\times$ which suffices to bring an equation to 1.

Boole's method was therefore only of historical interest. Its disadvantages result from the lack of a sign for negation—Boole had to write $1-x$ for $x_{\shortmid}$—and from the interpretation of the logical "or" as an exclusive "or". The inadequacy of Boole's language led to logically uninterpretable expressions in the course of calculating logical equations according to the model of arithmetic.

Schröder starts his discussion of alternatives with Jevons's method, which he considered as "without art" (*kunstlos*), although it was the "nearest at hand or most unsophisticated". Jevons proposed this "Crossing-off procedure" ("Ausmusterungsverfahren") in his *Pure Logic* (Jevons *1864*). According to Schröder it consisted in (Schröder *1890a*, 560)

> writing down for all classes mentioned in the formulation of the problem all the possible cases which can be thought of in respect to the presence or absence of one in relation to another, then crossing off all cases which are excluded from the thinkable combinations by the data of the problem as inadmissable, and trying to pick out the answers to the questions posed by the problem from the remaining ones.

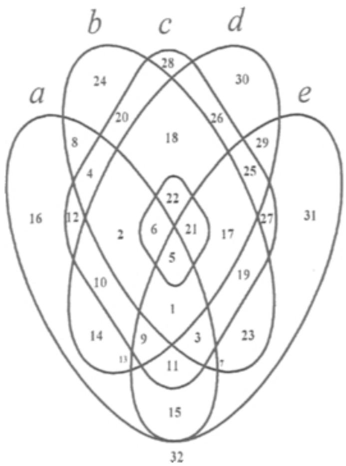
Schröder applies Jevons's method to Boole's problem (Schröder *1890a*, 562–655). It contains five class symbols. Therefore $2^5 = 32$ combinations would have to be considered, of which eleven were valid. Schröder criticizes the complexity of the combinatorial method, which grows with the square of the number of occurring class symbols. He also claims that the procedure is not really calculatory, but that it is based instead on a "mental comparison" of combinations and premises (*ibid.*, 567–568).

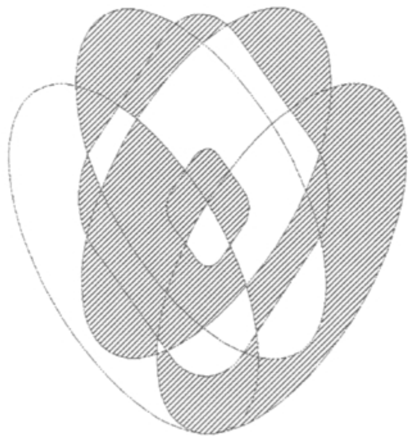
Schröder discusses the graphical extension of this method presented by John Venn in his *Symbolic Logic* (Venn *1881*, cf. Schröder *1890a*, 569–573). Venn symbolized the relations between the extensions of classes associated with two or three class symbols by circles, by ellipses with four class symbols, and by ellipses together with a ring in the form of a rhombus with five class symbols. The procedure is similar to Jevons's crossing-off method, since the fields not present according to the data of the problem are erased by hatching them. In regard to the complexity of the problems which admit of treatment, Venn's procedure is even more restrictive than Jevons's, because the schemes for more than five class symbols become rather intricate.[29] Schröder admits that Venn's method has the advantage that every logical problem which can be presented in an intuitive form, could be solved as soon as it was symbolized with the help of the graphical scheme. The scheme proposed by Venn for five class symbols is shown in the figure on the left hand side below. In the figure on the right hand side the solution of the Boolean problem is given.[30] The exterior field 32 has to be hatched as well. As an expository

[29] Schröder *1890a*, 569. Today, however, graphical procedures have been developed to handle greater complexity.

[30] Schröder uses the following algorithm for numbering the fields of the Venn diagram:

convenience, this is omitted here.[31]





While Schröder was critical of the methods of Jevons and Venn, he praised those of MacColl and Peirce for being equal to his own concerning their efficiency (Schröder *1890a*, 560). He illustrated the relation between the different methods with the following metaphor: While he himself combines the various clues of premises to one single knot (i. e. the united equation), and then hacks it through, Peirce separates every clue into thin threads and cuts them individually and binds them together again if necessary. Jevons, on the other hand, would immediately make chaff. If one modifies Peirce's procedure by separating clues only as far as needed to isolate symbols for eliminations and unknowns, it would come close to MacColl's. Schröder acknowledges that the variations of MacColl and Peirce are natural and simple, but criticizes their length (Schröder *1890a*, 573).

---

	a	b	c	d	e
1	1	1	1	1	1
2	1	1	1	1	0
3	1	1	1	0	1
4	1	1	1	0	0
5	1	1	0	1	1
6	1	1	0	1	0
7	1	1	0	0	1
8	1	1	0	0	0
			⋮		
32	0	0	0	0	0

I would like to thank Peter Bernhard (Erlangen) for sharing this information. For a comprehensive discussion of graphical representations in logic, especially Euler diagrams cf. Bernhard 2001.

[31] Schröder corrects the solution given by Venn (Venn *1881*, 281) by hatching field 24. Venn acknowledged this correction in the second edition of his *Symbolic Logic* (Venn *1894*, 352, n. 1).

Schröder reconstructs Peirce's method as a sequence of six steps which he called "processes" (Schröder *1890a*, 574–584). He follows Peirce's own exposition (cf. Peirce *1880*, 37–42).

(1) In an initial step, the premises are expressed as subsumptions.

(2) Then every subject (the term on the left hand side of a subsumption) is developed as a sum, every predicate (the term on the right hand side) as a product, using the schemes

$$\begin{array}{lrcl} 44_{+} & f(x) & = & f(1)x + f(0)x_{\shortmid} \\ 44_{\times} & f(x) & = & \{f(0)+x\}\{f(1)+x_{\shortmid}\}\ . \end{array}$$

(3) In the third step, all complex subsumptions are reduced, e. g., the subsumption

$$s + s' + s'' + \ldots \Subset pp'p'' \ldots$$

into the subsumptions

$$\begin{array}{l} s \Subset p, s \Subset p', s \Subset p'', \ldots \quad , \\ s' \Subset p, s' \Subset p', s' \Subset p'', \ldots \quad , \\ s'' \Subset p, s'' \Subset p', s'' \Subset p'', \ldots \quad , \\ \vdots \end{array}$$

(4) In the fourth step, the necessary eliminations are made.

(5) In the fifth step, all the terms, where the unknown x can be found at subject or predicate position, are picked up and finally

(6) united in the last step. With the help of the resulting formula the unknown can now be calculated.

Schröder sees the advantage of Peirce's method over Boole's in the fact that it operates with subsumptions and not with equations, and that it preserves the subject-predicate structure which "thoroughly matches the judging functions of ordinary reasoning" (Schröder *1890a*, 584)—but these are advantages in Schröder's method as well. Peirce's method has the further advantage that it is not necessary to bring the equations to zero on the right hand side and then unite them as one single equation.

Schröder closes his considerations on the class calculus (Schröder *1890a*, 589–592) with a discussion of MacColl's method.[32] He states that MacColl invented this method independently, but nevertheless rather belatedly, in order to solve the problems of the Boolean calculus. It differs, however, from the modified Boolean method (i.e., Schröder's method) not as much as MacColl himself thought.

[32]On Hugh MacColl see Astroh/Reid (eds.) *1998*. On the relation between Schröder and MacColl cf. Peckhaus *1998*.

Schröder stresses that he agreed with Venn who had written in his assessment (Venn *1881*, 37; *1894*, 492) that MacColl's symbolical method is "practically identical with those of Peirce and Schröder."

This assessment is likely if one regards MacColl's own evaluation of the differences between his system and those of Boole and Jevons. In the third part of the series of papers on "The Calculus of Equivalent Statements", he lists the points of difference (MacColl *1878*, 27):

> (1) With me every single letter, as well as every combination of letters, always denotes a *statement*.
>
> (2) I use a symbol (the symbol :) to denote that the statement following it is true provided the statement preceding it be true.
>
> (3) I use a special symbol—namely, an accent—to express denial; and this accent, like the minus sign in ordinary algebra, may be made to affect a multinomial statement of any complexity.

If one relates implication and subsumption, the latter being the class logical equivalent of the former, MacColl presents important modifications of the calculi of Boole and Jevons, which were later also introduced by Schöder. It is noteworthy that MacColl doesn't mention his use of the inclusive "or". The reason may be that MacColl, in the context of the quoted passage, parallelled the calculi of Boole and Jevons, using Jevons's *Pure Logic* (Jevons *1864*) where the exclusive "or" had already been replaced by the inclusive "or".

It is a matter of course that Schröder recognized that MacColl's formulas arise from the propositional calculus, the "calculus of equivalent statements", in which the symbols 0 and 1 are not class symbols but interpreted as truth values. Schröder discusses this method in his more general class logic, because MacColl also treated the Boolean class logical example 5 (cf. MacColl *1878*, 23–25).

According to Schröder's analysis, MacColl's solution is based on the two equations named "rule 22" (cf. MacColl *1878*, 19):

$$xf(x) = xf(1); \; x'f(x) = x'f(0) \; .$$

Schröder had discussed these equations as "theorems of MacColl" at an earlier stage in his *Vorlesungen* (Schröder *1890a*, § 19, p. 420).[33]

Schröder regards it an advantage of MacColl's procedure that the premises are not united. In this respect, he states, MacColl is a precursor of Peirce. But he denies further advantages over his or Peirce's method, e.g., as regards printing economy, superior exposition, and greater user-comfort (Schröder *1890a*, 591–592).

It is curious that Schröder returns to MacColl's method in the second volume of his *Vorlesungen*, which is devoted to the calculus of propositions. There he changes his assessment that MacColl's solution is not really original (Schröder *1891*, 391). Schröder then states that MacColl's method, as presented above, is

[33] In MacColl's notation the apostrophe denotes negation. MacColl himself says that he used the implications $xf(x) : f(1)$, $x'f(x) : f(0)$ named as rule 23.

only a scheme. In fact, says Schröder, MacColl had used another method, which was indeed original and advantageous. Schröder sketches it as follows (Schröder *1891*, 304–305):

In a given product of propositions $F(x, y)$ y is to be eliminated and x is to be calculated. The following four implications are used

$$\begin{array}{lcl|lcl} xy & \preccurlyeq & F(1,1) & x_{|}y & \preccurlyeq & F(0,1) \\ xy_{|} & \preccurlyeq & F(1,0) & x_{|}y_{|} & \preccurlyeq & F(0,0) \end{array}$$

By addition and using the theorems $x = xy + xy_{|}$ and $x = x_{|}y + x_{|}y_{|}$, y can be eliminated, resulting in

$$x \preccurlyeq F(1,1) + F(1,0) \mid x_{|} \preccurlyeq F(0,1) + F(0,0) \; .$$

By contraposition the solutions can be given:

$$F_{|}(1,1)F_{|}(1,0) \preccurlyeq x_{|} \mid F_{|}(0,1)F_{|}(0,0) \preccurlyeq x \; .$$

8 QUASI-AXIOMATICS

Schröder's logic, especially his calculi of domains and of classes, is usually presented in an axiomatic way. It appears that such an axiomatical interpretation could be traced back to Schröder himself since, in the first part of his posthumously published *Abriß der Algebra der Logik* (*1909*, *1910*), an axiomatic form is chosen.

In this comprehensive presentation of the theories given in the first two volumes of the *Vorlesungen* it is claimed (*1909*, *1966*, vol. 3, 666) that logic deals with "domains" which

> form in respect to the relation to each other the object of a "theory of domains." The kind of thing these are will be stipulated by certain general propositions, the so-called "axioms," which should be valid for all things to be taken into account, and for all domains as meanings of the general symbols.

All attributes of any domain $a, b, c, \ldots$ are given in a set of seven axioms, two of them split into the dual forms for logical addition and logical multiplication. The basic relation in such domains is non-symmetrical "subsumption" which is binary, reflexive and transitive but besides this, arbitrary. It is designated by the sign $\preccurlyeq$ ("sub").[34] Furthermore, a negated subsumption sign $\not\preccurlyeq$ is used. The axiomatic system runs as follows (680):

[34] The distinction is made between a "primary subsumption," i. e., the incorporation of a class symbol in another, and the "secondary subsumption" which stands for implication. Confusing of the two sorts of subsumption while using them simultaneously is claimed to be impossible in practice. Cf. Schröder *1909*, § 22, especially p. 680; cf. also § 11, 667–668, and § 84, 716–717.

I	$a \preccurlyeq a$	identity or tautology axiom,
II	$(a \preccurlyeq b)(b \preccurlyeq c) \preccurlyeq (a \preccurlyeq c)$	subsumption rule,
III	$(a \preccurlyeq b)(b \preccurlyeq a) = (a = b)$	equality definition,
$\mathrm{IV}_{\times}$	$0 \preccurlyeq a$	zero postulate,
IV_{+}	$a \preccurlyeq 1$	one postulate,
V	$1 \not\preccurlyeq 0$	existence postulate,
$\mathrm{VI}_{\times}$	$(x \preccurlyeq a)(x \preccurlyeq b) = (x \preccurlyeq ab)$	definition of logical product,
VI_{+}	$(a \preccurlyeq y)(b \preccurlyeq y) = (a + b \preccurlyeq y)$	definition of logical sum,
$\mathrm{VII}_{\times,+}$	$(a + z)(\bar{a} + z) = z = az + \bar{a}z$	negation or distribution principle.

Up to now, this axiomatic system has been the basis of axiomatic presentations of Schröder's calculus.[35] As early as 1904 the American postulation theorist Edward Vermilye Huntington had condensed a set of 10 postulates from Schröder's class calculus, using the dyadic relation "within" $\olessthan$ and had discussed the independence, completeness, and consistency of these postulates (*1904*, 297).

Both axiomatic systems are doubtless written in the spirit of David Hilbert's axiomatic programme. Nevertheless, it is questionable whether Schröder himself had anticipated Hilbert's notion of "axiom" during his lifetime. In the face of such doubts, one has to deal with the fact that the system of the *Abriß der Algebra der Logik* was published under Schröder's name, although seven years after his death. Responsible for this publication was the German grammar-school professor Karl Eugen Müller, who had been entrusted by the *Deutsche Mathematiker-Vereinigung* to check Schröder's extensive manuscript *Nachlaß* for material worth publishing, especially material suitable for completing the unfinished *Vorlesungen*. In his preface to the compendium, Müller reports that he found among the papers "only short sketches, and some remarks to an '*Abriß*' of the algebra of logic, but no realized presentation of any part of that *Abriß*." Müller had compiled the *Abriß* "leaning if possible upon the manuscript stuff and the chief work, but—according to Schröder's explicitly stated opinion—regarding recent research."[36] It is a reasonable assumption that the axiomatic form is also one of Müller's additions. Further evidence reveals Müller's own first contribution to symbolic logic on the foundations of the calculus of domains (*1900*). There he stresses that his presentation differs from Schröder's in regard to the hypothetical axiomatical foundations of the calculus (*1900*, 2), and he compiles a set of 9 axioms quite different from that of the later *Abriß* (*1900*, 20). Schröder, however, stands at more than one step apart from modern axiomatics.

An examination of Schröder's writings shows that he was inspired by the traditional notion of "axiom" associated with Aristotelian first principles, and Eu-

[35]Thus was, e.g., quoted by Randall R. Dipert (*1978*, 132–134), and from there has found its way into the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* (Houser *1994*). In the equality definition (axiom III), Houser incorrectly writes negated subsumption signs for both equality signs (*ibid.*, 611).

[36]Müller *1909/1966*, vol. 3, 653, cf. also Müller *1905*. For Müller's biography, his editorial work; and for the fate of Schröder's *Nachlaß* cf. Peckhaus *1988*.

clidean postulates. Schröder's growing reluctance to found his logical theories on axioms becomes obvious.

8.1 The "One and Only Axiom" (1873)

In his *Lehrbuch der Arithmetik und Algebra* published in *1873*, Schröder treats pure mathematics as the theory of numbers. Natural numbers are introduced by the countability of things, each thing being a unit. A natural number is defined as a sum of units. Schröder's theory of numbers is based on the "one and only axiom," the "axiom of the inherence of the signs." He insists that this axiom is presupposed in every deductive science and that it gives the certainty "that in all our arguments and inferences the signs inhere in our memory—and even more on the paper. [...] Without this principle," he continues (16–17),

> which is derived by induction or generalization from a very rich experience, every deduction would indeed be illusory, since every deduction begins when—after having sufficiently clothed the basic features of things into signs—the investigation of the things has made room for the investigation of their signs.

It should be stressed that axioms of such kind that make it possible at all to set up systems of propositions in mathematics or logic were not as unusual at that time as the heavy criticism of Gottlob Frege and Benno Kerry might suggest.[37] Similar axioms can be found in Dedekind's and Hilbert's early foundational studies.[38] Such axioms are not formal because of their empirical origin.

8.2 "Axiomatics" in the "Operationskreis des Logikkalkuls" (1877)

Without having completed his large-scale number theory project (four volumes were planned, only one was published), Schröder switched his interests toward logic. His *Operationskreis des Logikkalkuls* consists of 40 numbered propositions, of which the first 20 concern the direct logical connectives addition and multiplication and the second half their inverses. One definition and two axioms are presupposed in this set of propositions (*1877a*, 5):

i The definition of the equality of class symbols.

ii Axiom: every class symbol is equal to itself.

[37] cf. Frege *1884*, VIII; Kerry *1890*, 333–336.

[38] Dedekind presupposes such conditions when writing about mental practices: "It occurs very often that on some occasion several things a, b, c, ... considered under a common aspect are put together in the mind" (*1888*, § 1), and Hilbert explicitly formulates an "axiom of the existence of an intelligence" which runs as follows: "I have the ability to think ***things***, and to designate them by simple signs $(a, b, \dots X, Y, \dots)$ in such a completely characteristic way that I can always recognize them again without doubt. My thinking operates with these designated things in certain ways, according to certain laws, and I am able to recognize these laws through self-observation, and to describe them perfectly" (Hilbert *1905*, 219).

iii Axiom: If two class symbols are equal to a third, they are also equal to one another.

Schröder states that 13 of the theory's propositions "must be stated for the present as formal axioms."[39] In his *Operationskreis* Schröder uses the term "axiom" for unproved or unprovable theorems of his calculus. Such axioms also cover definitions that introduce schematic signs for, e.g., operations, and define the characteristics of such operations directly or implicitly. Such axiomatic definitions can be connected with postulates that claim the existence of objects of the calculus. Schröder connects, e.g., the axiomatic definitions of logical sum and logical product to the axiomatic postulate that addition and multiplication of class symbols lead again to class symbols, and that these operations can always be realized. Schröder stresses that the theorems of the algebra of logic are intuitive, that they are directly evident. The assertions called "axioms" could be regarded as implications, directly given together with the definitions. They are not empirical, but formal, i. e., derived from evident intuitions (*1877a*, 4).

8.3 Criticism in the "Vorlesungen über die Algebra der Logik" (1890–1895)

The terminological inexactness in his early writings inclined Schröder in the course of time to abandon the notion of "axiom" from his theory. In the first volume of his *Vorlesungen* Schröder distinguishes the following kinds of sentences: Definitions, i. e., explanations of terms, postulates, principles, and axioms (only "principles" is italicized) and theorems (*1890a*, 165-166). Schröder remarks that although a logic textbook should explain what definitions, postulates, axioms, and theorems are, he will here (which means in the complete *Vorlesungen*) abstain from doing so. We have therefore to check Schröder's use of these sorts of propositions in order to determine their status. *Theorems* are propositions derived from definitions and principles. *Principles* give the features of logical symbols which cannot be derived from other propositions of the calculus. In the "identical calculus" there are only three principles: identity, which is given by the reflexivity of the subsumption relation, the inference of subsumption, which asserts the transitivity of the subsumption relation, and the principle $\mathrm{III}_\times$: If $bc = 0$, then $a(b+c) \Subset ab + bc$ (293). The latter is a consequence of Schröder's proof that the second subsumption of the law of distributivity is independent from the propositions of the identical calculus without negation. *Definitions* introduce atomic expressions of the calculus, such as equality defined as antisymmetry of the subsumption relation [Def. (1), 184], "identical zero" ("nothing") and "identical one" ("all") [Def. $(2_{\times,+})$, 184], "identical multiplication" (conjunction) and "identical addition" (adjunction) [Def. $(3_{\times,+})$, 196] with modified versions [Def. $(4_{\times,+})$, 202; Def. $(5_{\times,+})$, 205], and negation [Def. (6), 302].

[39]Schröder *1877a*, 5. Randall R. Dipert writes (*1978*, 87): "The *Operationskreis* was one of the first serious attempts to axiomatize the Boolean calculus," an opinion which is questioned in what follows.

Contrary to the kinds of propositions just mentioned, *postulates* are not formalized. They are responsible for the connections between the formal system and perception in applications of logic. Schröder writes (212):

> As soon as we want to give a meaning to those symbols included in our "domains" [i.e., manifolds of indetermined elements], i.e., to claim that there are real domains accessible to perception which have the respective attributes, *we add* to our definitions certain *postulates*, i.e., we assert that demands for giving evidence of the existence of some domains can generally be accomplished although in this respect we can only refer to perception.

Although Schröder speaks in the beginning of "'*principles*' or 'axioms,'" he never uses the notion of "axiom" throughout the pages of the *Vorlesungen*. That this was not without reason becomes clear from the third volume. In this volume (Schröder *1895a*), devoted to the "algebra and logic of [binary] relatives," he again changes his terminology, skipping all kinds of propositions mentioned above, and founding his theory on 29 "conventional stipulations" which can also form, as he claims, the foundation of the complete logic (*1895a*, 16). Later Schröder stresses, referring to Charles Sanders Peirce's early paper "Description of a notation for the logic of relatives":[40]

> Apart from the fundamental conventions compiled in § 3 [Schröder adds in a footnote: "to be rigorous it should be inserted after 'conventions': and the few so-called principles of general logic, which can be regarded (generally but not formally) as being contained in these conventions."], we indeed do not need a further "principle" in the theory. And if the question is raised about the axiomatic foundations of our discipline of the algebra and logic of relatives, I can agree to Mr Peirce [...]. The foundations are of the same rank, they are nothing else, as the known "principles" of general logic. Contrary to geometry, logic and arithmetic do not need any real "axioms."

Schröder accepts the possibility of a formal conception of geometry. In such cases, however, the geometrical axioms are assertions of mere assumptions. The question about the fulfillment and validity of geometrical propositions in some domains of thinking is not an object of research. These geometrical propositions can only claim "relative truth" under the condition that the presuppositions become true. Schröder judges (67):

> Commonly, and in my opinion justly, this is not done. The geometrical axioms are on the contrary taught, presented and accepted with the claim of real validity, truth, be it for our subjective space perception, be it for reality which is thought to be objectively fundamental for space perception.

[40] Schröder *1895a*, 7, referring to Peirce *1870*.

In this conception axioms are not analytical judgments, which say nothing. They were ***psychologically*** essential for thinking because of the nature of our space perception, but they were not *logically* essential for thinking. "Geometry is more than a mere branch of logic; it is the most elementary limb in the great series of physical sciences. Arithmetic is a contrary case" (67). Concerning his own practice, in using the notion of "axiom", Schröder writes, quoting Peirce (68):

> Already in vol. 1 I have therefore abstained from using the name "axiom" for the "principles" necessary in that course. And these "principles" were only definitions in disguise—"are mere substitutes for definitions of the universal logical relations." As far as the general logical relations can be defined—says *Peirce* with justice—it is possible to get by without any "principles" in logic (all axioms may be dispensed with).

In a certain sense Schröder comes back to his early conception of the notion of "axiom" proposed in his *Lehrbuch der Arithmetik und Algebra*. *Axioms* are formulated with the claim to be valid in reality. They therefore concern not only formal structure but bind this structure to the human capacity to gain knowledge of the world of experiences. In the programme of an algebra and logic of relatives they are relevant for applications of formal structures to possible representatives of theses structures. They restrict the possible translations of schematic symbols and formal operations. For Schröder geometry is formulated in this same way, but he does not deny the possibility of "formal" (not empirical, not Euclidean) geometries. Their foundations were, however, not axiomatical but based on definitions or conventions.[41]

In its last state of development the algebra of logic presupposes a general logic which states the general laws dominating every formal system. If they can be formalized, these principles can be included into the system of stipulated conventions, but then they loose their status as principles. The principles of general logic are usually the principle of identity, the principle of (excluded) contradiction and the principle of sufficient reason, Leibniz's necessary truths. They are not axioms, thus do not concern the relation between the thinking subject and the real world, but provide the conditions for every activity of thought. Schröder removes the notion of "axiom" from mathematical terminology and places it at the intersection between the philosophy of mind and the philosophy of science.

[41] Thus Schröder's use of the notion of "axiom" is in accordance with the position which van der Waerden calls "classical axiomatics" and which is characterized by two criteria: (1) the objects the classical axioms are referring to are determined and known in advance, (2) he who states the axioms considers them as true (van der Waerden *1967*, 1–2).

9 LOGIC OF RELATIVES AS AN ORGANON FOR A UNIVERSAL LANGUAGE

The development of Schröder's scientific activities is consistent with his early algebraic programme. With the big plan to create an absolute algebra in the background he devotes himself for the present to the analysis of one model of absolute algebra: logic. In the *Operationskreis des Logikkalkuls* (*1877a*) Schröder concentrated on the duality between logical addition and logical multiplication and with this on the identity of the algebraic structures of these operations. In the *Vorlesungen* Schröder distinguished between the object of logic and its structure. He called the calculus an "auxiliary discipline" ("Hülfsdisziplin") that precedes or goes along with proper logic. The third volume with the subtitle *Algebra und Logik der Relative* provides a further generalization. Schröder does not hesitate to stress the twofold character of the theory of relatives consisting of an algebra and a logic of relatives. In the only published first part he presents the algebraic section. The logic of relatives that would have linked his theory to absolute algebra was planned for the second part that remained unfinished.

9.1 Basic Concepts of the Algebra and Logic of Relatives

As early as September 1894, Schröder sent a "Note über die Algebra der binären Relative" to the editorial office of the *Mathematische Annalen*; it was printed in this journal the next year (*1895b*). This note announced the third volume of his *Vorlesungen*, which would be published "soon, almost at the same time as the last part of the second volume". In this volume he promised to found the discipline of the algebra of binary relatives following Charles S. Peirce without presupposing the logic of the first two volumes (*1895b*, 144). In this note introductory on the algebra of binary relatives, however, he preferred to build on the results of the first two volumes of the *Vorlesungen*. Just at the beginning he compiled a group of 31 "fundamental determinations" from which all theorems of the new discipline should follow according to the "principles of general logic."[42]

The discipline starts from the *domain of thought* ("Denkbereich") 1^1 consisting of "specified" elements $A, B, C, \ldots$ which are disjoint to each other and different from nothing (0). The adjunctive connection of these elements can be represented as an "identical sum", in Schröder's symbolism:

$$1^1 = A + B + C + \ldots = \sum_i i \,.$$

Any two "general" elements i and j of the first domain of thought can be symbolized as a pair of elements $i : j$ standing in a certain relation to each other. i and j may be equal to each other. The totality of these pairs of elements forms the

[42] Schröder *1895b*, 145; cf. *1895a*, 17–42.

second domain of thought:

$$1^2 = \sum_{ij} i : j \ .$$

The general form of a *binary relative* a is represented as the logical sum of the pairs of elements in the second domain of thought:

$$a = \sum_{ij} a_{ij}(i : j) \ ,$$

with the indices i and j ranging independently of each other over the elements of the domain of thought 1^1 and the coefficients being restricted to the values 1 and 0.

Subsumption is the basic logical operation, as in the class and propositional calculus. In class logic $a \Subset b$ means "a is subordinated or identical with b" (cf. Schröder *1890a*, 169–170.) This can also be interpreted as the subsumption of propositions: "If a is valid, then b is valid" (*1895b*, 146). *Equality* (*identity*, "complete sameness" [*völlige Einerleiheit*]) is reduced to subsumption:[43]

$$(a = b) = (a \Subset b)(b \Subset a) \ .$$

As in volumes 1 and 2, Schröder uses the symbols 0 and 1 for representing the value ranges of "nothing" and "all".[44] Even *adjunction* ("+") and *conjunction* ("·") is taken over from the earlier work, as is negation, but now marked by the overstroke ("$\bar{a}$") common at that time, not by the negation stroke in postposition ("$a_{\shortmid}$"). Schröder takes over from the propositional calculus the sign for the product of propositions ("$\prod_i a_i$ means [...] that the proposition a_i is valid for *all* elements i") and the one for the sum of propositions ("$\sum_i a_i$ means that a_i is valid for *some* i, with other words, that there is *at least one* i for which it holds") (Schröder *1895b*, 146). The following signs are added in the calculus of relatives:

- The relative module 1' ("Einsap") stands for the set of all individual *self*-relatives $[(i = j)(i : j)]$ of the domain of thought 1^2. The relative module 0' ("Nullap") represents the set of all individual *alio*-relatives $[(i \neq j)(i : j)]$ of the same domain of thought.

- The *relative multiplication* $a; b$ ("a of b") stands for the *composition* of two relatives. "*Amans benefactoris*", e. g., can be interpreted as "lover *of* a benefactor (of –)."

- *Relative addition* $a \,\dot{+}\, b$ ("a piu b")[45] is defined by relative multiplication: $a \,\dot{+}\, b = \overline{\bar{a}; \bar{b}}$. The expression thus stands for something that is not a non-a

[43]It is remarkable that in the paper on pasigraphy Schröder counts identity "=", not subsumption, among the five fundamental concepts of general logic (*1898a*, 150).

[44]For distinguishing "identical 0" and "identical 1" from the respective numerals Schröder later (*1898a*, 152) chose to indicate numerals with a dot on top: $\dot{0}, \dot{1}$.

[45]"Piu" is taken from the Italian word for "+".

of a non-b, i.e., it is an a that is something of everything, except of b. In his pasigraphy paper (*1898a*, 153) Schröder gives the following example: If t means "factor of ...", then $t \dotplus 0$ means, given a restriction of the domain of thought on natural numbers, something that is factor of every number, i.e., the and only the number 1.

- The *converse* $\breve{a}$ of a relative a is that binary relative that comprehends all those binary relatives that are converse to the relative contained in a. The cause of something, e.g., is the converse of the effect of something.

After having investigated the independence of these concepts, Schröder later arrives at a system of "five categories or fundamental concepts of general logic including arithmetic" with seven symbols (Schröder *1898a*, 150):

$$\left|\begin{array}{c|c|c|c|c} = & \cdot & {}^{-} & {}^{\smile} & ; \\ 1' & \Pi & & & \end{array}\right|$$

This system of basic concepts can be minimized by reducing conversion to relative multiplication ("Nachschrift" to Schröder *1898a*, 162):

$$(i \mathrel{\underline{\in}} a; j) = (j \mathrel{\underline{\in}} \breve{a}; i)$$

9.2 Applications

In his "Note über die Algebra der binären Relative" Schröder had already illustrated the power of his method by applying it to an example from the mathematics discussed at his time. He symbolized those propositions from Richard Dedekind's theory of chains[46] that served for the foundation of complete induction (theorem 59). The theorem of complete induction reads in the language of the logic of relatives:[47]

$$\{a; (a_0; b)c + b \mathrel{\underline{\in}} c\} \mathrel{\underline{\in}} (a_0; b \mathrel{\underline{\in}} c),$$

with "$a_0; b$" ("a-chain of b"') standing for Dedekind's expression "chain of b (in respect to a)". Schröder sees the advantage of his presentation in extending the scope of Dedekind's theorems going beyond the validity for definite mappings and "systems", now covering all binary relatives. In addition Schröder shows that the theory of chains can be simplified at some places using his symbolism.

The aim of this example is evident. The possibilities of the new symbolism as a tool for an alternative presentation of (here, mathematical) connections, thereby demonstrating its advantages in respect to brevity, clarity and simplicity of proofs. This is also exactly the aim Schröder pursued in his two papers "Ueber zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze" (*1898c*) and "Die selbständige

[46]This theory is formulated in Dedekind's *Was sind und was sollen die Zahlen?* (*1888*), Schröder used the second edition of 1893.

[47]Schröder *1895a*, 355; *1895b*, 156.

Definition der Mächtigkeiten 0, 1, 2, 3 und die explizite Gleichzahligkeitsbedingung" (*1898d*), published in the *Nova Acta Leopoldina*, where he applies the logic of relatives to Cantorian set theory. Schröder lectured on the results of the first paper in a talk "Ueber G. Cantorsche Sätze" given at the convention of the *Deutsche Naturforscher und Ärzte* on 24 September 1894 in Frankfurt a. M.[48] In the first paragraph of the published version (*1898c*) Schröder compares the definition of infinity (I) in Dedekind's *Was sind und was sollen die Zahlen?* with the one given by Peirce three years before Dedekind in the—according to Schröder's judgement—abstruse treatise "On the Algebra of Logic. A Contribution to the Philosophy of Notation" (Peirce *1885*). After reformulating these definitions in the language of the logic of relatives, it becomes evident that these definitions do not formally coincide, but that they can be translated into each other. Peirce's definition proves to be shorter and simpler. Schröder obviously tried to show with this application that his symbolism could serve as a criterion for simplicity and economy of mathematical theorems and definitions.

Schröder's considerations in the following sections are of greater systematic significance. There he discusses Cantor's propositions A to E from the first paper of his "Beiträge zur Begründung der transfiniten Mengenlehre" (Cantor *1895*, 484). Above all, Schröder's proof of the equivalence theorem B caused sensation (Schröder *1898c*, § 4, 336–344). It reads in Cantor's formulation (Cantor *1895*, 484):

> If two sets M and N are of such a kind that M is equivalent with a part N_1 of N and N with a part M_1 of M, then M and N are equivalent, as well.

If "a" and "b" stand for sets, "a_1" and "b_1" for parts (subsets) of a and b, then "$a \sim b$" says that the sets a and b have the same power. The index form "$a \underset{z}{\sim} b$" means: "relative z *maps* the set a *one-to-one* to the set b" (Schröder *1898c*, 309). In Schröder's pasigraphic transcription the equivalence theorem runs:

$$(a \underset{x}{\sim} b_1 \Subset b \underset{y}{\sim} a_1 \Subset a) \Subset (a \sim b \sim b_1 \sim a_1 \sim a)\ .$$

Almost at the same time, in winter 1896/97, Felix Bernstein found also a proof of the equivalence theorem, which was first published by Émile Borel (Borel *1898*, 103–107). The theorem remained connected with the names of Schröder and Bernstein,[49] until Alwin Reinhold Korselt published evidence, found as early as 1902, that Schröder's proof was based on an implicit and incorrect presupposition (Korselt *1911*). In May 1902 Schröder admitted this fault, and states in a letter to Korselt that he "leaves the honor of having proved G. Cantor's theorem to Mr F. Bernstein alone."[50]

[48]The talk is listed in the congress proceedings (Wangerin/Taschenberg, eds., *1897*, 43). A short abstract appeared in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Schröder *1901c*).

[49]Cf. the few comprehensive contemporary presentations of set theory, such as Schoenflies *1900* (16, note 2) and Hessenberg *1906* (522).

[50]Schröder to Korselt, dated 25 May 1902; quoted according to Korselt *1911*, 295.

In the last paragraph (§ 5) of his paper on Cantorian propositions, Schröder discusses further results from Cantor's theory of ordered sets. In his résumé (189), he admits that in the algebra of logic it is not easy to reach a result for something a priori evident; on the other hand it proves to be possible—here Schröder takes up an idea from a correspondence of Aurel Voss—

> to provide far more insights that are accessible for verbal thinking and for which the hitherto common mathematical forms of expression seem to be not sufficient for gaining them.[51]

"With this, the new *Peirceian* discipline," Schröder writes, "has had [...] its opportunity to stand a little acid test. G. *Cantor*'s theory, as well." Schröder was sure that Cantorian set theory could be completely "presented pasigraphically with the designation capital of our algebraic logic" (*1898c*, 361).

Also in letters to Felix Klein, to whom he had offered several papers with applications of the algebra of relatives to set theory for publication in the *Mathematische Annalen*, Schröder made some propaganda for his tools, stressing especially the short time period he needed to develop a notational system for set theory standing on the same level as Cantor's own, if it was not superior to it. He writes:

> Mr G. Cantor—I'm far from comparing my modest talents with his genius—was occupied with the topic of his research for 20 years; although I always thought it a desideratum to go further into it, I found the time to do so only after the publication of his last paper in the *Annalen* which was published in November last year. Having now, in a certain sense, caught up with him in the shortest period of time, it might justified to compare my instrument with a "bicycle", with which the most sprightly pedestrian can be caught up quickly (whether it also applies for clearing the way is another question which can only be decided by the future).[52]

In the second paper published in the *Nova Acta Leopoldina* (*1898d*). Schröder provides a further example of his attempts to deal with concepts of set theory using the means of the algebra of logic. There Schröder gives a logical definition of the concept of number (*Anzahl*), especially of the powers 0, 1, 2 und 3 of sets, and this both, "independently", i.e., without presupposing the definition of the respective lower power (365–369), and "recursively" (*rekurrierend*), i. e., recurring to the use of the definition of the successor relation ("set a contains exactly one element more than set b").

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[51] Schröder *1898c*, 361.

[52] Schröder's letter to Felix Klein, dated Karlsruhe, 16 March 1896, Klein Papers, *Staats- und Universitätsbibliothek Göttingen*, Cod. Ms. F. Klein 11. Schröder's correspondence with Felix Klein, editor of the *Mathematische Annalen*, and Paul Carus, editor of *The Monist*, are published in Peckhaus *1990/91*.

In the pasigraphy paper (*1898a*) Schröder also illustrates the "implications of our new logic of relatives," by presenting pasigraphically some of the most important basic concepts of arithmetic: the concept of set, the numbers 0, 1 und 2, the relations of equinumerosity and power equality, the finiteness, the actual infinite, the concepts of function and substitution, the concept of order[53] as well as the relation greater than, the successor relation, factor relations, and the notion of a prime (Schröder *1898a*, 155–159). In the papers mentioned Schröder does not aim at a systematic construction of arithmetic or set theory. He is interested in a clear demonstration of the possibility to represent the basic concepts of arithmetic and set theory with the help of the algebra of relatives. Other examples serve the same goal, e. g. from geometry ("z is a point") and from the domain of human relationships, "which form a not unimportant chapter in the Corpus juris for our students of jurisprudence."[54]

The *Algebra und Logik der Relative* represents the attempt to extend the programme of absolute algebra to a foundational programme for all scientific disciplines that can be formalized or that work with formal means. This extended programme was twofold. It consisted of absolute algebra as general theory of connecting operations and the logic of relatives as general logical theory. The logic of relatives provided the notational system, i. e., the formal language which could be applied to various fields given a suitable interpretation of its schematic letters and relative operations. Roger D. Maddux has remarked that the early representatives of the logic of relations, Augustus De Morgan, Charles S. Peirce and Ernst Schröder, were in their relevant work "certainly interested in deducing complicated formulæ from simpler ones, but they were not particularly interested in the axiomatic approach to the calculus of relations," an approach later chosen by Alfred Tarski (Maddux *1991*, 438). This is certainly true, but Schröder's efforts should not be reduced to compiling a logical formulary. The catalogue of formal expressions was a side effect, which occasioned Peirce's criticism of 1911 that Schröder had systematically developed the algebra of relatives, but his development "brought out its glaring defect of involving hundreds of merely formal theorems without any significance, and some of them quite difficult."[55] This reformulation was for Schröder only a means to an end however. His aims reached further. His translation of Dedekind's theory of chains, e. g., abetted him in reaching the "*final* goal: to come to a strictly logical *definition* of the *relative* concept '*number of –*' ['*Anzahl von –*'] from which all propositions referring to this concept

[53]Schröder also reported on the concept of order at the International Congress of Philosophy in Paris 1900 (Schröder *1901b*).

[54]Schröder *1898a*, 159. The analysis of human relationships was not unusual in logical discussions of that time. Alexander Macfarlane devotes numerous studies to this subject (e. g., *1879*, *1880*, *1881*) trying to develop a logical "calculus of relationship" without using, however, the means of a logic of relations. Leibniz had already applied his combinatorics to this subject (*1666*, Probl. II, Nos. 16 et seq.).

[55]Peirce writes these words in the historical survey of his article "Relatives" in the second edition of the *Dictionary of Philosophy and Psychology*. Quoted from Peirce *1933*, 404–409. Cf. Maddux *1991*, 422–423.

can be deduced purely deductively" (Schröder *1895a*, 349–350). So Schröder's system comes close, at least in its objectives, to Frege's logicism, although it is commonly regarded as an antipode.

9.3 Logic of Relatives and Pasigraphy

The theory of relatives grew into the main tool for achieving step (4) in the programme of an absolute algebra, i. e., to decide, "what geometrical, physical, or generally reasonable meaning these numbers and operations can have, what real substratum they can be given" (Schröder *1873*, 294). In 1891 the first part of the second volume of the *Vorlesungen* appeared, devoted to the calculus of propositions. Schröder planned to insert considerations on the calculus of relatives into the second part. In the course of his deeper and deeper studies of Peirce's logic, the algebra and logic of relatives grew beyond the planned size. Schröder reported on it in a text that separates the two parts of volume two. He writes that after the completion of the first half of volume two in June 1891 he had hoped to publish the second half with the logic of relatives in the autumn of the same year, but (Schröder *1905*, XXIV):

> It is true, seldom in my life an estimation of mine failed to the same extent as then, when I judged the extension and the seriousness of the gaps in my manuscript. This was due to the fact that the only writing that seemed to be useful, Mr Peirce's paper on relatives, that became indeed the main basis of my volume three, has only a size of 18 pages in print (that could be printed on half the number of my pages), and that I thought that I could get away with a largely reproducing report. I became aware of the enormous significance of this paper when I worked at it in detail.

This discovery led to a real blockade. This can be inferred from a letter he wrote to Peirce's student Christine Ladd-Franklin in September 1893:[56]

> As to my book I am ignorant whether I already had told you, that the chapter on relatives develops or dwells into a *third* volume, over half of which would now be ready for print—that by the by will cost me a very pinching pecuniary sacrifice (of over 1000 Mark certainly). From subjective reasons I cannot attach myself on the easier work of ending and polishing the second vol. before being at least roughly throughout the third one. The difficulties here to overcome are such, however, that as compared with it, my vols. 1 and 2 will prove a mere *child's play*. Still I venture to hope that both will be accomplished next year.

[56] Schröder to Christine Ladd-Franklin, dated Karlsruhe, 17 September 1893, Ladd-Franklin Papers, Columbia University Library, Butler Library, New York, Box 5.

With Peirce's logic of relatives he had the tool in his hands that appeared to be capable of bringing him close to the realization of his dream of a scientific universal language. Schröder took the advantage at the First International Congress of Mathematicians at Zurich in 1898 to give a report on his ideas of a universal language and to propagate a new discipline which he called, following some precursors in the 17th century, "pasigraphy" or "general script" (Schröder *1898a*, English *1898b*). About the aim of the new discipline he informed his fellow mathematicians as follows (Schröder *1898b*, 45):

> [...] the aim of this novel branch of Science is nothing less than the ultimate establishment of a scientific Language, entirely free from national peculiarities, and through its very construction conveying the foundation of exact and true philosophy.

If this pasigraphy is applied to other branches of science the problem arises of (*ibid.*, 46)

> expressing *all* the notions which it comprises, adequately and in the concisest possible way, through a minimum of *primitive notions*, say "categories," by means of purely logical operations of general applicability, thus remaining the same for every branch of science and being subject to the laws of ordinary Logic, but which latter will present themselves in the shape of a "calculus ratiocinator." For the categories and the operations of this "lingua characteristica" or "scriptura universalis" easy signs and simple symbols, such as letters, are to be employed, and—unlike the "words" of common language—they are to be used with absolute consistency (with perfect "Konsequenz," as we Germans say, or mathematical strictness, "Strenge").

Even in pure mathematics (but, of course, not in formal algebra), the pasigraphy can be applied. Therefore pure mathematics appears, according to Schröder's opinion, "only one branch of general Logic" (*1898b*, 46). This position can be supported with the example of arithmetic which, in pasigraphical respects (*ibid.*, 47),

> can do without any peculiar categories or primitive notions—those of general logic sufficing to compose all its notions (such as multitude, number, finiteness, limes, function, Abbildung or one-to-one correspondence, addition, etc.).

Schröder's pasigraphy is indeed quite close to Leibniz's *characteristica universalis*, the algebra and logic of relatives representing the syntactical part, the *calculus ratiocinator*.

9.4 Schröder, Frege and Leibniz

Jean van Heijenoort's distinction (*1967b*) between logic as calculus, as exemplified in the work of George Boole, and logic as language, as paradigmatically represented by Gottlob Frege's logicism, builds on Frege's reference to the Leibnizian distinction of *calculus ratiocinator* and *lingua characterica* (better "characteristica universalis").[57] Van Heijenoort, thus, interprets this distinction as standing for two different approaches to logic. According to van Heijenoort it is the *lingua characterica* aspect that constitutes the universality of logic. And this universality is regarded as typical to the Fregean approach. It is represented by quantification theory, which provides a vocabulary that the propositional calculus lacks. Whereas in the Boolean propositional calculus the proposition is reduced to a mere truth value, in the Fregean logic (van Heijenoort *1967b*, 325),

> with the introduction of predicate letters, variables, and quantifiers, the proposition becomes articulated and can express meaning. The new notation allows the symbolic rewriting of whole tracts of scientific knowledge, perhaps all of it, a task that is altogether beyond the reach of the propositional calculus. We now have a *lingua* not simply a calculus.

For Leibniz, however, *calculus ratiocinator* and *characteristica universalis* are two constituents of the prior programme of a general science. They clearly support each other.[58] Leibniz was convinced that all human thoughts could be reduced to a few, so to speak, primitive thoughts. If it were possible to map these primitive thoughts unambiguously to a list of characters, then either no one using these characters in reasoning and writing would ever err, or he or she would recognize these errors with the help of most simple checks. For Leibniz the *ars characteristica* was therefore a true organon or means of a general science which encloses all of human reasoning. A possible first step on the way into such general characteristic would be to use arbitrarily chosen letters according to the model of mathematics. This notation allows "calculating with concepts" according to sets of rules, each of them forming a *calculus ratiocinator*.

The *characteristica universalis* presupposes a complete list of simple thoughts. These simple thoughts can then be unambiguously designated with characters. This designation programme is the more easily realizable the smaller the list of simple thoughts is. A complete realization of the programme may be utopian, but a partial realization may be seen in the symbolic system of mathematics. The restrictions are practical restrictions due to the limited powers of man, but no restrictions in principle. The *calculus ratiocinator* serves for mechanically deducing all possible truths from the list of simple thoughts. It forms the syntactic part of the universal *lingua rationalis*. The characteristics gives the semantical part.

[57]There is no "lingua characterica" in Leibniz's works. Leibniz spoke of "lingua generalis", "lingua universalis", "lingua rationalis", "lingua philosophica", the terms all meaning basically the same. He also introduces the terms "characteristica" viz. "characteristica universalis" representing his general theory of signs. Frege obviously took the term "lingua characterica" from Friedrich Adolf Trendelenburg who uses the expression "lingua characterica universalis" (*1857*, reprinted 1867, 6). Cf. Patzig *1976*, 10, n. 8; Peckhaus *1997*, 178–181; on Trendelenburg's influence in the history of logic cf. *ibid.*, ch. 4, Vilkko *2002*, ch. 4.

[58]Cf., e. g., Leibniz's short tract "Fundamenta calculi ratiocinatoris," written presumably during his stay in Vienna between May 1688 and February 1689, Leibniz *1999*, no. 192.

Going back to the work of Frege and Schröder, it becomes immediately evident that both authors relate their logical systems to the tradition of Leibniz, both put their logical systems into the logical tradition of Leibniz, and both rely heavily on Adolf Trendelenburg's account of Leibniz's theory of signs (Trendelenburg *1857*, cf. above, n. 4). Like Leibniz, both regard their logics as combinations of calculus and universal characteristics. Schröder remarks in the opening of the *Der Operationskreis des Logikkalkuls* that Leibniz's ideal of a calculus had been realized by Boole, although this had not been sufficiently recognized after 25 years (Schröder *1877a*). Frege goes into more details in his *Begriffsschrift*. He writes (*1879*, V; Beaney *1997*, 50):

> Leibniz too recognized—perhaps overestimated—the advantages of an appropriate symbolism. His conception of a universal characteristic, a *calculus philosophicus* or *ratiocinator*, was too grandiose for the attempt to realize it to go further than the bare preliminaries.

Frege is right in stressing the utopian character of the Leibnizian idea, but he nevertheless suggests trying to realize it, at least in parts (*1879*, VI; Beaney *1997*, 50):

> But even if this great aim cannot be achieved at the first attempt, one need not despair of a slow, step by step approach. If the problem in its full generality appears insoluble, it has to be limited provisionally; it can then, perhaps, be dealt with by advancing gradually. Arithmetical, geometrical and chemical symbols can be regarded as realizations of the Leibnizian conception in particular fields. The *Begriffsschrift* offered here adds a new one to these—indeed, the one located in the middle, adjoining all the others.

The fact that both saw themselves in the Leibnizian tradition gave rise to a controversy. When Schröder reviewed Frege's *Begriffsschrift* in detail, he took the opportunity to advertise his own algebraic logic in the Boolean tradition as the better alternative.[59] Schröder starts his review as follows (Schröder *1880*, 81):

> This really strange publication—obviously the original work of an ambitious thinker of purely scientific direction of thought—follows a tendency which is of course highly sympathetic for the reviewer who also

[59]According to Hans Sluga's influential assessment, this review is infamous for its hidden polemics, which was said to have been to a great extent responsible for Frege's ineffectiveness in his time: "It tore the book apart—in the politest possible way. There was a good deal of hurt vanity in Schröder's words" (Sluga *1980*, 68). A close reading of this review suggests, however, that we should assign Sluga's assessment to the category of logical folklore (cf. Vilkko *1998*). The review is polemical in parts, no doubt, but this style of argument may be ignored as typical of the time in which it was written. If this is done, one becomes aware that the reviewer takes a favorable view of Frege's book, taking advantage, however, of advertising the new subject symbolic logic as such. This made it possible to direct the interest of the readers not only to Frege's logic, but also to the algebraic direction initiated by Boole. So the review was some sort of comparative advertising, and it is clear that Schröder, as an algebraist, had to come to the conclusion that the algebra of logic was the better realization of common goals.

> tried his hand at related subjects. For it promises to step closer towards the Leibnizian ideal of a pasigraphy, which is still far from its realization despite of the great importance attached to it by the ingenious philosopher.

"Pasigraphy" means "general script".[60] Schröder here takes up a notion from the discussion on universal languages in the Baroque period. Schröder continues (*1880*, 82),

> Frege's "concept script" promises too much in its title—strictly speaking: that the contents do not at all conform with it. Instead of tending toward the side of a "general characteristics" it rather tends definitely—maybe unconsciously for the author—toward the side of Leibniz's "*calculus ratiocinator*"; and the little work makes an attempt in this direction that I would call commendable, even if a great deal of what it aims at has already been done by another party, in fact in an undoubtedly more appropriate manner—as I will show.

It is not difficult to guess that this "other side" refers to the Boole–Schröderian logic.

Frege responded to these reproaches not only in the published talk "Ueber den Zweck der Begriffsschrift" (*1882*), but also in a paper entitled "Boole's rechnende Logik und die Begriffsschrift" written about 1880/81. He sent it to several mathematical and philosophical journals, but received only refusals. In this paper Frege also expressed his great respect for Leibniz. "Leibniz had scattered such a wealth of germs of thought, that in this respect hardly anyone can match himself against him" (Frege *1983*, 9). Among the ideas seemingly dead and buried in the works of Leibniz, but that might presumably rise from the ashes some day, Frege counts the idea of a *lingua characterica* most closely connected to a *calculus ratiocinator*. According to Frege, Leibniz saw the main advantage of a language in which the concept is composed of its parts and not the word of its sounds in the practicability of some sort of calculation. Frege stresses that of all the expectations Leibniz had in this respect, this one can be shared with greatest confidence (*ibid.*, 9–10). In his small work *Begriffsschrift*, he had attempted "a reapproach to the Leibnizian idea of a lingua characterica" (*ibid.*, 11). In this project he deals with subjects similar to those Boole had dealt with before him (12), but Boole had tried only to develop a technique which allows logical problems to be solved in a systematic way, similar to an algebra which is a technique for the elimination and calculation of unknowns (13). Contrary to this, Frege had the expression of contents in mind: "The goal of my attempts is a *lingua characterica* first for mathematics, and not a *calculus* restricted to logic" (13). In the beginning, Frege is quite modest in his goals. He is looking for a symbolic language that concerns only mathematics. It should, however, go beyond a simple calculus, i. e., a system of rules stipulating how to go from given propositions to other propositions without

[60] For Schröder's conception of pasigraphy cf. Peckhaus *1990/91*.

changing truth values. Frege calls such a system of rules a purely logical calculus. He regards mathematical operations like addition or division as operations completely reducible to logical operations. Furthermore, logic serves in constituting the concepts of such a mathematics. Frege thus becomes the founder of logicism, i. e. the direction in the philosophy of mathematics that assumes that all concepts of mathematics (i. e. arithmetic and analysis, but not geometry) can be reduced to purely logical concepts. If this programme had succeeded, i. e., if all mathematical theorems had really been expressible exclusively with the help of logical expressions, an important aspect of Leibniz's *characteristica universalis* would indeed have been achieved: the demand to keep the number of means for expression as small as possible.

As with Frege, in his main logical work, the *Vorlesungen über die Algebra der Logik* (vol. 1, *1890a*), Schröder identifies as the main goal of Leibniz that of finding an adequate and general designation of the nature of concepts, in such a way that the analysis into their elements would be possible. Then they could be treated by calculation (*Schröder 1890a*, 95). Schröder correctly sees the significance of the *characteristica universalis* and the "ideal of a scientific classification and systematic designation of everything that can be designated" (*ibid.*). However, he stresses that a realization of this ideal would presuppose the complete knowledge of the fundamental operations and the laws according to which they can be applied. Logic has to prepare the ground (*ibid.*). This remark motivates Schröder's focus on the calculus, i. e. the calculation with concepts; but he integrates his logic into the comprehensive semantics of his "absolute algebra", thus also realizing an important aspect of the *characterica universalis* (cf. *Peckhaus 1997*, 254–283).

It should be noted that both, Frege and Schröder, criticized the ensuing system as a mere *calculus ratiocinator*. Each claimed, on the other hand, that his own system was the better realization of the Leibnizian idea of a *characteristica universalis*. Both accepted that a universal language would require both elements, and both aimed at such a language. In requiring an external semantics, Schröder's algebraic attitude seems to be closer to the original Leibnizian idea of a *lingua rationalis*.

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PEIRCE'S LOGIC

Risto Hilpinen

1 INTRODUCTION

The emergence of formal or mathematical logic in the 19th and the early 20th century was the outcome of two parallel and partly independent lines of development whose key figures were Charles S. Peirce and Gottlob Frege. In his Preface to the English translation of Louis Couturat's *L'Algebrè de la logique*, Philip E. B. Jourdain [1914] characterized these developments in terms of G. W. Leibniz's distinction between a *lingua characterica* (*characteristica universalis*), a universal language of thought, and a *calculus ratiocinator*, a calculus of reasoning. According to Jourdain [1914, p. viii],

> The *calculus ratiocinator* aspect of symbolic logic was developed by Boole, de Morgan, Jevons, Venn, C. S. Peirce, Schröder, Mrs. Ladd-Franklin, and others; the *lingua characteristica* aspect was developed by Frege, Peano, and Russell. Of course there is no hard and fast boundary-line between the domains of these two parties. Thus Peirce and Schröder early began to work at the foundations of arithmetic with the help of the calculus of relations; and thus they did not consider the logical calculus merely as an interesting branch of algebra. Then Peano paid particular attention to the calculative aspect of his symbolism. Frege has remarked that his own symbolism is meant to be a *calculus ratiocinator* as well as a *lingua characteristica*, but the using of Frege's symbolism as a calculus would be rather like using a three-legged stand-camera for what is called "snap-shot" photography...

Jean van Heijenoort [1967a] and Jaakko Hintikka [1988], [1997, pp. 14–15] have generalized Leibniz's distinction into a distinction between two contrasting conceptions about language and its relation to the world: language as the universal medium (the universalist tradition), and the model-theoretic view of language, and characterized Frege as a representative of the former position and Peirce, Ernst Schröder, and their followers as holding the model-theoretic view. According to the universalist conception, the interpretation of our language is given or fixed in advance; for example, in Frege's system the quantifiers binding individual variables are regarded as ranging over all objects, not just the objects of some selected "universe of discourse" which may vary from context to context (cf. [Goldfarb, 1979,

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pp. 351–2]). Bertrand Russell's assertion that "logic is concerned with the real world just as truly as zoology, though with its more general and abstract features" [1919, p. 169] is an expression of this view. According to the model-theoretic or "calculus" tradition, on the other hand, the interpretation of a language can be varied, and individual terms and variables range over a "universe of discourse" which need not have any independent ontological import: the universe of discourse comprises only what the language users agree to consider in a certain context ([van Heijenoort, 1967a, p. 325], [Hintikka, 1988, pp. 2–3]).

Most of Peirce's work in logic, for example, his early work on Boolean algebra and the logic of relations, falls clearly in the calculus tradition. The same holds for his later work, for example, the logic of existential graphs (discussed in section 5 below). Peirce himself observed that the system of existential graphs was "not intended to serve as a universal language for mathematicians or other reasoners, like that of Peano" (CP 4.424; see the note attached to the bibliography). He also denied that it was intended "as calculus, or apparatus by which conclusions can be reached and problems solved with greater facility than by more familiar systems of expression" The latter statement is not inconsistent with the model-theoretic conception of language (or the "calculus" view in the van Heijenoort-Hintikka sense): in the quoted passage Peirce simply notes that the main purpose of his logical work is the analysis of logical inference rather than practical facilitation of reasoning (cf. [Hintikka, 1988, p. 16]; [Shin, 2002, pp. 164–65]). According to Peirce, "a logical universe is, no doubt, a collection of *logical* subjects, but not necessarily of meta-physical Subjects, or 'substances'." (CP 4.546.) Here Peirce is following De Morgan [1846] and Boole [1854/1958]. De Morgan [1846] introduced the expression "universe of discourse" to refer to the universe of logical subjects, and according to Boole, every discourse involves "an assumed or expressed limit within which the subjects of its operation are confined," in other words, a universe of discourse which is "in the strictest sense the ultimate subject of the discourse." [Boole, 1854/1958, p. 42]. In the same way, when Peirce says that "the Universe is a Subject of every proposition," although "in another sense one assertion may have several individual subjects" (CP 4.552, n. 1; 4.553, n. 2), he is referring to a universe of discourse which may be more or less limited (CP 2.517), and not to Frege's (absolute) totality of all objects.

2 THE DEVELOPMENT OF QUANTIFICATION THEORY

Gottlob Frege and Charles Peirce were the first logicians who construed quantifiers as variable binding operators. They invented quantification theory independently of each other, at approximately the same time. Frege's monograph *Begriffschrift* appeared in 1879 [Frege, 1879/1977], and Peirce published his first papers on quantification theory a few years later [Peirce, 1883c; Peirce, 1885], apparently without any knowledge of Frege's work. In the *Begriffschrift* Frege presented a system of second-order predicate logic which included the first complete formalization first-order logic and propositional logic. *Begriffschrift* was Frege's first publication in logic, and the system presented there seems to have issued from its author's mind in full-fledged form, without having been being significantly influenced by the work of the other logicians of the time. (For Frege's philosophical background, see Sluga [1980, Ch. 2].) Peirce's papers on quantification theory, on the other hand, were an outcome of some twenty years' work on the algebra of logic.

In one of his first papers in logic, 'On an Improvement in Boole's Calculus of Logic' [1867], Peirce improved George Boole's logical algebra by regarding logical sum, x + y, where x and y can be interpreted as classes or propositions, as an inclusive disjunction (or union); in this way Boolean algebra received its contemporary form (CP 3.3-6, WCSP 2, 12-14). Stanley Jevons [1864] had made a similar proposal a few years earlier. In Boole's original system, x + y was well-defined only if x and y were mutually exclusive classes or propositions [Boole, 1854/1958, pp. 55–57]; [Kneale and Kneale, 1962/1984, pp. 410–11]. In his 1870 paper 'Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic,' influenced and inspired by Boole's and Augustus De Morgan's work, Peirce developed a logical algebra of relations. Peirce [1870] called relational expressions "relative terms", briefly "relatives", but later thought that De Morgan's [1864] expression "logic of relations" was preferable (CP 3.574). According to Peirce, a two-place relational expression ("a dual relative term"), such as 'lover', is a common name denoting an ordered pair of objects. He called such a pair "an individual relative", and took a "general relative" to be the aggregate or logical sum of such individual relatives (CP 3.328; WCSP 3, 452-453); thus Peirce in effect identified two-place relations with sets of ordered pairs, and relations generally with sets of ordered n-tuples of singular objects. The converse of a two-place relation (dual relative) is obtained by reversing the order of the members of each pair; thus the converse of 'lover' is 'loved' (CP 3.330). The basic Boolean operations on absolute terms, product (intersection), sum (union), and complementation, can be applied to relations in the same way as to non-relational terms: for example, the product of the relations *lover* (L) and *benefactor* (B) consists of all pairs of individuals such that the first individual is a lover and a benefactor of the second. Peirce called these operations non-relative or internal multiplication and addition (product and sum), and expressed them by ',' and '+'; thus, if 'L' denotes the relation of being a lover, and 'B' means 'benefactor', 'L,B' means a lover and benefactor (of someone), and 'L+B' denotes any pair of individuals such that the first individual is a lover or a benefactor of the second. Of greater interest are in this context the concepts of *relative* product and sum. Peirce expressed the former by concatenation and the latter by a dagger sign, '†'. The relative product of L and B, 'LB' ('lover of a benefactor'), denotes all pairs of individuals such that the first individual is a lover of a benefactor of the second individual, and 'L†B' means a lover of everything but benefactors (CP 3.332). Relative products are familiar from mathematics (for example, a composition of functions is a relative product) as well as everyday life. De Morgan, who had introduced the concept of relative product, used family relations as examples: for example, the relation of being an uncle is a relative product of the relations 'brother' and 'parent' [De Morgan, 1966, p. 225]. Peirce's work on the logic of relations was an extension of De Morgan's work, and he improved De Morgan's somewhat inconvenient notation. Peirce [1870] used the sign '—<' as a sign of class inclusion (CP 3.66) and for conditional or "hypothetical" propositions (Peirce [1880a; 1885]; CP 3.175, 3.375). Below I shall use the arrow '$\rightarrow$' in the latter sense.

The definitions of relative product and sum require a quantifier: for example, an individual x is a lover of a benefactor of another individual y means that there is some z such that x loves z and z is a benefactor of y. Quantifiers appeared first in their contemporary form in 'Note B' included in *Studies in Logic*, a collection papers on logic edited by Peirce [1883a]. (CP 3.328-358, WCSP 4, 453–466.) Peirce introduced quantifiers as follows: Let $a_1, a_2, a_3, \ldots$ be all the individual objects in the universe, and let the *value* of general term F for an individual a_i be

$$[F]_i = 1 \text{ if and only if } a_i \text{ is } F, \text{ othrwise } [F]_i = 0.$$

(The terminology used here differs in inessential ways from Peirce's terminology.) In the same way, the value of a relative term (relation) L (for example, lover) for a pair of individuals $(a_i, a_j)$is

$$[L]_{ij} = 1 \text{ if and only if } a_i \text{ loves } a_j, \text{ and}$$
$$[L]_{ij} = 0 \text{ in the opposite case.}$$

It is clear that the statement that some individual is F is true (about the universe of discourse in question) if and only if

$$(1) \quad \sum_i [F]_i > 0,$$

where $\sum$ has its usual arithmetical meaning. In the same way, the universal proposition 'Everything is F' is logically equivalent to the arithmetical statement

$$(2) \quad \prod_i [F]_i > 0,$$

where $\prod$ has its usual arithmetical meaning. The proposition that something loves something is equivalent to

$$(3) \quad \sum_i \sum_j [L]_{ij} > 0,$$

and

$$(4) \quad \prod_i \sum_j [L]_{ij} > 0$$

means that everything is a lover of something.

In general, Peirce observed,

> Any proposition whatever is equivalent to saying that some complexus of aggregates [sums] and products of numerical coefficients is greater than zero. (CP 3.351.)

Consequently we can "naturally omit, in writing the inequalities, the > 0 which terminates them all" (ibid.); the resulting abbreviated forms of (3) and (4) are

$$(5) \quad \sum_i \sum_j [L]_{ij}$$

and

$$(6) \quad \prod_i \sum_j [L]_{ij}.$$

Here the brackets have lost their original function of assigning values to terms: if (5) and (6) are propositions rather than numbers (values of formulas), (1) and (6) can be written simply as

$$(7) \quad \sum_i F_i$$

and

$$(8) \quad \prod_i \sum_j L_{ij}.$$

Many of Peirce's most interesting insights and results in quantification theory and propositional logic appear in his paper [Peirce, 1885]. The paper is divided into three parts, propositional logic ("Non-Relative Logic", CP 3.365–391), first-order logic ("First-Intentional Logic of Relatives", CP 3.392–397), and second-order logic and set theory ("Second-Intentional Logic", CP 3.398-403). In this paper Peirce no longer regarded (5)–(8) as abbreviations of propositions about the values of terms, but as complete propositions. He observed that

> $\sum_i x_i$ and $\prod_i x_i$ are only *similar* to a sum and product; they are not strictly of that nature, because the individuals of the universe may be innumerable. (CP 3.393, WCSP 5, 180.)

In (7) and (8), $\prod$ and $\sum$ have been transformed into quantifiers over individuals (or the indices of individuals) and the subscript indices are the individual variables bound by the quantifiers. Peirce's notation for quantifiers was adopted by other logicians; for example, by Schröder [1895]), Löwenheim [1913], and Skolem [1920], whose work was dependent on Peirce's contributions to the algebra of logic. The signs $\prod$ and $\sum$ were used a quantifier symbols in some elementary logic books until the 1950's. The contemporary counterparts of (7) and (8),

$$(9) \quad \exists x(Fx)$$

and

(10) $\forall x \exists y (Fxy)$

are mere notational variants of Peirce's formulas. (In this respect Peirce, always interested in questions about notation and terminology, was more successful than Frege, whose two-dimensional notation found no followers.) The use of the word 'quantifier' in its present sense also originated with Peirce (1885, CP 3.396; WCSP 5, 183). With the device of quantifiers, various operations on relations can be formally defined; for example,

$$(11)\quad LB_{ik} =_{df.} \sum_j (L_{ij} B_{jk})$$

and

$$(12)\quad (L \dagger B)_{ik} =_{df.} \prod_j (L_{ij} + B_{jk}).$$

In the contemporary notation, the right-hand side of (12) can be written as

$$\forall y(\neg Byz \to Lxy).$$

Peirce [1885] observed that the attempts to introduce the distinction between *some* and *all* (that is, quantifiers) into Boolean algebra were "more or less complete failures until Mr. Mitchell showed how it was to be effected." (CP 3.393, WCSP 5, 178.) Peirce was referring to the work of his student O. H. Mitchell [1883]. However, Mitchell did not construe quantifiers are variable binding operators, but used them as subscripts attached to a term (that is, as indices) to show whether a given term (predicate) holds for some or for all of the members of a given universe of discourse. By attaching more than one such index to a predicate it is possible to refer simultaneously to more than one universe of discourse or more than one "dimension" of the logical universe, for example, state that a predicate F holds for every person at some time [Mitchell, 1883, pp. 87–88], [Dipert, 1994, pp. 530–32]. Mitchell's device was not helpful for studying the logical dependence relations among propositions with complex and nested quantifier phrases and relational predicates.

Peirce did not present quantification theory as an axiomatic system, but he formulated some inference rules for quantifiers, for example, the principles which make it possible to transform any formula into a prenex normal form, that is, move the quantifiers to the left (outside the scopes) of the propositional (Boolean) connectives. (Instead of Peirce's notation, I shall use below $\wedge$ as the sign of a logical product (conjunction), $\vee$ as the sign of a logical sum (disjunction), $\to$ for a conditional, and $\leftrightarrow$ for a biconditional.)

$$(13)\quad \prod_i F_i \wedge \prod_j F_j \leftrightarrow \prod_i \prod_j (F_i \wedge F_j)$$

(14) $\sum_i F_i \wedge \prod_j F_j \leftrightarrow \sum_i \prod_j (F_i \wedge F_j)$

(15) $\sum_i F_i \wedge \sum_j F_j \quad \leftrightarrow \sum_i \sum_j (F_i \wedge F_j)$

Peirce called the string of quantifiers at the beginning of a prenex formula the "quantifying part" of the formula or simply the "Quantifier" (CP 3.396, WCSP 5, 183), and the rest of the formula its "Boolian" part. He observed that if the quantifiers are of the same type (both universal or both existential), their order is irrelevant:

(16) $\prod_i \prod_j F_{ij} \leftrightarrow \prod_j \prod_i F_{ij}$

(17) $\sum_i \sum_j F_{ij} \leftrightarrow \sum_j \sum_i F_{ij}$

(18) $\sum_i \prod_j (F_i \wedge F_j) \leftrightarrow \prod_j \sum_i (F_i \wedge F_j)$

However, an exchange rule analogous to (18) does not hold when "i and j are not separated" (CP 3.396), that is, when F is a relational expression; instead we have only

(19) $\sum_i \prod_j F_{ij} \rightarrow \prod_j \sum_i F_{ij}$;

the converse of formula (19) does not hold.

In the 1885 paper 'On the Algebra of Logic' Peirce also formulated some principles of second-order logic, or as he called it, "Second-Intentional Logic" (CP 3.398-403, WCSP 5, 185-190). He suggested that the concept of identity (I_{ij}) can be defined by a second-order formula

(20) $I_{ij} =_{\text{df.}} \prod_X ((X_i \wedge X_j) \vee (\neg X_i \wedge \neg X_j))$,

which is a form of Leibniz's principle of the identity of indiscernibles, and formulated some principles concerning the "relation of a quality, character, fact, or predicate to its subject" (CP 3.398, WCSP 5, 185), that is, the concept of predication or class membership.

Peirce's algebra of logic influenced the work of Ernst Schröder [1895] and Leopold Löwenheim [1913], and through Schröder and Löwenheim the later developments in the model-theoretic tradition in logic and the theory of relation algebras [Skolem, 1920; Tarski, 1941]. For discussions of these developments, see [Moore, 1987; Hintikka, 1988], and [Maddux, 1991].

3 PROPOSITIONAL LOGIC

The explicit use of truth-values in logic appears for the first time in Peirce's paper [1885], where he outlines a decision procedure for propositional logic [Church, 1956, p. 25, n. 67]. According to Peirce, the validity (logical truth) of propositional ("Boolian") formulas can be determined by investigating whether the formula could possibly express a false proposition. He expresses the truth of a formula X by writing '$(X) = \mathbf{v}$' (for verum), and falsity by '$(X) = \mathbf{f}$' (for falsum), and lays down the following procedure:

> to find out whether a formula is necessarily [i.e., logically] true substitute **f** and **v** for the letters and see whether it can be supposed false by any such assignment of values. (CP 3.387, WCSP 5, 175.)

This method is based on the model-theoretic conception of logical truth as truth under all interpretations (in the present case, assignments of truth-values to propositional letters), and it resembles the method of one-sided (signed) semantic tableaux. Like the tableau methods, it consists in a systematic search for a counter-example to a given formula or inference. For example, to prove the formula

(21) $(P \to Q) \to ((Q \to R) \to (P \to R))$,

Peirce considers the assumption that the formula is false, and shows that an analysis of this assumption leads to a contradiction (CP 3.387). A conditional $X \to Y$ is false if X is true and Y is false, otherwise $X \to Y$ is true. Thus the proof of formula (21) looks as follows:

(21.1) $((P \to Q) \to ((Q \to R) \to (P \to R))) = \mathbf{f}$

(21.2) $(P \to Q) = \mathbf{v}$ (from 1)

(21.3) $((Q \to R) \to (P \to R)) = \mathbf{f}$ (from 1)

(21.4) $(Q \to R) = \mathbf{v}$ (from 3)

(21.5) $(P \to R) = \mathbf{f}$ (from 3)

(21.6) $(P) = \mathbf{v}$ (from 5)

(21.7) $(R) = \mathbf{f}$ (from 5).

Peirce observes that according to (21.6) and (21.7), we get from (21.2) and (21.4)

(21.8) $(\mathbf{v} \to Q) = \mathbf{v}$

and

(21.9) $(Q \to \mathbf{f}) = \mathbf{v}$, which cannot be satisfied together: (21.8) can hold only if

$$(Q) = \mathbf{v},$$

and (21.9) holds only if

$$(Q) = \mathbf{f}.$$

Each of the lines (21.2)–(21.9) is obtained from the preceding lines by the rule that $(X \to Y) = \mathbf{f}$ only if $(X) = \mathbf{v}$ and $(Y) = \mathbf{f}$. Since the attempt to describe a possible "state of things" in which (21) would be false leads to a contradiction, the formula (21) is true under every assignment of values, that is, logically true. It is clear that this procedure resembles a proof by means of a semantic tableau [Hintikka, 1955], [Hodges, 1983, pp. 21–25]. In Smullyan's [1968] tableau method with signed formulas, the first 7 steps of Peirce's proof look as follows:

(21.1′) $\mathbf{f}(P \to Q) \to ((Q \to R) \to (P \to R))$

(21.2′) $\mathbf{t}(P \to Q)$

(21.3′) $\mathbf{f}((Q \to R) \to (P \to R))$

(21.4′) $\mathbf{t}(Q \to R)$

(21.5′) $\mathbf{f}(P \to R)$

(21.6′) $\mathbf{t}P$

(21.7′) $\mathbf{f}R$.

In a tableau proof, Peirce's concluding steps (21.8)–(21.9) are replaced by

(21.8′) $\mathbf{f}P$ or $\mathbf{t}Q$ (from 21.2′)

and

(21.9′) $\mathbf{f}Q$ or $\mathbf{t}R$ (from 21.4′),

representing four alternative branches of the tableau. All four branches close (contain a contradiction); thus no assignment of values to the propositional letters can make (21) true.

Formula (21) is one Peirce's five "icons" (axioms) of the algebra of logic. Four of these icons (called the 1st, the 2nd, the 3rd, and the 5th icon; cf. CP 3.378–384) can be represented as follows:

(I1) $P \to P$

(I2) $(P \to (Q \to R)) \to (Q \to (P \to R))$

(I3) $(P \to Q) \to ((Q \to R) \to (P \to R))$

(I5) $((P \to Q) \to P) \to P$.

Peirce seemed to regard these icons as axiom schemata [Turquette, 1964, p. 98] and the formulas (I1)–(I5) given above as the instances of such schemata. For example, concerning (I5) he observes that "a fifth icon is required for the principle of excluded middle and other propositions connected with it," and notes that the formula (I5) is "one of the simplest formulae of this kind." (CP 3.384.) He also calls the icons "examplars of algebraic proceedings" (CP 3.385). In the proofs based on these icons, Peirce accepted the *modus ponens* rule (the rule of detachment), but he did not distinguish rules of inference and axioms (or axiom schemata) from each other. For example, he observed that according to (I2), we can infer from an instance of the first icon (I1, "the formula of identity"),

(22) $(P \to Q) \to (P \to Q)$,

"the *modus ponens* of hypothetical inference" (CP 3.377),

(23) $P \to ((P \to Q) \to Q)$.

It is clear that the derivation of (23) from (I1) and (I2) requires the application of the inference rule *modus ponens*.

(I2)–(I3) and (I5) are purely implicational formulae, in fact, they constitute a complete system of implicational logic ([Hiż, 1997, p. 264], cf. [Wajsberg, 1937/1967, pp. 287–94]). The character of the fourth icon differs from that of (I1)–(I3) and (I5); its function is to introduce the concept of negation into the system:

> We must ... enlarge the notation so as to introduce negation. We have already seen that if a is true, we can write $x \to a$, whatever x may be. Let b be such that that we can write $b \to x$ whatever x may be. Then b is false. We have here a fourth icon, which gives a new sense to several formulae. (CP, 3.381, WCSP 5, 172.)

Peirce observes here that if b is a false proposition, we can write $b \to x$ for any x. In other words, if f represents a false proposition,

(I4) $f \to P$

holds for any formula P. This is Peirce's "fourth icon". But Peirce's remarks also show that the negation of P can be defined as

(24) $P \to f$,

where f represents any false proposition (in other words, f is a propositional constant for falsity): (24) is true if and only if P is false. He had already adopted this definition of negation in an earlier paper published in 1880 ([Peirce, 1880a]; CP 3.192-193; WCSP 4, 176.) Peirce says that the fourth icon "gives new sense to to several formulae," for example, according to (I2), from

$$P \to (Q \to f)$$

one can infer

$$Q \to (P \to f),$$

in other words, "if from the truth of x the falsity of y follows, then from the truth of y the falsity of x follows." (CP 3.381.) Adding (I4) to the other four icons yields a complete axiomatization of classical propositional logic ([Prior, 1962, p. 303], [Wajsberg, 1937/1967]). (I5) is usually called "Peirce's Law." It distinguishes intuitionistic logic from classical propositional logic: adding Peirce's law to the axioms of the former produces an axiom system for the classical propositional logic ([Beth, 1962, p. 18, 128], [Hiż, 1997, pp. 266–67])

In the discussion of his axiom system for propositional logic (the "icons of logical algebra"), Peirce observed that "the general formulae given above are not convenient in practice," and that "we may dispense with them altogether" (CP 3.387; WCSP 5, 175). He then proceeded to describe the truth-value analysis outlined at the beginning of this section. Peirce understood the value of semantical (or model-theoretic) methods long before the systematic development of model theory. He applied the same methodology to quantification theory; for example, to find out whether two first-order formulas were logically equivalent, he investigated whether they would be false under the same circumstances. (See CP 4.546, 4.580.)

In a paper written in 1880, 'A Boolian Algebra with One Constant,' Peirce observed that "every logical notation hitherto proposed has an unnecessary number of signs," and showed that all truth-functions can be defined by means of a single connective, interpreted as 'neither P nor Q,' that is, the denial of the disjunction of P and Q. (CP 4.12-20). Nowadays it is usually called the NOR-connective and expressed by the sign '$\downarrow$' [Fitting, 1996, p. 14]. The sufficiency of the NOR connective for propositional logic was rediscovered by Sheffer [1913], and it has sometimes been called the "Sheffer stroke." In 'Minute Logic' (1902, CP 4.264) Peirce also showed that $P \downarrow Q$ can be defined by means of another truth-functional connective, $P \uparrow Q$, which means that P and Q are not both true (negation of a conjunction). This connective is nowadays called 'NAN' or 'NAND' [Fitting, 1996, p. 14]. Thus Peirce showed that all truth-functional connectives can be defined by means of NOR or NAN, in other words, each of them is a sufficient constant for propositional logic.

Some 19th century logicians, for example, Stanley Jevons [1870]) and Allan Marquand [1883; 1886], designed and built mechanical logic machines for evaluating the validity of syllogistic inferences. Peirce was interested in these machines [Peirce, 1887], and especially in the work of Marquand, who had studied logic under Peirce in the 1880's at Johns Hopkins University. In a letter to Marquand dated December 30, 1886, Peirce made a proposal which turned out to be important to the development of general purpose computers. He suggested that "it is by no means hopeless make a machine for really difficult mathematical problems. But you would have to proceed step by step. I think electricity would be the best thing to rely on." Then Peirce drew two figures illustrating relay switching circuits; in the first figure, the switches were connected serially, and in the second, in parallel.

Peirce continued:

> Let A, B, C be three keys or other points where the circuit may be open or closed. As in Fig. 1. [with the switches connected in series], there is a circuit only if all are closed; in Fig. 2. [with the switches connected in parallel] there is a circuit if any one is closed. This is like multiplication & addition in Logic. (WCSP 5, 422.)

Peirce understood that logic could be applied to the design of switching circuits and that this could lead to the development of a general logical machine. His letter may have inspired Marquand to design a logic machine based on electromagnetic relay switches [Burks and Burks, 1988a, p. 340; 1988b], but that machine was never constructed.

4 SEMIOTICS, LOGIC, AND PRAGMATISM

Peirce regarded logic as a branch of semiotics, the general theory of signs. Sometimes he used the word 'logic' in a wide sense to mean to the entire area of general semiotics (CP 1.444, 2.93). He divided semiotics into three parts: (i) speculative grammar (*grammatica speculativa*), "the science of the general conditions of signs being signs;" (ii) critical logic (logic in a narrow sense), or the study of "formal conditions of the truth of the symbols," which "treats of the reference of symbols in general to their objects," and (iii) formal rhetoric (also called "speculative rhetoric"), which investigates the "formal conditions of the force of symbols, or their power of appealing to a mind" (CP 1.444, 1.559). According to Peirce, the third area of semiotics is not concerned with psychological questions, but with logical and methodological issues.

The preceding sections have discussed Peirce's contributions to what he called critical logic. Peirce's speculative grammar includes a classification and characterization of signs, which he regarded as a foundation of logic and philosophy. According to Peirce, a sign relation is a triadic relation: it connects a sign (a *representamen*) to its *object*, what the sign represents or stands for, and its *interpretant*, another sign which serves to articulate the meaning of the given sign. For example, the word 'cat' is a sign. Any cat is an object of the sign, and any other sign which refers to cats, for example, the Spanish word 'gato', a cat-picture, or an idea of a cat, is an interpretant of the sign 'cat'. This fundamental triad leads to a complex characterization of signs: First, signs can be classified on the basis of the nature of the sign itself; secondly, on the basis the relationship between a sign and its object, and thirdly, on the basis of its relation to an interpretant. In each case, Peirce divides signs into three types which correspond to his three basic ontological categories, quality or possibility (firstness), existence (secondness), and law (thirdness).

According to the first division, based on the nature of the sign itself, a sign can be a mere quality (a *qualisign*), or an existing (actual) object or event (a *sinsign*), or a sign can be a law, for example, a convention adopted by people (a *legisign*). A legisign is not a singular object, but a general *type*, and is capable of functioning as a sign only through its instances or *tokens* (*replicas*). The distinction between a type and its tokens is based on Peirce's first division of signs.

Peirce's second division gives rise to the familiar trichotomy icon - index - symbol. A sign can refer to its object (or objects) in different ways. If the relationship between a sign and its object depends on similarity (in some respect) or on some relation analogous to similarity so that it makes sense to say that an object fits or does not fit the sign, the sign is called an *icon* (iconic sign). If a sign is a sign of some object on the basis of some existential connection, for example, contiguity or causal dependence, the sign is called an *index* of the object. If something is an object of the sign simply because the latter is regarded or conventionally interpreted as a sign of the object, the sign is called a *symbol.* (See CP 2.247–249, 2.274–2.302.) All linguistic signs are symbols; but some symbols can be regarded as symbolic substitutes for iconic signs and have iconic interpretants; such signs may be called iconic symbols. There are also symbols which "act very much like indices", for example, demonstrative pronouns and proper names. Signs of this kind may be termed indexical symbols. It should be observed that the sign-object relation is relative to what Peirce calls the *ground* of representation: the ground determines how a sign represents its object (or objects) and determines the identity of the represented objects (CP 2.228). For example, a black square is, on the ground of its color, an icon any black object, and on the ground of its shape, and an icon of any square object. In the same way, a pillar of smoke rising from a chimney is an index of the fire which causes it, and, on a different ground, an index of the location of the house where it is coming from. Relativized in this way, iconicity and indexicality are objective sign relations, independent of social conventions.

According to Peirce's third division, a sign can be a *rhema*, a *dicent sign* (a *proposition*), or an *argument.* A rhema (often spelled "rheme") is a sign interpreted as a sign of qualitative possibility, that is, as a sign "understood as representing such and such kind of possible object," whereas a dicent sign (a proposition) is interpreted as a sign of "actual existence" (CP 2.250–251). A predicative expression (considered as a part of a proposition) is an example of a rhema. According to Peirce, an argument is a sign understood as a sign of law:

> The interpretant of an Argument represents it as an instance of a general class of Arguments, which class as a whole will always tend to the truth. It is this law, in some shape, which the argument urges; and this "urging" is the mode of representation proper to arguments. (CP 2.253.)

Peirce defined a rhema or a predicate in the same way as Frege, that is, as an incomplete or "unsaturated" expression (or sign) which becomes a propositional sign when it is completed by one or several proper names:

> By a *rhema*, or *predicate*, will here be meant a blank form of proposition which might have resulted by striking out certain parts of a proposition, and leaving a blank in the place of each, the parts being stricken out in such a way that if each blank were filled with a proper name, a proposition (however nonsensical) would thereby be recomposed. (CP 4.560; cf. [Frege, 1891/1997, 17/139].)

Peirce sometimes compares logical analysis with analysis in chemistry and in other natural sciences; in CP 3.469 he observes:

> A chemical atom is quite like a relative [i.e., a relational predicate] in having a definite number of loose ends or "unsaturated bonds", corresponding to the blanks of the relative, whereas the chemical molecule is a medad, like a complete proposition.

Frege's use of the expression 'unsaturated' (ungesättigt) in this context was also derived from chemistry (cf. [Sluga, 1980, p. 139]).

Peirce regarded the question about the nature of propositions as one of the central questions logic, and criticized logicians and philosophers for confusing a proposition with the act of asserting a proposition (CP 5.85 and 2.309). An assertion or "affirmation" is a speech act, a proposition is a sign which is "capable of being asserted" (CP 2.252, NE, vol. 4, 248). A proposition is a possible assertion, and the logical properties of propositions reflect this possibility. In an assertive speech act the speaker or the utterer of a proposition assumes responsibility for its truth so that if the proposition turns out to be false, the utterer is subject to certain penalties, as in the case of perjury. Peirce thought that in order to understand the nature of an assertive speech act, it is best to consider very formal assertions in which the assertive force is "magnified", for example, an affidavit or an oath made before an authorized magistrate (CP 5.30–31, [Peirce, 1903/1997, p. 116]). Frege characterized assertions in a similar way, by saying that the utterer of a genuine assertion is "responsible" or "answerable" ("verantwortlich") for its truth (see [Frege, 1903/1984, 371–72/281]).

Peirce accepted the traditional view that every proposition consists of a subject (or subjects) and a predicate:

> A proposition consists of two parts, the predicate which excites something like an image or dream in the mind of the interpreter, and the subject, or subjects, each of which serves to identify something which the predicate represents. (MS 280, p. 32)

He interpreted the subject-predicate analysis of propositions in a new way and applied it to complex as well as simple (singular) sentences, including sentences involving quantifier phrases and modal expressions. By a subject of a proposition Peirce means a *logical subject*, that is,

> every part of a proposition which might be replaced by a proper name, and still leave the proposition a proposition. (CP 4.438.)

In Peirce's classification of signs, the subject of a proposition is either an index or an indexical symbol: its function is to direct the interpreter's attention to a certain object or objects, whereas the predicate of a proposition is an iconic symbol, and has an iconic sign as its interpretant. Thus a proposition gives information about the object indicated by its subject by representing the icon signified by its predicate as an icon of the object. Peirce defines a proposition as a sign which "separately, or independently, indicates its object." (MS 517; NE 4, p. 242.) Thus a proposition must consist of two signs, an indexical sign or a number of indexical signs (the subject) which identify the object or objects of the proposition, and an iconic sign, the predicate of the proposition. A proposition is true if and only if its subject and its predicate have the same object, that is, if and only if the object identified by the subject is an object of the predicate. Peirce's definition of truth was essentially the same as the scholastic view that a (simple) proposition is true if and only if its subject and predicate "supposit (refer to) for the same thing." (See [Ockham, 1980, p. 8, 86–87], [Kaufmann, 1994, p. 177]).

According to Peirce, any part of a proposition which "describes the quality or character of the fact [expressed by the proposition]" should be regarded as belonging to predicate of the proposition, whereas of the subject consists of the part or parts which "distinguish this fact from others like it" (CP 5.473). Strictly speaking, only the indexical parts of a proposition should be regarded as belonging to its subject:

> To include anything in the subject which might be separated from it and left in the predicate is a positive fault of analysis. To say for example that "All men" is the subject of the proposition "All men are mortal" is incorrect. The true analysis is that "Anything" is the subject and "_ is mortal or else is not a man" is the predicate. So in "Some cat is blue-eyed" the subject is not "some cat" but "something," the predicate being "_ is a blue-eyed cat." [Peirce, 1903/1997, p. 181]

According to this analysis, the subject of a quantified sentence is the quantifier. As was observed earlier, Peirce knew that sentences with several nested quantifiers can be translated into a prenex normal form; thus, according to Peirce's analysis of propositions, the string of quantifiers at the beginning of the proposition should be regarded as the subject of the proposition, and the "Boolian" part of the sentence can be regarded as the predicate.

Unlike a proper name or a pronoun, a quantifier does not point to a definite singular object: quantifiers and quantifier phrases are *indeterminate* indexical signs. Peirce calls indeterminate signs *precepts* (CP 2.330). A precept does not denote a definite singular object, but tells the utterer and the interpreter what they have to do (or what they may do) in order to find a singular object to which the predicate of a given proposition can be regarded as being applicable; thus an indeterminate indexical symbol can be interpreted or "explicated" as representing more than one singular object. (CP 2.330, 2.336; [Peirce, 1903/1997, p. 176].) Peirce distinguished between two main forms of indeterminacy, *indefiniteness*, expressed by an existential quantifier, and *generality*, expressed by a universal quantifier (MS 283, CP 5.448n.). He analyzed the meaning of indeterminate indices in terms of the actions of the utterer and the interpreter; this analysis may be called the "pragmatic interpretation of the logical subject" (CP 2.328–2.331). Peirce's pragmatic analysis of the meaning of quantifier phrases is based the use of a proposition in an assertion, and it resembles the game-theoretical interpretation of quantifiers. (Cf. [Brock, 1980], Hilpinen [1982], [1995, pp. 292–94]; for game-theoretical semantics, see [Hintikka and Sandu, 1997].) By asserting a certain proposition, the utterer accepts responsibility for it and subjects himself to certain penalties in case the proposition turns out to be false. Thus the utterer is essentially a defender of any proposition that he may assert. On the other hand, the interpreter is interested in detecting any falsehood asserted by the utterer, since, as Peirce notes, "the affirmation of a proposition may determine a judgment to the same effect in the mind of the interpreter to his cost" (MS 517; NE, Vol. 4, 249). Consequently the utterer and the interpreter have opposite attitudes with respect to the verification of any proposition asserted by the former. Peirce describes the situation as follows:

> The utterer is essentially a defender of his own proposition, and wishes to interpret it so that it will be defensible. The interpreter, not being so interested, and being unable to interpret it fully without considering to what extreme it may reach, is relatively in a hostile attitude, and looks for the interpretation least defensible. (MS 9, 3–4.)

The interpreter of (an assertive utterance of) a proposition may also be called its "opponent" or "falsifier", and the utterer, its "advocate" or "verifier" (MS 515, 25; CP 3.480; [Hintikka and Sandu, 1997, p. 363]). Given this asymmetry in the roles of the utterer and interpreter, the meaning of different types of indeterminate indices can be explained as follows. An indeterminate index is indefinite if and only if the utterer of the proposition may select the object which the index should be regarded as representing; that is, if the utterer is free to choose the interpretation of the subject-term. An existential quantifier signifies the utterer's choice or "move" in the language-game. On the other hand, the utterer of a universally quantified sentence

> [allows] his opponent [i.e., the interpreter] a choice as to what singular object he will instance to refute the proposition, as in "Any man you please is mortal." (MS 515, 25)

In other words, a universal quantifier transfers the choice of the singular to the interpreter. For example, the proposition "Some woman is adored by every Catholic" means, according to Peirce, that

> a well-disposed person with sufficient means [i.e., the utterer] could find an index whose object should be a woman such that allowing an ill-disposed person [the interpreter] to select an index whose object should be a Catholic, that Catholic would adore that woman. [Peirce, 1903/1997, p. 176]

If an indeterminate index is a complex quantifier phrase involving several quantifiers, each existential quantifier indicates the utterer's choice of a singular object and each universal quantifier the interpreter's choice. Peirce observes that

> whichever of the two makes his choice of the object he is to choose, after the other has made his choice, is supposed to know what that choice was. This is an advantage to the defence or attack, as the case may be. (MS 9, paragraph 3)

In other words, the players of a semantic game (the utterer and the interpreter) do not make their choices independently of each other (cf. MS 9, §2); thus Peirce takes the semantical games for quantifiers to be games with perfect information. This means that the quantifiers in a complex indeterminate index are always linearly ordered and do not branch. (For the logic partially ordered (branching) quantifiers, see [Hintikka, 1996, Chs. 3–4], and [Hintikka and Sandu, 1997, pp. 366–369].)

According to Peirce, precepts (indeterminate indices) include, in addition to ordinary quantifiers, propositional connectives and modal operators (which Peirce takes to be quantifiers over possible "cases" or "hypothetical states of the universe"). The games for propositional connectives involve the utterer's and the interpreter's choices between the subsentences of a given sentence: the disjunction sign means that the utterer of a proposition is free to choose the disjunct to be analyzed, and the conjunction sign transfers the choice of the sentence (conjunct) to the interpreter. The negation sign reverses the roles of the advocate (the verifier) and the opponent (the falsifier) of the proposition so that the utterer begins to play the role of the opponent of the sentence and the interpreter assumes the role of the advocate (CP 3.480-482; [Brock, 1980, pp. 61–63]). Modal expressions are precepts for making choices among hypothetical states of the world (briefly, possible worlds), for example, an utterer's assertion that P is necessary means that the interpreter has the right to attempt to describe or otherwise indicate a (possible) state of the world in which P does not hold. Peirce's observation in 'On the Algebra of Logic' (1885) that

> the whole expression of a proposition consists of two parts, a pure Boolian expression referring to an individual and a Quantifying part saying what individual this is (CP 3.393; WCSP 5, 178),

may be interpreted as follows: the "Quantifying part" is a precept which tells how an object (or objects) of the proposition can be found, and the "Boolian expression" determines what the object (or objects) can be expected to be like.

If the truth of a proposition is defined as the utterer's ability to defend it successfully against the interpreter's attack, this analysis of quantifier phrases gives quantified sentences correct truth-conditions and is essentially similar to the game-theoretical interpretation of quantifiers. Peirce's concept *defensibility against attack* resembles the game-theoretical analysis of truth, according to which a sentence is true if and only if its utterer has a winning strategy in the game associated with it [Hintikka and Sandu, 1997, p. 364].

To say that a proposition contains an indeterminate index or precept does not mean that the index denotes an indeterminate or "ambiguous" object; the indeterminacy concerns the manner of reference, that is, in the semantic relationship between a sign and its objects. Peirce explained the nature of this relationship by means of the semantical games associated with sentences containing indeterminate signs. In this way the concept of indeterminate reference gets a clear meaning. Peirce's analysis is "pragmatic" in the sense that the explanation of the semantic relationship between a sign and its object (or objects) refers to the users of signs, the actions of the utterer and the interpreter. It is also pragmatic in the sense of the Pragmatic Maxim: the meaning of indeterminate indices (precepts) is explained in terms of their "practical effects".

5 EXISTENTIAL GRAPHS

Consider a proposition, for example, an indicative sentence

(25) Oscar is an orange cat

The deletion of the proper name 'Oscar' from (25) turns it into an "unsaturated" expression (a rhema)

(26) _____ (is) an orange cat,

where the line _____ indicates an empty place which can be filled by a proper name or a demonstrative pronoun or some other indexical sign to transform (26) into a proposition. In the same way, the deletion of proper names from sentence

(27) Vera loves Oscar

produces the rhematic signs (predicates)

(28) _____ loves Oscar,

(29) Vera loves _____,

and

(30) _____loves _____.

(26) and (28)–(30) are incomplete (rhematic) signs, not propositions, and uttering them would not produce complete (true or false) assertions. However, suppose that someone asserts just (26), that is,

(31) An orange cat!

How can such an utterance be interpreted? Perhaps the most natural interpretation is to take (31) to be an incomplete expression of the existential proposition

(32) There is an orange cat,

or "An orange cat, there!" According to this interpretation, the lines attached to the predicates '(is) an orange cat' and 'loves' are no longer signs of the empty spaces in incomplete expressions, but play the role of individual variables bound by existential quantifiers (CP 4.439). This reinterpretation of the expressions (26) and (28)–(30) led to Peirce's diagrammatic notation for quantification theory, the theory of existential graphs. Interpreted in this way, the lines attached to predicative expressions are called *lines of identity.*

Peirce divided the system of existential graphs into three parts, Alpha, Beta, and Gamma. The Alpha part is propositional logic, Beta consists of the graphs for quantification theory, and Gamma includes graphs for modal logic.

The system of Alpha graphs contains propositional symbols (for example, $P, Q, R, \ldots$) and means for expressing the negation of a proposition and the conjunction of two (or more than two) propositions. The graphs are thought of as being written (or "scribed") on a sheet of paper, called the sheet of assertion: whatever is written on the sheet is thought of as being asserted as true. The act of writing a graph on the sheet is regarded as an act or asserting the proposition expressed by the graph (CP 4.431). Writing two (or more than two) graphs on the sheet amounts to asserting both (or all of them), that is, asserting their conjunction. Thus the juxtaposition of propositions represents their conjunction. Peirce observes that if we wish to assert a conditional 'if P, then Q' ($P \rightarrow Q$),

> it must be drawn on the sheet of assertion, and in this graph the expressions P and Q must appear, and yet neither P nor Q must be drawn on the sheet of assertion [because neither is asserted]. How is this to be managed? (CP 4.435.)

Peirce's answer is to draw on the sheet a closed line called *cut* or *sep* (from the Latin word *saepes*, for fence) which cuts off its contents from the main sheet. If the material or "Philonian" conditional is expressed in the way shown in figure 1, the cut functions as a sign for negation: the graph in figure 1 says that it is not the case that P is true and Q is false.

In figure 1, the propositions P and Q are not scribed on the sheet of assertion; they are not (unconditionally) asserted. However, P and Q appear on the sheet in the sense that they are subgraphs of the entire graph on the sheet. Peirce calls an area cut off from the main sheet or from another area the *close* of the cut, and the area where the cut is made the *place* of the cut. The area where a graph is inscribed can also be called the *context* of the graph [Sowa, 1993, pp. 3–4]. In a linear representation, the function of the cut can be expressed (for example) by brackets: thus

(33) $[P[Q]]$

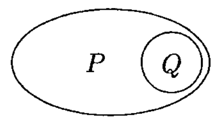


Figure 1. A graph for a conditional $P \rightarrow Q$.

conveys the same information as figure 1 (CP 4.378). The order or arrangement of the graphs in a given area is irrelevant; thus (33) and '[[Q]P]' are instances or replicas of the same graph (i.e., graph-type). This representation of a material conditional is more "iconic" than the formula '$P \rightarrow Q$' in the sense that it shows the meaning of the conditional by showing what it excludes. Since conjunction and negation form a complete set of connectives for propositional logic, this notation is sufficient for propositional logic.

The separation of the area enclosed by a cut (or any odd number of cuts) from the main part of the sheet can be emphasized by shading (or coloring) the enclosed area. Using this device makes the graphs easier to "read" (See CP 4.618-624; MS 514, see [Sowa, 2001].) A double cut corresponds to a double negation; thus a double cut leads through a shaded area back to an unshaded area, and more generally, any even number of cuts leads from an unshaded part of the sheet back to an unshaded area. Following John Sowa's [1993, pp. 3–4] terminology, I shall call the areas separated from the sheet of assertion by an even number of cuts (the unshaded areas) *positive contexts*, and the areas enclosed by an odd number of cuts *negative contexts*.

Before Peirce developed the theory of existential graphs, he made experiments with graphs ("entitative graphs", CP 4.434) in which the juxtaposition of two graphs on the main sheet represents a disjunction, not a conjunction. All truth-functions can be defined in terms of negation and disjunction; thus the resulting system is sufficient for propositional logic, but Peirce considered such graphs to be inferior to existential graphs because, as he put it, "unnaturalness and aniconicity haunt every part of the system of entitative graphs" (ibid.). In the system of entitative graphs, several graphs scribed on the sheet are not asserted by the utterer (or graphist), but rather presented for consideration as alternative propositions, at least one of which is true. The interpretation of the sheet as a sheet of assertion leads naturally to the system of existential graphs.

To obtain the expressive power of first-order quantification theory, it suffices to attach 1, 2, or more than 2 lines of identity to the predicates $P, Q, R, \ldots$, depending on the number of empty places attached to the predicate symbols (i.e., depending on whether they are monadic (1-place), dyadic (2-place), or many-place predicates). The lines of identity are "heavy lines" (CP 4.444), and are distinguished in this way from the lines indicating cuts. Peirce called predicative expressions (rhemata) the *spots* of a graph, and referred to the points to which the lines of identity are attached as *hooks*. In the case of many-place predicates, it is important to distinguish different hooks from each other, for example, the hook on the left side of the predicate 'parent' can be regarded as a place to be filled by an individual term denoting a parent, and that on the right-hand side as a place for a term indicating a child. Each line of identity corresponds to a variable bound by an existential quantifier; Peirce's system does not contain any free variables. For example, if we let the letters A and F stand for the predicates '(is an) artist' and 'forger', the graphs in figure 2 (below) represent the propositions "Someone is a forger" and "Someone is an artist", and together they form a graph for their conjunction.

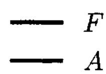


Figure 2. A graph for $\exists x Fx \wedge \exists x Ax$.

By joining the two lines of identity together we obtain the proposition 'Some artist is a forger':

Figure 3. A graph for $\exists x(Fx \wedge Ax)$.

A graph in which a line of identity attached to a predicate P crosses a cut around P states that something is not P; thus the graph in figure 4. states that some forger is not an artist:

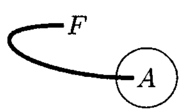


Figure 4. A graph for $\exists x(Fx \wedge \neg Ax)$.

A cut around this graph, that is, the negation of the graph in figure 4, yields the universal generalization 'Every forger is an artist' (figure 5).

It may again be observed that the representation in figure 5 is "iconic" in the sense that graph clearly shows the meaning of a universal generalization by what it excludes, viz. the existence of a forger who is not an artist.

The statement that someone is an artist and some other individual is a forger is expressed by means of a cut across the line of identity joining the two predicates, as shown in figure 6.

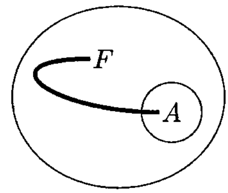


Figure 5. A graph for $\forall x(Fx \rightarrow Ax)$.

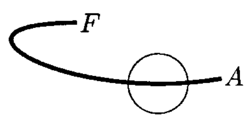


Figure 6. A graph for $\exists xFx \wedge \exists xAx \wedge \neg x = y$.

In this way it is possible to express the concepts of identity and nonidentity (difference) in the theory of graphs. The expressive power of Beta graphs is the same as that of first-order quantification theory with identity. The application of several predicates to a single individual is expressed by a branching line of identity, called by Peirce a *ligature* (CP 4.407-408); for example, the graph in Figure 7. states that an artist who is not a forger loves (L) a cat (C).

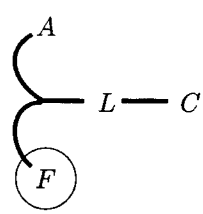


Figure 7. A graph for "An artist who is not a forger loves a cat."

Figure 8 shows another example, "If a farmer owns a donkey, then he beats it." [Sowa, 2000, p. 276]. Propositions of this kind ("donkey sentences") have been widely discussed by linguists and philosophers (cf. [Kamp and Reyle, 1993, p. 1]).

The interpretation of existential graphs proceeds "endoporeutically" or "endogenously", as Peirce put it (MS 650, 18-19, MS 477; [Roberts, 1973, p. 39]), from outside in, which means that the interpretation of a line of identity which

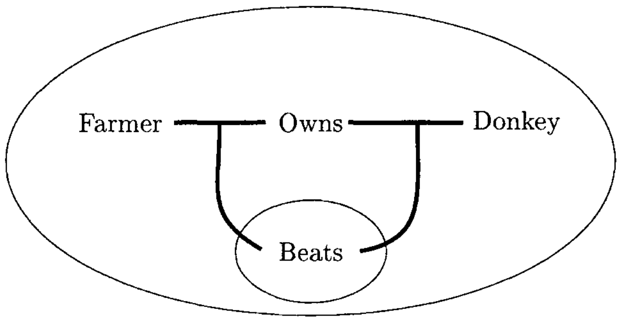
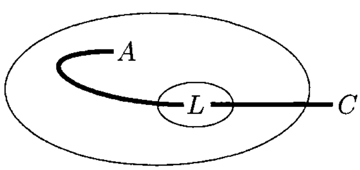
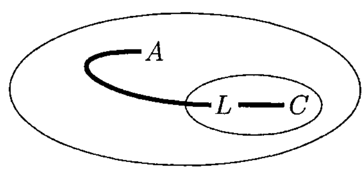


Figure 8. A graph for "If a farmer owns a donkey, then he beats it."

crosses cuts is determined by its outermost part. Thus in figure 9 (below), graph G1 represents the proposition that some cat is loved by every artist, in other words, there is a cat such that it is not the case that some artist fails to love it, whereas G2 is the proposition that every artist loves some cat, that is, it is not the case that there is an artist who does not love a cat.



G1



G2

Figure 9. Graphs for $\exists y(Cy \wedge \forall x(Ax \rightarrow Lxy))$ (G1) and $\forall x(Ax \rightarrow \exists y(Cy \wedge Lxy))$ (G2).

The endoporeutic interpretation of the graphs is required by Peirce's pragmatic or game-theoretical reading of logical constants: graphs are analyzed from the outside in. If a line of identity crosses cuts, but its outermost part lies (unenclosed) on the sheet of assertion, the utterer of the proposition has the right to select an individual in order to try to verify the proposition, but if the outermost part of a line is once enclosed, the interpreter or "a person who might be hostile or sceptical to the proposition" (MS 503, 3; cf. [Roberts, 1973, p. 50]) has the right to select an individual for the purpose of falsifying the proposition. More generally, a line of identity whose outermost part is in a positive context (enclosed by an even number of cuts) represents an existential quantifier, and an oddly enclosed line can be regarded as representing a universal quantifier, whereas the juxtaposition of graphs in a positive context represents the choice of a subgraph by the interpreter (a conjunction), and juxtaposition in a negative context represents the choice of a subgraph by the utterer (a disjunction).

Peirce's system of graphs contains two special graphs, an empty graph (i.e., an empty sheet of assertion), and a "pseudograph", the negation of an empty graph or a cut which has an empty graph as its content. A pseudograph is equivalent to the constant f representing falsity, and the empty graph represents a tautology. Moreover, a line of identity written on the sheet without being attached to any predicative expression is regarded as a graph for the proposition 'Something exists'.

The most interesting feature of existential graphs is the system of rules of proof for the graphs. A proof of a conclusion C from a conjunction of premises P is a series of steps by which a graph for P can be transformed into a graph for C. As special case, the proof of the logical truth of C can be defined as the proof of C from an empty graph. The empty graph (that is, a blank sheet of assertion) can be regarded as an "axiom" of Peirce's system ([Roberts, 1992, p. 647]; [Sowa, 1993, p. 4]). The rules of proof or inference rules are "Rules of Permission" which allow a graphist to perform the required transformations. The rules are of two kinds: (a) Rules of Insertion, which allow the graphist to insert (add) something on the sheet of assertion, and (b) Rules of Erasure, which allow the graphist to erase a graph or parts of a graph from the sheet of assertion. According to Peirce, rules of this kind are optimal for the purposes of logical analysis, because

> an omission and an insertion appear to be indecomposable transformations and the only indecomposable transformations. That is, if A can be transformed by insertion into AB, and AB by omission in B, the transformation of A into B can be decomposed into an insertion and an omission. (CP 4.464.)

The system Alpha (that is, the inference rules of propositional logic) can be specified by five rules concerning permissible transformations. The present formulation is based on Roberts [1973, pp. 40–45]; [1992, p. 647] and [Sowa, 1993, p. 4]. (Essentially similar formulations can be found in [Zeman, 1964, p. 14]; [Barwise and Hammer, 1994, p. 84], and [Shin, 2002, p. 81]. For Peirce's original formulation of the rules, see CP 4.492, 4.505-508.)

(R1) Rule of Erasure ("Erasure in Even"). Any graph G may be erased in a positive context.

(R2) Rule of Insertion ("Insertion in Odd"). Any graph G may be inserted in a negative context.

(R3) Rule of Iteration. If an instance of a graph G occurs in a context c, another instance of G may be inserted in c or in any context nested in c.

(R4) Rule of Deiteration. An occurrence of a graph G that could have been derived by iteration may be erased.

(R5) Rule of Double Cut ("Rule of Biclosure"). Two cuts may be drawn around or removed from any graph, provided that no graph occurs between the cuts in question.

Peirce's rules of erasure are analogous to the elimination rules of natural deduction systems, and the insertion rules function as introduction rules. For example, (R1) allows the graphist to eliminate conjunctions (by erasing conjuncts), and (R2) and (R5) together make it possible to introduce disjunctions and conditionals. Rule (R1) permits the deletion of conjuncts from the consequent and disjuncts from the antecedent of a conditional, and (R2) allows the graphist to add conjuncts to the antecedent and disjuncts to the consequent of a conditional. (R2) makes it also possible to introduce assumptions in a proof. Here is a very simple example, a proof of '$P \to P$':

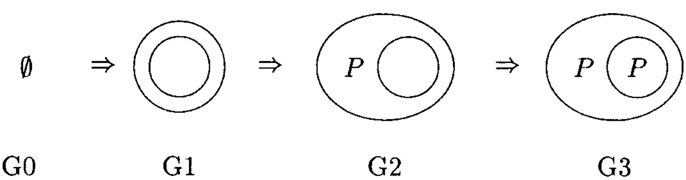


Figure 10. A proof of $P \to P$.

'$\emptyset$' symbolizes an empty graph (a blank sheet of assertion) and '$\Rightarrow$' an application of a rule of inference. G1 is obtained from G0 by (R5), G2 from G1 by (R2), and G3 from G2 by (R3), the Rule of Iteration. It is clear that the first step of any proof from an empty set of premises (a blank sheet of assertion) is the introduction of a double cut. Figure 11 shows another simple example, a proof of Q from $\neg P$ and $P \vee Q$. (Note that the disjunction must be represented as a negation of a conjunction, that is, a juxtaposition of two cuts inside a cut).

The first step in the proof is justified by (R4), the Rule of Deiteration: $\neg P$ may be erased from the negative context in G1b, because it could have been introduced by R3 from G1a. G3 is obtained by (R1) and the conclusion by (R5).

In the Beta graphs, an unattached line of identity (a line with two loose ends) is interpreted as the proposition that something exists. Such a line on an otherwise blank sheet is an axiom of the Beta system (in addition of the blank sheet itself). The rules of inference are the rules (R1)-(R5) listed above, interpreted as being applicable to Beta graphs. Thus the rules of Erasure, Insertion, Iteration, and

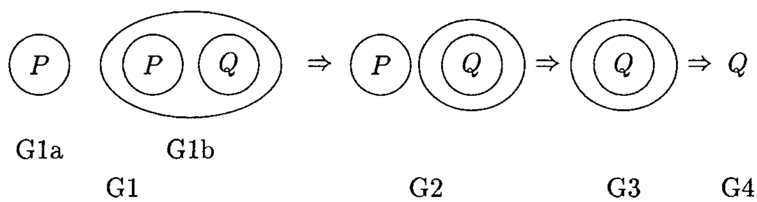


Figure 11. Proof of Q from $\neg P$ and $P \vee Q$.

Deiteration can be applied to lines of identity and graphs with lines of identity, and must be supplemented with clauses concerning the treatment of lines of identity. (See [Roberts, 1973, pp. 56–60] [Roberts, 1992, pp. 647–48], [Shin, 2002, pp. 139–41].) For example, in a positive context, (R1) permits the breaking of a line into two (by erasing a part of the line); thus the graph in figure 2 (see above) can be derived from that in figure 3, and the inference of $\exists x Fx \rightarrow \exists x Ax$ from $\forall x(Fx \rightarrow Ax)$ consists of a single application of rule (R1), shown in figure 12.

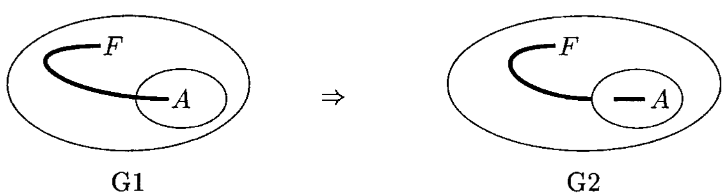


Figure 12. $\forall x(Fx \rightarrow Ax) \Rightarrow \exists x Fx \rightarrow \exists x Ax$.

According to Peirce, the points on a cut (that is, on the line indicating a cut) are considered to be outside the cut (CP 4.501; [Roberts, 1973, p. 54]); thus graph G2 in figure 12 is equivalent to the graph in figure 13.

According to (R2), the Rule of Insertion, two unattached ends of lines of identity may be joined together in a negative context (oddly enclosed area), and (R3) (Rule of Iteration) is interpreted as allowing the extension of a line of identity inwards through a nest of cuts. Rule (R5), the Rule of Double Cut, is supplemented by the clause that a line of identity without any branching may pass through the two cuts to be introduced or removed. Figure 14 shows a simple example of the application of Beta rules, a Beta proof of the syllogism Barbara, the inference of $\forall x(Cx \rightarrow Ax)$ from $\forall x(Bx \rightarrow Ax)$ and $\forall x(Cx \rightarrow Bx)$. (See [Roberts, 1973, p. 61], [Sowa, 2001, pp. 20–22], CP 4.571.)

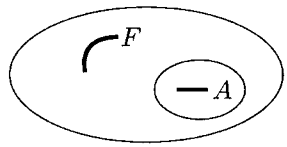


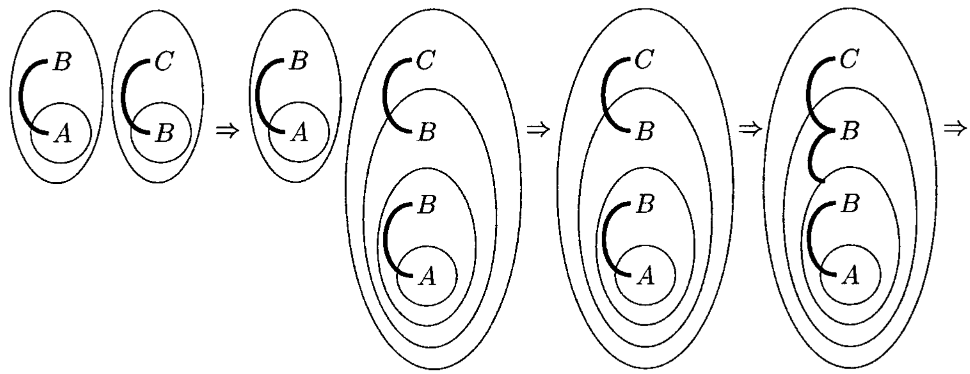
Figure 13. $\exists x Fx \rightarrow \exists x Ax$.

G1a and G1b are the two premises of the inference. G2 is obtained from G1 by the Rule of Iteration (R3): in G2, a second instance of G1a has been scribed inside the double cut in G1b. G3 is obtained from G2 by (R1), that is, by erasing G2a. G4 and G5 are obtained from G3 by extending the line of identity connecting C and B inwards and by joining it with the line of identity connecting B and A; these transformations are justified by the Beta version of the Rule of Iteration (R3) and the permission to join the loose ends of two lines of identity in a negative context (R2). G6 is obtained from G5 be deiterating the thrice enclosed occurrence of B (R4), and G6 is transformed into G7 be erasing the double cut around A (R5). Finally, the conclusion G8 is obtained from G7 by R1 (by erasing an occurrence of B in a positive context). Another example is provided by the graphs in figure 9 (see above): Graph G1 (that is, $\exists y(Cy \wedge \forall x(Ax \rightarrow Lxy))$) can be transformed into G2 ($\forall x(Ax \rightarrow \exists y(Cy \wedge Lxy))$): The Beta version of (R3) permits the iteration of C inside the double cut in G1, and (R1) permits the erasure of the outer instance of C and the retraction of the line of identity hooked to it, yielding G2 (cf. CP 4.566; [Hardwick and Cook, 1977, pp. 103–104]). On the other hand, G2 cannot be transformed into G1, because a new instance of C cannot be inserted into the positive context outside the double cut.

Detailed formulations and discussions of the Beta rules and examples of their application can be found in Roberts [1973, pp. 56–63], [1992, pp. 647–56] and in Shin [1999, pp. 275–76], [2002, pp. 134–150]. Zeman [1964, pp, 124–139] and Roberts [1973, pp. 139–150] have shown that if Peirce's Beta rules are formulated and interpreted in suitable way, they constitute a complete system of first-order logic with identity.

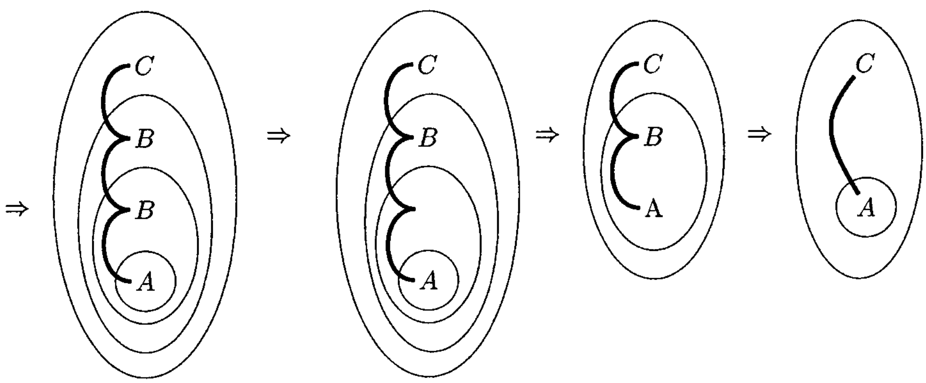
Peirce was not satisfied with the Beta system, because it was restricted to extensional logic. In his Gamma graphs Peirce attempted to develop logical graphs for modalities. If a graphist wishes assert a modal proposition, for example, 'Possibly P,' he has to scribe on the sheet of assertion an expression in which P appears without writing the proposition P itself on the sheet. Peirce uses for this purpose the device of a *broken cut* shown in figure 15:

A broken cut indicates that the graph inside the cut is possibly false. In a linear representation we can let (for example) braces play the role of a broken cut; thus,



G1a G1b G2a G2b

G1 G2 G3 G4



G5 G6 G7 G8

Figure 14. A proof of $\forall x(Cx \rightarrow Ax)$ from $\forall x(Bx \rightarrow Ax)$ and $\forall x(Cx \rightarrow Bx)$.

using the brackets for an ordinary cut (for negation),

$[\{P\}]$

means that P is not possibly false, i.e., that it is necessary that P, and

$\{[P]\}$

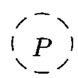


Figure 15. A broken cut

means that it is possible that P. Like the ordinary (unbroken) cut, the broken cut indicates a change of context: an unbroken cut leads from a positive to a negative context (or conversely), and a broken cut on a sheet of assertion leads from the context of actuality to a context of (negative) possibility. In his work on the Gamma graphs Peirce also considered different kinds of possibility and different speech act types, and represented them by different colors or "tinctures" ([Peirce, 1906], CP 4.554; cf. [Roberts, 1973, pp. 92–98]; [Zeman, 1997b]). Peirce formulated for Gamma graphs some inference rules analogous to the Alpha rules (CP 4.514-516), but the system remained a sketch; Peirce was not able to develop a satisfactory account of modal graphs. However, in his work on the Gamma graphs he anticipated interesting later developments in modal logic, for example, he observed that in the case of modal graphs, it is not enough to work with a single sheet of assertion representing a universe of existent individuals; instead "we have a book of separate sheets, tacked together at points, if not otherwise connected" (CP 4.512); these sheets represent different possible universes of discourse or possible worlds, and the sheets may contain cuts where "we pass into worlds which, in the imaginary worlds of the outer cuts, are themselves represented to be imaginary and false." (Ibid.) In this characterization Peirce comes to close to the contemporary analysis of modal propositions in terms of an accessibility relation between possible worlds. (Cf. Zeman [1986, p. 9]; [1997a, pp. 409–11].) Peirce's system of Gamma graphs can be interpreted and complemented in such a way that it forms the basis of interesting systems of modal logic; such systems have been developed and studied by Zeman [1964, Ch. III]; [van den Berg, 1993]; [Øhrstrøm, 1996], [Øhrstrøm and Hasle, 1995, pp. 320–43], and others.

Peirce called the system of existential graphs, together theory of logical analysis based on it, his "chef d'oeuvre", and thought that it "ought to be the logic of the future." [Roberts, 1973, p. 11, 110]. An outline of the system was published in Peirce [1906], but existential graphs became more widely known among logicians and philosophers only after the publication of several articles about them in the fourth volume of the *Collected Papers* in 1933 (CP 4.347–584). The first commentators did not share Peirce's view about the significance of existential graphs, but expressed doubts about their practical value and analytical power ([Quine, 1934], see [Roberts, 1973, pp. 12–13]). Apart from some specialized studies on Peirce's work [Zeman, 1964; Roberts, 1973; Thibaud, 1975], logicians and philosophers did not express much interest in the logic of existential graphs. The situation changed in the 1980s and 1990s when logicians and computer scientists became interested in reasoning by means of nonlinguistic (or quasi-linguistic) representations, and began to develop heterogeneous logical systems, that is, logics having both linguistics and nonlinguistic elements, such as diagrams, tables, charts, etc. [Barwise and Hammer, 1994, p. 88]. Since the early 1980s Peirce's system of existential graphs has inspired a great deal of research in this field, for example, John Sowa's [1984; 1993] system of conceptual graphs. In recent years logicians and linguists have developed independently of Peirce graphical representation systems which are closely related to Peirce's graphs, for example, Hans Kamp's discourse representation structures are structurally analogous to Peirce's existential graphs. (See [Kamp and Reyle, 1993, Ch. 1]; Sowa [2000, pp. 278–79], [1997, pp. 427–36].)

6 MODALITIES AND POSSIBLE WORLDS

Peirce was aware of the limitations of extensional logic (propositional logic and quantification theory), for example, he argued that conditional sentences cannot always be represented as truth-functional ("Philonian") conditionals (cf. [Read, 1992]). This is obvious in the case of subjunctive and counterfactual conditionals, but the failure of extensionality is not restricted to counterfactual conditionals. Peirce observed that if the statements

(34) There is some married woman who will commit suicide in case her husband fails in business.

and

(35) There is some married woman who will commit suicide if all married men fail in business.

are translated into the language of extensional (first-order) logic, they turn out to be logically equivalent (CP 4.546), even though (34) seems logically stronger than (35): if (34) is true, then (35) is true, but (according to Peirce) the converse does not hold. Peirce made this observation is his [1906] paper where he discussed the theory of existential graphs, and concluded from the example that the Beta graphs were not adequate for the representation of conditionals (cf. CP 4.580). The same point can be made by using a slightly simpler example (CP 4.580): "if nothing is real but existing things," the statement

(36) There is a man who commits suicide if he does bankrupt

turns out to be logically equivalent to

(37) There is a man who commits suicide if every man goes bankrupt.

Peirce supports this observation by the fact that an attempt to translate these statements into the language of extensional (first-order) logic leads to logically equivalent sentences. The most plausible translations of (36) and (37) into the customary notation of first-order logic look as follows:

(38) $\exists x(Bx \to Sx)$

and

(39) $(\exists x)(\forall y By \to Sx)$,

where 'Bx' means that x goes bankrupt (or fails in business), and 'Sx' means that x commits suicide, and the domain of discourse is restricted to men. (The same formulas serve as first-order translations of (34)–(35) if 'Bx' is taken to mean that x is a married woman whose husband fails in business, and the domain of discourse consists of women.) It is easy to see that (38) and (39) are logically equivalent: they are false under exactly the same circumstances. According to this interpretation,

> the proposition that there is a man who if *he* goes bankrupt will commit suicide is false only in case, taking any man you please, he *will* go bankrupt, and *will not* commit suicide. That is, it is falsified only if every man goes bankrupt without suiciding. But this is the same as the state of things under which the other proposition is false; namely, that every man goes broke while no man suicides. (CP 4.580.)

Peirce notes that "the equivalence of these two propositions is the absurd result of admitting no reality but existence," and of "assuming that there is but one kind of subjects which are either existing things or else quite fictitious." (CP 4.546) According to Peirce, statements (36)–(37) should be read as modal propositions:

(40) There is some businessman who under all possible conditions would commit suicide or else he would not have gone bankrupt.

(41) There is some man who under all possible conditions would commit suicide or else not all men would have gone bankrupt,

or briefly,

(42) $\exists x\Box(Bx \to Sx)$

and

(43) $\exists x\Box(\forall y By \to Sx)$,

where $\Box$ is a necessity operator which represents quantification over possible "states of things" or possible worlds (courses of events). Peirce observes that the conditionals (40)–(41) must be understood as *de re* modal propositions, statements about the behavior of the same individual under different conditions; thus the proper interpretation of (34)–(35) and (36)–(37) requires quantification into a modal context (cf. CP 4.546). Here (40) does not follow from (41): the latter is true and the former is false if, for example, there are two businessmen, Smith and Jones, such that under the actual conditions only Smith goes bankrupt and neither Smith nor Jones commits suicide, but in other (unactualized) states of things in which both go bankrupt, Smith commits suicide. Under such circumstances Smith instantiates (41), but not (40). (For an analysis of (34)–(35) by means of existential graphs, see [Øhrstrøm and Hasle, 1995, pp. 324–25].)

According to Peirce, conditional propositions can be regarded as modal propositions, and he accepted the scholastic view that modal propositions are *quantified* propositions of a special kind: "the necessary (or impossible) proposition is a sort of universal proposition; the possible (or contingent, in the sense of not necessary) proposition, a sort of particular proposition" (CP 2.382). As was observed earlier in the discussion of the pragmatic (or game-theoretical) interpretation of quantifiers, modalities function as quantifiers over possible circumstances or courses of events; thus Peirce accepted a version of the possible worlds analysis of modalities. In the system of existential graphs this semantical view is expressed by the assumption that the analysis of modal propositions requires a "book" of sheets of assertion, not just a single sheet. In one of his unpublished manuscripts Peirce explains the semantics of counterfactual conditionals by means of the following example:

> To say that if Napoleon had been in his best trim he would have won the battle of Waterloo, so far as it means anything, means that taking all the different possible courses of events that might reasonably be admitted as such by taking into consideration the variations of power shown by Napoleon during his life, while external circumstances remain substantially as they were, every such possible course of events would either be one in which Napoleon was not in his best trim or would be one in which he would have won the battle of Waterloo. (MS 284, 29)

If "external circumstances" are regarded as the circumstances independent of (or external to) the antecedent of the conditional, this account of the meaning of conditional statements resembles the characterization of subjunctive and counterfactual conditionals as "variably strict" conditionals (cf. [Lewis, 1973, pp. 13–19]), according to which the possible worlds relevant to the semantic evaluation of a counterfactual depend on its antecedent.

7 MANY-VALUED LOGIC

In his paper 'On the Algebra of Logic' ([Peirce, 1885]; CP 3.365; WCSP 5, 166) Peirce observed:

> According to ordinary logic, a proposition is either true or false, and no further distinction is recognized. This is the descriptive conception, as the geometers would say; the metric conception would be that every proposition is more or less false, and the question is one of amount.

Peirce did not attempt to develop a logic of degrees of falsity, but in his (unpublished) notes entitled 'Triadic Logic' (Logic Notebook, 1909, MS 339; partly reproduced in Fisch and Turquette 1966), he outlined a three-valued semantics for propositional connectives. He denoted the three values by the letters V (verum), F (falsum), and L, or "the limit," which he took to mean that the sentence "is not capable of the determination V or the determination F." According to Peirce, triadic logic is

> that logic which, though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not P, but is at the limit between P and not P. (MS 339, 344r.)

Peirce considered several unary 3-valued truth-functions, including a 3-valued concept of negation defined as follows (Peirce expressed it by means of a bar above the negated formula; thus it may be called the bar-operator):

P	T	L	F
$-P$	F	L	T

Peirce defined several binary 3-valued connectives, for example, his Zeta-operator Z has the following value matrix:

Z	V	L	F
V	V	L	F
L	L	L	F
F	F	F	F

In other words, if we represent V, L, and F by $1, \frac{1}{2}$, and 0, the value of $\mathrm{Z}(P,Q), V(\mathrm{Z}(P,Q))$ is $\min(V(P), V(Q))$; thus Z can be regarded as a generalization of the concept of conjunction. In the same way, Peirce's Theta-operator (Θ) can be regarded as a 3-valued disjunction:

Θ	V	L	F
V	V	V	V
L	V	L	L
F	V	L	F

Peirce's value-matrices for the bar-negation, Zeta, and Theta correspond to S. C. Kleene's "strong senses" of negation, $\wedge$, and $\vee$ [Kleene, 1952, pp. 332–335]. Systems of 3-valued logic were developed later, independently of Peirce, by Jan Łukasiewicz, Emil Post, and others. (See Łukasiewicz [1920/1967], [1930/1967]; [Post, 1921]. For discussions of Peirce's triadic logic, see [Fisch and Turquette, 1966]; Turquette [1967; 1969].)

8 THEORY OF REASONING: DEDUCTION, INDUCTION, AND ABDUCTION

In addition to his pioneering work in deductive logic, Peirce developed a new theory of non-deductive (non-demonstrative) inference as a part of his theory of scientific method. One of his most significant contributions to this area was his account of induction and abduction as the two main forms of non-deductive argument. Thus he divided arguments into three main classes: deductive, abductive, and inductive arguments.

In his early papers, written in the 1860s and 1870s, Peirce called the main types of non-demonstrative reasoning induction and hypothesis (or hypothetic inference), and derived this division from the structure of Aristotelian syllogisms ([Hilpinen, 2000]; for Aristotle's theory of deduction, see [Smith, 1989]). A syllogism is an argument form involving two premises and a conclusion, in which both premises and the conclusion contain two general terms or concepts, for example, in the following way:

$$(44)\quad \begin{array}{ccc} A & - & B \\ B & - & C \\ \hline A & - & C \end{array}$$

where the predicate of each proposition is written before the subject and the terms 'predicate' and 'subject' are understood in the traditional way (as terms, that is, general concepts), not in the way discussed above in section 4. The syllogisms in which the terms are arranged in the way shown above are said to belong to (or exemplify) the first figure. For example, if both the premises and the conclusion are universal sentences, (44) assumes the form called "Barbara":

$$(45)\quad \begin{array}{l} \text{All } B\text{s are } A \\ \text{All } C\text{s are } B \\ \hline \text{All } C\text{s are } A. \end{array}$$

The predicate of the conclusion (A) is called the major term, the subject of the conclusion (C) is called the minor term, and the term shared by the premises (B) is called the middle term. The premise containing the major term is called the major premise and the premise containing the minor term, the minor premise.

Peirce called the major premise "Rule", the minor premise "Case", and the conclusion "Result" ([Peirce, 1878]; CP 2.619-20; WCSP 3, 323–24). Thus Barbara is a deduction of a Result from a Rule and a Case. Peirce applied this terminology also to inferences in which the minor premise and the conclusion are singular sentences or statements about a restricted number (a sample) of cases, for example:

$$(46)\quad \begin{array}{ll} \text{All } B\text{s are } A & \text{(Rule)} \\ \text{These } C\text{s are } B & \text{(Case)} \\ \hline \text{These } C\text{s are } A & \text{(Result)} \end{array}$$

$$(47)\quad \begin{array}{ll} \text{All } B\text{s are } A & \text{(Rule)} \\ s \text{ is } B & \text{(Case)} \\ \hline s \text{ is } A. & \text{(Result)} \end{array}$$

According to Peirce ([Peirce, 1878]; CP 2.620; WCSP 3, 324), Barbara is "nothing but the application of a rule." Each of the three propositions in a syllogism of this form can be viewed as a conclusion inferred from the other two propositions; thus we obtain three possible inference forms: the logically binding inference of the Result from a Rule and a Case (a deduction); secondly, an inference of a Rule from a Case and a Result, called an *induction*; and as a third form, an inference from a Rule and a Result to a Case, designated as a *hypothesis* or *hypothetic inference.* The forms obtained from (46) can be schematized as follows:

Induction:

$$(48)\quad \begin{array}{ll} \text{These } C\text{s are } B & \text{(Case)} \\ \text{These } C\text{s are } A & \text{(Result)} \\ \hline \text{All } B\text{s are } A. & \text{(Rule)} \end{array}$$

Hypothesis (hypothetic inference):

$$(49)\quad \begin{array}{ll} \text{All } B\text{s are } A & \text{(Rule)} \\ \text{These } C\text{s are } A & \text{(Result)} \\ \hline \text{These } C\text{s are } B. & \text{(Case)} \end{array}$$

Peirce gave the following example of the relationship between the three forms of reasoning ([Peirce, 1878], CP 2.619-2.644; WCSP 3, 325-26):

Deduction

Rule	All the beans from this bag are white.
Case	These beans are from this bag
Result	These beans are white.

Induction

Case	These beans are from this bag.
Result	These beans are white.
Rule	All the beans from this bag are white.

Hypothesis

Rule	All the beans from this bag are white.
Result	These beans are white.
Case	These beans are from this bag.

This account of induction was based on Aristotle's definition of induction (*epagoge*) in *Prior Analytics* and other writings ([Peirce, 1865/1982]; WCSP 1, 163, 176-180). According to Aristotle (*An. Pr* B 23, 68b32-35, Robin Smith's (1989) translation),

> deduction proves the first extreme [the major term] to belong to the third [the minor term] term through the middle, while induction proves the first extreme to belong to the middle through the third.

Thus induction can be regarded as an inference to the major premise of a (deductive) syllogism. Peirce's schema for induction has this form: in (48), the conclusion states that A (the first or major extreme) belongs to all Bs (the middle), in other words, all Bs are A and this is proved by means of C (the minor extreme or third) [Peirce, 1865/1982, p. 180]. In the same way, Peirce's syllogistic characterization of hypothetic inference resembles Aristotle's account of the inference form called *apagoge.* In this case the major premise (all Bs are A) is well known or obvious to the inquirer, for example, to use Aristotle's example, it is well known that all science is teachable; but it is unclear (not known or understood) whether all C is A, for example, whether virtue is teachable. In this situation the hypothesis that C (virtue or justice) is B would make it possible to know that all C is A, or provide an explanation why all C is A. (Cf. *An. Pr* B 25, 69a20-36; [Smith, 1989, p. 223].)

Thus Peirce arrives to the classification of inference forms presented in Table 1 (CP 2.623).

In 'A Theory of Probable Inference' (1883b) Peirce describes the methodological function of the main forms of reasoning as follows (CP 2.713-2.714; WCSP 4, 423).

> We naturally conceive of science as having three tasks –(1) the discovery of Laws, which is accomplished by induction; (2) the discovery

Table 1. Peirce's Classification of Inference Forms (1878)

Inference		
Deductive or Analytic	Synthetic	
	Induction	Hypothesis

> of Causes, which is accomplished by hypothetic inference; and (3) the production of Effects, which is accomplished by deduction. It appears to me highly useful to select a system of logic which shall preserve all these natural conceptions.
>
> It may be added that, generally speaking, the conclusions of Hypothetic Inference cannot be arrived at inductively, because their truth is susceptible of direct observation in single cases. Nor can the conclusions of Inductions, on account of their generality, by reached by hypothetic inference.

In this passage Peirce adds to the syllogistic model a methodological characterization of the main forms of reasoning. Originally he distinguished deductive reasoning from induction and hypothesis on logical grounds, on the basis of the form of the premises and the conclusion and the logical relationship among them, but here all forms of reasoning, including deduction, are characterized on the basis of their methodological role in inquiry. The purpose of deductive reasoning is "the prediction of effects"; induction should lead to the discovery of laws, and hypothetic inference to the discovery of causes. This methodological approach led to Peirce's mature theory of inference, presented in papers written after 1890.

In a number of papers written in the early 1900's, Peirce began to use the term 'abduction' instead of 'hypothesis', and usually called the three main forms of reasoning 'deduction', 'induction', and 'abduction' (1903/1997, 217-18; CP 5.145). (Sometimes he called abductive reasoning 'retroduction'; cf. [Peirce, 1908]; CP 6.470; 7.97) He also adopted a new view of the character of each form of reasoning, and distinguished them from each other on the basis of their function in the process of inquiry. The function of abduction is to provide (tentative) explanations for phenomena; deductive reasoning is required for deducing testable consequences from explanatory hypotheses, and the task of inductive reasoning is the verification (or falsification) of theories and hypotheses. Peirce describes the role of the three forms of reasoning in inquiry as follows (MS 475, 1903; CP 5.590):

> If we are to give the names of Deduction, Induction and Abduction to the three grand classes of inference, then Deduction must cover every attempt at mathematical demonstration, whether it is to relate to single occurrences or to "probabilities," that is, to statistical ratios; Induction must mean the operation that induces an assent, with or

> without quantitative modification, to a proposition already put forward, this assent or modified assent being regarded as the provisional result of a method that must ultimately bring the truth to light; while Abduction must cover all the operations by which theories and conceptions are engendered.

Peirce illustrated the nature of abduction by the following schema ([Peirce, 1903/1997], 245; CP 5.189):

(50)
The surprising fact, C, is observed.
But if A were true, C would be a matter of course.
Hence, there is reason to suspect that A is true

According to Peirce's new theory, the difference between abduction and induction does not depend on the nature of the conclusion of the inference, but rather on the *modality* of its acceptance. The conclusion of an abduction is a tentative conjecture; Peirce observed that it might be preferable to speak about "conjecturing" rather than "inferring" an abductive conclusion (CP 5.189). Abduction justifies merely an "interrogative" attitude towards a hypothesis (CP 6.469; 6.524). Induction is an attempt to confirm (verify) or disconfirm (falsify) a proposition reached in an abductive stage of inquiry. Peirce's new view of induction as the "logic of verification" is essentially different from the usual (and traditional) conceptions, including his earlier view of induction as inference to a general hypothesis (a "Rule") (cf. [Hilpinen, 2000, pp. 118–119]).

In his 1906 manuscript 'Prolegomena to an Apology for Pragmaticism' (MS 293; NE 4, 313-30), Peirce distinguished induction from a form of reasoning called "probable deduction", and divided non-necessary reasoning into three classes "according to the different ways in which it may be valid":

> probable deduction; experimental reasoning which I now call Induction; and processes of thought capable of producing no conclusion more definite than a conjecture, which I now call Abduction (NE 4, 319).

This characterization differs from that given in Peirce (1883b) (quoted above), according to which the same conclusion cannot be reached by induction and by hypothetic inference.

According to the methodological characterization of the main forms of reasoning, the function of deductive reasoning is to derive testable consequences from an abductively conjectured hypothesis or theory. If deduction is understood in this way, it need not be a logically necessary argument form. Thus Peirce distinguished between two kinds of "deductive" reasoning, logically *necessary* deduction and *probable deduction.* The following inference forms are examples of probable deductions ([Peirce, 1883b, p. 127, 134], 134; CP 2.695, 2.700):

Table 2. Peirce's Classification of the Main Forms of Inference (1906)

Methodological characterization	Logical characterization (based on the relationship between the premises and the conclusion)	
	Necessary (Analytic)	Non-necessary (Synthetic)
Deduction	Necessary Deduction	Probable Deduction
Induction	———	Varieties of Induction
Abduction	———	Abduction(Conjecture)

$$(51)\quad \frac{\begin{array}{l}\text{The proportion } r \text{ of } M\text{'s are } P\\ \text{(This) } S \text{ is } M\end{array}}{\begin{array}{l}\text{Therefore, with probability } r\\ \text{This } S \text{ is } P,\end{array}}$$

or

$$(52)\quad \frac{\begin{array}{l}\text{The proportion } r \text{ of the } M\text{'s are } P\\ S, S', \ldots \text{ are taken at random from among the } M\text{'s}\end{array}}{\begin{array}{l}\text{Hence, probably and approximately}\\ \text{The proportion } r \text{ of the } S\text{'s are } P.\end{array}}$$

Peirce called schema (52) "statistical deduction" (CP 2.700). He presumably regarded (51) and (52) as forms of deduction on the basis of their formal similarity to necessary deductions: necessary deductions are limiting cases of (51) and (52) in which the probability is one. The two main forms of "deductive" reasoning (necessary and probable deductions) are analogous from the methodological standpoint as well: both are forms of *direct* inference (inference from a population to a sample) (cf. [Levi, 1997, pp. 43–44]), and can be used for deriving testable consequences from a general hypothesis.

In (51) and (52), the qualifier "probably" characterizes the relationship between the premises and the conclusion of a probable or statistical deduction, and is not regarded as part of the conclusion. This interpretation agrees with Peirce's remark:

> The conclusion of the statistical deduction is here regarded as being "The proportion r of the S's are P's", and the words "probably about" as indicating the modality with which this conclusion is drawn and held for true (CP 2.720n).

For discussions of Peirce's views on probabilistic reasoning, see [Putnam, 1992, pp. 61–67] and [Levi, 1997].

Peirce's classification of the main types of reasoning is summarized in Table 2.

Peirce distinguished several different forms of deduction and induction. As regards necessary deduction, Peirce made an interesting distinction between two kinds of necessary reasoning, *corollarial* and *theorematic* (or "theoremic") reasoning [Hintikka, 1983; Levy, 1997]. He called this distinction his "first real discovery about mathematical procedure" (1902; NE 4, 49):

> Corollarial deduction is where it is only necessary to imagine any case in which the premisses are true in order to perceive immediately that the conclusion holds in that case.... Theorematic deduction is deduction in which it is necessary to experiment in imagination upon the image of the premiss in order for the result of such experiment to make corollarial deductions to the truth of the conclusion.(NE 4, 38.)

This distinction is derived from the distinction between the subsidiary (or auxiliary) constructions and the "demonstration" (*apodeixis*) in the proofs in Euclid's geometry (CP 4.616). Peirce generalized the geometrical distinction to all deductive reasoning. A corollary (i.e., a proposition established by corollarial deduction) is a proposition deduced directly from certain premises without the use of any "construction" other than what is required for the understanding of the proposition, whereas a "theorem" (a conclusion of theorematic deduction) is

> a proposition pronouncing, in effect, that were a general condition which it describes fulfilled, a certain result which it describes in a general way, except so far as it may refer to some object or set of objects supposed in the condition, will be impossible, this proposition being capable of demonstration from propositions previously established, but not without imagining something more that what the condition supposes to exist... (NE 4, 288–89).

This passage suggests that in a theorematic deduction, it is necessary to consider other objects (individuals) than those needed to instantiate the premises of the argument. Peirce's distinction of considerable philosophical interest, for example, it helps to solve (part of) the problem of "logical incontinence": how can anyone fail to see the logical consequences of the premises one is aware of? [Hintikka 1983, 114]. It is not hard to see how this might happen in the case of theorematic deductions. (However, this leaves open the question about the possibility of logical incontinence in corollarial reasoning.) The distinction can also be used for explicating Kant's distinction between analytic and synthetic necessary reasoning: essentially theorematic reasoning can be regarded as "synthetic" in an interesting logical sense of the word. (Cf. Hintikka [1983, p. 114], [1973, pp. 136–43, 173–78]; see also [Levy, 1997, pp. 105–106].)

The varieties of induction include "rudimentary induction", induction from the fulfillment of predictions, and quantitative induction. In rudimentary induction, objects of a certain kind are assumed not exist on the basis of the evidence that such objects have not been observed. A second form of induction is based on the verification of the predictions made on the basis of a hypothesis. It consists in

> studying what effect that hypothesis, if embraced, must have in modifying our expectations in regard to future experience. Thereupon we make experiments, or quasi-experiments, in order to find out how far these new conditional expectations are going to be fulfilled. In so far as they greatly modify our former expectations of experience and in so far as we find them, nevertheless, to be fulfilled, we accord the hypothesis a due weight in determining all out future conduct and thought (CP 7.115).

The third main form of inductive reasoning, "statistical" or "quantitative" induction, consists in estimating the value of a certain quantity in a population on the basis of the information about a sample drawn from the population (CP 7.120–121). These inference forms have the following feature:

> The conclusion is justified not by there being any necessity of its being true or approximately true but by its being the result of a method which if steadily persisted in must bring the reasoner to the truth of the matter or must cause his conclusion in its changes to converge to the truth as its limit (CP 7.110).

According to Peirce, induction and the inductive method are justified by their self-corrective character: "Although the conclusion at any stage of the investigation may be more or less erroneous, yet the further application of the same method must correct the error." [1903/1997, 218.]

Abduction is distinguished from other forms of synthetic reasoning by the modality of abductive conclusions: as was observed above, an abduction leads to a "conjecture" and can justify only an "interrogative" attitude towards a proposition. According to Peirce,

> Induction shows that something actually is operative, Abduction merely suggests that something *may be.* ([Peirce 1903/1997, 230]; CP 5.171.)

Another distinctive feature of abductive reasoning is that only abduction (unlike induction or deduction) is capable of introducing new ideas and concepts into discourse. According to Peirce 9[1903/1997, 230]; CP 5.171),

> [abduction] is the only logical operation which introduces any new idea; for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis.

It is clear that Peirce refers in this passage to statistical or quantitative induction.

Peirce was interested in developing a "logic of abduction", that is, rules for good abductions. This was an important aspect of his pragmatism. An abduction is an inference which leads to a conjectured explanation, thus the logic of abduction may be expected to include conditions of adequacy for explanatory hypotheses as well as rules for discovering explanatory hypotheses. For example, Peirce put forward the following rules of abduction:

(RA1) The hypothesis (the "conclusion" of an abduction) must be capable of being subjected to empirical testing,

He called (RA1) "the first rule of abduction." Another rule of abduction is:

(RA2) The hypothesis must explain the surprising facts.

Peirce observes that an explanation may be a deductive explanation which renders the facts "necessary" or it may make the facts "natural chance results, as the kinetic theory of gases does". (CP 7.220.) These rules have counterparts in more recent theories of explanation, for example, in Carl G. Hempel and Paul Oppenheim's account. (RA1) and (RA2) correspond to Hempel and Oppenheim's "logical conditions of adequacy" for scientific explanations [Hempel and Oppenheim 1948/65, 247-248].

Peirce's logic of abduction also contains rule which may be called the *Principle of Economy*:

(RA3) In view of the fact that the hypothesis is one of innumerable possibly false ones, in view, too, of the enormous expensiveness of experimentation in money, time, energy, and thought, is the consideration of economy. Now economy, in general, depends upon three kinds of factors: cost, the value of the thing proposed, in itself, and its effect upon other projects. Under the head of cost, if a hypothesis can be pout to the test of experiment with very little expense of any kind, that should be regarded as giving it precedence in the inductive procedure. (CP 7.220 n.18.)

This principle is quite different in character from (RA1) and (RA2) and from Hempel and Oppenheim's logical conditions of adequacy for explanations. It is a *strategic* or *pragmatic* rule for selecting explanatory hypotheses for experimental testing, that is, for targets of inductive reasoning. It can nevertheless be regarded as a logical rule (in a wide sense of the word 'logical').

Peirce's distinction between abduction and induction has sometimes been associated with the logical empiricists' distinction between the context of discovery (the discovery or invention of an explanatory hypothesis) and the context of justification (the confirmation or disconfirmation of a hypothesis by empirical evidence) [Reichenbach 1938]. Many logical empiricists regarded only the latter as a proper subject of logical and philosophical investigation, and thought that the study of the discovery of hypotheses belongs to psychology rather than logic. It is clear the Peirce's rules of abduction can be said to "justify" a hypothesis in the way in which abductive reasoning can justify its conclusions: a good abduction justifies a hypothesis as a potential explanation worthy of further empirical testing. In Peirce's words, we can say that abduction justifies an interrogative attitude towards a hypothesis.

Peirce's idea of abduction as the logic of discovery was resurrected by Norwood Russell Hanson [1958], who used Peirce's term "retroduction" to refer to the reasoning "from surprising data to an explanation" [Hanson, 1958, p. 85]. Isaac

Levi has discussed and analyzed the interrogative character of abductive inferences and suggested that the "conclusion" of an abduction (in Peirce's sense) is "the construction of potential answers to a question under study" ([Levi, 1996, p. 161]; see also [Levi, 1991, pp. 71–77]). Peirce's account of abduction and induction as the main forms of non-demonstrative reasoning has inspired Levi's theory of inquiry and belief revision, articulated in several recent publications (Levi [1997; 1991; 1996; 2000]). In contemporary methodology, abduction is generally recognized as a distinctive form of reasoning, and models of abductive reasoning are being studied in applied logic, cognitive science and artificial intelligence, and in the theory of diagnostic reasoning. (See [Josephson and Josephson, 1994; Magnani *et al.*, 1999; Gabbay *et al.*, 2000; Flach and Kakas, 2000].) For discussions of Peirce's mature theory of abductive reasoning, see Tomis Kapitan [1997] and Jaakko Hintikka [1998].

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I shall follow the usual practice of citing from the *Collected Papers of Charles Sanders Peirce* (Peirce 1931–35, 1958) by volume number and paragraph number, preceded by 'CP'. Peirce (1976) will be abbreviated 'NE', followed by volume number and page numbers, and the chronological edition of Peirce's writings, Peirce (1982-2000), by 'WCSP', followed by volume number and page numbers. References to the microfilm edition of Peirce's manuscripts (Widener Library, Harvard University), catalogued in Robin (1967), will be indicated by 'MS', followed by the manuscript number. The page numbers used in the references to the manuscripts are those used by Peirce (and contained in the microfilm edition of the manuscripts).

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FREGE'S LOGIC

Peter M. Sullivan

"What is distinctive about my conception of logic is that I begin by giving pride of place to the content of the word 'true'..."

(PW 253/NS 273)

1 LIFE AND WORK

The German mathematician, logician and philosopher, Friedrich Ludwig *Gottlob* Frege, was born in Wismar on the Baltic on 8 November 1848. Surprisingly little is known of his life.[1] He remained in Wismar until 1869 when he entered university at Jena. After four semesters he moved to continue his studies at Göttingen; there he studied mathematics, physics, and a little philosophy. He gained his Ph.D. from Göttingen at the end of 1873 following submission of a dissertation, *On a Geometrical Representation of Imaginary Forms in the Plane*, and immediately applied to return to Jena. On successful examination of his *Habilitationsschrift*, *Methods of Calculation based on an Extension of the Concept of Quantity*, he was appointed *Privatdozent* (lecturer) in mathematics in May 1874. Twice promoted, he remained a member of the Jena mathematics department until, in 1918, he retired to the country at Bad Kleinen, just south of his home town, where he died in 1925. Frege married in 1887, but the couple had no children, and his wife died in early middle-age. He was survived only by his adopted son, Alfred, and his philosophical *Nachlass*. Alfred was killed in the Second World War; the *Nachlass* was destroyed in a bombing raid on Munster in March 1945; luckily, his then editor, Heinrich Scholz, had kept transcript copies.

During his lifetime Frege's work made little impression. However, the few people he did significantly influence — Husserl, Peano, Russell, Wittgenstein, and Carnap — themselves became enormously influential, so that indirectly he can be said to have shaped a whole philosophical tradition. Since about 1950 his work has been studied more at first hand, and never more so than now. But Frege was dead long before his importance was generally recognized. In contrast to some other philosophers, not much of Frege's character comes through in his writing; what does come through, all too clearly in his later work, is bitterness, disillusion,

[1] For many years the fullest account was Bynum's "On the life and work of Gottlob Frege", which accompanies his translation of *Begriffsschrift* in CN; Kreiser [2001] is the first full biography.

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resentment over the neglect of his work and envy of the acclaim achieved by others. Some of the examples his chooses in his writings smack of German nationalism, and a private diary from 1924 shows anti-democratic prejudice and a hostility to Jews striking even in a generally anti-Semitic environment. He was probably not a pleasant man to know.

Frege's major publications represent three stages in the project that occupied the core of his working life. Subsequently termed 'logicism', this project aimed to demonstrate "that the laws of arithmetic are analytic and consequently a priori" (Gl §87), or in Frege's later formulation, that "arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition" (Gg §0/BL 29). In *Begriffsschrift* (1879) Frege set out the system of logic without which rigorous demonstration of the logicist thesis could not be so much as attempted, and illustrated the power of this system by establishing general results about sequences, including a generalization of the principle of mathematical induction. In *Grundlagen* (1884) Frege first argued the need for his project, by exposing unclarity and confusion in prevalent views of the foundations of arithmetic, and then sketched informally his proposed construction. Its formal execution was left to his *Grundgesetze* (1893, 1903). Part I of that work sets out and elucidates his system of logic, importantly revised since *Begriffsschrift*. Part II, straddling volumes (i) and (ii), pursues the formal construction sketched in *Grundlagen*: basic laws of the theory of natural numbers are established, though, strangely, Frege does not specially highlight or isolate a group of axioms for arithmetic, and offers no treatment of addition and multiplication. Part III of *Grundgesetze* addresses the theory of real numbers. Its first part is a philosophical argument in prose which seeks to motivate his construal of the real numbers, as *Grundlagen* had motivated the construction of the natural numbers presented in Part II, by criticism of rival views. Its second part begins the formal construction, but is left incomplete at the end of volume (ii). A third volume was planned but never appeared.

The story that explains that is well known. On 16 June 1902 Russell sent to Frege a letter explaining the contradiction derivable from his Basic Law V, the 'naïve' axiom of set existence which is *Grundgesetze*'s principal addition to the system of *Begriffsschrift*. With volume (ii) already at the press, Frege had time only to acknowledge the flaw and propose a fix. The fix, a restriction on Basic Law V explained in an Appendix to volume (ii), does not work. The revised axiom remains contradictory, though whether Frege ever realized that is not clear. On the other hand, it is too weak for Frege's central proofs to go through, something Frege plainly did recognize [Dummett, 1991, p. 5]. Dating from 1906, two years before Zermelo's axiomatization, Frege's response to that realization now seems absurdly sweeping: "Set theory in ruins" (PW 176/NS 191). But the description applied well enough to Frege's own project.

From that low point Frege immediately continued, however, "my concept-script in the main not dependent on it" (PW 176/NS 191). And when, in a fragment composed within a month, he with desperate resilience posed the question, "What may I regard as the result of my work?", his answer begins, "It is almost all tied up with the concept-script" (PW 184/NS 200). The logicist project had failed. But the system of inference Frege had devised as a tool for this project in *Begriffsschrift*, and the insights into the nature of logic it embodied, survived the collapse. Though, as already remarked, it took some time to be appreciated, that became the common view: if the Appendix to *Grundgesetze* registers logical philosophy's most notorious failure, *Begriffsschrift* is unquestionably its greatest success. There is no exaggeration in Kneale and Kneale's verdict that "the deductive system or calculus which [Frege] elaborated is the greatest single achievement in the history of the subject" [Kneale and Kneale, 1984, p. 444], making 1879 "the most important date in the history of [logic]" (*ibid.*, p. 511).

Our concern here is with Frege's logical achievement, rather than with the fate of logicism. Much the greater part of this essay is therefore devoted to *Begriffsschrift*, in which that logical achievement is already made, later works being drawn upon principally to illuminate themes present in, though not made explicit in, that work. Only in a final section will I turn briefly to consider a development distinctive of his later work, his influential distinction between sense and reference. It will, though, be important throughout to remember that his logical system was, for Frege, in the first instance a necessary tool of the logicist project.

2 BEGRIFFSSCHRIFT (I): THE SYSTEM

2.1 Introduction

From one point of view, Frege's logic *needs* no explanation. The system of logic he presents in *Begriffsschrift* simply *is* modern logic. Furthermore, it needs no subtle or questionable exegesis to recognize it as such: anyone with a basic grounding in contemporary quantificational logic can immediately recognize in *Begriffsschrift* a version of what he has learned. There are, of course, differences of emphasis, and some points are explained differently from what one would now expect. Some of these divergences are important and we will need below to set them out, and to ask whether they signal important differences in conception of the ground, role or nature of a logical system. Even so, it is remarkable that the differences most likely to cause a modern reader to stumble are wholly trivial matters of notation. Reading the work of Frege's predecessors often seems to involve picking out familiar features in a largely foreign landscape. No one has that experience in reading *Begriffsschrift*.

No doubt that was part of what led to Dummett's remark that *Begriffsschrift* "is astonishing because it has no predecessors: it appears to have been born from Frege's brain unfertilized by external influences" [Dummett, 1981, p. xxxv], with its suggestion that, from another point of view, Frege's invention of modern logic *admits of* no explanation. To say that is not, of course, to refuse Frege's systematization of logic its place in the general nineteenth-century trend in pure mathematics towards rigour in foundations, a trend exemplified in Weierstrass's reformulation of the calculus and Dedekind's foundations for number theory, and culminating in Hilbert's axiomatization of Euclidean geometry. Nor is it to deny that quantificational logic was an idea 'whose time had come': it can hardly be coincidence that Peirce was led, independently and only four years later, to introduce a notation for quantifiers (his terminology) and bound variables (which he called 'indices') into his study of the logic of relations.[2] But whereas Peirce's innovations arrive piecemeal and in response to particular inadequacies of the Boolean framework he was developing, in *Begriffsschrift* modern logic appears to spring forth fully formed. The work's list of 'firsts' is remarkable: the first complete presentation of truth-functional propositional logic; the first representation of generality through quantifiers and variables, allowing the first formulation of reasoning involving multiple nested generality; the first formal system of logic, in which correctness of inference is to be confirmable by syntactic criteria; the first mathematically significant employment of higher-order logic, in the reduction of inductive to explicit definitions. But the *most* remarkable feature of *Begriffsschrift* is that all of these arrive *at once*. The very integration of the work obstructs any detailed reconstruction of how its elements arose in Frege's thought. And the extant pre-*Begriffsschrift* writings in any case provide no basis for that kind of genetic explanation. There is scarcely a sentence in what he published before 1879 that, without strained hindsight, would lead one to expect that Frege's next major work would be in logic, let alone to anticipate its form.

In *Begriffsschrift* itself, too, Frege says remarkable little — indeed, almost perversely little — to motivate his innovations. It is the work of a mathematician. It addresses itself to the truth, whilst hardly caring to address itself to any actual reader. Perhaps Frege with hopeless naïvety imagined that his system would speak for itself. Or perhaps his failure to engage and encourage readers familiar with existing treatments of logic is an indication that even he did not fully appreciate the huge gulf separating his achievement from his predecessors'. However that may be, the uniformly uncomprehending response *Begriffsschrift* evoked prompted him to remedy the omission in a number of expository and comparative articles. To understand the points he then made, though, we must begin, as he did, by setting out the elements of his system. This is done in the following section with minimal comment; subtleties of motivation and disputable questions of interpretation are reserved for subsequent discussion.

2.2 Elements of the system of Begriffsschrift

2.2.1 Judgement and content

Any sentence of the concept-script has the overall form

$$\vdash \Gamma$$

in which we must distinguish three elements.

[2] See [Kneale and Kneale, 1984, 430 ff.] and the essays collected in Section 15 of [Ewald, 1996].

(i) Γ stands in for a symbol giving the specific content of the sentence, which must be a *judgeable content*, that is, the content of a possible judgement.

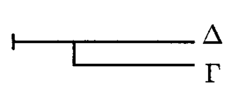
(ii) The horizontal *content stroke* that precedes it may be prefixed only to symbols with such a content. Frege describes the content stroke as "[tying] the symbols which follow it into a whole" (Bs §2), thereby implying that a symbol with judgeable content will always be complex. He adds that it "serves also to relate any sign to the symbols which follow it" (§2); that is, it has an important role, shortly to be explained, in demarcating scope.

(iii) Finally the vertical *judgement stroke* serves to effect assertion: by attaching the judgement stroke to '— Γ' one records one's judgement of the truth of the content expressed by Γ.

To emphasize that no assertion is yet made by a symbol '—Γ' Frege suggests that it might be paraphrased by a nominalization, 'the circumstance that Γ' or 'the proposition that Γ' (§2). Continuing in that vein, a reading for '$\vdash \Gamma$' would then need to incorporate a verb-phrase to counter the nominalization, for instance, 'the circumstance that Γ is a fact' (§3). These are no more than suggestive readings. The invariable role of the judgement stroke is to effect assertion. No device of natural language has that role: that is, no part or aspect of a natural language sentence invariably indicates that the sentence is used to make an assertion. And because a symbol '— Γ' has a role complementary to that of the judgement stroke (because it is what remains of an asserted sentence when the judgement stroke is omitted), the same point transfers to it. There is thus no natural language equivalent for either the judgement stroke or the content stroke to which their use might be held responsible.

2.2.2 Truth-functions

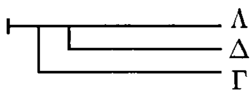
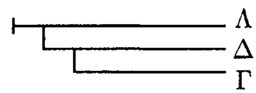
Compound formulae of the concept-script exploit the two-dimensionality of the page. Component sub-formulae, or letters indicating them, are arranged vertically on the right. To their left a network of lines formed from content strokes connected by Frege's symbols for logical operations structures the component sub-formulae unambiguously into a whole. Frege chooses material implication and negation as his primitives for propositional logic. An implication $\Gamma \rightarrow \Delta$ is written



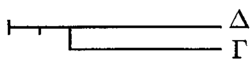
so that the contrast between

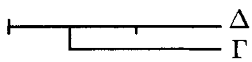
$$\Gamma \rightarrow (\Delta \rightarrow \Lambda) \text{ and } (\Gamma \rightarrow \Delta) \rightarrow \Lambda$$

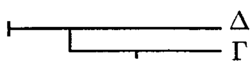
is clearly made without need for brackets by

 and 

Negation is symbolized by a short vertical stroke beneath the content stroke of the negated formula, again avoiding the need for supplementary indication of (or conventions governing) scope. Thus

 corresponds to $\neg(\Gamma \to \Delta)$

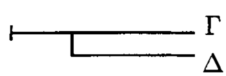
 to $(\Gamma \to \neg\Delta)$

 to $(\neg\Gamma \to \Delta)$

Implication and negation are defined by Frege in the now familiar truth-functional way. Where Γ and Δ stand for judgeable contents Frege tabulates the four possibilities as follows:

1. Γ is affirmed and Δ is affirmed
2. Γ is affirmed and Δ is denied
3. Γ is denied and Δ is affirmed
4. Γ is denied and Δ is denied.

Then



"stands for the judgement that *the third of these possibilities does not occur, but one of the other three does*" (§5);

means "[Γ] does not occur".

One might wonder whether it is significant that Frege gives the cases as those in which Γ is "affirmed" and "denied", rather than those in which it is true and false; but there seems to be nothing in this. He immediately goes on to use "must be affirmed" and "is to be denied" as indifferent alternatives, and it seems clear that what "must be affirmed" and what "is to be denied" are simply what is true and what is false. And in a related paper Frege tabulates the cases by "correct" (*richtig*) and "false" (*falsch*) (PW 11/NS 12). On the other hand, nothing in Frege's presentation shows that he regarded the four "possibilities" or "cases" as full-fledged *truth-possibilities*, on the understanding that Wittgenstein later gave to this notion, rather than, say, possibilities-for-all-we-know or even possibilities-for-all-so-far-said.

Frege explains how other truth-functions are expressed by his primitives but introduces no defined symbols for them. Similarly, he explains that other functions might have been chosen as primitive and implication defined. In each case he is clear that the choice is merely a pragmatic one. Implication is taken as primitive to provide the simplest formulation of his chosen inference rule, *modus ponens* (cf. PW 37/NS 42). And in restricting his language to an austere primitive vocabulary Frege is clearly aware of the trade-off between long-windedness in object-language proofs and conciseness of metatheory.[3]

2.2.3 Functions and generality

Frege divides his symbols into non-alphabetic characters with fixed meaning and letters used to convey generality. Greek capitals he uses, as above, to formulate generalizations about the system. Generality within the system is expressed by italic and by gothic letters. To modern eyes Frege's gothic letters appear in his formulae as bound variables preceded by an occurrence of the same letter in connection with his universal quantifier symbol – written as a concavity in the content stroke – serving explicitly to demarcate scope; his italic letters appear as free variables. Frege's official explanation is different: italic letters are introduced as part of an "abbreviation" of formulae where "the content of the whole judgement constitutes the scope of the gothic letter" (§11), so are themselves to be understood as implicitly bound by a quantifier with maximum scope. In accordance with that explanation Frege first introduces the expression of generality by means of gothic variables. But this official explanation does not fit Frege's practice: an italic variable figures in his axiom governing the quantifier; and in the inferences it permits italic variable formulae cannot be treated as mere abbreviations.[4] Taking a licence from that fact, I will here reverse the explanations, as Frege himself typically does in informal presentations (PW 11/NS 12; PW 52/NS 58; CN 207/BaA 92; CN 98-9/BaA 104-5).

If a symbol Γ is a component of a proposition $\Phi(\Gamma)$ then we can conceive of a range of propositions derived from the first by replacing Γ by other symbols.[5]

[3]Restricting the number of primitives, Frege says, makes it easier "to survey the state of a science", though "the result is more cumbersome formulae" (PW 36/NS 40).

[4](i) *Principia* takes $\neg$ and $\vee$ as primitive but then formulates its propositional axioms using $\rightarrow$. That, as we saw, is not Frege's way: "you don't lay down as primitive the sentences which hold for [defined] signs, you derive them from their meanings" (PW 36/NS 40). (ii) The possibility of drawing inferences to and from quantified formulae depends on treating italic variables as 'dummy constants' are treated in many modern formulations.

[5]That this is possible is part of what it means to call Γ a component of $\Phi(\Gamma)$: that nothing is said by the result of replacing 'mat' by 'carpet', 'rug', etc. in 'Fermat was a genius' is what shows that 'mat' is not a component of that proposition.

Each member of this range of propositions displays a common pattern, $\Phi(\ldots)$, completed differently in each case; so to conceive $\Phi(\Gamma)$ as one member of that range is to think of it as dividing into a constant component shared by them all and a replaceable component distinctive of it. Frege says, "I call the first component a function, the second its argument" (Bs §9). Or more generally,

> If, in an expression (whose content need not be assertible), a simple or complex symbol occurs in one or more places, and we imagine it as replaceable by another [symbol] (but the same one each time) at all or some of these places, then we call the part of the expression that shows itself invariant [under such a replacement] a function and the replaceable part its argument. (§9)

If, now, the argument is replaced by a variable, we may form the sentence

$$\vdash \Phi(x)$$

which expresses "the judgement that the function is a fact whatever we may take as its argument" (§11).

In this account of the division of a proposition into function and argument, and the consequent explanation of the truth of a generalization as consisting in the truth of each of its possible instances – that is, of each proposition in which the same function is completed by an appropriate argument – we have the essence of Frege's contribution to logic. The immense power of this contribution is owed to two features.

First, the account is iterable. Having, for instance, arrived at the judgement

$$\vdash \quad 2 + x = x + 2$$

by considering the symbol '3' in

$$— \quad 2 + 3 = 3 + 2$$

as replaceable by others, we may again consider '2' as replaceable by other symbols to form the judgement

$$\vdash y + x = x + y.$$

Thus Frege's account of the significance of a generalization, applied repeatedly to its own results, fixes the significance of a proposition involving any number of expressions of generality.

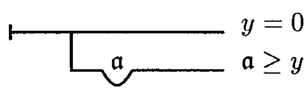
Secondly, the content of a judgement arrived at in this way "can occur as part of a judgement" (§11). We then need some device to indicate how much of the whole constitutes the function whose argument is indicated by a variable, so as to distinguish clearly the true conditional with generalized antecedent,

$$\text{if } x \geq y, \text{ whatever } x \text{ might be, then } y = 0$$

from the falsehood

$$x \geq y \rightarrow y = 0.$$

As Frege explains,

> the generality to be expressed by means of the x must not govern the whole... but must be restricted to [its antecedent]. I designate this by supplying the content stroke with a concavity in which I put a gothic letter which also replaces the x:
>
> 
>
>
> I thus restrict the scope of the generality designated by the gothic letter to the content, into whose content stroke the concavity has been introduced... So our judgement is given the following expression
>
> 
>
>
> (PW 19-20/NS 21-2, with simplified example.)

These two features mesh to yield the result that "the scope of one gothic letter can include that of another" (§11). In this account of nested generality lies the distinctive power of quantificational logic.

Frege's conception of function-argument division is in two respects still more general than so far indicated. First, having by regarding Γ as replaceable in $\Phi(\Gamma,\Delta)$ arrived at the function $\Phi(\xi,\Delta)$, we may in turn regard Δ as replaceable: "in this way, functions of two or more arguments arise" (§9). Secondly, since Φ is itself a symbol occurring in $\Phi(\Gamma)$, and since we may also "think of it as replaceable by other symbols..., we can consider $\Phi(\Gamma)$ as a function of the argument Φ" (§10, letters modified). As an argument, Φ is replaceable by a variable to form generalizations, so that the resulting language is of second order.

2.2.4 Identity

In some small respects Frege's explanation of the symbolism so far considered falls short of contemporary standards of explicitness. His treatment of identity, by comparison, is a mess.

Frege employs the congruence sign '$\equiv$' in contrast to the ordinary identity sign to signal its intended metalinguistic interpretation. $\Gamma \equiv \Delta$ is to express that "the two names [Γ and Δ] have the same content. Thus with [its] introduction... a bifurcation is necessarily introduced into the meaning of every symbol, the same symbols standing at times for their contents, at times for themselves" (Bs §8). By his practice Frege shows that he sees in this ambiguity no threat to the cogency of inference.

If symbols Γ and Δ have the same content, one might think one of them enough: one might think, that is, "that we have absolutely no need for different symbols of the same content, and thus no [need for a] symbol of identity of content either" (§8). Frege responds to this thought with a geometrical example involving two complex terms designating a single point. Switching to a simpler example he used later, let a, b and c be lines connecting the vertices of a triangle with the mid-points of the opposite sides. Then 'the intersection of a and b' and 'the intersection of $b$and c' have the same content, designating the same point, yet the difference between them is no "indifferent matter of form". Rather, the two names are "associated with different modes of determination" of that single content. That a single item is determined in those two ways is the content of a judgement, whose expression demands distinct names associated with those modes of determination.

The above reasoning, which constitutes almost all of Frege's motivation and explanation of his symbol for identity of content, is strangely disconnected from his employment of the symbol in the remainder of *Begriffsschrift.* At the forefront of his mind in this explanation in Chapter 1 are identities framed with complex singular terms; yet no such terms are introduced in the formal developments of Chapters 2 and 3. It is perhaps that fact that prevented Frege from examining his reasoning too critically. If it is granted, as Frege assumes without argument, that the content of a complex term is simply the thing it stands for, then, since it is evident both that the complexity of the term is *somehow* implicated in what is expressed by an identity statement involving it[6] and that this complexity is not in general a feature of the thing that is its content (a point, in Frege's example), there is some initial plausibility in inferring that what is expressed by the identity statement concerns the term rather than the thing. But had Frege actually introduced complex terms he would of course have gone on to use them in other than identity statements — plain statements, as we might call them. It would then have become obvious that there is just as much reason to hold that the complexity of a term *somehow* contributes to what is expressed by a plain statement involving it as there is in the supposedly special case of identity statements, but also that its contribution to a plain statement cannot be accounted for by supposing that statement to be about the term rather than its content — cannot, that is, without rendering altogether otiose the notion of the content of the term. The apparently plausible reasoning for a metalinguistic construal of identity statements would then have evaporated.

Be that as it may, Frege's actual uses of his identity sign gave him reason enough to doubt his account of its meaning. Those uses fall into two. First, it is used in the

[6] I mean this to be understood in a vague and intuitive way. What is evident is only (e.g.) that the fact that lines a and b are mentioned on one but not the other side of 'point $x =$ the intersection of a and b' has *something* to do with why the sentence is suited to express a non-trivial judgement, not any theoretically committed diagnosis of exactly what it has to do with it.

formulation of definitions, where the metalinguistic construal has some credibility (though as Frege recognized, since these definitions merely introduce convenient abbreviations, the idea that distinct symbols for the same content are needed to capture the content of important judgements would then fall by the wayside). But secondly, and essentially to Frege's purpose, the identity sign occurs between bound variables, for instance in expressing that a relation is many-one: in that context a metalinguistic construal is patently incoherent (since the role of 'x' and 'y' in '$(Rzx \wedge Rzy) \rightarrow x = y$' is not to stand for objects, the role of the final clause cannot be to maintain that they stand for the same object).

However ill-motivated, Frege's metalinguistic account of the identity sign licenses its use between different categories of expressions: while only things of the same type as Boston can be said to have six million inhabitants, nouns, adjectives and sentences can equally well be said to contain six letters. That Frege exploits this licence, counting as well formed sentences in which the identity sign is flanked by (variables standing in for) sentences as well as (...) singular terms, does not therefore indicate, as some have held,[7] that he recognized no fundamental difference of type between the two. By the same token, his account appears equally to license formulations in which different types of expression stand on either side of a single identity sign; no such formulation occurs in *Begriffsschrift* — Frege having, at the least, no call to assert such a thing — but it is unclear whether, or how, the language excludes them.

In the Preface of *Begriffsschrift* Frege curiously remarks that "[he] noticed only later" that $\neg\neg a \rightarrow a$ and $a \rightarrow \neg\neg a$ "can be combined into the single formula" $\neg\neg a \equiv a$ (CN 107/BaA xiv). How might this economy — a rather obvious one, if a genuine one — have escaped his notice? The answer seems to be that what dawned on Frege "only later" was that one of his principal uses of the identity sign rendered it, in effect, a biconditional. In any case, within a year the opposite economy had come to seem preferable: "I no longer regard [identity] as a primitive sign, but would define it by means of others" (PW36/NS 40) — he does not say how. These passages are an important counterbalance to the criticisms made above. They show quite plainly that, at the time of the publication of *Begriffsschrift*, Frege's thought was still in flux. Problems with his account of identity are the most obvious symptom of matters still to be resolved, but it is essential to realize that they are no more than a symptom. Underlying them are tensions in Frege's conception of how the complexity of an expression relates to complexity in its content, tensions that in due course will pull the notion of content apart.

2.2.5 Axioms and rules[8]

The system of logic presented in *Begriffsschrift* is axiomatic. In one way, this is hardly worth remarking. Euclid's *Elements* provided a model for the whole

[7] E.g. [Baker and Hacker, 1984, pp. 149–50].

[8] These are not Frege's own terms, but importing them does no damage to his understanding of the notions. His own terminology, which we will encounter in the course of the discussion, is, to say the least, less crisp.

nineteenth-century trend toward rigour of which *Begriffsschrift* was a part, and indeed no other model for the systematic and rigorous development of a science was available. But we should note immediately that in adopting that model Frege was according to logic the status of a science, and that this marks a distance between his conception and those both of his most important predecessor, Kant, and his most important successor, Wittgenstein. For neither of these did logic constitute a body of knowledge, nor therefore something revealingly presented as a deductively organized body of knowledge. For both — though, of course, in their different ways — logic was an adjunct or an auxiliary to science rather than a science itself. For Wittgenstein the so-called 'truths' of logic are empty, a by-product of genuine, scientific truths. For Kant they are a merely formal constraint, conformity to which is only ever a necessary condition, but never of itself a sufficient ground, for truth. In each case this characterization contributes to an explanation of the *a priority* of logic: not consisting of genuine truths, logic is not answerable to reality for its truth, so no issue arises of how its conformity to reality can be known. In counting logic a science Frege, it seems, cannot follow them in that line of thought, so we will need to ask what he might put in its place.[9]

After Gentzen it comes readily to contrast Frege's axiomatic approach, on which logic is directed towards establishing a body of truths of its own, with one according to which logic's concern is with rules of inference by which non-logical propositions may be derived from others. This 'natural deduction' approach is perhaps pre-echoed in Wittgenstein's contrast between proof *in* and proof *by* logic (TLP 6.1263), and can in turn yield a sense in which logical truths — those derivable from the null set of assumptions — appear as a degenerate case, a by-product of something whose proper rationale lies elsewhere (cf. [Dummett 1981, p.434]). But despite these points of similarity this contrast is orthogonal to the one just drawn. Frege was well aware that the burden of logic could be differently apportioned between axioms and rules (though the idea of assigning it to rules alone never occurred to him). The previous contrast has to do with whether logic carries *any* burden, not with how its burden is distributed.

In modern notation, but retaining Frege's use of letters, the axioms of *Begriffsschrift* are these (the numbers on the right are their numbers in Frege's development).

1. $a \to (b \to a)$ (1)

2. $(c \to (b \to a)) \to ((c \to b) \to (c \to a))$ (2)

3. $(d \to (b \to a)) \to (b \to (d \to a))$ (8)

4. $(b \to a) \to (\neg a \to \neg b)$ (28)

5. $\neg\neg a \to a$ (31)

6. $a \to \neg\neg a$ (41)

[9] §4.3 considers the question. The contrast with Kant is developed in §4.1.

7. $c \equiv d \to (f(c) \to f(d))$ (52)

8. $c \equiv c$ (54)

9. $\forall \mathfrak{a} f(\mathfrak{a}) \to f(c)$ (58)

These, together with the "principles of pure thought" already introduced and embodied in "rules for the application of our symbols", constitute the "kernel" of his system.

Frege's recommendation of this approach consists in a recitation of the virtues of axiomatic systems in general (Bs §13). The kernel contains "in embryonic form" the content of a "boundless number of laws" which could not be individually enumerated. The value of deducing other laws from this kernel is in general "not to make them more certain... but to bring out the relations of judgements to one another". In aiming to display these relations "it seems natural to deduce the more complex of these judgements from the simpler ones", Frege says, but neither here nor elsewhere in *Begriffsschrift* does he offer any absolute characterization (e.g. self-evidence) that an axiom must meet.[10]

Frege claims completeness for his axioms, i.e. that they imply "all the other" laws of thought, but he offers no semantic explanation of this claim nor any reason to think it true: one could not discern from his discussion here whether Frege took the question of completeness to be amenable to systematic investigation.[11] The question was answered much later. Łukasiewicz in 1934 proved the completeness for classical propositional logic of Frege's axioms 1–2 and 4–6 (with Frege's rules), and thereby the redundancy of 3.[12] And with a natural construal of his rules regarding the quantifier, addition of the remaining axioms yields a complete systematization of first-order logic with identity (and a systematization of higher-order logic no more incomplete than it has to be).

In his Preface (CN 107/BaA xiii; cf. PW 37/NS 42) Frege claims to have employed only one mode of inference, *modus ponens*, but his more careful statement qualifies this: "... at least in all cases where a new judgement is derived from more than one single judgement" (§6, CN 119). The further rules this qualification allows for are all connected with his gothic and italic notations for generality, and Frege is at no great pains to separate out inference rules from his explanation of the notation and conventions governing it. These additional rules include:

(i) Instantiation: "from a [universally quantified] judgement [$\forall \mathfrak{a} \Phi \mathfrak{a}$] we can always derive an arbitrary number of *judgements with less general content*

[10]However, both *Grundlagen* (§5) and *Grundgesetze* (Appendix, BL 127) require that axioms be self-evident.

[11]PW 37–8/NS42–3 says rather more on the matter: "The fundamental principle of reducing the number of primitive laws as far as possible wouldn't be fully satisfied without a demonstration that the few left are also sufficient". The kind of "demonstration" Frege goes on to sketch is, however, broadly experimental – one picks on some difficult theorems, and shows that the axioms will yield them – hence nothing in the nature of what would now be recognized as a completeness proof.

[12]See Bynum's Introduction, CN 73. Kneale and Kneale [1984, p. 490] give the proof.

[$\Phi(\Gamma)$] by putting something different each time in place of the gothic letter" (§11, CN 130).

(ii) Uniform replacement of bound variables.

(iii) Substitution: "other substitutions [than (ii)] are permitted only if the concavity follows immediately after the judgement stroke..." (CN 130).

(iv) Generalization: "An italic letter may always be replaced by a gothic letter which does not yet occur in the judgement; when this is done, the concavity must be placed immediately after the judgement stroke" (CN 132).

(v) Confinement: "...from $\Gamma \to \Phi a$ we can derive $\Gamma \to \forall \mathfrak{a}\Phi\mathfrak{a}$ if Γ is an expression in which a does not occur and a stands only in argument places of Φa" (CN 132, notation altered).

The inexplicitness of (i) and (iii) contrasts notably with the syntactic specification of the remaining rules.

In the case of (iii) this has led to a common complaint that Frege relies on but does not supply principles governing substitution.[13] If the aim is a formal system in the modern sense this complaint is justified, but there are other ways of looking at the matter. Since the letters replaced in substitution are by Frege's account generalized variables, it might be regarded as an inference licensed by his non-syntactic instantiation rule (i). But Frege's comments on the tables by which he records substitutions suggests another more interesting possibility. These tables, he says, "serve to make [the original] proposition... more easily recognizable in the more complex form in which it appears [in its substitution instance]" (Bs §15). Recognizing the legitimacy of a substitution instance is on this account a matter of discerning within it, by disregarding inessential complexity, a pattern whose general validity has already been affirmed. The underlying ground of substitution inferences would then be the account of function extraction given in §9, which explicitly allows that the elements conceived as variable in extracting a common pattern may be complex.

This suggestion comes under strain, however, in those few cases (theorems 77, 93) where Frege appeals to (direct consequences of) his quantifier axiom, $\forall \mathfrak{a} f\mathfrak{a} \to fc$, in justifying second-order inferences, substituting (as we would say) second-level predicates for first-level predicate variables. It would surely be a misdiagnosis of this to hold that Frege cited a first-order axiom when he needed a second-order one. Rather, his citing the axiom in these cases shows that he did not understand it as first order, but instead as the type-neutral principle that if a function holds of every argument it holds of any. It is indeed overwhelmingly the natural view that there are type-neutral logical principles, e.g. that there is a

[13]A reason for the attention this has received is the strength of the principle Frege tacitly invokes: in the context of his higher-order system free substitution of formulae for predicate letters is equivalent to the comprehension schema, $\exists F(Fx \leftrightarrow Ax)$, F not free in A. See [Boolos, 1985, pp. 167 and 171].

single principle of Barbara, exemplified by properties of any level, to the effect that if wherever one property applies a second does, and wherever the second applies a third does, then wherever the first applies so does the third. The alternative, hierarchical conception, which allows only for an open series of such principles with no logical connection between them, and thereby renders the laws of thought essentially unsurveyable, is one that is hard to state consistently, let alone to accept.[14] In just sketching it I spoke of '*such* principles', but the conception allows no meaning to this phrase, since it precludes any logical commonality between principles holding at different levels of the hierarchy. That predicament is typical, and avoidable only by contenting oneself with a merely syntactic generalization.[15] The grounds of the hierarchical conception that forces this problematic alternative on us are certainly present in *Begriffsschrift*, but Frege had not yet drawn the consequence. The immediate relevance of this point is that those grounds are intrinsically connected with §9's account of function extraction. So, if the previous paragraph was right in suggesting that this account underlies Frege's confident use of substitution, there is no avoiding the conclusion that his practice and his account of it are here in conflict.

By contrast, Frege seems presciently conscious of the danger Lewis Carroll exposed in confusing the status of axioms and rules. He begins the formal development of Chapter II by distinguishing, on the one hand, those "principles of thought" which, because they appear as "rules for the application of our symbols", form the "basis" of the concept-script and thus "cannot be expressed in [it]", from on the other "judgements of pure thought" that are to be "stated in [these] symbols" (Bs §13). No clearer acknowledgement of Carroll's point could be looked for. At the same time, Frege plainly holds that the contrast has only an intra-systematic significance: it is essential to a systematization of the "principles of thought" that it respects the contrasting and complementary roles of axioms and rules, but it is not essential to a given principle of thought that it should figure systematically in one of those roles rather than the other. So Frege holds that a theorem may "express the truth implicit in" a rule, and conversely that, pragmatic reasons apart, "we could make a special mode of inference of judgements expressed in formulae" (§6, CN119-20; cf. PW 37/NS 42). Principles of thought, he says, are "transformed into" rules and "correspond to" laws (§13). It would thus be a mistake to reason that recognition of the validity of a rule cannot constitute, or be grounded in, a judgement — on the ground that an expression

[14] It is, Frege says, "one of the requirements of reason [that it] must be able to embrace all first principles in a survey" (Gl §5). It is not clear how Frege thought this demand could be met within his hierarchical conception.

[15] A formulation of type-theory will typically attach numerical subscripts to variables as indices of their order, and then assert the validity of certain formulae for all values of these subscripts. But the syntactic complexity of a variable 'x_n' in this kind of formulation reflects no semantic complexity: we cannot suppose, as it were, that in 'x_n' the 'x' signifies generality while the subscripted 'n' specifies its range, for then 'x' would be the unrestricted variable which type-theory prohibits. Hence the kind of generalization mentioned, which exploits the complexity of 'x_n', is irremediably syntactic.

of this judgement would then have to be included in a full list of the premises on which rests a conclusion drawn in accordance with the rule, so starting the Carroll regress.[16] That train of reasoning overlooks Frege's careful distinction between a principle of thought and the role accorded it in any particular systematization, or equivalently, it conflates judgement *tout court* with judgement in such and such a particular system.

If the contrast of axioms and rules has only an intra-systematic import we should not expect any basic contrast in the kind of justification offered for them, and Frege's text bears this out. Frege describes the validity of his principal inference rule, *modus ponens*, as an obvious consequence of the truth-functional explanation of the conditional, and confirms it by truth-tabular reasoning (Bs §6, CN 117). Precisely parallel reasoning is presented to confirm the truth of the axioms governing the conditional in §§14 and 16. It would be hard to imagine a clearer example than these sections present of the grounding of axioms and rules in a semantic explanation of the symbols they involve.

Despite this, it is characteristic of a recent line of interpretation to seek a different account of the reasoning Frege apparently offers in such passages. To construe them as offering genuine reasoning, it is said, is to adopt a meta-perspective, an external perspective *on* logic – a stance characteristic of twentieth-century studies in logic but one quite foreign to Frege, who countenances no "real" or "serious" meta-perspective [Ricketts 1986]. What force this thought has will depend on the understanding of "real" or "serious" – something we will have to return to (§4.2). But if the thought were construed in such a way as to rule out an external perspective on a particular and in various respects arbitrary systematization of logic it would be patently absurd. The concept-script is, after all, an invented language, so one that trivially stands in need of an explanation not conducted in it. That much externality is, presumably, too cheaply had to count as a "real" meta-perspective.[17] But the example of Carroll's regress discussed above shows that the two are not always clearly separated.

2.3 A formal system?

The above discussion invites the question, to what extent Frege presents in *Begriffsschrift* a formal system. Only a preliminary answer is offered here: many of the issues that have to be touched on will need fuller discussion in later sections.[18]

[16]Ricketts reasons in this way in his [1986, p. 83].

[17]Though some examples of this line of interpretation would suggest otherwise. Weiner, for instance, argues that the axiom $a \rightarrow (b \rightarrow a)$ cannot be justified by the definition of the conditional because the conditional is primitive in *Begriffsschrift* [1990, p. 72]. That reasoning recognizes *no* distinction between what is definable and what happens to be defined or not in a particular systematization, and thus no distinction between the trivial externality mentioned in the text and a "real" meta-perspective. For further discussion see §4.2.2 below.

[18]These issues became prominent in the study of Frege largely as a consequence of van Heijenoort [1967]. Their most sustained discussion below is in §4.2 and §4.3.1.

It will be helpful to begin by distinguishing different component ideas in the modern notion of a formal system. The first and simplest contrasts 'formal' with 'natural', so that, for instance, a formal language is an invented rather than an encountered language, the meaning of whose symbols is to be fixed by stipulation rather than discovered through a survey of their antecedent use. A second component requires formality in the specification of operations with, i.e. within, the system. In this connection a formal specification may be more determinately understood as one referring only to syntactic features of elements of the system. A third component has to do with the completeness of those specifications: a formal system is according to this third idea a fully and precisely delineated object, whose features are amenable to mathematical, metalogical investigation.

2.3.1 An invented system

That *Begriffsschrift* offers a formal system according to the first of these ideas is, as lately remarked, trivial. Frege never tires of contrasting his concept-script with "language" (by which he means natural language), and at the centre of many of these contrasts lies the idea of authorial control that allows him to give reflection in the concept-script to *everything* relevant to his logical purposes and *only* that. A typical and simple example is provided by his remarks on his conditional stroke and the natural language conditional. The meaning of his invented symbol is first completely fixed by the truth-functional specification reported above. Only then does Frege ask how well it might be rendered by 'if' (Bs §5; cf. CN 95/BaA 102); the converse question, of how well 'if' is rendered by Frege's conditional stroke, is not raised at all, since it is no part of his purpose to provide a rendering for any natural language construction, nor to seek to be faithful to any meanings except those he lays down. It is typical, too, that in answering the secondary question Frege gives a careless misrepresentation of 'if', suggesting that it always imports the suggestion of a causal connection between antecedent and consequent (Bs §5; cf. his similarly inaccurate remark on the contrast between 'and' and 'but' in §7, CN 123): '*if*' (or '*wenn*') is not his concern; he pays just enough attention to the complexities of its meaning to dismiss them.

2.3.2 An incompletely formalized system

Matters become more complicated when we turn to the second component idea distinguished above. In crediting *Begriffsschrift* with the first formal system of logic my Introduction already took a stance on this question, and it is certainly true that what we find in the book sufficiently resembles a modern formulation of a system of logic to justify that stance. But here it becomes relevant to attend to points of dissimilarity.

A formalization of logic, as now standardly conceived, involves two components: first a formalization of a language, typically through an inductive definition of its well-formed formulae, and second a formalization of inference, specification of rules of inference whose correct application turns only on syntactic features of formulae determined by the first component. We noted in §2.2.5 that Frege departs from this model in the second component: his rules of substitution and instantiation are not syntactically specified. But as reflection on the instantiation rule makes specially obvious, that is forced by an earlier departure from the now standard model. This rule permits inference from a generalization $\forall \mathfrak{a} \Phi \mathfrak{a}$ to any "judgement with less general content". The permission could be expressed syntactically only if we had a syntactic characterization of formulae expressing such judgements. But we do not.

A formalization of a language itself typically has three stages. First it is said what combinations of basic vocabulary constitute atomic formulae. Secondly recursive operations are specified for constructing in turn more complex formulae from the basis provided by atomic formulae. Thirdly a closure principle states that only formulae constructible through those operations from that basis count as well formed. The method not only delineates which strings of symbols count as well-formed formulae of the language, but for each such string fixes a syntactic articulation, embodying its constructional history (i.e. the story of how, by application of operations to the atomic basis, it comes to be counted well formed),[19] and thereby relating it determinately to other such strings (e.g. a conjunction to its conjuncts, or a generalization to the class of formulae that count as instances of it). It is to those syntactic relations that rules of inference are keyed. If, now, we compare to this model Frege's specification of his concept-script described above, we find that we have (as nearly as possible — a qualification returned to below) a correspondence with the second stage of the model, but only that. Frege specifies no basic non-logical vocabulary for his language, nor how atomic formulae are to be formed from such vocabulary. And because that first stage is missing the third must be too: his concept-script is not a closed or precisely circumscribed language.

That comment on the third stage is needed to ensure that the remark about the first stage is not misconstrued. A formal language will typically include as primitive non-logical vocabulary singular terms, functors and predicates; but alternatives are possible. Quine, for instance, recommends a language without primitive terms, and lack of concern with mathematical applications may prompt one to exclude functors from the primitive vocabulary. The point made about Frege is not comparable to such cases. It is not that Frege formally specifies a language that excludes one or more categories of primitive vocabulary, but rather that he fails to specify formally what primitive vocabulary is included.

This omission is explained by a difference in aim. The concept-script Frege develops is not intended as an isolated and circumscribed logical calculus, but as the logical core of a language variously expandable to incorporate the rigorous development of the various special sciences (or equivalently, as the common logical core of a family of such expansions). Frege's most general notion of a concept-script is that of a notation whose elements and structure perfectly reflect the composition and inter-relation of its subject matter, so that operating with the notation will

[19]This 'history' metaphor is a very natural one (see [Dummett, 1981, Ch. 2]), but the only aspects of it to be taken seriously are successiveness and finiteness.

guide and assist, instead of obscuring and misleading, thought about it (Bs Preface, CN 105). He regards the special notations of arithmetic, geometry and chemistry as imperfect realizations of this Leibnizian ideal within their restricted domains (*ibid.*). His own concept-script, he says,

> adds a new domain to these; indeed, the one situated in the middle adjoining all the others. Thus from this starting point... we can begin to fill in the gaps in the existing formula languages, connect their hitherto separate domains to the province of a single formula language and extend it to fields which up to now have lacked such a language. (CN 105–6)

That vision is one that allows for "a slow, stepwise approximation" (CN 105). It is because *Begriffsschrift* offers only its first essential step, and only because of that, that the book is subtitled "a formula language *of pure thought*" (cf. CN 104), a title he later acknowledged could be misleading. Frege's description of the logical core of the concept script thus has to provide in advance for its expansion by incorporation of primitive vocabulary special to particular sciences. It is for that reason that principles are on occasion stated with a generality not exploited within *Begriffsschrift* itself.[20] For the same reason, principles relating the logical core of the language to its as yet unspecified expansions — again, the rule of instantiation is the simplest example — cannot be given a purely syntactic formulation.

That Frege accommodated himself to this fact does not show that he placed no value on the syntactic specification of rules. That is, on the contrary, an essential part of his explanation of how gapless proofs in the concept-script are to make explicit every assumption on which a conclusion rests. Since assumptions may be buried in rules of inference, this aim requires that the inference modes allowable in a correct proof must be strictly circumscribed. But even then one must guard against the possibility that "something intuitive [should] creep in unnoticed" (CN 104) in recognizing that the transition between such and such particular premises and conclusion exemplifies one of the specified modes of inference. It is to meet that last point that Frege requires of a gapless proof that its correctness be checkable by syntactic criteria; and the impossibility of formulating such criteria for inferences conducted in natural language is the chief inadequacy in it that motivates the construction of the concept-script (*ibid.*; cf. CN 84–5/BaA 108). Moreover, reasoning parallel to that just given supports his conception of the concept-script as an open language, able to expand to incorporate its applications. If logic were formulated instead as a closed calculus its application to reasoning in the sciences would necessarily be as an external object of comparison — "laws of logic [would be] applied externally, like a plumb line"[21] — and that process

[20]For instance, the principle governing function extraction in Bs 9 begins, "If, in an expression (whose content need not be assertible), a simple or complex symbols occurs in one or more places and we imagine it as replaceable by another..."; the parenthesis provides for extraction of a functor from a complex term, though no such cases arise in the remainder of the book.

[21]We will need to return more than once to this highly suggestive simile. It is most fully discussed in §3.2.3.

of comparison would provide another opening for "something intuitive", as well as straightforward mistakes, to creep in (cf. CN 85–6/BaA 109). Thus the same motivation of explicitness, the aim to close off every entry point for appeals to intuition or unexamined obviousness, which by the first train of reasoning requires syntactic specification of rules, by the second precludes it.

Two points combine to resolve the apparent tension. First, as Frege conceives it, syntactic purity is not an end but only a means. The end is that the correctness of a proof, and completeness in recording the assumptions on which it rests, should be immediately manifest without further presumptions. That these matters can be certified by a machine sensitive only to syntax, and therefore incapable of harbouring or invoking further presumptions, is one way of guaranteeing that, but nothing in the first train of reasoning given above dictates that it is the only way, or that any departure from syntactic purity will undermine the core aim of a *demonstratio ad occulos*. Secondly, the departure required by the second train of reasoning above is in any case a slight one. Rules that must relate to as yet unspecified expansions of the core language cannot be keyed to syntactic features. They can, however, be keyed to features which, when the expansion is made, will be given unambiguous syntactic representation. Thus although the full generality of the instantiation rule (to persist with that simple example) cannot be captured syntactically, what one might call its operative sub-rule at any stage in the expansion of the language — the rule licensing inference from a generalization to any instance of it formulable in the language as so far expanded — can be.

We should dwell here on a presumption of that point, prefigured by a qualification which earlier in the section I said we would have to return to. It was strictly inaccurate — and indeed, could make no very clear sense — to say, as I did, that Frege gives us an equivalent for the second stage in the now standard pattern for specifying a formal language but no equivalent for the first stage. How could one detail purely syntactically the inductive generation of complex formulae from an atomic basis if that basis is not itself syntactically characterized, since the inductive rules have to be keyed to features of their basis? In permitting that inaccuracy the earlier discussion took for granted a point parallel to the one just made, that the 'operative sub-rule' of a formation rule will be syntactically characterizable, though its full generality will not be captured that way. In each case the fully general principle is stated by reference to the *content* of expressions, not their syntactic form, but the correctness of any invocation of the principle actually made is to be confirmable syntactically. The presumption now relevant is that this approach is feasible *only if* those features of content receive unique and unambiguous representation in syntactic structure. So, although Frege's specification of his system does not supply it with primitive vocabulary and an atomic basis, it does make stringent demands on the form such a basis can take. Expansion of the concept-script to incorporate the contents of judgements specific to a particular science demands something reasonably called an analysis of those contents, that is, identification of, and unambiguous syntactic representation of, the logically relevant features of those contents to which the general principles of the concept-script relate.

Further elaboration of those demands is a task for later. But as a corrective to recent abstraction it may be helpful to witness a simple example where the presumption described here operates at a crucial point in Frege's presentation of his system. *Begriffsschrift* §9, on the extraction of functions, begins as follows:

> Let us suppose that the circumstance that hydrogen is lighter than carbon dioxide is expressed in our formula language. Then, in place of the symbol for hydrogen, we can insert the symbol for oxygen or nitrogen... [so that] 'oxygen' or 'nitrogen' enters into the relations in which 'hydrogen' stood before.

The presumption here — obvious enough to be overlooked — is first that *there will be* a "symbol for hydrogen" to be replaced, and secondly that its replacement by other symbols, e.g. 'oxygen', or 'nitrogen', will yield expression of judgements in which 'oxygen' or 'nitrogen' "enters into the relations in which 'hydrogen' stood before". What guarantees these presumptions? *Only* that the circumstance is expressed "*in our formula language*". That those and similar presumptions are met is indeed, as we will see (in §4.3.1), central to what Frege understands *expression* of a judgement in the concept-script to be.

2.3.3 Conceptions of logic as a system, and of its application

We turn finally and briefly to the third component idea in the modern notion of a formal system, that of a precisely delineated object amenable to metalogical investigation. We noted that Frege conceives his concept-script as open-ended, so trivially not such an object. But the easiness of that observation allows it correspondingly little force: we can evade the point altogether by posing our questions instead about each expansion (including the 'null' expansion) of his core language.

Two points then stand out clearly. First, the possibility of metalogical investigation is essentially provided for by Frege's presentation: that possibility is inseparable from the ideal that a gapless proof should be checkable by syntactic criteria. Further, Frege's demonstrations of the truth of his axioms and the soundness of his rules by reference to the semantic explanation of his symbolism both undeniably begin to exploit that possibility (though by later standards this beginning is certainly a modest one) and confirm that it is not restricted to a proof-theoretic approach. Secondly, though, it was clearly not Frege's principal aim in formalizing logic to further metalogical investigation. In the twentieth century it came to be seen as the principal or even the only purpose of formalizing modes of reasoning that it allows for a precise study of the powers of such reasoning; but for Frege formalization served the aim of rigour *in the conduct* of reasoning, not *the study of* it. His concept-script is for him a tool rather than an object of logical investigation. His aim in devising it is to embody and express the perspective of a reasoner, not to describe it. However, these two straightforward observations hardly begin to touch the delicate question, whether addition of ideas characteristic of the twentieth-century approach would complement or instead tend to undermine Frege's understanding of logic.

He held logic to be, as we noted, a science. Its principles are thus not schemata, but fully interpreted, genuine truths. His logical symbolism, the concept-script in which those principles are formulated, is therefore a genuine language, and because the principles of logic are universal in their application this language will be potentially all-embracing. Application is to be by subsumption and instantiation: logical principles will be applied in chemistry, for instance, not by holding up a logical formalism against chemical discourse as an external object of comparison, but by introducing special chemical vocabulary into the logical language. *Nothing* — no subject matter, no region of discourse — falls outside the governance of logic and the universal framework of thought it provides; so the idea of 'non-logical' or 'extra-logical' thought — the idea Wittgenstein gestured at by talking of "placing ourselves, with propositions, outside logic" (TLP 4.121) — is straightforwardly misbegotten.

This summary contrasts in obvious ways with the modern understanding of a system of logic. Such a system comes pre-equipped with its own primitive vocabulary, though internally to the system no particular significance is fixed for this vocabulary. Application of the system, which may be thought of in two ways, involves assigning an interpretation to its primitives. If the aim is to exploit the system in conducting ground-level reasoning about some subject matter, an interpretation is in effect a temporary meaning for or reading of the primitive vocabulary, though that species of direct application is scarcely engaged in outside of elementary training in logic. More typically the logician's first concern is with the resources of the system however its primitives are interpreted, and for consideration of that kind of question the thinner notion of interpretation, as semantic valuation, is appropriate (for instance, instead of being interpreted as bearing a meaning which determines it to be true of such and such objects, a predicate symbol is immediately assigned the set of those objects as its interpretation). Results obtained through that approach could then be applied to particular subject matters — by imposing an interpretation in the first sense — without grinding through the ground-level reasoning. On either understanding, however, a metalogical stance is involved in applying the system: on the second, that is trivially so; but on the first, too, application involves an assessment, external to the system, that the readings or meanings temporarily assigned to primitives behave in ways parallel to those in which the internal laws of the system determine those primitives to behave.

When explaining (in §2.3.2) why Frege formulated his Begriffsschrift as an open-endedly expansible language we noted one reason he would have had for rejecting this model of the application of a logical system: namely, that unexamined assumptions may lie in the comparison between the system and the subject matter about which conclusions are eventually to be drawn. The contrasts just now remarked do not disclose any other reason. In particular, although Frege does not employ the notion of a range of possible interpretations in his account of the application of logic, this does not show that he failed to grasp, or had any reason to be hostile to, that notion.[22] More centrally, the modern conception's resort to a *meta*logical stance does not put it in conflict with the conclusion just now attributed to Frege, that the idea of *extra*-logical thought is misbegotten. The two are simply not the same: in adopting a stance external to a given systematization of logic one betrays no ambition to stand "outside logic", in the sense Wittgenstein gestured at. To think otherwise would again be to conflate a particular logical formalism with logic *tout court*.[23]

3 BEGRIFFSSCHRIFT (II): GUIDING CONCEPTIONS

3.1 Introduction

Begriffsschrift received six reviews, their authors ranging from minor figures now forgotten to such leading exponents of the algebraic logic as Schröder and Venn. None showed any proper appreciation of the originality or significance of Frege's work.[24]

The blame is largely Frege's own. While Frege sets out his logical innovations with clarity, economy and rigour he offers remarkably little by way of discussion of the philosophical conceptions that motivate them or illustration of their mathematical fertility. Probably the most striking example of the second point occurs in his comments on theorem 81, a generalization of mathematical induction (the theorem states that every f-successor of x has any f-hereditary property had by x). Here Frege confines to a footnote the single unelaborated remark that "Bernouillian induction is based on this", while in the main text expounding in detail how, with the supplementary assumption that removing one bean from a heap leaves a heap, the theorem could be used to prove "that a single bean or even no bean at all would be a heap of beans" (Bs §27, CN 177). One is almost tempted to imagine that Frege had anticipated and set out to provoke Poincaré's famous quip about logic's sterility; but if that reaction is too fanciful, it is anyway no surprise if Frege's readers found the Sorites too little reward for the effort of reaching theorem 81. Regarding the first point, while Frege does mention in *Begriffsschrift*

[22]On how Frege's mathematical work would have made him familiar with this notion, see [Tappenden, 1997].

[23]Although this is a simple point, it is one with wide-ranging consequences for recently influential approaches to Frege. It is taken up more fully in §§4.2.2-3.

[24]A partial and, perhaps, a somewhat dishonourable exception is Schröder. Clearly offended by Frege's neglect of existing work in logic, not least his own, the general drift of Schröder's review is that Frege, working in naïve isolation, has achieved no more than to reinvent in cumbrous and eccentric form the Boolean wheel. But Schröder was too good a logician for his irritation to have altogether hidden from him the inadequacy of that verdict, so he excepts from it "what is said on pages 15–22 about 'function' and 'generality' and... the supplement beginning on p. 55 [i.e. Part III]" (CN 221) — the intellectual centres of Frege's book! Regrettably, Schröder did not choose to explain this qualification.

some of his guiding thoughts, for instance the replacement of subject and predicate by the notions of function and argument, and the separation of the logical from the psychological embodied in the notion of conceptual content, these are so little highlighted or elaborated that they stand out only in retrospect.

Such a perspective is provided by a series of five papers and addresses in which Frege sought to introduce his notation and to respond to criticisms and misunderstandings of *Begriffsschrift*,[25] along with a draft Introduction to an uncompleted early textbook on logic (PW 1-8/NS1-8). In the course of distinguishing logic from psychology in the first pages of that draft Frege sets out his conception of its "essence" and "subject-matter"

> ...there is a sharp divide between these disciplines, and it is marked by the word 'true'. Psychology is only concerned with truth in the way every other science is, in that its goal is to extend the domain of truths; but in the field it investigates it does not study the property 'true' as, in its field, physics focuses on the properties 'heavy', 'warm' etc. *This is what logic does. It would not perhaps be beside the mark to say that the laws of logic are nothing other than an unfolding of the content of the word 'true'. Anyone who has failed to grasp the meaning of this word — what marks it off from others — cannot attain to any clear idea of what the task of logic is.* (PW 3/NS 3, my emphasis.)

This statement, the first of a series of parallel statements running through to Frege's very last writings (PW 128/NS 139; CP 351/KS 342), must set the agenda for this section. An account of the thoughts guiding Frege's presentation of logic in *Begriffsschrift* must show how they relate to this conception of the essence of the subject.

3.2 Putting truth first

3.2.1 Irrelevance of psychology

To hold that logic is distinguished by the character of its concern with the notion of truth does not imply that logic will aim at a direct explication or definition of this notion. Indeed, to judge from the use Frege immediately makes of this thought, the reverse is the case. His idea is that a prior grasp of the notion of truth, of what it involves and what it excludes, should guide the development of logic, enabling it to free itself of irrelevant associations to pursue its proper course. This approach, of *putting truth first*, is most simply exemplified in Frege's dismissal of the crudely subjectivist idea that a law of inference might hold good for some people, or for people at some time or in some circumstance, yet not for

[25]These are: "On the scientific justification of a conceptual notation", "On the aim of the Conceptual Notation", and "Applications of the Conceptual Notation" (all in CN/BaA); also "Boole's logical calculus and the Concept-script" and "Boole's logical formula-language and my Concept-script" (both in PW/NS).

others. That idea, he says, "is utterly contrary to the nature of a law of logic, since it contrary to the sense of the word 'true', which excludes any reference to a knowing subject" (PW 5/NS 5). *How*, or *why* the notion of truth excludes any such reference is not further discussed. *That* it does so is, Frege clearly assumes, obvious to anyone who grasps the notion, and as such it is a legitimate *datum* of Frege's project.

The same straightforward strategy is apparent in the way Frege turns a traditional characterization of logic to his anti-psychologistic ends.

> Logic is only concerned with those grounds of judgement that are truths. To make a judgement because we are cognisant of other truths as providing a justification for it is known as *inferring*. There are laws governing this kind of justification, and to set up these laws of valid inference is the goal of logic. (PW 3/NS 3)

Justification is distinct from explanation. To explain why someone holds a judgement is to trace its causal origin. Providing that kind of explanation is a task for psychology. But that kind of explanation of a judgement tells us nothing at all about whether the judgement is justified: it has "no inherent relation to truth whatsoever" (PW 2/NS 2). Since that is logic's concern, psychological investigation can contribute nothing to logic.

Confusion on this can be fostered by the tradition of calling logical principles 'laws of thought'. In so far as this term is legitimate at all, it must be construed along the lines of 'laws of the road', prescriptions to which conduct is held responsible, rather than 'laws of planetary motion', descriptions responsible to the facts. Psychological laws are concerned with the description of how thinking in fact takes place, but that is no concern of logic, which determines how thinking *should* take place if it is to reach the truth (PW 4–5/NS 4–5). It would be wrong, however, to say that for Frege logic is a prescriptive rather than a descriptive discipline. That is the best one could get if one insisted on sticking with the phrase 'laws of thought', but that is only to make the best of a bad job. Prescriptions must always rest on a basis of descriptive theory: dietary advice, for instance, is given on the basis of nutritional theory. Similarly logic in its prescriptive aspect rests on its more fundamental characterization as comprising *laws of truth* (PW 128/NS 139).

3.2.2 Conceptual content

What other propositions follow from a proposition, and what other propositions it follows from, depend on what it means, its content; more specifically, we can say that the inferential powers of a proposition depend on the circumstance that must obtain if the proposition is to be true. Putting these two simple thoughts together already gives us Frege's equation: the content of a proposition, so far as it can concern logic, is the circumstance that must obtain if the proposition is to be true – that is, its truth-condition (Bs §2; PW 7–8/NS 8).

Frege acknowledges that there will, in an intuitive sense, be more to the meaning of a proposition than this. Propositions may differ in tone, in the kind of emotional reaction or psychological associations they are intended to promote, in the kinds of clues they offer regarding the speaker's reasons for putting them forward, etc. But unless these differences affect the circumstances in which the proposition would be true, they are irrelevant to logic, and no part of its 'content', as this must be conceived for the purposes of logic. The conception of content that is required by logic is called, in the *Begriffsschrift*, *conceptual content.*

As its subsequent history amply demonstrates the notion of a truth-condition is flexible, so that Frege's equation does not yet adequately fix the notion of conceptual content. One might hope to sharpen things by observing Frege's repeated insistence that "a difference is only logically significant if it has an effect on possible inferences" (PW 33n./NS 37n.). More explicitly,

> ...the contents of two judgements can differ in two ways: first, it may be the case than [all] the consequences which can be derived from the first judgement combined with certain others can always be derived also from the second judgement combined with the same others; secondly, this may not be the case... [In a case of the first kind] I call the part of the content which is the *same* in both *conceptual content.* (Bs §3)

This suggests a criterion of the form: φ and ψ have the same conceptual content iff (Γ, φ entails χ iff Γ, ψ entails χ). But while something of that form must be accepted as a truth about the identity of contents there seems no way of converting it into a workable criterion of individuation. If 'entails' were understood in a broadly truth-theoretic fashion, the result would be that all judgements necessarily alike in truth-value, and so for instance all true arithmetical judgements, would have the same content — something Frege plainly did not intend. On the other hand, to understand 'entails' in a proof-theoretic way necessarily implicates the language in which the judgements are expressed and proofs conducted. Then, since any merely verbal difference can obstruct a proof — since, for instance, $a = b \vdash a = b$ but not $a = a \vdash a = b$, however 'merely verbal' the difference between 'a' and 'b' — the criterion could relate only to a language from which merely verbal differences are excluded, a language in which sentences and contents are paired one-to-one. In effect, then, we could know when we may apply the criterion only by already knowing what results it should yield. This does not make the above principle vacuous: the relation it states between inference and content must be respected by an account of either. But it does disqualify the principle as a criterion: inferential equivalence is not settled in advance of identity of content.

3.2.3 Conceptual notation

Since the inferential powers of a proposition depend on its conceptual content, a language in which inferences are to be conducted must be so designed as to give clear expression to the all aspects of the conceptual content of a proposition, while suppressing any irrelevant, non-logical subtleties of its meaning or the way it is expressed in natural language. It must be able to represent differently any two propositions which differ in conceptual content, while providing a single, unambiguous expression in place of two propositions of ordinary language which, while they might differ in superficial respects, share a single conceptual content. The isolation of a logical core of meaning from its psychological trappings is only a first step to this end.

> But even when we have completely isolated what is logical in some form or phrase from the vernacular or in some combination of words, our task is still not complete. What we obtain will generally turn out to be complex; we have to analyse this, for here as elsewhere we only attain full insight by pressing forwards until we arrive at what is absolutely simple. (PW 6/NS 6)

The conceptual content of a proposition is determined by its constituent concepts and how these are put together. In both respects the presentation of content in ordinary language is far from explicit:

> ... verbal language... leaves a great deal to guesswork, even if only of the most elementary kind. There is only an imperfect correspondence between the way words are concatenated and the structure of the concepts. The words 'lifeboat' and 'deathbed' are similarly constructed though the logical relations of the constituents are different. So the latter isn't expressed at all, but is left to guesswork. Speech often only indicates by inessential marks or by imagery what a concept-script should spell out in full. (PW 12–13/NS 13)

As Frege understands it, ordinary grammar does not need to be explicit about the way concepts combine to form a more complex concept; the needs of ordinary communication will be met if, when a speaker puts words together, a hearer can reliably enough pick up intuitively what connection is intended between them. A coffee table is a table for serving coffee on; a seminar room is a room for conducting seminars in; a concept-script is a script for the clear expression of concepts; an omelette pan is a pan for cooking omelettes. No one would suppose that it was a pan for expressing omelettes, or for conducting omelettes in, or for serving omelettes on. Ordinary language and its grammar are driven by practicalities; there being no practical need to exclude these alternatives, the language simply does not do so.

The demands of science are very different. If principles of inference are to be formulated, the same grammatical structure must indicate the same conceptual structure. Frege's examples[26] illustrate the failure of that condition. There are parallels between such pairs (a death bed is a bed as a lifeboat is a boat) but also

[26] Actually, these particular examples are his translators', but no less apt for that.

failures of parallel (x's deathbed is the bed that x died in, but y's lifeboat is not the boat that y lived in). The commonplaceness of these examples demonstrates that there is no prospect of formulating grammatical principles of correct inference that would be sound across a natural language (cf. CN 84-5/BaA 108).

The remedy Frege proposed is "a script that compounds a concept out of its constituents" (PW 9/NS 10) rather than a word out of its sounds. An immediate and profound consequence of this is that expressing a conceptual content in the Begriffsschrift will involve settling *how* it is compounded out of its constituents. Thus, conceptual analysis becomes a preliminary to perspicuous representation or expression of a content, which is itself a requirement of codifying principles of valid inference. Not only the overt, visible features of the grammar of an ordinary sentence will be changed when its content is rendered in the Begriffsschrift; as well as that, conceptual connections that are left implicit in ordinary language, including those that involve the analytical definition of superficially simple expressions, will need to be rendered explicit. It is a standard thought to contrast, on the one hand, the narrow, formal methods of logic with, on the other, a broadly reflective, philosophical consideration of unobvious interconnections of meaning, as constituting two very different approaches to the evaluation of reasoning. On Frege's account this contrast is largely spurious. Certainly logic can lay down formal rules, and these will be quite general, and so have nothing to tell us about the significance of particular non-logical concepts. But, first, the application of these rules, and so their very point, cannot be automatic in natural language; and second, the material to which they are to be applied cannot be rendered in a language permitting such automatic or formal application of rules without engaging in the kind of task the standard contrast assigns to philosophy. It is because Boole never imagined that his job included anything so demanding as this that Frege says he showed "no concern about content whatsoever" (PW 12/NS 13):

> I believe almost all errors made in inference to have their roots in the imperfection of concepts. Boole presupposes logically perfect concepts as ready to hand, and hence the most difficult part of the task as having been already discharged. (PW 34–5/NS 39)

The above discussion follows Frege in presenting the *need* for analysis as a consequence of a feature of natural language, its lack of explicitness about conceptual structure. It by no means follows that natural language is the *object* of analysis. To reiterate what was just quoted, Frege's concern is with *content*.

It would be possible to hold that conceptual interconnections not apparent on the surface of ordinary grammar are embodied in some non-obvious level of linguistic structure, and hence to regard the task of making such connections explicit as one of uncovering deep structure. Some such view is presumably involved in talk, common amongst philosophers, of 'the logical structure of natural language'. Not only does Frege not employ that notion, but, as we have seen, what he does say on the matter suggests he would regard it as empty. The "logical relations of the constituents" which a concept-script must "spell out" are, he says, "*not* *expressed at all*" in ordinary language (PW 13/NS 13, my emphasis). There is on that view no room to enquire by what unobvious means they *are* expressed.

We reach the same conclusion by noting what Frege does *not* say, an omission that on the account just sketched would be a serious lacuna. If analysis aims at uncovering deep structure it becomes a species of investigation in need of an epistemology: what kind of access does one have to that non-obvious realm of fact, and by what criteria are some aspects of natural language and its use taken as guides to its nature while others are dismissed as misleading? Questions like these have been held to leave Frege in a "quandary" [Baker and Hacker, 1984, p. 74]; Frege's silence on such matters shows rather that the questions derive from a conception of analysis he did not share.

One might try to reproduce the "quandary" by transposing those questions from ordinary language sentences to their contents, now platonistically construed: what kind of access does Frege claim to *that* realm, so as to be able to conform expressions of the concept-script to its structure? But again the challenge misfires, presupposing that for him 'expressing a content' has the same logical grammar (as one might say) as 'describing a planet' — presupposing, that is, that the entity to be expressed or described is settled independently of the attempt to express or describe it, and so constitutes a standard by which to judge the adequacy of the expression or description. But while a faulty description of a planet still describes that planet, there is no such thing as a faulty expression of a content which even so expresses that content.[27]

The challenges just countered presuppose a recipe for the discernment and expression of content that begins, 'First catch your fish...'. Somehow, it is imagined, one lays hold of a content, and then, guided obscurely by its features, one manages to put together an expression of it. How far Frege's thought lies from any such model is apparent in his discussion of how the Boolean tradition had failed to realize the true value of the formalization of inference. These logicians, Frege says, had sought to mechanize the process of inference, casting the syllogism in the form of a calculation.

> But we can only derive any real benefit from doing this, if the content is not just indicated but is constructed out of its constituents by means of the same logical signs as are used in the computation. In that case, the calculation must quickly bring to light any flaw in the concept formations. (PW 35/NS 39)

Note first here that a content appears as something to be constructed, rather than encountered and copied. Secondly, that the formal skeleton of the construction is to be chosen to accord with the systematization of inference — itself, recall (§2.2.2 above), one of several equally legitimate systematizations and adopted over others largely on pragmatic grounds. And thirdly, that the adequacy of the construction is

[27]It is tempting to qualify this remark by "in a certain sense": cf. "In a certain sense, once cannot make mistakes in logic" (TLP 5.473). The certain sense will be elaborated through discussion of the notion of expression in §4.3.1 below.

confirmed, not by comparison with a content given in another way — i.e. otherwise than through the constructed expression — but intrasystematically, through the inferences it facilitates. (To forestall a possible misunderstanding, none of these points is inconsistent with the platonism about content to which Frege undoubtedly subscribed, though more emphatically in his later writings. They do not imply that a content first comes into being with the construction of an expression for it, nor that its existence or nature is in any other way dependent on a person's grasping or expressing it. The significance of the three points is methodological rather than ontological: they indicate that Frege's platonism about content provides no explanatory ground for its representation in the concept-script.[28])

Before leaving this passage we should ask how Frege himself understands the benefit of a construction of content integrated in the way he describes with an inferential calculus. Frege's answer is presented in a simile we encountered above (§2.3.2). When laws of logic are appealed to in the evaluation of reasoning in ordinary language those laws are, Frege suggestively says, "applied externally, like a plumb-line" (CN 85/BaA 109). The evaluation then involves an external comparison between the laws (or a formulation of them) and the discourse to be evaluated. Logical laws then "furnish little protection" against the intrusion of error or unexamined assumptions. Indeed,

> [t]hese laws have failed to defend even great philosophers from mistakes, and have helped just as little in keeping higher mathematics free from error, because they have always remained *external to content.* (CN 86/BaA 109–10, my emphasis)

The implied opposite is a scheme for the expression and evaluation of reasoning in which logical laws are *internal to content.* That ideal demands "a system of symbols from which every ambiguity is banned, which has a strict logical form from which the content cannot escape" (CN 86/BaA 110).

The point about ambiguity we have met before: a mode of linguistic composition must reflect a single mode of conceptual composition if syntactic inference rules are to be possible. But the image of content 'escaping' from — 'giving the slip to', or 'leaching out of' — linguistic structures gives a new twist to that point: ambiguity detracts from Frege's ideal because an ambiguous sign — standing in need of something external to it to disambiguate it on a particular occasion of use — cannot of itself *completely, exhaustively, embody* the content it is used to express. When content *is* so embodied in a system of signs the forms through which a content is constructed out of its constituents are also the forms by which it stands, *immediately and of itself,* in inferential relations to others. The prosecution of inference in such a system then displays immediately "inner relationships" (CN 87/BaA 111) between contents. Most centrally it displays that entailment is an internal relation holding solely in virtue of the conceptual composition of the contents it relates, or in other words, that it is a relation *internal to content.*

[28] A similar though broader point is a major theme of [Ricketts, 1986].

That he conceives the benefit of a system of inference in this way is another example of Frege's putting truth first in setting the goal of logic. Logic's immediate concern is to set out laws of valid inference, laws in accordance with which the truth or falsehood of one proposition constrains that of another. But if logic is to be a source of understanding those laws must draw into play an appreciation of how each of the propositions, severally, is determined as true or false. If those central properties were supposed simply to attach themselves to propositions in some not further explicable way, then logical laws would yield no understanding of inference. Likewise, if there were a story to be told about how individual propositions come to be true or false, but this story had no connection to logical laws — if the story made reference only to features of propositions distinct from those to which logical laws are keyed — there would again be no yield of understanding. Logical laws will display an "inherent relation to truth" (cf. PW 2/NS 2), and so serve to "unfold" that notion for us, only if the structural features of propositions to which the laws relate are those through which they are determined as true or false.

The understanding of inference that is yielded when that condition is met is, in a broad sense, that of a semantic explanation of validity: one proposition is shown to entail another in virtue of their being so composed of common elements that any way of making the first true also makes the second true. Conceptual content, which Frege distinguishes from other aspects of meaning as containing everything and only what has significance for inference, is therefore required to subserve that style of explanation of validity. One respect in which the notion, as explained in these early writings, fails to meet that condition will be considered later (§5). For the present, though, we should note how close these elementary reflections on Frege's earliest account of the essence of logic have brought us to an equation long maintained by Michael Dummett, namely, that the notion of content called for by Frege's understanding of logic is that of semantic value.

3.3 Putting judgements first

3.3.1 A thought is something that gives rise to a question of truth

Before writing *Begriffsschrift* Frege had studied only a very little philosophy (see CN 3, nn. 3 & 4). Yet with his narrowness of focus, and a remarkable conciseness of thought, he seems to have extracted from that study a profound diagnosis of the failure of a tradition extending over several centuries to make any significant advance on the logical problems that concerned him. The diagnosis was that the tradition had reversed a crucial priority, and Frege set down his correction of that reversal as a central plank of his new approach:

> I start out from judgements and their contents, not from concepts. (PW 16/NS 17)

Since the contents of judgements are those that give rise to a question of their truth, putting judgements first is again to put truth first. Because truth is the central concept of logic any notions employed in developing the theory of logic must have an eye to that goal and respect its priority. The notion of truth must be allowed to carve out the domain of logical concern and to order its exposition. Reflecting on the course of his work in 1919 Frege wrote:

> What is distinctive of my conception of logic is that I begin by giving pride of place to the content of the word 'true', and then immediately go on to introduce a thought as that to which the question 'Is it true?' is in principle applicable. (PW 252/NS 273)

This distinctive conception had been central to Frege's work from the very beginning. We find a compressed statement of it in one of the earliest surviving fragments of his writings, which also represents one of Frege's few direct engagements with the tradition he was rejecting. Six of its seventeen short remarks read as follows.

1. The connections which constitute the essence of thinking are of a different order from associations of ideas.
2. The difference is not a matter of the presence of some ancillary thought from which the connections in the former case [that of thinking] derive their status.
3. In the case of thinking, it is not really ideas that are connected, but things, properties, concepts, relations.
4. In language the distinctive character of thought finds expression in the copula or personal ending [=finite inflexion] of the verb.
5. A criterion for whether a mode of connection constitutes a thought is that it makes sense to ask whether it is true or untrue. Associations of ideas are neither true nor untrue.
6. What true is, I hold to be indefinable. (PW 174/NS 189)

The target of these remarks, identified by Dummett,[29] is the Introduction to the *Logik* published in 1874 by Lotze, Frege's teacher (in philosophy of religion) at Göttingen. Some few of their details can be explained, as Dummett shows, only by reference to the specifics of Lotze's text. But to understand their general drift, and what is involved in the reversal of tradition that they encapsulate, it is, I think, appropriate to go further back to examples of that tradition both more central and more familiar.

3.3.2 The way of ideas as an example of what Frege opposes

Descartes famously put forward a general principle concerning ideas: whatever we clearly and distinctly perceive is true. In addition he advanced claims about the

[29] Dummett [1981b, p. 77] convincingly argues for a pre-*Begriffsschrift* dating of this fragment.

clarity and distinctness of particular ideas, for instance that the idea of God is the clearest and most distinct idea we possess. Ignoring the question whether this claim about the idea of God is true, what follows from it? In conjunction with the general principle it seems that it ought to imply 'God is true'. But what could that mean? Maybe we could make *some* sense of it, as saying that God is true to his promises, or truthful, or a true support of the righteous, or something of the kind. But in the sense of 'true' that seemed to be involved in the general principle, one having nothing to do with honesty or loyalty or consistency or genuineness, we can make very little sense of the conclusion at all.

The complaint may well seem unsympathetic. What may be clearly and distinctly perceived, and so what may by Descartes' principle be true, are presumably not things like God or matter or extension or whatever, but ideas of these. But that, on the face of it, doesn't much improve things. I have ideas, for instance, of the colour purple, of the number twenty, of Tony Blair, of justice; and I have the idea of purple being between red and blue, of twenty being even, of Tony Blair presiding in Cabinet, of justice's relation to fairness. Is there any one thing it could mean to call all of these true?

Descartes gives over a good part of his Third Meditation to attempting to clarify when something can be called 'true' and when it cannot, and to distinguishing some of the different things that might be meant in calling anything 'true'. He has to do this because he starts with a notion of 'ideas', and then tries to make the notion of truth somehow fit together with it. What an idea is, it seems, is too transparent to need any kind of explanation.

Descartes' way of thinking of the mind perhaps made that assumption seem more than ever inevitable, but the particularities of that are irrelevant here. I have introduced him into the discussion to illustrate something more general, namely, the contortions inherent to any framework that settles upon some such notion as that of an idea, presumed to be understood independently of the notion of truth, as a starting point, and only subsequently attempts to locate the notion of truth in that framework. Further illustration of those contortions can most easily draw on a sketch of the way of ideas too generic to attribute to Descartes (though parts of it more nearly fit Hume).

We have, according to this sketch, ideas, which are images or representations of things. These ideas can be compounded one with another to form complex ideas. They can be regarded as, so to say, mere ideas, present to entertain the mind; or we can think of them in their representative role. It is only when ideas are thought of in the second way that they may be assessed as true or false, and they are then true or false according as there is or is not something in the world outside the mind that they accurately represent.

If my idea is of a golden mountain then, by the above, it is true if there is a golden mountain. But I might have the idea of a golden mountain, figuring in a representational role — that is, an idea that is genuinely an idea *of* a golden mountain, and not merely a piece of inner wallpaper — without the slightest inclination to suppose that there actually is one. And I would not be to blame for that. To the extent that falsehood carries a suggestion of blame, it is natural to want to acquit my idea of falsehood without at the same time divesting it of representational intent. One suggestion for doing this would be to say: my idea in itself is neither true nor false, what would be true or false — in this case false — would be the compound idea that results from attaching the idea of *existence* to that of a golden mountain.

How many things are wrong with this? To begin with, if there were such an idea of existence to be connected with that of a golden mountain, I could surely have the new compound idea, that of an existent golden mountain, without imagining for a moment that there is one, and so again not thinking anything false. Secondly, realizing that the idea of existence won't do what was asked of it, one can begin to doubt whether there really is such an idea. As Hume said, surely *whenever* I think of a golden mountain, I am already thinking of an existent golden mountain: any mountain is an existent mountain; and I of course realize that; so when I think of a mountain I am thereby thinking of an existent mountain. Thirdly, if the idea of existence needed to be added to make the idea of a golden mountain true or false, it ought to be needed too with other ideas. But if I have the idea of twenty being even, I don't need to augment it to anything along the lines of 'There is a twenty being even' to have a true idea. Fourthly... There are just too many things wrong with this to count.

The notion that ideas can be taken as a basis for a view of how we think about the world and represent it to ourselves is a source of special problems in epistemology. But from a logical point of view it is just one manifestation, the 17th century version, of the idea that concepts come first — an idea so prevalent before Frege that it went completely unquestioned. Anyone writing a logic book did not have to think at all about its basic organization. It would consist first of a theory of concepts; second, of a theory of judgements, complexes of concepts; and third of a theory of inference, complexes of complexes of concepts. Frege was the first to recognize clearly the central flaw of this scheme. On the one hand, by separating the theory of judgements as distinct from the theory of inference, it acknowledged that some kinds of complexes of concepts formed, as it were, natural wholes: although they might be further compounded with others of the same kind, they were not just lost in the mixture. But on the other hand the basic conception that drives the scheme — that logic deals with concepts and what you get when you put them together — makes no intrinsic provision for certain results of combining concepts to have a different status from others.

Starting from concepts, and then admitting complexes of these — for instance, by appeal to the mind's unrestricted ability to compound and combine any of its perceptions, as in Hume and the way of ideas generally — one will never arrive at a satisfactory understanding of how combinations of concepts come to be true. The reason is simple: not all combinations of concepts are of a kind to be true. The result of taking concepts as basic and *only later* trying to make the notion of truth fit into the theory is that it will never fit properly. Frege's radical alternative, as we saw, is to take the basic notion of logic, the notion of truth, as coming first, and to allow it to carve out the domain of the theory and to dictate the order of explanations it offers.

3.3.3 The context principle: a glance forwards

The way of ideas had supposed its ideas, or concepts, to provide a self-explanatory basis for the construction of representations of the world. To reverse its order of explanation will involve giving up on that supposed self-explanatoriness. It will require, that is, that we arrive at a new way of thinking of the conceptual elements of judgements.

Part of the point of the renowned 'context principle' that Frege was to formulate in his next major work, *Die Grundlagen der Arithmetik*, is to acknowledge this consequence.[30] This principle is one of three that, Frege says, he adhered to in developing the argument of this book. It is, at least at its first occurrence, an instruction:

> never to ask for the meaning of a word in isolation, but only in the context of a proposition. (Gl p. x)

If this instruction is not observed, Frege warns,

> one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the first principle as well [the first principle being: always to separate sharply the psychological from the logical, the subjective from the objective]. (*ibid.*)

If we cannot think of individual concepts, or the meanings of simple words, as self-explanatory starting points, we must have another way of thinking of them. Since judgements, and the truth of judgements, are the starting point required in logic, we have to think of word-meanings and concepts in terms of that. So we must think of the meaning of a word, from the beginning, as the contribution it makes to the meaning of a sentence in which it figures; and likewise we must think of a concept as an ingredient in a judgement. The pattern of thought here again follows the same lines as in the case of truth. If you leave it until too late in the explanation to introduce the notion, you will not be able to find a proper place for it. Similarly with word meanings and concepts: unless you think of these from the beginning as *essentially* ingredients of complex wholes — the meanings of sentences, and the judgements expressed in sentences — it will be impossible to explain later how they can combine to constitute such wholes, or how they can contribute to fixing the logically relevant character of the wholes in which they figure.

[30]The term 'the context principle' is not a description used by Frege. It is a label attached by Michael Dummett, but one that has stuck and is now widely recognized.

3.3.4 The extraction of concepts

This order or priority is respected in the account given in *Begriffsschrift* §9 of how a content divides into function and argument. A brief sketch of that account was given above (§2.2.3). But given its centrality to Frege's logical theory — it is, as remarked above, the single respect by which Frege's quantificational logic exceeds its Aristotelian and Boolean predecessors — we should dwell on it here.

Take the simplest of Frege's examples,

Cato killed Cato.

Suppose we think of 'Cato' as replaceable at its first occurrence. Then the expression divides into the function 'ξ killed Cato', expressing the concept of killing Cato, with the argument 'Cato'. Suppose now we think of 'Cato' as replaceable at its second occurrence. We then have a different function of the same argument: 'Cato killed ξ'. And so on.

On this account, then, a single sentence can be viewed as falling into function and argument in a variety of ways. In the case of 'Cato killed Cato' each of these four ways (as well as others) is available:

1. 'ξ killed ζ' as function with 'Cato', 'Cato' as arguments
2. 'ξ killed Cato' as function with 'Cato' as argument
3. 'Cato killed ξ' as function with 'Cato' as argument
4. 'ξ killed ξ' as function with 'Cato' as (the *single*) argument.

To view the sentence in any one of these ways is to view it as exemplifying a pattern shared by a class of sentences that can be produced from it by varying the argument expression. So, for instance, some members of the corresponding classes of sentences are:

(i) Brutus killed Caesar, Caesar killed Caesar, Caesar killed Brutus...

(ii) Brutus killed Cato, Caesar killed Cato, Cicero killed Cato...

(iii) Cato killed Brutus, Cato killed Caesar, Cato killed Cicero...

(iv) Brutus killed Brutus, Caesar killed Caesar, Cicero killed Cicero...

These classes are distinct, though not of course disjoint: 'Cato killed Cato' is a member of all of them (that was our starting point); also, classes (ii), (iii), and (iv) are each of them included in class (i).

Frege says that how we view the sentence as dividing into function and argument is a matter of indifference "so long as argument and function are completely determinate" (Bs §9, CN128). In terms of our example, that amounts to saying that, so long as our attention is directed solely onto the single sentence about

Cato, it does not matter what class of sentences we regard it as belonging to. And this is, indeed, obvious. Our attention is on the single member, and it is just as much a member of any of these classes as of any other.

The division of the sentence into function and argument acquires "a substantive significance" (eine inhaltliche Bedeutung) when "the argument becomes indeterminate" (*ibid.*). In the terms just introduced, that will be when our purposes demand that the sentence we are considering be counted as one of a range of connected sentences, each exemplifying a common pattern. Above all this is important in connection with inferences that involve generality. Roughly speaking, 'Someone killed Cato' will be true if some one (at least) of the sentences in class (ii) is true. Thus, when we infer, 'Cato killed Cato; so someone killed Cato', it becomes essential that the premise be viewed as a member of that class, namely (ii), the truth of any one of whose members verifies the conclusion. A slightly more complex example is the inference

Cato killed Cato; so Cato killed someone who killed himself

or in symbols

$$Kcc \vdash \exists x(Kcx \wedge Kxx).$$

In order to appreciate this inference, we must first see its premise as dividing into function and argument in the way indicated in (3), $Kc\xi$, and then in the way indicated in (4), $K\xi\xi$.

The pattern picked out by the division into function and argument indicated in (4) is, of course, there to be picked out. But it need have been no part of one's first apprehension of the sentence that it exemplifies this pattern. Similarly, when thinking of the sentence as dividing in this way you come to regard it as a member of a class of related sentences, the class illustrated in (iv); yet it need have played no role in your first understanding of the sentence that it was a member of this class. Indeed, it might never have occurred to you to think of this particular subclass of (i) as at all distinguished: you might not have been familiar with any way of picking out just this subclass of the sentences saying of someone or other that they killed someone or other. Each of the members of the now distinguished class of sentences (iv) says of some individual that he killed himself; otherwise put, in each of these sentences the concept of suicide is expressed. It is the fact that this concept is expressed in each of them that distinguishes the class. Thus anyone who comes to be able to distinguish this class for the first time can be thought of as acquiring a grasp of the distinguishing mark of the class, in this case, a grasp of the concept of suicide.[31] Given that the division indicated in (4) is not something that needs to be appreciated to understand 'Cato killed Cato', this concept can be new to such a person. To him the concept of suicide will be a novel concept; it will a conceptual achievement to recognize that 'Cato killed Cato' belongs to a class of sentences, (iv), in each of which this concept is expressed.

[31] This way of expressing matters — on which a predicate or propositional function appears as the characteristic mark of a class of propositions that are its values — is that of Wittgenstein's *Tractatus*.

Frege's simple example illustrates how recognizing a new way of dividing a sentence can be a way of acquiring a new concept. But the distance between understanding the predicate 'ξ killed ζ' and grasping the concept of suicide as expressed by 'ξ killed ξ' is admittedly slight. If someone had the notion of killing, and a general grasp of reflexive constructions, then, it might be thought, that person had a very little way to go to understand the notion of suicide. Somewhat more impressive examples are easily multiplied. From the fact that Duncan was a king and that Macbeth killed Duncan we might infer

$$\exists x(\mathrm{King}(x) \wedge \ \mathrm{Killed}(\mathrm{Macbeth}, x))$$

If we imagine the place of 'Macbeth in this sentence as variable, we extract the predicate

$$\exists x(\mathrm{King}(x) \wedge \ \mathrm{Killed}(\xi, x))$$

expressing the concept of *regicide.* (A hundred other specialized concepts of killing and being killed are derivable in similar ways.)

As Frege stresses in "Boole's logical calculus and the concept-script", however, the full value of this method of concept formation becomes apparent in mathematics. We can make do with one of Frege's simplest examples from that field.[32]

Suppose we already understand $>$ and a predicate Mxy meaning 'x is a multiple of y'.[33] Then these, together with the standard logical vocabulary, suffice for understanding the sentence

$$13 > 1 \wedge \forall n(M(13, n) \rightarrow (n = 1 \vee n = 13)).$$

Now of the many ways of dividing this sentence into function and argument it may occur to us to consider the predicate

$$\xi > 1 \wedge \forall n(M(\xi, n(\rightarrow (n = 1 \vee n = \xi)),$$

that is, to consider the results of replacing '13', at each of its occurrences, by another numeral. Every such sentence says of a number that it is larger than one and is a multiple only of itself and one; otherwise put, that it is *prime.* The concept of a prime number is thus *extractable from* the sentence we started with. Yet it was no part of the conceptual demand placed on anyone to understand that sentence that they should already have this concept. So, for such a person as we described, recognizing the possibility of dividing the sentence into function and argument in this way is a way of acquiring the concept of a prime number.

The concept *prime* will be a new concept to one who acquires it in this way. But Frege stresses that this method of discerning different ways of dividing a sentence yields concepts that are new in a strong sense, a sense he hopes to explain graphically in "Boole's logical calculus and the concept-script".

[32]PW 23/NS25, example 9. I choose this example because it is a favourite of Dummett's, and through his influence has come to be a standard feature of discussions of these matters.

[33]Frege employs his definition of the ancestral to define 'x is a multiple of y' as meaning 'x belongs to the series $0 + y, 0 + y + y \ldots$'. By means of this and other slight differences he reduces the presupposed arithmetical notions to addition alone. These differences are irrelevant to the present point.

3.3.5 Boolean composition versus Fregean decomposition

The Boolean tradition followed more or less in the lines of 'the way of ideas' in forming complex concepts from their simple ingredients. 'More or less', because in contrast to (e.g.) Hume's vague and open-ended understanding of combination, it circumscribed strictly the ways in which a complex concept could be formed from simple ingredients: Boolean combinations are complexes formed with the truth operators *and*, *or*, and *not*, corresponding to the *intersection*, *union* and *complement* operations of the theory of classes or aggregates.

Many computerized library catalogues allow readers to conduct what are known for that reason as 'Boolean searches'. Each book in the stock will be catalogued against several 'keywords' which can be thought of as expressing concepts under which the book falls. Search instructions then combine these keywords by Boolean operators. It may well be that some such instruction — one entered by a laterally thinking philosopher on sabbatical — has never been given to the computer before, so that it responds with a selection from the stock never before effected. In that weak sense these Boolean combinations can yield new concepts. But it is natural to think that the computer can never draw a genuinely novel division amongst the books in the library stock: it isn't that clever.

Frege shares this natural thought, and hopes to substantiate it by reference to Venn diagrams, by which any Boolean combination of concepts can be represented.

> If we look at what we have in the diagrams, we notice that in both cases the boundary of the concept, whether it is one formed by logical multiplication [conjunction] or addition [disjunction], is made up of parts of the boundaries of the concepts already given. This holds for any concept formation that can be represented by the Boolean notation. This feature of the diagrams is naturally an expression of something inherent in the situation itself, but which is hard to express without recourse to imagery. In this sort of concept formation one must ... assume as given a system of concepts, or, speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate the surface in a new way. (PW 33–34/NS 37–8)

The contrast Frege intends is with examples such as the concept *prime*, arrived at in the manner explained above. That concept, he holds, draws a *new line* through the domain of numbers; it is in no sense just tracing a line that is already present but which, until a complex term is devised, merely lacks a verbal expression.

How persuasive is this contrast? Frege's metaphor is, I think, intuitively compelling, so much so that one might hope to confirm the genuine novelty of the concept *prime* by, as it were, superimposing a plot of the 'output' concept on those of the 'input' concepts, and in that way displaying that "there is no question...of using the boundary lines we already have to form the boundar[y] of the new [one]" (PW 34/NS 38–9). But of course, in this case no such simple confirmation is possible: 'prime' draws a boundary through the numbers (N); the input concepts to its construction, 'larger than' and 'multiple', are relations, whose Venn-like representation requires a different domain ($N \times N$). The superimposition experiment cannot be run, and one wonders whether that is the source of the impression of novelty.

To give the Boolean a fair run, let us switch to Frege's next example, 'x and y are co-prime', simplifying its definition somewhat to $\forall z \neg(z > 1 \wedge Mxz \wedge Myz)$. To avoid collapsing distinct variables, and so to maintain the usual Boolean correlations (between conjunction and intersection, and so on), the regions representing satisfaction of the definition's 'input' conditions should be plotted on a domain of triples (N^3). Retracing those lines straightforwardly gives a region representing satisfaction of $\neg(z > 1 \wedge Mxz \wedge Myz)$, call it A. How, from this point, are we to trace the region B representing $\{(x, y, z) : x \text{ and } y \text{ are coprime}\}$? Of course there is such a region, and what it includes can be given a Boolean description (infinite, if need be) by reference to A. But — to continue Frege's metaphor — rather than re-tracing the boundaries so far drawn, this description *delves within* them. Less metaphorically, how B relates to A or to any region so far plotted is conditional, not just on the conceptual specification of those regions, but on what (if anything) lies within them, their membership. More metaphorically again, Frege's treatment of functions and generality provides for one to *reach inside* a complex conceptual specification to formulate that condition, whereas the Boolean can only treat specifications already reached as indivisible bricks for further construction.[34]

So Frege's conclusion is thus far borne out:

> ...even if its form [i.e. the form of Boole's notation] made it better suited to reproduce a content than it is, the lack of a representation of generality corresponding to mine would make true concept formation — one that didn't use already existing boundary lines — impossible. (PW 35/NS 39)

But where, exactly, does the novelty lie?

The introduction of relations, with the consequent need (in Peirce's terminology) to index the positions of their *relata*, might be thought the crucial step. But while this is indeed necessary for genuine novelty it is not sufficient. Purely Boolean compoundings of relations no more yield genuinely new concepts than do the same operations on properties. The reason is that these operations are blind to the complexity of their operands. In this 'calculus of domains', propositions, properties and relations are treated indifferently as species of *things*, so that the only internal relations recognized by the calculus are those directly constituted

[34]The simplest possible example should make this point vivid. Take a root domain of two objects, a and b, and let R be the region of $\{a, b\}^2$ representing satisfaction of Rxy. Then a Boolean description of S, representing satisfaction of $\exists x Rxy$, will run: $(a, a) \in S$ and $(b, a) \in S$ iff $(a, a) \in R$ or $(b, a) \in R$; $(a, b) \in S$ and $(b, b) \in S$ iff $(a, b) \in R$ or $(b, b) \in R$. Depending on the exact membership of R we will have either $R = S$ or $R \subset S$. To provide for both possibilities S must be represented as a distinct region from R. Trivially, that region cannot be plotted by tracing the only line we have to hand, that surrounding R.

by its compounding operations, and never those that hold in virtue also of the inner complexity of what is compounded. Work with the calculus thus reduces to a matter "simply [of] taking out of the box again what one has just put into it" (as Frege later put it, Gl §88), all too readily sustaining "the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot" (PW 34/NS 38). This essentially trivial work of to-ing and fro-ing could as well be done by a machine, because it imposes no need to *refer back* to the *content* of the concepts from which the compounding begins (cf. PW 35/NS 39). It is only with Frege's representation of generality, which for the first time begins genuinely to exploit the complexity marked by the indexing of relations, that the need arises to be guided by content: it is the fully articulated content of a complex judgement or condition, and not merely the abstract and formal sequence of operations by which it was compounded, that determines the availability of a given division of it into function and argument.

The ramifications of this involvement with content run deep and wide, and we will, in effect, be pursuing them throughout the whole of §4. Our overall aim will be to trace the connections between Frege's central logical innovation, the *Begriffsschrift* §9 account of function-extraction that inaugurated quantificational logic, and what might at first seem remote aspects of his philosophy of logic. Perhaps the clearest indication that there are important connections to be traced is that the contentions just discussed in connection with Boole recur, almost identically, in *Grundlagen* (§§88–9): there, their target is Kant. First, though, we should consolidate the results of the present discussion by making explicit the position it dictates on an issue of recurrent exegetical dispute.

3.3.6 Must what is analysed be already complex? The distinction between analysis and decomposition

The previous section's talk of the formation of genuinely new concepts implies a contrast with those concepts that anyone would already need to possess to arrive at the new concept in the way described: for instance, the concepts *larger than,* and *multiple of*, needed to arrive in the way outlined at the concept *prime.* Similarly, talk of discerning a new and optional pattern in a sentence implies a contrast with an old and obligatory pattern, one that had to be recognized for the sentence to be understood at all. Our discussion thus commits us to reject a line of thought pursued by several commentators on Frege.

These commentators have taken Frege's reversal of the traditional order of priority between concepts and judgements to entail that, for him, judgements have no intrinsic structure *in themselves*, but rather have a pattern read into them by the way we regard them as falling into function and argument.[35] This may indeed seem to be the doctrine expressed by certain of Frege's remarks:

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[35] Sluga [1980] holds that this was Frege's consistent view; Baker and Hacker [1984, pp. 154ff.] regard it as one of two contradictory views, to both of which Frege was committed.

> I start out from judgements and their contents, and not from concepts... I only allow the formation of concepts to proceed from judgements. (PW 16/NS 17)

We must, it seems, have the judgement to hand first, before concept formation can begin. But if that is so, the judgement cannot in turn consist of concepts: it cannot already be a complex with concepts as its ingredients.

However, the same source makes plain that this is not a necessary part of Frege's view:

> And so instead of putting a judgement together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement. Of course, if the expression of the content of possible judgement is to be analysable in this way, it must already be itself articulated. We may infer from this that at least the properties and relations that are not further analysable must have their own simple designations. But it does not follow from this that the ideas of these properties and relations are formed apart from objects: on the contrary, they arise *simultaneously with* the first judgement in which they are ascribed to things. (PW 17/NS 18–9, my emphasis)

Expressed in terms of words, as Frege expresses it in *Grundlagen*, the picture that emerges from this passage is roughly this. We cannot arrive at an understanding of how a word might be meaningful except by conceiving of the word as an element in a sentence, and of its meaning as a contribution towards the meaning of a sentence. But to say this is not to deny the obvious fact that sentences consist of words. The sentence 'Socrates is wise' means what it does (partly) because it contains the words 'is wise'. But it is inconceivable that anyone should acquire an understanding of these words except by learning how they serve, in such a sentence as this, to ascribe a property to an object. The understanding of words is *simultaneous with* the ability to employ them in the expression of judgement.

Even in the passage just quoted, though, there is some line of defence for the contention that Frege thought of judgeable contents as 'in themselves' unstructured. For Frege does not say there, 'if a judgeable content is to be analysable in this way, it must be already articulated', but rather, 'if *the expression of* a judgeable content...'. That sentences are already articulated would be hard to deny; but perhaps Frege accepted this, while maintaining the contrary view of the contents they express.[36] We will be better placed to close off this line of defence when we have considered more closely how Frege understands the notion of expression that it crucially involves (§4.3.1). We will then see that it contradicts Frege's central thoughts about the rationale and value of a Begriffsschrift.

[36] Giving verbal expression to a judgeable content or thought would then be the means by which we read a particular structure into it. Travis [2000, pp. 85–88] recommends this understanding of Frege.

Anticipating that, it remains here only to explain an important terminological distinction introduced by Dummett [1981a, Ch. 15] in the course of his definitive resolution of any apparent tension between Frege's commitments *both* to the priority of judgements *and* to their having intrinsic structure (a resolution that this section simply takes over from him). The distinction is that between *analysis* and *decomposition.*

Analysis is directed towards making explicit the 'old' structure of a sentence or the content it expresses, the structure that is 'obligatory' in the sense that it must be appreciated by anyone who understands the sentence. Decomposition is the dividing of an already grasped structure into function and argument in a new way, such as is exemplified by the case of the concept *prime.* Two other contrasts follow from this basic one.

First, where a sentence is at all complex, detailing its analysis will be a several stage process. Speaking again in terms of the historical metaphor, analysis must retrace, in reverse order, the steps that were taken in building up the sentence and the content it expresses from primitive elements by means of the constructions it exemplifies. Decomposition, by contrast, takes the result of the already completed process of construction, and extracts from it in a single step a predicate or concept not among the elements from which the sentence and its content were originally built up.

Secondly, there are, as we have now often seen, various ways of decomposing a sentence — various predicates that can be extracted from it, each of which will yield a new concept. But this variety in one process is no obstacle to uniqueness in another: the analysis of a sentence may nonetheless be unique. Indeed, we have every reason to suppose that it must be: given that the sentence is unambiguous, we should expect there to be one and only one account of how it is constructed so as to bear that unique meaning. A unique content has a unique composition.

4 BROADER ISSUES

With the elements of Frege's system in place and his basic thoughts about it identified, we can move on in this section to explore some of their ramifications in Frege's philosophy of logic. The focus remains on Frege's logical doctrines, as initially presented in *Begriffsschrift*, rather than his subsequent work in support of the logicist thesis in the philosophy of mathematics. I will, however, allow myself to draw more freely than hitherto on Frege's later writings.

4.1 Logic, content and objects: Frege's engagement with Kant

4.1.1 Content and objectual bearing

In our discussion to date the notion of *content* has been that of a logical purification of the notion of meaning. Not every use Frege makes of the notion can be understood in this way, however. For instance, Frege at one point admits hesitation over whether one "can talk of the content of sentences of pure logic at all", without of course implying that the assertions made in *Begriffsschrift* are perhaps meaningless (PW 38/NS 43). A little earlier, in justifying his choice of material implication as a primitive for propositional logic, he contrasts the propositional operators as having "more" or "less" content according as they exclude more or fewer cases (PW 36/KS 40–1); Frege clearly doesn't want to suggest that 'and' is more meaningful than 'or' — whatever that could mean. And in another early writing he carries this thought to its conclusion: a concept that excludes nothing, such as the concept of self-identity, "can no longer have any content at all...; for any content can only consist in a certain delimitation of the extension" (PW 63/NS 71). In all of these examples another aspect of the notion of content is to the fore: content is, or implies, commitment; for one's judgement to have content is for it to impose demands on how things are, and for it to have more content is for the demands imposed to be more stringent. Another way of emphasizing this aspect of the notion would be to say that having content involves having external bearing. This brings out that a concern with content will involve us with ontological issues.

The most important passage of *Begriffsschrift* to manifest this aspect of the notion of content occurs in the provocative introduction to Part III, on the General Theory of Sequences. The results to be established in this theory will illustrate, Frege says,

> how pure thought (regardless of any content given through the senses or even *a priori* through an intuition) is able, all by itself, to produce from the content which arises from its own nature judgements which at first glance seem to be possible only on the grounds of some intuition. We can compare this to condensation by which we succeed in changing air, which appears to be nothing to the childlike mind, into a visible drop-forming fluid. (Bs §23)

The target of the passage is, evidently, the fundamental Kantian duality of sensibility and understanding:

> Without sensibility no object would be given to us, without understanding no object would be thought. Thoughts without content are empty, intuitions without concepts are blind...These two powers or capacities cannot exchange their functions. The understanding can intuit nothing, the senses can think nothing. Only through their union can knowledge arise. (CPR A51/B75)

When, in *this* setting, Frege suggests that content can "arise from" the "nature" of "pure thought", the notion of content carries a more specific commitment than the somewhat vague notion of external bearing just canvassed. The Kantian thesis he opposes is that thought can have relation to objects only through its involvement with intuition. Frege's counter-claim must therefore mean that thought's own nature suffices to give it *objectual* bearing. Regrettably, rather than trying to elaborate or justify that claim in a way that could begin to meet a Kantian's inevitable incomprehension, Frege resorts to a patronizing and insulting metaphor. (That is why I called the passage provocative.)

That very fact might lead one to wonder how thoroughly Frege could have engaged with his target here. A suggestion that would give specific form to that doubt is that Frege was in the process of introducing a novel conception of an object — tying the notion, as it had not been previously tied, to the semantic category of singular terms, and in the process untying it from the metaphysical issues that circle around the notion in Kant.[37] But Frege certainly writes as if he intends his disagreement with Kant to be a genuine one, and by his own lights that will be so only if there is, in the background of that disagreement, a shared core understanding of the notion of an object. That there is such a shared understanding will therefore be taken as a working presumption for this section, to be confirmed by it.

4.1.2 The general notion of an object

We can best turn straightaway to Kant for that background:

> What, then, is to be understood when we speak of an object corresponding to, and consequently distinct from, our knowledge? It is easily seen that this object must be thought of only as something in general $= x$, since outside our knowledge we have nothing which we could set over against this knowledge as corresponding to it.
>
> Now we find that the thought of the relation of all knowledge to its object carries with it an element of necessity; the object is viewed as that which prevents our modes of knowledge from being haphazard or arbitrary, and which determines them *a priori* in some fashion. For in so far as they relate to an object, they must necessarily agree with one another, that is, must possess that unity which constitutes the conception of an object. (CPR A104–5)
>
> ... knowledge consists in the determinate relation of given representations to an object; and an *object* is that in the concept of which the manifold of a given intuition is *united*. (CPR B137)

The notion of an object, as it is understood in the context of the doctrine Frege is rejecting, is that of a common focus of distinct representations, and the presumed source of the constraints of discipline, coherence and consistency amongst these representations. Unless Frege was just changing the topic, his counter-claim, that

[37] For the suggestion, see [Dummett, 1981, pp. 471 ff.], and the explanatory note at p. xx. For some textual support for it see Gl §89.

objectual bearing arises from the nature of thought itself, ought to connect with that notion.

There are, though, more and less committed ways of spelling out this notion of an object — that is, ways of elaborating the notion that draw on more or less of the full Kantian story of how the objectual bearing of representations is secured. Most minimally, the notion involves the possibility of identifying something as that on which distinct representations each bear, so that these representations can be thought of as giving different 'angles' on a single thing. Most committedly, that possibility is grounded in the common object's being located, as one thing amongst others, in a framework in which the representations themselves also have their location: for an object of outer sense, the framework is space, and the different 'angles' made possible by locating the object and its representations are literally angles. Between these lies the idea that representations have objectual bearing through application, in some specific form or another, of "the concept of an object in general" (cf. B128), that is to say, by taking on, through application of the categories, a specific judgmental form (being "determined *a priori* in some fashion": A104), forms through which they stand in constraining relations of coherence and consistency to each other. In claiming that thought can have content independently of any involvement with intuition Frege, it seems, must be intending to separate the most committed elaboration of the notion of objectual bearing from the rest, and to hold that the less committed understandings can be sustained by logic alone. In other words, Frege's claim must be some version of this idea: that objectual bearing consists in structural features internal to the nature of thought, and "unfolded" by the laws of logic.

4.1.3 The fragmentation of the notion of analyticity

To anyone steeped in the Kantian scheme, and so to any likely contemporary reader of *Begriffsschrift*, that claim would have been simply unintelligible. The cost of the analyticity of logic, in this scheme, is that it does not constitute a body of knowledge. The principle of non-contradiction, Kant says, is "without content and merely formal" (A152/B191). Logic — "pure general logic", in the complex typology he presents — "abstracts from all content ... and deals with nothing but the mere form of thought" (A54/B78). This abstraction is not just topic-neutrality — a withdrawal (in contrast with "special" logic) from the peculiarities of specific subject-matters — but a withdrawal too (in contrast with "transcendental logic") from every requirement imposed by thought's having *any* subject matter. Logic concerns only interconnections between thoughts, and not at all the relations of thought to its objects. Since truth is (if only nominally) to be understood as thought's agreement with its object, logic's exclusive concern with the internal organization of thought implies that fulfilling its requirements is only a necessary and never a sufficient condition of truth: "no one can venture with the help of logic alone to judge regarding objects, or to make any assertion" (A60/B85). The logical faculty, on Kant's conception, is manipulative one: it sorts and re-arranges. What logic shuffles around are concepts, but *it is all one to logic* whether the shuffling is a "mere play of concepts" or a reorganization of genuine knowledge. The reason is that the shuffling operations that Kant took to belong to logic are insensitive to how the shuffled concepts apply to objects, and even to whether the conditions are met for them to apply to objects at all.[38] In the terms that Frege used making the same point against Boole, they show *no concern with content.*

Formal and philosophical features coincide to mark the limit of what Kant recognized as logic — effectively propositional and monadic predicate logic — as a profound boundary.[39] In the first place, this much of logic is trivial in the precise sense of being effectively decidable. 'Problem'-solving in this logic is handle-cranking, making it fit work for a machine. Connected with that is Kant's understanding of the idea that logic is analytic: namely, that it is not ampliative but merely explicative; that it yields no extension to, but at most a clarification of, existing knowledge. Indeed, although Kant did not use anything like the modern notion of decidability, it is clearly presupposed by his notion of analyticity, because without it the two criteria of analyticity that he does present won't coincide: they jointly imply that a predicate concept (or conclusion) is *already thought* (A7/B11, also B15) in thinking a subject concept (or premise) if the predicate (conclusion) can be extracted by logic from the subject (premise) (A151/B191); and but for decidability that would imply that I might have no means of determining exactly *what* I have thought in thinking as given subject (or premise). And connected with that in turn is logic's lack of involvement with objects. The "something else X" (A8) which Kant says is needed to ground the connection of subject and predicate in an ampliative or synthetic judgement is *for him* the same "something $= x$" that we met in his explanation of the notion of an object. Kant's form of expression here seems almost prescient. In formalized reasoning, the register of involvement with objects is the use of variables. And although we now routinely use quantified variables in formalizing syllogistic reasoning, there is no need to do so. Variables serve to keep track of sameness and difference of argument places in nested quantification. But any monadic formula is equivalent to one without nesting (i.e. one in which no variable falls within the scope of another). The variables then have no real work to do, and can be dropped.

With the introduction into logic of Frege's account of function extraction these coinciding marks are in part reversed, and for the rest severely disrupted. The root of the decidability of syllogistic logic is that it shares the straightforwardly linear complexity of truth-functional logic. Because our minds are finite, the compoundings involved in reaching any judgement are finite, and the linearity ensures that they can be finitely unpacked and run through. We saw in connection with Boole (§3.3.5) how Fregean function extraction disrupts this simple structure: metaphorically put, it allows the line representing a judgement's dependence on its elements

[38]The clearest illustration of this is general logic's inability to distinguish universal and singular judgements (A71/B96).

[39]In the discussion of the next few paragraphs I am specially indebted to conversations with Michael Potter, as well as to his discussion of Kant in [Potter, 2000].

to turn back on itself to fasten onto so far unexploited complexity. The immediate implication of the loss of decidability is that Frege's logic is ampliative, capable of yielding new knowledge (Gl §§88, 91). That invites, but does not compel, the conclusion that logic is synthetic. As we just noted, Kant's analytic/synthetic distinction is premised on a coincidence of criteria. When the loss of decidability pushes the criteria apart, that puts the notion up for grabs. What we can most usefully attempt to do in the face of that fragmentation, though, is not to adjudicate which successor notion might best claim the title, but to keep track of the several components it used to combine. The currently most relevant of these is Kant's idea that the possibility of ampliative or synthetic thinking depends on its involvement with objects.[40] Here, too, we need to be prepared for a separation. I just remarked that the "something else X" that Kant spoke of in explaining the notion of a synthetic judgement was *for him* the same "something $= x$" that gives the notion of an object. But this identification is itself a matter of doctrine. The second x is the variable that registers employment of the notion of an object. The first X is a reference to intuition, the ability we have to gain knowledge of objects through the manner in which we are affected by them. The doctrine that identifies them is that any genuine, knowledge-expressing use of the notion of an object must be grounded in intuition: the very doctrine Frege aims to deny. To make that denial even discussable we need to separate out the different dependencies that, in describing Kant's opposed view, it was reasonable enough to run together in talk of 'involvement with objects'.

The first separating move, which would be obvious if it weren't obscured by the looming difficulty of subsequent moves, is to distinguish semantic from epistemological involvement. A judgement might be said to be 'involved with elephants' if it *speaks of* elephants, so that its truth depends on how things are with elephants. On the other hand, it might be said to be 'involved with elephants' if it *derives from* a familiarity with elephants — more fancily, if it is epistemologically grounded in a form of awareness of elephants. In a parallel way, a judgement might be said on the one hand to be involved with objects if it speaks of objects, if it employs the notion of an object, so that its truth depends on how objects are; or on the other it could be so described if it is grounded in a form of awareness of objects. We could then say that, according to Frege, logic is involved with or depends on objects in the first way but not the second. That, as I said, would be the obvious separation to make, if one didn't already feel threatened by the difficulty of explaining how the first kind of dependence can hold without the second. How could anyone make a judgement that speaks about elephants, or even have any grasp of the notion of an elephant, unless the judgement is grounded in some form of awareness of elephants?

The short and obvious answer to that question is that they couldn't. Now no one on the present scene imagines that the corresponding question about objects is exactly analogous to that one. For both Kant and Frege, the concept of an object is an *a priori* concept, so that what has to be explained is the genuine use of the

[40]The 'predicate-contained-in-subject' strand of the notion will be taken up in §4.3.1 below.

concept, not its acquisition. But that point doesn't make the question any less daunting until we make another separating move, distinguishing ways in which a judgement might 'speak of' objects.

First, this might mean that that the judgement speaks of *specific* objects, either of specific individuals or specific kinds of objects, and so has determinate ontological commitments. Or secondly, it might mean only that the judgement employs the general concept of an object. Understood in the light of the first of these, Frege's claim that content "arises out of" pure thought would suggest that thinking has the power somehow to dredge up out of itself the materials to think about. It is certainly true that Kant would reject that idea, but it takes no specifically Kantian presumptions to make it seem mysterious. It is also true that Frege was himself working *towards* that idea: it is a very rough but nonetheless accurate description of the position of his *Grundlagen*, which aims to extract the determinate ontology of arithmetic out of logic alone. But however that ambition may have already coloured the way Frege presents his claim in *Begriffsschrift*,[41] what he needs there is only the second understanding, that logic alone can sustain a genuine employment of the general concept of an object.

4.1.4 The 'concept of an object in general' belongs to logic

Now that we have thinned down Frege's claim we can begin to see how to justify it. Recall that the core notion of an object, that we are supposing is shared between Kant and Frege, is that of a common focus of representations whose recognition introduces constraints of coherence between them. When thinking of Kant, it is most natural to illustrate this notion in connection with intuitive representations. For instance, ideas of redness and greenness, considered merely as "modes of consciousness" – varieties of inner wallpaper (in the terms of §3.3.2) – are so far neither consistent nor inconsistent; but when these ideas are "referred to" an object, and so take on the form of a judgement that this object is both red and green, they clash. But that connection with intuition can be seen as an inessential aspect of the illustration. Its essential pattern recurs, in connection with conceptual representations, in the examples we gave (in §§3.3.4–5) of Fregean function extraction. Extracting the concept *prime* from the representation,

$$13 > 1 \wedge \forall n(M(13,n) \rightarrow (n = 1 \vee n = 13))$$

is a matter of recognizing its component representations as having a common bearing, thereby rearticulating each of them in the form of a judgement regarding that thing; and through that rearticulation inferential relations are recognized between this judgement and others. Even the simplest of Frege's examples manifests the same pattern. Inferring from 'Cato killed Cato' that Cato committed suicide is a matter of recognizing the common bearing of the concepts of killer and victim.

[41] With characteristic care Frege reports that the logicist construction of Gg was "a design that [he] had in view as early as [his Bs] of 1897 and announced in [his Gl] of 1884" (Gg p. viii/BL 5).

Generalizing what these examples show, we can say that reasoning turning on this kind of rearticulation of representations involves a grasp of the objectual bearing of those representations, or in other words, that it manifest the general concept of an object. And because the standards for such reasoning are set out in the laws of Frege's logic, we can conclude that "the concept of an object in general" belongs to logic.

(For simplicity, and in order that the main point should stand out, I have spoken here as if function extraction is always the extraction of a first-level function. In the broader setting it is of course not true that conceiving X as an element of a judgement replaceable by others, so articulating the judgement as a function of X, is equivalent to recognizing X as an object (see §2.2.3). The simplification is innocent, however, because the essential groundedness of the Fregean hierarchy of levels is dictated by his account of function extraction: conceiving an object a as replaceable by others in a judgement $F(a)$ brings recognition of a first-level function $F(\xi)$, which may in turn be conceived as replaceable by others yielding a second-level function $(\varphi)a$, which in turn... and so on. Thus, of whatever level X itself is, articulation of a judgement as a function of X rests ultimately on the recognition of objects.)

4.1.5 Understandings of the 'purity' of pure thought

So far, I hope, so good. But how far is that? A very natural response to the conclusion just reached is that it is weaker that the conclusion Frege meant to maintain, and consequently cannot tell against Kant in the way Frege clearly intended that his conclusion should do. In the remainder of §4.1 I intend first to diagnose, and secondly to defuse this complaint.

One momentarily attractive way of formulating the complaint would be to say: perhaps we have shown the notion of an object to be a logical one, one that the laws of logic essentially employ; but have we shown it to be a *purely* logical one, one that logic on its own can sustain? But this way of pressing the complaint rests, I think, on a misconceived suspicion. The suspicion, which is worth exposing, runs along these lines:

> 'Frege says that content — objectual bearing, as you understand it — arises out of "pure thought". But when you came to justify this — or your 'thinned down' version of it — you appealed to examples of reasoning about numbers and whether or not they are prime, about people killing each other, and such like. It's not news that the notion of an object is in play in such reasoning. But it's not *logic* that puts it there. The notion of an object isn't a contribution from pure thought, but enters with the material it's applied to, the particular subject matters of your examples.'

This suspicion is misconceived because it mis-parses such notions as 'the laws of pure thought'. Properly understood, this is a reference to laws of thought that hold purely in virtue of the nature of thought, and so hold of thought of *whatever* more particular variety it might be. The suspicion takes it in a diametrically opposite way, as referring to the laws of one special variety of thought, thought that is 'pure'.[42] Because of that the stress this way of pressing the problem places on what counts as 'purely' logical is ineffective.[43]

While that is the right way to rebut the complaint from within a Fregean viewpoint, it perhaps doesn't offer much of a diagnosis of it. The idea that underlies the complaint's talk of 'purity' is that logic cannot be beholden to any conditions external to it, and that it therefore shouldn't take a stand on any questions it cannot settle (in something like the way that pure geometry shouldn't take a stand on the shape of space).[44] To that idea we then add the fact that questions can arise in the course of a particular application of a logical principle that logic itself provides no means of settling. For instance, before endorsing the inference that some theologians are German from the premises that Leo Sachse was a theologian and a German, it might occur to you to wonder whether there actually was such a person as Sachse, or whether Frege just invented the name to use in logic examples (cf. PW 53–67/NS60–75); and if a question like that arises it is certainly true that we can't look to logic to answer it. So by the first thought, logic shouldn't take a stand on such questions. And that is naturally enough heard to require that logic should be neutral on such matters: to require, that is, that a transition of representations could be everything it ought to be from a logical point of view independently of how such questions are answered, and so independently of whether the transition constitutes a valid inference between genuine judgements or whether it remains, in Kant's phrase, a mere "play of concepts", or in Frege's, "a mere game with words" (PW 60/NS 67).

Frege resists that conclusion without denying the premises. His claim that the notion of an object belongs to logic implies that *there can be no gap* between those features of a transition of thought that make it subject to logical assessment and those that give it objectual bearing. But in holding that he doesn't of course imagine that logic can pronounce on whether there was such a person as Leo Sachse. Things like that are presupposed in the application of logical principles. "The rules of logic always presuppose that the words we use are not empty, that our sentences express judgements, that one is not playing a mere game with words" (*ibid.*). Nor do those presuppositions show that logic is beholden to conditions external to it. *We* make the presupposition in taking a logical rule to be applicable

[42]Note, I don't deny that there is such a variety of thought, and even less do I deny that Frege thought so: to pure thought belongs the task of setting out the laws that hold for *all* thinking, including, but of course not specially, itself.

[43]Another way of rebutting the complaint would be to say that it recognizes no distinction between, on the one hand, the general notion of an object's being a purely logical one, and on the other, there being purely logical objects: that is, the possibility of the second separating move of §4.1.3 is overlooked.

[44]So far Frege would agree. Cf. PW 7/NS 7: "If we were to heed those who object that logic is unnatural, we would run the risk of becoming embroiled in interminable disputes about what it natural, disputes which are quite incapable of being resolved within the province of logic, and which therefore have no place in logic at all".

to a particular transition. If the presupposition fails, *we* are in the wrong; but the rule, being inapplicable to the case, is untouched. On Frege's view the purity of logic is protected, not by having it pronounce on specific transitions independently of conditions of their genuine meaningfulness, but by not expecting it to pronounce at all on specific transitions: unless every condition on genuine meaningfulness is satisfied, we are simply not in the arena of logical assessment.

4.1.6 Dimensions of the generality of logic

The considerations of the last two paragraphs, even if they are accepted, are likely to seem pedestrian. I have run through them principally to illustrate how what to the Kantian perspective is an issue of huge importance and difficulty can seem in a Fregean framework almost to disappear. For the Kantian there is one big question, of how the framework concepts interconnected in the principles of logic can be guaranteed any application to reality so as to yield real knowledge; and this question calls for one big answer, as big as the Transcendental Deduction. On Frege's account the issue of real application seems instead to dissipate into innumerable little questions. Severally, these little questions call for nothing philosophically ambitious; indeed, like the question whether there was such a person as Sachse, they are hardly philosophical matters at all; and it is not obvious how the answers to them could collectively amount to anything of philosophical substance. What Frege appears to have done, one might say, is to redraw the boundary between logic and its application. By making the notion of an object into a logical one, he has left no room for the *general* question, how logic connects with objects. But it's hard to believe that this redrawing of the map should persuade us that there never really was a more significant issue than the sum of all the little questions for which Frege does leave room. So the effect of the pedestrian considerations just run through, I think, is to make the question pressing, whether there is any point in the Fregean framework at which a more likely successor to Kant's one big question can emerge.

An indication that there is lies in the fact that the remarks made above about the presuppositions of logical rules, while true enough to Frege's view, would apply equally to a view of the methods and aims of logic that he clearly would not share. According to this view[45] the business of logic is to describe certain forms that judgements might take, and to spell out the consequences of any judgement's being of those forms. For instance, classical propositional logic describes a possible form for a conditional judgement, as a compound of two judgements that is true if either the first of them is false or the second true. It then spells out that from the truth of such a judgement, along with the truth of its first component, the truth of the second component must follow. But, according to this view, that is *all* logic does. This view holds, as we just saw that Frege would also hold, that it is no part of the business of logic to say whether any particular episode of

[45]An advocate of the view would of course present it in a better light: see [Travis, 2000, Ch. 7].

judgement exemplifies the forms it describes. And the view then sees no problem in generalizing that point: if, regarding *each* episode of judgement, logic should take no stand on which of the forms it describes that episode exemplifies, then logic should take no stand on whether *any* judgement exemplifies those forms.

It is in the ease with which it makes that generalization that this view departs from Frege's. By the generalization logical necessity is reduced to a kind of if-then-ish necessity: *if* one speaks or thinks in accordance with one of the forms logic describes, *then* logic will spell out what follows from one's so doing. One might call this the 'fortnight theory' of necessity: *if* one happens to speak in terms of fortnights, saying for instance that one's holiday lasted a fortnight, *then* one is constrained in how one should describe the same holiday in terms of weeks; but of course nothing constrains one to speak in those terms in the first place. In just the same way, the view we are considering allows no sense in which it might be held necessary *that* one should speak or think in accordance with the forms logic describes.

Because it allows no such sense this view cannot accommodate the two dimensions to Frege's understanding of the generality of logic:

> ...the task we assign to logic is only that of saying what holds *with the utmost generality* for *all thinking*, whatever its subject matter. We must assume that the rules for our thinking and for our holding something to be true are prescribed by the laws of truth. The former are given along with the latter. Consequently we can also say: logic is the science of the most general laws of truth. (PW 128/NS 139, my emphasis)

The first dimension of generality has to do with the general content of logical laws, and can be acknowledged by the fortnight theory: *any* fortnight, without exception, contains two weeks. The second dimension, which the fortnight theory cannot accommodate, has to do with the sphere of authority of these laws. They hold for *all* thinking, not merely for thinking that happens to take one of the forms described. One falls within the ambit of logical laws not by thinking or speaking in these or those particular ways, but just by thinking at all. Or to put it in still another way, the demands logical laws impose are categorical: one cannot evade them by adopting a style of thought or a range of structuring concepts other than those the laws anticipate, in the way one can evade a hypothetical imperative by falsifying its antecedent; the laws have authority for any thought just by virtue of its being thought, as a categorical imperative is supposed to have authority for any will just by virtue of its being will.

It is in Frege's argument for this claim, if he has one, that we should look to find his response to the most plausible successor, within his framework, to Kant's one big question. *If* we had such an argument, we could combine it with the earlier conclusion to yield the following train of thought: by having the structures that subject it to the laws set out in Frege's logic, thought has bearing on objects; but

— as will then have been shown — *all* thought, just *qua* thought, is necessarily subject to those laws; therefore thought, of its very nature, has objectual bearing.[46]

4.1.7 A circular but non-trivial argument: logic can, because it must, 'put truth first'

So the question becomes, does Frege have such an argument? Attention to the fortnight theory is again a useful way of thinking about what kind of argument we might expect. That theory moved by a generalization from something Frege accepts to a position he rejects. The premise of the move was that it is no business of logic to pronounce on whether its laws are applicable to a given bit of thinking. The conclusion was that these laws might have no application at all. Now, barring a minor quibble,[47] this would be a sound move provided that 'thinking' or 'thought' meant the same throughout. Because Frege resists the move we should conclude that, for him, they do not mean the same. If the conclusion were speaking, as the premise does, of actual episodes of human cogitation, then Frege would accept it but dismiss it as irrelevant: it is indeed not a question that logic can settle whether human beings ever get up to anything capable of logical assessment; but just for that reason, when Frege speaks of 'thought' in logic he is not speaking of actual human performances. So whatever kind of argument we can expect from Frege will be neither premised on nor even sensitive to a survey of the actual or imaginable variety of human thinking. It will instead be an essentialist one, premised on a conception of what thought *must* be understood to be *for the purposes of logic*, just as Frege's notion of content is a notion of what meaning has to be for the purposes of logic. This means that, as well as disappointing philosophers of an anthropological bent, any argument we can hope to find in Frege is pretty well bound to be circular: the argument will be to the effect that thoughts, as logic must conceive of them, are necessarily subject to the laws of logic; and an argument to that effect could hardly be anything else.

Not all circles are trivial, however, and this one is at any rate not immediately so. The connection we need to establish is between how logic must conceive of thoughts and the applicability to them of Frege's logic laws; or more particularly,

[46] As one might expect, given the redrawing of the map I mentioned, this bears at most a distant relation to Kant's own way with the question in the Transcendental Deduction (though it is just feasible to read Section 17 of the B Deduction as including this argument). But there is, I think, a non-accidental similarity to the argument of the Metaphysical Deduction. "The same understanding, through the same operations by which in concepts, by means of analytical unity, it produced the logical form of a judgement, also introduces a transcendental content into its representations, by means of the synthetic unity of the manifold in intuition in general" (A79/B105). Suppressing its reference to intuition this central passage presents a version of the conclusion reached earlier, that there is no gap between judgements' displaying the forms on which inferential relations turn and their having objectual bearing. That all thought necessarily displays those forms – the second premise of the canvassed train of thought, for which we now need from Frege an argument — is the claim with which Kant presents his table of judgements.

[47] The quibble alleges a shift from $\forall\Diamond$ (for any bit of thinking, it is possible for all logic says that...) to $\Diamond\forall$. The move is not generally valid, of course, but why might it be objectionable here?

why thoughts as logic must conceive them will display the kind of structure whose discernment and exploitation in reasoning constitutes, in the way earlier argued, recognition of their objectual bearing. That kind of structure, we saw, is what makes it possible for the thought to be conceived now as a function of this constituent and then of that. So establishing the connection boils down to answering the question, why must logic conceive thoughts as displaying function-argument composition? The answer then falls out of Frege's statement of the "essence" and "subject matter" of logic. Logic's concern is to set down laws of inference, or as Frege more often says, laws of truth. So while there might be other viewpoints or other concerns that would lead one to dissect thoughts in other ways, the structures that logic discerns in thoughts will be those through which they are determined as true or false; the constituents it finds in them will be constituents on which their truth or falsity depends. But that is just what function-argument composition amounts to: for a thought $P(a)$ be a function of an argument a *just is* for P to be true or false according as a is or is not P. So, by in that way giving to the notion of truth the centrality Frege insisted it must have, we complete the circle that is, as I said, is all we could expect.

The notions it interrelates are surely substantial enough to make this circle an interesting one. A more delicate question concerns what the circularity implies for the train of thought I canvassed earlier, and claimed to be the most plausible candidate in the Fregean framework to be counted a response to Kant's one big question. With the circularity made explicit, that train of thought would now run:

> In having the structures that subject it to the laws of Fregean logic thought has bearing on objects; but all thought, *in displaying the structure of truth*, is subject to those laws; so thought, of its nature, has objectual bearing.

There are different understandings of the nature of the gap we were to be helped across by Kant's one big answer. But, on any understanding of it, it seems plain that Frege, in presupposing that thought of its nature displays the structure of genuine truth, and not merely some schematic or necessary condition of it, is taking himself, without anything Kant would recognize as an argument, to be on the right side of it.

Some have held that Frege didn't address Kant's question because he meant something different by the notion of an object; others that, while he was using Kant's notion, he just helped himself to it. Neither of these seems to be true. What Frege helped himself to, as he maintained that logic must do, is the notion of truth.

Earlier in this discussion (§4.1.6) we had difficulty understanding how, in Frege's scheme, the innumerable little questions that arise in the application of logic could add up to one big question, or whether behind all the little presuppositions one makes in applying logic there lurks one big one. What emerges is that there is, but that it is one that Frege recognized. As he expressed it in one his earliest writings:

> Logic becomes possible only with the conviction that there is a difference between truth and untruth. (PW 175/NS 190)

We might re-express the thought of this passage: logic *can* presuppose possession of the notion of truth, because it *must* do so. Kant's question asks how the forms that logic interconnects can be assured of any real content. Frege's response is that the forms of his logic must be conceived, from the beginning, as involved with content.[48] Anything less would not be logic at all.

4.2 Metaperspectives: some distinctions

4.2.1 Fregean and Wittgensteinian internalisms

In "Objectivity and Objecthood" Ricketts argues that commitment to the objectivity of judgement is a starting point in Frege's conception of logic, one that "needs no securing and admits of no deeper explanation" [1986, p. 72]. There is an evident affinity between that contention and the conclusion just reached, but our route to it also highlights important differences. According to Ricketts, the commitment precludes any "metaperspective" from which are raised such questions as "How does language hook onto reality?" or "How do we know that the ontological presuppositions of our discourse are satisfied?" [1986, p. 66]. As he explains in a related paper, "[Frege's] conception of judgement commits [him] to taking the statements of language more or less at face value. There is no standpoint from which to ask whether the thoughts expressed by the statements of language really represent reality, whether they are really true or false. Similarly, there is no standpoint from which to ask whether the statements of language really do express thoughts." [1985, p. 8]. As we have approached the matter here, what rules out the first of these 'standpoints' is the centrality of the concept of truth in fixing what, for the purposes of logic, thought must be understood to be: any notion

[48]There is an important sense, then, in which one cannot explain Frege's presumption of the notion of truth: logic's possession of that notion is, for him (as I have several times stressed since §3.1), a *datum*. But one can, I think, say something to explain why he did not have the kind of reason Kant had for resisting the presumption.

Speaking at a high level of abstraction one can say that Kant and Frege share a hierarchical conception of judgement. For Kant, concepts are "predicates of possible judgements", and a judgement is the subsumption of a lower by a "higher" representation (A68–9/B93–4). For Frege a judgement is the subsumption by a concept of level n of an entity (or entities) of level $n-1$. The essential distinction is that, for Kant, logic is indifferent to the value of n: the predicational structure present in a judgement, on which the application to it of logical principles turns, is *the same* at any level. If a judgement is to have any real content, it must, sooner or later, connect up with intuition ("Judgement is therefore the mediate knowledge of an object, that is, the representation of a representation of it. In every judgement there is a concept which holds of many representations, and among them of a given representation that is immediately related to an object" — *ibid.*), but whether this condition is met sooner, or whether only later, is not dictated by the logical structure of the judgement. From *logic's* point of view, then, the Kantian hierarchy of judgement is not essentially grounded. In Frege, by contrast, logically distinct principles, applicable in virtue of distinct judgemental structures, operate at different levels of the hierarchy. Thus the internal structure of a judgement *does not leave open* the question 'how far down', nor therefore the question whether, the judgement connects with objects.

whose connection with truth remained an open question could not be the notion of thought that is properly logic's concern. That reason does not however extend to rule out the second standpoint, with its question whether 'the statements of language really do express thoughts'. Of course, if a 'statement of language' were to be understood here simply as the expression of a thought, then the question would be tautologously excluded without any contribution from Frege's conception of logic. But if it is not so understood — if 'statements of language' are presumed to be independently identifiable, for instance by their occurrence in actual human exchange — then the second question remains open: no contribution from Frege's conception of logic can, or should, close it.

That the first of these supposed questions *is* closed off is something that might reasonably be meant by saying that Frege's conception of logic allows no "*serious*" or "*real* metaperspective" ([1986, pp. 76 & 78], my emphasis both times). To hold that the second is likewise closed off would be to hold that any such little presupposition as one might make in the actual application of logical principles – for instance, the presupposition that there was a person, Sachse – cannot on another occasion be treated instead as an open question. Patently, there is nothing philosophically serious about such a question, nor is there anything contentious about the availability of the standpoint to be adopted in addressing it. So the effect of running together the first and second kinds of question as equally closed off by the 'no metaperspective' doctrine Ricketts attributes to Frege is to retract what seemed to be the important qualifications on this doctrine emphasized above. What then inclines Ricketts to make this move?

A passage from his discussion quoted in part above reads more fully:

> From the perspective Frege acquires in starting from judgements and their contents, the distinction between objective and subjective exhibited in our linguistic practice needs no securing and admits of no deeper explanation. [Ricketts, 1986, p. 72]

Why this incongruous mention of "our linguistic practice"? To the account developed above, this will appear either as a distracting change of topic, or else as an allusion to something tacitly presumed to play the very securing and explaining role which, one imagined, it was the intention of the passage to maintain that *nothing* can play. The former is hardly to be expected, and the latter is confirmed when, later in the essay, Ricketts speaks of "those features of our linguistic practice that fund Frege's conception of logic".[49] But what (if anything) should be said to 'fund' Frege's conception of logic is the notion of truth. Ricketts's framework of interpretation allows only two possible roles for this notion. The first 'external' role is that of a metalinguistic predicate used to describe or stipulate semantic relations between the expressions of a language and the things they can be used to describe. To hold that the *fundamental* grasp of the notion lies in appreciation of that role

[49][Ricketts, 1986, p. 92]; it seems that by 'funding' Ricketts intends something like grounding, supporting, or justifying. Compare his remark, "Passages like these should not be taken to fund the platonist reading of Frege" [1986, p. 72].

for it would be, as Ricketts clearly sees, incompatible with the universalist aspects of Frege's logical conception — though, as we will see below, this point does not prohibit use of the notion in that role. The 'internal' alternative he recommends, the only alternative he thinks there is, casts truth as a device of compendious redescription. "Frege's language for talking about judgement" — for instance, the language of 'truth', 'thought' and 'expression' employed when Frege holds that an assertion gives expression to a judgement, the recognition of the truth of a thought — is, Ricketts holds, "a means for systematically redescribing selected features of our linguistic practices" [1986, p. 72]. Whatever significance attaches to the terms of the redescription must therefore be sustained by our antecedent grasp of what is redescribed: "it would be a mistake to think that we have any understanding of what an act of judgement is apart from [that] given by the formula that judgements are what assertions manifest" [1986, p. 71].[50] Specifically, the significance of 'true' in 'It is true that the sea is salty' is "parasitic on" the equivalence of that statement with 'The sea is salty' [1986a, p.174]. Because the notion of truth is in that way sustained entirely from within our linguistic practice it cannot turn around on itself to criticize or assess that practice. Ricketts thus sees no option but to attribute to Frege an "uncritical attitude" [1986a, p. 187] which "commits him to taking the statements of language more or less at face value" [1985, p. 8].

Even if one allowed his premises the conclusion Ricketts draws is, in allowing no important difference between the two kinds of questions distinguished above, too narrowly internalistic. The position those premises do plausibly dictate is that described by Wittgenstein in §136 of the *Philosophical Investigations*. At bottom, unsecured and admitting of no deeper explanation, lies the language game of assertion. The notion of truth "belongs to", but having no independent anchor it cannot — cannot, that is, without danger of generating a philosophical illusion — be said to "fit" the propositions that are moves in this game. To hold that a proposition is something assessable as true or false is to enunciate a rule of this game, one that records the internal connection between two notions: to treat something as a proposition *is* to treat it as so assessable. That being so, candidates for the first status are not admitted or rejected according as they meet or fail to meet a criterion provided by the second. But this does not imply that every candidate is admitted. While as a whole the game has no external arbiter, it has multiple aspects, and its own internal standards of coherence between, and ways of negotiating between, ways of coping with threats of collision between, these various aspects. These are already enough to separate the two kinds of questions we have been considering. The game is feasible only if players "do not come to blows" (§240), and over a given question of our second kind they well might.[51] This tempers, to a greater extent than Ricketts acknowledges,

[50] No such 'formula' occurs in the passages of Frege that Ricketts refers to. Nor, I think, does any other passage assign this kind of priority to the notion of assertion over that of judgement.

[51] That is to say, they might well do so *if* the game did not allow enough flexibility and sideways movement for differences over some 'little' presuppositions to be negotiated.

the supposed commitment to "taking the statements of language more or less at face value".

Much more importantly, however, Ricketts's premises are mistaken. For Frege, the notion of truth is neither a "metalogical" one, one external to logic, nor one internal to "linguistic practice". *It is internal to, because constitutive of, logic.* Acknowledging the primitiveness and priority of this notion means accepting, not only that it is indefinable, but further that it is "parasitic on" *nothing.* Truth no more needs the domesticating, practice-internal grounding that Ricketts offers it than it needs the external metaphysical justification he denies it. To grasp the notion is, as Ricketts would agree, to acknowledge a norm for judgement. And since judgement is something to be done, there is no harm in re-expressing that by saying that truth is a norm governing a practice. But, if we do say that, then we must insist that what counts as engagement in that practice is precisely what is spelled out in the laws of truth that 'unfold' the content of its governing norm. To participate in the practice, and to acknowledge and fall subject to the authority of its laws, is one and the same. Neither can be understood antecedently to the other. There can therefore be no presumption of a practice independently characterized — for instance, a practice of human linguistic exchange — that to engage in *it* will be to fall subject to the laws of truth, nor that a compendious redescription of *its* accepted norms will unfold that notion. Grasp of the kind of norm truth is excludes any such essential reference to "linguistic practice", as much as, and in the same way as, it excludes reference to human psychological processes.[52]

4.2.2 'Extra-logical' and extra-systematic thought

The 'internal'/'external' dichotomy that structures Ricketts's discussion leaves no place for the notion of truth as Frege understands it, and therefore involves a misrepresentation of his views. Underlying it, I think, is a neglect of distinctions emphasized in §§2.2.5 & 2.3.3 above. Consider, for instance, the slide that occurs in the following passage.

> Frege puts forward his begriffsschrift as a formulation of the principles of valid reasoning. In developing a conception of logic that supports this identification Frege addresses the issues raised by the logocentric predicament... However, in the end, this conception of logic is unsatisfactory. For there are deep tensions between Frege's official construal of the content of the axioms of the begriffsschrift, and his view of judgement that underlies the identification of the begriffsschrift as logic. [Ricketts, 1985, p. 3]

Here Frege's Begriffsschrift appears first as *a* formulation of logic, but is then *identified with* logic. By that means the "logocentric predicament" — that "in order to give an account of logic, we must presuppose and employ logic" (Sheffer, quoted in

[52]That Frege endlessly complains against the second while hardly mentioning the first is indicative only of the kind of misunderstanding he encountered amongst his contemporaries.

[Ricketts, 1985, p. 3]) — is transferred from logic itself to a particular formulation of it. If the predicament precludes any "real" or "substantive" metaperspective on logic, this is now taken to confine thought within that formulation. The possibility of genuine thought and reasoning about that formulation which, nonetheless, "presupposes and employs logic", is closed down. So the only alternative that Ricketts can conceive to a metaphysical externalism altogether insensitive to the predicament is the practice-internalism he develops and attributes to Frege. Every remark or passage in Frege which at first sight seems to exemplify the closed down possibility has then to be assigned a different role. Ricketts's preferred description of that different role is "elucidation".[53] Elucidations do not comment on, explain or justify aspects of a practice, however much their surface assertoric form may suggest that they do; rather, they are prompts, hints or clues that serve (somehow) to induct one into the practice. Elucidations are therefore not subject to standards of assessment that would properly be applied to assertions, standards of clarity, consistency or truth. The only relevant assessment of an elucidation invokes a pragmatic, or, more straightforwardly, a causal standard: a 'good' elucidation is one that in fact has the intended effect of inducting a beginner into the practice.[54] That being the role of an elucidation it has, for any given reader, no lasting value: once the effect is achieved the instrument can be thrown away as completely as, and with no more loss than, we abandon the babbling and imitative games through which we came to speak English.

As one might expect, the effects of this reading are most apparent in connection with axioms and basic rules. These are, trivially, end points of justification — unjustified justifiers — within the system they define; and so, equally trivially, any justification they can be given will be extra-systematic. If the system and logic are identified, this triviality becomes the baffling claim that their justification must be 'extra-logical'. But since logic is the framework of *all* justification, an 'extra-logical' justification is nothing at all. Hence those justifications that Frege appears to offer must be something else: they are elucidations.

One simple example of this already encountered (§2.2.5) is Weiner's mystifying contention that arguments explicitly presented by Frege as premised on the definition of the conditional stroke cannot be such, because within the system of

[53]Frege does employ such a notion, but much more narrowly than Ricketts's view entails. In Frege elucidations are introductory remarks aimed at securing common understanding of simple and primitive notions, such as that of a concept, which cannot be defined (see e.g. CP 182/KS 167–8; CP 300/KS 288). It would be a mistake of the kind illustrated in the text of this section to transfer what Frege says about such absolutely indefinable notions to notions that happen to be chosen as primitives of, hence not capable of formal definition within, a particular formulation of logic. One way of bringing out the mistake would be to note that the notion of a concept, Frege's favourite example of something indefinable in the first, system-independent sense, *is* straightforwardly definable in the second sense, within the later system of *Grundgesetze*: $\forall x.\varphi x = -\varphi x$ holds just in case φ is a function whose value is always a truth-value, i.e. a concept.

[54]Ricketts gives every impression of intending this quite strictly. Were we so constituted that a bang on the head would have the same effect, there would be nothing to chose between the two methods. Compare Davidson's off-putting remark about the effectiveness of metaphor [1981, p. 217]), whose relevance here Roger White alerted me to.

Begriffsschrift the conditional stroke, being a primitive, has no definition:

> ... if the justification of a logical law requires an argument in which the definition of the conditional figures, this definition must be expressible in Begriffsschrift. But no such definition is expressible in Begriffsschrift. Frege's symbol for the conditional, the conditional stroke, is a primitive symbol in his language. [Weiner, 1990, p. 72]

What makes this contention mystifying is that it first invokes a notion of an argument whose identity is independent of the specifics of its formulation (it is supposed to be the *same* argument, premised on the same definition, that would, *per impossibile*, be expressible in the Begriffsschrift), but then immediately reverses that to maintain that the existence of an argument is equivalent to the availability of one specific formulation. Without that confusion, the contention reduces to the triviality that an extra-systematic argument cannot at the same time be intra-systematic.

More revealing of motivations that can encourage that confusion is Ricketts's construal of Frege's attitude to the justification of basic laws.

> ... characteristically, the logical law with which [a particular application of logic within science] begins will be inferred from simpler, more perspicuous general truths. There is, as far as Frege is concerned, nothing to be said about the justification for our recognition of those basic laws of logic to be truths... [quotation omitted] Moreover, the maximal generality of these laws precludes their inference on the basis of truths of any other discipline. [Ricketts, 1986, p. 81]

Here, evidently, the salient aspect of the logocentric predicament is the threat of circularity. If one supposed that there was something to be said in justification of the axioms and rules of the Begriffsschrift, and so took at face value those passages in which Frege appears to say it, that would be, Ricketts suggests, tantamount to interpreting Frege as attempting to infer the most general laws of truth from truths of one particular science — presumably, the science of logical semantics.[55] But the maximal generality of those laws consists in their setting the standards for judgement and inference in every science, including that one. So the imagined inference would be immediately and transparently circular, and the interpretation of Frege as engaging in it is therefore to be rejected.

Now there is indeed, in Frege's view and in fact, no non-circular justification of logical principles. It is a commonplace that a demonstration of the soundness of *modus ponens* (for instance) will itself most naturally employ *modus ponens*. But how, if at all, does this bear on the passages for which Ricketts recommends an elucidatory interpretation? Consider the following typical instance.

> A result of the definition [of $\rightarrow$] given in §5 is that from the two judgements '$\vdash \Delta \rightarrow \Gamma$' and '$\vdash \Delta$' the new judgement '$\vdash \Gamma$' follows. Of the

[55]See on this point [Stanley, 1996, p. 58].

> four cases enumerated above [i.e. the truth possibilities as listed in §2.2.2] the third is excluded by '$\vdash \Delta \to \Gamma$' and the second and fourth by '$\vdash \Delta$', so that only the first remains. (Bs §6)[56]

Does the commonplace show that this, construed as a genuine piece of reasoning, is circular? In one very obvious sense, No. The reasoning demonstrates the soundness of Frege's rule, that from formulae '$\vdash \Delta \to \Gamma$' and '$\vdash \Delta$' the formula '$\vdash \Gamma$' is inferable. And that *that* rule (call it R) is not employed in the demonstration is obvious. On a different method of counting, of course, rule R is *modus ponens*, and *modus ponens* is (tacitly) involved in the demonstration. So in suggesting that the reasoning would be circular Ricketts is employing this second method of counting, on which rules have an identity independent of any particular formal implementation — yet this is in the service of an interpretation according to which logic is *identified with* one particular formal implementation! It is perfectly coherent to count rules in the second, Ricketts's preferred, way. What is not coherent is to combine that way of counting with a classification of rules as primitive or derivable according to the location in a particular deductive calculus of their formal implementations — for that would lead to a single rule being counted both primitive and derivable.[57] Similarly, it is perfectly coherent, in line with the second way of counting, to hold that Frege's rule R is (i.e. that it implements) *modus ponens*; that is, indeed, one reasonable way of formulating the conclusion of the reasoning of Bs §6 just quoted. What is not coherent is to employ that very conclusion to dismiss, or condemn to elucidatory reinterpretation, the reasoning that warrants it.

In recommending that such passages as that recently quoted from Bs §6 should be understood as presenting exactly the kind of genuine reasoning they appear to present, I am attributing to Frege commitment to a semantic metaperspective and an ineliminable use of a truth-predicate – commitments of which Ricketts is keen to relieve Frege, believing them to be incompatible with the fundamental role played by the objectivity of judgement in his conception of logic. In each case a metaphysically innocent bystander is mistaken by Ricketts for something far more ambitious and condemned in its stead. The next two-sections will demonstrate the innocence of these bystanders; I'll then ask what the real villain, if any, might be.

4.2.3 A semantic metaperspective is a perspective on a language, not on language

According to Ricketts's elucidatory reading "Frege's stipulations, examples and commentary function like foreign language instruction to put his readers in a position to know what would be affirmed by the assertion of any begriffsschrift formula" [1986a, p. 176–7]. Indeed so. They put a reader in a position to know that by

[56] As before, I have changed Frege's unobviously Greek letters to obviously Greek ones.

[57] I do not intend to condemn any remark that seems to mix methods of counting, e.g. 'Disjunctive syllogism is a derived rule in S_1 but primitive in S_2'. That is an innocent shorthand, comparable to 'I have the same book at home, but I keep it on the top shelf'.

telling him it; and to tell such things is to adopt a semantic metaperspective. Ricketts resists this conclusion. He chooses always to emphasize the "upshot" of the commentary rather than its content, the abilities induced rather than the thoughts conveyed ([1986a, p. 177]; cf. [1986, p. 87]). And he does so out of a presumption that, were any thoughts conveyed by the commentary, they could only be understood as "incipient theorizing about a relation between words and things" [1986, p. 176], theorizing of a kind that invites or addresses the philosophical problematic of "how language hooks onto reality" [1986, p. 66]. Yet his own reference to foreign language instruction should have made plain the mistake in this. That the French 'cheval' means *horse* is a plain (non-philosophical) semantic truth, stating a relation between words (better, a word) and things (better, some things). It no more invites or contributes to philosophical theorizing about the grounding of that relation than the statement that the battle of Bannockburn was fought in 1314 invites or contributes to debate over the reality of the past. Ricketts, it seems, has no stable conception of what semantics involves. In the reasoning we are now considering he attributes to a semantic metaperspective metaphysical ambitions it does not have, and in consequence refuses to recognize as such the unambitiously semantic remarks by which Frege introduces us to his Begriffsschrift.[58] Yet in the previous discussion, concerning the justification of logical laws, we saw him cast semantics as merely one special science amongst others. It is quite right, but also quite obvious, that nothing can be both of the things that on occasion Ricketts takes a semantic metaperspective to be. The unambitious idea sometimes associated with it is that of a statement of the meaning of the expressions of *a* language (e.g. French, or Frege's formal language) from outside it (e.g. in English, or in German). The ambitious and dubious idea drawn upon at others aims at explaining meaning, or language in general, from outside. Running the two together under the title of a semantic metaperspective makes for too easy a target; where one hopes for a searching examination of the second one finds instead only an inflation of the first.

A clear and central example of this inflation is Ricketts's contention that, if a semantic metaperspective were available, so that statements "pairing linguistic expressions with the items meant by [them]" were possible, then "Frege's remarks about truth-value determination [would be convertible] into a genuine theory containing the resources for defining a concept of correctness or truth" [1986, p. 91]. This is not so. A recursive characterization of truth conditions – into which form, as Ricketts here concedes, Frege's "remarks about truth-value determination" are

[58]See [Weiner, 1990, p. 198] for a striking example of the same thing. She there maintains that the question "To what does the word 'Pluto' refer?" is "not a question about language or semantics", on the ground that "it does not call for investigation into... the nature of our relations to the external world". True, it does not. But 'Pluto' is a word, and the question does ask about that word. Weiner is apt to deny that asking after the meaning of a word, e.g. 'Venus', manifests concern with the relation of the word with something "out there". Yet if the demonstrative "out there" were accompanied by a gesture towards the heavens, that denial would be trivially false: out *there* is precisely where one must look, with a telescope if need be, to find what is meant by 'Venus'. So what are we to imagine is the gesture accompanying Weiner's use of the words? (Or are the words themselves the gesture?)

readily cast – exploits the notion of truth in giving the content of the formulae of the language it treats. It *can* do this *only because* it does not simultaneously purport to define the notion of truth (see [Dummett, 1959, p. 7] and [Davidson, 1984, *passim*]). Frege's commitment to the indefinability of truth does not therefore preclude a semantic metaperspective; instead, his acceptance of the primitiveness and priority of this notion provides for one.[59]

4.2.4 Eliminable and ineliminable truth-predicates

The statement, it is true that the sea is salty, makes use of a truth-predicate; but since what is thereby stated is as well stated, the sea is salty, the use is eliminable (cf. PW 251/NS 271). A statement may be said to make ineliminable use of a truth-predicate if it is not in that kind of way equivalent to a statement in which the notion of truth does not explicitly figure. This is not a precise explanation. What counts as 'that kind of way', and hence how stringently or generously equivalence should be understood, depends variously on the point of the original statement, on what is in question in it and hence on what can be presumed not to be in question. Is 'The sentence "The sea is salty" is true' equivalent to 'The sea is salty'? for most purposes, where the significance of the quoted piece of English is rightly presumed not to be in question, it of course is. But there are other contexts in which that *is* in question, so where one's aim would be undermined by substituting the second for the first (e.g. in the course of deriving a semantic theorem governing that sentence).

Much the same holds of the relation between the statement that an axiom is true and the axiom itself. Ricketts would accept, I think, that the use of a truth-predicate in the extra-systematic assertion that '$a \rightarrow (b \rightarrow a)$' is true is not ineliminable, since what is there affirmed is as well affirmed by the intra-systematic assertion of the axiom.[60] But this holds only for some contexts. When, for instance, the first occurs as conclusion of an extra-systematic argument providing a semantic validation of the axiom, one's aim would be undermined by replacing it by the second. This illustrates that the question whether the first and second are rightly counted equivalent, and so whether the first's use of a truth-predicate is eliminable, has only an unhelpful answer: in contexts where one's

[59]As is well known, other attitudes towards the same recursion are possible, and Ricketts's mention of "*a* concept of... truth" perhaps suggests one of these. If the content of the formulae treated by the recursion is already known, then it can yield a characterization of truth as restricted to the language to which they belong — this being *a* concept of truth, truth-in-that-given-L. *The* concept of truth is evidently not a construction from such concepts, since it is presupposed in their characterization. In affirming this I take myself to be agreeing with Ricketts, and with at least part of what he intends in ascribing a universalist conception of logic to Frege. (The agreement was registered above, when I held that the *fundamental* grasp of the notion of truth does not lie in appreciation of its role in a semantic theory.) I differ from Ricketts in denying that it is any part of a semantic metaperspective to suppose otherwise.

[60]The course of his argument at [Ricketts, 1986, p. 83] (paralleled by [Ricketts, 1986a, p. 176] and [Ricketts, 1985, p. 7]) strongly suggests this; contrasting statements of the truth of an axiom and the soundness of a rule, he suggests that ineliminable use of a truth-predicate arises only with the second.

aim demands attention to the boundary between the system and extra-systematic remarks on it, the two are importantly non-equivalent, and the first's use of a truth-predicate ineliminable; otherwise they are effectively equivalent. Still less helpfully, *qua* extra-systematic, but only *qua* extra-systematic, the first makes an essential, extra-systematic use of a truth-predicate.

The same holds again of the statement that an inference rule is sound. The point is perhaps less obvious in this case, since it has to do, not only with one intra-/extra-systematic boundary, but with a relation between systems.[61] But once it is allowed that effective equivalence can hold across the first kind of boundary, there is no reason to resist its holding across the second.[62] Ricketts does resist this, holding, for instance, that "there is no single logical law corresponding to the rule of instantiation for first-level variables... Nothing in the nearest approximation to a single corresponding logical law, 'If $\forall x Fx$ then Fy', captures the notion of an instance" [Ricketts, 1985, p. 7]. The cited ground is true: it is essential to a rule's functioning *as a rule* that its generality be metalinguistic; the metalinguistic notion of an instance, by which we effect that generality, is thus not captured by any object-language generalization. But this point grounds only the same kind of unhelpful observation we made in connection with axioms, namely, that the extra-systematic *role* of the use of a truth-predicate in asserting that a rule is sound will not be shared by any intra-systematic statement which is otherwise equivalent to that assertion.

In short, the use of a truth-predicate should be counted ineliminable just in case the role of the statement employing it is extra-systematic. One cannot therefore *first* address such an understanding of the truth-predicate, subsequently turning the results to criticism of the perspective adopted in extra-systematic reasoning. There is no issue of the intelligibility of this truth-predicate that is, in the way Ricketts's approach supposes, independent of or prior to the issue of the availability of that perspective or the coherence of that reasoning. Thus, given the above defence of the latter, the former poses no issue at all.

4.2.5 The possibility of a metaperspective, or its necessity?

If, then, an ineliminable truth-predicate is another innocent bystander, what is the real villain? Indeed, is there a real villain? I believe there is, though it emerges only when the above unclarities are dispelled. Ricketts overtly targets the possibility of an ineliminable explicit use of truth-predicate. What he is most concerned to combat is, almost oppositely, the ineliminable necessity for a perhaps implicit appeal to a truth-predicate.

[61]For instance, between the system of Bs., in which '$(d \to (b \to a)) \to (b \to (d \to a))$' is an axiom, and the later system of Gg., in which interchangeability of subcomponents is a rule (Gg 12 & 48, BL 53 & 107).

[62]See §2.2.5 above. When, as Frege anticipates (Bs. 6, CN 119-20), we "make a special rule of inference", R in S_2, out of "[a] judgement expressed in [a] formula[...]", A in S_1, a chain of effective equivalence holds between 'R is sound' (a meta-S_2 statement), 'A is true' (meta-S_1), and A itself.

This emerges, for instance, in his insistence that,

> [f]rom Frege's viewpoint, the most basic assessment to which a judgement is subject invokes no mention of the judgement, no metaperspective... Our appreciation that judgements are subject to assessment as correct or incorrect is not manifested by the use of predicates 'true' or 'false', but rather in the assertive employment of language in the construction of lines of reasoning. [Ricketts, 1986a, p. 174]

This insistence is importantly true to Frege (see PW 252/NS 272, quoted at Ricketts [1986, p. 84]). The point it makes is, however, one that holds good *just as much* when a truth-predicate is employed as when it is not. For, if the truth-predicate is eliminable, effecting no genuine ascent to a perspective on the language to whose sentences it is applied, then trivially it is not by its use that the statements one makes are marked as subject to logical assessment; but equally, if there *is* genuine ascent, then the truth-predicate *belongs to*, so does not serve to invoke a perspective on, the language to which one's statements then belong. In neither case does the basic assessment of a judgement, one's acceptance or rejection of the judgement, consist in ascribing to it or withholding from it a truth-predicate. Equally, in neither case do one's reasons for accepting or rejecting the judgement run *via* reasons for ascribing or withholding that predicate. So, to identify the real villain, we should ask: against what picture of judgement *would* Ricketts's insistence be apposite?[63]

Such a picture would be one according to which *any* judgement, *whatever* conceptual resources it internally employs, so far stands inertly in waiting for endorsement or rejection by ascription to it of truth or falsity. The judgement itself, as it were, never reaches so far as to determine how things are as being in accord with it or not — it does not, in Wittgenstein's graphic phrase, "reach right up to reality" (cf. TLP 2.1511). Instead, it provides only a potential object for that determination, which determination thus assumes the form of a comparison between (what we first called) the judgement and how things are. The ineliminable need

[63]Other similarly emphatic passages present us with the same question. Thus, in "Logic and truth in Frege" Ricketts insists that "in the use of the sentence 'Sea water is salty' to assert that sea water is salty, nothing is predicated of what is asserted. In general, in an assertion nothing is predicated of the thought expressed by the asserted sentence" [Ricketts, 1996, p. 133]. The continuation of the passage makes clear that its talk of something being predicated is to be understood by reference to the behaviour of a "regular predicate" [Ricketts, 1996, p. 134]. That explanatory point is important: without it, Ricketts's claims would contradict Frege's remark that "what distinguishes [truth] from all other predicates is that predicating it is always included in predicating anything whatever" (PW 129/NS 140; cf. CP 354/KS 345). But *with* that explanation, one wonders what the issue might be. For, if we call a 'ground-level' judgement such as is expressed by 'The sea is salty' a judgement of level 1, and then say that a judgement which refers to and ascribes a property to a judgement of level n is a judgement of level $n+1$, then Ricketts's emphasized claims come to this: a level n judgement is not of level $n+1$. Indeed so. Whoever thought otherwise?

for a truth-predicate is the necessity, if anything is *to be* true or false, for that comparison to be effected.[64]

This picture of judgement has only to be formulated to be recognized as incoherent. On the one hand, ascription of the truth-predicate is itself portrayed as a species of judgement; on the other, it is held to be what *any* judgement needs to get so far as being true or false; yet if that general requirement really held, it would apply as much to the species of judgement proposed to meet it as to any other.[65] The picture can, however, have a pervasive and destructive influence without ever being made explicit; we found confirmation of that in discussion of the way of ideas in §3.3.2 above, for Descartes and Hume are committed to just such a picture. Further, unless an alternative conception can be made out, the admitted incoherence of this picture will not be enough to enable us to break free of it and reject it. Other diagnoses are possible. Their general form is that the incoherence of the picture is a reflection of our predicament, and displays the impossibility of properly conceiving how judgement stands to the world from within the only standpoint we have, that of judgement itself.[66] Such diagnoses are an intelligible response to the difficulty explored in §3.3 of locating the notion of truth in a framework developed independently of it. So the necessary alternative conception will be, as Frege recognized, one that 'puts truth first'.

As traditionally conceived logic formulates its laws by reference to the forms of judgements. In conjunction with the incoherent picture of judgement just sketched this traditional conception would imply that the laws of logic are *not yet* laws of truth. Ricketts is properly sensitive to the opening up of this gap, and the threat that it poses to the immediate, presuppositionless applicability of logic, as well as to its universality ([1985, p. 6], [1986a, p. 175], [1986, p. 75]). It is, indeed, a central strength of his reading that it allows one to recognize a common form that this threat takes in alternatives to Frege's understanding of logic that are, on the face of it, very different from each other. Two examples from earlier in our discussion will serve to illustrate this point. For the first, recall the contrast drawn in §2.3.3 between Frege's understanding of logical principles as contentful generalizations that subsume their applications, and the modern view of logic as an uninterpreted calculus, to be applied by assigning an interpretation to its schematic principles; the modern view, we then observed, unavoidably involves a metalogical stance, adopted in judging that the subject-matter of the intended application suitably parallels that of the calculus to be applied. For a second instance consider

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[64] "...I read Frege...[as arguing] that were truth a property, then a truth-predicate *would be required* to make a predication implicit in every assertion explicit" ([Ricketts, 1996, p. 134]; my emphasis). This is, I think, the passage that most clearly displays Ricketts's real concern with the (supposed) *necessity* for a truth-predicate, rather than its possibility.

[65] I intend this to recall Ricketts's understanding of Frege's argument for the indefinability of truth, as targeting a confused attempt to explain the genus, judgement, by appeal to a species of it. See his [1986, pp. 7-9] and [1996, p. 131].

[66] To bring out the continental tinge of this form of diagnosis, one might express it thus: the comparison *does have* to be effected, but all that we can offer to effect it falls as far short of doing so as what it supplements; it is just 'more text'.

again the Kantian conception of logic as concerned only with the interconnections of forms of thought (§4.1.3), and therefore as incompetent to address the 'one big question' whether these forms have any real application (§4.1.6). These post-Fregean and pre-Fregean views are, of course, importantly different from each other, but comparing each of them to Frege's conception brings out a crucial similarity between them. According to each of them, the application of logic raises a real question which must be addressed and settled *before* logical principles can be brought to bear, and which therefore cannot be settled *by* bringing to bear those principles. Those principles cannot therefore be, as in Frege's conception they are, the framework of *all* justification — the universal framework within which any real question is to be addressed. To put the point more in Ricketts's terms, on both pre-Fregean and post-Fregean conceptions the application of logic presupposes a judgement *on* logic, and is in that way *essentially* metaperspectival. The truth in the 'no metaperspective' slogan is, therefore, that it is a distinctive feature of Frege's conception, and one required by his universalist understanding of logical principles as authoritative for all thinking, that logic does not in that way *require* a metaperspective. There is, though, no sound route from that point to the conclusion that Frege's conception of logic *precludes* a metaperspective.

Just as importantly, Frege's conception gives no ground for rejecting the traditional idea that logic is a formal discipline, that its concern is with the forms of judgements. In the reasoning just outlined, that idea led to the opening of a threatening gap — the gap that requires a metaperspective — only in conjunction with an admittedly incoherent picture of judgement. By 'putting truth first', so correcting the second component, Frege is free to retain the first. For him there is no opposition between conceiving of logic as relating to the forms of judgement and counting them laws of truth: the forms of judgement to which logical laws relate *are* 'forms of truth' — or less metaphorically, forms through which a judgement is fixed as true or false in accordance with its composition. It is true and important that, for Frege, the generality of logical laws is substantive, that its principles generalize over every object, every concept, and so forth. And from that it trivially follows that logical principles do not, in the sense of the phrase just employed, 'generalize over' the forms of statements [Ricketts, 1986, p. 76]: the content of the principle '$Fa \vee \neg Fa$' is that any object either has or lacks any property, not that every statement ascribing a property to an object is true or false. But it is wholly consistent with this to acknowledge that logic's generalizations are formal in a way that those of physics are not: the range within which a generalization of physics holds good must be specified through some special concept, whereas grasp of the range of a logical generalization is equivalent to appreciating a form of judgement; understanding the notion *all objects*, for instance, is equivalent to appreciating the general way in which a symbol for an object contributes to the content of a proposition in which it occurs.[67]

[67]This is an area in which Wittgenstein's thought in the *Tractatus* is deeply indebted to Frege. The point just made is the same as the one he expresses, "A formal concept is already given with an object that falls under it" (TLP 4.12721). The consequence Wittgenstein immediately draws,

The point is worth insisting on for its epistemological ramifications. Ricketts warns that Frege's description of logic as "the science of the most general laws of truth" (PW 128/NS 139) should not — despite the plainness of Frege's assertions to this effect (e.g. PW 3/NS3, PW 128/NS139, CP 351/KS342) — be understood to suggest that logic is concerned with truth in the kind of way that physics is concerned with weight or heat [Ricketts, 1986, p. 75]. Rather, "[t]o say that the laws of logic are the most general laws of truth is to say that they are the most general truths" [Ricketts, 1986, p. 80]. Logic, then, is concerned with truth only in the way that every science is, namely, in being concerned to advance truths; it is distinguished from other sciences only by the generality of the truths it advances. This forced and patently revisionary interpretation, by the way that it ousts the notion of truth from the centrally defining position Frege's formulation accords it and shunts it instead to an eliminable periphery, illustrates how deeply Ricketts is led, by his 'no metaperspective' reading, to depart from Frege's conception. It also makes a mystery of the possibility of knowing logical principles. For, after all, a more general truth is, by and large, harder to come to know than a less general truth. To confirm, for instance, that all primates are so-and-so calls for a more compendious investigation than is needed to discover that all humans are so-and-so; to know the same of all animals, or all living things, or all material objects, will be progressively harder still. How are we to suppose that at the limit of this kind of progression the epistemological situation flips, so that the most ambitious claim is the most readily established? Frege's commitment to the a priority of logic is intelligible only if he recognized a distinction of kind, and not merely of degree, between the most general laws of truth and laws of special sciences.

There is of course more than this to be said about Frege's understanding of the epistemology of logic. But to say it we must first revert to the issue of the importance he attached to his logical language, the Begriffsschrift.

4.3 Language, Knowledge and Mind

We encountered above (§3.2.3) Frege's ideal of a language that completely embodies the content it is used to express, so that the conduct of inference in that language immediately displays "inner relationships" (CN 87/BaA 111) between contents. The purpose of this section is to elaborate that ideal, so as to display its importance in Frege's logical epistemology.

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that "one cannot, therefore, introduce both the objects which fall under a formal concept, and the formal concept itself, as primitive ideas" (*ibid.*) is in two ways importantly Fregean. First, the principle by which the consequence is drawn is Frege's prohibition on multiple definitions (cf. TLP 5.457: "what Frege [in Gg.] said about the introduction of signs by definitions holds, *mutatis mutandis*, for the introduction of primitive signs also".) Second, the consequence itself — that because the notion *all objects* is already settled by the role of any symbol for an object, there is no room to attempt to settle it again, by fixing a 'domain of quantification' — is implicit in Frege, and explains what to modern eyes appears as an omission in his account.

4.3.1 A "lingua characterica":[68] thought laid open to view

Philosophers asking whether or how language matters to thought, and hence whether or how attention to language is important for the philosophy of thought, are apt to fasten onto two questions. First, does thinking, the activity, depend in any way on using language? Second, do thoughts, the contents one thinks, depend for their existence or identity on having an expression in language? Frege's answers to these two questions are clear, though less helpful for understanding his project than the prominence of the questions might lead one to hope.

The first question may be understood in an occurrent or dispositional way, and on both understandings Frege answers it positively. Regarding the occurrent understanding he says, "we think in words" even if, beyond childhood, we no longer need to speak the words aloud (CN 84/BaA 107). As a matter of human necessity, "thinking is tied up with what is perceptible to the senses" (PW 269/NS 288). This is, though, *only* a human necessity: "there is no contradiction in supposing there to exist beings that can grasp the same thought as we do without needing to clad it in a form that can be perceived by the senses. But still, for us men there is this necessity" (*ibid.*). It is important to note that after each of the passages just cited, from the beginning and end of his career, Frege turned immediately to warn against the dangers for thought posed by the logical defectiveness of the everyday language on which it ordinarily depends; this strongly suggests that he regarded thinking's reliance on language as, however real, a cause for regret. A similar attitude accompanies his response to the dispositional version of the first question, whether our ability to have certain kinds of thought, or even to think at all, depends on our acquisition of language. On this Frege says, "without symbols we would scarcely lift ourselves to conceptual thinking" (CN 84/BaA 107). But this does not for him imply that we should enquire into the nature of conceptual thought through attention to this condition of its origin (cf. Gl p. vii). The "logical disposition in man" is only one of the factors shaping ordinary language, which is therefore "not constructed from a logical blueprint" (PW 269/NS 288).

Frege's platonistic response to the second question, most simply expressed in his statement that "thoughts are independent of our thinking as such" (PW 133/NS 145), has been sufficiently emphasized above.[69] But we have already noted, too, that there is no simple route from this ontological thesis to a methodological conclusion (see §3.2.3).

Neither the dependence Frege admits in relation to the first of these questions, nor the independence he insists upon in relation to the second, explains why he should have been so greatly concerned with language, nor the form that this concern took. For that, the more helpful thought is a more obvious one. Whether or not thinking or the contents of thought depend on language, *science*, as a collective, developing, rational and objective endeavour requires language (CN 83–9/BaA106-14; cf. also PW 133 & 136–7/NS 144 & 148, Gg p. xiv/BL 17). Further, those

[68] For an explanation of this odd piece of Latin, see NS 9, fn. 2.

[69] Evidence for the unqualified character of Frege's commitment to the mind-independence and language-independence of thoughts is amassed in [Burge, 1992].

features of science impose particular requirements on a language that could serve as an adequate vehicle for it. Frege connects in a single flowing discussion the seemingly most mundane of these requirements (e.g. the permanence of written symbols, that allows us to "review a train of thought many times over without fear that it will change",[70] CN 85/BaA 109) with the most stringent (e.g. explicitness about logical composition, and the exclusion of all ambiguity — since the possibility of re-checking would be of diminished value were "subtle differences of meaning", which "easily escape the eye of the examiner" (CN 86/BaA 109), to render it uncertain whether it is a *single* train of thought that is *re*-checked). Frege's concern with language is thus not descriptive but constructive: he does not seek to detail the features of something actual or given, but to specify how a language would have to be to meet the needs of science — and above all, of course, those of the science of logic.

Language's role as the vehicle of objective science imposes an ideal of the complete and explicit expression of thought. In early writings Frege resorted to the quasi-Leibnizian '*lingua characteristica*' to encapsulate this ideal; I will instead use 'language' in a constrained sense.[71] To articulate what it is for such a language to give fully adequate expression to a content is inevitably, at the same time, to sharpen one's understanding of the nature of the contents expressed. It is this that gives the project the importance it has for Frege. Contrasting this project with Boole's work he said: "Right from the start I had in mind *the expression of a content*" (PW 12/NS 13). Our aim must be to understand what that involves.

Let us say, first, that among systems of signs for the encoding and transmission of thoughts a *language* is a system whose elements *express* thoughts; and secondly, that a sign *expresses* a thought just in case whatever is internally involved in understanding the thought is assured through one's understanding of the sign. To draw the intended distinction a claim that such-and-such is 'assured though one's understanding of the sign' must be taken to call for a richer relation than merely the first's being a necessary condition of the second. Imagine an *ad hoc* signalling arrangement in which someone dispatched to a hill will indicate by raising his left or right arm that there is or that there isn't a lake in the valley beyond. A necessary condition of grasping the thought his gesture conveys is that one possess the concept of a lake. Since, in the circumstances, one does not understand the gesture unless one takes it to convey this thought, possession of that concept becomes derivatively a condition of understanding the gesture. But it would be wrong in such a case to hold that possession of the concept is 'assured through' understanding the gesture. This richer relation will hold, and a sign express a thought, only when necessary conditions of grasping the thought are immediately and non-derivatively necessary conditions of understanding the sign.

[70]Perhaps this idea was drawn from Schröder's "Axiom of symbolic stability" (in his 1873 *Lehrbuch*), mocked by Frege at Gl p. viii. (It would, one has regretfully to say, be entirely typical of Frege to identify a source only when criticizing it.)

[71]Frege himself typically used the unqualified term 'language' to refer to ordinary language.

Dummett long ago emphasized that for a system of signs to express the thoughts associated with them it is not enough for the system to be compositional.[72] Suppose, abstractly, a domain of thoughts to be arrayed in three dimensions, so that, for instance, the thought that Edinburgh is north of Leeds is singled out by a sign XYZ giving its co-ordinates. A necessary condition of grasp of that thought is that one should be able to understand what it is, in general, for a place to be such that Edinburgh lies to its north. This is assured *non-derivatively* by one's understanding of the sign XYZ only if there is some part or aspect of the sign whose understanding amounts to grasp of that condition; and this requirement will be met for each such case only if, of the indefinitely many abstractly available, the co-ordinate system exploited is (to speak loosely) that of subject-verb-object. In any compositional system understanding the sign for a given thought assures the satisfaction of necessary (though insufficient) conditions for understanding signs for a range of thoughts related to it by the system; but those will be necessary conditions of understanding the thought itself only if thoughts related to each other by the system are related in themselves, that is, only if the composition of the sign reflects that of the thought. A sign so composed is a *sentence*. On our understanding of the notions involved, then, it is analytic that the sentences of a language display the structure of the thoughts they express.

The notions of a thought and of a sentence are constrained by their role in the expression relation. It demands, in the first place, a species of essentialism about thoughts: the notion of expression gets no grip except on the assumption of a separation between what is and what is not an intrinsic or internal feature of a thought and its understanding; also, since what it is for a sentence to express a particular thought is explained by reference only to those internal features, they are presumed sufficient to identify it as the thought it is. So the very notion that thoughts may be expressed rules out the interpretative suggestions considered in §3.3.6, that thoughts are initially grasped as unarticulated wholes, or that a thought has no intrinsic structure but can be ascribed a structure only relatively to a way of conceiving it or to a way of giving it verbal expression. Without the presumption that a thought has a unique intrinsic structure Frege's requirement on a *lingua characteristica*, that it should "combine a concept out of its constituents" (PW9/NS 9–10) and so "*peindre non pas les paroles, mais les pensées*" (PW 13/NS 14) is no requirement at all. In the second place, the relation of expression involves a kind of transparency which applies somewhat differently to the its *relata*: a thought is transparent in the sense that whatever belongs internally to it is open to the understanding, so that to grasp it at all is to be in a position to grasp it completely; a sentence, correspondingly, is a transparent vehicle of thought, in that one's understanding of the sentence makes available to one all that belongs internally to the thought. (An expected consequence will be that, if two sentences

[72]See, for instance, [Dummett, 1981a, pp. 43–8]. This section owes a great deal to that passage and to other discussions in Dummett of the relation between the notions of language and expression.

express the same thought, this will be evident to someone who understands them both.)

Only a language in which thoughts are expressed provides for the "formation of concepts" according to the model described in *Begriffsschrift* §9. On the account of this given in §3.3.4 above, formation of a concept amounts to the explicit recognition of a pattern or feature present unacknowledged in the thought. A concept is thus an internal feature possessed in common by a range of thoughts. To "imagine" some element of a given thought as variable or replaceable by others is to arrive at a conception of a range of thoughts each characterized by what was supposed invariant in the original thought. For instance, if, in the thought that $2^4 = 16$, "the 4 is... treated as replaceable... we get the concept 'logarithm of 16 to the base 2': $2^x = 16$" (PW 17/NS 18). The vehicle of this imagining is the sentence. It is through imagining '4' as replaceable by other signs, thus through conceiving the given sentence as belonging to a range of sentences each sharing the pattern '$2^x = 16$', that one arrives at the concept common to the range of thoughts each expressed by one of these sentences. Because the sentence is a transparent expression of the thought, to reconceive the sentence as dividing into variable and invariant parts now in this way and now in that is to reconceive what was expressed by the sentence. The sentence in that way constitutes a standpoint of reflection, allowing for the explicit acknowledgement of what was already contained in the thought it expresses. This understanding of the formation of concepts is thus the successor in Frege to the containment ingredient of Kant's notion of analyticity.

This point is decisive in allowing Frege vastly to extend the notion of analyticity to embrace the validities of his new logic while remaining faithful, as he intended, to a central part of its meaning.[73] And of course, this faithfulness is not merely a matter of deference.[74] Frege's attempted construction of arithmetic within his logic could show arithmetic to be analytic (Gl §87) only if that logic is itself analytic. More generally, the construction could solve the epistemological problems of arithmetic only if Frege's logic is itself epistemologically unproblematic in the kind of way that the Kantian analytic clearly is. Frege was clearly aware that his expansion of logic pulled apart Kant's criteria of analyticity.[75] And he could hardly have been unaware that, should the question be raised, with what right his new logic should succeed to the notion, his proposed replacement definition of it — that

[73]Gl §3, fn. 1; also §88. In the latter text, what is most revealing of this intention is not Frege's suggestion that Kant "did have some inkling of the wider sense in which [he uses] the term", but that he chooses first to emphasize what (by his lights) is the minor problem of the non-exhaustiveness of Kant's way of drawing the analytic-synthetic distinction, instead of the far more serious problem of its inconsistency (again, of course, by Frege's lights).

[74]It is, though, at least that. There is no irony in Frege's expression of reluctance to "pick petty quarrels with a genius to whom we must all look up with grateful awe" (Gl §89). Excepting Leibniz, in whose work Frege highlights (Gl §6) only the same kind of minor lapse in rigour he elsewhere identifies in Euclid (CN 85/BaA 108–9), Kant is the only philosopher of those he takes issue with in *Grundlagen* whom Frege invariably discussed with respect.

[75]"The conclusions we draw... extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic" (Gl §88).

a truth is analytic if provable by (that very) logic, supplemented only by definitions (Gl §3) — would be of no use at all in answering it. To establish continuity with Kant's understanding of analyticity was therefore essential to the epistemological significance of Frege's logicist project. The strand in Kant's understanding of the notion which Frege seeks to modify and preserve, rather than to reject,[76] is the containment ingredient: conclusions drawn by logic from the definitions fundamental to his logicist construction are, he says, "contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house" (Gl §88). As the surrounding discussion makes plain, it is *Begriffsschrift* §9's account of concept formation that is to give substance to this metaphor.

The epistemological role of that account requires that alternative ways of conceiving a thought should be ways of *re*conceiving it, the possibility of which is provided for in its structure. The structure of a thought cannot therefore be equated with, nor the notion understood solely by reference to, its inferential relations with other thoughts. It is right, for instance, that recognizing the validity of 'P, $a = b \vdash P(a/b)$' "goes hand in hand with" conceiving its first premise as a predication of a [Ricketts, 1986, p. 85]. But the epistemological compulsion of the inference depends on the fact that the possibility of so conceiving it is implicit in one's understanding of that premise. That in which it is implicit is the structure of the thought, which must therefore be conceived as grounding, rather than as constituted by, the thought's inferential connections with others.

One might be moved to resist this conclusion — and so to hold that acknowledgement of an inferential transition amounts to, rather than rests on, discernment of a structural relation between the thoughts involved — by the concern that logic would otherwise be cast in a metatheoretical role, so that one's focus in inference would be on the thoughts themselves rather than on whatever subject-matter it is they concern.[77] Locutions used above in sketching Frege's account — such as 'reconceiving the thought' — may be heard as carrying the same suggestion. But what was said above about transparency should counter these worries. The possi-

[76]It may be helpful here to recall in summary form the results of previous sections on how Frege's logic compares with the various strands of the Kantian notion of analyticity. For Kant analytic knowledge is decidable and so merely explicative; Frege's logic is undecidable, hence ampliative (§4.1.3 above; see again Gl §88). For Kant logic, being analytic, has no involvement with objects; for Frege, the notion of an object is a logical notion (§4.1.4 above; see also Gl §89). For Kant, logic's independence from intuition allows it absolute universality, though at the cost of any assurance that it has real application; for Frege, too, logic's universal authority for all thought shows it independent of intuition (Gl §14), but we saw that Frege has available only a circular, though non-trivially circular, argument that his logic has this universality (§4.1.7). The remaining strand of Kant's notion, that the predicate (conclusion) of an analytic judgement (inference) is already contained in its subject (premise), therefore represents Frege's best hope of claiming continuity with the Kantian notion.

[77]Ricketts is clearly moved by this concern: he remarks, of a non-Fregean understanding of the formal character of logic, that it "insinuates that logic is somehow concerned with judging itself" [1986, p. 81]. §4.2.5 above defended a different sense in which the general laws of logic *are* concerned with forms of judgement; and §4.2.3 maintained, in opposition to Ricketts, the innocence of a (semantic) metaperspective. I nonetheless agree that it would be a misrepresentation of ordinary inference to have it adopting that perspective.

bility of conceiving the thought (note that I do not say conceiving *of* the thought) in any of the ways its structure allows *comes with* the thought. To actualize such a possibility is not to adopt a new perspective, but reflectively to exploit the perspective on things which one's thought provides. Similar remarks will apply to a sentence understood, as here, as a transparent vehicle of thought. To understand the sentence *is* to grasp the thought it expresses *is* to conceive things as being in such-and-such a way, for instance, such that $2^4 = 16$. To imagine the sentence dividing in a hitherto unnoticed way into variable and invariant parts *is* to reconceive that thought *is* to reconceive things as being *that* way, for instance, such that 4 is a logarithm of 16 to the base 2. In this, neither the sentence nor the thought obtrudes itself as the object of consideration, or obstructs one's view of things. Quite the reverse. The sentence understood, the thought, *constitutes* one's view of things. That is what was meant earlier when I said that sentences of a language which expresses thoughts present a standpoint of reflection.

I earlier called a sentence the 'vehicle' of concept-formation, meaning that it is through one's recognition of a structural aspect of the sentence expressing it that one comes to recognize explicitly the corresponding aspect of a thought. As we observed, to fill this role the sentence must in some way reflect the structure of the thought, but this is so far a relatively weak requirement. It says nothing about the possibility of specifying independently the manner in which it does so, nor therefore which aspects of the sentence those are whose discernment will amount to the formation of concepts. It therefore demands nothing of the kind of language that can satisfy this conceptual purpose except that it should be, in the sense that we have adopted, a language.

The demands of science, and in particular those of the "gapless" proofs called for in a fully explicit deductive science, are more stringent. What we have been speaking of as concept-formation is involved in deductive proof. It is involved, for instance, in the proof mentioned in §3.3.3, from the premise that Cato killed Cato, that therefore Cato killed someone who killed himself, thus:

$$Kcc; \therefore Kcc \land Kcc, \therefore \exists x(Kcx \land Kxx).$$

Thus the requirement that the correctness of a proof be syntactically determinable includes the requirement that the legitimacy of concept-formation also be so. That is, there must be syntactically specifiable operations, which one might now call operations of open-sentence formation, meeting the condition that any open-sentence so formed expresses a condition asserted to hold, by the sentence from which it is formed, of the item whose designation is replaced by a variable, and likewise asserted to hold, by any sentence formed from the open-sentence by inserting a constant, of the item designated by that constant. Hence a language suited for inference must not only reflect the structure of the thoughts it expresses, but must do so perspicuously, that is, through features of syntax to which a mechanical proof checker could be tuned.[78]

[78]It hardly needs saying that a natural language does not work like that: while one might take

In asking after the significance for Frege of thought's being expressed in this way, in a conceptual notation integrated with an inferential calculus,[79] we should distinguish specific and general questions. If one asks why it matters that a given train of thought leading to a specific conclusion should be so expressed, then the question is naturally answered in terms of formal rigour. To establish the conclusion through a "gapless" proof forces explicit recognition of "every 'axiom', every 'assumption', 'hypothesis', or whatever you wish to call it" (Gg p. vii/BL 3) on which it rests. And this, by making clear the ultimate grounds on which it may be known, allows one "to judge the epistemological nature" (*ibid.*; cf. Gl §3) of the conclusion proved. The benefit of the proof thus lies, not so much in the certainty that it confers on the conclusion, but in what it shows about the nature or ground of that certainty. (These matters have been mentioned above, and will be more thoroughly examined in the following section.)

But one can also ask, why it matters that *any* train of thought should be so expressed. To answer that involves recognizing an ideal in which thought's workings are laid fully open to view. For the specific question, the importance of the notation's permitting syntactic criteria of correct inference lies in its closing an entry point for tacit, unacknowledged presuppositions. For the general question, the important point is not so much that no tacit presupposition is possible, but that none is necessary. When a train of thought is presented in the form of a gapless proof it is presented completely. No presupposition is necessary to recognize in its development the exposition of logically correct thought that is not made explicit in the notation itself. What makes this train of thought correct, and so what correct thought is, is fully apparent in its expression. From this there follows another contrast between the specific and general questions. The benefits promised by a gapless proof cited in answer to the specific question of course depend on the actual production of such a proof for a given conclusion. What matters for the general question is just the possibility that trains of thought should be so expressed.

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a little persuading of the general claim that syntactic criteria of correct inference are unavailable in a natural language (CN 84-5/BaA 108), the failure of *this* condition is glaring. It follows that a natural language is unsuited to science. But from Frege's observations about natural language, reported in §3.2.3, a stronger conclusion follows. We there saw him maintain that, in a natural language, "the logical relations of the constituents" in a thought are "not expressed at all, but... left to guesswork" (PW 12-3/NS 13). If this were true we should have to conclude, not only that natural language does not reflect logical structure in the perspicuous manner required by science, but that it does not reflect it at all. The conclusion would then be that, in the sense of 'expression' here adopted, the sentences of a natural language do not *express* thoughts. Or to put the point most starkly, in the sense of 'language' that matters for logic, a natural language is, in Frege's view, not a *language* at all. (I should perhaps stress that, although Frege did adopt this attitude to natural language, he was not compelled to do so by anything central to his logical thought: his descriptive observations about natural language can be questioned without challenging anything in his constructive interest in language. What his logical thought *does* imply (§4.2.1) is that there can be no presumption that human languages function in the way that logic requires.)

[79] Frege says that in Leibniz's mind "the idea of a *lingua characterica*... had the closest possible links with that of a *calculus ratiocinator*" (PW 9/NS 9). So too, as we saw in §3.2.3, in Frege's own thought.

That possibility already both establishes, and manifests what is involved in, the objectivity of thought.

It is, I believe, the presence of this ideal that led Frege to begin, when reflecting on the significance of his work, "It is almost all tied up with the concept-script" (PW 184/NS 200). It is present, too, when he says that, while a proof of the kind that ordinarily satisfies a mathematician may be evidently correct, his own aim is "a matter of gaining an insight into the nature of this 'being evident' " (Gg p. viii/BL 4). The concerns of the specific question are clear in the context of this remark: the proofs Frege offers will make plain, as ordinary 'gappy' proofs do not, the precise grounds on which a given truth may be known. But Frege's phrasing points to something more general: his proofs will make plain, not only what is involved in knowing this or that given truth, but also, as one might say, what knowing *anything* properly involves.[80]

In a short fragment offered by Frege as "a key to the understanding of [his] results" (PW 251/NS 271) Frege wrote:

> If our language were logically more perfect, we would perhaps have no further need of logic, or we might read it off from the language. (PW 252/NS 272)

Here by 'logic' Frege does not mean the laws of truth, but the philosophical work involved in the struggle to come explicitly to recognize these laws, to understand the notions involved in them, and the nature of their authority. "Only after our logical work has been completed shall we possess a more perfect instrument" (*ibid.*). This instrument, the Begriffsschrift, is to be one that allows logic, in that sense, to be "read off": it will manifest of itself what thought, conceived as reflectively acknowledging the authority of the laws of truth, properly is.

4.3.2 Frege's qualified Euclideanism

We saw in §2.2.5 that Frege adopted and endorsed a Euclidean model for the exposition of his science of the laws of truth. We are now in a position to enter important qualifications that would have to be made to any ascription to Frege of the epistemological doctrines which, in philosophers' estimations at least, ordinarily accompany that model.

In the Introduction to *Grundgesetze* Frege famously wrote:

> The question why and with what right we acknowledge a law of logic to be true, logic can answer only by reducing it to another law of logic. Where that is not possible, logic can give no answer. (Gg p. xvii/BL 15)

[80]It is a characteristic thought of rationalism that a clearly articulated instance of knowledge will make plain, to reflective consideration, what it is to know. Compare the opening argument of Descartes' Third Meditation.

It can give no answer because "logic is concerned only with those grounds of judgement which are truths" (PW 3/NS 3), i.e. prior known truths from which the judgement may be inferred. It falls to logic to set up the laws governing this kind of inferential justification.

> But if there are any truths recognized by us at all, this cannot be the only form that justification takes. There must be judgements whose justification rests on something else, if they stand in need of justification at all.
>
> And this is where epistemology comes in. (*ibid.*)

When epistemology comes in, what does it say? In this context Frege gives no answer to the question. More than that, though, it is hard to see how he has left room for *any* answer. For, first, epistemology is to enter in response to a question, with what *right* we acknowledge a truth, not a question about what in fact causes us to acknowledge it: its concern is with justification, not explanation (cf. §3.2.1). So it seems that whatever epistemology might contribute will amount to a consideration on the basis of which it is rational to make the judgement in question. But is that not just to say in other words that it will contribute a truth from which the judgement might be inferred — so starting another round of the kind of inferential justification that was supposed to be completed before epistemology made its distinctive contribution? Secondly, if "those grounds of judgement that are truths" are *logic's* concern, that seems to leave to epistemology grounds that are not truths. Yet how could anything except a truth confer a rational right to judgement? Thirdly, when, within logic, inferential justification has run its course (when the process of reducing one law to another has been taken as far as it can be), the residue will be "general laws, which themselves neither need nor admit of proof" (Gl §3). So to the above questions about how epistemology *can* make a contribution at this point (since the judgements admit of no proof) is added another about why it should *have* to (since they need none).

So long as we think simply of a two-fold distinction between justification and explanation we can find no space for epistemology to occupy. This confirms Gottfried Gabriel's contention [1996, p. 342] that we must recognize in Frege a third kind of consideration, which in *Grundgesetze* is spoken of as giving "a *reason* for *our taking* [something] to be true" (Gg p. xvii/BL 15, my emphases).[81] A "reason for something's *being* true" (*ibid.*, again my emphasis) is a truth from which that thing may be inferred. The justificatory relation that holds between the reason and what it is a reason for is simply that the first entails the second, a relation that holds solely in virtue of the content of the two independently of either's relation to a knowing subject. The task of mapping that kind of justificatory relation between truths falls to logic and is effected through the construction of proof. A "reason for our taking something to be true" contrasts with this, since it essentially concerns not only interrelations amongst the domain of truths, but a subject's relation to

[81] Burge [1992, p. 363] also emphasizes that Frege operates with a three-fold distinction.

that domain. On the other hand it contrasts equally with the merely causal explanation provided by "psychological laws of takings-to-be-true" (Gg p. xvi/BL 13). The clearest point of contrast here is that a *reason* for our taking something to be true can exist only when that thing *is* true, whereas psychological explanation embraces false judgement as readily as true. Thus, if we say that justification by proof is logic's province, and explanation by cause psychology's, we reserve to epistemology "reasons for our taking something to be true". But now, what are these epistemological reasons (as I will call them) actually like?

We have several times (most recently in §4.3.1) encountered the idea that a fully explicit proof, by exposing every assumption on which a conclusion rests, allows one "to judge [its] epistemological nature" (Gg p. vii/BL 3). It is in *Grundlagen*'s discussion of that theme that this later terminology of a reason for our taking something to be true is most nearly pre-figured.[82] Frege there maintains that, "the distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement" (Gl §3). To assign a proposition to one or another of these categories on the basis of the criteria Frege there spells out[83] is to make "a judgement about the ultimate ground on which rests the justification for our holding it to be true" (*ibid.*). These judgements are, it seems, the most plausible candidates to be Frege's epistemological reasons. First, while these judgements are most naturally described as being about reasons, rather than as giving reasons, they could not be the first without also being the second: to warrant that there is a proof (of such and such a kind) for A is to warrant A. Secondly, they avoid the problem just sketched, that any reason contributed by epistemology would seem to add another link to a supposedly already completed chain of inferential justification: these judgements are not further links in the chain, but comments on the whole chain — as it were, from above.[84] And thirdly, they are, on that account, reasons *for* a subject: whereas logic's inferential justification simply lays out propositions in relations of entailment, these judgements draw essentially on a reflective review of the structures laid out.

Unfortunately, though, this identification doesn't at first help with understanding how 'epistemology comes in' *at the points where proof runs out*, that is, with primitive truths. If an epistemological reason for A is or derives from a reflective overview of an ideally rigorous proof of A, then where A does not admit of proof there will be no such reason.[85] The situation is not much changed even if, unlike

[82]Burge [1992, p. 365] notes that the terminological similarity is more striking in the German.

[83]They are: if the proof of a proposition reveals it as resting on particular matters of fact it is a posteriori, otherwise it is a priori; and within the a priori, if it rests on laws of some special science, it is synthetic, but if it rests only on general laws of logic it is analytic.

[84]Gabriel on that account calls the predicates ('analytic', 'a priori', etc.) ascribed in these judgements, "proof-theoretical meta-predicates for judgements and entire sciences" [1996, p. 339].

[85]Dummett [1991, p. 24] notes that Frege's stipulations in Gl §3 leave 'analytic', 'a priori', etc. undefined for primitive truths. If Frege thought of ascription of these terms primarily as supplying epistemological reasons, this bit of "uncharacteristic carelessness" is perhaps explicable.

Frege, we admit the propounding of a primitive proposition as a degenerate case of proof. Since a reflective overview of a proof must involve appreciating its completeness, a review of a one-link 'chain' of inference beginning and ending with A might issue in the report: A is self-evident. But while A's self-evidence is doubtless a reason to believe it, it is surely impossible to conceive this reason as deriving from reflection on its 'proof'. It seems, then, that the best sense we can make of how, according to Frege, 'epistemology comes in' works *least* well just where we thought it had to come in. And that strongly suggests that that thought derives from a perspective that is not Frege's. While 'epistemology comes in' when inferential justification runs out, it does not follow that it will bear uniquely on those isolated points *at which* inferential justification runs out. Instead, it seems that epistemological considerations have to do, for Frege, with reflective appreciation of the adequacy of a completed structure of justification. This gives us a strong reason to reconsider how much of the usual epistemological baggage of Euclideanism Frege carries.

The Euclidean model for the exposition of a science represents it as grounded in a specified group of self-evident truths from which all of its other truths are derived by self-evidently sound principles of inference. Frege clearly and centrally endorses this model in each of his major publications (Bs §13, Gl §§1–3, Gg pp. i-vii/BL 2–4). Moreover, though it is not prominent in *Begriffsschrift*, Frege later showed no reservations about the notion of self-evidence it involves (e.g. Gl §5, Gg Appendix/BL 127). As Burge importantly observed, however, "Although he alludes to self-evidence frequently, he almost never appeals to it in justifying his own logical theory or logical axioms" [Burge, 1998, p. 327]. To that observation we should add, first, that Frege's clearest and fullest recommendations of the Euclidean model are made in relation to a mathematical science *not yet* demonstrated to be purely logical (since its Euclidean exposition is to subserve that demonstration), with the result that the accounts he then gives of the model's virtues cannot be specially sensitive to its application within logic. And secondly, that Frege adopts the model in logic in full awareness of various disanalogies between that and the model's original, geometrical home — disanalogies that would make any simple transfer of epistemological doctrines associated with the model in geometry seriously misleading about the structure, and hence nature, of logical knowledge. By briefly recalling some of those disanalogies I hope to make clearer why it is that Frege, whilst certainly admitting the notion of self-evidence, could hardly call on it to carry a foundational burden.

A central rationale of "the deductive mode of presentation" is that "it teaches us to recognize [the] kernel" of a science, that contains, in undeveloped form, the content of all of its laws (Bs §13). By requiring that every transition be made "according to acknowledged logical laws" (Gg p. vii/BL 3), and that every proposition used without proof be "expressly declared as such", we can "see distinctly what the whole structure rests upon" (Gg p. vi/BL 2). Where proof involve definitions, retracing them will identify "the ultimate building blocks of a discipline", the indefinable notions it essentially employs, "[whose] properties... contain, as it were in a nutshell, its whole contents" (CP 113/KS 104). In geometry this is a matter of separating out those notions and truths that carry the special content of the science from those general principles of thought which serve to explicate that content, of "cutting cleanly away" what is synthetic and based on intuition from what is analytic (Gl §90).[86] This ideal is most clearly illustrated in Hilbert's axiomatization which, by making this cut cleanly as Euclid had not, showed Kant to have been wrong in holding that geometrical inference must be guided *throughout* by intuition (CPR A717/B745). Within logic, though, there is no cut to make. There is no real distinction between what carries its burden and what explicates it: only pragmatic considerations guide the decision whether a principle of thought should be embodied in an axiom or a rule (§2.2.5), and similar considerations guide the choice of primitives (§2.2.2). So in *Begriffsschrift* Frege is careful to specify that the "kernel" includes rules as well as axioms. That restores the letter of the above formulations, but it does not close the distance between the significance they can have in relation to geometry and in logic. There would be, and Frege would recognize, an evident artificiality in claiming that a particular axiomatization of logic displays "what the whole structure rests upon", or that the notions employed in it are logic's "ultimate building blocks". So, if a question is raised about ultimate justification, it would be an inappropriate transfer from the geometrical model to suppose that this question automatically concentrates itself into one about our ground for accepting logical axioms.

The rationale just described of a Euclidean exposition of geometry requires that the axioms be independent (CP 113/KS 104). The possibility of "assum[ing] the contrary of some one or other of the geometrical axioms" shows that they "are independent of one another and of the primitive laws of logic" (Gl §14). This requirement is inapplicable in logic. "We have only to try denying" one of its laws and "complete confusion ensues" (*ibid.*).[87] Frege's thought here is not that, because the contrary of a logical law is inconsistent, everything will follow, with the consequence that no judgement is better grounded than any other. It is not that *anything* goes, but that *nothing* goes: "Even to think at all seems no longer possible" (*ibid.*). This contention is similar to one of Russell's, though Frege's stronger conclusion — not just that "arguments from the supposition of the falsity of an axiom are here subject to special fallacies" [Russell, 1937, p. 15], but that nothing would any longer count as an argument — implies a commitment to the thoroughgoing unity of logic. Again, this indicates that it would be a mistake to regard questions of ultimate justification as bearing severally on the axioms of logic.

If, though, one *were* to press the question in relation specific axioms, Frege's answer would be clear: "The truth of a logical law is immediately evident of itself, from the sense of its expression" (CP 405/KS 393). We saw in §2.2.5 that

[86] I am taking for granted Frege's view of geometry as synthetic a priori.

[87] The quotation is in fact from a claim about the basic propositions of arithmetic; but since this claim is intended to demonstrate arithmetic's intimate connection with the laws of logic, Frege would clearly hold the same about them.

Frege presents the truth of axioms and the soundness of rules in his *Begriffsschrift* formulation of logic as consequences of the meanings assigned to the expressions occurring in them; and we confirmed in §4.2.2 that there is no call the take the reasoning that draws out those consequences as anything other than the genuine reasoning it seems to be. But we also saw, in connection with the 'fortnight' theory of §4.1.6, that we should not inflate these observations about Frege's presentation of a particular formulation of logic into an account of the authority of logic itself. Such an account, we saw, could not accommodate Frege's understanding of the universality of logic. To present logic's authority as grounded *only* in the meanings assigned to logical expressions leaves untouched the question Wittgenstein memorably posed:

> What use can the all-embracing world-mirroring logic have for such special twiddles and manipulations? (TLP 5.511)

Or more prosaically, and in terms closer to those of our earlier discussion: *given* that the acknowledgement of such-and-such principles is drawn immediately in train by the employment of such-and-such concepts, how is it that the possibility and value of employing those concepts is itself intrinsic to the nature of thought, so that the principles consequent to their employment have authority for *all* thinking (cf. PW 128/NS 139)? That the compulsion of a logical law can be traced to the meanings of the logical constants Frege would acknowledge as a truth; it is, so far, common ground between him and various twentieth-century movements concerned to deflate or demystify logical necessity. Where those later movements depart from Frege is in recommending that it is the *whole* truth. But that is exactly the attitude that would be encouraged if we imagined that Frege's epistemology of logic lies in what he has to say about the evidence that attaches to logical axioms.

It is in no respect false to say that, for Frege, logic is properly presented according to the Euclidean model, with theorems derived by self-evidently sound principles from self-evidently true axioms. It is, though, so far incomplete as to be seriously misleading. Above all it is misleading in its suggestion that self-evidence is a feature whose discernment is independent of, and stands in a grounding relation to, the construction of chains of inferential justification within logic. If that were so, then one would expect that, when epistemology comes in, its role will be to illuminate that feature and how it is to be recognized. As we saw at the beginning of this section, that expectation would be disappointed. If that seems to reveal a gap in Frege's account the most likely cause is, I think, an ill-fitting import from the geometrical home of the Euclidean model. In that case the expectation is appropriate, because the source of knowledge drawn on in acknowledging its primitive truths (PW 273/NS 292–3) is distinct from that involved in their inferential development: there is thus both room for and a need for an independent account of how that source yields knowledge. In the Euclidean development of logic things stand very differently. Here self-evidence is evidence to reason,[88] and *what reason is* is only laid out in the development of logic. Epistemology comes in, not to offer

[88] Both Burge [1998] and Gabriel [1996] stress the objectivity of Frege's epistemological notions;

an unnecessary and impossible substitute for the exposition of reason which logic itself provides, but as a reflective appreciation of the adequacy of that exposition to its guiding norm of truth. If all has gone well, it will find in that exposition the nature of thought laid out. It was presumably with that ideal in view that Frege suggested, as logic's task, "the investigation of mind: of *mind*, not of minds" (CP 369/KS 359).

5 SENSE AND REFERENCE

In an essay of 1892 Frege presented his famous distinction between sense and reference — the reference of an expression being the thing it is used to speak about, and its sense embodying a particular mode of presentation of that thing, or, as one might say, a particular conception of it. If one were to judge by the nature of his influence in the philosophy of language one might count this distinction Frege's most momentous contribution to philosophy. For his logical thought it is, in my view, less central. But, unlike most other important developments distinctive of his post-*Begriffsschrift* work, whose discussion belongs to the philosophy of mathematics, the distinction of sense and reference responds to pressures internal to his *Begriffsschrift* account of the contents of judgements and the logical relations amongst them. So some account of it must be given here.

5.1 *A forced 'split' in the notion of content*

§3.3.4 explained Frege's account of how concepts are formed through the recognition of patterns, illustrating that account with his example of the concept *suicide*, formed through recognizing the pattern 'ξ killed ξ' in the sentence 'Cato killed Cato'. I remarked that the model would yield indefinitely many specialized concepts of killing and being killed. One such, it seems, is the concept *patricide*, formed perhaps in rearticulating

Oedipus killed Oedipus' father

by conceiving *Oedipus* as replaceable at each of its occurrences, so highlighting the pattern

ξ*killed* ξ*'s father*, or $K(\xi, f(\xi))$.

Now exactly *what* is being rearticulated here? Or equivalently, *what* are we thinking of as occurring twice, and as replaceable at each of those occurrences? The theory of *Begriffsschrift* allows no satisfactory answer to that question, as the collision between the following two lines of thought confirms.

First line of thought: "In the sentence 'Oedipus killed Oedipus' father' are two occurrences of 'Oedipus', and noting that is finding a pattern in the sentence.

Burge in particular brings out (pp. 3309–40) that the objective ideal they appeal to cannot be understood independently of logic.

Equally, though, there are in the sentence three occurrences of 'i', and noting *that* is also finding a pattern in it: 'Oedξpus kξlled Oedξpus' father'. Finding that second pattern does not, of course, amount to forming a concept. In noting the three occurrences of 'i' we are fastening onto an inessential aspect of the complexity of the sentence, a complexity in the sign that is irrelevant to its having the content it does. Finding a pattern in the sentence is a means of concept formation only if, in rearticulating the sentence in accordance with that pattern, we are thereby rearticulating the content it expresses. Concept formation is preparatory to inference, and a move with that logical significance cannot relate to the sentence alone. The first noted pattern in the sentence, consisting in the double occurrence of 'Oedipus' in it, will therefore be as logically irrelevant as the second unless it reflects a complexity in the content expressed."

Second line of thought: "The concept ξ *killed* ζ, expressed in

(a) Oedipus killed Oedipus' father,

is equally expressed in

(b) Oedipus killed Laius.

The role of the concept in (b) is to combine with the contents of 'Oedipus' and 'Laius' to form a judgement true iff the first killed the second. Since the role in judgement that a concept has is internal to it — because judgements 'come first' (§3.3) — that must also be its role in (a). So the content of 'Oedipus' father' in (a) must be such that the judgement (a) expresses is true iff it was killed by the content of 'Oedipus' — less tortuously, it must be the thing killed by Oedipus iff (a) is true: namely, Laius."

Now, the example of patricide *is*, by Frege's lights, a genuine case of concept formation.[89] By the first line of thought that requires that the double occurrence of 'Oedipus' in the sentence reflect a double occurrence of its content within that of the sentence. We must, as it were, be able to discern the content of 'Oedipus' on the 'being-done-to' side of the circumstance the sentence presents as well as on the 'doing-to' side, which is to say that we must be able to discern the content of 'Oedipus' within the content of 'Oedipus' father'. But the second line of thought tells us that the content of 'Oedipus' is Oedipus and that of 'Oedipus' father' is Laius.[90] Together, then, the two lines of thought require that we must be able to discern Oedipus within Laius. But he is not there to discern.

The two lines of thought thus represent opposed pressures on the notion of content. The first requires that content be complex wherever there is logically relevant complexity in its expression. The second entails that a logically complex expression can share its content with a logically simple one. Both lines of thought

[89] For conformation, see the examples given in "Applications of the Conceptual Notation", where a constituent of a complex term is replaced by a variable (CN 208/BaA 92–3); we noted in §2.3.2 that the account of function extraction given in *Begriffsschrift* §9 provides for this.

[90] The content of a term is the thing to which a property is ascribed by a sentence containing the term. For confirmation, see, e.g., Bs §9 on the term 'the number 20'.

were central to Frege's thinking, and he was not willing to give up either of them. The only way of keeping them both was to allow the notion of content to "split" (Gg p. x/BL 6-7): he had to recognize a kind of content that satisfied the demands of the first, and a distinct kind of content satisfying the second. The first kind of content is *sense*: "As the thought is the sense of the whole sentence, so a part of the thought is the sense of part of the sentence" (PW 192/NS 209); "We can regard a sentence as a mapping of a thought: corresponding to the whole-part relation of a thought and its parts we have, by and large, the same relation for the sentence and its parts" (PW 255/NS 275); and correspondingly in the case of complex terms, "the sense of '3', the sense of '+', and the sense of '5' are parts of the sense of '3+5' " (PMC 149/WB 231). The second is *reference*: "Things are different in the domain of reference. We cannot [despite the fact that 'Sweden' is part of 'the capital of Sweden'] say that Sweden is part of the capital of Sweden" (PW 255/NS 275).

The collision outlined forces a split in the *Begriffsschrift* notion of conceptual content. Beyond the points about complexity just specified, it does not determine the precise form it should take. Those points do, however, have one specially natural development, as we can see by connecting them to naïve, untutored reactions to inferences involving our two sample sentences.

(1) (a) Oedipus killed Oedipus' father. ∴ (c) Patricide was committed.

(2) (b) Oedipus Killed Laius. ∴ (c) Patricide was committed.

A first reaction to this pair draws (unknowingly, of course) on *Begriffsschrift* §2's equation of the content of a judgement with the circumstance that must obtain for it to be true. Observing that, if things are as they are stated to be in (a), then they are as stated to be in (b), and vice versa, so that any consequence of things' being as stated in one is thereby a consequence of the other, it concludes that, because inference (1) is clearly valid, (2) must be too. Against that, however, stands the idea that inference (2) draws not just on 'what we are told' in its premise, but on 'additional knowledge' of the things its premise is about. This second reaction compares inference (2) to an inference such as

(3) G. W. Bush is President. ∴ An oil-man is President,

and says that, just as (3) is invalid because there is 'nothing in the premise' about Bush's being an oil-man, so (2) is invalid because its premise 'contains nothing' about Laius' being Oedipus' father. That was indeed the situation, but it is an aspect of the situation that is brought out only in inference (1).

Independently of other commitments, there is something to be said for both of these untutored reactions, but other commitments will force a choice. In particular, a notion of valid inference as certifiable by reference to the expressions of judgements has clearly to ally itself with the second of them. It thereby adopts a notion of content as having a complexity necessarily aligning with the complexity of its expression (so justifying the idea that there is 'nothing in' the content of (b) about fatherhood, because 'father' does not occur in (b)) — by the above, content as sense. But it also, in doing that, adopts a general, intuitive view of how this kind of content relates to the notion of content that informs the first reaction. Where that first reaction equated the content of a judgement with the situation judged to obtain, the second must regard its kind of content, sense, as capturing a specific aspect of the situation. What by the first reaction's lights is a single situation can present itself differently, and to grasp the sense of a sentence, a *thought*, will be to grasp one particular way in which it is presented. Because what follows from a thought and what does not must be settled by what is internal to it, it must be settled by the particular mode in which it presents a situation. If, like (1), a putative inference turns on an aspect of the situation other than that captured in the premise, it is invalid — which is just to say that recognition of the logical consequences of a thought must be made possible by one's grasp of the thought, in understanding an expression of it, and cannot require any independent means of adverting to the situation it presents or to the things involved in that situation. So this second reaction will find that, while it is not ruled out that there might be other purposes for which it would be reasonable to classify (a) and (b) as 'saying the same thing', that notion of a single 'thing said' will be of no use to it in understanding inference.

Those untutored reactions, by the way they bring to bear on inference the opposed pressures that forced the split in Frege's early notion of content, already cast sense as a "mode of presentation" (CP 158/KS 144). They do not settle *what* is presented. In elaborating them I have aimed to describe, not the relation between sense and reference, but that between sense and the early notion of conceptual content. For that purpose it was reasonable to begin by taking the thing that can be differently presented, and whose different modes of presentation are contained in the senses of different sentences, as a 'situation' or 'circumstance'. But although we *began* with that idea the conclusion very soon reached was that the notion of a situation — the supposed single 'thing said' by such non-equivalent sentences as (a) and (b) — is one for which logic will have no use. That is a conclusion that Frege drew: the notion does not figure in his later theory.

For what a mode of presentation presents, reference, we need to turn back to an earlier discussion. We saw, in §3.2.3, how Frege's thought about the understanding of inference that a Begriffsschrift exposition of it is to provide points towards equating the inferentially relevant content of an expression with its semantic value, but signalled there a reservation over how far the early notion of conceptual content could meet that condition. The semantic value of an expression is the entity associated with it to represent its contribution to determining as true or false any sentence in which it figures. In a standard semantic theory for a language such as Frege's Begriffsschrift expressions differing essentially in their internal complexity are assigned values of the same category (simple and complex terms are assigned objects as their values; n–place predicates, basic and logically structured, are assigned n-ary relations; and so on). The fact that the conceptual content of an expression was in *Begriffsschrift* required, albeit inadequately, to embody its inferentially relevant complexity therefore stood in the way of its equation with the expression's semantic value. But now that *that* role for conceptual content is taken over, and more adequately realized, by the notion of sense, nothing obstructs the equation of the other component that Frege discerned within his early notion of content, *reference*, with semantic value. The account that results is Dummett's: the reference of an expression is its semantic value, as this notion is understood in a classical two-valued semantics; its sense is the manner in which its reference is presented to one who understands the expression. Dummett [1981, chs. 5 & 6], [1981a, ch. 7] compellingly develops and defends this account; it would be pointless for me to attempt here what could only be a summary of its merits.

5.2 The argument of 'On Sense and Reference' and its limitations

It will be more useful to indicate briefly how the way I have motivated the sense-reference distinction here compares to Frege's introduction of it in "On Sense and Reference".

Frege there begins with a criticism of his *Begriffsschrift* account of the content of non-trivial identity statements, and continues by arguing that that inadequate account was, even so, responding to a genuine need. He then proposes the sense-reference distinction as meeting this need.

The interpretation of Frege's negative argument is uncertain. Recalling the *Begriffsschrift* proposal that an identity '$a = b$' states that the signs 'a' and 'b' designate the same thing, he continues:

> But this is arbitrary. Nobody can be forbidden to use any arbitrarily producible event or object as a sign for something. In that case the sentence $a = b$ would no longer refer to the subject matter, but only to its mode of designation; we would express no proper knowledge by its means. (CP 157/KS 143).

According to a natural and widely accepted reading, Frege's criticism is that the proposal misrepresents the intention of an identity statement: in affirming that Hesperus is Phosphorus our concern is with astronomy, not linguistics. On this reading Frege allows that an identity statement, on the proposal criticized, would express a truth, but complains that it would be the wrong truth. Makin [2000, pp. 94–101] has forcefully argued that this reading cannot account for Frege's conclusion, that on the proposed account an identity would express "*no* proper knowledge", and contends instead that the proposal's involvement with the *arbitrary* matter of linguistic designation precludes its giving the content of what for Frege must be, to have a role in logic, an *objective* judgement. The stress that my exposition placed on the thought that a complexity with logical significance must be a complexity of content, and cannot relate to the sentence alone, indicates sympathy with the spirit of Makin's approach. But we need not try to resolve the matter here. As §2.2.4 made clear, Frege's *Begriffsschrift* account offers a local solution to a general problem: even if the proposal could not be faulted at all in its application to identities, it would still be patently inadequate.

Frege goes on to argue, by a geometrical example essentially like that given in *Begriffsschrift* §8, that an identity statement may contain real knowledge. He has pointed out that this will be so — that a true identity '$a = b$' will have a "cognitive value" distinct from that of '$a = b$' — only if 'a' and 'b' differ in the manner in which they designate their common reference. He concludes:

> It is natural, now, to think of there being connected with a sign..., besides that which the sign designates, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. (CP 158/KS 144)

The following paragraph notes that the proposal so far relates only to singular terms. (Its application to sentences is given later in the essay, and to other categories of expressions in other writings.) The central inadequacy of the *Begriffsschrift* account is, however, already overcome: the new proposal does not relate only to some particular occurrences of singular terms.

The point about cognitive value is therefore already a general one, and can best be explained by drawing on the distinction's application to predicates as well as singular terms. The content of an expression, for logic's purposes, must be what matters to inference: its contribution to fixing the truth or falsity of a sentence in which it figures. Thus the content of a term 'a' will be the object it stands for, that of a predicate 'F', speaking roughly, the class of objects it is true of.[91] To understand an expression is to know its content. But then it seems that one's understanding of '$F(a)$' — involving knowledge of which object 'a' stands for, and which objects 'F' is true of — includes everything it takes to work out whether it is true: if one knows as much as one has to know to understand a sentence, there can be no "cognitive value" in learning that it is true (or, as the case may be, false). This paradoxical conclusion is avoided by recognizing that there are different *ways of knowing* what the content of an expression is. One *might* know what the content of 'F' is by knowing that its content is the class $\{a, b, c\}$, and so have nothing to learn by being told that '$F(a)$' is true. But equally one might know the content of 'F' by knowing that it is the class $\{x : Fx\}$, and then there plainly is something to learn. In summary, what in setting up the problem was called the 'content' of the expression 'for the purposes of logic' is the entity that is its content, its reference; in the sense of the expression is contained a particular way in which that reference may be presented, so that to understand the expression, to grasp its sense, is to know in that particular way what its reference is.

Frege is surely right to present the considerations reported here, those he offers at the beginning of "On sense and reference", only as naturally inviting his theory, and not as compelling its adoption. Were it intended in the second way, there

[91] "...in any sentence we can substitute *salva veritate* one concept-word for another if they have the same extension, so that it is also the case in inference, and where the laws of logic are concerned, that concepts differ only in so far as their extensions are different" (PW 118). Frege even so distinguishes a concept, as something essentially predicative, from its extension, an object. The text puts things roughly by ignoring this second point.

would be various grounds of resistance. For instance, suppose it accepted that the point about cognitive value forces us to recognize distinctions in sense between expressions referring to the same thing. As Travis [2000, pp. 36–42] stresses, this falls far short of recognizing distinguishable *senses*. Accepting that there are various ways of knowing what an expression stands for does not imply that there is *any* generally feasible or useful principle for counting these 'ways', let alone dictate one such principle, any more than would be the case in accepting that there are different ways of running a race or of writing a novel. (People differ in outlook and character, and in the shape of their skulls; that is a long way short of founding a science of personality-types, or phrenology.) Or again, supposing that point is overcome by through some principle of individuation for senses that allows us to regard them as genuine objects, one might still resist Frege's platonistic construal of these objects. When the notion of sense is motivated by the need to recognize different ways in which someone might know a reference, or different ways in which an expression might present it, then senses are more naturally regarded as constitutively dependent on minds or on language. The references of expressions appear, on this account, as independent items in reality, to which language or minds stand in external relations. But senses are not in that way further things for language or mind to be related to: they are objectifications of language's, and minds', relation to reality. This view need not deny that there are senses that have never been grasped, or senses that are not the sense of any actual expression, but it will insist that the existence of any sense consists in the possibility of its being grasped, or in the possibility that an expression should be so used as to have that sense. In short, senses may be objects, but it would be mythology to count them as self-subsistent objects [Dummett, 1986].

Whatever the merits of these points, it would be a misunderstanding to oppose them to the reasoning Frege presents at the beginning of "On sense and reference".[92] That reasoning is, of course, intended to be persuasive. But as Makin very clearly brings out [2000, p. 6], it is intended to be persuasive *within* an objectivist (or, as he says, "propositionalist") framework, and is not offered in justification of the framework itself. That the kinds of contents of judgements that are properly logic's concern are objective, that they have an intrinsic structure (§3.3.6), that they are adequately individuated by their internal properties (§4.3.1), that they stand one to another in logical relations determined solely by these internal properties (§3.2.3), and so forth — all this constitutes *a continuing background* to the reasoning that motivates the sense–reference distinction. One might go so far as to say that, in comparison with the significance of that continuing background, the sense-reference distinction is a matter of detail: it is a refinement in Frege's understanding of how his objectivist commitments are to be worked out, in how precisely the objective contents of judgement are to be conceived as constituted, forced by a specific inadequacy in *Begriffsschrift*.

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[92] I am *not* attributing this misunderstanding to either Travis or Dummett: I am envisaging a use of their points that they do not make.

That is why I have been able to postpone discussion of the distinction until this final section: as, I hope, earlier sections go some way towards confirming, what is most original and important in Frege's logical thought can be developed independently of it. And that is also why I chose to introduce the distinction by highlighting incompatible pressures that Frege's account of inference placed on his early understanding of the complexity of contents. There are no working notes or jottings from the relevant period to answer the question, exactly what problem was in the forefront of Frege's mind when he first formulated the sense-reference distinction; and it is in any case far from clear, in view of the various uses to which he then put the distinction, whether that would be a good question. But the problem highlighted at the beginning of this section is one that, given his defining logical concerns, and the objectivist framework within which he pursued them, Frege *had* to address. The remarkable thing about *Begriffsschrift* is that there were not more such problems.

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My greatest debt in this essay, which anyone familiar with his work will recognize on pretty well every page, is to the writings of Michael Dummett. Only a fool would aim to write about Frege without incurring such a debt. And only someone with the mind of an archivist could attempt to record it in detail. I can only hope that this single acknowledgement, by its patent inadequacy, begins to be adequate. Peter Geach's work, too, has had an effect that I could not pin down in footnotes: his influence has more to do with what I think important in Frege and what not than with the reading of any particular passage or theme.

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CN *Conceptual Notation and related articles*, ed. T. W. Bynum. Clarendon Press: Oxford 1972.

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Gl *Die Gundlagen der Arithmetik.* Breslau 1884. Reprinted with a translation by J. L. Austin as *The Foundations of Arithmetic.* Blackwell: Oxford 1978.

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