Lecture Notes in Statistics 216

Corinne Berzin
Alain Latour
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Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion

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To the memory of Mario Wschebor, our friend and outstanding mathematician (1939–2011)

Foreword

This book is devoted to some stochastic models that present scale invariance. It is structured around three issues: probabilistic properties, statistical estimation and simulation of processes and estimators. The interested reader can be either a specialist of probability, who will find here a friendly presentation of statistics tools, or a statistician, who will have the occasion to tackle the most recent theories in probability in order to develop central limit theorems in this context. Both will certainly be interested in the last part on simulation, which, to my knowledge, is highly original. Algorithms are described in great detail, with concern of procedures that is not usually seen in mathematical treaties. The theoretical part is also partly original and finds its origin in previous work of the first and third authors, which they improve and extend here.

Models under study are fractional Brownian motions (fBm) and processes that derive from them through stochastic differential equations. Their use for modeling financial markets is by now well established in the presence of long-range dependence: fBm with parameter H larger than $\frac{1}{2}$ may then be preferred to Brownian motion. Other applications are not as standard as this first one, even if some of them, such as the description of network traffic, have played a central role in the development of the theory, as well as in its extension to multifractal analysis. The diversity of applications will certainly develop with time. I had the pleasure to work separately with the first and third authors on questions that arose from the description of bone micro-architecture. The tissue of a bone may be seen as a porous medium with some scale invariance, and these models by fBm have been tested by different authors. Fractional Brownian motions can also be used to model environmental phenomena, such as, for instance, diffusion of pollution in a lake or a river. It is certainly difficult to predict which application will prove to be really efficient for practical issues, but one has a strong motivation to develop theoretical studies and statistical tools in order to identify parameters, which is done here.

Even if this book does not directly deal with applications, it could not have been written if the authors did not have some of them in mind. Since the choice of terms can be an indicator of their preferences, it looks significant that they call “local variance” what is usually called “volatility”, by reference to the financial market!

They borrow this denomination from Mario Wschebor, who passed away recently and marked deeply the first and third authors through longstanding collaboration. The influence of Wschebor can in particular be felt through the presence of harmonic analysis all along the book: the authors use what they call mollified versions of processes and would be called filtered versions in signal processing. Such approximations of processes are central in the work of Wschebor.

Most fields of mathematics are, more than ever these days, the object of collective work. A huge number of mathematicians have contributed to the study of fractal Brownian motions, stochastic integrals and properties of estimators of parameters. It is one of the merits of the present book to rely on this very rich literature but guide the readers in such a way that issues and proofs are easily accessible. They let them profit from their intuition and vision but refer also carefully to previous work. It is impossible to enumerate here all contributors, but two major scientific figures, who also died recently, deserve to be mentioned first and foremost: Benoît Mandelbrot and Paul Malliavin. Even if they were opponents in many aspects of scientific life, and specifically concerning the use of Brownian motion in finance, their contributions add up for the greatest happiness of specialists of such random processes.

Everyone now agrees on the fact that Mandelbrot taught us how to see fractal patterns everywhere in natural objects. FBm was first introduced by Kolmogorov, but Mandelbrot is really the first to have seen how it connected with fractal analysis and could be used as a model for different kinds of phenomena. Recall that roughly speaking the fractional Brownian b_H has the property that

$$|b_H(t + \Delta t) - b_H(t)| \simeq (\Delta t)^H.$$

The parameter H, which lies between 0 and 1, is called the Hurst exponent in honor of the physicist Harold E. Hurst. The latter, who, as a hydrologist, studied the flow of Nile during the first half of the last century, remarked that the rescaled difference between the maximum and the minimum values of this flow during a length of time T behaves like T^H, with H approximately 0.7. Mandelbrot has described in detail how he got interested in the discovery of Hurst, and was led to the definition of the fBm, in his book "Fractales, hasard et finance". Even if his interest for finance was already present, environmental issues have been clearly evoked from the beginning to justify the use of fBm as a model.

Malliavin, who contributed deeply to analysis and probability, introduced powerful theoretical tools for the study of functionals of Gaussian processes as part of what is called, "Malliavin calculus". They may be very useful to establish central limit theorems and are indirectly used in this book through the "Fourth Moment Theorem", which is due to Nualart and Peccati (2005) and has led to a considerable literature. Roughly speaking, for functionals that belong to some Wiener chaos (in particular for quadratic variations, which are currently used in signal processing), the consideration of fourth moments is sufficient to prove central limit theorems. This is systematically used in this book, while it not so well known from statisticians. It leads the authors to rapid and nice proofs, which deserve to serve as models for the future.

Last but not least, the reader will also find elegant and new proofs for almost sure convergence. This is only one example of the many contributions of the authors (others concern the back and forth between discrete and continuous models, for instance) that he/she will discover all along this book. Not to mention again the last part, whose approach is likely to change practices in computational statistics. But now it is time to start reading. As a conclusion let me say what a pleasure it is for me to recommend this.

Orléans, France
November 23, 2013

Aline Bonami

Reference

Nualart, D., & Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability, 33*(1), 177–193.

Preface

The use of diffusion models driven by fractional noise has become popular for more than two decades. The reasons that produced this situation have been varied in nature. We can mention, among others, those that come from mathematics and other from the applications.

With respect to the first group, it should be noted that fractional Brownian motion (fBm) has interesting properties. First, it is self-similar. This property implies that such a process is, from the standpoint of its distribution, invariant with respect to scale transformations. Moreover the fractional noise, the process of increments of the fBm taking in a mesh of equally spaced points, satisfies a strong dependence condition that is a notion away from independence and mixing. Using this last property, it has been possible to model natural phenomena, which exhibit temporal correlations tending to zero so slowly that their sum tends to infinity.

With regard to the applications, we should mention that fractional models have become popular for modeling real-life events such as the value assets in financial markets, models of chaos in quantum physics, river flow along the time, irregular images, weather events and contaminant transport problems, among others.

The fBm is a mean zero Gaussian process with stationary increments and whose covariance function is uniquely determined by the Hurst's parameter, which we denote by H and that is between zero and one. The value $H = \frac{1}{2}$ is important because the associated process results in the Brownian motion (Bm). The parameter H determines the smoothness of the fBm trajectories. More regular are the trajectories as closer to one is the parameter. The exact opposite happening if H is near zero.

In the forties and fifties of the twentieth century, in the study of Bm, the introduction of the stochastic integral by Kiyosi Itô and Paul Levy was the key to the definition of diffusion processes. This important event led to the development of a whole area of probability and mathematics. Similarly, the introduction of several definitions of stochastic integrals with respect to fBm, from the 1990s, has led to the definition of pseudo-diffusion processes driven by this noise. As in the case of a Bm, the introduction of these processes significantly enriched the theory and the horizon of their applications.

In these notes we develop estimation techniques for the parameter H and the local variance (volatility) of the pseudo-diffusion. The estimation of the two parameters is made simultaneously. We will use the observation of the process in a discrete mesh of points. Then the study of the asymptotic properties will be done when the mesh's norm tends to zero.

We start by defining the second order increments of the process. Using these increments, we build the order p variations. These variations allow the definition of an estimator of the parameter H for all its range. The reason to work with second order increments, instead of the first order increments, is that variations built through them are asymptotically Gaussian in all the range $0 < H < 1$, instead of $0 < H < \frac{3}{4}$ that is the case for the variations constructed by using first order increments. From the asymptotic normality of the variations, we deduce that the estimators of H are asymptotically Gaussian, for all their possible values.

After estimating the Hurst parameter, we study the local variance estimation in four pseudo-diffusion models. For each of them, we construct a local variance estimator and study its asymptotic normality. If we do not know in advance the value of the Hurst parameter, this procedure will reduce the rate of convergence in the central limit theorem (CLT) for the estimator of the variance.

Then we assume H known and try to estimate local variance functionals for more general models. For instance in the case where the variance is not constant, for this estimation procedure we will recover the lost speed, noted in the previous paragraph.

Finally, one of the main purposes of these notes is to provide a set of tools for computational statistics: efficient simulation of the processes, assessment of the goodness of fit of the estimators and the selection of the best estimator in each of the presented situations.

We will develop this program once the asymptotic properties of estimators have been studied. We will discuss simulations and their computer implementations as well as some of the codes developed.

We note that the study of the asymptotic normality of the estimators we construct has been dramatically simplified using the techniques of the CLT for nonlinear functional Gaussian processes. These new techniques have been developed since 2005, from the seminal article of D. Nualart and G. Peccati by various authors. We can mention some of them: O.E. Barndorff-Nielsen, H. Biermé, A. Bonami, J.M. Corcuera, M. Kratz, C. Ludeña, I. Nourdin, S. Ortiz-Latorre, M. Podolskij, M. Taqqu and C. Tudor, and the two authors mentioned above.

The notes are aimed at a mixed audience. They can be used in a graduate course in statistics of Gaussian processes, as well as a reference book for researchers in the field and as a guide for those interested in the applications of fractional models.

The book has eight chapters. Chapter 1 contains the motivation for the study that we will realize in the text. It begins pointing out two types of research problems in the estimation of Brownian diffusions. In first place one considers the situation when the Brownian trajectory is observed smoothed by a convolution filter, tending to the Dirac's delta distribution when some specific parameter tends to zero. A CLT for the increments of Brownian motion is established. This last theorem has as a consequence a stable CLT for the quadratic variation of a general diffusion. Then we sketch the same type of study when the process is observed in a discrete mesh of points. The chapter ends considering the convergence of the number of crossings for the smoothed fBm towards the local time of this last process, when the smoothing parameter tends to zero. We should point out the relationship between this result and the theorem proved for the quadratic variation of a diffusion.

In Chap. 2, we introduce the basic tools that we will use. We define: the fBm, Hermite's polynomials and the complex Itô-Wiener chaos. Also we give some preliminaries about the stochastic integration with respect to the fBm and the chapter concludes with the hypothesis and notations that we will use in what follows.

Chapter 3 contains the statements and demonstrations of some of the main theoretical results. The different estimators of H are defined, studying after their asymptotic properties. Then the local variance estimator is introduced, and the simultaneous estimation of H and of the local variance is considered. Some tests of hypothesis are defined and the asymptotic behavior of the test function is obtained, both under the null hypothesis and under contiguous alternatives. The chapter ends studying the estimator of a functional of the local variance.

Chapter 4 presents a deep study by simulation to evaluate the performance of the estimators and the tests. First, we give some information about the computing environment and random generators. Afterwards the Durbin-Levinson algorithm is implemented to efficiently simulate the fBm. Finally in some subsections, we explore the goodness of fit of the estimators and the quality of the hypothesis tests.

Chapter 5 contains the proofs of the results of Chap. 3. In Chap. 6, there are some complementary results. Chapter 7 shows using tables and graphs the results of the simulation experiments of Chap. 4. Chapter 8 includes some Pascal procedures and function used in Chap. 4.

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Contents

Acronyms

Bm	Brownian motion
fBm	fractional Brownian motion
fBn	fractional Brownian noise
SDE	Stochastic differential equation
CLT	Central limit theorem
ODE	Ordinary differential equation
i.i.d.	Independent identically distributed

Notations

Throughout the document, we use the following notations:

$\hat{f}$	:	Fourier transform of function f
$f * g$	:	convolution of functions f and g
$f^{*(\ell)}$	:	ℓ-th convolution of function f with itself
$\int^x f(u)\,\mathrm{d}u$	:	a primitive of f
$\mathbb{1}_A$	:	characteristic function of set A
$\#A$	:	cardinality of set A
$\boldsymbol{C}$	:	a generic constant; its value may change during a proof
$\boldsymbol{C}(\omega)$	:	a generic constant depending on ω, a trajectory; its value may change during a proof
$\mathbb{N}$	:	$\{x \in \mathbb{Z} : x \geqslant 0\}$
$\mathbb{N}^*$	:	$\{x \in \mathbb{Z} : x > 0\}$
$\mathbb{R}^*$	:	$\{x \in \mathbb{R} : x \neq 0\}$
$\mathbb{R}^{+*}$	:	$\{x \in \mathbb{R} : x > 0\}$
log	:	Naperian logarithm
$\mathrm{E}[X]$	:	expected value of the random variable X
$\mathcal{N}(\mu, \sigma^2)$	:	the normal distribution with mean μ and variance σ^2
N	:	standard Gaussian random variable
$\lVert N \rVert_k^k$	:	for real $k > 0$, we note $\lVert N \rVert_k^k = \mathrm{E}[\lvert N \rvert^k]$
$\lVert C(\cdot) \rVert_k^k$	:	for real $k \geqslant 1$, denotes the integral $\int_0^1 \lvert C(u) \rvert^k \,\mathrm{d}u$ for a measurable function C
$\boldsymbol{Y}^\top$	:	transpose of vector $\boldsymbol{Y}$
$L^1(\mathbb{R})$	:	complex absolutely integrable functions with respect to Lebesgue measure on $\mathbb{R}$
$L^2(\mathbb{R}^k)$	:	complex square-integrable functions with respect to Lebesgue measure on $\mathbb{R}^k$

$L^{\infty}(\mathbb{R})$	:	essentially bounded complex functions with respect to Lebesgue measure on $\mathbb{R}$
$L^2_e(\mathbb{R}^k)$	:	subspace of $L^2(\mathbb{R}^k)$ made of complex-valued function ψ such that $\psi(-x) = \overline{\psi(x)}$, $\forall x \in \mathbb{R}^k$
$\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$	:	scalar product in $L^2(\mathbb{R})$
$\\|\cdot\\|_2$	:	the norm induced by the scalar product in $L^2(\mathbb{R})$
$\\|\cdot\\|_{\infty}$	:	the norm associated with $L^{\infty}(\mathbb{R})$ space
$L^2_s(\mathbb{R}^k)$	:	the subspace of symmetrical functions of $L^2_e(\mathbb{R}^k)$
$L^k(\Omega, P)$ or $L^k(\Omega)$	:	functions from Ω in $\mathbb{R}$ k-integrable with respect to the underlying probability P
$L^2(P^H)$	:	L^2 space for the probability measure generated by the fBm
$H(W)$	:	the subspace of random variables in $L^2(\Omega)$ measurable with respect to the Brownian motion W
$\mathcal{H}_0$	:	the space of real constants
H_p	:	Hermite polynomials
$\phi(x)\,\mathrm{d}x$	:	standard Gaussian measure
$L^p(\phi(x)\,\mathrm{d}x)$	:	functions from $\mathbb{R}$ to $\mathbb{R}$ p-integrable with respect to the standard Gaussian measure ($p \in \mathbb{R}^{+*}$)
$\lfloor x \rfloor$	:	the integer part of the positive real number x
λ	:	the Lebesgue measure
$(2n-1)!!$	:	product of all odd positive integers up to $2n-1$, i.e. $\prod_{j=1}^{n}(2j-1)$
C^k	:	k-times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$
$\dot{h}, \ddot{h}$	:	first and second derivatives of h
$\overset{k\cdot}{h}$	:	kth derivative of h, $k \in \mathbb{N}$
$C([0,1], \mathbb{R})$	:	the space of continuous real functions from $[0, 1]$ to $\mathbb{R}$
$\overset{Law}{=}$	:	equality in law
$\xrightarrow[n\to\infty]{\text{Law}}$	:	convergence in law as $n \to \infty$
$\xrightarrow[n\to\infty]{\text{P}}$	:	convergence in probability as $n \to \infty$
$\xrightarrow[n\to\infty]{\text{a.s.}}$	:	almost-sure convergence as $n \to \infty$
$\xrightarrow[\varepsilon\to 0]{\text{Law}}$	:	convergence in law as $\varepsilon \to 0$
$\xrightarrow[\varepsilon\to 0]{\text{P}}$	:	convergence in probability as $\varepsilon \to 0$
$\xrightarrow[\varepsilon\to 0]{\text{a.s.}}$	:	almost-sure convergence as $\varepsilon \to 0$
$\xrightarrow[n\to\infty]{L^2(P^H)}$	:	convergence in $L^2(P^H)$ as $n \to \infty$

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Chapter 1
Introduction

1.1 Motivation

The book is mainly addressed to study nonparametric estimation for fractional diffusions. We can defined these processes as the solution of the following stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \sigma(X(t))\ \mathrm{d}b_H(t) + b(X(t))\ \mathrm{d}t,$$

where σ and b are functions smooth enough to ensure existence and uniqueness of the process, b_H is a fractional Brownian motion (fBm) with Hurst parameter H, see Sect. 2.2.1, page 30, for its definition. We will only consider solutions of these equations whenever $H \geqslant 1/2$, the case $H = 1/2$ corresponds to Brownian diffusions.

The framework for the statistical inference is here the infill one, that means that we use the observations taken in a fixed interval, refining the mesh.

In the literature two types of infill estimation have been considered. The first one, studied in these notes, consists in observing the process in a regular mesh (with step equal to $\frac{k}{n}$), i.e. $\{X(\frac{k}{n})\}_{k=1}^{n}$, the asymptotic considered in that case is when $n \to \infty$, or as the step tends to 0. The second one consists in observing a mollified version of the process $X^{\varepsilon}(t) = \varphi_{\varepsilon} * X(t)$, where $*$ stands for the convolution product and where $\varphi_{\varepsilon}(\cdot) = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$, φ being a smooth probability density function. In that case, the considered asymptotic is when $\varepsilon \to 0$.

The inference is directed to look for estimators of the Hurst parameter H and of the local variance $\sigma(x)$. The estimation of the drift function b usually requires that the underlying process $X(t)$ is ergodic and moreover that the estimation takes place in an infinite interval framework.

The following sections aim to give to the reader some insight about the types of results we attend. First, we consider for sake of completeness in first place the case of Brownian diffusions and mollified observations. Then, in the penultimate section,

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we will compare this method with the case where the observations are given in a uniform mesh, establishing similarities and differences between the two procedures.

To build our estimators, we need Central Limit Theorems (CLT) and this is how we begin our study.

1.2 CLT for Non-linear Functionals of Gaussian Processes

The article of Breuer and Major (1983) is considered now as an important classic work. The authors proved a CLT for non-linear functionals of a stationary Gaussian process $\{X(t)\}_{t\in\mathbb{R}+}$. They considered occupation functionals of the form

$$T_t = \int_0^t F(X(s))\, \mathrm{d}s,$$

for some function F, belonging to $L^2(\phi(x)\,\mathrm{d}x)$, where $\phi(x)\,\mathrm{d}x$ stands for the standard Gaussian measure. This result was extended in Chambers and Slud (1989) to general functionals into the Itô-Wiener chaos. This last work allows getting CLT for functionals that depend on an infinite number of coordinates. However, their method requires the existence of the spectral density of the process $X(t)$.

1.3 Main Result

Let $\{X(t)\}_{t\in\mathbb{R}+}$ be a zero mean stationary Gaussian process with covariance $r(t) = \mathrm{E}[X(t)X(0)]$, such that $r(0) = 1$. We assume also that X has a spectral density f in $L^1(\mathbb{R})$.

The process X has the following spectral representation:

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}(f(\lambda))^{1/2}\,\mathrm{d}W(\lambda), \tag{1.1}$$

where W is a complex centered Gaussian random measure on $\mathbb{R}$ with Lebesgue control measure, such that $W(-A) = \overline{W(A)}$ a.s. for any Borel set A of $\mathbb{R}$. Moreover, let φ be an even continuous function on a bounded support included in $[-\frac{1}{2}, \frac{1}{2}]$.

Let us define $\psi(x) = \varphi*\varphi(x)$, with support in $[-1, 1]$. We suppose that the norm $L^2(\mathbb{R})$ of φ is equal to one. Then $\psi(0) = \frac{1}{2\pi}\|\hat{\varphi}\|_2^2 = \|\varphi\|_2^2 = 1$. Let us introduce the approximated stationary Gaussian process

$$X^{\varepsilon}(t) = \int_{-\infty}^{\infty} e^{it\lambda}(f * \hat{\psi}_{\varepsilon}(\lambda))^{1/2}\,\mathrm{d}W(\lambda),$$

where $\hat{\psi}_\varepsilon(\lambda) = \frac{1}{2\pi\varepsilon}\left|\hat{\varphi}(\frac{\lambda}{\varepsilon})\right|^2$, and $\varepsilon > 0$. The covariance function of $X^\varepsilon(t)$ is $r_\varepsilon(t) = r(t)\psi(\varepsilon t)$.

In the following, we use Hermite polynomials, denoted by H_p. We have:

$$\exp(tx - \tfrac{1}{2}t^2) = \sum_{p=0}^{+\infty} \frac{H_p(x)t^p}{p!}.$$

Hermite polynomials form an orthogonal system for the standard Gaussian measure $\phi(x)\,\mathrm{d}x$. If $h \in L^2(\phi(x)\,\mathrm{d}x)$ then there exist coefficients h_p such that $h(x) = \sum_{p=0}^{+\infty} h_p H_p(x)$.

Mehler's formula (see Breuer and Major 1983) gives a simple form to compute the covariance between two L^2 functions of Gaussian random variables. In fact, if $k \in L^2(\phi(x)\,\mathrm{d}x)$ and is written as $k(x) = \sum\limits_{p=0}^{+\infty} k_p H_p(x)$ and if (X, Y) is a Gaussian random vector with correlation ρ and unit variance then

$$\mathrm{E}\,[h(X)k(Y)] = \sum_{p=0}^{+\infty} h_p k_p p!\rho^p.$$

We obtain by using the Mehler's formula Proposition 1.1.

Proposition 1.1. *Let $\ell \in \mathbb{N}^*$. If $r^\ell \in L^1(\mathbb{R})$ then*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left[\frac{1}{\sqrt{t}}\int_0^t \{H_\ell(X(s)) - H_\ell(X^\varepsilon(s))\}\,\mathrm{d}s\right]^2 = 0.$$

Proof. Note that if $r^\ell \in L^1(\mathbb{R})$ and if $f^{*(\ell)}$ denotes the ℓ-order convolution of f with itself, the inversion formula for the Fourier Transform implies that $f^{*(\ell)}$ is bounded and continuous.

By using Mehler's formula we get

$$\begin{aligned}
&\mathrm{E}\left[\frac{1}{\sqrt{t}}\int_0^t \{H_\ell(X(s)) - H_\ell(X^\varepsilon(s))\}\,\mathrm{d}s\right]^2 \\
&= 2\ell!\int_0^t \left(1 - \tfrac{s}{t}\right)\left(r^\ell(s) + r_\varepsilon^\ell(s) - 2\rho_\varepsilon^\ell(s)\right)\mathrm{d}s \\
&= 2\ell!\left[\int_0^t \left(1 - \tfrac{s}{t}\right)\left(r_\varepsilon^\ell(s) - r^\ell(s)\right)\mathrm{d}s + 2\int_0^t \left(1 - \tfrac{s}{t}\right)\left(r^\ell(s) - \rho_\varepsilon^\ell(s)\right)\mathrm{d}s\right],
\end{aligned}$$

where $\rho_\varepsilon(s) = \mathrm{E}\,[X(0)X^\varepsilon(s)]$. Let us study each term separately. For the first, we have $|r_\varepsilon(s)|^\ell \leqslant |r(s)|^\ell$.

By the dominated convergence theorem and by using the fact that r^ℓ is integrable, we get that

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\int_0^t \left(1-\tfrac{s}{t}\right)(r_\varepsilon^\ell(s)-r^\ell(s))\,\mathrm{d}s = \lim_{\varepsilon\to 0}\int_0^\infty (r_\varepsilon^\ell(s)-r^\ell(s))\,\mathrm{d}s = 0.$$

For the second term we have

$$\int_0^t \left(1-\tfrac{s}{t}\right)(r^\ell(s)-\rho_\varepsilon^\ell(s))\,\mathrm{d}s$$
$$= 2\int_0^t \left(1-\tfrac{s}{t}\right)\left(\int_{-\infty}^\infty \cos\lambda s(f^{*(\ell)}(\lambda)-g_\varepsilon^{*(\ell)}(\lambda))\,\mathrm{d}\lambda\right)\mathrm{d}s,$$

where $g_\varepsilon^{*(\ell)}(\lambda)$ is the ℓ-th convolution of the function

$$g_\varepsilon(\lambda) = (f*\hat{\psi}_\varepsilon(\lambda))^{1/2}(f(\lambda))^{1/2},$$

with itself.

Fubini's theorem gives

$$\begin{aligned}
&\int_0^t \left(1-\tfrac{s}{t}\right)\left(\int_{-\infty}^\infty \cos\lambda s(f^{*(\ell)}(\lambda)-g_\varepsilon^{*(\ell)}(\lambda))\,\mathrm{d}\lambda\right)\mathrm{d}s\\
&\quad= \int_{-\infty}^\infty \frac{1-\cos\lambda t}{t\lambda^2}(f^{*(\ell)}(\lambda)-g_\varepsilon^{*(\ell)}(\lambda))\,\mathrm{d}\lambda\\
&\quad= \int_{-\infty}^\infty \frac{1-\cos\lambda}{\lambda^2}\left(f^{*(\ell)}(\tfrac{\lambda}{t})-g_\varepsilon^{*(\ell)}(\tfrac{\lambda}{t})\right)\mathrm{d}\lambda.
\end{aligned}$$

The case with $\ell=1$ is easy. Since the function f is bounded and continuous, by using

$$g_\varepsilon\left(\tfrac{\lambda}{t}\right) = \left(f*\hat{\psi}_\varepsilon(\tfrac{\lambda}{t})\right)^{1/2}\left(f(\tfrac{\lambda}{t})\right)^{1/2} \to (f*\hat{\psi}_\varepsilon(0))^{1/2}(f(0))^{1/2},$$

when $t\to\infty$, we get

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\int_{-\infty}^\infty \frac{1-\cos\lambda}{\lambda^2}\left(f(\tfrac{\lambda}{t})-g_\varepsilon(\tfrac{\lambda}{t})\right)\mathrm{d}\lambda = 0.$$

For $\ell>1$, we must study the behavior of the function $f^{*(\ell)}(\cdot)-g_\varepsilon^{*(\ell)}(\cdot)$, in a neighborhood of zero. Since $f^{*(\ell)}(\cdot)$ is continuous it holds

$$\lim_{t\to\infty} f^{*(\ell)}(\tfrac{\lambda}{t}) = f^{*(\ell)}(0).$$

Now let us consider the behavior of $g_\varepsilon^{*(\ell)}(\frac{\lambda}{t})$. We have

$$g_\varepsilon^{*(\ell)}(\tfrac{\lambda}{t}) = \int_{-\infty}^{\infty} g_\varepsilon\left(\tfrac{\lambda}{t} - \lambda_1\right) g_\varepsilon^{*(\ell-1)}(\lambda_1)\ \mathrm{d}\lambda_1.$$

But $g_\varepsilon\left(\frac{\lambda}{t} - \cdot\right)$ converges towards $g_\varepsilon(\cdot)$ in $L^1(\mathbb{R})$ when $t \to \infty$; this is a consequence of the continuity of the translation operator in $L^1(\mathbb{R})$. For fixed ε, we have

$$g_\varepsilon^{*(2)}(\lambda) \leqslant \left\| f * \hat{\psi}_\varepsilon \right\|_\infty \int_{-\infty}^{\infty} (f(\lambda - \lambda_1))^{1/2} (f(\lambda_1))^{1/2}\, \mathrm{d}\lambda_1 \leqslant \left\| f * \hat{\psi}_\varepsilon \right\|_\infty,$$

and for $k > 2$,

$$\begin{aligned} g_\varepsilon^{*(k)}(\lambda) &\leqslant \|g_\varepsilon^{*(k-1)}\|_\infty \int_{-\infty}^{\infty} (f * \hat{\psi}_\varepsilon(\lambda))^{1/2} (f(\lambda))^{1/2}\, \mathrm{d}\lambda \leqslant \|g_\varepsilon^{*(k-1)}\|_\infty \\ &\leqslant \left\| f * \hat{\psi}_\varepsilon \right\|_\infty. \end{aligned}$$

The duality between $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ implies $g_\varepsilon^{*(\ell)}(\frac{\lambda}{t}) \xrightarrow[t\to\infty]{} g_\varepsilon^{*(\ell)}(0)$. Since $f^{*(\ell)}(\cdot)$ is bounded, we get

$$\int_0^t \left(1 - \tfrac{s}{t}\right) (r^\ell(s) - \rho_\varepsilon^\ell(s))\, \mathrm{d}s \xrightarrow[t\to\infty]{} (f^{*(\ell)}(0) - g_\varepsilon^{*(\ell)}(0)) 2 \int_0^\infty \frac{1 - \cos\lambda}{\lambda^2}\, \mathrm{d}\lambda.$$

But now we have

$$g_\varepsilon^{*(\ell)}(0) = \int_{-\infty}^{\infty} g_\varepsilon(\lambda_{\ell-1}) g_\varepsilon^{*(\ell-1)}(\lambda_{\ell-1})\ \mathrm{d}\lambda_{\ell-1}.$$

First, by using a subsequence if needed, Fatou's lemma gives

$$\liminf_{\varepsilon\to 0} g_\varepsilon^{*(\ell)}(0) \geqslant f^{*(\ell)}(0). \tag{1.2}$$

Then, we get

$$\begin{aligned} I^\ell &= \int_{-\infty}^{\infty} g_\varepsilon(\lambda_{\ell-1}) g_\varepsilon^{*(\ell-1)}(\lambda_{\ell-1})\ \mathrm{d}\lambda_{\ell-1} \\ &= \int_{\mathbb{R}^{\ell-1}} g_\varepsilon(\lambda_{\ell-1}) g_\varepsilon(\lambda_{\ell-1} - \lambda_{\ell-2}) \cdots g_\varepsilon(\lambda_2 - \lambda_1) g_\varepsilon(\lambda_1)\, \mathbf{d}\boldsymbol{\lambda} \end{aligned}$$

where $\mathbf{d\lambda} = \mathrm{d}\lambda_{\ell-1}\,\mathrm{d}\lambda_{\ell-2}\ldots\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1$. Yielding, using Schwarz's inequality

$$I^\ell \leqslant \left[\int_{\mathbb{R}^{\ell-1}} f*\hat{\psi}_\varepsilon(\lambda_{\ell-1}) f*\hat{\psi}_\varepsilon(\lambda_{\ell-1}-\lambda_{\ell-2})\cdots f*\hat{\psi}_\varepsilon(\lambda_2-\lambda_1) f*\hat{\psi}_\varepsilon(\lambda_1)\,\mathbf{d\lambda}\right]^{1/2}$$
$$\times\left[\int_{\mathbb{R}^{\ell-1}} f(\lambda_{\ell-1}) f(\lambda_{\ell-1}-\lambda_{\ell-2})\cdots f(\lambda_2-\lambda_1) f(\lambda_1)\,\mathbf{d\lambda}\right]^{1/2}.$$

The properties of the convolution entail

$$\begin{aligned} I^\ell &\leqslant [(f*\hat{\psi}_\varepsilon)^{*(\ell)}(0)]^{1/2}[f^{*(\ell)}(0)]^{1/2} \\ &= [(f^{*(\ell)} * \hat{\psi}_\varepsilon^{*(\ell)})(0)]^{1/2}[f^{*(\ell)}(0)]^{1/2} \to f^{*(\ell)}(0), \end{aligned} \tag{1.3}$$

the last line is a consequence of the continuity of $f^{*(\ell)}$. Then, (1.2) and (1.3) allow getting $\lim\limits_{\varepsilon\to 0} g_\varepsilon^{*(\ell)}(0) = f^{*(\ell)}(0)$, and the results of the proposition are established.

Remark 1.2. The results of the Proposition 1.1 were proved in Berman (1992) for $\ell = 2$.

Let F be a function in $L^2(\phi(x)\,\mathrm{d}x)$. It has the following Hermite expansion:

$$F(x) = \sum_{n=0}^{\infty} c_n H_n(x) \quad \text{where} \quad c_n = \frac{1}{n!}\int_{-\infty}^{\infty} F(x)H_n(x)\phi(x)\,\mathrm{d}x.$$

In the last expression H_n is the Hermite polynomial of degree n, see the definition given in Sect. 1.3, page 2. Moreover, the norm of F in $L^2(\phi(x)\,\mathrm{d}x)$ satisfies $\|F\|_2^2 = \sum\limits_{n=0}^{\infty} c_n^2 n!$. See Mehler's formula given in Sect. 1.3, page 2.

We define the Hermite rank of F as the smallest n such that the coefficient c_n is different from zero.

We have the following well-known Breuer and Major (1983) result (see also Chambers and Slud 1989).

Theorem 1.3. *Let us assume that F belongs to $L^2(\phi(x)dx)$, whose Hermite rank is $\ell \geqslant 1$ and suppose also that $r^\ell \in L^1(\mathbb{R})$. Then*

$$S_t = \frac{1}{\sqrt{t}}\int_0^t F(X(s))\,\mathrm{d}s \xrightarrow[t\to\infty]{Law} N(0,\sigma^2(F)),$$

where

$$\sigma^2(F) = 2\sum_{k=\ell}^{\infty} c_k^2 k! \int_0^\infty r^k(s)\,\mathrm{d}s.$$

Remark 1.4. We will give here a proof of this result based on Proposition 1.1 and a CLT for m-dependent process. We give a standard type proof commonly used before the modern and powerful approach based on the Fourth Moment Theorem and developed by Nualart and Peccati (2005) became available. After this standard proof, we therefore propose a new one based on this innovative approach. However, we will only give a sketch as we will later have the opportunity to exploit these new techniques in the proofs of Theorem 3.4 and Lemma 5.10.

Proof. Let us define $F_M(x) = \sum_{n=\ell}^{M} c_n H_n(x)$ and $S_t^M = \frac{1}{\sqrt{t}} \int_0^t F_M(X(s))\,\mathrm{d}s$. The Mehler's formula entails

$$\operatorname{var}\left[S_t - S_t^M\right] = 2 \sum_{k=M+1}^{\infty} c_k^2 k! \int_0^t (1 - \tfrac{s}{t}) r^k(s)\,\mathrm{d}s$$

$$\xrightarrow[t\to\infty]{} 2 \sum_{k=M+1}^{\infty} c_k^2 k! \int_0^\infty r^k(s)\,\mathrm{d}s < \delta \qquad (1.4)$$

if $M > M(\delta)$. Hence we only need to prove the asymptotic normality for S_t^M. Let us introduce the process $S_t^{M\varepsilon} = \dfrac{1}{\sqrt{t}} \displaystyle\int_0^t F_M(X^\varepsilon(s))\,\mathrm{d}s$. Using Proposition 1.1 and recalling that if $r^\ell \in L^1(\mathbb{R})$ then $r^k \in L^1(\mathbb{R})$, $\forall k > \ell$, it yields:

$$\lim_{\varepsilon\to 0} \lim_{t\to\infty} \operatorname{var}\left[S_t^M - S_t^{M\varepsilon}\right] = 0.$$

It only remains to prove that $S_t^{M\varepsilon}$ is asymptotically Gaussian.

A stationary sequence $\{X_t\}_{t\in I}$ ($I = \mathbb{R}^+$ or $\mathbb{N}$) is m-dependent if X_i is independent of X_j whenever $|i - j| > m$. The CLT is a consequence of Hœffding and Robbins (1948) theorem for m-dependent sequences, that we will show in what follows. To apply the aforementioned theorem, we write $S_t^{M\varepsilon}$ in the following form

$$S_t^{M\varepsilon} = \frac{1}{\sqrt{\lfloor t \rfloor}} \sum_{i=1}^{\lfloor t \rfloor} X_i + o_{L^2}(1),$$

where $(X_i)_{i\in\mathbb{N}}$ are zero mean and stationary $(\lfloor \frac{1}{\varepsilon} \rfloor + 1)$-dependent random variables, having a second moment and defined as

$$X_i = \sum_{k=\ell}^{M} c_k \int_i^{i+1} H_k(X^\varepsilon(s))\,\mathrm{d}s.$$

All the moments of these random variables exist by Breuer and Major (1983, Diagram Formula Lemma, p. 432).

Moreover, let us note that

$$\sigma_\varepsilon^2 = \lim_{t\to\infty} \operatorname{var}[S_t^{M\varepsilon}] = 2\sum_{k=\ell}^{M} c_k^2 k! \int_0^{\frac{1}{\varepsilon}} r_\varepsilon^k(s)\,\mathrm{d}s,$$

and $\sigma_\varepsilon^2 \to \sigma^2(F_M)$ when $\varepsilon \to 0$. Besides, $\sigma^2(F_M) \to \sigma^2(F)$ when $M \to \infty$, assuring in this form the required convergence.

We will give a proof of the CLT for m-dependent random variables.

The following result is a particular case of the Lindeberg's theorem, see Billingsley (1995, Theorem 27.2, p. 359 and Lyapounov's condition (27.16), p. 362).

Theorem 1.5. *Let $\{X_{n,i}\}_{i=1,\ldots,K(n);n\in\mathbb{N}}$ be a triangular array of zero mean and i.i.d. random variables and assume $\mathrm{E}[X_{n,1}]^2 = 1$ and $\lim_{n\to\infty} K(n) = +\infty$. Furthermore, suppose that $|X_{n,1}|^{2+\delta}$ is integrable for some positive δ and that $\mathrm{E}\Big[|X_{n,1}|^{2+\delta}\Big] \le \boldsymbol{C}$. Let us define $S_n = \frac{1}{\sqrt{K(n)}} \sum_{i=1}^{K(n)} X_{n,i}$. Then:*

$$S_n \xrightarrow[n\to\infty]{Law} N(0,1).$$

Now, we have all the ingredients to prove Hœffding and Robbins (1948) theorem. Let $\{X_i\}_{i\in\mathbb{N}}$ a zero mean stationary m-dependent sequence ($m \in \mathbb{N}^*$), having finite second moment. We can define

$$\sigma^2 = \mathrm{E}\left[X_1^2\right] + 2\sum_{i=2}^{m+1} \mathrm{E}[X_1 X_i],$$

and also assume that $\sigma > 0$.

Theorem 1.6. *Let $\{X_i\}_{i\in\mathbb{N}^*}$ a stationary m-dependent sequence of zero mean random variables such that $|X_{n,1}|^{2+\delta}$ is integrable for some positive δ. Let $S_n = \frac{1}{\sqrt{n}} \sum\limits_{i=1}^{n} X_i$ then*

$$S_n \xrightarrow[n\to\infty]{law} N(0,\sigma^2).$$

Remark 1.7. It is possible to get rid of the hypothesis that the random variable X_1 has a finite moment of order $2+\delta$ and replace it by the existence of its second order moment. We refer the reader to Orey (1958, Corollary, p. 546). In this article the author state a CLT for centered m-dependent variables with finite second moment satisfying some Lindeberg-like conditions obviously fulfilled in the stationary case.

Proof. Let $p(n) = \lfloor n^\alpha \rfloor$ and $q(n) = \lfloor n^\beta \rfloor$ with $1 > \alpha > \beta > 0$. The first sequence allows us to decompose the interval of integers $[1,n]$ in large blocks and the second one in small blocks. Thus, we define the following intervals:

$$I_1 = [1, p(n)], \qquad J_1 = [p(n)+1, p(n)+q(n)]$$
$$I_2 = [p(n)+q(n)+1, 2p(n)+q(n)], \quad J_2 = [2p(n)+q(n)+1, 2p(n)+2q(n)]$$

and

$$\begin{aligned} I_k &= [(k-1)p(n) + (k-1)q(n) + 1, kp(n) + (k-1)q(n)], \\ J_k &= [kp(n) + (k-1)q(n) + 1, kp(n) + kq(n)], \end{aligned}$$

for $k \geqslant 3$. Let $K(n) = \left\lfloor \dfrac{n}{p(n)+q(n)} \right\rfloor$ thus

$$K(n) \times (p(n) + q(n)) \leqslant n.$$

Then we have two disjoint sets of indices $H_1 = \bigcup_{j=1}^{K(n)} I_j$ and $H_2 = \bigcup_{j=1}^{K(n)} J_j$. In this form $[1, n] = H_1 \cup H_2 \cup H_3$, with H_3 having a number of elements less or equal to $p(n) + q(n)$. The definitions of $p(n)$ and of $q(n)$ imply that $\lim_{n\to\infty} \frac{K(n)p(n)}{n} = 1$. Now, let

$$S_n = \frac{1}{\sqrt{n}} \sum_{i\in H_1} X_i + \frac{1}{\sqrt{n}} \sum_{i\in H_2} X_i + \frac{1}{\sqrt{n}} \sum_{i\in H_3} X_i.$$

First, we show that the last two terms tend to zero in probability. In fact, we only prove it for the second one. The proof for the third term is easier because it involves indices belonging to only one block. Now, using independence and stationarity, we have

$$\begin{aligned} \mathrm{E}\left[\frac{1}{\sqrt{n}} \sum_{i\in H_2} X_i\right]^2 &= \frac{K(n)}{n} \mathrm{E}\left[\sum_{i\in J_1} X_i\right]^2 \\ &= \frac{K(n)}{n} \left(q(n)\mathrm{E}\left[X_1^2\right] + 2\sum_{i=2}^{m+1} (q(n) - (i-1))\mathrm{E}\left[X_1 X_i\right]\right) \\ &\leqslant \boldsymbol{C}\, \frac{K(n)}{n}\, q(n)\, \mathrm{E}\left[X_1^2\right] \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Let us see the asymptotic normality of the first term. Let $\sigma_n^2 = \mathrm{E}\left[\sum_{i\in I_j} X_i\right]^2$. We have

$$\sigma_n^2 = p(n) \left(\mathrm{E}\left[X_1^2\right] + 2\sum_{i=2}^{m+1} \left(1 - \tfrac{i-1}{p(n)}\right) \mathrm{E}\left[X_1 X_i\right]\right).$$

In this form

$$\frac{1}{\sqrt{n}} \sum_{i \in H_1} X_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{K(n)} \sum_{i \in I_j} X_i = \sqrt{\frac{\sigma_n^2 K(n)}{n}} \left(\frac{1}{\sqrt{K(n)}} \sum_{j=1}^{K(n)} X_{n,j} \right).$$

The random variables $X_{n,j} = \frac{1}{\sigma_n} \sum_{i \in I_j} X_i$ are independent and identically distributed, with mean 0 and variance 1. Furthermore,

$$\frac{\sigma_n^2 K(n)}{n} = \frac{K(n)p(n)}{n} \left(\mathrm{E}\left[X_1^2\right] + 2 \sum_{i=2}^{m+1} \left(1 - \frac{i-1}{p(n)}\right) \mathrm{E}\left[X_1 X_i\right] \right) \to \sigma^2.$$

Theorem 1.5 allows to conclude if we prove that for some positive δ, $\mathrm{E}\left[|X_{n,1}|^{2+\delta}\right] \leqslant \boldsymbol{C}$. For this purpose we use an old inequality of Ibragimov and Linnik (1971, Lemma 18.5.1, p. 340) restated here.

Lemma 1.8. *Let $\{X_i\}_{i \in \mathbb{N}^*}$ a stationary uniformly mixing sequence of zero mean random variables such that $|X_1|^{2+\delta}$ is integrable for positive $\delta < 1$. If $\sigma_n^2 = \mathrm{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] \to +\infty$, there exists a constant $\boldsymbol{C}$ such that*

$$\mathrm{E}\left[\left|\sum_{i=1}^n X_i\right|^{2+\delta}\right] \leqslant \boldsymbol{C} \sigma_n^{2+\delta}.$$

To conclude the proof of Hœffding and Robbins (1948) theorem, let us remark that a stationary m-dependent sequence of random variables is uniformly mixing. In fact, according to Doukhan (1994, p. 17), the ϕ-mixing coefficients of a stationary m-dependent sequence are such that $\phi(n) = 0$ for $n > m$. For the definition of the uniformly mixing see Doukhan (1994, p. 3 and 16) and Ibragimov and Linnik (1971, Definition 17.2.2, p. 308).

As announced in Remark 1.4, we give a sketch of the new proof based on Nualart and Peccati result.

As we explained in the previous proof, it is sufficient to demonstrate the CLT for the functional S_t^M. Furthermore, we can show that S_t^M and $S_{\lfloor t \rfloor}^M$ are equivalent in $L^2(\Omega)$. The key comes from the fact that this functional $S_{\lfloor t \rfloor}^M$ can be decomposed into a finite number of Wiener chaos.

More precisely, using the spectral decomposition of the process X in the chaos of order 1, see (1.1) and Itô's formula (see Breuer and Major (1983, p. 30)), we can decompose $S_{\lfloor t \rfloor}^M$ in the multiple chaos in the following way:

$$S_{\lfloor t \rfloor}^M = \sum_{k=\ell}^{M} I_k(h_{\lfloor t \rfloor, k}),$$

where the operator definition of I_k is given later by (5.5), page 80, and the function $h_{\lfloor t \rfloor,k} \in L_s^2(\mathbb{R}^k)$ (cf. notations of Sect. 2.2.2 and Slud 1994) is defined by the following equality:

$$h_{\lfloor t \rfloor,k}(\lambda_1, \ldots, \lambda_k) = c_k\, k!\, \frac{1}{\sqrt{\lfloor t \rfloor}} \int_0^{\lfloor t \rfloor} \exp(i(\lambda_1 + \cdots + \lambda_k)s)\, \sqrt{f(\lambda_1)} \cdots \sqrt{f(\lambda_k)}\, \mathrm{d}s.$$

To establish the convergence of $S_{\lfloor t \rfloor}^M$, we use Theorem 1 of Peccati and Tudor (2005).

By Mehler's formula, since $r^\ell \in L^1(\mathbb{R})$, it is easy to see that

$$\mathrm{E}(S_{\lfloor t \rfloor}^M)^2 = 2 \sum_{k=\ell}^{M} c_k^2 k! \int_0^{\lfloor t \rfloor} \left(1 - \frac{s}{\lfloor t \rfloor}\right) r^k(s)\, \mathrm{d}s \xrightarrow[t \to +\infty]{} \sigma^2(F_M).$$

This latter convergence gives the required conditions appearing in the beginning of this latter theorem. So we will just verify condition (i). In other words, let s and k fixed such that $s = 1, \ldots, k-1$ and k such that $k = \sup(\ell, 2), \ldots, M$. We need to show that all the contractions of $h_{\lfloor t \rfloor,k}$ of order s tend to zero. These contractions are defined by (5.6).

It is sufficient to establish that $\lim\limits_{t \to +\infty} A_{t,k,s} = 0$, with

$$A_{t,k,s} = \frac{1}{\lfloor t \rfloor^2} \int_0^{\lfloor t \rfloor} \int_0^{\lfloor t \rfloor} \int_0^{\lfloor t \rfloor} \int_0^{\lfloor t \rfloor} r^s(u_1 - v_1) r^s(u_2 - v_2) \\ r^{k-s}(u_1 - u_2) r^{k-s}(v_1 - v_2)\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}v_1\, \mathrm{d}v_2.$$

With a convenient change of variables we get

$$|A_{t,k,s}| \leqslant \frac{1}{\lfloor t \rfloor} \int_{-\lfloor t \rfloor}^{\lfloor t \rfloor} \int_{-\lfloor t \rfloor}^{\lfloor t \rfloor} \int_{-\infty}^{+\infty} |r(x)|^s\ |r(x-z)|^s\ |r(y)|^{k-s}\ |r(y-z)|^{k-s}\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$$

We split the indices intervals into two parts, B_N and B_N^c, where we defined for a fixed positive real number N,

$$B_N = \{(x, y) \in \mathbb{R}^2,\ |x| > N \text{ or } |y| > N\}.$$

Applying Hölder inequality with $p = \frac{k}{s} > 1$ and $q = \frac{k}{k-s} > 1$ to both terms corresponding to B_N and B_N^c. Because $|r| \leq 1$, it follows that

$$|A_{t,k,s}| \leqslant \boldsymbol{C}\left[\left(\int_{-\infty}^{+\infty}|r(z)|^k\,\mathrm{d}z\right)^{2-\frac{s}{k}}\left(\int_{|x|>N}|r(x)|^k\,\mathrm{d}x\right)^{\frac{s}{k}} + \frac{N^2}{\lfloor t\rfloor}\left(\int_{-\infty}^{+\infty}|r(z)|^k\,\mathrm{d}z\right)\right].$$

Consequently, since for all $k \geqslant \ell$, we have $r^k \in L^1(\mathbb{R})$, we showed that:

$$\overline{\lim_t}\,|A_{t,k,s}| \leqslant \boldsymbol{C}\left(\int_{|x|>N}|r(x)|^k\,\mathrm{d}x\right)^{\frac{s}{k}} \xrightarrow[N\to+\infty]{} 0,$$

since $s > 0$. This completes the new proof.

1.4 Brownian Motion Increments

In this section, we will show some applications of Theorem 1.3 to the increments of the Bm.

Let $X(t)$ be a standard Bm. We can assume that $X(t)$ is defined in terms of its harmonizable representation (see Hunt 1951)

$$X(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\exp(it\lambda)-1}{i\lambda}\,\mathrm{d}W(\lambda).$$

To verify that $X(t)$ is actually a Bm, given that it is centered and Gaussian, it is enough to compute the variance of the increments. Thus, for $h > 0$,

$$\begin{aligned}
\mathrm{E}\,[X(t+h)-X(t)]^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\exp(i(h+t)\lambda)-\exp(it\lambda)}{i\lambda}\right|^2\mathrm{d}\lambda \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos h\lambda}{\lambda^2}\,\mathrm{d}\lambda = \frac{2}{\pi}h\int_0^{\infty}\frac{1-\cos\lambda}{\lambda^2}\,\mathrm{d}\lambda \\
&= \frac{2}{\pi}h\int_0^{\infty}\frac{\sin^2 u}{u^2}\,\mathrm{d}u = h.
\end{aligned}$$

We want to study the asymptotic behavior, as $\varepsilon \to 0$, of the random variables

$$\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}},$$

when we consider it, for almost all ω, as a random variable in the probability space $([0,\ 1], \mathcal{B}, \lambda)$ with the Lebesgue measure λ, $\mathcal{B}$ being the Borel sets. Let us denote by Φ the distribution of a standard Gaussian random variable.

Wschebor (1992) showed Theorem 1.9, a remarkable result.

Theorem 1.9. *For almost all ω one has*

$$\lim_{\varepsilon\to 0}\lambda\left\{s\leqslant 1:\ \frac{X(s+\varepsilon)(\omega)-X(s)(\omega)}{\sqrt{\varepsilon}}\leqslant x\right\}=\Phi(x).$$

Proof. Let $G:\mathbb{R}\to\mathbb{R}$ be a bounded and continuous function. Consider the sequence

$$\int_0^1 G\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s.$$

Initially, we will show that the above sequence tends to $\mathrm{E}\,[G(N)]$ in $L^2(\Omega)$, where N denotes a standard Gaussian random variable. To do so, let us compute its limit variance, when $\varepsilon\to 0$. Set $\tilde{G}=G-\mathrm{E}\,[G(N)]$. We must prove

$$\begin{aligned}\operatorname{var}&\left[\int_0^1 G\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s\right]\\&=\mathrm{E}\left[\int_0^1 \tilde{G}\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s\right]^2\xrightarrow[\varepsilon\to 0]{}0.\end{aligned}\tag{1.5}$$

Now

$$\begin{aligned}\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}&=\frac{1}{\sqrt{2\varepsilon\pi}}\int_{-\infty}^{\infty}\frac{\exp(i(s+\varepsilon)\lambda)-\exp(is\lambda)}{i\lambda}\,\mathrm{d}W(\lambda)\\&=\frac{1}{\sqrt{2\varepsilon\pi}}\int_{-\infty}^{\infty}\exp(is\lambda)\frac{\exp(i\varepsilon\lambda)-1}{i\lambda}\,\mathrm{d}W(\lambda).\end{aligned}$$

Let $\varepsilon\lambda=u$ into the stochastic integral. We get

$$\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(i\tfrac{s}{\varepsilon}u)\frac{\exp(iu)-1}{iu}\,\mathrm{d}W(u).$$

If our interest is to observe the process on the scale $s=\varepsilon v$, we obtain that there exists a stationary Gaussian process $Y(v)$, such that the following equality in distribution holds

$$Y(v)=\frac{X(\varepsilon v+\varepsilon)-X(\varepsilon v)}{\sqrt{\varepsilon}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(ivu)\frac{\exp(iu)-1}{iu}\,\mathrm{d}W(u).$$

Let us compute the covariance of process Y.

$$r(v) = \mathrm{E}\left[Y(v)Y(0)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ivu)\left|\frac{\exp(iu)-1}{iu}\right|^2 \mathrm{d}u$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(ivu)\frac{(1-\cos u)}{u^2}\,\mathrm{d}u = 1-|v|\,,$$

for $|v| \leqslant 1$ and zero if $|v|\geqslant 1$. Such a process is named Slepian's process. Let us return to the computation of the variance

$$\mathrm{E}\left[\int_0^1 \tilde{G}\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s\right]^2 = \mathrm{E}\left[\varepsilon\int_0^{1/\varepsilon} \tilde{G}(Y(v))\,\mathrm{d}v\right]^2.$$

Since the function $\tilde{G}$ is bounded and continuous, it has a Hermite expansion that converges in $L^2(\phi(x)\,\mathrm{d}x)$, i.e. $\tilde{G} = \sum_{k=1}^{\infty} \tilde{G}_k H_k$. Under this form, using Mehler's formula it follows that

$$\begin{aligned}
\mathrm{E}\left[\varepsilon\int_0^{1/\varepsilon} \tilde{G}(Y(v))\,\mathrm{d}v\right]^2 &= 2\varepsilon\int_0^{1/\varepsilon}(1-\varepsilon u)\mathrm{E}\left[\tilde{G}(Y(0))\tilde{G}(Y(u))\right]\mathrm{d}u \\
&= 2\varepsilon\sum_{k=1}^{\infty}\tilde{G}_k^2 k!\int_0^1 (1-\varepsilon u)(1-u)^k\,\mathrm{d}u \\
&= O(\varepsilon).
\end{aligned}$$

Under a more precise form

$$\frac{1}{\varepsilon}\mathrm{E}\left[\varepsilon\int_0^{1/\varepsilon} \tilde{G}(Y(v))\,\mathrm{d}v\right]^2 \to 2\sum_{k=1}^{\infty}\tilde{G}_k^2 k!\int_0^1 (1-u)^k\,\mathrm{d}u = 2\sum_{k=1}^{\infty}\tilde{G}_k^2\frac{k!}{k+1} = \sigma_{\tilde{G}}^2. \tag{1.6}$$

Then we have

$$\int_0^1 G\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \mathrm{E}\left[G(N)\right],$$

in $L^2(\Omega)$. If $\varepsilon_n = n^{-a}$ for $a > 1$, then $\sum_{n=1}^{\infty}\varepsilon_n < \infty$. The Borel-Cantelli lemma assures us that under this sequence the convergence is for a.s. in ω. A more delicate analysis is required to prove

$$\int_0^1 G\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right)\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \mathrm{E}\left[G(N)\right],$$

for a.s. in ω, whereof we deduce Theorem 1.9 result. Before completing the proof, let us recall that the Levy's theorem about the modulus of continuity of the Bm (see Karatzas and Shreve 1991), implies that for $\delta > 0$ it holds

$$\sup_{t \in [0,\,1]} |X(t+h) - X(t+h')| \leqslant \boldsymbol{C}\, \left|h - h'\right|^{1/2-\delta}. \tag{1.7}$$

Let us consider that G is continuous and Lipchitz. The class of functions with these properties determines the weak convergence. Let ε such that $\varepsilon_{n+1} < \varepsilon < \varepsilon_n$, then

$$\sup_{\varepsilon_{n+1}<\varepsilon<\varepsilon_n} \left| \int_0^1 G\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s - \int_0^1 G\left(\frac{X(s+\varepsilon_n)-X(s)}{\sqrt{\varepsilon_n}}\right) \mathrm{d}s \right|$$

$$\begin{aligned}
&\leqslant \sup_{\varepsilon_{n+1}<\varepsilon<\varepsilon_n} \boldsymbol{C} \int_0^1 \left| \frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}} - \frac{X(s+\varepsilon_n)-X(s)}{\sqrt{\varepsilon_n}} \right| \mathrm{d}s \\
&\leqslant \boldsymbol{C} \left[\left(\frac{1}{\sqrt{\varepsilon}} - \frac{1}{\sqrt{\varepsilon_n}} \right) \sup_{0 \leqslant s \leqslant 1} |X(s+\varepsilon) - X(s)| \right. \\
&\qquad\qquad \left. + \frac{1}{\sqrt{\varepsilon_n}} \sup_{0 \leqslant s \leqslant 1} |X(s+\varepsilon) - X(s+\varepsilon_n)| \right] \\
&\leqslant \boldsymbol{C}(\omega) \left\{ \frac{\varepsilon^{1/2-\delta}}{\varepsilon_n^{1/2}} \left[\left(\frac{\varepsilon_n}{\varepsilon} \right)^{1/2} - 1 \right] + \frac{(\varepsilon_n - \varepsilon)^{1/2-\delta}}{\varepsilon_n^{1/2}} \right\} \\
&\leqslant \boldsymbol{C}(\omega) \left\{ \varepsilon_n^{-\delta} \left[\left(\frac{\varepsilon_n}{\varepsilon_{n+1}} \right)^{1/2} - 1 \right] + \frac{(\varepsilon_n - \varepsilon_{n+1})^{1/2-\delta}}{\varepsilon_{n+1}^{1/2}} \right\} \\
&\leqslant \boldsymbol{C}(\omega) \left[\frac{1}{n^{1-a\delta}} + \frac{(n+1)^{a\delta}}{n^{1/2-\delta}} \right],
\end{aligned}$$

this last term tends to zero if $(a+1)\delta < 1/2$. This choice is always possible by taking δ small enough. In the third inequality, we used inequality (1.7).

Using Theorem 1.3 we can show the following theorem (see Berzin-Joseph and León 1997):

Theorem 1.10. *Let $\tilde{G}$ a continuous function belonging to $L^4(\phi(x)\,\mathrm{d}x)$ then*

$$S_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{G}\left(\frac{X(s+\varepsilon)-X(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s \xrightarrow[\varepsilon \to 0]{Law} \sigma_{\tilde{G}} W(t),$$

where $W(t)$ is another standard Bm, $\tilde{G} = G - \mathrm{E}[G(N)]$ and $\sigma_{\tilde{G}}$ is given by (1.6). Moreover if $\tilde{G}$ Hermite rank is greater than or equal to two the two Bm, W and X, are independent.

Proof. The one-dimensional convergence is obtained from Theorem 1.3 and from a change of variables in the stochastic integral as in the proof of the previous theorem. In fact, from the equality in distribution follows

$$S_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{G}\left(\frac{X(s+\varepsilon) - X(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s = \sqrt{\varepsilon} \int_0^{t/\varepsilon} \tilde{G}(Y(u))\, \mathrm{d}u \xrightarrow[\varepsilon \to 0]{\text{Law}} N(0, t\sigma_{\tilde{G}}^2).$$

Consider $t_1 < t_2 \leqslant t_3 < t_4$, some points of the interval $[0, 1]$. If $\varepsilon \leqslant t_3 - t_2$ then $S_{t_2}^\varepsilon - S_{t_1}^\varepsilon$ and $S_{t_4}^\varepsilon - S_{t_3}^\varepsilon$ are independent. Since the distribution of $S_{t_2}^\varepsilon - S_{t_1}^\varepsilon$ is the same as the one of $S_{t_2 - t_1}^\varepsilon$ and each of the variables converges to a Gaussian variable, then

$$(S_{t_2}^\varepsilon - S_{t_1}^\varepsilon, S_{t_4}^\varepsilon - S_{t_3}^\varepsilon) \xrightarrow[\varepsilon \to 0]{\text{Law}} \sigma_{\tilde{G}}(W(t_2) - W(t_1), W(t_4) - W(t_3)).$$

If $t_2 = t_3$, the same result is obtained by removing a subinterval of size $\frac{\varepsilon}{2}$ in each interval and using the 1-dependence, the two removed terms tend to zero in $L^2(\Omega)$. The whole procedure can be repeated for any n-vector of increments. In this form the finite dimensional convergence follows.

For the tightness, we need to prove

$$\mathrm{E}\left[S_{t_2}^\varepsilon - S_{t_1}^\varepsilon\right]^4 = \mathrm{E}\left[S_{t_2 - t_1}^\varepsilon\right]^4 = \mathrm{E}\left[\sqrt{\varepsilon} \int_0^{(t_1 - t_2)/\varepsilon} \tilde{G}(Y(u))\, \mathrm{d}u\right]^4 \leqslant \boldsymbol{C}\ (t_2 - t_1)^2.$$

To obtain this bound, set $t = t_2 - t_1$ to simplify the notation. If $t < \varepsilon$ the bound is immediate by Jensen's inequality. Consider then $t \geqslant \varepsilon$ and let $N(\varepsilon) = \lfloor t/\varepsilon \rfloor$. Then

$$\mathrm{E}\left[S_t^\varepsilon\right]^4 \leqslant \boldsymbol{C} \left(\mathrm{E}\left[\sqrt{\varepsilon} \sum_{i=0}^{N(\varepsilon)-1} \int_i^{i+1} \tilde{G}(Y(u))\, \mathrm{d}u\right]^4 + \mathrm{E}\left[\sqrt{\varepsilon} \int_{\lfloor t/\varepsilon \rfloor}^{t/\varepsilon} \tilde{G}(Y(u))\, \mathrm{d}u\right]^4 \right).$$

For the second term, Jensen's inequality entails

$$\mathrm{E}\left[\sqrt{\varepsilon} \int_{\lfloor t/\varepsilon \rfloor}^{t/\varepsilon} \tilde{G}(Y(u))\, \mathrm{d}u\right]^4 \leqslant \boldsymbol{C}\ \varepsilon^2 \leqslant \boldsymbol{C}\ t^2.$$

Defining the set of indices $I = \{0 \leqslant i_1, i_2, i_3, i_4 \leqslant (N(\varepsilon) - 1)\}$, then for the first term, we have the following decomposition

$$\mathrm{E}\left[\sqrt{\varepsilon} \sum_{i=0}^{N(\varepsilon)-1} \int_i^{i+1} \tilde{G}(Y(u))\, \mathrm{d}u\right]^4$$

$$= \varepsilon^2 \sum_I \mathrm{E}\left[\int_{i_1}^{i_1+1} \int_{i_2}^{i_2+1} \int_{i_3}^{i_3+1} \int_{i_4}^{i_4+1} \tilde{G}(Y(u_1))\tilde{G}(Y(u_2))\tilde{G}(Y(u_3))\tilde{G}(Y(u_4))\mathbf{du}\right],$$

where $\mathbf{du} = \mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\,\mathrm{d}u_4$. We can assume without loss of generality that $i_1 \leqslant i_2 \leqslant i_3 \leqslant i_4$. We need to consider the following cases:

- If $i_4 - i_3 \geqslant 2$ then by using the independence, these terms are zero.
- If $0 \leqslant i_4 - i_3 \leqslant 1$ then i_4 depends on i_3 and one has:
  - If $i_3 - i_2 \geqslant 2$ the independence implies that the term of interest is equal to

$$\mathrm{E}\left[\int_{i_1}^{i_1+1}\int_{i_2}^{i_2+1} \tilde{G}(Y(u_1))\tilde{G}(Y(u_2))\,\mathrm{d}u_1\,\mathrm{d}u_2\right]\times \mathrm{E}\left[\int_{i_3}^{i_3+1}\int_{i_4}^{i_4+1} \tilde{G}(Y(u_3))\tilde{G}(Y(u_4))\,\mathrm{d}u_3\,\mathrm{d}u_4\right]$$

  - If $i_2 - i_1 \geqslant 2$ the corresponding terms vanish.
  - If $0 \leqslant i_2 - i_1 \leqslant 1$ the sum is over two indexes and because

$$\mathrm{E}\left[\int_{i}^{i+1} \tilde{G}(Y(u))\,\mathrm{d}u\right]^2 \leqslant \boldsymbol{C},$$

we obtain that this sum over the corresponding indices is less than or equal to

$$\mathbf{C}\varepsilon^2 N^2(\varepsilon) \leqslant \boldsymbol{C}t^2.$$

The remaining cases can be treated in a similar fashion.

Let us prove now the last result of the theorem: if the function $\tilde{G}$ has a Hermite rank greater than or equal to two, we obtain that W is independent of X.

To prove this let us consider the following vector process defined in $C[0,\,1]\times C[0,\,1]$, $\boldsymbol{X}^{\varepsilon}(t) = (X(t), S_t^{\varepsilon})$. Each coordinates process is tight hence the vector process is also tight. Let us denote by $\boldsymbol{Y}(t)$ any continuous limit point for the sequence. By construction, the following vector

$$((S_{t_1}^{\varepsilon}, X(t_1)), (S_{t_2}^{\varepsilon} - S_{t_1+\varepsilon}^{\varepsilon}, X(t_2) - X(t_1+\varepsilon)), \ldots, (S_{t_m}^{\varepsilon} - S_{t_{m-1}+\varepsilon}^{\varepsilon}, X(t_m) - X(t_{m-1}+\varepsilon))),$$

has all its coordinates independent and moreover converges when $\varepsilon \to 0$ to $(\boldsymbol{Y}(t_1), \boldsymbol{Y}(t_2) - \boldsymbol{Y}(t_1), \ldots, \boldsymbol{Y}(t_m) - \boldsymbol{Y}(t_{m-1}))$. Then we deduce that $\boldsymbol{Y}$ is an independent increment process. Moreover process $\boldsymbol{Y}$ has finite second moment then it must be Gaussian.

Thus to prove the asymptotical independence we need only to compute the following covariance

$$\mathrm{E}\left[(S_{t_m}^{\varepsilon} - S_{t_{m-1}}^{\varepsilon})(X(t_m) - X(t_{m-1}))\right] = 0,$$

because the function $\tilde{G}$ has Hermite rank equal to 2. Consequently, all the limit points have the same Gaussian distribution and its two coordinates are independent.

Remark 1.11. The fact of obtaining the asymptotic independence between the original Bm and the limit process, implies that we have a stronger convergence. This convergence is a particular case of stable convergence that we will reconsider later.

We can obtain Theorems 1.9 and 1.10 for more general diffusion processes. The same type of study was undertaken by Perera and Wschebor (1998) in a more general form than ours.

We will begin with the Bm with drift. Let $b : \mathbb{R} \to \mathbb{R}$ be a continuous function. Also assume that $\mathrm{E}_x\Big[\exp(\int_0^t b^2(X(s))\,\mathrm{d}s)\Big] < \infty$, where E_x is the expectation with respect to Bm such that $X(0) = x$, for all $x \in \mathbb{R}$. The following SDE

$$\mathrm{d}Z(t) = \mathrm{d}X(t) + b(Z(t))\,\mathrm{d}t \quad Z(0) = x,$$

admits a unique weak solution that can be expressed through the Girsanov's formula (see Karatzas and Shreve 1991). In first place, we have the exponential martingale

$$M(t) = \exp\left(\int_0^t b(X(s))\,\mathrm{d}X(s) - \frac{1}{2}\int_0^t b^2(X(s))\,\mathrm{d}s\right) \quad \text{with} \quad \mathrm{E}_x[M(t)] = 1.$$

And in second place if $H : C[0,t] \to \mathbb{R}$ is a measurable and integrable functional, we have the Girsanov's formula

$$\mathrm{E}_x[H\{Z(s) : 0 \leqslant s \leqslant t\}] = \mathrm{E}_x[M(t)H\{X(s) : 0 \leqslant s \leqslant t\}].$$

We can obtain the two following results.

Corollary 1.12. *For almost all ω one has*

$$\lim_{\varepsilon\to 0} \lambda\left\{s \leqslant 1 : \frac{Z(s+\varepsilon)(\omega) - Z(s)(\omega)}{\sqrt{\varepsilon}} \leqslant x\right\} = \Phi(x).$$

Proof. Let G be a continuous and bounded real function and consider

$$\Delta = \left\{\omega : \int_0^1 G\left(\frac{Z(s+\varepsilon)(\omega) - Z(s)(\omega)}{\sqrt{\varepsilon}}\right)\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{E}\left[G(N)\right]\right\},$$

and

$$\tilde{\Delta} = \left\{\omega : \int_0^1 G\left(\frac{X(s+\varepsilon)(\omega) - X(s)(\omega)}{\sqrt{\varepsilon}}\right)\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{E}\left[G(N)\right]\right\}.$$

By using Girsanov's formula we get

$$P_x(\Delta) = \mathrm{E}_x[\mathbb{1}_\Delta(\omega)] = \mathrm{E}_x\big[M(t)\mathbb{1}_{\tilde{\Delta}}(\omega)\big] = \mathrm{E}_x[M(t)] = 1.$$

The third equality is a consequence of Theorem 1.9, i.e. $P_x\{\tilde{\Delta}\} = 1$.

Remark 1.13. Let G be a continuous function such that $|G(x)| \leqslant \sum_{i=0}^m a_i\,|x|^i$ for a certain $m \geqslant 0$ and f another continuous function. One can show by using the above result that a.s. in ω

$$\int_0^t G\left(\frac{Z(s+\varepsilon)(\omega) - Z(s)(\omega)}{\sqrt{\varepsilon}}\right) \mathrm{d}s \xrightarrow[\varepsilon\to 0]{} t\mathrm{E}\,[G(N)].$$

and

$$\int_0^t f(Z(s))G\left(\frac{Z(s+\varepsilon)(\omega) - Z(s)(\omega)}{\sqrt{\varepsilon}}\right) \mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \mathrm{E}\,[G(N)] \int_0^t f(Z(s))\,\mathrm{d}s.$$

Corollary 1.14. *Let G a continuous an even function belonging to $L^4(\phi(x)\,\mathrm{d}x)$, such that G has a Lipchitz derivative then*

$$\tilde{S}_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{G}\left(\frac{Z(s+\varepsilon) - Z(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s \xrightarrow[\varepsilon\to 0]{\text{Law}} \sigma_{\tilde{G}} W(t),$$

where $W(t)$ is a standard Bm and $\tilde{G} = G - \mathrm{E}\,[G(N)]$ and $\sigma_{\tilde{G}}$ is defined by (1.6).

Remark 1.15. This last Bm is the same as in Theorem 1.10. Hence, for this class of functions $\tilde{G}$, the second conclusion of the theorem is in force and a fortiori W is also independent of the process Z.

Proof. We can write

$$\begin{aligned}
\tilde{S}_t^\varepsilon &= S_t^\varepsilon + \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[\tilde{G}\left(\frac{Z(s+\varepsilon) - Z(s)}{\sqrt{\varepsilon}}\right) - \tilde{G}\left(\frac{X(s+\varepsilon) - X(s)}{\sqrt{\varepsilon}}\right)\right] \mathrm{d}s \\
&= S_t^\varepsilon + I_t^\varepsilon.
\end{aligned}$$

Using a Taylor expansion of first order, we get that there exists a real number α, $0 < \alpha < 1$, such that

$$I_t^\varepsilon = \int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon) - Z(s)}{\sqrt{\varepsilon}} - \frac{\alpha}{\sqrt{\varepsilon}} \int_s^{s+\varepsilon} b(Z(u))\,\mathrm{d}u\right)\left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} b(Z(u))\,\mathrm{d}u\right) \mathrm{d}s.$$

Then the Lipchitz property of function $\dot{\tilde{G}}$ yields

$$I_t^\varepsilon = \int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon) - Z(s)}{\sqrt{\varepsilon}}\right)\left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} b(Z(u))\,\mathrm{d}u\right) \mathrm{d}s + J_t^\varepsilon,$$

where

$$|J_t^\varepsilon| \leqslant \sqrt{\varepsilon} \int_0^t \sup_{s \leqslant u \leqslant s+\varepsilon} b^2(Z(u))\, \mathrm{d}s \xrightarrow[\varepsilon\to 0]{} 0, \qquad \text{a.s. in } \omega.$$

Moreover,

$$\begin{aligned}
&\int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right)\left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} b(Z(u))\,\mathrm{d}u\right)\mathrm{d}s \\
&\qquad = \int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right) b(Z(s))\,\mathrm{d}s \\
&\qquad\qquad + \int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right)\frac{1}{\varepsilon}\int_s^{s+\varepsilon} [b(Z(u))-b(Z(s))]\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

Obtaining in the first place:

$$\begin{aligned}
&\left|\int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right)\frac{1}{\varepsilon}\int_s^{s+\varepsilon} [b(Z(u))-b(Z(s))]\,\mathrm{d}u\,\mathrm{d}s\right| \\
&\leqslant \sup_{0\leqslant s\leqslant t}\ \sup_{s\leqslant u\leqslant s+\varepsilon} |b(Z(u))-b(Z(s))| \int_0^t \left|\dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right)\right| \mathrm{d}s \xrightarrow[\varepsilon\to 0]{} 0,
\end{aligned}$$

a.s. in ω, because the last integral is bounded thanks to Remark 1.13.

And in the second place, as a consequence of the same remark, $\dot{\tilde{G}}$ being an odd function, we have

$$\int_0^t \dot{\tilde{G}}\left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right) b(Z(s))\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \mathrm{E}\,[\dot{\tilde{G}}(N)] \int_0^t b(Z(s))\,\mathrm{d}s = 0,$$

a.s. in ω. Theorem 1.10 yields Corollary 1.14 and Remark 1.15.

The study we have previously addressed concerning the oscillation of the Bm and other diffusion processes allows us to build a nonparametric estimator of the quadratic variation for a general one dimensional diffusion process.

The observed process will be the solution of the SDE

$$\mathrm{d}Z(s) = \sigma(Z(s))\,\mathrm{d}X(s) + b(Z(s))\,\mathrm{d}s \quad Z(0) = x.$$

We assume that functions σ and b satisfy the hypotheses of existence and uniqueness. The estimator of the quadratic variation of Z in the interval $[0, t]$ is

$$\hat{V}^\varepsilon(t) = \int_0^t \left(\frac{Z(s+\varepsilon)-Z(s)}{\sqrt{\varepsilon}}\right)^2 \mathrm{d}s.$$

Now we prove Theorem 1.16.

Theorem 1.16. *Let σ be a continuously differentiable function satisfying $\sigma(x) > 0$ and b a continuous function then*

$$\hat{V}^{\varepsilon}(t) \xrightarrow[\varepsilon \to 0]{} \int_0^t \sigma^2(Z(s))\,\mathrm{d}s = V(t), \qquad a.s.\ in\ \omega.$$

Moreover, there exists W, a standard Bm independent of Z such that

$$\frac{1}{\sqrt{\varepsilon}}(\hat{V}^{\varepsilon}(t) - V(t)) \xrightarrow[\varepsilon \to 0]{\text{Law}} \frac{2}{\sqrt{3}} \int_0^t \sigma^2(Z(s))\ \mathrm{d}W(s).$$

Proof. Let us define function $F_\sigma(x) = \int^x \frac{1}{\sigma(u)}\,\mathrm{d}u$, thus $\dot{F}_\sigma(x) = \frac{1}{\sigma(x)}$ and $\ddot{F}_\sigma(x) = -\frac{\dot{\sigma}(x)}{\sigma^2(x)}$. The function F_σ allows us to introduce the process $Y(t) = F_\sigma(Z(t))$. By using Itô's formula, see (Karatzas and Shreve, 1991, Chap. 3) we get

$$\mathrm{d}Y(t) = \mathrm{d}X(t) + \mu(Y(t))\,\mathrm{d}t, \qquad Y(0) = F_\sigma(x),$$

where

$$\mu(x) = \frac{b(F_\sigma^{-1}(x))}{\sigma(F_\sigma^{-1}(x))} - \frac{1}{2}\dot{\sigma}(F_\sigma^{-1}(x)).$$

We will assume that the function μ satisfies the technical conditions ensuring that $\mathrm{E}_x\Big[\exp(\int_0^t \mu^2(X(s))\,\mathrm{d}s)\Big] < \infty$.
There exists a real number $\alpha(s,\varepsilon)$, $0 < \alpha(s,\varepsilon) < 1$, such that

$$\begin{aligned}\hat{V}^{\varepsilon}(t) &= \int_0^t \{\dot{\widehat{F_\sigma^{-1}}}\,(Y(s) + \alpha(s,\varepsilon)\,[Y(s+\varepsilon) - Y(s)])\}^2 \left(\frac{Y(s+\varepsilon) - Y(s)}{\sqrt{\varepsilon}}\right)^2 \mathrm{d}s \\ &\approx \int_0^t \sigma^2(Z(s)) \left(\frac{Y(s+\varepsilon) - Y(s)}{\sqrt{\varepsilon}}\right)^2 \mathrm{d}s \xrightarrow[\varepsilon \to 0]{} \mathrm{E}\left[N^2\right] \int_0^t \sigma^2(Z(s))\,\mathrm{d}s\end{aligned}$$

a.s. in ω. The last limit is a consequence of Remark 1.13.

The second assertion of the theorem is more involved, we will give only a sketch of the proof. First, we point out that

$$\frac{1}{\sqrt{\varepsilon}}(\hat{V}^{\varepsilon}(t) - V(t)) \approx \frac{1}{\sqrt{\varepsilon}} \int_0^t \sigma^2(Z(s)) H_2\left(\frac{Y(s+\varepsilon) - Y(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s.$$

Defining as before $\tilde{S}_t^{\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \int_0^t H_2\left(\frac{Y(s+\varepsilon)-Y(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s$, we consider first a fixed discretization of the integral say,

$$\sum_{i=0}^{\lfloor nt\rfloor-1} \sigma^2(Z(\tfrac{i}{n}))\left(\tilde{S}^{\varepsilon}_{\frac{i+1}{n}} - \tilde{S}^{\varepsilon}_{\frac{i}{n}}\right) \xrightarrow[\varepsilon\to 0]{\text{Law}} \frac{2}{\sqrt{3}} \sum_{i=0}^{\lfloor nt\rfloor-1} \sigma^2(Z(\tfrac{i}{n}))\left(W(\tfrac{i+1}{n}) - W(\tfrac{i}{n})\right),$$

by Corollary 1.14. Moreover, by Remark 1.15, one has

$$\frac{2}{\sqrt{3}} \sum_{i=0}^{\lfloor nt\rfloor-1} \sigma^2(Z(\tfrac{i}{n}))(W(\tfrac{i+1}{n}) - W(\tfrac{i}{n})) \xrightarrow[n\to\infty]{\text{Law}} \frac{2}{\sqrt{3}} \int_0^t \sigma^2(Z(s))\, \mathrm{d}W(s).$$

To finish the proof we need to show that

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \mathrm{E}\left[\frac{1}{\sqrt{\varepsilon}} \int_0^t \sigma^2(Z(s)) H_2\left(\frac{Y(s+\varepsilon) - Y(s)}{\sqrt{\varepsilon}}\right) \mathrm{d}s - \sum_{i=0}^{\lfloor nt\rfloor-1} \sigma^2(Z(\tfrac{i}{n}))\left(\tilde{S}^{\varepsilon}_{\frac{i+1}{n}} - \tilde{S}^{\varepsilon}_{\frac{i}{n}}\right)\right]^2 = 0.$$

We do not prove this last fact. A complete proof is given in Berzin-Joseph and León (1997, pages 577–578).

1.5 Other Increments of the Bm

Consider the function $\varphi(u) = \mathbb{1}_{[-1,0]}(u)$, such a function is of bounded variation. Set $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon}\varphi(\frac{\cdot}{\varepsilon})$. Defining $X^\varepsilon = \varphi_\varepsilon * X$, where $*$ denotes the convolution between functions or measures and X is again a standard Bm. The process X^ε is almost everywhere differentiable and it holds

$$\sqrt{\varepsilon}\dot{X}^\varepsilon(t) = \frac{X(t+\varepsilon) - X(t)}{\sqrt{\varepsilon}}.$$

This fact allows us to formulate the problem of the previous section for more general functions φ. We will make this in what follows leaving the details as exercises. Let φ a bounded support density with a continuous derivative. Again, let us define $X^\varepsilon = \varphi_\varepsilon * X$.

The outline of the approach is the following.

1. Write the harmonizable expression for X^ε.
2. Show that process $\sqrt{\varepsilon}\dot{X}^\varepsilon(\varepsilon u)$ is equal in distribution to a stationary Gaussian process $Y(u)$. Determine its spectral density and its covariance.

3. Let us define $\sigma^2 = \text{var}\left[\sqrt{\varepsilon}\dot{X}^{\varepsilon}(\varepsilon u)\right] = \text{var}\,[Y(u)]$ and compute this last constant.
4. Repeat the same steps of the proof of Theorem 1.9 to show that, for all t and almost all ω

$$\lambda\left\{s \leqslant t : \frac{\sqrt{\varepsilon}\dot{X}^{\varepsilon}(s)}{\sigma} \leqslant x\right\} \xrightarrow[\varepsilon\to 0]{} t\Phi(x).$$

5. In such a case, what is the value of the constant $\sigma^2_{\tilde{G}}$?
6. Prove the corresponding Theorem 1.10.

1.6 Discretization

Let us consider the Bm X, observed in a uniform mesh of $[0,\ 1]$. Using the strong law of large numbers for independent array of random variables the first result is that for all continuous function G such that $\mathrm{E}\left[G^4(N)\right] < \infty$,

$$S_n^G(t) = \frac{1}{n}\sum_{i=0}^{\lfloor nt\rfloor - 1} G\left(\sqrt{n}\left[X\left(\tfrac{i+1}{n}\right) - X\left(\tfrac{i}{n}\right)\right]\right) \xrightarrow[n\to\infty]{\text{a.s.}} t\mathrm{E}\,[G(N)].$$

Moreover, defining $\tilde{G}(x) = G(x) - \mathrm{E}\,[G(N)]$ the Lindeberg's CLT and Donsker's invariance principle (see Billingsley 1995) yield

$$\sqrt{n}S_n^{\tilde{G}}(t) \xrightarrow[n\to\infty]{\text{Law}} \sigma_{\tilde{G}}W(t),$$

in the Skorohod's space $\mathscr{D}[0,\ 1]$, where $\sigma^2_{\tilde{G}} = \sum_{k=1}^{\infty}\tilde{G}_k^2 k!$. Again, the Bm W turns out to be independent of X if the function G does not have a first order coefficient in the Hermite basis.

These two results can be extended by using the absolutely continuity of the measures given by Girsanov's formula to the Bm with drift, let

$$\mathrm{d}Y(t) = \mathrm{d}X(t) + b(Y(t))\,\mathrm{d}t.$$

To get the asymptotical independence, G ought to be an even function.

For a general diffusion $\mathrm{d}Z(s) = \sigma(Z(s))\,\mathrm{d}X(s) + b(Z(s))\,\mathrm{d}s$, the same procedure of change of variables leads us to the following two results:

$$\sum_{i=0}^{\lfloor nt\rfloor - 1}\left(Z(\tfrac{i+1}{n}) - Z(\tfrac{i}{n})\right)^2 \xrightarrow[n\to\infty]{\text{a.s.}} \int_0^t \sigma^2(Z(s))\,\mathrm{d}s$$

and

$$\sqrt{n}\left[\sum_{i=0}^{\lfloor nt\rfloor-1}\left(Z(\tfrac{i+1}{n})-Z(\tfrac{i}{n})\right)^2-\int_0^t\sigma^2(Z(s))\,\mathrm{d}s\right]\xrightarrow[n\to\infty]{\text{Law}}\sqrt{2}\int_0^t\sigma^2(Z(s))\,\mathrm{d}W(s).$$

As we can see, things seem easier in this case of discretization. However, the difficulty is more or less the same.

Now, our interest turns to the fractional models. Two main difficulties arise. First, we have to estimate two things, the Hurst parameter H and also the local variance $\sigma(x)$. Second, the underlying process, actually the fBm, has not the independence properties of the Bm and some more involved CLT are needed. In the next section we will illustrate these matters with some preliminaries examples.

1.7 Crossings and Local Time for Smoothing fBm

In this section we consider an estimation problem seemingly far from what we have seen in previous sections. The problem consists in approaching the local time of the fBm b_H, by means of the number of crossings of a mollified version of this process $b_H^\varepsilon(t)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\varphi(\frac{t-s}{\varepsilon})b_H(s)\,\mathrm{d}s=\int_{-\infty}^{\infty}\varphi_\varepsilon(t-s)b_H(s)\,\mathrm{d}s$, where φ is a probability density function of bounded variation with a compact support. We will denote by $\dot{\varphi}$ its continuous derivative.

Let us see some properties of this process. The spectral representation for the fBm (see Hunt 1951) yields the following formula

$$\begin{aligned}\dot{b}_H^\varepsilon(t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot{\varphi}_\varepsilon(t-s)(e^{is\lambda}-1)\frac{1}{|\lambda|^{H+\frac{1}{2}}}\,\mathrm{d}W(\lambda)\,\mathrm{d}s\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_\varepsilon(t-s)i\lambda e^{is\lambda}\frac{1}{|\lambda|^{H+\frac{1}{2}}}\,\mathrm{d}s\,\mathrm{d}W(\lambda)\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{it\lambda}i\lambda\hat{\varphi}_\varepsilon(-\lambda)\frac{1}{|\lambda|^{H+\frac{1}{2}}}\,\mathrm{d}W(\lambda)\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{it\lambda}i\lambda\hat{\varphi}(-\varepsilon\lambda)\frac{1}{|\lambda|^{H+\frac{1}{2}}}\,\mathrm{d}W(\lambda)\\&\overset{Law}{=}\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon^{1-H}}\int_{-\infty}^{\infty}e^{i\frac{t}{\varepsilon}\lambda}i\lambda\hat{\varphi}(-\lambda)\frac{1}{|\lambda|^{H+\frac{1}{2}}}\,\mathrm{d}W(\lambda),\end{aligned}$$

the last equality is in law. As a consequence, if we define $Y_\varepsilon(t) = \varepsilon^{1-H}\dot{b}^\varepsilon_H(\varepsilon t)$, then Y_ε is a zero mean Gaussian stationary process whose spectral density is

$$f_H(\lambda) = \frac{1}{2\pi}\,|\hat{\varphi}(\lambda)|^2\,\frac{1}{|\lambda|^{2H-1}}.$$

Observe that this function belongs to $L^2(\mathbb{R})$ whenever $H < 3/4$. Moreover,

$$\mathrm{E}\left[Y_\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{\varphi}(\lambda)|^2\,\frac{1}{|\lambda|^{2H-1}}\,\mathrm{d}\lambda = \sigma_Y^2.$$

Hence, we can introduce the unit variance process $Z_\varepsilon(t) = Y_\varepsilon(t)/\sigma_Y$.

The problem mentioned above about the convergence of the crossings for the process b^ε_H towards the local time for b_H, has its origins in Wschebor (1992) work on the Bm. The problem can be precisely formulated as follows. Given that the process b^ε_H is differentiable, the random variable: number of crossings in $[0, T]$ of level u of the process b^ε_H, defined as

$$N_T^{b^\varepsilon_H}(u) = \#\{t \leqslant T : b^\varepsilon_H(t) = u\}$$

is well defined and has a first moment. If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function the area formula (Azaïs and Wschebor, 2009, Chapter 3) allows writing

$$\int_{-\infty}^{\infty} h(u)N_T^{b^\varepsilon_H}(u)\,\mathrm{d}u = \int_0^T h(b^\varepsilon_H(t))|\dot{b}^\varepsilon_H(t)|\,\mathrm{d}t.$$

Besides a result similar to the one given in Sect. 1.5 is obtained in Azaïs and Wschebor (1996).

Indeed, almost surely

$$\begin{aligned}\varepsilon^{1-H}\int_{-\infty}^{\infty} h(u)N_T^{b^\varepsilon_H}(u)\,\mathrm{d}u &= \varepsilon^{1-H}\int_0^T h(b^\varepsilon_H(t))\left|\dot{b}^\varepsilon_H(t)\right|\mathrm{d}t\\ &\xrightarrow[\varepsilon\to 0]{} \sqrt{\frac{2}{\pi}}\,\sigma_Y\int_0^T h(b_H(t))\,\mathrm{d}t = \sqrt{\frac{2}{\pi}}\,\sigma_Y\int_{-\infty}^{\infty} h(u)\mathcal{L}_T(u)\,\mathrm{d}u,\end{aligned} \quad (1.8)$$

where $\mathcal{L}_T(u)$ is the local time of level u for b_H, that exists and is continuous (see Berman 1970).

The rate of convergence in the almost sure convergence result (1.8), that constitutes the following theorem, was obtained in Berzin and León (2005). To state such a result, let us introduce first the function $\tilde{G}(x) = \sqrt{\frac{\pi}{2}}\,|x| - 1$. By defining

$$J_\varepsilon(T) = \frac{1}{\sqrt{\varepsilon}}\left(\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-H}}{\sigma_Y}\int_{-\infty}^{\infty} h(u) N_T^{b_H^\varepsilon}(u)\,\mathrm{d}u - \int_{-\infty}^{\infty} h(u)\mathcal{L}_T(u)\,\mathrm{d}u\right),$$

we get Theorem 1.17.

Theorem 1.17. *Suppose that* $\frac{1}{4} < H < \frac{3}{4}$ *and that* $h \in C^4$ *such that* $|\overset{4\cdot}{h}(x)| \leqslant P(|x|)$ *where* P *is a polynomial. Then, there exists a Bm* W *independent of* b_H *and a constant* $C_{H,\varphi}$ *such that,*

$$J_\varepsilon(T) = \frac{1}{\sqrt{\varepsilon}}\int_0^T h(b_H^\varepsilon(t))\tilde{G}(Z_\varepsilon(\tfrac{t}{\varepsilon}))\,\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\int_0^T \{h(b_H^\varepsilon(t)) - h(b_H(t))\}\,\mathrm{d}t \tag{1.9}$$

$$\xrightarrow[\varepsilon\to 0]{Law} C_{H,\varphi}\int_0^T h(b_H(t))\,\mathrm{d}W(t).$$

Note that as indicated in the notations, $\overset{k\cdot}{h}$ *is the* k^{th} *derivative of* h.

Proof. We only give a sketch of the proof. A complete demonstration can be found in Berzin and León (2005).

Let us begin with the second term in the expression for $J_\varepsilon(T)$. By using a Taylor expansion we have

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\int_0^T [h(b_H^\varepsilon(t)) - h(b_H(t))]\,\mathrm{d}t \\
&\qquad = \frac{\varepsilon^H}{\sqrt{\varepsilon}}\int_0^T \dot{h}(b_H(t))\frac{b_H^\varepsilon(t) - b_H(t)}{\varepsilon^H}\,\mathrm{d}t \\
&\quad + \frac{\varepsilon^{2H}}{2\sqrt{\varepsilon}}\int_0^T \ddot{h}(\theta(\varepsilon,t)b_H(t) + \{1-\theta(\varepsilon,t)\}b_H^\varepsilon(t))\left(\frac{b_H^\varepsilon(t) - b_H(t)}{\varepsilon^H}\right)^2\mathrm{d}t,
\end{aligned}$$

where $0 < \theta(\varepsilon,t) < 1$. First, in Berzin and León (2005), it is shown that

$$\begin{aligned}
&\mathrm{E}\left[\int_0^T \dot{h}(b_H(t))\frac{b_H^\varepsilon(t) - b_H(t)}{\varepsilon^H}\,\mathrm{d}t\right]^2 \\
&\qquad = \left(O(\varepsilon) + O(\varepsilon^{2H})\right)\mathbb{1}_{H<1/2} + o(1)\mathbb{1}_{H\geqslant 1/2}.
\end{aligned}$$

Second, if $\overset{3\cdot}{h}$ is continuous and if $|h(x)| \leqslant P(|x|)$, it holds

$$\int_0^T \ddot{h}(\theta(\varepsilon,t)b_H(t) + (1-\theta(\varepsilon,t))b_H^\varepsilon(t))\left(\frac{b_H^\varepsilon(t) - b_H(t)}{\varepsilon^H}\right)^2\mathrm{d}t = O_P(1).$$

Hence, the whole term tends to zero whenever $\frac{1}{4} < H$.

This implies that the weak limit will be provided by the asymptotic behavior of first term of the sum in equality (1.9). To study this convergence we proceed according to the following steps:

1. Let us define $S_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{G}(Z_\varepsilon(\frac{s}{\varepsilon}))\,\mathrm{d}s$. Given that $H < \frac{3}{4}$, Theorem 1.3 implies that there exists a Brownian motion $W(t)$ and a constant $C_{H,\varphi}$ such that the finite dimensional distributions of S_t^ε converge towards the finite dimensional distribution of $C_{H,\varphi} W(t)$ as $\varepsilon \to 0$.
The tightness in this convergence is far from trivial. It was established in a recent article, Cohen and Wschebor (2010).
2. Given that function $\tilde{G}$ is an even function then by an argument of Gaussian convergence into the Wiener chaos it follows that process W is independent of b_H.
3. Let n be a positive integer and define $t_i = \frac{i}{n}$. The weak convergence entails the following one

$$\sum_{i=0}^{\lfloor nT \rfloor - 1} h(b_H^\varepsilon(t_i))(S_{t_{i+1}}^\varepsilon - S_{t_i}^\varepsilon) \xrightarrow[\varepsilon \to 0]{\text{Law}} C_{H,\varphi} \sum_{i=0}^{\lfloor nT \rfloor - 1} h(b_H(t_i))\,(W(t_{i+1}) - W(t_i))\,.$$

Moreover the asymptotic independence yields

$$\lim_{n\to\infty} C_{H,\varphi} \sum_{i=0}^{\lfloor nT \rfloor - 1} h(b_H(t_i))\,\{W(t_{i+1}) - W(t_i)\} = C_{H,\varphi} \int_0^T h(b_H(t))\,\mathrm{d}W(t),$$

this last convergence is in $L^2(\Omega)$.
4. To conclude it is necessary to prove the following result

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \mathrm{E}\left[\frac{1}{\sqrt{\varepsilon}} \int_0^T h(b_H^\varepsilon(t))\tilde{G}\left(Z_\varepsilon\left(\tfrac{t}{\varepsilon}\right)\right)\mathrm{d}t - \sum_{i=0}^{\lfloor nT \rfloor - 1} h(b_H^\varepsilon(t_i))(S_{t_{i+1}}^\varepsilon - S_{t_i}^\varepsilon)\right]^2 = 0.$$

This results is a non-trivial computation completely developed in Berzin and León (2005).

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Chapter 2
Preliminaries

2.1 Introduction

The use of Brownian diffusions for modeling environmental phenomena, financial markets, physical and molecular interaction and biological issues, has been very successful in the past decades. However, in the various branches of application persistence or rather strong dependence is often encountered. This last property is sometimes interpreted as a slow rate in the convergence to zero of the covariances, when the delay goes to infinity. Moreover, from a physical point of view when the diffusion of pollution particles is observed on the water surface, for some substances, the behavior of the trajectories of the particles seems more regular than in a Brownian case.

These issues lead physicists and probabilitists to introduce new models aimed to solve or better model the above phenomena. The first authors to define such models were Mandelbrot and Van Ness (1968) who introduce the fractional Brownian motion (fBm). This is a stationary increments and autosimilar Gaussian process whose covariance depends on a parameter H (the Hurst parameter) such that $0 < H < 1$.

More recently, there has been a renewed interest in this process and in the possibility of defining a stochastic calculus by using the trajectories of such process as integration measure. Various authors reached this goal. One of the first intents was the work of Lin (1995) who built the integral by means of Riemann sums in the case when $H > \frac{1}{2}$. We must point out that the case when $H = \frac{1}{2}$ corresponds to the Bm, and the integral results the Îto's integral.

When the bases for the integration are completed it is natural to extend the notion of stochastic differential equation (SDE) driven by a fractional noise. A very complete study of these equations was realized by Nualart and Răşcanu (2002).

This work has three main goals, all are of statistical nature. First, we define an estimator of the parameter H, through the observation of one trajectory, on a regular grid of points. We use the k-variations of the order two increments. These

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variations allow the estimation of the H parameter all over its range, leading also to consistency and asymptotically normality. If the first order increments were used, we would only have the asymptotic normality for $H \in]0, \frac{3}{4}[$.

Second, we consider four models of SDE allowing our method the simultaneous estimation of H and the local variance $\sigma^2(x)$. This estimation procedure leads to a loss of convergence rate for the Central Limit Theorem (CLT) for the estimator of σ^2. We also consider the case where H is supposed to be known and in this case our method leads us to define a test of hypothesis for certain functionals of the function $\sigma^2(x)$ and, as a bonus, we can made an evaluation of the asymptotic power of the test.

The third goal consists in the realization of a deep simulation study of the performance of our estimators. To achieve this task we simulate the fBm with the help of the Durbin-Levinson algorithm. Then the different models are simulated using an Euler's finite difference schema. Afterwards, for each of the four models, the estimators of the parameters are computed and then we assess the quality of each estimator and we conclude by comparing their performance.

We must indicate that to demonstrate the asymptotic normality of our estimators, we use the technique of the CLT for functionals that belong to the Wiener Chaos. This method has been developed by Nualart and Peccati (2005), Nourdin and Peccati (2010), Peccati and Tudor (2005), among others. Application of these tools leads to an enormous simplification in the computations.

2.2 Fractional Brownian Motion, Stochastic Integration and Complex Wiener Chaos

2.2.1 Preliminaries on Fractional Brownian Motion and Stochastic Integration

In this section some properties and notions related to fBm are presented. The fBm of Hurst parameter H is a mean zero Gaussian process b_H, with stationary increments whose covariance function is

$$\mathrm{E}\left[b_H(t)b_H(s)\right] = \frac{1}{2}\, v_{2H}^2 \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right],$$

where $v_{2H}^2 = [\Gamma(2H+1)\sin(\pi H)]^{-1}$. Let us point out that the Bm corresponds to the case where $H = \frac{1}{2}$. This process is autosimilar. In fact, by using the above covariance, one readily gets $b_H(\alpha t) \overset{Law}{=} \alpha^H b_H(t)$,where "$\overset{Law}{=}$" denotes the equality in law of the processes.

There exists a harmonizable representation of this process (see Hunt 1951)

$$b_H(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\exp(i\lambda t) - 1]\, |\lambda|^{-H-\frac{1}{2}}\, \mathrm{d}W(\lambda),$$

where W is a complex white noise. The variance of the increments Γ_H results

$$\Gamma_H(t-s) = \mathrm{E}\left[(b_H(t) - b_H(s))^2\right] = v_{2H}^2 (t-s)^{2H}, \quad t \geqslant s.$$

Proposition 2.1. *The trajectories of b_H are continuous, with Hölder coefficient $h < H$.*

Proof. Since we have $\mathrm{E}\left[|b_H(t) - b_H(s)|^p\right] = C_p\,|t-s|^{pH}$, the Kolmogorov continuity criterion implies that $|b_H(t) - b_H(s)| \leqslant C(\omega)\,|t-s|^h$ for $0 < h < \frac{pH-1}{p}$. The result follows by taking p large enough. □

The fractional Brownian noise (fBn) is defined as the following stationary discrete time Gaussian process

$$X_n^H = b_H(n+1) - b_H(n).$$

Computing the covariance

$$r_H(n) = \mathrm{E}\left[X_n^H X_0^H\right] = \frac{1}{2} v_{2H}^2 n^{2H}\left[\left|1 + \tfrac{1}{n}\right|^{2H} - 2 + \left|1 - \tfrac{1}{n}\right|^{2H}\right] \approx C_H n^{2(H-1)}$$

when $n \to \infty$.

Thus for $H > \frac{1}{2}$, we have $\sum_{n=1}^{\infty} |r_H(n)| = +\infty$. This phenomena has been interpreted, in the literature, by saying that the fBn exhibits a long range dependence whenever $\frac{1}{2} < H < 1$. The spectral density for the fBn can be obtained from

$$r_H(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{in\lambda}|\exp(i\lambda) - 1|^2\,|\lambda|^{-2H-1}\,\mathrm{d}\lambda,$$

by using the Poisson's summation formula, this yields

$$r_H(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\lambda}\,|\exp(i\lambda) - 1|^2 \sum_{k=-\infty}^{\infty} \frac{1}{|\lambda + 2\pi k|^{2H+1}}\,\mathrm{d}\lambda.$$

Let us define for a process X with time parameter t in $[0,\ 1]$ the p-variation index as

$$I(X, [0,\ 1]) = \inf\left\{p > 0 : \sup_{\pi} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^p < \infty\right\}$$

where π is the set of all the finite increasing sequences $\{t_i\}_{i=1}^n$ in the interval $[0,\ 1]$.

A process X is a semi-martingale if $I(X, [0,\ 1]) \in [0,\ 1] \cup \{2\}$.

Proposition 2.2. *The fBm is not a semi-martingale for $H \neq \frac{1}{2}$.*

Proof. Considering the following sum for the fBm

$$V_{n,p} = \sum_{k=1}^{n} \left|b_H(\tfrac{k}{n}) - b_H(\tfrac{k-1}{n})\right|^p = n^{(-pH+1)}\ \frac{1}{n}\sum_{k=0}^{n-1} |X_k^H|^p,$$

where the last equality holds by using autosimilarity. If $p = \frac{1}{H}$, the ergodic theorem implies that

$$V_{n,1/H} \xrightarrow[n\to\infty]{\text{a.s.}} v_{2H}^{1/H}\ \mathrm{E}\left[|N|^{1/H}\right]$$

and we also have the convergence in $L^1(\Omega)$. Moreover $V_{n,p}$ tends to zero or to infinity in probability, whenever $p > \frac{1}{H}$ or $p < \frac{1}{H}$ respectively. Hence this implies that $I(b_H, [\,0,\ 1\,]) = \frac{1}{H}$ and the result follows. $\square$

Now we are ready to introduce the stochastic integral with respect to fBm. Several types of stochastic integrals with respect to b_H can be defined, we chose to work with the notion of pathwise integrals.

Definition 2.3. Let $\{u(t) : t \in [0, T]\}$ a process with integrable trajectories. The symmetric pathwise integral with respect to b_H is defined as

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^T u(t)\big[b_H(t+\varepsilon) - b_H(t-\varepsilon)\big]\,\mathrm{d}t,$$

whenever that limit exists in probability. The integral will be denoted as $\int_0^T u(t)\,\mathrm{d}^s b_H(t)$.

Remark 2.4. Two other notions can be introduced. The forward integral

$$\int_0^T u(t)\,\mathrm{d}b_H^-(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^T u(t)\big[b_H(t+\varepsilon) - b_H(t)\big]\,\mathrm{d}t,$$

and the backward integral

$$\int_0^T u(t)\,\mathrm{d}b_H^+(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^T u(t)\big[b_H(t) - b_H(t-\varepsilon)\big]\,\mathrm{d}t.$$

In Lin (1995), another pathwise definition is given. Let Z be a continuous process, with $Z(0) = 0$ and zero quadratic variation. An important example is $b_H + V$, where V is a continuous process with finite variation with initial value equal to zero and $H > \frac{1}{2}$. Lin (1995) shows Theorem 2.5.

Theorem 2.5. *For any function $\phi \in C^1$ and a sequence of partitions $\Delta^n = \{0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t\}$ of $[0,\ t]$ with $|\Delta^n| \xrightarrow[n\to\infty]{} 0$,*

$$\lim_{|\Delta^n|\to 0} \sum_{i=1}^{n} \phi(Z(t_i))[Z(t_i) - Z(t_{i-1})] = \int_0^{Z(t)} \phi(u)\,\mathrm{d}u.$$

The limit is taken in probability.

Proof. The proof goes easily by using the facts that the function $\Phi(x) = \int_0^x \phi(u)\,\mathrm{d}u$ is twice differentiable, that its second derivative is locally bounded, and the following Taylor expansion

$$\begin{aligned}\Phi(Z(t)) &= \sum_{i=1}^{n} [\Phi(Z(t_i)) - \Phi(Z(t_{i-1}))] \\ &= \sum_{i=1}^{n} \phi(Z(t_i))[Z(t_i) - Z(t_{i-1})] + \sum_{i=1}^{n} \tfrac{1}{2}\dot{\phi}(\xi_i)[Z(t_i) - Z(t_{i-1})]^2,\end{aligned}$$

where ξ_i is between $Z(t_i)$ and $Z(t_{i-1})$. □

The above theorem allows to define $\int_0^t \phi(Z(s))\,\mathrm{d}Z(s)$ as the limit of the Riemann sums in probability. Another formulation is a sort of fundamental theorem of calculus i.e.,

$$\Phi(Z(t)) - \Phi(Z(0)) = \int_0^t \phi(Z(s))\,\mathrm{d}Z(s).$$

By using this observation, we can search for the solution of the following SDE

$$X(t) = c + \int_0^t \sigma(X(s), Z(s))\,\mathrm{d}Z(s), \quad \sigma \in C^1,\ Z(s) = b_H(s) + V(s), \tag{2.1}$$

where V is as before, a continuous process with finite variation and initial value equal to zero and $H > \frac{1}{2}$. To solve that equation, let g be the unique solution of the following ordinary differential equation (ODE)

$$\begin{aligned}\frac{\mathrm{d}g(t)}{\mathrm{d}t} &= \sigma(g(t), t) \\ g(0) &= c.\end{aligned} \tag{2.2}$$

The solution of Eq. (2.1) is $X(t) = g(Z(t))$. An important example is the fractional version of the Black-Scholes SDE defined in Cutland et al. (1995). This process is the solution of the SDE

$$\mathrm{d}X(t) = X(t)(\sigma\,\mathrm{d}b_H(t) + \mu\,\mathrm{d}t), \quad \sigma,\ \mu \in \mathbb{R},\ H > \tfrac{1}{2}.$$

Here the ODE is $\mathrm{d}g(t)/\mathrm{d}t = g(t)$ and $g(0) = c$. It yields that the solution of (2.1) is

$$X(t) = c\exp(Z(t)) = c\exp(\sigma b_H(t) + \mu t).$$

We can now explore the relationship between the three notions of stochastic integrals with respect to $\mathrm{d}^s b_H$, $\mathrm{d}b_H^-$ and $\mathrm{d}b_H$. The following discussion is based on Biagini et al. (2008, Section 5.5 of Chapter 5). Let us begin with the following definition.

Definition 2.6. The process $(f(s), 0 \leqslant s \leqslant t \leqslant 1)$ is said to be a bounded quadratic variation process if there are constants $p \geqslant 1$ and $0 < C_p < \infty$ such that for any partition $\Delta^n = \{0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t\}$,

$$\sum_{i=1}^{n} \mathrm{E}^{1/p}[|f(t_i) - f(t_{i-1})|^{2p}] \leqslant C_p.$$

Let us examine the following example. Consider $f : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function with bounded first derivative. Then $f(b_H(s))$ is a bounded quadratic variation process for $H > 1/2$. In fact, let Δ^n be a partition of $[0, t]$,

$$\begin{aligned}
&\sum_{i=1}^{n} \mathrm{E}^{1/p}\left[|f(b_H(t_i)) - f(b_H(t_{i-1}))|^{2p}\right] \\
&\quad = \sum_{i=1}^{n} \mathrm{E}^{1/p}\Bigg[\Bigg|\left\{\int_0^1 \dot{f}(b_H(t_{i-1}) + \theta(b_H(t_i) - b_H(t_{i-1})))\,\mathrm{d}\theta\right\} \\
&\qquad\qquad \times [b_H(t_i) - b_H(t_{i-1})]\Bigg|^{2p}\Bigg] \leqslant C_p \sum_{i=1}^{n} |t_i - t_{i-1}|^{2H} \leqslant C_p t.
\end{aligned}$$

The following theorem states that the definition of the integral does not depend on the point where the integrand is evaluated.

Theorem 2.7. *Let $(f(b_H(s)), 0 \leqslant s \leqslant t \leqslant 1)$ be a bounded quadratic variation process. Let $\Delta^n = \{0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t\}$ be a sequence of partitions of $[0, t]$ such that $|\Delta^n| \to 0$ as $n \to \infty$ and*

$$\sum_{i=1}^{n} f(b_H(t_{i-1}))[b_H(t_i) - b_H(t_{i-1})],$$

converges, for $H > \frac{1}{2}$, to a random variable G in $L^2(P^H)$ (the L^2 space for the probability measure generated by the fBm). Then

$$\sum_{i=1}^{n} f(b_H(t_i))[b_H(t_i) - b_H(t_{i-1})],$$

also converges to G *in* $L^2(P^H)$.

Proof. For simplicity's sake, let us write $f(t)$ instead of $f(b_H(t))$. The result follows by showing that

$$\sum_{i=1}^{n}[f(t_i) - f(t_{i-1})][b_H(t_i) - b_H(t_{i-1})] \xrightarrow[n\to\infty]{L^2(P^H)} 0.$$

Using Hölder's inequality, we get

$$\left\{\mathrm{E}\left[\sum_{i=1}^{n}[f(t_i) - f(t_{i-1})][b_H(t_i) - b_H(t_{i-1})]\right]^2\right\}^{1/2}$$

$$\leqslant \sum_{i=1}^{n} \left\{\mathrm{E}\left[\{f(t_i) - f(t_{i-1})\}^2\{b_H(t_i) - b_H(t_{i-1})\}^2\right]\right\}^{1/2}$$

$$\leqslant \sum_{i=1}^{n} \left(\mathrm{E}\left[\{f(t_i) - f(t_{i-1})\}^{2p}\right]\right)^{1/(2p)} \left(\mathrm{E}\left[\{b_H(t_i) - b_H(t_{i-1})\}^{2q}\right]\right)^{1/(2q)}$$

$$\leqslant \left\{\sum_{i=1}^{n} \left(\mathrm{E}\left[(f(t_i) - f(t_{i-1}))^{2p}\right]\right)^{1/p}\right\}^{1/2}$$

$$\times \left\{\sum_{i=1}^{n} \left(\mathrm{E}\left[(b_H(t_i) - b_H(t_{i-1}))^{2q}\right]\right)^{1/q}\right\}^{1/2} \leqslant C_p \left\{\sum_{i=1}^{n} |t_i - t_{i-1}|^{2H}\right\}^{1/2}$$

The last term goes to 0 as n goes to infinity because $H > 1/2$. Thus the result follows. □

As pointed out in Biagini et al. (2008, page 137), a more general result can be proved. If for a choice of $\xi_i \in [t_i, t_{i-1}]$, the sum:

$$\sum_{i=1}^{n} f(\xi_i)(b_H(t_i) - b_H(t_{i-1}))$$

converges to $\int_0^t f(u)\, \mathrm{d}^s\, b_H(u)$, it converges for any other choice.

If $f \in C^1$ and $H > \frac{1}{2}$ the notions of symmetric pathwise integral $\int_0^t f(b_H(u))\, \mathrm{d}b_H^s(u)$ and integral $\int_0^t f(b_H(s))\, \mathrm{d}b_H^-(s)$, are the same. To obtain such a result, let us compute first the covariation between $Y(s) = f(b_H(s))$ and $b_H(s)$. For the definition of the covariation, see (Biagini et al., 2008, Chap. 5, p. 124). By definition the absolute value of this covariation is

$$\begin{aligned}
|[Y, B_H]_t| &= \left| \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t [Y(u+\varepsilon) - Y(u)][b_H(u+\varepsilon) - b_H(u)] \, \mathrm{d}u \right| \\
&\leqslant \boldsymbol{C}(\omega) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t [b_H(u+\varepsilon) - b_H(u)]^2 \, \mathrm{d}u \\
&\leqslant \boldsymbol{C}(\omega) \lim_{\varepsilon \to 0} \frac{\varepsilon^{2H-2\delta}}{\varepsilon} = 0,
\end{aligned}$$

where we used the fact that

$$\sup \left\{ \left| \dot{f}(z) \right| : |z| \leqslant \sup_{0 \leqslant s \leqslant 2t} |b_H(s)| \right\}$$

is a finite random variable and the modulus of continuity of b_H, see Proposition 2.1. The following equality gives the result

$$\int_0^t f(b_H(u)) \, \mathrm{d}^s \, b_H(u) = \int_0^t f(b_H(s)) \, \mathrm{d}b_H^-(s) + \frac{1}{2}[Y, b_H]_t.$$

Let us prove now for $H > \frac{1}{2}$ the equality between the two definitions of pathwise integrations that is, $\mathrm{d}b_H^-$ and $\mathrm{d}b_H$. Let f be a continuously differentiable function, the following equality holds

$$\begin{aligned}
\int_0^t f(b_H(s)) \, \mathrm{d}b_H^-(s) &= \lim_{|\Delta^n| \to 0} \sum_{i=1}^n f(b_H(t_{i-1}))[b_H(t_i) - b_H(t_{i-1})] \\
&= \int_0^t f(b_H(s)) \, \mathrm{d}b_H(s).
\end{aligned}$$

Indeed, consider the step function $f_\Delta(s) = \sum_{i=1}^n f(b_H(t_{i-1})) \mathbb{1}_{(t_{i-1};t_i]}(s)$; $f_\Delta(s)$ converges boundedly almost surely to $f(b_H(s))$ when $|\Delta^n| \to 0$. Moreover

$$\begin{aligned}
\int_0^t f_\Delta(s) \, \mathrm{d}b_H^-(s) &= \lim_{\varepsilon \to 0} \int_0^t f_\Delta(s) \frac{b_H(s+\varepsilon) - b_H(s)}{\varepsilon} \, \mathrm{d}s \\
&= \lim_{\varepsilon \to 0} \sum_{i=1}^n f(b_H(t_{i-1})) \int_{t_{i-1}}^{t_i} \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \mathrm{d}b_H(u) \, \mathrm{d}s \\
&= \lim_{\varepsilon \to 0} \sum_{i=1}^n f(b_H(t_{i-1})) \left[\int_{t_{i-1}}^{t_{i-1}+\varepsilon} \frac{u - t_{i-1}}{\varepsilon} \, \mathrm{d}b_H(u) \right. \\
&\qquad \left. + \int_{t_{i-1}+\varepsilon}^{t_i} \mathrm{d}b_H(u) + \int_{t_i}^{t_i+\varepsilon} \frac{t_i - u + \varepsilon}{\varepsilon} \, \mathrm{d}b_H(u) \right] \\
&= \sum_{i=1}^n f(b_H(t_{i-1}))[b_H(t_i) - b_H(t_{i-1})].
\end{aligned}$$

The last equality follows from an integration by parts of the first and third terms of the second last equality because both tend to 0 as ε goes to 0.

Taking the limit in both sides when $|\Delta^n| \to 0$, the result follows.

2.2.2 Complex Wiener Chaos

The discussion in this section comes from Major (1981). Let W be a complex centered Gaussian random measure on $\mathbb{R}$ with Lebesgue control measure $\mathrm{d}x$ such that, for any Borel set A of $\mathbb{R}$ we have $W(-A) = \overline{W(A)}$ almost surely. We consider complex-valued functions ψ defined on $\mathbb{R}$ for almost every $x \in \mathbb{R}$,

$$\overline{\psi(x)} = \psi(-x).$$

We write $L^2_e(\mathbb{R})$ for the real vector space of the functions that are square integrable with respect to the Lebesgue measure on $\mathbb{R}$. Endowed with the scalar product of $L^2(\mathbb{R})$, which we also note

$$\langle \psi, \varphi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi(x)\overline{\varphi(x)}\,\mathrm{d}x,$$

$L^2_e(\mathbb{R})$ is a real separable Hilbert space. Moreover, for any $\psi \in L^2_e(\mathbb{R})$, one can define its stochastic integral with respect to W as

$$I_1(\psi) = \int_{\mathbb{R}} \psi(x)\,\mathrm{d}W(x).$$

Then $I_1(\psi)$ is a real centered Gaussian variable with variance given by $\|\psi\|_2^2$, where $\|\cdot\|_2$ is the norm induced by the scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$. To introduce the k-th Itô-Wiener integral, with $k \geqslant 1$, we consider the complex functions belonging to

$$L^2_e(\mathbb{R}^k) = \{\psi \in L^2(\mathbb{R}^k) : \psi(-x) = \overline{\psi(x)}\}.$$

The inner product in the real Hilbert space of complex functions of $L^2_e(\mathbb{R}^k)$ is given by

$$\langle \psi, \varphi \rangle_{L^2(\mathbb{R}^k)} = \int_{\mathbb{R}^k} \psi(x)\overline{\varphi(x)}\,\mathrm{d}x.$$

The space $L^2_s(\mathbb{R}^k)$ denotes the subspace of functions of $L^2_e(\mathbb{R}^k)$ a.e. invariant under permutations of their arguments. By convention $L^2_s(\mathbb{R}^k) = \mathbb{R}$ for $k = 0$. Let us define $H(W)$ the subspace of random variables in $L^2(\Omega)$ measurable with respect to W. The k-Itô-Wiener integral I_k is defined in such a way that $(k!)^{-1/2} I_k$ is an isometry between $L_s^2(\mathbb{R}^k)$ and its range $\mathcal{H}_k \subset H(W)$, so that we have the orthogonal decomposition

$$H(W) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

where $\mathcal{H}_0$ is the space of real constants. Each $Y \in H(W)$ has a $L^2(\Omega, P)$ convergent decomposition

$$Y = \sum_{k=0}^{\infty} I_k(\psi_k), \quad \psi_k \in L_s^2(\mathbb{R}^k).$$

2.3 Hypothesis and Notation

We give the definitions of the fBm and of the Hermite polynomials. Mehler's formula is recalled. The covariance function at different scales of time of the second order increments of the fBm is also brought in this section as well as the definition of a functional variation of the fBm, for a general function including the definition of the absolute k-power variation. Finally, the definitions of the associated asymptotic variances are also presented in different scales of time.

Let $\{b_H(t), t \in \mathbb{R}\}$ be a fBm with Hurst parameter H such that $0 < H < 1$, see for instance Samorodnitsky and Taqqu (1994, Chapter 7). The covariance function of this centered Gaussian process is:

$$\mathrm{E}\left[b_H(t)b_H(s)\right] = \frac{1}{2}\, v_{2H}^2 \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$$

where $v_{2H}^2 = [\Gamma(2H+1)\sin(\pi H)]^{-1}$.

Here, let us recall that Hermite polynomials, denoted by H_p, are defined by

$$\exp(tx - \tfrac{1}{2}t^2) = \sum_{p=0}^{+\infty} \frac{H_p(x)t^p}{p!}.$$

Hermite polynomials form an orthogonal system for the standard Gaussian measure $\phi(x)\,\mathrm{d}x$. If $h \in L^2(\phi(x)\,\mathrm{d}x)$ then there exist coefficients h such that $h(x) = \sum_{p=0}^{+\infty} h_p H_p(x)$.

Also recall that Mehler's formula (see Breuer and Major 1983) gives a simple form to compute the covariance between two L^2 functions of Gaussian random variables. In fact, if $k \in L^2(\phi(x)\,\mathrm{d}x)$ and is written as $k(x) = \sum_{p=0}^{+\infty} k_p H_p(x)$ and if (X, Y) is a Gaussian random vector with correlation ρ and unit variance then

$$\mathrm{E}\left[h(X)k(Y)\right] = \sum_{p=0}^{+\infty} h_p k_p p! \rho^p. \tag{2.3}$$

We define the Hermite rank of k as the smallest p such that the coefficient k_p is different from 0.

Let g be a function in $L^2(\phi(x)\,\mathrm{d}x)$ such that

$$g(x) = \sum_{p=1}^{+\infty} g_p H_p(x), \text{ with } \|g\|_{2,\phi}^2 = \sum_{p=1}^{+\infty} g_p^2 p! < +\infty.$$

Let A_g be the set $\{p : p \geqslant 2 \text{ and } g_p \neq 0\}$.

Let Z be a random process on the interval $[0,\ 1]$. For an integer $n \geqslant 2$, let

$$\Delta_n Z(i) = \frac{n^H}{\sigma_{2H}}\,\delta_n Z(i), \quad i = 0, 1, \ldots, n-2,$$

where δ_n is given by

$$\delta_n Z\,(i) = \left[Z\left(\tfrac{i+2}{n}\right) - 2\,Z\left(\tfrac{i+1}{n}\right) + Z\left(\tfrac{i}{n}\right)\right],$$

and where

$$\sigma_{2H}^2 = v_{2H}^2\left(4 - 2^{2H}\right).$$

Also, if Y_n is a random variable defined on the subset $\{0, 1, \ldots, n-2\}$, we define the random variable Y_n^* on the interval $[0,\ 1]$ by

$$Y_n^*(u) = Y_n(i) \text{ if } u \in \left[\tfrac{i}{n-1}, \tfrac{i+1}{n-1}\right[.$$

Thus the process $\Delta_n b_H$ is a centered stationary Gaussian process with variance 1. Its covariance function is given by $\rho_H(i-j)$ for $i, j = 0, 1, \ldots, n-2$, where for any real number x, $\rho_H(x)$ is

$$\rho_H(x) = \frac{1}{2\,(4-2^{2H})}\Big[-6\,|x|^{2H} + 4\,|x+1|^{2H} - |x+2|^{2H} - |x-2|^{2H} + 4\,|x-1|^{2H}\Big].$$

In the following, σ_g^2 stands for the following summation:

$$\sigma_g^2 = \sum_{p=1}^{+\infty} g_p^2 p! \left(\sum_{r=-\infty}^{+\infty} \rho_H^p(r) \right).$$

Note that since $\sum_{r=-\infty}^{+\infty} \rho_H(r) = 0$, then

$$\sigma_g^2 = \sum_{p=2}^{+\infty} g_p^2 p! \left(\sum_{r=-\infty}^{+\infty} \rho_H^p(r) \right).$$

More generally, for $x \in \mathbb{R}$ and $b, c \in \mathbb{R}^*$, we define

$$\begin{aligned}\rho_{b,c}(x) = \frac{1}{2\,(4-2^{2H})}(bc)^{-H} \Big[&- |x|^{2H} + 2\,|x-b|^{2H} - |x-2b|^{2H} \\ &+2\,|x+c|^{2H} - 4\,|x+c-b|^{2H} + 2\,|x+c-2b|^{2H} - |x+2c|^{2H} \\ &+2\,|x+2c-b|^{2H} - |x+2c-2b|^{2H} \Big] \\ = \rho_{c,b}(-x)&\end{aligned}$$

and note that $\rho_{1,1}(x) = \rho_H(x)$. With this definition, we get

$$\mathrm{E}[\Delta_{bn} b_H(i) \Delta_{cn} b_H(j)] = \rho_{b,c}(ci - bj).$$

For $k, \ell \in \mathbb{N}^*$, we also define

$$\rho_g(k, \ell) = \frac{1}{\sqrt{k\ell}} \sum_{p=1}^{+\infty} g_p^2 p! \left(\sum_{s=0}^{k-1} \sum_{r=-\infty}^{+\infty} \rho_{k,\ell}^p(kr + \ell s) \right).$$

Since $\rho_{b,c}(x) = \rho_{b/c,1}(x/c)$ it follows that $\rho_g(k,k) = \sigma_g^2$.

For all $m \in \mathbb{N}^*$, for all $\boldsymbol{k} = (k_1, \ldots, k_m) \in (\mathbb{N}^*)^m$ and for all $\boldsymbol{d} = (d_1, \ldots, d_m) \in \mathbb{R}^m$, we denote by $\sigma_{g,m}^2(\boldsymbol{k}, \boldsymbol{d})$ the following sum:

$$\sigma_{g,m}^2(\boldsymbol{k}, \boldsymbol{d}) = \sum_{i=1}^{m} \sum_{j=1}^{m} d_i\, d_j\, \rho_g(k_i, k_j).$$

For $n \in \mathbb{N}^*$ and $t \in [0,\ 1]$, let

$$S_{g,n}(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor nt \rfloor - 2} g(\Delta_n b_H(i)), \tag{2.4}$$

with $S_{g,n}(t) = 0$ if $\lfloor nt \rfloor \leqslant 1$ and where $\lfloor x \rfloor$ denotes the integer part of the positive real number x.

Remark 2.8. We chose to work with the double increment operator rather than the simple increment operator.

In fact, one of our goals is to study the asymptotic behavior of functionals of the fBm increments in order to estimate its Hurst parameter H. If we use the simple ones and if the Hermite rank of the functional is two, we get to distinguish between three cases, $0 < H < \frac{3}{4}$, $H = \frac{3}{4}$ and $\frac{3}{4} < H < 1$.

It does not really make sense to tackle the estimation problem of H with such distinctions. When $H < \frac{3}{4}$ we get a Gaussian limit and if $H > \frac{3}{4}$, the convergence takes place in the second order Wiener chaos, and more generally in the ℓth order Wiener chaos ($\ell \geqslant 1$), ℓ being the Hermite rank of the functional. Finally, if $H = \frac{3}{4}$, a Gaussian limit is obtained through a convenient normalization.

We can refer for this case study to one of the first papers on the subject Guyon and León (1989) and Corcuera et al. (2006). This case study has also been considered by Berzin and León (2005). A classification of the possible limits is provided for the different values of H according to the Hermite rank. These results are obtained in the more general context of functionals of the regularization derivative, the regularization being obtained by the convolution of the fBm with a kernel φ. In the particular case where $\varphi = \mathbb{1}_{[-1,\,0]}$, this derivative is just the first order increments of the fBm.

The idea of working with higher order differences to diminish the long memory effect is not new. Istas and Lang (1997) is one of the pioneer works on the subject; it uses the filter notion.

León and Ludeña (2007) is one of the first papers working with the double increments. A Gaussian limit is obtained for all the H ranks, $0 < H < 1$.

Note that in Berzin and León (2005) previously cited, this latter convergence is obtained for the second derivative of the smoothed fBm. The particular case based on the kernel $\varphi = \mathbb{1}_{[-1,\,0]} * \mathbb{1}_{[0,\,1]}$ leads to the double increments of the fBm.

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Chapter 3
Estimation of the Parameters

3.1 Introduction

The first theorem of this chapter establishes the almost sure convergence for the k-power second order increments of the fBm toward the k-th moment of a standard normal distribution. Then we give the rate of this convergence in law. Moreover, for a general functional variation of the fBm, see (2.4), page 40, including the absolute k-power variation, the result remains true. This allows us to propose several estimators of the Hurst parameter H of a fBm using classical linear regression. The first one, $\hat{H}_k$, uses the function $|x|^k$, and the second one, $\hat{H}_{\log}$, uses the Napierian logarithm and both lead to unbiased consistent estimators.

A Central Limit Theorem (CLT) is also obtained for both estimators. These estimators are linked in the sense that if $k(n)$ is a sequence of positive numbers converging to zero with n, and if $\hat{H}_{k(n)}$ denotes the corresponding estimator of the H parameter, we establish that the asymptotic behaviors of $\hat{H}_{k(n)}$ and of $\hat{H}_{\log}$ are the same.

The same techniques can be used to provide simultaneous estimators of parameter H and of the local variance σ, in four particular simple models all driven by a fBm. As before, a regression model can be written and least squares estimators of H and of σ are defined. These estimators are built on the second order increments of the stochastic process solution of the proposed model. We prove their consistency and a CLT is given for both of them.

Furthermore, we consider testing the hypothesis $\sigma_n = \sigma$ against an alternative in the four previous models.

Finally, we propose functional estimation of the local variance of general stochastic differential equation (SDE). This estimation is based on the observation of the second order increments of the solution of such an SDE. We highlight that to show the convergence in these models, it is sufficient to prove it in the special case where the solution process is the fBm.

C. Berzin et al., *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lecture Notes in Statistics 216,
DOI 10.1007/978-3-319-07875-5_3,

3.2 Estimation of the Hurst Parameter

We propose several estimators of the H parameter for fBm, through the observation of one trajectory, on a regular grid of time points.

We study two estimators, say $\hat{H}_k$ and $\hat{H}_{\log}$, respectively built with the k-power and the Napierian logarithm of the modulus of the second order increments of fBm. Working with the order two increments allows the estimation of H over all its range, $]\,0,\ 1\,[$, and provides consistent and asymptotic normal estimators.

We also give the explicit link between $\hat{H}_{k(n)}$ and $\hat{H}_{\log}$, $k(n)$ being a sequence of positive numbers converging to zero when n goes to infinity. We state properties and a CLT for the estimator $\hat{H}_{k(n)}$.

3.2.1 Almost Sure Convergence for the Second Order Increments

We present the almost sure convergence in law for the second order increments of the fBm, seen as a variable on $([\,0,\ 1\,], \lambda)$, where λ is the Lebesgue measure.

Theorem 3.1. *For all* $0 < H < 1$, *almost surely for all* $k \in \mathbb{N}^*$,

$$\frac{1}{n-1}\sum_{i=0}^{n-2}(\Delta_n b_H(i))^k \underset{n\to+\infty}{\longrightarrow} \mathrm{E}[N]^k\,.$$

Corollary 3.2 is a direct consequence of Theorem 3.1.

Corollary 3.2. *For all* $0 < H < 1$, *almost surely*

$$(\Delta_n b_H)^* \xrightarrow[n\to\infty]{Law} N.$$

The above convergence is in law, the random variable $(\Delta_n b_H)^*$ *is seen as a variable on* $([\,0,\ 1\,], \lambda)$ *where* λ *is the Lebesgue measure.*

From Theorem 3.1 and Corollary 3.2, we deduce Corollary 3.3.

Corollary 3.3. *For all* $0 < H < 1$, *almost surely for all* $k \in \mathbb{R}^{+*}$,

$$\frac{1}{n-1}\sum_{i=0}^{n-2}|\Delta_n b_H(i)|^k \underset{n\to+\infty}{\longrightarrow} \mathrm{E}[|N|^k]\,.$$

3.2.2 Convergence in Law of the Absolute k-Power Variation

We establish out the finite-dimensional convergence in law for a g functional variation of the fBm, function g being centered and such that $g \in L^2(\phi(x)\,\mathrm{d}x)$. The obtained limit is a cylindrical centered Gaussian process.

Theorem 3.4. *For all* $0 < H < 1$,

$$S_{g,\cdot n}(1) \xrightarrow[n\to\infty]{\text{Law}} X,$$

where X is a cylindrical centered Gaussian process with covariance $\rho_g(k,\ell) = \mathrm{E}[X(k)X(\ell)]$, $k,\ \ell \in \mathbb{N}^*$.

The above convergence is in the sense of finite-dimensional distributions.

Remark 3.5. If g has a finite expansion with respect to the Hermite basis, then $\mathrm{E}\big[S_{g,n}(1)\big]^4 \leqslant \boldsymbol{C}$, for n large enough.

Remark 3.6. $S_{g,2^n+\cdot}(1) \xrightarrow[n\to\infty]{\text{Law}} X$, where X is a cylindrical centered Gaussian process with covariance $\rho_g(k,\ell)$ defined by

$$\rho_g(k,\ell) = 2^{(k-\ell)/2} \sum_{p=1}^{+\infty} g_p^2\, p! \left(\sum_{r=-\infty}^{+\infty} \rho^p_{1,2^{\ell-k}}(r) \right)$$

when $k \leqslant \ell$, and then X is a stationary process.

Remark 3.7. If $k \in \mathbb{N}^*$ is fixed, $S_{g,kn}(1) \xrightarrow[n\to\infty]{\text{Law}} \sigma_g N$.

The two following lemmas can be used to show that $\rho_g(k,\ell)$ is a covariance function and are proved in Sect. 5.2.2.

Lemma 3.8. *For all $m \in \mathbb{N}^*$, for all $\boldsymbol{k} \in (\mathbb{N}^*)^m$ and for all $\boldsymbol{d} \in \mathbb{R}^m$,*

$$\sigma^2_{g,m}(\boldsymbol{k},\boldsymbol{d}) = \lim_{n\to+\infty} \mathrm{E}\left[\sum_{i=1}^m d_i\, S_{g,k_i n}(1)\right]^2,$$

and then $\sigma^2_{g,m}(\boldsymbol{k},\boldsymbol{d}) \geqslant 0$.

Lemma 3.9. *For all $k,\ell \in \mathbb{N}^*$, $\rho_g(k,\ell) = \rho_g(\ell,k)$.*

3.2.3 Estimators of the Hurst Parameter

We chose to work with the centered functions g_k and $g_{\log}$, respectively defined by $g_k(x) = |x|^k - \mathrm{E}[|N|^k]$ and by $g_{\log}(x) = \log(|x|) - \mathrm{E}[\log(|N|)]$, in the definition of functional variation of the fBm. This choice allows us, via a regression model, to propose two estimators of the parameter H, say $\hat{H}_k$ and $\hat{H}_{\log}$. Their properties are studied here.

A third estimator of H, say $\hat{H}_{k(n)}$, links the two previous estimators and its properties are also studied; more, a CLT is given.

For $n \in \mathbb{N}^* - \{1\}$ and for $k \in \mathbb{R}^{+*}$, let us define

$$M_k(n) = \frac{1}{n-1} \sum_{i=0}^{n-2} |\delta_n b_H(i)|^k . \tag{3.1}$$

Thanks to Corollary 3.3,

$$\left(\frac{n^H}{\sigma_{2H} \, \|N\|_k} \right)^k M_k(n) \xrightarrow[n \to +\infty]{a.s.} 1.$$

Then,

$$kH \, \log(n) - k \, \log(\sigma_{2H} \, \|N\|_k) + \log(M_k(n)) \xrightarrow[n \to +\infty]{a.s.} 0.$$

Thus

$$\log(M_k(n)) = -kH \, \log(n) + k \, \log(\sigma_{2H} \, \|N\|_k) + o_{a.s.}(1). \tag{3.2}$$

Let $n_i = r_i n$, $r_i \in \mathbb{N}^*$, $i = 1, \ldots, \ell$. Equation (3.2) can be written as a classical linear regression equation:

$$Y_i = a X_i + k \, b_k + \xi_i, \; i = 1, \ldots, \ell,$$

where $a = H$ and for $i = 1, \ldots, \ell$, $Y_i = \log(M_k(n_i))$, $X_i = -k \log(n_i)$ and $b_k = \log(\sigma_{2H} \, \|N\|_k)$.

Hence, the least squares estimator $\hat{H}_k$ of H is given by

$$\hat{H}_k = -\frac{1}{k} \sum_{i=1}^{\ell} z_i \, \log(M_k(n_i)), \tag{3.3}$$

where for $i = 1, \ldots, \ell$,

$$z_i = \frac{y_i}{\sum_{i=1}^{\ell} y_i^2} \text{ and } y_i = \log(r_i) - \frac{1}{\ell} \sum_{i=1}^{\ell} \log(r_i). \tag{3.4}$$

Note the following property

$$\sum_{i=1}^{\ell} y_i = 0 \text{ and } \sum_{i=1}^{\ell} z_i \, y_i = 1. \tag{3.5}$$

Corollary 3.10 follows from Theorem 3.4.

Corollary 3.10. *For $k \in \mathbb{R}^{+*}$,*

(1) $\hat{H}_k$ is an asymptotically unbiased strongly consistent estimator of H.
(2) Furthermore,

$$\sqrt{n}\left(\hat{H}_k - H\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0, \sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\,\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right)\right),$$

where

$$g_k(x) = \frac{|x|^k}{\mathrm{E}[|N|^k]} - 1 = \sum_{p=1}^{\infty} g_{2p,k} H_{2p}(x), \tag{3.6}$$

with

$$g_{2p,k} = \frac{1}{(2p)!} \prod_{i=0}^{p-1} (k - 2i)\,. \tag{3.7}$$

Remark 3.11. As in Berzin and León (2007) and Cœurjolly (2001), for $k = 2$, the variance $\sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\,\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right)$ is minimal. This fact is shown in Sect. 5.2.3, after the proof of Corollary 3.10.

Remark 3.12. For $k = 2$ and $r_i = 2^{i-1}$, for $i = 1, \ldots, \ell$, the asymptotic variance of $\sqrt{n}\,\hat{H}_k$ is

$$\left(\frac{6}{\log(2)}\right)^2 \frac{1}{\ell^2\,(\ell^2-1)^2} \left(2 \sum_{i<j;\,i,j=1}^{\ell} 2^{-j}\,(2i-(\ell+1))\,(2j-(\ell+1)) \times \right.$$
$$\left. \sum_{r=-\infty}^{+\infty} \rho^2_{1,2^{j-i}}(r) + \sum_{i=1}^{\ell} 2^{-i}(2i-(\ell+1))^2 \sum_{r=-\infty}^{+\infty} \rho^2_H(r)\right).$$

Now, let us define

$$M_{\log}(n) = \frac{1}{n-1} \sum_{i=0}^{n-2} \log\left(|\delta_n b_H(i)|\right). \tag{3.8}$$

Lemma 3.8 following Theorem 3.4, also entails that

$$\frac{1}{n-1}\sum_{i=0}^{n-2} g_{\log}\left(\Delta_n b_H(i)\right) \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

as well as it converges to 0 in $L^2(\Omega)$, the function $g_{\log}$ is defined by (3.11).

Thus

$$\frac{1}{n-1}\sum_{i=0}^{n-2} \log\left(|\Delta_n b_H(i)|\right) \xrightarrow[n\to\infty]{\mathrm{P}} \mathrm{E}[\log(|N|)],$$

i.e.

$$M_{\log}(n) = -H\ \log(n) + \log(\sigma_{2H}) + \mathrm{E}[\log|N|] + o_p(1). \tag{3.9}$$

Proceeding as before the least squares estimator $\hat{H}_{\log}$ of H is given by

$$\hat{H}_{\log} = -\sum_{i=1}^{\ell} z_i\ M_{\log}(n_i). \tag{3.10}$$

Theorem 3.4 leads the following corollary.

Corollary 3.13. *(1)* $\hat{H}_{\log}$ *is an unbiased weakly consistent estimator of* H*.*
(2) Furthermore,

$$\sqrt{n}\left(\hat{H}_{\log} - H\right) \xrightarrow[n\to\infty]{\mathrm{Law}} \mathcal{N}\left(0, \sigma^2_{g_{\log},\ell}\left(\boldsymbol{r}, \frac{\boldsymbol{z}}{\sqrt{\boldsymbol{r}}}\right)\right),$$

where

$$g_{\log}(x) = \log(|x|) - \mathrm{E}[\log(|N|)] = \sum_{p=1}^{\infty} g_{2p,\log} H_{2p}(x), \tag{3.11}$$

with

$$g_{2p,\log} = \frac{(-1)^{p-1}}{2p(2p-1)!!}. \tag{3.12}$$

Remark 3.14. As shown in Sect. 5.2.3 after the proof of Corollary 3.13, the variance $\sigma^2_{g_{\log},\ell}\left(\boldsymbol{r}, \frac{\boldsymbol{z}}{\sqrt{\boldsymbol{r}}}\right)$ is always greater than $\sigma^2_{g_2,\ell}\left(\boldsymbol{r}, \frac{1}{2}\left(\frac{\boldsymbol{z}}{\sqrt{\boldsymbol{r}}}\right)\right)$ and $\sigma^2_{g_4,\ell}\left(\boldsymbol{r}, \frac{1}{4}\left(\frac{\boldsymbol{z}}{\sqrt{\boldsymbol{r}}}\right)\right)$.

Remark 3.15. In the case where $r_i = 2^{i-1}$, for $i = 1, \dots, \ell$, the asymptotic variance of $\sqrt{n}\,\hat{H}_{\log}$ is

$$\left(\frac{3}{\log(2)}\right)^2 \frac{1}{\ell^2\,(\ell^2-1)^2}\left(2\sum_{i<j;i,j=1}^{\ell} 2^{-j+1}\,(2i-(\ell+1))\,(2j-(\ell+1))\times\right.$$
$$\sum_{p=1}^{+\infty}(2p)!\left(\frac{1}{p(2p-1)!!}\right)^2\sum_{r=-\infty}^{+\infty}\rho_{1,2^{j-i}}^{2p}(r)+\sum_{i=1}^{\ell}2^{-i+1}(2i-(\ell+1))^2\times$$
$$\left.\sum_{p=1}^{+\infty}(2p)!\left(\frac{1}{p(2p-1)!!}\right)^2\sum_{r=-\infty}^{+\infty}\rho_H^{2p}(r)\right).$$

We can link the two estimators $\hat{H}_k$ and $\hat{H}_{\log}$. For this, let $k(n)$ be a sequence of positive numbers converging to zero as n tends to infinity and let $\hat{H}_{k(n)}$ be the corresponding estimator, say

$$\begin{aligned}\hat{H}_{k(n)} &= -\sum_{i=1}^{\ell} z_i\,\frac{\log(M_{k(n_i)}(n_i))}{k(n_i)}\\ \text{where, }\ M_{k(n)}(n) &= \frac{1}{n-1}\sum_{i=0}^{n-2}|\delta_n b_H(i)|^{k(n)}.\end{aligned}\tag{3.13}$$

We have the following corollary.

Corollary 3.16. *If $k(n) = o(1/\sqrt{n})$ then $\hat{H}_{k(n)}$ is an asymptotically unbiased weakly consistent estimator of H and the asymptotic behaviors of $\sqrt{n}\left(\hat{H}_{k(n)} - H\right)$ and $\sqrt{n}\left(\hat{H}_{\log} - H\right)$ are the same.*

3.3 Estimation of the Local Variance

In this section, we give two kinds of results concerning the estimation of the local variance σ.

First we provide simultaneous estimators of parameters H and σ in four simple SDE driven by a fBm. These estimators come from a regression model and are built on the second order increments of the stochastic process solution of the SDE. We study their properties and a CLT is obtained.

The estimation procedure leads to a loss of convergence rate for the CLT for the estimator of function σ. However, if H is known, an other estimator of σ is proposed, giving the actual convergence rate for the CLT. Then we propose an hypothesis test on σ in the context of the four previous models. Finally, we propose functional estimation of the function σ in a general pseudo-diffusion driven by a fBm.

3.3.1 Simultaneous Estimation of the Hurst Parameter and of the Local Variance

We propose simultaneous estimators of the parameter H and of the local variance σ for solutions of the SDE:

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t.$$

Four cases are considered: depending on the form of functions σ and μ: $\sigma(x) = \sigma$ or σx and $\mu(x) = \mu$ or μx.

Using results of Sects. 3.2.1 and 3.2.2, we obtain consistent estimators of H and σ. Observing the second order increments of X at several scales of the parameter time, we obtain regression models that give least squares estimators of H and σ. A CLT is stated for both of them.

As a bonus, if H is supposed to be known, we propose an other estimator of H based on the absolute k-power of the second order increments of X. A CLT is also stated, the rate of convergence being better than in the case where we perform simultaneous estimation.

We would like to provide simultaneous estimators of H and σ in the four following models. For $H > \frac{1}{2}$ and $t \geqslant 0$

$$\mathrm{d}X(t) = \sigma\,\mathrm{d}b_H(t) + \mu\,\mathrm{d}t, \tag{3.14}$$

$$\mathrm{d}X(t) = \sigma\,\mathrm{d}b_H(t) + \mu X(t)\,\mathrm{d}t, \tag{3.15}$$

$$\mathrm{d}X(t) = \sigma X(t)\,\mathrm{d}b_H(t) + \mu X(t)\,\mathrm{d}t, \tag{3.16}$$

$$\mathrm{d}X(t) = \sigma X(t)\,\mathrm{d}b_H(t) + \mu\,\mathrm{d}t, \tag{3.17}$$

with $X(0) = c$.

The solutions of these equations are respectively:

$$(3.14): \quad X(t) = \sigma b_H(t) + \mu t + c,$$

$$(3.15): \quad X(t) = \sigma b_H(t) + \exp(\mu t)\left[\sigma\mu\left(\int_0^t b_H(s)\exp(-\mu s)\,\mathrm{d}s\right) + c\right],$$

$$(3.16): \quad X(t) = c\exp(\mu t + \sigma b_H(t)),$$

$$(3.17): \quad X(t) = \exp(\sigma b_H(t))\left(c + \mu\int_0^t \exp(-\sigma b_H(s))\,\mathrm{d}s\right).$$

For (3.14), see Lin (1995) and for (3.16), as detailed in Sect. 2.2.1 by (2.2), see Cutland et al. (1995) and Klingenhöfer and Zähle (1999).

We consider the problem of estimating simultaneously H and $\sigma > 0$. Suppose X is observed on a grid $\{\frac{i}{n}, i = 0, 1, \ldots, n\}$ so that the increments $\delta_n X(i)$, for $i = 0, 1, \ldots, n-2$, can be computed.

For models (3.16) and (3.17) we will suppose that $c \neq 0$. For model (3.17) we will make the additional hypothesis that μ and c have the same sign or that μ is eventually null.

From now on, we shall note for each $n \in \mathbb{N}^* - \{1\}$ and $i \in \{0, 1, \ldots, n-2\}$,

$$\varGamma_n X(i) = \begin{cases} \varDelta_n X(i), & \text{for the first two models} \\ \dfrac{\varDelta_n X(i)}{X(\frac{i}{n})}, & \text{for the other two.} \end{cases} \tag{3.18}$$

In a similar way, we define $\gamma_n X(i)$ substituting δ_n to $\varDelta_n$ in the last expression.

For a real number $k \geqslant 1$, let us denote

$$A_k^X(n) = \frac{1}{\sigma^k \, \|N\|_k^k} \left(\frac{1}{n-1} \sum_{i=0}^{n-2} |\varGamma_n X(i)|^k \right) - 1. \tag{3.19}$$

Corollary 3.3 allows us to state the following theorem.

Theorem 3.17. *(1) For each real $k \geqslant 1$,*

$$A_k^X(n) \xrightarrow[n\to\infty]{a.s.} 0.$$

(2) Furthermore

$$\frac{n-1}{\sqrt{n}} A_k^X(n) = S_{g_k, n}(1) + o_{a.s.}(1),$$

where the function g_k is defined by (3.6).

At this step, we can propose estimators of H and σ, by observing $\gamma_n X(i)$ at several scales of the parameter n, i.e. $n_i = r_i \, n$, $r_i \in \mathbb{N}^*$, $i = 1, \ldots, \ell$. In this aim, let us define

$$M_k^X(n) = \frac{1}{n-1} \sum_{i=0}^{n-2} |\gamma_n X(i)|^k . \tag{3.20}$$

Using assertion (1) of Theorem 3.17, we get

$$\left(\frac{n^H}{\sigma \, \sigma_{2H} \, \|N\|_k} \right)^k M_k^X(n) \xrightarrow[n\to+\infty]{a.s.} 1,$$

from which we obtain

$$\log(M_k^X(n)) = -k\,H\,\log(n) + k\,\log(\sigma\,\sigma_{2H}\,\|N\|_k) + o_{a.s.}(1). \tag{3.21}$$

The following regression model can be written, for each scale n_i:

$$Y_i = aX_i + k\,b_k + \xi_i,\; i = 1,\dots,\ell,$$

where $a = H$, $b_k = \log(\sigma\,\sigma_{2H}\,\|N\|_k)$ and for $i = 1,\dots,\ell$, $Y_i = \log(M_k^X(n_i))$, $X_i = -k\,\log(n_i)$. Hence, the least squares estimators $\hat{H}_k$ of H and $\hat{B}_k$ of b_k are defined as

$$\hat{H}_k = -\frac{1}{k}\sum_{i=1}^{\ell} z_i\,\log(M_k^X(n_i)), \tag{3.22}$$

and

$$\hat{B}_k = \frac{1}{k}\left(\frac{1}{\ell}\sum_{i=1}^{\ell}\log(M_k^X(n_i))\right) + \hat{H}_k\left(\frac{1}{\ell}\sum_{i=1}^{\ell}\log(n_i)\right), \tag{3.23}$$

where z_i are defined by (3.4).

Finally, we propose as an estimator of σ

$$\hat{\sigma}_k = \frac{\exp(\hat{B}_k)}{\sigma_{2\hat{H}_k}\,\|N\|_k}. \tag{3.24}$$

Theorems 3.17 and 3.4 imply the following results for any H in the interval $]\,\frac{1}{2},\;1\,[$.

Theorem 3.18. *For each real $k \geqslant 1$,*

(1) $\hat{H}_k$ is a strongly consistent estimator of H and

$$\sqrt{n}\left(\hat{H}_k - H\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0, \sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\,\left(z/\sqrt{\boldsymbol{r}}\right)\right)\right),$$

where the function g_k is defined by (3.6) and the coefficients $g_{2p,k}$ by (3.7).

(2) $\hat{\sigma}_k$ is a weakly consistent estimator of σ and

$$\frac{\sqrt{n}}{\log(n)}\left(\hat{\sigma}_k - \sigma\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0, \sigma^2\,\sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \tfrac{1}{k}\,\left(z/\sqrt{\boldsymbol{r}}\right)\right)\right).$$

Remark 3.19. As in Corollary 3.10, the asymptotic variance $\sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\left(z/\sqrt{\boldsymbol{r}}\right)\right)$ is minimal for $k = 2$ and then the best estimators for H and σ in the sense of minimal variance are obtained for $k = 2$.

Theorem 3.17 also provides estimators for σ when H is known. Indeed, for each real $k \geqslant 1$ we set,

$$\tilde{\sigma}_k = \frac{\left(\frac{1}{n-1}\sum_{i=0}^{n-1}|\Gamma_n X(i)|^k\right)^{1/k}}{\|N\|_k}$$

where $\Gamma_n X$ is given by (3.18).

Theorem 3.20 follows from Theorem 3.17 and Remark 3.7.

Theorem 3.20. *For each real $k \geqslant 1$, if H is known, $\frac{1}{2} < H < 1$, then*

(1) $\tilde{\sigma}_k$ is a strongly consistent estimator of σ and
(2)

$$\sqrt{n}(\tilde{\sigma}_k - \sigma) \xrightarrow[n\to\infty]{Law} \mathcal{N}\left(0, \frac{\sigma^2}{k^2}\sigma^2_{g_k}\right),$$

where the function g_k is defined by (3.6).

Remark 3.21. Note that the rate of convergence in assertion (2) is $\sqrt{n}$ instead of $\sqrt{n}/\log(n)$ as it is in assertion (2) in Theorem 3.18. This is due to the fact that here H is known.

Remark 3.22. The variance $\sigma^2_{g_k}/k^2$ is minimal for $k = 2$ and then the best estimator for σ in the sense of minimal variance is obtained for $k = 2$.

This fact will be shown in Sect. 5.3.1 after the proof of Theorem 3.20.

3.3.2 Hypothesis Testing

Tests of hypothesis on σ are proposed, for the four models proposed in Sect. 3.3.1, where parameter H is supposed to be known.

We test the hypothesis $\sigma_n = \sigma$ against $\sigma_n = \sigma + \frac{1}{\sqrt{n}}(d + F(\sqrt{n}))$, where d is positive constant and F a positive function tending to zero with n. An evaluation of the asymptotic power of the test is made.

Let us consider the four stochastic differential equations, for known H, $H > \frac{1}{2}$, $t \geqslant 0$ and $n \in \mathbb{N}^*$,

$$\mathrm{d}X_n(t) = \sigma_n\ \mathrm{d}b_H(t) + \mu_n\ \mathrm{d}t, \tag{3.25}$$

$$\mathrm{d}X_n(t) = \sigma_n\ \mathrm{d}b_H(t) + \mu_n\ X_n(t)\ \mathrm{d}t,$$

$$\mathrm{d}X_n(t) = \sigma_n\, X_n(t)\ \mathrm{d}b_H(t) + \mu_n\, X_n(t)\ \mathrm{d}t, \tag{3.26}$$

$$\mathrm{d}X_n(t) = \sigma_n\, X_n(t)\ \mathrm{d}b_H(t) + \mu_n\ \mathrm{d}t, \tag{3.27}$$

with $X_n(0) = c$.

We consider testing the hypothesis

$$H_0 : \sigma_n = \sigma,$$

against the alternatives

$$H_n : \sigma_n = \sigma + \frac{1}{\sqrt{n}}\,(d + F(\sqrt{n})),$$

where σ, d are positive constants, F is a positive function such that $F(\sqrt{n})$ converges to 0 as $n \to \infty$ and μ_n is supposed bounded, possibly except for model (3.25). For models (3.26) and (3.27) we will suppose that $c \neq 0$. For model (3.27) we will make the additional hypothesis that μ_n and c have the same sign or that μ_n is eventually null.

The reason why we must choose this sequence of alternatives H_n, is that we are interested in the asymptotic behavior of the test. If the alternative is fixed, the two hypotheses are well separated. Then at the end when n goes to infinity, our test always chooses one of the two hypotheses. However for a sequence of alternatives tending to the hypothesis H_0, with a rate of convergence similar to the one of the CLT, it would be more difficult to choose. A good result to discriminate between one of the hypotheses can be understood as a proof of the quality of the test.

By Sect. 3.3.1 for each model there exists an unique solution to the stochastic equation, say X_n. We are interested in observing the following functionals

$$F_n = \sqrt{n}\left[\sqrt{\frac{\pi}{2}}\frac{1}{n-1}\sum_{i=0}^{n-2}|\varGamma_n X_n(i)| - \sigma\right],$$

where $\varGamma_n X_n$ is defined by (3.18), where we replaced X by X_n.

Using Corollary 3.3 and Remark 3.7, we can prove the following theorem.

Theorem 3.23. *Suppose that H is known with $1/2 < H < 1$, then*

$$F_n \xrightarrow[n\to\infty]{\text{Law}} \sigma_{g_1}\,\sigma\, N + d,$$

where the function g_1 is defined by (3.6).

Remark 3.24. There is an asymptotic bias d, and the larger is the bias the easier is discriminating between the two hypotheses.

Remark 3.25. X_n plays the role of X, in Sect. 3.3.1, with $\sigma_n = \sigma$ and $\mu_n = \mu$.

3.3.3 Functional Estimation of the Local Variance

We propose functional estimation of the local variance σ in the following model: $\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t$. Some regularity conditions need to be satisfied by the functions σ and μ.

At this state we give an outline of the proof of the results. Indeed we do the remark that when $\mu \equiv 0$, the solution for the previous SDE can be expressed as $X(t) = K(b_H(t))$, where K is solution of an ordinary differential equation (ODE).

Then we explain how in that case results concerning functional estimation for σ can be held by considering the particular case where the solution process of the SDE is a fBm. In the case where the function μ is not necessarily null, we use the Girsanov's theorem.

We consider the following equation with respect to b_H:

$$X(t) = c + \int_0^t \sigma(X(u))\,\mathrm{d}b_H(u) + \int_0^t \mu(X(u))\,\mathrm{d}u, \tag{3.28}$$

for $t \geqslant 0$, $H > 1/2$ and positive σ. We want to estimate σ.

In this aim, we consider the following assumptions on the coefficients μ and σ:

(H1) – σ is a Lipchitz function on $\mathbb{R}$ of class C^1, bounded and bounded away from zero.
– There exists some constant η, $1/H - 1 < \eta \leqslant 1$, and for every $N > 0$, there exists $M_N > 0$ such that

$$|\dot{\sigma}(x) - \dot{\sigma}(y)| \leqslant M_N\ |x-y|^{\eta},\ \forall\ |x|,\ |y| \leqslant N.$$

(H2) – μ is C^1, bounded and Lipchitz function on $\mathbb{R}$.

Remark 3.26. Hypotheses (H1) and (H2) require that σ is bounded and bounded away from zero and that μ is C^1 and bounded. These two last assumptions can be relaxed, ensuring that there exists an unique process solution of the stochastic equation (3.28).

Furthermore, X will almost-surely have $(H-\delta)$-Hölder continuous trajectories on all compact set included in $\mathbb{R}^+$ (see Nualart and Răşcanu 2002).

Using similar arguments to the ones of Theorem 3.1 we can prove the following theorem.

Theorem 3.27. *Let $\frac{1}{2} < H < 1$, under hypotheses (H1) and (H2), almost surely for all continuous function h and for all real $k \geqslant 1$ then,*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(X(\tfrac{i}{n}))\,\frac{|\Delta_n X(i)|^k}{\mathrm{E}[|N|^k]} \underset{n\to+\infty}{\longrightarrow} \int_0^1 h(X(u))\ [\sigma(X(u))]^k\,\mathrm{d}u.$$

Remark 3.28. If $\mu \equiv 0$, hypotheses (H1) and (H2) can be replaced by $\sigma \in C^1$.

Moreover, we can also obtain the following theorem giving the convergence rate in the last theorem.

Theorem 3.29. *Let us suppose that $\frac{1}{2} < H < 1$, $h \in C^2$, $\sigma \in C^2$, σ is bounded and bounded away from zero and $\sup\{|\ddot{\sigma}(x)|, |\ddot{h}(x)|\} \leqslant P(|x|)$, where P is a polynomial, then under hypotheses (H1) and (H2) and for all real $k \geqslant 1$,*

$$\sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(X(\tfrac{i}{n}))\,\frac{|\Delta_n X(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(X(u))\ [\sigma(X(u))]^k\,\mathrm{d}u\right]$$

$$\xrightarrow[n\to\infty]{Law} \sigma_{g_k}\int_0^1 h(X(u))\ [\sigma(X(u))]^k\,\mathrm{d}\hat{W}(u),$$

where g_k is defined by (3.6) and $\hat{W}$ is a standard Brownian motion independent of b_H.

Remark 3.30. If $\mu \equiv 0$, hypotheses (H1) and (H2) can be relaxed and convergence becomes stable convergence.

We give here an outline of the proofs of Theorems 3.27 and 3.29 in order to state two other quite interesting theorems.

On the one hand, we consider the case where $\mu \equiv 0$ and prove Remarks 3.28 and 3.30 in this case.

On the other hand, we consider the case where μ is not necessarily null. We then prove Theorems 3.27 and 3.29 using Remarks 3.28 and 3.30 and Girsanov's theorem given in Decreusefond and Üstünel (1999).

Indeed, in the case where $\mu \equiv 0$ and $\sigma \in C^1$, as seen in the introduction, X is solution of ODE (2.2). More precisely, since b_H has zero quadratic variation when $H > \frac{1}{2}$, Lin (1995) proved that the solution for the SDE (3.28) can be expressed as $X(t) = K(b_H(t))$, for $t \geqslant 0$, where $K(t)$ is the solution of the ODE

$$\dot{K}(t) = \sigma(K(t)); \quad K(0) = c, \tag{3.29}$$

see also (2.2). We then need the two following lemmas for which proofs are provided in Chap. 6 (pages 110 and 111).

Lemma 3.31. *In model (3.28), if $H > \frac{1}{2}$, $\mu \equiv 0$ and $\sigma \in C^1$, then for $i = 0, 1, \ldots, n-2$,*

$$\Delta_n X(i) = \sigma(X(\tfrac{i}{n}))\,\Delta_n b_H(i) + a_n(i),$$

with

$$|a_n(i)| \leqslant \boldsymbol{C}(\omega)\,\left(\tfrac{1}{n}\right)^{H-\delta}, \quad \textit{for any } \delta > 0.$$

This lemma allows us to enunciate the following one.

Lemma 3.32. *In model (3.28), let $H > \frac{1}{2}$, $\mu \equiv 0$ and $\sigma \in C^1$. Then almost surely, for all continuous function h and for all real $k \geqslant 1$, we have*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(X(\tfrac{i}{n}))\,\{|\Delta_n X(i)|^k - [\sigma(X(\tfrac{i}{n}))]^k\,|\Delta_n b_H(i)|^k\} = o\left(\tfrac{1}{\sqrt{n}}\right).$$

Thus if we choose $f = h \circ K \cdot (\sigma \circ K)^k$ in following theorem, that will be enough to obtain Remark 3.28.

Theorem 3.33. *Let $0 < H < 1$, almost surely for all continuous function f and for all real $k > 0$ then,*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} f(b_H(\tfrac{i}{n}))\,\frac{|\Delta_n b_H(i)|^k}{\mathrm{E}[|N|^k]} \xrightarrow[n\to\infty]{\text{a.s.}} \int_0^1 f(b_H(u))\,\mathrm{d}u.$$

Now let us remark that almost surely for all C^1 function f and for all $H > \frac{1}{2}$, one has

$$\left(\int_0^1 f(b_H(u))\,\mathrm{d}u - \frac{1}{n-1}\sum_{i=0}^{n-2} f(b_H(\frac{i}{n}))\right) = o(\tfrac{1}{\sqrt{n}}).$$

Thus using once again Lemma 3.32 and last equality, Remark 3.30 will ensue from the following theorem.

Theorem 3.34. *Let us suppose $\frac{1}{2} < H < 1$, $f \in C^2$ and $\left|\ddot{f}(x)\right| \leqslant P(|x|)$, where P is a polynomial, then for all real $k > 0$,*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-2} f(b_H(\tfrac{i}{n}))\,g_k(\Delta_n b_H(i))$$

stably converges as n goes to infinity toward

$$\sigma_{g_k}\int_0^1 f(b_H(u))\,\mathrm{d}\hat{W}(u).$$

Here $\hat{W}$ is still a standard Brownian motion independent of b_H and g_k is defined by (3.6).

Remark 3.35. Let g be a general function with four moments with respect to the standard Gaussian measure, even, or odd, with Hermite rank greater than or equal to one and such that $A_g \neq \emptyset$ (for the definition of A_g, see Sect. 2.3). It can be proved that, under the same hypotheses on H and f,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-2} f(b_H(\tfrac{i}{n}))\, g(\Delta_n b_H(i))$$

stably converges as n goes to infinity toward $\sigma_g \int_0^1 f(b_H(u))\, \mathrm{d}\hat{W}(u)$. Furthermore if $f \in C^4$ and $\left|\overset{4\cdot}{f}(x)\right| \leqslant P(|x|)$, this result is still valid under the weaker hypothesis that $H > \frac{1}{4}$ and under the supplementary condition, in case where g is odd, that g has Hermite rank greater than or equal to three.

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Chapter 4
Simulation Algorithms and Simulation Studies

4.1 Introduction

In this chapter, we present the basic ideas for the simulation of a stationary Gaussian process from which we deduce the simulation of a fBm and the simulation of processes driven by a fBm.

Our approach is based on the Durbin-Levinson's algorithm. Since the process formed by the first order increments of a fBm is a stationary Gaussian one, we first simulate the increments of the process and then, by a simple "integration", we obtain a trajectory of the fBm. For models defined by differential equations, first an observation of the fBm is generated and then, it is transformed according to differential equation.

Simulating these processes, we can explore the statistical properties of the estimators defined in the previous chapter from an empirical point of view. We study the distribution of the estimators of H and of σ. Special attention is devoted to the construction of a confidence interval for H. Some simulation results concern the estimation of the parameters of a pure fBm, some others are for the parameters of models that are excited by a fBm.

In the first two simulation studies, the uniform generator is based on three linear congruential generators (cf. Press et al. 2007, p. 196). Random normal deviates are obtained by Box and Muller's method (see Knuth 1981, p. 104).

In the other simulations studies, uniform deviates are obtained by a linear congruential generator given in Langlands et al. (1994, p. 36) and for the normal deviates, we use Algorithm M described in Knuth (1981). It is a very fast generator. Pascal programs are given in Chap. 8.

C. Berzin et al., *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lecture Notes in Statistics 216,
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4.2 Computing Environment

The computation resources required for the simulation studies are quite important. As we will see in Sect. 4.5.1.1, the simulation studies are supported by a design of experiment in which 10,000 trajectories are simulated. An important time machine is needed to achieve this goal. We decided to use compiled code to perform these computations. The Pascal language was retained. We wrote the programs in the Apple's Macintosh environment with the Mac OS X operating system. Two compilers were used: the GNU Pascal compiler (see http://www.gnu-pascal.de/gpc/h-index.html) and the Free Pascal compiler (see http://www.freepascal.org/).

4.3 Random Generators

For the first two simulation studies, concerning the uniform generator, the reader is referred to Press et al. (2007). For the third simulation study, in conjunction with the following linear congruential generator:

$$x_{i+1} = (a\, x_i + c) \mod m$$

with

$$a = 142\,412\,240\,584\,757 = (4 \times 35\,603\,060\,146\,189) + 1, \quad m = 2^{48}, \quad c = 11,$$

(see Langlands et al. 1994, p. 36), we use Marsaglia's algorithm for the normal deviates.

This algorithm is very fast, easily implemented and described in details in Knuth (1981, p. 122). A Pascal implementation is given in Chap. 8, page 160. Implementation of the congruential generator is not straightforward: clever programming is required to avoid overflows. Again, see Chap. 8, page 160 for a Pascal implementation.

There are good reasons to prefer Marsaglia's algorithm to Box and Muller's method. In the following, we will shortly describe the approach. To increase the procedure performance, some programming ingenuity was also brought by Knuth who asserts that the final version of this algorithm "*is a very pretty example of mathematical theory intimately interwoven with programming ingenuity—a fine illustration of the art of computer programming!*"

Marsaglia's algorithm is primarily aimed at generating X, the absolute value of a standard Gaussian variable with distribution function F given by (4.1).

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}\, \mathrm{d}t, \quad x > 0. \tag{4.1}$$

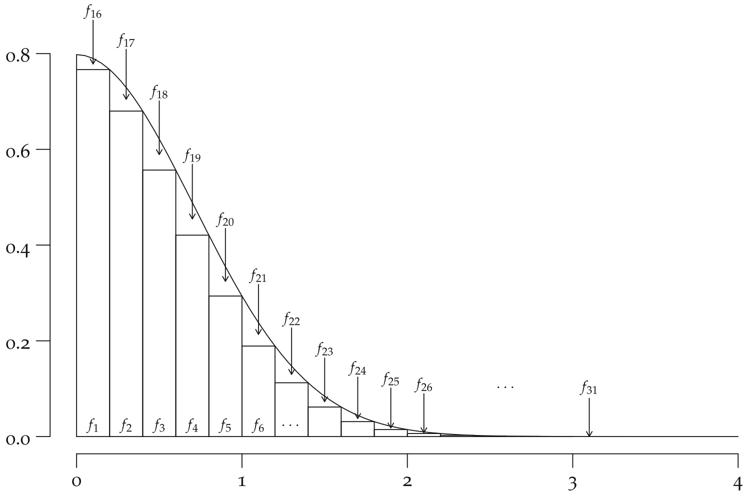

Fig. 4.1 Density function of X broken into 31 parts. The area of each part is the probability p_j of selecting the associated distribution. Knuth, Donald K, The Art of Computer Programming, Volume 2: Seminumerical Algorithms, 2nd Edition, ©1981. Reprinted by permission of Pearson Education, Inc., Upper Saddle River, NJ

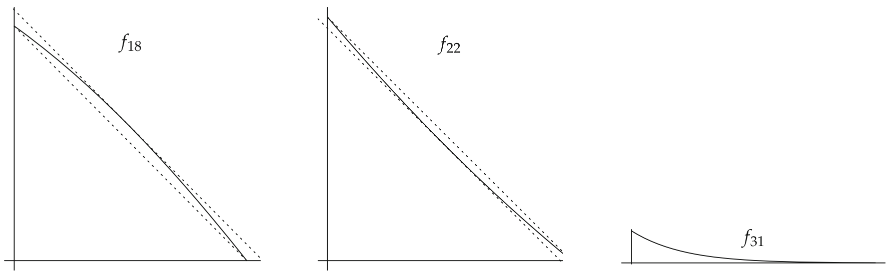

Fig. 4.2 Wedge-shaped densities (f_{18} and f_{22}) and the tail of the distribution (f_{31}). Knuth, Donald K, The Art of Computer Programming, Volume 2: Seminumerical Algorithms, 2nd Edition, ©1981. Reprinted by permission of Pearson Education, Inc., Upper Saddle River, NJ

A negative sign is then given to this absolute value with probability $\frac{1}{2}$ to get a normal deviate. The distribution function F can be seen as a mixture of several distributions:

$$F(x) = \sum_{i=1}^{31} p_i F_i(x)$$

where $F_1, \dots, F_{31}$ are appropriate distributions and $p_1, \dots, p_{31}$ are probabilities. More precisely, to generate X, first we chose F_j with probability p_j and then we generate a random deviate according to this distribution.

The density of X is represented in Fig. 4.1. There are three types of distributions: rectangular ($f_1, \dots, f_{15}$), wedge-shaped ($f_{16}, \dots, f_{30}$) and the tail (f_{31}). Magnified views of these different types are presented in Fig. 4.2. Obviously, rectangular parts correspond to uniform variables. We easily see that

$$p_j = \sqrt{\frac{2}{25\pi}} e^{-j^2/50}, \quad j = 1, \ldots, 15$$

and also note that $\sum_{j=1}^{15} p_j \approx 0.92$. It shows that 92 % of the normal deviates are generated using an uniform generator that does not require important computing resources.

For nearly linear densities like f_{18} or f_{22}, a very efficient algorithm, based on a rejection approach, has been designed, see Algorithm L in Knuth (1981, p. 121). More computer time is required only when density f_{31} is chosen. But this distribution needs to be treated with probability ≈ 0.00270.

Based on Walker's alias method, Knuth (1981, exercise 7, p. 134), the random choice of the distribution f_j, $j = 1, \ldots, 31$, is cleverly done. In fact, the choice of f_j among $\{f_1, \ldots, f_{31}\}$ corresponds to a random experiment, whose outcome is C and that can be described in the following way.

Let $\Omega = \{f_1, \ldots, f_{31}\}$. Let U be an uniform variate and define C as

$$C = \begin{cases} f_1, & \text{if } 0 \leqslant U < p_1; \\ f_2, & \text{if } p_1 \leqslant U < p_1 + p_2; \\ \vdots & \\ f_{31}, & \text{if } p_1 + p_2 + \cdots + p_{30} \leqslant U < 1; \end{cases} \tag{4.2}$$

where and $p_1 + p_2 + \cdots + p_{31} = 1$.

As mentioned by Knuth (1981, page 115), there is a best possible way to do the comparisons of U against the various values of $p_1 + p_2 + \cdots + p_s$, as implied in (4.2) and known as Walker's alias method (see Kronmal and Peterson 1979).

We do recommend the reading of Knuth (1981, p. 119–123).

4.4 Simulation of a Stationary Gaussian Process and of the fBm

Based on the ideas of the Durbin-Levinson algorithm, we tackle the problem of simulating a stationary Gaussian process. Since the increments of a fBm is a stationary Gaussian process, we simulate a trajectory of the increments and by a simple "integration", we obtain a trajectory of the fBm.

First, we considered the standardized process of the simple differences of a fBm:

$$\Delta_n^{(1)} b_H(i) = \frac{n^H}{v_{2H}} \left(b_H(\tfrac{i}{n}) - b_H(\tfrac{i-1}{n}) \right),$$

for $i = 1, 2, \ldots, n$. This sequence is a centered stationary Gaussian vector. The covariance function is denoted by $\gamma_H(i-j)$, for $i, j = 1, 2, \ldots, n$, and for $x \in \mathbb{R}$,

$$\gamma_H(x) = \frac{1}{2}\left[|x+1|^{2H} - 2\,|x|^{2H} + |x-1|^{2H}\right].$$

Our primary aim is to simulate $(Y_1, \ldots, Y_n)^\top$ a vector with a covariance structure identical to that of $(\Delta_n^{(1)} b_H(i))_{i\in\{1,\ldots,n\}}$.

The idea consists in writing

$$\begin{aligned} Y_{k+1} &= \phi_{k,1} Y_k + \cdots + \phi_{k,k} Y_1 + a_{k+1} \\ &= \hat{Y}_{k+1} + a_{k+1} \end{aligned}$$

where coefficients $\phi_{k,j}$, $j = 1, \ldots, k$, are chosen at each step k to minimize:

$$\mathrm{E}[(Y_{k+1} - \hat{Y}_{k+1})^2] = \mathrm{E}[a_{k+1}^2].$$

The covariance matrix of $\boldsymbol{Y}_k = (Y_1, \ldots, Y_k)^\top$ is a Tœplitz symmetrical matrix given by $\boldsymbol{\Gamma}_k$:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \gamma_H(0) & \gamma_H(1) & \gamma_H(2) & \ldots & \gamma_H(k-1) \\ \gamma_H(1) & \gamma_H(0) & \gamma_H(1) & \ldots & \gamma_H(k-2) \\ \gamma_H(2) & \gamma_H(1) & \gamma_H(0) & \ldots & \gamma_H(k-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_H(k-1) & \gamma_H(k-2) & \gamma_H(k-3) & \ldots & \gamma_H(0) \end{bmatrix}$$

The Durbin-Levinson algorithm has been originally designed to recursively predict the value of time series at time $(\nu+1)$ given the values at times $1, \ldots, \nu$. An easy adaptation of this algorithm can be done to simulate observations of a fBm on a regular grid on the interval $[\,0,\ 1\,]$.

The Durbin-Levinson algorithm allows to find the $\phi_{k,j}$, $j = 1, \ldots, k$ coefficients in a recurrent way. A description of the algorithm follows:

Initialization. For $k = 1$, let

$$v_0 = \gamma_H(0); \quad \phi_{11} = \frac{\gamma_H(1)}{\gamma_H(0)}; \quad v_1 = v_0(1 - \phi_{11}^2).$$

Recurrence. For $k = 2, \ldots, n$

$$\phi_{k,k} = \frac{1}{v_{k-1}}\left[\gamma_H(k) - \sum_{j=1}^{k-1} \phi_{k-1,j}\,\gamma_H(k-j)\right],$$

$$\begin{bmatrix} \phi_{k,1} \\ \vdots \\ \phi_{k,k-1} \end{bmatrix} = \begin{bmatrix} \phi_{k-1,1} \\ \vdots \\ \phi_{k-1,k-1} \end{bmatrix} - \phi_{k,k} \begin{bmatrix} \phi_{k-1,k-1} \\ \vdots \\ \phi_{k-1,1} \end{bmatrix}$$

$$v_k = v_{k-1}[1 - \phi_{k,k}^2]$$

To simulate n successive observations of the process, we suggest to use the following recurrence:

$$Y_1 = a_1;$$

$$Y_{k+1} = a_{k+1} + \sum_{j=1}^{k} \phi_{k,j} Y_{k+1-j}, \quad k = 1, \ldots, n-1$$

where $\{a_k\}_{k=1,\ldots,n}$ is a sequence of independent variables $\mathcal{N}(0; v_{k-1})$, $k = 1, \ldots, n$. We assert that for all n, vector $(Y_1, \ldots, Y_n)^\top$ is $\mathcal{N}(\mathbf{0}_n; \boldsymbol{\Gamma}_n)$ (see Brockwell and Davis 1991).

Finally, to obtain a simulated trajectory of the fBm process, we let

$$\begin{cases} Z_0 = 0, & \text{if } k = 0; \\ Z_k = Z_{k-1} + Y_k, & \text{if } k > 0, \end{cases}$$

and $b_H(\frac{k}{n}) = \frac{v_{2H}}{n^H} Z_k$, $0 \leqslant k \leqslant n$. In fact, we simulate the simple increment process. Some trajectories are exhibited in Fig. 4.3.

For the models defined by a differential equation as in Sect. 3.3.1, first a trajectory of the fBm is generated and then, it is transformed according to differential equation. See Fig. 4.4.

4.5 Simulation Studies

In this section, we report three simulation studies. First, we describe the design of experiment. Then, referring to tables and figures of Chap. 7, a discussion of the results follows.

In the first simulation study, we empirically assess the estimation qualities of the different proposed estimators of H. A paragraph is devoted to the construction of a confidence interval. We compare the confidence interval based on the empirical distribution fractiles with the confidence interval based on a normal approximation.

In the second simulation study, we are interested in the joint estimation of H and σ for models defined in terms of a differential equation (see (3.14)–(3.17)).

In the third study, we want to assess the power of a test on σ. As in the two previous studies, discussion is based on tables and graphics presented in Chap. 7.

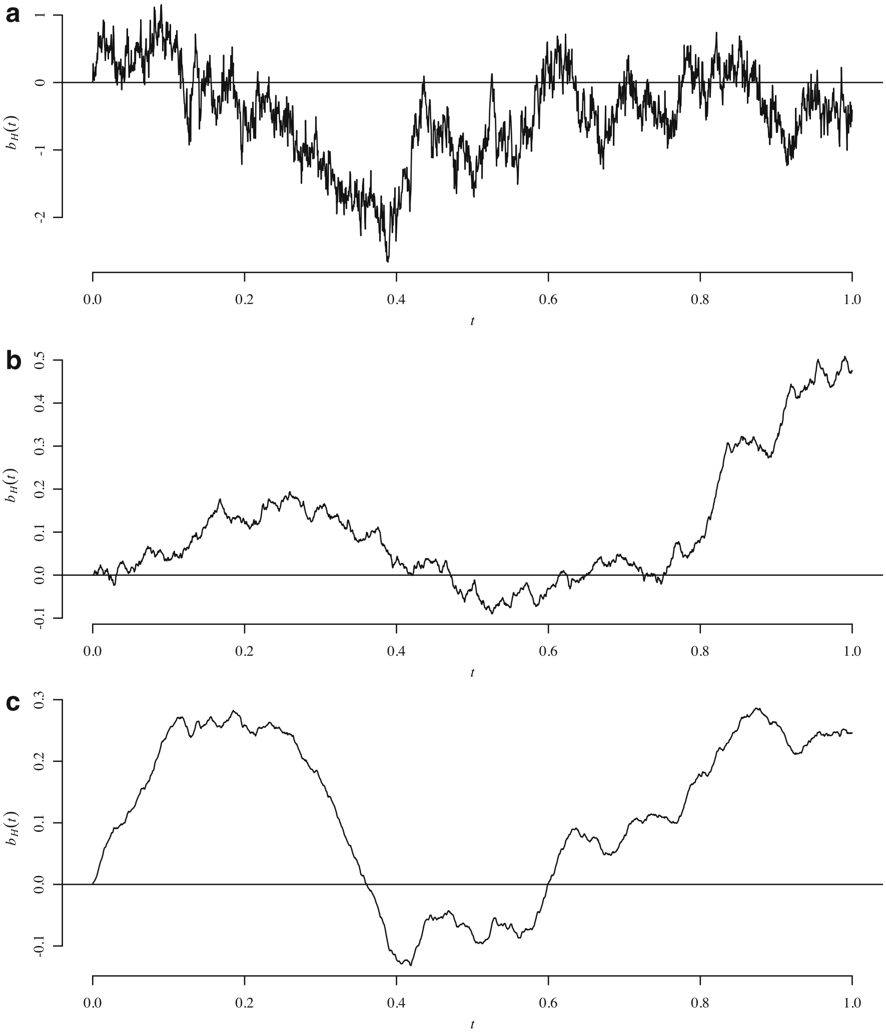

Fig. 4.3 FBm observed on a grid of 1/2,048-th on the interval $[0,\ 1]$: (**a**) with an Hurst parameter equal to 0.25; (**b**) with an Hurst parameter equal to 0.75; (**c**) with an Hurst parameter equal to 0.9

4.5.1 Estimators of the Hurst Parameter and the Local Variance Based on the Observation of One Trajectory

4.5.1.1 Design of the Experiment

In order to assess the quality of the estimation procedures, we used some reference values for H:

$$H \in \mathcal{H} = \{0.05,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.95\}.$$

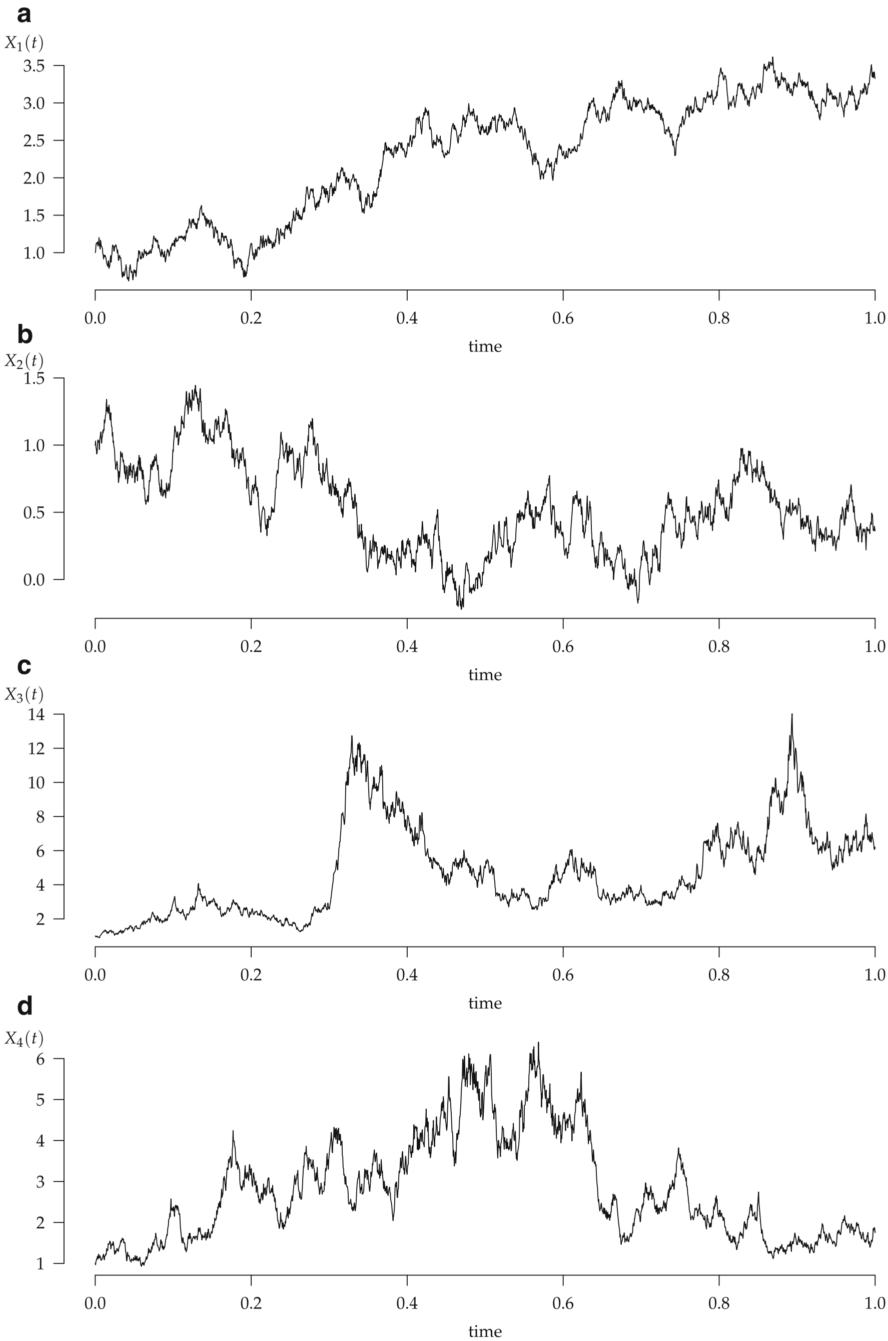

Fig. 4.4 Processes driven by a fBm using $H = \frac{1}{2}$ observed on a grid of 1/2,048-th on the interval $[0,\ 1]$ (**a**) Model 1; (**b**) Model 2; (**c**) Model 3; (**d**) Model 4 with $\mu = 2, \sigma = 2$ and $c = 1$

We computed the estimators $\hat{H}_k$, for $k = 1, 2, 3$ and 4; the estimator $\hat{H}_{\log}$ was also computed. The value of ℓ appearing in (3.3) and (3.10) was set to $\ell = 2, 3, 4$ and 5.

Simulation programs are written in Pascal and compiled using GNU Pascal 3.4.5 compiler under Mac OS X, 10.4.8.

For each value of $H \in \mathcal{H}$, 10,000 trajectories were simulated. To all these trajectories, we applied the estimation procedure for the values of k and ℓ previously given. The trajectories were observed at a higher resolution of 1/2,048-th.

Let us explain with some details the computation done for the case where $\ell = 2$. We got five subcases; estimation can be done for different choices of n_1 and n_2 as used in Eq. (3.3): these are $(64, 128)$, $(128, 256) \ldots (1{,}024, 2{,}048)$. Note that these n_i's are powers of 2. So, the number of points used in $M_k(n_2)$ is twice the number of points used in $M_k(n_1)$. Here and for future references n_2 is said to be the maximum number of points used in the summation for a given subcase.

All the tables and figures referred in this section are displayed in Chap. 7, starting on page 123.

4.5.1.2 Empirical Distributions of the Estimators of the Hurst Parameter

For this first study, basic statistics concerning the empirical distributions obtained by simulation appear in Tables 7.1–7.5 (pages 125–129). A close look to these tables can help to understand the following.

First, we know that $\hat{H}_k$ is biased while $\hat{H}_{\log}$ is not. If we compare the empirical biases $\hat{\mu}_{\hat{H}_{\log}} - H$ to $\hat{\mu}_{\hat{H}_k} - H$, we realize that they are quite the same. This assertion is illustrated taking $H = \frac{1}{2}$. There are no major differences between this case and the others. This can be seen in Figs. 7.2 and 7.3, pages 130 and 131. This remark also applies to the other values of H, as suggested by Tables 7.1–7.5. Note that the scales are all the same for an easy comparison between graphs.

The estimator $\hat{H}_{\log}$ is unbiased, nevertheless, as indicated by Fig. 7.3, it has the largest standard error. As we noted previously, all the other parameters being equal, the standard error is better with $k = 2$. Let us look at the particular case where $k = 2$, $H = 0.05$, $H = \frac{1}{2}$ and $H = 0.95$.

Obviously and as expected, a higher resolution in observing the trajectory produces a better estimation of H. For $\ell \geqslant 3$, results are quite similar. Sure, there is some gain in using a higher resolution of 1/2,048-th compare to 1/1,024-th.

Figure 7.1 on page 124 shows the distribution of $\hat{H}_2$ using a resolution of 1/2,048-th and $\ell = 5$. As we can see, the empirical distribution is very close to the normal distribution.

4.5.1.3 Confidence Intervals for H Using the Fractiles

Let us consider the optimal case with $k = 2$. Using these simulations results, we can give a confidence interval for H given an observed value $\hat{H}_{2;\mathrm{obs}}$. The idea

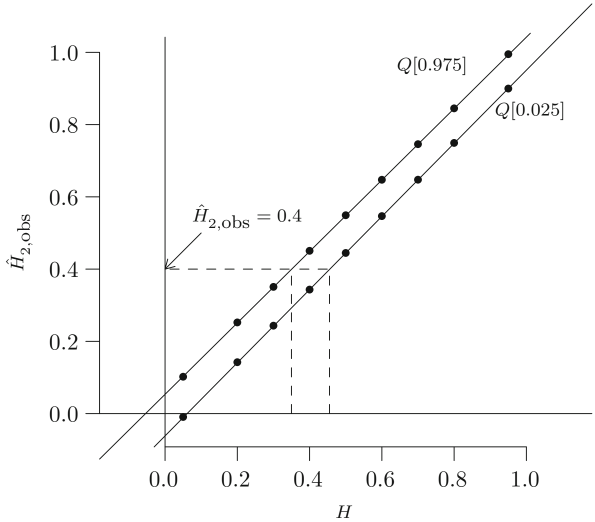

Fig. 4.5 Regression lines for $Q_{0.025}(H)$ and $Q_{0.975}(H)$ with $n = 2{,}048$, $k = 2$ and $\ell = 5$

is the following one. Suppose we would like to have a $1 - \alpha$ confidence interval. Let $Q_\beta(H)$ be the β-fractile of the sample distribution of $\hat{H}_2$, i.e. $Q_\beta(H)$ is such that $\Pr(\hat{H}_2 < Q_\beta(H) \mid H) = \beta$ and let

$$H_1 = \inf_H \{H : Q_{1-\alpha/2}(H) \geqslant \hat{H}_{2;\text{obs}}\} \quad \text{and} \quad H_2 = \sup_H \{H : Q_{\alpha/2}(H) \leqslant \hat{H}_{2;\text{obs}}\}.$$

To illustrate the procedure, we plotted the values of $Q_{0.025}(H)$ and $Q_{0.975}(H)$: see Fig. 4.5.[1] The points are quite close to straight lines.

For the observed estimated value $\hat{H}_{2;\text{obs}}$ of H, let us denote by $I_{1-\alpha}(\hat{H}_{2;\text{obs}})$ the interval

$$\begin{aligned} &I_{1-\alpha}(\hat{H}_{2;\text{obs}}) \\ &= \{H : Q_{1-\alpha/2}(H) \geqslant \hat{H}_{2;\text{obs}}\} \cap \{H : Q_{\alpha/2}(H) \leqslant \hat{H}_{2;\text{obs}}\} = [H_1,\ H_2]. \end{aligned} \tag{4.3}$$

Let $H^\star$ be the actual value of H. If $H^\star \notin I_{1-\alpha}(\hat{H}_{2;\text{obs}})$, then

$$Q_{1-\alpha/2}(H^\star) < \hat{H}_{2;\text{obs}} \text{ or } \hat{H}_{2;\text{obs}} < Q_{\alpha/2}(H^\star).$$

Note that $\forall H,\ Q_{1-\alpha/2}(H) > Q_{\alpha/2}(H)$, so

$$\{\hat{H}_2 : Q_{1-\alpha/2}(H) < \hat{H}_2\} \cap \{\hat{H}_2 : \hat{H}_2 < Q_{\alpha/2}(H)\} = \emptyset$$

[1] The procedure can be done for any confidence level.

and we get that

$$\Pr(\{\hat{H}_2 : Q_{1-\alpha/2}(H^\star) < \hat{H}_2\} \cup \{\hat{H}_2 : \hat{H}_2 < Q_{\alpha/2}(H^\star)\}) = \alpha$$

because the probability of both events is $\alpha/2$. We conclude that the probability that the actual value is recovered by the interval $I_{1-\alpha}(\hat{H}_{2;\text{obs}})$ is $1-\alpha$.

In general, the fractiles $Q_{1-\alpha/2}(H)$ and $Q_{\alpha/2}(H)$ can be estimated using the empirical distribution we got by simulation. Let us give an example with $\alpha = 0.05$. For $H = 0.3$, we got 0.2434 and 0.3508 respectively for $\tilde{Q}_{0.025}(0.3)$ and $\tilde{Q}_{0.975}(0.3)$; for $H = 0.5$, we got 0.4447 and 0.5491 respectively for $\tilde{Q}_{0.025}(0.5)$ and $\tilde{Q}_{0.975}(0.5)$. The regression line equations are:

$$\hat{Q}_{0.025}(H) = -0.06008 + 1.011H$$

$$\hat{Q}_{0.975}(H) = +0.05362 + 0.9901H$$

Using these regression lines we get:

H	$\hat{Q}_{0.025}(H)$	$\hat{Q}_{0.975}(H)$
0.3	0.2432	0.3507
0.5	0.4453	0.5487

These values are almost the same as the previous ones.

Now suppose that we observe $\hat{H}_2 = 0.4$. The smallest value of H such that $\hat{H}_2 = 0.4$ is recovered by $I_{0.95}(H)$ is 0.3498 whilst the greatest one is 0.4552. So, the "95% confidence interval" is[2]: $[0.3498,\ 0.4552]$.

Based on the previous simulation results we computed the coefficients of the regression lines and we wrote another simulation program to assess the confidence level of the procedure. In other words, we used other simulations to compute the values of $\hat{H}_2$ and check if H was recovered by the confidence interval. In all cases, the empirical confidence level was very close to 0.95. In fact, if the real covering probability is 0.95, the estimated value should be between 0.9365 and 0.9635 in 95 % of the cases. As we can see, there are only three cases, indicated by a "*" in Table 7.6, for which the value is outside this interval. So, the approximate procedure is quite satisfactory.

[2]These values are obtained using the following equations:

$$\text{UL}(\hat{H}_2) = 0.9893\,\hat{H}_2 + 0.05944$$

$$\text{LL}(\hat{H}) = 1.0010\,\hat{H}_2 - 0.05415$$

where UL and LL stand for upper and lower limits respectively.

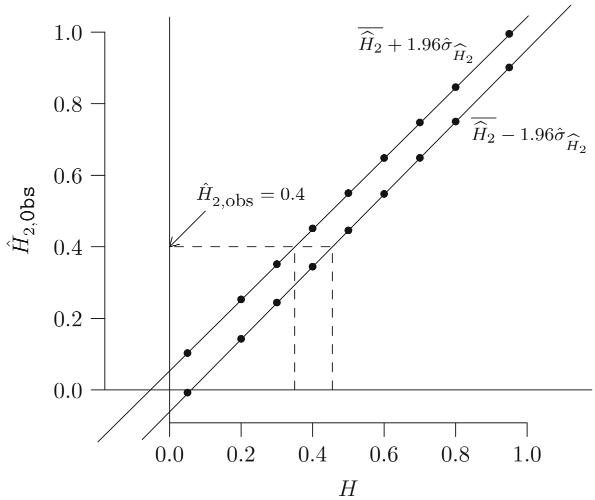

Fig. 4.6 Regression lines for the lower and upper limits of the interval using the normal approximation ($n = 2{,}048$, $k = 2$ and $\ell = 5$)

4.5.1.4 Confidence Intervals for H Using the Normal Approximation

The procedure corresponding to Fig. 4.5 can be compared with the one based on the normal approximation. From the first simulation study, for all the empirical distributions we can compute the standard deviation of $\hat{H}_2$. In Fig. 4.6, we plotted the values of

$$\overline{\hat{H}_2} \pm 1.96\hat{\sigma}_{\hat{H}_2}$$

as functions of H. The two straight lines are very close to the ones we get in Fig. 4.5. In the same way these regression lines can be used to determine confidence intervals. So we conducted another simulation study to assess the recovering probability of the proposed confidence interval. As one may expect, the results are as good as in the previous method. There are only two cases, again indicated by the symbol "*" in Table 7.7, for which the value is outside the interval $[0.9365,\ 0.9635]$. So, the approximate procedure is quite satisfactory.

4.5.2 Estimation of σ

In (3.14)–(3.17), $\{b_H(t),\ t \in \mathbb{R}\}$ is a fBm with parameter $0 < H < 1$. Note that if $H > \frac{1}{2}$, these processes are solutions of some specific stochastic differential equations. See Eq. (2.2) and Sect. 3.3.1 for details.

The parameter H is estimated using $\hat{H}_2$. To estimate σ, we use: $\hat{\sigma}_2$ as estimator of σ defined by

$$\hat{\sigma}_2 = \frac{\exp(\hat{B}_2)}{\sigma_{2\hat{H}_2}} \tag{4.4}$$

where

$$\hat{B}_2 = \frac{1}{2}\left(\frac{1}{\ell}\sum_{i=1}^{\ell}\log(M_2^X(n_i))\right) + \hat{H}_2\left(\frac{1}{\ell}\sum_{i=1}^{\ell}\log(n_i)\right).$$

4.5.3 Estimators of H and σ Based on the Observation of $X(t)$

We already studied the performance of $\hat{H}_2$, when we directly observe $b_H(t)$. Here, we study the distribution of $\hat{H}_2$ and the distribution of $\hat{\sigma}_2$ when trajectories are generated according to models proposed in Sect. 3.3.1: we observe $X(t)$ instead of $b_H(t)$.

The design of the experiment is the same. Simulations where done using $\mu = 2$, $\sigma = 2$ and $c = 1$ in (3.14)–(3.17). Simulation of the four processes is quite straightforward: first $b_H(t)$ is simulated and then it is transformed into $X(t)$.

For each value of $H \in \mathcal{H}$, 10,000 trajectories were simulated, for a total of 90,000. To all these trajectories we applied the estimation procedure for the values of k and ℓ previously given. The trajectories were observed at an highest resolution of 1/2,048-th.

All the basic statistics concerning the empirical distributions obtained by simulation appear in Tables 7.8–7.15. A close look at these tables can help to understand the following. Graphical representations are also provided in Figs. 7.4–7.11.

4.5.3.1 The First and Second Models

As indicated in Tables 7.8 and 7.10, the estimation of H by $\hat{H}_2$ in the case of the first two models leads to results almost identical to the results we got for the simple fBm process. All previous comments made previously for the simple fBm process still apply.

There is a major problem concerning the estimation of σ. (See Tables 7.9 and 7.11). The reader should bear in mind that when we estimate H, the computed value of $\hat{H}_2$ may be negative or greater than 1. Obviously, with a low resolution, if the actual value of H is close to 0, there is a quite important probability that $\hat{H}_2 \leqslant 0$. In the same way, if the actual value of H is close to 1, there is also a quite important probability that $\hat{H}_2 \geqslant 1$. For example, look at Table 7.9, with $H = 0.05$, $\ell = 2$ and a 128-point resolution, only 59.2 % of the estimated values $\hat{H}_2$ were in the interval $]\,0,\ 1\,[$; in all the other cases we got unacceptable values.

Looking at (4.4) we see that for H close to 0:

$$\hat{\sigma}_2 = \frac{\exp(\hat{B}_2)}{\sigma_{2\hat{H}_2}} \equiv \sqrt{\frac{\pi \hat{H}_2}{3}} \exp(\hat{B}_2)$$

and when $\hat{H}_2 < 0$, we dare say that there is nothing to do and we cannot estimate σ. In this case our computer program returns $\hat{\sigma}_2 = 0$ considered as a missing value.

In the case where $\hat{H}_2 > 1$, we suggest to use the limit value we get as $H \to 1$ which is:

$$\hat{\sigma}_2 \equiv \frac{1}{2}\sqrt{\frac{\pi}{\ln 2}} \exp(\hat{B}_2). \tag{4.5}$$

We know that for $H > \frac{1}{2}$, model 1 is the solution of a stochastic differential equation. So, if we are only interested in this case, we do not have to consider what happens for low values of H. An interesting point is the fact that if a 2,048-point resolution is used, with $\ell \in \{3, 4, 5\}$, about 95 % of the trajectories produce admissible values for $\hat{H}$.

The bias of $\hat{\sigma}_2$ is positive, but for $\ell = 4$ or 5, it is not so important. Let us mention that results seem to be slightly better for model 2 for low values of H. Over all the 10,000 trajectories, the average value of $\hat{\sigma}_2$ is a little higher than the actual value which is $\sigma = 2$. For this model, if H is not too close to 0, the estimation of σ seems to be acceptable as soon as the maximum resolution is 1,024 points and $\ell \geqslant 4$. As we may expect with a 2,048-point resolution, we have a quite small standard error decreasing with H. It is important to note that for $H = 0.95$, we got very interesting results, considering the correction proposed in Eq. (4.5).

4.5.3.2 The Third and Fourth Models

With models 3 and 4, the estimations are not as good as they are for the other two models. First, the average of $\hat{H}_2$ is often negative when $H = 0.05$, even with a 2,048-point resolution and $\ell = 5$. Otherwise, the bias is important for values of H less than 0.5. In general, the averages of $\hat{H}_2$ and the standard errors are similar for both models 3 and 4.

Concerning the estimation of σ, when $H = 0.05$, $\hat{\sigma}_2$ has a very poor performance. For example, with a 2,048-point resolution and $\ell = 5$, we have $\overline{\hat{\sigma}}_2 = 1.3 \times 10^{11}$ (In Tables 7.13 and 7.15 the notation $1.4^{(24)}$ stands for 1.4×10^{24}). In fact, for $H = 0.05$, the percentage of admissible estimated values is never higher than 50 %. Even with $H = 0.2$, we get major problems. We see in Tables 7.13 and 7.15 that the bias is important for values of H less than 0.5. Bias are better for $H \in \{0.6,\ 0.7,\ 0.8,\ 0.95\}$, but never as good as they are for models 1 and 2.

4.5.4 Hypothesis Testing

To study the performance of test on σ when the value of H is known, we choose to simulate 10,000 trajectories for each model. As in Sect. 4.5.3, the values of the parameters were: $\mu = 2$, $\sigma = 2$, $c = 1$. Here again, we consider different resolutions: $n \in \{128, 256, 512, 1{,}024, 2{,}048\}$. Five different values were used under H_1: $\sigma_i = 2 + 1/2^{8-i}$. So, under H_1, we look at values of σ in the interval $[2, 2.125]$.

First, let us see if the asymptotic law provides good critical points when we want to test:

$$H_0 : \sigma_1 = \sigma_0.$$

The results are presented in Table 7.16, on page 150. As we may expect for high values of n, the level is very close to 5 %, at least for the first two models. In that case too, even for values of n, as low as 128, the empirical level is closer to 6 %.

Things are not so nice for models 3 and 4. If we accept that 7 % is not so far from 5 %, with a resolution of 1,024 or 2,048 points, the level seems to be acceptable. But for resolutions of 128, 256 or 512 points, the level may be far from what we expect. In some cases, they are higher than 10 %, and can be as high as 15.9 %. So, some prudence is required when working with models 3 and 4.

The power of the test is also assessed. The test size being far from 5 % in some cases, we prefer to use the empirical distributions of $\hat{\sigma}_2$ to design a test that have a size close to the level. The performance is quite the same for the four models. See Figs. 7.12–7.19.

When the asymptotic distribution is used to assess the power function, it turns out that this approximation is quite good. It seems that the power is very slightly overestimated in the case of models 1 and 2, while it is very slightly underestimated in the case of models 3 and 4.

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Chapter 5
Proofs of All the Results

5.1 Introduction

In this chapter dedicated to the proofs of the various results, we explore the properties of three kinds of estimators for the Hurst parameter of the fBm. These estimators are built on the second order increments of fBm that allows estimation all over the range of parameter H in $]\,0,\ 1\,[$. We prove a CLT for simultaneous estimators of the Hurst parameter H and of the local variance σ in the four following models: $\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t$, where $\sigma(x) = \sigma$ or σx and $\mu(x) = \mu$ or μx.

When H is supposed to be known, test of hypotheses on σ are proposed.

Finally, functional estimation is considered for function σ in the following model: $\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t$, where functions σ and μ verify technical hypotheses.

In this chapter we used the techniques of the CLT for functionals that belong to Wiener chaos and more precisely the one of the Peccati-Tudor theorem.

5.2 Estimation of the Hurst Parameter

The aim of this section is to establish properties for three kinds of estimators for H, the Hurst parameter. To reach this objective, we first prove an almost sure convergence for the absolute k-power variation of the fBm, using the Borel-Cantelli Lemma. Then, we prove a CLT for the rate of this convergence, using the Peccati-Tudor theorem. It allows some insight into the properties of a first estimator, say $\hat{H}_k$ of H.

Indeed, with the same work tools a more general CLT is proved, establishing convergence in law for a g functional variation of the fBm, see (2.4) page 40, the function g belongs to $L^2(\phi(x)\,\mathrm{d}x)$, $\phi(x)\,\mathrm{d}x$ standing for the standard Gaussian

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measure. The particular case where g is $g(x) = |x|^k$ giving previous CLT and $\hat{H}_k$, we chose to bring out function g equal to $g(x) = \log(|x|)$. This choice gives rise to a second estimator of H, say $\hat{H}_{\log}$, and permits to establish its properties.

Finally, we link the estimators $\hat{H}_k$ and $\hat{H}_{\log}$ proposing a third estimator, $\hat{H}_{k(n)}$, built as the first one but with a sequence $k(n)$ converging to zero with n instead of fixed k. We establish that the asymptotic behavior of $\hat{H}_{\log}$ and of $\hat{H}_{k(n)}$ are the same by showing that their associated functional is equivalent in L^2.

5.2.1 *Almost Sure Convergence for the Second Order Increments*

We prove the almost sure convergence in law for the increments of the fBm. In this aim, we consider A_n, the centered k-power of the fBm increments. We link A_n to the $g_{(k)}$ functional variation of the fBm, that is $S_{g_{(k)},n}(1)$, where function $g_{(k)}$ is, $g_{(k)}(x) = x^k - \mathrm{E}[N^k]$. Then, because the function $g_{(k)}$ has a finite expansion in terms on the Hermite basis, we announce that using Remark 3.5 proved later in Sect. 5.2.2 the fourth moment of $S_{g_{(k)},n}(1)$ is bounded. Since A_n is equivalent to $S_{g_{(k)},n}(1)/\sqrt{n}$, we show that the last remark implies that the fourth moment for A_n is bounded by $\boldsymbol{C}/n^2$. Finally, the Borel-Cantelli Lemma yields the required result.

Proof of Theorem 3.1. For all $k \in \mathbb{N}^*$ and $n \in \mathbb{N}^* - \{1\}$, let:

$$A_n = \frac{1}{n-1} \sum_{i=0}^{n-2} (\Delta_n b_H(i))^k - \mathrm{E}[N]^k.$$

Now defining

$$g_{(k)}(x) = x^k - \mathrm{E}[N]^k = \sum_{p=1}^{k} g_{p,(k)} H_p(x), \tag{5.1}$$

one obtains

$$A_n = \frac{\sqrt{n}}{n-1} S_{g_{(k)},n}(1),$$

and then

$$\mathrm{E}[A_n]^4 \leqslant \frac{n^2}{(n-1)^4} \mathrm{E}[S_{g_{(k)},n}(1)]^4.$$

The function $g_{(k)}$ has a finite expansion with respect to the Hermite basis. Applying Remark 3.5 page 45 to $g = g_{(k)}$, one obtains

$$\mathrm{E}[A_n]^4 \leqslant \boldsymbol{C}\frac{1}{n^2}, \text{ for } n \text{ large enough.}$$

The Borel-Cantelli lemma yields Theorem 3.1. □

5.2.2 Convergence in Law of the Absolute *k*-Power Variation

We prove the convergence in law for a g functional variation of the fBm, say $S_{g,n}(1)$, see (2.4), page 40, function g being centered, belonging to $L^2(\phi(x)\,\mathrm{d}x)$, $\phi(x)\,\mathrm{d}x$ standing for the standard Gaussian measure. Note that in the particular case where g is $g_k(x) = |x|^k - \mathrm{E}[|N|^k]$, the result deals with the absolute k-power variation convergence.

More precisely, we prove that the variation $S_{g,\ell n}(1)$ seen as a variable with parameter ℓ converges to a cylindrical centered Gaussian process X with covariance $\rho_g(\ell, m)$. In this aim, using the Mehler's formula, we first compute the asymptotic variance of the random variable defined as a linear combination of variables of the type $S_{g,\ell_i n}(1)$, that are g functional variation seen at different scales of time. Then we prove that function ρ_g is actually a covariance function.

Finally we prove a CLT for this linear combination. The work tool to build this proof is based on the Peccati-Tudor theorem; it consists in decomposing the functional we are interested in into a sum of functionals belonging to distinct Wiener chaos. Then we prove that the p-th contractions of functions defining each functional tend to zero in L^2. This fact and the finiteness of the asymptotic variance ensuring the required convergence.

To complete the proof, we show how the Peccati-Tudor's Theorem permits to bound the fourth moment of $S_{g,n}(1)$ in the case where the function g possesses a finite expansion with respect to the Hermite basis.

Proof of Lemma 3.8. For any choice of $m \in \mathbb{N}^*$, $\boldsymbol{k} \in (\mathbb{N}^*)^m$ and $\boldsymbol{d} \in \mathbb{R}^m$, we have

$$\mathrm{E}[\sum_{i=1}^m d_i S_{g,k_i n}(1)]^2 = \sum_{i=1}^m \sum_{j=1}^m d_i d_j \mathrm{E}[S_{g,k_i n}(1) S_{g,k_j n}(1)].$$

For fixed $k \in \mathbb{N}^*$ and $\ell \in \mathbb{N}^*$ and by Mehler's formula (2.3), we get

$$\mathrm{E}[S_{g,kn}(1) S_{g,\ell n}(1)] = \sum_{p=1}^{+\infty} g_p^2 p! \frac{1}{\sqrt{k\ell}} \left(\frac{1}{n} \sum_{i=0}^{kn-2} \sum_{j=0}^{\ell n-2} \rho_{k,\ell}^p(\ell i - kj) \right).$$

To conclude the proof we need Lemma 5.1 proved in the Chap. 6 (see page 111).

Lemma 5.1. *For all $k, \ell, p \in \mathbb{N}^*$,*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{kn-2} \sum_{j=0}^{\ell n-2} \rho_{k,\ell}^{p}(\ell i - kj) = \sum_{s=0}^{k-1} \sum_{r=-\infty}^{+\infty} \rho_{k,\ell}^{p}(kr + \ell s).$$

Since $\rho_{k,\ell}(x)$ is a correlation, we have $|\rho_{k,\ell}(x)|^p \leqslant |\rho_{k,\ell}(x)|$ for all $p \in \mathbb{N}^*$. So proving a lemma similar to Lemma 5.1 for $p = 1$, replacing function $\rho_{k,\ell}$ by $|\rho_{k,\ell}|$, we get the following bound for large enough values of n,

$$\left| \frac{1}{n} \sum_{i=0}^{kn-2} \sum_{j=0}^{\ell n-2} \rho_{k,\ell}^{p}(\ell i - kj) \right| \leqslant \left(\sum_{s=0}^{k-1} \sum_{r=-\infty}^{+\infty} |\rho_{k,\ell}(kr + \ell s)| \right) + 1 < +\infty.$$

The last summation finiteness comes from the fact that, $\rho_{k,\ell}(x)$ is equivalent to

$$\frac{-1}{4 - 2^{2H}} (k\ell)^{2-H} H(2H-1)(2H-2)(2H-3)\,|x|^{2H-4}$$

for $|x|$ large enough. Since $\|g\|_{2,\phi}^2 = \sum_{p=1}^{+\infty} g_p^2 p! < +\infty$, the dominated convergence theorem and Lemma 5.1 entail that

$$\mathrm{E}\big[S_{g,kn}(1) S_{g,\ell n}(1)\big] \underset{n\to+\infty}{\longrightarrow} \rho_g(k,\ell) \text{ thus } \mathrm{E}[\sum_{i=1}^{m} d_i S_{g,k_i n}(1)]^2 \underset{n\to+\infty}{\longrightarrow} \sigma_{g,m}^2(\boldsymbol{k}, \boldsymbol{d}),$$

this yields Lemma 3.8. □

Proof of Lemma 3.9. For each function f such that $\sum_{r=-\infty}^{+\infty} |f(r)| < +\infty$, we shall use the identity, for $m \in \mathbb{N}^*$,

$$\sum_{r=-\infty}^{+\infty} f(r) = \sum_{r=-\infty}^{+\infty} \sum_{u=0}^{m-1} f(mr + u). \tag{5.2}$$

We shall denote for fixed k, ℓ and $p \in \mathbb{N}^*$, $\delta_{k,\ell} = \rho_{k,\ell}^p$. We shall prove the following identity

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{s=0}^{k-1} \sum_{r=-\infty}^{+\infty} \delta_{k,\ell}(kr + \ell s),$$

that will be sufficient to prove Lemma 3.9. Knowing that for $x \in \mathbb{R}$, $\delta_{k,\ell}(x) = \delta_{\ell,k}(-x)$, we have

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{r=-\infty}^{+\infty} \left(\sum_{s=0}^{\ell-1} \delta_{k,\ell}(\ell r - ks) \right).$$

Applying identity (5.2) to the summation on the right-hand member, for function $f(r) = \sum_{s=0}^{\ell-1} \delta_{k,\ell}(\ell r - ks)$ and for $m = k$, we get

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{u=0}^{k-1} \left(\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{k,\ell}(k(\ell r - s) + \ell u) \right).$$

Making the change of variable $s - \ell = -v - 1$ in the last summation, one obtains

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{u=0}^{k-1} \left(\sum_{r=-\infty}^{+\infty} \sum_{v=0}^{\ell-1} \delta_{k,\ell}(k(\ell r + v) - k\ell + k + \ell u) \right).$$

Finally applying (5.2) once again to the summation on the right-hand member, for function $f(r) = \delta_{k,\ell}(kr - k\ell + k + \ell u)$ and for $m = \ell$, we get

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{u=0}^{k-1} \sum_{r=-\infty}^{+\infty} \delta_{k,\ell}(kr - k\ell + k + \ell u).$$

Now, making the change of variables $r = i + \ell - 1$ in the last summation, we have

$$\sum_{r=-\infty}^{+\infty} \sum_{s=0}^{\ell-1} \delta_{\ell,k}(\ell r + ks) = \sum_{u=0}^{k-1} \sum_{i=-\infty}^{+\infty} \delta_{k,\ell}(ki + \ell u),$$

so we proved that $\rho_g(\ell, k) = \rho_g(k, \ell)$ and Lemma 3.9 follows. □

Proof of Theorem 3.4 and Remark 3.5. For any $m \in \mathbb{N}^*$, $\boldsymbol{k} \in (\mathbb{N}^*)^m$ and $\boldsymbol{d} \in \mathbb{R}^m$, let us define

$$S_{g,\boldsymbol{k}n}(1) = \sum_{i=1}^{m} d_i S_{g,k_i n}(1).$$

We want to prove that

$$S_{g,\boldsymbol{k}n}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0;\ \sigma^2_{g,m}(\boldsymbol{k}, \boldsymbol{d})).$$

Let

$$S_{g_M,\boldsymbol{k}n}(1) = \sum_{i=1}^{m} d_i S_{g_M,k_i n}(1) \ \text{ with } \ g_M(x) = \sum_{\ell=1}^{M} g_\ell H_\ell(x),$$

where $M \geqslant 1$ is a fixed integer. We will prove that

$$S_{g_M,\boldsymbol{k}n}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0; \sigma^2_{g_M,m}(\boldsymbol{k}, \boldsymbol{d})).$$

In this aim, using the chaos representation for the fBm (see Hunt 1951), we can write for $t > 0$

$$b_H(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [\exp(i\lambda t) - 1]\,|\lambda|^{-H-\frac{1}{2}}\,\mathrm{d}W(\lambda).$$

Thus, for $j = 0, 1, \ldots, n-2$

$$\Delta_n b_H(j) = \int_{-\infty}^{+\infty} f^{(n)}(\lambda, j)\,\mathrm{d}W(\lambda), \tag{5.3}$$

where we defined function $f^{(n)}$ by

$$f^{(n)}(\lambda, j) = \frac{n^H}{\sigma_{2H}\sqrt{2\pi}} \exp(i\lambda \tfrac{j}{n})[\exp(i\tfrac{\lambda}{n}) - 1]^2\,|\lambda|^{-H-\frac{1}{2}}. \tag{5.4}$$

Now, since $\int_{\mathbb{R}} \left|f^{(n)}(\lambda, j)\right|^2 \mathrm{d}\lambda = 1$, using Itô's formula, see Major (1981, p. 30), for fixed $\ell \geqslant 1$,

$$H_\ell(\Delta_n b_H(j)) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^{(n)}(\lambda_1, j)\ldots f^{(n)}(\lambda_\ell, j)\,\mathrm{d}W(\lambda_1)\ldots \mathrm{d}W(\lambda_\ell).$$

To get the asymptotic behavior of $S_{g_M,\boldsymbol{k}n}(1)$, we use notations introduced in Sect. 2.2.2 and in Slud (1994).
For $\ell \in \mathbb{N}^\star$ and $f_\ell \in L^2_{\mathrm{s}}(\mathbb{R}^\ell)$, we define

$$I_\ell(f_\ell) = \frac{1}{\ell!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_\ell(\lambda_1, \ldots, \lambda_\ell)\,\mathrm{d}W(\lambda_1)\ldots \mathrm{d}W(\lambda_\ell), \tag{5.5}$$

and for $p = 1, \ldots, \ell$, we write $f_\ell \otimes_p f_\ell$ for the p-th contraction of f_ℓ defined as

$$\begin{aligned} f_\ell \otimes_p f_\ell(\lambda_1, \ldots, \lambda_{2\ell-2p}) = \int_{\mathbb{R}^p} & f_\ell(\lambda_1, \ldots, \lambda_{\ell-p}, x_1, \ldots, x_p) \\ & f_\ell(\lambda_{\ell-p+1}, \ldots, \lambda_{2\ell-2p}, -x_1, \ldots, -x_p)\,\mathrm{d}x_1 \ldots \mathrm{d}x_p. \end{aligned} \tag{5.6}$$

With these notations, one gets

$$S_{g_M,\boldsymbol{k}n}(1) = \sum_{\ell=1}^{M} I_\ell(h_\ell^{(n,\boldsymbol{k})}), \tag{5.7}$$

where function $h_\ell^{(n,\boldsymbol{k})}$ is

$$h_\ell^{(n,\boldsymbol{k})}(\lambda_1, \ldots, \lambda_\ell) = g_\ell \ell! \sum_{i=1}^{m} d_i \frac{1}{\sqrt{nk_i}} \sum_{j=0}^{nk_i-2} f^{(nk_i)}(\lambda_1, j)\ldots f^{(nk_i)}(\lambda_\ell, j).$$

To obtain convergence of $S_{g_M,\boldsymbol{k}n}(1)$, we will use Theorem 1 of Peccati and Tudor (2005).
Lemma 3.8 page 45 gives the required conditions appearing in the beginning of this latter theorem. So we will just verify condition (i) while proving the following lemma.

Lemma 5.2. *For fixed ℓ and p, such that $\ell \geqslant 2$ and $p = 1, \ldots, \ell - 1$,*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n,\boldsymbol{k})} \otimes_p h_\ell^{(n,\boldsymbol{k})}(\lambda_1, \ldots, \lambda_{\ell-p}, \mu_1, \ldots, \mu_{\ell-p}) \right|^2 \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{\ell-p}\, \mathrm{d}\mu_1 \ldots, \mathrm{d}\mu_{\ell-p} = 0.$$

Proof of Lemma 5.2.

$$\begin{aligned}
&\int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n,\boldsymbol{k})} \otimes_p h_\ell^{(n,\boldsymbol{k})}(\lambda_1, \ldots, \lambda_{\ell-p}, \mu_1, \ldots, \mu_{\ell-p}) \right|^2 \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{\ell-p}\, \mathrm{d}\mu_1 \ldots, \mathrm{d}\mu_{\ell-p} \\
&\quad = (\ell!)^4 g_\ell^4 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \sum_{i_3=1}^{m} \sum_{i_4=1}^{m} d_{i_1} d_{i_2} d_{i_3} d_{i_4} \frac{1}{\sqrt{nk_{i_1}}} \frac{1}{\sqrt{nk_{i_2}}} \frac{1}{\sqrt{nk_{i_3}}} \frac{1}{\sqrt{nk_{i_4}}} \times \\
&\qquad \sum_{j_1=0}^{nk_{i_1}-2} \sum_{j_2=0}^{nk_{i_2}-2} \sum_{j_3=0}^{nk_{i_3}-2} \sum_{j_4=0}^{nk_{i_4}-2} \rho_{k_{i_1},k_{i_2}}^{\ell-p}(k_{i_2} j_1 - k_{i_1} j_2) \times \\
&\qquad \rho_{k_{i_3},k_{i_4}}^{\ell-p}(k_{i_4} j_3 - k_{i_3} j_4) \rho_{k_{i_1},k_{i_3}}^{p}(k_{i_3} j_1 - k_{i_1} j_3) \rho_{k_{i_2},k_{i_4}}^{p}(k_{i_4} j_2 - k_{i_2} j_4).
\end{aligned}$$

Now, since $\rho_{k,\ell}(x)$ is a correlation, $p \geqslant 1$ and $\ell - p \geqslant 1$, we just have to prove that for fixed $k_1, k_2, k_3, k_4 \in \mathbb{N}^*$, $\lim\limits_{n\to+\infty} A_n = 0$, where we defined

$$\begin{aligned}
A_n = \frac{1}{n^2} \sum_{j_1=0}^{nk_1-2} \sum_{j_2=0}^{nk_2-2} \sum_{j_3=0}^{nk_3-2} \sum_{j_4=0}^{nk_4-2} &\left|\rho_{k_1,k_2}(k_2 j_1 - k_1 j_2)\right| \times \\
&\left|\rho_{k_3,k_4}(k_4 j_3 - k_3 j_4)\right| \left|\rho_{k_1,k_3}(k_3 j_1 - k_1 j_3)\right| \left|\rho_{k_2,k_4}(k_4 j_2 - k_2 j_4)\right|.
\end{aligned}$$

We split the indices intervals into two parts, B_N and B_N^c, where we defined for a fixed positive real number N,

$$\begin{aligned}
B_N = \{(j_1, j_2, j_3, j_4) \in \mathbb{N}^4, |k_2 j_1 - k_1 j_2| > N \text{ or } |k_3 j_1 - k_1 j_3| > N \\
\text{or } |k_4 j_2 - k_2 j_4| > N\}.
\end{aligned}$$

We can write A_n as the sum of two terms corresponding to B_N and B_N^c respectively.

For the first term, we use the fact that, as already seen in proof of Lemma 5.1 (page 78), for all $k, \ell \in \mathbb{N}^*$ and for n large enough $\frac{1}{n} \sum_{i=0}^{kn-2} \sum_{j=0}^{\ell n-2} |\rho_{k,\ell}(\ell i - kj)| \leqslant \boldsymbol{C}$ and that, $|\rho_{k,\ell}| \leqslant 1$. Furthermore for N large enough and $|x| \geqslant N$, we use the bound $|\rho_{k,\ell}(x)| \leqslant \boldsymbol{C}\,|x|^{2H-4} \leqslant \boldsymbol{C}\,N^{2H-4}$.
For the second term, we bound each of the four functions $|\rho_{k,\ell}|$ by 1, so that for all N large enough we get

$$\overline{\lim_n} A_n \leqslant \boldsymbol{C}\left(N^{2H-4} + N^3 \overline{\lim_n}(\tfrac{1}{n})\right) \leqslant \boldsymbol{C}\,N^{2H-4},$$

and since $0 < H < 1$ then $\lim_{n\to+\infty} A_n = 0$ and Lemma 5.2 follows. □

Hence, we proved that

$$S_{g_M,\boldsymbol{k}n}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0; \sigma^2_{g_M,m}(\boldsymbol{k},\boldsymbol{d})).$$

Furthermore, $\sum_{p=M+1}^{\infty} g_p^2 p! \underset{M\to+\infty}{\longrightarrow} 0$, so we get

$$\lim_{M\to+\infty} \sup_{n\geqslant 1} \mathrm{E}[S_{g_M,\boldsymbol{k}n}(1) - S_{g,\boldsymbol{k}n}(1)]^2 = 0.$$

Now, since

$$\mathcal{N}(0;\ \sigma^2_{g_M,m}(\boldsymbol{k},\boldsymbol{d})) \xrightarrow[M\to\infty]{\text{Law}} \mathcal{N}(0;\ \sigma^2_{g,m}(\boldsymbol{k},\boldsymbol{d})),$$

applying Lemma 1.1 of Dynkin (1988), Theorem 3.4 is proved.
Now, Remark 3.5 page 45 follows from the following argumentation. First we establish the following inequalities:

$$\mathrm{E}[\sum_{\ell=1}^{M} I_\ell(h_\ell^{(n,\boldsymbol{k})})]^4 \leqslant M^3 \sum_{\ell=1}^{M} \mathrm{E}[I_\ell(h_\ell^{(n,\boldsymbol{k})})]^4 \leqslant \boldsymbol{C}.$$

The last inequality follows from (v) of Theorem 1 of Peccati and Tudor (2005) and from the fact that for each $\ell \in \{1, \ldots, M\}$, one has $\mathrm{E}[I_\ell(h_\ell^{(n,\boldsymbol{k})})]^2 \leqslant \boldsymbol{C}$, for n large enough.

So Remark 3.5 follows by considering equality (5.7) and noting that $S_{g_M,n}(1) = S_{g_M,\boldsymbol{k}n}(1)$ for $m = 1 = d_1 = k_1$. □

5.2.3 Estimators of the Hurst Parameter

In Sect. 5.2.1, as a corollary of the main result, we have shown the almost sure convergence of the absolute k-power variation of the fBm. This convergence permits then to write a classical linear regression equation and to propose the least squares estimator $\hat{H}_k$ of the parameter H. This estimator being thus an asymptotically unbiased strongly consistent estimator of H. To prove the asymptotic normality of this estimator, results proved in previous Sect. 5.2.2 are useful.

Indeed, in Sect. 5.2.2, we have proved that the g functional variation of the fBm, say $S_{g,\cdot n}(1)$, converges in law to a cylindrical Gaussian process. The function g being very general, centered and belonging to $L^2(\phi(x)\,\mathrm{d}x)$, the idea consists in choosing two particular functions of that type, say, $g_k(x) = |x|^k - \mathrm{E}[|N|]^k$ and $g_{\log}(x) = \log|x| - \mathrm{E}[\log|N|]$. The first one concerns $\hat{H}_k$. The fact that this estimator is equivalent to a linear combination of functionals of the type $S_{g_k,\cdot n}(1)$ will ensure its asymptotic normality. In the same way, since the functional $S_{g_{\log},\cdot n}(1)$ converges in law to a Gaussian process, this implies the convergence in probability of the Napierian logarithm of the modulus of the second order increments of the fBm. So, as for the estimator $\hat{H}_k$, a least squares estimator of the parameter H will be proposed, $\hat{H}_{\log}$, leading to an unbiased weakly consistent estimator of H.

As for the previous estimator, this new estimator $\hat{H}_{\log}$ is equivalent to a linear combination of functionals of the form $S_{g_{\log},\cdot n}(1)$ leading again to a CLT. As a remark, we then prove that in the class of estimators $\hat{H}_k$, the best estimator in terms of minimal variance is obtained for $k = 2$ and that the asymptotic variance for the second estimator $\hat{H}_{\log}$ is always greater than the one obtained for $\hat{H}_k$ in the case where $k = 2$ or $k = 4$.

Finally we link the two estimators defined above by introducing a third estimator, say $\hat{H}_{k(n)}$, built as $\hat{H}_k$, except that $k(n)$ is a sequence of positive numbers converging to zero as n goes to infinity more rapidly than the sequence $\frac{1}{\sqrt{n}}$. We prove that the corresponding functional is equivalent in L^2 to the one built to get $\hat{H}_{\log}$. We obtain that the asymptotic behaviors of estimators $\hat{H}_{k(n)}$ and $\hat{H}_{\log}$ are the same. We also prove that the estimator $\hat{H}_{k(n)}$ is asymptotically unbiased for the parameter H.

Proof of Corollary 3.10.

(1) Using (3.3) and (3.2) we get

$$\hat{H}_k = -\frac{1}{k}\sum_{i=1}^{\ell} z_i\,[-kH\log(r_i n) + kb_k] + o_{a.s.}(1),$$

and property (3.5) gives

$$\hat{H}_k = H + o_{a.s.}(1).$$

We proved that $\hat{H}_k$ is a strongly consistent estimator of H.

Now, let us see that $\hat{H}_k$ is an asymptotically unbiased estimator of H. By (3.3)

$$\mathrm{E}[\hat{H}_k] = -\frac{1}{k}\sum_{i=1}^{\ell} z_i \,\mathrm{E}[\log(M_k(r_i n))],$$

where $M_k(n)$ is defined by (3.1). Since

$$\frac{1}{n-1}\sum_{i=0}^{n-2} |\Delta_n b_H(i)|^k = \left(\frac{n^H}{\sigma_{2H}}\right)^k M_k(n), \tag{5.8}$$

by property (3.5), one has

$$\mathrm{E}[\hat{H}_k] = -\frac{1}{k}\sum_{i=1}^{\ell} z_i \,\mathrm{E}\left[\log\left(\frac{1}{(r_i n - 1)}\sum_{j=0}^{r_i n-2} |\Delta_{r_i n} b_H(j)|^k\right)\right] + H. \tag{5.9}$$

Hence, it is enough to prove that

$$\mathrm{E}\left[\log\left(\frac{1}{n-1}\sum_{i=0}^{n-2} |\Delta_n b_H(i)|^k\right)\right] \underset{n\to+\infty}{\longrightarrow} k\log\left(\|N\|_k\right).$$

To prove the last convergence we just have to prove the following one:

$$\mathrm{E}\left[\log\left(\int_0^1 |(\Delta_n b_H)^*|^k(u)\,\mathrm{d}u\right)\right] \underset{n\to+\infty}{\longrightarrow} k\log\left(\|N\|_k\right). \tag{5.10}$$

For this, let us notice that since log is a concave function and $\log(x) \leqslant x$ when $x > 0$, by Jensen's inequality we have

$$\begin{aligned}\int_0^1 \log\left(|(\Delta_n b_H)^*|^k(u)\right)\mathrm{d}u &\leqslant \log\left(\int_0^1 |(\Delta_n b_H)^*|^k(u)\,\mathrm{d}u\right)\\ &\leqslant \int_0^1 |(\Delta_n b_H)^*|^k(u)\,\mathrm{d}u.\end{aligned}$$

Thus, if we let

$$X_n = \int_0^1 |(\Delta_n b_H)^*|^k(u)\,\mathrm{d}u + k\left|\int_0^1 \log\left(|(\Delta_n b_H)^*|(u)\right)\mathrm{d}u\right|,$$

we have shown that

$$\left|\log\left(\int_0^1 |(\Delta_n b_H)^*|^k(u)\,\mathrm{d}u\right)\right| \leqslant X_n. \tag{5.11}$$

Now, since $|N|^k$ and $\log(|N|)$ are in $L^1(\Omega)$, the same result is true for X_n, say

$$X_n \in L^1(\Omega). \tag{5.12}$$

Furthermore using Lemma 3.8 page 45, it is easy to see that

$$X_n \xrightarrow[n\to+\infty]{L^1} \mathrm{E}[|N|^k] + k\,|\mathrm{E}[\log(|N|)]|\,. \tag{5.13}$$

Finally, using Corollary 3.3 page 44, we get:

$$\log\Big(\int_0^1 |(\Delta_n b_H)^*|^k\,(u)\,\mathrm{d}u\Big) \xrightarrow[n\to+\infty]{a.s.} k\log\left(\|N\|_k\right). \tag{5.14}$$

Hence, (5.11)–(5.14) yield (5.10).

(2) Formula (5.8) entails that

$$\mathrm{E}[M_k(n)] = \sigma_{2H}^k n^{-kH}\,\|N\|_k^k\,.$$

As in Berzin and León (2007), let us define

$$A_k(n) = \frac{M_k(n) - \mathrm{E}[M_k(n)]}{\mathrm{E}[M_k(n)]}.$$

With this definition, the Taylor expansion of the log function gives

$$\begin{aligned}\log\left(M_k(n)\right) &= \log\left(\mathrm{E}[M_k(n)]\right) + \log\left(1 + A_k(n)\right)\\ &= -kH\log(n) + kb_k + A_k(n)\\ &\quad + A_k^2(n)\left[-\tfrac{1}{2} + \varepsilon(A_k(n))\right].\end{aligned} \tag{5.15}$$

Let us see that

$$A_k^2(n)\left[-\tfrac{1}{2} + \varepsilon(A_k(n))\right] = o_P\left(\tfrac{1}{\sqrt{n}}\right). \tag{5.16}$$

By the definition of g_k (see (3.6)), one has

$$A_k(n) = \frac{\sqrt{n}}{n-1} S_{g_k,n}(1), \tag{5.17}$$

and by Lemma 3.8 page 45,

$$\mathrm{E}[\sqrt{n}A_k^2(n)] = \frac{n^{3/2}}{(n-1)^2}\mathrm{E}[S_{g_k,n}^2(1)] = O\left(\frac{1}{\sqrt{n}}\right), \tag{5.18}$$

so $\sqrt{n}A_k^2(n) = o_p(1)$ and then (5.16) is proved.

Using (5.15)–(5.18), we obtain

$$\log\left(M_k(n)\right) = -kH\log(n) + k\log(\sigma_{2H}) + \log\left(\mathrm{E}[|N|^k]\right) + \frac{1}{\sqrt{n}} S_{g_k,n}(1) + o_P\left(\tfrac{1}{\sqrt{n}}\right). \quad (5.19)$$

Thus, by (3.3) and property (3.5), we have

$$\hat{H}_k = H - \frac{1}{k}\sum_{i=1}^{\ell} \frac{z_i}{\sqrt{r_i n}} S_{g_k, r_i n}(1) + o_P\left(\tfrac{1}{\sqrt{n}}\right).$$

Thus

$$\sqrt{n}\left(\hat{H}_k - H\right) = -\frac{1}{k}\sum_{i=1}^{\ell} \frac{z_i}{\sqrt{r_i}} S_{g_k, r_i n}(1) + o_p(1).$$

Theorem 3.4 gives the required result.
The computation of the coefficients $g_{2p,k}$ is explicitly made in Berzin and León (2007) and Cœurjolly (2001).

□

Proof of Remark 3.11. Let us note that $g_{2,k} = \frac{k}{2}$. Then for $i, j \in \mathbb{N}^*$,

$$\rho_{g_k}(i,j) = \tfrac{k^2}{4}\rho_{g_2}(i,j) + \rho_{g'_k}(i,j),$$

where $g'_k(x) = \sum_{p=2}^{\infty} g_{2p,k} H_{2p}(x)$ which belongs to $L^2(\phi(x)\,\mathrm{d}x)$.
Then,

$$\sigma^2_{g_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right) = \sigma^2_{g_2,\ell}\left(\boldsymbol{r}, \frac{1}{2}\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right) + \sigma^2_{g'_k,\ell}\left(\boldsymbol{r}, \frac{1}{k}\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right)$$
$$\geqslant \sigma^2_{g_2,\ell}\left(\boldsymbol{r}, \frac{1}{2}\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right),$$

since the last term in the above equality is positive by Lemma 3.8 (see page 45). □

Proof of Corollary 3.13.

(1) By (3.10) and (3.9) and using property (3.5) we get

$$\hat{H}_{\log} = -\sum_{i=1}^{\ell} z_i\left(-H\log(r_i n) + \log(\sigma_{2H}) + \mathrm{E}[\log|N|]\right) + o_p(1)$$
$$= H + o_p(1).$$

We have proved that $\hat{H}_{\log}$ is a weakly consistent estimator of H.

Let us show now that $\hat{H}_{\log}$ is unbiased.

Since by (3.8)

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\log\left(|\Delta_n b_H(i)|\right) = M_{\log}(n) + H\log(n) - \log(\sigma_{2H}) \tag{5.20}$$

one has,

$$\mathrm{E}[M_{\log}(n)] = \mathrm{E}[\log(|N|)] - H\log(n) + \log(\sigma_{2H}), \tag{5.21}$$

and by (3.10) and property (3.5), we get

$$\mathrm{E}[\hat{H}_{\log}] = -\sum_{i=1}^{\ell} z_i\,\mathrm{E}[M_{\log}(r_i n)] = H, \tag{5.22}$$

and $\hat{H}_{\log}$ is an unbiased estimator of H.

(2) Now, (3.10), (5.20), (5.21), (3.11) and (5.22) entail that

$$\begin{aligned}\hat{H}_{\log} &= -\sum_{i=1}^{\ell} z_i\left(M_{\log}(r_i n) - \mathrm{E}[M_{\log}(r_i n)]\right) - \sum_{i=1}^{\ell} z_i\,\mathrm{E}[M_{\log}(r_i n)]\\ &= -\sum_{i=1}^{\ell}\frac{z_i}{\sqrt{r_i n}} S_{g_{\log}, r_i n}(1) + H - \sum_{i=1}^{\ell}\frac{z_i}{\sqrt{r_i n}(r_i n - 1)} S_{g_{\log}, r_i n}(1).\end{aligned}$$

Thus by Lemma 3.8 (see page 45)

$$\sqrt{n}\left(\hat{H}_{\log} - H\right) = -\sum_{i=1}^{\ell}\frac{z_i}{\sqrt{r_i}} S_{g_{\log}, r_i n}(1) + o_p(1).$$

Theorem 3.4 gives the required result.
Explicit computation of coefficients in the Hermite expansion of function $g_{\log}$ can be found in Berzin and León (2007). □

Proof of Remark 3.14. Since $g_{2,\log} = \dfrac{1}{2}$ (see equality (3.12)), then for $i, j \in \mathbb{N}^*$,

$$\rho_{g_{\log}}(i,j) = \frac{1}{4}\rho_{g_2}(i,j) + \rho_{g'_{\log}}(i,j),$$

where $g'_{\log}(x) = \sum_{p=2}^{\infty} g_{2p,\log} H_{2p}(x)$ which belongs to $L^2(\phi(x)\,\mathrm{d}x)$.
Then, as in the proof of Remark 3.11,

$$\sigma^2_{g_{\log},\ell}\left(\boldsymbol{r}, \boldsymbol{z}/\sqrt{\boldsymbol{r}}\right) \geqslant \sigma^2_{g_2,\ell}\left(\boldsymbol{r}, \frac{1}{2}\left(\boldsymbol{z}/\sqrt{\boldsymbol{r}}\right)\right).$$

In the same way, knowing that $g^2_{2p,\log} = \dfrac{1}{16} g^2_{2p,4}$, for $p = 1, 2$ (see equalities (3.7) and (3.12)), we have

$$\sigma^2_{g_{\log},\ell}(\boldsymbol{r}, z/\sqrt{\boldsymbol{r}}) \geqslant \sigma^2_{g_4,\ell}\left(\boldsymbol{r}, \tfrac{1}{4}(z/\sqrt{\boldsymbol{r}})\right).$$

□

Proof of Corollary 3.16. Let $S(n) = \dfrac{1}{k(n)} S_{g_{k(n)},n}(1)$, where

$$g_{k(n)}(x) = \frac{|x|^{k(n)}}{\mathrm{E}\left[|N|^{k(n)}\right]} - 1 = \sum_{p=1}^{\infty} g_{2p,k(n)} H_{2p}(x).$$

In a similar way as in Berzin and León (2007) and Cœurjolly (2001), we have Lemma 5.3 proved in Chap. 6.

Lemma 5.3.

$$S(n) - S_{g_{\log},n}(1) \xrightarrow[n\to+\infty]{L^2} 0.$$

Let us show now that from the definition of $M_{k(n)}(n)$ given by (3.13), one has

$$\frac{\log(M_{k(n)}(n))}{k(n)} = -H\log(n) + \log(\sigma_{2H}) \\ + \mathrm{E}[\log(|N|)] + \frac{1}{\sqrt{n}} S(n) + o_P\left(\tfrac{1}{\sqrt{n}}\right). \tag{5.23}$$

Indeed, as in the proof of Corollary 3.10 (see (5.19)), we get

$$\frac{\log(M_{k(n)}(n))}{k(n)} = -H\log(n) + \log(\sigma_{2H}) \\ + \frac{\log\left(\mathrm{E}[|N|^{k(n)}]\right)}{k(n)} + \frac{1}{\sqrt{n}} S(n) + o_P\left(\tfrac{1}{\sqrt{n}}\right).$$

Since

$$\frac{\log\left(\mathrm{E}[|N|^{k(n)}]\right)}{k(n)} - \mathrm{E}[\log(|N|)] = O(k(n)), \tag{5.24}$$

and $k(n) = o\left(\frac{1}{\sqrt{n}}\right)$, (5.23) holds.

A proof similar to the one given for Corollary 3.10 leads to

$$\sqrt{n}\left(\hat{H}_{k(n)} - H\right) = -\sum_{i=1}^{\ell} \frac{z_i}{\sqrt{r_i}} S(r_i n) + o_p(1),$$

where $\hat{H}_{k(n)}$ is defined in (3.13). From Lemma 5.3 and Theorem 3.4 we can conclude that $\hat{H}_{k(n)}$ is asymptotically normal.
Let us show now that $\hat{H}_{k(n)}$ is an asymptotically unbiased estimator of H. As in the proof of Corollary 3.10 (see (5.9) and (5.10)), it is enough to prove that

$$\frac{1}{k(n)} \mathrm{E}\left[\log\left(\int_0^1 |(\Delta_n b_H)^*|^{k(n)}(u)\,\mathrm{d}u\right)\right] \xrightarrow[n\to\infty]{} \mathrm{E}[\log(|N|)]. \tag{5.25}$$

To show this convergence, we use the fact that the log function is concave and then, by Jensen's inequality,

$$\begin{aligned} k(n)\mathrm{E}[\log(|N|)] &= \mathrm{E}\left[\int_0^1 \log\left(|(\Delta_n b_H)^*|^{k(n)}(u)\right)\mathrm{d}u\right] \\ &\leqslant \mathrm{E}\left[\log\left(\int_0^1 |(\Delta_n b_H)^*|^{k(n)}(u)\,\mathrm{d}u\right)\right] \\ &\leqslant \log\left(\mathrm{E}\left[\int_0^1 |(\Delta_n b_H)^*|^{k(n)}(u)\,\mathrm{d}u\right]\right) \\ &= \log\left(\mathrm{E}\left[|N|^{k(n)}\right]\right). \end{aligned}$$

Dividing by $k(n)$ on both sides of the inequality and using (5.24), we get (5.25). $\square$

5.3 Estimation of the Local Variance

Using techniques of Sect. 5.2 we proved a CLT for the vector $(\hat{H}_k, \hat{\sigma}_k)$, respectively estimators of the Hurst parameter H and of the local variance σ in the four following models:

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t,$$

where $\sigma(x) = \sigma$ or σx and $\mu(x) = \mu$ or μx.

These two estimators are based on the second order increments of the process X solution of the previous SDE and come from a linear regression model.

Then we propose hypothesis testing on σ, that is we test if $\sigma_n = \sigma + \frac{1}{\sqrt{n}}(d + F(\sqrt{n}))$ and we assess the asymptotic power of the test. Working tools are those of Sect. 5.2.

Finally, we proposed functional estimation for function σ in the following model: $\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t$, where functions σ and μ verify some technical conditions. Here, the techniques we use are the Girsanov's theorem and techniques implemented in Sect. 5.2.

5.3.1 *Simultaneous Estimators of the Hurst Parameter and of the Local Variance*

Here we propose simultaneous estimators of the parameter H and of the local variance σ through the observation of one trajectory of the X process on a regular grid of points. The X process is solution of one of the four following models:

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t,$$

where functions σ and μ are respectively defined by $\sigma(x) = \sigma$ or σx and $\mu(x) = \mu$ or μx and $H > \frac{1}{2}$.

As in Sect. 5.2.3 the idea consists in using the absolute k-power variation of such a process (eventually normalized). We prove a lemma establishing the almost sure equivalence between the second order increments of X (eventually normalized by X evaluated at the grid points) and of σ times the increments of the fBm. This lemma also provides a regression model that leads to simultaneous estimators of H and of σ, say $\hat{H}_k$ and $\hat{\sigma}_k$. Thus the same techniques are used to show that estimator $\hat{H}_k$ is a strongly consistent estimator of H and to obtain its asymptotic normality. Then we prove that $\sigma\sqrt{n}\hat{H}_k$ and $\frac{\sqrt{n}}{\log(n)}\hat{\sigma}_k$ are equivalent in probability, leading to a degenerate Gaussian limit for vector $(\hat{H}_k, \hat{\sigma}_k)$.

Finally, in the case where the parameter H is supposed to be known, the lemma cited above gives rise to a strongly consistent estimator of parameter σ based on the absolute k-variations of process X. Moreover, a CLT is shown giving a rate of convergence in $\sqrt{n}$ instead of $\frac{\sqrt{n}}{\log(n)}$ as obtained before.

Proof of Theorem 3.17. We need Lemma 5.4 proved in Chap. 6, page 116.

Lemma 5.4. *For* $i = 0, 1, \ldots, n-2$,

$$\Gamma_n X(i) = \sigma \Delta_n b_H(i) + a_n(i),$$

with $|a_n(i)| \leqslant \boldsymbol{C}(\omega)\left(\frac{1}{n}\right)^{H-\delta}$, *for any* $\delta > 0$.

Remark 5.5. Indeed, for the first model one has, $a_n(i) = 0$ and for the second one, $|a_n(i)| \leqslant \boldsymbol{C}(\omega)\left(\frac{1}{n}\right)^{1-\delta}$.

We write

$$\frac{n-1}{\sqrt{n}} A_k^X(n) = S_{g_k,n}(1) + \frac{n-1}{\sqrt{n}\sigma^k \|N\|_k^k} \left(\|(\Gamma_n X)^*\|_k^k - \|\sigma(\Delta_n b_H)^*\|_k^k \right).$$

Let us prove now that $\|(\Gamma_n X)^*\|_k^k - \|\sigma(\Delta_n b_H)^*\|_k^k = o_{a.s.}\left(\frac{1}{\sqrt{n}}\right)$.

Using Hölder's inequality, one can show the following inequality.

$$\begin{aligned} \left| \|f\|_k^k - \|g\|_k^k \right| &\leqslant \left\| |f|^k - |g|^k \right\|_1 \\ &\leqslant 2^{k-1} k \|f-g\|_k \left[\|g\|_k^{k-1} + \|f-g\|_k^{k-1} \right]. \end{aligned} \tag{5.26}$$

Applying previous inequality to $f = (\Gamma_n X)^*$ and to $g = \sigma(\Delta_n b_H)^*$ and using Lemma 5.4 and Corollary 3.3, one has

$$\begin{aligned} &\left| \|(\Gamma_n X)^*\|_k^k - \|\sigma(\Delta_n b_H)^*\|_k^k \right| \\ &\quad \leqslant 2^{k-1} k \|(\Gamma_n X)^* - \sigma(\Delta_n b_H)^*\|_k \\ &\qquad \times \left[\|\sigma(\Delta_n b_H)^*\|_k^{k-1} + \|(\Gamma_n X)^* - \sigma(\Delta_n b_H)^*\|_k^{k-1} \right] \\ &\quad \leqslant \boldsymbol{C}(\omega) \|a_n^*\|_k \left[\boldsymbol{C}(\omega) + \|a_n^*\|_k^{k-1} \right] \\ &\quad \leqslant \boldsymbol{C}(\omega) \left(\frac{1}{n}\right)^{H-\delta} = o_{a.s.}\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

for δ small enough, i.e. for $0 < \delta < H - \frac{1}{2}$.
Thus assertion (2) follows. Assertion (1) follows from Corollary 3.3 (see page 44). □

Proof of Theorem 3.18.

(1) Using (3.22) and (3.21) we obtain

$$\hat{H}_k = -\frac{1}{k} \sum_{i=1}^{\ell} z_i \left[-kH \log(r_i n) + k b_k \right] + o_{a.s.}(1),$$

and property (3.5) gives

$$\hat{H}_k = H + o_{a.s.}(1).$$

We proved that $\hat{H}_k$ is a strongly consistent estimator of H.

Now by using definitions of $A_k^X(n)$ and of $M_k^X(n)$ (see (3.18)–(3.20)), one obtains

$$A_k^X(n) = n^{kH}\exp(-kb_k)M_k^X(n) - 1.$$

With this definition and using a Taylor expansion of the logarithm function one has

$$\begin{aligned}\log\left(M_k^X(n)\right) &= \log\left(n^{-kH}\exp(kb_k)\right) + \log\left(1 + A_k^X(n)\right)\\ &= -kH\log(n) + kb_k + A_k^X(n)\\ &\quad + \left(A_k^X(n)\right)^2\left[-\tfrac{1}{2} + \varepsilon(A_k^X(n))\right].\end{aligned} \tag{5.27}$$

Let us verify that

$$\left(A_k^X(n)\right)^2\left[-\tfrac{1}{2} + \varepsilon(A_k^X(n))\right] = o_P\left(\tfrac{1}{\sqrt{n}}\right). \tag{5.28}$$

By assertion (2) of Theorem 3.17, we know that

$$\sqrt{n}A_k^X(n) = \frac{n}{n-1}S_{g_k,n}(1) + o_{a.s.}(1), \text{ where} \tag{5.29}$$

the function g_k is defined by (3.6), and by Lemma 3.8,

$$\mathrm{E}[S_{g_k,n}^2(1)] = O(1), \tag{5.30}$$

so $\sqrt{n}(A_k^X(n))^2 = o_P(1)$ and then (5.28) is proved.
Using (5.27)–(5.30), we obtain

$$\log(M_k^X(n)) = -kH\log(n) + kb_k + \frac{1}{\sqrt{n}}S_{g_k,n}(1) + o_P\left(\tfrac{1}{\sqrt{n}}\right). \tag{5.31}$$

Thus (3.22), (5.31) and property (3.5) entail that

$$\hat{H}_k = H - \frac{1}{k}\sum_{i=1}^{\ell} z_i \frac{1}{\sqrt{r_i n}}S_{g_k,r_i n}(1) + o_P\left(\tfrac{1}{\sqrt{n}}\right).$$

Then

$$\sqrt{n}\left(\hat{H}_k - H\right) = -\frac{1}{k}\sum_{i=1}^{\ell}\frac{z_i}{\sqrt{r_i}}S_{g_k,r_i n}(1) + o_P(1).$$

Theorem 3.4 gives the required result. The computation of the coefficients in the Hermite expansion of function g_k is explicitly made in Berzin and León (2007) and Cœurjolly (2001).

(2) Let us see that $\hat{B}_k$ is a weakly consistent estimator of b_k. By using (3.23) and (5.31), one has

$$\hat{B}_k - b_k = \frac{1}{\ell}(\hat{H}_k - H)\sum_{i=1}^{\ell}\log(n_i) + \frac{1}{\sqrt{n}}\frac{1}{k\ell}\sum_{i=1}^{\ell}\frac{1}{\sqrt{r_i}}S_{g_k,n_i}(1) + \frac{1}{\sqrt{n}}o_P(1).$$

Thus

$$\hat{B}_k - b_k = \log(n)(\hat{H}_k - H) + \frac{1}{\ell}(\hat{H}_k - H)\sum_{i=1}^{\ell}\log(r_i) + \frac{1}{\sqrt{n}}\frac{1}{k\ell}\sum_{i=1}^{\ell}\frac{1}{\sqrt{r_i}}S_{g_k,r_i n}(1) + \frac{1}{\sqrt{n}}o_P(1).$$

Using assertion (1) of Theorem 3.18 and (5.30), we obtain

$$\frac{\sqrt{n}}{\log(n)}\left(\hat{B}_k - b_k\right) = \sqrt{n}\left(\hat{H}_k - H\right) + o_P(1), \tag{5.32}$$

and then using again assertion (1) of Theorem 3.18 we proved that $\hat{B}_k$ is a weakly consistent estimator of b_k.

Now using the order one Taylor expansion of the exponential function, equality (5.32) and the first point of Theorem 3.18, we finally get

$$\frac{\sqrt{n}}{\log(n)}\left(\exp(\hat{B}_k) - \exp(b_k)\right) = \exp(b_k)\sqrt{n}\left(\hat{H}_k - H\right) + o_P(1). \tag{5.33}$$

Thus if we get back to the definition of $\hat{\sigma}_k$ (see (3.24)) and use (5.33), we get

$$\begin{aligned}
&\frac{\sqrt{n}}{\log(n)}(\hat{\sigma}_k - \sigma) \\
&= \sigma\exp(-b_k)\frac{\sqrt{n}}{\log(n)}\left\{\exp(\hat{B}_k) - \exp(b_k) + \exp(\hat{B}_k)\left(\frac{1}{\sigma_{2\hat{H}_k}} - \frac{1}{\sigma_{2H}}\right)\sigma_{2H}\right\} \\
&= \sigma\sqrt{n}\left(\hat{H}_k - H\right) + \frac{\sigma}{\sigma_{2\hat{H}_k}}\exp(-b_k)\exp(\hat{B}_k)\frac{\sqrt{n}}{\log(n)}\left(\sigma_{2H} - \sigma_{2\hat{H}_k}\right) + o_P(1).
\end{aligned}$$

At this step of the proof, we are going to show that

$$\frac{\sqrt{n}}{\log(n)}(\hat{\sigma}_k - \sigma) = \sigma\sqrt{n}\left(\hat{H}_k - H\right) + o_P(1). \tag{5.34}$$

Using the fact that $\hat{B}_k$ is a weakly consistent estimator of b_k it is enough to prove the following convergence

$$\frac{\sqrt{n}}{\log(n)}\left(\sigma_{2H}-\sigma_{2\hat{H}_k}\right)\xrightarrow[n\to\infty]{\mathrm{P}}0,$$

which is the same as showing

$$\frac{\sqrt{n}}{\log(n)}\left(\sigma_{2H}^2-\sigma_{2\hat{H}_k}^2\right)\xrightarrow[n\to\infty]{\mathrm{P}}0.$$

The last convergence is an immediately consequence of the following fact

$$\frac{\sqrt{n}}{\log n}\left(\hat{H}_k-H\right)=o_P(1),$$

which follows from assertion (1) of Theorem 3.18.
Thus by equality (5.34) and assertion (1) of Theorem 3.18, the proof of assertion (2) is completed.

□

Proof of Theorem 3.20.

(1) Assertion (1) follows from the first assertion of Theorem 3.17.
(2) Assertion (2) of Theorem 3.17 and Remark 3.7 imply that $\sqrt{n}\left(\left[\frac{\tilde{\sigma}_k}{\sigma}\right]^k-1\right)$ converges weakly to $\sigma_{g_k}N$ that yields assertion (2).

Remark 3.22 page 53 follows from the fact that since $g_{2,k}=\frac{k}{2}$ (see equality (3.7)), one has

$$\begin{aligned}\frac{\sigma_{g_k}^2}{k^2}&=\frac{1}{k^2}\sum_{n=1}^{\infty}g_{2n,k}^2(2n)!\sum_{r=-\infty}^{+\infty}\rho_H^{2n}(r)\\&\geqslant\frac{2}{k^2}g_{2,k}^2\sum_{r=-\infty}^{+\infty}\rho_H^2(r)=\frac{1}{2}\sum_{r=-\infty}^{+\infty}\rho_H^2(r)=\frac{1}{4}\sigma_{g_2}^2.\end{aligned}$$

□

5.3.2 Hypothesis Testing

We consider the following four stochastic models, for known parameter $H>\frac{1}{2}$:

$$\mathrm{d}X_n(t)=\sigma_n(X_n(t))\,\mathrm{d}b_H(t)+\mu_n(X_n(t))\,\mathrm{d}t,$$

where the functions σ_n and μ_n are respectively defined by $\sigma_n(x) = \sigma_n$ or $\sigma_n x$ and $\mu_n(x) = \mu_n$ or $\mu_n x$. We test H_0: $\sigma_n = \sigma$ again the alternatives: H_n: $\sigma_n = \sigma + \frac{1}{\sqrt{n}}(d + F(\sqrt{n}))$, where σ, d are positive constants, and F a positive function tending to zero with n.
Note that under hypothesis H_0 (and $\mu_n = \mu$) the studied model is the one of Sect. 5.3.1. We observe the absolute variation of such a process X_n. Note that this is equivalent to let $k = 1$ in the previous section. A lemma similar to Lemma 5.4 is proposed, replacing σ by σ_n. This allows, using what we did in last section for the estimation of σ in the case where H is known and using results of Sect. 5.2.1 to show a CLT for theses variations of the process X.

We show that there is an asymptotic bias d, and the larger is the bias the easier is discriminating between the two hypotheses.

Proof of Theorem 3.23. We need the following lemma proved in Chap. 6.

Lemma 5.6. *For* $i = 0, 1, \ldots, n-2$,

$$\Gamma_n X_n(i) = \sigma_n \Delta_n b_H(i) + a_n(i),$$

with

$$|a_n(i)| \leqslant \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{H-\delta}, \text{ for any } \delta > 0.$$

Now we write F_n as

$$F_n = \sigma \frac{n}{n-1} S_{g_1,n}(1) + d + G_n \text{ , where,}$$

$$\begin{aligned} G_n = d\left(\frac{1}{n-1}\sum_{i=0}^{n-2}\sqrt{\frac{\pi}{2}}\,|\Delta_n b_H(i)| - 1\right) \\ + F(\sqrt{n})\sqrt{\frac{\pi}{2}}\frac{1}{n-1}\sum_{i=0}^{n-2}|\Delta_n b_H(i)| \\ + \sqrt{\frac{\pi}{2}}\frac{\sqrt{n}}{n-1}\sum_{i=0}^{n-2}\left(|\Gamma_n X_n(i)| - |\sigma_n \Delta_n b_H(i)|\right). \end{aligned}$$

Note that $F(\sqrt{n})$ tends to zero with n. This fact and Corollary 3.3 ensure that the first two terms almost surely tend to 0. From Lemma 5.6 and the fact that $H > \frac{1}{2}$ we see that the third term almost surely tends to 0 and finally $G_n = o_{a.s.}(1)$. Remark 3.7 page 45 yields the result. □

5.3.3 Functional Estimation of the Local Variance

In case where $H > \frac{1}{2}$, we consider the following model:

$$\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}b_H(t) + \mu(X(t))\,\mathrm{d}t,$$

where functions σ and μ verify technical hypotheses, ensuring the existence and the uniqueness of the solution of such a SDE. Our aim is to propose functional estimation for function σ.

First, for a continuous function h, we propose a demonstration of an almost sure convergence result for the random variable

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(X(\tfrac{i}{n}))\,|\Delta_n X(i)|^k\,,$$

Δ_n standing for the operator of the second order increments. We decompose this objective into two cases $\mu \equiv 0$ and $\mu \not\equiv 0$.

In the case where $\mu \equiv 0$, the solution for the SDE is $X(t) = K(b_H(t))$, where the function K is solution of an ODE. Thus we show that proving the required result is equivalent to prove it for the fBm, this is done using techniques of Sect. 5.2.1. Then in the case where $\mu \not\equiv 0$, the idea consists in applying the Girsanov theorem and the last convergence result obtained when $\mu \equiv 0$.

Second, we show an asymptotic result and get the rate of convergence. Here again, we decompose the analysis into the same two cases. As before, in the case where $\mu \equiv 0$, we show that it is enough to consider the case of the fBm, that it is done and we show the stable convergence of the required functional.

To achieve this goal, the working tools are those of Sect. 5.2.2, the use of the chaos representation of the fBm increments and the decomposition of the functional in the multiple Wiener chaos. Then the Peccati-Tudor theorem allows to obtain a CLT. In the case where $\mu \not\equiv 0$, we still use the Girsanov theorem and the stable convergence showed before.

Proof of Theorem 3.27. First, we suppose that the function $\mu \equiv 0$ and that $\sigma \in C^1$. As mentioned in Sect. 3.3.3, we have to prove Theorem 3.33. Suppose for the moment that it is done, then Remark 3.28 page 56 is true. To conclude the proof of this theorem we just have to get back to model (3.28) where μ is not necessarily identically null. As in Berzin and León (2008), with the additional hypotheses (H1) and (H2) on μ and σ, we can apply Girsanov's theorem (see Theorem 4.9 of Decreusefond and Üstünel 1999). That is, for G a measurable and bounded real function defined on the space $C([0,\ 1], \mathbb{R})$ of continuous real functions, we have the following equality:

$$\mathrm{E}[G(X)] = \mathrm{E}[G(K(b_H))\Lambda], \tag{5.35}$$

where Λ is the Radon-Nikodym derivative and K is solution of the ODE (3.29).

Let us define the set of trajectories

$$\Delta = \Big\{ x \in C([0,\ 1], \mathbb{R}) : \text{ for any continuous function } h \text{ and for all real } k \geqslant 1,$$

$$\lim_{n\to+\infty} \frac{1}{n-1} \sum_{i=0}^{n-2} h(x(\tfrac{i}{n})) \frac{|\Delta_n x(i)|^k}{\mathrm{E}[|N|^k]} = \int_0^1 h(x(u))\,[\sigma(x(u))]^k\,\mathrm{d}u \Big\} .$$

If we choose G as $\mathbb{1}_\Delta$, using (5.35) we obtain

$$\mathrm{E}[\mathbb{1}_\Delta(X)] = \mathrm{E}[\mathbb{1}_\Delta(K(b_H))\Lambda] = \mathrm{E}[\Lambda] = 1,$$

with the help of Remark 3.28, *i.e.* $P(K(b_H) \in \Delta) = 1$, thus Theorem 3.27 follows.
□

Proof of Theorem 3.33. We need Lemma 5.7.

Lemma 5.7. *For all* $0 < H < 1$*, for all interval* $[a,\ b] \subset [0,\ 1]$ *and for all* $k \in \mathbb{N}^*$*, almost surely one has*

$$\int_a^b \left[(\Delta_n b_H)^*\right]^k (u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} (b-a)\mathrm{E}[N]^k.$$

Remark 5.8. Note that in the case where $a = 0$ and $b = 1$ we get Theorem 3.1.

We will show this lemma after the proof of this theorem.
By a density argument, this lemma implies that if we take intervals of $[0,\ 1]$ with rational extremities, then almost surely for any real $k > 0$ and any interval $[a,\ b] \subset [0,\ 1]$,

$$\int_a^b \left|(\Delta_n b_H)^*\right|^k (u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} (b-a)\mathrm{E}\Big[|N|^k\Big].$$

Again, by a density argument, if we approximate continuous function by stepwise functions, we get that almost surely, for any continuous function h and for any real $k > 0$,

$$\int_0^1 h(u)\left|(\Delta_n b_H)^*\right|^k (u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} \left(\int_0^1 h(u)\,\mathrm{d}u\right)\mathrm{E}[|N|^k].$$

To conclude the proof of this theorem, let us consider the following equality:

$$\int_0^1 h(u)\left|(\Delta_n b_H)^*\right|^k (u)\,\mathrm{d}u$$
$$= \frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\,|\Delta_n b_H(i)|^k$$

$$+\frac{1}{n-1}\sum_{i=0}^{n-2}\left(h(\tfrac{i}{n-1})-h(\tfrac{i}{n})\right)|\Delta_n b_H(i)|^k$$
$$+\frac{1}{n-1}\sum_{i=0}^{n-2}(n-1)\left(\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}(h(u)-h(\tfrac{i}{n-1}))\,\mathrm{d}u\right)|\Delta_n b_H(i)|^k .$$

By Corollary 3.3 page 44 and since h is uniformly continuous on any compact, the two above last terms almost surely tend to zero. This yields Theorem 3.33 by taking $h=f\circ b_H$. □

Proof of Lemma 5.7. Let us suppose that $0<a<b<1$. The cases where $0=a<b<1$ or where $0<a<b=1$ would be treated in the same fashion. Note that the case where $a=0$ and $b=1$ has been treated in Theorem 3.1.

For n large enough, $a\geqslant\frac{2}{n}$, $b\leqslant 1-\frac{3}{n}$ and $b-a\geqslant\frac{3}{n}$, so that

$$\int_a^b\left[(\Delta_n b_H)^*\right]^k(u)\,\mathrm{d}u=\sum_{i=0}^{n-2}(\Delta_n b_H(i))^k\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}\mathbb{1}_{[a,b]}(u)\,\mathrm{d}u$$
$$=\begin{cases}\displaystyle\sum_{i=\lfloor na\rfloor-1}^{\lfloor na\rfloor}(\Delta_n b_H(i))^k\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}\mathbb{1}_{[a,b]}(u)\,\mathrm{d}u\\ \displaystyle\quad+\frac{1}{n-1}\sum_{i=0}^{\lfloor nb\rfloor-2}(\Delta_n b_H(i))^k-\frac{1}{n-1}\sum_{i=0}^{\lfloor na\rfloor-2}(\Delta_n b_H(i))^k\\ \displaystyle\quad-\frac{1}{n-1}\sum_{i=\lfloor na\rfloor-1}^{\lfloor na\rfloor}(\Delta_n b_H(i))^k\\ \displaystyle\quad+\sum_{i=\lfloor nb\rfloor-1}^{\lfloor nb\rfloor}(\Delta_n b_H(i))^k\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}\mathbb{1}_{[a,b]}(u)\,\mathrm{d}u\end{cases}\tag{5.36}$$
$$=T_1+T_2+T_3+T_4,$$

where T_i, $i=1,\dots,4$ are the four terms of (5.36). Using that the trajectories of b_H are $(H-\delta)$-Hölder continuous, see Proposition 2.1, in other words for any $\delta>0$, $u,v\geqslant 0$,

$$|b_H(u+v)-b_H(u)|\leqslant \boldsymbol{C}(\omega)\,|v|^{H-\delta},\tag{5.37}$$

one obtains that $\forall i\in\{0,\dots,n-2\}$, $|\Delta_n b_H(i)|^k\leqslant\boldsymbol{C}(\omega)n^{\delta k}$, for any $\delta>0$. Thus for n large enough, $\sup\{|T_i|,\ i=1,\ 3,\ 4\}\leqslant\boldsymbol{C}(\omega)n^{\delta k-1}$. With δ small enough, that is $\delta k<1$, we proved that T_1, T_3 and T_4 converge to zero with n. To prove Lemma 5.7, we need to show that T_2 tends to $(b-a)\mathrm{E}[N^k]$. In fact, it is a consequence of the following convergence. For any a such that $0<a<1$, for all $k\in\mathbb{N}^*$, almost surely one has

$$\frac{1}{n}\sum_{i=0}^{\lfloor na\rfloor-2}(\Delta_n b_H(i))^k \underset{n\to+\infty}{\longrightarrow} a\mathrm{E}[N]^k,$$

which is equivalent to

$$\frac{1}{na}\sum_{i=0}^{\lfloor na\rfloor-2} g_{(k)}(\Delta_n b_H(i)) \underset{n\to+\infty}{\longrightarrow} 0,$$

where function $g_{(k)}$ is defined by (5.1).
This last convergence is a consequence of Remark 5.11 forthcoming on page 101, where $g = g_{(k)} \in L^4(\phi(x)\,\mathrm{d}x)$, with Hermite rank $\geqslant 1$ and of the Borel-Cantelli lemma. This yields Lemma 5.7. □

Proof of Theorem 3.29. We first suppose that $\mu \equiv 0$ and that the function σ still verifies hypotheses given in Theorem 3.29, except H_1 and H_2.

As mentioned in Sect. 3.3.3, we still have to prove Theorem 3.34. Suppose for the moment that it is done, then Remark 3.30 is true. So if we consider $Y = K(b_H)$ where K is as before, a solution of the ODE (3.29), then

$$M_n(Y) = \sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\frac{|\Delta_n Y(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(Y(u))\,[\sigma(Y(u))]^k\,\mathrm{d}u\right],$$

stably converges toward

$$M(Y,\hat{W}) = \sigma_{g_k}\int_0^1 h(Y(u))\,[\sigma(Y(u))]^k\,\mathrm{d}\hat{W}(u).$$

To complete the proof of this theorem, we have to consider model (3.28) where μ is not necessarily identically null. As in Berzin and León (2008), with the additional hypotheses (H1) and (H2) on μ and σ we can apply the Girsanov's theorem. That is, let F be a continuous and bounded real function then applying once again equality (5.35) to $G = F \circ M_n$, one gets

$$\mathrm{E}[F(M_n(X))] = \mathrm{E}[F(M_n(K(b_H)))\Lambda].$$

Using the property of stable convergence (see Aldous and Eagleson 1978) and last convergence, we have

$$\mathrm{E}[F(M_n(K(b_H)))\Lambda] \underset{n\to\infty}{\longrightarrow} \mathrm{E}[F(M(Y,\hat{W}))\Lambda],$$

and using again Girsanov's theorem, we have

$$\mathrm{E}[F(M(X,\hat{W}))] = \mathrm{E}[F(M(Y,\hat{W}))\Lambda],$$

that yields Theorem 3.29. □

Proof of Theorem 3.34. We prove Remark 3.35 page 57 in the case where $f \in C^4$, $\left|\overset{4\cdot}{f}(x)\right| \leqslant P(|x|)$, $H > \frac{1}{4}$ and for general function $g \in L^4(\phi(x)\,\mathrm{d}x)$, with Hermite rank $\geqslant 1$ and we suppose that A_g is not an empty set (see Sect. 2.3 for definition). Furthermore, we will suppose that g is even, or odd with Hermite rank greater than or equal to three. A proof similar to the last one could easily be given to obtain the other cases described in the Remark 3.35. It is sufficient to adapt forthcoming Lemma 5.9 to the new hypotheses that is proved in Chap. 6.

The proof will proceed in several steps. Let us define

$$T_n(f) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-2} f(b_H(\tfrac{i}{n}))g(\Delta_n b_H(i)).$$

On the one hand, we prove in forthcoming Lemma 5.10 that $\left(b_H, S_{g,n}\right)$ stably converge to $\left(b_H, \sigma_g \hat{W}\right)$. We will show this lemma after the proof of this theorem.

On the other hand, we will consider a discrete version of $T_n(f)$, defining

$$T_n^{(m)}(f) = \sum_{\ell=0}^{m-1} f(b_H(\tfrac{\ell}{m}))\frac{1}{\sqrt{n}} \sum_{i=\left\lfloor \frac{n\ell}{m}\right\rfloor -1}^{\left\lfloor \frac{n(\ell+1)}{m}\right\rfloor -2} g(\Delta_n b_H(i)).$$

The stable convergence of $(b_H, S_{g,n})$ implies that

$$T_n^{(m)}(f) \xrightarrow[n\to\infty]{} T^{(m)}(f) = \sigma_g \sum_{\ell=0}^{m-1} f(b_H(\tfrac{\ell}{m}))\left(\hat{W}(\tfrac{\ell+1}{m}) - \hat{W}(\tfrac{\ell}{m})\right).$$

Furthermore, it is easy to show that $T^{(m)}(f)$ is a Cauchy sequence in $L^2(\Omega)$. Using the asymptotic independence between b_H and $\hat{W}$, it follows that

$$T^{(m)}(f) \xrightarrow[m\to\infty]{} \sigma_g \int_0^1 f(b_H(u))\,\mathrm{d}\hat{W}(u).$$

To conclude, that is to prove the convergence of $T_n(f)$, it is sufficient to prove the following Lemma 5.9 for which a proof is given in Chap. 6.

Lemma 5.9. *Let $f \in C^4$, $\left|\overset{4\cdot}{f}(x)\right| \leqslant P(|x|)$, $H > \frac{1}{4}$ and function $g \in L^4(\phi(x)\,\mathrm{d}x)$, let $g(x) = \sum_{p=1}^{+\infty} g_p H_p(x)$. Furthermore we will suppose that g is even, or odd with Hermite rank greater than or equal to three, then*

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{E}[T_n(f) - T_n^{(m)}(f)]^2 = 0.$$

□

Proof of Lemma 5.10. We only make the hypothesis that $g \in L^4(\phi(x)\,\mathrm{d}x)$, with Hermite rank $\geqslant 1$ and we suppose that A_g is not an empty set (See Sect. 2.3 for definition). We shall prove the following lemma.

Lemma 5.10. *For* $0 < H < 1$,

(1)

$$S_{g,n} \xrightarrow[n\to\infty]{} \sigma_g \hat{W}.$$

(2) Furthermore,

$$\left(b_H, S_{g,n}\right) \xrightarrow[n\to\infty]{} \left(b_H, \sigma_g \hat{W}\right),$$

where $\hat{W}$ *is a standard Brownian motion independent of* b_H.

The convergence in (1) and (2) is stable.

Remark 5.11. If $g \in L^4(\phi(x)\,\mathrm{d}x)$, with Hermite rank $\geqslant 1$, then for all t such that $0 \leqslant t \leqslant 1$, one has $\mathrm{E}\left[S_{g,n}(t)\right]^4 \leqslant \boldsymbol{C}$.

Remark 5.12. In Theorems 3 and 2 of Corcuera et al. (2006, 2009) and in Theorems 1 and 2 of León and Ludeña (2004, 2007), the result is proved for an even function g or for the function $g(x) = |x|^p - \mathrm{E}\,|N|^p$, $p > 0$. In both cases, as mentioned in Remark 2.8, page 41, the result stands for values of H such that $0 < H < \frac{3}{4}$, since these authors consider the first order increments of b_H. However, in Theorem 3 of León and Ludeña (2007) working with second order increments of the discrete sample b_H, the authors obtain analogous results for the whole interval $0 < H < 1$, considering function g such as even polynomials or polynomials of absolute values.

(1) For $m \in \mathbb{N}^*$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m \leqslant 1$, let $\boldsymbol{t} = (t_1, \ldots, t_m)$ and

$$S_g(n\boldsymbol{t}) = \sum_{i=1}^{m} \alpha_i \left(S_{g,n}(t_i) - S_{g,n}(t_{i-1})\right),$$

where

$$\alpha_i = \frac{d_i}{\sqrt{\sum_{i=1}^m d_i^2 (t_i - t_{i-1})}},$$

while $d_1, \ldots, d_m \in \mathbb{R}$. We want to prove that

$$S_g(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0; \sigma_g^2).$$

We consider $S_{g_M}(n\boldsymbol{t})$ where $g_M(x) = \sum_{\ell=1}^{M} g_\ell H_\ell(x)$, where $M \geqslant 1$ is a fixed integer. We will prove that

$$S_{g_M}(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0; \sigma_{g_M}^2).$$

As in the proof of Theorem 3.4, the chaos representation of the fractional Brownian motion increments (see (5.3)) allows us to write $S_{g_M}(n\boldsymbol{t})$ in the multiple Wiener chaos:

$$S_{g_M}(n\boldsymbol{t}) = \sum_{\ell=1}^{M} I_\ell(h_\ell^{(n,\mathbf{t})}),$$

where $h_\ell^{(n,\mathbf{t})}$ is

$$h_\ell^{(n,\mathbf{t})}(\lambda_1,\ldots,\lambda_\ell) = g_\ell \ell! \sum_{i=1}^{m} \alpha_i \frac{1}{\sqrt{n}} \sum_{j=\lfloor nt_{i-1}\rfloor-1}^{\lfloor nt_i\rfloor-2} f^{(n)}(\lambda_1, j)\cdots f^{(n)}(\lambda_\ell, j),$$

and where I_ℓ is given by (5.5) and $f^{(n)}$ by equality (5.4).

First, let us compute the variance of $S_{g_M}(n\boldsymbol{t})$.

$$\begin{aligned}
\mathrm{E}[S_{g_M}(n\boldsymbol{t})]^2 &= \sum_{\ell=1}^{M} \frac{1}{\ell!} \int_{\mathbb{R}^\ell} \left| h_\ell^{(n,\mathbf{t})}(\lambda_1,\ldots,\lambda_\ell)\right|^2 \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_\ell \\
&= \sum_{\ell=1}^{M} \ell! g_\ell^2 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \alpha_{i_1}\alpha_{i_2} \left(\frac{1}{n} \sum_{j_1=\lfloor nt_{i_1-1}\rfloor-1}^{\lfloor nt_{i_1}\rfloor-2} \sum_{j_2=\lfloor nt_{i_2-1}\rfloor-1}^{\lfloor nt_{i_2}\rfloor-2} \rho_H^\ell(j_1-j_2) \right).
\end{aligned}$$

Now for $\ell \geqslant 1$,

$$\begin{aligned}
&\frac{1}{n} \sum_{s_1=\lfloor nt_{i-1}\rfloor-1}^{\lfloor nt_i\rfloor-2} \sum_{s_2=\lfloor nt_{j-1}\rfloor-1}^{\lfloor nt_j\rfloor-2} \rho_H^\ell(s_1-s_2) \\
&\qquad \xrightarrow[n\to+\infty]{} \begin{cases} (t_i - t_{i-1}) \sum_{r=-\infty}^{+\infty} \rho_H^\ell(r), & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{5.38}$$

A proof of the latter convergence is obtained in the case where $i = j$ by considering Lemma 5.1 page 78. To prove it for $i \neq j$, once again, we use Lemma 5.1, showing that for $k \geqslant 2$, $\frac{1}{n}\sum_{a=0}^{n}\sum_{b=n}^{nk}|\rho_H(a-b)|$ tends to zero as n goes to infinity. Thus

$$\lim_{n\to+\infty} \mathrm{E}[S_{g_M}(n\boldsymbol{t})]^2 = \left(\sum_{i=1}^{m}\alpha_i^2(t_i - t_{i-1})\right)\left(\sum_{\ell=1}^{M}\ell! g_\ell^2 \sum_{r=-\infty}^{+\infty}\rho_H^\ell(r)\right) = \sigma_{g_M}^2.$$

To conclude the proof of (1), Theorem 1 of Peccati and Tudor (2005) is used and as in the proof of Theorem 3.4, it is enough to prove that for fixed ℓ and p, $\ell \geqslant 2$ and $p = 1, \ldots, \ell - 1$, $\lim_{n\to+\infty} B_n = 0$, where B_n is

$$B_n = \int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n,\mathbf{t})} \otimes_p h_\ell^{(n,\mathbf{t})}(\lambda_1, \ldots, \lambda_{\ell-p}, \mu_1, \ldots, \mu_{\ell-p})\right|^2 \\ \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{\ell-p}\, \mathrm{d}\mu_1 \ldots, \mathrm{d}\mu_{\ell-p},$$

remembering that we defined the p-th contractions $\otimes_p$ in (5.6).
Now we compute B_n and we get

$$B_n = (\ell!)^4 g_\ell^4 \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{i_3=1}^{m}\sum_{i_4=1}^{m}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{i_4} \times \\ \frac{1}{n^2}\sum_{j_1=\lfloor nt_{i_1-1}\rfloor-1}^{\lfloor nt_{i_1}\rfloor-2}\ \sum_{j_2=\lfloor nt_{i_2-1}\rfloor-1}^{\lfloor nt_{i_2}\rfloor-2}\ \sum_{j_3=\lfloor nt_{i_3-1}\rfloor-1}^{\lfloor nt_{i_3}\rfloor-2}\ \sum_{j_4=\lfloor nt_{i_4-1}\rfloor-1}^{\lfloor nt_{i_4}\rfloor-2} \\ \rho_H^{\ell-p}(j_1-j_2)\rho_H^{\ell-p}(j_3-j_4)\rho_H^{p}(j_1-j_3)\rho_H^{p}(j_2-j_4).$$

Using the same arguments as in the proof of Theorem 3.4, it is easy to see that for N large enough, one has

$$\overline{\lim_{n\to\infty}} B_n \leqslant \boldsymbol{C} N^{2H-4},$$

and then $\lim_{n\to+\infty} B_n = 0$.
Hence, we proved that

$$S_{g_M}(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0;\sigma_{g_M}^2),$$

where $\boldsymbol{t} = (t_1, \ldots, t_m)$ and $S_{g_M}(n\boldsymbol{t}) = \sum_{i=1}^{m}\alpha_i\left(S_{g_M,n}(t_i) - S_{g_M,n}(t_{i-1})\right)$.

Furthermore, using that $\sum_{p=M+1}^{+\infty} g_p^2 p! \underset{M\to+\infty}{\longrightarrow} 0$, we can prove that

$$\lim_{M\to+\infty} \sup_{n\geqslant 1} \mathrm{E}[S_g(n\mathbf{t}) - S_{g_M}(n\boldsymbol{t})]^2 = 0,$$

and since

$$\mathcal{N}(0;\sigma_{g_M}^2) \xrightarrow[M\to\infty]{\text{Law}} \mathcal{N}(0;\sigma_g^2),$$

by applying Lemma 1.1 of Dynkin (1988), we proved that

$$S_g(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0;\sigma_g^2).$$

To obtain assertion (1) about the convergence of process $S_{g,n}$, we just have to prove the tightness of the sequence of this process. We need the following lemma.

Lemma 5.13. *Let G a function in $L^4(\phi(x)\,\mathrm{d}x)$ with Hermite rank $m \geqslant 1$ and let $\{X_i\}_{i=1}^{\infty}$ a stationary Gaussian sequence with mean 0, variance 1 and covariance function r such $\sum_{i=0}^{\infty} |r(i)|^m < +\infty$.*
For $I \geqslant 1$,

$$\mathrm{E}\left[\frac{1}{\sqrt{I}}\sum_{i=1}^{I} G(X_i)\right]^4 \leqslant \boldsymbol{C}.$$

Proof of Lemma 5.13. Since $\sum_{i=0}^{\infty} |r(i)|^m < +\infty$, $\forall 0 < \varepsilon < \frac{1}{3}$, $\exists j = j(\varepsilon) \in \mathbb{N}$, such that $\forall i \geqslant j$, $|r(i)| \leqslant \varepsilon < \frac{1}{3}$.
Let $i^\star = I - \lfloor I/j \rfloor j$. We have

$$\left|\sum_{i=1}^{I} G(X_i)\right| \leqslant \sum_{i=1}^{i^\star} \left|\sum_{k=0}^{\lfloor I/j\rfloor} G(X_{i+kj})\right| + \sum_{i=i^\star+1}^{j} \left|\sum_{k=0}^{\lfloor I/j\rfloor-1} G(X_{i+kj})\right|.$$

Jensen's inequality leads to

$$\left(\sum_{i=1}^{I} G(X_i)\right)^4 \leqslant 8\left[(i^\star)^3 \sum_{i=1}^{i^\star}\left(\sum_{k=0}^{\lfloor I/j\rfloor} G(X_{i+kj})\right)^4 \right.$$
$$\left. +(j-i^\star)^3 \sum_{i=i^\star+1}^{j}\left(\sum_{k=0}^{\lfloor I/j\rfloor-1} G(X_{i+kj})\right)^4\right].$$

Consequently

$$\mathrm{E}\left[\left(\sum_{i=1}^{I} G(X_i)\right)^4\right] \leqslant 8j^3 \left\{ \sum_{i=1}^{i^\star} \mathrm{E}\left[\left(\sum_{k=0}^{\lfloor I/j \rfloor} G(X_{i+kj})\right)^4\right] \right.$$
$$\left. + \sum_{i=i^\star+1}^{j} \mathrm{E}\left[\left(\sum_{k=0}^{\lfloor I/j \rfloor -1} G(X_{i+kj})\right)^4\right]\right\}.$$

First, note that by Proposition 3.1 (ii) of Taqqu (1977), since $0 < \varepsilon < \frac{1}{3}$, we get $G \in \bar{\mathcal{G}}_4(\varepsilon)$ (the notation $\bar{\mathcal{G}}_p(\varepsilon)$ for $p \geqslant 2$ can be found in Taqqu (1977, Definition 3.2, p. 209)).

The process $\{X_{i+kj}, k \geqslant 0\}$ is ε-standard Gaussian, see Taqqu (1977, Definition 3.1, p. 209), with covariance function $r(kj)$. Applying Proposition 4.2 (i) of Taqqu (1977) with $m \geqslant 1$, $p = 4$, $G \in \bar{\mathcal{G}}_4(\varepsilon)$, $N = \lfloor I/j \rfloor + 1$ and also with $N = \lfloor I/j \rfloor$ we get

$$\mathrm{E}\left[\left(\sum_{i=1}^{I} G(X_i)\right)^4\right] \leqslant 8j^4 K(\varepsilon, G)(\lfloor I/j \rfloor + 1)^2 \left(\sum_{k=0}^{\infty} |r^m(kj)|\right)^2$$
$$\leqslant 8j^2 K(\varepsilon, G)(I + j)^2 \left(\sum_{i=0}^{\infty} |r^m(i)|\right)^2$$

So, we showed, since $I \geqslant 1$, that

$$\mathrm{E}\left[\frac{1}{\sqrt{I}} \sum_{i=1}^{I} G(X_i)\right]^4 \leqslant 8j^2 K(\varepsilon, G)(j+1)^2 (\sum_{i=0}^{\infty} |r^m(i)|)^2.$$

To complete the proof of this lemma, it is sufficient to note that the right-hand side expression of the previous inequality is uniform in I, that ε is fixed and that $j(\varepsilon)$ is consequently a fixed integer. $\square$

Now for any $t > s$, we have

$$\mathrm{E}\big[S_{g,n}(t) - S_{g,n}(s)\big]^4 = \frac{1}{n^2} \mathrm{E}\left[\sum_{i=\lfloor ns \rfloor -1}^{\lfloor nt \rfloor -2} g(\Delta_n b_H(i))\right]^4$$
$$= \frac{1}{n^2} \mathrm{E}\left[\sum_{i=1}^{\lfloor nt \rfloor - \lfloor ns \rfloor} g(\Delta_n b_H(i))\right]^4,$$

since the process b_H has stationary increments.

We apply Lemma 5.13 to $g \in L^4(\phi(x)\,\mathrm{d}x)$, $m = 1$ and to the process $\{\Delta_n b_H(i)\}_{i=1}^{\infty}$ with covariance $r = \rho_H$ satisfying $\sum_{i=0}^{\infty} |\rho_H(i)| < +\infty$.
The last finiteness comes from equivalence (6.2), page 116.
We get

$$\mathrm{E}\big[S_{g,n}(t) - S_{g,n}(s)\big]^4 \leqslant \boldsymbol{C} \left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^2. \tag{5.39}$$

Now, let fixed $t_1 < t < t_2$.
If $t_2 - t_1 \geqslant \frac{1}{n}$, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&\mathrm{E}\big[(S_{g,n}(t_2) - S_{g,n}(t))^2 (S_{g,n}(t) - S_{g,n}(t_1))^2\big] \\
&\leqslant \boldsymbol{C} \left(\frac{\lfloor nt_2 \rfloor - \lfloor nt \rfloor}{n}\right)\left(\frac{\lfloor nt \rfloor - \lfloor nt_1 \rfloor}{n}\right) \\
&\leqslant \boldsymbol{C} \left(\frac{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}{n}\right)^2 \leqslant \boldsymbol{C}\,(t_2 - t_1)^2.
\end{aligned}$$

Now, if $t_2 - t_1 < \frac{1}{n}$, two cases occur. If t_1 and t_2 are in the same interval, that is $t_1, t_2 \in (\frac{k}{n}, \frac{k+1}{n})$, then t_1 and t are in the same interval, and $S_{g,n}(t) - S_{g,n}(t_1) = 0$. Otherwise, t_1 and t_2 are in contiguous intervals and in this case, then t_1 and t are in the same interval, or t and t_2 are in the same interval. Then in both cases, we have $(S_{g,n}(t) - S_{g,n}(t_1))(S_{g,n}(t_2) - S_{g,n}(t)) = 0$.

The tightness of process $S_{g,n}$ follows by Theorem 15.6. in Billingsley (1968) and assertion (1) follows.

Note that if $g \in L^4(\phi(x)\,\mathrm{d}x)$, with Hermite rank greater than or equal to one, the bound given in (5.39) for $s = 0$ implies that for all $0 \leqslant t \leqslant 1$, $\mathrm{E}[S_{g,n}(t)]^4 \leqslant \boldsymbol{C}$. Consequently, Remark 5.11 follows.

(2) We can suppose that $g(x) = \sum_{\ell=2}^{+\infty} g_\ell H_\ell(x)$. Indeed, since $\sum_{r=-\infty}^{+\infty} \rho_H(r) = 0$, it follows that $\frac{1}{\sqrt{n}} \sum_{i=0}^{[n\cdot]-2} \Delta_n b_H(i)$ tends to zero in L^2 as n tends to infinity.
Let $c_0, \ldots, c_m$, be real constants. As before, it is enough to establish the limit distribution of

$$\sum_{j=0}^{m} c_j b_H(t_j) \; + \; S_{g_M}(n\boldsymbol{t}).$$

As in the proof of part (1), Theorem 1 of Peccati and Tudor (2005) allows us to conclude the convergence of finite dimensional distributions of $(b_H(t), S_{g,n}(t))$. Indeed it is enough to remark that $\sum_{j=0}^{m} c_j b_H(t_j)$ belongs to the first Wiener chaos and

then is a Gaussian random variable with finite variance and that $S_{g_M}(nt)$ belongs to the superior order one.
Furthermore the tightness of the sequence of processes $(b_H, S_{g,n})$ follows from that of the sequence of process $S_{g,n}$ proved in part (1) and implies convergence of $(b_H, S_{g,n})$. Thus assertion (2) of Lemma 5.10 follows. Then, this convergence ensures stable convergence in part (1) and (2) of this Lemma (see Proposition 1 (B) of Aldous and Eagleson (1978)). □

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Chapter 6
Complementary Results

6.1 Introduction

In this chapter, we prove seven lemmas required in the detailed proofs of the results. The first two are related to the functional estimation seen in Sect. 5.3.3. Indeed, in that section we explained that in the case where $\mu \equiv 0$, the solution of the SDE is $X(t) = K(b_H(t))$ where K is a solution of an ODE and thus we assert that proving results enunciated in Remarks 3.28 and 3.30 is equivalent to prove them for the fBm. These lemmas give in an explicit manner how the increments of X can be approximated by those of the fBm. The proofs required the use of the modulus of continuity for the fBm and other results proved in Sect. 5.2.1.

The third lemma is a straightforward calculation of the asymptotic variance of the random variable defined as a linear combination of variables of the type $S_{g,\ell_i n}(1)$, used in Sect. 5.2.2.

The fourth lemma is concerned by Sect. 5.2.3 where we link $\hat{H}_{k(n)}$ with $\hat{H}_{\log}$. In this lemma we proved that the corresponding functionals are equivalent in L^2. For this aim we show that the Hermite coefficients for function $\frac{g_{k(n)}}{k(n)}$ converge to those of function $g_{\log}$.

In the fifth lemma, we prove the almost sure equivalence between the second order increments of X and of σ times the increments of the fBm, referred to in Sect. 5.3.1. Giving the explicit solution for each of the four models and using the modulus of continuity for the fBm lead to the proof.

A similar lemma is then demonstrated in the case where we do hypotheses testing seen in Sect. 5.3.2 replacing σ by σ_n and the techniques are the same that for previous lemma.

Finally in last and seventh lemma, we get back to functional estimation seen in Sect. 5.3.3 where μ is supposed to be null and where we want to prove the stable convergence for a functional of the fBm. This lemma is a step in this progression. More precisely, we prove the L^2 equivalence between the looked for functional

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and its approximation. That is done using regression techniques and straightforward calculus of expectations.

6.2 Proofs

Proof of Lemma 3.31. Recall that $H > \frac{1}{2}$, $\mu \equiv 0$ and $\sigma \in C^1$. Using that $X(t) = K(b_H(t))$, for $t \geqslant 0$, where K is the solution of the ODE (3.29), for $i = 0, 1, \ldots, n-2$ one has,

$$\Delta_n X(i) = \frac{n^H}{\sigma_{2H}} \left(K(b_H(\tfrac{i+2}{n})) - K(b_H(\tfrac{i+1}{n})) \right) + \frac{n^H}{\sigma_{2H}} \left(K(b_H(\tfrac{i}{n})) - K(b_H(\tfrac{i+1}{n})) \right).$$

The Taylor expansion for the function K gives

$$\Delta_n X(i) = \sigma(X(\tfrac{i+1}{n}))\Delta_n b_H(i) + \frac{1}{2}\frac{n^H}{\sigma_{2H}} \left\{ \left(b_H(\tfrac{i+2}{n}) - b_H(\tfrac{i+1}{n})\right)^2 \right.$$
$$\left. \ddot{K}\left(b_H(\tfrac{i+1}{n}) + \theta_1 \left[b_H(\tfrac{i+2}{n}) - b_H(\tfrac{i+1}{n})\right]\right) \right\}$$
$$+ \frac{1}{2}\frac{n^H}{\sigma_{2H}} \left\{ \left(b_H(\frac{i}{n}) - b_H(\tfrac{i+1}{n})\right)^2 \right.$$
$$\left. \ddot{K}\left(b_H(\tfrac{i+1}{n}) + \theta_2 \left[b_H(\tfrac{i}{n}) - b_H(\tfrac{i+1}{n})\right]\right) \right\},$$

where θ_1 (resp. θ_2) is a point between $b_H(\frac{i+1}{n})$ and $b_H(\frac{i+2}{n})$ (resp. between $b_H(\frac{i+1}{n})$ and $b_H(\frac{i}{n})$).

Using the modulus of continuity of b_H (see (5.37)), one has

$$\Delta_n X(i) = \sigma(X(\tfrac{i}{n}))\Delta_n b_H(i)$$
$$+ \left[\sigma(X(\tfrac{i+1}{n})) - \sigma(X(\tfrac{i}{n}))\right] \Delta_n b_H(i) + O_{a.s.}((\tfrac{1}{n})^{H-\delta}).$$

Using the Taylor expansion of $\dot{K}$, one obtains

$$\sigma(X(\tfrac{i+1}{n})) - \sigma(X(\tfrac{i}{n})) = \dot{K}(b_H(\tfrac{i+1}{n})) - \dot{K}(b_H(\tfrac{i}{n}))$$
$$= \left(b_H(\tfrac{i+1}{n}) - b_H(\tfrac{i}{n})\right) \ddot{K}\left(b_H(\tfrac{i}{n}) + \theta\left[b_H(\tfrac{i+1}{n}) - b_H(\tfrac{i}{n})\right]\right),$$

where θ is a point between $b_H(\frac{i}{n})$ and $b_H(\frac{i+1}{n})$.

Once again, using the modulus of continuity of b_H, we finally get the result. □

Proof of Lemma 3.32. Let

$$A_n(h) = \tfrac{1}{n-1}\sum_{i=0}^{n-2} h(X(\tfrac{i}{n}))\left\{|\Delta_n X(i)|^k - \left[\sigma(X(\tfrac{i}{n}))\right]^k |\Delta_n b_H(i)|^k\right\}.$$

Recall that $\mu \equiv 0$, $\sigma \in C^1$ and $H > \frac{1}{2}$ so that $X(t) = K(b_H(t))$, for $t \geqslant 0$, where K is the solution of the ODE (3.29).

We will prove that almost surely, for all continuous function h and for all real $k \geqslant 1$, $A_n(h) = O((\frac{1}{n})^{H-\delta}) = o\left(\frac{1}{\sqrt{n}}\right)$ for δ small enough.

On the one hand

$$|A_n(h)| \leqslant \boldsymbol{C}(\omega)\left\| |(\Delta_n X)^*|^k - \left[(\sigma \circ X(\tfrac{\cdot}{n}))^*\right]^k |(\Delta_n b_H)^*|^k \right\|_1.$$

On the other hand, we apply the second part of inequality (5.26) to $f = (\Delta_n X)^*$ and to $g = (\sigma \circ X(\frac{\cdot}{n}))^*(\Delta_n b_H)^*$. Thus, using Lemma 3.31 and Corollary 3.3, we finally get

$$\begin{aligned}
|A_n(h)| &\leqslant \boldsymbol{C}(\omega)\,\|a_n^*\|_k\left[\|(\Delta_n b_H)^*\|_k^{k-1} + \|a_n^*\|_k^{k-1}\right]\\
&\leqslant \boldsymbol{C}(\omega)(\tfrac{1}{n})^{H-\delta}\left(\boldsymbol{C}(\omega) + (\tfrac{1}{n})^{(H-\delta)(k-1)}\right)\\
&\leqslant \boldsymbol{C}(\omega)(\tfrac{1}{n})^{H-\delta},
\end{aligned}$$

that yields lemma. □

Proof of Lemma 5.1. We need to prove the following lemma. For fixed $p, k, \ell \in \mathbb{N}^*$, let us denote by $\delta_{k,\ell}$ the expression $\rho^p_{k,\ell}$.

Lemma 6.1. *For all $k, \ell \in \mathbb{N}^*$,*

$$\begin{aligned}
&\sum_{i=0}^{kn}\sum_{j=0}^{\ell n}\delta_{k,\ell}(\ell i - kj) = \sum_{r=0}^{\ell n}\left(n - \lfloor \tfrac{r}{\ell}\rfloor\right)\delta_{k,\ell}(kr) + \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell n}\delta_{k,\ell}(kr)\\
&+ \sum_{r=\ell}^{\ell n}\left(\lfloor n - \tfrac{r}{\ell}\rfloor + 1\right)\delta_{\ell,k}(kr) + \left(n\sum_{r=1}^{\ell-1}\delta_{\ell,k}(kr)\right)\mathbb{1}_{\{\ell\geqslant 2\}}\\
&+ \left[\sum_{s=1}^{k-1}\left(\sum_{r=0}^{\ell(n-1)}\left(n - \lfloor \tfrac{r}{\ell}\rfloor - 1\right)\delta_{k,\ell}(kr+\ell s) + \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)}\delta_{k,\ell}(kr+\ell s)\right.\right.\\
&\left.\left. + \left(n\sum_{r=1}^{\ell-1}\delta_{\ell,k}(kr-\ell s)\right)\mathbb{1}_{\{\ell\geqslant 2\}} + \sum_{r=\ell}^{\ell n}\left(\lfloor n - \tfrac{r}{\ell}\rfloor + 1\right)\delta_{\ell,k}(kr-\ell s)\right)\right]\mathbb{1}_{\{k\geqslant 2\}}.
\end{aligned}$$

Proof of Lemma 6.1.

$$\sum_{i=0}^{kn}\sum_{j=0}^{\ell n}\delta_{k,\ell}(\ell i-kj)=\sum_{i=0}^{kn}\sum_{j=0}^{\ell n}\delta_{1,\frac{\ell}{k}}(\tfrac{\ell}{k}i-j),$$

since for $x\in\mathbb{R}$, $\delta_{k,\ell}(x)=\delta_{1,\frac{\ell}{k}}(\frac{x}{k})$.

Let $i=kz+s$, with $0\leqslant s\leqslant k-1$. We get

$$\begin{aligned}\sum_{i=0}^{kn}\sum_{j=0}^{\ell n}\delta_{k,\ell}(\ell i-kj)&=\sum_{z=0}^{n}\sum_{j=0}^{\ell n}\delta_{1,\frac{\ell}{k}}(\ell z-j)+\left(\sum_{s=1}^{k-1}\sum_{z=0}^{n-1}\sum_{j=0}^{\ell n}\delta_{1,\frac{\ell}{k}}(\ell z-j+\tfrac{\ell}{k}s)\right)\mathbb{1}_{\{k\geqslant 2\}}\\&=S_1+S_2.\end{aligned}$$

We study S_1 and S_2 separately.

We suppose $k\geqslant 2$, in this case $S_2=\sum\limits_{s=1}^{k-1}T_s$ where for $s=1,\dots,k-1$,

$$T_s=\sum_{i=0}^{n-1}\sum_{j=0}^{\ell n}\delta_{1,\frac{\ell}{k}}(\ell i-j+\tfrac{\ell}{k}s).$$

Now, we study T_s for fixed s, $1\leqslant s\leqslant k-1$.

$$T_s=\sum_{i=0}^{n-1}\sum_{j=0}^{\ell i}\delta_{1,\frac{\ell}{k}}(\ell i-j+\tfrac{\ell}{k}s)+\sum_{i=0}^{n-1}\sum_{j=\ell i+1}^{\ell n}\delta_{1,\frac{\ell}{k}}(\ell i-j+\tfrac{\ell}{k}s).$$

Making the changes of variables $r=\ell i-j$ in the first summation and $r=j-\ell i$ in the second one and using that $\delta_{1,\frac{\ell}{k}}(-x)=\delta_{\frac{\ell}{k},1}(x)$, one gets

$$T_s=\sum_{i=0}^{n-1}\sum_{r=0}^{\ell i}\delta_{1,\frac{\ell}{k}}(r+\tfrac{\ell}{k}s)+\sum_{i=0}^{n-1}\sum_{r=1}^{\ell(n-i)}\delta_{\frac{\ell}{k},1}(r-\tfrac{\ell}{k}s).$$

Inverting the indices of summation for the first summation, one obtains

$$\begin{aligned}T_s=&\sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)}\sum_{i=\frac{r}{\ell}}^{n-1}\delta_{1,\frac{\ell}{k}}(r+\tfrac{\ell}{k}s)+\sum_{\substack{r=0\\ \frac{r}{\ell}\notin\mathbb{N}}}^{\ell(n-1)}\sum_{i=[\frac{r}{\ell}]+1}^{n-1}\delta_{1,\frac{\ell}{k}}(r+\tfrac{\ell}{k}s)\\&+\left(\sum_{i=0}^{n-1}\sum_{r=1}^{\ell-1}\delta_{\frac{\ell}{k},1}(r-\tfrac{\ell}{k}s)\right)\mathbb{1}_{\{\ell\geqslant 2\}}+\sum_{i=0}^{n-1}\sum_{r=\ell}^{\ell(n-i)}\delta_{\frac{\ell}{k},1}(r-\tfrac{\ell}{k}s),\end{aligned}$$

and then

$$T_s = \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)} \left(n - \lfloor \tfrac{r}{\ell} \rfloor\right) \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s) + \sum_{\substack{r=0\\ \frac{r}{\ell}\notin\mathbb{N}}}^{\ell(n-1)} \left(n - \lfloor \tfrac{r}{\ell} \rfloor - 1\right) \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s)$$

$$+ \left(n \sum_{r=1}^{\ell-1} \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s)\right) \mathbb{1}_{\{\ell\geqslant 2\}} + \sum_{i=0}^{n-1} \sum_{r=\ell}^{\ell(n-i)} \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s).$$

Inverting the indices of summation for the last summation, ensures that

$$T_s = \sum_{r=0}^{\ell(n-1)} \left(n - \lfloor \tfrac{r}{\ell} \rfloor - 1\right) \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s) + \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)} \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s)$$

$$+ \left(n \sum_{r=1}^{\ell-1} \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s)\right) \mathbb{1}_{\{\ell\geqslant 2\}} + \sum_{r=\ell}^{\ell n} \sum_{i=0}^{\lfloor n-\frac{r}{\ell} \rfloor} \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s).$$

Finally, we proved that

$$T_s = \sum_{r=0}^{\ell(n-1)} \left(n - \lfloor \tfrac{r}{\ell} \rfloor - 1\right) \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s) + \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)} \delta_{1,\frac{\ell}{k}}(r + \tfrac{\ell}{k}s)$$

$$+ \left(n \sum_{r=1}^{\ell-1} \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s)\right) \mathbb{1}_{\{\ell\geqslant 2\}} + \sum_{r=\ell}^{\ell n} \left(\left\lfloor n - \frac{r}{\ell} \right\rfloor + 1\right) \delta_{\frac{\ell}{k},1}(r - \tfrac{\ell}{k}s).$$

Thus using that for $x \in \mathbb{R}$ and $k, \ell \in \mathbb{N}^*$, $\delta_{k,\ell}(x) = \delta_{1,\frac{\ell}{k}}(\frac{x}{k}) = \delta_{\frac{k}{\ell},1}(\frac{x}{\ell})$, we obtain

$$S_2 = \left(\sum_{s=1}^{k-1} T_s\right) \mathbb{1}_{\{k\geqslant 2\}} = \left[\sum_{s=1}^{k-1} \left(\sum_{r=0}^{\ell(n-1)} \left(n - \lfloor \tfrac{r}{\ell} \rfloor - 1\right) \delta_{k,\ell}(kr + \ell s)\right.\right.$$

$$+ \sum_{\substack{r=0\\ \frac{r}{\ell}\in\mathbb{N}}}^{\ell(n-1)} \delta_{k,\ell}(kr + \ell s) + \left(n \sum_{r=1}^{\ell-1} \delta_{\ell,k}(kr - \ell s)\right) \mathbb{1}_{\{\ell\geqslant 2\}}$$

$$\left.\left. + \sum_{r=\ell}^{\ell n} \left(\lfloor n - \tfrac{r}{\ell} \rfloor + 1\right) \delta_{\ell,k}(kr - \ell s)\right)\right] \mathbb{1}_{\{k\geqslant 2\}}.$$

Now,

$$S_1 = \sum_{i=0}^{n} \sum_{j=0}^{\ell n} \delta_{1,\frac{\ell}{k}}(\ell i - j) = \sum_{i=0}^{n} \sum_{j=0}^{\ell i} \delta_{1,\frac{\ell}{k}}(\ell i - j) + \sum_{i=0}^{n-1} \sum_{j=\ell i+1}^{\ell n} \delta_{1,\frac{\ell}{k}}(\ell i - j).$$

Making the same changes of variables as in the computation concerning T_s, we obtain

$$S_1 = \sum_{r=0}^{\ell n} \left(n - \left\lfloor \tfrac{r}{\ell} \right\rfloor\right) \delta_{k,\ell}(kr) + \sum_{\substack{r=0 \\ \frac{r}{\ell} \in \mathbb{N}}}^{\ell n} \delta_{k,\ell}(kr)$$

$$+ \sum_{r=\ell}^{\ell n} \left(\left\lfloor n - \tfrac{r}{\ell} \right\rfloor + 1\right) \delta_{\ell,k}(kr) + \left(n \sum_{r=1}^{\ell-1} \delta_{\ell,k}(kr)\right) \mathbb{1}_{\{\ell \geqslant 2\}},$$

and Lemma 6.1 follows. □

Using Lemma 6.1 and the fact that $\rho_{k,\ell}(x)$ is equivalent to $\boldsymbol{C}\, |x|^{2H-4}$ for $|x|$ large enough (see the proof of Lemma 3.8), so that $\sum\limits_{s=0}^{k-1} \sum\limits_{r=-\infty}^{+\infty} |r|\, |\delta_{k,\ell}(kr + \ell s)| < +\infty$, we get

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{(kn-2)} \sum_{j=0}^{(\ell n-2)} \delta_{k,\ell}(\ell i - kj)$$

$$= \sum_{r=0}^{+\infty} \delta_{k,\ell}(kr) + \sum_{r=\ell}^{+\infty} \delta_{\ell,k}(kr) + \left(\sum_{r=1}^{\ell-1} \delta_{\ell,k}(kr)\right) \mathbb{1}_{\{\ell \geqslant 2\}}$$

$$+ \left[\sum_{s=1}^{k-1} \left(\sum_{r=0}^{+\infty} \delta_{k,\ell}(kr + \ell s) + \left(\sum_{r=1}^{\ell-1} \delta_{\ell,k}(kr - \ell s)\right) \mathbb{1}_{\{\ell \geqslant 2\}}\right.\right.$$

$$\left.\left. + \sum_{r=\ell}^{+\infty} \delta_{\ell,k}(kr - \ell s)\right)\right] \mathbb{1}_{\{k \geqslant 2\}}$$

$$= \sum_{s=0}^{k-1} \left(\sum_{r=0}^{+\infty} \delta_{k,\ell}(kr + \ell s) + \sum_{r=1}^{+\infty} \delta_{\ell,k}(kr - \ell s)\right).$$

Using that for $x \in \mathbb{R}$, $\delta_{k,\ell}(x) = \delta_{\ell,k}(-x)$, one gets Lemma 5.1. □

Proof of Lemma 5.3. Lemma 3.8 page 45 gives the asymptotic behavior of $\mathrm{E}[S_{g_{\log},n}(1)]^2$, that is

$$\mathrm{E}[S_{g_{\log},n}(1)]^2 \underset{n \to +\infty}{\longrightarrow} \sigma^2_{g_{\log}}. \tag{6.1}$$

Now, (3.7) in Corollary 3.10 gives the expression of the Hermite coefficients, $g_{2p,k(n)}$, of $g_{k(n)}$, let

$$g_{2p,k(n)} = \frac{1}{(2p)!} \prod_{i=0}^{p-1} (k(n) - 2i),$$

and as in the proof of Lemma 3.8, Mehler's formula (2.3) allows us to compute $\mathrm{E}[S(n)]^2$, and

$$\mathrm{E}[S(n)]^2 = \sum_{p=1}^{\infty} \left(\frac{g_{2p,k(n)}}{k(n)} \right)^2 (2p)! \left(\frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \rho_H^{2p}(i-j) \right).$$

On the one hand, by Lemma 5.1,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \rho_H^{2p}(i-j) = \sum_{r=-\infty}^{+\infty} \rho_H^{2p}(r).$$

Furthermore, the expression of the Hermite coefficients, $g_{2p,\log}$, of function $g_{\log}$ is given in (3.12) in Corollary 3.13 and we observe that

$$\frac{g_{2p,k(n)}}{k(n)} \underset{n \to +\infty}{\longrightarrow} \frac{1}{(2p)!} \prod_{i=1}^{p-1} (-2i) = g_{2p,\log}.$$

On the other hand for large enough n

$$\left| \frac{g_{2p,k(n)}}{k(n)} \right| = \frac{1}{(2p)!} \prod_{i=1}^{p-1} (2i - k(n)) \leqslant \frac{1}{(2p)!} \prod_{i=1}^{p-1} (2i) = |g_{2p,\log}|.$$

Now we consider the following inequality. Let f be an even function, then for $n \geqslant 1$, one has

$$\sum_{i=0}^{n} \sum_{j=0}^{n} f(i-j) = 2 \sum_{i=1}^{n} (n - (i-1)) f(i) + (n+1) f(0),$$

and then

$$\left| \sum_{i=0}^{n} \sum_{j=0}^{n} f(i-j) \right| \leqslant 2n \sum_{i=0}^{+\infty} |f(i)|.$$

Thus we obtain the following inequality

$$\left| \frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \rho_H^{2p}(i-j) \right| \leqslant 2 \sum_{i=0}^{+\infty} \left| \rho_H^{2p}(i) \right| \leqslant 2 \sum_{i=0}^{+\infty} |\rho_H(i)| < +\infty.$$

The last finiteness provides from the fact that, as for $\rho_{k,\ell}$ in the proof of Lemma 3.8, it can be seen that $\rho_H(x)$ is equivalent to

$$-1/(4-2^{2H})\,|x|^{2H-4}\,H(2H-1)(2H-2)(2H-3) \tag{6.2}$$

for large values of $|x|$. Thus $|\rho_H(x)|$ is bounded from above by $\boldsymbol{C}\,|x|^{2H-4}$.

Since $\|g_{\log}\|^2_{2,\phi} < +\infty$, we finally proved that

$$\mathrm{E}[S(n)]^2 \underset{n\to+\infty}{\longrightarrow} \sigma^2_{g_{\log}}. \tag{6.3}$$

To achieve the proof of Lemma 5.3, we compute $\mathrm{E}[S(n)S_{g_{\log},n}(1)]$ by Mehler's formula and we proceed as for $\mathrm{E}[S(n)]^2$ to obtain

$$\mathrm{E}[S(n)S_{g_{\log},n}(1)] \underset{n\to+\infty}{\longrightarrow} \sigma^2_{g_{\log}}, \tag{6.4}$$

(6.1), (6.3) and (6.4) give the required result. □

Proof of Lemma 5.4. We shall proof this lemma for the third model. The other models could be treated in a similar way.

For $i = 0, 1, \ldots, n-2$, one has

$$\begin{aligned}\Gamma_n X(i) = \frac{n^H}{\sigma_{2H}}\Bigg[\Bigg\{&\exp\left(\frac{2\mu}{n} + \sigma\left[b_H(\tfrac{i+2}{n}) - b_H(\tfrac{i}{n})\right]\right) - 1\Bigg\} - \\ &2\left\{\exp\left(\frac{\mu}{n} + \sigma\left[b_H(\tfrac{i+1}{n}) - b_H(\tfrac{i}{n})\right]\right) - 1\right\}\Bigg].\end{aligned}$$

By the Taylor expansion of the exponential function one gets

$$\begin{aligned}\Gamma_n X(i) = \sigma\Delta_n b_H(i) + \frac{n^H}{2\sigma_{2H}}&\left[\frac{2\mu}{n} + \sigma\left(b_H(\tfrac{i+2}{n}) - b_H(\tfrac{i}{n})\right)\right]^2 \times \\ &\exp\left(\theta\left[\frac{2\mu}{n} + \sigma\left(b_H(\tfrac{i+2}{n}) - b_H(\tfrac{i}{n})\right)\right]\right) - \\ \frac{n^H}{\sigma_{2H}}&\left[\frac{\mu}{n} + \sigma\left(b_H(\tfrac{i+1}{n}) - b_H(\tfrac{i}{n})\right)\right]^2 \times \\ &\exp\left(\theta'\left[\frac{\mu}{n} + \sigma\left(b_H(\tfrac{i+1}{n}) - b_H(\tfrac{i}{n})\right)\right]\right),\end{aligned}$$

with $0 < \theta < 1$ and $0 < \theta' < 1$.

Using the modulus of continuity of b_H (see (5.37)), one obtains

$$\Gamma_n X(i) = \sigma\Delta_n b_H(i) + a_n(i), \text{ and}$$

$$\begin{aligned}|a_n(i)| &\leqslant \boldsymbol{C}(\omega)\left[\left(\tfrac{1}{n}\right)^{H-\delta}+\left(\tfrac{1}{n}\right)^{2-H}\right]\\ &\leqslant \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{H-\delta}.\end{aligned}$$

Remark 6.2. For the fourth model, we prove that

$$\Delta_n X(i) = \sigma X(\tfrac{i}{n})\Delta_n b_H(i) + O_{a.s.}((\tfrac{1}{n})^{H-\delta}).$$

If μ and c are of the same sign or if $\mu = 0$, then $|X(\frac{i}{n})| \geqslant |c|\exp(-a(\omega)) > 0$, where $a(\omega) = \sigma \sup_{t\in[0,\,1]}|b_H(t)(\omega)|$.

Thus $\Gamma_n X(i) = \sigma\Delta_n b_H(i) + O_{a.s.}((\frac{1}{n})^{H-\delta})$.

□

Proof of Lemma 5.6. The proof is based on the proof of Lemma 5.4. It consists in bounding the expression $a_n(i)$ appearing in this lemma, with σ_n and μ_n taking the role of σ and μ, using the fact that both are bounded. □

Proof of Lemma 5.9. First we compute $\mathrm{E}[T_n(f)]^2$. In this aim, we decompose this expectation into two terms S_1 and S_2 where

$$S_1 = \frac{1}{n}\sum_{\substack{i,j=0\\ i\neq j}}^{n-2}\mathrm{E}[f(b_H(\tfrac{i}{n}))f(b_H(\tfrac{j}{n}))g(\Delta_n b_H(i))g(\Delta_n b_H(j))]$$

$$S_2 = \frac{1}{n}\sum_{i=0}^{n-2}\mathrm{E}[f^2(b_H(\tfrac{i}{n}))g^2(\Delta_n b_H(i))].$$

Let us consider S_1. We fix $i, j \in \{0,1,\ldots,n-2\}$, $i \neq j$ and we consider the change of variables

$$\begin{aligned}b_H(\tfrac{i}{n}) &= Z_{1,n}(i,j) + A_{1,n}(i,j)\Delta_n b_H(i) + A_{2,n}(i,j)\Delta_n b_H(j),\\ b_H(\tfrac{j}{n}) &= Z_{2,n}(i,j) + B_{1,n}(i,j)\Delta_n b_H(i) + B_{2,n}(i,j)\Delta_n b_H(j),\end{aligned}$$

with $(Z_{1,n}(i,j), Z_{2,n}(i,j))$ a zero mean Gaussian vector independent of $(\Delta_n b_H(i), \Delta_n b_H(j))$ and

$$A_{1,n}(i,j) = \frac{\mathrm{E}[b_H(\frac{i}{n})\Delta_n b_H(i)] - \rho_H(i-j)\mathrm{E}[b_H(\frac{i}{n})\Delta_n b_H(j)]}{1-\rho_H^2(i-j)},$$

$$A_{2,n}(i,j) = \frac{\mathrm{E}[b_H(\frac{i}{n})\Delta_n b_H(j)] - \rho_H(i-j)\mathrm{E}[b_H(\frac{i}{n})\Delta_n b_H(i)]}{1-\rho_H^2(i-j)}.$$

Two similar formulas hold for $B_{1,n}(i,j)$ and $B_{2,n}(i,j)$.

A straightforward computation shows that since $|i-j| \geqslant 1$, then $(1-\rho_H^2(i-j)) \geqslant \boldsymbol{C} > 0$ and then for $i \neq j$ we get

$$\max_{k=1,2} |A_{k,n}(i,j), B_{k,n}(i,j)| \leqslant \boldsymbol{C} n^{-H}. \tag{6.5}$$

Writing the Taylor expansion of f one has,

$$f(b_H(\tfrac{i}{n})) = \sum_{k=0}^{3} \frac{1}{k!} \overset{k\cdot}{f}(Z_{1,n}(i,j)) \left[A_{1,n}(i,j)\Delta_n b_H(i) + A_{2,n}(i,j)\Delta_n b_H(j)\right]^k$$
$$+ \frac{1}{4!} \overset{4\cdot}{f}(\theta_{1,n}(i,j))[A_{1,n}(i,j)\Delta_n b_H(i) + A_{2,n}(i,j)\Delta_n b_H(j)]^4,$$

with $\theta_{1,n}(i,j)$ between $b_H(\frac{i}{n})$ and $Z_{1,n}(i,j)$.

A similar formula holds for $f(b_H(\frac{j}{n}))$.

We can decompose S_1 as the sum of 25 terms. We use the notations J_{j_1,j_2} for the corresponding sums, where $j_1, j_2 = 0, \ldots, 4$ are the subscripts involving $\overset{j_1\cdot}{f}$ and $\overset{j_2\cdot}{f}$. We only consider J_{j_1,j_2} with $j_1 \leqslant j_2$. Then we obtain the following

(A) One term of the form

$$J_{0,0} = \frac{1}{n} \sum_{\substack{i,j=0\\ i\neq j}}^{n-2} \mathrm{E}[f(Z_{1,n}(i,j)) f(Z_{2,n}(i,j))] \mathrm{E}[g(\Delta_n b_H(i)) g(\Delta_n b_H(j))].$$

We will denote by $a_n(i,j) = \mathrm{E}[f(Z_{1,n}(i,j)) f(Z_{2,n}(i,j))]$ and let

$$\beta(k) = \mathrm{E}[g(\Delta_n b_H(0)) g(\Delta_n b_H(k))] = \sum_{p=1}^{+\infty} g_p^2 p! \rho_H^p(k).$$

With these notations and making the change of variable $(i-j) = k$ in last summation one obtains

$$J_{0,0} = \left(\frac{1}{n} \sum_{i=0}^{n-2} \mathrm{E}[f^2(b_H(\tfrac{i}{n}))] \right) \left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty} \beta(k) \right)$$
$$+ \frac{1}{n} \sum_{i=0}^{n-2} \sum_{\substack{k=i-n+2\\ k\neq 0}}^{i} \left(a_n(i,i-k) - \mathrm{E}[f^2(b_H(\tfrac{i}{n}))] \right) \beta(k)$$

$$-\frac{1}{n}\sum_{i=0}^{n-2}\sum_{k=-\infty}^{i-n+1}\mathrm{E}[f^2(b_H(\tfrac{i}{n}))]\beta(k)-\frac{1}{n}\sum_{i=0}^{n-2}\sum_{k=i+1}^{+\infty}\mathrm{E}[f^2(b_H(\tfrac{i}{n}))]\beta(k)$$
$$=(1)+(2)+(3)+(4).$$

Since $\sum_{k=-\infty}^{+\infty}|\beta(k)|<+\infty$ (see equivalence (6.2)), it is obvious that (3) and (4) tend to zero when n goes to infinity. Furthermore using inequality (6.5), we can prove that for $k\neq 0$, $\left|a_n(i,i-k)-\mathrm{E}[f^2(b_H(\frac{i}{n}))]\right|\leqslant \boldsymbol{C}\left|\frac{k}{n}\right|^H$. Now, since $H<1$, $\sum_{k=-\infty}^{+\infty}|k|^H|\beta(k)|<+\infty$ (see again equivalence (6.2)), so that (2) tends to zero with n. Thus we proved that

$$\lim_{n\to+\infty}J_{0,0}=\left(\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\beta(k)\right)\left(\int_0^1\mathrm{E}[f^2(b_H(u))]\,\mathrm{d}u\right). \tag{6.6}$$

(B) Two terms of the form $J_{0,1}\equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V)=\mathcal{N}(0,\Sigma)$ then $\mathrm{E}[Ug(U)g(V)]=0$ for g even or odd.

(C) Two terms of the form

$$J_{0,2}=\frac{1}{2n}\sum_{\substack{i,j=0\\i\neq j}}^{n-2}\mathrm{E}[f(Z_{1,n}(i,j))\ddot{f}(Z_{2,n}(i,j))]\times$$
$$\mathrm{E}[g(\Delta_n b_H(i))g(\Delta_n b_H(j))\,(B_{1,n}(i,j)\Delta_n b_H(i)$$
$$+B_{2,n}(i,j)\Delta_n b_H(j))^2].$$

Since $|\rho_H(i-j)|\leqslant 1$, g is even, or odd with Hermite rank greater than or equal to three, then

$$\left|\mathrm{E}[g(\Delta_n b_H(i))g(\Delta_n b_H(j))\,(B_{1,n}(i,j)\Delta_n b_H(i)+B_{2,n}(i,j)\Delta_n b_H(j))^2]\right|$$
$$\leqslant \boldsymbol{C}\left(\max_{k=1,2}B_{k,n}^2(i,j)\right)|\rho_H(i-j)|.$$

Using (6.5), and since $\displaystyle\frac{1}{n}\sum_{i=0}^{n}\sum_{j=0}^{n}|\rho_H(i-j)|\leqslant 2\sum_{i=0}^{+\infty}|\rho_H(i)|<+\infty$, we get

$$J_{0,2}=O(n^{-2H})=o(1).$$

(D) Two terms of the form $J_{0,3}\equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V)=\mathcal{N}(0,\Sigma)$ then $\mathrm{E}[(aU+bV)^3g(U)g(V)]=0$ for any two constants a and b and for function g even or odd.

(E) Two terms of the form

$$J_{0,4} = \frac{1}{4!n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathbb{E}[f(Z_{1,n}(i,j)) \overset{4\cdot}{f}(\theta_{2,n}(i,j)) g(\Delta_n b_H(i)) g(\Delta_n b_H(j))$$
$$\times [B_{1,n}(i,j)\Delta_n b_H(i) + B_{2,n}(i,j)\Delta_n b_H(j)]^4].$$

Therefore

$$|J_{0,4}| \leqslant \boldsymbol{C} \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \left(\max_{k=1,2} B_{k,n}^4(i,j) \right).$$

Finally using (6.5) once again, one obtains

$$J_{0,4} = O(n^{-(4H-1)}) = o(1),$$

since $H > \frac{1}{4}$.

Using the same type of arguments as for (C), (D), (E) we can prove that the other terms are all $o(1)$. Thus using equality (6.6) we proved that

$$\lim_{n \to +\infty} S_1 = \left(\sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \beta(k) \right) \left(\int_0^1 \mathrm{E}[f^2(b_H(u))] \, \mathrm{d}u \right).$$

Let us now consider S_2. Similar computations, holding i fixed and doing a regression of $b_H(\frac{i}{n})$ on $\Delta_n b_H(i)$, give that

$$\lim_{n \to +\infty} S_2 = \beta(0) \left(\int_0^1 \mathrm{E}[f^2(b_H(u))] \, \mathrm{d}u \right).$$

Thus we proved that

$$\lim_{n \to +\infty} \mathrm{E}[T_n(f)]^2 = \sigma_g^2 \left(\int_0^1 \mathrm{E}[f^2(b_H(u))] \, \mathrm{d}u \right). \tag{6.7}$$

Now let us compute $\mathrm{E}[T_n^{(m)}(f)]^2$. We decompose the last expression into two terms: $S_1 + S_2$, where

$$S_1 = \sum_{\substack{\ell_1,\ell_2=0\\ \ell_1\neq\ell_2}}^{m-1} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell_1}{m}\rfloor-1}^{\lfloor \frac{n(\ell_1+1)}{m}\rfloor-2} \sum_{j=\lfloor \frac{n\ell_2}{m}\rfloor-1}^{\lfloor \frac{n(\ell_2+1)}{m}\rfloor-2} \mathrm{E}[f(b_H(\tfrac{\ell_1}{m}))f(b_H(\tfrac{\ell_2}{m}))\times$$

$$g(\Delta_n b_H(i))g(\Delta_n b_H(j))],$$

and

$$S_2 = \sum_{\ell=0}^{m-1} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell}{m}\rfloor-1}^{\lfloor \frac{n(\ell+1)}{m}\rfloor-2} \sum_{j=\lfloor \frac{n\ell}{m}\rfloor-1}^{\lfloor \frac{n(\ell+1)}{m}\rfloor-2} \mathrm{E}[f^2(b_H(\tfrac{\ell}{m}))g(\Delta_n b_H(i))g(\Delta_n b_H(j))].$$

First we look at the first term. For fixed $\ell_1 \neq \ell_2$ and i, j (in this case necessarily i is different from j), we use the regression of $(b_H(\frac{\ell_1}{m}), b_H(\frac{\ell_2}{m}))$ on $(\Delta_n b_H(i), \Delta_n b_H(j))$. We can prove in the same way as before that

$$\lim_{n\to+\infty} S_1 = \sum_{\substack{\ell_1,\ell_2=0\\ \ell_1\neq\ell_2}}^{m-1} \mathrm{E}[f(b_H(\tfrac{\ell_1}{m}))f(b_H(\tfrac{\ell_2}{m}))]\times$$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell_1}{m}\rfloor-1}^{\lfloor \frac{n(\ell_1+1)}{m}\rfloor-2} \sum_{j=\lfloor \frac{n\ell_2}{m}\rfloor-1}^{\lfloor \frac{n(\ell_2+1)}{m}\rfloor-2} \beta(i-j) = 0$$

(last equality follows from convergence seen in (5.38)).

Then for the second term S_2, for fixed ℓ, i, j, using a regression of $b_H(\frac{\ell}{m})$ on $(\Delta_n b_H(i), \Delta_n b_H(j))$ if $i \neq j$ and on $\Delta_n b_H(i)$ otherwise, as before similar straightforward calculations show that

$$\lim_{n\to+\infty} \mathrm{E}[T_n^{(m)}(f)]^2 = \lim_{n\to+\infty} S_2 = \sigma_g^2 \left(\frac{1}{m} \sum_{\ell=0}^{m-1} \mathrm{E}[f^2(b_H(\tfrac{\ell}{m}))] \right),$$

and then

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{E}[T_n^{(m)}(f)]^2 = \sigma_g^2 \left(\int_0^1 \mathrm{E}[f^2(b_H(u))]\,\mathrm{d}u \right). \tag{6.8}$$

To conclude the proof of lemma we have to compute $\mathrm{E}[T_n(f)T_n^{(m)}(f)]$.

$$\mathrm{E}[T_n(f)T_n^{(m)}(f)] =$$

$$\sum_{\ell=0}^{m-1} \frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=\lfloor \frac{n\ell}{m}\rfloor-1}^{\lfloor \frac{n(\ell+1)}{m}\rfloor-2} \mathrm{E}[f(b_H(\tfrac{\ell}{m}))f(b_H(\tfrac{i}{n}))g(\Delta_n b_H(i))g(\Delta_n b_H(j))].$$

For fixed ℓ, i, j, using a regression of $(b_H(\frac{i}{n}), b_H(\frac{\ell}{m}))$ on $(\Delta_n b_H(i), \Delta_n b_H(j))$ if $i \neq j$ and on $\Delta_n b_H(i)$ otherwise, as before similar straightforward calculations show that

$$\lim_{n\to+\infty} \mathrm{E}[T_n(f)T_n^{(m)}(f)] = \sigma_g^2 \left(\sum_{\ell=0}^{m-1} \int_{\frac{\ell}{m}}^{\frac{\ell+1}{m}} \mathrm{E}[f(b_H(u))f(b_H(\tfrac{\ell}{m}))] \, \mathrm{d}u \right),$$

so that

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{E}[T_n(f)T_n^{(m)}(f)] = \sigma_g^2 \left(\int_0^1 \mathrm{E}[f^2(b_H(u))] \, \mathrm{d}u \right). \tag{6.9}$$

(6.7)–(6.9) yield lemma. □

Chapter 7
Tables and Figures Related to the Simulation Studies

7.1 Introduction

In this chapter we collect all the graphics and tables to which we refer in the text. They are presented to help the understanding of the different comments concerning the simulation results.

First, we display three graphics showing the empirical distribution of $\hat{H}_2$ obtained with a resolution of 1/2,048-th for different values of H.

Then, in Tables 7.1–7.5, we give the empirical mean and standard deviation of the estimators of H in the case of a fBm. Graphical representations are presented on pages 130–131.

Tables 7.6 and 7.7 present some results concerning the estimated covering probability of the confidence intervals we developed in Sects. 4.5.1.3 and 4.5.1.4, pages 67 and 70.

A series of Tables 7.8–7.15, followed by a series of graphics, Figs. 7.4–7.11 present results about the simultaneous estimation of H and σ for models excited by an fBm.

Table 7.16 gives the observed empirical level of the test on σ. Figures 7.12–7.19 present the empirical and the asymptotic power function of the test.

C. Berzin et al., *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lecture Notes in Statistics 216,
DOI 10.1007/978-3-319-07875-5_7,

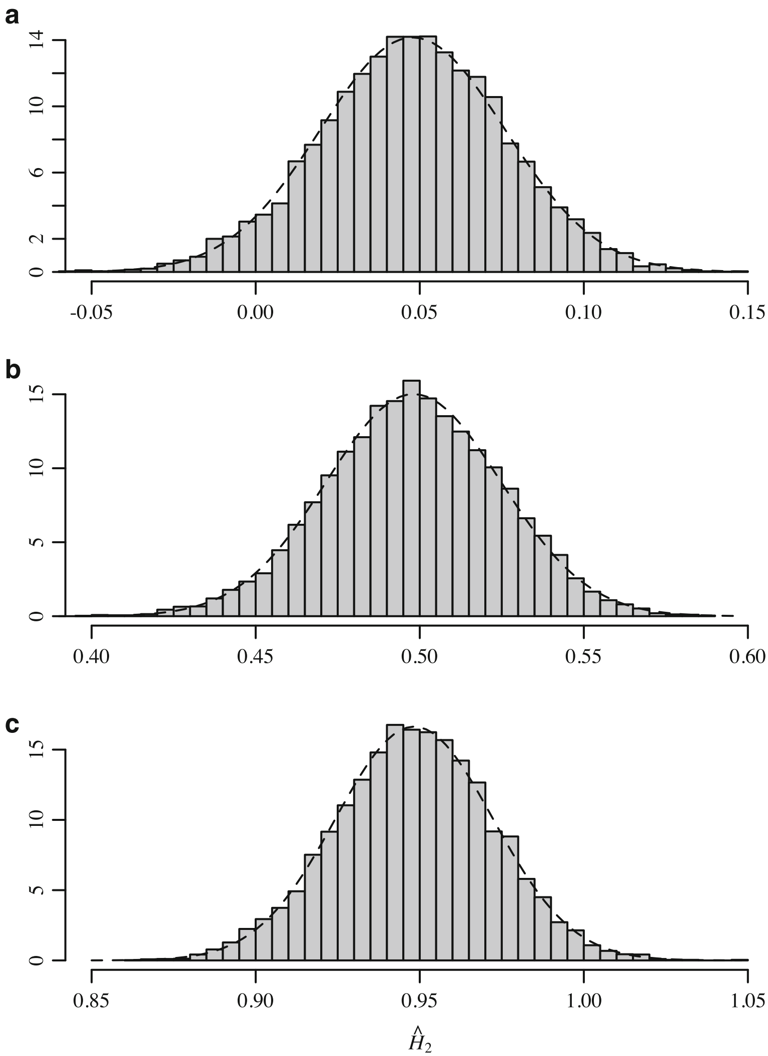

Fig. 7.1 Empirical distribution of $\hat{H}_2$ using a resolution of 1/2,048-th and $\ell = 5$, for (**a**) $H = 0.05$, (**b**) $H = 0.50$ and (**c**) $H = 0.95$. Superimposed are the normal densities with empirical means and standard errors

Table 7.1 Estimated mean and standard deviation of $\hat{H}_1$ for different values of H

	$\hat{H}_1$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.045	0.044	0.050	0.049	0.049	0.045	0.047	0.050	0.049	0.046	0.048	0.049	0.047	0.048
	Std. err.	0.199	0.138	0.099	0.070	0.050	0.096	0.066	0.048	0.035	0.060	0.042	0.030	0.043	0.030
0.20	Mean	0.198	0.195	0.200	0.199	0.199	0.196	0.197	0.199	0.199	0.197	0.198	0.199	0.198	0.199
	Std. err.	0.187	0.133	0.093	0.067	0.048	0.092	0.066	0.046	0.034	0.058	0.042	0.030	0.041	0.030
0.30	Mean	0.296	0.295	0.300	0.298	0.300	0.295	0.297	0.299	0.299	0.297	0.298	0.299	0.297	0.298
	Std. err.	0.182	0.128	0.090	0.065	0.047	0.091	0.065	0.045	0.034	0.058	0.042	0.029	0.041	0.029
0.40	Mean	0.396	0.395	0.399	0.399	0.400	0.396	0.397	0.399	0.400	0.397	0.398	0.399	0.397	0.398
	Std. err.	0.174	0.124	0.087	0.062	0.045	0.090	0.063	0.045	0.033	0.057	0.040	0.029	0.040	0.028
0.50	Mean	0.497	0.496	0.498	0.500	0.500	0.497	0.497	0.499	0.500	0.497	0.498	0.499	0.498	0.498
	Std. err.	0.167	0.118	0.083	0.059	0.043	0.087	0.062	0.044	0.032	0.056	0.040	0.028	0.040	0.028
0.60	Mean	0.597	0.596	0.598	0.601	0.600	0.597	0.597	0.599	0.600	0.597	0.598	0.600	0.598	0.599
	Std. err.	0.160	0.112	0.080	0.056	0.041	0.085	0.060	0.042	0.031	0.055	0.039	0.027	0.039	0.028
0.70	Mean	0.696	0.697	0.697	0.701	0.700	0.696	0.697	0.699	0.700	0.696	0.698	0.699	0.697	0.698
	Std. err.	0.151	0.106	0.075	0.053	0.039	0.082	0.059	0.041	0.030	0.053	0.038	0.027	0.038	0.027
0.80	Mean	0.796	0.796	0.797	0.801	0.800	0.796	0.797	0.799	0.801	0.796	0.798	0.799	0.797	0.799
	Std. err.	0.143	0.102	0.073	0.050	0.037	0.079	0.057	0.040	0.029	0.052	0.037	0.026	0.037	0.027
0.95	Mean	0.946	0.947	0.947	0.951	0.951	0.946	0.947	0.949	0.951	0.946	0.948	0.950	0.947	0.949
	Std. err.	0.131	0.092	0.066	0.045	0.033	0.075	0.053	0.038	0.027	0.050	0.036	0.025	0.036	0.026

Table 7.2 Estimated mean and standard deviation of $\hat{H}_2$ for different values of H

	$\hat{H}_2$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.041	0.043	0.049	0.046	0.054	0.042	0.046	0.047	0.050	0.044	0.046	0.049	0.045	0.048
	Std. err.	0.189	0.130	0.092	0.066	0.046	0.091	0.063	0.044	0.031	0.057	0.040	0.028	0.041	0.028
0.20	Mean	0.192	0.193	0.199	0.197	0.204	0.192	0.196	0.198	0.200	0.194	0.196	0.200	0.195	0.198
	Std. err.	0.178	0.127	0.088	0.062	0.044	0.089	0.063	0.044	0.031	0.057	0.040	0.028	0.040	0.028
0.30	Mean	0.292	0.291	0.301	0.296	0.304	0.291	0.296	0.298	0.300	0.294	0.296	0.300	0.295	0.298
	Std. err.	0.171	0.121	0.085	0.059	0.044	0.089	0.062	0.043	0.030	0.056	0.039	0.027	0.040	0.027
0.40	Mean	0.392	0.392	0.399	0.398	0.403	0.392	0.395	0.398	0.401	0.394	0.396	0.400	0.395	0.398
	Std. err.	0.164	0.117	0.082	0.057	0.042	0.086	0.061	0.043	0.030	0.055	0.039	0.027	0.039	0.027
0.50	Mean	0.495	0.491	0.498	0.499	0.503	0.493	0.495	0.499	0.501	0.495	0.497	0.500	0.496	0.498
	Std. err.	0.153	0.111	0.078	0.055	0.041	0.081	0.059	0.042	0.029	0.053	0.038	0.026	0.037	0.027
0.60	Mean	0.594	0.592	0.598	0.599	0.603	0.593	0.595	0.599	0.601	0.594	0.597	0.600	0.596	0.598
	Std. err.	0.147	0.105	0.074	0.053	0.039	0.080	0.057	0.040	0.028	0.052	0.036	0.026	0.037	0.026
0.70	Mean	0.695	0.692	0.698	0.699	0.702	0.694	0.695	0.699	0.701	0.695	0.697	0.700	0.696	0.698
	Std. err.	0.139	0.101	0.070	0.050	0.037	0.076	0.055	0.039	0.027	0.050	0.036	0.025	0.036	0.025
0.80	Mean	0.795	0.793	0.798	0.800	0.802	0.794	0.795	0.799	0.801	0.795	0.797	0.800	0.796	0.798
	Std. err.	0.131	0.095	0.066	0.047	0.035	0.074	0.053	0.037	0.026	0.049	0.035	0.024	0.035	0.025
0.95	Mean	0.948	0.943	0.947	0.951	0.951	0.945	0.945	0.949	0.951	0.946	0.947	0.950	0.947	0.948
	Std. err.	0.115	0.086	0.060	0.042	0.031	0.068	0.050	0.035	0.024	0.046	0.033	0.023	0.034	0.024

Table 7.3 Estimated mean and standard deviation of $\hat{H}_3$ for different values of H

	$\hat{H}_3$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.032	0.046	0.043	0.045	0.053	0.039	0.045	0.044	0.049	0.041	0.045	0.047	0.042	0.046
	Std. err.	0.188	0.133	0.095	0.066	0.047	0.091	0.064	0.046	0.032	0.057	0.041	0.029	0.041	0.029
0.20	Mean	0.181	0.199	0.195	0.193	0.204	0.190	0.197	0.194	0.199	0.192	0.196	0.197	0.193	0.197
	Std. err.	0.181	0.128	0.091	0.065	0.045	0.091	0.064	0.046	0.032	0.058	0.041	0.029	0.041	0.029
0.30	Mean	0.284	0.300	0.294	0.293	0.305	0.292	0.297	0.294	0.299	0.294	0.296	0.297	0.294	0.297
	Std. err.	0.176	0.124	0.089	0.064	0.044	0.091	0.064	0.046	0.032	0.058	0.041	0.029	0.041	0.029
0.40	Mean	0.388	0.400	0.394	0.394	0.405	0.394	0.397	0.394	0.399	0.395	0.396	0.397	0.394	0.397
	Std. err.	0.167	0.119	0.086	0.060	0.042	0.087	0.063	0.044	0.031	0.056	0.040	0.028	0.040	0.028
0.50	Mean	0.489	0.501	0.495	0.494	0.505	0.495	0.498	0.494	0.499	0.495	0.496	0.497	0.495	0.498
	Std. err.	0.161	0.114	0.081	0.057	0.040	0.085	0.061	0.043	0.030	0.054	0.039	0.027	0.038	0.027
0.60	Mean	0.587	0.603	0.595	0.593	0.604	0.595	0.599	0.594	0.599	0.596	0.597	0.597	0.595	0.598
	Std. err.	0.155	0.108	0.078	0.055	0.038	0.083	0.058	0.042	0.029	0.053	0.038	0.027	0.038	0.027
0.70	Mean	0.689	0.703	0.696	0.693	0.703	0.696	0.699	0.695	0.698	0.697	0.697	0.697	0.696	0.698
	Std. err.	0.145	0.103	0.075	0.052	0.036	0.079	0.056	0.042	0.028	0.051	0.037	0.027	0.037	0.026
0.80	Mean	0.787	0.804	0.796	0.794	0.803	0.796	0.800	0.795	0.799	0.797	0.798	0.797	0.796	0.798
	Std. err.	0.140	0.098	0.070	0.049	0.034	0.078	0.054	0.040	0.026	0.050	0.036	0.026	0.036	0.026
0.95	Mean	0.938	0.953	0.948	0.945	0.951	0.945	0.950	0.946	0.948	0.947	0.948	0.948	0.947	0.949
	Std. err.	0.128	0.089	0.063	0.045	0.031	0.073	0.051	0.037	0.025	0.049	0.035	0.025	0.035	0.025

Table 7.4 Estimated mean and standard deviation of $\hat{H}_4$ for different values of H

	$\hat{H}_4$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.031	0.034	0.044	0.047	0.047	0.033	0.039	0.046	0.047	0.036	0.042	0.046	0.039	0.044
	Std. err.	0.200	0.143	0.104	0.074	0.051	0.097	0.070	0.050	0.035	0.061	0.044	0.031	0.044	0.031
0.20	Mean	0.184	0.178	0.195	0.198	0.197	0.181	0.186	0.197	0.198	0.185	0.191	0.197	0.188	0.193
	Std. err.	0.191	0.137	0.100	0.071	0.049	0.096	0.069	0.050	0.035	0.061	0.044	0.031	0.043	0.031
0.30	Mean	0.286	0.279	0.295	0.299	0.297	0.282	0.287	0.297	0.298	0.286	0.291	0.297	0.289	0.293
	Std. err.	0.188	0.135	0.097	0.069	0.047	0.095	0.068	0.049	0.035	0.061	0.043	0.031	0.043	0.030
0.40	Mean	0.388	0.378	0.395	0.399	0.397	0.383	0.386	0.397	0.398	0.386	0.391	0.397	0.389	0.393
	Std. err.	0.180	0.130	0.093	0.066	0.046	0.093	0.067	0.048	0.034	0.060	0.043	0.030	0.042	0.030
0.50	Mean	0.490	0.482	0.494	0.500	0.497	0.486	0.488	0.497	0.499	0.488	0.492	0.497	0.491	0.494
	Std. err.	0.169	0.126	0.090	0.064	0.044	0.089	0.065	0.048	0.033	0.058	0.041	0.030	0.041	0.029
0.60	Mean	0.592	0.581	0.595	0.600	0.597	0.586	0.588	0.598	0.599	0.588	0.592	0.598	0.591	0.594
	Std. err.	0.166	0.120	0.085	0.060	0.042	0.088	0.064	0.046	0.032	0.058	0.041	0.029	0.040	0.029
0.70	Mean	0.693	0.682	0.694	0.701	0.698	0.687	0.688	0.697	0.699	0.689	0.692	0.698	0.691	0.694
	Std. err.	0.159	0.115	0.080	0.057	0.041	0.086	0.062	0.044	0.030	0.056	0.040	0.028	0.040	0.028
0.80	Mean	0.793	0.783	0.793	0.800	0.799	0.788	0.788	0.796	0.799	0.789	0.792	0.797	0.791	0.794
	Std. err.	0.150	0.110	0.077	0.054	0.038	0.083	0.060	0.042	0.030	0.055	0.039	0.028	0.039	0.028
0.95	Mean	0.945	0.934	0.943	0.950	0.949	0.940	0.939	0.946	0.949	0.940	0.942	0.948	0.942	0.945
	Std. err.	0.137	0.102	0.071	0.049	0.035	0.078	0.058	0.040	0.028	0.053	0.038	0.026	0.038	0.027

Table 7.5 Estimated mean and standard deviation of $\hat{H}_{\log}$ for different values of H

	$\hat{H}_{\log}$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.059	0.046	0.053	0.045	0.056	0.053	0.050	0.049	0.051	0.052	0.049	0.051	0.051	0.050
	Std. err.	0.273	0.191	0.136	0.095	0.067	0.128	0.089	0.064	0.045	0.078	0.055	0.039	0.054	0.038
0.20	Mean	0.209	0.196	0.204	0.199	0.203	0.202	0.200	0.201	0.201	0.202	0.200	0.202	0.201	0.201
	Std. err.	0.262	0.186	0.132	0.091	0.064	0.126	0.088	0.063	0.044	0.077	0.055	0.039	0.054	0.038
0.30	Mean	0.311	0.299	0.301	0.299	0.305	0.305	0.300	0.300	0.302	0.303	0.300	0.301	0.302	0.301
	Std. err.	0.257	0.183	0.125	0.090	0.062	0.124	0.088	0.062	0.044	0.077	0.054	0.039	0.053	0.038
0.40	Mean	0.409	0.399	0.401	0.400	0.404	0.404	0.400	0.401	0.402	0.403	0.400	0.402	0.402	0.401
	Std. err.	0.250	0.176	0.123	0.089	0.062	0.123	0.086	0.061	0.043	0.076	0.053	0.038	0.053	0.037
0.50	Mean	0.507	0.506	0.500	0.498	0.502	0.507	0.503	0.499	0.500	0.505	0.501	0.500	0.503	0.501
	Std. err.	0.243	0.173	0.121	0.087	0.061	0.120	0.085	0.061	0.042	0.074	0.053	0.037	0.052	0.037
0.60	Mean	0.607	0.604	0.602	0.599	0.603	0.606	0.603	0.600	0.601	0.604	0.602	0.601	0.603	0.602
	Std. err.	0.237	0.168	0.118	0.085	0.059	0.118	0.084	0.059	0.041	0.073	0.052	0.036	0.051	0.036
0.70	Mean	0.708	0.703	0.703	0.698	0.703	0.706	0.703	0.701	0.700	0.705	0.702	0.701	0.703	0.702
	Std. err.	0.228	0.163	0.117	0.082	0.057	0.116	0.082	0.059	0.041	0.072	0.052	0.036	0.050	0.036
0.80	Mean	0.811	0.804	0.800	0.800	0.801	0.808	0.802	0.800	0.801	0.805	0.802	0.801	0.804	0.801
	Std. err.	0.224	0.158	0.113	0.078	0.055	0.113	0.081	0.056	0.040	0.071	0.051	0.035	0.050	0.035
0.95	Mean	0.960	0.954	0.952	0.951	0.951	0.957	0.953	0.952	0.951	0.955	0.952	0.951	0.954	0.952
	Std. err.	0.216	0.154	0.108	0.076	0.052	0.111	0.078	0.055	0.038	0.070	0.049	0.034	0.049	0.035

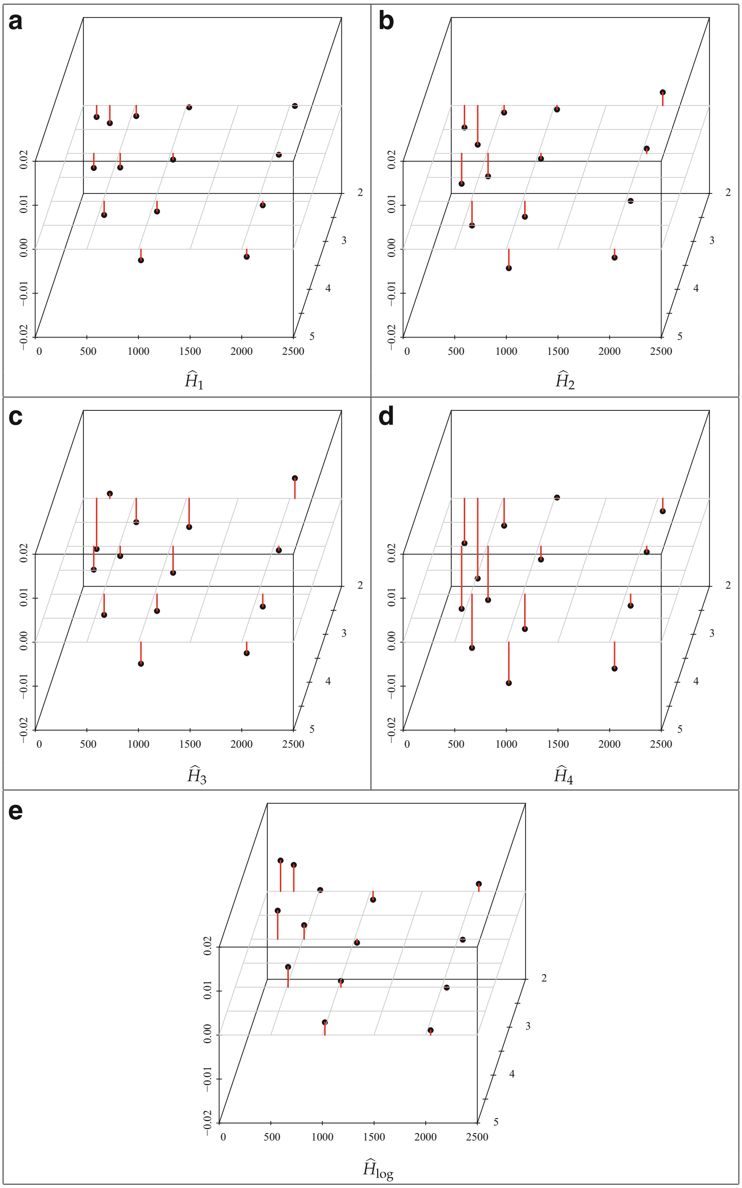

Fig. 7.2 3D-diagrams showing the difference between H and the empirical mean of $\hat{H}_k$ for values of $k = 1, \ldots, 4$ and $\hat{H}_{\log}$, given $H = \frac{1}{2}$. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5. Note that the z-scale goes from -0.02 to $+0.02$

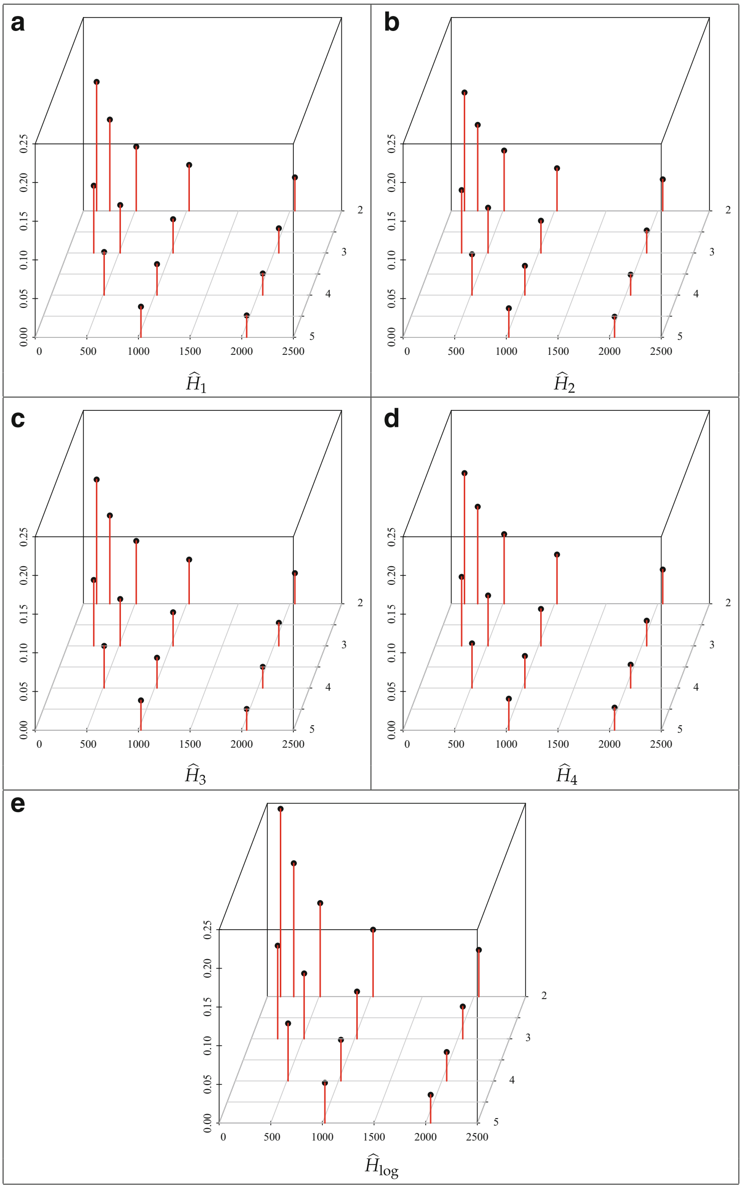

Fig. 7.3 3D-diagrams showing the empirical standard error of $\hat{H}_k$ for values of $k = 1, \ldots, 4$ and $\hat{H}_{\log}$, given $H = \frac{1}{2}$. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5. Note that the z-scale goes from 0.00 to $+0.25$

Table 7.6 Estimated covering probability of the confidence interval based on $\hat{Q}_{0.025}(H)$ and $\hat{Q}_{0.975}(H)$

		n				
H	ℓ	128	256	512	1,024	2,048
0.05	2	0.9537	0.9529	0.9498	0.9502	0.9536
	3		0.9587	0.9510	0.9468	0.9599
	4			0.9546	0.9514	0.9521
	5				0.9532	0.9512
0.2	2	0.9473	0.9521	0.9468	0.9459	0.9498
	3		0.9457	0.9455	0.9439	0.9564
	4			0.9448	0.9463	0.9488
	5				0.9476	0.9486
0.3	2	0.9444	0.9526	0.9536	0.9436	0.9507
	3		0.9468	0.9482	0.9446	0.9550
	4			0.9461	0.9467	0.9500
	5				0.9465	0.9478
0.4	2	0.9392	0.9510	0.9479	0.9429	0.9539
	3		0.9375	0.9458	0.9463	0.9552
	4			0.9387	0.9442	0.9496
	5				0.9443	0.9479
0.5	2	0.9386	0.9520	0.9544	0.9434	0.9569
	3		0.9393	0.9456	0.9469	0.9510
	4			0.9417	0.9458	0.9498
	5				0.9433	0.9473
0.6	2	0.9395	0.9514	0.9505	0.9463	0.9591
	3		0.9378	0.9473	0.9493	0.9508
	4			0.9423	0.9484	0.9508
	5				0.9417	0.9505
0.7	2	0.9389	0.9521	0.9515	0.9476	0.9628
	3		0.9390	0.9467	0.9525	0.9522
	4			0.9395	0.9511	0.9484
	5				0.9420	0.9501
0.8	2	0.9428	0.9509	0.9546	0.9522	*0.9648
	3		0.9432	0.9522	0.9540	0.9548
	4			0.9418	0.9518	0.9516
	5				0.9435	0.9509
0.95	2	0.9442	0.9571	0.9577	0.9568	*0.9692
	3		0.9426	0.9571	0.9609	0.9539
	4			0.9372	0.9529	0.9540
	5				*0.9362	0.9517

Table 7.7 Estimated covering probability of the confidence interval based on the normal approximation using estimated values of $\hat{\sigma}_{\hat{H}_2}$

		n				
H	ℓ	128	256	512	1,024	2,048
0.05	2	0.9606	0.9555	0.9493	0.9504	0.9588
	3		0.9585	0.9514	0.9574	0.9549
	4			0.9573	0.9528	0.9531
	5				0.9565	0.9497
0.2	2	0.9521	0.9517	0.9488	0.9486	0.9577
	3		0.9562	0.9502	0.9504	0.9527
	4			0.9547	0.9516	0.9516
	5				0.9549	0.9482
0.3	2	0.9493	0.9520	0.9444	0.9483	0.9569
	3		0.9521	0.9498	0.9497	0.9501
	4			0.9520	0.9505	0.9518
	5				0.9546	0.9476
0.4	2	0.9481	0.9508	0.9421	0.9459	0.9582
	3		0.9527	0.9486	0.9484	0.9479
	4			0.9532	0.9503	0.9502
	5				0.9531	0.9508
0.5	2	0.9519	0.9530	0.9479	0.9517	0.9600
	3		0.9519	0.9452	0.9488	0.9473
	4			0.9536	0.9485	0.9527
	5				0.9554	0.9469
0.6	2	0.9462	0.9495	0.9450	0.9532	0.9615
	3		0.9497	0.9512	0.9486	0.9487
	4			0.9540	0.9541	0.9479
	5				0.9539	0.9553
0.7	2	0.9475	0.9553	0.9482	0.9475	0.9609
	3		0.9485	0.9533	0.9487	0.9460
	4			0.9524	0.9508	0.9448
	5				0.9538	0.9500
0.8	2	0.9516	0.9543	0.9489	0.9529	*0.9656
	3		0.9573	0.9534	0.9519	0.9506
	4			0.9568	0.9535	0.9514
	5				0.9562	0.9510
0.95	2	0.9436	0.9617	0.9571	0.9568	*0.9663
	3		0.9504	0.9620	0.9599	0.9543
	4			0.9554	0.9588	0.9549
	5				0.9542	0.9533

Table 7.8 Estimated mean and standard deviation of $\hat{H}_2$ for different values of H under model 1

	$\hat{H}_2$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.038	0.042	0.049	0.046	0.051	0.040	0.046	0.048	0.049	0.043	0.046	0.049	0.044	0.047
	Std. err.	0.187	0.131	0.092	0.065	0.047	0.090	0.063	0.044	0.032	0.057	0.040	0.028	0.040	0.028
0.2	Mean	0.188	0.192	0.199	0.198	0.200	0.190	0.195	0.198	0.199	0.193	0.196	0.199	0.194	0.197
	Std. err.	0.177	0.126	0.088	0.063	0.044	0.089	0.063	0.044	0.032	0.057	0.040	0.028	0.040	0.028
0.3	Mean	0.291	0.292	0.299	0.298	0.300	0.291	0.296	0.299	0.299	0.294	0.297	0.299	0.295	0.298
	Std. err.	0.174	0.122	0.085	0.060	0.042	0.088	0.062	0.044	0.031	0.056	0.040	0.028	0.039	0.028
0.4	Mean	0.390	0.392	0.400	0.398	0.400	0.391	0.396	0.399	0.399	0.394	0.397	0.399	0.395	0.398
	Std. err.	0.164	0.117	0.082	0.057	0.041	0.085	0.061	0.043	0.030	0.055	0.039	0.027	0.039	0.027
0.5	Mean	0.493	0.491	0.501	0.499	0.500	0.492	0.496	0.500	0.499	0.494	0.497	0.500	0.496	0.498
	Std. err.	0.158	0.111	0.078	0.055	0.039	0.083	0.059	0.042	0.029	0.053	0.038	0.027	0.038	0.026
0.6	Mean	0.595	0.592	0.600	0.599	0.599	0.594	0.596	0.600	0.599	0.595	0.597	0.599	0.597	0.598
	Std. err.	0.149	0.106	0.074	0.053	0.037	0.080	0.057	0.040	0.029	0.052	0.037	0.026	0.037	0.026
0.7	Mean	0.695	0.691	0.701	0.700	0.699	0.693	0.696	0.700	0.700	0.695	0.698	0.700	0.697	0.698
	Std. err.	0.142	0.101	0.070	0.050	0.035	0.078	0.055	0.039	0.028	0.051	0.036	0.025	0.036	0.025
0.8	Mean	0.798	0.792	0.800	0.801	0.799	0.795	0.796	0.800	0.800	0.796	0.798	0.800	0.797	0.798
	Std. err.	0.133	0.095	0.066	0.047	0.033	0.074	0.053	0.037	0.026	0.049	0.035	0.025	0.035	0.025
0.95	Mean	0.948	0.943	0.949	0.951	0.949	0.945	0.946	0.950	0.950	0.946	0.948	0.950	0.947	0.948
	Std. err.	0.121	0.085	0.058	0.043	0.030	0.070	0.050	0.035	0.025	0.048	0.033	0.023	0.034	0.024

Table 7.9 Estimated mean and standard deviation of $\hat{\sigma}$ for different values of H under model 1

	$\hat{\sigma}$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	7.795	5.497	4.148	3.115	2.671	3.522	2.868	2.431	2.181	2.606	2.267	2.110	2.226	2.067
	Std. err.	8.841	5.498	3.444	2.253	1.615	2.539	1.833	1.360	1.059	1.540	1.182	0.930	1.156	0.909
	% admis.	59.3	63.4	70.5	76.0	85.7	67.7	77.1	85.8	93.3	77.7	87.6	95.3	86.0	94.8
0.2	Mean	3.733	2.897	2.510	2.252	2.181	2.256	2.152	2.102	2.062	2.054	2.040	2.037	2.001	2.009
	Std. err.	4.245	2.789	1.913	1.356	0.966	1.492	1.157	0.849	0.646	0.955	0.734	0.549	0.703	0.534
	% admis.	85.5	93.2	98.6	99.9		97.7	99.9			99.9				
0.3	Mean	3.202	2.605	2.405	2.217	2.143	2.202	2.133	2.097	2.055	2.058	2.039	2.036	2.009	2.009
	Std. err.	3.524	2.333	1.641	1.187	0.841	1.361	1.027	0.763	0.584	0.867	0.659	0.499	0.633	0.482
	% admis.	94.8	98.9				99.9								
0.4	Mean	2.848	2.501	2.363	2.185	2.123	2.167	2.123	2.093	2.051	2.049	2.040	2.036	2.006	2.012
	Std. err.	2.871	2.082	1.493	1.052	0.775	1.236	0.958	0.713	0.547	0.810	0.625	0.470	0.596	0.457
	% admis.	98.6	99.9												
0.5	Mean	2.745	2.395	2.332	2.176	2.101	2.153	2.104	2.100	2.048	2.048	2.038	2.042	2.011	2.013
	Std. err.	2.573	1.811	1.371	0.986	0.712	1.155	0.880	0.685	0.515	0.757	0.586	0.451	0.561	0.433
	% admis.	99.7													
0.6	Mean	2.644	2.348	2.269	2.154	2.084	2.151	2.092	2.082	2.043	2.052	2.032	2.033	2.015	2.009
	Std. err.	2.278	1.648	1.221	0.901	0.659	1.075	0.832	0.639	0.491	0.722	0.562	0.432	0.537	0.417
	% admis.														
0.7	Mean	2.542	2.278	2.244	2.144	2.074	2.125	2.078	2.086	2.044	2.042	2.031	2.039	2.012	2.011
	Std. err.	2.003	1.488	1.108	0.831	0.618	1.019	0.786	0.608	0.467	0.696	0.538	0.415	0.522	0.402
	% admis.														
0.8	Mean	2.487	2.235	2.197	2.134	2.060	2.132	2.067	2.074	2.042	2.049	2.028	2.034	2.020	2.010
	Std. err.	1.784	1.328	1.005	0.770	0.568	0.968	0.741	0.572	0.439	0.667	0.516	0.400	0.504	0.390
	% admis.														
0.95	Mean	2.295	2.243	2.092	2.059	2.107	2.128	2.076	2.016	2.043	2.058	2.025	2.013	2.024	2.014
	Std. err.	1.361	1.132	0.821	0.643	0.525	0.862	0.676	0.508	0.415	0.627	0.480	0.374	0.486	0.371
	% admis.														

Table 7.10 Estimated mean and standard deviation of $\hat{H}_2$ for different values of H under model 2

	$\hat{H}_2$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	0.039	0.046	0.048	0.047	0.053	0.043	0.047	0.048	0.050	0.045	0.047	0.049	0.046	0.049
	Std. err.	0.185	0.133	0.090	0.064	0.046	0.089	0.063	0.044	0.032	0.057	0.040	0.028	0.040	0.028
0.2	Mean	0.192	0.196	0.197	0.199	0.204	0.194	0.197	0.198	0.201	0.195	0.197	0.200	0.196	0.199
	Std. err.	0.177	0.126	0.086	0.062	0.044	0.088	0.063	0.044	0.031	0.057	0.040	0.028	0.040	0.028
0.3	Mean	0.293	0.297	0.298	0.299	0.304	0.295	0.297	0.298	0.301	0.296	0.298	0.300	0.297	0.299
	Std. err.	0.169	0.123	0.083	0.059	0.042	0.087	0.063	0.043	0.031	0.056	0.039	0.028	0.040	0.028
0.4	Mean	0.394	0.397	0.398	0.399	0.404	0.396	0.398	0.399	0.401	0.397	0.398	0.400	0.397	0.399
	Std. err.	0.165	0.118	0.081	0.058	0.041	0.085	0.062	0.042	0.030	0.055	0.039	0.027	0.039	0.028
0.5	Mean	0.493	0.498	0.498	0.499	0.503	0.496	0.498	0.499	0.501	0.497	0.498	0.500	0.497	0.499
	Std. err.	0.158	0.113	0.077	0.054	0.039	0.083	0.061	0.040	0.029	0.054	0.038	0.026	0.038	0.027
0.6	Mean	0.596	0.599	0.599	0.600	0.603	0.597	0.599	0.599	0.601	0.598	0.599	0.600	0.598	0.600
	Std. err.	0.149	0.107	0.074	0.052	0.037	0.081	0.059	0.039	0.029	0.053	0.037	0.026	0.038	0.027
0.7	Mean	0.696	0.699	0.700	0.700	0.702	0.697	0.699	0.700	0.701	0.698	0.700	0.700	0.699	0.700
	Std. err.	0.142	0.101	0.070	0.049	0.035	0.078	0.056	0.038	0.028	0.052	0.036	0.025	0.037	0.026
0.8	Mean	0.799	0.800	0.800	0.800	0.801	0.799	0.800	0.800	0.801	0.800	0.800	0.801	0.800	0.800
	Std. err.	0.135	0.096	0.066	0.046	0.033	0.075	0.054	0.037	0.027	0.051	0.036	0.025	0.036	0.026
0.95	Mean	0.967	0.954	0.952	0.951	0.951	0.961	0.953	0.952	0.951	0.958	0.953	0.951	0.956	0.952
	Std. err.	0.121	0.086	0.060	0.041	0.030	0.071	0.051	0.035	0.025	0.048	0.034	0.024	0.035	0.025

Table 7.11 Estimated mean and standard deviation of $\hat{\sigma}$ for different values of H under model 2

	$\hat{\sigma}$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	3.787	3.007	2.458	2.268	2.254	2.317	2.187	2.101	2.108	2.102	2.063	2.060	2.038	2.040
	Std. err.	4.371	2.998	1.820	1.328	0.994	1.550	1.176	0.837	0.652	0.987	0.738	0.554	0.713	0.545
	% admis.	86.4	93.8	98.8	99.9		98.3	99.9			99.9				
0.2	Mean	3.787	3.007	2.458	2.268	2.254	2.317	2.187	2.101	2.108	2.102	2.063	2.060	2.038	2.040
	Std. err.	4.371	2.998	1.820	1.328	0.994	1.550	1.176	0.837	0.652	0.987	0.738	0.554	0.713	0.545
	% admis.	86.4	93.8	98.8	99.9		98.3	99.9			99.9				
0.3	Mean	3.159	2.726	2.374	2.226	2.221	2.256	2.172	2.095	2.098	2.097	2.063	2.059	2.040	2.041
	Std. err.	3.361	2.525	1.600	1.171	0.883	1.386	1.062	0.757	0.587	0.884	0.674	0.509	0.647	0.503
	% admis.	95.5	98.9				99.9								
0.4	Mean	2.961	2.603	2.334	2.204	2.195	2.239	2.167	2.092	2.092	2.098	2.066	2.057	2.045	2.044
	Std. err.	3.099	2.154	1.450	1.055	0.808	1.272	0.991	0.706	0.551	0.824	0.632	0.476	0.605	0.472
	% admis.	98.9	99.9												
0.5	Mean	2.763	2.532	2.291	2.170	2.168	2.215	2.156	2.076	2.079	2.091	2.061	2.047	2.040	2.039
	Std. err.	2.599	1.979	1.332	0.952	0.737	1.189	0.943	0.654	0.516	0.790	0.604	0.446	0.580	0.450
	% admis.	99.9													
0.6	Mean	2.687	2.460	2.269	2.162	2.142	2.216	2.150	2.082	2.076	2.100	2.066	2.051	2.052	2.044
	Std. err.	2.347	1.737	1.239	0.891	0.673	1.131	0.889	0.630	0.498	0.758	0.580	0.436	0.562	0.436
	% admis.														
0.7	Mean	2.574	2.397	2.239	2.143	2.119	2.196	2.138	2.078	2.068	2.094	2.065	2.049	2.052	2.043
	Std. err.	2.034	1.555	1.132	0.816	0.618	1.066	0.836	0.596	0.469	0.724	0.555	0.418	0.539	0.419
	% admis.														
0.8	Mean	2.520	2.357	2.216	2.131	2.097	2.199	2.137	2.081	2.061	2.106	2.072	2.049	2.064	2.048
	Std. err.	1.816	1.428	1.035	0.752	0.569	1.004	0.794	0.572	0.443	0.700	0.542	0.403	0.529	0.413
	% admis.														
0.95	Mean	2.587	2.308	2.186	2.114	2.072	2.300	2.153	2.089	2.057	2.196	2.098	2.057	2.142	2.070
	Std. err.	1.547	1.162	0.879	0.647	0.501	0.914	0.716	0.529	0.407	0.665	0.514	0.382	0.516	0.398
	% admis.														

Table 7.12 Estimated mean and standard deviation of $\hat{H}_2$ for different values of H under model 3

	$\hat{H}_2$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	−0.375	−0.223	−0.086	−0.020	0.033	−0.299	−0.154	−0.053	0.006	−0.227	−0.107	−0.024	−0.172	−0.070
	Std. err.	2.651	2.354	2.193	1.947	1.859	1.540	1.372	1.266	1.141	1.021	0.930	0.837	0.752	0.676
0.2	Mean	0.523	0.487	0.441	0.393	0.352	0.505	0.464	0.417	0.373	0.484	0.441	0.395	0.462	0.418
	Std. err.	0.633	0.437	0.273	0.172	0.102	0.341	0.228	0.144	0.088	0.219	0.146	0.091	0.156	0.103
0.3	Mean	0.505	0.437	0.391	0.359	0.343	0.471	0.414	0.375	0.351	0.443	0.395	0.364	0.421	0.381
	Std. err.	0.333	0.209	0.126	0.080	0.051	0.179	0.107	0.065	0.041	0.116	0.069	0.042	0.082	0.048
0.4	Mean	0.513	0.458	0.434	0.418	0.415	0.486	0.446	0.426	0.417	0.467	0.437	0.422	0.454	0.430
	Std. err.	0.230	0.142	0.093	0.062	0.042	0.123	0.075	0.048	0.032	0.080	0.048	0.030	0.056	0.034
0.5	Mean	0.564	0.530	0.514	0.507	0.508	0.547	0.522	0.510	0.508	0.535	0.517	0.509	0.527	0.514
	Std. err.	0.183	0.121	0.081	0.056	0.039	0.099	0.064	0.043	0.029	0.064	0.041	0.028	0.045	0.029
0.6	Mean	0.643	0.618	0.607	0.603	0.606	0.630	0.612	0.605	0.605	0.622	0.609	0.605	0.617	0.608
	Std. err.	0.161	0.109	0.076	0.053	0.037	0.088	0.059	0.041	0.028	0.057	0.038	0.026	0.041	0.027
0.7	Mean	0.733	0.713	0.704	0.702	0.706	0.723	0.709	0.703	0.704	0.716	0.706	0.704	0.712	0.706
	Std. err.	0.149	0.102	0.071	0.049	0.035	0.083	0.056	0.039	0.027	0.054	0.036	0.025	0.039	0.026
0.8	Mean	0.830	0.812	0.805	0.802	0.805	0.821	0.808	0.803	0.803	0.815	0.806	0.804	0.811	0.805
	Std. err.	0.139	0.097	0.067	0.047	0.032	0.080	0.054	0.037	0.026	0.054	0.036	0.025	0.039	0.026
0.95	Mean	1.002	0.968	0.956	0.952	0.955	0.985	0.962	0.954	0.954	0.975	0.959	0.954	0.968	0.957
	Std. err.	0.149	0.092	0.061	0.042	0.030	0.094	0.055	0.036	0.025	0.065	0.037	0.024	0.048	0.027

Table 7.13 Estimated mean and standard deviation of $\hat{\sigma}$ for different values of H under model 3

	$\hat{\sigma}$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	$1.4^{(24)}$	$3.5^{(29)}$	$4.6^{(26)}$	$1.9^{(24)}$	$1.0^{(43)}$	$6.3^{(17)}$	$1.1^{(16)}$	$3.6^{(15)}$	$6.1^{(24)}$	$1.6^{(13)}$	$2.6^{(13)}$	$1.1^{(13)}$	$7.0^{(10)}$	$1.3^{(11)}$
	Std. err.	$6.6^{(25)}$	$2.3^{(31)}$	$1.5^{(28)}$	$6.5^{(25)}$	$3.9^{(44)}$	$3.2^{(19)}$	$3.1^{(17)}$	$1.5^{(17)}$	$2.5^{(26)}$	$6.1^{(14)}$	$1.4^{(15)}$	$5.4^{(14)}$	$3.1^{(12)}$	$6.0^{(12)}$
	% admis.	50.0	51.8	53.8	54.8	56.3	40.8	43.7	48.2	49.9	39.4	43.6	47.1	38.9	43.5
0.2	Mean	$8.7^{(04)}$	$1.2^{(05)}$	$1.4^{(03)}$	$6.8^{(02)}$	$1.8^{(01)}$	$1.3^{(03)}$	$2.3^{(02)}$	$7.0^{(01)}$	$2.3^{(01)}$	$2.1^{(02)}$	$6.0^{(01)}$	$2.8^{(01)}$	$7.8^{(01)}$	$3.3^{(01)}$
	Std. err.	$4.3^{(06)}$	$6.9^{(06)}$	$7.4^{(04)}$	$2.5^{(04)}$	$2.5^{(01)}$	$4.2^{(04)}$	$3.9^{(03)}$	$1.1^{(03)}$	$1.6^{(02)}$	$3.2^{(03)}$	$4.8^{(02)}$	$1.7^{(02)}$	$5.3^{(02)}$	$1.1^{(02)}$
	% admis.	80.8	88.0	95.9	99.1		95.1	99.1			99.7				
0.3	Mean	64.871	16.322	7.038	4.450	3.480	16.828	7.367	4.693	3.600	9.687	5.537	4.017	7.186	4.716
	Std. err.	380.29	38.570	8.655	3.855	1.644	45.302	7.488	2.882	1.364	13.462	3.453	1.581	6.310	2.106
	% admis.	94.2	98.4	99.9			99.9								
0.4	Mean	10.515	4.739	3.330	2.673	2.478	5.169	3.355	2.714	2.438	3.963	2.933	2.540	3.417	2.733
	Std. err.	29.527	5.260	2.468	1.360	0.924	5.819	1.959	1.063	0.678	2.696	1.142	0.670	1.635	0.789
	% admis.	98.7	99.9												
0.5	Mean	5.074	3.329	2.627	2.339	2.274	3.357	2.618	2.309	2.209	2.861	2.408	2.231	2.623	2.318
	Std. err.	6.381	2.814	1.597	1.049	0.762	2.325	1.212	0.788	0.551	1.332	0.764	0.520	0.912	0.554
	% admis.	99.8													
0.6	Mean	3.826	2.834	2.440	2.240	2.217	2.842	2.386	2.197	2.145	2.532	2.246	2.145	2.381	2.191
	Std. err.	3.718	2.006	1.370	0.942	0.698	1.635	0.974	0.697	0.505	1.019	0.645	0.462	0.727	0.478
	% admis.														
0.7	Mean	3.334	2.660	2.331	2.194	2.191	2.639	2.297	2.150	2.124	2.400	2.186	2.115	2.285	2.146
	Std. err.	2.806	1.755	1.167	0.839	0.638	1.376	0.898	0.633	0.473	0.891	0.595	0.435	0.648	0.446
	% admis.														
0.8	Mean	3.109	2.566	2.302	2.163	2.171	2.573	2.271	2.139	2.110	2.368	2.171	2.107	2.263	2.135
	Std. err.	2.349	1.530	1.081	0.785	0.584	1.280	0.829	0.606	0.454	0.854	0.569	0.423	0.631	0.433
	% admis.														
0.95	Mean	3.685	2.578	2.259	2.141	2.147	2.892	2.319	2.141	2.105	2.573	2.214	2.111	2.409	2.167
	Std. err.	4.707	1.467	0.913	0.666	0.519	2.185	0.865	0.565	0.419	1.324	0.608	0.408	0.926	0.468
	% admis.														

Here, $1.4^{(24)}$ stands for $1.4 \times 10^{(24)}$

Table 7.14 Estimated mean and standard deviation of $\hat{H}_2$ for different values of H under model 4

	$\hat{H}_2$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	−0.460	−0.195	−0.015	−0.105	0.209	−0.328	−0.105	−0.060	0.052	−0.221	−0.096	0.017	−0.176	−0.033
	Std. err.	2.725	2.414	2.207	1.986	1.760	1.555	1.392	1.266	1.136	1.031	0.937	0.835	0.756	0.675
0.2	Mean	0.508	0.493	0.445	0.386	0.350	0.501	0.469	0.415	0.368	0.483	0.442	0.393	0.460	0.418
	Std. err.	0.627	0.427	0.280	0.170	0.099	0.334	0.225	0.147	0.086	0.215	0.146	0.093	0.153	0.103
0.3	Mean	0.497	0.438	0.388	0.357	0.339	0.467	0.413	0.373	0.348	0.441	0.394	0.361	0.419	0.379
	Std. err.	0.333	0.207	0.126	0.080	0.049	0.176	0.108	0.065	0.040	0.113	0.068	0.041	0.079	0.048
0.4	Mean	0.506	0.461	0.430	0.416	0.410	0.483	0.445	0.423	0.413	0.465	0.435	0.419	0.452	0.428
	Std. err.	0.224	0.144	0.092	0.063	0.042	0.119	0.075	0.048	0.032	0.076	0.048	0.031	0.053	0.033
0.5	Mean	0.560	0.529	0.511	0.504	0.503	0.544	0.520	0.507	0.504	0.533	0.514	0.506	0.525	0.511
	Std. err.	0.183	0.122	0.083	0.056	0.040	0.097	0.065	0.043	0.030	0.062	0.041	0.028	0.044	0.029
0.6	Mean	0.639	0.618	0.604	0.599	0.601	0.628	0.611	0.602	0.600	0.620	0.607	0.601	0.614	0.605
	Std. err.	0.161	0.109	0.076	0.053	0.038	0.088	0.059	0.041	0.029	0.057	0.038	0.027	0.040	0.027
0.7	Mean	0.728	0.713	0.702	0.699	0.700	0.720	0.707	0.700	0.699	0.714	0.704	0.700	0.710	0.703
	Std. err.	0.148	0.100	0.072	0.050	0.036	0.083	0.056	0.040	0.027	0.054	0.036	0.026	0.038	0.026
0.8	Mean	0.825	0.811	0.801	0.798	0.800	0.818	0.806	0.799	0.799	0.812	0.803	0.800	0.808	0.802
	Std. err.	0.138	0.095	0.068	0.047	0.034	0.079	0.053	0.038	0.027	0.052	0.035	0.025	0.037	0.025
0.95	Mean	0.986	0.967	0.953	0.949	0.950	0.977	0.960	0.951	0.949	0.969	0.956	0.950	0.963	0.954
	Std. err.	0.137	0.087	0.062	0.043	0.031	0.082	0.051	0.036	0.025	0.056	0.035	0.025	0.041	0.026

Table 7.15 Estimated mean and standard deviation of $\hat{\sigma}$ for different values of H under model 4

	$\hat{\sigma}$														
		$\ell = 2$					$\ell = 3$				$\ell = 4$			$\ell = 5$	
H	n_i max	128	256	512	1,024	2,048	256	512	1,024	2,048	512	1,024	2,048	1,024	2,048
0.05	Mean	$1.4^{(24)}$	$3.5^{(29)}$	$4.6^{(26)}$	$1.9^{(24)}$	$1.0^{(43)}$	$6.3^{(17)}$	$1.1^{(16)}$	$3.6^{(15)}$	$6.1^{(24)}$	$1.6^{(13)}$	$2.6^{(13)}$	$1.1^{(13)}$	$7.0^{(10)}$	$1.3^{(11)}$
	Std. err.	$6.6^{(25)}$	$2.3^{(31)}$	$1.5^{(28)}$	$6.5^{(25)}$	$3.9^{(44)}$	$3.2^{(19)}$	$3.1^{(17)}$	$1.5^{(17)}$	$2.5^{(26)}$	$6.1^{(14)}$	$1.4^{(15)}$	$5.4^{(14)}$	$3.1^{(12)}$	$6.0^{(12)}$
	% admis.	50.0	51.8	53.8	54.8	56.3	40.8	43.7	48.2	49.9	39.4	43.6	47.1	38.9	43.5
0.2	Mean	$2.3^{(04)}$	$6.0^{(03)}$	$2.4^{(05)}$	$4.4^{(01)}$	$2.0^{(01)}$	$6.4^{(02)}$	$9.0^{(02)}$	$1.1^{(02)}$	$1.9^{(01)}$	$1.4^{(02)}$	$1.0^{(02)}$	$2.6^{(01)}$	$6.5^{(01)}$	$3.7^{(01)}$
	Std. err.	$4.6^{(05)}$	$3.0^{(05)}$	$1.7^{(07)}$	$1.6^{(02)}$	$9.2^{(01)}$	$1.1^{(04)}$	$6.6^{(04)}$	$4.1^{(03)}$	$2.6^{(01)}$	$1.2^{(03)}$	$4.0^{(03)}$	$1.3^{(02)}$	$2.4^{(02)}$	$4.3^{(02)}$
	% admis.	79.9	88.5	95.4	99.1		94.8	99.2	99.9		99.6				
0.3	Mean	58.595	16.341	6.762	4.306	3.326	15.288	7.301	4.559	3.485	9.132	5.446	3.897	6.879	4.615
	Std. err.	558.96	48.765	7.721	2.998	1.552	33.580	7.697	2.595	1.281	11.175	3.399	1.457	5.376	2.038
	% admis.	93.6	98.5	99.9			99.9								
0.4	Mean	9.152	4.811	3.191	2.624	2.362	4.882	3.314	2.635	2.358	3.800	2.878	2.457	3.301	2.667
	Std. err.	16.421	5.396	2.327	1.372	0.889	4.416	1.919	1.041	0.665	2.229	1.095	0.657	1.412	0.753
	% admis.	98.8	99.9												
0.5	Mean	4.839	3.273	2.577	2.285	2.177	3.238	2.567	2.251	2.132	2.774	2.352	2.162	2.549	2.254
	Std. err.	5.696	2.806	1.596	1.042	0.751	2.088	1.200	0.771	0.537	1.217	0.748	0.505	0.838	0.538
	% admis.	99.8													
0.6	Mean	3.670	2.821	2.364	2.164	2.122	2.764	2.344	2.126	2.062	2.465	2.192	2.070	2.315	2.128
	Std. err.	3.461	2.042	1.302	0.921	0.692	1.518	0.975	0.670	0.503	0.948	0.628	0.455	0.673	0.463
	% admis.														
0.7	Mean	3.208	2.602	2.285	2.122	2.097	2.564	2.253	2.092	2.041	2.337	2.135	2.045	2.223	2.086
	Std. err.	2.701	1.651	1.178	0.822	0.641	1.303	0.858	0.626	0.463	0.834	0.574	0.429	0.603	0.428
	% admis.														
0.8	Mean	2.977	2.518	2.235	2.091	2.083	2.490	2.217	2.071	2.029	2.295	2.113	2.032	2.193	2.070
	Std. err.	2.240	1.483	1.060	0.754	0.595	1.195	0.801	0.591	0.439	0.787	0.546	0.412	0.577	0.411
	% admis.														
0.95	Mean	3.122	2.494	2.205	2.075	2.069	2.615	2.253	2.081	2.030	2.397	2.151	2.044	2.276	2.102
	Std. err.	3.263	1.274	0.903	0.660	0.527	1.529	0.759	0.553	0.408	0.958	0.544	0.397	0.691	0.419
	% admis.														

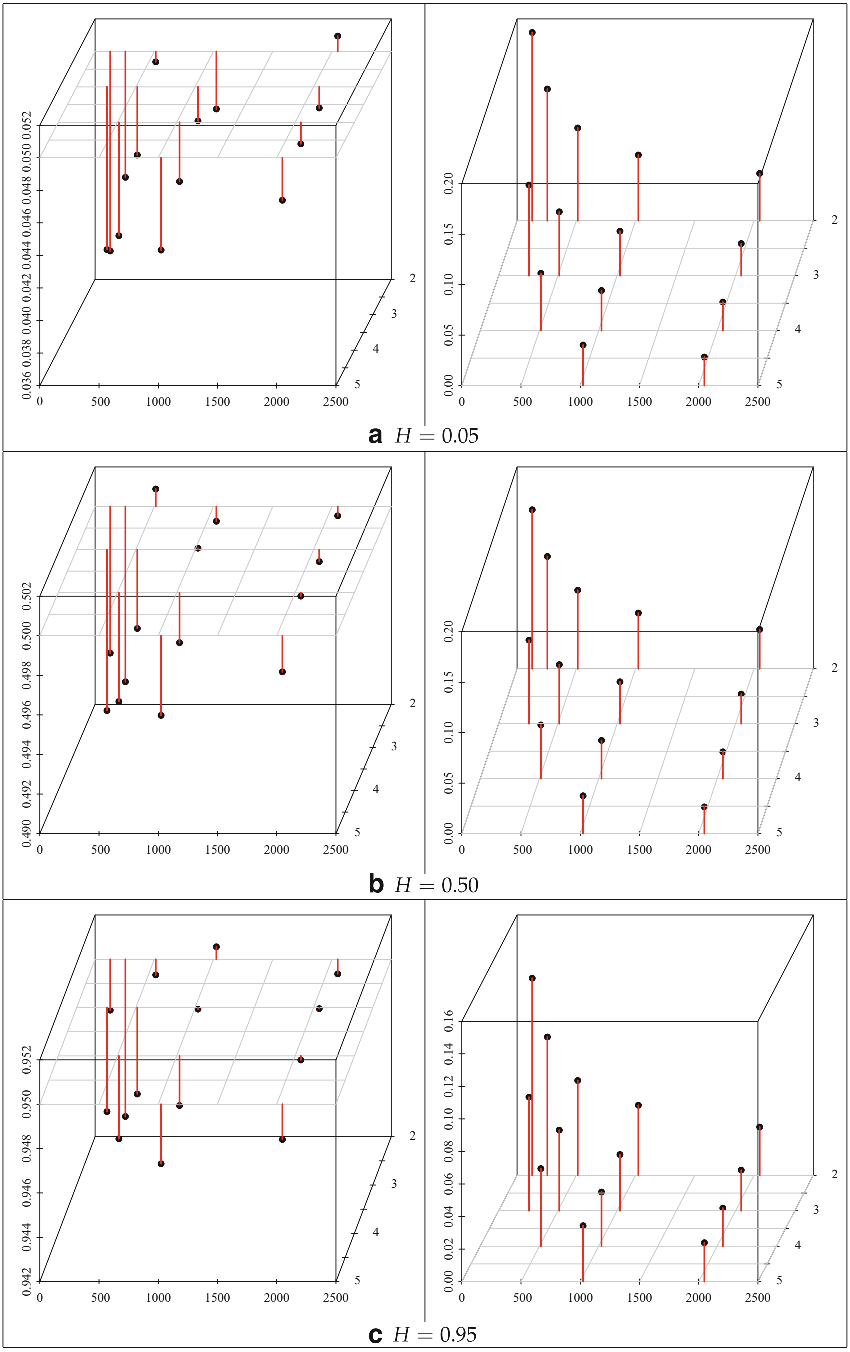

Fig. 7.4 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{H}_2$ for model 1. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

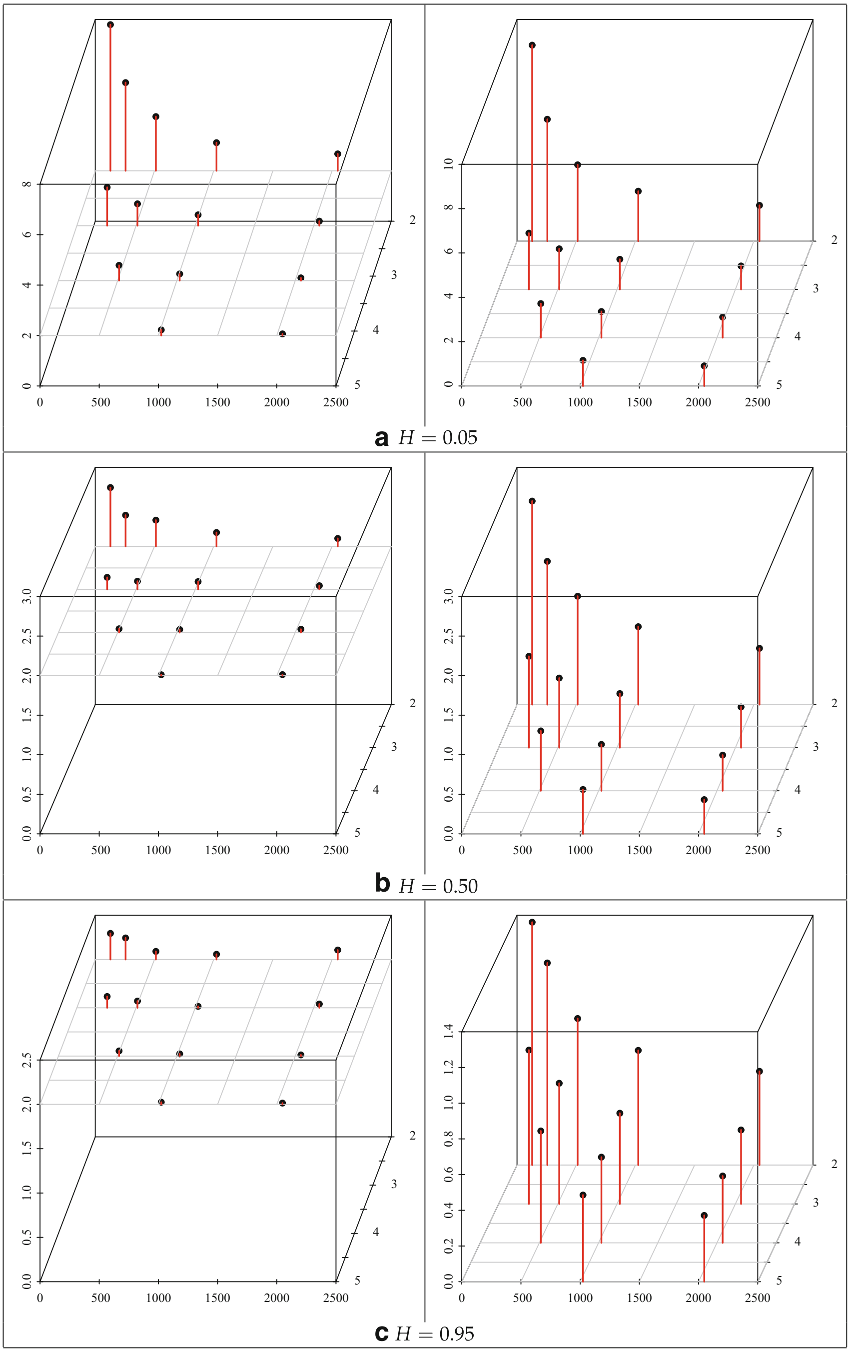

Fig. 7.5 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{\sigma}$ for model 1. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

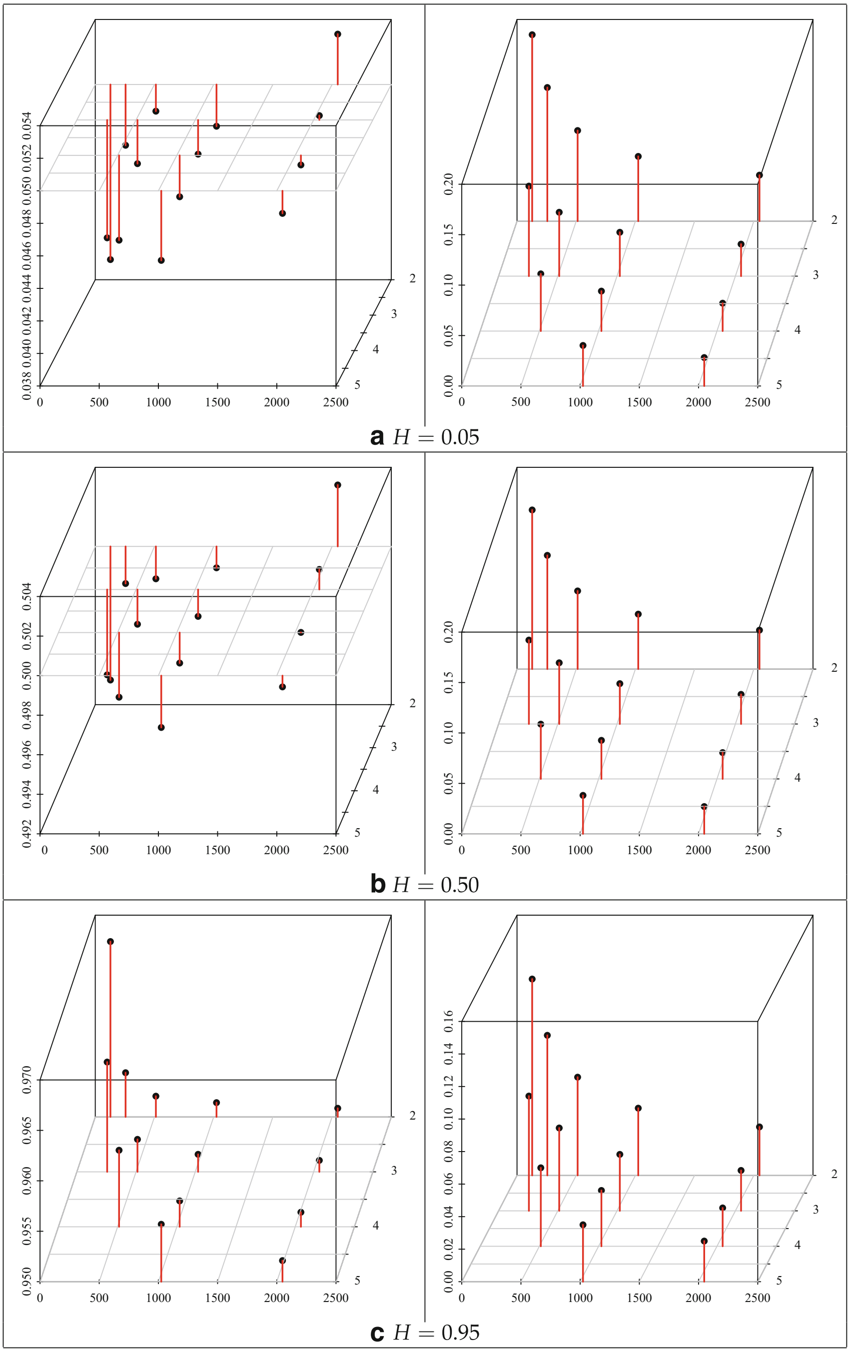

Fig. 7.6 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{H}_2$ for model 2. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

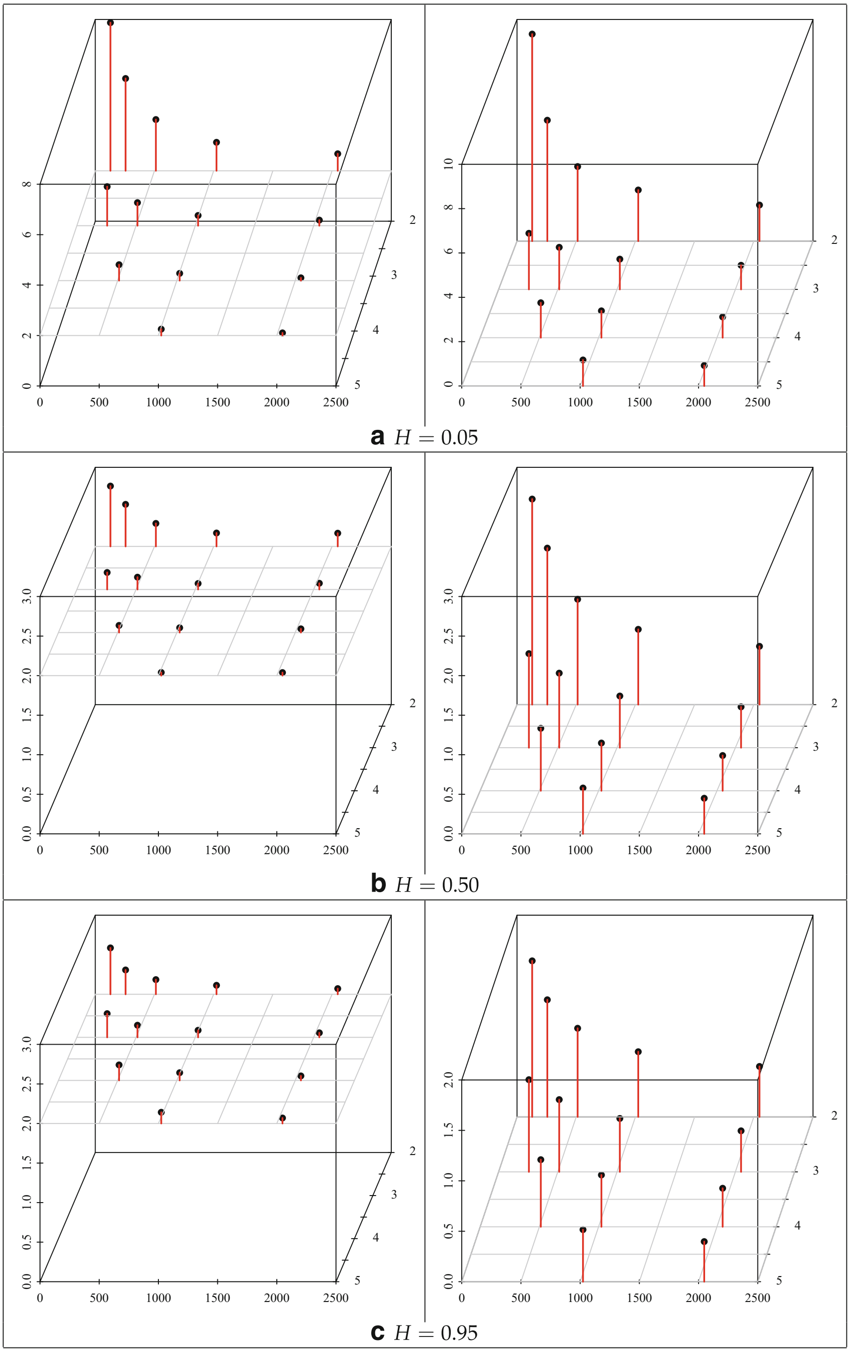

Fig. 7.7 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{\sigma}$ for model 2. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

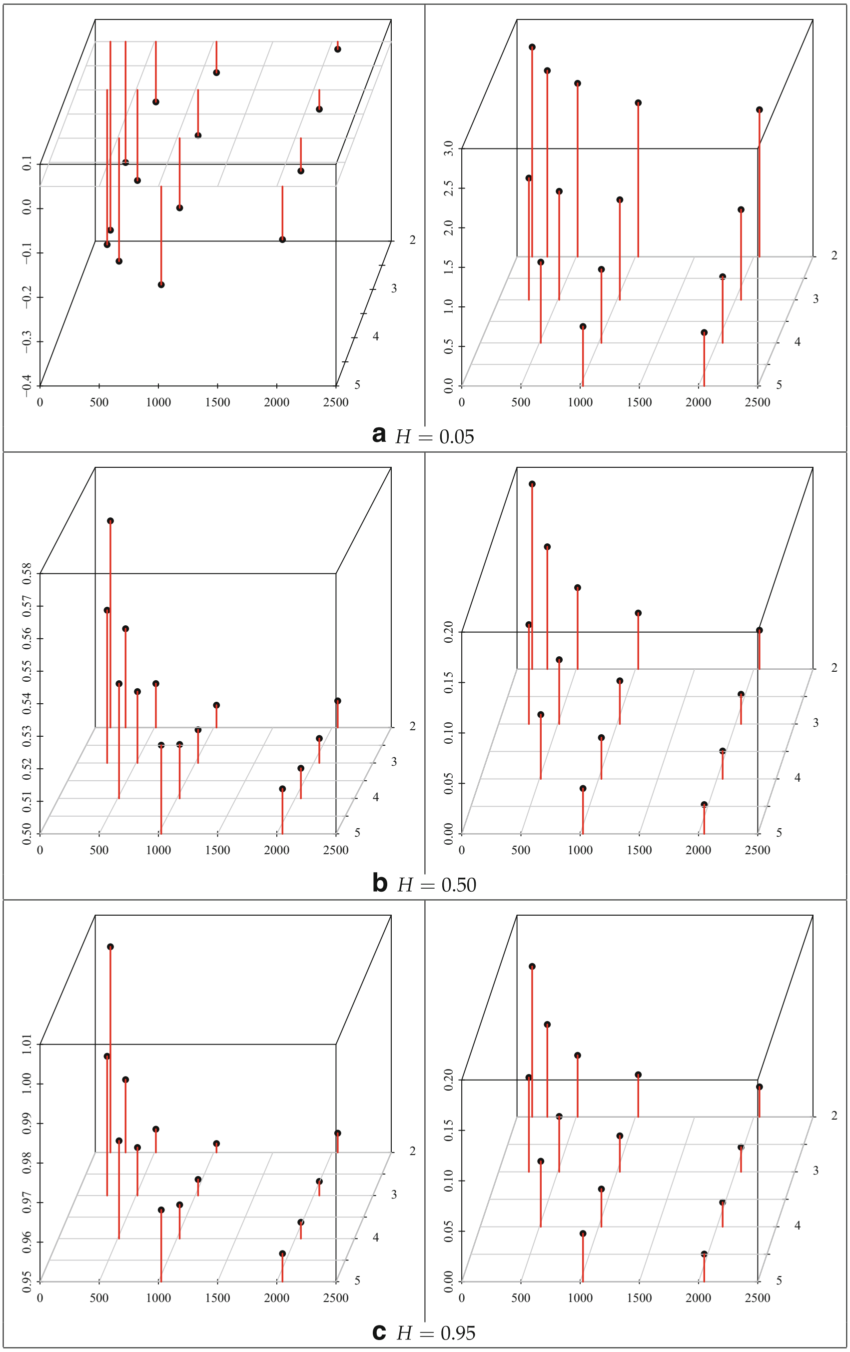

Fig. 7.8 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{H}_2$ for model 3. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

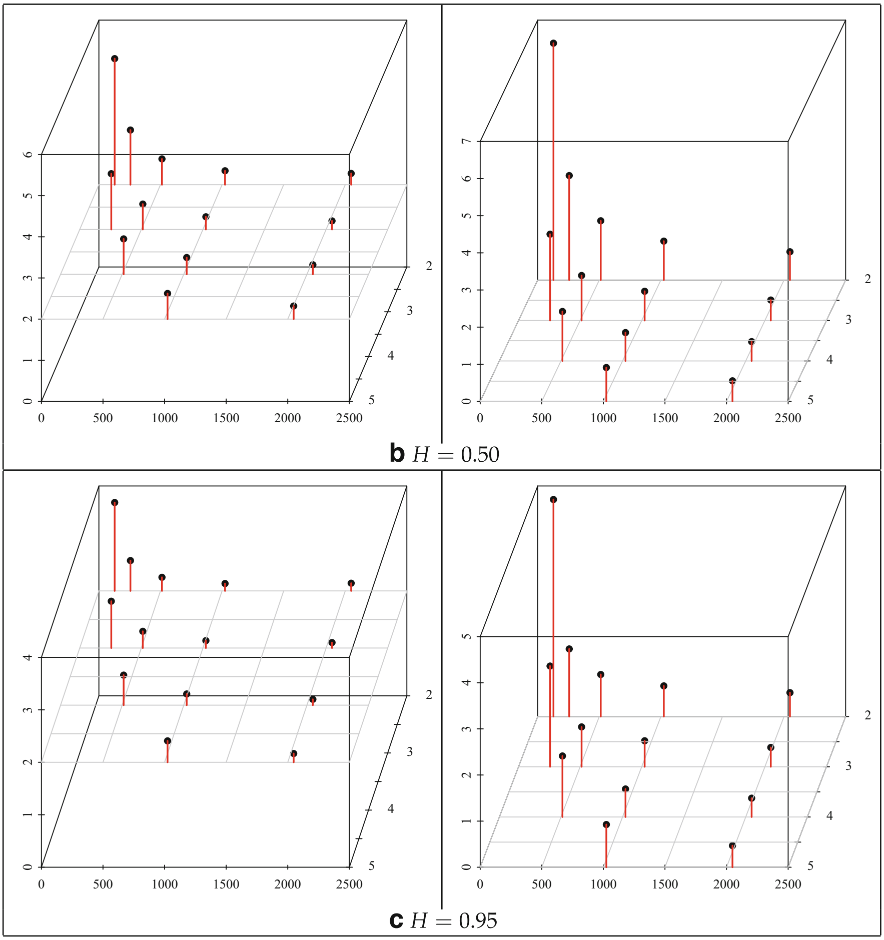

Fig. 7.9 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{\sigma}$ for model 3. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

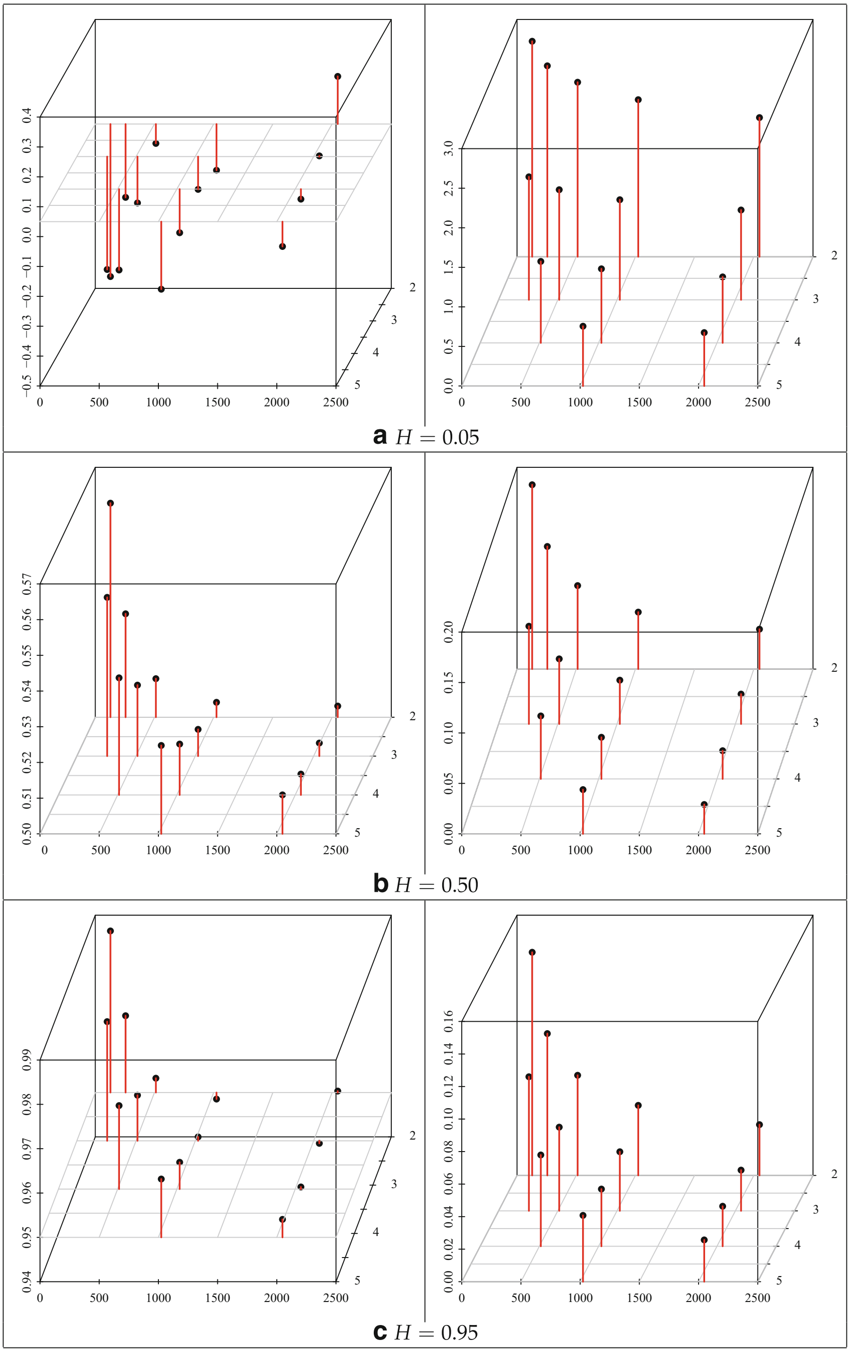

Fig. 7.10 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{H}_2$ for model 4. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

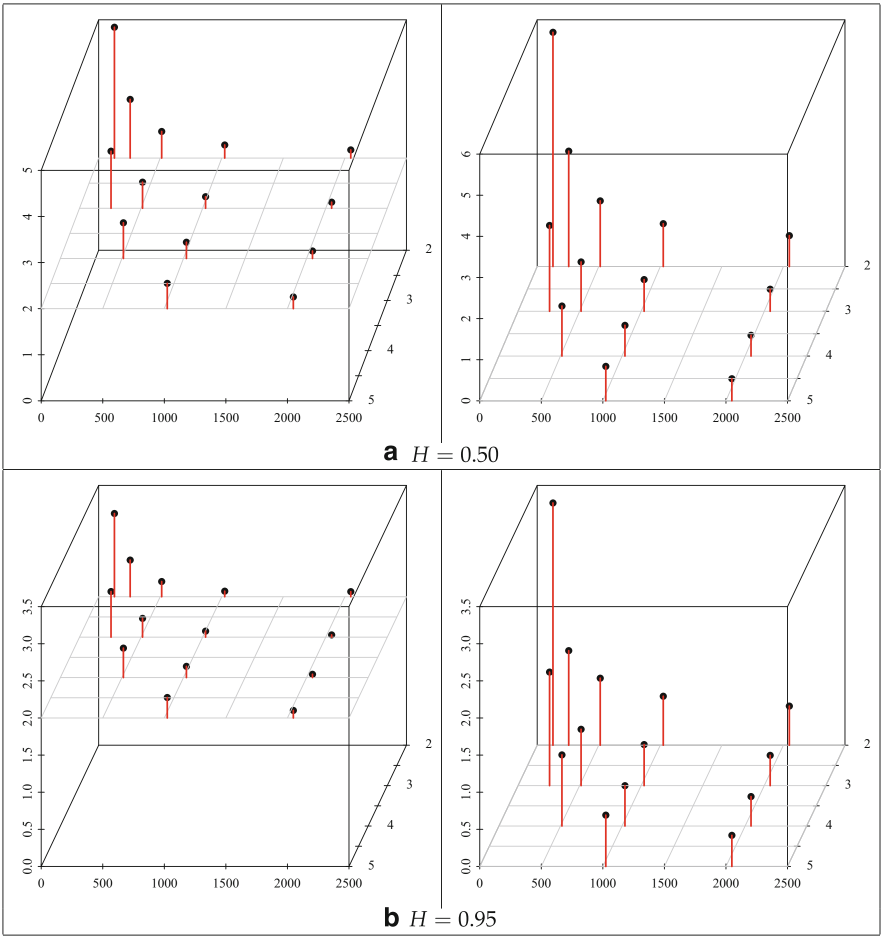

Fig. 7.11 3D-diagrams of the difference between the empirical mean and real value and of the standard error of $\hat{\sigma}$ for model 4. The maximum number of observations of the process used in estimation is 2^{7+j}, $j = 0, \ldots, 4$. The number of points in the regression, ℓ, varies from 2 to 5

Table 7.16 Observed level for a theoretical level of 5 %

Model 1						Model 2				
	n					n				
H	128 (%)	256 (%)	512 (%)	1,024 (%)	2,048 (%)	128 (%)	256 (%)	512 (%)	1,024 (%)	2,048 (%)
0.55	6.0	5.8	5.5	5.6	5.2	6.0	6.1	5.9	5.8	5.2
0.65	5.8	5.2	5.3	5.1	4.9	6.0	5.7	5.7	5.5	5.2
0.75	6.3	5.7	5.9	5.2	5.6	5.9	5.9	5.3	5.3	5.3
0.85	6.0	5.7	5.2	5.0	4.9	6.2	5.7	5.9	5.8	5.6
0.95	5.6	5.9	5.5	5.5	5.3	6.8	6.1	5.9	5.8	5.8
Model 3						Model 4				
	n					n				
H	128 (%)	256 (%)	512 (%)	1,024 (%)	2,048 (%)	128 (%)	256 (%)	512 (%)	1,024 (%)	2,048 (%)
0.55	15.9	10.7	8.9	7.3	6.3	13.9	10.6	8.2	6.8	6.0
0.65	12.1	9.3	7.9	6.2	5.9	10.1	7.7	6.9	6.8	5.7
0.75	11.0	8.4	7.4	6.4	6.1	9.2	7.3	6.1	5.6	5.7
0.85	11.0	8.7	6.7	6.1	5.8	9.0	7.1	6.7	5.5	5.5
0.95	15.4	9.6	7.4	6.6	5.8	11.9	8.9	6.9	5.7	5.5

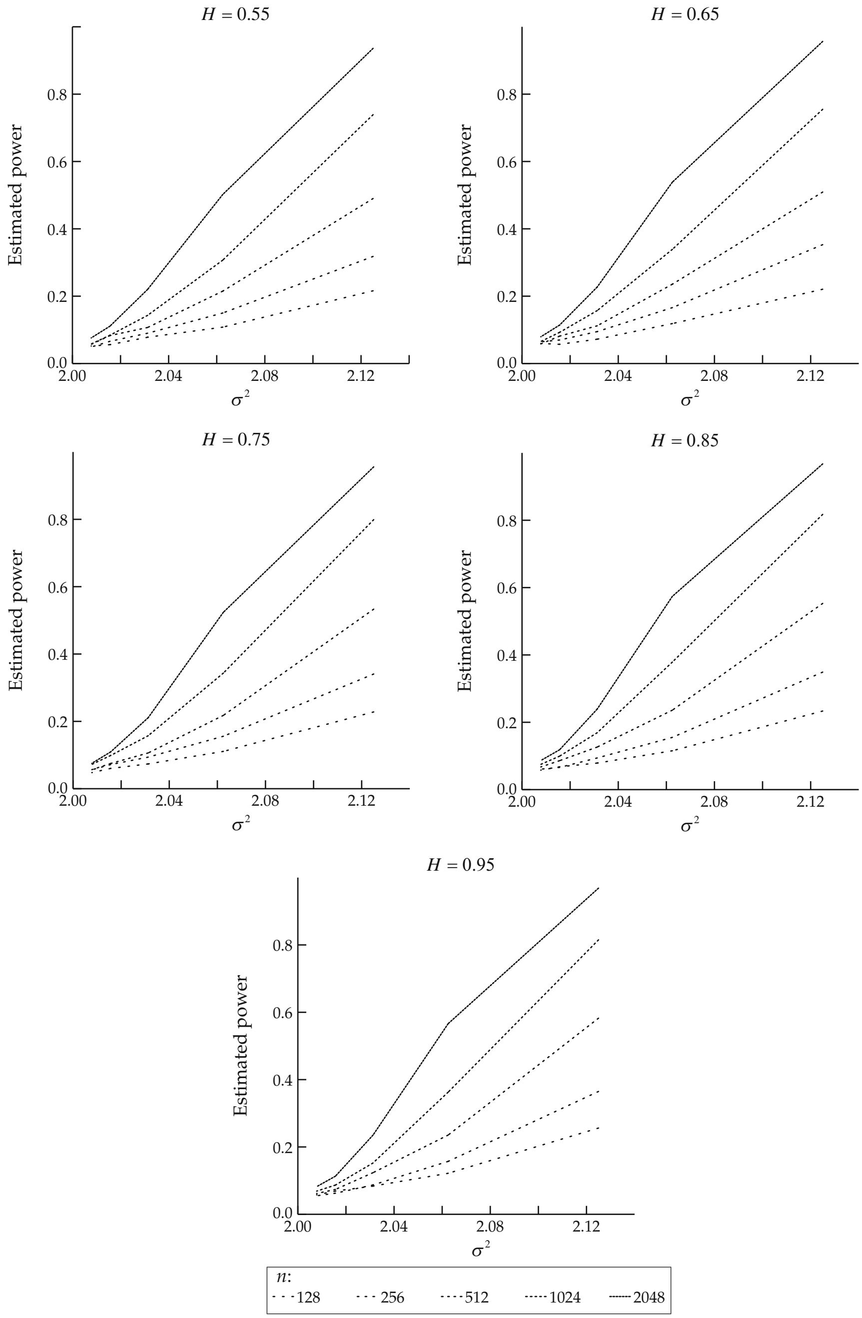

Fig. 7.12 Empirical power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (1)

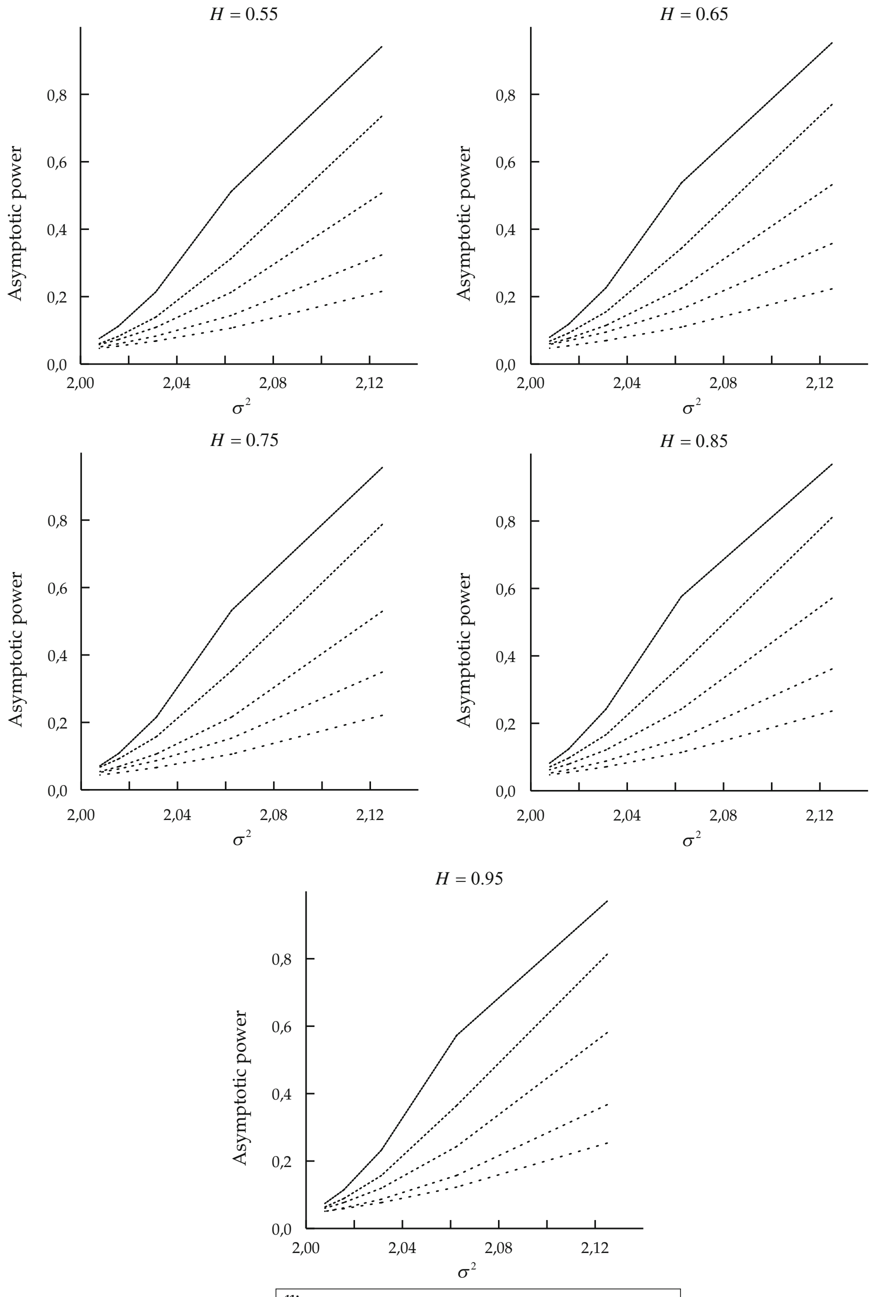

Fig. 7.13 Asymptotic power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (1)

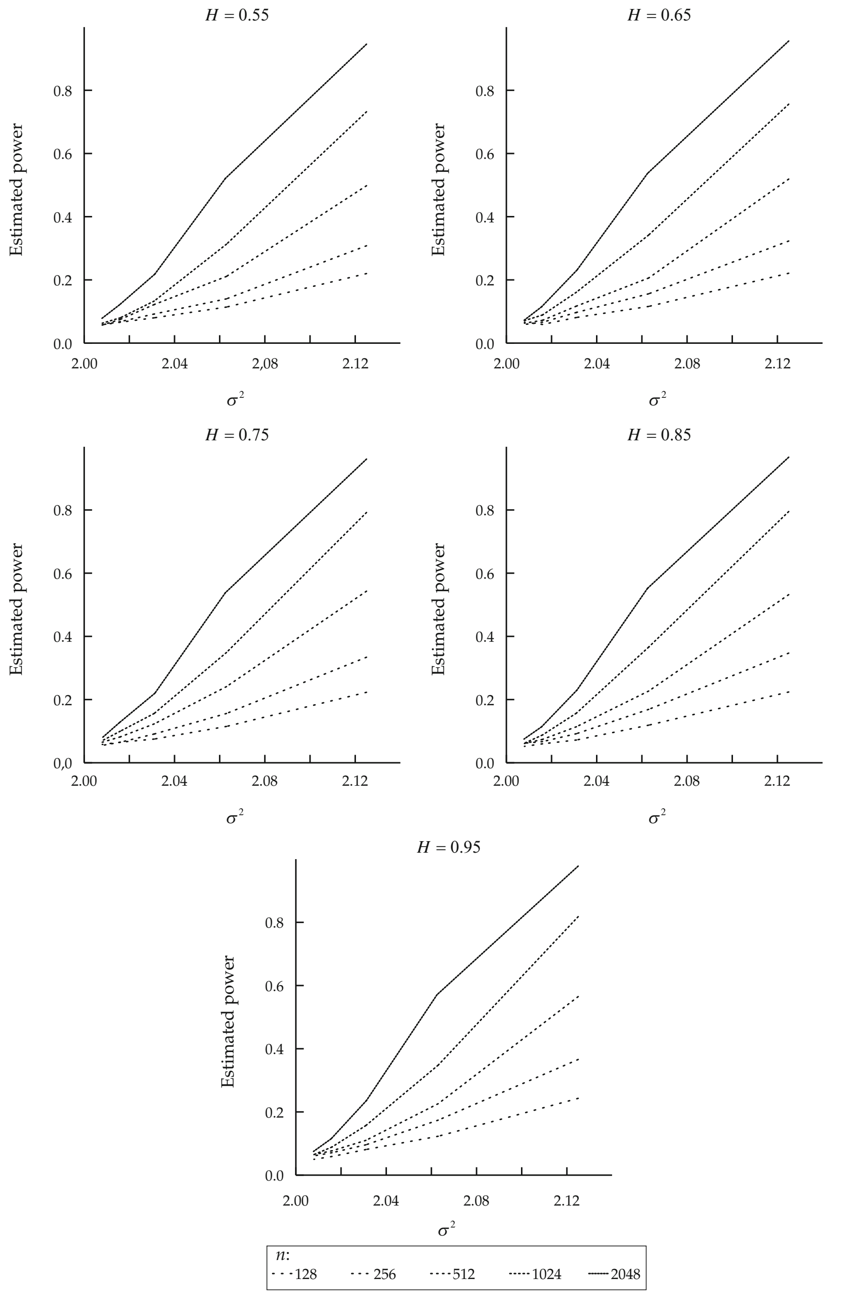

Fig. 7.14 Empirical power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (2)

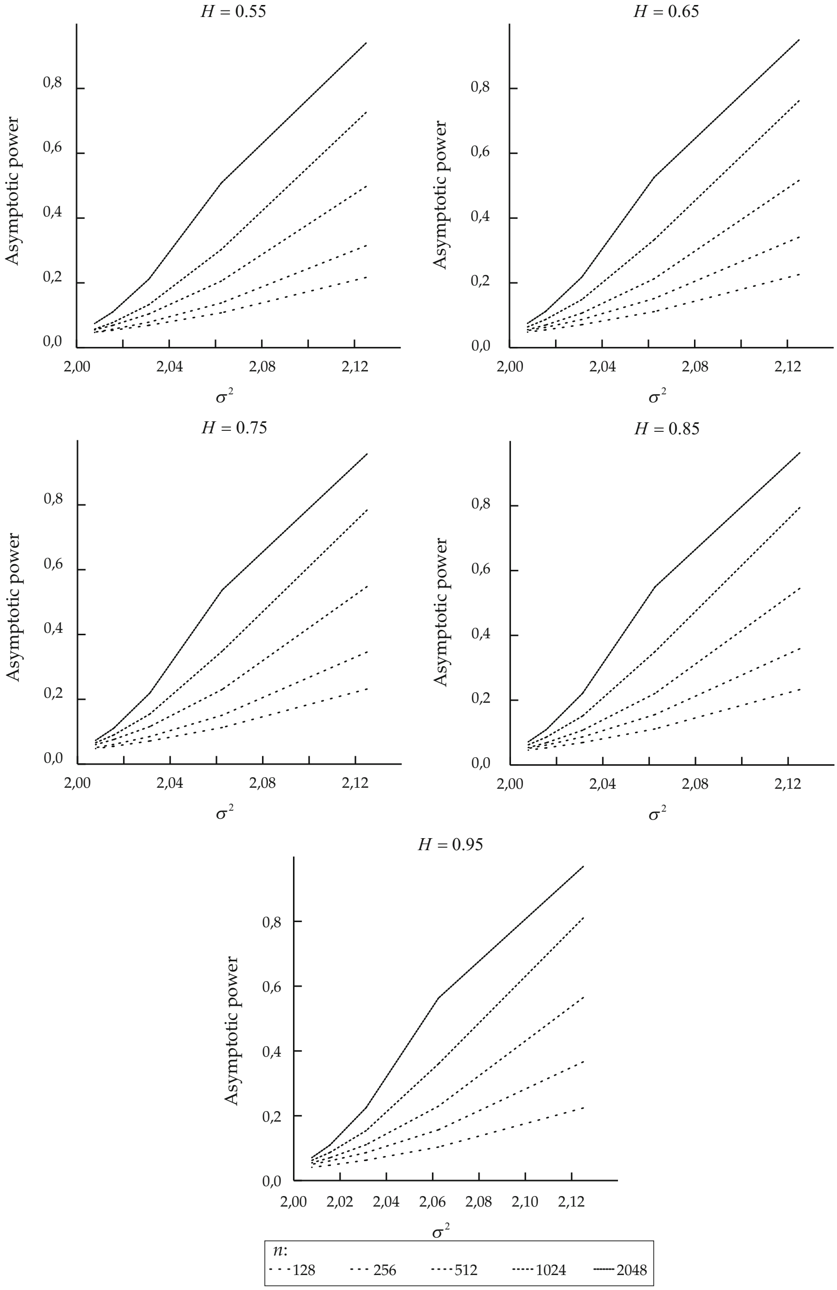

Fig. 7.15 Asymptotic power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (2)

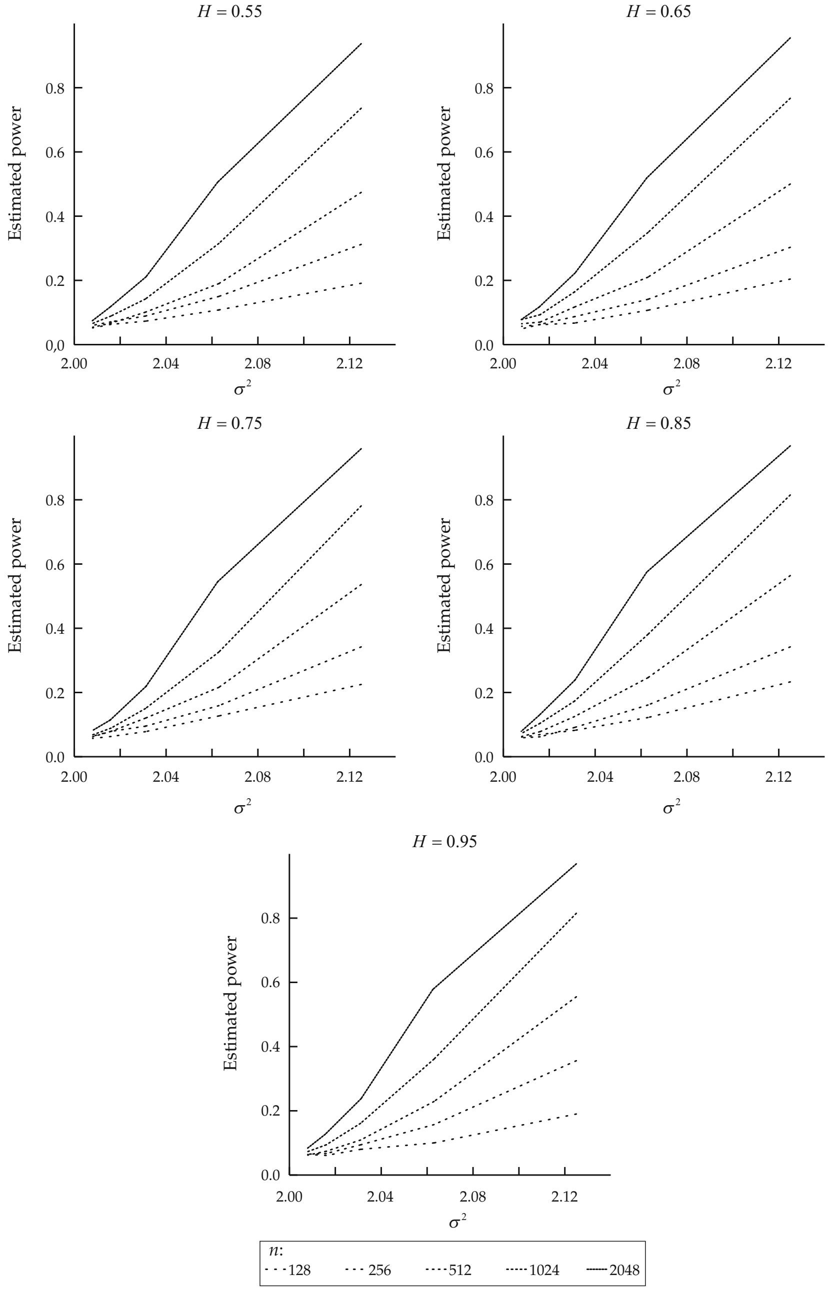

Fig. 7.16 Empirical power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (3)

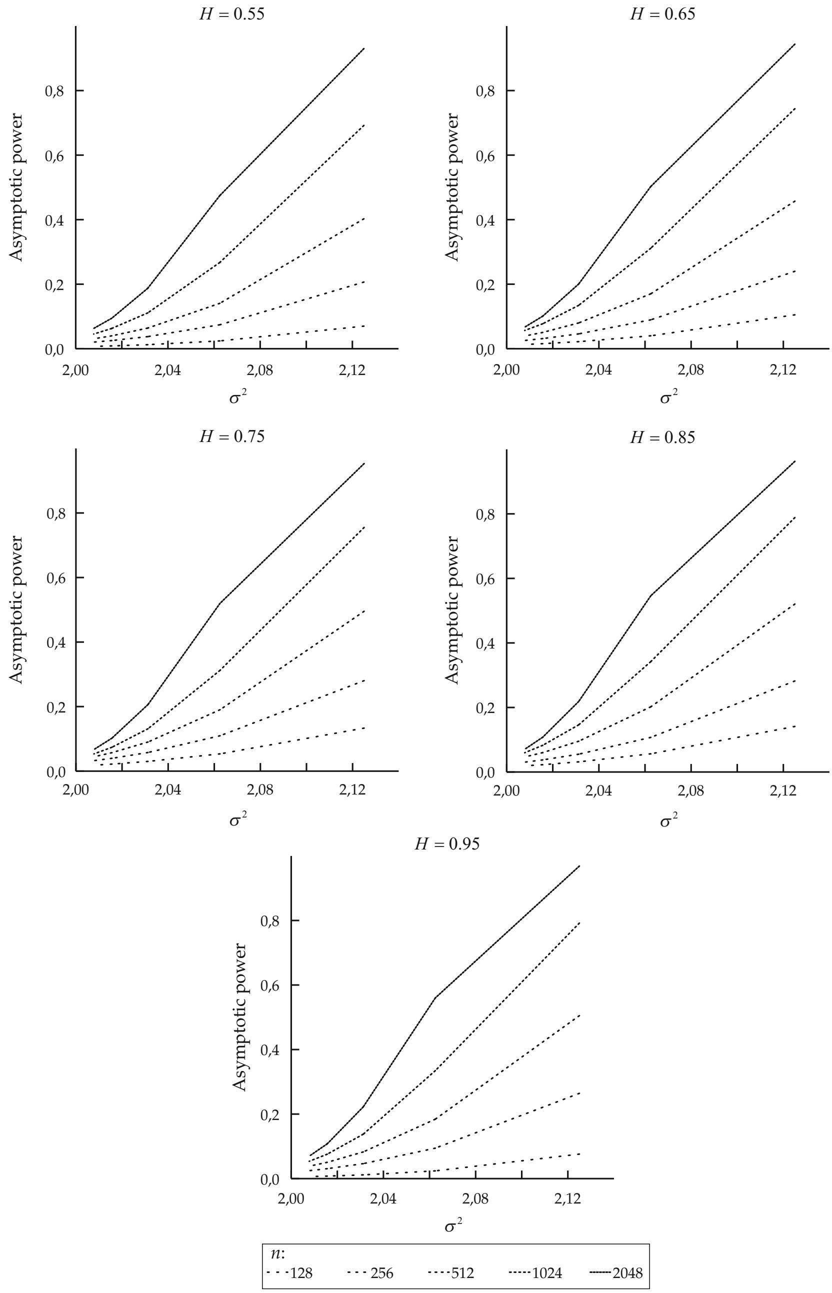

Fig. 7.17 Asymptotic power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (3)

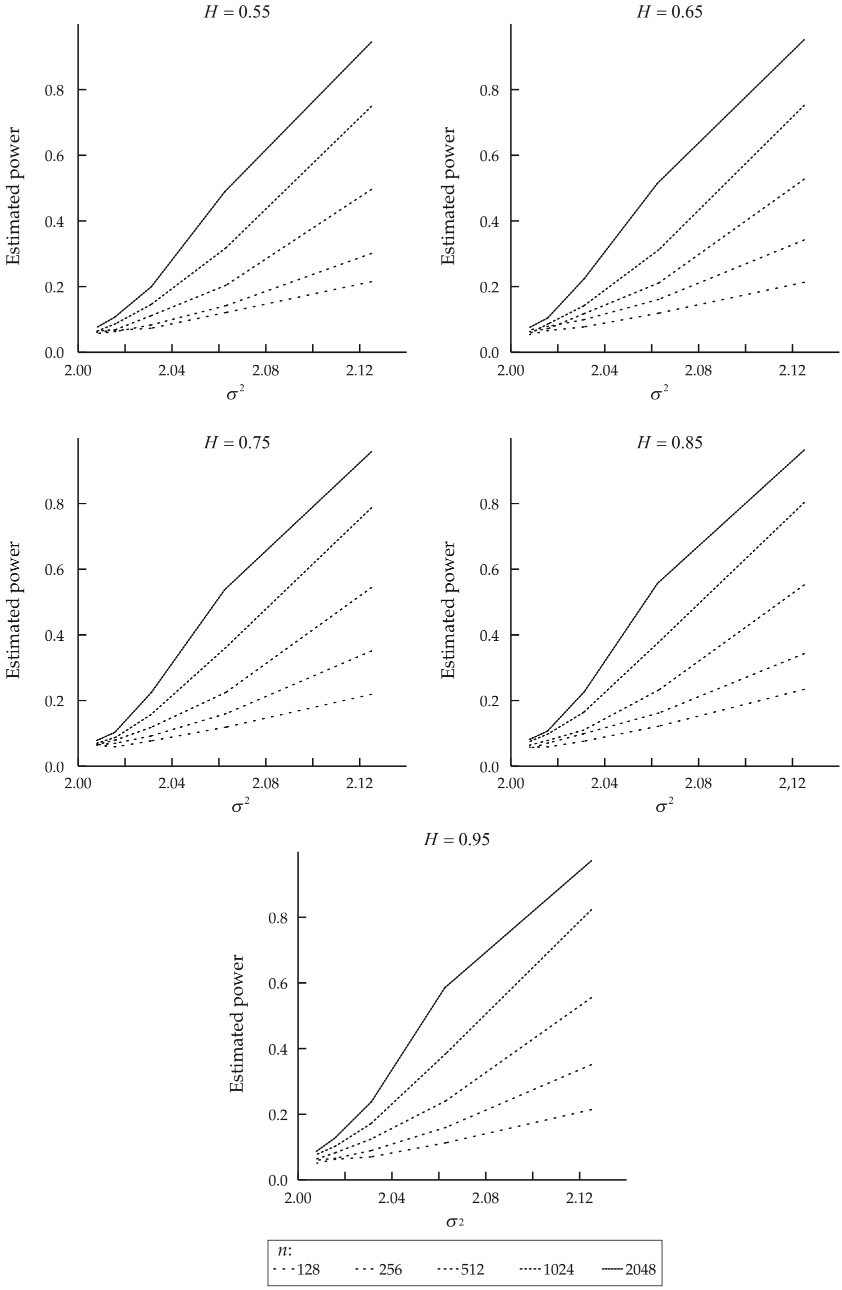

Fig. 7.18 Empirical power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (4)

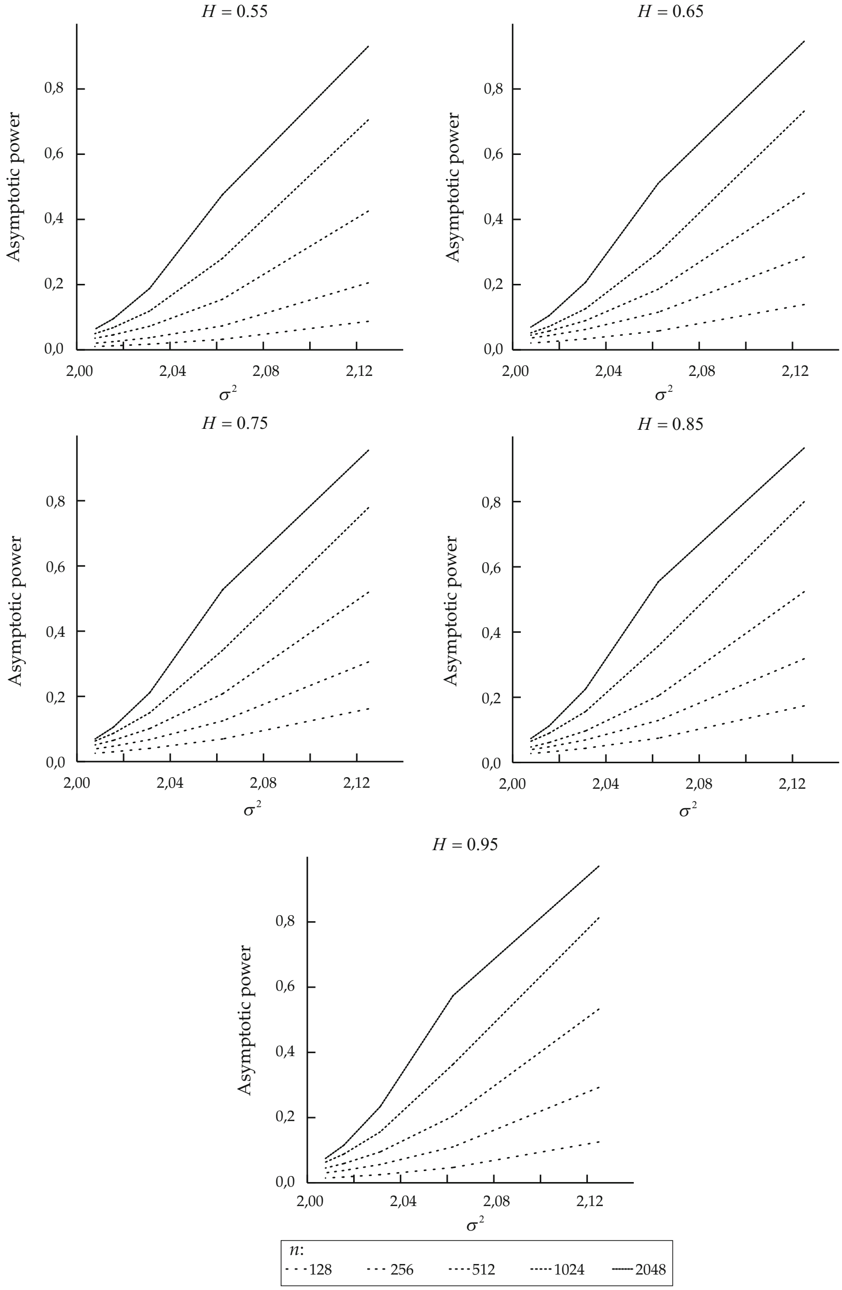

Fig. 7.19 Asymptotic power functions for $H_0 : \sigma = 2$ against $H_1 : \sigma > 2$, data generated according to model (4)

Chapter 8
Some Pascal Procedures and Functions

In this chapter, we give the important Pascal procedures used in the simulation studies: the uniform and the normal generators. These are the basic functions use in the procedure DurbinSim written to simulate a trajectory of a Gaussian stationary process. If the increments are simulated, the function Somme is used to get the trajectory. We also give the procedure Model that control the simulation of the four different models defined by a stochastic differential equation considered in the text.

- **Minimal interface for the procedures and functions**

```
unit SimLib;
interface
   uses Math;
   const
      maxLag = 2051;
      maxnObs = 2051;
   type
      CovSeries = array[0..maxLag] of extended;
      TimeSeries = array[0..maxnObs] of extended;

   {Variables for the random generators:}
   var {GenNorm}
      ChoixDeU : integer;
      U1,U2 : extended;
   {Var for the uniform random deviates:}
   var zRanLong0, zRanLong1, zRanLong2, zRanLong3 : longint;
      xRanLong : array[0..3] of longint;
   function RandomLong: extended;
   function GenNorm: extended;
   procedure Model(var data : TimeSeries; nObs,k:integer; sigma,mu,c:extended);
   procedure Somme(var data:TimeSeries; nObs:integer;consNorm:extended);
```

C. Berzin et al., *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lecture Notes in Statistics 216,
DOI 10.1007/978-3-319-07875-5_8,

```
    procedure DurbinSim(var g : CovSeries; n:integer; var data:↘
    →TimeSeries);

implementation
```

• **Simulation of an uniform random deviate**

```
{=========================================================}
function RandomLong: extended;
{=========================================================}
    begin
        zRanLong0 := (53*xRanLong[0])+11;
        zRanLong1 := (53*xRanLong[1])+(15372*xRanLong[0]);
        zRanLong2 := (53*xRanLong[2])+(15372*xRanLong[1])+(6238*↘
        →xRanLong[0]);
        zRanLong3 := (53*xRanLong[3])+(15372*xRanLong[2])+(6238*↘
        →xRanLong[1])+(32*xRanLong[0]);
        xRanLong[0] := zRanLong0 mod 16384;
        zRanLong1 := zRanLong1+(zRanLong0 div 16384);
        xRanLong[1] := zRanLong1 mod 16384;
        zRanLong2 := zRanLong2+(zRanLong1 div 16384);
        xRanLong[2] := zRanLong2 mod 16384;
        zRanLong3 := zRanLong3+(zRanLong2 div 16384);
        xRanLong[3] := zRanLong3 mod 64;
        randomLong := (xRanLong[3]*0.015625)+(xRanLong↘
        →[2]*0.9536743164e-06)+(xRanLong[1]*0.5820766091e-10)+(↘
        →xRanLong[0]*0.3552713679e-14);
    end;
```

• **Simulation of a Gaussian random deviate**

```
{=========================================================}
    Function GenNorm: extended;
{=========================================================}
        Const
            h = 0.2;
            pTab1 : array [0..31] of extended =
                (0.000000000000000, 0.848737394964225, ↘
                →0.969988979312695, 0.855031042869243, ↘
                →0.994279264213257,
                 0.995158709535307, 0.932743754634730, ↘
                 →0.923403371004114, 0.727370667776133, ↘
                 →1.000000000000000,
                 0.691084371368807, 0.454074788431763, ↘
                 →0.286649987773989, 0.173862006191176, ↘
                 →0.101317780262144,
                 0.056727659672807, 0.067274921216759, ↘
                 →0.160512263075615, 0.235534083919509, ↘
                 →0.285402151320344,
                 0.307583983879102, 0.303895853638795, ↘
                 →0.279521143090448, 0.241484838292606, ↘
                 →0.197052219293015,
```

```
         0.152447607174975,  0.112113015387796,  0.078528353915769,  0.052464259464977,  0.033469912108469,
         0.020407207450255,  0.086393474024337);
     qTab1 : array[1..15] of extended =
        (0.235644147632296,  0.206187909621112,  0.233909635992696,  0.201150730180674,  0.200972968516138,
         0.214421162303383,  0.216590069172608,  0.274962971233747,  0.200000000000000,  0.289400264693972,
         0.440456077050082,  0.697715013187760,  1.150337583129480,  1.973987186479330,  3.525616976860300);
     yTab1 : array[0..31] of extended =
        (0.000000000000000,  -0.922203858334308,  -5.864218524383640,  -0.579605702892703,  -33.160537849598600,
         -39.511299426995400,  -2.573701533795050,  -1.611080965596300,  0.666403220927925,  0.000000000000000,
         0.352574031666183,  -0.166350547221432,  0.919632723666887,  0.357909693660353,  -0.022548077181653,
         0.187972156661961,  0.585574544365309,  0.961759474018363,  -0.061620558605705,  0.120122303237360,
         1.311156305828320,  0.312686670456606,  1.122406843612470,  0.536326958119326,  0.750917799630877,
         0.564026387403180,  0.174746106806203,  0.382955882744852,  -0.011073832304279,  0.393074212064835,
           0.195833532544245,  0.781087378084986);
     zTab1 : array[0..31] of extended =
        (0.200000000000000,  1.322203858334310,  6.664218524383640,  1.379605702892700,  34.960537849598600,
         41.311299426995400,  2.973701533795050,  2.611080965596300,  0.733596779072076,  0.000000000000000,
         0.647425968333817,  0.366350547221432,  0.280367276333114,  0.242090306339648,  0.222548077181654,
         0.212027843338040,  0.214425455634692,  0.238240525981638,  0.261620558605705,  0.279877696762641,
         0.288843694171683,  0.287313329543394,  0.277593156387527,  0.263673041880674,  0.249082200369123,
         0.235973612596820,  0.225253893193797,  0.217044117255148,  0.211073832304279,  0.206925787935166,
```

```
            0.204166467455756,  0.218912621915015);
        sTab1 : array[1..16] of extended =
            (0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0);
        dTab1 : array[16..30] of extended =
            (0.505033500668897,  0.772956831816995,  0.876424317297063,  0.939211242857670,  0.986086815609050,
            0.995154501317651,  0.986748014243518,  0.979211358622571,  0.972273916173362,  0.965752340045163,
            0.959530972921599,  0.953534096080232,  0.947710264937407,  0.942023401989513,  0.936447524949834);
        eTab1 : array[16..30] of extended =
            (25.000000000000000,  12.500000000000000,  8.333333333333330,  6.250000000000000,  5.000000000000000,
            4.063773106920830,  3.367796140933380,  2.858295913510080,  2.469455364849140,  2.163169664002580,
            1.915849911234920,  1.712111865433360,  1.541494082536800,  1.396634659266460,  1.272202427870800);
    var
        j: integer;
        u, v, x, f: extended;
        negatif, rejet: boolean;
    Procedure interchanger (Var u, v: extended);
        Var
            t: extended;
    Begin
        t := u;
        u := v;
        v := t;
    End;
Begin
    {M1}
    u := 2 * RandomLong;
    negatif := u < 1;
    {M2}
    u := u - trunc(u);
    u := 32 * u;
    j := trunc(u);
    f := u - j;
     {Walker's alias method is used}
     If f >= pTab1[j] Then
        Begin
            x := yTab1[j] + f * zTab1[j];
        End
    Else If (j <= 15) Then {An uniform distribution}
        x := sTab1[j] + f * qTab1[j]
    Else If ((16 <= j) And (j <= 30)) Then
```

```
          Begin {A wedge-shaped distribution}
            rejet := true;
            While rejet Do
              Begin
                u := RandomLong;
                v := RandomLong;
                If u > v Then
                  Interchanger(u, v);
                x := sTab1[j - 15] + h * u;
          {(U,V) is in the upper left corner of the unit square}
                rejet := v > dTab1[j];
                If rejet Then
                  rejet := (v > (u + eTab1[j] * (exp((sqr(sTab1[j - 14]) - sqr(x)) / 2) - 1)))
              End;
          End
        Else {Tail distribution is required}
          Begin
            rejet := true;
            While rejet Do
              Begin
                u := RandomLong;
                v := RandomLong;
                x := sqrt(9 - 2 * ln(v));
                rejet := ((u * x) >= 3);
              End
          End;
        {M9}
        If negatif Then
          x := -x;
        GenNorm := x
    End;
```

• **Simulation of a stationary process using the Durbin-Levinson's algorithm**

```
{=====================================================}
procedure DurbinSim(var g : CovSeries; n:integer; var data:TimeSeries);
{=====================================================}
     Var
       v : CovSeries;
       phi : array[1..2] of TimeSeries;
       temp : extended;
       i,j,pred, actu : integer;
Begin
    data[0] := 0;
    v[0] :=g[0];
    data[1] := GenNorm*sqrt(v[0]);
    phi[1,1] := g[1]/g[0];
    v[1] := v[0]*(1-sqr(phi[1,1]));
    temp := phi[1,1]*data[1];
    data[2] := temp+GenNorm*sqrt(v[1]);
    pred :=1;
    actu :=2;
```

```
    for i:=3 to n do
        begin
        {Computation of the ph[i-1,j] coefficients }
            temp:=g[i-1];
            for j:=1 to i-2 do
                    temp := temp-phi[pred,j]*g[i-1-j];
            temp:=temp/v[i-2];
            phi[actu,i-1]:=temp;
            for j:=1 to i-2 do
                    phi[actu,j] := phi[pred,j]-temp*phi[pred,i-1-j];
            v[i-1]:=v[i-2]*(1-sqr(phi[actu,i-1]));
            temp:= 0;
            for j:=1 to i-1 do
                    temp:=temp+phi[actu,i-j]*data[j];
            data[i]:=temp+sqrt(v[i-1])*GenNorm;
            j:=pred;
            pred:=actu;
            actu:=j
        end;
End;
```

• **Integration of the increments of a fBm producing a trajectory**

```
{===========================================================}
procedure Somme(var data:TimeSeries; nObs:integer;consNorm:extended);
{===========================================================}
Var i:integer;
    begin
        for i:= 1 to nObs do
            data[i]:=data[i-1]+data[i];
        for i:= 0 to nObs do
            data[i]:=consNorm*data[i];
    end;
```

• **Simulation of the four models defined by an SDE**

```
{===========================================================}
procedure Model(var data : TimeSeries; nObs, k:integer; sigma, mu, c:extended);
{===========================================================}
    const debug = false;
    var cumul,atom : extended;
        i : integer;
    begin
        cumul :=0;
        case k of
        0 : begin
              end;
        1 : begin
                  for i:= 0 to nObs do
                      begin
                          data[i] := sigma*data[i]+mu*i/nObs+c
```

```
                end;
            end;
      2 : begin
              for i:= 0 to nObs do
                begin
                  atom := data[i]*exp(-mu*i/nObs)/nObs;
                  cumul:=cumul+atom;
                  data[i] := sigma*data[i]+exp(mu*i/nObs)*(sigma*mu*cumul+c)
                end;
            end;
      3 : begin
              for i:= 0 to nObs do
                begin
                  data[i] := exp(sigma*data[i]+mu*i/nObs)*c
                end;
            end;
      4 : begin
              for i:= 0 to nObs do
                begin
                  atom := exp(-sigma*data[i])/nObs;
                  cumul:=cumul+atom;
                  data[i] := exp(sigma*data[i])*(c+mu*cumul)
                end;
            end;
        end;
      if debug then for i:=0 to nObs do
        writeln(data[i]:24:20);
  end;
```

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C. Berzin et al., *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lecture Notes in Statistics 216,
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