Harmonic Analysis and Special Functions on Symmetric Spaces

Gerrit Heckman
Henrik Schlichtkrull

PERSPECTIVES IN MATHEMATICS
S. HELGASON, EDITOR

Harmonic Analysis and Special Functions on Symmetric Spaces

PERSPECTIVES IN MATHEMATICS, Vol. 16

S. Helgason, editor

Harmonic Analysis and Special Functions on Symmetric Spaces

Gerrit Heckman
Department of Mathematics, KUN
Nijmegen, The Netherlands

Henrik Schlicktkrull
Department of Mathematics and Physics
The Royal Veterinary and Agricultural University
Frederiksberg, Denmark

ACADEMIC PRESS
San Diego New York Boston
London Sydney Tokyo Toronto

This book is printed on acid-free paper. ♾

Academic Press, Inc.
A Division of Harcourt Brace & Company
525 B Street, Suite 1900, San Diego, California 92101-4495

United Kingdom Edition published by
Academic Press Limited
24-28 Oval Road, London NW1 7DX

Library of Congress Cataloging-in-Publication Data

Heckman, Gerrit.
Harmonic analysis and special functions on symmetric spaces / Gerrit Heckman, Henrik Schlichtkrull.
p. cm. -- (Perspectives in mathematics : vol. 14)
Includes bibliographical references and index.
ISBN 0-12-336170-2
1. symmetric spaces. 2. Harmonic analysis. I. Schlichtkrull, Henrik, date. II. Title. III. Series.
QA649.H42 1994
516.3' 62--dc20 94-26553
CIP

PRINTED IN THE UNITED STATES OF AMERICA
94 95 96 97 98 99 BC 9 8 7 6 5 4 3 2 1

With love to
Elise, Jesse, and Michael,
our children

This Page Intentionally Left Blank

Contents

This Page Intentionally Left Blank

Preface

In the summer of 1991, M. Duflo, J. Faraut, and J. Waldspurger organized a summer school in Luminy (France) for Ph.D. students in the field of Lie groups. Subsequently this initiative has become an annual event, held in one of the European countries under the name of "European School of Group Theory". In the following years the school took place in Twente (the Netherlands) and in Trento (Italy), and this year it will be in Sønderborg (Denmark). During the two-week session of the school four series of main lectures are given, each by a specialist in some area within the theory of Lie groups. A set of lecture notes is furnished by the lecturers.

This book consists of two major parts, containing the notes for lectures given at the summer schools in Luminy (GH) and in Twente (HS). These two sets of lecture notes were written and can be read totally independently of each other. The idea of publishing them together came up only after they were finished. A shorter third part by one of us (GH) is added, in order to explain the connection between the two topics. It provides the direct motivation for our choice of publishing these notes together.

The theory of harmonic analysis has always been intimately connected with the theory of special functions. This is apparent, for example, on the 2-sphere S^2, where the harmonic analysis with respect to the action of the orthogonal group essentially is contained in the classical theory of spherical functions (the spherical harmonics). In spherical coordinates these spherical functions are the Legendre polynomials $P_n(\cos\theta)$. Also the very root of harmonic analysis, the Fourier theory on S^1 and $\mathbb{R}$, is of course based on the trigonometric functions.

The two main parts of this book both have their origin more generally in the theory of harmonic analysis and spherical functions on Riemannian symmetric spaces G/K, as developed by Harish-Chandra, S. Helgason, and others. In both parts we search for generalizations of this theory, but the directions of generalization are quite different.

The first part deals with a generalization of the elementary spherical functions from the point of view of special functions. For example, the elementary spherical functions on the k-sphere $S^k = \mathrm{SO}(k+1)/\mathrm{SO}(k)$, $(k = 1, 2, 3, \dots)$, are given by the Gegenbauer (or ultraspherical) polynomials $C_n^\alpha(x)$ with $\alpha = (k-1)/2$ (in case $k = 2$ they specialize to the Legendre polynomials $P_n(x) = C_n^{1/2}(x)$, as mentioned above). Here $x = \cos\theta \in [-1;1]$ is the height function on S^k, and n is the degree of the polynomial. In connection with harmonic analysis the basic property of the Gegenbauer polynomials is that they are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. In fact this weight function is integrable and the Gegenbauer polynomials $C_n^\alpha(x)$ are naturally defined for all values of $\alpha > -\frac{1}{2}$, but they appear "in nature" only for $\alpha \in \frac{1}{2}\mathbb{N}$. More generally the elementary spherical functions on a Riemannian symmetric space of rank one can all be expressed in terms of the classical (Gaussian) hypergeometric functions (in the compact case the Jacobi polynomials), which make sense for more general values of the parameters than those resulting from the harmonic analysis on G/K. A similar phenomenon is seen for Riemannian symmetric spaces of higher rank. The structure of a Riemannian symmetric space is described by a (restricted) root datum together with certain multiplicities attached to the roots. In the lecture notes it will be explained that one can introduce a theory of Jacobi polynomials and hypergeometric functions in several variables associated with a root system R and a multiplicity parameter k on R. The number of variables is the rank of the root system, and the root multiplicities are allowed to be arbitrary real (and nonnegative). When the root multiplicities do correspond to those of a Riemannian symmetric space, then these Jacobi polynomials and hypergeometric functions are exactly the elementary spherical functions of the two associated Riemannian symmetric spaces of the compact and noncompact type, respectively, expressed in suitable coordinates.

In the second part we generalize the harmonic analysis on G/K in a different direction; the differential structure is now allowed to be pseudo-Riemannian. More precisely we develop the harmonic analysis on semisimple symmetric spaces G/H. An example of such a non-Riemannian symmetric space is the one sheeted hyperboloid

$$\mathrm{H} = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots x_n^2 - x_{n+1}^2 = 1\}$$

with the action of the Lorentz group $G = \mathrm{SO}_e(n,1)$. Another example, referred to as the group case, is a semisimple group G viewed as a homogeneous space for $G \times G$ via the G-action from the left and the right. The harmonic analysis on G/H is concerned with the spectral decomposition of $L^2(G/H)$ as a representation space for G. In the group case, as well as in the Riemannian case, this problem was ultimately solved by Harish-Chandra, and it was a primary motivation for his work on semisimple Lie groups. The discrete part of the decomposition of $L^2(G/H)$ is fairly well understood in general from the work of Flensted-Jensen [107] and Oshima and Matsuki [166]. In contrast this part of the book deals with the decomposition of the most continuous part L^2_{mc} of $L^2(G/H)$. In the Riemannian case we have $L^2_{\mathrm{mc}} = L^2(G/H)$, but in the group case L^2_{mc} is in general a proper subspace of $L^2(G/H)$; it is the space of wave packets for the minimal principal series, and it decomposes as the direct integral of these representations. Our purpose is to explain how this decomposition can be generalized to the case of an arbitrary semisimple symmetric space G/H. In order to reach this goal we first have to develop some basic theory of semisimple symmetric spaces and the corresponding principal series representations – in fact the development of this theory composes most of these lecture notes. One of the complications in comparison with the group and Riemannian cases is that the decomposition of L^2_{mc} is not multiplicity free in general; the multiplicity is equal to the cardinality of a factor space $\mathcal{W}$ of a certain Weyl group.

The analogs for G/H of the elementary spherical functions on G/K are called Eisenstein integrals. The Eisenstein integrals, which are K-invariant (where K is a maximal compact subgroup of G), are particularly simple. These are the "spherical functions" which are needed for the harmonic analysis of the K-invariant functions on G/H. The presentation of the harmonic analysis on G/H, which we give in Part II, is simplified by considering primarily the K-invariant case.

Finally, in Part III, we draw a connection between the two generalizations of the spherical function theory on G/K by examining whether the theory of the K-invariant Eisenstein integrals developed in Part II can be integrated in the theory of generalized hypergeometric functions as developed in Part I. Indeed this seems to be the case; the K-invariant Eisenstein integrals can be expressed in terms of hypergeometric functions corresponding to a root system and a multiplicity parameter k determined from the structure of G/H. This observation opens up some interesting problems with which the book is brought to its end.

As mentioned above, the main part of this book was written as lecture notes for courses meant for Ph. D. students. The participants (who were at varying phases of their education) were encouraged before the session of the summer school to prepare by studying some prerequisites. For the lectures on hypergeometric functions these were the theory of root systems and Weyl groups as can be found for example in [7] or in various text books on semisimple Lie theory. Moreover some basic knowledge of the gamma function and the Gaussian hypergeometric function is assumed (as for example in the standard text book by Whittaker and Watson [74]). For the last chapter some familiarity with the structure theory and the analysis of Riemannian symmetric spaces is also needed. For this material the two text books by Helgason [35, 36] are the standard reference (as an alternative one could read Part II, and then return to this chapter). Some knowledge of the theory of spherical functions (as in [36]) could in fact also be useful for understanding the motivation behind the theory developed in the first four chapters of Part I. For the lectures in the second part of the book the suggested preparation was the first five chapters of the textbook by Knapp [130]. In order to reach a deeper understanding of the material in the final lectures some knowledge of the Riemannian symmetric space theory is an advantage (see the above mentioned books by Helgason).

The summer school in Luminy was organized by M. Duflo, J. Faraut, and J.L. Waldspurger, and the one in Twente by E. van den Ban, G. van Dijk, G. Heckman, G. Helminck, and T. Koornwinder. We are grateful to these people for the establishment of the schools and for inviting us to give lectures there. In addition the second author is grateful to E. van den Ban for the permission to present here (for the first time in print) several results of their joint work. Finally, we both express our gratitude to Sigurdur Helgason for his interest and enthusiasm, which paved the way for the realization of this project.

March 1994

G.J. Heckman
H. Schlichtkrull

Part I: Hypergeometric and Spherical Functions

Gerrit Heckman

Contents, Part I

Introduction

The theory of the hypergeometric function

$$F(\alpha,\beta,\gamma;z) = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\,2!}z^2 + \cdots$$

was developed mainly in the 19th century by the work of Euler, Gauss, Kummer, Riemann, Schwarz, and Klein. In the 20th century the theory of semisimple Lie groups has come to flourish, and as observed by E. Cartan and V. Bargmann some hypergeometric functions (Jacobi polynomials) appear as spherical functions on (compact) rank one symmetric spaces. Explicit calculations for the root systems A_2 and BC_2 by Koornwinder in his thesis (1975) made it plausible that spherical functions on higher rank symmetric spaces are part of a hypergeometric function theory in several variables. These hypergeometric functions can be thought of as "spherical functions" corresponding to arbitrary complex root multiplicities. Subsequently such a hypergeometric function theory associated with a root system was established by the joint work of Opdam and the author.

The hypergeometric theory is exposed from Chapter 1 to Chapter 4. The first three chapters are elementary algebraic in nature, and study the hypergeometric differential operators and the associated Jacobi polynomials. In comparison with the theory of spherical functions the surprising new concept is that of shift operator. It is at this level (of differential operators) that the c-function (rather a variant the $\widetilde{c}$-function) enters in a natural way. Chapter 4 is of a more analytic nature.

Chapter 5 deals with elementary spherical functions not only corresponding to the trivial K-type but also to an arbitrary one-dimensional K-type. Whereas the former were the natural example from which the hypergeometric theory was generalized, it turns out that the latter are easily expressible as hypergeometric functions.

CHAPTER 1

The hypergeometric differential operators

1.1. Differential-reflection operators for root systems

Let E be a real vector space of finite dimension, endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$. For $\alpha \in E$ with $\alpha \neq 0$ we write

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \in E \tag{1.1.1}$$

for the covector of α and

$$r_\alpha\colon\ E \to E, \quad r_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha \tag{1.1.2}$$

for the orthogonal reflection in the hyperplane perpendicular to α.

Definition 1.1.1. A *root system* R in E is a finite set of nonzero vectors in E spanning E with $r_\alpha(\beta) \in R$ and $(\beta, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

Note that we do not require R to be reduced. The standard reference for the theory of root systems (structure, classification, and tables) will be [7]. The group $W = W(R)$ generated by the reflections r_α, $\alpha \in R$ is called the Weyl group of R. Let $P = \{\lambda \in E; (\lambda, \alpha^\vee) \in \mathbb{Z}\ \ \forall\alpha \in R\}$ be the weight lattice of R. We write $\mathbb{R}[P]$ for the group algebra over $\mathbb{R}$ of the free abelian group P. For each $\lambda \in P$ let e^λ denote the corresponding element of $\mathbb{R}[P]$, so that $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$, $(e^\lambda)^{-1} = e^{-\lambda}$, and $e^0 = 1$, the identity element of $\mathbb{R}[P]$. The elements e^λ, $\lambda \in P$ form an $\mathbb{R}$-basis of $\mathbb{R}[P]$. The Weyl group W of R acts on P and hence also on $\mathbb{R}[P]$: $w(e^\lambda) = e^{w\lambda}$ for $w \in W$, $\lambda \in P$. It is easy to see that for $\alpha \in R$ the operator

$$\Delta_\alpha = \frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-r_\alpha)\colon\ \mathbb{R}[P] \to \mathbb{R}[P] \tag{1.1.3}$$

is a well-defined endomorphism of $\mathbb{R}[P]$. Clearly $\Delta_{-\alpha} = -\Delta_\alpha$ and $w\Delta_\alpha w^{-1} = \Delta_{w\alpha}$ for $\alpha \in R$, $w \in W$. For $\xi \in E$ the partial derivative

$$\partial_\xi \colon \mathbb{R}[P] \to \mathbb{R}[P] \tag{1.1.4}$$

is a linear operator defined by $\partial_\xi(e^\lambda) = (\lambda, \xi)e^\lambda$. Clearly the map $\xi \mapsto \partial_\xi$ is linear, and $w\partial_\xi w^{-1} = \partial_{w\xi}$ for $\xi \in E$, $w \in W$.

Definition 1.1.2. A (real) *multiplicity function* on R is a map $R \to \mathbb{R}$, denoted by $\alpha \mapsto k_\alpha$ and such that $k_{w\alpha} = k_\alpha$ for $w \in W$, $\alpha \in R$. Given a multiplicity function on R, and $R_+ \subset R$ a fixed set of positive roots we write for $\xi \in E$

$$D_\xi = D_\xi(k) = \partial_\xi + \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi)\Delta_\alpha \colon \mathbb{R}[P] \to \mathbb{R}[P]. \tag{1.1.5}$$

Clearly the map $\xi \mapsto D_\xi$ is a linear map: $E \to \mathrm{End}(\mathbb{R}[P])$. Note that D_ξ is independent of the choice of $R_+ \subset R$, which in turn implies that $wD_\xi w^{-1} = D_{w\xi}$ for $w \in W$, $\xi \in E$.

Remark 1.1.3. The operator (1.1.5) is a global analog of differential-reflection operators associated to finite real reflection groups by Dunkl [17, 32]. However, in the infinitesimal case (where the definition (1.1.3) of Δ_α is replaced by $\Delta_\alpha = 2\alpha^{-1}(1 - r_\alpha)$) the operators D_ξ, $\xi \in E$ commute, whereas in the global case

$$[D_\xi, D_\eta] = -\tfrac{1}{4} \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta \{(\alpha, \xi)(\beta, \eta) - (\alpha, \eta)(\beta, \xi)\} r_\alpha r_\beta \tag{1.1.6}$$

for $\xi, \eta \in E$. This formula can be derived along the same lines as in [17, 32]. Operators of the form (1.1.3) appeared in the work of Demazure on Schubert varieties [15, 16], and their infinitesimal analogs were introduced by Bernstein, Gel'fand, and Gel'fand [6, 38].

For $k = (k_\alpha)$ a multiplicity function on R we write

$$\rho(k) = \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha \in E, \tag{1.1.7}$$

$$\delta(k)^{\frac{1}{2}} = \prod_{\alpha \in R_+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^{k_\alpha}. \tag{1.1.8}$$

Lemma 1.1.4. *If $k_\alpha \in \mathbb{Z}_+$ for $\alpha \in R\backslash\frac{1}{2}R$, and $k_\alpha \in 2\mathbb{Z}$, $k_\alpha + k_{2\alpha} \in \mathbb{Z}_+$ for $\alpha \in R \cap \frac{1}{2}R$ then $\delta(k)^{\frac{1}{2}} \in \mathbb{R}[P]$.*

Proof. For S an orbit of W in $R\backslash\frac{1}{2}R$ it is easily seen that

$$\rho_S := \tfrac{1}{2} \sum_{\alpha \in S\cap R_+} \alpha \in P_+,$$

where $P_+ = \{\lambda \in P; (\lambda, \alpha^\vee) \in Z_+ \ \forall \alpha \in R_+\}$ is the set of dominant weights (for this statement we can assume that R is reduced and irreducible). Writing $k_S = \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha$ for $\alpha \in S$ (with the convention $k_{\frac{1}{2}\alpha} = 0$ if $\frac{1}{2}\alpha \notin R$) we conclude from the assumptions on k that $k_S \in \mathbb{Z}_+$, and hence $\rho(k) = \sum\limits_S k_S \rho_S \in P_+$. Write (with $k_{\frac{1}{2}S} = k_{\frac{1}{2}\alpha}$ for $\alpha \in S$)

$$\begin{aligned}
\delta(k)^{\frac{1}{2}} &= e^{\rho(k)} \prod_{\alpha\in R_+} (1-e^{-\alpha})^{k_\alpha} \\
&= e^{\rho(k)} \prod_S \prod_{\alpha \in S\cap R_+} (1-e^{-\frac{1}{2}\alpha})^{k_{\frac{1}{2}S}} (1-e^{-\alpha})^{k_S - \frac{1}{2}k_{\frac{1}{2}S}} \\
&= e^{\rho(k)} \prod_S \prod_{\alpha \in S\cap R_+} (1-e^{-\frac{1}{2}\alpha})^{k_S + \frac{1}{2}k_{\frac{1}{2}S}} (1+e^{-\frac{1}{2}\alpha})^{k_S - \frac{1}{2}k_{\frac{1}{2}S}},
\end{aligned}$$

where the product goes over W-orbits S in $R\backslash\frac{1}{2}R$. Since $k_S + \frac{1}{2}k_{\frac{1}{2}S}$, $k_S - \frac{1}{2}k_{\frac{1}{2}S} \in \mathbb{Z}_+$ and $(k_S + \frac{1}{2}k_{\frac{1}{2}S}) - (k_S - \frac{1}{2}k_{\frac{1}{2}S}) = k_{\frac{1}{2}S} \in 2\mathbb{Z}$ the lemma follows. □

For $F = \sum a_\lambda e^\lambda \in \mathbb{R}[P]$ with $a_\lambda \in \mathbb{R}$ and $a_\lambda \neq 0$ for only finitely many $\lambda \in P$ we write

$$\overline{F} = \sum a_{-\lambda} e^\lambda, \tag{1.1.9}$$

$$CT(F) = a_0. \tag{1.1.10}$$

Here CT denotes the constant term.

Proposition 1.1.5. *If the multiplicity function $k = (k_\alpha)$ on R satisfies the conditions of Lemma 1.1.4 then*

$$(F,G)_k := CT(F\overline{G}\delta(k)^{\frac{1}{2}}\overline{\delta(k)^{\frac{1}{2}}}) \qquad F, G \in \mathbb{R}[P] \tag{1.1.11}$$

defines a positive definite symmetric bilinear form on $\mathbb{R}[P]$.

Proof. Clearly the formula (1.1.11) defines a symmetric bilinear form on $\mathbb{R}[P]$. Since the standard bilinear form $(F, G) = CT(F\overline{G})$ on $\mathbb{R}[P]$ is positive definite the form (1.1.11) is positive definite as well because $\mathbb{R}[P]$ has no zero divisors. □

Note that the inner product (1.1.11) is independent of the choice of $R_+ \subset R$. Consider the torus $T = iE/2\pi i Q^\vee$ where $Q^\vee$ is the coroot lattice spanned by $R^\vee$. An element $F = \sum a_\lambda e^\lambda \in \mathbb{R}[P]$ can be considered as a Fourier polynomial on T by $F(t) = \sum a_\lambda e^{(\lambda, \log t)}$, where $\log t \in iE$ is a representative of $t \in T$. With this notation the inner product (1.1.11) can be rewritten as

$$(F, G)_k = \int_T F(t)\overline{G(t)}|\delta(k,t)|dt, \tag{1.1.12}$$

where dt is the normalized Haar measure on T. From this formula it is obvious how to define $(F, G)_k$ for $k_\alpha \geq 0\ \forall \alpha \in R$ (the precise restriction on the multiplicity function $k = (k_\alpha)$ is that $|\delta(k, \cdot)| \in L^1(T, dt) \in L'(T, dt)$ which is a slightly more general condition).

Theorem 1.1.6. *For all* $\xi \in E$ *the operator* $D_\xi(k)$: $\mathbb{R}[P] \to \mathbb{R}[P]$ *given by equation* (1.1.5) *is symmetric with respect to the inner product* (1.1.11) *on* $\mathbb{R}[P]$, *i.e.,*

$$(D_\xi(k)F, G)_k = (F, D_\xi(k)G)_k \qquad \forall F, G \in \mathbb{R}[P]. \tag{1.1.13}$$

Proof. Observe that for the standard inner product $(F, G) = CT(F\overline{G})$ we have $(\partial_\xi F, G) = (F, \partial_\xi G)\ \forall F, G \in \mathbb{R}[P]$. Indeed this follows from $CT(\partial_\xi(F\overline{G})) = 0$ and the fact that ∂_ξ is a derivation of $\mathbb{R}[P]$ (note that $\partial_\xi \overline{G} = -\overline{\partial_\xi G}$). Hence the adjoint D_ξ^* of D_ξ with respect to the inner product (1.1.11) is given by

$$D_\xi^* = (\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}})^{-1} \circ \{\partial_\xi + \tfrac{1}{2}\sum_{\alpha>0} k_\alpha(\alpha,\xi)(1-r_\alpha) \circ \frac{1+e^\alpha}{1-e^\alpha}\} \circ (\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}}).$$

First observe that

$$\begin{aligned}\tfrac{1}{2}\sum_{\alpha>0} k_\alpha(\alpha,\xi)(1-r_\alpha) \circ \frac{1+e^\alpha}{1-e^\alpha} &= \tfrac{1}{2}\sum_{\alpha>0} k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(-1-r_\alpha) \\ &= -\sum_{\alpha>0} k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}} + \tfrac{1}{2}\sum_{\alpha>0} k_\alpha(\alpha,\xi)\Delta_\alpha.\end{aligned}$$

If we write $\mathbb{R}[P]^W$ for the space of W-invariants in $\mathbb{R}[P]$ then it is clear that

$$\Delta_\alpha \circ F = F \circ \Delta_\alpha \qquad \forall F \in \mathbb{R}[P]^W, \quad \forall \alpha \in R$$

as endomorphisms of $\mathbb{R}[P]$.

Since

$$\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}} = \prod_{\alpha \in R}(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^{k_\alpha} \in \mathbb{R}[P]^W$$

and

$$\begin{aligned}(\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}})^{-1} \circ \partial_\xi \circ (\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}}) &= \partial_\xi + \sum_{\alpha \in R} k_\alpha(\tfrac{1}{2}\alpha, \xi)\frac{e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha}}{e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}} \\ &= \partial_\xi + \sum_{\alpha > 0} k_\alpha(\alpha, \xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\end{aligned}$$

we find $D_\xi^* = D_\xi$. $\square$

1.2. The constant term of differential operators in $\mathbb{D}_{\mathfrak{R}}$

Consider the algebra $\mathfrak{R} = \mathfrak{R}(R)$ (with 1) generated by the functions

$$\frac{1}{1-e^{-\alpha}} \qquad \text{for } \alpha \in R_+ \tag{1.2.1}$$

(viewed as a subalgebra of the quotient field of $\mathbb{R}[P]$). Note that for $\alpha \in R_+$

$$\frac{1}{1-e^{\alpha}} = 1 - \frac{1}{1-e^{-\alpha}}\,.$$

Hence $\mathfrak{R}$ is independent of the choice of R_+, and the Weyl group W acts on $\mathfrak{R}$ in a natural way.

The symmetric algebra SE has a double interpretation: we write $p \in SE$ if we consider p as a polynomial function on $E^* = \mathrm{Hom}\,(E, \mathbb{R})$, and $\partial_p \in SE$ if we consider ∂_p as a constant coefficient differential operator on E. Let $\mathbb{D}_{\mathfrak{R}} := \mathfrak{R} \otimes SE$ denote the algebra of differential operators on E with coefficients in $\mathfrak{R}$. Note that W acts on $\mathbb{D}_{\mathfrak{R}}$ in an obvious way: $w(f \otimes \partial_p) = w(f) \otimes \partial_{w(p)}$. We write $\mathbb{D}_{\mathfrak{R}}^W = \{P \in \mathbb{D}_{\mathfrak{R}}; w(P) = P \ \forall w \in W\}$ for the subalgebra of W-invariants in $\mathbb{D}_{\mathfrak{R}}$. If $\mathbb{R}_\Delta[P] = \cup \Delta^j \mathbb{R}[P]$ (union over integral j) denotes the localization of $\mathbb{R}[P]$ along the Weyl denominator

$$\Delta = \prod_{\alpha \in R_+ \setminus \frac{1}{2}R} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) \in \mathbb{R}[P] \tag{1.2.2}$$

then $\mathbb{R}_\Delta[P]$ has an obvious structure as a faithful (left) $\mathbb{D}_{\mathfrak{R}}$-module.

For $P, Q \in \mathbb{D}_{\mathfrak{R}}$ and $w, v \in W$ it is easy to check that

$$P \otimes w \cdot Q \otimes v = Pw(Q) \otimes wv \tag{1.2.3}$$

defines on $\mathbb{D}R_{\mathfrak{R}} := \mathbb{D}_{\mathfrak{R}} \otimes \mathbb{R}[W]$ the structure of an associative algebra. We call $\mathbb{D}R_{\mathfrak{R}}$ the algebra of *differential-reflection operators* on E with coefficients in $\mathfrak{R}$. Note that $\mathbb{R}_\Delta[P]$ has a $\mathbb{D}R_{\mathfrak{R}}$-module structure by

$$P \otimes w \cdot F = P \cdot w(F).$$

With this notation we view $D_\xi(k) \in \mathbb{D}R_{\mathfrak{R}}$. Define a linear map β: $\mathbb{D}R_{\mathfrak{R}} \to \mathbb{D}_{\mathfrak{R}}$ by

$$\beta(\sum_w P_w \otimes w) = \sum_w P_w \in \mathbb{D}_{\mathfrak{R}}. \tag{1.2.4}$$

Then it is obvious that

$$P \cdot F = \beta(P) \cdot F$$

for $P \in \mathbb{D}R_{\mathfrak{R}}$ and $F \in \mathbb{R}_\Delta[P]^W$.

Lemma 1.2.1. *For $P = \sum_w P_w \otimes w \in \mathbb{D}R_{\mathfrak{R}}$ we have $[P, 1 \otimes v] = 0$ if and only if $v(P_w) = P_{vwv^{-1}}$ $\forall w \in W$.*

Proof. Using the definition (1.2.3) we have $(1 \otimes v).P = \sum_w v(P_w) \otimes vw$ and $P.1 \otimes v = \sum_w P_w \otimes wv = \sum_w P_{vwv^{-1}} \otimes vw$. □

Lemma 1.2.2. *If we write $\mathbb{D}R_{\mathfrak{R}}^{1\otimes\mathbb{R}[W]} = \{P \in \mathbb{D}R_{\mathfrak{R}}; [P, 1 \otimes v] = 0 \;\; \forall v \in W\}$ then*

$$\beta\colon \mathbb{D}R_{\mathfrak{R}}^{1\otimes\mathbb{R}[W]} \to \mathbb{D}_{\mathfrak{R}}^W$$

is an algebra homomorphism.

Proof. If $P = \sum\limits_w P_w \otimes w \in \mathbb{D}R_{\mathfrak{R}}^{1\otimes R[W]}$ then for all $v \in W$

$$v(\beta(P)) = \sum_w v(P_w) = \sum_w P_{vwv^{-1}} = \sum_w P_w = \beta(P).$$

Hence $\beta(P) \in \mathbb{D}_{\mathfrak{R}}^W$. For $P = \sum P_w \otimes w \in \mathbb{D}R_{\mathfrak{R}}$, $Q = \sum Q_v \otimes v \in \mathbb{D}R_{\mathfrak{R}}^{1\otimes\mathbb{R}[w]}$ we have

$$\begin{aligned} P.Q &= \sum_{w,v}(P_w \otimes w).(Q_v \otimes v) = \sum_{w,v} P_w w(Q_v) \otimes wv \\ &= \sum_{w,v} P_w Q_{wvw^{-1}} \otimes wv = \sum_{w,v} P_w Q_v \otimes vw \end{aligned}$$

and hence $\beta(PQ) = \sum_{w,v} P_w Q_v = \beta(P)\beta(Q)$. □

Proposition 1.2.3. *If $\xi_1, \dots, \xi_n \in E$ is an orthonormal basis then the operator $\sum_1^n D_{\xi_j}(k)^2 \in \mathbb{D}R_{\mathfrak{R}}$ is given by*

$$(1.2.5) \qquad \sum_1^n D_{\xi_j}^2 = L(k) - \sum_{\alpha>0} \frac{k_\alpha(\alpha,\alpha)}{e^\alpha - e^{-\alpha}}\Delta_\alpha + \tfrac{1}{4} \sum_{\alpha,\beta>0} k_\alpha k_\beta (\alpha,\beta)\Delta_\alpha\Delta_\beta$$

with

$$(1.2.6) \qquad L(k) = \sum_1^n \partial_{\xi_j}^2 + \sum_{\alpha>0} k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\partial_\alpha .$$

In particular $\sum_1^n D_{\xi_j}^2$ is independent of the choice of the orthonormal basis $\xi_1, \dots, \xi_n$ for E. Moreover $\sum_1^n D_{\xi_j}^2 \in \mathbb{D}_{\mathfrak{R}}^{1\otimes\mathbb{R}[W]}$ and $L(k) = \beta(\sum_1^n D_{\xi_j}^2) \in \mathbb{D}_{\mathfrak{R}}^W$.

Proof. Since

$$\begin{aligned} &\partial_\xi \circ \Delta_\alpha + \Delta_\alpha \circ \partial_\xi \\ &\quad = \partial_\xi\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\right)(1-r_\alpha) + \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\{\partial_\xi \circ (1-r_\alpha) + (1-r_\alpha)\circ\partial_\xi\} \\ &\quad = \frac{-2(\alpha,\xi)}{e^\alpha - e^{-\alpha}}\Delta_\alpha + \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\{2\partial_\xi - (\partial_\xi \circ r_\alpha + r_\alpha \circ \partial_\xi)\} \end{aligned}$$

we get (using $\partial_\alpha \circ r_\alpha + r_\alpha \circ \partial_\alpha = 0$)

$$\sum_1^n (\alpha,\xi_j)\{\partial_{\xi_j} \circ \Delta_\alpha + \Delta_\alpha \circ \partial_{\xi_j}\} = \frac{-2(\alpha,\alpha)}{e^\alpha - e^{-\alpha}}\Delta_\alpha + 2\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\partial_\alpha$$

and (1.2.5) follows immediately. □

Definition 1.2.4. Suppose F is a face of the Weyl chamber E_+ in E corresponding to R_+, and let $R_F = \{\alpha \in R;\ (\alpha,\xi) = 0\ \forall\xi \in F\}$ denote the corresponding standard parabolic subsystem of R. Let

$$\rho_F(k) = \tfrac{1}{2} \sum_{\alpha \in R_+ \setminus R_F} k_\alpha \alpha \tag{1.2.7}$$

be the orthogonal projection of $\rho(k)$ onto the $\mathbb{R}$-span of F. Then there exist unique algebra homomorphisms

$$\gamma'_F, \gamma_F(k)\colon\ \mathbb{D}_{\mathfrak{R}(R)} \to \mathbb{D}_{\mathfrak{R}(R_F)} := \mathfrak{R}(R_F) \otimes SE \tag{1.2.8}$$

characterized by

$$\gamma'_F\left(\frac{1}{1-e^{-\alpha}}\right) = \gamma_F(k)\left(\frac{1}{1-e^{-\alpha}}\right) = \begin{cases} \dfrac{1}{1-e^{-\alpha}} & \text{for } \alpha \in R_F,\ \alpha > 0 \\ 1 & \text{for } \alpha \in R_+ \setminus R_F \end{cases}$$

and

$$\gamma'_F(\partial_\xi) = \partial_\xi, \quad \gamma_F(k)(\partial_\xi) = \partial_\xi - (\rho_F(k),\xi) \quad \text{for } \xi \in E.$$

In other words $\gamma_F(k)(\partial_p) = \partial_q$ with $q(\lambda) = p(\lambda - \rho_F(k))$ and we have formally

$$\gamma_F(k)(P) = e^{\rho_F(k)} \circ \gamma'_F(P) \circ e^{-\rho_F(k)} \quad \text{for } P \in \mathbb{D}_{\mathfrak{R}}. \tag{1.2.9}$$

The operator $\gamma_F(k)(P) \in \mathfrak{R}(R_F) \otimes SE$ is called the *k-constant term* of the differential operator $P \in \mathbb{D}_{\mathfrak{R}}$ along the face F. Note that both mappings (1.2.8) are equivariant for the action of $W(R_F)$. Hence we have

$$\gamma_F(k)\colon\ \mathbb{D}_{\mathfrak{R}(R)}^{W(R)} \to \mathbb{D}_{\mathfrak{R}(R_F)}^{W(R_F)}. \tag{1.2.10}$$

In the special case $F = E_+$ we simply write $\gamma(k) = \gamma_{E_+}(k)$. Note that there is a transitivity of k-constant terms (writing also $\gamma_F(k) = \gamma_{R_F,R}(k)$)

$$\gamma_{R_F,R_G}(k) \circ \gamma_{R_G,R}(k) = \gamma_{R_F,R}(k) \tag{1.2.11}$$

if F is a face of G and G a face of E_+.

Example 1.2.5. For $L(k)$ the operator given by (1.2.6) we have

$$\gamma'(L(k)) = \sum_1^n \partial_{\xi_j}^2 + \sum_{\alpha>0} k_\alpha \partial_\alpha = \sum_1^n \partial_{\xi_j}^2 + 2\partial_{\rho(k)}$$

and hence

$$\gamma(k)(L(k)) = \sum_1^n \partial_{\xi_j}^2 - (\rho(k), \rho(k)) \in SE^W. \tag{1.2.12}$$

For F a codimension one face of E_+ with $R_F = \{\pm\alpha\}$ or $R_F = \{\pm\frac{1}{2}\alpha, \pm\alpha\}$ we have (with the convention $k_{\frac{1}{2}\alpha} = 0$ if $\frac{1}{2}\alpha \notin R_F$)

$$\gamma'_F(L(k)) = \sum_1^n \partial_{\xi_j}^2 + 2\partial_{\rho_F(k)} + \{\tfrac{1}{2}k_{\frac{1}{2}\alpha}\frac{1+e^{-\frac{1}{2}\alpha}}{1-e^{-\frac{1}{2}\alpha}} + k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\}\partial_\alpha$$

and hence

$$\begin{aligned}&\gamma_F(k)(L(k)) = \\ &\sum_1^n \partial_{\xi_j}^2 + \{\tfrac{1}{2}k_{\frac{1}{2}\alpha}\frac{1+e^{-\frac{1}{2}\alpha}}{1-e^{-\frac{1}{2}\alpha}} + k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}}\}\partial_\alpha - (\rho_F(k), \rho_F(k)).\end{aligned} \tag{1.2.13}$$

Lemma 1.2.6. *Consider the algebra $\mathbb{R}(x) \otimes \mathbb{R}[\theta]$ of ordinary differential operators on the line with rational coefficients (here $\theta = x\frac{d}{dx}$). If $P \in \mathbb{R}(x) \otimes \mathbb{R}[\theta]$ is invariant under the substitution $x \mapsto x^{-1}$, and P commutes with the operator*

$$L(k_1, k_2) = \theta^2 + \{k_1 \frac{1+x^{-1}}{1-x^{-1}} + 2k_2 \frac{1+x^{-2}}{1-x^{-2}}\}\theta$$

then P is a polynomial in $L(k_1, k_2)$.

Proof. By induction on the order of P. Write $P = \sum_0^N a_j \theta^j$ with $a_j \in \mathbb{R}(x)$ and $a_N \neq 0$. Then we have

$$0 = [L(k_1, k_2), P] = [\theta^2, a_N]\theta^N + \text{ terms of order } \leq N,$$

and since $[\theta^2, a_N] = \theta[\theta, a_N] + [\theta, a_N]\theta = 2\theta(a_N)\theta + \theta^2(a_N)$ we conclude $\theta(a_N) = 0$ or equivalently $a_N \in \mathbb{R}$ is constant. Because P is invariant under substitution $x \mapsto x^{-1}$ (which transforms θ into $-\theta$) we also have $N \in 2\mathbb{Z}$. Now $Q := P - a_N L(k_1, k_2)^{\frac{1}{2}N}$ satisfies again the conditions of the lemma and the order of Q is strictly less than the order of P. $\square$

Let Q_+ be the cone spanned over $\mathbb{Z}_+$ by R_+ or equivalently by the simple roots $\alpha_1, \dots, \alpha_n \in R_+$. Write

$$\mu \leq \lambda \iff \lambda - \mu \in Q_+ \tag{1.2.14}$$

for the usual partial ordering on $E_c = \mathbb{C} \otimes_{\mathbb{R}} E$. An element of the algebra of differential operators $\mathbb{R}[[e^{-\alpha_1}, \dots, e^{-\alpha_n}]] \otimes SE$ can be written as a formal infinite sum

$$P = \sum_{\mu \leq 0} e^{\mu} \partial_{p_\mu} \tag{1.2.15}$$

with multiplication derived from $\partial_p \circ e^{\mu} = e^{\mu} \partial_q$ where $q \in SE$ is obtained from $p \in SE$ by $q(\lambda) = p(\lambda + \mu)$. Expanding $(1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots$ for $\alpha \in R_+$ (either formally or as a convergent power series on E_+) we can view $\mathbb{D}_{\mathfrak{R}}$ as a subalgebra of $\mathbb{R}[[e^{-\alpha_1}, \dots, e^{-\alpha_n}]] \otimes SE$. For example the operator $L(k)$ has the expansion

$$L(k) = \sum_1^n \partial_{\xi_j}^2 + 2\partial_{\rho(k)} + 2 \sum_{\alpha > 0} k_\alpha \sum_{j \geq 1} e^{-j\alpha} \partial_\alpha \tag{1.2.16}$$

and we have $\gamma'(P) = \partial_{p_0}$.

Lemma 1.2.7. *For $P \in \mathbb{R}[[e^{-\alpha_1}, \dots, e^{-\alpha_n}]] \otimes SE$ a differential operator of the form* (1.2.15) *we have $[L(k), P] = 0$ if and only if the polynomials $p_\mu \in SE$ satisfy the recurrence relations*

$$\begin{aligned}&(2\lambda + 2\rho(k) + \mu, \mu) p_\mu(\lambda) \\ &\quad = 2 \sum_{\alpha > 0} k_\alpha \sum_{j \geq 1} \{(\lambda + \mu + j\alpha, \alpha) p_{\mu + j\alpha}(\lambda) - (\lambda, \alpha) p_{\mu + j\alpha}(\lambda - j\alpha)\}.\end{aligned}$$

Proof. An easy formal computation, left to the reader. $\square$

Corollary 1.2.8. *Write $\mathbb{D}_{\mathfrak{R}}^{L(k)}$ for the algebra of all differential operator $P \in \mathbb{D}_{\mathfrak{R}}$ with $[L(k), P] = 0$. Then the k-constant term*

$$(1.2.17) \qquad \gamma(k)\colon\ \mathbb{D}_{\mathfrak{R}}^{L(k)} \to SE$$

is an injective algebra homomorphism. In particular $\mathbb{D}_{\mathfrak{R}}^{L(k)}$ is a commutative algebra. Moreover if $P \in \mathbb{D}_{\mathfrak{R}}^{L(k)}$ is a differential operator of order N then the symbol of P of order N has constant coefficients, and $\gamma(k)(P)$ is a polynomial of degree N whose homogeneous component of degree N equals the Nth order symbol of P.

Proof. The first statement is clear from the previous lemma. The last statement is clear from the recurrence relation since $\deg(p_\mu) < \deg(p_0) = \deg(\gamma(k)(P))$ for $\mu < 0$. □

Theorem 1.2.9. *If $\mathbb{D}(k) := \{P \in \mathbb{D}_{\mathfrak{R}}^W; [L(k), P] = 0\}$ denotes the commutant of $L(k)$ in $\mathbb{D}_{\mathfrak{R}}^W$ then the map*

$$(1.2.18) \qquad \gamma(k)\colon\ \mathbb{D}(k) \to SE^W$$

is an injective algebra homomorphism.

Proof. It remains to show that $\gamma(k)(P) \in SE^W$ for $P \in \mathbb{D}(k)$. Factor $\gamma(k)$ through $\gamma_F(k)$ where F is a codimension one face of E_+ (cf. (1.2.11)). Then $\gamma_F(k)(P)$ is invariant under $W(R_F)$ by (1.2.10), and commutes with $\gamma_F(k)(L(k))$ given by (1.2.13). Applying Lemma 1.2.6 we conclude that $\gamma(k)(P)$ is invariant under $W(R_F)$. Since $W(R)$ is generated by the subgroups $W(R_F)$ as F runs over all codimension one faces of E_+ we conclude that $\gamma(k)(P) \in SE^W$. □

In the next section we will see that the map (1.2.18) is an isomorphism onto.

1.3. The Jacobi polynomials

Since each W-orbit in P meets P_+ in exactly one point it follows that the monomial symmetric functions

$$(1.3.1) \qquad M(\lambda) = \sum_{\mu \in W\lambda} e^\mu$$

form an $\mathbb{R}$-basis for $\mathbb{R}[P]^W$ as λ varies over P_+.

Definition 1.3.1. The *Jacobi polynomials* $P(\lambda,k) \in \mathbb{R}[P]^W$ are defined by

$$P(\lambda,k) = \sum_{\mu\in P_+,\mu\leq\lambda} c_{\lambda\mu}(k)M(\mu), \quad c_{\lambda\lambda}(k) = 1 \tag{1.3.2}$$

and

$$(P(\lambda,k), M(\mu))_k = 0, \quad \forall\mu \in P_+, \ \mu < \lambda. \tag{1.3.3}$$

Note that the Jacobi polynomials are defined whenever the inner product (1.1.11) is defined. Indeed $P(\lambda,k)$ is equal to $M(\lambda)$ minus the orthogonal projection of $M(\lambda)$ onto $\mathrm{span}\{M(\mu);\ \mu < \lambda,\ \mu \in P_+\}$. Clearly the Jacobi polynomials also form an $\mathbb{R}$-basis of $\mathbb{R}[P]^W$.

Example 1.3.2. For R of type BC_1, say $R = \{\pm\lambda_1, \pm 2\lambda_1\}$ with $P_+ = \mathbb{Z}_+\lambda_1$, we have $\mathbb{R}[e^{\lambda_1}, e^{-\lambda_1}]^W \cong \mathbb{R}[x]$ with $x = \frac{1}{2}(e^{\lambda_1}+e^{-\lambda_1})$.
Then the weight function $\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}}$ becomes

$$\begin{aligned}\delta(k)^{\frac{1}{2}}\overline{\delta(k)}^{\frac{1}{2}} &= (2-e^{\lambda_1}-e^{-\lambda_1})^{k_1+k_2}(2+e^{\lambda_1}+e^{-\lambda_1})^{k_2}\\ &= 2^{k_1+2k_2}(1-x)^{k_1+k_2}(1+x)^{k_2}\end{aligned}$$

and for the corresponding weight measure we get (cf. (1.1.12))

$$|\delta(k,t)|dt = \frac{2^{k_1+2k_2}}{2\pi}(1-x)^{k_1+k_2-\frac{1}{2}}(1+x)^{k_2-\frac{1}{2}}dx.$$

Hence up to normalization the Jacobi polynomials $P(\lambda,k)$, $\lambda \in P_+$ for R of type BC_1 are the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{Z}_+$ with

$$\alpha = k_1 + k_2 - \tfrac{1}{2}, \quad \beta = k_2 - \tfrac{1}{2}.$$

The case of the Gegenbauer polynomials occurs for $\alpha = \beta \iff k_1 = 0$, or equivalently for R of type A_1. See [19, Vol 2].

Example 1.3.3. In case $k_\alpha = 0\ \forall\alpha \in R$ the Jacobi polynomials $P(\lambda,k)$ specialize to the monomial symmetric functions $M(\lambda)$. In case R is reduced and $k_\alpha = 1\ \forall\alpha \in R$ the Jacobi polynomials $P(\lambda,k)$ become the Weyl characters $Ch(\lambda) := \Delta^{-1}.\sum\varepsilon(w)e^{w(\lambda+\rho)}$ where Δ is the Weyl denominator (1.2.2) and $\rho = \rho_R = \frac{1}{2}\sum_{\alpha>0}\alpha$.

Definition 1.3.4. A linear operator L: $\mathbb{R}[P]^W \to \mathbb{R}[P]^W$ is called *triangular* if

$$L(M(\lambda)) = \sum_{\mu \in P_+, \mu \leq \lambda} a_{\lambda\mu} M(\mu) \qquad \forall \lambda \in P_+. \tag{1.3.4}$$

Proposition 1.3.5. *If L: $\mathbb{R}[P]^W \to \mathbb{R}[P]^W$ is triangular and symmetric with respect to the inner product $(\cdot,\cdot)_k$ then the Jacobi polynomials $P(\lambda, k)$, $\lambda \in P_+$ are eigenfunctions of L.*

Proof. Since L is triangular we have using (1.3.2)

$$L(P(\lambda,k)) = \sum_{\mu \leq \lambda} c_{\lambda\mu}(k) L(M(\mu)) = \sum_{\nu \leq \lambda} b_{\lambda\nu} M(\nu)$$

with the coefficients $b_{\lambda\nu}$ given by $b_{\lambda\nu} = \sum_{\nu \leq \mu \leq \lambda} c_{\lambda\mu}(k) a_{\mu\nu}$. Using that L is symmetric we get

$$\begin{aligned}(L(P(\lambda,k)), M(\mu))_k &= (P(\lambda,k), L(M(\mu)))_k \\ &= \sum_{\nu \leq \mu} a_{\mu\nu} (P(\lambda,k), M(\nu))_k = 0\end{aligned}$$

if $\mu < \lambda$. Hence $L(P(\lambda,k)) = a_{\lambda\lambda} P(\lambda,k)$. $\square$

Corollary 1.3.6. *All symmetric triangular linear operators on $\mathbb{R}[P]^W$ are simultaneously diagonalized by the Jacobi polynomials, and therefore commute with each other.*

Proposition 1.3.7. *A differential operator $P \in \mathbb{D}_{\mathfrak{R}}$ is completely determined by the corresponding endomorphism $P \in \mathrm{Hom}(\mathbb{R}[P]^W, \mathbb{R}_\Delta[P])$.*

Proof. We expand $P = \sum_{\mu \leq 0} e^\mu \partial_{p_\mu}$ as in Section 1.2. Let $r_1, \dots, r_n \in W$ be the simple reflections corresponding to the simple roots $\alpha_1, \dots, \alpha_n \in R_+$. Suppose $\mu = \sum_1^n m_j \alpha_j \leq 0$ or equivalently $m_j \in \mathbb{Z}_-$ for $j = 1, \dots, n$. Knowing $P(M(\lambda))$ for $\lambda \in P_+$ means that we know the polynomial $p_\mu \in SE$ on

$$\begin{aligned}\{\lambda \in P_+;\ \lambda + \mu &\not\leq r_j(\lambda) \quad \text{for } j = 1, \dots, n\} \\ &= \{\lambda \in P_+;\ (\lambda, \alpha_j^\vee)\alpha_j \not\leq -\mu \quad \text{for } j = 1, \dots, n\} \\ &= \{\lambda \in P_+;\ (\lambda, \alpha_j^\vee) \geq 1 - m_j \quad \text{for } j = 1, \dots, n\}.\end{aligned}$$

Since the latter set is Zariski dense in E we can recover the polynomial $p_\mu \in SE$. $\square$

For $\lambda \in P_+$ we write

$$C(\lambda) = \{\mu \in P;\ w\mu \leq \lambda\ \ \forall w \in W\} \tag{1.3.5}$$

for the integral convex hull of $W\lambda$.

Proposition 1.3.8. *For fixed $\lambda \in P_+$ the linear space*

$$\{F = \sum_\mu a_\mu e^\mu \in \mathbb{R}[P];\ a_\mu = 0 \text{ unless } \mu \in C(\lambda)\} \tag{1.3.6}$$

is invariant under the operators $D_\xi(k)$ for $\xi \in E$.

Proof. This is clear since the space (1.3.6) is easily seen to be invariant under both ∂_ξ, $\xi \in E$ and Δ_α, $\alpha \in R$. $\square$

Proposition 1.3.9. *For $\xi \in E$ and $N \in \mathbb{Z}_+$ we put*

$$P_{\xi,N}(k) = \sum_{\eta \in W\xi} \beta(D_\eta(k)^N) \in \mathbb{D}_{\mathfrak{R}}^W. \tag{1.3.7}$$

Then $P_{\xi,N}(k)$: $\mathbb{R}[P]^W \to \mathbb{R}[P]^W$ is a symmetric triangular operator. Moreover $\gamma(k)(P_{\xi,N}(k)) \in SE$ is a polynomial on E^* of degree $\leq N$ with homogeneous component of degree N equal to $\lambda \mapsto \sum_\eta (\eta, \lambda)^N$.

Proof. Since $wD_\eta(k) = D_{w\eta}(k)w$ it is clear that

$$D_{\xi,N}(k) = \sum_{\eta \in W\xi} D_\eta(k)^N \in \mathbb{D}R_{\mathfrak{R}} \tag{1.3.8}$$

is a differential-reflection operator which commutes with $1 \otimes \mathbb{R}[W]$. Hence $D_{\xi,N}(k) \in \mathrm{End}(\mathbb{R}[P])$ leaves the subspace $\mathbb{R}[P]^W$ invariant, and on this subspace $D_{\xi,N}(k)$ and $P_{\xi,N}(k)$ coincide. The operator $D_{\xi,N}(k)$ is symmetric on $\mathbb{R}[P]$ by Theorem 1.1.6, and $D_{\xi,N}(k)$ is triangular on $\mathbb{R}[P]^W$ by the previous proposition. Hence $P_{\xi,N}(k)$: $\mathbb{R}[P]^W \to \mathbb{R}[P]^W$ is triangular and symmetric. The second statement on the homogeneous component of degree N of $\gamma(k)(P_{\xi,N}(k))$ is trivial. $\square$

Proposition 1.3.10. *With the notation* (1.2.6) *the operator* $L(k) \in \mathbb{D}_{\mathfrak{R}}^W$ *leaves the space* $\mathbb{R}[P]^W$ *invariant, and is a symmetric triangular operator on* $\mathbb{R}[P]^W$.

Proof. Using Proposition 1.2.3 the same arguments work as in the proof of the previous proposition. □

Corollary 1.3.11. *For* $\xi \in E$ *and* $N \in Z_+$ *we have* $P_{\xi,N}(k) \in \mathbb{D}(k)$.

Proof. From the previous two propositions and Corollary 1.3.6 it follows that $P_{\xi,N}(k)$ and $L(k)$ commute as operators on $\mathbb{R}[P]^W$. But then $P_{\xi,N}(k)$ and $L(k)$ also commute as elements of $\mathbb{D}_{\mathfrak{R}}$ by Proposition 1.3.7. □

Theorem 1.3.12. *The Harish-Chandra homomorphism*

$$\gamma(k)\colon\ \mathbb{D}(k) \to SE^W \tag{1.3.9}$$

is an isomorphism of (commutative) algebras. Here $\mathbb{D}(k)$ *is the commutant of* $L(k)$ *in* $\mathbb{D}_{\mathfrak{R}}^W$, *and* $\mathbb{D}_{\mathfrak{R}} = \mathfrak{R} \otimes SE$ *is the algebra of differential operators on* E *with coefficients in the algebra* $\mathfrak{R}$ *generated by the functions* $(1-e^{-\alpha})^{-1}$, $\alpha \in R_+$.

Proof. It remains to be shown by Theorem 1.2.9 that the map (1.3.9) is surjective. This follows by induction on the degree from Proposition 1.3.9, Corollary 1.3.11, and Theorem 1.2.9, since the polynomials $\lambda \mapsto \sum_\eta (\eta, \lambda)^N$ with the sum over $W\xi$ generate the algebra SE^W as ξ ranges over E and N over $\mathbb{Z}_+$. □

Corollary 1.3.13. *For* $\lambda, \mu \in P_+$ *with* $\lambda \neq \mu$ *we have*

$$(P(\lambda, k), P(\mu, k))_k = 0.$$

Proof. For $P \in \mathbb{D}(k)$ we have for $\lambda \in P_+$

$$P(P(\lambda, k)) = \gamma(k)(P)(\lambda + \rho(k)).P(\lambda, k), \tag{1.3.10}$$

and because of the Harish-Chandra isomorphism (1.3.9) the algebra $\mathbb{D}(k)$ of operators on $\mathbb{R}[P]^W$, symmetric with respect to $(\cdot, \cdot)_k$, is sufficiently rich to separate the points of $P_+ + \rho(k)$. □

Remark 1.3.14. Consider the $\mathbb{C}$-vector space

$$K = \{k = (k_\alpha);\ k_\alpha \in \mathbb{C},\ k_{w\alpha} = k_\alpha \quad \forall \alpha \in R, w \in W\} \tag{1.3.11}$$

of complex-valued multiplicity functions on R. The results of Section 1.2 immediately generalize to the case $k \in K$ (replace $\mathbb{R}[P]$ by $\mathbb{C}[P]$, etc.). The construction of the operator $P_{\xi,N}(k)$ also goes through for $k \in K$. However, for the proof of Theorem 1.3.12 we need the inner product $(\cdot,\cdot)_k$ which imposes a Zariski dense restriction on $k \in K$ (cf. Proposition 1.1.5 for the algebraic description or (1.1.12) for the analytic description of $(\cdot,\cdot)_k$). Nevertheless, since the operator $P_{\xi,N}(k)$ depends polynomially on $k \in K$ (of degree $\leq N$) it follows that the Harish-Chandra isomorphism

$$\gamma(k)\colon\ \mathbb{D}_c(k) \xrightarrow{\simeq} SE_c^W \tag{1.3.12}$$

holds for all $k \in K$, where $D_c(k) = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{D}(k)$ and $E_c = \mathbb{C} \otimes_{\mathbb{R}} E$.

Remark 1.3.15. Let $z_j = M(\lambda_j)$ be the monomial symmetric functions corresponding to the fundamental weights $\lambda_1, \ldots, \lambda_n \in P_+$. Then it is well known (see [7]) that

$$\mathbb{R}[P]^W \cong \mathbb{R}[z_1, \ldots, z_n], \tag{1.3.13}$$

and we can view the commutative algebra $\mathbb{D}(k)$ also as a subalgebra of the Weyl algebra $\mathbb{A}_n = \mathbb{R}[z_1, \ldots, z_n, \partial_{z_1}, \ldots, \partial_{z_n}]$.

Notes for Chapter 1

The results of this chapter were obtained in a series of four papers [34, 30, 58, 59] by transcendental and computer algebra methods. The computer algebra part was removed in [31]. The elementary approach to Theorem 1.3.12 as given here was derived in [33]. Previously Theorem 1.3.12 was found by Koornwinder for R of type A_2 and BC_2 [42], and for R of type A_n in [68, 12, 49].

CHAPTER 2

The periodic Calogero-Moser system

2.1. Quantum integrability for the Calogero-Moser system

We write

$$\mathfrak{h} := E_c = \mathbb{C} \otimes_{\mathbb{R}} E, \quad \mathfrak{a} := E, \quad \mathfrak{t} := iE \tag{2.1.1}$$

and view these as (abelian) Lie algebras of the complex torus

$$H := \mathfrak{h}/2\pi i Q^\vee \tag{2.1.2}$$

and its two real forms

$$A := \mathfrak{a}, \quad T = \mathfrak{t}/2\pi i Q^\vee, \tag{2.1.3}$$

respectively. Write also

$$\exp\colon \ \mathfrak{h} \twoheadrightarrow H \tag{2.1.4}$$

for the canonical map and

$$\log\colon \ H \to \mathfrak{h} \tag{2.1.5}$$

for the multivalued inverse. Then

$$\exp\colon \ \mathfrak{a} \twoheadrightarrow A, \quad \exp\colon \ \mathfrak{t} \twoheadrightarrow T \tag{2.1.6}$$

are both surjective and

$$\log\colon \ A \to \mathfrak{a} \tag{2.1.7}$$

is a singlevalued inverse. Viewing H as an affine algebraic variety the algebra $\mathbb{C}[P]$ is just the ring of regular functions on H, or equivalently the ring of holomorphic functions on H with moderate growth at infinity.

Writing

$$H^{\mathrm{reg}} = \{h \in H;\ \Delta(h) \neq 0\} = \{h \in H;\ wh \neq h\ \forall w \in W, w \neq e\} \tag{2.1.8}$$

we view $\delta(k;h)^{\frac{1}{2}}$ for $k \in K$ as a Nilsson class function on H^{reg} (see [13] for the concept of Nilsson class functions).

Theorem 2.1.1. *We have for all $k \in K$ the equality of differential operators on H^{reg}*

$$\begin{aligned}\delta(k;h)^{\frac{1}{2}} \circ \{L(k)+(\rho(k),\rho(k))\} \circ \delta(k;h)^{-\frac{1}{2}} \\ = \sum_1^n \partial_{\xi_j}^2 + \sum_{\alpha>0} \frac{k_\alpha(1-k_\alpha-2k_{2\alpha})(\alpha,\alpha)}{\left(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}\right)^2}.\end{aligned} \tag{2.1.9}$$

Proof. Clearly we have for $\xi \in \mathfrak{h}$

$$\begin{aligned}\delta^{-\frac{1}{2}} \circ \partial_\xi \circ \delta^{\frac{1}{2}} &= \partial_\xi + \tfrac{1}{2}\partial_\xi(\log\delta) \\ \delta^{-\frac{1}{2}} \circ \partial_\xi^2 \circ \delta^{\frac{1}{2}} &= \partial_\xi^2 + \partial_\xi(\log\delta) \circ \partial_\xi + \delta^{-\frac{1}{2}}\partial_\xi^2(\delta^{\frac{1}{2}})\end{aligned}$$

and if we write $\Box = \sum_1^n \partial_{\xi_j}^2$ we get

$$\sum_1^n \partial_{\xi_j}(\log\delta)\partial_{\xi_j} = \sum_{\alpha>0} k_\alpha \frac{e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha}}{e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}}\partial_\alpha \tag{2.1.10}$$

$$\begin{aligned}\delta^{-\frac{1}{2}}\Box(\delta^{\frac{1}{2}}) &= \sum_{\alpha>0} \frac{-k_\alpha(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})^2} \\ &+ \sum_{\alpha,\beta>0} \tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}+e^{-\frac{1}{2}\beta})}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}-e^{-\frac{1}{2}\beta})}.\end{aligned} \tag{2.1.11}$$

Observe that the right-hand side of (2.1.10) is precisely the first-order term of the differential operator $L(k)$. We rewrite the second term on the right-hand side of (2.1.11) as

$$\begin{aligned}&(\rho(k),\rho(k)) + \sum_{\alpha,\beta>0} \tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\left\{\frac{(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}+e^{-\frac{1}{2}\beta})}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}-e^{-\frac{1}{2}\beta})} - 1\right\} \\ &= (\rho(k),\rho(k)) + \sum_{\alpha,\beta>0} \tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{2(e^{\frac{1}{2}(\alpha-\beta)}+e^{-\frac{1}{2}(\alpha-\beta)})}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}-e^{-\frac{1}{2}\beta})} \\ &= (\rho(k),\rho(k)) + \sum_{\alpha>0} \frac{k_\alpha(k_\alpha+2k_{2\alpha})(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})^2} \\ &\quad + {\sum_{\alpha,\beta}}' \tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{2(e^{\frac{1}{2}(\alpha-\beta)}+e^{-\frac{1}{2}(\alpha-\beta)})}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})(e^{\frac{1}{2}\beta}-e^{-\frac{1}{2}\beta})},\end{aligned}$$

where the $\sum'_{\alpha,\beta}$ denotes the sum over all pairs $\alpha, \beta \in R_+$ for which α and β are not multiples of each other. The formula (2.1.9) follows if we show that the $\sum'_{\alpha,\beta}$ term vanishes identically. Note that this term is a W-invariant function, and that its product with the Weyl denominator Δ is holomorphic on all of H. From this we conclude that it belongs to $\mathbb{C}[P]^W$, and we can deduce that it vanishes by taking the constant term γ' along A_+. □

Corollary 2.1.2. *For all $k, l \in K$ we have*

$$\begin{aligned}(2.1.12)\quad &\sum_1^n \partial_{\xi_j}^2 + \sum_{\alpha>0} k_\alpha \frac{e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha}}{e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha}}\partial_\alpha + \sum_{\alpha>0} \frac{l_\alpha^2(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})^2} \\ &= \delta(m-k)^{\frac{1}{2}} \circ \{L(m)+(\rho(m),\rho(m))-(\rho(k),\rho(k))\} \circ \delta(m-k)^{-\frac{1}{2}}\end{aligned}$$

with $m \in K$ satisfying $m_\alpha(1-m_\alpha-2m_{2\alpha}) = l_\alpha^2 + k_\alpha(1-k_\alpha - 2k_{2\alpha})$.

Proof. Indeed we have by (2.1.9)

$$\begin{aligned}&\delta(k)^{\frac{1}{2}} \circ \{L(k) + (\rho(k),\rho(k)) + \sum_{\alpha>0} \frac{l_\alpha^2(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})^2}\} \circ \delta(k)^{-\frac{1}{2}} \\ &= \sum_1^n \partial_{\xi_j}^2 + \sum_{\alpha>0} \frac{(l_\alpha^2 + k_\alpha(1-k_\alpha-2k_{2\alpha}))(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2} \\ &= \delta(m)^{\frac{1}{2}} \circ \{L(m)+(\rho(m),\rho(m))\} \circ \delta(m)^{-\frac{1}{2}}\end{aligned}$$

with $m \in K$ given by $m_\alpha(1-m_\alpha-2m_{2\alpha}) = l_\alpha^2+k_\alpha(1-k_\alpha - 2k_{2\alpha})$. □

Remark 2.1.3. The operator $L(k)$ is the standard second-order hypergeometric operator. The operator (2.1.12) is like the Riemann-Papperitz operator in the one-variable case, which indeed is equal up to conjugation by a suitable function to a standard second-order hypergeometric operator [66, 64, 74].

Definition 2.1.4. The periodic *Calogero-Moser potential* with coupling constant $g^2 \in K$ (the 2 is a square) is the function

$$(2.1.13)\qquad \sum_{\alpha>0} \frac{g_\alpha^2}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2}.$$

Example 2.1.5. For the root system R of type A_n we have $\mathfrak{a} = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1};\ \sum x_j = 0\}$ and the Calogero-Moser potential becomes

$$g^2 . \sum_{i<j} \frac{1}{4 \sinh^2 \frac{1}{2}(x_i - x_j)}. \tag{2.1.14}$$

On the space $\mathfrak{t} = i\mathfrak{a}$ this potential corresponds to a system of $n+1$ points on the circle $\mathbb{R}/2\pi\mathbb{Z}$ whose potential is proportional to the sum of the inverse squares of the pairwise distances.

Theorem 2.1.6. *For $g \in K$ consider the Schrödinger operator*

$$S(g) = -\tfrac{1}{2} \sum_{1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha>0} \frac{g_\alpha^2}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2} \in \mathbb{D}_{\mathfrak{R}}^W \tag{2.1.15}$$

associated with the Calogero-Moser potential (2.1.13). *Then the (unshifted) constant term (cf.* (1.2.8)*)*

$$\gamma' \colon \mathbb{D}_{\mathfrak{R}}^{W,S(g)} \longrightarrow S\mathfrak{h}^W \tag{2.1.16}$$

is an isomorphism of commutative algebras. Here

$$\mathbb{D}_{\mathfrak{R}}^{W,S(g)} = \{P \in \mathbb{D}_{\mathfrak{R}}^W;\ [P, S(g)] = 0\} \tag{2.1.17}$$

is the algebra of quantum integrals for $S(g)$ in $\mathbb{D}_{\mathfrak{R}}^W$.

Proof. Observe that the map $P \in \mathbb{D}_{\mathfrak{R}}^W \mapsto \delta(k)^{-\frac{1}{2}} \circ P \circ \delta(k)^{\frac{1}{2}} \in \mathbb{D}_{\mathfrak{R}}^W$ is an automorphism of $\mathbb{D}_{\mathfrak{R}}^W$. Taking

$$g_\alpha^2 = -\tfrac{1}{2} k_\alpha (1 - k_\alpha - k_{2\alpha})(\alpha, \alpha) \tag{2.1.18}$$

we deduce from Theorem 2.1.1 that the map

$$P \in \mathbb{D}_{\mathfrak{R}}^{W,S(g)} \mapsto \delta(k)^{-\frac{1}{2}} \circ P \circ \delta(k)^{\frac{1}{2}} \in \mathbb{D}_{\mathfrak{R}}^{W,L(k)} \tag{2.1.19}$$

is an isomorphism of algebras. Since the diagram

$$\begin{tikzcd}
\mathbb{D}_{\mathfrak{R}}^{W,S(g)} \arrow[rr, "P \mapsto \delta^{-\frac{1}{2}} \circ P \circ \delta^{\frac{1}{2}}"] \arrow[dr, "\gamma'"'] & & \mathbb{D}_{\mathfrak{R}}^{W,L(k)} \arrow[dl, "\gamma(k)"] \\
& S\mathfrak{h}^W &
\end{tikzcd}$$

is commutative the theorem follows immediately from the Harish-Chandra isomorphism (1.3.9). □

The expansion of the operator $S(g)$ on A_+ analogous to the expansion (1.2.16) for the operator $L(k)$ becomes

$$S(g) = -\tfrac{1}{2}\sum_{1}^{n}\partial_{\xi_j}^2 + \sum_{\alpha>0} g_\alpha^2 \sum_{j\geq 1} je^{-j\alpha}, \tag{2.1.20}$$

and the analog of Lemma 1.2.7 is now

Lemma 2.1.7. *For the operator $P = \sum_{\mu\leq 0} e^\mu \partial_{p_\mu}$ we have $[S(g), P] = 0$ if and only if the polynomials $p_\mu \in S\mathfrak{h}$ satisfy the recurrence relations*

$$(2\lambda+\mu,\mu)p_\mu(\lambda) = -2\sum_{\alpha>0} g_\alpha^2 \sum_{j\geq 1} j\{p_{\mu+j\alpha}(\lambda-j\alpha)-p_{\mu+j\alpha}(\lambda)\}.$$

Proof. An easy computation. □

Corollary 2.1.8. *If $p_o \in S\mathfrak{h}^W \cong \mathbb{C}[\mathfrak{h}^*]^W$ is a polynomial in $\lambda \in \mathfrak{h}^*$ independent of $g \in K$ then $p_\mu \in \mathbb{C}[K \times \mathfrak{h}^*]$ are polynomials in both the multiplicity function $g \in K$ (even in g_α^2) and the spectral parameter $\lambda \in \mathfrak{h}^*$. More precisely, if $\deg(p_0) \leq N$ for some $N \in \mathbb{N}$ then also $\deg(p_\mu) \leq N$ $\forall \mu \leq 0$. Here $\deg(p_\mu)$ means the degree of p_μ as element of $\mathbb{C}[K \times \mathfrak{h}^*]$.*

Proof. This follows from the above recurrence relations by induction on μ with respect to the partial ordering $\leq$. □

2.2. Classical integrability for the Calogero-Moser system

Consider the algebra of differential operators

$$\mathfrak{D} := \mathbb{C}[K] \otimes \mathfrak{R} \otimes S\mathfrak{h} \tag{2.2.1}$$

where the multiplicity parameter $g \in K$ is considered as an indeterminate commuting with $\mathfrak{R} \otimes S\mathfrak{h}$. For $N \in \mathbb{N}$ we put

$$\begin{aligned}\mathfrak{D}_N = \{\sum_i q_i \otimes f_i \otimes \partial_{p_i} \in \mathfrak{D};\\ q_i \otimes p_i \in \mathbb{C}[K \times \mathfrak{h}^*] \text{ has degree } \leq N \forall i\}.\end{aligned} \tag{2.2.2}$$

Then it is easily seen that

$$\mathfrak{D} = \bigcup_{N \geq 0} \mathfrak{D}_N, \quad \mathfrak{D}_{N_1}.\mathfrak{D}_{N_2} \subset \mathfrak{D}_{N_1+N_2} \tag{2.2.3}$$

gives a filtration of the algebra $\mathfrak{D}$. Since

$$[\mathfrak{D}_{N_1}, \mathfrak{D}_{N_2}] \subset \mathfrak{D}_{N_1+N_2-1} \tag{2.2.4}$$

the associated graded $\mathrm{gr}(\mathfrak{D})$ is commutative, and inherits a Poisson bracket $\{\cdot,\cdot\}$ from the commutator bracket $[\cdot,\cdot]$ in $\mathfrak{D}$.

Proposition 2.2.1. *With the above notation we have*

$$gr(\mathfrak{D}) \cong \mathbb{C}[K] \otimes \mathfrak{R} \otimes \mathbb{C}[\mathfrak{h}^*] \tag{2.2.5}$$

as functions space on $K \times H^{\mathrm{reg}} \times \mathfrak{h}^*$ (pointwise multiplication), and the Poisson bracket is given by

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left\{ \frac{\partial f_1}{\partial y_i}\frac{\partial f_2}{\partial x_i} - \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial y_i} \right\}. \tag{2.2.6}$$

Here $x_1, \ldots, x_n$ are linear coordinates on $\mathfrak{h}$ and $y_1, \ldots, y_n$ the dual coordinates on $\mathfrak{h}^*$ (so $\frac{\partial}{\partial x_i}$ acts on $\mathfrak{R}$ and $\frac{\partial}{\partial y_i}$ on $\mathbb{C}[\mathfrak{h}^*]$).

Proof. Easily verified. □

Theorem 2.2.2. *The Hamiltonian*

$$H(g) = \tfrac{1}{2}(\lambda,\lambda) + \sum_{\alpha>0} \frac{g_\alpha^2}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2} \tag{2.2.7}$$

as function on $T^*A = A \times \mathfrak{a}^*$ is completely integrable with integrals in $\mathbb{C}[K] \otimes \mathfrak{R} \otimes \mathbb{C}[\mathfrak{h}^*]$. More precisely this means that for each $p \in \mathbb{C}[\mathfrak{h}^*]^W$ homogeneous of degree N there exists an integral I_p for $H(g)$ with

$$I_p = p(\lambda) + (\text{terms of degree} \leq N-1 \text{ in } \lambda), \tag{2.2.8}$$

and all the integrals I_p Poisson commute among each other.

Proof. This is clear from Theorem 2.1.6, Corollary 2.1.8, and the previous proposition. □

Notes for Chapter 2

The complete integrability for the inverse square potential of three particles on a line was found already in 1866 by Jacobi [41]. Marchioro rediscoverd this fact and discussed the classical and quantum mechanical scattering problem [51]. Calogero subsequently studied the quantum scattering problem for an arbitrary number of particles on the line [9]. Moser proved the classical integrability (still in case R of type A_n) by giving a Lax representation [53]. Generalizing the method of Moser partial results on the classical integrability were obtained by Olshanetsky and Perelomov for classical root systems R [56]. Theorem 2.2.2 for general R goes back to Opdam, and our exposition follows his paper [59]. In our form Theorem 2.1.1 is due to [57], but the conjugation of the operator $L(k)$ with $\delta^{\frac{1}{2}}$ was previously carried out by Gangolli in order to obtain uniform estimates of spherical functions [22]. For a nice introduction to the various concepts of classical mechanics we refer the reader to [2].

CHAPTER 3

The hypergeometric shift operators

3.1. Algebraic properties of shift operators

Rather than the operator $L(k)$ given by (1.2.6) we will now use the *modified operator*

$$ML(k) = L(k) + (\rho(k), \rho(k)) \in \mathbb{D}(k) \tag{3.1.1}$$

which maps under the Harish-Chandra isomorphism (1.3.9) onto the Laplace operator $\sum_1^n \partial_{\xi_j}^2 \in S\mathfrak{h}^W$.

Definition 3.1.1. We say that $l \in K$ is integral if $l_\alpha \in \mathbb{Z}\ \forall \alpha \in R \backslash \frac{1}{2}R$ and $l_\alpha \in 2\mathbb{Z}\ \forall \alpha \in R \cap \frac{1}{2}R$. Note that $\rho(l) \in P$ if $l \in K$ is integral. An operator $D(k) \in \mathbb{C}[K] \otimes \mathbb{C}_\Delta[P] \otimes S\mathfrak{h}$ is called a *shift operator* with integral shift $l \in K$ if

$$D(k) \circ ML(k) = ML(k{+}l) \circ D(k) \qquad \forall k \in K, \tag{3.1.2}$$

and on A_+ the operator $D(k)$ has an expansion of the form

$$D(k) = \sum_{\mu \leq 0} e^{-\rho(l)+\mu} \partial_{p_\mu} \tag{3.1.3}$$

with $p_\mu \in \mathbb{C}[K \times \mathfrak{h}^*]$ (by expanding: $(1-e^{-\alpha})^{-1} = 1 + e^{-\alpha} + \cdots\ \forall \alpha \in R_+$). We write $\mathbb{S}(l,k)$ for the $\mathbb{C}[K]$-module of all shift operators with integral shift $l \in K$.

We substitute a formal series on A_+ with leading exponent $\lambda \in \mathfrak{h}^*$

$$\Phi'(\lambda,k) = \sum_{\mu \leq \lambda} \Gamma'_\mu(\lambda,k) e^\mu, \quad \Gamma'_\lambda(\lambda,k) = 1 \tag{3.1.4}$$

into the differential equation

$$ML(k)(\Phi'(\lambda,k)) = (\lambda+\rho(k),\lambda+\rho(k))\Phi'(\lambda,k). \tag{3.1.5}$$

Then the coefficients $\Gamma'_\mu(\lambda,k)$ are given by Freudenthal type recurrence relations

$$\begin{aligned}\{(\lambda+\rho(k),\lambda+\rho(k)) - (\mu+\rho(k),\mu+\rho(k))\}\Gamma'_\mu(\lambda,k)\\ = 2\sum_{\alpha>0} k_\alpha \sum_{j\geq 1}(\mu+j\alpha,\alpha)\Gamma'_{\mu+j\alpha}(\lambda,k)\end{aligned} \tag{3.1.6}$$

which can be solved uniquely if

$$2(\lambda+\rho(k),\kappa) - (\kappa,\kappa) \neq 0 \qquad \text{for all } \kappa > 0. \tag{3.1.7}$$

In order to get rid of the shift over $\rho(k)$ we can reformulate the above by substituting a formal series

$$\Phi(\lambda,k) = \sum_{\kappa\leq 0}\Gamma_\kappa(\lambda,k)e^{\lambda-\rho(k)+\kappa}, \quad \Gamma_0(\lambda,k) = 1 \tag{3.1.8}$$

with leading exponent $\lambda-\rho(k)$ into the differential equation

$$ML(k)(\Phi(\lambda,k)) = (\lambda,\lambda)\Phi(\lambda,k). \tag{3.1.9}$$

Now the coefficients $\Gamma_\kappa(\lambda,k)$ satisfy the Harish-Chandra type recurrence relations

$$\begin{aligned}-(2\lambda+\kappa,\kappa)\Gamma_\kappa(\lambda,k)\\ = 2\sum_{\alpha>0} k_\alpha \sum_{j\geq 1}(\lambda-\rho(k)+\kappa+j\alpha,\alpha)\Gamma_{\kappa+j\alpha}(\lambda,k),\end{aligned} \tag{3.1.10}$$

which can be solved uniquely if

$$2(\lambda,\kappa) + (\kappa,\kappa) \neq 0 \qquad \text{for all } \kappa < 0. \tag{3.1.11}$$

Observe that $\Phi'(\lambda,k) = \Phi(\lambda+\rho(k),k)$ and $\Gamma'_{\lambda+\kappa}(\lambda,k) = \Gamma_\kappa(\lambda+\rho(k),k)$. The conditions (3.1.7) and (3.1.11) mean that λ lies outside a locally finite set of affine hyperplanes in $\mathfrak{h}^*$ which become more and more dense in the direction of $\mathfrak{a}^*_+$.

Proposition 3.1.2. *If $k \in K$ with $k_\alpha \geq 0$ $\forall\alpha$ is generic then the series* (3.1.4) *terminates for $\lambda \in P_+$ and $P(\lambda, k) = \Phi'(\lambda, k)$ for all $\lambda \in P_+$.*

Proof. Immediate, since for $k_\alpha \geq 0$ generic (e.g., irrational) the conditions (3.1.7) are satisfied for all $\lambda \in P_+$. □

Proposition 3.1.3. *For $\lambda \in \mathfrak{h}^*$ satisfying* (3.1.11) *and $D(k) \in \mathbb{S}(l, k)$ a shift operator with shift $l \in K$ we have*

$$D(k)(\Phi(\lambda, k)) = \eta(k, \lambda)\Phi(\lambda, k+l) \tag{3.1.12}$$

for some $\eta(\cdot, \cdot) \in \mathbb{C}[K \times \mathfrak{h}^*]$.

Proof. Immediate from Definition 3.1.1. Note that in the notation (3.1.3) we have $\eta(k, \lambda) = p_o(\lambda - \rho(k))$. □

Corollary 3.1.4. *For $\lambda \in P_+$ and $D(k) \in \mathbb{S}(l, k)$ we have*

$$D(k)(P(\lambda, k)) = \eta(k, \lambda + \rho(k)).P(\lambda - \rho(l), k+l) \tag{3.1.13}$$

with $\eta(k, \lambda + \rho(k)) = 0$ if $\lambda - \rho(l) \notin P_+$. In particular shift operators are W-invariant differential operators on H^{reg} which map $\mathbb{C}[P]^W$ into itself, and hence they can also be viewed as elements of some Weyl algebra $\mathbb{A}_n$ (cf. Remark 1.3.15).

Proof. Immediate since $\Phi(\lambda, k) = \Phi'(\lambda - \rho(k), k)$. □

Definition 3.1.5. Let $l \in K$ be integral. Then the mapping

$$\eta = \eta(l) = \eta(l, k)\colon\ \mathbb{S}(l, k) \to \mathbb{C}[K \times \mathfrak{h}^*] \tag{3.1.14}$$

defined by

$$\eta(l, k)(D(k))(\lambda) = p_0(\lambda - \rho(k)), \tag{3.1.15}$$

where $D(k) \in \mathbb{S}(l, k)$ has expansion (3.1.3), is called the *Harish-Chandra mapping* for shift operators with shift $l \in K$. Note that for shift operators with shift $l = 0$ (i.e., operators commuting with $L(k)$) we see that $\eta(0, k) = \gamma(k)$ becomes the Harish-Chandra mapping of Theorem 1.3.12.

Proposition 3.1.6. *The mapping* (3.1.14) *is injective, and a shift operator of order N is mapped onto a polynomial of degree N.*

Proof. The proof is analogous as for Corollary 1.2.8. For a differential operator $D = \sum_{\mu\leq 0} e^{-\rho(l)+\mu}\partial_{p_\mu}$ we have $D \in \mathbb{S}(l,k)$ if and only if the polynomials $p_\mu \in \mathbb{C}[\mathfrak{h}^*]$ satisfy the recurrence relations

$$(2\lambda+2\rho(k)+\mu,\mu)p_\mu(\lambda) = 2\sum_{\alpha>0}\sum_{j\geq 1}\{(k_\alpha+l_\alpha)(\lambda+\mu-\rho(l)+j\alpha,\alpha)p_{\mu+j\alpha}(\lambda) \\ -k_\alpha(\lambda,\alpha)p_{\mu+j\alpha}(\lambda-j\alpha)\}$$

and the proposition follows easily. □

We have bilinear mappings

$$\begin{aligned}\Pi_{l_1,l_2}\colon \mathbb{S}(l_1,k)\times\mathbb{S}(l_2,k) &\to \mathbb{S}(l_1+l_2,k)\\ (D_1(k),D_2(k)) &\mapsto D_1(k+l_2)\circ D_2(k)\end{aligned} \tag{3.1.16}$$

and for the corresponding Harish-Chandra mappings this yields

$$\eta(l_1+l_2,k)(\Pi_{l_1,l_2}(D_1,D_2)) = \eta(l_1,k+l_2)(D_1).\eta(l_2,k)(D_2). \tag{3.1.17}$$

In particular $\mathbb{S}(l,k)$ is a right $\mathbb{S}(0,k)$-module, and in view of the Harish-Chandra isomorphism for $\mathbb{D}(k) = \mathbb{S}(0,k)$ we conclude that the image of the Harish-Chandra mapping (3.1.14) is a $\mathbb{C}[\mathfrak{h}^*]^W$-module in $\mathbb{C}[\mathfrak{h}^*]$.

Proposition 3.1.7. *For $D = D(k) \in \mathbb{S}(l,k)$ we put*

$$\widetilde{D}(k) = \delta(l-k)\circ D^*(k-l)\circ\delta(k) \tag{3.1.18}$$

viewed as differential operator on H^{reg}. Here the asterisk signifies formal transpose as differential operator on A with respect to the Haar measure da: $(D_1D_2)^* = D_2^*D_1^*$ and $(\partial_p)^* = \partial_{p^*}$ with $p^*(\lambda) = p(-\lambda)$. Then we have $\widetilde{D} \in \mathbb{S}(-l,k)$ and $\eta(-l,k)(\widetilde{D}) = \eta(l,k-l)(D)^*$.

Proof. Indeed $\widetilde{D}$ has the correct asymptotic expansion on A_+. From Theorem 2.1.1 it follows that operator $ML(k)$ is symmetric with respect to the measure $\delta(k;a)da$ on A, or equivalently

$$ML(k) = \delta(-k)\circ ML^*(k)\circ\delta(k).$$

Hence we have

$$\begin{aligned}\widetilde{D}(k)\circ ML(k) &= \delta(l-k)\circ D^*(k-l)\circ\delta(k)\circ\delta(-k)\circ ML^*(k)\circ\delta(k)\\ &= \delta(l-k)\circ\{ML(k)\circ D(k-l)\}^*\circ\delta(k)\\ &= \delta(l-k)\circ\{D(k-l)\circ ML(k-l)\}^*\circ\delta(k)\\ &= \delta(l-k)\circ ML^*(k-l)\circ\delta(k-l)\circ\delta(l-k)\circ D^*(k-l)\circ\delta(k)\\ &= ML(k-l)\circ\widetilde{D}(k),\end{aligned}$$

which implies that $\widetilde{D}(k)\in\mathbb{S}(-l,k)$. If $D(k-l)=\sum_{\mu\le 0} e^{-\rho(l)+\mu}\partial_{p_\mu}$ then

$$\begin{aligned}\widetilde{D}(k) &= e^{2\rho(l)-2\rho(k)}(1+\cdots)\circ\{\sum_{\mu\le 0}\partial^*_{p_\mu}\circ e^{-\rho(l)+\mu}\}\circ e^{2\rho(k)}(1+\cdots)\\ &= \sum_{\mu\le 0} e^{\rho(l)+\mu}\partial_{q_\mu}\end{aligned}$$

with $q_0(\lambda){=}p_0^*(\lambda{+}2\rho(k){-}\rho(l)){=}p_0(-\lambda{-}2\rho(k){+}\rho(l))$. Hence $\eta(-l,k)(\widetilde{D})(\lambda)$ $= q_0(\lambda-\rho(k)) = p_0(-\lambda-\rho(k-l)) = \eta(l,k-l)(D)(-\lambda)$. $\square$

Proposition 3.1.8. *Suppose R of type BC_n with root multiplicities $(k_s,k_m,k_l)\in\mathbb{C}^3$ corresponding to the short, medium, and long roots, respectively. Then we have*

$$\begin{aligned}\Delta_s^{-1+2k_s+2k_l}\circ &ML(k_s,k_m,k_l)\circ\Delta_s^{1-2k_s-2k_l}\\ &= ML(1-k_s-2k_l,k_m,k_l),\end{aligned}\tag{3.1.19}$$

where $\Delta_s = \prod_{\alpha>0,\alpha\text{ short}}(e^{\frac12\alpha}-e^{-\frac12\alpha}) = \delta(1,0,0)^{\frac12}$.

Proof. Just apply Corollary 2.1.2 with k and m given by $k=(k_s,k_m,k_l)$ and $m=(1-k_s-2k_l,k_m,k_l)$. Then indeed $l_m=l_l=0$ and l_s is given by $l_s^2 = m_s(1-m_s-2m_l)-k_s(1-k_s-2k_l)=0$. $\square$

Corollary 3.1.9. *With the notation of the previous proposition suppose that $G_+(k_s,k_m,k_l)$ is a shift operator with shift $(0,0,1)$. Then the operator*

$$\begin{aligned}&E_-(k_s,k_m,k_l)\\ &\quad=\Delta_s^{3-2k_s-2k_l}\circ G_+(1-k_s-2k_l,k_m,k_l)\circ\Delta_s^{-1+2k_s+2k_l}\end{aligned}\tag{3.1.20}$$

is again a shift operator with shift $(-2,0,1)$.

Proof. Indeed the operator (3.1.20) has the correct asymptotic expansion on A_+. Using (3.1.19) we have

$$\begin{aligned}
&E_-(k_s,k_m,k_l)ML(k_s,k_m,k_l)\\
&= \Delta_s^{3-2k_s-2k_l}\circ G_+(1-k_s-2k_l,k_m,k_l)\\
&\qquad\qquad \circ\Delta_s^{-1+2k_s+2k_l}\circ ML(k_s,k_m,k_l)\\
&= \Delta_s^{3-2k_s-2k_l}\circ G_+(1-k_s-2k_l,k_m,k_l)\\
&\qquad\qquad \circ ML(1-k_s-2k_l,k_m,k_l)\circ\Delta_s^{-1+2k_s+2k_l}\\
&= \Delta_s^{3-2k_s-2k_l}\circ ML(1-k_s-2k_l,k_m,k_l+1)\\
&\qquad\qquad \circ G_+(1-k_s-2k_l,k_m,k_l)\circ\Delta_s^{-1+2k_s+2k_l}\\
&= ML(k_s-2,k_m,k_l+1)\circ\Delta_s^{3-2k_s-2k_l}\\
&\qquad\qquad \circ G_+(1-k_s-2k_l,k_m,k_l)\circ\Delta_s^{-1+2k_s+2k_l}\\
&= ML(k_s-2,k_m,k_l+1)\circ E_-(k_s,k_m,k_l)
\end{aligned}$$

and the statement follows. □

3.2. The construction of the fundamental shift operators

Assume that S is a W-orbit of inmultiplyable roots in R (i.e., $2S\cap R=\varnothing$). Writing $S_+ = S\cap R_+$ the function

$$\Delta_S = \prod_{\alpha\in S_+}(e^{\frac{1}{2}\alpha}-e^{-\frac{1}{2}\alpha})\in\mathbb{Z}[P] \tag{3.2.1}$$

transforms under W according to a character ε_S and every $F\in\mathbb{C}[P]$ which transforms under W according to ε_S is divisible (inside $\mathbb{C}[P]$) by Δ_S. We write 1_S for the multiplicity function on R which is 1 on S and 0 outside S.

Definition 3.2.1. For $N:=\#S_+$ and $\xi\in\mathfrak{a}$ later to be specified the operators

$$G_{S,+}(k) = \beta\left(\frac{1}{\Delta_S}\circ\sum_{w\in W}\varepsilon_S(w)D_{w\xi}(k)^N\right) \tag{3.2.2}$$

$$G_{S,-}(k) = \beta\left(\sum_{w\in W}\varepsilon_S(w)D_{w\xi}(k-1_S)^N\circ\Delta_S\right) \tag{3.2.3}$$

are called the *raising* and *lowering operators* associated with S.

Proposition 3.2.2. *The raising and lowering operators* (3.2.2) *and* (3.2.3) *are differential operators in* $\mathbb{C}[K]\otimes\mathbb{C}_\Delta[P]\otimes S\mathfrak{h}$ *which map* $\mathbb{C}[P]^W$ *into itself, and hence they can also be viewed as elements of the Weyl algebra* $\mathbb{A}_n$ *(cf. Remark 1.3.15). Moreover on* A_+ *they have asymptotic expansions of the form*

$$G_{S,+}(k) = \sum_{\mu\leq 0} e^{-\rho_S+\mu}\partial_{p_\mu} \tag{3.2.4}$$

$$G_{S,-}(k) = \sum_{\mu\leq 0} e^{\rho_S+\mu}\partial_{q_\mu} \tag{3.2.5}$$

with $\rho_S = \rho(1_S) = \frac{1}{2}\sum_{\alpha\in S_+}\alpha$.

Proof. This is obvious. □

Proposition 3.2.3. *For all* $F, G \in \mathbb{R}[P]^W$ *we have*

$$(G_{S,+}(k)F, G)_{k+1_S} = (F, G_{S,-}(k+1_S)G)_k. \tag{3.2.6}$$

Proof. Indeed we have for $F, G \in \mathbb{R}[P]^W$

$$\begin{aligned}(G_{S,+}(k)F, G)_{k+1_S} &= \sum_{w\in W}\varepsilon_S(w)\left(\frac{1}{\Delta_S}D_{w\xi}(k)^N F, G\right)_{k+1_S}\\ &= \sum_{w\in W}\varepsilon_S(w)(D_{w\xi}(k)^N F, \Delta_S G)_k = (F, G_{S,-}(k+1_S)G)_k\end{aligned}$$

by Theorem 1.1.6. □

Corollary 3.2.4. *There exist polynomials* $\eta_{S,+}$ *and* $\eta_{S,-}$ *in* $\mathbb{C}[K\times\mathfrak{h}^*]$ *such that*

$$G_{S,+}(k)(P(\lambda,k)) = \eta_{S,+}(k,\lambda+\rho(k)).P(\lambda-\rho_S, k+1_S) \tag{3.2.7}$$

$$G_{S,-}(k)(P(\lambda,k)) = \eta_{S,-}(k,\lambda+\rho(k)).P(\lambda+\rho_S, k-1_S) \tag{3.2.8}$$

and the degree of $\eta_{S,+}$ *and* $\eta_{S,-}$ *is* $\leq N$ *as polynomials in* $\lambda\in\mathfrak{h}^*$ *and the homogeneous part of degree* N *is independent of* $k\in K$ *and given by*

$$\lambda \mapsto \sum_{w\in W}\varepsilon_S(w)(w\xi,\lambda)^N. \tag{3.2.9}$$

Proof. In view of the expansion (3.2.4) it follows that $G_{S,+}(k)(P(\lambda,k))$ is a linear combination of monomial symmetric functions $M(\mu)$ with $\mu \leq \lambda - \rho_S$. Using (3.2.6) and (3.2.5) we get

$$(G_{S,+}(k)(P(\lambda,k)), M(\mu))_{k+1_S} = (P(\lambda,k), G_{S,-}(k)(M(\mu)))_k = 0$$

if $\mu < \lambda - \rho_S$. Hence $G_{S,+}(k)(P(\lambda,k))$ is a multiple of the Jacobi polynomial $P(\lambda-\rho_S, k+1_S)$. Moreover the scalar multiple $\eta_{S,+}(k, \lambda+\rho(k))$ is given by $p_0(\lambda)$ using (3.2.4). Hence $\eta_{S,+} \in \mathbb{C}[K \times \mathfrak{h}^*]$ and the last statement is clear from (3.2.2). A similar argument works for $G_{S,-}(k)$. □

Corollary 3.2.5. *The operators $G_{S,+}(k)$ and $G_{S,-}(k)$ are shift operators with shift 1_S and -1_S, respectively, and*

$$\eta_{S,\pm}(k,\lambda) = \eta(G_{S,\pm}(k))(k,\lambda)$$

is just the Harish-Chandra mapping for $G_{S,\pm}(k)$.

Proof. Clear from the above and Definition 3.1.5. □

By composing shift operators as in (3.1.16) we conclude from Proposition 3.1.7, Corollary 3.1.9, and the results of this section that $\mathbb{S}(l,k) \neq 0$ for each integral $l \in K$.

3.3. Theory of the constant term for shift operators

We start by discussing the rank one situation R of type BC_1. Say $R = \{\pm\alpha, \pm 2\alpha\}$ with $(\alpha,\alpha) = 1$ and put $k_1 = k_\alpha$, $k_2 = k_{2\alpha}$. Then the modified operator $ML(k)$ becomes

$$ML = \theta^2 + \left\{ k_1 \frac{1+x^{-1}}{1-x^{-1}} + 2k_2 \frac{1+x^{-2}}{1-x^{-2}} \right\} \theta + (\tfrac{1}{2}k_1 + k_2)^2 \tag{3.3.1}$$

with the identification $\mathbb{C}[P] = \mathbb{C}[x, x^{-1}]$ and $\theta = x\frac{d}{dx}$.

Proposition 3.3.1. *The operators*

$$G_+ = \frac{1}{x - x^{-1}}\theta \tag{3.3.2}$$

$$G_- = (x - x^{-1})\theta + (k_1 + 2k_2 - 1)(x + x^{-1}) + 2k_1 \tag{3.3.3}$$

$$E_+ = \frac{1+x^{-1}}{1-x^{-1}}\theta + (k_2 - \tfrac{1}{2}) \tag{3.3.4}$$

$$E_- = \frac{1-x^{-1}}{1+x^{-1}}\theta + (k_1 + k_2 - \tfrac{1}{2}) \tag{3.3.5}$$

are shift operators for (3.3.1) *with shifts* $(0,1),(0,-1),(2,-1),(-2,1)$, *respectively.*

Proof. In our notation $\Delta = (x-x^{-1})$ is the Weyl denominator. The reflection operator r: $\mathbb{C}[x,x^{-1}]$ is given by $r(x^j) = x^{-j}$ and the differential-reflection operator becomes

$$D(k_1,k_2) = \theta + \left\{ \tfrac{1}{2}k_1 \frac{1+x^{-1}}{1-x^{-1}} + k_2 \frac{1+x^{-2}}{1-x^{-2}} \right\} (1-r)$$

in accordance with (1.1.5). Taking $\xi = \frac{1}{2}\alpha$ in Definition 3.2.1 yields

$$\begin{aligned}
G_+ &= \beta\left(\frac{1}{x-x^{-1}} \circ D(k_1,k_2) \right) = \frac{1}{x-x^{-1}}\theta, \\
G_- &= \beta(D(k_1,k_2-1) \circ (x-x^{-1})) \\
&= \beta((x-x^{-1})\theta + (x+x^{-1}) + 2\left(\tfrac{1}{2}k_1 \frac{1+x^{-1}}{1-x^{-1}} + (k_2-1)\frac{1+x^{-2}}{1-x^{-2}} \right)(x-x^{-1})) \\
&= (x-x^{-1})\theta + (k_1+2k_2-1)(x+x^{-1}) + 2k_1,
\end{aligned}$$

the desired expressions for G_+ and G_-. Using (3.1.20) the operator E_- is given by

$$\begin{aligned}
E_- &= (x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{3-2k_1-2k_2} \circ G_+ \circ (x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{-1+2k_1+2k_2} \\
&= \frac{(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2}{(x-x^{-1})}\theta + \tfrac{1}{2}(-1+2k_1+2k_2)\frac{(x^{\frac{1}{2}} - x^{-\frac{1}{2}})(x^{\frac{1}{2}} + x^{-\frac{1}{2}})}{(x-x^{-1})} \\
&= \frac{1-x^{-1}}{1+x^{-1}}\theta + k_1 + k_2 - \tfrac{1}{2}
\end{aligned}$$

and the operator $E_+ = -\widetilde{E}_-$ is derived from E_- using (3.1.18). □

Corollary 3.3.2. *The Harish-Chandra mapping for the rank one shift operators of the previous proposition becomes*

$$\begin{aligned}
\eta(G_+) &= \theta - (\tfrac{1}{2}k_1 + k_2), \quad \eta(G_-) = \theta + (\tfrac{1}{2}k_1 + k_2 - 1) \\
\eta(E_+) &= \theta - (\tfrac{1}{2}k_1 + \tfrac{1}{2}), \quad \eta(E_-) = \theta + (\tfrac{1}{2}k_1 - \tfrac{1}{2}).
\end{aligned}$$

Proof. Immediate from Definition 3.1.5 and the previous proposition. □

Corollary 3.3.3. *For $l=(l_1,l_2)\in 2\mathbb{Z}\times\mathbb{Z}$ we write*

$$(l_1,l_2)=\tfrac{1}{2}l_1(2,-1)+(\tfrac{1}{2}l_1+l_2)(0,1)=\varepsilon_1 N_1(2,-1)+\varepsilon_2 N_2(0,1)$$

with $N_1=|\frac{1}{2}l_1|$, $N_2=|\frac{1}{2}l_1+l_2|\in\mathbb{Z}_+$ and $\varepsilon_1,\varepsilon_2\in\{\pm 1\}$. The differential operator of order $N=N_1+N_2$ defined by

$$\begin{aligned} G(l)=G(l,k) &:= E_{\varepsilon_1}(k_1+2\varepsilon_1(N_1-1),k_2-\varepsilon_1(N_1-1)+\varepsilon_2 N_2)\circ\cdots \\ &\cdots\circ E_{\varepsilon_1}(k_1,k_2+\varepsilon_2 N_2)\circ G_{\varepsilon_2}(k_1,k_2+\varepsilon_2(N_2-1))\circ\cdots\circ G_{\varepsilon_2}(k_1,k_2) \\ &=\prod_{j=0}^{N_1-1} E_{\varepsilon_1}(k_1+2\varepsilon_1 j,k_2-\varepsilon_1 j+\varepsilon_2 N_2)\circ\prod_{j=0}^{N_2-1} G_{\varepsilon_2}(k_1,k_2+\varepsilon_2 j)\end{aligned}$$

is a shift operator for (3.3.1) with shift $l=(l_1,l_2)$. Moreover

$$G(l)=(x^{\frac{1}{2}}-x^{-\frac{1}{2}})^{-l_1}(x-x^{-1})^{-l_2}\theta^N+\text{lower order terms}, \tag{3.3.6}$$

and

$$r(l,k,\theta):=\eta(G(l))\in\mathbb{C}[k_1,k_2,\theta] \tag{3.3.7}$$

is a polynomial of degree N which can be calculated explicitly from Corollary 3.3.2 as a product of N linear factors.

Proof. Obvious. □

Proposition 3.3.4. *Every rank one shift operator $D(k)$ with integral shift $l=(l_1,l_2)\in 2\mathbb{Z}\times\mathbb{Z}$ is of the form*

$$D(k)=G(l,k)P(ML(k))$$

with P a polynomial in one variable (independent of $k\in K$).

Proof. Suppose $D(k)=a\theta^N+\cdots$ has order N. Looking at the $(N+1)^{\text{st}}$ order part of the equation

$$D(k)\circ ML(k)=ML(k+l)\circ D(k)$$

yields a first-order differential equation for a of the form

$$2\theta(a) = a\left(-l_1\frac{1+x^{-1}}{1-x^{-1}} - 2l_2\frac{1+x^{-2}}{1-x^{-2}}\right) \iff$$
$$\frac{1}{a}\theta(a) = -\tfrac{1}{2}l_1\frac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}} - l_2\frac{x+x^{-1}}{x-x^{-1}}$$

which has as its solution

$$a = c(x^{\frac{1}{2}}-x^{-\frac{1}{2}})^{-l_1}(x-x^{-1})^{-l_2} \quad \text{with } c \in \mathbb{C}[K],\ c \neq 0.$$

It remains to be shown that $N - |\frac{1}{2}l_1| - |\frac{1}{2}l_1+l_2| \in 2\mathbb{Z}_+$. Indeed then the proposition follows by induction on N using (3.3.6).

Using Corollary 3.1.4 it follows that $D(k)$ when expressed in the coordinate $z = x+x^{-1}$ lies in fact in the Weyl-algebra $\mathbb{C}[k, z, \frac{d}{dz}]$. Since $\theta = (x-x^{-1})\frac{d}{dz}$ and $x^{\frac{1}{2}}-x^{-\frac{1}{2}} = (z-2)^{\frac{1}{2}}$, $x^{\frac{1}{2}}+x^{-\frac{1}{2}} = (z+2)^{\frac{1}{2}}$ we get

$$a\theta^N = c(z-2)^{\frac{1}{2}(N-l_1-l_2)}(z+2)^{\frac{1}{2}(N-l_2)}\frac{d^N}{dz^N} + \cdots$$

which in turn implies

$$N-l_1-l_2, N-l_2 \in 2\mathbb{Z}_+.$$

Because $\widetilde{D}(k)$ is also in the Weyl algebra (cf. Proposition 3.1.7) we have

$$N+l_1+l_2, N+l_2 \in 2\mathbb{Z}_+.$$

Observe that $|\frac{1}{2}l_1| + |\frac{1}{2}l_1+l_2| = \max(|l_1+l_2|, |l_2|)$ and the desired relation $N-|\frac{1}{2}l_1| - |\frac{1}{2}l_1+l_2| \in 2\mathbb{Z}_+$ follows. □

Corollary 3.3.5. *For $l = (l_1, l_2) \in 2\mathbb{Z}\times\mathbb{Z}$ the space $\mathbb{S}(l)$ of shift operators for the operator* (3.3.1) *is a free rank one (right)* $\mathbb{S}(0)$*-module with generator $G(l)$ given in Corollary 3.3.3. In particular the generators $G(l,k)$ satisfy*

$$G(l+m, k) = G(l, k+m) \circ G(m, k) = G(m, k+l) \circ G(l, k)$$

for $l = (l_1, l_2)$, $m = (m_1, m_2) \in 2\mathbb{Z} \times \mathbb{Z}$ with $\operatorname{sign}(\frac{1}{2}l_1) = \operatorname{sign}(\frac{1}{2}m_1)$ *and* $\operatorname{sign}(\frac{1}{2}l_1+l_2) = \operatorname{sign}(\frac{1}{2}m_1+m_2)$.

Proof. Obvious. □

We are now in a position to describe the image of the Harish-Chandra mapping

$$\eta\colon\ \mathbb{S}(l) \to C[K \times \mathfrak{h}^*] \tag{3.3.8}$$

in the case of arbitrary rank root systems. Similarly to the results of Section 1.2 the crucial ingredient will be the asymptotic behavior of a shift operator along codimension one walls of A_+. This reduces the situation to rank one. Therefore the above computations with rank one shift operators are not merely illustrative but basic for understanding the higher rank situation.

For R a possibly nonreduced root system we write $R^0 = R\backslash\frac{1}{2}R$ for the corresponding reduced root system of inmultiplyable roots, and let $\alpha_1, \dots, \alpha_n$ be the simple roots in R^0_+. Write

$$R^0 = S_1 \cup S_2 \cup \ldots \cup S_{m_0}$$

as a disjoint union of W-orbits. For $k \in K$ we write k_i for the restriction of the multiplicity function from R to $(S_i \cup \frac{1}{2}S_i) \cap R$.

Theorem 3.3.6. *For $l \in K$ integral and $D \in \mathbb{S}(l)$ a shift operator with shift l the polynomial $\eta(D) \in \mathbb{C}[K \times \mathfrak{h}^*]$ is of the form*

$$\eta(D)(k, \lambda) = \left\{ \prod_{i=1}^{m_0} \prod_{\alpha \in S_{i,+}} r(l_i, k_i, (\lambda, \alpha^\vee)) \right\} p(k, \lambda) \tag{3.3.9}$$

with $p \in \mathbb{C}[K \times \mathfrak{h}^*]^W$ and $r(l_i, k_i, \theta)$ the polynomial defined by (3.3.7).

Proof. We have from Definition 1.2.4 and Definition 3.1.5 that

$$\eta(D)(k, \lambda) = \gamma(k)(e^{\rho(l)} \circ D).$$

For F a face of A_+ we put

$$\eta_F(D) = \gamma_F(k)(e^{\rho_F(l)} \circ D).$$

Then $\eta_F(D)$ is a shift operator for $\gamma_F(k)(ML(k))$ with shift l_F the restriction of l to $R_F = \{\alpha \in R; (\alpha, \xi) = 0\ \forall \xi \in F\}$. Indeed

$$\begin{aligned}
\eta_F(D) \circ \gamma_F(k)(ML(k)) &= \gamma_F(k)(e^{\rho_F(l)} \circ D(k) \circ ML(k)) \\
&= \gamma_F(k)(e^{\rho_F(l)} \circ ML(k+l) \circ e^{-\rho_F(l)} \circ e^{\rho_F(l)} \circ D(k)) \\
&= \gamma_F(k)(e^{\rho_F(l)} \circ ML(k+l) \circ e^{-\rho_F(l)}) \circ \gamma_F(k)(e^{\rho_F(l)} \circ D(k)) \\
&= \gamma_F(k+l)(ML(k+l)) \circ \eta_F(D).
\end{aligned}$$

Suppose F is a codimension one face (a wall) of A_+ with $R_F^0 = \{\pm\alpha_j\}$ for some simple root α_j in R_+^0. If $i \in \{1, \dots, m_0\}$ with $\alpha_j \in S_i$ then we conclude from Proposition 3.3.4 that $\eta(D)$ is divisible (as a polynomial) by $r(l_i, k_i, (\lambda, \alpha_j^\vee))$ and the remainder is invariant under the reflection r_j. Since r_j leaves the set $R_+^0 \backslash \{\alpha_j\}$ invariant the expression

$$\prod_{i=1}^{m_0} \prod_{\substack{\alpha \in S_{i,+} \\ \alpha \neq \alpha_j}} r(l_i, k_i, (\lambda, \alpha^\vee))$$

is also invariant under r_j. Hence the rational function

$$p(k, \lambda) := \frac{\eta(D)(k, \lambda)}{\prod_{i=1}^{m_0} \prod_{\alpha \in S_{i,+}} r(l_i, k_i, (\lambda, \alpha^\vee))}$$

is W-invariant in λ with its set of poles P contained in the set of hyperplanes

$$\bigcup_{i=1}^{m_0} \bigcup_{\substack{\alpha \in S_{i,+} \\ \alpha \text{ not simple in } R_+^0}} \{r(l_i, k_i, (\lambda, \alpha^\vee)) = 0\}.$$

Hence P is empty or equivalently $p(k, \lambda)$ is a polynomial. $\square$

Theorem 3.3.7. *For $l \in K$ integral the space $\mathbb{S}(l)$ of shift operators with shift l is a free rank one (right) $\mathbb{S}(0)$-module generated by an operator $G(l) = G(l, k)$ with*

$$\eta(G(l))(k, \lambda) = \prod_{i=1}^{m_0} \prod_{\alpha \in S_{i,+}} r(l_i, k_i, (\lambda, \alpha^\vee)). \tag{3.3.10}$$

Here $R^0 = S_1 \cup \ldots \cup S_{m_0}$ is the disjoint union of W-orbit in $R^0 = R \backslash \frac{1}{2}R$. The generators $G(l)$ are differential operators of order

$$\sum_{\alpha \in R_+^0} \max(|l_\alpha|, |l_{\frac{1}{2}\alpha} + l_\alpha|) \tag{3.3.11}$$

and satisfy the relations

$$G(l+m, k) = G(l, k+m) \circ G(m, k) = G(m, k+l) \circ G(l, k) \tag{3.3.12}$$

if $l, m \in K$ are both integral with $|l_\alpha| + |m_\alpha| = |l_\alpha + m_\alpha|$ and $|l_{\frac{1}{2}\alpha} + l_\alpha| + |m_{\frac{1}{2}\alpha} + m_\alpha| = |l_{\frac{1}{2}\alpha} + m_{\frac{1}{2}\alpha} + l_\alpha + m_\alpha|$.

Proof. This follows from the previous theorem, Corollary 3.1.9, and the construction of the fundamental shift operators in Section 3.2. $\square$

Remark 3.3.8. By Theorem 3.3.6 (and Proposition 3.1.6) the fundamental shift operators $G_\pm(k)$ of Section 3.2 depend on $\xi \in \mathfrak{a}$ only up to a multiplicative constant. In view of the identity (with $N = \#S_+$)

$$\sum_{w \in W} \varepsilon_S(w)(w\xi, \lambda)^N = c \cdot \prod_{\alpha \in S_+} (\xi, \alpha) \cdot \prod_{\alpha \in S_+} (\lambda, \alpha^\vee) \tag{3.3.13}$$

for some $c \in \mathbb{C}^\times$, and because the leading symbol of order N of $G_\pm(k)$ is given by

$$\Delta_S^{\mp 1} \cdot \sum_{w \in W} \varepsilon_S(w)(w\xi, \cdot)^N$$

(independent of k as should) we choose $\xi \in \mathfrak{a}$ such that

$$c \cdot \prod_{\alpha \in S_+} (\xi, \alpha) = 1. \tag{3.3.14}$$

With this choice of $\xi \in \mathfrak{a}$ the Harish-Chandra mapping of the operators $G_\pm(k)$ becomes

$$\eta_+(k, \lambda) = \prod_{\alpha \in S_+} ((\lambda, \alpha^\vee) - \tfrac{1}{2}k_{\frac{1}{2}\alpha} - k_\alpha) \tag{3.3.15}$$

$$\eta_-(k, \lambda) = \prod_{\alpha \in S_+} ((\lambda, \alpha^\vee) + \tfrac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha - 1) \tag{3.3.16}$$

and $\eta_-(k, \lambda) = (-1)^N \eta_+(k - 1_S, -\lambda)$ in accordance with Proposition 3.1.7 (since $G_-(k) = (-1)^N \widetilde{G}_+(k)$).

3.4. Raising and lowering operators

Definition 3.4.1. Let $R = \bigcup_{i=1}^m S_i$ be the disjoint union of W-orbits in R and define $e_i \in K$ by $e_{i,\alpha} = \delta_{ij}$ for $\alpha \in S_j$. Let $B = \{b_1, \ldots, b_m\}$ be the following basis of K

$$b_i = \begin{cases} e_i & \text{if } 2S_i \cap R = \varnothing \\ 2e_i - e_j & \text{if } 2S_i = S_j \text{ for some } j. \end{cases} \tag{3.4.1}$$

Note that $l \in K$ is integral (Definition 3.1.1) if and only if $l \in \mathbb{Z}.B$. A shift operator with shift $l \in \mathbb{Z}.B$ is called a *raising operator* if $l \in \mathbb{Z}_+.B$ and a *lowering operator* if $l \in \mathbb{Z}_-.B$.

Definition 3.4.2. The meromorphic functions $\widetilde{c}, c$: $\mathfrak{h}^* \times K \to \mathbb{C}$ are defined by

$$\widetilde{c}(\lambda, k) = \prod_{\alpha>0} \frac{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha})}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha)} \tag{3.4.2}$$

and

$$c(\lambda, k) = \frac{\widetilde{c}(\lambda, k)}{\widetilde{c}(\rho(k), k)} \tag{3.4.3}$$

with the convention that $k_{\frac{1}{2}\alpha} = 0$ if $\frac{1}{2}\alpha \notin R$.

Theorem 3.4.3. *For $l \in \mathbb{Z}_-.B$ there exists a lowering operator $G_-(l) = G_-(l, k)$ with shift l whose image under the Harish-Chandra mapping is given by*

$$\eta(G_-(l))(k, \lambda) = \frac{\widetilde{c}(\lambda, k+l)}{\widetilde{c}(\lambda, k)}. \tag{3.4.4}$$

Proof. For S a W-orbit in R^0 we take $G_-(-1_S, k) := G_-(k)$ in the notation of Remark 3.3.8. Using the functional equation $\Gamma(z+1) = z\Gamma(z)$ for the Γ-function relation (3.4.4) follows in case $l = -1_S$ from (3.3.16).

For $\alpha \in R$ we write

$$\widetilde{c}_\alpha(\lambda, k) = \frac{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha})}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha)} \tag{3.4.5}$$

and using the duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$ for the Γ-function we get for $\frac{1}{2}\alpha, \alpha \in R$

$$\begin{aligned} &\widetilde{c}_{\frac{1}{2}\alpha}(\lambda, k)\widetilde{c}_\alpha(\lambda, k) \\ &= \frac{\Gamma((\lambda, \alpha^\vee))\Gamma((\lambda, \alpha^\vee) + \frac{1}{2})}{2^{k_{\frac{1}{2}\alpha}}\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha}+\frac{1}{2})\Gamma((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha)}. \end{aligned} \tag{3.4.6}$$

Hence we have

$$\frac{\widetilde{c}_{\frac{1}{2}\alpha}(\lambda, k_{\frac{1}{2}\alpha}-2, k_\alpha+1)\widetilde{c}_\alpha(\lambda, k_{\frac{1}{2}\alpha}-2, k_\alpha+1)}{\widetilde{c}_{\frac{1}{2}\alpha}(\lambda, k)\widetilde{c}_\alpha(\lambda, k)} = 4((\lambda, \alpha^\vee) + \tfrac{1}{2}k_{\frac{1}{2}\alpha}-\tfrac{1}{2}),$$

which implies that for R of type BC_n we should take (in the notation of Corollary 3.1.9) for the lowering operator with shift $(-2,0,1)$ the operator $G_-((-2,0,1),k)=4^nE_-(k)$ (cf. Corollary 3.3.2). This proves the existence of the lowering $G_-(l)$ for each $l=-b_i$ with $i=1,\dots,m$. By composing these lowering operators as in (3.1.16) the theorem follows by induction on $-\sum_{\alpha>0} l_\alpha$. □

Corollary 3.4.4. *For $l\in\mathbb{Z}_+.B$ the differential operator*

$$G_+(l,k) := \delta(-l-k)\circ G_-^*(-l,k+l)\circ\delta(k) \tag{3.4.7}$$

is a raising operator with shift l and

$$\eta(G_+(l))(k,\lambda)=\frac{\tilde{c}(-\lambda,k)}{\tilde{c}(-\lambda,k+l)}. \tag{3.4.8}$$

The order of $G_+(l)$ as a differential operator is equal to $\sum_{\alpha>0} l_\alpha$.

Proof. Immediate from Proposition 3.1.7 and the previous theorem. □

For a reduced root system R the lowering operators $G(l)$ given by (3.3.10) and $G_-(l)$ given by (3.4.4) for $l\in\mathbb{Z}_-.B$ coincide and the raising operators $G(l)$ and $G_+(l)$ for $l\in\mathbb{Z}_+.B$ only differ by a possible sign $(-1)^{\sum_{\alpha>0} l_\alpha}$. In case R is nonreduced the shift operators $G(l)$ and the lowering and raising operators $G_-(l)$, $G_+(l)$ can differ in addition by some factors of 4 (cf. the proof of Theorem 3.4.3).

3.5. The L^2-norm of the Jacobi polynomials

With the help of shift operators we can compute the L^2-norm of the Jacobi polynomials.

Proposition 3.5.1. *For $l\in\mathbb{Z}_+.B$ and $k\in K$ with $k_\alpha\geq 0$ and $k_\alpha-l_\alpha\geq 0$ we have for $\lambda\in P_+$*

$$\begin{aligned}\frac{|P(\lambda,k)|_k^2}{|P(\lambda+\rho(l),k-l)|_{k-l}^2}\\ =(-1)^{\sum_{\alpha>0} l_\alpha}\frac{\tilde{c}(\lambda+\rho(k),k-l)\tilde{c}(-(\lambda+\rho(k)),k)}{\tilde{c}(\lambda+\rho(k),k)\tilde{c}(-(\lambda+\rho(k)),k-l)}.\end{aligned} \tag{3.5.1}$$

Proof. Replacing λ by $\lambda+\rho(l)$ and k by $k-l$ in (3.1.13) yields

$$G_+(l,k-l)(P(\lambda+\rho(l),k-l)) = \eta(G_+(l))(k-l,\lambda+\rho(k))P(\lambda,k).$$

Hence we get

$$\begin{aligned}
|P(\lambda,k)|_k^2 &= \frac{1}{\eta(G_+(l))(k-l,\lambda+\rho(k))}(G_+(l,k-l)P(\lambda+\rho(l),k-l),P(\lambda,k))_k \\
&= \frac{(-1)^{\sum_{\alpha>0} l_\alpha}}{\eta(G_+(l)(k-l,\lambda+\rho(k))}(P(\lambda+\rho(l),k-l),G_-(-l,k)P(\lambda,k))_{k-l} \\
&= (-1)^{\sum_{\alpha>0} l_\alpha}\cdot\frac{\eta(G_-(-l))(k,\lambda+\rho(k))}{\eta(G_+(l))(k-l,\lambda+\rho(k))}|P(\lambda+\rho(l),k-l)|_{k-l}^2
\end{aligned}$$

and the proposition follows from (3.4.4) and (3.4.8). $\square$

We write

$$c^*(\lambda,k) = \prod_{\alpha>0}\frac{\Gamma(-(\lambda,\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}-k_\alpha+1)}{\Gamma(-(\lambda,\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}+1)} \tag{3.5.2}$$

which is equivalent to (using $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$)

$$\frac{\widetilde{c}(\lambda,k)}{c^*(\lambda,k)} = \prod_{\alpha>0}\frac{\sin\pi((\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha)}{\sin\pi((\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha})}. \tag{3.5.3}$$

Corollary 3.5.2. *We can rewrite* (3.5.1) *as*

$$\frac{|P(\lambda,k)|_k^2}{|P(\lambda+\rho(l),k-l)|_{k-l}^2} = \frac{c^*(-(\lambda+\rho(k)),k)\widetilde{c}(\lambda+\rho(k)),k-l)}{\widetilde{c}(\lambda+\rho(k),k)c^*(-(\lambda+\rho(k)),k-l)}, \tag{3.5.4}$$

which has the advantage over (3.5.1) *that each of the four functions on the right hand side has no poles for* $\lambda\in P_+$.

Proof. Obvious. $\square$

Corollary 3.5.3. *For* $k\in K$ *integral with* $k_\alpha\geq 1$ *we have*

$$|P(\lambda,k)|_k^2 = |W|\frac{c^*(-(\lambda+\rho(k)),k)}{\widetilde{c}(\lambda+\rho(k),k)}. \tag{3.5.5}$$

Proof. Take $k=l$ in (3.5.4) and use that $\widetilde{c}(\lambda+\rho(k),0)=1$, $c^*(-(\lambda+\rho(k)),0)$ $=1$ together with $|P(\lambda+\rho(k),0)|_0^2 = |M(\lambda+\rho(k))|_0^2 = |W|$. $\square$

Corollary 3.5.4. *For $k \in K$ real with $k_\alpha \geq 0$ we have*

$$\frac{|P(\lambda,k)|_k^2}{|P(0,k)|_k^2} = \frac{c^*(-(\lambda+\rho(k)),k)\widetilde{c}(\rho(k),k)}{\widetilde{c}(\lambda+\rho(k),k)c^*(-\rho(k),k)}. \tag{3.5.6}$$

Proof. For $k \in K$ integral with $k_\alpha \geq 1$ this is immediate from (3.5.5). However, both sides of (3.5.6) are rational functions of $k \in K$ and therefore (3.5.6) remains valid for real $k_\alpha \geq 0$. □

Theorem 3.5.5. *For $k \in K$ real with $k_\alpha \geq 0$ we have*

$$|P(\lambda,k)|_k^2 = |W| \cdot \frac{c^*(-(\lambda+\rho(k)),k)}{\widetilde{c}(\lambda+\rho(k),k)}. \tag{3.5.7}$$

Proof. In view of Corollary 3.5.4 it suffices to show the theorem for $\lambda = 0$. Clearly the function

$$k \in K \mapsto f(k) = \frac{\widetilde{c}(\rho(k),k)}{c^*(-\rho(k),k)} \int_T \prod_{\alpha>0} |e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}|^{2k_\alpha} dt$$

is holomorphic on the domain $\{k \in K; \mathrm{Re}(k_\alpha) > 0\}$. Moreover it is periodic with period lattice $\mathbb{Z}.B$ using (3.5.4) and (3.5.6). Now fix $k \in K$ integral with $k_\alpha \geq 1$. Then the function

$$z \in \mathbb{C} \mapsto f(zk) \tag{3.5.8}$$

is holomorphic on the half plane $\{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$ and periodic with period lattice $\mathbb{Z}$. We claim that

$$|f(zk)| \leq e^{a \,\mathrm{Re}\,(z) + b \log|z| + c} \tag{3.5.9}$$

for some $a, b, c \in \mathbb{R}$. Together with the periodicity this implies that the function (3.5.8) is of moderate growth at infinity and hence equal to a constant. Taking $z = 1$ we conclude from Corollary 3.5.3 that

$$|f(zk)| = |W|. \tag{3.5.10}$$

As k varies over integral points in K with $k_\alpha \geq 1$ the theorem follows by continuity. It remains to check (3.5.9). From Stirling's asymptotic expansion

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} \, e^{-z+(z-\frac{1}{2})\log z}(1+O(\frac{1}{z})), \tag{3.5.11}$$

valid for $|\arg z| < \pi$, we get for $a > 0$, b arbitrary

$$\Gamma(az+b) = (2\pi)^{\frac{1}{2}} e^{(-a+a\log a)z+(az+b-\frac{1}{2})\log z+(b-\frac{1}{2})\log a}(1+O(\frac{1}{z})),$$

which in turn implies

$$\widetilde{c}_\alpha(\rho(zk), zk) = e^{\widetilde{a}_\alpha z+\widetilde{b}_\alpha - k_\alpha z \log z}(1+O(\frac{1}{z}))$$

$$c^*_\alpha(-\rho(zk), zk) = e^{a^*_\alpha z+b^*_\alpha+d^*_\alpha \log z - k_\alpha z\log z}(1+O(\frac{1}{z}))$$

for some $\widetilde{a}_\alpha, \widetilde{b}_\alpha, a^*_\alpha, b^*_\alpha, d^*_\alpha \in \mathbb{R}$. (Note that $d^*_\alpha = 0$ unless α is simple in R^0_+ in which case $d^*_\alpha = -\frac{1}{2}$.) Hence we get

$$\frac{\widetilde{c}(\rho(zk), zk)}{c^*(-\rho(zk), zk)} = e^{Az+B\log z+C}(1+O(\frac{1}{z}))$$

for some $A, B, C \in \mathbb{R}$, and the desired estimate (3.5.9) follows easily. □

Corollary 3.5.6. *For $k \in K$ real with $k_\alpha \geq 0$ we have*

$$\int_T |\delta(k,t)|dt = \prod_{\alpha>0} \frac{\Gamma((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha+1)\Gamma((\rho(k),\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}-k_\alpha+1)}{\Gamma((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+1)\Gamma((\rho(k),\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}+1)}. \tag{3.5.12}$$

Proof. Specializing (3.5.7) for $\lambda = 0$ yields

$$\int_T |\delta(k,t)|dt = |W| \cdot \prod_{\alpha>0} \frac{\Gamma((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha)\Gamma((\rho(k),\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}-k_\alpha+1)}{\Gamma((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha})\Gamma((\rho(k),\alpha^\vee)-\frac{1}{2}k_{\frac{1}{2}\alpha}+1)} \tag{3.5.13}$$

and taking the limit for $k \to 0$ gives

$$\prod_{\alpha>0} \frac{(\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}+k_\alpha}{(\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{1}{2}\alpha}} = |W|. \tag{3.5.14}$$

Now relation (3.5.12) follows by combining (3.5.13) and (3.5.14). □

Example 3.5.7. In case R is of type BC_n formula (3.5.12) can be rewritten in the form (see [48])

$$\begin{aligned}\int_0^1 \cdots \int_0^1 (t_1 \dots t_n)^{x-1}\{(1-t_1)\dots(1-t_n)\}^{y-1}|\Delta(t)|^{2z}dt_1\dots dt_n \\ = \prod_{j=1}^{n} \frac{\Gamma(1+jz)\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)}{\Gamma(1+z)\Gamma(x+y+(n+j-2)z)},\end{aligned} \tag{3.5.15}$$

where $\Delta(t) = \Delta(t_1, \dots, t_n) = \prod_{i<j}(t_i - t_j)$ is the discriminant. This is Selberg's multivariable B-integral formula [69].

Example 3.5.8. In case R is irreducible and reduced with $k = k_\alpha\ \forall \alpha \in R$ formula (3.5.12) takes the form

$$\int_T |\Delta(t)|^{2k} dt = \prod_{j=1}^{n} \binom{kd_j}{k}, \tag{3.5.16}$$

where $d_1 \leq d_2 \leq \dots \leq d_n$ are the primitive degrees of R. Indeed $\rho(k) = k\rho$ with $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ and $(\rho, \alpha^\vee) = ht(\alpha^\vee)$. Hence

$$\begin{aligned}\int_T |\Delta(t)|^{2k} dt &= \prod_{\alpha>0} \frac{(kht(\alpha^\vee)+k)!(kht(\alpha^\vee)-k)!}{(kht(\alpha^\vee))!(kht(\alpha^\vee))!} \\ &= \prod_{j=1}^{n} \frac{(kd_j)!\ 0!}{k!\ (kd_j-k)!} = \prod_{j=1}^{n} \binom{kd_j}{k}\end{aligned}$$

since the partition of positive roots by height is conjugate to the partition formed by the exponents $m_1, \dots, m_n$ ($m_j = d_j - 1$). See [47, 48]. For R of type A_n formula (3.5.16) was conjectured by Dyson [18] and proved by Gunson [26], Wilson [75], and Good [25].

3.6. The value of Jacobi polynomials at the identity

Let $\mathbb{C}_\pi[\mathfrak{a}]$ denote the localization of $\mathbb{C}[\mathfrak{a}]$ along the polynomial

$$\pi = \prod_{\alpha\in R_+^0} (\alpha, \cdot) \in \mathbb{C}[\mathfrak{a}]. \tag{3.6.1}$$

The Euler operator on $\mathfrak{a}$ is defined by

$$E = \sum_{1}^{n} (\xi_i, \cdot)\partial_{\xi_i} \in \mathbb{C}[\mathfrak{a}] \otimes U\mathfrak{h}. \tag{3.6.2}$$

For $D \in \mathbb{C}_\Delta[P] \otimes U\mathfrak{h}$ we have a convergent expansion

$$D = \sum_{N \in \mathbb{Z}, N \geq N_0} D_N \tag{3.6.3}$$

for some $N_0 \in \mathbb{Z}$ and $D_N \in \mathbb{C}_\pi[\mathfrak{a}] \otimes U\mathfrak{h}$ with $[E, D_N] = ND_N$. If $D_{N_0} \neq 0$ then we say that D has lowest homogeneous degree equal to $LHD(D) := N_0$ and $LHP(D) := D_{N_0}$ is called the lowest homogeneous part of D.

Suppose $D_1, D_2 \in \mathbb{C}_\Delta[P] \otimes U\mathfrak{h}$ with $LHD(D_1) = N_1$, $LHD(D_2) = N_2$. Then it is obvious that

$$LHD(D_1 D_2) = LHD(D_1) + LHD(D_2) \tag{3.6.4}$$

and

$$LHP(D_1 D_2) = LHP(D_1)\, LHP(D_2). \tag{3.6.5}$$

Example 3.6.1. For the operator $L(k) \in \mathfrak{R} \otimes U\mathfrak{h}$ we have $LHD(L(k)) = -2$ and

$$LHP(L(k)) = \sum_{1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha > 0} \frac{k_\alpha}{(\alpha, \cdot)} \partial_\alpha. \tag{3.6.6}$$

Proposition 3.6.2. *The elements of* $\mathbb{C}_\pi[\mathfrak{a}] \otimes U\mathfrak{h}$

$$\begin{cases} e = e(k) = \frac{1}{2} \sum_{1}^{n} (\xi_j, \cdot)^2 \\ h = h(k) = E + \frac{1}{2}n + \sum_{\alpha > 0} k_\alpha \\ f = f(k) = -\frac{1}{2}\, LHP(L(k)) \end{cases} \tag{3.6.7}$$

satisfy the $sl(2)$ *commutation relations*

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h. \tag{3.6.8}$$

Proof. An easy calculation. □

Proposition 3.6.3. *If $D \in \mathbb{D}(k) = \mathbb{S}(0,k)$ is a differential operator of order N then $LHD(D) = -N$.*

Proof. By Corollary 1.2.8 D has the form

$$D = \partial_p + \text{terms of order} < N$$

for some nonzero polynomial $p \in S\mathfrak{h}$ homogeneous of degree N. Hence $LHD(D) \leq -N$.

On the other hand

$$ad(f)(LHP(D)) = -\tfrac{1}{2}[LHP(L(k)), LHP(D)] = 0$$

by (3.6.5), and since $LHP(D))$ is a differential operator of order $\leq N$ we also get

$$ad(e)^{N+1}(LHP(D)) = 0.$$

Hence $LHD(D) \geq -N$ by standard $sl(2)$-representation theory. □

Theorem 3.6.4. *If $l \in \mathbb{Z}_- \cdot B$ then we have*

$$LHD(G_-(l)) = 0, \tag{3.6.9}$$

where $G_-(l)$ is the lowering operator given by Theorem 3.4.3.

Proof. By (3.1.16) the operator $G_+(-l, k+l)G_-(l,k) \in \mathbb{S}(0,k)$ and has order $-2\sum_{\alpha>0} l_\alpha$. Here G_+ is given by Corollary 3.4.4. Hence using (3.6.4) and Proposition 3.6.3 we get

$$LHD(G_-(l,k)) + LHD(G_+(-l,k+l)) - 2\sum_{\alpha>0} l_\alpha = 0. \tag{3.6.10}$$

On the other hand using (3.4.7) and (3.6.4) we get

$$LHD(G_-(l,k)) - LHD(G_+(-l,k+l)) + 2\sum_{\alpha>0} l_\alpha = 0 \tag{3.6.11}$$

since $LHD(\delta(l)) = 2\sum_{\alpha>0} l_\alpha$. The theorem follows immediately from these equations. □

Corollary 3.6.5. *For $F \in C^\infty(A)^W$ and $l \in \mathbb{Z}_- \cdot B$ we have*

$$G_-(l)(F)(e) = G_-(l)(1)(e) \cdot F(e). \tag{3.6.12}$$

Proof. Since $G_-(l)$ lies in the Weyl algebra $\mathbb{C}[z_1, \dots, z_n, \frac{\partial}{\partial z_1}, \dots \frac{\partial}{\partial z_n}]$ we conclude that $G_-(l)(F) \in C^\infty(A)^W$. If we write

$$G_-(l) = \sum_{N_1,\dots,N_n} a_{N_1,\dots,N_n} \partial_{\xi_1}^{N_1} \dots \partial_{\xi_n}^{N_n}$$

with $a_{N_1,\dots,N_n} \in \mathbb{C}_\Delta[P]$ then by Theorem 3.6.4 we get

$$LHD(a_{N_1,\dots,N_n}) \geq N_1 + \dots + N_n.$$

Hence (3.6.12) follows from $G_-(l)(F)(e) = \lim_{t\to 0} G_-(l)(F)(\exp t\xi)$ for some $\xi \in \mathfrak{a}$ with $\pi(\xi) \neq 0$. □

Theorem 3.6.6. *For $k \in K$ with $k_\alpha \geq 0$ $\forall \alpha \in R$ we have*

$$P(\lambda, k; e) = \frac{\widetilde{c}(\rho(k), k)}{\widetilde{c}(\lambda+\rho(k), k)}. \tag{3.6.13}$$

Proof. We apply (3.6.12) with $F = P(\lambda, k; h)$ equal to a Jacobi polynomial. Using (3.1.13) and Theorem 3.4.3 we get

$$G_-(l,k)(P(\lambda, k; h)) = \frac{\widetilde{c}(\lambda+\rho(k), k+l)}{\widetilde{c}(\lambda+\rho(k), k)} P(\lambda-\rho(l), k+l; h) \tag{3.6.14}$$

and since $1 \equiv P(0, k; h)$ we also have

$$G_-(l,k)(1) = \frac{\widetilde{c}(\rho(k), k+l)}{\widetilde{c}(\rho(k), k)} P(-\rho(l), k+l; h). \tag{3.6.15}$$

Here $l \in \mathbb{Z}_- \cdot B$. Hence we have from (3.6.12), (3.6.14), (3.6.15)

$$P(\lambda, k; e) = \frac{\widetilde{c}(\lambda+\rho(k), k+l)\widetilde{c}(\rho(k), k)}{\widetilde{c}(\lambda+\rho(k), k)\widetilde{c}(\rho(k), k+l)} \cdot \frac{P(\lambda-\rho(l), k+l; e)}{P(-\rho(l), k+l; e)}$$

and taking $k = -l \in \mathbb{Z}_+ \cdot B$ this yields

$$P(\lambda, k; e) = \frac{\widetilde{c}(\rho(k), k)}{\widetilde{c}(\lambda+\rho(k), k)}.$$

Since both sides are rational functions of $k \in K$ (use $\lambda \in P_+$) the extension to $k \in K$ with $k_\alpha \geq 0$ $\forall \alpha \in R$ is immediate. □

Corollary 3.6.7. *For $k \in K$ and $l \in \mathbb{Z}_- \cdot B$ we have*

$$G_-(l,k)(1)(e) = \frac{\widetilde{c}(\rho(k+l),k+l)}{\widetilde{c}(\rho(k),k)}. \tag{3.6.16}$$

Proof. Clear from (3.6.13) and (3.6.15). □

Notes for Chapter 3

Shift operators are multivariable generalizations of the familiar identity

$$\frac{d}{dz}F(\alpha,\beta,\gamma;z) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z).$$

That shift operators should exist for higher rank root systems was first hinted at by Koornwinder who found a shift operator for R of type BC_2 [42]. A systematic study of shift operators was made by Opdam in his thesis [58, 59]. We have followed these papers closely with a simplified treatment of the existence of shift operators in Section 3.2 due to [33]. The results of Sections 3.5 and 3.6 are due to Opdam as well [58, 60]. Corollary 3.5.4 was obtained before in [30] and Corollary 3.5.6 had been conjectured in [48]. Proposition 3.6.2 was inspired by [27].

CHAPTER 4

The hypergeometric function

4.1. The hypergeometric differential equations

Everything we have presented so far is essentially formal algebra, but now we will start a more analytic study.

Definition 4.1.1. Fix $\lambda \in \mathfrak{h}^*$ and $k \in K$. The system of differential equations

$$D(u) = \gamma(D, k, \lambda)u \qquad \forall D \in \mathbb{D}(k) \tag{4.1.1}$$

is called the system of *hypergeometric differential equations* with spectral parameter $\lambda \in \mathfrak{h}^*$ and multiplicity parameter $k \in K$ associated with the root system R. Here $u = u(h)$ is some scalar valued function depending on the variable h in (an open subset of) $H^{\text{reg}} = \{h \in H;\ \Delta(h) \neq 0\}$, and $\gamma(D, k, \lambda)$ denotes the value at $\lambda \in \mathfrak{h}^*$ of the polynomial $\gamma(k)(D)$, which is the image under the Harish-Chandra isomorphism $\gamma(k)\colon \mathbb{D}(k) \to S\mathfrak{h}^W$ of the differential operator $D \in \mathbb{D}(k)$.

In view of Chevalley's theorem (stating that $S\mathfrak{h}^W$ is itself a polynomial algebra) the system of hypergeometric differential equations (4.1.1) is just the simultaneous eigenvalue problem for the commuting algebra $\mathbb{D}(k)$ of differential operators on H. We write $U\mathfrak{h}$ for the translation invariant differential operators on H and $S\mathfrak{h}$ for the polynomial functions on $\mathfrak{h}^*$ (but clearly $U\mathfrak{h} \cong S\mathfrak{h}$ are canonically isomorphic). An element $\partial_q \in U\mathfrak{h}$ is called harmonic if $\partial_q(p) = 0$ for all W-invariant polynomials p only with $p(0) = 0$. The harmonics in $U\mathfrak{h}$ are denoted by $H\mathfrak{h}$. The dimension d of $H\mathfrak{h}$ is equal to the order $|W|$ of the Weyl group W. A well known result of Chevalley shows that

$$U\mathfrak{h} \cong H\mathfrak{h} \otimes U\mathfrak{h}^W. \tag{4.1.2}$$

For $\lambda \in \mathfrak{h}^*$, $k \in K$ we write

$$I(\lambda, k) = \{P \in \mathbb{D}(k);\ \gamma(P, k, \lambda) = 0\} \tag{4.1.3}$$

and with this notation the system (4.1.1) gets the form $P(u) = 0$, $\forall P \in I(\lambda, k)$.

Proposition 4.1.2. *We have an isomorphism of left $\mathfrak{R}$-modules*

$$\mathfrak{R} \otimes U\mathfrak{h} \cong \{\mathfrak{R} \otimes H\mathfrak{h}\} \oplus \{\mathfrak{R} \otimes H\mathfrak{h} \cdot I(\lambda, k)\}. \tag{4.1.4}$$

Proof. To simplify the notation we write U, H, and I instead of $U\mathfrak{h}$, $H\mathfrak{h}$, and $I(\lambda, k)$. Let U^j denote the homogeneous elements in U of degree j, and $U_j = \bigoplus_{i \leq j} U^i$ the elements of degree $\leq j$. Also write $H^j = H \cap U^j$, $H_j = H \cap U_j$, and $I_j = I \cap \{\mathfrak{R} \otimes U_j\}$. We prove by induction on j that

$$\mathfrak{R} \otimes U_j \cong \{\mathfrak{R} \otimes H_j\} \oplus \{\sum_{i \geq 1} \mathfrak{R} \otimes H_{j-1} \cdot I_i\} \tag{4.1.5}$$

as left $\mathfrak{R}$-modules. The case $j = 0$ is clear. Now suppose $j \geq 1$. By (4.1.2) we can write $\partial_q \in U^j$ as

$$\partial_q = \sum_i \partial_{q_i} \partial_{p_i}$$

with $\partial_{q_i} \in H^{j-j_i}$ and $\partial_{p_i} \in U^{j_i}$ Weyl group invariants. By Corollary 1.2.8 and Theorem 1.3.12 there exists $P_i \in I_{j_i}$ with

$$P_i - (\partial_{p_i} - p_i(\lambda)) \in \mathfrak{R} \otimes U_{j_i - 1}.$$

Since $\partial_{q_i}(\mathfrak{R} \otimes U_{j_i-1}) \subset \mathfrak{R} \otimes U_{j-1}$ we get

$$\partial_q - \{\sum_i p_i(\lambda)\partial_{q_i} + \sum_i \partial_{q_i} \cdot P_i\} \in \mathfrak{R} \otimes U_{j-1}$$

and using the induction hypothesis we have

$$\mathfrak{R} \otimes U_j \cong \{\mathfrak{R} \otimes H_j\} + \{\sum_{i \geq 1} \mathfrak{R} \otimes H_{j-1} \cdot I_i\}. \tag{4.1.6}$$

It remains to be shown that the sum is direct. This follows again by induction on j by taking the jth order symbol in (4.1.6) and using (4.1.2). $\square$

Corollary 4.1.3. *Let* $J(\lambda,k) = \mathfrak{R}\otimes U\mathfrak{h}\cdot I(\lambda,k)$ *be the left ideal of* $\mathfrak{R}\otimes U\mathfrak{h}$ *generated by* $I(\lambda,k)$. *Then we have a direct sum decomposition of left* $\mathfrak{R}$-*modules*

$$\mathfrak{R}\otimes U\mathfrak{h} \cong \{\mathfrak{R}\otimes H\mathfrak{h}\} \oplus J(\lambda,k). \tag{4.1.7}$$

Definition 4.1.4. Fix a basis $\{q_1,\ldots,q_d\}$ of homogeneous harmonics with $q_1 \equiv 1$ and $\deg(q_i) \leq \deg(q_{i+1})$. The map

$$A\colon\ \mathfrak{R}\otimes U\mathfrak{h} \to gl(d,\mathfrak{R}), \tag{4.1.8}$$

defined by the requirement (use (4.1.7))

$$P\circ\partial_{q_i} + \sum_{j=1}^{d} A_{ij}(P)\partial_{q_j} \in J(\lambda,k), \tag{4.1.9}$$

is a morphism of left $\mathfrak{R}$-modules.

Proposition 4.1.5. *For all* $\xi,\eta\in\mathfrak{h}$ *we have*

$$[\partial_\xi + A(\partial_\xi), \partial_\eta + A(\partial_\eta)] = 0. \tag{4.1.10}$$

Proof. Using the Leibniz rule we get

$$A(\partial_\xi\circ P) + A(P)A(\partial_\xi) = \partial_\xi(A(P)) \tag{4.1.11}$$

for $\xi\in\mathfrak{h}$ and $P\in\mathbb{R}\otimes U\mathfrak{h}$. Hence

$$\begin{aligned}
&[\partial_\xi{+}A(\partial_\xi), \partial_\eta{+}A(\partial_\eta)] \\
&\quad = [\partial_\xi, A(\partial_\eta)] + [A(\partial_\xi),\partial_\eta] + [A(\partial_\xi), A(\partial_\eta)] \\
&\quad = \partial_\xi(A(\partial_\eta)) - \partial_\eta(A(\partial_\xi)) + [A(\partial_\xi), A(\partial_\eta)] \\
&\quad = A([\partial_\xi,\partial_\eta]) + A(\partial_\eta)A(\partial_\xi) - A(\partial_\xi)A(\partial_\eta) + [A(\partial_\xi), A(\partial_\eta)] = 0. \quad \square
\end{aligned}$$

Definition 4.1.6. The system of first-order differential equations

$$(4.1.12) \qquad (\partial_\xi + A(\partial_\xi))U = 0 \qquad \forall \xi \in \mathfrak{h}$$

with $U = (u_1, \dots, u_d)^t$ is called the *matrix form* of the hypergeometric differential equations (4.1.1).

Proposition 4.1.7. *If u is a solution of* (4.1.1) *then $U = (\partial_{q_1}u, \dots, \partial_{q_d}u)^t$ is a solution of* (4.1.12). *Conversely, if $U = (u_1, \dots, u_d)^t$ is a solution of* (4.1.12) *then $u = u_1$ is a solution of* (4.1.1) *and $u_j = \partial_{q_j}u_1$.*

Proof. Suppose u is a solution of (4.1.1), i.e., $P(u) = 0\ \forall P \in J(\lambda, k)$. If we write $U = (\partial_{q_1}u, \dots, \partial_{q_d}u)^t$ then it follows from (4.1.9) that $(P + A(P))(U) = 0\ \forall P \in \mathfrak{R} \otimes U\mathfrak{h}$. In particular U is a solution of (4.1.12).

Now suppose $U = (u_1, \dots, u_d)^t$ is a solution of (4.1.12). Using (4.1.11) and induction on the order of differential operators it is easy to see that $(P + A(P))(U) = 0\ \forall P \in \mathfrak{R} \otimes U\mathfrak{h}$. Since $A_{1j}(P) = 0$ for $P \in J(\lambda, k)$ we get $P(u_1) = 0\ \forall P \in J(\lambda, k)$.

Moreover $u_j = \partial_{q_j}(u_1)$ because $A_{1j}(\partial_{q_i}) = -\delta_{ij}$. $\square$

Corollary 4.1.8. *Locally on H^{reg} the solution space of the system of hypergeometric differential equations* (4.1.1) *has dimension $d = |W|$ and consists of holomorphic functions. More precisely a local solution u of* (4.1.1) *near a point $h_0 \in H^{\mathrm{reg}}$ is completely determined by its harmonic derivatives*

$$\partial_{q_1}(u)(h_0) = u(h_0), \dots, \partial_{q_d}(u)(h_0)$$

at the point h_0, which can be freely prescribed.

4.2. Regular singular points at infinity

The central subgroup C of H is defined by

$$(4.2.1) \qquad C = \{h \in H;\ h^\alpha = 1\ \ \forall \alpha \in R\}$$

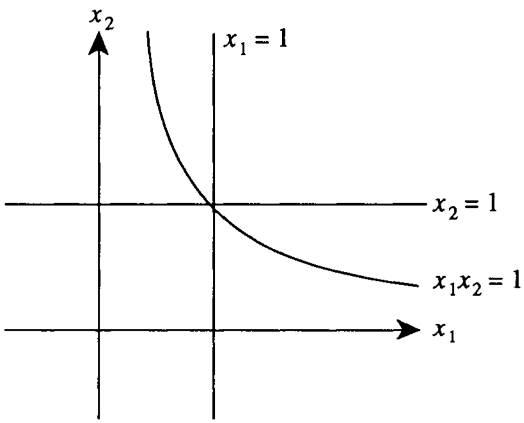
with the notation $h^\alpha = e^{\alpha(\log h)}$, and the torus H/C has rational character lattice equal to the root lattice Q of R.

Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots in R_+, and put $x_j = e^{-\alpha_j}$ considered as function on H or H/C, $j = 1, \dots, n$. The map

$$x = (x_1, \dots, x_n)\colon\ H/C \to \mathbb{C}^n \tag{4.2.2}$$

is injective with image $(\mathbb{C}^\times)^n$. Hence (4.2.2) defines a partial compactification of H/C, and using the action of the Weyl group W this can be extended to a smooth global compactification of H/C. This is nothing but the toroidal compactification corresponding to the decomposition of $\mathfrak{a}$ into Weyl chambers (see for example [11, 55]). Note that the positive chamber A_+ is mapped by (4.2.2) onto $(0,1)^n$.

Example 4.2.1. For R of type A_2 the image of (4.2.2) has the picture



The point $(1,1)$ is the image of the identity element, and the curves $x_1 = 1$, $x_2 = 1$, $x_1x_2 = 1$ are the image of $\{h \in H;\ \Delta(h) = 0\}$.

Let $\{\eta_1, \dots, \eta_n\}$ be a basis of $\mathfrak{a}$ such that $\alpha_i(\eta_j) = \delta_{ij}$. In the coordinates (4.2.2) the differentiation ∂_{η_j} becomes $-x_j\frac{\partial}{\partial x_j}$ for $j = 1, \dots, n$ and the matrix form of the hypergeometric differential equations (4.1.12) becomes

$$x_j\frac{\partial U}{\partial x_j} = A_jU \qquad \text{for } j = 1, \dots, n. \tag{4.2.3}$$

It is important to note that $A_j(x) = A(\partial_{\eta_j}) \in gl(d)$ is a matrix whose entries are convergent power series on the polydisc $\{x \in \mathbb{C}^n;\ |x_j| < 1,\ j =$

$1, \ldots, n\}$. This means that the system (4.2.3) has regular singular points along the divisor $x_1 x_2 \ldots x_n = 0$ and is in normal form.

With the notation $a^\mu = e^{\mu(\log a)}$ for $\mu \in \mathfrak{h}^*$, $a \in A$ the *Harish-Chandra series* $\Phi(\lambda, k; a)$ is defined by

$$\Phi(\lambda, k; a) = \sum_{\kappa \leq 0} \Gamma_\kappa(\lambda, k) a^{\lambda - \rho(k) + \kappa} \tag{4.2.4}$$

with $\Gamma_\kappa(\lambda, k)$ defined by the recurrence relations

$$\Gamma_0(\lambda, k) = 1, \tag{4.2.5}$$

$$\begin{aligned} &-(2\lambda+\kappa, \kappa)\Gamma_\kappa(\lambda, k) \\ &\qquad = 2 \sum_{\alpha>0} k_\alpha \sum_{j \geq 1} (\lambda - \rho(k) + \kappa + j\alpha, \alpha) \Gamma_{\kappa + j\alpha}(\lambda, k). \end{aligned} \tag{4.2.6}$$

Note that these recurrence relations can be solved uniquely if

$$2(\lambda, \kappa) + (\kappa, \kappa) \neq 0 \qquad \forall \kappa < 0. \tag{4.2.7}$$

Lemma 4.2.2. *Let $U \subset \mathfrak{h}^* \times K$ be a bounded domain and $d(\lambda, k)$ a holomorphic function on $\overline{U}$ such that the function $(\lambda, k) \mapsto d(\lambda, k)\Gamma_\kappa(\lambda, k)$ is holomorphic on $\overline{U}$ for all $\kappa \leq 0$. (This means that $d(\lambda, k)$ has to be divisible by those linear functions $\lambda \mapsto (2\lambda+\kappa, \kappa)$ for which the right-hand side of (4.2.6) is not divisible by $\lambda \mapsto (2\lambda+\kappa, \kappa)$ and whose zero locus intersects $\overline{U}$.) For $a \in A_+$ fixed there exists a constant $M = M_{U,a} > 0$ such that*

$$|d(\lambda, k)\Gamma_\kappa(\lambda, k)| \leq M a^\kappa \qquad \forall \kappa \leq 0,\ (\lambda, k) \in \overline{U}. \tag{4.2.8}$$

Proof. With $\alpha_1, \ldots, \alpha_n \in R_+$ simple and $\mu = \sum_1^n m_i \alpha_i \in \mathfrak{a}^*$ consider $N(\mu) = \sum_1^n |m_i|$ as a norm on $\mathfrak{a}^*$. Choose $c_1 > 0$ such that

$$|(\lambda - \rho(k) + \kappa, \alpha)| \leq c_1(1 + N(\kappa))$$

$\forall (\lambda, k) \in \overline{U}$, $\kappa \leq 0$, $\alpha \in R_+$. Choose $N_1 \in \mathbb{N}$ and $c_2 > 0$ such that

$$|(2\lambda + \kappa, \kappa)| \geq c_2 N(\kappa)^2$$

$\forall \lambda \in \mathfrak{h}^*$: $\exists k \in K$ with $(\lambda, k) \in \overline{U}$, $\forall \kappa \leq 0$ with $N(\kappa) \geq N_1$. Hence if $\kappa \leq 0$ with $N(\kappa) \geq N_1$, we get (with $c = 2c_1c_2^{-1}$)

$$|d(\lambda,k)\Gamma_\kappa(\lambda,k)| \leq cN(\kappa)^{-1} \sum_{\alpha>0} |k_\alpha| \sum_{j\geq 1} |d(\lambda,k)\Gamma_{\kappa+j\alpha}(\lambda,k)|, \tag{4.2.9}$$

$\forall (\lambda, k) \in \overline{U}$. Choose $N_2 \in \mathbb{N}$ such that

$$c \sum_{\alpha>0} |k_\alpha| \sum_{j\geq 1} a^{j\alpha} \leq N_2$$

$\forall k \in K$: $\exists \lambda \in \mathfrak{h}^*$ with $(\lambda, k) \in \overline{U}$. Let $N = \max(N_1, N_2) \in \mathbb{N}$. Finally choose $M > 0$ such that

$$|d(\lambda,k)\Gamma_\kappa(\lambda,k)| \leq Ma^\kappa$$

$\forall (\lambda, k) \in \overline{U}$, and $\forall \kappa \leq 0$ with $N(\kappa) \leq N$. We now prove (4.2.8) by induction on $N(\kappa)$. Let $\kappa \leq 0$ with $N(\kappa) > N$ and suppose (4.2.8) is true for all $\mu \leq 0$ with $N(\mu) < N(\kappa)$. Using (4.2.9) we get

$$\begin{aligned} |d(\lambda,k)\Gamma_\kappa(\lambda,k)| &\leq cN(\kappa)^{-1} \sum_{\alpha>0} |k_\alpha| \sum_{j\geq 1} Ma^{j\alpha+\kappa} \\ &\leq N(\kappa)^{-1} MNa^\kappa \leq Ma^\kappa. \quad \square \end{aligned}$$

Corollary 4.2.3. *With the above notation the series*

$$\sum_{\kappa\leq 0} d(\lambda,k)\Gamma_\kappa(\lambda,k)a^{\lambda-\rho(k)+\kappa} \tag{4.2.10}$$

converges absolutely and uniformly on $\overline{U} \times a\overline{A_+}$. Hence it defines an analytic function on $U \times A_+$.

Corollary 4.2.4. *For $\lambda \in \mathfrak{h}^*$ satisfying (4.2.7) and $k \in K$ arbitrary the Harish-Chandra series (4.2.4) converges to an analytic function on A_+. As a function of the spectral parameter $\lambda \in \mathfrak{h}^*$ it is meromorphic with simple poles along hyperplanes of the form $\{(2\lambda+\kappa, \kappa) = 0\}$ for $\kappa < 0$. Moreover for $\lambda_0 \in \mathfrak{h}^*$ with $(2\lambda_0+\kappa, \kappa) = 0$ for precisely one $\kappa = \kappa_0 < 0$ we have*

$$\begin{aligned} &\{(2\lambda+\kappa_0, \kappa_0)\Phi(\lambda, k; a)\}_{\lambda=\lambda_0} \\ &= \{(2\lambda+\kappa_0, \kappa_0)\Gamma_{\kappa_0}(\lambda,k)\}_{\lambda=\lambda_0} \cdot \Phi(\lambda_0+\kappa_0, k; a). \end{aligned} \tag{4.2.11}$$

Proof. Take $d(\lambda, k) = (2\lambda+\kappa_0, \kappa_0)$ in the previous corollary and U a small neighborhood of $(\lambda_0, k) \in \mathfrak{h}^* \times K$ (some $k \in K$). Now the recurrence relations (4.2.6) for the coefficients of the Harish-Chandra series (4.2.4) were derived from the differential equation

$$ML(k)\Phi(\lambda, k; a) = (\lambda, \lambda)\Phi(\lambda, k; a). \tag{4.2.12}$$

Observe that with our choice of $d(\lambda, k)$ we have

$$\{d(\lambda, k)\Gamma_\kappa(\lambda, k)\}_{\lambda=\lambda_0} = 0 \qquad \forall\kappa \leq 0 \text{ with } \kappa \not\leq \kappa_0.$$

Hence $\{d(\lambda, k)\Phi(\lambda, k; a)\}_{\lambda=\lambda_0}$ is a multiple of $\Phi(\lambda_0+\kappa_0, k; a)$. □

Since the algebra $\mathbb{D}(k)$ is commutative it is immediate that the Harish-Chandra series (4.2.4) is in fact a solution of the full system (4.1.1) of hypergeometric differential equations. With the equivalence of (4.1.1) and (4.2.3) in mind we can therefore say that the exponents at infinity of (4.1.1) are of the form

$$w\lambda - \rho(k) \qquad \text{for } w \in W \tag{4.2.13}$$

and the Harish-Chandra series $\Phi(w\lambda; k; a)$ are the series solutions of (4.1.1) with leading exponent $w\lambda - \rho(k)$, $w \in W$.

Proposition 4.2.5. *The Harish-Chandra series $\Phi(\lambda, k; a)$ is a meromorphic function on $\mathfrak{h}^* \times K \times A_+$ with simple poles along hyperplanes of the form*

$$(\lambda, \alpha^\vee) = j \qquad \text{for some } \alpha \in R_+, \text{ some } j \in \mathbb{N} = \{1, 2, \dots\}. \tag{4.2.14}$$

Proof. The fact that for certain $\lambda \in \mathfrak{h}^*$ (cf. (4.2.7)) the recurrence relations break down is the phenomenon of logarithmic terms caused by the differences of exponents being integers. In our notation this amounts to

$$\lambda - w\lambda \in Q \qquad \text{for some } w \in W,\ w \neq 1. \tag{4.2.15}$$

However, the only $w \in W$ in (4.2.15) which matter are those for which (4.2.15) is a codimension one condition on $\lambda \in \mathfrak{h}^*$, i.e., $w = r_\alpha$ for some $\alpha \in R$. Hence the condition (4.2.15) becomes

$$(\lambda, \alpha^\vee) \in \mathbb{Z} \qquad \text{for some } \alpha \in R, \tag{4.2.16}$$

and combined with (4.2.7) the proposition follows. □

Apparently for those $\kappa < 0$ not of the form $\kappa = -j\alpha$ for some $\alpha \in R_+$ and $j \in \mathbb{N}$ the right-hand side of (4.2.6) is divisible by the linear function $(2\lambda+\kappa, \kappa)$.

Corollary 4.2.6. *If $(\lambda, \alpha^\vee) \notin \mathbb{Z}$ for all $\alpha \in R$ then the Harish-Chandra series*

$$\Phi(w\lambda, k; a) \qquad \text{for } w \in W \tag{4.2.17}$$

are a basis for the solution space of the system of hypergeometric differential equations (4.1.1) *on* A_+.

Proof. This follows from Corollary 4.1.8 and the above since Harish-Chandra series with different leading exponents are clearly linearly independent over $\mathbb{C}$. □

4.3. The monodromy representation

The system of hypergeometric differential equations (4.1.1) is invariant under W, and hence can be viewed as a system on the space $W\backslash H \cong \mathbb{C}^n$ (cf. Remark 1.3.15). As such it has singular points at infinity and along the discriminant $D = 0$, where $D(z) = \Delta(h)^2$ with $z_j = M(\lambda_j)$. We start by describing the fundamental group of the regular orbit space $W\backslash H^{\text{reg}} \cong \mathbb{C}^n\backslash\{D{=}0\}$. Fix a base point $a_0 \in A_+$ and let $z_0 = Wa_0$ the corresponding point in $\mathbb{C}^n$.

Definition 4.3.1. Let $\{\alpha_1, \dots, \alpha_n\}$ be the basis of simple roots for $R_+^0 = R_+\backslash\frac{1}{2}R$, and let $r_j \in W$ denote the corresponding simple reflections. For $j = 1, \dots, n$ define curves G_j, L_j in H^{reg} by

$$G_j(t) = \exp\{(1-t)\log a_0 + t r_j \log a_0 + \varepsilon(t) 2\pi i \alpha_j^\vee\} \tag{4.3.1}$$

$$L_j(t) = \exp\{\log a_0 + 2\pi i t \alpha_j^\vee\}, \tag{4.3.2}$$

where $t \in [0,1]$ and ε: $[0,1] \to [0, \frac{1}{2})$ a continuous function with $\varepsilon(0) = \varepsilon(1) = 0$ and $\varepsilon(\frac{1}{2}) > 0$ (for example take $\varepsilon(t) = \frac{1}{4}\sin \pi t$).

Note that $\Pi_1(H, a_0) \cong 2\pi i Q^\vee$ is a free abelian group on the generators $L_1, \dots, L_n$. Write $g_1, \dots, g_n, l_1, \dots, l_n \in \Pi_1(W\backslash H^{\text{reg}}, z_0)$ for the corresponding closed curves in $W\backslash H^{\text{reg}}$ with base point z_0.

Theorem 4.3.2. *The fundamental group $\Pi_1(W\backslash H^{\text{reg}}, z_0)$ has a presentation with generators $g_1, \dots, g_n, l_1, \dots, l_n$, and relations*

$$g_i g_j g_i \dots = g_j g_i g_j \quad 1 \le i \ne j \le n,\ m_{ij} \text{ factors on both sides} \tag{4.3.3}$$

$$l_i l_j = l_j l_i \quad 1 \le i, j \le n \tag{4.3.4}$$

$$g_i l_j = l_j l_i^r g_i l_i^{-r} \quad 1 \le i \ne j \le n,\ n_{ij} = -2r \text{ even} \tag{4.3.5}$$

$$g_i l_j = l_j l_i^{r+1} g_i l_i^{-r} \quad 1 \le i \ne j \le n,\ n_{ij} = -(2r+1) \text{ odd,} \tag{4.3.6}$$

where $m_{ij} = (\alpha_i, \alpha_j^\vee)(\alpha_i^\vee, \alpha_j)$ is the order of $r_i r_j \in W$ and $n_{ij} = (\alpha_i, \alpha_j^\vee)$ are the Cartan integers.

Remark 4.3.3. For $x \in Q^\vee$ of the form $x = m_1\alpha_1^\vee + \dots + m_n\alpha_n^\vee$ we write $l_x = l_1^{m_1} \dots l_n^{m_n} \in \Pi_1(W\backslash H^{\text{reg}}, z_0)$. Then it is easy to see that

$$l_x l_y = l_y l_x \quad \text{for all } x, y \in Q^\vee \tag{4.3.7}$$

$$g_j l_x = l_x g_j \quad \text{if } (x, \alpha_j) = 0 \tag{4.3.8}$$

$$g_j l_{r_j(x)} = l_x g_j \quad \text{if } (x, \alpha_j) = 1. \tag{4.3.9}$$

Remark 4.3.4. Suppose R is irreducible with highest root α_0. If $r_{\alpha_0} = r_{i_1} \dots r_{i_p} \in W$ is a reduced expression then let $g_0 \in \Pi_1(W\backslash H^{\text{reg}}, z_0)$ be defined by

$$l_{\alpha_0^\vee} = g_0 g_{i_1} \cdots g_{i_p}. \tag{4.3.10}$$

One can show that $\Pi_1(W\backslash H^{\text{reg}}, z_0)$ has another presentation with generators $g_0, g_1, \dots, g_n$ and relations

(4.3.11)

$$g_i g_j g_i \dots = g_j g_i g_j \dots \quad 0 \le i \ne j \le n,\ m_{ij} \text{ factors on both sides,}$$

where $m_{ij} = (\alpha_i, \alpha_j^\vee)(\alpha_i^\vee, \alpha_j)$ as before. Note that the situation is similar as for the affine Weyl group, which on the one hand has a Coxeter presentation on $(n+1)$ generators and on the other hand is a semidirect product of the finite Weyl group and its translation lattice $2\pi i Q^\vee$.

We do not prove the above results, but instead make some references to the literature. The presentation (4.3.11) is due to Nguyên Viêt Dung and was inspired by the work of Brieskorn [54, 8]. Theorem 4.3.2 is due to Van der Lek and Looijenga. See [44] for a description of the results and [45] for the proofs. The work of Van der Lek was inspired by Deligne's paper on braid groups [14].

Proposition 4.3.5. *If $\alpha_j(Q^\vee) = \mathbb{Z}$ then $l_j g_j^{-1}$ and g_j are conjugate inside $\Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$. If R^0 is irreducible and α_j a long simple root then $l_j g_j^{-1}$ and g_0 are conjugate inside $\Pi_1(W\backslash H^{\mathrm{reg}}, z_o)$.*

Proof. Suppose $(x, \alpha_j) = 1$ for some $x \in Q^\vee$. Using (4.3.9) we get

$$l_x g_j l_{-x} = l_x l_{-r_j(x)} g_j^{-1} = l_j g_j^{-1},$$

which proves the first statement. For the second statement observe that we can choose a sequence $j_1 = j, j_2, \dots, j_p \in \{1, \dots, n\}$ with

$$\begin{aligned}
&\beta_i^\vee = \alpha_{j_1}^\vee + \dots + \alpha_{j_i}^\vee \\
&(\beta_i^\vee, \alpha_{j_{i+1}}) = -1 \iff r_{j_{i+1}}(\beta_i^\vee) = \beta_{i+1}^\vee \\
&\beta_p^\vee = \alpha_0^\vee .
\end{aligned}$$

Now $r_{\beta_{i+1}} = r_{j_{i+1}} r_{\beta_i} r_{j_{i+1}}$ and $l(r_{\beta_{i+1}}) = l(r_{\beta_i}) + 2$. Hence the expression $r_{\alpha_0} = r_{j_p} \dots r_{j_2} r_{j_1} r_{j_2} \dots r_{j_p}$ is reduced, and using (4.3.9) it is easily seen that

$$g_0 = l_{\beta_p^\vee} g_{j_p}^{-1} \dots g_{j_1}^{-1} \dots g_{j_p}^{-1} = g_{j_p} \dots g_{j_2} (l_j g_j^{-1}) g_{j_2}^{-1} \dots g_{j_p}^{-1},$$

which proves the second statement. □

Denote by $V(\lambda, k)$ the local solution space of (4.1.1) around the point $a_0 \in A_+$ or equivalently on A_+ by analytic continuation. We write

$$M(\lambda, k)\colon \Pi_1(W\backslash H^{\mathrm{reg}}, z_0) \to GL(V(\lambda, k)) \tag{4.3.12}$$

for the monodromy representation. Assuming that $\lambda \in \mathfrak{h}^*$ satisfies the condition

$$(\lambda, \alpha^\vee) \notin \mathbb{Z} \qquad \forall \alpha \in R \tag{4.3.13}$$

it follows from Corollary 4.2.6 that the Harish-Chandra series $\Phi(w\lambda, k; a)$, $w \in W$ are a basis for the solution space $V(\lambda, k)$ and

$$M(\lambda, k)(l_x)\Phi(w\lambda, k; a) = e^{2\pi i(w\lambda - \rho(k), x)}\Phi(w\lambda, k; a), \tag{4.3.14}$$

which implies that the Harish-Chandra series $\Phi(w\lambda, k; a)$, $w \in W$ are the up to a constant unique simultaneous eigenvectors for the monodromy operators $M(\lambda, k)(l_x)$, $x \in Q^\vee$. Using (4.3.8) it is clear that the two-dimensional subspace

$$\mathrm{span}\{\Phi(w\lambda, k; a), \Phi(r_j w\lambda, k; a)\} \tag{4.3.15}$$

of $V(\lambda, k)$ is invariant under the monodromy operator $M(\lambda, k)(g_j)$.

Theorem 4.3.6. *If $\lambda \in \mathfrak{h}^*$ satisfies* (4.3.13) *then the solution*

$$\tilde{c}(w\lambda, k)\Phi(w\lambda, k; a) + \tilde{c}(r_j w\lambda, k)\Phi(r_j w\lambda, k; a) \tag{4.3.16}$$

is an eigenvector for the monodromy operator $M(\lambda, k)(g_j)$ with the eigenvalue 1.

Proof. Observe that the system (4.1.1) can be brought in the equivalent form (4.2.3) in which it has regular singular points at infinity. Taking boundary values along hyperplanes at infinity is an operation that commutes with monodromy along these hyperplanes. This allows induction on the dimension n, and the situation ultimately reduces to rank one. In this case $R_+ = \{\frac{1}{2}\alpha, \alpha\}$ and the differential equation (4.1.1) becomes the ordinary hypergeometric differential equation with solution $F(a, b, c; z)$ with $a = (\lambda+\rho(k), \alpha^\vee)$, $b = (-\lambda+\rho(k), \alpha^\vee)$, $c = \frac{1}{2}+k_{\frac{1}{2}\alpha}+k_\alpha$, and $z = \frac{1}{2}-\frac{1}{4}(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})$. The theorem follows in this case from Kummer's identity

$$F(a, b, c; z) = c(\lambda, k)\Phi(\lambda, k; z) + c(-\lambda, k)\Phi(-\lambda, k; z) \tag{4.3.17}$$

by analytic continuation of z along the negative real axis. Here

$$c(\lambda, k) = \frac{2^{2a}\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}$$
$$\Phi(\lambda, k; z) = 2^{-2a}(-z)^{-a}F(a, 1+a-c, 1+a-b; z^{-1})$$

and the same expressions for $c(-\lambda, k)$ and $\Phi(-\lambda, k; z)$ with a and b interchanged in these formulas. □

Corollary 4.3.7. *For $k \in K$ let $k' \in K$ be defined by*

$$k'_\alpha = 1-k_\alpha \text{for } \alpha \in R^0, \quad k'_\alpha = -k_\alpha \text{for } \alpha \in R \cap \tfrac{1}{2}R. \tag{4.3.18}$$

If $\lambda \in \mathfrak{h}^*$ satisfies (4.3.13) *then the solution*

$$\tilde{c}(w\lambda, k')\Phi(w\lambda, k; a) + \tilde{c}(r_j w\lambda, k')\Phi(r_j w\lambda, k; a) \tag{4.3.19}$$

is an eigenvector for the monodromy operator $M(\lambda,k)(g_j)$ *with eigenvalue* $-e^{2\pi i(k_{\frac{1}{2}\alpha_j}+k_{\alpha_j})}$.

Proof. Since the right-hand side of (2.1.9) is invariant under the substitution $k\mapsto k'$ we get

$$\delta(k;a)^{\frac{1}{2}}\Phi(w\lambda,k;a)=\delta(k';a)^{\frac{1}{2}}\Phi(w\lambda,k';a) \tag{4.3.20}$$

and since

$$\delta(k';a)^{-\frac{1}{2}}\delta(k;a)^{\frac{1}{2}}=\Delta(a)\delta(k;a) \tag{4.3.21}$$

transforms by the factor $-e^{-2\pi i(k_{\frac{1}{2}\alpha_j}+k_{\alpha_j})}$ under monodromy along the loop g_j, the result follows from Theorem 4.3.6. $\square$

Corollary 4.3.8. *The monodromy operators given by* $M(\lambda,k)(g_j)$ *and* $M(\lambda,k)(l_jg_j^{-1})$ *satisfy in End*$(V(\lambda,k))$ *the quadratic relations*

$$(M(\lambda,k)(g_j)-1)(M(\lambda,k)(g_j)+e^{2\pi i(k_{\frac{1}{2}\alpha_j}+k_{\alpha_j})})=0 \tag{4.3.22}$$

$$(M(\lambda,k)(l_jg_j^{-1})-1)(M(\lambda,k)(l_jg_j^{-1})+e^{2\pi ik_{\alpha_j}})=0. \tag{4.3.23}$$

In particular the monodromy representation (4.3.12) *of the affine braid group* $\Pi_1(W\backslash H^{\mathrm{reg}},z_0)$ *factors through a representation of the affine Hecke algebra.*

Proof. Relation (4.3.22) is immediate from Theorem 4.3.6 and Corollary 4.3.7. Relation (4.3.23) can be derived along the same lines by working in Theorem 4.3.6 and Corollary 4.3.7 with the loop $l_jg_j^{-1}$ instead of g_j. Note that in the rank one reduction the loop $l_jg_j^{-1}$ goes once around the point $z=1$ in the negative direction whereas the loop g_j goes once around $z=0$ in the negative direction. The exponents of the hypergeometric function $F(a,b,c;z)$ at the point $z=0$ are $0,1-c$ and at the point $z=1$ are $0,c-a-b$. With the notation as in the proof of Theorem 4.3.6 we have $1-c=\frac{1}{2}-k_{\frac{1}{2}\alpha}-k_\alpha$ and $c-a-b=\frac{1}{2}-k_\alpha$ and (4.3.23) follows. Note that in case $\alpha_j(Q^\vee)=\mathbb{Z}$ we have $k_{\frac{1}{2}\alpha_j}=0$ and (4.3.22) and (4.3.23) are compatible in accordance with Proposition 4.3.5. The last statement that the monodromy representation factors through a representation of the affine Hecke algebra follows from Proposition 4.3.5 and the definition of the Hecke algebra associated with a Coxeter group (see [7]). $\square$

Corollary 4.3.9. *If $\lambda \in \mathfrak{h}^*$ satisfies* (4.3.13) *then the solution*

$$\widetilde{F}(\lambda, k; a) = \sum_{w \in W} \widetilde{c}(w\lambda, k)\Phi(w\lambda, k; a) \tag{4.3.24}$$

is a simultaneous eigenvector for the monodromy operators $M(\lambda, k)(g_j)$ with eigenvalue 1 for $j = 1, \dots, n$. In other words the function (4.3.24) *has an analytic continuation from A_+ to a single-valued W-invariant function on $U \cap H^{\mathrm{reg}}$, where U is a W-invariant tubular neighborhood of A in H.*

Proof. Clear from Theorem 4.3.6. □

Proposition 4.3.10. *Suppose $\lambda \in \mathfrak{h}^*$ satisfies both* (4.3.13) *and*

$$(\lambda, \alpha^\vee) + \tfrac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha \notin \mathbb{Z} \qquad \forall \alpha \in R. \tag{4.3.25}$$

Then the monodromy representation (4.3.12) *is irreducible.*

Proof. If both (4.3.13) and (4.3.25) hold then $\widetilde{c}(w\lambda, k) \neq 0$ for all $w \in W$. Now it is clear from Theorem 4.3.6 that the two-dimensional representation on the space (4.3.15) of the group generated by $M(\lambda, k)(l_x)$, $x \in Q^\vee$ and $M(\lambda, k)(g_j)$ is irreducible. From this it easily follows that the full representation (4.3.12) is irreducible. □

Theorem 4.3.11. *The system* (4.1.1) *has regular singular points along the discriminant $D = 0$. Moreover the exponents along the image of the subtorus $\{a \in A;\ a^\alpha = 1\}$ in $W\backslash H \cong \mathbb{C}^n$ are of the form 0 and $\frac{1}{2} - k_{\frac{1}{2}\alpha} - k_\alpha$ both with multiplicity equal to $\frac{1}{2}d$, $d = |W|$.*

Proof. In case $k_\alpha = 0\ \forall \alpha \in R$ this is obvious. Indeed viewing the system on H (rather than $W\backslash H$) the points $\{h \in H;\ \Delta(h) = 0\}$ are just regular points, and hence on $W\backslash H$ the points $\{D{=}0\}$ become regular singular points. Observe that for $l \in \mathbb{Z}_- \cdot B$ the lowering operator $G_-(l)$ of Theorem 3.4.3 lies in the Weyl algebra $\mathbb{C}[z_1, \dots, z_n, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}]$ and satisfies

$$G_-(l)\widetilde{F}(\lambda, k; a) = \widetilde{F}(\lambda, k{+}l; a). \tag{4.3.26}$$

Hence for $\lambda \in \mathfrak{h}^*$ satisfying (4.3.13) and (4.3.25) we conclude from Proposition (4.3.25) that

$$G_-(l)\colon\ V(\lambda, k) \to V(\lambda, k{+}l)$$

is a linear isomorphism. In particular if the system is regular singular along $D = 0$ for some (λ, k) then it remains regular singular along $D = 0$ for $(\lambda, k+l)$. The conclusion is that the system (4.1.1) is regular singular along $D = 0$ for all $(\lambda, l) \in \mathfrak{h}^* \times K$ with λ satisfying (4.3.13) and $l \in \mathbb{Z}_- \cdot B$. However, this is a Zariski-dense subset of $\mathfrak{h}^* \times K$, and since the coefficients of the system (4.1.1) are polynomial in $(\lambda, k) \in \mathfrak{h}^* \times K$ the first statement follows. The second statement follows from the single differential equation

$$ML(k)u = (\lambda, \lambda)u$$

contained in (4.1.1). $\square$

Remark 4.3.12. If $\alpha(Q^\vee) = 2\mathbb{Z}$ for some $\alpha \in R$ then (the image under the map $H \to W\backslash H \cong \mathbb{C}^n$ of) the variety $\{h \in H;\ h^\alpha = 1\}$ has two connected components, $\{h \in H;\ h^{\frac{1}{2}\alpha} = 1\}$ and $\{h \in H\ h^{\frac{1}{2}\alpha} = -1\}$. Along the former, going through the identity element, the system (4.1.1) has exponents 0 and $\frac{1}{2} - k_{\frac{1}{2}\alpha} - k_\alpha$, and along the latter 0 and $\frac{1}{2} - k_\alpha$. In both cases each exponent has multiplicity $\frac{1}{2}d$. This is in accordance with Corollary 4.3.8.

Corollary 4.3.13. *The function* (4.3.24) *has an analytic continuation to a single-valued W-invariant holomorphic function on a W-invariant tubular neighborhood U of A in H.*

Proof. Clear from Corollary 4.3.9 and Theorem 4.3.11. $\square$

Theorem 4.3.14. *The function $\widetilde{F}(\lambda, k; h)$ given by* (4.3.24) *is a holomorphic function of*

$$(\lambda, k, h) \in \mathfrak{h}^* \times K \times U$$

with U a W-invariant tubular neighborhood of A in H. It satisfies

$$\widetilde{F}(w\lambda, k; h) = \widetilde{F}(\lambda, k; h) \qquad \textit{for all } w \in W \tag{4.3.27}$$

$$\widetilde{F}(\lambda, k; wh) = \widetilde{F}(\lambda, k; h) \qquad \textit{for all } w \in W \tag{4.3.28}$$

and $(\lambda, k, h) \in \mathfrak{h}^* \times K \times U$.

Proof. Everything is clear except that the word holomorphic should be replaced by meromorphic with simple poles along hyperplanes of the form $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for some $\alpha \in R$. Using (4.3.27) it is clear that the simple poles along hyperplanes of the form $(\lambda, \alpha^\vee) = 0$ for some $\alpha \in R$ are all removable. Fix $\alpha \in R_+$, $j \in \mathbb{N}$ and put $\kappa_0 = -j\alpha < 0$. Let $\lambda_0 \in \mathfrak{h}^*$ with $(2\lambda_0+\kappa_0, \kappa_0) = 0 \iff (\lambda_0, \alpha^\vee) = j$ but λ_0 on none of the other hyperplanes $(2\lambda+\kappa, \kappa) = 0$ with $\kappa \neq \kappa_0$, $\kappa \in \mathbb{Z}\beta$ for some $\beta \in R$. We claim that for $a \in A_+$ the residue

$$\operatorname*{Res}_{\lambda_0}\{\widetilde{F}(\lambda, k; a)\} := \lim_{\lambda \to \lambda_0} \{(2\lambda+\kappa_0, \kappa_0)\widetilde{F}(\lambda, k; a)\}$$

of $\widetilde{F}(\lambda, k; a)$ along $(2\lambda+\kappa_0, \kappa_0) = 0$ vanishes at λ_0. If we can prove this the theorem will follow from Hartogs extension theorem.

Using (4.3.11) we get

$$\operatorname*{Res}_{\lambda_0}\{\widetilde{F}(\lambda, k; a)\} = \sum_{w \in W,\ w(\alpha)<0} d(w, \lambda_0, k)\Phi(w\lambda_0, k; a)$$

as a sum of $\frac{1}{2}d$ Harish-Chandra series with coefficients

$$d(w, \lambda_0, k) = \lim_{\lambda \to \lambda_0} (2\lambda+\lambda+\kappa_0, \kappa_0)\{\widetilde{c}(wr_\alpha\lambda, k)\Gamma_{\kappa_0}(wr_\alpha\lambda, k)+\widetilde{c}(w\lambda, k)\}$$

being holomorphic in (λ_0, k). On the other hand, we have that the residue $\operatorname{Res}_{\lambda_0}\{\widetilde{F}(\lambda, k; a)\} \in V(\lambda_0, k)$ remains a solution which is a simultaneous eigenvector of $M(\lambda_0, k)(g_j)$ with eigenvalue 1. Arguing as in the proof of Theorem 4.3.6 this leads by rank one reduction to a contradiction unless all coefficients $d(w, \lambda_0, k) = 0$. $\square$

4.4. The hypergeometric function

Definition 4.4.1. The function

$$F(\lambda, k; a) = \sum_{w \in W} c(w\lambda, k)\Phi(w\lambda, k; a) \tag{4.4.1}$$

is called the *hypergeometric function* associated with R. Here $c(\lambda, k)$ is defined by (3.4.3).

Theorem 4.4.2. *With $S \subset K$ defined by*

$$S = \{\text{pole locus of the meromorphic function } \frac{1}{\tilde{c}(\rho(k), k)}\} \tag{4.4.2}$$

the hypergeometric function $F(\lambda, k; a)$ is a holomorphic function on

$$\mathfrak{h}^* \times (K \backslash S) \times U \tag{4.4.3}$$

with U a W-invariant tubular neighborhood of A in H and satisfies

$$F(w\lambda, k; a) = F(\lambda, k; a) \qquad \text{for all } w \in W \tag{4.4.4}$$

$$F(\lambda, k; wa) = F(\lambda, k; a) \qquad \text{for all } w \in W \tag{4.4.5}$$

and (λ, k, a) in the set (4.4.3).

Proof. This is immediate from Theorem 4.3.14 and the definition of the c-function. □

Remark 4.4.3. From the definition of the $\tilde{c}$-function it is easy to see that the open set $K \backslash S$ contains the closed set

$$\{k \in K; Re(k_{\frac{1}{2}\alpha} + k_\alpha) \geq 0 \qquad \forall \alpha \in R^0\} \tag{4.4.6}$$

which in turn contains the closed set $\mathbb{C}_+ \cdot B$ with B the basis of Definition 3.4.1 and $\mathbb{C}_+ = \{z \in \mathbb{C}; Re(z) \geq 0\}$.

Proposition 4.4.4. *We have $F(\lambda, 0; e) = 1$ for all $\lambda \in \mathfrak{h}^*$.*

Proof. Since $\tilde{c}(\lambda, 0) = 1$ we have

$$\tilde{F}(\lambda, 0; a) = \sum_{w \in W} a^{w\lambda}$$

for $a \in A$ and $\lambda \in \mathfrak{h}^*$. Using (3.5.14) we have

$$\lim_{k \to 0} \tilde{c}(\rho(k), k) = |W|$$

and hence

$$F(\lambda, 0; a) = \frac{1}{|W|} \sum_{w \in W} a^{w\lambda}$$

from which the proposition follows immediately. □

Theorem 4.4.5. *For $l \in \mathbb{Z} \cdot B$ and $k \in K$ with $k, k+l \notin S$ we have*

$$F(\lambda, k; e) = F(\lambda, k+l; e). \tag{4.4.7}$$

Proof. For $l \in \mathbb{Z}_- \cdot B$, $k \in K$ with $k, k+l \notin S$ we apply Corollary 3.6.5 with $F = F(\lambda, k; a)$. From (4.3.26) we get

$$G_-(l)F(\lambda, k; a) = \frac{\widetilde{c}(\rho(k+l), k+l)}{\widetilde{c}(\rho(k), k)} F(\lambda, k+l; a) \tag{4.4.8}$$

and the theorem follows from (3.6.12) and Corollary 3.6.7. □

Corollary 4.4.6. *For $k \in \mathbb{Z} \cdot B$ with $k_{\frac{1}{2}\alpha} + k_\alpha \geq 0$ $\forall \alpha \in R^0$ we have*

$$F(\lambda, k; e) = 1 \tag{4.4.9}$$

for all $\lambda \in \mathfrak{h}^*$.

Proof. This is clear from the previous remark, proposition, and theorem. □

Observe that for $\lambda \in P_+$ and $k_\alpha \geq 0$ $\forall \alpha \in R$ we have

$$F(\lambda + \rho(k), k; a) = c(\lambda + \rho(k), k) P(\lambda, k; a). \tag{4.4.10}$$

For the normalization problem at the identity element of Jacobi polynomials as given in Theorem 3.6.6 the extension from integral K to real k was obvious (see the last sentence of the proof of Theorem 3.6.6). For the normalization problem at the identity of the hypergeometric function as given by (4.4.9) the extension from integral k to real k is more subtle. The analysis in this case has been carried out by Opdam [61, 62]. The result is as follows. For the proof we refer to these papers.

Theorem 4.4.7. *For $\lambda \in \mathfrak{h}^*$ and $k \in K \backslash S$ we have*

$$F(\lambda, k, e) = 1. \tag{4.4.11}$$

Notes for Chapter 4

The hypergeometric function studied in this chapter is a generalization of the spherical function for a real semisimple Lie group in a sense which will be explained in the next chapter. Essentially all the results obtained here go back in the case of spherical functions to the pioneering papers [28, 24]. Whereas in the case of spherical functions on a real semisimple Lie group one has both differential and integral operators at hand we have in the case of hypergeometric functions only the differential equations available. Certain results which are fairly obvious from the integral aspect require here longer proofs. Of course one expects the integral theory of spherical functions to have an appropriate generalization to the context of hypergeometric functions but this remains a project for future research. See [5] for the root system A_2.

The theory of differential equations in several variables with regular singular points was developed most elegantly by Deligne [14, 50] and its applicability to the situation of spherical functions was stressed in [10]. The observation that the monodromy representation of the affine braid group factor through a representation of the affine Hecke algebra as stated in Corollary 4.3.8 was made in [34, 30, 31]. This seems to be a new result even in the case of spherical functions on a real semisimple Lie group.

The hypergeometric function of this chapter was introduced in [34] under the assumption of the existence of the hypergeometric differential, which was only known at that time for some root systems. Subsequently the hypergeometric function was constructed in [30] from its monodromy using the Riemann-Hilbert correspondence, at least for generic parameters. The analytic continuation in the parameters was exhibited in [59], and from this the existence of the hypergeometric differential equations was derived. Later the existence of the hypergeometric differential equations was proved by elementary means in [33].

CHAPTER 5

Spherical functions of type χ on a Riemannian symmetric space

5.1. The Harish-Chandra isomorphism

Suppose G_c is a connected simply connected complex semisimple Lie group and $G \subset G_c$ a real form. Let $K \subset G$ be a maximal compact subgroup and

$$\chi\colon K \to \mathbb{C}^\times \tag{5.1.1}$$

a one-dimensional representation. Of course if the symmetric space G/K is irreducible then we have necessarily $\chi = 1$ unless G/K is of Hermitian type in which case the set of such χ's is parameterized by $\mathbb{Z}$. For $f \in C^\infty(G)$ and $X_1, \dots X_N \in \mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of G) we put

$$\begin{aligned}&(X_1 \dots X_N f)(g)\\ &\quad = \left\{ \frac{\partial^N}{\partial t_1 \dots \partial t_N} f(g \exp t_1 X_1 \dots \exp t_N X_N) \right\}_{t_1 = \dots = t_N = 0}\end{aligned} \tag{5.1.2}$$

which means that the elements of the Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$ are considered as left-invariant differential operators on G.

We write

$$C^\infty(G/K;\chi) = \{f \in C^\infty(G);\ f(gk) = \chi(k)^{-1} f(g)\ \forall k \in K\} \tag{5.1.3}$$

and think of these functions geometrically as sections in a homogeneous line bundle $L(\chi) \to G/K$. For $f \in C^\infty(G/K;\chi)$ and $z \in U(\mathfrak{g})^K :=$ {invariants in $U(\mathfrak{g})$ for the adjoint action of K} it is clear that $zf \in C^\infty(G/K;\chi)$. Moreover zf depends only on the class D_z of z modulo $U(\mathfrak{g})^K \cap \sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi(X))$. Here we write $\mathfrak{k}$ for the Lie algebra of K and $\chi\colon \mathfrak{k} \to \mathbb{C}$ for the Lie algebra representation associated with (5.1.1).

Proposition 5.1.1. *The natural map identifies*

$$\mathbb{D}(\chi) := U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap \sum_{X \in \mathfrak{k}} U(\mathfrak{g})(X + \chi(X)) \tag{5.1.4}$$

as the algebra of all G-invariant differential operators on sections in the homogeneous line bundle $L(\chi) \to G/K$.

Proof. See [70, Thm 2.1] or [71, Prop 2.1]. □

Denote

$$\begin{aligned} &C^\infty(G//K;\chi) \\ &\quad = \{f \in \mathbb{C}^\infty(G);\ f(k_1 g k_2) = \chi(k_1 k_2)^{-1} f(g)\ \forall k_1, k_2 \in K\} \end{aligned} \tag{5.1.5}$$

for the subspace of (5.1.3) of spherical functions of type χ. It is clear that $D_z f \in C^\infty(G//K;\chi)$ for $f \in C^\infty(G//K;\chi)$ and $D_z \in \mathbb{D}(\chi)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $G = K \exp(\mathfrak{p})$ be the Cartan decomposition, and choose $\mathfrak{a} \subset \mathfrak{p}$, $A = \exp(\mathfrak{a})$ a maximal split torus.

Proposition 5.1.2. *The map $f \in C^\infty(G) \mapsto \operatorname{res}(f) := f|_A \in C^\infty(A)$ defines a linear bijection*

$$\operatorname{res}\colon\ C^\infty(G//K;\chi) \to C^\infty(A)^W. \tag{5.1.6}$$

Proof. This is immediate from the Cartan decomposition and the Chevalley isomorphism $C^\infty(\mathfrak{p})^K \to C^\infty(\mathfrak{a})^W$. □

Notation 5.1.3. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system and $m(\alpha)$, $\alpha \in \Sigma$ the corresponding root multiplicity. We put

$$R = 2\Sigma, \quad k_{2\alpha} = \tfrac{1}{2} m(\alpha), \tag{5.1.7}$$

which implies that $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma_+} m(\alpha)\alpha = \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \alpha = \rho(k)$. Moreover the root system R of type C_n will always be considered as being of type BC_n with $k_s = 0$ (with k_s the multiplicity of the short roots in BC_n as in Proposition 3.1.8).

Theorem 5.1.4. *For each $D_z \in \mathbb{D}(\chi)$ there exists a unique differential operator* $\mathrm{rad}(D_z) \in (\mathfrak{R} \otimes U\mathfrak{a})^W$ *such that*

$$\mathrm{res}(D_z f) = \mathrm{rad}(D_z)\,\mathrm{res}(f) \qquad \forall f \in C^\infty(G//K;\chi). \tag{5.1.8}$$

The differential operator $\mathrm{rad}(D_z)$ *is called the radial part of the differential operator* $D_z \in \mathbb{D}(\chi)$. *The mapping*

$$\mathrm{rad}\colon\ \mathbb{D}(\chi) \to (\mathfrak{R} \otimes U\mathfrak{a})^W \tag{5.1.9}$$

is an injective homomorphism of algebras. Let $Ch\colon (S\mathfrak{a})^W \xrightarrow{\sim} (S\mathfrak{p})^K$ *be the Chevalley isomorphism and* $Sym\colon S\mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$ *the symmetrizer (which is a G-equivariant linear bijection). For $p \in (S\mathfrak{a})^W$ homogeneous of degree N the operator rad($D_{Sym(Ch(p))}$) is a differential operator of order N with leading symbol of order N having constant coefficients and equal to ∂_p.*

Proof. This follows from [10, Sections 2 and 3] except that their ring $\mathfrak{R}$ is slightly bigger than ours. However, with the convention for R of type C_n as in Notation 5.1.3 it follows from the sequel that our ring $\mathfrak{R}$ suffices. □

Proposition 5.1.5. *For the Casimir operator $\Omega \in U(\mathfrak{g})$ the differential operator*

$$\mathrm{rad}(D_\Omega) - L(k) \in \mathfrak{R} \tag{5.1.10}$$

has order 0 (i.e. is a function in $\mathfrak{R}$). Here the inner product on $\mathfrak{a}$ is obtained from the Killing form on $\mathfrak{g}$.

Proof. By the previous theorem $\mathrm{rad}(D_\Omega) - L(k)$ has order ≤ 1. Using the integral formula for the Cartan decomposition we have for $f_1, f_2 \in C_c^\infty(G//K;\chi)$

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}dg = \frac{1}{|W|}\int_A f_1(a)\overline{f_2(a)}|\delta(k,a)|da. \tag{5.1.11}$$

The symmetry $(\Omega f_1, f_2) = (f_1, \Omega f_2)$ for the Casimir operator is obvious. The symmetry of the operator $L(k)$ with respect to the measure $|\delta(k,a)|da$ follows from Theorem 2.1.1. Hence $\mathrm{rad}(D_\Omega) - L(k)$ is also symmetric with respect to the measure $|\delta(k,a)|da$. But a first-order differential operator being symmetric with respect to a smooth measure has order 0. □

Corollary 5.1.6. *If $\chi = 1_K$ is trivial then* $\mathrm{rad}(D_\Omega) = L(k)$.

Proof. Apply the operator $\mathrm{rad}(D_\Omega) - L(k)$ to the function 1_A and observe that $1_A = \mathrm{res}(1_G)$ with $1_G \in C^\infty(G//K)$. Hence $\mathrm{rad}(D_\Omega)1_A = \mathrm{res}(\Omega 1_G) = 0$. □

Theorem 5.1.7. *Suppose that G/K is an irreducible Hermitian symmetric space (which is equivalent with the fact that R is of type BC_n (or C_n) and $k_l = \frac{1}{2}$, either from the classification* [35, pp.532–534] *or from the theory of strongly orthogonal roots). Choose a generator χ_1 for the rank one lattice of one-dimensional characters of K and say $\chi = \chi_l = \chi_1^l$ for some $l \in \mathbb{Z}$. Then the radial part of the Casimir operator is given by*

$$\begin{aligned} &\mathrm{rad}(D_\Omega) \\ &= L(k) + \sum_{\substack{\alpha\in R_+ \\ \alpha \text{ short}}} l^2 \left\{ \frac{(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2} - \frac{(2\alpha,2\alpha)}{(e^{\alpha}-e^{-\alpha})^2} \right\} + \frac{\Omega_{\mathfrak{m}}(\chi|_M)}{\chi|_M}, \end{aligned} \tag{5.1.12}$$

where $M = Z_K(\mathfrak{a})$, $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $(\chi|_M)^{-1}\Omega_{\mathfrak{m}}(\chi|_M)$ is the scalar by which the Casimir operator $\Omega_{\mathfrak{m}}$ of $\mathfrak{m}$ (with respect to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$) acts on the one-dimensional representation $\chi|_M$ of M.

Before proving the theorem (in Section 5.3) we start by giving some corollaries.

Corollary 5.1.8. *We have*

$$\begin{aligned} &\mathrm{rad}(D_\Omega + (\rho(k),\rho(k))) \\ &= \prod_{\substack{\alpha\in R_+ \\ \alpha \text{ short}}} (e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})^{\pm|l|} \circ ML(m_\pm) \circ \prod_{\substack{\alpha\in R_+ \\ \alpha \text{ short}}} (e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})^{\mp|l|} \\ &\qquad + \frac{\Omega_{\mathfrak{m}}(\chi|_M)}{\chi|_M} \end{aligned} \tag{5.1.13}$$

with multiplicity function $m_\pm \in K \simeq \mathbb{C}^3$ given by

$$m_\pm = (k_s \mp |l|, k_m, k_l \pm |l|), \quad k_l = \tfrac{1}{2}. \tag{5.1.14}$$

Here the $\pm$ sign indicates that both possibilities are valid.

Proof. Apply (2.1.12) to (5.1.12). Observe that the equations

$$m_l(1-m_l) = -l^2 + k_l(1-k_l), \quad m_s(1-m_s-2m_l) = l^2 + k_s(1-k_s-2k_l)$$

have as a possible solution: $m_l = \frac{1}{2} \pm |l|$, $m_s = k_s \mp |l|$ which in turn implies that

$$\delta(m_\pm - k)^{\frac{1}{2}} = (\Delta_s^{-1}\Delta_l)^{\pm|l|} = \prod_{\alpha>0, \alpha \text{ short}} (e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})^{\pm|l|}.$$

Hence (5.1.13) follows. □

Definition 5.1.9. The Harish-Chandra homomorphism

$$\gamma_{HC}\colon \mathbb{D}(\chi) \to S\mathfrak{a} \tag{5.1.15}$$

is defined by $\gamma_{HC}(D_z) = \gamma(k)(\mathrm{rad}(D_z))$. Indeed it is a homomorphism since both the k-constant term $\gamma(k)$ (see Definition 1.2.4) and the radial part (see Theorem 5.1.4) are algebra homomorphisms.

Theorem 5.1.10. *The Harish-Chandra homomorphism γ_{HC} is an isomorphism*

$$\gamma_{HC}\colon \mathbb{D}(\chi) \to (S\mathfrak{a})^W \tag{5.1.16}$$

of commutative algebras.

Proof. The statement follows by induction on the order of differential operators from the last part of Theorem 5.1.4 once we have proved that $\gamma_{HC}(D_z) \in S\mathfrak{a}^W$ $\forall D_z \in \mathbb{D}(\chi)$. For this observe that $\mathrm{rad}(D_\Omega)$ commute with $\mathrm{rad}(D_z)$ $\forall D_z \in \mathbb{D}(\chi)$. Indeed Ω lies in the center of $U\mathfrak{g}$ and rad is a homomorphism. Hence in case $\chi = 1_\kappa$ we conclude $\gamma_{HC}(D_z) \in S\mathfrak{a}^W$ from Theorem 1.2.9, Theorem 5.1.4, and Corollary 5.1.6.

Now suppose G/K is an irreducible Hermitian symmetric space and keep the notation as in Theorem 5.1.7. Again we have

$$[\mathrm{rad}(D_z), \mathrm{rad}(D_\Omega)] = 0 \qquad \forall D_z \in \mathbb{D}(\chi)$$

which is equivalent (using (5.1.13)) to

$$\left[\prod_{\substack{\alpha\in R_+ \\ \alpha \text{ short}}} (e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})^{\mp|l|} \circ \mathrm{rad}(D_z) \circ \prod_{\substack{\alpha\in R_+ \\ \alpha \text{ short}}} (e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})^{\pm|l|}, ML(m_\pm)\right] = 0$$

and since (use (1.2.9) with $F = A_+$)

$$\gamma(m_\pm)\Bigg(\prod_{\substack{\alpha\in R_+\\ \alpha\ \text{short}}}(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})^{\mp|l|}\circ \mathrm{rad}(D_z)\circ \prod_{\substack{\alpha\in R_+\\ \alpha\ \text{short}}}(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})^{\pm|l|}\Bigg)$$
$$= \gamma(k)(\mathrm{rad}(D_z))$$

we conclude $\gamma_{HC}(D_z) \in S\mathfrak{a}^W\ \forall D_z \in \mathbb{D}(\chi)$ as before. □

Corollary 5.1.11. *In case $\chi = 1_K$ we have*

(5.1.17) $$\mathrm{rad}\colon\ \mathbb{D}(\chi) \xrightarrow{\simeq} \mathbb{D}(k)$$

and in case G/K is an irreducible Hermitian symmetric space we have

(5.1.18) $$\mathrm{rad}\colon\ \mathbb{D}(\chi_l) \xrightarrow{\simeq} \prod_{\substack{\alpha\in R_+\\ \alpha\ \text{short}}}(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})^{\pm|l|}\circ \mathbb{D}(m_\pm)\circ \prod_{\substack{\alpha\in R_+\\ \alpha\ \text{short}}}(e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})^{\mp|l|}.$$

Proof. Clear from (the proof of) the previous theorem. □

5.2. Elementary spherical functions as hypergeometric functions

Elementary spherical functions can be defined in various ways: by integral or differential equations or via representation theory. With the preparations of the previous section the approach with differential equations is the most convenient.

Definition 5.2.1. A spherical function $\varphi \in C^\infty(G//K;\chi)$ of type χ is called *elementary* with *spectral parameter* $\lambda \in \mathfrak{h}^*$ ($\mathfrak{h}$ is the complexification of $\mathfrak{a}$) if

(5.2.1) $$D_z\varphi = \gamma_{HC}(D_z)(\lambda)\varphi \qquad \forall D_z \in \mathbb{D}(\chi)$$

and φ is normalized by

(5.2.2) $$\varphi(e) = 1.$$

The function φ is uniquely characterized by these two conditions and we write $\varphi = \varphi_{\chi,\lambda}$. In case $\chi = 1_K$ we also write $\varphi_{\chi,\lambda} = \varphi_\lambda$ and in case G/K is an irreducible Hermitian symmetric space with $\chi = \chi_l$ for $l \in \mathbb{Z}$ we put $\varphi_{\chi,\lambda} = \varphi_{l,\lambda}$.

Theorem 5.2.2. *In case $\chi = 1_K$ we have*

$$\text{res}(\varphi_\lambda) = F(\lambda, k; \cdot) \tag{5.2.3}$$

and in case G/K is an irreducible Hermitian symmetric space we have

$$\text{res}(\varphi_{l,\lambda}) = \prod_{\substack{\alpha>0 \\ \alpha \text{ short}}} \left(\frac{e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha}}{2} \right)^{\pm|l|} F(\lambda, m_\pm; \cdot) \tag{5.2.4}$$

with $k \in K$ given by (5.1.7) *and $m_\pm$ given by* (5.1.14).

Proof. This is immediate from Chapter 4 and Corollary 5.1.11. □

Suppose $U \subset G_c$ is the Lie group with Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Then U/K is the compact dual symmetric space for the noncompact symmetric space G/K.

Corollary 5.2.3. *The elementary spherical function $\varphi_{\chi,\lambda}$ which is an analytic function on G extends to a holomorphic function on G_c (and by restriction gives an elementary spherical function for the compact pair (U, K)) if and only if in case $\chi = 1_K$ we have (by choosing λ in its orbit $W\lambda$ such that* $\text{Re}(\lambda, \alpha^\vee) \geq 0$ *for all $\alpha \in R_+$)*

$$\lambda \in \rho(k) + P_+ \tag{5.2.5}$$

and in case G/K is an irreducible Hermitian symmetric space we have

$$\lambda \in \rho(m_+) + P_+, \tag{5.2.6}$$

where $\rho(m_+) = \rho(k) + |l|\rho_s$ with $\rho_s = \frac{1}{2} \sum\limits_{\substack{\alpha>0 \\ \alpha \text{ short}}} \alpha$.

Proof. Just apply Theorem 5.2.2. We write

$$C^\infty(U//K; \chi) = \{f \in C^\infty(U);\ f(k_1 u k_2) = \chi(k_1 k_2)^{-1} f(u)\ \ \forall k_1, k_2 \in K\}$$

and

$$T = \exp(i\mathfrak{a}) / \exp(i\mathfrak{a}) \cap K = \exp(i\mathfrak{a})/\text{points of order 2.}$$

In case $\chi = 1_K$ the restriction map

$$\text{res: } C^\infty(U//K) \xrightarrow{\sim} C^\infty(T)^W$$

is a linear bijection, and the result follows from (4.4.10) since (apart from normalization) the Jacobi polynomials are the only hypergeometric functions which are holomorphic on the full complex torus H.

In case G/K is an irreducible Hermitian symmetric space observe that the restriction map

$$\text{res: } C^\infty(U//K;\chi) \to C^\infty(\exp(i\mathfrak{a}))^W$$

defines a linear bijection

$$\text{res: } C^\infty(U//K;\chi_l)) \xrightarrow{\sim} \prod_{\substack{\alpha>0 \\ \alpha \text{ short}}} (e^{\frac{1}{2}\alpha}+e^{-\frac{1}{2}\alpha})^{|l|} \cdot C^\infty(T)^W$$

and (5.2.6) follows similarly from (5.2.4). $\square$

We write

$$E(\chi,\lambda) = \{f \in C^\infty(G/K;\chi); D_z f = \gamma_{HC}(D_z)(\lambda) f \quad \forall D_z \in \mathbb{D}(\chi)\}$$
$$L(\chi,\lambda)\colon\ G \to \mathrm{Aut}(E(\chi,\lambda)),\ \text{ or : } \mathfrak{g} \to \mathrm{End}(E(\chi,\lambda))$$

for the eigenspace representation of G on the space of smooth functions which transform on the right under K according to χ^{-1} and are simultaneous eigenfunctions of the invariant differential operators. For $\delta \in \widehat{K}$ let $E(\chi,\lambda)_\delta$ denote the δ-isotypical component of $E(\chi,\lambda)$. Then it is clear (cf. Definition 5.2.1) that $\dim E(\chi,\lambda)_\chi = 1$.

Proposition 5.2.4. *Any subrepresentation V of the eigenspace representation $E(\chi,\lambda)$ contains the elementary spherical function $\varphi_{\chi,\lambda}$ of type χ.*

Proof. Suppose $V \subset E(\chi,\lambda)$ is a subrepresentation and let $f \in V$, $f \neq 0$. Replacing f by $L(\chi,\lambda)(g)f$ for some $g \in G$ we can assume that $f(e) \neq 0$. Then the function

$$\varphi\colon\ g \mapsto \int_K f(kg)\chi(k)dk$$

again lies in V and is spherical of type χ. Here dk is the normalized Haar measure on K. Moreover $\varphi(e) = f(e)$ and we conclude that $\varphi = f(e)\varphi_{\chi,\lambda}$. $\square$

Corollary 5.2.5. *The subrepresentation $V(\chi,\lambda)$ of $E(\chi,\lambda)$ generated by the elementary spherical function $\varphi_{\chi,\lambda}$ is irreducible. The representation $V(\chi,\lambda)$ is called the spherical representation of type χ with parameter λ.*

Corollary 5.2.6. *The center $\mathfrak{Z}$ of $U(\mathfrak{g})$ acts on $V(\chi,\lambda)$ by a scalar.*

Proposition 5.2.7. *Suppose $\mathfrak{b}\subset\mathfrak{m}$ is a Cartan subalgebra and $\mathfrak{a}\oplus\mathfrak{b}\subset\mathfrak{g}$ the corresponding full Cartan subalgebra. Write $\Sigma(\mathfrak{g},\mathfrak{a}),\Sigma(\mathfrak{m},\mathfrak{b}),\Sigma(\mathfrak{g},\mathfrak{a}\oplus\mathfrak{b})$ for the various root systems, which implies that with compatible positive systems the corresponding ρ-vectors satisfy $\rho(\mathfrak{g},\mathfrak{a}\oplus\mathfrak{b})=\rho(\mathfrak{g},\mathfrak{a})+\rho(\mathfrak{m},\mathfrak{b})$, where $\rho(\mathfrak{g},\mathfrak{a})=\rho(k)$. Then the central character of $V(\chi,\lambda)$ is given by $\chi|\mathfrak{m}+\rho(\mathfrak{m},\mathfrak{b})+\lambda$.*

Proof. This follows from the description of the natural map $\mathfrak{Z}\to\mathbb{D}(\chi)$ in terms of the Harish-Chandra isomorphisms for $\mathfrak{Z}$ and $\mathbb{D}(\chi)$ respectively. For details we refer to [37]. □

Corollary 5.2.8. *If and only if the conditions of Corollary 5.2.3 hold then the spherical representation $V(\chi,\lambda)$ is an irreducible finite dimensional representation with highest weight $\chi|\mathfrak{m}+\lambda-\rho(k)$.*

Proof. Indeed the central character and the highest weight differ by $\rho=\rho(\mathfrak{g},\mathfrak{a}\oplus\mathfrak{b})$. □

Remark 5.2.9. The above corollary is a reformulation of the Cartan-Helgason theorem [36, Chap. 5, Thm. 4.1] in case $\chi=1_K$ and a theorem of Schlichtkrull [67, Thm. 7.2] in case G/K is an irreducible Hermitian symmetric space which give necessary and sufficient conditions for the highest weight of a finite dimensional irreducible representation of $\mathfrak{g}$ in order that the representation has one-dimensional K-types.

Proposition 5.2.10. *If the conditions of Corollary 5.2.3 hold then the dimension $d(\chi,\lambda)$ of the finite dimensional spherical representation $V(\chi,\lambda)$ of type χ is given by*

$$d(\chi,\lambda)=4^{n|l|}\cdot\frac{\widetilde{c}(\rho(m_+),m_+)c^*(-\rho(k),k)}{\widetilde{c}(\rho(k),k)c^*(-\rho(m_+),m_+)}\cdot\frac{\widetilde{c}(\rho(m_+),m_+)c^*(-\rho(m_+),m_+)}{\widetilde{c}(\lambda,m_+)c^*(-\lambda,m_+)}. \tag{5.2.7}$$

Proof. By the Schur orthogonality relations we have

$$\int_U |\varphi_{\chi,\lambda}(u)|^2 du = d(\chi,\lambda)^{-1} \cdot \int_U du$$

and the formula follows from the integral formula for the Cartan decomposition, Theorem 5.2.2, Theorem 4.4.7, Theorem 3.6.6, and Theorem 3.5.5. □

Example 5.2.11. In case $\chi = 1_K$ formula (5.2.7) becomes

$$d(\lambda) = \frac{\widetilde{c}(\rho(k),k)c^*(-\rho(k),k)}{\widetilde{c}(\lambda,k)c^*(-\lambda,k)} \tag{5.2.8}$$

and was derived by Vretare [72, 73].

Example 5.2.12. Suppose G/K is an irreducible Hermitian symmetric space and $\chi = \chi_l$ as before. The smallest dimensional representation containing the K-type χ has parameter $\lambda = \rho(m_+)$ and its dimension $d(l,\rho(m_+))$ is given by (use the transcription from (3.5.12) to Selberg's integral (3.5.15) as in [48, p.993])

$$d(l,\rho(m_+)) = \prod_{i=1}^{|l|}\prod_{j=1}^{n} \frac{k_s + 1 + i + (n+j-2)k_m}{i + (j-1)k_m}. \tag{5.2.9}$$

Remark 5.2.13. Considering a compact Lie group as a symmetric space formula (5.2.8) boils down to Weyl's dimension formula. However, it does not seem clear (without using the classification of symmetric spaces) how to derive (5.2.7) from Weyl's dimension formula [63].

5.3. Proof of Theorem 5.1.7

The proof of Theorem 5.1.7 given here will be similar to the proof of Corollary 5.1.6. In view of Theorem 5.2.2 a natural choice is to replace the function 1_A in the proof of Corollary 5.1.6 by

$$\prod_{\substack{\alpha>0 \\ \alpha \text{ short}}} \left(\frac{e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha}}{2}\right)^{|l|} \tag{5.3.1}$$

and to verify in an independent way that this function is the restriction to A of an elementary spherical function of type χ_l. Moreover this spherical function is an eigenfunction of the Casimir operator Ω with eigenvalue (in the notation from below) $(|l|\chi_1, |l|\chi_1 + 2\rho(k) + 2\rho_{\mathfrak{m}}) = (\rho(\mathfrak{m}_+), \rho(\mathfrak{m}_+)) - (\rho(k), \rho(k)) +$ (value of $\Omega_{\mathfrak{m}}$ on $|l|\chi_1$). Hence (5.1.13) or equivalently (5.1.12) follows.

We recall some structure theory for an irreducible Hermitian symmetric space G/K and its Cartan dual U/K (keep in mind that both G and U are real forms of the simply connected complex semisimple group G_c). Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ which is also a Cartan subalgebra for $\mathfrak{g}$. The root system $\Sigma(\mathfrak{g}, \mathfrak{t}) = \Sigma_c \cup \Sigma_n$ decomposes into compact roots $\Sigma_c = \Sigma(\mathfrak{k}, \mathfrak{t})$ and noncompact roots $\Sigma_n = \Sigma(\mathfrak{p}, \mathfrak{t})$. Let χ_1 be a generator for the orthocomplement of Σ_c in the weight lattice of $\Sigma(\mathfrak{g}, \mathfrak{t})$. In comparison with Theorem 5.1.7 we change from a global multiplicative to an infinitesimal additive notation. Choose a positive system $\Sigma_+(\mathfrak{g}, \mathfrak{t})$ such that $(\alpha, \chi_1) \geq 0$ $\forall \alpha \in \Sigma_+(\mathfrak{g}, \mathfrak{t})$. There exists a unique simple noncompact root α_1 in $\Sigma_+(\mathfrak{g}, \mathfrak{t})$. Let $\gamma_1, \ldots, \gamma_n$ be the strongly orthogonal roots in $\Sigma_{n,+}$: $\gamma_1 = \alpha_1$ and γ_j is the smallest root in $\Sigma_{n,+}$ strongly orthogonal to $\gamma_1, \ldots, \gamma_{j-1}$. Let V be the subspace of $i\mathfrak{t}^*$ spanned over $\mathbb{R}$ by the γ's and π: $i\mathfrak{t}^* \to V$ the orthogonal projection.

The following result is due to C.C. Moore [52, Thm 2].

Theorem 5.3.1. *There are two possibilities for $\pi(\Sigma_+(\mathfrak{g}, \mathfrak{t}))$ except for 0:*
Case I : $\pi(\Sigma_+)\backslash 0 = \{\gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k);\ 1 \leq i \leq n,\ 1 \leq k < j \leq n\}$
Case II: $\pi(\Sigma_+)\backslash 0 = \{\frac{1}{2}\gamma_i, \gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k);\ 1 \leq i \leq n,\ 1 \leq k < j \leq n\}$.
Furthermore the nonzero projections of the compact roots have the form $\frac{1}{2}\gamma_i$ or $\frac{1}{2}(\gamma_j - \gamma_k)$, and the projections of the noncompact roots have the form $\frac{1}{2}\gamma_i, \gamma_i$ or $\frac{1}{2}(\gamma_j + \gamma_k)$.

Proposition 5.3.2. *We have $\pi(\chi_1) = \frac{1}{2}(\gamma_1 + \cdots + \gamma_n)$.*

Proof. Since χ_1 and $\sum_{\alpha \in \Sigma_{n,+}} \alpha$ are multiples of each other we conclude that $\pi(\chi_1)$ and $\frac{1}{2}(\gamma_1 + \cdots \gamma_n)$ are also multiples. The proposition follows since $(\pi(\chi_1), \gamma_1^\vee) = (\chi_1, \gamma_1^\vee) = (\chi_1, \alpha_1^\vee) = 1$. □

Consider the finite dimensional irreducible representation $V(\chi_l)$ of $\mathfrak{g}$ with highest weight $\chi_l = l\chi_1$ for some $l \in \mathbb{N}$. Let v_+ be a highest weight vector. Then it is obvious that v_+ is also a spherical vector for K of type χ_l.

Hence $V(\chi_l)$ is a spherical representation of type χ_l and the corresponding elementary spherical function on the compact form U is given by

$$\varphi(u) = (v_+, u \cdot v_+) \qquad \text{for } u \in U. \tag{5.3.2}$$

Here $(\cdot,\cdot)$ is the Hermitian inner product on $V(\chi_l)$ invariant under U and normalized by $(v_+, v_+) = 1$. It remains to be shown that the restriction of the function (5.3.2) to a maximal split torus for (U, K) is given by (5.3.1). Using the Cayley transform (see [43]) this will be reduced to the computation for $sl(2)$, and this will be straightforward.

More precisely, let $\mathfrak{s}$ be the subalgebra of $\mathfrak{g}_c$ isomorphic to the direct sum of n copies of $sl(2,\mathbb{C})$ corresponding to the strongly orthogonal roots.

Lemma 5.3.3. *The $\mathfrak{s}$-submodule V' of $V(\chi_l)$ generated by the highest weight vector v_+ is isomorphic to the n-fold tensor product $V(l)^{\otimes n}$, where $V(l)$ is the irreducible $sl(2)$-module with highest weight l of dimension $l+1$.*

Proof. This is immediate from Proposition 5.3.2. □

Proposition 5.3.4. *Let*

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis for $sl(2)$. Let $V(l)$ be the finite dimensional irreducible representation of $sl(2)$ with highest weight l and basis $v_0, v_1, \dots, v_l$ satisfying (see [40, Section 7]*)*

$$\begin{aligned} hv_j &= (l-2j)v_j \\ yv_j &= (j+1)v_{j+1} \\ xv_j &= (l-j+1)v_{j-1}. \end{aligned}$$

Let $(\cdot,\cdot)$ be the $su(2,\mathbb{C})$-invariant Hermitian inner product on $V(l)$ normalized by $(v_0, v_0) = 1$. Then we have

$$(v_j, v_j) = \binom{l}{j} \qquad \text{for } j = 0, \dots, l. \tag{5.3.3}$$

Consider the element (cf. [43, p.272]*)*

$$c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

which satisfies

$$c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} c^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Hence conjugation by c (= Cayley transform) maps the compact Cartan subgroup $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ of the group $SL(2,\mathbb{R})$ onto the diagonal subgroup $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ of $SU(2,\mathbb{C})$. Then

$$(5.3.4)\quad c(v_0) = \left(\frac{1}{\sqrt{2}}\right)^2 \exp(iy)v_0 = \left(\frac{1}{\sqrt{2}}\right)^l (v_0 + iv_1 + i^2v_2 + \cdots + i^lv_l)$$

and hence

$$(5.3.5)\quad \left(v_0, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v_0\right) = (cv_0, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} cv_0) = (\cos\theta)^l.$$

Proof. Easy and left to the reader. □

Remark 5.3.5. Harish-Chandra has given a formula for the radial part of the Casimir operator acting on τ-spherical functions where τ is just any double representation of K, see [29, Lemma 22]. Of course (5.1.12) could also have been derived from Harish-Chandra's formula, but this still requires some work since (5.1.12) is more explicit.

5.4. Integral representations

In this section we assume that G/K is an irreducible Hermitian symmetric space and $\chi = \chi_l$, $l \in \mathbb{Z}$ as before. Let $G = KAN$ be the Iwasawa decomposition corresponding to $R_+ = 2\Sigma_+$ as in Notation 5.1.3. Write $g \in G$ as $g = kan = k(g)a(g)n(g)$ correspondingly.

Proposition 5.4.1. *The elementary spherical function $\varphi_{l,\lambda}$ of type $\chi = \chi_l$ with parameter $\lambda \in \mathfrak{h}^*$ has the integral representation*

$$(5.4.1)\qquad \varphi_{l,\lambda}(g) = \int_K a(gk)^{\lambda-\rho}\chi_l(k(gk)^{-1}k)dk.$$

Proof. This formula is analogous to Harish-Chandra's integral formula for the usual spherical function in the case $\chi = 1_K$ and the proof goes along the same lines [23, 36]. See also [67, 70] where the formula is explicitly stated. □

Consider the following integral transformations.

Spherical Fourier transform (also Harish-Chandra transform):

For $f \in C_c^\infty(G//K;\chi_l)$ we put for $\lambda \in \mathfrak{h}^*$ (recall $\mathfrak{h} = \mathcal{A}_c$)

$$\mathcal{H}f(\lambda) = \int_G f(g)\varphi_{-l,-\lambda}(g)dg. \tag{5.4.2}$$

Clearly $\mathcal{H}f$ is a holomorphic function of $\lambda \in \mathfrak{h}^*$.

Abel transform (of Harish-Chandra):

For $f \in C_c^\infty(G)$ we put for $a \in A$

$$\mathcal{A}f(a) = a^\rho \int_N f(an)dn. \tag{5.4.3}$$

Clearly $\mathcal{A}f \in C_c^\infty(A)$.

(Euclidean) Fourier transform:

For $f \in C_c^\infty(A)$ we put for $\lambda \in \mathfrak{h}^*$

$$\mathcal{F}f(\lambda) = \int_A f(a)a^{-\lambda}da. \tag{5.4.4}$$

Then $\mathcal{F}f \in \mathcal{P}(\mathfrak{h}^*)$, the space of Paley-Wiener functions on $\mathfrak{h}^*$.

Theorem 5.4.2. *We have a commutative diagram*

$$\begin{array}{ccc} & \mathcal{P}(\mathfrak{h}^*)^W & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ C_c^\infty(G//K;\chi_l) & \underset{\mathcal{A}}{\longrightarrow} & C_c^\infty(A)^W \end{array}$$

Proof. Indeed we have

$$\begin{aligned}
\mathcal{H}f(\lambda) &= \int_G f(g)\{\int_K a(gk)^{-\lambda-\rho}\chi_{-l}(k(gk)^{-1}k)\,dk\}\,dg \\
&= \iint_{G\times K} \{f(g)a(gk)^{-\lambda-\rho}\chi_{-l}(k(gk)^{-1}k)\}\,dg\,dk \\
&= \iint_{G\times K} \{f(gk^{-1})a(g)^{-\lambda-\rho}\chi_{-l}(k(g)^{-1}k)\}\,dg\,dk \\
&= \iint_{G\times K} \{f(g)\chi_l(k)a(g)^{-\lambda-\rho}\chi_{-l}(k(g)^{-1}k)\}\,dg\,dk \\
&= \iint_{G\times K} \{f(g)a(g)^{-\lambda-\rho}\chi_l(k(g))\}\,dg\,dk \\
&= \int_G \{f(g)a(g)^{-\lambda-\rho}\chi_l(k(g))\}\,dg \\
&= \iiint_{K\times A\times N} \{f(kan)a^{-\lambda-\rho}\chi_l(k)a^{2\rho}\}\,dk\,da\,dn \\
&= \iiint_{K\times A\times N} \{f(an)a^{-\lambda+\rho}\}\,dk\,da\,dn \\
&= \int_A \{a^\rho \int_N f(an)\,dn\}a^{-\lambda}\,da = \mathcal{F}\mathcal{A}f(\lambda). \quad \square
\end{aligned}$$

Proposition 5.4.3. *For $a \in A$ we put*

$$C(a) = \exp(\text{convex hull of } W \log a). \tag{5.4.5}$$

If $f \in C_c^\infty(G//K;\chi_l)$ with $\operatorname{supp}(f) \subset KC(a)K$ *for some $a \in A$ then* $\operatorname{supp}(\mathcal{A}f) \subset C(a)$.

Proof. This is immediate from $|\mathcal{A}f| \leq \mathcal{A}(|f|)$ and the corresponding result for $\mathcal{A}$: $C_c^0(G//K) \to C_c^0(A)^W$, which is a corollary of Kostant's convexity theorem (in fact only of the inclusion part of this theorem) [36, Chap. IV, §10], [1]. $\square$

5.5. The Plancherel theorem and the Paley-Wiener theorem for spherical functions of type χ in the standard case

Definition 5.5.1. A real multiplicity function $k = (k_\alpha)$ on a root system R is said to be standard if

$$\tfrac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha > 0 \qquad \forall\alpha \in R. \tag{5.5.1}$$

Lemma 5.5.2. *If the multiplicity function $k = (k_\alpha)$ on R is standard then $\exists C, N > 0$ such that*

$$\left|\frac{1}{c(\lambda, k)}\right| \leq C(1+|\lambda|)^N \quad \text{if } \mathrm{Re}(\lambda) \in \mathrm{Cl}(\mathfrak{a}_+^*). \tag{5.5.2}$$

Proof. This is immediate from the expression for the c-function as a product of Γ-factors and Stirling's formula, cf. [36, Chap. IV, Proposition 7.2]. □

Corollary 5.5.3. *With G/K an irreducible Hermitian symmetric space and $R = 2\Sigma$ of type BC_n (cf. Notation 5.1.3) the multiplicity function*

$$m_- = (k_s + |l|, k_m, k_l - |l|), \qquad k_l = \tfrac{1}{2} \tag{5.5.3}$$

given by (5.1.14) *is standard if and only if $|l| < k_s + 1$.*

Proof. Indeed $\frac{1}{2}(k_s+|l|) + k_l - |l| > 0 \iff |l| < k_s + 2k_l = k_l + 1$. □

We now recall the classical Paley-Wiener theorem. For this we need the notion of supporting function. Let C be a compact convex set in A. The supporting function $H_C\colon \mathfrak{a}^* \to \mathbb{R}$ is defined by

$$H_C(\xi) = \sup\{\langle\xi, X\rangle;\ X \in \log(C)\},$$

and C can be recovered from H_C by

$$\log(C) = \{X \in \mathfrak{a};\ \langle\xi, X\rangle \leq H_C(\xi)\ \ \forall\xi \in \mathfrak{a}^*\}.$$

For this result and the Paley-Wiener theorem see for example [39, p.105 and p.181].

Theorem 5.5.4. (Euclidean Paley-Wiener theorem): *Let $C \subset A$ be a compact convex set with supporting function $H = H_C$. Then the Fourier transform (5.4.4) maps the space of C_c^∞-functions on A with support in C onto the space of entire functions on $\mathfrak{h}^*$ ($\mathfrak{h}$ is the complexification $\mathfrak{a}_c$) satisfying*

$$\forall N \in \mathbb{N},\ \exists C_N > 0 \ s.t.\ |\mathcal{F}f(\lambda)| \leq C_N(1+|\lambda|)^{-N} e^{H(-\mathrm{Re}(\lambda))}. \tag{5.5.4}$$

Note that (5.4.4) differs from the usual Fourier transform by a factor i: $\mathcal{F}f(i\lambda) = \widehat{f}(\lambda)$. We write $\mathcal{P}_C(\mathfrak{h}^*)$ for the Paley-Wiener space of entire functions on $\mathfrak{h}^*$ satisfying (5.5.4).

From now on we assume that $|l| \leq k_s+1$. In this case a proof of the Plancherel theorem and the Paley-Wiener theorem for the spherical Fourier transform can be established along the following lines:

Step I: For $f \in C_0^\infty(G//K;\chi_l)$ with $\mathrm{supp}(f) \subset KC(a)K$ for some $a \in A$ the function $\mathcal{H}f \in \mathcal{P}_{C(a)}(\mathfrak{h}^*)^W$.

Step II: For $F \in \mathcal{P}_{C(a)}(\mathfrak{h}^*)^W$ for some $a \in A$ we define the normalized wave packet operator $\mathcal{J}$ by

$$\mathcal{J}f(g) = \int_{\lambda \in i\mathfrak{a}^*} F(\lambda)\varphi_{l,\lambda}(g) \frac{d\lambda}{4^{n|l|}c(\lambda, m_-)c(-\lambda, m_-)}, \tag{5.5.5}$$

where $d\lambda$ is the regularly normalized Lebesgue measure on $i\mathfrak{a}^*$. Then $\mathcal{J}F \in C_c^\infty(G//K;\chi_l)$ with $\mathrm{supp}(\mathcal{J}f) \subset KC(a)K$.

Step III: The linear operator

$$\mathrm{res} \circ \mathcal{J} \circ \mathcal{H} \circ \mathrm{res}^{-1}: C_c^\infty(A)^W \to C_c^\infty(A)^W \tag{5.5.6}$$

preserves (or possibly diminish) support, and hence is a differential operator by the theorem of Peetre [65].

Step IV: The operator (5.5.6) commutes with the algebra $\mathrm{rad}(\mathbb{D}(\chi_l))$ which amounts to a system of differential equations for the coefficients of the differential operator (5.5.6). More precisely the differential operator (5.5.6) behaves at infinity in A_+ like a constant coefficient differential operator from which the full operator (5.5.6) can be recovered (cf. Lemma 1.2.7).

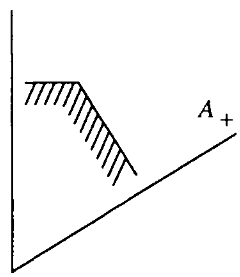
Step V: By a scaling argument we conclude that the operator (5.5.6) equals $|W|.\mathrm{Id}$.

We now comment on the above outline with more details. Step I is immediate from Theorem 5.4.2, Proposition 5.4.3, and the classical Paley-Wiener theorem. Step II follows by shifting the integration over $i\mathfrak{a}^*$ into the complex space $\mathfrak{h}^*$ in the direction of the negative chamber. Using the explicit expression (5.2.4) for the elementary spherical function as a hypergeometric function the arguments are exactly the same as in the Helgason-Gangolli proof of the spherical Paley-Wiener theorem. The crucial point is that (since we are in the standard case) under the integration shift we do not encounter poles of the function $c(-\lambda, m_-)^{-1}$. For details we refer to [36, Chap. IV, Section 7.2]. Combining Step I and Step II we conclude that the linear operator (5.5.6) leaves the space of functions $f \in C_c^\infty(A)^W$ with $\mathrm{supp}(f) \subset C(a)$ invariant for all $a \in A$.

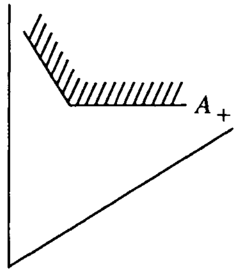
For $f_1, f_2 \in C_c^\infty(G//K; \chi_l)$ we have

$$(5.5.7) \qquad \int_G \mathcal{J} \circ \mathcal{H} f_1(g) \overline{f_2(g)} dg = \int_{i\mathfrak{a}^*} \mathcal{H} f_1(\lambda) \overline{\mathcal{H} f_2(\lambda)} \frac{d\lambda}{4^{n|l|} |c(\lambda, m_-)|^2}$$

which implies that the operator (5.5.6) is formally symmetric with respect to the measure $|\delta(k, a)| da$. Leaving invariant supports of the form



we conclude by symmetry that supports of the form



are left invariant as well. Combining these two we conclude that the operator (5.5.6) preserves supports.

The steps III, IV, V are a variation on Rosenberg's proof of the spherical Plancherel formula [36, Chap. IV, Section 7.3] and were found by van den Ban and Schlichtkrull in their study of the Plancherel decomposition for a pseudo-Riemannian symmetric space [3]. We refer to this paper or to the other part of this book for details. Assuming $|l| < k_s + 1$ we arrive at:

Conclusion 5.5.5. (The inversion formula). *The inversion of the spherical Fourier transform is given by* $\frac{1}{|W|}\mathcal{J}$ *where* $\mathcal{J}$ *is the normalized wave packet operator* (5.5.5).

Corollary 5.5.6. (The Paley-Wiener theorem). *The spherical Fourier transform maps the space* $C_c^\infty(C)$ *bijectively onto the space* $\mathcal{P}_C(\mathfrak{h}^*)$ *for any* W*-invariant compact convex set* $C \subset A \simeq \mathfrak{a}$.

Corollary 5.5.7. (The Plancherel theorem). *The spherical Fourier transform extends to a unitary isometry*

$$\mathcal{H}\colon\ L^2(G//K;\chi_l) \to L^2\Big(i\mathfrak{a}^*, \frac{d\lambda}{|W|4^{n|l|}|c(\lambda,m_-)|^2}\Big).$$

Proof. Use the inversion formula for $f = f_1 * \widetilde{f_1}$, $\widetilde{f_1}(g) = \overline{f_1(g^{-1})}$. $\square$

Notes for Chapter 5

The theory of spherical functions (corresponding to the trivial K-type) is a beautiful part of harmonic analysis going back to the work of Gel'fand, Godement (for the abstract setting), and Harish-Chandra (in the concrete setting for a Riemannian symmetric space). The theory has been exposed in textbooks [36, 23] to which we refer for further reading.

The main point of this chapter is that the theory of spherical functions corresponding to one-dimensional K-types admits a treatment as explicit and of the same level of difficulty as for the trivial K-type. The work of this chapter was motivated by [20, 21] where (among other things) the rank one situation was worked out. For example formula (5.2.4) in the rank one case can be found in [21, Theorem 2.1]. For nontrivial K-types Theorem 5.1.10 is due to Shimeno, whose proof is along the same lines as the corresponding result in case $\chi = 1_K$ (using the integral formula (5.4.1), see [70]). Our proof is somewhat different and purely algebraic.

This Page Intentionally Left Blank

Part II:
Harmonic Analysis on Semisimple Symmetric Spaces

Henrik Schlichtkrull

Contents, Part II

LECTURE 1

Introduction

In these lectures my goal will be to explain some recent joint work with Erik van den Ban on harmonic analysis on semisimple symmetric spaces. In the first lecture I intend to give some motivation and background information. The following seven lectures will be more precise on definitions and statements, though I will have to omit many details.

Harmonic analysis, in its commutative and noncommutative forms, is currently one of the most important and powerful areas in mathematics. It may be defined broadly as the attempt to decompose functions by superposition of some particularly simple functions, as in the classical theory of Fourier decompositions. To be more explicit, let X be a space acted on by a group G. Assume that this action leaves invariant a positive measure dx on X. Then there is a natural unitary representation ℓ (the *regular representation*) of G on the Hilbert space $L^2(X)$ of square integrable functions on X. The aim of *harmonic analysis on* X is to decompose this representation into irreducible subrepresentations. Under mild assumptions on G such a decomposition is possible within direct integral theory; this is known as the "abstract Plancherel formula." However, X and G will usually have more structure, and then a more explicit form of the decomposition is desirable. Typically, G will be a Lie group and X will be the homogeneous space G/H, where H is a closed subgroup. Very often there will be some differential operators on X which commute with the action of G (hence called *invariant differential operators*), and which are essentially selfadjoint operators on $L^2(X)$. Then G preserves their spectral decomposition, and thus the solution of the spectral problem for these operators will lead to decompositions of ℓ into subrepresentations, which at best happen to be irreducible, and at least give a first step toward the complete decomposition. The spectral theory of the invariant differential operators thus becomes an important tool in the harmonic analysis (sometimes harmonic analysis is simply defined this way).

From an explicit decomposition of a function f on X a *Fourier transform* is obtained. As is well known from classical analysis, such a transform is extremely useful for example in solving differential equations. The differential equations of primary interest happen to be those which are invariant under the transformation group G (or G could be chosen such that it preserves a given differential equation of particular interest). Thus the theories of harmonic analysis and of invariant differential operators on X are closely related. When Sophus Lie developed his theory of transformation groups he was motivated by the intent to apply it to differential equations. Thus, to him the group was a tool in the study of the differential equations. Since then the mathematical focus has been shifted somewhat. The space G/H has become at least as fundamental as its invariant differential operators, which primarily serve as a tool for the harmonic analysis on G/H; in some sense this is the opposite of Lie's way of thinking (see [122, 154]).

Before I continue describing the goal of the lectures, I would like to give some simple examples.

Example 1.1. The Euclidean spaces. The most familiar examples of harmonic analysis are of course the ordinary theories of Fourier analysis on the torus group $\mathbf{T}$ and on Euclidean space $\mathbf{R}^n$. For example in the latter case, $X = \mathbf{R}^n$ is viewed as a homogeneous space of itself, $G = \mathbf{R}^n$ (acting by translations), with trivial subgroup H. The invariant measure is Lebesgue measure, and the invariant differential operators are just the differential operators with constant coefficients. Their eigenfunctions are the exponential functions, and hence their spectral decomposition is exactly the decomposition of functions by superposition of exponential functions (plane waves), as obtained in the classical *inversion formula*,

$$f(x) = c \int_{\mathbf{R}^n} \hat{f}(\lambda) e^{i\lambda \cdot x} \, d\lambda,$$

where $f \in C_c^\infty(X)$ and

$$\hat{f}(\lambda) = \int_X f(x) e^{-i\lambda \cdot x} \, dx,$$

and c is a nonzero constant. The Fourier transform enables us to pick out the irreducible components of the regular representation: the Fourier transform extends to an isometry of $L^2(X)$ onto $L^2(\mathbf{R}^n)$ and gives a decomposition of ℓ as the direct integral over $\lambda \in \mathbf{R}^n$ of the one-dimensional representations π_λ defined by $\pi_\lambda(a) = e^{i\lambda\cdot a}$ $(a \in \mathbf{R}^n)$,

$$\ell \simeq \int_{\mathbf{R}^n}^{\oplus} \pi_\lambda \, d\lambda,$$

the *Plancherel decomposition* for $\mathbf{R}^n$ (with respect to the group action of $G = \mathbf{R}^n$).

In the case of $\mathbf{T}$ the decomposition of the regular representation is obtained similarly from the theory of Fourier series; ℓ decomposes as the direct sum (over $\mathbf{Z}$) of all the one-dimensional representations of $\mathbf{T}$. However, since G is abelian and H is trivial in both cases, these examples are really too simple to reveal the complexities encountered in general.

Example 1.2. Euclidean space revisited. When $n \geq 2$ a more sophisticated way of looking at $\mathbf{R}^n$ is to view it as a homogeneous space of the nonabelian group $G = \mathrm{M}(n)$ of all its *motions* (isometries); then $H = \mathrm{O}(n)$ is the orthogonal group leaving the origin fixed, and G is the semidirect product of H and $\mathbf{R}^n$. In this case it is easily seen that the only invariant differential operators are the polynomials in the Laplacian L. Since all the exponential functions $e^{i\lambda\cdot x}$ with a given length of λ are eigenfunctions for L with the same eigenvalue, it is natural from the point of view of spectral theory of L to change the interpretation of the Fourier transform as follows: Instead of viewing $\hat{f}$ as an L^2-function in $\lambda \in \mathbf{R}^n$ we shall now view it as an $L^2(\mathbf{S}^{n-1})$-valued function on $\mathbf{R}^+$ by means of the polar coordinates $\lambda = p\omega$, $(p > 0, \omega \in B = \mathbf{S}^{n-1})$:

$$\hat{f}(p,\omega) = \int_X f(x) e^{-ip\omega\cdot x} dx.$$

Let $\mathcal{H} = L^2_{L^2(B)}(\mathbf{R}^+, p^{n-1}dp)$ be the space of $L^2(B)$-valued functions ϕ on $\mathbf{R}^+$ which are square integrable with respect to the measure $p^{n-1}dp$, then $\hat{f} \in \mathcal{H}$, and the Fourier transform maps $L^2(X)$ isometrically onto $\mathcal{H}$. The decomposition of the regular representation ℓ can now be read as follows. For each $p \in \mathbf{R}$ we define a representation π_p of $H \times \mathbf{R}^n$ on $L^2(B)$ by

$$\pi_p(k,y)\varphi(\omega) = e^{ipy\cdot\omega}\varphi(k^{-1}\omega), \quad (y \in \mathbf{R}^n, k \in \mathrm{O}(n)).$$

This is easily seen to give a unitary representation of the semidirect product group G. One can prove that it is irreducible for $p \neq 0$, and that $\pi_p \simeq \pi_{-p}$. Next we define a unitary representation π of G on $\mathcal{H}$ by $(\pi(g)\phi)(p) =$

$\pi_{-p}(g)(\phi(p))$, then π is equivalent with the direct integral of the π_{-p}. Let $\mathbf{1} \in L^2(\mathbf{S}^{n-1})$ denote the distinguished vector given by $\mathbf{1}(\omega) = 1$, then we have

$$\hat{f}(p) = \int_G f(gH)\pi_{-p}(g)\mathbf{1}\, dg = \pi_{-p}(f)\mathbf{1}$$

for $f \in C_c^\infty(X)$, from which it follows that the Fourier transform is a G-equivariant map from $C_c^\infty(X)$ into $\mathcal{H}$. It follows from the above that the Fourier transform extends to an isometry of $L^2(X)$ onto $\mathcal{H}$, and we have

$$\ell \simeq \pi \simeq \int_{\mathbf{R}^+}^{\oplus} \pi_{-p}\, p^{n-1}\, dp,$$

the Plancherel decomposition for $\mathbf{R}^n$ with respect to the group action of $G = M(n)$. The inversion formula can be reformulated as follows: for $f \in C_c^\infty(X)$ we get

$$f(gH) = c \int_{\mathbf{R}^+} \langle \hat{f}(p) | \pi_{-p}(g)\mathbf{1}\rangle\, p^{n-1}\, dp,$$

where $\langle\cdot|\cdot\rangle$ is the sesquilinear form on the Hilbert space $L^2(B)$.

Note that we have got an essentially different theory of harmonic analysis on the same space X by choosing another group G of transformations. For this reason it is more correct to speak of harmonic analysis on X *with respect to* G, rather than just on X.

Example 1.3. Compact homogeneous spaces. The classical theory of Fourier series on $\mathbf{T}$ has a far reaching generalization as follows. Let G be any compact topological group endowed with its normalized Haar measure.

Let me first recall the famous theorem of Peter and Weyl. Let $\hat{G}$ denote the set of equivalence classes of irreducible representations of G, and for $\delta \in \hat{G}$ let V_δ be a Hilbert space on which δ can be realized (I use the customary abuse of notation by not distinguishing a representation from its equivalence class). Let $\check{\delta}$ be the contragradient representation, realized on the dual space $V_{\check{\delta}} = V_\delta^*$. There is a natural map from $V_\delta \otimes V_\delta^*$ into $L^2(G)$, the *matrix coefficient* map, defined by $v \otimes v^* \mapsto \dim(\delta)^{1/2}\langle v, \check{\delta}(\cdot)v^*\rangle$. It is easily seen that this map is a $G \times G$-homomorphism of the tensor product into $L^2(G)$ with the left times right action, and it follows from the Schur orthogonality relations that it is an isometry. Identifying $V_\delta \otimes V_\delta^*$ with its matrix coefficient image the *Peter-Weyl theorem* states that we have the orthogonal direct sum decomposition

$$L^2(G) = \oplus_{\delta\in\hat{G}} V_\delta \otimes V_\delta^*.$$

This gives the decomposition of the regular (left times right) representation of $G \times G$ on $L^2(G)$ into irreducible subrepresentations (harmonic analysis on G with respect to $G \times G$).

Let H be a closed subgroup of G, then it easily seen that the homogeneous space G/H inherits an invariant measure from the Haar measure on G. It was observed by Cartan (in the Lie case) and Weyl (in general) that the Peter-Weyl theorem has the following generalization,

$$L^2(G/H) = \oplus_{\delta\in\hat{G}} V_\delta \otimes (V_\delta^*)^H, \tag{1.1}$$

where $(V_\delta^*)^H$ is the space of $\check{\delta}(H)$-fixed vectors in V_δ^*. The decomposition is orthogonal and equivariant for the G-action (G acts on the tensor products by its action on the first factors), and thus it gives the decomposition of ℓ (harmonic analysis on G/H with respect to G). Its derivation from the Peter-Weyl theorem as formulated above is immediate, once we observe that $L^2(G/H)$ may be identified with the space of right H-invariant functions in $L^2(G)$. Note that the decomposition only contains the representations δ for which $(V_\delta^*)^H \neq 0$, or equivalently, for which $V_\delta^H \neq 0$. If $\dim V_\delta^H \leq 1$ for all δ, the decomposition of ℓ is said to be *multiplicity free.*

Example 1.4. The spheres. Let X be the n-sphere $\mathbf{S}^n$, viewed as the homogeneous space $\mathrm{O}(n+1)/\mathrm{O}(n)$. This is a particular example of the situation in the previous example. In this case the harmonic analysis on $\mathbf{S}^n$ with respect to $\mathrm{O}(n)$ is classical: it is the theory of *spherical harmonics.* Since it is probably familiar to most readers, it may serve as a good example. Recall that a spherical harmonic (of degree k) on $\mathbf{S}^n$ is the restriction of a harmonic homogeneous polynomial (of degree k) on $\mathbf{R}^{n+1}$. Equivalently, it is an eigenfunction for the Laplace operator on X (with the eigenvalue $-k(n-1+k)$). Let H_k be the space of spherical harmonics of degree k, then H_k (as an eigenspace for L) is G-invariant, and we have the orthogonal decomposition

$$L^2(\mathbf{S}^n) = \oplus_{k=0}^{\infty} H_k.$$

In fact each H_k is irreducible, and this decomposition is thus an explicit form of (1.1) for this case, with a multiplicity free decomposition. The one-dimensional subspace $H_k^{\mathrm{O}(n)}$ of H_k is the space of *zonal* spherical harmonics of degree k. Note that the decomposition is realized as a spectral decomposition for the invariant differential operator L, in accordance with the view on harmonic analysis suggested earlier.

In the examples above there is an essential difference between the non-compact $\mathbf{R}^n$ and the compact $\mathbf{S}^n$. In the former case the Plancherel decomposition is a direct integral over a continuous parameter, and in the latter case it is a direct sum over a discrete parameter. In general one expects a combination of these phenomena, such that the decomposition of ℓ will invoke both continuous and discrete parameters.

A class of homogeneous spaces, for which the program of harmonic analysis via spectral decomposition of invariant differential operators is particularly compelling, is the class of symmetric spaces. A *symmetric pair* may be defined as a pair (G, H) with a Lie group G, for which there is an involution σ of G such that $G^\sigma_e \subset H \subset G^\sigma$, where G^σ is the subgroup of fixed points for σ and G^σ_e denotes its identity component. A *symmetric space* is a space X for which there exists a symmetric pair such that $X = G/H$. The map $gH \mapsto \sigma(g)H$ of X to itself is then called the *symmetry* around the origin $o = eH$. By parallel transport there are symmetries around all other points of X as well.

One can prove that a connected smooth manifold X is a symmetric space if and only if there exists on it an affine connection, for which the reflexion in geodesics around any point x extends to an affine diffeomorphism S_x of X. If X is such a manifold with a given point of origin it can be realized as the symmetric space corresponding to a certain canonically determined symmetric pair $(G(X), H(X))$ of subgroups of the group of affine transformations of X ($G(X)$ is the group of "displacements" generated by all the products $S_xS_y, (x, y \in X)$, and $H(X)$ is the stabilizer of the origin). Note however, that if $X = G/H$ is a symmetric space, then G may differ from $G(X)$. The same space with the same symmetries and the same point of origin may thus correspond to several symmetric pairs, as in Examples 1.1 and 1.2 above, where $\mathbf{R}^n$ is the symmetric space corresponding to the symmetric pairs $(\mathbf{R}^n, \{0\})$ and $(\mathrm{M}(n), \mathrm{O}(n))$, respectively. In this case $(G(X), H(X))$ is the former pair.

In these lectures I shall only consider harmonic analysis on symmetric spaces. Clearly Examples 1.1 and 1.2 mentioned above fall into this category; the symmetry around a point is the reflexion in the point.

Example 1.5. **The group case.** Let $'G$ be a Lie group, let $G = {}'G \times {}'G$, and define $\sigma: G \to G$ by $\sigma(x, y) = (y, x)$. Then $H = G^\sigma$ is the diagonal, and via the mapping $(x, y) \mapsto xy^{-1}$ we have that the symmetric space G/H is isomorphic to $'G$, viewed as a homogeneous space for the left times right action of $'G \times {}'G$. This example, referred to as the *group case* in the following, shows some of the scope of the program of harmonic analysis on all symmetric spaces: it contains as a subprogram that of doing harmonic analysis on all Lie groups.

In fact, I shall restrict attention even further than just to symmetric spaces; they will also be required to be *semisimple* or, slightly more general, *reductive.* In order to explain these notions, I have to discuss some of the geometric structure of X a bit. Let $\mathfrak{g}$ be the Lie algebra of G, and let σ denote also the involution of $\mathfrak{g}$ obtained from that of G by differentiation. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces for σ, then $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{q}$ may be identified with the tangent space of X at o. Associated with the affine connection on X there is a canonical 2-form, the *Ricci curvature tensor* (or the Ricci *form*), on the tangent space TX. It is G-invariant, and at o it is given by

$$r(X, Y) = \mathrm{Tr}_{\mathfrak{q}}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$$

for $X, Y \in \mathfrak{q}$. The space X is called *semisimple* if this form is nondegenerate and symmetric (the latter property actually implies that r is a constant multiple of the restriction of the Killing form $B(\cdot,\cdot)$ of $\mathfrak{g}$ to $\mathfrak{q} \times \mathfrak{q}$.) In Example 1.5 we have that r can be identified with the Killing form of the Lie algebra $'\mathfrak{g}$ of $'G$, and thus $'G$ is semisimple as a symmetric space for $'G \times {}'G$ if and only if it is a semisimple Lie group. It is clear that the Ricci tensor gives rise to a G-invariant *pseudo-Riemannian structure* on a semisimple symmetric space X.

A symmetric pair (G, H) is called a *semisimple symmetric pair* if G is semisimple. One can prove that a symmetric space X is semisimple if and only if there is a semisimple symmetric pair (G, H) with G acting on X by affine transformations, such that X is the symmetric space G/H (in particular, if X is semisimple, then the group $G(X)$ of displacements is semisimple). Again it is noted that the same space X with the same symmetries may correspond to several symmetric pairs (G, H), among which only some are semisimple. In the following, when I speak of a semisimple symmetric space G/H, it is to be understood that (G, H) is a semisimple symmetric pair.

As motivation for restricting the attention to semisimple symmetric spaces it is noted that an irreducible symmetric space (one that has no nontrivial invariant "subsymmetric spaces") is either semisimple or one-dimensional. Note, however, that none of the spaces mentioned in Examples 1.1, 1.2, and 1.4 are semisimple, since the Ricci tensor in these cases is the trivial 2-form. For this reason it is sometimes more convenient to extend focus a bit and consider *reductive* symmetric spaces. By definition, in a reductive symmetric space every invariant subsymmetric space has an invariant complementary subsymmetric space. Equivalently, a symmetric space is reductive if it is a symmetric space G/H for a symmetric pair with G reductive (a *reductive symmetric pair*). The pairs in Examples 1.1 and 1.4 are reductive, whereas that in Example 1.2 (where G is a solvable Lie group) is not reductive. Note that reductive symmetric spaces are only slightly more general than semisimple symmetric spaces, since any reductive group G is the product of its semisimple part and its center.

Example 1.6. Hyperbolic spaces. Let p and q be positive integers, and let $X = X_{p,q}$ be the *real hyperbolic space*

$$\{x \in \mathbf{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}$$

(if $p = 1$ it is also required that $x_1 > 0$ to get only one sheet of the hyperboloid), then X is the symmetric space corresponding to the pair $(\mathrm{SO}_e(p,q), \mathrm{SO}_e(p-1,q))$ (the involution of G is given by $\sigma(g) = IgI$ where I is the diagonal matrix with diagonal entries $1, -1, \dots, -1$). Thus X is a semisimple symmetric space except if $p = q = 1$ (in which case $X \simeq \mathbf{R}$ is reductive). It has a pseudo-Riemannian structure of index $(p-1, q)$. Similarly, one can define hyperbolic spaces over the complex and quaternion fields; when viewed as real manifolds they (or rather, their projective images) correspond to the symmetric pairs $(\mathrm{SU}(p,q), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(p-1,q))$ and $(\mathrm{Sp}(p,q), \mathrm{Sp}(1) \times \mathrm{Sp}(p-1,q))$ (when formulated suitably, the construction can be given a sense even for the Cayley octonions, but only when (p, q) is $(2,1)$ or $(1,2)$, where one gets that G is the exceptional group $G = \mathrm{F}_{4(-20)}$).

Example 1.7. Symmetric spaces of $\mathrm{SL}(2, \mathbf{R})$. Let $G = \mathrm{SL}(2, \mathbf{R})$. There are two (nonconjugate) involutions of G, given by

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad \text{and} \quad \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

To these involutions correspond three symmetric spaces: G/G^θ, G/G^σ, and G/G^σ_e. The first two can be realized within $\mathfrak{sl}(2, \mathbf{R})$ as the spaces $\{Y \mid B(Y, Y) = \epsilon\}$, with $\epsilon = \pm 1$, respectively; the action of G is then the adjoint action. It follows that they are equal to the spaces $X_{1,2}$ and $X_{2,1}$ of the previous example. The third is a double cover of the second (here the action does not factor through the adjoint map).

Example 1.8. Riemannian symmetric spaces. Let G be a connected linear semisimple Lie group, and let θ be the Cartan involution of G. Then the fixed point group $K = G^\theta$ is a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, then the Killing form $B(\cdot, \cdot)$ is positive definite on $\mathfrak{p}$. Thus G/K is a semisimple symmetric space, and its structure is Riemannian.

As it is apparent from the title, the goal of these lectures is to do harmonic analysis on semisimple symmetric spaces. Looking back at the definition I gave of harmonic analysis, it should first of all be noted that a semisimple symmetric space does carry an invariant measure associated to the nondegenerate 2-form r (I shall return to this measure in the next lecture). Moreover, it is also encouraging for the mentioned program of obtaining spectral decompositions that the algebra $\mathbf{D}(G/H)$ of all the invariant differential operators on G/H is known to be commutative, and that the formally self-adjoint ones among these operators are essentially self-adjoint operators on $L^2(G/H)$ (I shall return to these points in Lecture 4). Thus they will have a simultaneous G-invariant spectral decomposition.

The program of finding an explicit Plancherel decomposition for a general semisimple symmetric space G/H is too ambitious a task for these lectures. In fact, as a consequence of Example 1.5 it would necessarily extend Harish-Chandra's work on harmonic analysis for semisimple groups. Indeed such a result does not exist in the mathematical literature of today (though a result like that has been announced by Oshima and Sekiguchi) — only special cases have been treated. Most of the known examples are spaces of *rank one* (the notion of the rank of G/H will be defined in the next lecture). In particular, all the spaces mentioned in Example 1.6 are of rank one, and for these spaces the above-mentioned "Plancherel program" has been carried out. The basic idea is to introduce a kind of polar coordinates on X, in which the radial part of the Laplace-Beltrami operator L (which exists on any semisimple symmetric space, thanks to the pseudo-Riemannian structure) becomes a singular ordinary differential operator, to which a general theory of Weyl, Kodaira, and Titchmarsh can be applied. However, this theory is not applicable in higher rank, since one cannot reduce in any way to an ordinary differential operator. (See the notes at the end for more details and a list of references.)

Apart from the cases mentioned above, the harmonic analysis program has also been carried out in the class of Riemannian symmetric spaces (see Example 1.8). In this case explicit inversion and Plancherel formulae are known from the work of Harish-Chandra and Helgason. I shall return to this case later, as a motivating example.

In these lectures I shall consider general semisimple symmetric spaces, but with a more moderate goal than the full decomposition of ℓ. I shall now describe this goal. It is known from Harish-Chandra's work on the group case mentioned in Example 1.5 that ℓ decomposes into several series of representations, the most famous of which are the "discrete series" and the "(minimal) principal series." The former enters discretely into the decomposition of ℓ (as in Example 1.3) and the latter enters as a direct integral over a continuous parameter (as in Examples 1.1 and 1.2). A similar phenomenon is expected (and indeed seen in the cases where the program has been carried out) for the general semisimple symmetric space. In short, the goal of these lectures will be the generalization of the (minimal) principal series part, which will be called "the most continuous part" of the decomposition (the reason for this terminology is that in general one expects several series of representations, each parametrized with a continuous parameter running in a finite dimensional real vector space, and the series that we shall consider here are those for which this parameter space has the highest dimension).

Even this goal is out of reach in eight lectures, at least with full attention to details, but at least we shall reach the stage where the main theorem concerning this decomposition can be stated (Theorem 7.1). In lectures 2–6 leading up to this, the basic structure of G/H and the related representations of G will be developed. Finally, Lectures 7 and 8 will be devoted to a sketch of the proof of the main theorem.

In the notes at the end some historical remarks are given, together with references for the skipped proofs. In particular, the notes to this lecture contain some hints about the discrete series for G/H. I am not going to explain this series any further during these lectures, since I shall not be using it.

LECTURE 2

Structure theory

In the Introduction I defined the notion of a semisimple symmetric space $X = G/H$. In this lecture I shall discuss some of the basic structure of X.

For simplicity it is assumed that the semisimple Lie group G is connected and linear, and that the subgroup H is connected (for various reasons one would actually like to consider a more general class, the so-called Harish-Chandra class, of reductive symmetric spaces, but I shall not do so here, since the generalization usually is rather straightforward). Let θ be the Cartan involution of G with corresponding maximally compact subgroup K, and with the corresponding Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $G = K \exp \mathfrak{p}$.

Recall that σ is the involution of G for which we have $H = G^\sigma_e$. In general it may not be the case that σ and θ commute, but this can always be accomplished by replacing σ with a conjugate $\sigma^g = \operatorname{Ad} g^{-1} \circ \sigma \circ \operatorname{Ad} g$ for some $g \in G$.

Proposition 2.1. *There exists $g \in G$ such that the conjugate involution σ^g commutes with θ.*

Proof. I shall not stop to prove Proposition 2.1 here. See the notes for references. $\square$

Replacing σ by a conjugate corresponds to replacing the chosen origin of X with another point. Since this does not affect the harmonic analysis on X with respect to G, we shall from now on assume that this has been done. At the same time H is replaced by a conjugate. Thus we assume that σ commutes with θ, and then we also have that $\sigma(K) = K$ and $\theta(H) = H$. Thus H is a connected linear reductive group and $K \cap H$ is a maximal compact subgroup. In particular, it follows that $K \cap H$ is connected.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g}$ induced by σ (I shall use the same symbol for an involution on G and its differential on $\mathfrak{g}$). Then we have that $\mathfrak{h}$ and $\mathfrak{q}$ are θ-invariant, and that $\mathfrak{k}$ and $\mathfrak{p}$ are σ-invariant. Moreover we have the joint decomposition

$$\mathfrak{g} = \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{k}\cap\mathfrak{q} \oplus \mathfrak{p}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q}. \tag{2.1}$$

Note also that since the two involutions commute, their product $\sigma\theta$ is also an involution. Hence we have three symmetric pairs: $(G,K) = (G,G^\theta)$, $(G,H) = (G,G_e^\sigma)$, and $(G,G_e^{\sigma\theta})$. For later purposes it will be useful with some names related to the latter pair: Let

$$G_+ = G_e^{\sigma\theta}, \quad \mathfrak{g}_+ = \mathfrak{g}^{\sigma\theta} = \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q}, \quad \text{and} \quad \mathfrak{g}_- = \mathfrak{k}\cap\mathfrak{q} \oplus \mathfrak{p}\cap\mathfrak{h}.$$

Since $\theta(G_+) = G_+$ we have that G_+ is a connected linear reductive group with the maximal compact subgroup $K\cap G_+ = K\cap H$.

Example 2.1. Let X be the real hyperbolic space $G/H = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$ as in Example 1.6. In this case $K = \mathrm{SO}(p)\times\mathrm{SO}(q)$ and the decomposition (2.1) of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p,q)$ is indicated in the following diagram, which shows where the matrices in each of the four subspaces have their nonzero entries.

$$\begin{array}{r|c|c|c|}
 & \overbrace{\qquad\quad}^{1} & \overbrace{\qquad\quad}^{p-1} & \overbrace{\qquad\quad}^{q} \\
\hline
1\{ & 0 & \mathfrak{k}\cap\mathfrak{q} & \mathfrak{p}\cap\mathfrak{q} \\
\hline
p-1\{ & \mathfrak{k}\cap\mathfrak{q} & \mathfrak{k}\cap\mathfrak{h} & \mathfrak{p}\cap\mathfrak{h} \\
\hline
q\{ & \mathfrak{p}\cap\mathfrak{q} & \mathfrak{p}\cap\mathfrak{h} & \mathfrak{k}\cap\mathfrak{h} \\
\hline
\end{array}$$

It follows that $\mathfrak{g}_+ \simeq \mathfrak{so}(p-1)\times\mathfrak{so}(1,q)$.

For the semisimple group G there are four important decompositions:

$$\begin{aligned}
&G = K\exp\mathfrak{p} && \text{(the Cartan decomposition),}\\
&G = KAK && \text{(the } KAK \text{ decomposition),}\\
&G = KAN && \text{(the Iwasawa decomposition),}\\
&G = \cup_{w\in W}\bar{N}\tilde{w}P && \text{(the Bruhat decomposition).}
\end{aligned}$$

(The KAK decomposition is sometimes also called the Cartan decomposition.) In this and the following lecture we shall be looking for related decompositions for the semisimple symmetric space G/H.

The Cartan decomposition $G = K \exp \mathfrak{p} \simeq K \times \mathfrak{p}$ implies that the symmetric space G/K as a manifold is diffeomorphic via the exponential map to the Euclidean space $\mathfrak{p}$. The direct analog of this, that $G/H \simeq \mathfrak{q}$, is false in general. For this reason the exponential map $\exp\colon \mathfrak{q} \to G/H$ is most useful locally around the origin. The following proposition may be seen as a generalization of the Cartan decomposition.

Proposition 2.2. *The map $(k, Y, X) \mapsto k \exp Y \exp X$ is a real analytic diffeomorphism of $K \times (\mathfrak{p} \cap \mathfrak{q}) \times (\mathfrak{p} \cap \mathfrak{h})$ onto G.*

It follows that G/H is diffeomorphic to the vector bundle $K \times_{K \cap H} \mathfrak{p} \cap \mathfrak{q}$ over $K/K \cap H$ (where $K \cap H$ acts on $\mathfrak{p} \cap \mathfrak{q}$ by the adjoint action).

Proof. Clearly the map is real analytic. We will now construct an inverse map. Let $g \in G$ be given. By $G = K \exp \mathfrak{p}$ there is a unique $S \in \mathfrak{p}$ such that $g \in K \exp S$.

Let us analyze the relation we want, that is $g \in K \exp Y \exp X$ with $Y \in \mathfrak{p} \cap \mathfrak{q}$ and $X \in \mathfrak{p} \cap \mathfrak{h}$. If we had this we would have

$$\exp 2S = (\theta g)^{-1} g = \exp X \exp 2Y \exp X, \tag{2.2}$$

and hence also

$$\exp 2\sigma S = \exp X \exp -2Y \exp X.$$

Eliminating Y this would imply

$$\exp 2\sigma S = \exp 2X \exp -2S \exp 2X. \tag{2.3}$$

We shall now solve this equation with respect to X. We use Lemma 2.3 below, which shows that if $T \in \mathfrak{p}$ is defined by

$$\exp 2T = \exp -S \exp 2\sigma S \exp -S, \tag{2.4}$$

then (2.3) is equivalent with

$$\exp 2X = \exp S \exp T \exp S. \tag{2.5}$$

This analysis shows how to obtain X. Given $g \in K \exp S$ we define X by (2.5), where T is defined by (2.4). Next we define Y by (2.2) and k by $g = k \exp Y \exp X$. It is easily verified that $g \mapsto (k, Y, X)$ is the inverse map we are looking for. $\square$

Lemma 2.3. *Let* $U, S \in \mathfrak{p}$ *be given, and let* $T \in \mathfrak{p}$ *be defined by the expression* $\exp 2T = \exp -S \exp U \exp -S$. *Then the equation* $\exp U = \exp X \exp -2S \exp X$ *has the unique solution* $X \in \mathfrak{p}$ *given by* $\exp X = \exp S \exp T \exp S$.

Proof. The proof is straightforward. □

We shall now see how the KAK-decomposition can be generalized to G/H. In the next lecture I will then take a look at the other decompositions.

First I would like to recall the restricted root theory for G/K. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ (such a space is called a *Cartan subspace* for G/K). It is unique up to conjugacy by K. The elements of $\operatorname{ad}\mathfrak{a}$ can be simultaneously diagonalized, with real eigenvalues (for this reason $\mathfrak{a}$ is said to be split). The nonzero eigenspaces

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\} \tag{2.6}$$

with $\alpha \in \mathfrak{a}^*$ nonzero are called the *root spaces* and the corresponding α's the *restricted roots*. The set of restricted roots, denoted $\Sigma(\mathfrak{a}, \mathfrak{g})$, is a root system (it satisfies the axioms of an abstract root system). Note however that in contrast to the diagonalization of a Cartan subalgebra of a complex Lie algebra where the root spaces are always one-dimensional, the root α has a *multiplicity* $m_\alpha = \dim \mathfrak{g}_\alpha$ which may exceed 1. Moreover, both α and 2α can be roots. The eigenspace $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. By maximality of $\mathfrak{a}$ we have $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. Denoting the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ by $\mathfrak{m}$, we have $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, and hence

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus_{\alpha \in \Sigma(\mathfrak{a},\mathfrak{g})} \mathfrak{g}_\alpha.$$

Choose a positive set $\Sigma^+(\mathfrak{a}, \mathfrak{g})$ for $\Sigma(\mathfrak{a}, \mathfrak{g})$, and let $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ denote the sums of the root spaces for the positive and negative roots, respectively, then we get the *Iwasawa* and *Bruhat decompositions* of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

A *regular element* $H \in \mathfrak{a}$ is an element for which $\alpha(H) \neq 0$ for all $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{g})$. A connected component of the set of regular elements is called an open *Weyl chamber*; in particular we have the *positive chamber* $\mathfrak{a}^+$ corresponding to $\Sigma^+(\mathfrak{a},\mathfrak{g})$, where the positive roots take positive values. Finally, the *Weyl group* $W(\mathfrak{a},\mathfrak{g})$ is defined as the quotient of the normalizer $N_K(\mathfrak{a})$ with the centralizer $M = Z_K(\mathfrak{a})$; it acts naturally on $\mathfrak{a}$ and coincides via this action with the reflection group of the root system $\Sigma(\mathfrak{a},\mathfrak{g})$. In particular, it acts simply transitively on the Weyl chambers as well as on the different choices for $\Sigma^+(\mathfrak{a},\mathfrak{g})$.

Let $A = \exp\mathfrak{a}$ and $A^+ = \exp\mathfrak{a}^+$, then the KAK decomposition says that every element $g \in G$ can be written as $g = k_1ak_2$ with $k_1,k_2 \in K$ and with $a \in A$. The $a \in A$ is uniquely determined up to conjugacy by $W(\mathfrak{a},\mathfrak{g})$; in particular it can be chosen in the closure $\overline{A^+}$ of A^+. This decomposition is the basis for the use of *polar coordinates* on G/K: the map $(kM,a) \mapsto kaK \in G/K$ maps $K/M \times \overline{A^+}$ onto G/K and it maps $K/M \times A^+$ diffeomorphically onto an open dense subset of G/K.

We now return to the setting of semisimple symmetric spaces. Let $\mathfrak{a}_q$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. Since $\mathfrak{g}_+ = \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q}$ is the Cartan decomposition of $\mathfrak{g}_+$, and $K\cap H$ is a maximal compact subgroup of G_+, we can apply the theory outlined above to G_+ and obtain that $\mathfrak{a}_q$ is unique up to conjugacy by $K\cap H$. Moreover, let $\Sigma(\mathfrak{a}_q,\mathfrak{g}_+)$ be the corresponding set of restricted roots, $\Sigma^+(\mathfrak{a}_q,\mathfrak{g}_+)$ a set of positive roots, $\mathfrak{a}_q^+$ the corresponding positive chamber, $A_q^+ = \exp\mathfrak{a}_q^+$ and $W_{K\cap H} = N_{K\cap H}(\mathfrak{a}_q)/Z_{K\cap H}(\mathfrak{a}_q) = W(\mathfrak{a}_q,\mathfrak{g}_+)$ the Weyl group. The KAK-decomposition applied to G_+ gives that $G_+ = (K\cap H)\overline{A_q^+}(K\cap H)$.

Theorem 2.4. *(KA_qH-decomposition.) Every element $g \in G$ has a decomposition as $g = kah$ with $k \in K$, $a \in A_q$ and $h \in H$. In this decomposition the a is unique up to conjugacy by $W_{K\cap H}$. The mapping*

$$(kZ_{K\cap H}(\mathfrak{a}_q),a) \mapsto kaH \in G/H \tag{2.7}$$

maps $K/Z_{K\cap H}(\mathfrak{a}_q) \times \overline{A_q^+}$ onto G/H, and it maps $K/Z_{K\cap H}(\mathfrak{a}_q) \times A_q^+$ diffeomorphically onto an open dense subset of G/H.

Proof. This follows from Proposition 2.2 combined with the KAK decomposition for G_+ and the Cartan decomposition $H = (H\cap K)\exp(\mathfrak{p}\cap\mathfrak{h})$. □

The map (2.7) is called *polar (*or *spherical) coordinates* on X.

Example 2.2. Let X be as in Example 2.1, that is

$$\begin{aligned} X &= \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q) \\ &= \{x \in \mathbf{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\} \end{aligned}$$

(with $x_1 > 0$ if $p = 1$). For $1 \le i, j \le p+q$ let E_{ij} denote the $(p+q) \times (p+q)$ matrix with 1 on the (i,j)th entry and zero on all other entries, and let $Y = E_{p+q,1} + E_{1,p+q}$. Then $\mathfrak{a}_q = \mathbf{R}Y$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. The centralizer of Y in $K \cap H$ consists of the elements of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $V \in \mathrm{SO}(p-1)$ and $W \in \mathrm{SO}(q-1)$. Hence $K/Z_{K\cap H}(\mathfrak{a}_q)$ can be identified with $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$, and the polar coordinate map is then given by

$$\begin{aligned} &\mathbf{S}^{p-1} \times \mathbf{S}^{q-1} \times \mathbf{R} \ni (v,w,t) \\ &\quad \mapsto x(v,w,t) = (v_1 \cosh t, \ldots, v_p \cosh t, w_1 \sinh t, \ldots, w_q \sinh t) \in X. \end{aligned}$$

Note that if $p = 1$ or $q = 1$ we should read $\mathbf{S}^0$ as $\{1\}$. Note also that there is a significant difference between the cases $q > 1$ and $q = 1$. In the former case we have $x(v,w,-t) = x(v,-w,t)$ and the map is a diffeomorphism of $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1} \times \mathbf{R}^+$ onto an open dense set, whereas in the latter case one has to use both signs on t in order to get an open dense set in X. In the terms of Theorem 2.4, the open chamber $\mathfrak{a}_q^+$ is different in the two cases. The explanation is that (as mentioned in Example 2.1) $\mathfrak{g}_+ = \mathfrak{so}(p-1) \times \mathfrak{so}(1,q)$, which means that $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$ and $W_{K\cap H}$ are trivial when $q = 1$, whereas otherwise $W_{K\cap H} \simeq \{\pm 1\}$.

It will be very important for us to be able to integrate over G/H. As mentioned in the Introduction, a semisimple symmetric space does have an invariant measure. This measure is unique up to scalar multiplication. The following theorem gives a formula for it in polar coordinates.

For $\alpha \in \mathfrak{a}_q^*$ we define $\mathfrak{g}_\alpha$ in analogy with (2.6) by

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H,Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}_q\},$$

and we denote by $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ the set of those nonzero α's for which $\mathfrak{g}_\alpha \neq 0$. As we shall soon discuss this set is a root system. In particular, this means that we can select a positive set $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$. Note that $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) \subset \Sigma(\mathfrak{a}_q, \mathfrak{g})$. We require that $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ is chosen such that it contains the set $\Sigma^+(\mathfrak{a}_q, \mathfrak{g}_+)$. Note also that $\sigma\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$, which shows that $\mathfrak{g}_\alpha$ decomposes as $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$ where $\mathfrak{g}_\alpha^\pm = \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm$. Let $m_\alpha = \dim \mathfrak{g}_\alpha$ be the *multiplicity* of α, and define $m_\alpha^\pm = \dim \mathfrak{g}_\alpha^\pm$, then $m_\alpha = m_\alpha^+ + m_\alpha^-$, and m_α^+ is the multiplicity of α as a member of $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$. Let

$$J(Y) = \prod_{\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})} \sinh^{m_\alpha^+} \alpha(Y) \cosh^{m_\alpha^-} \alpha(Y)$$

for $Y \in \mathfrak{a}_q$.

Theorem 2.5. *An invariant measure dx on $X = G/H$ is given by*

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_q^+} f(k \exp Y \cdot o) J(Y)\, dY\, dk,$$

where dY denotes a Lebesgue measure on $\mathfrak{a}_q$ and dk a Haar measure on K, and where the Jacobian $J(Y)$ is given above.

Proof. I give the proof only in the special case of the example below. □

Example 2.3. As before let X be the real hyperbolic space. On $\mathbf{R}^{p+q}$ the Lebesgue measure $dx = dx_1 \dots dx_{p+q}$ is invariant for $G = \mathrm{SO}_e(p, q)$. If we use the polar coordinates $(v, r) \in \mathbf{S}^{p-1} \times \mathbf{R}^+$ and $(w, s) \in \mathbf{S}^{q-1} \times \mathbf{R}^+$ on the first p and last q entries, respectively, we get

$$dx = dv\, dw\, r^{p-1} dr\, s^{q-1} ds,$$

where dv and dw are the rotation invariant measures on the two spheres. Restricting to the open set where $r > s$ we can write the pair (r, s) as $(\xi \cosh t, \xi \sinh t)$, and by computation of the Jacobian we have $dr\, ds = \xi d\xi\, dt$. Hence we get in these coordinates that

$$dx = dv\, dw\, \xi^{p+q-1} d\xi \cosh^{p-1} t \sinh^{q-1} t\, dt.$$

Now X is given by $\xi = 1$, and we get that the measure

$$dv\, dw\, \cosh^{p-1} t \sinh^{q-1} t\, dt$$

is invariant on X (along the way we have implicitly assumed that $p, q > 1$ but the argument is quite easily extended to the other cases as well). This result is in accordance with Theorem 2.5. Indeed, we have seen that $\mathfrak{a}_q = \mathbf{R}Y$ where $Y = E_{p+q,1} + E_{1,p+q}$. It is easily seen that $\Sigma(\mathfrak{a}_q, \mathfrak{g}) = \{\pm\alpha\}$ where $\alpha(Y) = 1$, and that the root space for α is the span of the vectors

$$X_i = -E_{1+i,1} + E_{1,i+1} + E_{p+q,i+1} + E_{1+i,p+q} \in \mathfrak{g}_-$$

for $i = 1, \ldots, p-1$ and the vectors

$$Z_j = E_{p+j,1} + E_{1,p+j} + E_{p+q,p+j} - E_{p+j,p+q} \in \mathfrak{g}_+$$

for $j = 1, \ldots, q-1$. Hence $m_\alpha^- = p-1$ and $m_\alpha^+ = q-1$.

I will end this lecture by giving some more details about $\Sigma(\mathfrak{a}_q, \mathfrak{g})$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_q$, then $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}_q$ by the maximality of $\mathfrak{a}_q$. Define the *Weyl group* of $\mathfrak{a}_q$ in $\mathfrak{g}$ by $W = W(\mathfrak{a}_q, \mathfrak{g}) = N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$. The first statement of the following proposition was mentioned earlier.

Theorem 2.6. *The set $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ is a root system. Its reflection group is naturally identified with $W(\mathfrak{a}_q, \mathfrak{g})$, and each element w in this group has a representative $\tilde{w} \in N_K(\mathfrak{a}_q)$ which at the same time also normalizes $\mathfrak{a}$.*

Proof. See the notes for a reference. □

The situation is thus that we have two root systems on $\mathfrak{a}_q$, $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ and the subset $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$. Correspondingly, we have two Weyl groups, $W = W(\mathfrak{a}_q, \mathfrak{g})$ and the subgroup $W_{K\cap H} = W(\mathfrak{a}_q, \mathfrak{g}_+)$. The quotient of these two groups turns out to be very important. If $\mathfrak{a}_q^+$ is a Weyl chamber for $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$, it contains in general several chambers for $\Sigma(\mathfrak{a}_q, \mathfrak{g})$, and these subchambers can be parametrized by $W/W_{K\cap H}$.

Example 2.4. Let X be as in Examples 2.1–2.3. We saw that $\Sigma(\mathfrak{a}_q, \mathfrak{g}) = \{\pm\alpha\}$, which is clearly a root system. The Weyl group is $W \simeq \{\pm 1\}$. We also saw that $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) = \Sigma(\mathfrak{a}_q, \mathfrak{g})$ if $q > 1$ and $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) = \emptyset$ if $q = 1$. In the latter case $W_{K\cap H}$ is strictly smaller than W.

Example 2.5. Let $G/H = \mathrm{SL}(n,\mathbf{R})/\mathrm{SO}_e(1,n-1)$. Here the involution is given by $\sigma(x) = J\theta(x)J$, where J is the diagonal matrix with entries $-1,1,\dots,1$. A maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$ is the space $\mathfrak{a}_q$ of diagonal matrices in $\mathfrak{g} = \mathfrak{sl}(n,\mathbf{R})$. Then $\mathfrak{a}_q$ is in fact at the same time maximal abelian in $\mathfrak{g}$. The restricted root system $\Sigma(\mathfrak{a}_q,\mathfrak{g})$ is then A_{n-1}, that is $\Sigma(\mathfrak{a}_q,\mathfrak{g}) = \{e_i - e_j \mid 1 \le i \ne j \le n\}$. All roots have multiplicity one in this case. The reflection group W is the corresponding group of permutations of the n entries.

It is easily seen that G_+ consists of the matrices

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in \mathrm{SL}(n,\mathbf{R}),$$

where $A \in GL(n-1,\mathbf{R})$ and $a^{-1} = \det A > 0$. Hence $\Sigma(\mathfrak{a}_q,\mathfrak{g}_+) = \{e_i - e_j \mid 2 \le i \ne j \le n\}$ and $W_{K\cap H}$ is the subgroup of W leaving the first entry fixed. Thus the quotient $W/W_{K\cap H}$ has n elements.

Example 2.6. The group case. Let G be $'G \times {'G}$ and H the diagonal, so that G/H is isomorphic to $'G$ by the map $(x,y)H \mapsto xy^{-1}$. I shall denote objects related to $'G$ with a $'$ in front of the symbol used for the similar object defined earlier for a group G. For example $'\theta$ is a Cartan involution for $'G$ and $'K$ is the corresponding maximal compact subgroup. This said, we have the following equalities: $\theta = {'\theta} \times {'\theta}$, $K = {'K} \times {'K}$ etc. A maximal abelian subspace $\mathfrak{a}_q$ of $\mathfrak{p}\cap\mathfrak{q}$ is obtained by letting $\mathfrak{a}_q = \{(-Y,Y) \mid Y \in {'\mathfrak{a}}\}$, where $'\mathfrak{a}$ is maximal abelian in $'\mathfrak{p}$, and its root system is

$$\Sigma(\mathfrak{a}_q,\mathfrak{g}) = \{\alpha \mid \exists \dot\alpha \in \Sigma({'\mathfrak{a}},{'\mathfrak{g}})\colon \alpha(-Y,Y) = \dot\alpha(Y)\}.$$

The map $\alpha \mapsto \dot\alpha$ is a bijection (the root space corresponding to α is $\mathfrak{g}_\alpha = {'\mathfrak{g}}_{-\dot\alpha} \times {'\mathfrak{g}}_{\dot\alpha}$, thus the multiplicity of α is twice the multiplicity of $\dot\alpha$). Hence $\Sigma(\mathfrak{a}_q,\mathfrak{g})$ is really a root system. Its Weyl group W is easily seen to consist of the elements w given by $w(-Y,Y) = (-\dot wY, \dot wY)$ for some $\dot w \in {'W}$. As a representative for w we can take any element $(x_1,x_2) \in K$ for which $x_1, x_2 \in {'K}$ both are representatives for $\dot w$. Clearly this element (x_1,x_2) normalizes $\mathfrak{a} = {'\mathfrak{a}} \times {'\mathfrak{a}}$; thus the final statements of Theorem 2.6 are verified for this case. In particular, if we take $x_1 = x_2$ we obtain a representative in $K \cap H$, and hence we have $W_{K\cap H} = W$ in this case.

LECTURE 3

Parabolic subgroups

In this lecture I shall begin by describing the parabolic subgroups of G related to G/H. As in the group case, parabolic subgroups are indispensable for the harmonic analysis; all the representations of G that enter in the decomposition of $L^2(G/H)$, except the discrete series, are (supposedly) constructed by means of induction from parabolic subgroups.

Recall that the *minimal parabolic subgroups* of G are the conjugates of the subgroup $P_0 = M_0A_0N_0$. Here A_0 and N_0 are the subgroups given in the Iwasawa decomposition $G = KA_0N_0$, and M_0 is the centralizer of A_0 in K (note the deviation from earlier notation; since we shall be dealing mainly with other parabolic subgroups than P_0, it is convenient to reserve M, A, and N for a better use). It follows from the Iwasawa decomposition that all minimal parabolic subgroups are conjugates of P_0 by elements from K.

Recall also that a *parabolic subgroup* of G is a subgroup containing a minimal parabolic subgroup, and that each parabolic subgroup P has a *Langlands decomposition* $P = M_1N = MAN \simeq M \times A \times N$, where N is nilpotent and $M_1 = MA$ is reductive, and where A is the vectorial part of the center of M_1.

The parabolic subgroups with which we shall be dealing mostly here are the so-called σ-minimal parabolic subgroups P. Before I introduce these, I need some notation from the previous lecture. Let $\mathfrak{a}_q$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. Given a set $\Sigma^+(\mathfrak{a}_q,\mathfrak{g})$ of positive roots for the root system of $\mathfrak{a}_q$ in $\mathfrak{g}$, let $\mathfrak{n} = \mathfrak{n}(\Sigma^+(\mathfrak{a}_q,\mathfrak{g}))$ be the sum of the root spaces corresponding to the roots in this set, and put $N = N(\Sigma^+(\mathfrak{a}_q,\mathfrak{g})) := \exp\mathfrak{n}$. Let M_1 denote the centralizer of $\mathfrak{a}_q$ in G, and put $P = P(\Sigma^+(\mathfrak{a}_q,\mathfrak{g})) := M_1N$. It is easily seen that M_1 normalizes N and hence P is a subgroup of G. By definition, a *σ-minimal* (or *minimal $\sigma\theta$-stable*) parabolic subgroup of G is a conjugate by an element from $K\cap H$ of $P(\Sigma^+(\mathfrak{a}_q,\mathfrak{g}))$ for some set $\Sigma^+(\mathfrak{a}_q,\mathfrak{g})$. It is clear that the θ-minimal parabolic subgroups are the minimal parabolic subgroups. The terminology is motivated by the following lemma.

Lemma 3.1. *The σ-minimal parabolic subgroups are parabolic subgroups satisfying the identity $\sigma\theta(P) = P$, and they are minimal among all parabolic subgroups P satisfying this identity.*

Proof. Only the first statements will be proved, since the last one will not be used.

Extend $\mathfrak{a}_q$ to a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$, then $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ consists of the nonzero restrictions to $\mathfrak{a}_q$ of the elements of $\Sigma(\mathfrak{a}_0, \mathfrak{g})$. Given a positive set $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ for $\Sigma(\mathfrak{a}_0, \mathfrak{g})$, the set $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ of its nonzero restrictions to $\mathfrak{a}_q$ is a positive set for $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ (and any positive set for $\Sigma(\mathfrak{a}_q, \mathfrak{g})$ is obtained by restriction from a (possibly several) $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ — the sets $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ and $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ are said to be *compatible*). It follows that $\mathfrak{n}(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$ is spanned by those root spaces from $\mathfrak{n}_0$ that correspond to roots with nonzero restrictions to $\mathfrak{a}_q$. The remaining root spaces are contained in $\mathfrak{m}_1$, the centralizer of $\mathfrak{a}_q$. It follows that $N_0 \subset P(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$. Since we also have $M_0 A_0 \subset M_1$ we conclude that $P_0 \subset P(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$. Hence $P = P(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$ is a parabolic subgroup. The identity $\sigma\theta(P) = P$ easily follows from the fact that the composed involution $\sigma\theta$ acts trivially on $\mathfrak{a}_q$.

By definition a σ-minimal parabolic subgroup is a $K \cap H$-conjugate of a subgroup of the form $P(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$, hence it is also a $\sigma\theta$-stable parabolic subgroup. □

Example 3.1. Let us again take a look at $X = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$. As in the previous lecture we have that $\mathfrak{a}_q = \mathbf{R}Y$ where $Y = E_{p+q,1} + E_{1,p+q}$. We then get that the centralizer M_1 consists of the matrices in $\mathrm{SO}_e(p,q)$ of the form

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \tag{3.1}$$

where $\epsilon = \pm 1$, $m \in \mathrm{SO}(p-1, q-1)$, and $t \in \mathbf{R}$. The root spaces generating $\mathfrak{n}$ were described earlier (Example 2.3). It follows easily that P is the subgroup of $G = \mathrm{SO}_e(p,q)$ leaving the space spanned by the vector $(1, 0, \ldots, 0, 1) \in \mathbf{R}^{p+q}$ invariant. Note that P is only minimal if $p = 1$ or $q = 1$.

Example 3.2. The group case. The parabolic subgroups of $'G \times 'G$ are given by $P = 'P_1 \times 'P_2$, where $'P_1, 'P_2$ are parabolic subgroups of $'G$. It is clear that P is $\sigma\theta$ stable if and only if $`P_1$ and $`P_2$ are *opposite*, that is, $`P_1 = `\bar{P}_2 := \theta(`P_2)$, and that $P$ is minimal among these if and only if in addition we have that $`P_1$ is minimal. Thus the minimal $\sigma\theta$-stable parabolic subgroups are the parabolic subgroups $`\bar{P}_0 \times `P_0$, where $`P_0$ is a minimal parabolic subgroup of $`G$. Comparing with Example 2.6 we see that these are exactly the parabolic subgroups we get from the construction above.

Fix $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ and let $P = P(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$. As in the proof of Lemma 3.1, let $P_0 = M_0 A_0 N_0$ be a minimal parabolic subgroup corresponding to a compatible choice $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ of positive roots, then $P_0 \subset P$. Note that we have $\sigma(\mathfrak{a}_0) = \mathfrak{a}_0$ by the maximality of $\mathfrak{a}_q$ (if $Y \in \mathfrak{a}_0$ then $Y - \sigma(Y)$ must belong to $\mathfrak{a}_q$, and it follows that $\sigma(Y) \in \mathfrak{a}_0$). Hence M_0 is also σ-stable.

Let $P = MAN$ be the Langlands decomposition of P, then $N = N(\Sigma^+(\mathfrak{a}_q, \mathfrak{g}))$ and $M_1 = MA$ is the centralizer of $\mathfrak{a}_q$. Since $\mathfrak{a}_q$ is σ-stable we have that M_1 is also σ-stable. Moreover, the vectorial part A is σ-stable as well (use that $\mathfrak{a}$ is the intersection of the kernels of all roots of $\Sigma(\mathfrak{a}_0, \mathfrak{g})$ that vanish on $\mathfrak{a}_q$), and so is M (use that $M = M_e M_0$). Since conjugation by $K \cap H$ preserves these properties it follows that the M_Q and the A_Q are σ-stable for any σ-minimal parabolic subgroup $Q = M_Q A_Q N_Q$.

In particular we have that A splits as the direct product $A = A_q A_h$ where $A_h = A \cap H$ and $A_q = \exp \mathfrak{a}_q$. We now have the following σ-stable subspaces of $\mathfrak{p}$,

$$\mathfrak{a}_q \subset \mathfrak{a} \subset \mathfrak{a}_0$$

with

$$\mathfrak{a}_q = \mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}_0 \cap \mathfrak{q} \qquad \text{and} \qquad \mathfrak{a}_h = \mathfrak{a} \cap \mathfrak{h} \subset \mathfrak{a}_0 \cap \mathfrak{h}. \tag{3.2}$$

In contrast to the case of minimal parabolic subgroups, the M-part of a σ-minimal parabolic subgroup is in general not compact. The following lemma shows that this is actually not a serious complication, from the symmetric space viewpoint. Note first that since M is σ-invariant, the homogeneous space $M/(M \cap H)$ is a symmetric space (note however that M, $M \cap H$, and their quotient may all be disconnected).

Lemma 3.2. *The symmetric space $M/(M \cap H)$ is compact.*

Proof. Let M_n be the connected normal subgroup of M which is maximal subject to the condition that $\{e\}$ is its only compact normal subgroup. If we prove that

$$M_n \subset H \tag{3.3}$$

and that

$$M = M_0 M_n, \tag{3.4}$$

then it follows that $M/(M \cap H) \simeq M_0/(M_0 \cap H)$ is a compact symmetric space.

To see (3.3) note that the Lie algebra $\mathfrak{m}_n$ of M_n is the Lie algebra generated by the intersection $\mathfrak{m} \cap \mathfrak{p}$. Since $\mathfrak{a}_q$ is maximal in $\mathfrak{p} \cap \mathfrak{q}$ we have $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{h}$ from which it follows that $\mathfrak{m}_n \subset \mathfrak{h}$. Since M_n is connected we conclude that it is contained in H.

Finally (3.4), which is valid for any parabolic subgroup, easily follows from the fact that $M = M_0 M_e$, where M_e is the identity component of M. □

Example 3.3. For the hyperbolic spaces, we saw in Example 3.1 that M_1 consists of all matrices in $\mathrm{SO}_e(p,q)$ of the form (3.1). The decomposition of M_1 as MA is indicated in this matrix product; in particular we have that $A = A_q$ (with an exception for the case $p = q = 2$). The group M has two components, corresponding to the two values of ϵ (with exceptions for $p = 1$ or $q = 1$, where ϵ is forced to be 1). The elements of $M \cap H$ are obtained by requiring $\epsilon = 1$. Thus $M/(M \cap H)$ has at most 2 elements.

As mentioned in the previous lecture the quotient $W/W_{K\cap H}$ is important. Note that we can identify $W/W_{K\cap H}$ naturally with the double quotient $(M \cap K)\backslash N_K(\mathfrak{a}_q)/N_{K\cap H}(\mathfrak{a}_q)$ because $W \simeq N_K(\mathfrak{a}_q)/(M \cap K)$, $W_{K\cap H} \simeq N_{K\cap H}(\mathfrak{a}_q)/(M \cap K \cap H)$ and $M \cap K$ is a normal subgroup of $N_K(\mathfrak{a}_q)$. It will be convenient to work with a fixed set of representatives in $N_K(\mathfrak{a}_q)$ for $W/W_{K\cap H}$. This set will be denoted $\mathcal{W}$. By Theorem 2.6 we may assume $\mathcal{W} \subset N_K(\mathfrak{a}_0)$.

Note that conjugation by an element w from $N_K(\mathfrak{a}_q)$ leaves M invariant, and that hence $M/(w(M \cap H)w^{-1}) = M/(M \cap wHw^{-1})$ is a symmetric space, corresponding to the restriction to M of the conjugate involution $\sigma^{w^{-1}}$. It follows from Lemma 3.2 that this space is also compact.

I now come to the heart of this lecture, which is the description of the orbits of P on G/H. This description may be seen as a generalization of the Iwasawa decomposition, from which it follows that the minimal parabolic subgroup P_0 has one orbit (acts transitively) on G/K. In general it turns out that the picture is much more complicated, as can be seen already in the group case (Example 3.2). Here $P = {}'\bar{P} \times {}'P$, and the description we are looking for is the description of the ${}'\bar{P} \times {}'P$ double cosets on ${}'G$. This picture is given by the Bruhat decomposition

$$ {}'G = \cup_{w \in {}'W} {}'\bar{P}\, \tilde{w}\, {}'P. $$

A description of the P-orbits on the general G/H will thus be a generalization of both the Iwasawa and the Bruhat decomposition at the same time.

It turns out that in general there is also a finite number of P-orbits on G/H, but here I shall in fact not give the full description of all these orbits. Only the *open* orbits will be described. In the group case we know from the Bruhat decomposition that there is exactly one such orbit, ${}'\bar{P}{}'P$. As we shall see in the following theorem, this corresponds to the fact that the quotient $W/W_{K\cap H}$ in this case is trivial (just as it is in the case of G/K). The theorem gives a one-to-one correspondence of the set of open P-orbits on G/H with $W/W_{K\cap H}$.

Theorem 3.3. *Let P be a σ-minimal parabolic subgroup of G with the Langlands decomposition $P = MAN$, and let $w \in N_K(\mathfrak{a}_q)$. The mapping*

$$ \varphi \colon M \times A_q \times N \ni (m, a, n) \mapsto manwH $$

gives a diffeomorphism of $M/(M \cap wHw^{-1}) \times A_q \times N$ onto the open subset PwH of G/H. Moreover, the union

$$ \cup_{w \in \mathcal{W}} PwH \tag{3.5} $$

is disjoint and dense in G/H. Its complement is a finite union of P-orbits.

Proof. Only the first statement will be proved.

It is easily verified that φ gives rise to a map Φ from $M/(M \cap wHw^{-1}) \times A_q \times N$ onto the subset $\Omega = PwH$ of G/H. Note that Ω only depends on the side class $(M \cap K)wN_{K\cap H}(\mathfrak{a}_q)$. It is also clear that for the proof of the first statement we may take $w = e$ (after that we can apply the statement for $w = e$ to the parabolic subgroup $w^{-1}Pw$).

To see that Φ: $M/(M \cap H) \times A_q \times N \to G/H$ is injective we need that $P \cap H = (M \cap H)A_h$. Let $man \in P \cap H$. Then $\sigma(m) \in M$ and $\sigma(a) \in A$, whereas $\sigma(n)$ is in the nilpotent part $\bar{N}$ of the opposite parabolic subgroup $\bar{P}$ (because σ reverses the sign on all the roots of $\mathfrak{a}_q$). Since $\sigma(man) = man$ and $P \cap \bar{P} = MA$ it follows that $n = \sigma(n) = e$. Moreover it also follows that $\sigma(a) = a$ and $\sigma(m) = m$. Thus $a \in A_h$ as claimed, and then $ma \in H$ implies that m also has to be in H (the identity $\sigma(m) = m$ only implies that m is in G^σ).

In order to finish the proof of the first statement it is sufficient to show that

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \mathfrak{h}. \tag{3.6}$$

Indeed, if G is a Lie group and H_1, H_2 closed subgroups whose Lie algebras satisfy $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$, then the map $h_1 \mapsto h_1H_2$ gives a diffeomorphism of $H_1/(H_1 \cap H_2)$ onto an open subset of G/H_2 (use translation by H_1 to reduce to a neighborhood of the origin).

Since $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ it suffices for (3.6) to prove that $\bar{\mathfrak{n}} \subset \mathfrak{n} + \mathfrak{h}$. Let $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ and $Y \in \mathfrak{g}^{-\alpha}$. Then $-\sigma\alpha$ is also in $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$, and hence $\sigma(Y) \in \mathfrak{g}^{-\sigma\alpha} \subset \mathfrak{n}$. Thus $Y = (Y + \sigma(Y)) - \sigma(Y) \in \mathfrak{h} + \mathfrak{n}$. $\square$

Example 3.4. In the case of the hyperbolic space X it follows from the theorem above that there is one open P orbit on X, unless when $q = 1$, where there are two. This can be seen directly as follows (for simplicity we assume that we are in the non-Riemannian case $p > 1$). Recall that $X = \{x \in \mathbf{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 1\}$ and that P is the subgroup of G leaving the space spanned by the vector $e_0 = (1, 0, \ldots, 0, 1)$ stable. Let Ω be the set of all elements $x \in X$ with $(x, e_0) = x_1 - x_{p+q} \neq 0$ (here $(\cdot,\cdot)$ denotes the standard $\mathrm{O}(p,q)$-invariant bilinear form on $\mathbf{R}^{p+q}$), then it is clear that Ω is open and dense in X, and moreover it is P-invariant. It can be seen that if $q > 1$ then P acts transitively on Ω, whereas if $q = 1$ it divides into the two P-orbits $\Omega_\pm = \{x \in X \mid (x, e_0) \gtrless 0\}$.

From the Iwasawa decomposition $G = KA_0N_0 \simeq K \times A_0 \times N_0$ one gets the important *Iwasawa projection* $H: G \to \mathfrak{a}_0$, defined by the requirement $g \in K \exp H(g) N_0$. Reformulating it in terms of the symmetric space G/K we have the map $gK \mapsto \mathbf{a}_0(gK) = -H(g^{-1}) \in \mathfrak{a}_0$ given by $g \in \exp \mathbf{a}_0(gK)\, N_0 K$. Since we have just generalized the Iwasawa decomposition to G/H it is natural also to look at the corresponding generalization of this projection. Let $P = MAN$ be a fixed σ-minimal parabolic subgroup and let Ω be the open subset PH of G/H. Then we define the *generalized Iwasawa projection* $\mathbf{a}: \Omega \to \mathfrak{a}_q$ by

$$g \in M \exp \mathbf{a}(gH)\, NH.$$

More generally, we can of course similarly define maps $\mathbf{a}_w: PwH \to \mathfrak{a}_q$ for each $w \in N_K(\mathfrak{a}_q)$, but let me for simplicity just concentrate on the trivial w.

Later on it will be useful to know some details about this map. More specifically, I shall need the following result. For any $\nu \in \mathfrak{a}_q^*$ let $H_\nu \in \mathfrak{a}_q$ be the dual element with respect to the Killing form (that is $\nu(Y) = B(Y, H_\nu)$ for all $Y \in \mathfrak{a}_q$). Recall from the previous lecture that m_α^- is the dimension of the -1 eigenspace of $\sigma\theta$ in $\mathfrak{g}^\alpha$.

Theorem 3.4. *Let $a \in A_q$ be fixed and let K_a be the open subset $\{k \in K \mid ka \in \Omega\}$ of K. The map*

$$K_a \ni k \mapsto \mathbf{a}(ka) \in \mathfrak{a}_q$$

is proper and has the image

$$\mathbf{a}(K_a a) = \operatorname{conv}(W_{K\cap H} \log a) + \Gamma^-, \tag{3.7}$$

where conv *denotes convex hull, and where Γ^- is the closed convex cone in $\mathfrak{a}_q$ spanned by the vectors H_α, where $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ with $m_\alpha^- \neq 0$.*

(Recall that a continuous map is called *proper* if the preimage of each compact set is compact.)

Before discussing the proof of this theorem, let me give some examples.

Example 3.5. Let σ be the Cartan involution so that $G/H = G/K$. Then $\Omega = G$ so that $K_a = K$, and moreover $m_\alpha^- = 0$ for all α so that $\Gamma^- = \{0\}$. The theorem then states that the map $k \mapsto H(ak)$ has the image

$$H(aK) = \operatorname{conv}(W_0 \log a),$$

where W_0 is the Weyl group of the root system $\Sigma(\mathfrak{a}_0, \mathfrak{g})$ (in this case the properness is obvious). This result is known as *Kostant's convexity theorem.*

Example 3.6. In the group case, the theorem comes down to the following result (for simplicity I omit the ''s).

Proposition 3.5. *Let* $\mathbf{b}$: $N_0A_0M_0\bar{N}_0 \to \mathfrak{a}_0$ *be the Bruhat projection defined by* $g \in N_0 \exp \mathbf{b}(g) M_0\bar{N}_0$. *Let* $a \in A_0$ *be fixed and let* $(K \times K)_a$ *be the open subset* $\{(k_1, k_2) \mid k_1ak_2 \in N_0A_0M_0\bar{N}_0\}$ *of* $K \times K$. *Then the map*

$$(K \times K)_a \ni (k_1, k_2) \mapsto \mathbf{b}(k_1ak_2) \in \mathfrak{a}_0$$

is proper and has the image

$$\mathbf{b}(KaK \cap N_0A_0M_0\bar{N}_0) = \mathrm{conv}(W_0 \log a) + \Gamma_0, \tag{3.8}$$

where Γ_0 *is the closed convex cone in* $\mathfrak{a}_0$ *spanned by the vectors* H_α *for* $\alpha \in \Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ *(the dual cone to the open positive chamber).*

Proof. Let me indicate a proof of the properness and the inclusion "$\subset$" of (3.8), independent of Theorem 3.4. I need the following two lemmata, whose proofs I omit. See the notes for references.

Lemma 3.6. *Let* n_j *and* $\bar{n}_j$ *be sequences in* N_0 *and* $\bar{N}_0$ *such that the sequence* $n_j\bar{n}_j$ *converges in* G. *Then each of the sequences* n_j *and* $\bar{n}_j$ *also converges.*

Lemma 3.7. *Let* H: $G \to \mathfrak{a}_0$ *be the Iwasawa projection. Then* $H(\bar{N}_0) \subset \Gamma_0$.

Let (k_{1j}, k_{2j}) be a sequence in $(K \times K)_a$ for which $\mathbf{b}(k_{1j}ak_{2j})$ stays inside a compact set. To get the properness in Proposition 3.5 we must prove that (k_{1j}, k_{2j}) has an accumulation point in $(K \times K)_a$. Write

$$k_{1j}ak_{2j} = n_ja_jm_j\bar{n}_j \in N_0A_0M_0\bar{N}_0,$$

then $a_j = \exp \mathbf{b}(k_{1j}ak_{2j})$. By passing to a subsequence we may assume that the sequences k_{1j}, k_{2j}, a_j, and m_j converge. Using that A_0M_0 normalizes N_0 it then follows from Lemma 3.6 that n_j and $\bar{n}_j$ also converge. Hence the limit of $k_{1j}ak_{2j}$ belongs to $N_0A_0M_0\bar{N}_0$. This proves the claim, and hence the properness of $\mathbf{b}$.

To prove that the left side of (3.8) is contained in the right side note that if $x = nbm\bar{n}$ then $\log b = H(x\bar{n}^{-1})$. Hence

$$\mathbf{b}(KaK \cap N_0A_0M_0\bar{N}_0) \subset H(aK\bar{N}) = H(aK) + H(\bar{N}).$$

Now use Example 3.5 together with Lemma 3.7. $\square$

Proof of Theorem 3.4. I shall only give part of the proof. The proof of the properness is based on the following observation:

$$(3.9) \qquad 2\mathbf{a}(ka) = \mathbf{b}(ka^2\sigma(k)^{-1}), \qquad (k \in K_a),$$

where $\mathbf{b}$ is the Bruhat projection (see Proposition 3.5 above). Indeed, if we write $ka = m\exp(\mathbf{a})nh$, then we have

$$\begin{aligned} ka^2\sigma(k)^{-1} &= ka\sigma(ka)^{-1} \\ &= m\exp(\mathbf{a})n\sigma(n)^{-1}\exp(\mathbf{a})\sigma(m)^{-1} \in N\exp(2\mathbf{a})m\sigma(m)^{-1}\bar{N}. \end{aligned}$$

Now $N \subset N_0$, $\bar{N} \subset \bar{N}_0$ and by (3.3) and (3.4) we have $m\sigma(m)^{-1} \in M_0$. This gives (3.9), and then the properness easily follows from Proposition 3.5.

By a similar computation, a weak version of the inclusion "$\subset$" of (3.7) can be obtained as follows. I am going to prove that

$$\mathbf{a}(ka) \in \mathrm{conv}(W\log a) + \Gamma,$$

where Γ is the closed convex cone in $\mathfrak{a}_q$ spanned by all the vectors H_α, $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$.

Choose an element $w \in W$ such that $w\log a$ is antidominant with respect to $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$, then $s\log a \in w\log a + \Gamma$ for all $s \in W$, and hence

$$\mathrm{conv}(W\log a) + \Gamma = w\log a + \Gamma.$$

By Theorem 2.6 there exists an element in W_0 which normalizes $\mathfrak{a}_q$ and acts as w there. Since $w\log a$ is also antidominant with respect to $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ we thus obtain

$$\mathrm{conv}(W_0\log a) + \Gamma_0 = w\log a + \Gamma_0.$$

It now follows from (3.9) and Proposition 3.5 (the part of it that was proved) that

$$\mathbf{a}(ka) \in w\log a + \Gamma_0.$$

It remains to be seen that $\Gamma_0 \cap \mathfrak{a}_q = \Gamma$, but this is quite easy. $\square$

Example 3.7. For the hyperbolic space X we found in Example 3.4 that $PH = \Omega = \{x \in X \mid x_1 - x_{p+q} \neq 0\}$ if $q > 1$ and $PH = \Omega_+ = \{x \in X \mid x_1 - x_{p+q} > 0\}$ if $q = 1$. It is now easily seen that $\mathbf{a}$: $PH \to \mathfrak{a}_q$ is given by $\mathbf{a}(x) = -\log|x_1 - x_{p+q}|Y$, and Theorem 3.4 can be verified for this case. Note the essential difference between the Riemannian ($p = 1$) and the non-Riemannian ($p > 1$) cases, and also between the cases $q = 1$ and $q > 1$.

LECTURE 4

Invariant differential operators

I shall now turn to another important matter for the harmonic analysis, the description of the invariant differential operators.

Let us for the moment consider any homogeneous space G/H of a Lie group G. Let $\mathbf{D}(G/H)$ be the set of invariant differential operators on G/H; this is a subalgebra of the algebra of all differential operators on X. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_c$, the complexification of $\mathfrak{g}$, and denote by $U(\mathfrak{g})^H$ the subalgebra of elements invariant for the adjoint action of H. The elements of $U(\mathfrak{g})$ act on G as left-invariant differential operators, by means of the action generated by

$$Xf(g) = \frac{d}{dt}\bigg|_{t=0} f(g \exp tX) \tag{4.1}$$

for $X \in \mathfrak{g}$ and $f \in C^\infty(G)$. Viewing functions on G/H as right H-invariant functions on G it follows that there is a natural action of the elements of $U(\mathfrak{g})^H$ on $C^\infty(G/H)$. It is easily verified that this action is an action of differential operators on G/H, and that a homomorphism of algebras $r\colon U(\mathfrak{g})^H \to \mathbf{D}(G/H)$ is thus obtained. It is clear that $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$ is an ideal (both left and right) in $U(\mathfrak{g})^H$, and that it is annihilated by r. Thus we have a homomorphism, also denoted r, from the quotient $U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h})$ into $\mathbf{D}(G/H)$.

Proposition 4.1. *Assume that $\mathfrak{h}$ has an H-invariant complement in $\mathfrak{g}$. Then r is an isomorphism of the algebra $U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}_c)$ onto $\mathbf{D}(G/H)$.*

Proof. Omitted. See the notes for a reference. $\square$

Assume now that G/H is a semisimple symmetric space. Then Proposition 4.1 applies, since $\mathfrak{q}$ is H-invariant.

A particularly important element of $\mathbf{D}(G/H)$ is the *Laplace-Beltrami operator* (or *Laplacian*) L on G/H. As on any pseudo-Riemannian manifold this is defined in local coordinates by

$$L = \frac{1}{\sqrt{|\det g|}} \sum_{i,j} \partial_j \sqrt{|\det g|} g^{ij} \partial_i,$$

where $g = g_{ij}$ is the pseudo-Riemannian structure and g^{ij} is the inverse matrix. It is an invariant differential operator, because the pseudo-Riemannian structure is invariant. On the other hand, we have in $U(\mathfrak{g})$ the *Casimir element* Ω defined by $\Omega = \sum_{i,j} \gamma^{ij} X_i X_j$ where X_i is a basis of $\mathfrak{g}$, and γ^{ij} the inverse matrix of $B(X_i, X_j)$. It can be seen that L and $r(\Omega)$ coincide, up to a positive scalar multiple.

Before I continue with the description of $\mathbf{D}(G/H)$ for the general semisimple symmetric space G/H, I will first give the description of $\mathbf{D}(G/K)$. The description of $\mathbf{D}(G/K)$ is based on the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$, and on the Poincaré-Birkhoff-Witt theorem. From these we get that

$$U(\mathfrak{g}) = (\mathfrak{n}_{0,c} U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}_c) \oplus U(\mathfrak{a}_0),$$

and hence we can define a map ${}'\gamma_0 \colon U(\mathfrak{g}) \to U(\mathfrak{a}_0)$ as the projection with respect to this decomposition. Since $\mathfrak{a}_0$ is abelian it is customary to identify its universal enveloping algebra with its symmetric algebra, and write $S(\mathfrak{a}_0)$ instead of $U(\mathfrak{a}_0)$. It is not difficult to see that the restriction of ${}'\gamma_0$ to $U(\mathfrak{g})^K$ is a homomorphism. Moreover, it is clear that ${}'\gamma_0$ annihilates $U(\mathfrak{g})\mathfrak{k}_c$, and hence it follows from the proposition above that ${}'\gamma_0$ gives rise to a homomorphism of $\mathbf{D}(G/K)$ into $S(\mathfrak{a})$. This homomorphism is called the *Harish-Chandra homomorphism.* We denote it also by ${}'\gamma_0$. Note that it depends on the choice we made for $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$, because $\mathfrak{n}_0$ depends on it.

It turns out that a modified version of ${}'\gamma_0$ is actually more fundamental than ${}'\gamma_0$ itself. Let $\rho_0 \in \mathfrak{a}_0^*$ be given by

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{a}_0, \mathfrak{g})} m_\alpha \alpha,$$

that is, half the trace of ad on $\mathfrak{n}_0$, and let T_{ρ_0} be the automorphism of $S(\mathfrak{a}_0)$ generated by $T_{\rho_0}(Y) = Y + \rho_0(Y)$, for $Y \in \mathfrak{a}_0$. We now define $\gamma_0 \colon U(\mathfrak{g})^H \to S(\mathfrak{a}_0)$ by $\gamma_0 = T_{\rho_0} \circ {}'\gamma_0$. This map is called the Harish-Chandra *iso*morphism because of the following theorem.

Theorem 4.2. *The map γ_0 is an algebra isomorphism of $\mathbf{D}(G/K)$ onto $S(\mathfrak{a}_0)^{W_0}$, the set of W_0-invariant elements in $S(\mathfrak{a}_0)$. It is independent of the choice of $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$.*

Proof. It remains to be seen that $\gamma_0(D)$ is W_0-invariant for all invariant differential operators D, and that γ_0 is bijective (the independence on $\Sigma^+(\mathfrak{a}_0, \mathfrak{g})$ is an easy consequence of the W_0-invariance).

The proof of the W_0-invariance is surprisingly complicated. One proof involves the *spherical functions* φ_λ on G/K (a reference to a different one can be found in the notes). Let me recall how these are defined. As in the previous section let $H\colon G \to \mathfrak{a}_0$ be the Iwasawa projection. Then

$$\varphi_\lambda(g) := \int_K e^{-(\lambda+\rho_0)H(g^{-1}k)}dk \tag{4.2}$$

for $\lambda \in \mathfrak{a}_{0,c}^*$ and $g \in G$. Clearly each φ_λ is a smooth function on G/K. I shall return to the importance of these functions soon. For the moment, let me note the following two facts:

(a) The spherical functions are eigenfunctions for $\mathbf{D}(G/K)$. In fact we have
$$D\varphi_\lambda = \gamma_0(D, \lambda)\varphi_\lambda$$
for all $D \in \mathbf{D}(G/K)$. This follows, because the integrand in (4.2) is already an eigenfunction with this eigenvalue (this is easily seen).

(b) We have $\varphi_{w\lambda} = \varphi_\lambda$ for all $w \in W_0$ (see [130, Prop 7.15]).

It follows from (a) and (b) that $\gamma_0(D, w\lambda) = \gamma_0(D, \lambda)$ as claimed.

The proof that γ_0 is bijective is too extensive to be given here. $\square$

Note that it follows immediately that $\mathbf{D}(G/K)$ is commutative. In fact, one can say more: from the theory of finite reflexion groups it follows that it is a polynomial ring in $\dim \mathfrak{a}$ algebraically independent generators.

We shall now generalize this result to G/H. By definition, a *Cartan subspace* for G/H is a maximal abelian subspace of $\mathfrak{q}$, consisting of semisimple elements. In particular, there exists a Cartan subspace $\mathfrak{a}_1$ containing $\mathfrak{a}_q$. Then $\mathfrak{a}_q = \mathfrak{a}_1 \cap \mathfrak{p}$. The elements of $\operatorname{ad}\mathfrak{a}_1$ can be simultaneously diagonalized, but in general there will be complex eigenvalues. In analogy with what we had for $\mathfrak{a}_0$ and $\mathfrak{a}_q$ we get a root system $\Sigma(\mathfrak{a}_{1c}, \mathfrak{g}_c)$ (but the complexified Lie algebras are needed), and corresponding to each choice of positive set $\Sigma^+(\mathfrak{a}_{1c}, \mathfrak{g}_c)$ an analog of the Iwasawa decomposition $\mathfrak{g}_c = \mathfrak{n}_1 \oplus \mathfrak{a}_{1c} \oplus \mathfrak{h}_c$, where $\mathfrak{n}_1$ is the sum of the root spaces corresponding to the positive roots. However, this decomposition will not in general correspond to a decomposition of the real Lie algebra $\mathfrak{g}$. Nevertheless, the construction of the Harish-Chandra homomorphism can be generalized to this setting: a map ${}'\gamma: U(\mathfrak{g}) \to U(\mathfrak{a}_1)$ is defined by projection with respect to the Iwasawa decomposition, and this gives rise to a homomorphism from $\mathbf{D}(G/H)$ to $S(\mathfrak{a}_1)$. As before we define $\gamma = T_{\rho_1} \circ {}'\gamma$, where $\rho_1 \in \mathfrak{a}_{1c}^*$ is half the trace of ad on $\mathfrak{n}_1$, and denoting by W_1 the Weyl group of $\Sigma(\mathfrak{a}_{1c}, \mathfrak{g}_c)$ we have:

Theorem 4.3. *The map γ is an algebra isomorphism of $\mathbf{D}(G/H)$ onto $S(\mathfrak{a}_1)^{W_1}$. It is independent of the choice of $\Sigma^+(\mathfrak{a}_{1c}, \mathfrak{g}_c)$.*

Proof. The proof consists of reduction to Theorem 4.2 by means of an important technique, called "duality". We have seen that $\mathbf{D}(G/H)$ is isomorphic via r to $U(\mathfrak{g})^H/(U(\mathfrak{g})\mathfrak{h}_c \cap U(\mathfrak{g})^H)$ (this isomorphism is implicit already in the construction of γ as a map from $\mathbf{D}(G/H)$).

Define
$$\mathfrak{g}^d = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q} \oplus i(\mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}) \subset \mathfrak{g}_c,$$
then $\mathfrak{g}^d$ is a real semisimple Lie algebra with the same complexification as $\mathfrak{g}$. Let
$$\mathfrak{k}^d = \mathfrak{k} \cap \mathfrak{h} \oplus i(\mathfrak{p} \cap \mathfrak{h}) = \mathfrak{h}_c \cap \mathfrak{g}^d$$
and
$$\mathfrak{p}^d = \mathfrak{p} \cap \mathfrak{q} \oplus i(\mathfrak{k} \cap \mathfrak{q}) = \mathfrak{q}_c \cap \mathfrak{g}^d,$$
then $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ is a Cartan decomposition of $\mathfrak{g}^d$ (by this I mean that the Killing form is negative definite on $\mathfrak{k}^d$ and positive definite on $\mathfrak{p}^d$). The pair $(\mathfrak{g}^d, \mathfrak{k}^d)$ is called the *noncompact Riemannian form* of the pair $(\mathfrak{g}, \mathfrak{h})$. Let
$$\mathfrak{a}_0^d = \mathfrak{a}_q \oplus i(\mathfrak{a}_1 \cap \mathfrak{k}) = \mathfrak{a}_{1c} \cap \mathfrak{g}^d,$$
then $\mathfrak{a}_0^d$ is a maximal abelian subspace of $\mathfrak{p}^d$. Since $\mathfrak{a}_0^d$ and $\mathfrak{a}_1$ have the same complexification, the root system $\Sigma(\mathfrak{a}_{1c}, \mathfrak{g}_c)$ is essentially the same as the root system $\Sigma(\mathfrak{a}_0^d, \mathfrak{g}^d)$ (the space $\mathfrak{a}_0^d$ is the subspace of $\mathfrak{a}_{1c}$ on which the roots are real), and their root spaces in $\mathfrak{g}_c$ are identical. Let (G^d, K^d) be a symmetric pair with $(\mathfrak{g}^d, \mathfrak{k}^d)$ as Lie algebras, then G^d/K^d is a Riemannian symmetric space. Using Theorem 4.2 on G^d/K^d we get the Harish-Chandra isomorphism γ_0^d of $U(\mathfrak{g}^d)^{K^d}/(U(\mathfrak{g}^d)^{K^d} \cap U(\mathfrak{g}^d)\mathfrak{k}_c^d)$ onto $S(\mathfrak{a}_0^d)^{W^d}$, where W^d is the Weyl group of $\Sigma(\mathfrak{a}_0^d, \mathfrak{g}^d)$. Since $U(\mathfrak{g}^d)^{K^d} = U(\mathfrak{g})^{\mathfrak{h}_c} = U(\mathfrak{g})^H$ and

$S(\mathfrak{a}_0^d)^{W^d} = S(\mathfrak{a}_1)^{W_1}$, it follows from the definition of γ_0^d that it is actually identical with γ. $\square$

As for $\mathbf{D}(G/K)$ it follows that $\mathbf{D}(G/H)$ is a polynomial algebra with $\dim \mathfrak{a}_1$ independent generators, and in particular it is commutative. In the terminology of the proof above we have actually that $\mathbf{D}(G/H) \simeq \mathbf{D}(G^d/K^d)$.

As another application of the technique of proof in Theorem 4.3 we get the following: all Cartan subspaces for G/H are conjugate under the complex group H_c (they are, however, in general not conjugate under H). In particular they have the same dimension; this dimension is called the *rank* of G/H. The dimension of the maximal abelian subspace $\mathfrak{a}_q$ of $\mathfrak{p} \cap \mathfrak{q}$ is called the *split rank* of G/H (because $\mathfrak{a}_q$ is a maximal subspace of $\mathfrak{q}$ for which $\mathfrak{g}$ splits over the reals). The rank is the number of generators for $\mathbf{D}(G/H)$.

Example 4.1. For the real hyperbolic space $X = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$ we have that the maximal abelian subalgebra $\mathfrak{a}_q = \mathbf{R}Y$ of $\mathfrak{p} \cap \mathfrak{q}$ defined earlier, is actually maximal abelian in $\mathfrak{q}$. Hence $\mathfrak{a}_1 = \mathfrak{a}_q$ is a Cartan subspace, and X has rank one as well as split rank one. In particular it follows from Theorem 4.3 that $\mathbf{D}(G/H)$ consists of all polynomials in the Laplacian.

Let $\mathfrak{Z}(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$, then $\mathfrak{Z}(\mathfrak{g}) \subset U(\mathfrak{g})^H$. Let $\mathbf{Z}(G/H)$ denote the subalgebra $r(\mathfrak{Z}(\mathfrak{g}))$ of $\mathbf{D}(G/H)$. Note that for $D = r(z) \in \mathbf{Z}(G/H)$ we have that the action of D on G/H can also be obtained from the *left* action of $\mathfrak{g}$ on G/H as follows. All the elements of $U(\mathfrak{g})$ act on G as right-invariant differential operators, by means of the action generated by

$$\ell(X)f(g) = \frac{d}{dt}\bigg|_{t=0} f(\exp -tX\, g)$$

for $X \in \mathfrak{g}$. Identifying functions on G/H with right H-invariant functions on G, this action gives a homomorphism, also denoted ℓ, from $U(\mathfrak{g})$ into the algebra of differential operators on G/H. Clearly, the restriction of ℓ to $\mathfrak{Z}(\mathfrak{g})$ maps into the invariant differential operators. In fact, it is not difficult to see that $\ell(z) = r(\check{z})$ for $z \in \mathfrak{Z}(\mathfrak{g})$, where $u \mapsto \check{u}$ is the principal antiautomorphism of $U(\mathfrak{g})$ determined by $X \mapsto -X$ for $X \in \mathfrak{g}_c$.

In general $\mathbf{Z}(G/H)$ is a proper subalgebra of $\mathbf{D}(G/H)$, but this is actually quite exceptional:

Lemma 4.4. *If G is a classical Lie group, or if the rank of G/H is one, then $\mathbf{Z}(G/H) = \mathbf{D}(G/H)$.*

Proof. By the same argument as in the proof of Theorem 4.3 we may assume that $H = K$. For the classical groups one proceeds case-by-case (see the references in the notes). If the rank of G/K is one it follows from Theorem 4.3 that $\mathbf{D}(G/H)$ is generated by the Laplace-Beltrami operator L, which equals a constant times $r(\Omega) \in \mathbf{Z}(G/H)$. □

As mentioned in the Introduction, the spectral theory for the invariant differential operators is an important tool for the harmonic analysis on $L^2(G/H)$. The operators $D \in \mathbf{D}(G/H)$ are of course unbounded as operators on $L^2(G/H)$; as their domain it is convenient to take the dense subset $C_c^\infty(X)$ of compactly supported smooth functions on X.

Recall that the formal adjoint D^* of $D \in \mathbf{D}(G/H)$ is the differential operator defined by

$$\int_{G/H} Df(x)\overline{g(x)}dx = \int_{G/H} f(x)\overline{D^*g(x)}dx$$

for $f, g \in C_c^\infty(X)$. Clearly we have $D^* \in \mathbf{D}(G/H)$. If $D = D^*$ then D is called formally self-adjoint (this means that D is a symmetric operator).

Proposition 4.5. *Let $D \in \mathbf{D}(G/H)$ be formally self-adjoint. Then D is essentially self-adjoint.*

Recall that an unbounded operator is called *essentially self-adjoint* if it has a self-adjoint closure.

Proof. I first need to recall some general representation theory. If $(\pi, \mathcal{H})$ is a representation of G on a Hilbert space $\mathcal{H}$, the space of C^∞-vectors for π is denoted $\mathcal{H}^\infty$ (by definition it is the space of vectors $v \in \mathcal{H}$ for which $g \mapsto \pi(g)v$ is smooth). It is a dense subspace of $\mathcal{H}$, and it carries a natural representation of $U(\mathfrak{g})$. It also has a natural Fréchet topology, with respect to which the action of $U(\mathfrak{g})$ is continuous.

Applying this to the representation $(\ell, L^2(X))$, it is easily seen that the space of C^∞ vectors for this representation is the space

$$L^2(X)^\infty = \{f \in C^\infty(X) \mid \ell(u)f \in L^2(G/H) \text{ for all } u \in U(\mathfrak{g})\},$$

with the topology induced by the seminorms $p_u(f) = \|\ell(u)f\|$.

Let me first note that $C_c^\infty(X)$ is dense in $L^2(X)^\infty$. This can be seen by a standard argument as follows: There exist functions $h_n \in C_c^\infty(G)$ with $h_n \geq 0$, $\int_G h_n(g)dg=1$ and whose support shrinks to $\{e\}$ as $n \to \infty$. Let $h_n * f$ be the convolution product of h_n with f defined by

$$(h_n * f)(x) = (\ell(h_n)f)(x) = \int_G h_n(g)f(g^{-1}x)dg,$$

then I claim that for any $f \in L^2(X)^\infty$ we have that $h_n * f \to f$ in $L^2(X)^\infty$ as $n \to \infty$, and that each $h_n * f$ is in the closure of $C_c^\infty(X)$ in $L^2(X)^\infty$. Both claims are easily seen, and they clearly imply the stated density of $C_c^\infty(X)$.

Obviously each $D \in \mathbf{D}(G/H)$ extends to an operator with domain $L^2(X)^\infty$. In fact, it can be seen that D maps $L^2(X)^\infty$ continuously into itself. I shall not attempt to prove this here, but only note that for $D \in \mathbf{Z}(G/H)$ this is clear because $\ell(U(\mathfrak{g}))$ is continuous on $L^2(X)^\infty$ (thus, by Lemma 4.4 all symmetric spaces of the classical groups or of rank one are covered). It follows from the continuity combined with the density of $C_c^\infty(X)$ that if $D \in \mathbf{D}(G/H)$ is formally self-adjoint then the extension to $L^2(X)^\infty$ is symmetric.

Now let

$$\mathrm{Dom}(\bar{D}) = \{f \in L^2(X) \mid Df \in L^2(X)\}$$

(where Df is defined in the distributional sense) and let $\bar{D}$ denote the extension of D to this domain. I claim that this extension is self-adjoint. First of all we have that $(\bar{D}f, g) = (f, \bar{D}g)$ for all $f, g \in \mathrm{Dom}(\bar{D})$, because this holds for $f, g \in L^2(X)^\infty$ and with h_n as above we have $h_n * f \in L^2(X)^\infty$ with $h_n * f \to f$ and $D(h_n * f) = h_n * Df \to Df$. This shows that $\bar{D}$ is symmetric, that is, $\bar{D} \subset \bar{D}^*$. Conversely, if f is in the domain of $\bar{D}^*$, we have by definition that $(\bar{D}^* f, g) = (f, Dg)$ for all $g \in \mathrm{Dom}(\bar{D})$, hence in particular for $g \in C_c^\infty(X)$. This shows that the distribution Df equals $\bar{D}^* f$, which is in $L^2(X)$, so $f \in \mathrm{Dom}(\bar{D})$. $\square$

It follows from Theorem 4.3 and Proposition 4.5 that the formally self-adjoint elements of $\mathbf{D}(G/H)$ admit a simultaneous spectral decomposition of $L^2(X)$ (see [156, Cor. 9.2]).

We have defined two Harish-Chandra isomorphisms,

$$\gamma_0\colon \mathbf{D}(G/K) \to S(\mathfrak{a}_0)^{W_0} \qquad \text{and} \qquad \gamma\colon \mathbf{D}(G/H) \to S(\mathfrak{a}_1)^{W_1},$$

but we shall actually need one more analogous map,

$$\gamma_q\colon \mathbf{D}(G/H) \to S(\mathfrak{a}_q)^W.$$

(Recall that $\mathfrak{a}_q$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, and that W is the reflection group of the root system $\Sigma(\mathfrak{a}_q, \mathfrak{g})$.) As the other maps it is defined by means of projection along a decomposition of $\mathfrak{g}$, followed by a ρ-shift. More precisely we have (see (3.6) and (3.2)) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a}_q \oplus \mathfrak{h}$, and define ${}'\gamma_q\colon \mathbf{D}(G/H) \to U(\mathfrak{a}_q)$ by

$$u - {}'\gamma_q(D) \in (\mathfrak{n} + \mathfrak{m})_c U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}_c,$$

where u is any element in $U(\mathfrak{g})^H$ with $r(u) = D$. Furthermore we define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})} m_\alpha \alpha \in \mathfrak{a}_q^* \tag{4.3}$$

and $\gamma_q = T_\rho \circ {}'\gamma_q$. We now have:

Lemma 4.6. *The map γ_q is an algebra homomorphism of $\mathbf{D}(G/H)$ into $S(\mathfrak{a}_q)^W$. It is independent of the choice of $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$.*

Remark. In general γ_q does not map onto $S(\mathfrak{a}_q)^W$.

Proof. Choose compatible positive sets of roots $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ and $\Sigma^+(\mathfrak{a}_{1c}, \mathfrak{g}_c)$, and let $\rho_m \in \mathfrak{a}_{1c}^*$ be half the trace of ad on $\mathfrak{n}_1 \cap \mathfrak{m}$. Using that $\mathfrak{n}$ is $\sigma\theta$-invariant it is easily seen that $\rho_1 = \rho + \rho_m$, or equivalently, that the restriction of $\rho_1 - \rho_m$ to $\mathfrak{a}_1 \cap \mathfrak{m} = \mathfrak{a}_1 \cap \mathfrak{k}$ vanishes.

Let $\lambda \in \mathfrak{a}_{qc}^*$ and $D \in \mathbf{D}(G/H)$. Then it is easily seen that ${}'\gamma_q(D)(\lambda) = {}'\gamma(D)(\lambda)$, and hence we get

$$\gamma_q(D)(\lambda) = \gamma(D)(\lambda - \rho_m).$$

Now every element $w \in W$ can be represented by an element $\tilde{w} \in N_{W_1}(\mathfrak{a}_q)$ (apply Theorem 2.6 to $\mathfrak{g}^d$). This element then also normalizes $\mathfrak{a}_1 \cap \mathfrak{m}$, and multiplying it with an element from the Weyl group of $\Sigma(\mathfrak{a}_{1c}, \mathfrak{m})$ we can obtain that it leaves ρ_m fixed. Now the W-invariance of $\gamma_q(D)$ follows from the W-invariance of $\gamma(D)$. A similar argument shows the independence on $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$. $\square$

The map γ_q is significant because of the following result. Let $S \subset \mathfrak{a}_q$ be a convex, compact $W_{K\cap H}$-invariant set, and put

$$X_S = \{kaH \in X \mid k \in K, \log a \in S\}.$$

Theorem 4.7. *Let $D \in \mathbf{D}(G/H)$ be nonzero, and assume that $\gamma_q(D)$ has the same degree as the order of D. Then we have*

$$\operatorname{supp} f \subset X_S \Longleftrightarrow \operatorname{supp} Df \subset X_S \tag{4.4}$$

for all $f \in C_c^\infty(X)$. In particular we have that D is injective on this class of functions.

Proof. Here I shall only give the proof of the nontrivial implication "$\Longleftarrow$" of (4.4) for the empty set $S = \emptyset$. The general case is only slightly more complicated. Note that the final statement of the theorem is obtained with $S = \emptyset$. I am going to use *Holmgren's uniqueness theorem*, which states the following (see [129, Thm. 5.3.1]):

Theorem 4.8. *Let ϕ be a real valued C^1 function on an open set $\Omega \subset \mathbf{R}^n$ and D a differential operator with analytic coefficients on Ω. Let x_0 be a point in Ω where the principal symbol $\sigma(D)$ of D satisfies*

$$\sigma(D)(d\phi(x_o)) \neq 0. \tag{4.5}$$

Then there exists a neighborhood $\Omega' \subset \Omega$ of x_0 such that every distribution $f \in \mathcal{D}'(\Omega)$ satisfying the equation $Df = 0$ and vanishing when $\phi(x) > \phi(x_0)$ must also vanish in Ω'.

The idea is to apply this at a point x_0 on the boundary of the support of f. If we can find a function ϕ with the property that $f(x) = 0$ when $\phi(x) > \phi(x_0)$ then a contradiction is reached.

Assume $\operatorname{supp} Df = 0$. I shall use the expansion of f as a sum of K-finite functions. Recall that this is given by

$$f = \sum_{\delta \in \hat{K}} f_\delta, \tag{4.6}$$

where $\hat{K}$ is the set of (equivalence classes of) irreducible K-representations, and where f_δ is the function given in terms of the character χ_δ by

$$f_\delta(x) = \dim \delta \int_K \overline{\chi_\delta(k)} f(k^{-1}x)dk,$$

which transforms on the left according to the K-type δ. The sum is absolutely convergent, and its terms are unique. It is easily seen that D can be applied termwise to the sum, hence $Df = 0$ implies that each term is annihilated by D. It follows from this analysis that we may assume f to be K-finite. Then the support of f is K-invariant, and it suffices to prove that $\operatorname{supp} f \cap A_qH = \emptyset$.

Let $m = \operatorname{order} D$, then $m = \deg \gamma_q(D)$ by the assumption on D. Let u_0 denote the homogeneous part of $\gamma_q(D)$ of degree m, then $u_0 \neq 0$. Note that u_0 is also the homogeneous part of $'\gamma_q(D)$ of degree $m = \deg {'\gamma_q}(D)$ for any choice of $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$.

Assume that $\operatorname{supp} f \cap A_qH$ is not empty, and let S' denote the set

$$S' = \{Y \in \mathfrak{a}_q \mid \exists w \in W\colon \exp(wY)H \in \operatorname{supp} f\}.$$

This set is clearly compact. Since $u_0 \neq 0$ there exists an antidominant $\lambda \in \mathfrak{a}_q^*$ with $u_0(\lambda) \neq 0$. Choose $Y_0 \in S'$ such that λ attains its maximum over S' in this point:

$$\lambda(Y) \leq \lambda(Y_0), \quad (Y \in S'). \tag{4.7}$$

Let $a_0 = \exp Y_0$. The point a_0H is going to be the x_0 in Holmgren's theorem.

As in the previous lecture, let Ω denote the open subset PH of $X = G/H$ and define $\mathbf{a}\colon \Omega \to \mathfrak{a}_q$ by $\mathbf{a}(manH) = \log a$ for $m \in M$, $a \in A_q$, $n \in N$. I claim that

$$f = 0 \quad \text{on} \quad \{x \in \Omega \mid \lambda(\mathbf{a}(x)) > \lambda(Y_0)\}, \tag{4.8}$$

which shows that $\phi(x) = \lambda(\mathbf{a}(x))$ is a suitable function for the application of Holmgren's theorem.

To prove (4.8) let $x = manH \in \Omega \cap \operatorname{supp} f$. Then $\mathbf{a}(x) = \log a$ and we must show that $\lambda(\log a) \leq \lambda(Y_0)$. To see that this holds, write

$$x = k\exp(Z)H, \quad (k \in K, Z \in \mathfrak{a}_q)$$

according to the $G = KA_qH$ decomposition. Then by Theorem 3.4 we have that $\log a = U + V$, where $U \in \operatorname{conv}(WZ)$ and $V \in \Gamma^-$. In particular, $\lambda(V) \leq 0$ by the antidominance of λ, and hence

$$\lambda(\log a) \leq \lambda(U) \leq \max_{w \in W} \lambda(wZ).$$

Now $\exp(Z)H = k^{-1}x$ and since the support of f is K-invariant it contains this point. Hence $wZ \in S'$ for all $w \in W$, and we conclude by (4.7) that

$$\lambda(\log a) \leq \lambda(Y_0).$$

This implies (4.8).

We still need to check the condition (4.5). The principal symbol $\sigma(D)$ is given at a point $x_0 \in X$ by

$$\sigma(D)(d\phi(x_0)) = \frac{1}{m!}D((\phi - \phi(x_0))^m)(x) \tag{4.9}$$

for $\phi \in C^\infty(X)$. In particular, let $\phi(x) = \lambda(\mathbf{a}(x))$. Regarding ϕ as a right H-invariant function on G, it follows immediately that for the right action defined by (4.1) we have $r(u)\phi = 0$ for $u \in U(\mathfrak{g})\mathfrak{h}_c$. Moreover, since $\mathbf{a}$ is left NM-invariant, and since $\mathfrak{n}$ and $\mathfrak{m}$ are normalized by A, we also have that $r(u)\phi(a) = 0$ for $a \in A_q, u \in (\mathfrak{n} + \mathfrak{m})_c U(\mathfrak{g})$. Hence $D\phi(aH) = r('\gamma_q(D))\phi(a)$. Applying the same reasoning to the function $(\phi - \phi(a))^m$ we obtain that

$$D((\phi - \phi(a))^m)(a) = r('\gamma_q(D))((\phi - \phi(a_0))^m)(a) = m!\, u_0(\lambda). \tag{4.10}$$

Combining (4.9) and (4.10) we obtain that $\sigma(D)(d\phi(a)) = u_0(\lambda)$ for all $a \in A_q$. In particular, (4.5) holds by the assumption on λ. Hence we can apply Holmgren's theorem and reach a contradiction. $\square$

Remark. Note that we only used the parts of Theorem 3.4 that were proved in the previous lecture.

LECTURE 5

Principal series representations

In this lecture I am going to consider the representations that enter in the decomposition of the most continuous part of $L^2(X)$. They constitute what is known as the principal series for G/H.

Let me first recall the *principal series* of representations for G. Let $P = MAN$ be any parabolic subgroup with the indicated Langlands decomposition, and let $(\xi, \mathcal{H}_\xi)$ be an irreducible unitary representation of M. For each element $\lambda \in \mathfrak{a}_c^*$ one defines a representation $(\pi_{\xi,\lambda}, \mathcal{H}_{\xi,\lambda})$ of G as follows. Let $\rho_P \in \mathfrak{a}^*$ be half the trace of ad on $\mathfrak{n}$. The Hilbert space $\mathcal{H}_{\xi,\lambda}$ is the completion of the space $C(\xi\colon\lambda)$ of continuous functions $f\colon G \to \mathcal{H}_\xi$ satisfying

$$(5.1)\quad f(gman) = a^{-\lambda-\rho_P}\xi(m^{-1})f(g), \quad (g \in G, m \in M, a \in A, n \in N),$$

with respect to the sesquilinear product

$$\langle f_1 | f_2 \rangle = \int_K \langle f_1(k) | f_2(k) \rangle dk.$$

The action $\pi_{\xi,\lambda}(g)$ of G is given by the left regular action

$$\pi_{\xi,\lambda}(g)f(x) = f(g^{-1}x).$$

It is easily seen that one gets a bounded representation of G this way (the representation is *induced* from the representation $\xi \otimes e^\lambda \otimes 1$ of MAN), and that the sesquilinear product defined above is G-invariant if λ is purely imaginary on $\mathfrak{a}$, so that the representation in that case becomes a unitary representation. It is also easily checked that the equivalence class of $\pi_{\xi,\lambda}$ only depends on the equivalence class of ξ.

Note that because $G = KMAN$ we have that restriction to K is a bijection of $C(\xi\colon\lambda)$ onto the space $C(K\colon\xi)$ of continuous functions $f\colon K \to \mathcal{H}_\xi$ satisfying

$$(5.2)\qquad f(km) = \xi(m^{-1})f(k), \qquad (k \in K, m \in M \cap K).$$

Using this picture it follows that $\mathcal{H}_{\xi,\lambda}$ is isomorphic to the space $L^2(K:\xi)$ of L^2 functions from K to $\mathcal{H}_\xi$ satisfying (5.2).

It turns out that the parabolic subgroups which are best suited for the study of G/H are the $\sigma\theta$-stable parabolic subgroups, and the simplest of these are the minimal ones, the σ-minimal parabolic subgroups. From now on I confine myself to the principal series representations induced from σ-minimal parabolic subgroups. However, not all $\pi_{\xi,\lambda}$ of these qualify for being "the principal series for G/H." Before I proceed with defining which ξ and λ qualify, let me for the purpose of motivation consider the "abstract" Plancherel decomposition of $L^2(X)$.

It is known (because G is a so-called type I group) that any unitary representation V of G on a separable Hilbert space $\mathcal{H}_V$ has a direct integral decomposition

$$V \simeq \int_{\pi\in\hat{G}}^{\oplus} V^\pi d\mu_V(\pi), \tag{5.3}$$

where $\hat{G}$ is the unitary dual (the set of equivalence classes of unitary irreducible representations) of G, $d\mu_V$ a Borel measure on $\hat{G}$ and V^π a (possibly infinite) multiple of π.

In particular this applies to the regular representation ℓ of G on $L^2(X)$. If we denote by m_π the multiplicity of π in ℓ^π we can thus write down the *abstract Plancherel decomposition*

$$\ell \simeq \int_{\pi\in\hat{G}}^{\oplus} m_\pi \pi \, d\mu(\pi). \tag{5.4}$$

The measure $d\mu$ (whose class is uniquely determined) is called the *Plancherel measure* for G/H, and m_π (which is unique almost everywhere) the *multiplicity* of π in $L^2(X)$. As mentioned in the Introduction, the aim of the harmonic analysis on X is to make this decomposition more explicit.

Let $(V, \mathcal{H}_V)$ be as above, and let $\mathcal{H}_V^\infty$ be the Fréchet space of C^∞ vectors for V. Its topological anti-dual is denoted $\mathcal{H}_V^{-\infty}$ and called the space of *distribution vectors* for V. It follows from the unitarity of V that

$$\mathcal{H}_V^\infty \subset \mathcal{H}_V \subset \mathcal{H}_V^{-\infty}.$$

One can prove that together with the decomposition (5.3) of the representation V (and the corresponding decomposition of $\mathcal{H}_V$) one also has compatible decompositions of the spaces $\mathcal{H}_V^\infty$ and $\mathcal{H}_V^{-\infty}$:

$$(5.5)\quad \mathcal{H}_V^\infty \simeq \int_{\pi\in\hat{G}}^{\oplus} (V^\pi)^\infty \, d\mu_V(\pi) \quad \text{and} \quad \mathcal{H}_V^{-\infty} \simeq \int_{\pi\in\hat{G}}^{\oplus} (V^\pi)^{-\infty} \, d\mu_V(\pi).$$

Thus each element $\delta \in \mathcal{H}_V^{-\infty}$ can be decomposed as

$$\delta \simeq \int_{\pi\in\hat{G}} \delta^\pi \, d\mu_V(\pi),$$

with distribution vectors $\delta^\pi \in (V^\pi)^{-\infty}$, which are uniquely determined almost everywhere. The δ^π are *cyclic distribution vectors* for V^π, in the sense that if $u \in (V^\pi)^\infty$ and $\delta^\pi(\pi(g^{-1})u) = 0$ for all $g \in G$ then $u = 0$.

We apply this to ℓ and δ_o, the Dirac measure of G/H at the origin:

$$(5.6)\qquad \delta_o \simeq \int_{\pi\in\hat{G}}^{\oplus} \delta_o^\pi \, d\mu(\pi).$$

Since δ_o is H-invariant it follows from the uniqueness of the δ_o^π that they (or at least almost all of them) are also H-invariant. Being cyclic vectors the δ_o^π must be nonzero, and hence it follows that *only the representations $\pi \in \hat{G}$ which have nonzero H-fixed distribution vectors contribute to the Plancherel decomposition of ℓ* (the remaining representations form a $d\mu$-null set). The space of H-invariant distribution vectors for V is denoted by $(\mathcal{H}_V^{-\infty})^H$, and the set of $\pi \in \hat{G}$ with $(\mathcal{H}_\pi^{-\infty})^H \neq 0$ is denoted $\hat{G}_H$. This gives the following refinement of (5.4):

$$(5.7)\qquad \ell \simeq \int_{\pi\in\hat{G}_H}^{\oplus} m_\pi \pi \, d\mu(\pi).$$

(In fact it is not clear whether the subset $\hat{G}_H$ of $\hat{G}$ is measurable; nevertheless (5.7) makes sense because $d\mu$ is concentrated on the (measurable) set where $m_\pi \neq 0$, and this set is contained in $\hat{G}_H$ because of (5.9) below).

Note that since δ_o^π is a cyclic vector for ℓ^π the map $u \mapsto \overline{\delta_o^\pi(\pi(g^{-1})u)}$ is a G-equivariant continuous linear injection of the space $(\ell^\pi)^\infty$ of smooth vectors for ℓ^π into $C^\infty(G/H)$. In fact this property of allowing an injection into $C^\infty(G/H)$ is characteristic for all of $\hat{G}_H$:

Lemma 5.1. *Let $\pi \in \hat{G}$. There is a bijective antilinear map from the space $(\mathcal{H}_\pi^{-\infty})^H$ of H-fixed distribution vectors for π onto the space of continuous equivariant linear maps from $\mathcal{H}_\pi^\infty$ to $C^\infty(G/H)$.*

Proof. For $v' \in \mathcal{H}_\pi^{-\infty}$ and $v \in \mathcal{H}_\pi^\infty$ define the "matrix coefficient" $T_{v,v'} \in C^\infty(G)$ by

$$T_{v,v'}(g) = v'(\pi(g^{-1})v), \tag{5.8}$$

then T is antilinear in v and linear in v'. It is clear that if v' is H-fixed then $v \mapsto \overline{T_{v,v'}}$ is a continuous equivariant linear map $\mathcal{H}_\pi^\infty \to C^\infty(G/H)$. Conversely, if such a map j: $\mathcal{H}_\pi^\infty \to C^\infty(G/H)$ is given, then an element $v' \in (\mathcal{H}_\pi^{-\infty})^H$ is obtained by letting $v'(v) = \overline{j(v)(e)}$. The proof is easily completed. □

Since $(\ell^\pi)^\infty$, which is an m_π-fold multiple of $\mathcal{H}_\pi^\infty$, can be embedded into $C^\infty(G/H)$ it follows that

$$m_\pi \leq \dim(\mathcal{H}_\pi^{-\infty})^H \tag{5.9}$$

for almost all π. Note that according to the lemma the multiplicity of π in $C^\infty(G/H)$ is $\dim(\mathcal{H}_\pi^{-\infty})^H$; since m_π is the multiplicity of π in $L^2(G/H)$ (hence by (5.5) also of $\mathcal{H}_\pi^\infty$ in $L^2(G/H)^\infty$), the statement in (5.9) is quite natural: the extra requirement of square integrability gives a smaller or equal multiplicity.

With these results in mind it is interesting that we have

Proposition 5.2. *The space $(\mathcal{H}_\pi^{-\infty})^H$ is finite dimensional for all $\pi \in \hat{G}$.*

Proof. (sketch) Fix a nonzero K-finite vector v in $\mathcal{H}_\pi^\infty$. It follows Lemma 5.1 and its proof that the map taking an element $v' \in (\mathcal{H}_\pi^{-\infty})^H$ to the matrix coefficient $T_{v,v'} \in C^\infty(G/H)$ given by (5.8) is injective. Since π is irreducible it has an infinitesimal character χ. Hence it follows that $T_{v,v'}$ is a K-finite eigenfunction for the center $\mathfrak{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$. In fact it can be shown that the space of functions f on G/H, which are K-finite of a given type and eigenfunctions for $\mathfrak{Z}(\mathfrak{g})$ with a given infinitesimal character χ, is finite dimensional. If G/H has split rank one this can be seen roughly as follows. Since f is an eigenfunction for L its restriction to $\mathfrak{a}_q$ satisfies a second-order ordinary differential equation, and hence lies in the two-dimensional solution space. It follows easily that all such functions f lie in a space of dimension at most twice the square of the dimension of the K-type. For spaces of higher split rank the argument is of a similar nature. The proposition follows from this. $\square$

Note that it can be proved that the decomposition (5.6) also can be written in the following fashion, which is less abstract because the integrand has its values in the distributions on G/H. There exist for each $\pi \in \hat{G}_H$ distribution vectors $\delta_i^\pi \in (\mathcal{H}_\pi^{-\infty})^H$, $(1 \leq i \leq m_\pi)$ such that

$$\delta_o = \int_{\pi \in \hat{G}_H} \sum_{i=1}^{m_\pi} T_{\delta_i^\pi, \delta_i^\pi} \, d\mu(\pi), \tag{5.10}$$

where $T_{v',v'}$ for $v' \in (\mathcal{H}_\pi^{-\infty})^H$ is the H-fixed distribution on G/H given by

$$T_{v',v'}(\phi) = v'(\pi(\overline{\phi^\vee})v') \tag{5.11}$$

for $\phi \in C_c^\infty(G)$, where $\phi^\vee(g) = \phi(g^{-1})$. (The expression (5.11) makes sense because $\pi(C_c^\infty(G))\mathcal{H}_\pi^{-\infty} \subset \mathcal{H}_\pi^\infty$.)

Example 5.1. If $H = K$ is compact the space $(\mathcal{H}_\pi^{-\infty})^H$ has dimension at most one. This can be seen as follows. First of all, the elements of $(\mathcal{H}_\pi^{-\infty})^K$ are K-finite (since they are actually K-fixed). It follows from the irreducibility of π that if v is any nonzero element in $(\mathcal{H}_\pi^{-\infty})^K$ then $\pi(U(\mathfrak{g}))v$ equals the space of all K-finite vectors in $\mathcal{H}_\pi$. In particular we have that $(\mathcal{H}_\pi^{-\infty})^K \subset \pi(U(\mathfrak{g}))v$. But for any element $a \in U(\mathfrak{g})$ we have that if $\pi(a)v$ is also K-fixed, then $\pi(a)v = \pi(a^\sharp)v$ where $a^\sharp = \int_K \mathrm{Ad}(k)(a)dk \in U(\mathfrak{g})^K$. This shows that $U(\mathfrak{g})^K$ acts irreducibly on $(\mathcal{H}_\pi^{-\infty})^K$. Since $U(\mathfrak{g})\mathfrak{k}$ clearly annihilates $(\mathcal{H}_\pi^{-\infty})^K$ this action passes to an irreducible action of $\mathbf{D}(G/K)$. Since $\mathbf{D}(G/K)$ is abelian it follows that the dimension of $(\mathcal{H}_\pi^{-\infty})^K$ is at most one.

As given above, the argument applies to the situation where G/K is a noncompact Riemannian symmetric space (K is maximal compact in G). In fact it applies to a compact symmetric space as well (where π is finite dimensional), because also in this case $\mathbf{D}(G/K)$ is abelian. This follows from Theorem 4.3.

It follows now from (5.9) that the decomposition of $L^2(G/K)$ is *multiplicity free*, that is, $m_\pi = 1$ for all $\pi \in \hat{G}_K$. Moreover the distributions $T_{v,v}$ in (5.10) are K-biinvariant eigenfunctions for $\mathbf{D}(G/K)$. Such a function is called a *spherical function* if it takes the value 1 at the origin. To a given eigenvalue homomorphism $\chi: \mathbf{D}(G/K) \to \mathbf{C}$ there corresponds one and only one spherical function $\phi = \phi_\pi$ (this follows easily from the fact that ϕ, as an eigenfunction for the elliptic operator L on G/K, is real analytic, because the Taylor series at o is determined from the set of all $(r(a)\phi)(o)$ where $a \in U(\mathfrak{g})$, and by integration of a over K as above these are determined by the $(r(a^\natural)\phi)(o))$. Thus (5.10) says that

$$\delta_o = \int_{\pi \in \hat{G}_K} \phi_\pi \, d\mu(\pi) \tag{5.12}$$

for some Borel measure $d\mu$ on $\hat{G}_K$.

Example 5.2. The group case $G = {}'G \times {}'G$. The unitary dual $\hat{G}$ is equal to the Cartesian product ${}'\hat{G} \times {}'\hat{G}$. Its elements are the representations $\pi = \pi_1 \otimes \pi_2$, where $\pi_1, \pi_2 \in {}'\hat{G}$. It is easily seen that the representation π belongs to $\hat{G}_H$ if and only if π_2 is the contragradient to π_1, and that the space $(\mathcal{H}_\pi^{-\infty})^H$ then has dimension 1. (For example one can use Lemma 5.1 combined with the following observation: The space of continuous G-equivariant linear maps $j: \mathcal{H}_{\pi_1}^\infty \otimes \mathcal{H}_{\pi_2}^\infty \to C^\infty({}'G)$ is in bijective correspondence with the space of continuous ${}'G$-equivariant bilinear pairings $\mathcal{H}_{\pi_1}^\infty \times \mathcal{H}_{\pi_2}^\infty \to \mathbf{C}$; the map j corresponding to a given pairing $(\cdot,\cdot)$ is the map that takes $u \otimes v$ to the matrix coefficient $g \mapsto (\pi_1(g^{-1})u, v) = (u, \pi_2(g)v)$ on ${}'G$.)

After this motivational digression it is time to return to the principal series. The conclusion we draw is that if we want the representations we have constructed to enter into the decomposition of $L^2(X)$, we should look for representations with nontrivial H-fixed distribution vectors.

As is easily seen, the C^∞ vectors for $\pi_{\xi,\lambda}$ are the smooth functions $f: G \to \mathcal{H}_\xi^\infty$ satisfying the transformation rule (5.1). Similarly, the distribution vectors for $\pi_{\xi,\lambda}$ are the $\mathcal{H}_\xi^{-\infty}$-valued distributions on G which satisfy (5.1). Recall from the previous lectures (see (3.2)) that for the σ-minimal parabolic subgroup $P = MAN$ we have $\mathfrak{a} = \mathfrak{a}_h \oplus \mathfrak{a}_q$, where $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. By means of this orthogonal decomposition $\mathfrak{a}_{q,c}^*$ is naturally viewed as a subspace of $\mathfrak{a}_c^*$. Since P is $\sigma\theta$-stable we have that $\sigma\theta\rho_P = \rho_P$, and hence $\rho_P \in \mathfrak{a}_q$ (it vanishes on $\mathfrak{a}_h$). Moreover, it then follows from the definition of ρ_P that it coincides with the element ρ defined in (4.3).

Recall also from Lecture 3 that $\cup_{w\in\mathcal{W}} Hw^{-1}P$ is the union of open $H\times P$ cosets in G. It follows that an H-fixed distribution vector for $\pi_{\xi,\lambda}$ restricts to a smooth $\mathcal{H}_\xi^{-\infty}$-valued function f on each open coset $Hw^{-1}P$, and this restriction is uniquely determined by the value $f(w^{-1})$. Moreover, this value has to satisfy

$$a^{-\lambda-\rho_P}\xi(m^{-1})f(w^{-1}) = f(w^{-1}ma) = f(w^{-1}maww^{-1}) = f(w^{-1})$$

for each $ma \in MA \cap wHw^{-1}$. Thus if the restriction of f to $Hw^{-1}P$ is nonzero, ξ must have a nonzero distribution vector fixed by $M\cap wHw^{-1} = w(M\cap H)w^{-1}$, and $\lambda+\rho_P$ must vanish on $\mathfrak{a}\cap\mathrm{Ad}(w)(\mathfrak{h}) = \mathfrak{a}_h$ (here it is used that w has been chosen according to Theorem 2.6, so that it normalizes $\mathfrak{a}_h$). Since $\rho_P = \rho \in \mathfrak{a}_q^*$ it follows that we must have $\lambda \in \mathfrak{a}_{q,c}^*$.

Lemma 5.3. *Let $w \in N_K(\mathfrak{a}_q)$, and let ξ be an irreducible unitary representation of M for which the space $(\mathcal{H}_\xi^{-\infty})^{w(M\cap H)w^{-1}}$ of $w(M\cap H)w^{-1}$ fixed distribution vectors is nonzero. Then this space is one-dimensional, and ξ is finite dimensional.*

Remark. Note that the dimension of the space $(\mathcal{H}_\xi^{-\infty})^{w(M\cap H)w^{-1}}$ depends only on the double coset $(M\cap K)wN_{K\cap H}(\mathfrak{a}_q)$ (but the space itself may vary).

Proof. It suffices to consider the trivial w. Recall Lemma 3.2 and its proof, according to which there is a normal subgroup M_n of M contained in H such that $M = M_0M_n$. It follows easily that if $(\mathcal{H}_\xi^{-\infty})^{M\cap H}$ is nonzero then $\xi|_{M_n}$ is trivial and $\xi|_{M_0}$ is irreducible. Hence $\dim\xi < \infty$ by the compactness of M_0. Moreover we then have

$$(\mathcal{H}_\xi^{-\infty})^{M\cap H} \simeq (\mathcal{H}_{\xi|_{M_0}}^{-\infty})^{M_0\cap H} \tag{5.13}$$

Under our general assumption on G that it is linear there exists a finite central subgroup F of M_0 such that $M_0 = (M_0)_eF$ (see [123, p. 435, Exercise A3]). It follows that also $\xi|_{(M_0)_e}$ is irreducible. Now according to Example 5.1, the space $(\mathcal{H}_{\xi|_{(M_0)_e}}^{-\infty})^{(M_0)_e\cap H}$ has dimension zero or one, and hence the same holds for the (possibly smaller) spaces in (5.13). □

Motivated by Lemma 5.3 and the preceding discussion we define *the principal series for G/H* (or the *H-spherical principal series*) related to the

σ-minimal parabolic subgroup $P = MAN$ as the series of representations $\pi_{\xi,\lambda}$ where ξ is a finite dimensional irreducible unitary representation of M having a nonzero $w(M \cap H)w^{-1}$ fixed vector for some $w \in \mathcal{W}$, and where $\lambda \in \mathfrak{a}_{q,c}^*$. The *unitary principal series* is the subseries with λ purely imaginary on $\mathfrak{a}_q$.

Note that I did not argue that these conditions on ξ and λ are necessary for the induced representation to have a nonzero H-fixed distribution vector, but only that if these conditions do not hold, such a distribution has to be more singular in the sense that it has to be concentrated on the nonopen $H \times P$ cosets. On the other hand, we shall see in the next lecture that the representations in the principal series for G/H really do have nonzero H-fixed distribution vectors.

Example 5.3. In continuation of Example 5.2 let $H = K$. By the definition above the principal series for G/K related to the minimal parabolic subgroup $P_0 = M_0A_0N_0$ is the *spherical principal series* consisting of the induced representations $\pi_{1,\lambda}$ where $1 \in \hat{M}_0$ denotes the trivial representation. In this case it is in fact clear from the definition that the induced representation $\pi_{\xi,\lambda}$ has a K-fixed vector (which is then unique up to scalar multiplication) if and only if ξ is the trivial representation. One K-fixed vector is the function $v \in C(1{:}\,\lambda)$ defined by $v = \mathbf{1}_\lambda(g) := e^{-(\lambda+\rho_0)H(g)}$, where H is the Iwasawa projection. The corresponding *spherical function* $\phi_{\pi_{1,\lambda}} = T_{v,v}$ is then given by

$$\begin{aligned} T_{v,v}(\varphi) = \langle \pi_{1,\lambda}(\varphi)v | v\rangle &= \int_K \left(\pi_{1,\lambda}(\varphi)v\right)(k)\overline{v(k)}dk \\ &= \int_K \int_G \varphi(g)\mathbf{1}_\lambda(g^{-1}k)dg\, dk = \int_G \varphi(g)\varphi_\lambda(g)\, dg, \end{aligned}$$

where $\varphi_\lambda(g) = \int_K \mathbf{1}_\lambda(g^{-1}k)dk$ (see (4.2)), and we get that

$$\phi_{\pi_{1,\lambda}} = \varphi_\lambda.$$

Example 5.4. Consider again the hyperbolic spaces $\mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$. Recall that a σ-minimal parabolic subgroup is the stabilizer in G of the line $\mathbf{R}(1,0,\ldots,0,1) \in \mathbf{R}^{p+q}$. The group M consists of the matrices of the form

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \epsilon \end{pmatrix},$$

where $m \in \mathrm{SO}(p-1, q-1), \epsilon = \pm 1$, and $M \cap H$ is the subgroup where $\epsilon = 1$ (if $p = 1$ or $q = 1$ then ϵ is always 1 and $M \cap H = M$). Thus the representations of M that we need for the principal series are the trivial representation, and the representation which assigns ϵ to the element above (if $p = 1$ or $q = 1$ this is also the trivial representation). We denote these by ξ_0 and ξ_1, respectively.

Let Ξ be the set

$$\Xi = \{x \in \mathbf{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0, x \neq 0\}$$

(if $p = 1$ it is also required that $x_1 > 0$, and if $q = 1$ that $x_{p+1} > 0$). Then G acts transitively on Ξ, and we get that $\Xi \simeq G/(M \cap H)N$.

For $\lambda \in \mathbf{C}$ and $i = 0, 1$ let $C_{i,\lambda}(\Xi)$ denote the space of continuous functions f on Ξ satisfying

$$f(vx) = \mathrm{sign}(v)^i |v|^{-\lambda-\rho} f(x)$$

for all $v \neq 0$, where $\rho = \frac{1}{2}(p+q-2)$. Then there is a natural representation of G on this space, and it can be seen that the Hilbert space norm

$$\|f\|^2 = \int_{\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}} |f(x)|^2 dx$$

is invariant if λ is purely imaginary. By this construction we get an explicit model for the principal series representation $\pi_{\xi_i,\nu}$ where $\nu \in \mathfrak{a}_{q,c}^*$ is given by $\nu(Y) = \lambda$.

The following result is clearly important.

Theorem 5.4. *Let $\pi_{\xi,\lambda}$ be a unitary principal series representation for G/H, and assume that $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma(\mathfrak{a}_q, \mathfrak{g})$. Then $\pi_{\xi,\lambda}$ is irreducible.*

Proof. See the notes for references to this theorem. □

Example 5.5. In the case of the hyperboloids one can show that the representations $\pi_{\xi_i,\lambda}$ constructed above are irreducible if $\lambda+\rho$ is not an integer. In particular, they are irreducible if λ is purely imaginary and nonzero. See the notes for references.

In general two principal series representations $\pi_{\xi,\lambda}$ and $\pi_{\xi',\lambda'}$ with different pairs (ξ,λ) and (ξ',λ') may well be equivalent. It is important to study these equivalences as well as the corresponding intertwining operators.

Let $s\in W=N_K(\mathfrak{a}_q)/(M\cap K)$, and let $\tilde{s}$ be a representative. Conjugation by $\tilde{s}$ preserves M, and hence from each representation $(\xi,\mathcal{H}_\xi)$ of M another representation denoted $(\tilde{s}\xi,\mathcal{H}_{\tilde{s}\xi})$ is obtained by letting $\mathcal{H}_{\tilde{s}\xi}=\mathcal{H}_\xi$ and $\tilde{s}\xi(m)=\xi(\tilde{s}^{-1}m\tilde{s})$. It is easily seen that the equivalence class of $\tilde{s}\xi$ only depends on s and the equivalence class of ξ. For this reason I shall often write $s\xi$ instead of $\tilde{s}\xi$. We shall see below that for generic λ we have $\pi_{\xi,\lambda}\simeq\pi_{s\xi,s\lambda}$.

When working with intertwining operators between the principal series it is convenient to be able also to switch between representations induced from different parabolic subgroups. Thus I write $\pi_{P,\xi,\lambda}$ for the principal series representation associated to the parabolic subgroup P, and $C(P{:}\,\xi{:}\,\lambda)$ for space denoted $C(\xi{:}\,\lambda)$ above. However, only the nilpotent part N of the parabolic subgroup $P=MAN$ will vary, and thus the space $C(K{:}\,\xi)$ of restrictions to K is the same for all P (it is the G-action which varies). Note that switching the P is basically a technical matter, because any two σ-minimal parabolic subgroups are related by conjugation, and there is an equivalence $\pi_{sPs^{-1},s\xi,s\lambda}\simeq\pi_{P,\xi,\lambda}$ obtained by the simple intertwining operator

$$R(s)\colon C(P{:}\,\xi{:}\,\lambda)\xrightarrow{\sim}C(sPs^{-1}{:}\,s\xi{:}\,s\lambda)$$

defined by $R(s)f(g)=f(g\tilde{s})$.

There is a well known set of intertwining operators between principal series representations, called the *standard intertwining operators.* Let me sketch the construction of these in case of the σ-minimal principal series. Let $P=MAN$ and $P'=MAN'$ be σ-minimal parabolic subgroups, and let ξ be a finite dimensional unitary representation of M and $\lambda\in\mathfrak{a}^*_{q,c}$. For $f\in C^\infty(P{:}\,\xi{:}\,\lambda)$ define

$$A(P'{:}\,P{:}\,\xi{:}\,\lambda)f(g)=\int_{\bar{N}\cap N'}f(g\bar{n})d\bar{n}, \tag{5.14}$$

where $d\bar{n}$ is a (suitably normalized) Haar measure on $\bar{N} \cap N'$. Disregarding the convergence of (5.14) it is easily checked that $A(P': P: \xi: \lambda)$ is intertwining from $\pi_{P,\xi,\lambda}$ to $\pi_{P',\xi,\lambda}$. The problem of convergence is serious, but at least the following holds.

Proposition 5.5. *There exists a constant $C \geq 0$ such that if $\langle \operatorname{Re} \lambda, \alpha \rangle > C$ for all roots $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ such that $\mathfrak{g}^\alpha \subset \bar{\mathfrak{n}} \cap \mathfrak{n}'$, then the integral (5.14) converges absolutely and defines a continuous intertwining operator from $C^\infty(P: \xi: \lambda)$ to $C^\infty(P': \xi: \lambda)$.*

Proof. The proof uses results from Chapter 7 of [130]. For the case $H = K$, where P and P' are minimal parabolic subgroups, see *loc. cit.*, Prop 7.8. Here $C = 0$. For the general case let $\rho_m \in \mathfrak{a}_0^*$ denote half the trace of ad on $\mathfrak{n}_0 \cap \mathfrak{m}$ (then ρ_m is zero on $\mathfrak{a}$), and for $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ let C_α denote the maximum of the $\langle \beta, \rho_m \rangle$ where $\beta \in \Sigma(\mathfrak{a}_0, \mathfrak{g})$ with $\beta|_{\mathfrak{a}_q} = \alpha$. Then $-C_\alpha$ is the minimum of these numbers. Hence if $\langle \operatorname{Re} \lambda, \alpha \rangle > C_\alpha$ we have $\langle \rho_m + \operatorname{Re} \lambda, \beta \rangle > 0$. We can now apply *loc. cit.*, Theorem 7.22 with $\lambda = \rho_m$. (The reason for taking $\lambda = \rho_m$ is that then $\phi_\lambda^M = 1$ in the notation of *loc. cit.* In the cited theorem f is assumed K-finite, but this is not needed when ξ is finite dimensional.) □

For parameters λ outside the domain of convergence of (5.14) given in Proposition 5.5, an intertwining operator can be constructed by means of analytic continuation. The result is as follows (see [172, pp. 78–79] for the notion of a Fréchet space valued analytic function).

Theorem 5.6. *Let $f \in C^\infty(K: \xi)$. Then $A(P': P: \xi: \lambda)f$, which is defined by the convergent integral (5.14) for λ in the region given in Proposition 5.5, extends to a meromorphic $C^\infty(K: \xi)$-valued function of λ in $\mathfrak{a}_{q,c}^*$. The operator $A(P': P: \xi: \lambda)$ thus obtained for generic λ is a continuous intertwining operator from $C^\infty(P: \xi: \lambda)$ to $C^\infty(P': \xi: \lambda)$.*

Proof. Too complicated to be given here. □

It follows easily from the definitions that we have

$$(5.15) \qquad R(s)A(s^{-1}Ps: P: \xi: \lambda) = A(P: sPs^{-1}: s\xi: s\lambda)R(s).$$

For generic λ this is a nonzero intertwining operator from $\pi_{P,\xi,\lambda}$ to $\pi_{P,s\xi,s\lambda}$. By Theorem 5.4 these representations are irreducible and must hence be equivalent.

LECTURE 6

Spherical distributions

In the previous lecture I defined the principal series of representations $\pi_{\xi,\lambda}$ for G/H. The motivation for the requirements on ξ and λ was the demand that $\pi_{\xi,\lambda}$ should have a nonzero H-fixed distribution vector (a *spherical distribution*). In this lecture I shall show that this is indeed the case by a rather explicit construction of some spherical distributions.

Let $P = MAN$ be a σ-minimal parabolic subgroup, ξ a finite dimensional unitary representation of M, λ an element in $\mathfrak{a}_{q,c}^*$, and $\pi_{\xi,\lambda}$ the corresponding principal series representation. Let $C^{-\infty}(\xi\colon\lambda)$ denote the space of $\mathcal{H}_\xi$-valued distributions on G satisfying the transformation rule (5.1). It is convenient to have a model for this space which is independent of λ. This is obtained by taking restrictions to K (it follows from the transformation rule that this makes sense also on distributions). Thus $C^{-\infty}(\xi\colon\lambda)$ is isomorphic to the space $C^{-\infty}(K\colon\xi)$ of $\mathcal{H}_\xi$-valued distributions on K satisfying the transformation rule (5.2). The space $C^{\infty}(K\colon\xi)$ is defined similarly. By definition $C^{-\infty}(K\colon\xi)$ is the topological antidual of $C^{\infty}(K\colon\xi)$; by means of the sesquilinear product on $\mathcal{H}_\xi$ and the normalized Haar measure on K we view the latter space as a subspace of the former.

Fix an element η in the one-dimensional space $\mathcal{H}_\xi^{M\cap H}$, and define a $\mathcal{H}_\xi$-valued function f_λ on the open set HP by

$$f_\lambda(hman) = a^{-\lambda-\rho}\xi(m^{-1})\eta$$

for $h \in H, m \in M, a \in A, n \in N$ and $\lambda \in \mathfrak{a}_q^*$. Since η is $M\cap H$ fixed it follows from Theorem 3.3 that this function is well-defined and smooth. We now extend f_λ to G by letting it equal to zero on the complement of HP. It is clear that f_λ satisfies (5.1), and also that f_λ is H-invariant. However, it is by no means clear that it is a distribution on G. For λ in a certain range, this is true. In fact it is even a continuous function.

Proposition 6.1. *If $\langle \operatorname{Re}\lambda+\rho, \alpha\rangle < 0$ for all $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ then f_λ belongs to the space $C(\xi\colon\lambda)^H$. As a $C(K\colon\xi)$-valued function of λ it is holomorphic on this domain.*

Proof. For the first statement it only remains to check the continuity. We must prove that $f_\lambda(x_n) \to 0$ for $x_n \in HP$ with $\lim x_n \notin HP$. By (5.1) and the continuity of the Iwasawa decomposition it suffices to have $x_n \in K$. As in Lecture 3, define $\mathbf{a}\colon PH \to \mathfrak{a}_q$ by $\mathbf{a}(manh) = \log a$ for $m \in M$, $a \in A_q$, $n \in N$, $h \in H$, then we have

$$\|f_\lambda(x)\| = e^{(\operatorname{Re}\lambda+\rho)\mathbf{a}(x^{-1})}\|\eta\|$$

for $x \in HP$. Now according to Theorem 3.4 the restriction of $\mathbf{a}$ to $PH \cap K$ is proper, and hence $\lim x_n \notin HP$ implies that the sequence $\mathbf{a}(x_n^{-1})$ will eventually exit any compact subset of $\mathfrak{a}_q$. According to the same theorem we also have that $\mathbf{a}(PH \cap K)$ is contained in the nonnegative span of the vectors H_α (defined by $\alpha = \langle H_\alpha, \cdot\rangle$) for $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$. Writing

$$\mathbf{a}(x_n^{-1}) = \sum_\alpha s_{n,\alpha} H_\alpha$$

we thus have $s_{n,\alpha} \geq 0$ for all α and $s_{n,\alpha} \to \infty$ for at least one α. It now follows from the assumption on λ that

$$(\operatorname{Re}\lambda + \rho)\mathbf{a}(x_n^{-1}) = \sum_\alpha s_{n,\alpha}\langle \operatorname{Re}\lambda + \rho, \alpha\rangle \to -\infty.$$

This shows the asserted continuity.

It is easily seen that the argument given above can be carried through also for the derivative of f_λ with respect to λ. The holomorphicity in λ follows. $\square$

Remark. Note that I only used the parts of Theorem 3.4 that were proved. Using Theorem 3.4 in its full strength one gets that the conclusions of Proposition 6.1 can be drawn for λ in the larger set, where $\langle \operatorname{Re}\lambda+\rho, \alpha\rangle < 0$ is required only for the positive roots α with nonzero multiplicity m_α^-.

Example 6.1. In the case of $G/H = G/K$ the function f_λ is identical with the function $\mathbf{1}_\lambda(g) = e^{-(\lambda+\rho_0)H(g)}$ defined previously (see Example 5.3). It is clear that it is holomorphic in λ on all of $\mathfrak{a}_0^*$ (this also follows from the remark above, since $m_\alpha^- = 0$ for all roots).

Example 6.2. Consider the real hyperbolic space X. In Example 5.4 $G/(M\cap H)N$ was identified with the space

$$\Xi = \{x \in \mathbf{R}^{p+q} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0, x \neq 0\}$$

(with $x_1 > 0$ if $p = 1$ and $x_{p+1} > 0$ if $q = 1$), and $C(\xi_i\colon\lambda)$ with the space of continuous functions on Ξ satisfying

$$f(vx) = \operatorname{sign}(v)^i |v|^{-\lambda-\rho} f(x) \tag{6.1}$$

for all $v \neq 0$. The function f_λ constructed above is the function on Ξ given by

$$f_\lambda(x) = \operatorname{sign}(x_1)^i |x_1|^{-\lambda-\rho}$$

for $i = 0, 1$ and $\lambda \in \mathbf{C}$. Clearly this is continuous if and only if $\operatorname{Re}\lambda + \rho \leq 0$, except for $p = 1$ where it is always continuous. Moreover, its restriction to $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$ is holomorphic in λ. Consider the case $p > 1$. In this case x_1 has $[-1; 1]$ as its range, and hence f_λ is not locally integrable if $\operatorname{Re}\lambda + \rho \geq 1$. Nevertheless it is well known (see for example [111, p. 50]) that the distributions $[-1; 1] \ni t \mapsto \operatorname{sign}(t)^i |t|^\mu$, which are locally integrable for $\operatorname{Re}\mu > -1$, can be given a sense beyond this range of μ's by means of analytic continuation. Indeed they extend meromorphically to $\mu \in \mathbf{C}$ with simple poles at $\mu = -1, -3, \ldots$ and $\mu = -2, -4, \ldots$, respectively, for $i = 0, 1$. It follows that f_λ extends meromorphically to a family of H-fixed distributions satisfying (6.1). For any function φ in the space $C^\infty(K\colon\xi_i)$, which can be identified with the space of smooth even (for $i = 0$) or odd (for $i = 1$) functions on $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$, we thus have that $\lambda \mapsto f_\lambda(\varphi)$ is the meromorphic function on $\mathbf{C}$, which is given by the convergent integral

$$f_\lambda(\varphi) = \int_{\mathbf{S}^{p-1}\times\mathbf{S}^{q-1}} \operatorname{sign}(x_1)^i |x_1|^{-\lambda-\rho} \varphi(x)\,dx$$

for $\operatorname{Re}\lambda + \rho < 0$. For example for $i = 0$ and $\varphi(x) \equiv 1$ we have

$$f_\lambda(1) = c\int_0^\pi |\cos\theta|^{-\lambda-\rho} \sin^{p-2}\theta\,d\theta = cB\Big(\frac{1-\lambda-\rho}{2}, \frac{p-1}{2}\Big) \tag{6.2}$$

for a constant c depending on the normalization of measures. Here B is the beta function $B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

As can be seen from the previous example, the function f_λ as we have defined it, will not in general be locally integrable outside the range of λ's given in Proposition 6.1. The example also shows that to overcome this obstacle (which was not present in Example 6.1) we have to invoke analytic continuation. Let me sketch one more example supporting this strategy.

Example 6.3. In the group case, $G = {}'G \times {}'G$ we have (see Example 3.2) that the minimal parabolic subgroup $P = {}'\bar{P}_0 \times {}'P_0$ in G is also a σ-minimal parabolic subgroup of G. The irreducible representations of $M = {}'M_0 \times {}'M_0$ are given by $\xi = {}'\xi \otimes {}'\xi'$ where ${}'\xi$ and ${}'\xi'$ are irreducible (necessarily finite dimensional) representations of ${}'M_0$, and since $M \cap H$ is the diagonal in M this ξ has a nonzero $M \cap H$-fixed vector if and only if ${}'\xi'$ is the contragradient ${}'\xi^\vee$ to ${}'\xi$ (see also Example 5.2). It is then natural to identify $\mathcal{H}_\xi = \mathcal{H}_{'\xi} \otimes \mathcal{H}_{'\xi}^*$ with the space $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_{'\xi}, \mathcal{H}_{'\xi})$. The subspace $\mathcal{H}_\xi^{M\cap H}$ is then identified with $\mathrm{Hom}_{'M_0}(\mathcal{H}_{'\xi}, \mathcal{H}_{'\xi}) = \mathbf{C}I$, where I is the identity map. Furthermore $\mathfrak{a} = \mathfrak{a}_0 = {}'\mathfrak{a}_0 \times {}'\mathfrak{a}_0$, and $\mathfrak{a}_q = \{(Y, -Y) \mid Y \in {}'\mathfrak{a}_0\}$. Hence the $\lambda \in \mathfrak{a}_{q,c}^*$ are given by $\lambda(Y, Z) = {}'\lambda(Y) - {}'\lambda(Z)$ with ${}'\lambda \in {}'\mathfrak{a}_{0,c}^*$ (but note that dominant λ's correspond to antidominant ${}'\lambda$'s). It follows that $C(P{:}\,\xi{:}\,\lambda)$ consists of the continuous functions $f{:}\,{}'G \times {}'G \to \mathrm{Hom}(\mathcal{H}_{'\xi}, \mathcal{H}_{'\xi})$ satisfying

$$f(gma\bar{n}, g'm'a'n) = (a^{-1}a')^{({}'\lambda - {}'\rho_0)}\, {}'\xi(m^{-1}) f(g, g')\, {}'\xi(m').$$

If in addition f is H-invariant we can view it as a function F on ${}'G$ by means of $F(x^{-1}y) = f(x, y)$. Hence $C(P{:}\,\xi{:}\,\lambda)^H$ may be identified with the space of continuous functions $F{:}\,{}'G \to \mathrm{Hom}(\mathcal{H}_{'\xi}, \mathcal{H}_{'\xi})$ satisfying

$$F(\bar{n}amxm'a'n) = (aa')^{({}'\lambda - {}'\rho_0)}\, {}'\xi(m) F(x)\, {}'\xi(m'). \tag{6.3}$$

Note that F is the kernel of an intertwining operator A from $C({}'P{:}\,{}'\xi{:}\,{}'\lambda)$ to $C({}'\bar{P}{:}\,{}'\xi{:}\,{}'\lambda)$ obtained from

$$A\varphi(x) = \int_{'K/'M_0} F(x^{-1}k)\varphi(k)\,dk = F(e) \int_{'\bar{N}} \varphi(x\bar{n})\,d\bar{n} \tag{6.4}$$

(the last equality follows from [130, Eq. (5.25)]), provided the integrals converge. Similar considerations on the level of distributions lead to the observation that *the H-fixed distribution vectors for $\pi_{\xi,\lambda}$ are the intertwining operators between the principal series for ${}'G$ corresponding to opposite* *minimal parabolic subgroups.* In particular it follows from the irreducibility of $\pi_{{}'\xi,{}'\lambda}$ for generic ${}'\lambda$ that $C^{-\infty}(\xi{:}\,\lambda)^H$ is one-dimensional for those λ.

The standard intertwining operators are obtained by defining F by $F(e) = 1$ together with (6.3). In this case this is exactly what the f_λ amounts to (taking $\eta = I$). Note that the condition in Proposition 6.1 for continuity in this case means that $\operatorname{Re}{}'\lambda - {}'\rho_0$ is strictly dominant, a slightly stronger condition than that of Proposition 5.5 for convergence of the defining integral (recall that the constant C in Proposition 5.5 is zero for the minimal parabolic). As we know from above (Theorem 5.6), the way to extend the standard intertwining operator to all of ${}'\mathfrak{a}^*_{0,c}$ is by analytic continuation.

As these examples indicate we have the following general result.

Theorem 6.2. *The map $\lambda \mapsto f_\lambda \in C^{-\infty}(K{:}\,\xi)$, initially defined when $\operatorname{Re}\lambda + \rho$ is strictly antidominant, extends to a meromorphic function on $\mathfrak{a}^*_{q,c}$. The distribution vectors $f_\lambda \in C^{-\infty}(\xi{:}\,\lambda)$ so obtained are H-fixed.*

Remark. Since $C^{-\infty}(K{:}\,\xi)$ is not a Fréchet space it is probably in order to discuss the notion of analyticity used here. A map h from a complex space to $C^{-\infty}(K{:}\,\xi)$ is called analytic if, locally, it is analytic into the Banach space of distributions of some finite order. (One can prove, along the lines of [172, p. 79], that h is analytic if and only if it is weakly analytic, that is, $\lambda \to h(\lambda)(\phi)$ is analytic for all test functions ϕ.)

It is clear that for λ in the initial domain, the support of f_λ is the closure of HP in G. In the proof we need also the H-fixed distribution vectors analogous to f_λ, but supported on the closure of the other open $H \times P$ double cosets on G. Let me discuss these before I give the proof of Theorem 6.2.

Recall that the open $H \times P$ double cosets on G are given by $Hw^{-1}P$ for our fixed set $\mathcal{W}$ of representatives $w \in N_K(\mathfrak{a}_q)$ for $W/W_{K\cap H}$. Recall also from the previous lecture that each H-fixed distribution vector for $\pi_{\xi,\lambda}$ restricts to a smooth function on these open sets. We can thus define an evaluation map ev_w from $C^{-\infty}(\xi{:}\,\lambda)^H$ to $\mathcal{H}_\xi$ by

$$\mathrm{ev}_w(f) = f(w^{-1}),$$

and then ev_w actually takes its values in the one-dimensional (cf. Lemma 5.3) space $\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$ Let $V(\xi)$ denote the formal sum

$$V(\xi) = \oplus_{w\in\mathcal{W}}\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$$

provided with the direct sum inner product. Thus by definition the summands are mutually orthogonal, even though this may not be the case inside $\mathcal{H}_\xi$ (for example if ξ is the trivial representation). For $\eta \in V(\xi)$ let η_w denote the w-component, now viewed as an element of $\mathcal{H}_\xi$. We can then collect all the maps ev_w into one map $\mathrm{ev}: C^{-\infty}(\xi:\lambda)^H \to V(\xi)$ defined by $\mathrm{ev}(f)_w = \mathrm{ev}_w(f)$. It turns out that for generic λ there is no element in $C^{-\infty}(\xi:\lambda)^H$ whose support is disjoint from all the open cosets $Hw^{-1}P$. More precisely we have the following.

Theorem 6.3. *Let $\pi_{\xi,\lambda}$ be a principal series representation for G/H. There is a countable set of complex hyperplanes in $\mathfrak{a}_{q,c}^*$ such that* ev *is injective when λ is in the complement of all these hyperplanes.*

Proof. This is based on an analysis similar to that of Bruhat (sketched in [130, Section 7.3], see also [112] for a more thorough sketch), which leads to the fact that for generic λ, the representation $\pi_{\xi,\lambda}$ in the minimal principal series is irreducible (as seen in Example 6.3 above, this is actually related to a special case). See the example below for an idea of the proof. $\square$

Example 6.4. Consider the real hyperboloids for the simplest case where $p > 2, q > 1$. In analogy with what we have seen earlier for continuous functions we have that $C^{-\infty}(\xi_i:\lambda)$ consists of the distributions f on Ξ satisfying (6.1). The only open $H \times P$ coset in G is HP (see Example 3.4). This corresponds to the subset $\Xi_0 = \{x \in \Xi \mid x_1 \neq 0\}$. The action of H on the complement $\Xi_1 = \{x \in \Xi \mid x_1 = 0\}$ is transitive (here $p > 2$ is used). By the general structure of distributions supported in a submanifold we have that if f has support on Ξ_1 then it is given uniquely by a distribution on Ξ_1 together with some transversal derivatives. If $f \in C^{-\infty}(\xi_i:\lambda)^H$ then the distribution on Ξ_1 must be H-fixed, and hence it is a constant. Thus it follows that f is the distribution

$$\varphi \mapsto \int_{\Xi_1} (P(\partial_{x_1})\varphi)(y)dy$$

for some polynomial P (where dy is the H-invariant measure on Ξ_1). The homogeneity in (6.1) now forces $P(v) = v^{-\lambda-\rho}$ for $v > 0$. This shows that $-\lambda - \rho$ has to be a nonnegative integer in order for such a distribution to exist. This proves Theorem 6.3 for this case. The cases $p = 2$ or $q = 1$ are similar.

Let me now turn to the construction of the analogs of f_λ for all the open double cosets $Hw^{-1}P$. It is convenient to collect all these together and at once define a linear map $j(\xi{:}\,\lambda) = j(P{:}\,\xi{:}\,\lambda)$ from $V(\xi)$ to $C(\xi{:}\,\lambda)^H$ by

$$(6.5) \qquad j(\xi{:}\,\lambda)(\eta)(hw^{-1}man) = a^{-\lambda-\rho}\xi(m^{-1})\eta_w \in \mathcal{H}_\xi$$

on $\cup_{w\in\mathcal{W}} Hw^{-1}P$, and by $j(\xi{:}\,\lambda)(\eta) = 0$ on the complement of this set. The f_λ constructed above is obtained by composing j with the embedding of $\mathcal{H}_\xi^{M\cap H}$ as a subspace of $V(\xi)$, and its analog supported on the closure of $Hw^{-1}P$ is similarly obtained by composition with the embedding of $\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$. The proof of Proposition 6.1 is easily generalized to show that we really do have $j(\xi{:}\,\lambda)\eta \in C(\xi{:}\,\lambda)^H$ for all η when $\operatorname{Re}\lambda + \rho$ is strictly antidominant. For such λ we then have that $\operatorname{ev} \circ j(\xi{:}\,\lambda)$ is the identity operator on $V(\xi)$, and if in addition λ is generic then it follows from Theorem 6.3 that $j(\xi{:}\,\lambda)$ is a bijection of $V(\xi)$ onto $C^{-\infty}(\xi{:}\,\lambda)^H$. We can now state the following extension of Theorem 6.2.

Theorem 6.4. *The map $\lambda \mapsto j(\xi{:}\,\lambda) \in \operatorname{Hom}(V(\xi), C^{-\infty}(K{:}\,\xi))$ initially defined for $\lambda \in \mathfrak{a}_{q,c}^*$ with $\operatorname{Re}\lambda + \rho$ strictly antidominant, extends to a meromorphic function on $\mathfrak{a}_{q,c}^*$. For generic λ the $j(\xi{:}\,\lambda)$ so obtained is a bijection from $V(\xi)$ onto $C^{-\infty}(\xi{:}\,\lambda)^H$, and* ev *is its inverse.*

We call the distributions $j(\xi{:}\,\lambda)\eta \in C^{-\infty}(K{:}\,\xi)$, where $\eta \in V(\xi)$, the *standard spherical distributions*, and $j(\xi{:}\,\lambda)$ the *standard spherical distribution map.*

Proof. The idea of the proof is as follows. From Theorem 6.3 we know that ev for generic λ is a bijection. If we can prove the existence of a meromorphic $\operatorname{Hom}(V(\xi), C^{-\infty}(K{:}\,\xi))$-valued function $J(\lambda)$ on all of $\mathfrak{a}_{q,c}^*$, which for generic λ gives rise to an inverse of ev, then we are done, because J has to coincide with j on the initial domain for j. The J is obtained in two steps.

The first step is to prove the existence of J on the opposite of the initial domain, that is, where $\operatorname{Re}\lambda-\rho>0$. This is obtained by means of the standard intertwining operator $A(\bar{P}\colon P\colon\xi\colon\lambda)\colon C^{-\infty}(P\colon\xi\colon\lambda)\to C^{-\infty}(\bar{P}\colon\xi\colon\lambda)$ (actually it was defined as a continuous operator between spaces of smooth functions, but the action is easily extended to distributions, with meromorphic dependence on λ), by defining $j^\circ(\xi\colon\lambda)=j^\circ(P\colon\xi\colon\lambda)$ by

$$j^\circ(P\colon\xi\colon\lambda)=A(\bar{P}\colon P\colon\xi\colon\lambda)^{-1}j(\bar{P}\colon\xi\colon\lambda). \tag{6.6}$$

By the equivariance of the intertwining operator we have

$$j^\circ(P\colon\xi\colon\lambda)\in\operatorname{Hom}(V(\xi),C^{-\infty}(P\colon\xi\colon\lambda)^H),$$

and this homomorphism is bijective for generic λ. But then $\mathrm{ev}\circ j^\circ$ is generically a bijection of $V(\xi)$ onto itself, and hence it has an inverse which is meromorphic in λ, and then we can take $J=j^\circ\circ(\mathrm{ev}\circ j^\circ)^{-1}$.

The second step consists of extending the existence of J from the domain $\operatorname{Re}\lambda-\rho>0$ to larger sets. This is done by multiplication with matrix coefficients of some special finite dimensional representations. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}_c$ containing $\mathfrak{a}_{0,c}$, choose a positive set of roots $\Sigma^+(\mathfrak{j},\mathfrak{g}_c)$ compatible with $\Sigma^+(\mathfrak{a}_0,\mathfrak{g})$, and let $\mu\in\mathfrak{j}^*$ be the highest weight of a finite dimensional representation (π_μ,V_μ) of G, with highest weight vector v_μ. One can show that M acts trivially on v_μ if μ restricts to zero on the complement of $\mathfrak{a}$. If it is furthermore assumed that the contragradient representation has a nonzero H-fixed vector v_H^*, it follows that the matrix coefficient $\psi(g)=v_H^*(\pi(g)v_\mu)$ is a real analytic function on G satisfying $\psi(hgman)=a^\mu\psi(g)$. Hence $f\in C^{-\infty}(\xi\colon\lambda+\mu)^H$ implies $\psi f\in C^{-\infty}(\xi\colon\lambda)^H$. Moreover, by the real analyticity we must have that ψ has no zeros on the open $H\times P$ cosets. Let Ψ be the operator on $V(\xi)$ given by multiplication with $\psi(w^{-1})$ on $\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$, and put $J_1(\lambda)=\psi J(\lambda+\mu)\Psi^{-1}$ for $\operatorname{Re}\lambda-\rho+\mu>0$, then it follows easily that $\mathrm{ev}\circ J_1(\lambda)=1$.

Finally one has to prove the existence of sufficiently many π_μ as above such that any point belongs to the domain $\operatorname{Re}\lambda-\rho+\mu>0$ for some such μ. See the notes for references to this fact. □

This also finishes the proof of Theorem 6.2. □

Note that Theorem 6.4 in the group case (see Example 6.3) gives the meromorphic continuation of the standard intertwining operators for opposite parabolic subgroups. However, these were actually used in the proof.

For the decomposition of $L^2(X)$ we are particularly interested in the imaginary values of λ, where $\pi_{\xi,\lambda}$ is unitary. Note however that these values are in the domain where the analytic continuation was necessary to obtain the standard spherical distribution map $j(\xi:\lambda)$. In particular, $j(\xi:\lambda)$ may have poles at imaginary points (this is for example the case for the real hyperboloids when $p+q$ is even and $p>1$ (see Example 6.2 above), where there is a pole at $\lambda=0$). This unpleasantness can be overcome by a suitable "renormalization." During the proof of Theorem 6.4 the operator $j^\circ(\xi:\lambda) \in \mathrm{Hom}(V(\xi), C^{-\infty}(P:\xi:\lambda)^H)$ was introduced by normalization of the standard spherical distribution map with the inverse of a standard intertwining operator (see (6.6)). This turns out to be a very fundamental operator.

Theorem 6.5. *Let (G,H) be as mentioned above. The meromorphic function $\lambda \mapsto j^\circ(P:\xi:\lambda)$ given by (6.6) has no singularities in $i\mathfrak{a}_q^*$.*

Proof. The proof will be briefly sketched in the next lecture (see the remark below Theorem 7.6). Below is an example (note however that the proof in the general case is quite different). □

We call $j^\circ(\xi:\lambda)$ the *normalized spherical distribution map.*

Example 6.5. In this example I shall prove Theorem 6.5 for the real hyperboloids X, when $q>1$ and ξ is the trivial M type $\xi_0=1$, except for the omission of the explicit evaluation of a certain integral. Since $q>1$ the space of H-fixed distribution vectors for $\pi_{1,\lambda}$ is one-dimensional for generic λ (see Example 6.4), and hence we have

$$j^\circ(P:1:\lambda) = h(\lambda) j(P:1:\lambda)$$

for some meromorphic function $h(\lambda)$. By the definition of j° we now have

$$j(\bar{P}:1:\lambda) = h(\lambda) A(\bar{P}:P:1:\lambda) j(P:1:\lambda). \tag{6.7}$$

The function h can be explicitly determined by applying the distributions in (6.7) to the test function $\varphi(x)=1$. In Example 6.2 we computed

$j(P\colon 1\colon \lambda)(1)$; analogously we get on the left side of (6.7)

$$j(\bar{P}\colon 1\colon \lambda)(1) = cB(\tfrac{1}{2}(\lambda - \rho + 1), \tfrac{1}{2}(p-1)). \tag{6.8}$$

In analogy with (6.4) we have that $A = A(\bar{P}\colon P\colon \xi\colon \lambda)$ has an integral kernel F_λ as follows,

$$Af(x) = \int_{\bar{N}} f(x\bar{n})d\bar{n} = \int_{K/M\cap K} F_\lambda(x^{-1}k)f(k)dk,$$

where F_λ is the continuous $\mathcal{H}_\xi$-valued function on G given by $F_\lambda(\bar{n}amn) = a^{\lambda-\rho}\xi(m)$ for $\lambda - \rho > 0$. Hence with $f_\lambda = j(P\colon 1\colon \lambda)$ we have

$$Af_\lambda(\varphi) = \int_K \int_{K/M\cap K} \varphi(k')F_\lambda(k'^{-1}k)f_\lambda(k)dk\,dk',$$

but here it should be noted that F_λ and f_λ are not both continuous at the same time. Fortunately for $\varphi = 1$ the above integral splits

$$Af_\lambda(1) = \int_K F_\lambda(k')dk' \int_{K/M\cap K} f_\lambda(k)dk, \tag{6.9}$$

and the two factors can be computed separately. The second factor is given by (6.2) with convergence for $\operatorname{Re}\lambda + \rho < 0$. The first factor is more complicated. It is not difficult to check that F_λ can be identified with the function on Ξ given by

$$x \mapsto |x_1 + x_{p+q}|^{\lambda-\rho}$$

(use that $x_1 + x_{p+q} = (1, 0, \ldots, 0, 1)\cdot x$) and hence

$$\int_K F_\lambda(k)dk = c\int_0^\pi \int_0^\pi |\cos\theta_1 + \cos\theta_2|^{\lambda-\rho} \sin^{p-2}\theta_1 \sin^{q-2}\theta_2\, d\theta_1\, d\theta_2.$$

This double integral is computable (see [182, Appendix A]). It converges for $\operatorname{Re}\lambda - \rho > 0$ and the value is a constant times

$$\frac{\Gamma(\lambda)\Gamma(\frac{1}{2}(\lambda-\rho+1))}{\Gamma(\frac{1}{2}(\lambda+\rho))\Gamma(\frac{1}{2}(\lambda+\rho-p+2))\Gamma(\frac{1}{2}(\lambda+\rho-q+2))}. \tag{6.10}$$

Note that as mentioned the domains where the two factors in (6.9) converge are disjoint. By combining equations (6.2, 6.7–6.10) one obtains

$$\begin{aligned} h(\lambda) &= c' \frac{\Gamma(\frac{1}{2}(-\lambda-\rho+p))\Gamma(\frac{1}{2}(\lambda+\rho-p+2))\Gamma(\frac{1}{2}(\lambda+\rho))}{\Gamma(\frac{1}{2}(-\lambda-\rho+1))\Gamma(\lambda)} \\ &= c' \frac{\pi}{\sin\frac{\pi}{2}(-\lambda-\rho+p)} \frac{\Gamma(\frac{1}{2}(\lambda+\rho))}{\Gamma(\frac{1}{2}(-\lambda-\rho+1))\Gamma(\lambda)}. \end{aligned}$$

We know already (see Example 6.2) that $j(P\colon 1\colon \lambda)$ has its only poles at the points where $\lambda+\rho$ is a positive odd integer and these poles are simple. It follows that the first gamma factor in the denominator of h will cancel all these poles. It is easily seen that the poles of the rest of h are not imaginary (the sine may give a pole at $\lambda = 0$, but this is killed by the $\Gamma(\lambda)$ in the denominator). Hence j° is regular on the imaginary axis, as asserted in Theorem 6.5.

LECTURE 7

The Fourier transform

The first topic of this lecture will be the definition of the *Fourier transform* on G/H. When that is given I will be ready to state the main theorem of these lectures, which is Theorem 7.1 below.

The Fourier transform $\hat{f}(\xi{:}\,\lambda) \in \mathrm{Hom}(V(\xi), C^\infty(\xi{:}\,-\lambda))$ is defined for functions $f \in C_c^\infty(G/H)$ by

$$\begin{aligned}\hat{f}(\xi{:}\,\lambda) &= \pi_{\xi,-\lambda}(f)j^\circ(\xi{:}\,-\lambda) \\ &= \int_{G/H} f(gH)\pi_{\xi,-\lambda}(g)j^\circ(\xi{:}\,-\lambda)\,d(gH),\end{aligned} \tag{7.1}$$

for a finite dimensional unitary representation ξ of M and $\lambda \in i\mathfrak{a}_q^*$. Here $j^\circ(\xi{:}\,\lambda)$ is the normalized spherical distribution map given by (6.6). The map $f \mapsto \hat{f}$ is G-equivariant:

$$(\ell(g)f)\hat{}\,(\xi{:}\,\lambda) = \pi_{\xi,-\lambda}(g)\hat{f}(\xi{:}\,\lambda).$$

Note the importance of Theorem 6.5 — without that $\hat{f}$ might not be defined on all of $i\mathfrak{a}_q$. The function $\hat{f}(\xi{:}\,\lambda)$ is analytic in λ, and more generally we can define $\hat{f}(\lambda)$ for $\lambda \in \mathfrak{a}_{q,c}^*$ by (7.1). This $\hat{f}$ is then meromorphic in λ.

Example 7.1. For the Riemannian symmetric space G/K the Fourier transform is usually defined by

$$\hat{f}(\lambda, kM) = \int_G f(g)e^{(\lambda-\rho_0)H(g^{-1}k)}dg, \tag{7.2}$$

where $f \in C_c^\infty(G/K)$, $\lambda \in \mathfrak{a}_{0,c}^*$, and $kM \in K/M_0$. Since $j(1{:}\,\lambda)$ in this case is the function $\mathbf{1}_\lambda(x) = e^{(-\lambda-\rho_0)H(x)}$ (see Example 6.1), this is equivalent with

$$\hat{f}(\lambda) = \pi_{1,-\lambda}(f)j(1{:}\,-\lambda) \in C^\infty(K/M),$$

that is, (7.1) with the unnormalized j. Here the normalization is unnecessary, because $j(1{:}\,-\lambda)$ is holomorphic.

As we shall see later (in Example 7.3 below), the normalized j is also significant in this case. The function $A(\bar{P}_0{:}\, P_0{:}\, 1{:}\, \lambda)\mathbf{1}_\lambda$ is clearly K-fixed, and hence it is a constant times the function $\bar{\mathbf{1}}_\lambda \in C(\bar{P}_0{:}\, 1{:}\, \lambda)$ whose restriction to K is the function 1. Denoting the constant by $\mathbf{c}(\lambda)$ we have

$$A(\bar{P}_0{:}\, P_0{:}\, 1{:}\, \lambda)\mathbf{1}_\lambda = \mathbf{c}(\lambda)\bar{\mathbf{1}}_\lambda.$$

By the definition of $A(\bar{P}_0{:}\, P_0{:}\, 1{:}\, \lambda)$,

$$\mathbf{c}(\lambda) = \int_{\bar{N}_0} e^{(-\lambda-\rho_0)H(\bar{n})}\, d\bar{n}$$

for $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$, $\alpha \in \Sigma^+(\mathfrak{a}_0, \mathfrak{g})$. This is the famous $\mathbf{c}$-function of Harish-Chandra. We thus get

$$j^\circ(1{:}\, \lambda) = \mathbf{c}(\lambda)^{-1}\mathbf{1}_\lambda,$$

and our Fourier transform is the one in (7.2) divided by $\mathbf{c}(-\lambda)$. It is known that $\mathbf{c}(\lambda) \neq 0$ on $i\mathfrak{a}_0^*$ (this follows for example from the Gindikin-Karpelevic formula for $\mathbf{c}(\lambda)$ (see [130, Section 7.5])), so that $j^\circ(1{:}\, \lambda)$ is regular on this set, as it should be according to Theorem 6.5.

Example 7.2. For the real hyperbolic space X with $q > 1$ where $V(\xi)$ is one-dimensional, we saw in the final example of the previous lecture that

$$j^\circ(1{:}\, \lambda) = h(\lambda)j(1{:}\, \lambda),$$

where h was explicitly computed. Recall that $j(1{:}\, \lambda)$ was identified with the distribution on Ξ given by

$$f_\lambda(y) = |y_1|^{-\lambda-\rho}, \quad (y \in \Xi),$$

for $\operatorname{Re}\lambda + \rho < 0$. It follows that the Fourier transform is given by

$$\begin{aligned} \hat{f}(\lambda)(y) &= h(-\lambda)\int_G f(gH)\,|(g^{-1}y)_1|^{\lambda-\rho}\, dg \\ &= h(-\lambda)\int_X f(x)\,|(x,y)|^{\lambda-\rho}\, dx \end{aligned} \tag{7.3}$$

(where $(\cdot,\cdot)$ is the standard $\mathrm{O}(p,q)$-invariant bilinear form on $\mathbf{R}^{p+q}$) for $\mathrm{Re}\,\lambda > \rho$, and by analytic continuation for other values of λ.

The theorem that I am now going to state shows how the Fourier transform is used to get a Plancherel decomposition of the part of $L^2(G/H)$ which is associated with the principal series of representations induced from σ-minimal parabolic subgroups. We shall have to work with the direct integral of these representations, so let me begin by making this explicit. At the same time, the multiplicities with which the representations are going to occur are also taken into account. The direct integral representation will be denoted

$$(\pi, \mathfrak{L}^2) \simeq \int^{\oplus}_{\xi \in \hat{M}_H, \lambda \in i\mathfrak{a}_q^*} (\pi_{\xi,-\lambda} \otimes 1, \mathcal{H}_{\xi,-\lambda} \otimes V(\xi)^*)\, d\lambda.$$

Here $\hat{M}_H$ denotes the set of (equivalence classes of) finite dimensional irreducible unitary representations of M having a nonzero $w(M \cap H)w^{-1}$ fixed vector for some $w \in N_K(\mathfrak{a}_q)$.

An explicit model for $(\pi, \mathfrak{L}^2)$ is obtained roughly as follows. Let $d\lambda$ be some Lebesgue measure on $i\mathfrak{a}_q^*$. Then $\mathfrak{L}^2$ is the Hilbert space consisting of the measurable functions F of the two variables $\xi \in \hat{M}_H$ and $\lambda \in i\mathfrak{a}_q^*$ with values

$$F(\xi\colon\lambda) \in \mathrm{Hom}(V(\xi), L^2(K\colon\xi)) \simeq L^2(K\colon\xi) \otimes V(\xi)^*,$$

satisfying

$$\sum_{\xi \in \hat{M}_H} \int_{i\mathfrak{a}_q^*} \dim(\xi) \|F(\xi\colon\lambda)\|^2\, d\lambda < +\infty, \tag{7.4}$$

with (7.4) as the square norm of F (of course one has to mod out the null space for the norm in order to get a proper Hilbert space). Furthermore π is the representation given by $(\pi(g)F)(\xi\colon\lambda) = (\pi_{\xi,-\lambda}(g) \otimes 1)F(\xi\colon\lambda)$.

In the following I shall also consider the subrepresentation

$$\int^{\oplus}_{\xi \in \hat{M}_H, \lambda \in i\mathfrak{a}_q^{*+}} \pi_{\xi,-\lambda} \otimes 1\, d\lambda$$

of π, where $\mathfrak{a}_q^{*+}$ is the positive chamber for the Weyl group $W = W(\mathfrak{a}_q, \mathfrak{g})$ in $\mathfrak{a}_q^*$. The reason for this is that we have the equivalences $\pi_{\xi,\lambda} \simeq \pi_{s\xi,s\lambda}$, $s \in W$.

Theorem 7.1. *For suitably normalized Lebesgue measure $d\lambda$ on $i\mathfrak{a}_q^*$ the following holds.*

(a) *If $f \in C_c^\infty(G/H)$ then $\hat{f} \in \mathfrak{L}^2$, and*

$$\|\hat{f}\|_{\mathfrak{L}^2}^2 = \sum_{\xi \in \hat{M}_H} \int_{i\mathfrak{a}_q^*} \dim(\xi) \|\hat{f}(\xi\colon\lambda)\|^2 \, d\lambda \leq \|f\|_{L^2(G/H)}^2.$$

In particular $f \mapsto \hat{f}$ extends uniquely to a G-equivariant continuous linear map $\mathfrak{F}$ from $(\ell, L^2(G/H))$ into $(\pi, \mathfrak{L}^2)$. Moreover:

(b) *This map $\mathfrak{F}$ is a partial isometry, that is, its restriction to the orthocomplement L^2_{mc} of its kernel in $L^2(G/H)$ is an isometry.*

(c) *We have the following decomposition:*

$$(\ell|_{L^2_{\mathrm{mc}}}, L^2_{\mathrm{mc}}) \simeq \int_{\xi \in \hat{M}_H, \lambda \in i\mathfrak{a}_q^{*+}}^{\oplus} (\pi_{\xi,-\lambda} \otimes 1, \mathcal{H}_{\xi,-\lambda} \otimes V(\xi)^*) \, d\lambda.$$

In particular the multiplicity of each $\pi_{\xi,-\lambda}$ in L^2_{mc} equals the dimension of $V(\xi)$.

The subspace L^2_{mc} is called the *most continuous part* of $L^2(G/H)$. As mentioned already in the Introduction I shall not be able to give a detailed proof of this theorem during these lectures — my primary goal was just to reach the point we have reached now, where it can be stated.

Example 7.3. The Riemannian symmetric spaces. As seen in Example 7.1 we have that our Fourier transform is $\mathbf{c}(-\lambda)^{-1}$ times the one given by (7.2). Moreover, $\hat{M}_H$ consists in this case only of the trivial representation $\xi = 1$. Translated in terms of (7.2) we get the following content of Theorem 7.1 in this case.

Let $\mathfrak{L}^2$ denote the L^2 space $L^2(\mathfrak{a}_0^* \times K/M_0, |\mathbf{c}(\lambda)|^{-2} d\lambda \, d(kM_0))$ with the representation $(\pi(g)F)(\lambda, kM_0) = F(\lambda, g^{-1}kM_0)$ (here $F(\lambda, \cdot)$ is extended to a function on G by means of $F(\lambda, kan) = a^{\lambda-\rho}F(kM_0)$). We then have:

(a) *If $f \in C_c^\infty(G/K)$ then $\hat{f} \in \mathfrak{L}^2$ and*

$$\|\hat{f}\|_{\mathfrak{L}^2}^2 \leq \|f\|_{L^2(G/K)}^2.$$

In particular, $f \mapsto \hat{f}$ extends uniquely to a G-equivariant continuous linear map $\mathfrak{F}$ from $(\ell, L^2(G/K))$ into $(\pi, \mathfrak{L}^2)$. Moreover:

(b) *This map $\mathfrak{F}$ is a partial isometry, that is, its restriction to the orthocomplement L^2_{mc} of $\ker \mathfrak{F}$ in $L^2(G/K)$ is an isometry.*

(c) *We have the following decomposition:*

$$(\ell|_{L^2_{\mathrm{mc}}}, L^2_{\mathrm{mc}}) \simeq \int_{\lambda \in i\mathfrak{a}_0^{*+}}^{\oplus} (\pi_{1,-\lambda}, L^2(K/M_0))\, d\lambda. \tag{7.5}$$

So much is the content of Theorem 7.1, but in this case one can actually say more:

(d) *We have $L^2_{\mathrm{mc}} = L^2(G/K)$, so that* (7.5) *gives the full decomposition of $L^2(G/K)$.*

The result can also be phrased as follows (see (5.12)):

$$\delta_0 = \int_{\lambda \in i\mathfrak{a}_0^{*+}} \varphi_{-\lambda} |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

that is,

$$f(e) = \int_{\lambda \in i\mathfrak{a}_0^{*+}} \tilde{f}(\lambda) |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

where

$$\tilde{f}(\lambda) = \int_{G/K} f(x) \varphi_{-\lambda}(x)\, dx$$

is the *spherical Fourier transform* of $f \in C_c^\infty(G/K)$. Note the significance of normalizing the Fourier transform: it will cause the cancellation of the terms $|\mathbf{c}(\lambda)|^{-2}$ from these formulas.

In contrast to the Riemannian case we do not have $L^2_{\mathrm{mc}} = L^2(X)$ in general, since discrete series may occur (since L^2_{mc} is given by a continuous integral, it has no irreducible subrepresentations). The following result shows that nevertheless we have that L^2_{mc} is quite big in $L^2(X)$.

Theorem 7.2. *If $f \in C_c^\infty(X)$ and $\hat{f} = 0$ then $f = 0$.*

In general there are in fact other obstacles than the discrete series which prevent L^2_{mc} from being equal to $L^2(G/H)$, but if the split rank of G/H is one this is not so. In this case there are only the most continuous series and the discrete series in the Plancherel decomposition of $L^2(G/H)$:

Theorem 7.3. *Assume that* $\dim \mathfrak{a}_q = 1$. *Then the orthocomplement of* L^2_{mc} *in* $L^2(G/H)$ *has a discrete decomposition (that is, it is the direct sum of its irreducible subrepresentations).*

Example 7.4. For the real hyperboloids with $q > 1$ we have from Theorems 7.1 and 7.3 that

$$\ell \simeq \sum_{j=0,1} \int_{i\mathbf{R}^+} \pi_{j,-\lambda}\, d\lambda + \sum \text{Discrete series},$$

and the Fourier transform is given explicitly by (7.3) for $j = 0$, and by a similar formula for $j = 1$. (A more explicit form of the decomposition will be given later, in Example 8.3.)

The first step in the proof of these theorems is to expand f as a sum of K-finite functions (as in (4.6)), and then prove a similar result for the functions transforming on the left according to a given K-type. For simplicity I will here only consider the trivial K-type, thus restricting myself to K-invariant functions on G/H. The analysis for other K-types is similar, but considerably more complicated.

For $f \in C_c^\infty(G/H)$ we have that $\hat{f}(\xi{:}\,\lambda)\eta$ for $\eta \in V(\xi)$ is the element in $C^\infty(K{:}\,\xi)$ given by

$$\hat{f}(\xi{:}\,\lambda)(\eta)(k) = \int_{G/H} f(gH) j^\circ(\xi{:}\,-\lambda)(\eta)(g^{-1}k)\,d(gH),$$

and if f is K-invariant it follows that this is a constant function. Now if ξ is irreducible and $C(K{:}\,\xi)$ contains a nonzero constant function it follows that ξ has a nonzero $M \cap K$-fixed vector, and then ξ must be the trivial representation of M (this follows from the facts that ξ also has a nonzero $w(M \cap H)w^{-1}$-fixed vector, and that $M = (M \cap K)(w(M \cap H)w^{-1})$ by Lemma 3.2). Thus for K-invariant functions on G/H we need only consider the principal series with the trivial M-type 1.

It follows from the definition of $V(\xi)$ that for $\xi = 1$ we have $V(\xi) \simeq \mathbf{C}^{\mathcal{W}}$. From now on I shall therefore replace $V(\xi)$ by $\mathbf{C}^{\mathcal{W}}$ whenever it is convenient. Thus for example, in place of (6.5) we have

$$j(1{:}\,\lambda)(\eta)(hw^{-1}man) = a^{-\lambda-\rho}\eta_w \in \mathbf{C}, \tag{7.6}$$

for $\eta \in \mathbf{C}^{\mathcal{W}}$, $w \in \mathcal{W}$. Let the functions $E(\lambda\colon\eta) = E(P\colon\lambda\colon\eta)$ and $E^\circ(\lambda\colon\eta) = E^\circ(P\colon\lambda\colon\eta)$ be defined on G/H by

$$E(\lambda\colon\eta)(gH) = \int_K j(1\colon\lambda)(\eta)(g^{-1}k)dk$$

and

$$E^\circ(\lambda\colon\eta)(gH) = \int_K j^\circ(1\colon\lambda)(\eta)(g^{-1}k)dk,$$

for $\eta \in \mathbf{C}^{\mathcal{W}}$ and $\lambda \in \mathfrak{a}^*_{q,c}$ (a priori E and E° are just distributions, but we shall see soon that they are actually analytic functions on G/H). These functions are K-invariant and we have for a K-invariant $f \in C_c^\infty(G/H)$ that its Fourier transform $\hat{f}(1\colon\lambda)$, from now on denoted just $\hat{f}(\lambda)$, is the linear form on $\mathbf{C}^{\mathcal{W}}$ given by

$$\hat{f}(\lambda)\eta = \int_X f(x)E^\circ(-\lambda\colon\eta)(x)dx, \qquad (\eta \in \mathbf{C}^{\mathcal{W}}).$$

The functions $E(\lambda\colon\eta)$ (and their counterparts for other K-types) are called *Eisenstein integrals* and similarly the $E^\circ(\lambda\colon\eta)$ are called *normalized Eisenstein integrals.* They are meromorphic functions of λ (in a suitable sense), and by Theorem 6.5 the normalized Eisenstein integral E° is nonsingular on $i\mathfrak{a}^*_q$. In the special case of $G/H = G/K$ the Eisenstein integrals are the spherical functions φ_λ, and the normalized Eisenstein integrals are the functions $\mathbf{c}(\lambda)^{-1}\varphi_\lambda$.

Just as the spherical functions are joint eigenfunctions for $\mathbf{D}(G/K)$ we have the following generalization. Recall from Lemma 4.6 that for $D \in \mathbf{D}(G/H)$ we defined $\gamma_q(D) \in S(\mathfrak{a}_q)^W$ by

$$u \in (\mathfrak{n}+\mathfrak{m})_c U(\mathfrak{g}) + T_{-\rho}\gamma_q(D) + U(\mathfrak{g})\mathfrak{h}_c, \tag{7.7}$$

where $u \in U(\mathfrak{g})^H$ with $r(u) = D$.

Proposition 7.4. *The K-invariant Eisenstein integrals are joint eigenfunctions for the invariant differential operators. More precisely, we have*

$$DE(\lambda\colon\eta) = \gamma_q(D\colon\lambda)E(\lambda\colon\eta) \tag{7.8}$$

for all $D \in \mathbf{D}(G/H)$, $\lambda \in \mathfrak{a}^*_{q,c}$ and $\eta \in \mathbf{C}^{\mathcal{W}}$. The equation (7.8) *also holds for the normalized Eisenstein integrals $E^\circ(\lambda\colon\eta)$.*

Remark. The non-K-invariant Eisenstein integrals will in general only be $\mathbf{D}(G/H)$-finite.

Proof. In fact already the function $gH \mapsto j(P\colon 1\colon\lambda)(g^{-1})$ satisfies the differential equation (7.8). To see this it suffices to consider the λ's where j is defined by a continuous function and then prove that the smooth restriction to the open $P \times H$ cosets satisfies this equation. Now this restriction is given by $namwh \mapsto a^{\lambda+\rho}\eta_w$. For $w = e$ it follows easily from (7.7) that this is an eigenfunction for D with eigenvalue $\gamma_q(D\colon\lambda)$. For other w's the independence of $\gamma_q(D)$ on the choice of positive system $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ can be used. Now (7.8) follows. The independence on $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ also implies that $gH \mapsto j(\bar{P}\colon 1\colon\lambda)(g^{-1})$ satisfies (7.8), and the intertwining property of $A(\bar{P}\colon P\colon 1\colon\lambda)$ then gives that so does $gH \mapsto j^\circ(P\colon 1\colon\lambda)(g^{-1})$, and hence also $E^\circ(\lambda\colon\eta)$. $\square$

Note that it follows from Proposition 7.4 that the Eisenstein integrals are analytic functions on X (viewed as functions on $K\backslash G$ they are eigenfunctions for the Laplace operator, which is elliptic).

An essential tool for the proof of Theorem 7.1 is the existence of *asymptotic expansions* for the Eisenstein integrals. The purpose of these are to determine the behavior of $E(\lambda\colon\eta)(a)$ when $a \in A_q$ tends to infinity. Let me begin by specifying what is meant by this. Fix a positive set $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ with corresponding parabolic subgroup P. Then $a \to \infty$ means that $\alpha(\log a) \to \infty$ for all $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$. Let $\mathfrak{a}_q^+$ be the open positive chamber in $\mathfrak{a}_q$ corresponding to $\Sigma^+(\mathfrak{a}_q, \mathfrak{g})$ and let $A_q^+ = \exp \mathfrak{a}_q^+$. Note that A_q^+ is different from the A_q^+ of the KA_qH-decomposition in Theorem 2.4; with the present definition of A_q^+ this decomposition can be written as

$$G = \cup_{w\in\mathcal{W}} K\overline{A_q^+}wH.$$

In order to control all the directions to infinity we must then consider the behavior as $a \to \infty$ of the functions $E(\lambda\colon\eta)(aw) = E(\lambda\colon\eta)(w^{-1}aw)$ for all $w \in \mathcal{W}$.

Regarding A_q^+ as a submanifold of X one can show that for each differential operator D on X there is a unique differential operator $\Pi(D)$ on A_q^+ such that $(Df)|_{A_q^+} = \Pi(D)(f|_{A_q^+})$ for all K-invariant functions $f \in C^\infty(X)$. The operator $\Pi(D)$ is called the *radial part* of D (see the notes for a reference). On A_q^+ we then have that the K-invariant Eisenstein integrals satisfy the differential equation

$$\Pi(D)\Phi = \gamma_q(D\colon\lambda)\Phi \tag{7.9}$$

for all $D \in \mathbf{D}(G/H)$. The first step is to consider formal power series solutions to this equation (actually taking $D = L$ would be sufficient here).

Proposition 7.5. *Let S denote the union of all the hyperplanes given by $\sigma_\mu = \{\lambda \in \mathfrak{a}_{q,c}^* \mid \langle 2\lambda - \mu, \mu\rangle = 0\}$, where $\mu \in \mathbf{N}\Sigma^+(\mathfrak{a}_q, \mathfrak{g}) \setminus \{0\}$. There exists, for $\lambda \notin S$, a unique formal series*

$$\Phi_\lambda(a) = a^{\lambda-\rho} \sum_{\mu\in\mathbf{N}\Sigma^+(\mathfrak{a}_q,\mathfrak{g})} a^{-\mu}\Gamma_\mu(\lambda)$$

on A_q^+ with $\Gamma_\mu(\lambda) \in \mathbf{C}$, $\Gamma_0 = 1$, which solves (7.9). The series converges absolutely and can be differentiated term by term.

For $R \in \mathbf{R}$ let

$$\mathfrak{a}_q^*(R) = \{\lambda \in \mathfrak{a}_{q,c}^* \mid \mathrm{Re}\langle\lambda, \alpha\rangle \le R \text{ for all } \alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})\}, \tag{7.10}$$

then the set $X_R = \{\mu \in \mathbf{N}\Sigma^+(\mathfrak{a}_q, \mathfrak{g}) \setminus \{0\} \mid \sigma_\mu \cap \mathfrak{a}_q^*(R) \neq \emptyset\}$ is finite. Let $p_R(\lambda)$ be the polynomial

$$p_R(\lambda) = \prod_{\mu\in X_R} \langle 2\lambda - \mu, \mu\rangle,$$

then $p_R(\lambda)\Phi_\lambda(a)$ is holomorphic as a function of λ in $\mathfrak{a}_q^*(R)$. Moreover it satisfies the following bound. There exists a constant $c > 0$ (depending on R) such that for each $\epsilon > 0$ the following holds. Let

$$A_q^{+\epsilon} = \{a \in A_q \mid \alpha(\log a) > \epsilon \text{ for all } \alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})\}.$$

There exists a constant C such that

$$|p_R(\lambda)\Phi_\lambda(a)| \le C(1 + |\lambda|)^c a^{\mathrm{Re}\,\lambda-\rho} \tag{7.11}$$

for all $a \in A_q^{+\epsilon}$ and all $\lambda \in \mathfrak{a}_q^*(R)$.

Remark. It is easily seen that there exists $R > 0$ such that $S \cap \mathfrak{a}_q^*(R)$ is empty. For this value of R it follows that $p_R = 1$ and that Φ_λ is holomorphic on $\mathfrak{a}_q^*(R)$. In particular we have that Φ_λ is holomorphic on the set where $\operatorname{Re}\lambda \leq 0$, and that the estimate (7.11) holds on this set without the polynomial factor $p_R(\lambda)$. However, the analogous statement for the other K-types is false.

Theorem 7.6. *There exists for each $s \in W$ a unique endomorphism-valued meromorphic function $\lambda \mapsto C^\circ(s\colon\lambda) \in \operatorname{End}(\mathbf{C}^{\mathcal{W}})$ on $\mathfrak{a}_{q,c}^*$ such that*

$$E^\circ(\lambda\colon\eta)(aw) = \sum_{s \in W} \Phi_{s\lambda}(a)(C^\circ(s\colon\lambda)\eta)_w$$

for $a \in A_q^+, w \in \mathcal{W}, \eta \in \mathbf{C}^{\mathcal{W}}$, as a meromorphic identity in $\lambda \in \mathfrak{a}_{q,c}^*$. Moreover we have

$$C^\circ(s\colon\lambda)^*\, C^\circ(s\colon -\bar{\lambda}) = 1 \tag{7.12}$$

for all $s \in W$ and $\lambda \in \mathfrak{a}_{q,c}^*$. In particular we have that $C^\circ(s\colon\lambda)$ is unitary for purely imaginary λ.

Proof. The proofs of these results are too long to be given here. See the references in the notes and the examples below. □

Remark. It follows from the remark above that Φ_λ is regular on $i\mathfrak{a}_q^*$. On the other hand, the final statement of Theorem 7.6 implies (by the Riemann boundedness theorem) that $C^\circ(s\colon\lambda)$ is also regular on this set. Hence we obtain from the expansion above that $E^\circ(\lambda\colon\eta)(aw)$ is regular on $i\mathfrak{a}_q^*$, for all a, w, and η as above. From this it can be seen, independently of Theorem 6.5, that $E^\circ(\lambda\colon\eta)$ is regular on $i\mathfrak{a}_q^*$. (Say there was a singularity at λ_0, then $\lambda \mapsto p(\lambda)E^\circ(\lambda\colon\eta)$ would be regular and nonzero (as a function on G/H) at λ_0 for a suitable polynomial p in λ with $p(\lambda_0) = 0$. However, on the dense set of the points $x = kaw$, with $k \in K$ and a, w as above, it would have to vanish at λ_0 by the regularity just obtained; being an eigenfunction for $\mathbf{D}(G/H)$, hence analytic, it would then have to vanish for all x, a contradiction.) For the non-K-invariant normalized Eisenstein integrals the statement of Theorem 7.6 is also valid, and the regularity on $i\mathfrak{a}_q^*$ can be derived (though not with the same ease) from (7.12), independently of Theorem 6.5. In fact, going backward the regularity in Theorem 6.5 is deduced from the regularity of the normalized Eisenstein integrals.

Example 7.5. Consider again the real hyperboloid with $q > 1$. It can be seen that the radial part of the Laplace operator is given by

$$\Pi(L)\Phi = J^{-1/2}[L_A(J^{1/2}\Phi) - L_A(J^{1/2})\Phi], \tag{7.13}$$

where $J(t) = \cosh^{p-1} t \sinh^{q-1} t$ is the Jacobian in Theorem 2.5 and $L_A = (d/dt)^2$ the Laplacian on A. It is convenient to introduce the function $\tilde{\Phi}_\lambda = J^{1/2}\Phi_\lambda$, which then satisfies the equation

$$\tilde{\Phi}_\lambda'' - d''\tilde{\Phi}_\lambda = (\lambda^2 - \rho^2)\tilde{\Phi}_\lambda, \tag{7.14}$$

where $d = J^{-1/2}(J^{1/2})''$. We have $d(t) = \sum_{n=0}^\infty d_n e^{-nt}$ for some constants d_n, explicitly given by

$$d_0 = \rho^2 \qquad \text{and} \qquad d_n = ((q-1)(q-3) + (-1)^n(p-1)(p-3))n.$$

In particular, we have $|d_n| \le c_0 n$ for some $c_0 > 0$. Now if

$$\tilde{\Phi}_\lambda(t) = e^{\lambda t} \sum_{m=0}^\infty \tilde{\Gamma}_m(\lambda) e^{-mt}$$

satisfies (7.14) then it follows by insertion that

$$m(m - 2\lambda)\tilde{\Gamma}_m(\lambda) = \sum_{n=1}^m d_n \tilde{\Gamma}_{m-n}(\lambda).$$

Hence if $\lambda \ne \frac{1}{2}, 1, \frac{3}{2}, \dots$ then the $\tilde{\Gamma}_m$ can be determined recursively.

Consider for simplicity only the case where the real number R in Proposition 7.5 is less than $1/2$. Then the set X_R is empty, and we get for $m > 0$ that

$$|\tilde{\Gamma}_m(\lambda)| \le \frac{c_0}{m(1-2R)} \sum_{n=1}^m n|\tilde{\Gamma}_{m-n}(\lambda)|.$$

Let $\epsilon > 0$ be arbitrary, then a straightforward induction shows that there exists for each m a constant $C_m > 0$ such that $|\tilde{\Gamma}_m(\lambda)| \le C_m e^{m\epsilon}$. I claim that C_m in fact can be chosen independently of m. To see this let C be the maximum of all the C_m for which $m < c_0(1-2R)^{-1}\sum_1^\infty ne^{-n\epsilon}$, and apply induction once more. We now have

$$|\tilde{\Gamma}_m(\lambda)| \le Ce^{m\epsilon} \tag{7.15}$$

for all m and all λ with $\operatorname{Re}\lambda \leq R$.

It follows immediately from (7.15) that the series for $\tilde{\Phi}_\lambda(t)$ converges uniformly on the set $t > \epsilon$, with the sum bounded by $C' e^{\operatorname{Re}\lambda t}$. It follows easily that Φ_λ is bounded by $C'' e^{(\operatorname{Re}\lambda - \rho)t}$. The result is also easily generalized to the situation where R is not necessarily less than $1/2$.

Now consider the statements of Theorem 7.6 for this case. Since $q > 1$ we have that $\mathcal{W}$ only consists of the trivial element 1. The first statement is then that there exist scalar-valued meromorphic functions $C^\circ_\pm(\lambda)$ such that

$$E^\circ(\lambda) = C^\circ_+(\lambda)\Phi_\lambda + C^\circ_-(\lambda)\Phi_{-\lambda}$$

on A^+. It follows immediately from the fact that $E^\circ(\lambda)$ satisfies a second-order ordinary differential equation on A, and that Φ_λ and $\Phi_{-\lambda}$ for generic λ are linearly independent solutions to the same equation, that $E^\circ(\lambda)$ is a linear combination of Φ_λ and $\Phi_{-\lambda}$. It remains to be seen that the functions $C^\circ_\pm(\lambda)$ are meromorphic. This follows easily from the meromorphicity of $E^\circ(\lambda)$ combined with the linear independence of Φ_λ and $\Phi_{-\lambda}$. Alternatively we have from the following proposition that $C^\circ_+(\lambda) = 1$, and in the following lecture (see Example 8.1) we shall derive an explicit expression for $C^\circ_-(\lambda)$, from which it also follows that it is meromorphic. The identity of (7.12) will likewise follow from this expression.

The following result shows that the normalization of E° is determined so that it has particularly simple asymptotics.

Proposition 7.7. *Let $\lambda \in \mathfrak{a}^*_{q,c}$ with $\operatorname{Re}\lambda$ strictly dominant, and let $w \in \mathcal{W}$. Then*

$$a^{\rho-\lambda} E^\circ(\lambda : \eta)(aw) \to \eta_w \tag{7.16}$$

as $a \to \infty$ in A^+_q. We have $C^\circ(1 : \lambda) = 1$ (the identity operator on $\mathbf{C}^{\mathcal{W}}$) for all λ.

Proof. The following formal computations can be justified. It is easily seen that $j(\bar{P} : 1 : \lambda)(g) = j(P : 1 : -\lambda)(\theta g)$, and hence by definition we have

$$\int_{\bar{N}} j^\circ(P : 1 : \lambda)(g\bar{n})\, d\bar{n} = j(P : 1 : -\lambda)(\theta g).$$

Integration over K gives

$$\int_{\bar N}\int_K j^\circ(P\colon 1\colon \lambda)(gk\bar n)\,dk\,d\bar n = \int_K j(P\colon 1\colon -\lambda)(\theta(g)k)\,dk.$$

On the left we apply the Iwasawa decomposition to $\bar n$, and on the right we rewrite the integral over K as an integral over $\bar N$, using [130, Eq. (5.25)]. The result is the equation

$$\int_{\bar N} e^{(-\lambda-\rho)H(\bar n)}\,d\bar n \int_K j^\circ(P\colon 1\colon \lambda)(gk)\,dk$$
$$= \int_{\bar N} j(P\colon 1\colon -\lambda)(\theta(g)\bar n)e^{(-\lambda-\rho)H(\bar n)}\,d\bar n.$$

Note that we now have $E^\circ(\lambda)(g^{-1})$ present on the left side. Now if $g = (aw)^{-1}$ with $w\in W$ and $a\in A_q^+$ we have $a\bar n a^{-1}\to e$ as $a\to\infty$, and hence the integral on the right behaves as follows

$$\int_{\bar N} j(P\colon 1\colon -\lambda)(w^{-1}a\bar n)e^{(-\lambda-\rho)H(\bar n)}\,d\bar n$$
$$= a^{\lambda-\rho}\int_{\bar N} j(P\colon 1\colon -\lambda)(w^{-1}a\bar n a^{-1})e^{(-\lambda-\rho)H(\bar n)}\,d\bar n$$
$$\sim a^{\lambda-\rho} j(P\colon 1\colon -\lambda)(w^{-1})\int_{\bar N} e^{(-\lambda-\rho)H(\bar n)}\,d\bar n.$$

Now the integrals over $\bar N$ cancel and we get (7.16). The final statement is an immediate consequence. □

Example 7.6. For G/K we have Harish-Chandra's famous asymptotic expansion for the spherical functions:

$$\varphi_\lambda(a) = \sum_{s\in W_0} \mathbf{c}(s\lambda)\Phi_{s\lambda}(a).$$

Hence the normalized c-functions are given by $C^\circ(s\colon\lambda) = \mathbf{c}(\lambda)^{-1}\mathbf{c}(s\lambda)$. In particular we have $C^\circ(1\colon\lambda) = 1$ as stated in Proposition 7.7. Since $\overline{\mathbf{c}(\lambda)} = \mathbf{c}(\bar\lambda)$ the statement of (7.12) comes down to the relation $\mathbf{c}(-s\lambda)\mathbf{c}(s\lambda) = \mathbf{c}(-\lambda)\mathbf{c}(\lambda)$, which follows from the Gindikin-Karpelevic product formula for $\mathbf{c}(\lambda)$ (see [130, Section 7.5]).

I would like to end this lecture by mentioning the following result. We have seen in Proposition 7.4 that the K-invariant Eisenstein integrals are solutions to the eigenequation (7.8). One can prove in analogy with Theorem 6.4 that the map $\eta \mapsto E(\lambda\colon\eta)$ for generic λ is a bijection of $\mathbf{C}^{\mathcal{W}}$ onto the space of K-invariant solutions to (7.8). See [167, Prop. 4.2] (the result is actually stated only for symmetric spaces of so-called G/K_ϵ-type, but the proof can be adapted to the general case of K-invariant functions on G/H).

LECTURE 8

Wave packets

In this final lecture I shall try to indicate some of the steps in the proof of Theorems 7.1, 7.2, and 7.3. The most important ingredient is the construction of a candidate for the "inverse" of the Fourier transform. As is well known, the inverse of the Euclidean Fourier transform

$$f \mapsto \mathcal{F}f(\lambda) = \hat{f}(\lambda) = \int_{\mathbf{R}^n} f(x) e^{-i\lambda\cdot x}\, dx$$

is given by the transform

$$\varphi \mapsto \mathcal{J}\varphi(x) = \int_{\mathbf{R}^n} \varphi(\lambda) e^{i\lambda\cdot x}\, d\lambda,$$

measures suitably normalized. One may regard $\mathcal{J}\varphi(x)$ as a superposition of the plane waves $e^{i\lambda\cdot x}$ with the amplitudes $\varphi(\lambda)$. For this reason it is called a *wave packet.*

In order to find the appropriate analog, recall first that to each λ corresponds a $|\mathcal{W}|$ dimensional space of "waves" on X, the Eisenstein integrals $E(\lambda\colon \eta)(x)$ (as in the previous lecture I only consider the K-invariant Eisenstein integrals). Hence the amplitude function φ has to be a $\mathbf{C}^{\mathcal{W}}$-valued function on $i\mathfrak{a}_q^*$. The wave with "amplitude" η is $E(\lambda\colon \eta)(x)$. As in the definition of the Fourier transform it is preferable to use here the normalized Eisenstein integral, because of the regularity on $i\mathfrak{a}_q^*$. This leads to the following definition of the wave packet corresponding to φ:

$$\mathcal{J}\varphi(x) = \int_{i\mathfrak{a}_q^*} E^\circ(\lambda\colon \varphi(\lambda))(x)\, d\lambda. \tag{8.1}$$

We first have to make this definition rigorous. For this we need an estimate of the normalized Eisenstein integral which is uniform in $\lambda \in i\mathfrak{a}_q^*$. At a later point we also need such an estimate on the set $-\mathfrak{a}_q^*(0)$ defined by

$$-\mathfrak{a}_q^*(0) = \{\lambda \in \mathfrak{a}_{q,c}^* \mid \langle \operatorname{Re}\lambda, \alpha\rangle \geq 0 \text{ for all } \alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})\}$$

(see also (7.10)). Since in general E° has poles on this set we first have to cancel these.

Proposition 8.1. *There exists a complex polynomial p° on $\mathfrak{a}^*_{q,c}$, which is a product of first-order polynomials of the form $\lambda \mapsto \langle\lambda, \alpha\rangle$ – constant, ($\alpha \in \Sigma(\mathfrak{a}_q, \mathfrak{g})$), such that $p^\circ(\lambda)E^\circ(\lambda{:}\,\eta)$ is holomorphic on a neighborhood of $-\mathfrak{a}^*_q(0)$, for all η.*

Moreover, there exist constants C, N, and s such that

$$|p^\circ(\lambda)E^\circ(\lambda{:}\,\eta)(a)| \leq C(1+|\lambda|)^N e^{(s+|\operatorname{Re}\lambda|)|\log a|}|\eta| \tag{8.2}$$

for all $\lambda \in -\mathfrak{a}^*_q(0)$, $\eta \in \mathbf{C}^{\mathcal{W}}$ and $a \in A_q$.

Remark. In particular it follows that there exists a constant R such that $E^\circ(\lambda{:}\,\eta)$ is holomorphic in λ on the set where $\langle\operatorname{Re}\lambda, \alpha\rangle \geq R$ for all $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})\}$.

Proof. This is derived by means of a functional equation for j, but it is too complicated to be given here. See the references in the notes, and the example below. $\square$

Example 8.1. The real hyperboloids, $q > 1$. It follows from (7.13) that the differential equation for the K-invariant Eisenstein integrals $E(\lambda)(\exp tY)$ and $E^\circ(\lambda)(\exp tY)$ is given by

$$J^{-1/2}[(J^{1/2}f)'' - (J^{1/2})''f] = (\lambda^2 - \rho^2)f, \tag{8.3}$$

where $J(t) = \cosh^{p-1} t \sinh^{q-1} t$. This differential equation is actually a well known equation; by the change of variables $z = -\sinh^2 t$ it becomes the *hypergeometric equation*

$$z(1-z)u'' + (c - (a+b+1)z)u' - abu = 0,$$

with $a = \frac{1}{2}(\rho+\lambda), b = \frac{1}{2}(\rho-\lambda), c = \frac{1}{2}q$. One can show that this equation has a unique solution which is regular at $z = 0$ with the value 1. This solution is called the *hypergeometric function* $F(a, b; c; z)$. It follows immediately that the unnormalized Eisenstein integral $E(\lambda)$ is given by

$$E(\lambda)(t) = E(\lambda)(0)F(a, b; c; -\sinh^2 t),$$

but it takes some effort to compute the constant $E(\lambda)(0)$ (see for example [105, Appendix B]). The normalized Eisenstein integral $E^\circ(\lambda)$ is more easily determined, because we know its asymptotic behavior from Proposition 7.7. It follows that

$$E^\circ(\lambda)(t) = [\lim_{s\to\infty} e^{(\rho-\lambda)s} F(a,b;c;-\sinh^2 s)]^{-1} F(a,b;c;-\sinh^2 t)$$

for $\operatorname{Re}\lambda > 0$. The limit is determined from the identity (see [104, p. 63, Eq. (17)]):

$$\begin{aligned} F(a,b;c;z) =& \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} F(b,1-c+b;1-a+b;z^{-1}) \\ &+ \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} F(a,1-c+a;1-b+a;z^{-1}). \end{aligned} \tag{8.4}$$

It follows that

$$E^\circ(\lambda)(t) = \frac{\Gamma\left(\frac{1}{2}(\lambda+\rho)\right)\Gamma\left(\frac{1}{2}(\lambda-\rho+q)\right)}{\Gamma(\lambda)\Gamma(\frac{1}{2}q)} 2^{\lambda-\rho} F(a,b;c;-\sinh^2 t).$$

In particular we can determine the poles from this expression; they are caused by the Γ-functions in the numerator (but some of them may be cancelled by the denominator). It is seen that there are only finitely many poles with positive real part (if $p < q+2$ there are none, otherwise they occur at $\rho-q, \rho-q-2, \dots$), and (in accordance with Theorem 6.5) no purely imaginary poles (because of the $\Gamma(\lambda)$ in the denominator). This establishes the first statement of Proposition 8.1 for this case. Note also that we get from (8.4) that

$$E^\circ(\lambda)(t) = \Phi_\lambda(t) + \Phi_{-\lambda}(t) C^\circ_-(\lambda),$$

where

$$\Phi_\lambda(t) = (2\sinh t)^{\lambda-\rho} F(\frac{1}{2}(\rho-\lambda), \frac{1}{2}(p-\rho-\lambda); 1-\lambda; -\sinh^{-2} t) \tag{8.5}$$

and

$$C^\circ_-(\lambda) = \frac{\Gamma(\frac{1}{2}(\rho+\lambda))\Gamma(-\lambda)\Gamma(\frac{1}{2}(q-\rho+\lambda))}{\Gamma(\frac{1}{2}(\rho-\lambda))\Gamma(+\lambda)\Gamma(\frac{1}{2}(q-\rho-\lambda))}$$

in accordance with Theorem 7.6.

The estimate (8.2) is harder to obtain, but it can be deduced from [132, Lemma 2.3] (in fact this gives a stronger estimate).

In particular, by combining Proposition 8.1 with Theorem 6.5, which implies that E° is not singular on $i\mathfrak{a}_q^*$, we get that

$$|E^\circ(\lambda\colon\eta)(a)| \le C(1+|\lambda|)^N e^{s|\log a|}|\eta| \tag{8.6}$$

for $\lambda \in i\mathfrak{a}_q^*$. This shows that the integral (8.1) converges provided $\varphi(\lambda)$ has a reasonable decay in λ, for example as a Schwartz function. Similar estimates for the derivatives of E° with respect to x show that $\mathcal{J}\varphi$ is smooth.

Let us now return to the Fourier transform. Recall that for K-invariant functions we have

$$\hat{f}(\lambda)\eta = \int_X f(x)E^\circ(-\lambda\colon\eta)(x)\,dx, \qquad \eta \in \mathbf{C}^{\mathcal{W}},$$

thus $\hat{f}(\lambda)$ is a linear form on $\mathbf{C}^{\mathcal{W}}$. It is actually more convenient to have a Fourier transform which takes its values in $\mathbf{C}^{\mathcal{W}}$. For this reason I define a new Fourier transform $\mathcal{F}f$ as follows,

$$\langle \mathcal{F}f(\lambda)|\eta\rangle = \langle f|E^\circ(-\bar{\lambda}\colon\eta)\rangle, \qquad f \in C_c^\infty(K\backslash G/H),$$

for all η, where the sesquilinear product $\langle\cdot|\cdot\rangle$ on the left side is the standard inner product on $\mathbf{C}^{\mathcal{W}}$, and on the right is given by

$$\langle f_1|f_2\rangle = \int_X f_1(x)\overline{f_2(x)}\,dx \tag{8.7}$$

for complex functions f_1, f_2 on X. It follows from (7.6) that

$$\overline{E^\circ(-\bar{\lambda}\colon\eta)(x)} = E^\circ(-\lambda\colon\bar{\eta})(x),$$

and hence $\mathcal{F}f(\lambda) \in \mathbf{C}^{\mathcal{W}}$ is simply the element for which $\hat{f}(\lambda)\eta = \mathcal{F}f(\lambda)\cdot\eta$ for all $\eta \in \mathbf{C}^{\mathcal{W}}$ (the dot denotes the standard bilinear product on $\mathbf{C}^{\mathcal{W}}$). Note that $\mathcal{F}f(\lambda)$ is meromorphic in $\lambda \in \mathfrak{a}_{q,c}^*$.

We can use Proposition 8.1 to obtain an estimate of $\mathcal{F}f$ for functions $f \in C_c^\infty(K\backslash G/H)$. Let p° be a polynomial on $\mathfrak{a}_{q,c}^*$ with the properties of this proposition and let $p(\lambda) = p^\circ(-\lambda)$. Then $p\mathcal{F}f$ is holomorphic on $\mathfrak{a}_q^*(0)$, and we have

$$|p(\lambda)\mathcal{F}f(\lambda)| \le C(1+|\lambda|)^N e^{r|\operatorname{Re}\lambda|} \tag{8.8}$$

for all $\lambda \in \mathfrak{a}_q^*(0)$, with constants C, N, r. Here N is independent of f, whereas C and r depend on f. However, r depends only on the size of the support of f. In fact we can take

$$r = \sup_{a \in \operatorname{supp} f \cap A_q} |\log a|. \tag{8.9}$$

There is an important duality between the transforms $\mathcal{F}$ and $\mathcal{J}$, expressed in the following lemma. As above let $\langle\cdot|\cdot\rangle$ denote the standard inner product on $\mathbf{C}^{\mathcal{W}}$. Furthermore let also

$$\langle\varphi_1|\varphi_2\rangle = \int_{i\mathfrak{a}_q^*} \langle\varphi_1(\lambda)|\varphi_2(\lambda)\rangle \, d\lambda, \tag{8.10}$$

for $\mathbf{C}^{\mathcal{W}}$-valued functions φ_1, φ_2 on $i\mathfrak{a}_q^*$.

Lemma 8.2. *Let $f \in C_c^\infty(X)$ be K-invariant and let φ be a $\mathbf{C}^{\mathcal{W}}$-valued Schwartz function on $i\mathfrak{a}_q^*$. Then*

$$\langle f|\mathcal{J}\varphi\rangle = \langle\mathcal{F}f|\varphi\rangle,$$

where $\langle\cdot|\cdot\rangle$ is defined by (8.7) *and* (8.10), *respectively.*

Proof. This is a straightforward application of Fubini's theorem. □

Now it is time to invoke the invariant differential operators. Recall from the previous lecture that we have

$$DE^\circ(\lambda\colon\eta) = \gamma_q(D\colon\lambda)E^\circ(\lambda\colon\eta)$$

for $D \in \mathbf{D}(G/H)$. It follows that

$$D\mathcal{J}\varphi(x) = \mathcal{J}(\gamma_q(D)\varphi) \quad \text{and} \tag{8.11}$$

$$\mathcal{F}(Df) = \gamma_q(D)\mathcal{F}f. \tag{8.12}$$

Here the first equality is obvious, but for the second one needs the following relation for the formal adjoint D^* of D. Define $v^* \in S(\mathfrak{a}_q)$ for $v \in S(\mathfrak{a}_q)$ by $v^*(\lambda) = v(-\lambda)$.

Lemma 8.3. *Let* $D \in \mathbf{D}(G/H)$. *Then* $\gamma_q(D^*) = \gamma_q(D)^*$.

Proof. Let $u \in U(\mathfrak{g})^H$ with $D = r(u)$. It is easily seen that $D^* = r(\check{u})$, where $v \mapsto \check{v}$ is the principal antiautomorphism of $U(\mathfrak{g})$. Let $s: S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map, then it is known (this is part of the proof of Proposition 4.1) that we can choose $u = s(v)$ for an element $v \in S(\mathfrak{q})^H$. Since $s(v)^\vee = \sigma(s(v))$ we obtain $D^* = r(\sigma(u))$. It follows immediately from the definition of γ_q that $\gamma_q(\sigma(u))^*$ equals the $\gamma_q(u)$ one would obtain from using the opposite positive system. Since $\gamma_q(D)$ is actually independent of the choice of positive roots, the lemma follows. $\square$

The equation for $\mathcal{F}(Df)$ can be used to improve the estimate (8.8).

Proposition 8.4. *Let* $p(\lambda)$ *be as above. Let* $f \in C_c^\infty(K\backslash G/H)$ *and* $n \in \mathbf{N}$. *There exists a constant* C *such that*

$$|p(\lambda)\mathcal{F}f(\lambda)| \leq C(1+|\lambda|)^{-n} e^{r|\operatorname{Re}\lambda|} \tag{8.13}$$

for all $\lambda \in \mathfrak{a}_q^*(0)$, $\eta \in \mathbf{C}^{\mathcal{W}}$. *Here* r *is given by* (8.9); *in particular it depends only on the size of the support of* f.

Proof. Just to give the idea, assume for simplicity that $\dim \mathfrak{a}_q = 1$. It is easily seen that $\gamma_q(L: \lambda) = \langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle$. By using suitably high powers of L we can obtain a D with $|\gamma_q(D: \lambda)| \geq (1+|\lambda|)^{N+n}$ for all λ. Applying (8.8) to Df and using (8.12) we get (8.13). $\square$

The purpose of the polynomial $p(\lambda)$ in (8.13) is to cancel the singularities of $\mathcal{F}f(\lambda)$. Hence p is not needed for $\lambda \in i\mathfrak{a}_q^*$ (because of Theorem 6.5), and it follows that $\mathcal{F}f(\lambda)$ is bounded by $C(1+|\lambda|)^{-n}$ for all n. Similar estimates for the derivatives with respect to λ imply that $\mathcal{F}f$ is actually a Schwartz function on $i\mathfrak{a}_q^*$. In particular it makes sense to apply $\mathcal{J}$ to $\mathcal{F}f$. This is important, because as mentioned, the wave packet transform is the candidate for the "inverse" of the Fourier transform on K-invariant functions. As we shall see below, it is actually *not* the inverse of $\mathcal{F}$ in general (it will only be the inverse of the restriction of $\mathcal{F}$ to L^2_{mc}). The main step in the proof of Theorems 7.1–7.3 consists of the following result, which shows that $\mathcal{J}$ is the inverse of $\mathcal{F}$ in a certain weak sense.

Theorem 8.5. *There exists an invariant differential operator* D *on* G/H *for which* $\deg \gamma_q(D) = \operatorname{order} D \neq 0$, *and a positive constant* c *such that*

$$D\mathcal{J}\mathcal{F}f = cDf \tag{8.14}$$

for all K-invariant $f \in C_c^\infty(X)$.

Note that it follows from Theorem 4.7 that D is injective as an operator on $C_c^\infty(X)$. Nevertheless, since $\mathcal{J}\mathcal{F}f$ in general does not have compact support, one cannot conclude from (8.14) that $\mathcal{J}\mathcal{F}f = cf$.

The rest of this final lecture will be spent on discussing the proof of this result, but before that let me indicate how it is applied to Theorems 7.1–7.3. First of all, Theorem 7.2 follows immediately, by means of the injectivity of D. To obtain Theorem 7.1 one has to introduce a notion of *Schwartz functions* on X. Without giving the details, let $\mathcal{C}(X)$ denote the space of such functions. A rather delicate refinement of the estimates for E° given above shows that $\mathcal{F}$ maps Schwartz functions on X to Schwartz functions on $i\mathfrak{a}_q^*$, and vice versa for $\mathcal{J}$, and these operations are continuous. Applying $\mathcal{F}$ on both sides of (8.14) we get by means of (8.12) that $\gamma_q(D)\mathcal{F}\mathcal{J}\mathcal{F}f = c\gamma_q(D)\mathcal{F}f$, and hence by division with $\gamma_q(D)$ (which is permissible on meromorphic functions)

$$\mathcal{F}\mathcal{J}\mathcal{F}f = c\mathcal{F}f. \tag{8.15}$$

(Note that the Schwartz estimates are implicitly used when $\mathcal{F}$ is allowed to operate on $\mathcal{J}\mathcal{F}f$.) Normalizing measures suitably we may assume $c = 1$. The relation (8.15) shows that $\mathcal{J}\mathcal{F}$ is idempotent, by Lemma 8.2 it is symmetric, and hence it is a projection operator. Now Theorem 7.1 (a) is obtained by

$$\|\mathcal{F}f\|^2 = \langle \mathcal{F}f|\mathcal{F}f\rangle = \langle \mathcal{J}\mathcal{F}f|f\rangle = \langle (\mathcal{J}\mathcal{F})^2 f|f\rangle = \|\mathcal{J}\mathcal{F}f\|^2 \le \|f\|^2, \tag{8.16}$$

and (b) by noting that (8.15) implies that the kernel of $\mathcal{F}$ is identical with the kernel of the projection operator $\mathcal{J}\mathcal{F}$; hence L^2_{mc} is the image of this projection, on which it is easily seen that $\mathcal{F}$ is isometrical,

$$\|\mathcal{F}\mathcal{J}\mathcal{F}f\|^2 = \|\mathcal{F}f\|^2 = \|\mathcal{J}\mathcal{F}f\|^2,$$

by (8.15) and (8.16). It follows that $\mathcal{F}$ embeds L^2_{mc} isometrically into the space

$$\int_{\lambda \in i\mathfrak{a}_q^*}^{\oplus} \mathcal{H}_{1,\lambda} \otimes \mathbf{C}^{\mathcal{W}}\, d\lambda$$

(recall that we only consider K-invariant functions for simplicity). The fact that there are nontrivial intertwining operators between $\pi_{1,\lambda}$ and $\pi_{1,s\lambda}$ for $s \in W$ results in the existence of a simple relation between $\mathcal{F}f(\lambda)$ and $\mathcal{F}f(s\lambda)$ for $s \in W$, which implies that the image of L^2_{mc} is completely determined by its restriction to only one chamber. This shows that L^2_{mc} is equivalent with a subrepresentation of the representation in the right-hand side of (c) in Theorem 7.1. The proof that it is actually equivalent with the full right-hand side requires a further analysis of the map $\mathcal{F}\mathcal{J}$, which I cannot give here.

Finally, let me sketch how to deduce Theorem 7.3 from Theorem 8.5. Let $D \in \mathbf{D}(G/H)$ be as in the latter theorem. As mentioned $\mathcal{F}$ and $\mathcal{J}$ extend continuously to Schwartz space, and hence (8.14) holds also for the K-invariant functions $f \in \mathcal{C}(X)$. In particular, if f is orthogonal to L^2_{mc}, which by definition means that $\mathcal{F}f = 0$, then $Df = 0$. If $\dim \mathfrak{a}_q = 1$ the space of smooth K-invariant functions annihilated by D has finite dimension, say d (they satisfy an ordinary linear differential equation on A_q). It follows that the subrepresentation of $L^2(G/H)$ generated by f is the sum of at most d irreducible subrepresentations of $L^2(G/H)$ (otherwise it could be written as the direct sum of $d+1$ nontrivial invariant subspaces; one of these would necessarily have no K-fixed vectors, and hence would be orthogonal to f, a contradiction).

The relation between $\mathcal{F}f(s\lambda)$ and $\mathcal{F}f(\lambda)$ mentioned above is the following.

Proposition 8.6. *We have*

$$\mathcal{F}f(s\lambda) = C^\circ(s\colon\lambda)\mathcal{F}f(\lambda)$$

for all $f \in C_c^\infty(K\backslash G/H)$, $s \in W$ *and* $\lambda \in \mathfrak{a}^*_{q,c}$.

Proof. This is easily obtained as a consequence of (7.12) and the relation

$$E^\circ(s\lambda\colon C^\circ(s\colon\lambda)\eta) = E^\circ(\lambda\colon\eta), \tag{8.17}$$

for $s \in W$, $\lambda \in \mathfrak{a}_{q,c}$, and $\eta \in \mathbf{C}^W$, of which I shall now sketch the proof.

Consider the distribution

$$R(s)A(s^{-1}Ps\colon P\colon 1\colon\lambda)j^\circ(P\colon 1\colon\lambda)(\eta) \in C^{-\infty}(1\colon s\lambda),$$

obtained by applying the intertwining operator of (5.15) to $j^\circ(P{:}\,1{:}\,\lambda)(\eta)$. Since the operator is intertwining this is an H-fixed distribution, so for generic λ it is given by

$$R(s)A(s^{-1}Ps{:}\,P{:}\,1{:}\,\lambda)j^\circ(P{:}\,1{:}\,\lambda)(\eta) = j^\circ(P{:}\,1{:}\,s\lambda)(B^\circ(s{:}\,\lambda)\eta),$$

for some endomorphism $B^\circ(s{:}\,\lambda)$ of $\mathbf{C}^{\mathcal{W}}$, meromorphic in λ. We evaluate this identity at $g^{-1}k$ and integrate over $k \in K$. On the right-hand side we obtain $E^\circ(s\lambda{:}\,B^\circ(s{:}\,\lambda)\eta)(gH)$. Let us compute the left-hand side by means of the formula (5.14) for the standard intertwining operator,

$$\begin{aligned}\int_K & R(s)A(s^{-1}Ps{:}\,P{:}\,1{:}\,\lambda)j^\circ(P{:}\,1{:}\,\lambda)(\eta)(g^{-1}k)\,dk \\ &= \int_K \int_{\bar{N}\cap s^{-1}Ns} j^\circ(P{:}\,1{:}\,\lambda)(\eta)(g^{-1}k\tilde{s}\bar{n})\,d\bar{n}\,dk,\end{aligned}$$

where $\tilde{s}$, the representative in K for s, immediately is swallowed by the K-integration. Disregarding all questions of convergence we exchange the order of the integrals. Furthermore we define $a(\bar{n}) \in A$ such that $\bar{n} \in KMa(\bar{n})N$, then it follows from (5.1) (with $\xi = 1$) that the double integral splits as the product of

$$\int_K j^\circ(P{:}\,1{:}\,\lambda)(\eta)(g^{-1}k)\,dk = E^\circ(\lambda{:}\,\eta)(gH)$$

and

$$\int_{\bar{N}\cap s^{-1}Ns} a(\bar{n})^{-\lambda-\rho}\,d\bar{n}.$$

Let $c(s{:}\,\lambda)$ denote the latter quantity, then we have obtained the identity

$$E^\circ(s\lambda{:}\,B^\circ(s{:}\,\lambda)\eta) = c(s{:}\,\lambda)E^\circ(\lambda{:}\,\eta). \tag{8.18}$$

Apart from the justification of these formal manipulations, which I skip, it remains to be seen that $c(s{:}\,\lambda)^{-1}B^\circ(s{:}\,\lambda) = C^\circ(s{:}\,\lambda)$ in order to have (8.17). By meromorphy we may assume that $s\,\mathrm{Re}\,\lambda$ is strictly dominant. Then it follows from Proposition 7.7 and Theorem 7.6, respectively, that the two sides of (8.18), evaluated at aw, behave like $a^{s\lambda-\rho}(B^\circ(s{:}\,\lambda)\eta)_w$ and $a^{s\lambda-\rho}c(s{:}\,\lambda)(C^\circ(s{:}\,\lambda)\eta)_w$, respectively, when $a \to \infty$ in A_q^+. From this the desired identity between B° and C° follows, since $c(s{:}\,\lambda)$ is not identically zero. $\square$

I am now ready to sketch the main steps leading to Theorem 8.5. Consider the integral (8.1) defining the wave packet $\mathcal{J}\varphi$. Inserting the expansion for $E^\circ(\lambda{:}\,\eta)$ from Theorem 7.6 we obtain

$$\begin{aligned}\mathcal{J}\varphi(aw) &= \int_{i\mathfrak{a}_q^*} E^\circ(\lambda{:}\,\varphi(\lambda))(aw)\,d\lambda \\ &= \int_{i\mathfrak{a}_q^*} \sum_{s\in W} \Phi_{s\lambda}(a)(C^\circ(s{:}\,\lambda)\varphi(\lambda))_w\,d\lambda.\end{aligned}$$

For $\varphi = \mathcal{F}f$ we can use Proposition 8.6 and obtain

$$\begin{aligned}\mathcal{J}\mathcal{F}f(aw) &= \int_{i\mathfrak{a}_q^*} \sum_{s\in W} \Phi_{s\lambda}(a)\mathcal{F}f(s\lambda)_w\,d\lambda \\ &= |W| \int_{i\mathfrak{a}_q^*} \Phi_{\lambda}(a)\mathcal{F}f(\lambda)_w\,d\lambda.\end{aligned} \tag{8.19}$$

We would like to use Cauchy's theorem on this integral in order to obtain

$$\mathcal{J}\mathcal{F}f(aw) = |W| \int_{i\mathfrak{a}_q^*} \Phi_{\lambda+\mu}(a)\mathcal{F}f(\lambda+\mu)_w\,d\lambda$$

for $\mu \in \mathfrak{a}_q^*$ antidominant, but of course this is not permitted since $\mathcal{F}f$ is only meromorphic. Recall however that $p\mathcal{F}f$ is holomorphic on a neighborhood of $\mathfrak{a}_q^*(0)$. We now need the following.

Lemma 8.7. *There exists an element $D \in \mathbf{D}(G/H)$ such that p divides $\gamma_q(D)$, and such that* $\deg\gamma_q(D) = \operatorname{order} D \neq 0$.

Proof. Roughly the idea is that $\prod_{s\in W} p(s\lambda)$ is a Weyl invariant polynomial, hence in the image of γ_q. (This is not quite good enough, however, since actually γ_q is not surjective on $S(\mathfrak{a}_q)^W$ in general.) $\square$

With D as in this lemma, we now apply (8.19) to Df instead of f. By means of (8.11–8.12) we then obtain

$$D\mathcal{J}\mathcal{F}f(aw) = \mathcal{J}\mathcal{F}Df(aw) = |W| \int_{i\mathfrak{a}_q^*} \Phi_{\lambda}(a)\gamma_q(D)(\lambda)\mathcal{F}f(\lambda)_w\,d\lambda,$$

and (since p divides $\gamma_q(D)$) the integrand has become holomorphic on a neighborhood $\mathfrak{a}_q^*(0)$. The estimates of Propositions 7.5 and 8.4 allow the use of Cauchy's theorem to conclude that

$$(8.20)\qquad D\mathcal{J}\mathcal{F}f(aw) = |W| \int_{i\mathfrak{a}_q^*} \Phi_{\lambda+\mu}(a)\gamma_q(D)(\lambda+\mu)\mathcal{F}f(\lambda+\mu)_w \, d\lambda$$

for μ antidominant. The strategy is now to let μ pass to infinity in this direction. It follows from the estimates that the integral is bounded by a constant times $a^\mu e^{r|\mu|}$, where r is given by (8.9). Now if $a \in A_q^+$ and $|\log a| > r$ then we can find an antidominant μ_0 such that $\mu_0(\log a) < -r|\mu_0|$. Taking μ proportional to μ_0 it follows that the integral tends to zero, so that $D\mathcal{J}\mathcal{F}f(aw) = 0$. The conclusion we reach is that $D\mathcal{J}\mathcal{F}f$ has compact support, and that this support has roughly the same size as the support of f.

Refining the argument given above it is actually possible to prove that if $S \subset \mathfrak{a}_q$ is a convex, compact, W-invariant set, then

$$(8.21)\qquad \operatorname{supp} f \subset X_S \Longrightarrow \operatorname{supp} D\mathcal{J}\mathcal{F}f \subset X_S.$$

(Recall that $X_S = K \exp SH$.) The next step in the proof of Theorem 8.5 consists of a strong improvement of this statement: we have actually

$$(8.22)\qquad \operatorname{supp} D\mathcal{J}\mathcal{F}f \subset \operatorname{supp} f,$$

that is, (8.21) holds for all compact $W_{K\cap H}$-invariant sets S. Let me sketch the proof of this under the simplifying assumption that $W_{K\cap H} = W$. Let $\mathfrak{S}$ denote the collection of all closed W-invariant sets $S \subset \mathfrak{a}_q$ for which (8.21) holds for all K-invariant $f \in C_c^\infty(X)$. We know that the *convex* closed W-invariant sets S belong to $\mathfrak{S}$. Now clearly $\mathfrak{S}$ is stable for taking intersections. Furthermore, if S belongs to $\mathfrak{S}$, then the closure S^c of its complement also belongs to $\mathfrak{S}$. To see this, first note that we may assume D is formally selfadjoint (otherwise we replace it by D^*D). Then if φ is any K-invariant smooth function with support in X_S we have by (8.21) that $\operatorname{supp} D\mathcal{J}\mathcal{F}\varphi \subset X_S$, and hence by Lemma 8.2 $\langle D\mathcal{J}\mathcal{F}f|\varphi\rangle = \langle f|D\mathcal{J}\mathcal{F}\varphi\rangle = 0$ for all K-invariant $f \in C_c^\infty(X)$ with support in X_{S^c}. Hence the latter condition implies that $D\mathcal{J}\mathcal{F}f$ vanishes on the interior of S, that is, S^c belongs to $\mathfrak{S}$, as claimed. Combining this with the property of intersections, it follows that the closure of a union of sets from $\mathfrak{S}$ again belongs to $\mathfrak{S}$. Now it is easily seen that any closed W-invariant set can be obtained by these operations starting with *convex* closed W-invariant sets. This establishes (8.22).

From (8.22) it follows by means of Peetre's theorem ([124, Thm. 1.4]) that the operator $D\mathcal{J}\mathcal{F}$ is a differential operator D' on G/H (more precisely, on the image of A_q^+ in $K\backslash G/H$). It remains to be seen that $D' = cD$ for some constant $c \neq 0$. This can be proved roughly as follows: Observe that D' commutes with all elements from $\mathbf{D}(G/H)$ (use (8.11–8.12)). This commutation relation may be seen as a system of differential equations on the coefficients of D'. One can show that this system has a regular singularity at infinity. In particular this implies that D' is uniquely determined by its asymptotic behavior. Using the asymptotic expansion in Theorem 7.6 one can analyze how D' behaves at infinity: $\mathcal{F}$ and $\mathcal{J}$ become the Euclidean Fourier and inverse Fourier transforms, respectively, and hence they cancel each other (up to a positive constant c) and we obtain $D' \sim cD$. As said, this implies $D' = cD$. This finishes my sketch of the proof of Theorem 8.5. $\square$

Example 8.2. For the Riemannian symmetric spaces G/K we have that φ_λ is holomorphic and $\mathbf{c}(\lambda)^{-1}$ has no poles in $-\mathfrak{a}_0^*(0)$. Hence we can take $p^\circ = 1$ in Proposition 8.1, and hence also $D = 1$ in Lemma 8.7 and in Theorem 8.5. It follows that $L^2_{\mathrm{mc}}(G/K) = L^2(G/K)$, as stated in (d) of Example 7.3.

Example 8.3. The real hyperboloids, $q > 1$. We have seen in Example 8.1 that the poles of $E^\circ(\lambda)$ with positive real part are located at $\lambda = \lambda_j = \rho - q - 2j$ where $j = 0, 1, \ldots$, say for $j \leq k$, and these poles are simple. Hence $\mathcal{F}f(\lambda)$ has poles at the negative of these locations (depending on f only some of them may occur). Instead of introducing the operator D in order to cancel these poles in (8.19) we can in this case perform the shift leading to (8.20), keeping track of the residues. Instead of (8.20) we obtain

$$\begin{aligned}(8.23) \qquad \mathcal{J}\mathcal{F}f(aw) =& |W| \int_{i\mathfrak{a}_q^*} \Phi_{\lambda+\mu}(a)\mathcal{F}f(\lambda+\mu)_w \, d\lambda \\ &+ 2\pi i |W| \sum_{j=0}^{k} \Phi_{-\lambda_j}(a) \operatorname*{Res}_{\lambda=-\lambda_j} \mathcal{F}f(\lambda).\end{aligned}$$

By [104, p. 64, (22)] the expression (8.5) for Φ_λ can be rewritten as

$$(8.24) \quad \Phi_\lambda(t) = (2\cosh t)^{\lambda-\rho} F(\frac{1}{2}(\rho-\lambda), \frac{1}{2}(q-\rho-\lambda); 1-\lambda; \cosh^{-2} t),$$

and this hypergeometric function becomes a polynomial in $\cosh^{-2} t$ exactly when $\lambda = -\lambda_j = q - \rho + 2j$ (the Taylor series for $F(a, b; c, z)$ at $z = 0$ (the Gauss summation formula) terminates when a or b is a negative integer). In particular, it is regular at $\cosh^{-2} t = 1$, and it follows that (8.24) for $\lambda = -\lambda_j$ extends to a K-invariant smooth function on X, which is of Schwartz type (because of the factor $(2\cosh t)^{-\lambda_j-\rho}$). Moreover, since $E^\circ(\lambda)$ has a simple pole at λ_j, its residue is also a smooth K-invariant eigenfunction, and hence it must be proportional to $\Phi_{-\lambda_j}$.

It follows from these remarks that the summation term in (8.23) is the restriction of a global Schwartz function on X. Let

$$\mathcal{D}f = -2\pi i \sum_{j=0}^{k} \operatorname*{Res}_{\lambda=-\lambda_j} \mathcal{F}f(\lambda)\, \Phi_{-\lambda_j}.$$

It is now easily seen that the operator $\mathcal{D}$ commutes with the Laplace operator, and also that it is symmetric (use the above-mentioned proportionality). Following the argumentation in the general proof above we then obtain that

$$(\mathcal{J}\mathcal{F} + \mathcal{D})f = cf,$$

the Plancherel formula for X; it shows how f is decomposed into its L^2_{mc} part $\mathcal{J}\mathcal{F}f$ and its discrete series part $\mathcal{D}f$. Note that if we insert Df instead of f in this equation we obtain (8.14), because $\operatorname{Res}_{\lambda=-\lambda_j} \mathcal{F}Df(\lambda) = \gamma_q(D)(-\lambda_j) \operatorname{Res}_{\lambda=-\lambda_j} \mathcal{F}f(\lambda) = 0$, so that $\mathcal{D}Df = 0$.

Notes

LECTURE 1. A readable introduction to the theory of semisimple symmetric spaces, with some more details on the geometric viewpoints, is given in the first chapter of [108]. Thorough treatments are given in the books [123], [124], [131, Chap. 9], [139], [177]. The example of the real hyperboloids (Example 1.6) has been treated thoroughly by several authors. See for example [170], [105], [178] (some further references can be found in the list of rank one symmetric spaces below). The account in [170] is particularly recommendable as a companion to these notes. In addition to these examples, other examples of harmonic analysis on particular semisimple symmetric spaces can be found, for example in [185] and [95] (see also the list below). Very much of the analysis done in the first of these lectures has been done in [167] for a class of semisimple symmetric spaces called K_ϵ-type.

Research on the program of harmonic analysis on general semisimple symmetric spaces was basically begun in the late 1970's and developed rapidly in the 1980's. An overview is given in [84]. Up to now, the part of the decomposition which is best understood is the discrete series. Below are given some hints and some references. These notes deal with the "opposite" part, the most continuous part. The basic references are the forthcoming papers [90,91], on which the final lectures (7 and 8) will be built. Finally, there are also series of representations that lie "between" the most continuous series and the discrete series. These series have only been studied quite recently, see [97], [101] and [98].

By definition, a *discrete series* representation of a locally compact group G with respect to a homogeneous space G/H is an irreducible representation π of G, which can be embedded as a subrepresentation of $L^2(G/H)$ (it is assumed that G/H has an invariant measure). Let G/H be a semisimple symmetric space. It is known from the pioneering work of Flensted-Jensen [107] that the discrete series is nonempty if the rank of G/H equals that of $K/(K \cap H)$ (here K is a σ-invariant maximal compact subgroup of G, see Prop. 2.1 for its existence). In the cited paper a construction of "most" of the discrete series is given. The basic tool in the construction is the duality (see the proof of Thm. 4.3). The construction was extended by Oshima and

Matsuki [166], who showed that the mentioned rank condition is also necessary for the existence of the discrete series (a significant simplification of their proof is given in [141]). The construction of Flensted-Jensen, Oshima, and Matsuki (see also [85], [86]) gives a series of subrepresentations π_λ of $L^2(G/H)$, whose span equals the span of the discrete series. For a few of the π_λ it remained an open problem whether they are irreducible (a priori they might decompose as finite sums of irreducibles) and nonzero. The irreducibility was settled by Vogan in [188], and Matsuki gave in [141] some necessary conditions for the nonvanishing, and announced them also to be sufficient. The final problem is whether there are equivalences among the π_λ. The answer is believed to be no, and this has been confirmed by Bien [93] in all cases except for a handful of "exceptional" symmetric spaces. Differently put, this means that *no irreducible representations occur more than once in* ℓ (the discrete series have multiplicity one).

In the group case the discrete series was known beforehand from Harish-Chandra [117]. For a noncompact (that is, G has no compact factors) Riemannian symmetric space there is no discrete series (also by Harish-Chandra).

As mentioned in the lecture, the basic method for finding the Plancherel decomposition in symmetric space of rank one is to use polar coordinates in which the Laplacian L becomes an ordinary singular differential operator of the type treated by Weyl, Kodaira, and Titchmarsh. For a Riemannian symmetric space G/K of rank one, the obvious way of obtaining polar coordinates comes from the Cartan decomposition $G = K \exp \mathfrak{a} K$, with the angular parameter being furnished by K and the radial parameter by $\mathfrak{a}$. Thus the system of coordinates is obtained from the fact that the regular orbits of K on G/K constitute a one-dimensional family. The generalization to non-Riemannian spaces G/H of rank one offers two possibilities: one can use the orbits of K or the orbits of H on X to obtain polar coordinates. In both cases the regular orbits constitute one-dimensional families. These two ways of introducing coordinates on X give rise to two essentially different ways of obtaining the spectral resolution of L on $L^2(X)$.

The first method has only been applied successfully to the hyperbolic spaces (Example 1.6; the first four blocks of the list below). It was introduced by Limić et al. in [138]. The second method works for all rank one spaces, but it has the drawback of being more complicated. A method of this kind was first used by Molchanov in the announcement [144] (with detailed proofs given in [147]). It is based on a study of spherical distributions on X (i.e., generalized functions on X, which are H-invariant and are eigenfunctions for L), and the final result is a decomposition of the Dirac measure concentrated at the origin in terms of positive definite spherical distributions (that is, an explicit version of (5.10)). Faraut [105] gives a careful exposition of both methods, applied to the classical hyperbolic spaces.

Below is a list of all the non-Riemannian semisimple symmetric spaces of rank one (up to covering), with some references to papers treating the Plancherel formula. (The list of references is not complete, and some of the papers contain full proofs of the theorems, whereas others are announcements only.)

The non-Riemannian semisimple symmetric spaces of rank one
$\mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$, $(p>1, q>0)$ *(the real hyperbolic spaces).* [115] $(p=2, q=2)$; [110], [175] $(p=3, q=1)$; [144], [181] $(p>1, q=1)$; [145] $(p>1, q$ odd); [138], [182], [170], [105], [147] and [152].
$\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(p-1,q))$, $(p>1, q>0)$ *(the complex hyperbolic spaces).* [143] $(p>1, q=1)$; [105], [152].
$\mathrm{Sp}(p,q)/(\mathrm{Sp}(1)\times\mathrm{Sp}(p-1,q))$, $(p>1, q>0)$ *(the quaternion hyperbolic spaces).* [105], [152].
$\mathrm{F}_{4(-20)}/\mathrm{Spin}(1,8)$, *(the octonion hyperbolic space).* [134], [152].
$\mathrm{SL}(n,\mathbf{R})/\mathrm{GL}_+(n-1,\mathbf{R})$, $(n>1)$. [150], [151] (n=2); [149] (n=3); [148], [185], [184], [152].
$\mathrm{Sp}(n,\mathbf{R})/(\mathrm{Sp}(1,\mathbf{R})\times\mathrm{Sp}(n-1,\mathbf{R}))$, $(n>1)$. [136], [152].
$\mathrm{F}_{4(4)}/\mathrm{Spin}(4,5)$. [152]

For all these spaces, the decomposition of $L^2(X)$ contains a discrete part and a continuous part (the discrete series and the principal series). In particular, one finds that the multiplicities are one, except when X is (or covers) the real hyperbolic space $\mathrm{SO}_e(p,1)/\mathrm{SO}_e(p-1,1)$ (cf. [184]), where the representations of the principal series have multiplicity 2.

LECTURE 2. One of the first systematic studies of semisimple symmetric spaces is the paper [92] by Berger, which gives a classification of these spaces. Proposition 2.1 is in that paper. A proof is easily derived from [123, Thm. III.7.1] (see [177, Prop. 7.1.1]). The Cartan decomposition of Proposition 2.2 is from Mostow [153], and Theorems 2.4 and 2.5 are from

Flensted-Jensen [106, 107]. The proof of Theorem 2.5 consists of a root space computation of the Jacobian J (see [177, Thm. 8.1.1]). Theorem 2.6 is from Rossmann [171], the proof (for the involution $\sigma\theta$) is also given in [177, Prop. 7.2.1] (see also [168]).

LECTURE 3. For details on parabolic subgroups in general, see for example [130, Sect. 5.5], [187, Sect. II,6], or [189, Sect. 1.2]. The σ-minimal parabolic subgroups are introduced in Rossmann [171], where Theorem 3.3 is given as Cor. 17. The proof can be found in this paper, and also in Matsuki [140], where also the nonopen $P \times H$ cosets are determined (for further material on these coset decompositions see also [142]). For the proof Lemma 3.1, and for more details about these parabolic subgroups, see [80, Sect. 2]. Theorem 3.4 is from van den Ban [77], where the full proof can be found. It was first found by Kostant [133] in the Riemannian case (cf. Example 3.5) (see also [124, Thm. IV,10.5]). Lemmas 3.6 and 3.7 are from Harish-Chandra [116, Lemmas 39, 43]; the proofs are based only on finite dimensional representation theory (for Lemma 3.7 see also [124, Cor. IV,6.6]). The proof of the properness of $\mathbf{a}$ based on Lemma 3.6 is taken from [167, Lemma 3.7]

LECTURE 4. For generalities on invariant differential operators, see [124, Chap. II] where Proposition 4.1 is given as Theorem 4.6. The result is independent of our assumption that H is connected, and then $U(\mathfrak{g})^H$ can also be replaced by $U(\mathfrak{g})^{H_e}$ (see [164, Lemma 2.1] or [81, Lemma 2.1]) so that in fact $\mathbf{D}(G/H) = \mathbf{D}(G/H_e)$. Theorem 4.2 is due to Harish-Chandra [116], for the proof see [124, Thm. II,5.17]. An alternative proof of the W_0-invariance (without the use of spherical functions) has been given in [137]. Theorem 4.3 and its proof are due to Helgason ([124, Thm. II,5.7]), but the "duality" used in the proof goes further back to Berger [92]. The use of this duality has been exploited by Flensted-Jensen (see [108] and references mentioned there). For example, the isomorphism between $\mathbf{D}(G/H)$ and $\mathbf{D}(G^d/K^d)$ appears as an algebraic isomorphism, but it can actually be given an analytic sense essentially by looking at G/H and G^d/K^d as submanifolds of the same complex symmetric space G_c/H_c. For the proof of Lemma 4.4, see [119] (see also [125]). Proposition 4.5 is from van den Ban [78]; the part of the proof given here is an elaboration of [170, Lemma 9]. Lemma 4.6 and Theorem 4.7 are from [89], where the full proof of the latter can be found. In the Riemannian case it goes back to Helgason [121].

LECTURE 5. For more details on general principal series representations and the standard intertwining operators, see [130, Chap. 7], and for more details on the specialization to the σ-minimal case, see [80, Sects. 3–4]. The abstract integral decomposition theory can be found in [103] (see also [169] for (5.5)). For details on the application to G/H, see van den Ban [78] (where the full proof of Proposition 5.2 can be found) and [105, Sect. 1], [185]. The theory of spherical functions on G/K (Example 5.1) can be found in [124, Chap. IV]. Lemma 5.3 is due to 'Olafsson [157] (for further discussions see also [88]). The irreducibility in Theorem 5.4 is due to Bruhat (see [96, p. 203, Thm. 4], and also [80, Prop. 3.7]). The idea of Bruhat's proof is sketched in [130, Sect. 7.3], and more thoroughly in [112]. See also [157, Thm. 3.7]. Proposition 5.5 and Theorem 5.6 are from [80, Sect. 4]; the proof of Theorem 5.6 is based on [130, Thm. 8.38], but matters simplify because ξ is finite dimensional. The proof of Theorem 5.4 for the real hyperboloids (Example 5.5) can be found, for example, in [182] (see also [146]).

LECTURE 6. The basic reference for this lecture is [80] (see also [157]). Proposition 6.1 and Theorem 6.2 are given in both these papers. The existence of the meromorphic extension of f_λ was originally announced in [162]. See also [167, Prop. 3.8] for a special case. Theorems 6.3 and 6.4 are [80, Thms. 5.1 and 5.10]. In the proof of Theorem 6.4 a finite dimensional representation π_μ is used. The existence of this representation with the properties mentioned in the proof follows from a general theorem on finite dimensional representations of G with both a nontrivial K-fixed vector and a nontrivial H-fixed vector (a $K \times H$-*spherical* representation). For $H = K$ these have been classified by Helgason, see [124, Thm. V,4.1]. For the generalization to arbitrary H, see [157] or [81] (the generalization is due to Hoogenboom). Theorem 6.5 is announced in [88, Thm. 2] under a certain Condition (F), which at the time of the announcement was needed for the proof of [80, Thm. 6.3] (the proof given in [80] has an error, which is easily repaired under the assumption of Condition (F) – see [88]). Since then van den Ban [82] has found a proof of this theorem independent of Condition (F). The proof of Theorem 6.5 will appear in [90].

LECTURE 7. The definition of the Fourier transform is given in [90]. Theorems 7.1–7.3 are announced in [88] (see also [84]), their proofs (sketched in Lecture 8) will appear in [91]. Note that as a consequence of Theorem 7.3

and the known results for the discrete series (see the notes to Lecture 1), the full decomposition of $L^2(G/H)$ is known for symmetric spaces G/H of split rank one. This class of spaces is considerably wider than that of symmetric spaces of rank one (a table is given in [168, p. 462]). In the Riemannian case (Ex. 7.3) the full Plancherel decomposition is obtained from Theorem 7.1. In this case, the result is well known and due to Harish-Chandra and Helgason (see [120], [121], [126]). A substantial simplification of the original proof was found by Rosenberg; this is the line of proof followed in [124, Thm. 7.5]. At the same time, a proof of Helgason's Paley-Wiener theorem for G/K (that is, the description of the image $\mathcal{F}(C_c^\infty(G/K))$) is obtained (see [124, Thm. 7.1]). The proof of Theorem 7.1 of these notes is built similarly; in particular, a Paley-Wiener theorem is also obtained for spaces G/H of split rank one. However, major complications arise from the fact that the spherical distribution in general is not holomorphic in the parameter λ, and from fact that $\dim V(\xi)$ can be larger than one in general (so that the Plancherel decomposition will have multiplicities larger than one). The Eisenstein integrals (unnormalized as well as normalized) are defined and studied in [81] (see also [90]); Propositions 7.4 and 7.7 are from [81]. In the group case the Eisenstein integrals were introduced by Harish-Chandra [117, I]. The asymptotic expansions in Proposition 7.5 and Theorem 7.6 will be proved in [91]. For G/K and ${}^{\backprime}G$, they were obtained by Harish-Chandra (see [124, Thm. 5.5] for the former case, and [189] for the latter). The relation (7.12), which is quite crucial, is established in [81, Thm. 16.3]. It was obtained by Harish-Chandra in the group case (the "Maass-Selberg relations" [117, III p. 153]), and in the Riemannian case by Helgason [121, Thm. 6.6]. For the general theory of radial parts of differential operators, see [124, Thm. II,3.6], and for the application to the Laplace operator in the KA_qH decomposition, see [106, Eq. (4.12)], where (7.13) is proved for arbitrary semisimple symmetric spaces.

LECTURE 8. Most of the material of this section will be published in [90] and [91]. The only exception is Proposition 8.1 which follows directly from [81, Prop. 10.3 and Cor. 16.2]. Example 8.1 shows that the K-invariant Eisenstein integrals on the real hyperboloids are hypergeometric functions; see [118] for the generalization to arbitrary semisimple symmetric spaces.

Part III:
Are K-invariant Eisenstein Integrals for G/H Hypergeometric Functions?

Gerrit Heckman

Contents, Part III

1. Introduction

At the Roskilde conference in honor of the 65th birthday of S. Helgason the last lecture was given by E.P. van den Ban, who spoke about his joint work with H. Schlichtkrull on the spectral decomposition (of the most continuous part) of $L^2(G/H)$ with G a reductive Lie group of Harish-Chandra class and H an open subgroup of the fixed points of some involution σ of G [190–192, 209]. In order to limit technicalities only the decomposition of the K-spherical part $L^2(G/H)^K = L^2(K\backslash G/H)$ was discussed. Here K is the maximal compact subgroup of G fixed by a Cartan involution θ of G commuting with σ. After the lecture J. Faraut brought up the question whether the spectral decomposition of $L^2(K\backslash G/H)$ could be described in terms of hypergeometric functions associated with a root system as developed by E.M. Opdam and the author (see [199] for a survey). In this paper we show that the answer to this question is yes.

In Section 2 we discuss the generalized Cartan decomposition in case G is replaced by a compact real form U. These results were obtained by B. Hoogenboom in his thesis [201]. It follows from these results that there exists a root system R and a multiplicity function k such that the radial part of the Laplace-Bertrami operator on G/H (acting on K-invariant functions) is exactly the same as the operator $L(k)$ associated with R and k. Since the algebra $D(G/H)$ of invariant differential operators on G/H is commutative it follows that the radial parts of an eigensystem of $D(G/H)$, plus invariance by K, gives rise to a commutative eigensystem for an algebra differential operators containing $L(k)$. Hence this system is a subsystem of the hypergeometric system (by the very definition of the latter). Since both systems are commutative and have finite dimensional solution spaces the conclusion is that K-invariant eigenfunctions for $D(G/H)$ are finite linear combinations of hypergeometric functions. In particular this applies to the K-invariant Eisenstein integrals.

In Section 3 we discuss the natural context of the spectral problem for the hypergeometric function associated with a root system. Partly this is a review of know results, and leads to a description of the spectral problem on noncompact vector groups. We give a little extra evidence that the spectral theory in the noncompact case works for all nonnegative values of the multiplicity parameter $k \in K$ rather than just some (integral or half-integral) group values of k (coming from harmonic analysis on $K\backslash G/H$)

by checking that the Maass-Selberg relations just go through. In the group case of G/H (not only for the trivial but for all K-types) the Maass-Selberg relations were obtained by E.P. van den Ban by a rank one reduction and certain manipulations with an integral formula [191, 209]. We also use rank one reduction, but instead of using an integral formula we use the Kummer relations for the Gaussian hypergeometric function. In Section 5 we end this paper with the discussion of some open problems.

We hope that the content of this paper will stimulate a further search for integral representations for the hypergeometric function associated with a root system, because these seem to be the main obstacle for dealing with the spectral problem [193, 195, 199, 210].

Finally I would like to thank E.P. van den Ban and H. Schlichtkrull for enlightening discussions about their work.

2. The generalized Cartan decomposition (after B. Hoogenboom)

Suppose U is a compact, connected and simply connected Lie group. Let $\theta, \sigma\colon U \to U$ be a pair of commuting involutions of U, and denote $K = U^\theta$, $H = U^\sigma$ the respective groups of fixed points. It is known that both K and H are connected (see [200], Chap. VII, Theorem 8.2). On the Lie algebra level we have the corresponding decompositions

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p} = \mathfrak{h} \oplus i\mathfrak{q} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap i\mathfrak{q}) \oplus (i\mathfrak{p} \cap \mathfrak{h}) \oplus i(\mathfrak{p} \cap \mathfrak{q}). \tag{2.1}$$

Choose a maximal abelian subspace $\mathfrak{a}_{pq} \subset \mathfrak{p} \cap \mathfrak{q}$, and put $\mathfrak{t}_{pq} = i\mathfrak{a}_{pq}$. Then $T_{pq} := \exp(\mathfrak{t}_{pq})$ is a torus in U maximal with respect to the conditions $\theta(t) = t^{-1}, \sigma(t) = t^{-1}$ for all $t \in T_{pq}$. We also have the restricted root system

$$\Sigma_{pq} := \Sigma(\mathfrak{g} = \mathfrak{u}_c, \mathfrak{a}_{pq}) \subset \mathfrak{a}^*_{pq} \tag{2.2}$$

with corresponding Weyl group

$$W_{pq} := W(\Sigma_{pq}) \cong N_K(\mathfrak{t}_{pq})/Z_K(\mathfrak{t}_{pq}) \cong N_H(\mathfrak{t}_{pq})/Z_H(\mathfrak{t}_{pq}). \tag{2.3}$$

Note that the group W_{pq} acts on T_{pq} in a natural way: $w(t) = ntn^{-1}$ if $w = nZ_K(t_{pq}) \in W_{pq}$, $n \in N_K(\mathfrak{t}_{pq})$, $t \in T_{pq}$. Since $w(t_1)w(t_2) = w(t_1t_2)$ we can consider the semidirect product group $T_{pq} \rtimes W_{pq}$ with multiplication defined by $(t_1, w_1)(t_2, w_2) = (t_1 w_1(t_2), w_1 w_2)$. Note that $T_{pq} \rtimes W_{pq}$ acts on T_{pq} by $(t_1, w) \cdot t_2 = t_1 w(t_2)$.

Definition 2.1. We put

$$N = \{(t = kh, w) \in \big(T_{pq} \cap N_K(\mathfrak{t}_{pq}) N_H(\mathfrak{t}_{pq})\big) \times W_{pq}; \\ w = kZ_K(\mathfrak{t}_{pq}) = h^{-1} Z_H(\mathfrak{t}_{pq})\} \tag{2.4}$$

viewed as a subset of the group $T_{pq} \rtimes W_{pq}$.

Lemma 2.2. *N is a subgroup of $T_{pq} \rtimes W_{pq}$.*

Proof. Suppose $(t_1 = k_1 h_1, w_1), (t_2 = k_2 h_2, w_2) \in N$. Then we have

$$\begin{aligned}(t_1, w_1)(t_2, w_2) &= (k_1 h_1 w_1(k_2 h_2), w_1 w_2) = (k_1 h_1 h_1^{-1} k_2 h_2 h_1, w_1 w_2) \\ &= (k_1 k_2 h_2 h_1, w_1 w_2) \in N,\end{aligned}$$

and hence also $(t = kh, w)^{-1} = (k^{-1} h^{-1}, w^{-1}) \in N$. $\square$

As a subgroup of $T_{pq} \rtimes W_{pq}$ the group N also acts on T_{pq}. Written out explicitly this becomes $(kh, w) \cdot t = kth$.

Lemma 2.3. *If $k \in K$, $h \in H$, and $t_1, t_2 \in T_{pq}$ with $kt_1 h = t_2$ then $t_2^4 = kt_1^4 k^{-1} = h^{-1} t_1^4 h$.*

Proof. Suppose $kt_1 h = t_2$ for $k \in K, h \in H, t_1, t_2 \in T_{pq}$. Then by applying θ we get $kt_1^{-1}\theta(h) = t_2^{-1}$ or equivalently $\theta(h) = t_1 k^{-1} t_2^{-1}$, and by applying σ we get $\sigma(k) t_1^{-1} h = t_2^{-1}$ or equivalently $\sigma(k) = t_2^{-1} h^{-1} t_1$. Applying $\theta\sigma = \sigma\theta$ yields $\sigma(k) t_1 \theta(h) = t_2$, and hence $t_2^{-1} h^{-1} t_1 \cdot t_1 \cdot t_1 k^{-1} t_2^{-1} = t_2$ or equivalently $t_2^3 = h^{-1} t_1^3 k^{-1}$. In turn we get $t_2^4 = t_2 \cdot t_2^3 = kt_1^4 k^{-1} = t_2^3 \cdot t_2 = h^{-1} t_1^4 h$. $\square$

Corollary 2.4. *If we denote*

$$L := T_{pq} \cap Z_K(\mathfrak{t}_{pq}) Z_H(\mathfrak{t}_{pq}) = N \cap (T_{pq} \times \{1\}), \tag{2.5}$$

then we have $T_{pq}[2] \subset L \subset T_{pq}[4]$, when for $m \in \mathbb{N}$ we write $T_{pq}[m] = \{t \in T_{pq}; t^m = 1\}$ for the points of order (divisor of) m.

Proof. Indeed $T_{pq}[2] = \{t \in T_{pq}; t = t^{-1}\} = \{t \in T_{pq}; t = \theta(t)\} = T_{pq} \cap K \subset Z_K(\mathfrak{t}_{pq})$, and similarly for H. Hence $T_{pq}[2] \subset L$ is clear. Now suppose $t = kh \in T_{pq}$ for some $k \in K, h \in H$. Applying the previous lemma with $t_1 = 1, t_2 = t$ yields $t^4 = 1$. Hence $L \subset T_{pq}[4]$. $\square$

In particular L (and hence N) is finite.

Lemma 2.5. *We have a short exact sequence of groups*

$$1 \longrightarrow L \longrightarrow N \longrightarrow W_{pq} \longrightarrow 1, \tag{2.6}$$

where $N \to W_{pq}$ is projection on the second factor.

Proof (Sketch). The only nontrivial point of this statement is that the map $N \to W_{pq}$ is onto. Since W_{pq} is generated by reflections it suffices to prove that each reflection lies in the image, and this reduces to an $su(2)$-computation (for more details see [201], Chap. 5 and Lemma 8.3). □

Theorem 2.6. (Generalized Cartan decomposition). *We have the decomposition $U = KT_{pq}H$. Moreover for $t_1, t_2 \in T_{pq}$ there exist $k_1, k_2 \in K$, $h_1, h_2 \in H$ with $k_1t_1h_1 = k_2t_2h_2$ if and only if $n \cdot t_1 = t_2$ for some $n \in N$.*

Proof (Sketch). The proof of the decomposition $U{=}KT_{pq}H$ is similar to the corresponding decomposition $G{=}KA_{pq}H$ in the noncompact case. Moreover the component in A_{pq} is now unique modulo the action by the reflection subgroup $N_{K\cap H}(\mathfrak{a}_{pq})/Z_{K\cap H}(\mathfrak{a}_{pq})$ of $W_{pq} = N_K(\mathfrak{a}_{pq})/Z_K(\mathfrak{a}_{pq})$ (see, e.g., [209], Theorem 2.4). For the proof of the second statement we restrict ourselves to the case that $t_1 \in T_{pq}$ is generic. Then $kt_1h = t_2$ with $k = k_2^{-1}k_1, h = h_1h_2^{-1}$, and we conclude by Lemma 2.3 that $k \in N_K(\mathfrak{t}_{pq})$, $h \in N_H(\mathfrak{t}_{pq})$ and $t = kh \in T_{pq}$. By the very definition of N this amounts to $n \cdot t_1 = t_2$ for some $n \in N$. □

Theorem 2.7. *For suitably normalized Haar measures we have*

$$du = J(t)\, dt\, dk\, dh \tag{2.7}$$

where $u = kth$ is the generalized Cartan decomposition, and the weight function J on T_{pq} is given by

$$J(t) = \prod_{\alpha\in\Sigma_{pq}^+} |t^\alpha - t^{-\alpha}|^{m_\alpha^+} \cdot |t^\alpha + t^{-\alpha}|^{m_\alpha^-}. \tag{2.8}$$

Here $m_\alpha^+ = \dim \mathfrak{g}_\alpha^+$, $m_\alpha^- = \dim \mathfrak{g}_\alpha^-$, $m_\alpha = \dim \mathfrak{g}_\alpha = m_\alpha^+ + m_\alpha^-$ are the multiplicities of the root spaces

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} = \mathfrak{u}_c; [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{a}_{pq}\}$$
$$\mathfrak{g}_\alpha^+ = \{X \in \mathfrak{g}_\alpha; \sigma\theta X = X\}, \quad \mathfrak{g}_\alpha^- = \{X \in \mathfrak{g}_\alpha; \sigma\theta X = -X\}.$$

Proof. This is a calculation of the Jacobian J entirely similar to the formula in the corresponding noncompact case (see [201], Chap. 9 and [209], Theorem 2.5). $\square$

Now the Jacobian J on T_{pq} is invariant under the action of the group N, i.e.,

$$J(n \cdot t) = J(t) \quad \forall n \in N, \forall t \in T_{pq}. \tag{2.9}$$

Note that $(t = kh, w) \cdot 1 = 1$ if and only if $t = 1$, or equivalently that $\mathrm{Stab}_N(1 \in T_{pq}) = N_{K\cap H}(\mathfrak{t}_{pq})/Z_{K\cap H}(\mathfrak{t}_{pq})$. Under the natural epimorphism $N \twoheadrightarrow W_{pq}$ this group maps bijectively onto the subgroup of W_{pq} generated by the reflections $r_\alpha \in W_{pq}$ for which $m_\alpha^+ \geq 1$. The invariance (2.9) near $t = 1$ amounts to $m_{w\alpha}^+ = m_\alpha^+, m_{w\alpha}^- = m_\alpha^-$ $\forall w \in N_{K\cap H}(\mathfrak{t}_{pq})/Z_{K\cap H}(\mathfrak{t}_{pq})$, $\forall \alpha \in \Sigma_{pq}$.

Now put in the notation of Corollary 2.4

$$T = L\backslash T_{pq},\ W = N/L \tag{2.10}$$

and let W act on T in the obvious way: $nL(Lt) = L(n \cdot t)$ $\forall n \in N, \forall t \in T_{pq}$. One should keep in mind that the identity element $1 \in T$ is not necessarily fixed by all $w \in W$. However from the existence of special points (a consequence of the fact that W is generated by $n = rk(\Sigma_{pq})$ reflections, cf. [194] Chap. 5, §3, Prop. 10) we conclude the existence of a point $t_0 \in T$ with $w(t_0) = t_0$ $\forall w \in W$. In fact choosing $t_0 \in T$ appropriately we can arrange that

$$J(tt_0) = \prod_{\alpha \in \Sigma_{pq}^+} \left|t^\alpha - t^{-\alpha}\right|^{n_\alpha^+} \cdot \left|t^\alpha + t^{-\alpha}\right|^{n_\alpha^-} \tag{2.11}$$

for suitable integers n_α^+, n_α^- with $\{n_\alpha^+, n_\alpha^-\} = \{m_\alpha^+, m_\alpha^-\}$ $\forall \alpha \in \Sigma_{pq}$. Moreover $n_\alpha^+ \geq n_\alpha^-$ $\forall \alpha \in \Sigma_{pq}^0 = \{\alpha \in \Sigma_{pq}; \frac{1}{2}\alpha \notin \Sigma_{pq}\}$, and $n_{w\alpha}^+ = n_\alpha^+, n_{w\alpha}^- = n_\alpha^-$ $\forall w \in W$, $\forall\ \alpha \in \Sigma_{pq}$. By abuse of notation we have not distinguished between $t \in T$ and its representative $t \in T_{pq}$. Note that $t_0^{4\alpha} = 1$ $\forall \alpha \in \Sigma_{pq}$.

Now there exists a new (possibly nonreduced) root system Σ with $\Sigma \subset \Sigma_{pq} \cup 2\Sigma_{pq}$ such that (2.11) can be rewritten in the form

$$J(tt_0) = \prod_{\alpha \in \Sigma^+} |t^\alpha - t^{-\alpha}|^{n_\alpha}, \tag{2.12}$$

by just using $(t^{\alpha}-t^{-\alpha})(t^{\alpha}+t^{-\alpha}) = (t^{2\alpha}-t^{-2\alpha})$. By W-invariance we have $n_{w\alpha} = n_{\alpha}$ $\forall w \in W$, $\forall \alpha \in \Sigma$.

Finally in order to match with the notation used in the theory of hypergeometric functions associated with a root system we put $R = 2\Sigma$ and $k_{2\alpha} = \frac{1}{2}n_{\alpha}$ for $\alpha \in \Sigma$. Then (2.12) becomes

$$J(tt_0) = \prod_{\alpha \in R} \left| t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha} \right|^{k_{\alpha}}, \tag{2.13}$$

and the point $t_0 \in T$ satisfies $t_0^{2\alpha} = 1 \quad \forall \alpha \in R$. Let V be an irreducible unitary representation of U having a nonzero K-fixed vector v_K and a nonzero H-fixed vector v_H. The matrix coefficient $U \ni u \mapsto (v_k, uv_H)$ is called a (K, H)-*intertwining function.* By restriction such a function gives a W-invariant Fourier polynomial on T. If V runs over the set of equivalence classes of irreducible (K, H)-spherical representations of U we obtain in this way an orthogonal basis of $L^2(K\backslash U/H, du) \simeq L^2(T, J(t\, t_0)dt)^W$.

By a further analysis one can show that the restriction of the (K, H)-intertwining functions are obtained from the basis of monomial symmetric functions on T by a triangular operation.

Conclusion 2.8. *The restriction of the (K, H)-intertwining functions on U to a split torus are Jacobi polynomials associated with the root system R and with multiplicity parameter $k = (k_{\alpha})_{\alpha \in R}$.*

3. The spectral problems for hypergeometric functions associated with a root system

We start by fixing some notation, cf. [199]. Let $\mathfrak{a}$ be a Euclidean space with inner product $(\cdot, \cdot)$. Let $R \subset \mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ be a possibly non-reduced root system, and $R^{\vee} = \{\alpha^{\vee} = \frac{2\alpha}{(\alpha,\alpha)}; \alpha \in R\} \subset \mathfrak{a}$ the dual root system (using the linear isomorphism $\mathfrak{a} \cong \mathfrak{a}^*$ coming from the inner product). The weight lattice

$$P = \{\lambda \in \mathfrak{a}^*; (\lambda, \alpha^{\vee}) \in \mathbb{Z} \quad \forall \alpha \in R\}$$

of R can be viewed as the character lattice of the complex torus $H = \mathfrak{h}/2\pi i Q^{\vee}$. Here $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a} = \mathfrak{a} \oplus \mathfrak{t}$, $\mathfrak{t} := i\mathfrak{a}$ and $Q^{\vee} = \mathbb{Z}R^{\vee} \subset \mathfrak{g}$ is the coroot lattice of R. Clearly $H = AT$ (unique decomposition) with $A := \mathfrak{a}$

a vector group and $T := \mathfrak{t}/2\pi i Q^\vee$ a torus. For $\lambda \in P$ we write e^λ for the corresponding character of H, and the value of e^λ at a point $h \in H$ is written as $h^\lambda \in \mathbb{C}^*$. Similarly for $\lambda \in \mathfrak{h}^*$ we write e^λ for the corresponding character of A, and the value of e^λ at a point $a \in A$ is denoted by a^λ. The group algebra $\mathbb{C}[P]$ of the abelian group P now becomes identified with the ring of Laurent polynomials on H.

Consider the algebra $\mathcal{R}$ (with unit 1) generated by the functions of the form $(1-e^{-\alpha})^{-1}$ for $\alpha \in R$. Note that the Weyl group W acts on $\mathcal{R}$. Clearly $\mathcal{R}$ is invariant under the algebra $S\mathfrak{h}$ of linear differential operators on H with constant coefficients. Hence the algebra $\mathcal{R} \otimes S\mathfrak{h}$ represents the algebra of differential operators on H with coefficients in $\mathcal{R}$. Since $(1-e^\alpha)^{-1} = 1-(1-e^{-\alpha})^{-1}$ $\mathcal{R}$ is generated by the functions $(1-e^{-\alpha})^{-1}$ with $\alpha \in R_+$. Here R_+ is a fixed set of positive roots corresponding to a positive chamber $\mathfrak{a}_+ \subset \mathfrak{a}$. Writing $(1-e^{-\alpha})^{-1} = 1+e^{-\alpha}+e^{-2\alpha} + \cdots$ $\forall \alpha \in R_+$ we can expand any differential operator $P \in \mathcal{R} \otimes S\mathfrak{h}$ in the form $P = \gamma'(P) + \cdots$ with $\gamma'(P) \in S\mathfrak{h}$. Clearly these formal expansions in $\mathcal{R} \otimes S\mathfrak{h}$ (viewed as subalgebra of $\mathbb{C}[\![e^{-\alpha_1}, \ldots, e^{-\alpha_n}]\!] \otimes S\mathfrak{h}$ where $\alpha_1, \ldots, \alpha_n$ are the simple roots in R_+) are convergent on the positive chamber $A_+ := \mathfrak{a}_+$. The element $\gamma'(P) \in S\mathfrak{h}$ is called the *constant term* of the differential operator P along A_+. For example the differential operator

$$L(k) := \sum_1^n \frac{\partial^2}{\partial \xi_j^2} + \sum_{\alpha > 0} k_\alpha \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \cdot \frac{\partial}{\partial \alpha} \tag{3.1}$$

has constant term along A_+ equal to

$$\gamma'\big(L(k)\big) = \sum_1^n \frac{\partial^2}{\partial \xi_j^2} + \sum_{\alpha>0} k_\alpha \frac{\partial}{\partial \alpha}. \tag{3.2}$$

Here $\xi_1, \ldots, \xi_n$ is an orthonormal basis of $\mathfrak{a}$, and $k = (k_\alpha)_{\alpha \in R} \in K$ is a multiplicity function on R, i.e., $k_\alpha \in \mathbb{C}$ $\forall \alpha \in R$ and $k_{w\alpha} = k_\alpha$ $\forall w \in W$, $\forall \alpha \in R$. We also define a map

$$\gamma(k) \colon \mathcal{R} \otimes S\mathfrak{h} \longrightarrow S\mathfrak{h} \tag{3.3}$$

by the formulae

$$\gamma(k)(P) = e^{\rho(k)} \circ \gamma'(P) \circ e^{-\rho(k)} \tag{3.4}$$

$$\rho(k) = \frac{1}{2} \sum_{\alpha>0} k_\alpha \alpha \in \mathfrak{h}^*, \tag{3.5}$$

and call it the k-constant term along A_+. Obviously both γ' and $\gamma(k)$ are algebra homomorphisms. For the operator $L(k)$ given by (3.1) we get

$$\gamma(k)\big(L(k)\big) = \sum_1^n \frac{\partial^2}{\partial\xi_j^2} - \big(\rho(k),\rho(k)\big). \tag{3.6}$$

The advantage of $\gamma(k)$ over γ' is that (3.6) is independent of the choice of the positive Weyl chamber A_+. We put

$$\mathbb{D}(k) = \{P \in \mathcal{R}\otimes S\mathfrak{h}; wPw^{-1} = P \quad \forall w \in W, [P, L(k)] = 0\}. \tag{3.7}$$

for the algebra of all W-invariant differential operators commuting with $L(k)$. The following theorem is a crucial result (due to Opdam [205]; see also [198, 199] for a simplified proof).

Theorem 3.1. *The k-constant term*

$$\gamma(k)\colon\ \mathbb{D}(k) \xrightarrow{\cong} S\mathfrak{h}^W \tag{3.8}$$

is an isomorphism of (commutative) algebras.

Hence the second-order operator $L(k)$ is part of a commutative set of n algebraically independent differential operators. The map (3.8) is called the generalized Harish-Chandra isomorphism, because for special values of $k \in K$ (referred to as the group values) the isomorphism (3.8) is intimately connected with Harish-Chandra's description of the algebra of invariant differential operators on a Riemannian symmetric space. The map (3.8) is natural in the sense that it is independent of the choice of A_+.

The purpose of this section is to discuss the various spectral problems associated with the commutative algebra $\mathbb{D}(k)$. For this we will impose the restriction (always satisfied for group values)

$$k_\alpha + k_{2\alpha} \geq 0 \quad \forall \alpha \in R. \tag{3.9}$$

Here we put $k_\beta = 0$ if $\beta \notin R$. Consider the functions

$$\mu(k) = \prod_{\alpha>0} \left|e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\right|^{2k_\alpha}, \tag{3.10}$$

$$\delta(k) = \prod_{\alpha>0} \left(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\right)^{2k_\alpha}. \tag{3.11}$$

Because of (3.9) the function $\mu(k)$ is a nonnegative continuous function on all of H, whereas the function $\delta(k)$ is viewed as a multivalued holomorphic function on

$$H^{\mathrm{reg}} := \{h \in H; h^\alpha \neq 1 \quad \forall \alpha \in R\} \tag{3.12}$$

obtained by analytic continuation of $\mu(k)$ on $A_+ \subset H^{\mathrm{reg}}$.

Proposition 3.2. *Let $k \in K$ vary subject to the condition (3.9). The function $\frac{\delta(k)}{\mu(k)}$ is locally constant on $T \cap H^{\mathrm{reg}}$. For $t \in T$ the function $\frac{\delta(k)}{\mu(k)}$ is locally constant on $At \cap H^{\mathrm{reg}}$ if and only if $t^2 \in C$. Here*

$$C := \{h \in H; h^\alpha = 1 \quad \forall \alpha \in R\} \subset T \tag{3.13}$$

is the central subgroup of H associated with R (C is a finite subgroup of H of order equal to the index $[P\colon Q]$ of Q in P).

Proof. If we write $h \in H^{\mathrm{reg}}$ as $h = at$ with $a \in A$, $t \in T$ then

$$\frac{\delta(k;h)}{\mu(k;h)} = \prod_{\alpha>0} \frac{\left(h^{\frac{1}{2}\alpha} - h^{-\frac{1}{2}\alpha}\right)^{k_\alpha}\left(h^{\frac{1}{2}\alpha} - h^{-\frac{1}{2}\alpha}\right)^{k_\alpha}}{\left(h^{\frac{1}{2}\alpha} - h^{-\frac{1}{2}\alpha}\right)^{k_\alpha}\left(\overline{h^{\frac{1}{2}\alpha} - h^{-\frac{1}{2}\alpha}}\right)^{k_\alpha}}$$
$$= \prod_{\alpha>0}\left(\frac{a^{\frac{1}{2}\alpha}t^{\frac{1}{2}\alpha} - a^{-\frac{1}{2}\alpha}t^{-\frac{1}{2}\alpha}}{a^{\frac{1}{2}\alpha}t^{-\frac{1}{2}\alpha} - a^{-\frac{1}{2}\alpha}t^{\frac{1}{2}\alpha}}\right)^{k_\alpha}$$

On the one hand, if $a = 1$ then $\frac{\delta(k;t)}{\mu(k;t)} = \prod_{\alpha>0}(-1)^{k_\alpha}$ is locally constant on $T \cap H^{\mathrm{reg}}$. On the other hand, if $t \in T$ is fixed then $\frac{\delta(k;at)}{\mu(k;at)}$ is independent of $a \in A_+$ if and only if

$$\frac{a^{\frac{1}{2}\alpha}t^{\frac{1}{2}\alpha} - a^{-\frac{1}{2}\alpha}t^{-\frac{1}{2}\alpha}}{a^{\frac{1}{2}\alpha}t^{-\frac{1}{2}\alpha} - a^{-\frac{1}{2}\alpha}t^{\frac{1}{2}\alpha}} = t^\alpha \quad \forall \alpha > 0 \Longleftrightarrow t^{2\alpha} = 1 \quad \forall \alpha > 0 \Longleftrightarrow t^2 \in C,$$

and the proposition is proved. $\square$

By algebraic manipulations (see [199, Theorem 2.1.1]) it is not hard to show that

$$\begin{aligned} &\delta(k)^{\frac{1}{2}} \circ \{L(k) + (\rho(k), \rho(k))\} \circ \delta(k)^{-\frac{1}{2}} \\ &= \sum_1^n \frac{\partial^2}{\partial \xi_j^2} + \sum_{\alpha>0} \frac{k_\alpha(1 - k_\alpha - 2k_{2\alpha})(\alpha,\alpha)}{(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})^2}. \end{aligned} \tag{3.14}$$

Now observe that the right-hand side of (3.14) is (formally) symmetric with respect to both Haar measures dt on T and da on A. In turn the following consequences can easily be derived.

Corollary 3.3. *With $k \in K$ subject to* (3.9) *define an inner product* $(\cdot,\cdot)_k$ *on* $C^\infty(T)^W$ *by*

$$(F,G)_k := \frac{1}{|W|}\int_T F(t)\overline{G(t)}\mu(k;t)dt, \tag{3.15}$$

where dt is the normalized Haar measure on T. Then the algebra $\mathbb{D}(k)$ leaves invariant $C^\infty(T)^W$ and is invariant under taking adjoints with respect to (3.15). *In fact under the generalized Harish-Chandra isomorphism* (3.8) *the adjoint corresponds to conjugation of $S\mathfrak{h}^W$ with respect to the real form $\mathfrak{a}$ of $\mathfrak{h}$.*

Corollary 3.4. *Let $k \in K$ satisfy* (3.9). *Fix $t \in T$ with $t^2 \in C$, and put $W(t) = \{w \in W; w(t) = t\}$ for the stabilizer of t in W. Define the inner product $(\cdot,\cdot)_k$ on $C^\infty(At)^{W(t)}$ by*

$$(F,G)_k = \frac{1}{|W(t)|}\int_A F(at)\overline{G(at)}\mu(k;at)da, \tag{3.16}$$

where da is a Haar measure on A (a natural choice could be the continuation of the normalized Haar measure dt on T). Then the algebra $\mathbb{D}(k)$ leaves invariant the space $C^\infty(At)^{W(t)}$ and is invariant under taking adjoints with respect to (3.16). *In fact under the generalized Harish-Chandra isomorphism* (3.8) *the adjoint corresponds to conjugation of $S\mathfrak{h}^W$ with respect to real form $\mathfrak{t} = i\mathfrak{a}$ of $\mathfrak{h}$.*

The spectral problem connected with Corollary 3.3 has an exact solution due to Opdam and the author [197, 206] which we briefly recall. For $\lambda \in P_+$ the monomial symmetric functions $M(\lambda) = \sum_{\mu \in W\lambda} e^\mu$ form a basis of $\mathbb{C}[P]^W$. For $\lambda, \mu \in \mathfrak{h}^*$ write $\mu \le \lambda$ if and only if $\lambda - \mu = k_1\alpha_1 + \cdots + k_n\alpha_n$, with $k_1, \ldots, k_n \in \mathbb{Z}_{\ge 0}$. Then $\le$ is a partial ordering on P_+ (which is only a total ordering in case $n = 1$).

Definition 3.5. For $\lambda \in P_+$ the Jacobi polynomials $P(\lambda, k) \in \mathbb{C}[P]^W$ are defined by the following two properties:

$$P(\lambda,k) = \sum_{\mu \in P_+, \mu \le \lambda} c_{\lambda\mu}(k)M(\mu), \quad c_{\lambda\lambda}(k) = 1 \tag{3.17}$$

$$\big(P(\lambda,k), M(\mu)\big)_k = 0 \quad \forall \mu \in P_+,\ \mu < \lambda. \tag{3.18}$$

Definition 3.6. The meromorphic functions $\widetilde{c}, c^*\colon \mathfrak{h}^* \times K \to \mathbb{C}$ are defined by

$$\widetilde{c}(\lambda, k) = \prod_{\alpha>0} \frac{\Gamma\big((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha}\big)}{\Gamma\big((\lambda, \alpha^\vee) + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_\alpha\big)} \tag{3.19}$$

$$c^*(\lambda, k) = \prod_{\alpha>0} \frac{\Gamma\big(-(\lambda, \alpha^\vee) - \frac{1}{2}k_{\frac{1}{2}\alpha} - k_\alpha + 1\big)}{\Gamma\big(-(\lambda, \alpha^\vee) - \frac{1}{2}k_{\frac{1}{2}\alpha} + 1\big)}. \tag{3.20}$$

Theorem 3.7. *For all $P \in \mathbb{D}(k)$ and $\lambda \in P_+$ we have*

$$P\big(P(\lambda, k)\big) = \gamma(k)(P)\big(\lambda + \rho(k)\big) \cdot P(\lambda, k). \tag{3.21}$$

Hence the Jacobi polynomials are a complete orthogonal basis of the Hilbert space $L^2(T, \mu(k; t)dt)^W$. Moreover their L^2-norms are given by

$$\big(P(\lambda, k), P(\lambda, k)\big)_k = \frac{c^*\big(-\big(\lambda + \rho(k)\big), k\big)}{\widetilde{c}(\lambda + \rho(k), k)}. \tag{3.22}$$

This gives the solution of the simultaneous spectral problem problem of $D(k)$ in the context of Corollary 3.3. The first part of this theorem is an easy consequence of Theorem 3.1 and Corollary 3.3. The formula for the L^2-norm of $P(\lambda, k)$ as an explicit product of Γ-factors can be derived using so-called shift operators [206, 199]. For group values of the parameter $k \in K$ formula (3.22) goes back to Vretare [211].

We now consider the spectral problem related to Corollary 3.4. If $\lambda \in \mathfrak{h}^*$ with $(\lambda, \alpha^\vee) \notin \mathbb{Z}\ \ \forall \alpha \in R$ then the system of differential equations

$$P(u) = \gamma(k)(P)(\lambda) \cdot u \quad \forall P \in \mathbb{D}(k) \tag{3.23}$$

has a basis of (formal) solutions of the form

$$\Phi(\mu, k) = \sum_{\nu \leq 0} \Gamma_\nu(\mu, k) e^{\mu - \rho(k) + \nu}, \tag{3.24}$$

where $\Gamma_0(\mu, k) = 1$ and $\Gamma_\nu(\mu, k)$ is defined by recurrence relations ($\nu < 0$) and μ runs over the orbit $W\lambda \subset \mathfrak{h}^*$. These are called the Harish-Chandra series. The (power) series

$$e^{-\mu + \rho(k)} \Phi(\mu, k) = \sum_{\nu \leq 0} \Gamma_\nu(\mu, k) e^\nu$$

(in the variables $e^{-\alpha_1}, \dots, e^{-\alpha_n}$) converges to a holomorphic function on the (polydisc) domain $A_+ \times T$.

Now fix $t \in T$ with $t^2 \in C$. If we specify $e^{\mu-\rho(k)}$ on A_+t by

$$e^{\mu-\rho(k)}(at) = a^{\mu-\rho(k)} = e^{(\mu-\rho(k),\log a)}$$

then the functions

$$\Phi(A_+t,\mu,k) = e^{\mu-\rho(k)} \sum_{\nu\leq 0} \Gamma_\nu(\mu,k)e^{\nu}, \quad \mu \in W\lambda \tag{3.25}$$

are a basis for the solution space of (3.23) on A_+t. Let $V(At,\lambda,k)$ denote the linear space of $W(t)$-invariant analytic solutions of (3.23) on At. Let $w_1 = 1, w_2, \dots, w_d \in W$ be representatives for W modulo $W(t)$ such that $\bigcup_1^d w_j(A_+)$ is dense in the chamber in A for $W(t)$ containing A_+. This latter chamber corresponds to $R_+ \cap R(t)$ where $R(t) = \{\alpha \in R; t^\alpha = 1\}$. Define a linear map

$$C(\lambda,k)\colon\ V(At,\lambda,k) \longrightarrow \mathbb{C}^d \tag{3.26}$$

by means of

$$\begin{aligned} &C(\lambda,k)(u) = (c_1,\dots,c_d)^t \Longleftrightarrow c_j = c(w_j,w_j) \text{ with} \\ &u|_{w_j(A_+)t} = \sum_{w\in W} c(w_j,w)\Phi(w_j(A_+)t,w\lambda,k) \quad \forall j. \end{aligned} \tag{3.27}$$

Proposition 3.8. *For $(\lambda,k) \in \mathfrak{h}^* \times K$ generic the map (3.26) is a linear injection.*

In fact what can be shown is the existence of a linear map (or matrix)

$$C^0(w{:}\lambda,k)\colon \mathbb{C}^d \longrightarrow \mathbb{C}^d, \quad w \in W \tag{3.28}$$

such that we have a commutative diagram

$$\begin{array}{ccc} V(At,\lambda,k) & = & V(At,w\lambda,k) \\ \Big\downarrow C(\lambda,k) & & \Big\downarrow C(w\lambda,k) \\ \mathbb{C}^d & \xrightarrow[C^0(w:\lambda,k)]{} & \mathbb{C}^d. \end{array}$$

Moreover the entries of the matrix $C^0(w{:}\lambda,k)$ are meromorphic in $(\lambda,k)\in\mathfrak{h}^*\times K$. Indeed this immediately implies the proposition since a solution $u\in V(At,\lambda,k)$ is completely determined on A_+t (and hence on all of At) by the numbers $c(1,w)$, $w\in W$ in (3.27). The existence of the matrix (3.28) is immediate from the trivial relation

$$C^0(w_1w_2{:}\,\lambda,k)=C^0(w_1{:}\,w_2\lambda,k)C^0(w_2{:}\,\lambda,k) \tag{3.29}$$

together with the construction of the matrix $C^0(r_i{:}\,\lambda,k)$ where $r_i\in W$ is a simple reflection (corresponding to A_+). For $j=1,\dots,d$ there are two possibilities: either $w_jr_iw_j^{-1}\in W(t)$ or equivalently $w_jr_i\neq w_{j'}$ for all $j'=1,\dots,d$, or $w_jr_iw_j^{-1}\notin W(t)$ or equivalently $w_jr_i=w_{j'}$ for some $j'=1,\dots,d$. Let $e_1,\dots,e_d$ denote the standard basis vectors for $\mathbb{C}^d$. In the former case the one-dimensional space $\mathbb{C}e_j$ is invariant under $C^0(r_i;\lambda,k)$, whereas in the latter case the two-dimensional space $\mathbb{C}e_j+\mathbb{C}e_{j'}$ is invariant under $C^0(r_i{:}\,\lambda,k)$. Moreover by taking boundary values the explicit computation of the matrix coefficients of $C^0(r_i{:}\,\lambda,k)$ reduces to the rank one case of the Gaussian hypergeometric function. The next result is also clear from the explicit form of these matrix coefficients.

Theorem 3.9. (Maass-Selberg relations). *For all $\lambda\in\mathfrak{h}^*$, $k\in K$ and $w\in W$ we have*

$$C^0(w{:}\,-\overline{\lambda},\overline{k})^*C^0(w{:}\,\lambda,k)=Id \tag{3.30}$$

with $\overline{\lambda}=\lambda_1-i\lambda_2$ the conjugate of $\lambda=\lambda_1+i\lambda_2\in\mathfrak{h}^*(\lambda_1,\lambda_2\in\mathfrak{a}^*)$ and the conjugate $\overline{k}\in K$ defined by $(\overline{k})_\alpha=\overline{k}_\alpha$ $\forall\alpha\in R$. Moreover the star * denotes the adjoint (= conjugate and transposed) matrix. In particular for $\lambda\in i\mathfrak{a}^*$ purely imaginary and $k\in K$ real-valued (a fortiori if (3.9) *holds) the matrix $C^0(w{:}\,\lambda,k)$ is unitary.*

Example 3.10. The case $t=1$ is a simple and illuminating example. Under condition (3.9) the space $V(A,\lambda,k)$ is one-dimensional, and generated by the (hypergeometric) function

$$\widetilde{F}(\lambda,k;\cdot)=\sum_{w\in W}\widetilde{c}(w\lambda,k)\Phi(w\lambda,k) \tag{3.31}$$

with $\widetilde{c}(\lambda,k)$ given by (3.19) and $\Phi(\mu,k)$ the Harish-Chandra series on A_+, cf. [199, Section 4.3]. With respect to this basis vector the matrix of $C(\lambda,k)$ just becomes $\widetilde{c}(\lambda,k)$, and in turn

$$C^0(w\colon\lambda,k)=\frac{\widetilde{c}(w\lambda,k)}{\widetilde{c}(\lambda.k)}. \tag{3.32}$$

The Maass-Selberg relations in this case follow from $\overline{\widetilde{c}(\lambda,k)}=\widetilde{c}(\overline{\lambda},\overline{k})$ and the fact that $\widetilde{c}(-\lambda,k)\widetilde{c}(\lambda,k)$ is a W-invariant function of $\lambda\in\mathfrak{h}^*$.

The standard Hermitian inner product on $\mathbb{C}^d$ can be transferred by the map $C(\lambda,k)$ to a Hermitian inner product on $V(At,\lambda,k)$, and for $\lambda\in i\mathfrak{a}^*$ purely imaginary and $k\in K$ real-valued it follows from the Maass-Selberg relations that this inner product only depends on the orbit $W\lambda$ (as does the system of differential equations (3.23) and the solution space $V(At,\lambda,k)$ on At). For $\lambda\in i\mathfrak{a}^*$ purely imaginary and $k\in K$ satisfying (3.9) this will be the canonical Hilbert space structure on $V(At,\lambda,k)$.

Conjecture 3.11. *The Hilbert space $L^2(At,\mu(k;a)da)^{W(t)}$ has a closed subspace L^2_{mc} (called the most continuous part of the Plancherel decomposition), which admits a direct integral decomposition*

$$L^2_{mc}=\int_{W\backslash\mathfrak{a}^*}^{\oplus}V(At,i\lambda,k)d\lambda. \tag{3.33}$$

Here $d\lambda$ is the regularly normalized Lebesgue measure on $\mathfrak{a}^*$. The orthocomplement of L^2_{mc} has lower spectral dimension, which can be rephrased by saying that it is annihilated by a suitable differential operator.

For group values of $k\in K$ this has been proved by van den Ban and Schlichtkrull [192] (see also [209]) by a variation of the Helgason-Gangolli-Rosenberg proof of the spherical Plancherel theorem on a Riemannian symmetric space. Their proof carries over to the situation of arbitrary k satisfying (3.9) except that a suitable integral representation for functions in $V(At,\lambda,k)$ is missing. In the group case these integral representations are precisely given by the (K-invariant) Eisenstein integrals [190, 191].

Example 3.12. For $t = 1$ we have $V(A, \lambda, k) = \mathbb{C}\widetilde{F}(\lambda, k; \cdot)$. If in addition condition (3.9) is sharpened to

$$k_\alpha \geq 0 \quad \forall \alpha \in R, \tag{3.34}$$

then L^2_{mc} should be equal to $L^2(A, \mu(k; a)da)^W$. The decomposition (3.33) can be written in the equivalent form

$$f(\cdot) = \int_{\mathfrak{a}^*_+} \left\{ \int_{A_+} f(a)\widetilde{F}(-i\lambda, k; a)\mu(k; a)da \right\} \widetilde{F}(i\lambda, k; \cdot) \frac{d\lambda}{|\widetilde{c}(i\lambda, k)|^2}. \tag{3.35}$$

For group values of $k \in K$ this is Harish-Chandra's spherical Plancherel theorem.

4. The case of the Gaussian hypergeometric function

In the previous section the existence of the matrix $C^0(w\colon \lambda, k)$, $w \in W$ was reduced to the rank one case. In this reduction one had to discuss two cases separately: either $w_j r_i w_j^{-1} \in W(t)$ or $w_j r_i w_j^{-1} \notin W(t)$. Since the former case is essentially covered by Example 3.10 we now look at the latter case. Consider three copies of the complex plane with coordinates x, y, and z connected by $2-4y = x+x^{-1}$, $z = 4y(1-y)$, and $2-4z = x^2+x^{-2}$, respectively.

Consider the following scheme of exponents.

points	-1	0	1	∞
exponents in the x-variable	0 $1-2k$	$\lambda+k$ $-\lambda+k$	0 $1-k$	$\lambda+k$ $-\lambda+k$
exponents in the y-variable	–	0 $\frac{1}{2}-k$	0 $\frac{1}{2}-k$	$\lambda+k$ $-\lambda+k$
exponents in the z-variable	–	0 $1-\gamma = \frac{1}{2}-k$	0 $\frac{1}{2}$	$\alpha = \frac{1}{2}\lambda+\frac{1}{2}k$ $\beta = -\frac{1}{2}\lambda+\frac{1}{2}k$

In the z-plane these are the exponents of the Gaussian hypergeometric equation with parameters α, β, γ. The set At equals $i\mathbb{R}_{>0}$ in the x-plane, and the system of differential equations (3.23) is the pull-back of the hypergeometric equation in the y-plane or z-plane. Let the functions

$$(4.1)\qquad \varphi_e(z) = 1+O(z-1), \quad \varphi_0(z) = (z-1)^{\frac{1}{2}}\big(1+O(z-1)\big)$$

be a basis for the solution space near $z=1$, and let

$$(4.2)\qquad \Phi_\alpha(z) = z^{-\alpha}\big(1+O(\frac{1}{z})\big), \quad \Phi_\beta(z) = z^{-\beta}\big(1+O(\frac{1}{z})\big)$$

be a basis for the solution space near $z=+\infty$ ($\alpha-\beta \notin \mathbb{Z}$). The Kummer relations give the connection between these two bases by analytic continuation along the interval $(1,\infty)$, and the outcome is [196, Section 2.9]:

$$(4.3)\qquad \varphi_e = \frac{\Gamma(\frac{1}{2})\Gamma(\beta-\alpha)}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\beta)}\Phi_\alpha + \frac{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)}{\Gamma(\frac{1}{2}-\beta)\Gamma(\alpha)}\Phi_\beta$$

$$(4.4)\qquad \varphi_0 = \frac{\Gamma(\frac{3}{2})\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\frac{1}{2}+\beta)}\Phi_\alpha + \frac{\Gamma(\frac{3}{2})\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\frac{1}{2}+\alpha)}\Phi_\beta.$$

With respect to the basis $\{\varphi_e,\varphi_0\}$ of $V(At,\lambda,k)$ the matrix of the operator $C(\lambda,k)\colon V(At,\lambda,k)\to\mathbb{C}^2$ takes the form

$$(4.5)\qquad C(\lambda,k) =: C(\alpha,\beta) = \begin{pmatrix} \frac{\Gamma(\frac{1}{2})\Gamma(\beta-\alpha)}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\beta)} & \frac{\Gamma(\frac{3}{2})\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\frac{1}{2}+\beta)} \\ \frac{\Gamma(\frac{1}{2})\Gamma(\beta-\alpha)}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\beta)} & -\frac{\Gamma(\frac{3}{2})\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\frac{1}{2}+\beta)} \end{pmatrix},$$

and a straightforward calculation yields

$$(4.6)\qquad \begin{aligned} C^0(\alpha,\beta)\colon &= C(\beta,\alpha)C(\alpha,\beta)^{-1} \\ &= \frac{2^{2(\alpha-\beta)}\Gamma(1-2\alpha)\Gamma(2\beta)}{\Gamma(\beta-\alpha)\Gamma(\beta-\alpha+1)}\begin{pmatrix} \frac{\sin\pi(\alpha+\beta)}{\sin\pi(\alpha-\beta)} & 1 \\ 1 & \frac{\sin\pi(\alpha+\beta)}{\sin\pi(\alpha-\beta)} \end{pmatrix}. \end{aligned}$$

In turn it is easy to check that

$$(4.7)\qquad C^0(\beta,\alpha)C^0(\alpha,\beta) = Id.$$

Together with $\overline{C^0(\alpha,\beta)} = C^0(\overline{\alpha},\overline{\beta})$ and $C^0(\alpha,\beta)^t = C^0(\alpha,\beta)$ this implies

$$(4.8)\quad C^0(\overline{\beta},\overline{\alpha})^*C^0(\alpha,\beta) = C^0(\beta,\alpha)^tC^0(\alpha,\beta) = C^0(\beta,\alpha)C^0(\alpha,\beta) = Id,$$

which proves the Maass-Selberg relations (3.30).

Recall that $\alpha = \frac{1}{2}\lambda+\frac{1}{2}k, \beta = -\frac{1}{2}\lambda+\frac{1}{2}k$. Now if $\lambda = k-1, k-3, k-5, \ldots$ ($\beta = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$) then φ_e is a multiple of Φ_α by (4.3). Similarly if $\lambda = k-2, k-4, k-6, \ldots$ ($\beta = 1, 2, 3, \ldots$) then φ_0 is a multiple of Φ_α by (4.4). If in addition $\lambda > 0$ then Φ_α becomes $\mu(k)$-square integrable on At. In fact the condition (3.9) can be weakened to $k \in K$ being real-valued and $\mu(k)$ being locally integrable on At. In this rank one situation this simply means $k \in \mathbb{R}$ rather than $k \geq 0$. By a similar reasoning as before we have: If $\lambda = k, k+2, k+4 (\beta = 0, -1, -2, \ldots)$ then φ_e is a multiple of Φ_β. If $\lambda = k+1, k+3, k+5 (\beta = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots)$ then φ_0 is a multiple of Φ_β. If in addition $\lambda < 0$ then Φ_β becomes $\mu(k)$-square integrable on At. The conclusion is that for $k \in [0,1]$ there are no $\mu(k)$-square integrable eigenfunctions on At, and the most continuous part $\mathcal{H}$ is equal to all of $L^2\big(At, \mu(k;a)da\big)$ in the notation of Conjecture 3.9. For $k \in \mathbb{R}$ arbitrary the most continuous part $\mathcal{H}$ will always have finite codimension in $L^2\big(At, \mu(k;a)da\big)$. Indeed this codimension, which is equal to the number of linearly independent $\mu(k)$-square integrable eigenfunctions, is given by $N \in \mathbb{N}$ if $k \in [-N, -N+1) \cup (N, N+1]$.

5. Open problems

Indeed compared with the case $t = 1$ the general situation of the spectral problem on At with $t \in T$ and $t^2 \in C$ gives rise to several complications caused by the fact that the space $V(At, \lambda, k)$ of $W(t)$-invariant analytic solutions of (3.23) on At need no longer be one-dimensional.

Question 5.1. For $(\lambda, k) \in \mathfrak{h}^* \times K$ generic the solution space $V(At, \lambda, k)$ has dimension $d := |W/W(t)|$. Is this also true for all $\lambda \in \mathfrak{h}^*$ as long as $k \in K$ is restricted by condition (3.9), or more generally as long as $\mu(k)$ is locally integrable on At?

Probably the answer to this question is yes. In fact one might even expect that a solution in $V(At, \lambda, k)$ is uniquely characterized by prescribing its $W(t)$-invariant W-harmonic derivatives at the point t.

Question 5.2. Is the connection problem on At between the origin t and the points at infinity explicitly solvable?

Although in the rank one case the answer is yes and given by formulas (4.3) and (4.4) I do not expect this to be possible in general, cf. [207].

Question 5.3. Is the subspace of $V(At, \lambda, k)$ of $\mu(k)$-square integrable eigenfunctions always at most one-dimensional?

I do not even know what answer to expect for this question. In fact if we relax condition (3.9) to the weight function (3.10) being locally integrable on At then I am inclined to believe that the answer is negative. The reason is the following. There is an analogy between this type of question and a corresponding question for the Hecke algebra associated with an affine Weyl group (see [203], Section 5.3). Let W_I, $I = \{0, 1, \ldots, n\}$ be an affine Weyl group with I indexing the nodes of a connected extended Dynkin diagram, and let W_J $(J \subset I)$ be a parabolic subgroup where J is obtained from I by deleting either two nodes with mark 1 or one node with mark ≤ 2. Here the marks come from the coefficients of the highest root. Indeed this is the analog of the condition $t^2 \in C$ in Proposition 3.2. Then the reflection representation gives a counter example that each square integrable representation of the relative Hecke algebra $H(W_I, W_J, q)$ is one-dimensional (see [202, 204]).

Recently Conjecture 3.11 has been solved by E.M. Opdam (see [208]) in the special case discussed in Example 3.12. The approach is very remarkable. Instead of having an integral representation (as for group values of $k \in K$) the desired estimates for the hypergeometric function can be obtained from the differential equation (by rewriting it as a Knizhnik-Zamolodchikov type differential equation).

Question 5.4. Can Conjecture 3.11 be solved in general by a similar approach?

References

Part I

[1] J.-Ph. Anker, *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi and Varadarajan*, J. Funct. Anal. **96** (1991), 331–349.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, Springer-Verlag, 1978.

[3] E.P. van den Ban and H. Schlichtkrull, *The most continuous part of the Plancherel decomposition for a reductive symmetric space (in preparation).*

[4] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947), 568–640.

[5] R.J. Beerends, *A transmutation property of the generalized Abel transform associated with root system A_2*, Indag. Math., N. S. **1** (1990), 155–168.

[6] I.N. Bernstein, I.M. Gel'fand, and S.I. Gel'fand, *Schubert cells and the cohomology of G/P*, Russian Math. Surveys **28** (1973), 1–26.

[7] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Masson, 1981.

[8] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.

[9] F. Calogero, *Solution of the one dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Physics **12** (1971), 419–436.

[10] W. Casselman and D. Miličič, *Asymptotic behavior of matrix coefficients of admissable representations*, Duke Math. J. **49** (1982), 869–930.

[11] V.I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.

[12] A. Debiard, *Polynomes de Tchébychev et de Jacobi dans un espace Euclidien de dimension p*, C. R. Acad. Sc. Paris **296** (1983), 529–532.

[13] P. Deligne, *Equations differentielles à Points Singuliers Reguliers*, LNM 163, Springer Verlag, 1970.

[14] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.

[15] M. Demazure, *Désingularisation des variétés de Schubert généralisés*, Ann. Sci. Ec. Norm. Sup. **7** (1974), 53–88.

[16] M. Demazure, *Une nouvelle formule des caractères*, Bull. Soc. Math. **98** (1974), 163–172.

[17] C.F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. AMS **311** (1989), 167–183.

[18] F.J. Dyson, *Statistical theory of the energy levels of complex systems I*, J. Math. Phys. **3** (1962), 140–156.

[19] A. Erdélyi, *Higher Transcendental Functions*, R.E. Krieger Publ. Co., Florida, 1985.

[20] M. Flensted-Jensen, *Spherical functions on a simply connected semisimple Lie group*, Amer. J. Math. **99** (1977), 341–361.

[21] M. Flensted-Jensen, *Spherical functions on a simply connected semisimple Lie group II. The Paley-Wiener theorem for the rank one case*, Math. Ann. **228** (1977), 65–92.

[22] R. Gangolli, *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. **93** (1971), 150–165.

[23] R. Gangolli and V.S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Ergebnisse der Mathematik 101, Springer-Verlag, 1988.

[24] S.G. Gindikin and F.I. Karpelevič, *Plancherel measure of Riemannian symmetric spaces of nonpositive curvature*, Dokl. Akad. Nauk SSSR **145** (1962), 252–255.

[25] I.J. Good, *Short proof of a conjecture of Dyson*, J. Math. Phys. **11** (1970), 1884.

[26] J. Gunson, *Proof of a conjecture of Dyson in the statistical theory of energy levels*, J. Math. Phys. **3** (1962), 752–753.

[27] Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120 (= Coll. Papers, Vol. 2, 243–276).

[28] Harish-Chandra, *Spherical functions on a semisimple Lie group, I-II*, Amer. J. Math. **80** (1958), 241–310 and 553–613 (= Coll. Papers, Vol. 2, 409–539).

[29] Harish-Chandra, *Differential Equations and Semisimple Lie groups*, Coll. Papers, Vol. 3, 57–120. (Unpublished, 1960).

[30] G.J. Heckman, *Root systems and hypergeometric functions II*, Comp. Math. **64** (1987), 353–373.

[31] G.J. Heckman, *Hecke algebras and hypergeometric functions*, Invent. Math. **100** (1990), 403–417.

[32] G.J. Heckman, *A remark on the Dunkl differential-difference operators*, Harmonic Analysis on Reductive groups, Proceedings, Bowdoin College 1989, Progr. Math. 101, Birkhäuser (1991), 181–191.

[33] G.J. Heckman, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350.

[34] G.J. Heckman, E.M. Opdam, *Root systems and hypergeometric functions I*, Comp. Math. **64** (1987), 329–352.

[35] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.

[36] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.

[37] S. Helgason, *Some results on invariant differential operators on symmetric spaces*, Amer. J. Math. **114** (1992), 789–811.

[38] H.L. Hiller, *Geometry of Coxeter Groups*, Research Notes in Mathematics 54, Pitman, Boston, 1982.

[39] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol I*, Springer-Verlag, 1983.

[40] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.

[41] C. Jacobi, *Problema trium corporum mutis attractionibus cubus distantiarum inverse proportionalibus recta linea se moventium*, Ges. Werke 4, Berlin (1866).

[42] T.H. Koornwinder, *Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent differential operators I-IV*, Indag. Math. **36** (1974), 48–66 and 358–381.

[43] A. Korányi and J.A. Wolf, *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. of Math. **81** (1965), 265–288.

[44] H. v.d. Lek, *Extended Artin groups*, Proc. Symp. Pure Math. **40** (1981), 117–122.

[45] H. v.d. Lek, *The homotopy type of complex hyperplane complements*, Thesis, Nijmegen (1983).

[46] I.G. Macdonald, *Spherical functions on a group of p-adic type*, Publications of the Ramanujan Institute No. 2, Madras, 1971.

[47] I.G. Macdonald, *The Poincaré series of a Coxeter group*, Math. Annalen **199** (1972), 161–174.

[48] I.G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Analysis **13** (1982), 988–1007.

[49] I.G. Macdonald, *Commuting differential operators and zonal spherical functions*, Algebraic groups, Utrecht 1986, LNM **1271**, 189–200.

[50] B. Malgrange, *Regular connections after Deligne*, in A. Borel (Ed.), Algebraic D-modules, Perspectives in Mathematics 2, Academic Press, 1987.

[51] C. Marchioro, *Solution of a three body scattering problem in one dimension*, J. Math. Physics **11** (1970), 2193–2196.

[52] C.C. Moore, *Compactifications of symmetric spaces II: The Cartan domains*, Amer. J. Math. **86** (1964), 358–378.

[53] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformation*, Adv. Math. **16** (1975), 197–220.

[54] Nguyên Viêt Dung, *The fundamental groups of the regular orbits of affine Weyl groups*, Topology **22** (1983), 425–435.

[55] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer Verlag, Berlin, 1988.

[56] M.A. Olshanetsky and A.M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras*, Invent. Math. **37** (1976), 93–108.

[57] M.A. Olshanetsky and A.M. Perelomov, *Quantum systems related to root systems, and radial parts of Laplace operators*, Funct. Anal. Appl. **12** (1978), 121–128.

[58] E.M. Opdam, *Root systems and hypergeometric functions III*, Comp. Math. **67** (1988), 21–49.

[59] E.M. Opdam, *Root systems and hypergeometric functions IV*, Comp. Math. **67** (1988), 191–209.

[60] E.M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1–18.

[61] E.M. Opdam, *An analogue of the Gauss summation formula for hypergeometric functions related to root systems*, Math. Z. **212** (1993), 313–336.

[62] E.M. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Comp. Math. **85** (1993), 333–373.

[63] J. Orloff, *private communication.*

[64] E. Papperitz, *Über verwandte s-Funktionen*, Math. Ann. **25** (1885), 212–238.

[65] J.Peetre, *Une caracterization abstraite des opérateurs différentiels*, Math. Scand. **7** (1959), 211–218; Rectification, Math. Scand. **8** (1960), 116–120.

[66] B. Riemann, *Beiträge zur Theorie der durch die Gauss'sche Reihe* $F(\alpha,\beta,\gamma;x)$ *darstellbaren Funktionen*, Ges. Werke (1857), 67–83.

[67] H. Schlichtkrull, *One-dimensional K types in finite dimensional representations of semisimple Lie groups: a generalization of Helgason's theorem*, Math. Scand. **54** (1984), 279–294.

[68] J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Publ. RIMS Kyoto Univ. **12** (1977), 455–459.

[69] A. Selberg, *Bemerkninger om et multipelt integral*, Norsk Mat. Tidsskr. **26** (1944), 71–78 (= Coll. Papers, Vol 1, 204–213).

[70] N. Shimeno, *Eigenspaces of invariant differential operators on a homogeneous line bundle of a Riemannian symmetric space*, Thesis, Tokyo (1989).

[71] G. Shimura, *Invariant differential operators on hermitian symmetric spaces*, Ann. of Math. **132** (1990), 237–272.

[72] L. Vretare, *Elementary spherical functions on symmetric spaces*, Math. Scand. **39** (1976), 343–358.

[73] L. Vretare, *On a recurrence formula for elementary spherical functions on symmetric spaces and its applications*, Math. Scand. **41** (1977), 99–112.

[74] E.T. Whittaker, G.N. Watson, *A course of modern analysis*, Cambridge Univ. Press, 1927.

[75] K. Wilson, *Proof of a conjecture of Dyson*, J. Math. Phys. **3** (1962), 1040–1043.

Part II

[76] W. Baldoni-Silva and A. W. Knapp, *Intertwining operators into* $L^2(G/H)$, in T. Kawazoe, T. Oshima and S. Sano (eds.), Repre-

sentation Theory of Lie Groups and Lie Algebras, Fuji-Kawaguchiko 1990 (1992), 95–115.

[77] E. P. van den Ban, *A convexity theorem for semisimple symmetric spaces*, Pac. J. Math. **124** (1986), 21–55.

[78] ———, *Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula*, Ark. för Mat. **25** (1987), 175–187.

[79] ———, *Asymptotic behaviour of matrix coefficients related to reductive symmetric spaces*, Indag. Math. **49** (1987), 225–249.

[80] ———, *The principal series for a reductive symmetric space I. H-fixed distribution vectors*, Ann. Sci. Éc. Norm. Sup. **4, 21** (1988), 359–412.

[81] ———, *The principal series for a reductive symmetric space II. Eisenstein integrals*, J. Funct. Anal. **109** (1992), 331–441.

[82] ———, *The action of intertwining operators on H-fixed generalized vectors in the minimal principal series of a reductive symmetric space* (to appear).

[83] E. P. van den Ban and P. Delorme, *Quelques proprietés des représentations sphériques pour les espaces symétriques réductifs*, J. Funct. Anal. **80** (1988), 284–307.

[84] E. P. van den Ban, M. Flensted-Jensen, and H. Schlichtkrull, *Basic harmonic analysis on pseudo-Riemannian symmetric spaces*, in E. Tanner and R. Wilson (Eds.), Noncompact Lie Groups and Some of Their Applications, Kluwer 1994.

[85] E. P. van den Ban and H. Schlichtkrull, *Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces*, J. reine und angew. Math. **380** (1987), 108–165.

[86] ———, *Local boundary data of eigenfunctions on a Riemannian symmetric space*, Invent. Math. **98** (1989), 639–657.

[87] ———, *Asymptotic expansions on symmetric spaces*, in W. Barker and P. Sally (Eds.), Harmonic Analysis on Reductive Groups, Progress in Mathematics 101, Birkhäuser (1991), 79–87.

[88] ———, *Multiplicities in the Plancherel decomposition for a semisimple symmetric space*, Contemporary Math. **145** (1993), 163–180.

[89] ———, *Convexity for invariant differential operators on semisimple symmetric spaces*, Compos. Math. **89** (1993), 301-313.

[90] ———, *Fourier transforms on semisimple symmetric spaces*, in preparation.

[91] ______, *The most continuous part of the Plancherel decomposition for a reductive symmetric space*, in preparation.

[92] M. Berger, *Les espaces symétriques non compacts*, Ann. Sci. École Norm. Sup. **74** (1957), 85–177.

[93] F. Bien, *$\mathcal{D}$-modules and spherical representations*, Princeton University Press, Princeton, New Jersey, 1990.

[94] N. Bopp, *Analyse sur un espace symétrique pseudo-Riemannien*, Thesis, Univ. Strassbourg (1987).

[95] N. Bopp and P. Harinck, *Formule de Plancherel pour* $\mathrm{GL}(n, \mathbf{R})/\mathrm{U}(p, q)$, J. reine und angew. Math. **428** (1992), 45–95.

[96] F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France **84** (1956), 97–205.

[97] J.-L. Brylinski and P. Delorme, *Vecteurs distributions $H-$invariants pour les séries principales géneralisées d'espaces symétriques réductifs et prolongement méromorphe d'integrales d'Eisenstein*, Invent. Math. **109** (1992), 619–664.

[98] J. Carmona and P. Delorme, *Base méromorphe de vecteurs distributions H-invariants pour les séries principales géneralisées d'espaces symétriques réductifs. Equation fonctionelle*, preprint (1992).

[99] P. Delorme, *Injection de modules sphériques pour les espaces symétriques réductifs dans certaines représentations induites*, Lect. Notes in Math. **1243** (1987), 108–143.

[100] ______, *Coefficients géneralises de séries principales sphériques et distributions sphériques sur $G_{\mathbf{C}}/G_{\mathbf{R}}$*, Invent. Math. **105** (1991), 305–346.

[101] ______, *Intégrales d'Eisenstein pour les espaces symétriques réductifs: temperance, majorations. Petite matrice B.*, preprint (1994).

[102] P. Delorme and M. Flensted–Jensen, *Towards a Paley–Wiener theorem for semisimple symmetric spaces*, Acta Math. **167** (1991), 127–151.

[103] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthiers-Villars, Paris, 1964.

[104] A. Erdélyi et al, *Higher Transcendental Functions,* Vol 1, McGraw–Hill, New York, 1953.

[105] J. Faraut, *Distributions sphériques sur les espaces hyperboliques*, J. Math. Pures Appl. **58** (1979), 369–444.

[106] M. Flensted-Jensen, *Spherical functions on a real semisimple Lie group. A method of reduction to the complex case*, J. Funct. Anal.

30 (1978), 106–146.

[107] ———, *Discrete series for semisimple symmetric spaces*, Ann. of Math. **111** (1980), 253–311.

[108] ———, *Analysis on Non-Riemannian Symmetric Spaces*, Regional Conference Series in Math. 61, Amer. Math. Soc., Providence, 1986.

[109] M. Flensted-Jensen, T. Oshima and H. Schlichtkrull, *Boundedness of certain unitarizable Harish-Chandra modules*, Adv. Stud. Pure Math. **14** (1988), 651–660.

[110] I. M. Gel'fand and M. I. Graev, *Application of the method of horospheres to the spectral analysis of functions in ordinary and in imaginary Lobachevskian spaces (Russian)*, Tr. Mosk. Mat. Obshch. **11** (1962), 243–308.

[111] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* Vol 1, Academic Press, New York, London, 1964.

[112] R. Godement, *Représentations induites des groupes de Lie*, Séminaire Bourbaki, Exposées **126, 131** (1956).

[113] P. Harinck, *Fonctions généralisées spheriques sur $G_{\mathbb{C}}/G_{\mathbb{R}}$*, Ann. Sci. Éc. Norm. Sup. **4, 23** (1990), 1–38.

[114] ———, *Fonctions généralisées spheriques induites sur $G_{\mathbb{C}}/G_{\mathbb{R}}$ et applications*, J. Funct. Anal. **103** (1992), 104–127.

[115] Harish-Chandra, *Plancherel formula for the 2×2 real unimodular group*, Proc. Nat. Acad. Sci. USA **38** (1952), 337–342.

[116] ———, *Spherical functions on a semisimple Lie group, I-II*, Amer. J. Math. **80** (1958), 241–310, 553–613.

[117] ———, *Harmonic analysis on real reductive groups, I. The theory of the constant term*, J. Funct. Anal. **19** (1975), 104–204; *II. Wave packets in the Schwartz space*, Invent. Math. **36** (1976), 1–55; *III. The Maass–Selberg relations*, Ann. of Math. **104** (1976), 117–201.

[118] G. Heckman, *Are K-invariant Eisenstein integrals for G/H hypergeometric functions?*, Part III of this book.

[119] S. Helgason, *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Math. **86** (1964), 565–601.

[120] ———, *A duality for symmetric spaces with applications to group representations*, Adv. in Math. **5** (1970), 1–154.

[121] ———, *The surjectivity of invariant differential operators on symmetric spaces I*, Ann. of Math. **98** (1973), 451–479.

[122] ——, *Invariant differential equations on homogeneous manifolds*, Bull. Amer. Math. Soc. **83** (1977), 751–774.

[123] ——, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, San Francisco, London, 1978.

[124] ——, *Groups and Geometric Analysis*, Academic Press, Orlando, 1984.

[125] ——, *Some results on invariant differential operators on symmetric spaces*, Amer. J. Math. **114** (1992), 789–811.

[126] ——, *Geometric Analysis on Symmetric Spaces*, American Mathematical Society, Providence, 1994.

[127] J. Hilgert, G. Ólafsson and B. Ørsted, *Hardy spaces on affine symmetric spaces*, J. Reine Angew. Math. **415** (1991), 189–218.

[128] K. Hiraoka, S. Matsumoto and K. Okamoto, *Eigenfunctions of the Laplacian on a real hyperboloid of one sheet*, Hiroshima Math. J. **7** (1977), 855–864.

[129] L. Hörmander, *Linear Partial Differential Operators*, Springer Verlag, Berlin, 1963.

[130] A. W. Knapp, *Representation theory of semisimple Lie groups*, Princeton University Press, Princeton, New Jersey, 1986.

[131] S. Kobayashi and K. Numizu, *Foundations of Differential Geometry*, Vol. II, Interscience, 1969.

[132] T. Koornwinder, *Jacobi functions and analysis on noncompact semisimple groups*, Special Functions: Group Theoretical Aspects and Applications, Reidel, 1984, pp. 1–84.

[133] B. Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. École Norm. Sup. **6** (1973), 413–455.

[134] M. T. Kosters, *Spherical distributions on rank one symmetric spaces*, Thesis, Univ. of Leiden (1983).

[135] M. T. Kosters and G. van Dijk, *Spherical distributions on the pseudo-Riemannian space $SL(n,R)/GL(n-1,R)$*, J. Funct. Anal. **68** (1985), 168–213.

[136] W. A. Kosters, *Harmonic analysis on symmetric spaces*, Thesis, Univ. of Leiden (1985).

[137] J. Lepowsky, *On the Harish-Chandra homomorphism*, Trans. Amer. Math. Soc. **208** (1975), 193–218.

[138] N. Limić, J. Niederle and R. Rączka, *Eigenfunction expansions associated with the second order invariant operator on hyperboloids and*

cones, III, J. Math. Phys. **8** (1967), 1079–1093.

[139] O. Loos, *Symmetric Spaces. I: General theory*, Benjamin, New York, Amsterdam, 1969.

[140] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.

[141] ———, *A description of discrete series for semisimple symmetric spaces II*, Adv. Studies in Pure Math **14** (1988), 531–540.

[142] ———, *Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups*, Adv. Studies in Pure Math **14** (1988), 541–559.

[143] S. Matsumoto, *The Plancherel formula for a pseudo-Riemannian symmetric space*, Hiroshima Math. J. **8** (1978), 181–193.

[144] V. F. Molchanov, *Harmonic analysis on a hyperboloid of one sheet*, Soviet Math. Dokl. **7** (1966), 1553–1556.

[145] ———, *Analogue of the Plancherel formula for hyperboloids*, Soviet Math. Dokl. **9** (1968), 1382–1385.

[146] ———, *Representations of pseudo-orthogonal groups associated with a cone*, Math. USSR Sb. **10** (1970), 333–347.

[147] ———, *The Plancherel formula for hyperboloids*, Proc. Stekl. Inst. Math. (1981), 63–83.

[148] ———, *The Plancherel formula for the tangent bundle of a projective space*, Soviet Math. Dokl. **24** (1981), 393–396.

[149] ———, *Plancherel formula for the pseudo-Riemannian space* $\mathrm{SL}(3,R)/\mathrm{GL}(2,R)$, Sib. Math. J. **23** (1983), 703–711.

[150] ———, *Harmonic analysis on pseudo-Riemannian symmetric spaces of the group $SL(2,R)$*, Math. USSR Sbornik **46** (1983), 493–506.

[151] ———, *Plancherel formula for pseudoriemannian symmetric spaces of the universal cover of SL(2,R)*, Sib. Math. J. **25** (1984), 903–917.

[152] ———, *Plancherel's formula for pseudo-riemannian symmetric spaces of rank 1*, Soviet Math. Dokl. **34** (1987), 323–326.

[153] G. D. Mostow, *Some new decomposition theorems for semisimple Lie groups*, Mem. Amer. Math. Soc. **14** (1955), 31–54.

[154] ———, *Continuous Groups*, Encyclopedia Britannica (1967).

[155] K.-H. Neeb, *Convexity theorems in harmonic analysis*, Semin. Sophus Lie **1** (1991), 143–151.

[156] E. Nelson, *Analytic vectors*, Ann. of Math. **70** (1959), 572–615.

[157] G. Ólafsson, *Fourier and Poisson transformation associated to a semisimple symmetric space*, Invent. Math. **90** (1987), 605–629.

[158] ———, *Causal symmetric spaces*, *Habilitationsschrift*, Univ. of Göttingen, 1990.

[159] ———, *Symmetric spaces of Hermitian type*, Diff. Geom. Appl. 1 (1991), 195-233.

[160] G. Ólafsson and B. Ørsted, *The holomorphic discrete series for affine symmetric spaces. I*, J. Funct. Anal. **81** (1988), 126–159.

[161] ———, *The holomorphic discrete series of an affine symmetric space and representations with reproducing kernels*, Trans. Am. Math. Soc. **326** (1991), 385–405.

[162] T. Oshima, *Poisson transformations on affine symmetric spaces*, Proc. Japan. Acad., A **55** (1979), 323–327.

[163] ———, *Asymptotic behavior of spherical functions on semisimple symmetric spaces*, Adv. Stud. Pure Math. **14** (1988), 561–601.

[164] ———, *A realization of semisimple symmetric spaces and construction of boundary value maps*, Adv. Stud. Pure Math. **14** (1988), 603–650.

[165] ———, *A method of harmonic analysis on semisimple symmetric spaces*, in M. Kashiwara and T. Kawai (eds.), Algebraic Analysis. Papers Dedicated to Professor Mikio Sato on the Occasion of his Sixtieth Birthday, Vol. 2, Academic Press, Boston.

[166] T. Oshima and T. Matsuki, *A description of discrete series for semisimple symmetric spaces*, Adv. Stud. Pure Math. **4** (1984), 331–390.

[167] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on a semisimple symmetric space*, Invent. Math. **57** (1980), 1–81.

[168] ———, *The restricted root system of a semisimple symmetric pair*, Adv. Stud. Pure Math. **4** (1984), 433–497.

[169] R. Penney, *Abstract Plancherel theorems and a Frobenius reciprocity theorem*, J. Funct. Anal. **18** (1975), 177–190.

[170] W. Rossmann, *Analysis on real hyperbolic spaces*, J. Funct. Anal. **30** (1978), 448–477.

[171] ———, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157–180.

[172] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[173] Z. Rudnick and H. Schlichtkrull, *Decay of eigenfunctions on semisimple symmetric spaces*, Duke Math. J. **64** (1991), 445–450.

[174] S. Sano, *Invariant spherical distributions and the Fourier inversion formula on $GL(n,\mathbb{C})/GL(n,\mathbb{R})$*, J. Math. Soc. Japan **36** (1984), 191–219.

[175] S. Sano and J. Sekiguchi, *The Plancherel formula for $SL(2,\mathbb{C})/SL(2,\mathbb{R})$*, Sci. Pap. Coll. Gen. Educ., Univ. Tokyo **30** (1980), 93–105.

[176] H. Schlichtkrull, *The Langlands parameters of Flensted-Jensen's discrete series for semisimple symmetric spaces*, J. Funct. Anal. **50** (1983), 133–150.

[177] ———, *Hyperfunctions and Harmonic Analysis on Symmetric Spaces*, Birkhäuser, Boston, 1984.

[178] ———, *Eigenspaces of the Laplacian on hyperbolic spaces: Composition series and integral transforms*, J. Funct. Anal. **70** (1987), 194–219.

[179] J. Sekiguchi, *Eigenspaces of the Laplace-Beltrami operator on a hyperboloid*, Nagoya Math. J. **79** (1980), 151–185.

[180] ———, *Fundamental groups of semisimple symmetric spaces*, Adv. Stud. Pure Math. **14** (1988), 519–529.

[181] T. Shintani, *On the decomposition of regular representation of the Lorentz group on a hyperboloid of one sheet*, Proc. Japan Acad. **43** (1967), 1–5.

[182] R. S. Strichartz, *Harmonic analysis on hyperboloids*, J. Funct. Anal. **12** (1973), 341–383.

[183] Y. L. Tong and S.P. Wang, *Geometric realization of discrete series for semisimple symmetric spaces*, Invent. Math. **96** (1989), 425–458.

[184] G. van Dijk, *On a class of generalized Gelfand Pairs*, Math. Z. **193** (1986), 581–593.

[185] G. van Dijk and M. Poel, *The Plancherel formula for the pseudo-Riemannian space* $\mathrm{SL}(n,\mathbf{R})/\mathrm{GL}(n-1,\mathbf{R})$, Compos. Math. **58** (1986), 371–397.

[186] ———, *The irreducible unitary $GL(n-1,\mathbf{R})$-spherical representations of $SL(n,\mathbf{R})$*, Compos. Math. **73** (1990), 1–30.

[187] V. S. Varadarajan, *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics 576, Springer Verlag, Berlin-Heidelberg-New York, 1977.

[188] D. Vogan, *Irreducibility of discrete series representations for semisimple symmetric spaces*, Adv. Studies in Pure Math. **14** (1988), 191–221.

[189] G. Warner, *Harmonic Analysis on Semisimple Lie Groups,* Vol. I-II, Springer Verlag, Berlin-Heidelberg-New York, 1972.

Part III

[190] E.P. van den Ban, *The principal series for a reductive symmetric space I. H-fixed distribution vectors*, Ann. Sci. Éc. Norm. Sup. **4, 21** (1988), 359–412.

[191] E.P. van den Ban, *The principal series for a reductive symmetric space II. Eisenstein integrals*, J. Funct. Anal. **109** (1992), 331–441.

[192] E.P. van den Ban and H. Schlichtkrull, *The most continuous part of the Plancherel decomposition for a reductive symmetric space*, preprint (to appear).

[193] R.J. Beerends, *A transmutation property of the generalized Abel transform associated with the root system A_2*, Indag. Math. N.S. **1** (1990), 155–168.

[194] N. Bourbaki, *Groupes et Algébres de Lie, Ch. 4, 5 et 6*, Masson, 1981.

[195] R. Brusse, G.J. Heckman, E.M. Opdam, *Variation on a theme of Macdonald*, Math. Z. **208** (1991), 1–10.

[196] A. Erdélyi, *Higher Transcendental Functions, Vol. 1*, Krieger Publ. Co., Florida, 1985.

[197] G.J. Heckman, *Root systems and hypergeometric functions II*, Comp. Math. **64** (1987), 353–373.

[198] G.J. Heckman, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350.

[199] G.J. Heckman, *Lectures on hypergeometric and spherical functions, Notes for the European school of group theory, Luminy 1991 (= Part 1 of this book).*

[200] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.

[201] B. Hoogenboom, *Intertwining functions on compact Lie groups*, CWI tract 5, Amsterdam, 1984.

[202] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. AMS **277** (1983), 623-653.

[203] I.G. Macdonald, *Spherical functions on a group of p-adic type*, Ramanujan Institute Publications, 1971.

[204] H. Matsumoto, *Analyse harmoniques dans les systèmes de Tits bornologiques de type affine*, Lect. Notes Math. Vol. 590, 1977.

[205] E.M. Opdam, *Root systems and hypergeometric functions IV*, Comp. Math. **67** (1988), 191–209.

[206] E.M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1–18.

[207] E.M. Opdam, *An analogue of the Gauss summation formula for hypergeometric functions related to root systems*, Math. Z. **212** (1993), 313-336.

[208] E.M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Preprint (1993).

[209] H. Schlichtkrull, *Harmonic Analysis on Semisimple Symmetric Spaces, Notes for the European school of group theory, Twente, 1992 (= Part 2 of this book).*

[210] N. Shimeno, *The Plancherel formula for spherical functions with one-dimensional K-type on a simply connected simple Lie group of hermitian type*, Preprint (1992).

[211] L. Vretare, *On a recurrence formula for elementary spherical functions on symmetric spaces and its applications*, Math. Scand. **41** (1977), 99–112.

Index

This Page Intentionally Left Blank