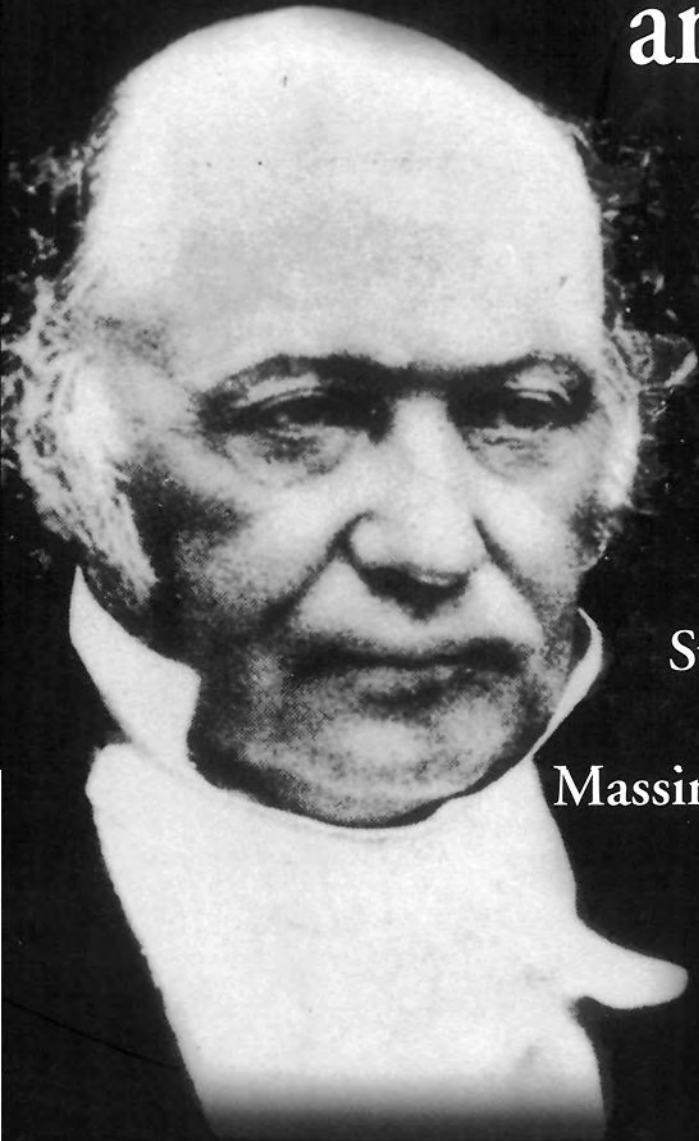


Proceedings of the Second Meeting

# Quaternionic Structures in Mathematics and Physics



Editors

Stefano Marchiafava

Paolo Piccinni

Massimiliano Pontecorvo

$$i^2=j^2=k^2=-1, ij=-ji=k, jk=-kj=i, ki=-ik=j$$

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Rome, Italy

6–10 September 1999

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**Proceedings of the Second Meeting**

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*This volume is dedicated to the memory of  
André Lichnerowicz and Enzo Martinelli*

## FOREWORD

Five years after the meeting "Quaternionic Structures in Mathematics and Physics", which took place at the International School for Advanced Studies (SISSA), Trieste, 5-9 September 1994, we felt it was time to have another meeting on the same subject to bring together scientists from both areas.

The second Meeting on Quaternionic Structures in Mathematics and Physics was held in Rome, 6-10 September 1999.

Like in 1994, also this time the Meeting opened a semester at The Erwin Schrödinger International Institute for Mathematical Physics of Vienna, that in Fall 1999 was dedicated to "Holonomy Groups in Differential Geometry", and many participants of this ESI program were invited speakers at the Quaternionic Meeting.

We thank D.V. Alekseevsky, K. Galicki, P. Gauduchon, S. Salamon, members of the Scientific Committee of this Second Meeting, and all the speakers for their contribution.

We gratefully acknowledge financial support from Progetto Nazionale di Ricerca MURST "Proprieta' Geometriche delle Varieta' reali e complesse" (both with local and national funds), Universita' di Roma "La Sapienza", Universita' di Roma Tre, Comitato Nazionale per la Matematica - C.N.R.. We acknowledge also the hospitality of both Departments of Mathematics of Universities "Roma La Sapienza" and "Roma Tre".

We hearty thank the valuable collaboration of Elena Colazingari in the organization of the Meeting. We are also grateful to Angelo Bardelloni for the technical preparation of the electronic version of these Proceedings, to Tiziana Manfroni and Paolo Marini for setting up the web site and to our colleague Francesco Pappalardi for TeX-nical support.

Finally, it gives us great pleasure to thank all the participants of the meeting for their interest and enthusiasm, and of course all the contributors of the present Proceedings.

Roma, March 2001

Stefano Marchiafava  
Paolo Piccinni  
Massimiliano Pontecorvo

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## INTRODUCTION TO THE CONTRIBUTIONS

During the five years, which passed after the first meeting "Quaternionic Structures in Mathematics and Physics" interest in quaternionic geometry and its applications continued to increase. A progress was done in constructing new classes of manifolds with quaternionic structures (quaternionic Kähler, hyper-Kähler, hyper-complex etc.), studying differential geometry of special classes of such manifolds and their submanifolds, understanding relations between the quaternionic structure and other differential-geometric structures, and also in physical applications of quaternionic geometry. Some generalizations of classical quaternionic-like structures (like HKT -structures and hyper-Kähler manifolds with singularities) naturally appeared and were studied. Some of these results are published in this proceedings.

A new simple and elegant construction of homogeneous quaternionic pseudo-Kähler manifolds is proposed by V. CORTES. It gives a unified description of all known homogeneous quaternionic Kähler manifolds as well as new families of quaternionic pseudo-Kähler manifolds and their natural mirror in the category of supermanifolds.

Generalizing the Hitchin classification of  $Sp(1)$ -invariant hyper-Kähler and quaternionic Kähler 4-manifolds, T. NITTA and T. TANIGUCHI obtain a classification of  $Sp(1)^n$ -invariant quaternionic Kähler metrics on  $4n$ -manifold. All of these metrics are hyper-Kähler.

I.G. DOTTI presents a general method to construct quaternionic Kähler compact flat manifolds using Bieberbach theory of torsion free crystallographic groups.

Using the representation theory, U. SEMMELMANN and G. WEINGART find some Weitzböck type formulas for the Laplacian and Dirac operators on a compact quaternionic Kähler manifold and use them for eigenvalue estimates of these operators. As an application, they prove some vanishing theorems, for example, they prove that odd Betti numbers of a compact quaternionic Kähler manifold with negative scalar curvature vanish.

A hyper-Kähler structure on a manifold  $M$  defines a family  $(J_t, \omega_t)$  of complex symplectic structures, parametrized by  $t \in \mathbb{C}P^1$ . R. BIELAWSKI gives a generalization of the hyper-Kähler quotient construction to the case when a holomorphic family  $G_t$ ,  $t \in \mathbb{C}P^1$  of complex Lie group is given, such that  $G_t$  acts on  $M$  as a group of automorphisms of  $(J_t, \omega_t)$ .

The existence of a canonical hyper-Kähler metric on the cotangent bundle  $T^*M$  of a Kähler manifold  $M$  was proved independently by D.Kaledin and B.Feix. In the present paper D. KALEDIN presents his proof in simplified form and obtains an explicit formula for the case when  $M$  is a Hermitian symmetric space.

A toric hyper-Kähler manifold is defined as the hyper-Kähler quotient of the quaternionic vector space  $\mathbb{H}^n$  by a subtorus of the symplectic group  $Sp(n)$ . H. KONNO determines the ring structure of the integral equivariant cohomology of a toric hyper-Kähler manifold.

M. VERBITSKY gives a survey of some recent works about singularities in hyper-Kähler geometry and their resolution. It is shown that singularities in a singular hyper-Kähler variety (in the sense of Deligne and Simpson) have a simple



structure and admit canonical desingularization to a smooth hyper-Kähler manifold. Some results can be extended to the case of the hypercomplex geometry.

Hypercomplex manifolds (which is the same as  $4n$ -manifolds with a torsion free connection with holonomy in  $GL(n, \mathbb{H})$ ) are studied by H. PEDERSEN. He describes three constructions of such manifolds: 1) via Abelian monopoles and geodesic congruences on Einstein-Weyl 3-manifolds, 2) as a deformation of Joyce homogeneous hypercomplex structures on  $G \times T^k$  where  $G$  is a compact Lie group and 3) as a deformation of the hypercomplex manifold  $\mathcal{V}_P(M)$ , associated with a quaternionic Kähler manifold  $M$  and an instanton bundle  $P \rightarrow M$  by the construction of Swann and Joyce.

M.L. BARBERIS describes a construction of left-invariant hypercomplex structures on some class of solvable Lie groups. It gives all left-invariant hypercomplex structures on 4-dimensional Lie groups. Properties of associated hyper-Hermitian metrics on 4-dimensional Lie groups are discussed.

D. JOYCE proposes an original theory of quaternionic algebra, having in mind to create algebraic tools for developing quaternionic algebraic geometry. Applications for constructing hypercomplex manifolds and study their singularities are considered.

Properties of hyperholomorphic functions in  $\mathbb{R}^4$  are studied by S-L. ERIKSSON-BIQUE. Hyperholomorphic functions are defined as solutions of some generalized Cauchy-Riemann equation, which is defined in terms of the Clifford algebra  $Cl(\mathbb{R}^{0,3}) \approx \mathbb{H} \oplus \mathbb{H}$ .

Other more general notion of hyper-holomorphic function on a hypercomplex manifold  $M$  is proposed and discussed in the paper by ST. DIMIEV, R. LAZOV and N. MILEV.

O.BIQUARD defines and studies quaternionic contact structures on a manifold. Roughly speaking, it is a quaternionic analogous of integrable CR structures.

Generalizing the ideas of A. Gray about weak holonomy groups, A.SWANN looks for  $G$ -structures which admit a connection with "small" torsion, such that the curvature of these connections satisfies automatically some interesting conditions, for example, the Einstein equation.

G. GRANTCHAROV and L. ORNEA propose a procedure of reduction which associates to a Sasakian manifold  $S$  with a group of symmetries a new Sasakian manifold and relate it to the Kähler reduction of the associated Kähler cone  $K(S)$ .

The geometry of circles in quaternionic and complex projective spaces are studied by S. MAEDA and T. ADACHI. The main problem is to find the full system of invariants of a circle  $C$ , which determines  $C$  up to an isometry, and to determine when a circle is closed.

Special 4-planar mappings between almost Hermitian quaternionic spaces are defined and studied by J. MIKEŠ, J. BĚLOHL'AVKOV'A and O. POKORN'A.

Some generalization of the flat Penrose twistor space  $\mathbb{C}^{2,2}$  is constructed and discussed by J. LAWRYNOWICZ and O. SUZUKI.

M. PUTA considers some geometrical aspects of the left-invariant control problem on the Lie group  $Sp(1)$ .

Quaternionic representations of finite groups are studied by G. SCOLARICI and L. SOLOMBRINO.

Quaternionic and hyper-Kähler manifolds naturally appear in the different physical models and physical ideas produce new results in quaternionic geometry. For

example, Rozanski and Witten introduce a new invariant of hyper-Kähler manifold as the weights in a Feynman diagram expansion of the partition function of a 3-dimensional physical theory. A variation of this construction, proposed by N. Hitchin and J. Sawon, gives a new invariant of links and a new relations between invariants of a hyper-Kähler manifold  $X$ , in particular, a formula for the norm of the curvature of  $X$  in terms of some characteristic numbers and the volume of  $X$ . These results are presented in the paper by J. SAWON.

The classical Atiyah-Hitchin-Drinfeld-Manin's monad construction of anti-self-dual connections over  $S^4 = \mathbb{H}P^1$  was generalized by M. Capria and S. Salamon to any quaternionic projective space  $\mathbb{H}P^n$ . Using representation theory of compact Lie groups, Y. NAGATOMO extends the monad construction to any Wolf space (i.e. compact quaternionic symmetric space).

A quaternionic description of the classical Maxwell electrodynamics is proposed in the paper by D. SWEETSER and G. SANDRI.

A. PRASTARO applies his theory of non commutative quantum manifolds to the category of quantum quaternionic manifolds and discusses the theorem of existence of local and global solutions of some partial differential equations.

A hyper-Kähler structure on a  $4n$ -manifold  $M$  is defined by a torsion free connection  $\nabla$  with holonomy group in  $Sp(n)$ . A natural generalization is a connection  $\nabla$  with a torsion  $T = (T_{ki}^j)$  and the holonomy in  $Sp(n)$ . If the covariant torsion  $\tau = g(T) = (g_{ij}T_{kl}^j)$ , where  $g$  is the  $\nabla$ -parallel metric on  $M$ , is skew-symmetric, then the manifold  $(M, \nabla, g)$  is called hyper-Kähler manifold with a torsion, or shortly HKT-manifold. If, moreover, the 3-form  $\tau$  is closed, it is called strongly HKT-manifold. Such manifolds appear in some recent versions of supergravity.

A purely mathematical survey of the theory of HKT-manifolds is given by Y. S. POON.

Some applications of HKT-geometry to physics is discussed in the paper by G. PAPADOPOULUS. He describes a class of brain configurations, which are approximations of solutions of 10 and 11 dimensional supergravitation.

A. VAN PROEYEN gives a review of special Kähler geometry (which can be mathematically defined as the geometry of Kähler manifolds together with a compatible, in some rigorous sense, flat connection), its physical meaning and connections to quaternionic and Sasakian geometry.

**Dmitri V. Alekseevsky**

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# Second Meeting on Quaternionic Structures in Mathematics and Physics

Roma, Italy  
September 6-10, 1999

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**Second Meeting on  
 Quaternionic Structures  
 in Mathematics and Physics**  
 Roma, 6-10 September 1999

**HYPERCOMPLEX STRUCTURES ON SPECIAL CLASSES OF  
 NILPOTENT AND SOLVABLE LIE GROUPS**

MARÍA LAURA BARBERIS

1. INTRODUCTION

A hypercomplex structure on a manifold  $M$  is a family  $\{J_\alpha\}_{\alpha=1,2,3}$  of complex structures on  $M$  satisfying the following relations:

$$(1.1) \quad J_\alpha^2 = -I, \quad \alpha = 1, 2, 3, \quad J_3 = J_1 J_2 = -J_2 J_1$$

where  $I$  is the identity on the tangent space  $T_p M$  of  $M$  at  $p$  for all  $p$  in  $M$ . A riemannian metric  $g$  on a hypercomplex manifold  $(M, \{J_\alpha\}_{\alpha=1,2,3})$  is called hyper-Hermitian when  $g(J_\alpha X, J_\alpha Y) = g(X, Y)$  for all vectors fields  $X, Y$  on  $M$ ,  $\alpha = 1, 2, 3$ .

Given a manifold  $M$  with a hypercomplex structure  $\{J_\alpha\}_{\alpha=1,2,3}$  and a hyper-Hermitian metric  $g$  consider the 2-forms  $\omega_\alpha$ ,  $\alpha = 1, 2, 3$ , defined by

$$\omega_\alpha(X, Y) = g(X, J_\alpha Y).$$

The metric  $g$  is said to be hyper-Kähler when  $d\omega_\alpha = 0$  for  $\alpha = 1, 2, 3$ .

It is well known (cf. [5]) that a hyper-Hermitian metric  $g$  is conformal to a hyper-Kähler metric  $\tilde{g}$  if and only if there exists an exact 1-form  $\theta \in \Lambda^1 M$  such that

$$(1.2) \quad d\omega_\alpha = \theta \wedge \omega_\alpha, \quad \alpha = 1, 2, 3$$

where, if  $g = e^f \tilde{g}$  for some  $f \in C^\infty(M)$ , then  $\theta = df$ .

A hypercomplex structure on a real Lie group  $G$  is said to be invariant if left translations by elements of  $G$  are holomorphic with respect to  $J_1, J_2$  and  $J_3$ .

Given  $\mathfrak{g}$  a real Lie algebra, a hypercomplex structure on  $\mathfrak{g}$  is a family  $\{J_\alpha\}_{\alpha=1,2,3}$  of endomorphisms of  $\mathfrak{g}$  satisfying the relations (1.1) and the following conditions:

$$(1.3) \quad N_\alpha = 0, \quad \alpha = 1, 2, 3$$

where  $I$  is the identity on  $\mathfrak{g}$  and  $N_\alpha$  is the Nijenhuis tensor corresponding to  $J_\alpha$ :

$$(1.4) \quad N_\alpha(x, y) = [J_\alpha x, J_\alpha y] - J_\alpha([x, J_\alpha y] + [J_\alpha x, y]) - [x, y]$$

for all  $x, y \in \mathfrak{g}$ . Clearly, if  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , a hypercomplex structure on  $\mathfrak{g}$  induces by left translations an invariant hypercomplex structure on  $G$ .

Two hypercomplex structures  $\{J_\alpha\}_{\alpha=1,2,3}$  and  $\{J'_\alpha\}_{\alpha=1,2,3}$  on  $\mathfrak{g}$  are said to be equivalent if there exists an automorphism  $\phi$  of  $\mathfrak{g}$  such that  $\phi J_\alpha = J'_\alpha \phi$  for  $\alpha = 1, 2, 3$ . The classification of the four-dimensional real Lie algebras carrying hypercomplex structures was done in [2], where the equivalence classes of hypercomplex structures were determined and the corresponding left invariant hyper-Hermitian metrics were studied. It turns out that all such metrics are conformal to hyper-Kähler metrics (cf. [4]).

In the present work we study some remarkable properties of a special hyper-Hermitian metric which corresponds to a four-dimensional solvable Lie group. We also sketch a procedure for constructing hypercomplex structures on certain nilpotent and solvable Lie groups, following the lines of [3].

**Acknowledgement** The author would like to thank the organizers of the meeting *Quaternionic Structures in Mathematics and Physics* for their kind invitation to take part in this event.

## 2. A SPECIAL HYPER-HERMITIAN METRIC

Consider the four-dimensional real Lie algebra  $\mathfrak{s} = \text{span} \{e_j\}_{j=1,\dots,4}$  with the following Lie bracket:

$$[e_3, e_4] = \frac{1}{2} e_2, \quad [e_1, e_2] = e_2, \quad [e_1, e_j] = \frac{1}{2} e_j, \quad j = 3, 4$$

and let  $g$  be the inner product with respect to which  $\{e_j\}_{j=1,\dots,4}$  is an orthonormal basis. It follows from [2] that  $g$  is hyper-Hermitian. Let  $\{e^j\}_{j=1,\dots,4} \subset \mathfrak{s}^*$  be the dual basis of  $\{e_j\}_{j=1,\dots,4}$  and write  $e^{ij\dots}$  to denote  $e^i \wedge e^j \wedge \dots$ . In this case we have the following equations for  $\omega_\alpha$ :

$$\omega_1 = -e^{12} + e^{34}, \quad \omega_2 = -e^{13} - e^{24}, \quad \omega_3 = e^{14} - e^{23}.$$

To calculate  $d\omega_\alpha$ ,  $\alpha = 1, 2, 3$ , we compute first:

$$de^1 = 0, \quad de^2 = -e^{12} - \frac{1}{2} e^{34}, \quad de^j = -\frac{1}{2} e^{1j}, \quad j = 3, 4$$

to obtain:

$$d\omega_1 = -\frac{3}{2} e^{134}, \quad d\omega_2 = \frac{3}{2} e^{124}, \quad d\omega_3 = \frac{3}{2} e^{123}$$

so that (1.2) is satisfied for  $\theta = -\frac{3}{2} e^1$ . We can therefore conclude that the left invariant hyper-Hermitian metric induced by  $g$  on the corresponding simply connected solvable Lie group  $S$  is conformally hyper-Kähler. We recall from [2] that  $g$  is neither symmetric nor conformally flat. The Levi-Civita connection  $\nabla^g$  is given as follows:

$$\nabla_{e_1}^g \equiv 0,$$

$$\nabla_{e_2}^g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ & 0 & \frac{1}{4} \\ & -\frac{1}{4} & 0 \end{pmatrix}, \quad \nabla_{e_3}^g = \begin{pmatrix} & & 1 & 0 \\ & & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & & \\ 0 & -\frac{1}{4} & & \end{pmatrix},$$

$$\nabla_{e_4}^g = \begin{pmatrix} & & 0 & \frac{1}{2} \\ & & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & & \\ -\frac{1}{2} & 0 & & \end{pmatrix}.$$

Using these formulas we calculate the curvature tensor  $R^g$ :  $R^g(e_1, v) = -\nabla_{[e_1, v]}^g$  for all  $v$  in  $\mathfrak{s}$  and

$$R^g(e_2, e_3) = \begin{pmatrix} & 0 & \frac{1}{8} \\ & -\frac{7}{16} & 0 \\ 0 & \frac{7}{16} & \\ -\frac{1}{8} & 0 & \end{pmatrix}, \quad R^g(e_2, e_4) = \begin{pmatrix} & & -\frac{1}{8} & 0 \\ & & 0 & -\frac{7}{16} \\ \frac{1}{8} & 0 & & \\ 0 & \frac{7}{16} & & \end{pmatrix},$$

$$R^g(e_3, e_4) = \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{1}{4} & 0 \\ & 0 & -\frac{7}{16} \\ & \frac{7}{16} & 0 \end{pmatrix}$$

and after some tedious calculations one can verify that the sectional curvature  $K$  satisfies  $K \leq -\frac{1}{4}$ .

It is possible to show that the connected component of the identity  $I_0(S, g)$  of the isometry group of  $(S, g)$  is a semidirect product  $S^1 \times S$ . To see this, one shows that every isometry  $f$  of  $S$  fixing the identity  $e$  of  $S$  satisfies that  $(df)_e$  is an automorphism of  $\mathfrak{s}$ . It follows from this fact that  $I_0(S, g)$  has no discrete co-compact subgroups and therefore, since  $S$  is solvable,  $S$  itself does not admit such a discrete subgroup.

### 3. HYPERCOMPLEX STRUCTURES ON CERTAIN NILPOTENT AND SOLVABLE LIE GROUPS

An *abelian* complex structure on a real Lie algebra  $\mathfrak{g}$  is an endomorphism of  $\mathfrak{g}$  satisfying

$$(3.1) \quad J^2 = -I, \quad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}.$$

The above conditions automatically imply the vanishing of the Nijenhuis tensor. By an abelian hypercomplex structure we mean a pair of anticommuting abelian complex structures. Our main motivation for studying abelian hypercomplex structures comes from the fact that such structures provide examples of homogeneous HKT-geometries (where HKT stands for hyper-Kähler with torsion, cf. [8]).

It was proved in [1] that if  $\dim[\mathfrak{g}, \mathfrak{g}] \leq 2$  then every hypercomplex structure on  $\mathfrak{g}$  must be abelian. To complete the classification of the Lie algebras  $\mathfrak{g}$  with  $\dim[\mathfrak{g}, \mathfrak{g}] \leq 2$  carrying hypercomplex structures (cf. [1]) it remained to give a characterization in the case when  $\mathfrak{g}$  is 2-step nilpotent and  $\dim[\mathfrak{g}, \mathfrak{g}] = 2$ : this is obtained by taking  $m = 2$  in Theorem 3.1 below.

It is a result of [7] that the only 8-dimensional non-abelian nilpotent Lie algebras carrying abelian hypercomplex structures are trivial central extensions of  $H$ -type Lie algebras. We show in [3] that this does not hold for higher dimensions: there exist 2-step nilpotent Lie algebras which are not of type  $H$  carrying such structures.

Let  $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$  be a two-step nilpotent Lie algebra endowed with an inner product  $\langle \cdot, \cdot \rangle$  and consider the orthogonal decomposition  $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ , where  $\mathfrak{z}$  is the center of  $\mathfrak{n}$  and  $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$ . Define a linear map  $j : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ ,  $z \mapsto j_z$ , where  $j_z$  is determined as follows:

$$(3.2) \quad \langle j_z v, w \rangle = \langle z, [v, w] \rangle, \quad \forall v, w \in \mathfrak{v}.$$

Observe that  $j_z$ ,  $z \in \mathfrak{z}$ , are skew-symmetric so that  $z \mapsto j_z$  defines a linear map  $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ . Note that  $\text{Ker}(j)$  is the orthogonal complement of  $[\mathfrak{n}, \mathfrak{n}]$  in  $\mathfrak{z}$ . In particular,  $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$  if and only if  $j$  is injective. Conversely, any linear map  $j : \mathbb{R}^m \rightarrow \mathfrak{so}(k)$  gives rise to a 2-step nilpotent Lie algebra  $\mathfrak{n}$  by means of (3.2). It follows that the center of  $\mathfrak{n}$  is  $\mathbb{R}^m \oplus (\bigcap_{z \in \mathbb{R}^m} \text{Ker } j_z)$  and  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathbb{R}^m$  where equality holds precisely when  $j$  is injective. We say that  $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$  is irreducible when  $\mathfrak{v}$  has no proper subspaces invariant by all  $j_z$ ,  $z \in \mathfrak{z}$ .

It follows that a two-step nilpotent Lie algebra carrying an abelian complex structure amounts to a linear map  $j : \mathfrak{z} \rightarrow \mathfrak{u}(k)$  (where  $\dim \mathfrak{v} = 2k$  and  $\mathfrak{u}(k)$  denotes the Lie algebra of the unitary group  $U(k)$ ). As a consequence of this we obtain the following result, where we denote by  $\mathfrak{sp}(k)$  the Lie algebra of the symplectic group  $Sp(k)$ :

*Theorem 3.1* ([3]). Every injective linear map  $j : \mathbb{R}^m \rightarrow \mathfrak{sp}(k)$  ( $m \leq k(2k + 1)$ ) gives rise to a two-step nilpotent Lie algebra  $\mathfrak{n}$  with  $\dim[\mathfrak{n}, \mathfrak{n}] = m$  carrying an abelian hypercomplex structure. Conversely, any two step nilpotent Lie algebra carrying an abelian hypercomplex structure arises in this manner.

Using the same idea as in the above theorem it is possible to construct hypercomplex structures on certain solvable Lie algebras. In fact, given a two step nilpotent Lie algebra  $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$  set  $\mathfrak{s} = \mathbb{R}a \oplus \mathfrak{n}$  with  $[a, z] = z$ ,  $\forall z \in \mathfrak{z}$ ,  $[a, v] = \frac{1}{2}v$ ,  $\forall v \in \mathfrak{v}$ , where the inner product on  $\mathfrak{v}$  is extended to  $\mathfrak{s}$  by decreeing  $a \perp \mathfrak{v}$  and  $\langle a, a \rangle = 1$ . This solvable extension of  $\mathfrak{n}$  has been studied by various authors ([6]). In the special case when  $\dim \mathfrak{z} \equiv 3 \pmod{4}$ ,  $\dim \mathfrak{v} = 4k$  and the the endomorphisms  $j_z$ ,  $z \in \mathfrak{z}$ , defined as in (3.2), belong to  $\mathfrak{sp}(k)$ , it can be shown that  $\mathfrak{s}$  carries a hypercomplex (hyper-Hermitian) structure. The procedure is analogous to that in the preceding theorem. It should be noted that these structures cannot be abelian and the corresponding metrics are not hyper-Kähler (since they are not flat).

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## TWISTOR QUOTIENTS OF HYPERKÄHLER MANIFOLDS

ROGER BIELAWSKI

ABSTRACT. We generalize the hyperkähler quotient construction to the situation where there is no group action preserving the hyperkähler structure but for each complex structure there is an action of a complex group preserving the corresponding complex symplectic structure. Many (known and new) hyperkähler manifolds arise as quotients in this setting. For example, all hyperkähler structures on semisimple coadjoint orbits of a complex semisimple Lie group  $G$  arise as such quotients of  $T^*G$ . The generalized Legendre transform construction of Lindström and Roček is also explained in this framework.

### INTRODUCTION

The motivation for this work stems from two problems. The first is the following question: when is a complex-symplectic quotient of a hyperkähler manifold hyperkähler? A good example is the hyperkähler structure on  $M = T^*G$ , where  $G$  is a complex semisimple Lie group (found by Kronheimer, cf. [3]). The complex symplectic quotients of  $M$  by  $G$  are precisely coadjoint orbits of  $G$ . These carry hyperkähler structures by the work of Kronheimer [13], Biquard [5] and Kovalev [12].

The second motivating problem is the generalized Legendre transform (GLT) construction of hyperkähler metrics due to Lindström and Roček [14]. Unlike the ordinary Legendre transform which produces  $4n$ -dimensional hyperkähler metrics with  $n$  commuting Killing vector fields, the GLT produces metrics without (usually) any Killing vector fields. The defining feature of these metrics is that their twistor space admits a holomorphic projection onto a vector bundle of rank  $n$  over  $\mathbb{C}P^1$ .

It turns out that in both of these problems there is a group-like object involved, namely a bundle of complex groups over  $\mathbb{C}P^1$  which act fiberwise on the twistor space  $Z$  of a hyperkähler manifold. This action is also Hamiltonian for the twisted symplectic form of  $Z$ . Thus, whenever we have such an action, we can form fiberwise complex-symplectic quotient of  $Z$  giving us (in good cases) a new twistor space. Similarly, in the case of the GLT, the projection onto the vector bundle  $V$  should be regarded as the moment map for an action of a bundle of abelian groups on  $Z$ , which preserve the twisted symplectic form.

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We call our bundles of groups over  $\mathbb{C}P^1$  *twistor groups*. The simplest definition of a twistor group is a *group in the category of spaces over  $\mathbb{C}P^1$  with a real structure*.

If a twistor group acts on the twistor space  $Z$  of a hyperkähler manifold  $M$ , we can interpret (in most cases) the resulting vector fields on  $Z$  as objects on  $M$ , namely either as higher rank Killing spinors (cf. [7]) or, in the  $E - H$  formalism (cf. [15]) as sections of  $E \otimes S^{2i+1}H$  ( $i > 1$ ) satisfying equations analogous to the Killing vector field equation (case  $i = 1$ ).

The main purpose of this paper is to introduce the concept of twistor groups and their actions and to give some interesting examples. We also prove results which can be viewed as new constructions of hyperkähler manifolds.

## 1. TWISTOR GROUPS AND THEIR ACTIONS

**1.1. Twistor groups.** Let  $X$  be a complex manifold. A *space over  $X$*  is a complex space  $Z$  together with a surjective holomorphic map (*projection*)  $\pi : Z \rightarrow X$ . We shall say that  $Z \xrightarrow{\pi} X$  is *smooth* if  $Z$  is smooth and  $\pi$  is a submersion.

The category of spaces over  $X$  is a category with products (fiber product) and a final object ( $X \xrightarrow{\text{Id}} X$ ). In any category with such properties we can define a *group* as an object  $\mathcal{G}$  together with morphisms defining group multiplication, inverse, and the identity. Thus we define:

*Definition 1.1.* A *group over  $X$*  is a group in the category of spaces over  $X$ .

More explicitly, a group over  $X$  is a space  $\mathcal{G} \xrightarrow{\pi} X$  together with fibrewise holomorphic maps  $\cdot : \mathcal{G} \times_X \mathcal{G} \rightarrow \mathcal{G}$  (*multiplication*),  $i : \mathcal{G} \rightarrow \mathcal{G}$  (*group inverse*) and  $1 : X \rightarrow \mathcal{G}$  (*identity section*) which commute with  $\pi$  and satisfy the group axioms. In particular, for each  $x \in X$   $(\pi^{-1}(x), \cdot, i|_{\pi^{-1}(x)}, 1(x))$  is a group.

*Remark 1.2.* Even if one is interested (as we are) primarily in smooth groups over  $X$ , one cannot avoid the singular ones, since a subgroup of a smooth group can be singular. In particular the stabilizers of smooth group actions can be singular.

We shall be interested mostly in the case when  $X = \mathbb{C}P^1$  and the spaces over  $\mathbb{C}P^1$  come equipped with an antiholomorphic involution (*real structure*) covering the antipodal map on  $\mathbb{C}P^1$ . The category of spaces with a real structure over  $\mathbb{C}P^1$  is also a category with products and a final object. Therefore we can define:

*Definition 1.3.* A *twistor group* is a group in the category of smooth spaces with a real structure over  $\mathbb{C}P^1$ .

*Remark 1.4.* Although the natural setting is the category of complex spaces rather than of manifolds, all our examples involve only smooth groups. In addition, the proofs are either simpler or work only for smooth groups.

Let us give a few examples of twistor groups.



*Example 1.5.* Let  $G$  be a complex Lie group equipped with an antiholomorphic involutive automorphism  $\sigma$ . Then  $G \times \mathbb{P}^1$  with the involution  $(\sigma, a)$ , where  $a$  is the antipodal map, is a twistor group which we shall call a *trivial twistor group* (with structure group  $G$ ) and denote by  $\underline{G}$ .

*Example 1.6.* Many nontrivial examples arise as twistor subgroups of  $\underline{G}$ . For example, if  $G$  acts fibrewise on a space  $Z$  with a real structure over  $\mathbb{C}P^1$ , then the stabilizer of any real section of  $Z$  is a twistor subgroup of  $\underline{G}$ . In particular, we can take the adjoint action of  $G$  on  $Z = \underline{\mathfrak{g}} \otimes V$ , where  $V$  is a vector bundle over  $\mathbb{C}P^1$  equipped with a real structure.

*Example 1.7.* Another important twistor group is constructed as follows. Let  $G$  be reductive and let  $\mathfrak{k}$  denote the Lie algebra of the maximal compact subgroup of  $G$ . Let  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{k}$  be a homomorphism of Lie algebras. For each element  $z = (a, b, c)$ ,  $a^2 + b^2 + c^2 = 1$ , of  $S^2 \simeq \mathbb{C}P^1$ , define a subalgebra  $\mathfrak{n}_z$  of  $\mathfrak{g}$  as the sum of negative eigenspaces of  $\text{ad}(a\rho(\sigma_1) + b\rho(\sigma_2) + c\rho(\sigma_3))$ , where  $\sigma_1, \sigma_2, \sigma_3$  denote the Pauli matrices. Now define  $\mathcal{N}$  as a twistor subgroup of  $\underline{G}$  whose fiber at  $z$  is the subgroup of  $G$  whose Lie algebra is  $\mathfrak{n}_z$ . It is straightforward to observe that the real structure of  $\underline{G}$  acts on  $\mathcal{N}$ . We also observe that each fiber of  $\mathcal{N}$  is the unipotent radical of the parabolic subgroup of  $G$  determined by  $\rho$ .

*Example 1.8.* A vector bundle over  $\mathbb{P}^1$  equipped with a (linear) real structure is an abelian twistor group. We observe that a line bundle  $\mathcal{O}(k)$  is a twistor group if and only if  $k$  is even.

The last example can be generalized as follows. Let  $\mathcal{G}$  be any twistor group. For an open subset  $U$  of  $\mathbb{C}P^1$  denote by  $\mathcal{G}_U$  the group over  $U$  obtained as the restriction of  $\mathcal{G}$  to  $U$ . Now suppose that we are given a covering  $\{U_i\}$  of  $\mathbb{C}P^1$ , invariant under the antipodal map and a fibrewise automorphism  $\{\phi_{ij}\}$  of  $\mathcal{G}_{U_i \cap U_j}$  for each nonempty intersection  $U_i \cap U_j$ . In addition we suppose that the family of  $\phi_{ij}$  is  $\tau$ -equivariant, where  $\tau$  is the real structure of  $\mathcal{G}$ . Then gluing together  $\mathcal{G}_{U_i}$  via the  $\phi_{ij}$  gives us a new twistor group, locally isomorphic to  $\mathcal{G}$ . We deduce the following:

**Proposition 1.9.** *Let  $\mathcal{G}$  be a twistor group. Then the isomorphism classes of twistor groups locally isomorphic to  $\mathcal{G}$  are in bijective correspondence with elements of (non-abelian) sheaf cohomology group  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{A})$ , where  $\mathcal{A}(U)$  is the group of automorphisms of  $\mathcal{G}_U$ .  $\square$*

The subscript  $\mathbb{R}$  denotes  $\tau$ -invariant elements.

In particular, if  $G$  is a complex Lie group with an antiholomorphic automorphism, then we can consider twistor groups which are locally isomorphic to  $\underline{G}$ . We shall call such twistor groups *locally trivial*. We have:

**Corollary 1.10.** *The isomorphism classes of locally trivial twistor groups with structure group  $G$  are in 1-1 correspondence with elements of  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\text{Aut}(G)))$ .  $\square$*

We shall call a twistor group  $\mathcal{G}$  *discrete*, if each fiber of  $\mathcal{G}$  is discrete. The following example shows that twistor groups need not be locally trivial.

*Example 1.11.* Let  $s$  be a real meromorphic section of  $\mathcal{O}(-2)$  having poles at a pair of antipodal points  $a, -1/\bar{a}$ . We have a (smooth) discrete twistor subgroup  $\mathcal{L}$  of  $\mathcal{O}(-2)$  defined as the subgroup of  $\mathcal{O}(-2)$  with fiber 0 at  $a$  and  $-1/\bar{a}$  and  $\mathbb{Z}s(x)$  at other points.

Notice that  $\mathcal{O}(-2)/\mathcal{L}$  (fibrewise quotient) is also a twistor group.

**1.2. Twistor Lie algebras.** Let  $\mathcal{G} \xrightarrow{\pi} X$  be a smooth group over  $X$ . Then the normal bundle to the identity section has a natural structure of a *Lie algebra over  $X$*  (i.e. a Lie algebra in the category of vector bundles over  $X$ ). We shall denote this space by  $\text{Lie}(\mathcal{G})$ . In particular  $\text{Lie}(\mathcal{G})$  is locally trivial as a vector bundle (cf. [8]).

Let us consider the structure of a twistor Lie algebra  $\mathcal{L}$  in a more detail. As observed in the previous section,  $\mathcal{L}$  is a locally trivial vector bundle and so it splits as  $\bigoplus \mathcal{O}(p_i)$  for some integers  $p_1, \dots, p_n$ . We choose coordinates  $e_1, \dots, e_n$  for  $\mathcal{L}$  over  $\zeta \neq \infty$  and  $\tilde{e}_1, \dots, \tilde{e}_n$  over  $\zeta \neq 0$ , so that  $\tilde{e}_i = \zeta^{-p_i} e_i$  over  $\mathbb{C}P^1 - \{0, \infty\}$ . The fibrewise Lie bracket is given by

$$(1) \quad [e_i, e_j]_{\zeta} = \sum_{k=1}^n f_k(\zeta) e_k,$$

$$(2) \quad [\tilde{e}_i, \tilde{e}_j]_{\zeta} = \sum_{k=1}^n \tilde{f}_k(\zeta) \tilde{e}_k,$$

for some holomorphic functions  $f_k = f_k^{ij}$ ,  $\tilde{f}_k = \tilde{f}_k^{ij}$ . Comparing the two expressions for the bracket over  $\zeta \neq 0, \infty$ , we see that  $f_k, \tilde{f}_k$  define a section of  $\mathcal{O}(p_i + p_j - p_k)$ . In particular if some of these sections are nonconstant (i.e. have zeros), then  $\mathcal{L}$  is locally nontrivial as a bundle of Lie algebras.

The preceding considerations imply the following fact, which will be useful later on.

**Proposition 1.12.** *Let  $\mathcal{G}$  be a smooth twistor group whose Lie algebra splits as a sum of line bundles of negative degrees. Then  $\mathcal{G}$  is nilpotent.*  $\square$

**1.3. Actions of twistor groups.** We now define actions of twistor groups or of groups over  $X$ . Once more, this is a tautological definition in any category with products and a final object. In our case an *action* of a group  $\mathcal{G} \xrightarrow{\pi} X$  on  $Z \xrightarrow{p} X$  is a holomorphic map

$$\cdot : \mathcal{G} \times_X Z \rightarrow Z$$

which commutes with the projections and which is a group action on each fiber. An action of a twistor group is required to respect the real structures  $\mathcal{G}$  and  $Z$  (i.e.  $\tau(g \cdot z) = \tau(g) \cdot \tau(z)$ ). Most notions related to (ordinary) group actions carry over to actions of groups over  $X$ . Thus, we shall say that the action of  $\mathcal{G}$  is free (resp.

locally free) if each fiber action is free (resp. locally free). We can define equivariant morphisms. We also observe that for smooth groups over  $X$  we have canonical notions of the *adjoint* and *coadjoint* action. Finally we can define orbits and stabilizers of sections of  $Z \xrightarrow{p} X$ .

*Remark 1.13.* In the case of an action on a twistor space of a hyperkähler manifold  $M$ , we shall also speak of  $\mathcal{G}$  acting on  $M$ . Similarly, if  $s$  is a twistor line corresponding to a point  $m$  in  $M$ , we can speak of the stabilizer of  $\mathcal{G}$  at  $m \in M$  etc.

We shall be particularly interested in the following types of actions.

*Definition 1.14.* Let a smooth twistor group  $\mathcal{G} \xrightarrow{\pi} \mathbb{C}P^1$  act on a twistor space  $Z \xrightarrow{p} \mathbb{C}P^1$ . We shall say that the action is *symplectic* (resp. *Hamiltonian*), if the action is symplectic on each fiber for the twisted symplectic form  $\omega$  on  $Z$  (resp. if it is symplectic and if there is a holomorphic map  $\mu : Z \rightarrow \text{Lie}(\mathcal{G})^* \otimes \mathcal{O}(2)$  which is the moment map for the twisted symplectic form  $\omega$  on each fiber).

*Example 1.15.* Let a compact Lie group  $K$  act on a hyperkähler manifold  $M$  by hyperkähler isometries and suppose that this action extends to the action of  $K^{\mathbb{C}}$  for each complex structure of  $M$ . Then the trivial group  $\underline{K}^{\mathbb{C}}$  acts symplectically on  $Z$ . Consequently, any twistor subgroup of  $\underline{K}^{\mathbb{C}}$  acts symplectically on  $Z$ . If the action of  $K$  is tri-Hamiltonian, then the action of  $\underline{K}^{\mathbb{C}}$  and any of its twistor group subspaces is Hamiltonian.

*Example 1.16.* Let  $M = \mathbb{H}$  so that  $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$ . Then the twistor group  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  acts on  $Z$  via fibrewise addition. This action is symplectic, but not Hamiltonian.

*Example 1.17. (Atiyah-Hitchin manifold).* This is an example of a Hamiltonian action of a twistor group (namely  $\mathcal{O}(-2)$ ) which does not arise from any hyperkähler group action. Let  $M$  be the hyperkähler manifold of strongly centered monopoles of charge 2, i.e. the double (or universal) cover of the Atiyah-Hitchin manifold. With respect to any complex structure it is biholomorphic to the space of degree 2 rational maps of the form  $p(z)/q(z)$  where  $q(z) = z^2 - c$  and  $p(z) = az + b$  with  $b^2 - ca^2 = 1$ . Let  $\beta$  denote any of the two roots of  $q$ . The complex symplectic form is given (on the set where  $c \neq 0$ ) by  $\omega = \frac{dp(\beta)}{p(\beta)} \wedge d\beta$ . The twistor space  $Z$  of  $M$  is essentially given by requiring that  $\beta$  is a section of  $\mathcal{O}(2)$  while  $p(\beta)$  is a value of a certain line bundle  $L^{-2}$  on  $\mathcal{O}(2)$  [2]. We define an action of  $\mathbb{C}$  on  $M$  by

$$\lambda \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \cosh \lambda\beta & \frac{\sinh \lambda\beta}{\beta} & 0 \\ \beta \sinh \lambda\beta & \cosh \lambda\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Here  $\beta = \pm\sqrt{c}$ . This action sends  $p(\beta)$  to  $e^{\lambda\beta}p(\beta)$  and so it respects the complex symplectic form  $\omega$ . By looking at the transition functions for the twistor space  $Z$  we conclude that this action extends to the fibrewise action of  $\mathcal{O}(-2)$  on  $Z$ . One can

check that the real structures are compatible with this action. This action is locally free and the orbits of twistor lines are isomorphic to the twistor groups  $O(-2)/\mathcal{L}$  defined in example 1.11. Finally, this action is Hamiltonian and the twistor lines of  $Z$  project via the moment map  $\mu : Z \rightarrow \mathcal{O}(4)$  to the *spectral curves* of the monopoles.

We have an obvious restriction on twistor groups which can act locally freely at any twistor line.

**Proposition 1.18.** *Suppose that a twistor group  $\mathcal{G}$  acts on a twistor space  $Z$  and that the action is locally free at some twistor line  $s$ . Then  $\text{Lie } \mathcal{G}$  is the sum of line bundles of degree at most one.*

*Proof.* We have an injective morphism  $\iota : \mathcal{L} \rightarrow \mathcal{O}(1) \otimes \mathbb{C}^n$  of vector bundles over  $\mathbb{C}P^1$ .  $\square$

For Hamiltonian actions the restriction is more severe.

**Proposition 1.19.** *Suppose that there is a Hamiltonian action of a twistor group  $\mathcal{G}$  on a twistor space  $Z$  which is locally free at some twistor line. Then  $\text{Lie } \mathcal{G}$  is the sum of line bundles of degree at most zero.*  $\square$

**1.4. Quotients and principal bundles.** An action of a twistor group  $\mathcal{G}$  on a space  $Z \xrightarrow{p} \mathbb{C}P^1$  defines an equivalence relation  $R \subset Z \times Z$ :  $(z_1, z_2) \in R \iff p(z_1) = p(z_2) =: \zeta$  and  $z_1, z_2$  are in the same orbit of  $\mathcal{G}_\zeta$ . Equivalently,  $R$  can be viewed as a subspace of  $Z \times_{\mathbb{C}P^1} Z$  over  $\mathbb{C}P^1$ . A quotient of  $Z$  by this relation is a topological space  $Z/\mathcal{G}$  which also has a natural structure of  $\mathbb{C}$ -ringed space. We define

*Definition 1.20.* An action of  $\mathcal{G}$  on a smooth  $Z$  is called *regular*, if  $Z/\mathcal{G}$  is a smooth space over  $\mathbb{C}P^1$  and the natural projection  $\pi : Z \rightarrow Z/\mathcal{G}$  is a submersion.

A theorem of Godement ([16], Part II, Thm. 3.12.2) gives us a necessary and sufficient condition for regularity:

**Proposition 1.21.** *An action of a twistor group  $\mathcal{G}$  on a smooth  $Z$  is regular if and only if  $R$  is a smooth closed subspace of  $Z \times_{\mathbb{C}P^1} Z$  over  $\mathbb{C}P^1$ . In particular, if the action is free, then it is regular if and only if the quotient  $Z/\mathcal{G}$  is Hausdorff, i.e.  $R$  is a smooth closed subspace of  $Z \times_{\mathbb{C}P^1} Z$  over  $\mathbb{C}P^1$ .*

Suppose now that the action of  $\mathcal{G}$  on  $Z$  is free and regular. Then we shall call  $Z$  a *principal  $\mathcal{G}$ -bundle* over  $T := Z/\mathcal{G}$ . We can classify these as follows:

**Proposition 1.22.** *Let  $T \xrightarrow{p} \mathbb{C}P^1$  be a smooth space over  $\mathbb{C}P^1$  equipped with a real structure. The set of isomorphism classes of principal  $\mathcal{G}$ -bundles over  $T$  is in bijective correspondence with elements of  $H_{\mathbb{R}}^1(T, \mathcal{O}(p^*\mathcal{G}))$ .*

*Proof.* Let  $P$  be a principal  $\mathcal{G}$ -bundle over  $T$  and let  $\pi_o : p^*\mathcal{G} \rightarrow T$  denote the “trivial”  $\mathcal{G}$ -bundle. Since the projection  $\pi : P \rightarrow T$  is a submersion, it admits local sections through every point. Therefore we have  $\mathcal{G}_{p(U_i)}$ -equivariant isomorphisms

$h_i : \pi^{-1}(U_i) \rightarrow \pi_o^{-1}(U_i)$  for some covering  $U_i$  of  $T$ . For each nonempty intersection  $U_i \cap U_j$  we have the transition function  $\phi_{ij} = h_j h_i^{-1}$  which is a  $\mathcal{G}$ -equivariant (fibrewise) automorphism of  $\pi_o^{-1}(U_i \cap U_j)$ . Let  $G$  denote a particular fiber of  $p^*\mathcal{G}$  and let  $\phi$  denote  $\phi_{ij}$  restricted to this fiber. Thus  $\phi : G \rightarrow G$  and  $\phi(gx) = g\phi(x)$  for any  $x \in G$ . Therefore  $\phi$  is determined by the value of  $\phi(1)$ . One checks that the cocycle conditions for local equivariant automorphisms of  $p^*\mathcal{G}$  and for local sections of  $p^*\mathcal{G}$  coincide, and this concludes the proof.  $\square$

**1.5. Orbits and homogeneous spaces.** The definition of an orbit of a twistor group as an orbit of a section is quite inadequate. We remark that in the case of an action  $\cdot : G \times M \rightarrow M$  of a Lie group  $G$  on a manifold  $M$ , an orbit can be defined in two ways: 1) as the image of a point  $m$  under the mapping  $\cdot m : G \rightarrow M$ , or 2) as a  $G$ -homogeneous  $G$ -invariant submanifold of  $M$ . The second definition is more suitable in the case of twistor groups.

*Definition 1.23.* Let a twistor group  $\mathcal{G} \xrightarrow{\pi} \mathbb{C}P^1$  act on a space  $Z$  over  $\mathbb{C}P^1$  equipped with a real structure  $\tau$ . An orbit of  $\mathcal{G}$  is a  $\tau$ -invariant  $\mathcal{G}$ -homogeneous subspace of  $Z$ .

Thus to know the structure of possible orbits we should classify the homogeneous  $\mathcal{G}$ -spaces. For example, if  $\mathcal{G}$  is vector bundle (with additive group structure), then  $\mathcal{G}$  acts transitively on any affine bundle  $A$  (equipped with an appropriate real structure) such that the linear part of its transition function coincides with the transition function of  $\mathcal{G}$ . Thus  $A$  is an orbit of  $\mathcal{G}$ .

Firts of all we have

**Proposition 1.24.** *Let  $\mathcal{H}$  be a closed twistor subgroup of a twistor group  $\mathcal{G}$ . Then  $\mathcal{G}/\mathcal{H}$  is a smooth homogeneous  $\mathcal{G}$ -space.*

*Proof.* Since the projection  $\mathcal{G} \rightarrow \mathbb{C}P^1$  is a submersion,  $\mathcal{G}$  admits local sections through every point. Let  $g(\zeta)$  be such a section over an  $U \subset \mathbb{C}P^1$ . Consider  $\text{Lie}(\mathcal{G})$  and its subalgebra  $\text{Lie}(\mathcal{H})$ . By taking a smaller  $U$ , we can assume that  $\text{Lie}(\mathcal{G})$  is trivial as a vector bundle over  $U$ . Therefore there is a subbundle  $V$  of  $\text{Lie}(\mathcal{G})|_U$  complementary to  $\text{Lie}(\mathcal{H})$ .  $V$  is also trivial and we have local section  $e_1, \dots, e_m$ , which provide a basis of  $V_\zeta$  for each  $\zeta$ . Then the map  $(\zeta, v_1, \dots, v_m) \mapsto g(\zeta) \exp_\zeta(v_1 e_1(\zeta) + \dots + v_m e_m(\zeta))$  composed with the projection  $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$  provides a local coordinate system on  $\mathcal{G}/\mathcal{H}$ . Here  $\exp_\zeta$  denotes the fibrewise exponential mapping. The fact that  $\mathcal{G}$  admits a local section through every point implies that so does  $\mathcal{G}/\mathcal{H}$  and hence the projection  $\mathcal{G}/\mathcal{H} \rightarrow \mathbb{C}P^1$  is a submersion.  $\square$

From Proposition 1.22 we already know homogeneous  $\mathcal{G}$ -spaces on which  $\mathcal{G}$  acts freely: they are given by elements of  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{G}))$ . In fact, the same argument allows to classify homogeneous  $\mathcal{G}$ -spaces which are locally isomorphic to  $\mathcal{G}/\mathcal{H}$ :

**Proposition 1.25.** *Let  $\mathcal{H}$  be a closed twistor subgroup of  $\mathcal{G}$ . The set of isomorphism classes of homogeneous  $\mathcal{G}$ -spaces which are locally isomorphic to  $\mathcal{G}/\mathcal{H}$  is in bijection with elements of  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(N(\mathcal{H})/\mathcal{H}))$ , where  $N(\mathcal{H})$  denotes the normalizer of  $\mathcal{H}$ .*

*Proof.* We proceed as in the proof of Proposition 1.22 obtaining transition functions  $\phi_{ij}$  for such a homogeneous space, which are  $\mathcal{G}$ -equivariant (fibrewise) automorphisms of  $\pi^{-1}(U_i \cap U_j)$ , where  $\pi$  denotes the projection  $\mathcal{G}/\mathcal{H} \rightarrow \mathbb{C}P^1$ . Let  $G$  and  $H$  denote a particular fiber of  $\mathcal{G}$  and  $\mathcal{H}$  and let  $\phi$  denote  $\phi_{ij}$  restricted to this fiber. Thus  $\phi : G/H \rightarrow G/H$  and  $\phi(gx) = g\phi(x)$  for any  $x \in G/H$ . Thus  $\phi$  is determined by the value of  $\phi(H)$ . Suppose that  $\phi(H) = pH$ . Then, since  $\phi(H) = \phi(hH) = h\phi(H) = hpH$ ,  $p \in N(H)$  and such  $\phi$ 's are in 1-1 correspondence with elements of  $N(H)/H$ . Once again the cocycle conditions for local equivariant automorphisms of  $\mathcal{G}/\mathcal{H}$  and for local sections of  $N(\mathcal{H})/\mathcal{H}$  coincide, and this concludes the proof.  $\square$

In general, there is no reason why a homogeneous  $\mathcal{G}$ -space should be locally isomorphic to a fixed  $\mathcal{G}/\mathcal{H}$ . We have, however:

**Proposition 1.26.** *Let a twistor group  $\mathcal{G}$  act transitively on a smooth space  $W$  in such a way that, for each  $\zeta \in \mathbb{C}P^1$ , the stabilizer of the action of the fiber  $G_\zeta$  on  $W_\zeta$  is a normal subgroup of  $G_\zeta$ . Then there exists a closed normal subgroup  $\mathcal{H}$  of  $\mathcal{G}$  such that  $W$  is locally isomorphic to  $\mathcal{G}/\mathcal{H}$ .*

*Proof.* Since  $W$  is smooth, the projection  $p$  on  $\mathbb{C}P^1$  is a submersion, and so local sections of  $W$  exist. Using the transitivity, it follows that locally  $p^{-1}(U)$  is  $\mathcal{G}_U$ -equivariantly isomorphic to  $\mathcal{G}_U/H(U)$ , where  $H(U)$  is a subgroup of  $\mathcal{G}_U$ . Consider the intersection of two such sets  $U_1$  and  $U_2$  and proceed as in the proof of the previous proposition. This time, on each fiber, we have a  $G$ -equivariant diffeomorphism  $\phi$  from  $G/H_1$  to  $G/H_2$ . As in the previous proof,  $\phi$  is completely determined by its value on  $H_1$ , say  $pH_2$ . It follows that  $p^{-1}H_1p = H_2$ , and since  $H_1$  is normal,  $H_1 = H_2$ . Therefore there is a subgroup  $\mathcal{H}$  of  $\mathcal{G}$  whose restriction to each  $U$  is  $H(U)$ .  $\square$

## 2. NEGATIVE TWISTOR GROUPS AND DEFORMATIONS OF HYPERKÄHLER STRUCTURES

Let a twistor group  $\mathcal{G}$  act regularly (i.e. the quotient is smooth) on a twistor space  $Z$  of a hyperkähler manifold  $M$  (i.e.  $Z/\mathcal{G}$  is smooth and the natural projection  $\pi : Z \rightarrow Z/\mathcal{G}$  is a submersion). The space  $Z/\mathcal{G}$  over  $\mathbb{C}P^1$  has an induced real structure and the pre-image  $\pi^{-1}(s)$  of any real section of  $Z/\mathcal{G}$  is a  $\mathcal{G}$ -orbit in the sense of definition 1.23. Therefore we define:

*Definition 2.1.* With the above assumptions, the (real-analytic) space of real sections of  $Z/\mathcal{G} \rightarrow \mathbb{C}P^1$  is called the *space of  $\mathcal{G}$ -orbits* and is denoted by  $M/\mathcal{G}$ .

We have a natural map  $\rho : M \rightarrow M/\mathcal{G}$  obtained by projecting the twistor lines corresponding to points of  $M$  to  $Z/\mathcal{G}$ . Consider a section  $s$  of  $Z/\mathcal{G}$  corresponding to such  $\pi(m)$ ,  $m \in M$ . Suppose that the action of  $\mathcal{G}$  is locally free. If  $L$  denotes  $\text{Lie}(\mathcal{G})$ , we have an exact sequence of vector bundles:

$$(3) \quad 0 \rightarrow L \rightarrow E \rightarrow T_s^V(Z/\mathcal{G}) \rightarrow 0,$$

where  $V$  denotes the vertical bundle and  $E$  is the normal bundle to the twistor line, i.e.  $O(1) \otimes \mathbb{C}^{2n}$ . It follows that the normal bundle  $N$  to  $s$  in  $Z/\mathcal{G}$  splits as the sum of line bundles of degree at least 1. Therefore  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(N)) = 0$  and the well-known theorem of Kodaira shows that any section of  $N$  can be integrated to a deformation the section  $s$ . This makes a neighbourhood of  $\rho(M)$  in  $M/\mathcal{G}$  into a smooth manifold of dimension  $\dim M - h^0(L) + h^1(L)$ .

We shall now restrict our attention to a special kind of twistor groups. We adopt the following definition

*Definition 2.2.* A twistor Lie algebra  $\mathcal{L}$  is called *negative* if it is a direct sum of line bundles of negative degree. A smooth twistor group is called *negative* if its Lie algebra is negative.

We also adopt the convention that the identity twistor group  $\text{Id} : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$  is also negative.

We have the following simple properties of negative twistor groups which are consequence of Proposition 1.12 and the considerations preceding it:

- Proposition 2.3.** (1) *A negative twistor group is nilpotent.*  
(2) *Any twistor group  $\mathcal{G}$  contains a unique maximal negative subgroup  $N(\mathcal{G})$ .*  
(3) *If  $\text{Lie}(\mathcal{G})$  is a direct sum of line bundles of nonpositive degree (e.g. there is a Hamiltonian action of  $\mathcal{G}$  on some twistor space), then  $N(\mathcal{G})$  is a normal subgroup of  $\mathcal{G}$  and  $\mathcal{G}/N(\mathcal{G})$  is a trivial group (with some structure group  $H$ ).  $\square$*

It is easy to classify smooth twistor groups whose Lie algebra is a fixed negative  $\mathcal{L}$ . First of all, there exists a unique twistor group  $\mathcal{G}$  with simply connected fibers and such that  $\text{Lie}(\mathcal{G}) = \mathcal{L}$ . Indeed, as a manifold  $\mathcal{G} = \mathcal{L}$  and, since every fiber of  $\mathcal{L}$  is a nilpotent Lie algebra, the fibrewise multiplication in  $\mathcal{G}$  is determined by the Campbell-Hausdorff formula.

By doing things once again fibrewise, we conclude that any other twistor group with the same Lie algebra is of the form  $\mathcal{G}/\mathcal{D}$  for some (fibrewise) discrete twistor subgroup of the center of  $\mathcal{G}$ .

We shall usually denote negative twistor groups by the letter  $\mathcal{N}$ .

For us, the importance of negative twistor groups follows from the following observation:

**Proposition 2.4.** *Let a negative twistor group  $\mathcal{N}$  act regularly a hyperkähler manifold  $M$ . Then  $\rho : M \rightarrow M/\mathcal{N}$  is an imbedding.*

*Proof.* To see that  $\rho$  is an immersion observe that  $d\rho$  at any point of  $M$ , i.e. at a section of  $Z$ , is given by the long exact sequence of cohomology induced by (3). Since  $H_{\mathbb{R}}^0(\mathbb{C}P^1, L) = 0$ ,  $d\rho$  is injective. Let us show that  $\rho$  is injective  $M/\mathcal{N}$  corresponds to  $\mathcal{N}$ -orbits in  $M$ . The map  $\rho$  assigns to a section of the twistor space  $Z$ , corresponding to a point  $m \in M$ , its  $\mathcal{N}$ -orbit. Thus  $\rho$  fails to be an imbedding if an orbit of a section of  $Z$  admits more than one section. Since such an orbit  $W$  admits a section,

it is of the form  $\mathcal{N}/\mathcal{H}$  for some closed twistor subgroup (not necessarily smooth)  $\mathcal{H}$  of  $\mathcal{N}$ . Since  $W$  admits two sections,  $\mathcal{N}$  contains two different copies of  $\mathcal{H}$  and so two different sections. Now  $\mathcal{N}$  is of the form  $\mathcal{G}/\mathcal{D}$  where  $\mathcal{G}$  is isomorphic to  $\text{Lie}(\mathcal{N})$  and repeating the argument implies that  $\mathcal{G}$  and so  $\text{Lie}(\mathcal{N})$  contains two sections which is impossible.  $\square$

Now suppose that the action of  $\mathcal{N}$  on  $Z$  is, in addition to being regular, almost free. Then, according to Proposition 1.22, the fibration  $Z \rightarrow Z/\mathcal{N}$  comes from a  $\tau$ -invariant element of  $H_{\mathbb{R}}^1(Z/\mathcal{G}, \mathcal{O}(\mathcal{N}))$ . Restricting this cohomology class to sections of  $Z/\mathcal{G}$  gives us a map

$$(4) \quad \Lambda : M/\mathcal{N} \rightarrow H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{N})).$$

At this point a remark about the structure of  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{N}))$  is in order. It is not a group, unless  $\mathcal{N}$  is abelian. It does have, however, a preferred element 1 (corresponding to  $\mathcal{N}$ ). In addition, it has a natural structure of a smooth manifold, with charts diffeomorphic to  $H_{\mathbb{R}}^1(\mathbb{C}P^1, \text{Lie}(\mathcal{N}))$ . We observe that the map  $\Lambda$  is a smooth, with the differential defined as follows. Let  $A$  be an  $\mathcal{N}$ -orbit in  $Z$ , corresponding to an element  $\pi(A)$  of  $M/\mathcal{N}$ . Then we have an exact sequence of vector bundles

$$(5) \quad 0 \rightarrow (T^V A)/\mathcal{N} \rightarrow (T_A^V Z)/\mathcal{N} \rightarrow T_{\pi(A)}^V(Z/\mathcal{N}) \rightarrow 0,$$

where the action of  $\mathcal{N}$  on  $T^V Z$  is the tangent action along the fibers. The differential of  $\Lambda$  at  $\pi(A)$  is then the induced map

$$(6) \quad d\Lambda_{\pi(A)} : H_{\mathbb{R}}^0(\pi(A), T_{\pi(A)}^V(Z/\mathcal{N})) \rightarrow H_{\mathbb{R}}^1(A/\mathcal{N}, (T^V A)/\mathcal{N}) \simeq H_{\mathbb{R}}^1(\mathbb{C}P^1, \text{Lie}(\mathcal{N})).$$

We observe that  $\Lambda^{-1}(1)$  corresponds to  $\mathcal{N}$ -orbits possessing a section and so, from the previous proposition, to  $M$ . In general, for any  $\lambda$ ,  $\Lambda^{-1}(\lambda)$  parameterizes orbits of the fixed type  $\lambda$ . We claim that  $M_\lambda := \Lambda^{-1}(\lambda)$  carries a natural hyperkähler structure, which should be viewed as a deformation of the hyperkähler structure of  $M$ . More precisely:

**Theorem 2.5.** *Let a negative twistor group  $\mathcal{N}$  act regularly, almost freely, and symplectically on a hyperkähler manifold  $M^{4n}$ . Then there exists a smooth neighbourhood  $U$  of  $M$  in  $M/\mathcal{N}$  such that  $\Lambda$  is a submersion on  $U$  and, for any  $\lambda \in H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{N}))$ ,  $M_\lambda := \Lambda^{-1}(\lambda) \cap U$  carries a natural hyperkähler structure. Furthermore, with respect to each complex structure,  $M_\lambda$  is isomorphic, as a complex symplectic manifold, to an open subset of  $M$ .*

*Remark 2.6.* A completely analogous result holds for hypercomplex manifolds.

*Proof.* We consider the vector bundle  $F = (T^V Z)/\mathcal{N}$  on  $Z/\mathcal{N}$ . Over a section obtained by projecting a twistor line  $s$  in  $Z$ , this bundle is just the normal bundle of  $s$ , and so  $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$ . By standard semi-continuity theorems,  $F$  is  $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$  when restricted to neighbouring sections, i.e. to a neighbourhood  $U$  of  $\rho(M) \simeq M$  in  $M/\mathcal{N}$ . Then (5) and (6) show that  $\Lambda$  is a submersion on  $U$ . Thus  $M_\lambda = \Lambda^{-1}(\lambda) \cap U$  is a



submanifold of  $M/\mathcal{N}$ . The tangent space  $T_p M_\lambda$  is the space of real sections of  $F|_{s(p)}$ , where  $s(p)$  is the section of  $Z/\mathcal{N}$  given by  $p$ . This is the same as  $(T_A^V Z)/\mathcal{N}$  where  $A$  is the  $\mathcal{N}$ -orbit in  $Z$ , whose projection is  $s(p)$ . We have an  $O(2)$ -valued complex-symplectic form on the fibers of  $(T_A^V Z)/\mathcal{N}$ , given by  $\tilde{\omega}([a], [b]) = \omega(a, b)$ , where  $\omega$  is the given form on  $Z$  and the representatives  $a, b$  are tangent to the same point of  $A$ . Since  $\mathcal{N}$  acts symplectically, this does not depend on the choice of point in  $A$ . We notice that on each fiber over  $\mathbb{C}P^1$  this is canonically isomorphic to  $\omega$  on this fiber. In particular  $\tilde{\omega}$  is nondegenerate and closed. Now, as in the proof of Theorem in [9], we obtain a hyperhermitian structure on  $M_\lambda$ . The above isomorphisms on each fiber give us local isomorphisms of complex structures (essentially  $(Z_\zeta \times N_\zeta)/N_\zeta \simeq Z_\zeta$ ) proving their integrability and proving the theorem.  $\square$

The above proof allows us to identify the twistor space of  $M_\lambda$ . Let  $W_\lambda$  be a principal  $\mathcal{N}$ -bundle over  $\mathbb{C}P^1$  corresponding to a  $\lambda \in H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{N}))$ . Let  $Z$  be the twistor space of  $M$  and consider the diagonal action of  $\mathcal{N}$  on  $Z \rightarrow W_\lambda$ . Then  $Z_\lambda = (Z \times_{\mathbb{C}P^1} W_\lambda)/\mathcal{N}$  is the twistor space of  $M_\lambda$  (i.e.  $M_\lambda$  is the family of sections of  $Z_\lambda$  with correct normal bundle). We observe that  $\mathcal{N}$  does not necessarily act on  $Z_\lambda$ . It acts only if  $\mathcal{N}$  is abelian. In general case, we obtain an action of another negative twistor group  $\mathcal{N}_\lambda$ , locally isomorphic to  $\mathcal{N}$  and obtained by gluing pieces of  $\mathcal{N}$  by inner automorphisms corresponding to local sections of  $\mathcal{N}$  determined by  $\lambda \in H_{\mathbb{R}}^1(\mathbb{C}P^1, \mathcal{O}(\mathcal{N}))$  (for  $\lambda$  close to 1, the Lie algebra of  $\mathcal{N}_\lambda$  must be negative).

### 3. TWISTOR QUOTIENTS

We now wish to associate a “quotient” to a hyperkähler manifold with a twistor group action. Essentially, this quotient is formed by taking the complex symplectic quotients along the fibers of the twistor space.

Let therefore a twistor group  $\mathcal{G} \xrightarrow{\pi} \mathbb{C}P^1$  act on the twistor space  $Z \xrightarrow{p} \mathbb{C}P^1$  of a hyperkähler manifold  $M$ . We suppose that this action is Hamiltonian with the moment map  $\mu : Z \rightarrow \mathcal{L}^* \otimes \mathcal{O}(2)$ . Here  $\mathcal{L} = \text{Lie } \mathcal{G}$ .

Let  $s$  be a twistor line in  $Z$ . Then  $\mu \circ s$  is a real section of  $\mathcal{L}^* \otimes \mathcal{O}(2)$ . Let  $S = (\mu \circ s)(\mathbb{C}P^1)$  and suppose that the fibrewise quotient of  $\mu^{-1}(S)$  by  $\text{Stab}(\mu \circ s)$  (stabilizer of coadjoint action) is a manifold (fibering over  $\mathbb{C}P^1$ ), which we denote by  $Z_{\text{red}}$ . It inherits the real structure, the twisted complex-symplectic form along the fibers and a real section  $\bar{s}$ , induced by  $s$ . Thus, if  $Z_{\text{red}}$  contains a real section (e.g.  $\bar{s}$ ) whose normal bundle is the sum of line bundles of degree 1, then  $Z_{\text{red}}$  is a twistor space of a pseudo-hyperkähler manifold, which we denote by  $M//\mathcal{G}$ . We shall call this construction the “twistor quotient”. If  $M$  has dimension  $4n$  and the complex dimension of the fiber of  $\text{Lie } \mathcal{G}$  is  $m$ , then  $M//\mathcal{G}$  has dimension  $4n - 4m$ .

What we need then are conditions which guarantee that  $Z_{\text{red}}$  has sections with correct normal bundle. First of all, if the action of  $\mathcal{G}$  is locally free at  $s$ , then the normal bundle of  $\bar{s}$  in  $Z_{\text{red}}$  is  $L^\perp/(L \cap L^\perp)$ , where  $L$  is the subbundle of the normal

bundle of  $s$  in  $Z$  generated by the Lie algebra of  $\mathcal{G}$  and the orthogonal is taken with respect to the twisted symplectic form. Let us make the following definition.

*Definition 3.1.* Let  $s$  be a twistor section of a twistor space  $Z \xrightarrow{p} \mathbb{C}P^1$  of a hyperkähler manifold  $M$  on which there is a locally free Hamiltonian action of a twistor group  $\mathcal{G}$ . We shall say that  $s$  is  $\mathcal{G}$ -admissible if  $L^\perp/(L \cap L^\perp)$  is the sum of line bundles of degree 1.

Thus, if  $s$  is  $\mathcal{G}$ -admissible and  $Z_{\text{red}}$  is a manifold in a neighbourhood of  $s$ , then  $Z_{\text{red}}$  is a twistor space of a pseudo-hyperkähler manifold. We now have:

**Proposition 3.2.** *Let  $\mathcal{H}$  be a twistor subgroup of a twistor group  $\mathcal{G}$  such that the quotient  $\text{Lie}(\mathcal{G})/\text{Lie}(\mathcal{H})$  is the sum of line bundles of degree 1. Suppose that we have a locally free Hamiltonian action of  $\mathcal{G}$  on a  $Z \xrightarrow{p} \mathbb{C}P^1$  with a moment map  $\mu$ . Then any  $\mathcal{G}$ -admissible twistor section  $s$  of  $Z$  such that  $\mu \circ s$  is  $\mathcal{G}$ -invariant, is  $\mathcal{H}$ -admissible.*

*Proof.* As above, let  $L$  denote the subbundle of the normal bundle of  $s$  generated by the action of  $\mathcal{G}$ . Since  $\mathcal{G}$  acts locally freely at  $s$ ,  $L \simeq \text{Lie}(\mathcal{G})$  as vector bundles. Furthermore, since  $\mu \circ s$  is  $\mathcal{G}$ -invariant,  $L \subset L^\perp$ . Let  $P$  denote the subbundle of  $L$  generated by  $\mathcal{H}$ . We have to show that  $P^\perp/P$  is the sum of line bundles of degree 1. We observe that it is enough to show that  $H^1((P^\perp/P)^*) = 0$ . Indeed, this implies that  $P^\perp/P$  is the sum of line bundles of degree at most 1, and since we also have the isomorphism  $P^\perp/P \simeq (P^\perp/P)^* \otimes \mathcal{O}(2)$  given by the  $\omega$ , all line bundle summands in  $P^\perp/P$  have degree 1.

To show that  $H^1((P^\perp/P)^*) = 0$  it is sufficient to show that the map  $H^0((P^\perp)^*) \rightarrow H^0(P^*)$  is surjective (as  $P^\perp$  is a subbundle of the normal bundle of  $s$  - sum of line bundles of degree 1 - therefore  $H^1((P^\perp)^*) = 0$ ). We have the following embeddings of vector bundles

$$P \hookrightarrow L \hookrightarrow L^\perp \hookrightarrow P^\perp.$$

We shall show that the dual of each of these maps is surjective on  $H^0$  by showing that  $H^1$  of each quotient vanishes. For the first one,  $H^1(L/P)^* = 0$  by our assumption. For the middle map this follows from the fact that  $L^\perp/L$  is the sum of  $\mathcal{O}(1)$ 's (by assumption,  $s$  is  $\mathcal{G}$ -admissible). For the last one, we have to show that  $H^1((P^\perp/L^\perp)^*) = 0$ . The form  $\omega$  and Serre duality show that this cohomology group is the same as  $H^0((P/L)^*)$  which again vanishes by our assumption.  $\square$

There are two cases, when a twistor section  $s$  is automatically  $\mathcal{G}$ -admissible: 1) if  $\mathcal{G}$  is trivial twistor group  $\underline{\mathcal{G}}$ ; and 2) if  $L \simeq \text{Lie}(\mathcal{G})$  is a Lagrangian subbundle of the normal bundle of  $s$ . This second condition holds, e.g., in the case of the generalized Legendre transform. We make several other remarks:

*Remark 3.3.* A necessary condition for  $\text{Lie}(\underline{\mathcal{G}})/\text{Lie}(\mathcal{H})$  to be the sum of  $\mathcal{O}(1)$ 's is that, as a vector bundle,  $\text{Lie}(\underline{\mathcal{H}}) = \sum \mathcal{O}(-p_i)$  with  $\sum p_i = d$ , where  $d$  is the fiber codimension of  $\mathcal{H}$  in  $\underline{\mathcal{G}}$ .

*Remark 3.4.* The sufficient condition of this proposition is particularly useful when dealing with abelian twistor groups. If both  $\underline{G}$  and  $\mathcal{H}$  are abelian (e.g. vector bundles), and the numerical condition of the previous remark is satisfied, then a generic embedding of  $\mathcal{H}$  into  $\underline{G}$  gives a twistor quotient. Thus, for example, if there is a locally free 3-Hamiltonian action of  $\mathbb{R}^3$  (effective or not) on a hyperkähler manifold  $M$  which extends to an action of  $\mathbb{C}^3$  with respect to each complex structure, then a generic embedding of  $\mathcal{O}(-2)$  (compatible with real structures) into  $\underline{\mathbb{C}^3}$  satisfies the condition of Proposition 3.2.

There also is a simple necessary condition in the setting of Proposition 3.2. Namely, since (in the notation of the proof of that proposition)  $L/P \hookrightarrow P^\perp/P$ , we need that  $\text{Lie}(\mathcal{G})/\text{Lie}(\mathcal{H})$  is the sum of line bundles of degree at most 1. Thus we shall not find twistor quotients by  $\mathcal{O}(-4)$  embedded into  $\underline{\mathbb{C}^3}$ .

Let us turn to examples.

*Example 3.5.* Santa Cruz [6] constructed twistor spaces of hyperkähler metrics on coadjoint orbits of complex semisimple Lie groups (see also [1]). He associates such a metric to any real section (spectral curve)  $s$  of  $\mathfrak{g} \otimes \mathcal{O}(2)$ , whose fibrewise stabilizers have constant dimension. Here  $\mathfrak{g}$  is the Lie algebra of a Lie group  $G$ . His construction can be interpreted as a twistor quotient of the hyperkähler metric on  $T^*G$  (cf. [3]) by the trivial twistor group  $\underline{G}$ , where the level set of the moment map is chosen to be  $s$ . In other words the resulting twistor space is  $\mu^{-1}(s)/\text{Stab}(s)$ , i.e. the fibrewise complex-symplectic quotient of the twistor space of  $T^*G$  by  $G$ .

*Example 3.6.* Many interesting metrics can be constructed as twistor quotients by the group  $\mathcal{N}$  defined in Example 1.7. Thus whenever we have an effective triholomorphic and isometric action of a compact Lie group  $G$  on a hyperkähler manifold  $M$  we can form a twistor quotient of  $M$  by  $\mathcal{N}$ . This is a reinterpretation of the construction given in [3].

In particular, the natural hyperkähler metric on the moduli space of  $SU(2)$ -monopoles of charge  $k$  can be obtained as such a quotient of  $T^*Gl(k, \mathbb{C})$ . Also the ALE spaces can be obtained as such quotients of coadjoint orbits with Kronheimer's metric [4].

It is possible to know the metric on the twistor quotient of  $M$ , if we know the metric on the deformations  $M_\lambda$  of Theorem 2.5. First of all, since a twistor group  $\mathcal{G}$ , by which we quotient, admits a chain of subgroups  $\mathcal{G} = \mathcal{G}_1 \subset \dots \subset \mathcal{G}_k$ , such that each subgroup is normal in the previous one and  $\mathcal{G}_i/\mathcal{G}_{i+1}$  is abelian for  $i < k-1$  and trivial for  $i = k-1$ , a twistor quotient by an arbitrary group reduces to twistor quotients by abelian twistor groups and to hyperkähler quotients. We shall, therefore, assume for the remainder of the section that  $\mathcal{G}$  is abelian. In this case the moment map  $\mu: Z \rightarrow \text{Lie}(\mathcal{G}) \otimes \mathcal{O}(2)'$  descends to  $Z/\mathcal{G}$ .

Let us choose local complex coordinates  $u_1, \dots, u_n, z_1, \dots, z_n$  in a fiber  $Z_\zeta$  of  $Z$ , so that  $u_1, \dots, u_k$  correspond to  $\mathcal{G}_\zeta$  and  $z_1, \dots, z_k$  give us the complex moment map for  $\mathcal{G}_\zeta$ . The remaining coordinates give complex coordinates on  $M//\mathcal{G}$

Since  $\mu$  descends to  $Z/\mathcal{G}$ , we have a corresponding map

$$\Phi : M//\mathcal{G} \rightarrow \Gamma(\text{Lie}(\mathcal{G}) \otimes O(2)).$$

$M//\mathcal{G}$  can be identified with  $\Phi^{-1}(v_0)$ , for some  $v_0 \in V = \Gamma(\text{Lie}(\mathcal{G}) \otimes O(2))$ . The coordinates on  $M/\mathcal{G}$  are  $u_{k+1}, \dots, u_n, z_1, \dots, z_n$  and the remaining (apart from  $z_1, \dots, z_k$ ) coordinates of  $V$ . Now, on each deformation  $\Lambda^{-1}(\lambda) \subset M/\mathcal{G}$  of Theorem 2.5 we have a Kähler potential

$$K_\lambda(u_1, \dots, u_n, z_1, \dots, z_n) =: K(u_{k+1}, \dots, u_n, z_{k+1}, \dots, z_n, v)$$

where  $v$  varies over  $V$ . The Kähler potential for  $M//\mathcal{G}$  is then

$$\bar{K}(u_{k+1}, \dots, u_n, z_{k+1}, \dots, z_n) := K(u_{k+1}, \dots, u_n, z_{k+1}, \dots, z_n, v_0).$$

#### 4. THE GENERALIZED LEGENDRE TRANSFORM

Lindström and Roček [14] found several constructions of hyperkähler metrics, in particular two based on the Legendre transform. The simpler one produces precisely hyperkähler metrics in  $4n$  dimensions with a local tri-Hamiltonian (hence isometric) action of  $\mathbb{R}^n$ . The second one, the generalized Legendre transform (GLT), produces metrics which generically don't have triholomorphic vector fields.

In the simplest case of 4-dimensional metrics, such a metric is associated to a real-valued function  $F$  on  $\mathbb{R}^{2k+1}$ ,  $k \geq 2$ , with coordinates  $w_0, \dots, w_{2k} \in \mathbb{C}$ ,  $w_{2k-i} = (-1)^{k+i}$  which satisfies the system of linear PDE's:

$$F_{w_i w_j} = F_{w_{i+s} w_{j-s}}$$

for all  $i, j, s$ . The hyperkähler metric lives on the submanifold of  $\mathbb{R}^{2k+1}$ , defined by the equations  $F_{w_i} = 0$ , for  $2 \leq i \leq 2k - 2$ . An example of a metric which can be constructed using the GLT is the Atiyah-Hitchin metric [11] or other  $SU(2)$ -monopole metrics [10].

In [11], Ivanov and Roček gave an interpretation of metrics constructed via GLT in terms of twistor spaces. They show that the twistor space  $Z$  of such a manifold  $M$  has a projection  $p$  onto  $O(2k)$  (or at least an open subset of it, invariant under the real structure). Moreover the kernel of  $dp$  is a Lagrangian subbundle of  $T^V Z$ . We can interpret this as saying that there is a local action of the twistor group  $O(-2k + 2)$  on  $Z$ . The projection onto  $O(2k)$  is the moment map, and  $O(2k)$  can be identified (if the fibers of  $p$  are connected) with  $Z/O(-2k + 2)$ . The vector space  $\mathbb{R}^{2k+1}$  is  $M/O(-2k + 2)$  and the equations  $F_{w_i} = 0$ , for  $2 \leq i \leq 2k - 2$ , which determine  $M$ , are equivalent to setting the  $\Lambda$  of (4) equal to zero.

A similar interpretation holds for higher dimensional hyperkähler metrics constructed via the generalized Legendre transform. This construction produces  $4n$ -dimensional metrics which admit a local Hamiltonian action of an  $n$ -dimensional abelian twistor group.

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## QUATERNIONIC CONTACT STRUCTURES

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This article is a survey on the notion of *quaternionic contact structures*, which I defined in [2]. Roughly speaking, quaternionic contact structures are quaternionic analogues of integrable CR structures.

### DEFINITION AND FIRST EXAMPLES

Let  $X$  be a manifold and  $V$  a distribution in  $X$ , so at each point  $x \in X$  we have a subspace  $V_x$  of  $T_x X$ . One can define a nilpotent Lie algebra structure on  $V_x \oplus (T_x X / V_x)$  by

$$[a, b] = \begin{cases} \pi_{T_x X / V_x}[a, b] & \text{if } a, b \in V_x, \\ 0 & \text{otherwise,} \end{cases}$$

where on the RHS we have the bracket of vector fields.

The Heisenberg algebra is defined as the vector space  $\mathbf{C}^m \oplus \mathbf{R}$  with a Lie bracket  $[\mathbf{C}^m, \mathbf{C}^m] \subset \mathbf{R}$  given by

$$\left[ \sum_1^m x_i e_i, \sum_1^m y_i e_i \right] = \text{Im} \sum_1^m \bar{x}_i y_i.$$

The same formula gives also the Lie bracket of the quaternionic Heisenberg algebra  $\mathbf{H}^m \oplus \text{Im}\mathbf{H}$ .

A contact structure on  $X^{2m+1}$  is a codimension 1 distribution  $V$  such that at each point  $x$  the nilpotent Lie algebra  $V_x \oplus T_x X / V_x$  is isomorphic to the Heisenberg algebra. Similarly, a quaternionic contact structure on  $X^{4m+3}$  is concerned with a codimension 3 distribution  $V$  such that at each point  $x$  the nilpotent Lie algebra  $V_x \oplus T_x X / V_x$  is isomorphic to the quaternionic Heisenberg algebra.

There is an equivalent, more concrete, description of such distributions: there exists a 1-form  $\eta = (\eta_1, \eta_2, \eta_3)$  with values in  $\mathbf{R}^3$ , such that  $V = \ker \eta$  and the three 2-forms

$(d\eta_i|_V)$  are the fundamental 2-forms of a quaternionic structure on  $V$ : this means that there exists a metric  $\gamma$  on  $V$  such that

$$(\dagger) \quad d\eta_i|_V = \gamma(I_i \cdot, \cdot),$$

and the  $I_i$  are complex structures on  $V$  satisfying the commutation relations of the quaternions  $I_1 I_2 I_3 = -1$ . The 1-form  $\eta$  is given only up to the action of  $SO_3$  on  $\mathbf{R}^3$  and to a conformal factor, thus we get a  $CSp_m Sp_1$ -structure on  $V$ .

**Definition 1.** A quaternionic contact structure on  $X^{4m+3}$  is the data of a codimension 3 distribution  $V$ , equipped with a  $CSp_m Sp_1$ -structure, such that the  $CSp_m Sp_1$ -structure and the contact form with values in  $\mathbf{R}^3$  satisfy the compatibility relation  $(\dagger)$ .

Let us point an important difference between contact structures and quaternionic contact structures: a quaternionic contact structure always define a (conformal) metric on the distribution, but a contact structure defines only a symplectic structure, and one needs to choose some compatible complex structure on the distribution (that is, a CR structure) in order to get a metric. That is why I consider quaternionic contact structures as a quaternionic analogue of CR structures.

The sphere  $S^{4m-1} \subset \mathbf{R}^{4m}$  has a canonical quaternionic contact structure, defined as follows: the flat manifold  $\mathbf{R}^{4m}$  is hyperkähler with three complex structures  $I_1, I_2, I_3$  satisfying  $I_1 I_2 I_3 = -1$ ; then on  $S^{4m-1}$  the contact form  $\eta$  with values in  $\mathbf{R}^3$  is

$$\eta_i = I_i dr,$$

where  $r$  is the radius in  $\mathbf{R}^{4m}$ ; the associated metric  $\gamma$  is the restriction to  $V = \ker \eta$  of the standard metric.

More generally, any 3-Sasakian manifold has a canonical quaternionic contact structure; as we shall see later, this is a very special case, since 3-Sasakian manifolds are rigid [3], but quaternionic contact structures come in infinite dimensional families.

## CONFORMAL INFINITIES OF EINSTEIN METRICS

Submanifolds of complex manifolds are integrable CR manifolds. The example of  $S^{4m-1} \subset \mathbf{R}^{4m}$  could suggest the same for quaternionic contact structures, but this is actually not true. A better interpretation of this example is to see  $S^{4m-1}$  as the boundary at infinity of the quaternionic hyperbolic space  $\mathbf{H}H^m$ . If we pick up a point  $*$  in  $\mathbf{H}H^m$ , we may identify  $\mathbf{H}H^m - \{*\} = \mathbf{R}_+^* \times S^{4m-1}$ , and the hyperbolic metric can be written as

$$g = dr^2 + \sinh^2(2r)\tilde{\gamma} + \sinh^2(4r)\eta^2,$$

where  $\tilde{\gamma}$  is the extension of  $\gamma$  to the whole  $TS^{4m-1}$  by 0 on the fibers of the map

$$\begin{array}{ccc} S^3 & \longrightarrow & S^{4m-1} \\ & & \downarrow \\ & & \mathbf{H}P^{m-1} \end{array}$$



At infinity, we recover  $\gamma$  as

$$\gamma = \lim_{r \rightarrow \infty} e^{-2r} g|_{\{r\} \times S_r};$$

note that this limit is infinite, except on  $V = \ker \eta$ ; so from this point of view, we should consider  $\gamma$  as a Carnot-Carathéodory metric, that is a metric which is infinite outside some distribution whose brackets generate the whole tangent space. Also, the change of base point  $\star$  induces a conformal change on the limit, so that only the conformal class  $[\gamma]$  of  $\gamma$  is defined: we call it the *conformal infinity* of  $g$ .

The first motivation for studying quaternionic contact structures is the following result on Einstein deformations of  $\mathbf{H}H^m$ .

**Theorem 2.** *If a quaternionic contact structure  $(V, \gamma)$  on  $S^{4m-1}$  is close enough to the standard one, then it is the conformal infinity of a complete Einstein metric  $g$ .*

More precisely,  $g$  has conformal infinity  $[\gamma]$  means that, near infinity, one has

$$g \sim dr^2 + e^{2r} \gamma + e^{4r} \eta^2.$$

Also, this theorem is a generalization of a theorem of Graham-Lee [7] on Einstein deformations of real hyperbolic space; for other rank one symmetric spaces, see [2] and the survey [1].

#### TWISTOR CONSTRUCTION FOR QUATERNIONIC CONTACT STRUCTURES

Recall that the twistor space of  $\mathbf{H}P^m$  is  $\mathbf{C}P^{2m+1}$ , with projection given in homogeneous coordinates by

$$[z_1 : \cdots : z_{2m+2}] \longrightarrow [z_1 + jz_2 : \cdots : z_{2m+1} + jz_{2m+2}].$$

One may realize  $\mathbf{H}H^m$  inside  $\mathbf{H}P^m$  as

$$\mathbf{H}H^m = \{[q_1 : \cdots : q_{m+1}], |q_1|^2 + \cdots + |q_m|^2 < |q_{m+1}|^2\}$$

and its twistor space  $T(\mathbf{H}H^m)$  is a domain in  $\mathbf{C}P^{2m+1}$ :

$$T(\mathbf{H}H^m) = \{[z_1 : \cdots : z_{2m+2}], |z_1|^2 + \cdots + |z_{2m}|^2 < |z_{2m+1}|^2 + |z_{2m+2}|^2\}.$$

Remark that the twistorial fibration restricts on the boundary to give

$$\begin{array}{ccc} \partial T(\mathbf{H}H^m) & \subset & \mathbf{C}P^{2m+1} \\ \downarrow & & \\ S^{4m-1} & \subset & \mathbf{H}P^m \end{array}$$

So we see that the sphere  $S^{4m-1}$  has some kind of twistor space which is a real hypersurface in  $\mathbf{C}P^{2m+1}$ . This generalizes to quaternionic contact structures in the following way.

**Theorem 3.** *If  $X^{4m-1}$  has a quaternionic contact structure, with  $4m - 1 > 7$ , then there is a twistor space  $T^{4m+1}$  with a projection*

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & T \\ & & \downarrow \pi \\ & & X \end{array}$$

such that

- (i)  $T$  has an integrable CR structure, and the fibers of  $\pi$  are holomorphic;
- (ii)  $T$  has a holomorphic contact structure, orthogonal to the fibers;
- (iii)  $T$  has a real structure compatible with the other structures.

One should precise what a holomorphic contact structure is for an integrable CR manifold  $M^{4m+1}$ : consider the bundle  $T'M = T_{\mathbf{C}}M/T^{0,1}M$ , which can be identified with  $T^{1,0}M \oplus \mathbf{C}R$  for some choice of a Reeb vector field  $R$ . When  $M$  is the boundary of a complex manifold,  $T'M$  is the restriction of the holomorphic tangent bundle to the boundary. In general, it has a canonical holomorphic structure defined by

$$X \in T^{0,1}M, \quad \sigma \in T'M, \quad \bar{\partial}_X \sigma = [X, \sigma];$$

this is well defined because of the integrability condition  $[T^{0,1}, T^{0,1}] \subset T^{0,1}$ . Now a holomorphic contact structure is a codimension 1 holomorphic distribution of  $T'M$ , given locally by a complex 1-form  $\eta^c$  such that  $d\eta^c$  is (complex) symplectic on the distribution.

This theorem generalizes a twistorial construction of LeBrun [11] for conformal 3-dimensional metrics. It is probable that, as in dimension 3, the converse of the theorem holds, that is a fibration by  $\mathbf{C}P^1$ 's satisfying the three conditions of the theorem is a twistor space of a quaternionic contact structure, but I have not completely checked this statement.

Let us now give an idea of the proof of theorem 3. In the construction of the twistor space of a conformal 3-dimensional metric, or in Salamon's construction [13] of the twistor space of a quaternionic-Kähler manifold, one uses the Levi-Civita connection to define a horizontal subspace for the fibration, and then the complex structure. In the case of a quaternionic contact structure, there is a priori no canonical connection; fortunately, the following theorem provides a connection, which is analogous to the Tanaka-Webster connection [15] in CR geometry.

**Theorem 4.** *If  $X^{4m-1}$  (for  $4m - 1 > 7$ ) has a quaternionic contact structure  $V$ , with a choice of metric  $\gamma$  on  $V$  in the conformal class, then there exists a unique connection  $\nabla$  on  $X$  and a unique supplementary subspace  $W$  of  $V$  in  $TX$ , such that*

- (i)  $\nabla$  preserves the decomposition  $V \oplus W$  and the metric;
- (ii) for  $A, B \in V$ , one has  $T_{A,B}^{\nabla} = -[A, B]_W$ ;
- (iii)  $\nabla$  preserves the  $Sp_{m-1}Sp_1$ -structure on  $V$ ;
- (iv) for  $R \in W$ , the endomorphism  $\cdot \rightarrow (T_{R,\cdot}^{\nabla})_V$  of  $V$  lies in the orthogonal of  $\mathfrak{sp}_1 \oplus \mathfrak{sp}_m$ ;

(v) the connection on  $W$  is induced by the natural identification of  $W$  with the subspace  $\mathfrak{sp}_1$  of the endomorphisms of  $V$ .

This theorem shows us that quaternionic contact structures, in dimension greater than 7, have some kind of integrability hidden in the definition, which enables to construct a natural connection preserving the  $Sp_{m-1}Sp_1$ -structure. This is why they are the quaternionic analogue of *integrable* CR structures. But in dimension 7 (see also below), some integrability condition probably remains to understand.

Given the connection of theorem 4, one can construct the CR structure on the twistor space in the usual way, but then integrability is a difficult task, because the connection has nonzero torsion, so one has to prove some complicated algebraic identities on torsion, in order to get theorem 3.

Now, we outline some steps for the proof of theorem 4. In particular, we want to explain what does not work in dimension 7. The main issue is to understand the derivation  $\nabla_A$  on  $V$  for  $A \in V$ ; this part of the connection and the supplementary subspace  $W$  are characterized by properties (i), (ii) and (iii): given some  $W$ , the properties (i) and (ii) define a unique connection on  $V$  along  $V$ ; property (iii) can be expressed saying that the fundamental 4-form  $\Omega = \sum_1^3 (d\eta_i|_V)^2$  is parallel; the method consists in decomposing  $\nabla\Omega$  in irreducible components under the action of  $Sp_{m-1}Sp_1$  (see the book [14]): some components vanish because  $\Omega$ , from its definition, is somehow closed, and the remaining obstructions correspond exactly to fixing a choice of  $W$ .

The proof does not work for dimension 7 because  $V$  is then 4-dimensional and condition (iii) becomes empty; instead of a condition on the connection, one rather needs some kind of selfduality condition on the curvature, but I have not done that.

One interesting point is that the supplementary subspace  $W$  can be described explicitly: choose 1-forms  $(\eta_1, \eta_2, \eta_3)$ , then  $W$  is generated by vector fields  $R_1, R_2, R_3$  such that

$$(\ddagger) \quad \eta_i(R_j) = \delta_{ij}, \quad (i_{R_i}d\eta_i)|_V = 0.$$

The space generated by such  $R_1, R_2, R_3$  a priori depends on the action of  $SO_3$  on the 1-forms; actually, it is fixed, and one has the stronger identity

$$(i_{R_i}d\eta_j + i_{R_j}d\eta_i)|_V = 0;$$

this property together with  $(\ddagger)$  is invariant under  $SO_3$ .

More generally, our quaternionic contact structures, after a choice of conformal factor, fit in the theory of Carnot-Carathéodory metrics with strong bracket generating hypothesis (see [8]), and this could lead to a slightly different proof of theorem 4: the supplementary subspace defined by  $(\ddagger)$  enables to define a metric on the whole tangent space; this metric becomes canonical after averaging over  $SO_3$ , and the orthogonal of  $V$  furnishes the canonical supplementary subspace  $W$ ; it remains to verify that the

connection satisfying (i) and (ii) also satisfies the integrability (iii). This approach could be interesting in order to understand the case of dimension 7.

### CONSTRUCTION OF QUATERNIONIC-KÄHLER METRICS

**Theorem 5.** *If  $X^{4m-1}$  (for  $4m - 1 > 7$ ) has a real analytic quaternionic contact structure, then it is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of  $X$ .*

Here are some comments on this theorem:

1. This theorem is the natural generalization to higher dimension of LeBrun's theorem [10] on filling of 3-dimensional real analytic conformal manifolds by selfdual Einstein metrics.
2. The theorem remains true in dimension 7, under the additional assumption of the existence of the twistor space of  $X$ . More generally, quaternionic contact structures in dimension 7 could be related to "symplectic quaternionic structures" in dimension 8, that is  $Sp_2Sp_1$ -structures such that the fundamental 4-form is closed.
3. LeBrun [9] has constructed an infinite dimensional family of complete quaternionic-Kähler deformations of  $\mathbf{H}H^m$ ; as we shall see below, these metrics have conformal infinities which are quaternionic contact structures, so the uniqueness statement implies that they coincide with the metric constructed by the theorem. Also, they provide an infinite dimensional family of examples of quaternionic contact structures.
4. Under the assumption of theorem 2, the quaternionic-Kähler metric usually does not coincide with the Einstein metric; instead, the quaternionic-Kähler metric gives a high order approximation of the Einstein metric at infinity. Obviously, an interesting problem is to understand which quaternionic contact structures can be filled by complete quaternionic-Kähler metrics (this problem is also unsolved for conformal metrics in dimension 3).

The basic construction in the proof of this theorem is the following. One has the twistor fibration

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & T \\ & & \downarrow \pi \\ & & X \end{array}$$

with the connection  $\nabla$ ; at least locally, one may complexify  $X$  as  $X^{\mathbf{C}}$  and extend the fibration  $T$  as a holomorphic fibration  $U$  on  $X^{\mathbf{C}}$ ,

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & U \\ & & \downarrow \pi \\ & & X^{\mathbf{C}} \end{array}$$

The complex dimension of  $U$  is  $4m$ ; the connection  $\nabla$  extends to a connection on  $U$  and the CR structure on  $T$  induces a complex endomorphism  $J$  with  $J^2 = -1$  on a  $(4m - 1)$ -dimensional distribution. Now one considers the distribution  $F$  which is a subspace of the horizontal distribution  $Hor^\nabla$  of  $\nabla$ :

$$F = T_J^{0,1} \cap Hor^\nabla.$$

One can see that  $F$  is actually a  $(2m - 1)$ -dimensional holomorphic integrable distribution, so there is a  $(2m + 1)$ -dimensional space of leaves  $N$ , which will be the twistor space of the manifold  $M$  that we want to construct.

In dimension 3, remark that  $U$  is the bundle of null directions in  $TX^{\mathbb{C}}$  and the leaves of the foliation are null geodesics, so one recovers LeBrun's construction.

The space  $U$  has two projections

$$\begin{array}{ccc} U & \xrightarrow{q} & N \\ p \downarrow & & \\ X^{\mathbb{C}} & & \end{array}$$

for  $x \in X^{\mathbb{C}}$ ,  $C_x = q(p^{-1}(x))$  is a holomorphic line in  $N$ , with normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^{2m}$ , so the space  $M^{\mathbb{C}}$  of deformations of these lines is  $4m$ -dimensional, when  $X^{\mathbb{C}}$  is only a  $(4m - 1)$ -dimensional submanifold; also,  $N$  has a holomorphic contact structure, and  $C_x$  is transverse to the contact distribution except for  $x \in X^{\mathbb{C}}$ ; this kind of situation is analyzed in a general context in the following proposition, from which the theorem follows.

**Proposition 6.** *Suppose that  $N$  is a  $(2m + 1)$ -dimensional complex manifold, with*

- (i) a holomorphic contact structure;*
- (ii) a family  $M^{\mathbb{C}}$  of holomorphic lines  $(C_m)_{m \in M}$  with normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^{2m}$ , such that  $C_m$  is transverse to the contact distribution except on a hypersurface  $S^{\mathbb{C}}$ ;*
- (iii) a real structure, compatible with the other structures;*

*then  $N$  is the twistor space of a quaternionic-Kähler metric on  $M - S$ , with conformal infinity a quaternionic contact structure on  $S$ ; the twistor space of this quaternionic contact structure is  $N|_S$ .*

#### EXPLICIT EXAMPLES

From theorem 5 and the quotient construction for quaternionic-Kähler metrics [4, 6], one may deduce that there is a quotient construction for groups acting on quaternionic contact structures. Actually, some known quotients of the quaternionic hyperbolic space by Hitchin and Galicki have a counterpart on their conformal infinities, as I shall now explain.

The isometry group of  $\mathbf{H}\mathbf{H}^2$  is  $Sp_{2,1}$ ; there is an action of  $\mathbf{R}$  on  $\mathbf{H}\mathbf{H}^2$  by

$$\begin{pmatrix} e^{ix} & & & \\ & \cosh(\ell x) & \sinh(\ell x) & \\ & \sinh(\ell x) & \cosh(\ell x) & \\ & & & \end{pmatrix} \in Sp_1 \times SO_{1,1} \subset Sp_{2,1}$$

and the quotient is Pedersen's selfdual Einstein metric [12] on the 4-ball, with conformal infinity the Berger sphere. This means that the quotient of  $S^7$  by this action is the 3-sphere with the Berger metric. Note that the action on  $S^7$  preserves the quaternionic contact structure, not the metric.

This example can be generalized to quotients of  $\mathbf{H}\mathbf{H}^m$  by subgroups of  $Sp_{m,1}$ , see [5]; at infinity, this gives quotients of  $S^{4m-1}$  by subgroups of  $Sp_{m,1}$ , some of which could probably be made explicit.

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A NEW CONSTRUCTION OF HOMOGENEOUS QUATERNIONIC  
MANIFOLDS AND RELATED GEOMETRIC STRUCTURES

VICENTE CORTÉS

ABSTRACT. Let  $V = \mathbb{R}^{p,q}$  be the pseudo-Euclidean vector space of signature  $(p, q)$ ,  $p \geq 3$  and  $W$  a module over the even Clifford algebra  $C\ell^0(V)$ . A homogeneous quaternionic manifold  $(M, Q)$  is constructed for any  $\mathfrak{spin}(V)$ -equivariant linear map  $\Pi : \wedge^2 W \rightarrow V$ . If the skew symmetric vector valued bilinear form  $\Pi$  is nondegenerate then  $(M, Q)$  is endowed with a canonical pseudo-Riemannian metric  $g$  such that  $(M, Q, g)$  is a homogeneous quaternionic pseudo-Kähler manifold. If the metric  $g$  is positive definite, i.e. a Riemannian metric, then the quaternionic Kähler manifold  $(M, Q, g)$  is shown to admit a simply transitive solvable group of automorphisms. In this special case ( $p = 3$ ) we recover all the known homogeneous quaternionic Kähler manifolds of negative scalar curvature (Alekseevsky spaces) in a unified and direct way. If  $p > 3$  then  $M$  does not admit any transitive action of a solvable Lie group and we obtain new families of quaternionic pseudo-Kähler manifolds. Then it is shown that for  $q = 0$  the noncompact quaternionic manifold  $(M, Q)$  can be endowed with a Riemannian metric  $h$  such that  $(M, Q, h)$  is a homogeneous quaternionic Hermitian manifold, which does not admit any transitive solvable group of isometries if  $p > 3$ .

The twistor bundle  $Z \rightarrow M$  and the canonical  $SO(3)$ -principal bundle  $S \rightarrow M$  associated to the quaternionic manifold  $(M, Q)$  are shown to be homogeneous under the automorphism group of the base. More specifically, the twistor space is a homogeneous complex manifold carrying an invariant holomorphic distribution  $\mathcal{D}$  of complex codimension one, which is a complex contact structure if and only if  $\Pi$  is nondegenerate. Moreover, an equivariant open holomorphic immersion  $Z \rightarrow \bar{Z}$  into a homogeneous complex manifold  $\bar{Z}$  of complex algebraic group is constructed.

Finally, the construction is shown to have a natural mirror in the category of supermanifolds. In fact, for any  $\mathfrak{spin}(V)$ -equivariant linear map  $\Pi : \vee^2 W \rightarrow V$  a homogeneous quaternionic supermanifold  $(M, Q)$  is constructed and, moreover, a homogeneous quaternionic pseudo-Kähler supermanifold  $(M, Q, g)$  if the symmetric vector valued bilinear form  $\Pi$  is nondegenerate.

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## INTRODUCTION

Let us start this introduction by recalling the notion of quaternionic manifold, see [A-M2]. A hypercomplex structure on a real vector space  $E$  consists of 3 complex structures  $(J_1, J_2, J_3)$  on  $E$  satisfying  $J_1 J_2 = J_3$ . It defines on  $E$  the structure of (left-) vector space over the quaternions  $\mathbb{H} = \{1, i, j, k\}$  such that multiplication by  $i$ ,  $j$  and  $k$  is given, respectively, by  $J_1$ ,  $J_2$  and  $J_3$ . The 3-dimensional subspace  $Q = \text{span}\{J_1, J_2, J_3\} \subset \text{End}(E)$  is what is called a quaternionic structure on  $E$ . A Euclidean scalar product  $\langle \cdot, \cdot \rangle$  on  $(E, Q)$  is called ( $Q$ -) Hermitian if  $Q$  consists of skew symmetric endomorphisms of  $(E, \langle \cdot, \cdot \rangle)$ . Now let  $M$  be a smooth manifold,  $\dim M > 4$ .



An almost quaternionic structure  $Q$  on  $M$  is a smooth field  $m \mapsto Q_m$  whose value at  $m \in M$  is a quaternionic structure on  $T_m M$ .  $Q$  is called a quaternionic structure and  $(M, Q)$  a **quaternionic manifold** if there exists a torsionfree connection on  $TM$  preserving the rank 3 subbundle  $Q \subset \text{End}(TM)$ . Now let  $g$  be a Riemannian metric on  $M$ , Hermitian with respect to an (almost) quaternionic structure  $Q$ . Then  $M$  with the structure  $(Q, g)$  is called an (almost) quaternionic Hermitian manifold. If, moreover, the Levi-Civita connection preserves  $Q$  then  $(M, Q, g)$  is said to be a **quaternionic Kähler manifold**. We remark that quaternionic Kähler manifolds represent one of the few basic Riemannian geometries, as defined by Berger's list of possible Riemannian holonomy groups, see [A1], [Bes], [Br1] and [S2]. For the possible holonomy groups of, not necessarily Riemannian, torsionfree connections see [Br2], [Schw] and references therein.

Next we review what is known about homogeneous quaternionic Kähler manifolds. First of all, quaternionic Kähler manifolds  $(M, Q, g)$  are Einstein manifolds, i.e.  $Ric = cg$ , see e.g. [A1], [Bes], [S2] and [Kak]. We discuss 3 cases depending on the sign of the constant  $c$  (which is the sign of the scalar curvature).

$c = 0$ ) Simply connected Ricci-flat quaternionic Kähler manifolds are hyper-Kähler manifolds, see [Bes] Ch. 14. They are Kähler manifolds with respect to 3 complex structures  $J_1, J_2$  and  $J_3 = J_1 J_2$ . It is a general fact that any homogeneous Ricci-flat Riemannian manifold is necessarily flat, see [A-K]. In particular, all homogeneous hyper-Kähler manifolds are flat. For hyper-Kähler manifolds of small cohomogeneity see [Bi1], [Bi2], [B-G], [D-S1], [D-S2], [K-S2], [Sw1], [Sw2] and [C4].

$c > 0$ ) It follows from Myer's theorem that any complete Einstein manifold of positive scalar curvature is necessarily compact. In particular, any complete quaternionic Kähler manifold of positive scalar curvature is compact. It was proven in [L-S] that for every  $n > 1$  there is only a finite number of such manifolds of dimension  $4n$  up to homothety, cf. [Bea], [G-S], [L1], [L4], [L5], [P-S]. The only known examples are, up to now, the Wolf spaces [W1]. These are precisely the homogeneous quaternionic Kähler manifolds of positive scalar curvature and are all symmetric of compact type, cf. [A2]. More generally, the Wolf spaces can be characterized as the compact quaternionic Kähler manifolds which admit an action of cohomogeneity  $\leq 1$  by a compact semisimple group of isometries and which are not scalar-flat, see [D-S3], cf. [A-P].

$c < 0$ ) Complete noncompact quaternionic Kähler manifolds of negative scalar curvature exist in abundance. In fact, it was proven in [L3] that the moduli space of complete quaternionic Kähler metrics on  $\mathbb{R}^{4n}$ ,  $n > 1$ , is infinite dimensional. For explicit constructions of, in general not complete, quaternionic Kähler manifolds see e.g. [G1], [G-L] and [L2].

What about homogeneous examples? First of all, the noncompact duals of the Wolf spaces are symmetric (and hence homogeneous) quaternionic Kähler manifolds of negative Ricci and nonpositive sectional curvature. Moreover, like any Riemannian symmetric space of noncompact type, these manifolds admit smooth quotients by discrete cocompact groups of isometries, see [Bo], [Fi-S]. The first examples of quaternionic

Kähler manifolds which are not locally symmetric were found by D.V. Alekseevsky in [A3]. An Alekseevsky space is a homogeneous quaternionic Kähler manifold of negative scalar curvature which admits a simply transitive splittable solvable group of isometries. It follows from Iwasawa's decomposition theorem that the noncompact duals of the Wolf spaces are precisely the symmetric Alekseevsky spaces. Besides these there are 3 series of nonsymmetric Alekseevsky spaces, see [A3], [dW-VP2] and [C2]. In [A3] it was conjectured that any noncompact homogeneous quaternionic Kähler manifold admits a transitive solvable group of isometries. This conjecture is still open: up to now, the only known examples of homogeneous quaternionic Kähler manifolds of negative scalar curvature are the Alekseevsky spaces. However, by the construction presented in this work we obtain many homogeneous quaternionic pseudo-Kähler manifolds (with indefinite metric) which do not admit any transitive action of a solvable Lie group. Moreover, in 2.6 we construct a family of noncompact homogeneous quaternionic Hermitian manifolds (with positive definite metric) with no transitive solvable group of isometries.

It is natural to ask for examples of compact locally homogeneous quaternionic Kähler manifolds. The following negative result was proven in [A-C4]. Let  $M$  be a compact quaternionic Kähler manifold or, more generally, a quaternionic Kähler manifold of finite volume. If the universal cover  $\tilde{M}$  is a homogeneous quaternionic Kähler manifold then it is necessarily symmetric. In particular, the only Alekseevsky spaces which admit smooth quotients of finite volume by discrete groups of isometries are the symmetric ones, this was as well proven in [A-C1] by a simpler method. Additionally, the symmetric Alekseevsky spaces can be characterized by the property of having nonpositive curvature, see [C2].

Given a simply transitive Lie group  $L$  of isometries acting on a Riemannian manifold  $M$ , there exists an algorithm to compute the full isometry group of  $M$  [A-W], cf. [Wo]. However, this algorithm involves the covariant derivatives (of all orders) of the curvature tensor and hence can only be applied effectively in very simple situations. If the simply transitive group  $L$  is splittable solvable and unimodular then the full isometry group is easily computed, see [G-W]. Unfortunately, the splittable solvable groups of isometries acting simply transitively on the Alekseevsky spaces are not unimodular. For that reason in [A-C1] a new algorithm was developed which completely avoids the curvature tensor and works also for nonunimodular splittable solvable groups. Using it the full isometry group of the nonsymmetric Alekseevsky spaces was determined [A-C1]. The Lie algebra of the full isometry group was previously described by the theoretical physicists de Wit, Vandarseyden and Van Proeyen by a different method, see [dW-V-VP].

In the following, we shortly comment on the attention paid to Alekseevsky's spaces in the physical literature. There is a concept of special geometry, which evolved in the theory of strings and supergravity, see e.g. [Z], [B-W], [dW-VP1], [G-S-T]. More specifically, special Kähler geometry is the geometry associated to  $N = 2$  supergravity in  $d = 4$  space time dimensions coupled to vector multiplets and was

first described in [dW-VP1], cf. [C-D-F], [St], [C3] and [Fr]. The (Kuranishi) moduli space of a Calabi-Yau 3-fold bears this particular geometry. Moreover, there is a construction (the “c-map”), related to Mirror Symmetry, which to any special Kähler manifold associates a quaternionic Kähler manifold, see [C-F-G] and [F-S]; for the general framework see [Cr]. Using the c-map, Cecotti [Ce] was the first to relate the classification problem for homogeneous special Kähler manifolds to Alekseevsky’s classification [A3]. In addition, he introduced Vinberg’s theory of  $T$ -algebras [V2] to describe the first nonsymmetric homogeneous special Kähler manifolds (the symmetric special Kähler manifolds were described in terms of Jordan algebras, see [C-VP] and [G-S-T]). Cecotti’s classification of homogeneous special Kähler manifolds was extended in [dW-VP2], [C3] and [A-C5]. In [C4], the hyper-Kählerian version of the c-map was used to construct a natural (pseudo-) hyper-Kähler structure on the bundle of intermediate Jacobians over the moduli space of gauged Calabi-Yau 3-folds.

In the last part of the introduction we describe the main components, results and the global structure of the present paper. The basic algebraic data of our construction are a pseudo-Euclidean vector space  $V$ , a module  $W$  over the even Clifford algebra  $\mathcal{C}\ell^0(V)$  and a  $\mathfrak{spin}(V)$ -equivariant linear map  $\Pi : \wedge^2 W \rightarrow V$ . Any such  $\Pi$  defines a  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{p} = \mathfrak{p}(\Pi) = \mathfrak{p}_0 + \mathfrak{p}_1$ , where  $\mathfrak{p}_0 = \text{Lie Isom}(V) = \mathfrak{o}(V) + V$  and  $\mathfrak{p}_1 = W$ . Here  $V$  acts trivially on  $W$  and  $\mathfrak{o}(V) \cong \mathfrak{spin}(V)$  acts via the inclusion  $\mathfrak{spin}(V) \subset \mathcal{C}\ell^0(V)$  on the  $\mathcal{C}\ell^0(V)$ -module  $W$ . The Lie bracket on  $\mathfrak{p}_1 \times \mathfrak{p}_1$  is given by  $\Pi$ . The Lie algebras  $\mathfrak{p}(\Pi)$  were introduced in [A-C2], where a basis for the vector space of  $\mathfrak{spin}(V)$ -equivariant linear maps  $\Pi : \wedge^2 W \rightarrow V$  was explicitly constructed. The Lie algebra  $\mathfrak{p}(\Pi)$  is called an extended Poincaré algebra of signature  $(p, q)$  if  $V \cong \mathbb{R}^{p,q}$  has signature  $(p, q)$ . A mirror symmetric version of this construction is obtained replacing  $\Pi : \wedge^2 W \rightarrow V$  by a  $\mathfrak{spin}(V)$ -equivariant linear map  $\Pi : \vee^2 W \rightarrow V$ . Here  $\vee^2 W = \text{Sym}^2 W$  denotes the symmetric square of  $W$ . The corresponding algebraic structure  $\mathfrak{p}(\Pi) = \mathfrak{p}_0 + \mathfrak{p}_1$  is now a super Lie algebra. It is called a superextended Poincaré algebra of signature  $(p, q)$ . For special signatures  $(p, q)$  of space time  $V$  these super Lie algebras play an important role in the physical literature since the early days of supersymmetry and supergravity, see e.g. [Go-L] and [O-S]; for more recent contributions see e.g. [F] and references therein. Notice that if  $(p, q) = (1, 3)$  then  $V = \mathbb{R}^{1,3}$  is Minkowski space and the even subalgebra  $\mathfrak{p}_0 = \text{Lie Isom}(\mathbb{R}^{1,3}) \subset \mathfrak{p}(\Pi)$  is the classical Poincaré algebra. The construction of all superextended Poincaré algebras (of arbitrary signature) was carried out in [A-C2]. In [A-C-D-S] the twistor equation is interpreted as the differential equation satisfied by infinitesimal automorphisms of a geometric structure modelled on the linear Lie supergroup associated to a superextended Poincaré algebra (for more on the twistor equation see e.g. [K-R] and references therein). In contrast with the case of superextended Poincaré algebras, to our knowledge, extended Poincaré algebras do not occur in the physical literature before the publication of [A-C2]: The first occurrence is [D-L].

Any extended Poincaré algebra  $\mathfrak{p}(\Pi)$ , as above, admits a derivation  $D$  with eigenspace decomposition  $\mathfrak{p}(\Pi) = \mathfrak{o}(V) + V + W$  and corresponding eigenvalues  $(0, 1, 1/2)$ . We

can extend  $\mathfrak{p}(\Pi)$  by  $D$  obtaining a new  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathfrak{g}_0 + \mathfrak{g}_1$ , where  $\mathfrak{g}_0 = \mathbb{R}D + \mathfrak{p}_0 = \mathbb{R}D + \mathfrak{o}(V) + V$  and  $\mathfrak{g}_1 = \mathfrak{p}_1 = W$ . The adjoint representation of the Lie algebra  $\mathfrak{g}$  is faithful and hence defines on  $\mathfrak{g}$  the structure of linear Lie algebra. Let  $G = G(\Pi) \subset \text{Aut } \mathfrak{g}$  denote the corresponding connected linear Lie group. It is the numerator of the homogeneous quaternionic manifolds  $M = M(\Pi) = G/K$  we are going to construct. To define the denominator  $K$  let  $E \subset V$  be a 3-dimensional Euclidean subspace and  $\mathfrak{k} = \mathfrak{k}(E) \subset \mathfrak{o}(V) \subset \mathfrak{g}$  the maximal subalgebra which preserves  $E$ , i.e.  $\mathfrak{k} = \mathfrak{o}(E) \oplus \mathfrak{o}(E^\perp)$ . Now we define  $K = K(E) \subset G$  to be the connected linear Lie group with Lie algebra  $\mathfrak{k}$ . Our main theorem is the following, see Thm. 8:

**Theorem A** *Let  $V$  be any pseudo-Euclidean vector space,  $E \subset V$  a Euclidean 3-dimensional subspace,  $W$  any  $\mathcal{C}^0(V)$ -module and  $\Pi : \wedge^2 W \rightarrow V$  any  $\mathfrak{spin}(V)$ -equivariant linear map. Let  $M = G/K$  be the homogeneous manifold associated to the Lie groups  $G = G(\Pi)$  and  $K = K(E) \subset G$  constructed above. Then the following is true:*

- 1) *There exists a  $G$ -invariant quaternionic structure  $Q$  on  $M$ .*
- 2) *If  $\Pi$  is nondegenerate (i.e. if  $W \ni s \mapsto \Pi(s \wedge \cdot) \in W^* \otimes V$  is injective) then there exists a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$  such that  $(M, Q, g)$  is a homogeneous quaternionic pseudo-Kähler manifold.*

We remark that for  $V = \mathbb{R}^{3,q}$  and  $W$  arbitrary the map  $\Pi$  can always be chosen such that the metric  $g$  in 2) becomes positive definite. In this special case we recover the 3 series of Alekseevsky spaces by a simple and unified construction, which completely avoids the technicalities of constructing complicated representations of Kählerian Lie algebras, see [A3] and [C2].

If in all constructions  $\Pi : \wedge^2 W \rightarrow V$  is replaced by a  $\mathfrak{spin}(V)$ -equivariant linear map  $\Pi : \vee^2 W \rightarrow V$  then we obtain the following analogue of Theorem A in the category of supermanifolds, see Thm. 17:

**Theorem B** *Let  $V$  be any pseudo-Euclidean vector space,  $E \subset V$  a Euclidean 3-dimensional subspace,  $W$  any  $\mathcal{C}^0(V)$ -module and  $\Pi : \vee^2 W \rightarrow V$  any  $\mathfrak{spin}(V)$ -equivariant linear map. Let  $M = G/K$  be the homogeneous supermanifold associated to the Lie supergroups  $G = G(\Pi)$  and  $K = K(E) \subset G$  constructed in 4. Then the following holds:*

- 1) *There exists a  $G$ -invariant quaternionic structure  $Q$  on  $M$ .*
- 2) *If  $\Pi$  is nondegenerate (i.e. if  $W \ni s \mapsto \Pi(s \vee \cdot) \in W^* \otimes V$  is injective) then there exists a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$  such that  $(M, Q, g)$  is a homogeneous quaternionic pseudo-Kähler supermanifold.*

*(For the definition of quaternionic structure, pseudo-Riemannian metric etc. on a supermanifold see the appendix.)*

Furthermore, we remark that replacing Euclidean 3-space  $E \cong \mathbb{R}^{3,0}$  by Lorentzian 3-space  $E \cong \mathbb{R}^{1,2}$  one obtains a para-quaternionic version of our construction.

Finally, we outline the structure of the paper: In section 1 we discuss extended Poincaré algebras. The basic definitions and formulas are given in 1.1. To our fundamental map  $\Pi : \wedge^2 W \rightarrow V$  and to an oriented Euclidean subspace  $E \subset V$ ,  $\dim E \equiv 3 \pmod{4}$ , we associate a canonical symmetric bilinear form  $b$  on  $W$  and study its properties in 1.2. Using the form  $b$ , in 1.3 we classify extended Poincaré algebras of signature  $(p, q)$ ,  $p \equiv 3 \pmod{4}$ , up to isomorphism.

In section 2 we construct the homogeneous quaternionic manifolds of Theorem A. The basic notions of quaternionic geometry are recalled in 2.2. First, see 2.1, we describe the structure of the Lie group  $G = G(\Pi)$  and the coset spaces  $M = G/K$ ,  $K = K(E)$ , where  $E \subset V$  is any pseudo-Euclidean subspace. The proof of Theorem A is given in 2.4. A crucial observation is that  $M = G/K$  contains the locally symmetric quaternionic pseudo-Kähler submanifold  $M_0 = G_0/K$ , where  $G_0 \subset G$  is the connected linear Lie group associated to  $\mathfrak{g}_0 \subset \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ . The first step consists in extending the  $G_0$ -invariant quaternionic and pseudo-Riemannian structures on  $M_0$  to  $G$ -invariant structures on  $M$ . Using the canonical symmetric bilinear form  $b$  on  $W$  introduced in 1.2 the pseudo-Riemannian metric is extended, in the nondegenerate case, to a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$ . The quaternionic structure is always extended by the beautifully simple formula (13) to a  $G$ -invariant almost quaternionic structure  $Q$  on  $M$ . To prove that  $Q$  is a quaternionic structure, we construct a  $G$ -invariant torsionfree connection  $\nabla$  on  $M$  which preserves  $Q$ . In the nondegenerate case  $\nabla$  is simply the Levi-Civita connection of the pseudo-Riemannian metric  $g$ . We use the description of invariant connections on homogeneous manifolds in terms of Nomizu maps, see 2.3.

Then we concentrate on the Riemannian case, see 2.5. The Riemannian manifolds  $M(\Pi)$  are classified up to isometry using results of 1.3. We show that all these manifolds admit a non-Abelian simply transitive splittable solvable group of isometries and hence are Alekseevsky spaces, see Thm. 9. Moreover, we explain how to obtain the 3 series of Alekseevsky spaces by specifying  $V$ ,  $W$ , and  $\Pi$ , see Thm. 10.

The quaternionic pseudo-Kähler manifolds  $(M(\Pi), Q, g)$  of Theorem A do not admit any transitive action of a solvable group if  $V = \mathbb{R}^{p,q}$  with  $p > 3$ , see Thm. 12. Moreover, if  $q = 0$ , then we can replace  $g$  by a  $G$ -invariant Riemannian and  $Q$ -Hermitian metric  $h$  such that  $(M(\Pi), h, g)$  are homogeneous quaternionic Hermitian manifolds with no transitive solvable group of isometries, see Thm. 11.

In section 3 we study the various bundles associated to the quaternionic manifolds  $M = M(\Pi) = G/K$ : The twistor bundle  $Z(M)$ , the canonical  $\mathrm{SO}(3)$ -principal bundle  $S(M)$  and the Swann bundle  $U(M)$ . We show that  $G$  acts transitively on  $Z(M)$  and  $S(M)$  and with cohomogeneity one on  $U(M)$ . In particular,  $Z = Z(M)$  is a homogeneous complex manifold of the group  $G = G(\Pi)$ . We exhibit a  $G$ -invariant holomorphic tangent hyperplane distribution  $\mathcal{D}$  on  $Z$  and prove that  $\mathcal{D}$  defines a complex contact structure on  $Z$  if and only if  $\Pi$  is nondegenerate. Moreover, we construct

an open  $G$ -equivariant holomorphic immersion  $Z \rightarrow \bar{Z}$  of  $Z$  into a homogeneous complex manifold of the complex algebraic group  $G^{\mathbb{C}} \subset \text{Aut}(\mathfrak{g}^{\mathbb{C}})$ , see Thm. 14. This immersion is a finite covering over an open  $G$ -orbit. In the nondegenerate case  $\bar{Z}$  is a homogeneous complex contact manifold of the group  $G^{\mathbb{C}}$  and the immersion is a morphism of complex contact manifolds.

In the final section 4 we extend our construction to the category of supermanifolds, proving Theorem B. We have aimed at a straightforward presentation, summarizing the needed supergeometric background in the appendix.

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## 1. EXTENDED POINCARÉ ALGEBRAS

**1.1. Basic facts.** Let  $V$  be a pseudo-Euclidean vector space with scalar product  $\langle \cdot, \cdot \rangle$ . There exists an orthonormal basis  $(e_i)$ ,  $i = 1, \dots, n = \dim V = p + q$ , of  $V$  such that  $\langle x, x \rangle = \sum_{i=1}^p (x^i)^2 - \sum_{j=p+1}^n (x^j)^2$  for all  $x = \sum_{i=1}^n x^i e_i \in V$ . Any such basis defines an isometry between  $V$  and the standard pseudo-Euclidean vector space  $\mathbb{R}^{p,q}$  of signature  $(p, q)$ . The isometry group of  $V$  is the semidirect product

$$\text{Isom}(V) = \text{O}(V) \ltimes V.$$

**Definition 1.** The Lie group  $\text{P}(V) := \text{Isom}(V)$  is called the **Poincaré group** of  $V$ . Its Lie algebra  $\mathfrak{p}(V) = \mathfrak{o}(V) + V$  is called the **Poincaré algebra** of  $V$ .

Next we recall some basic facts concerning the Clifford algebra  $\mathcal{C}\ell(V) = \mathcal{C}\ell^0(V) + \mathcal{C}\ell^1(V)$ , see [L-M]. Any unit vector  $x \in V$ ,  $\langle x, x \rangle = \pm 1$ , defines an invertible element  $x \in \mathcal{C}\ell(V)$ . The group  $\text{Pin}(V) \subset \mathcal{C}\ell(V)$  generated by all unit vectors is called the **pin group**. Its subgroup  $\text{Spin}(V) := \text{Pin}(V) \cap \mathcal{C}\ell^0(V)$  consisting of even elements is called the **spin group**. The **adjoint representation**

$$\text{Ad} : \text{Pin}(V) \longrightarrow \text{O}(V),$$

$$\text{Ad}(x)y = xyx^{-1} \in V, \quad x \in \text{Pin}(V), \quad y \in V,$$

induces a two-fold covering of the special orthogonal group:

$$\text{Spin}(V) \xrightarrow{2:1} \text{SO}(V).$$

In fact, if  $r_x \in O(V)$  denotes the reflection in the hyperplane  $x^\perp \subset V$  orthogonal to a unit vector  $x \in V$  then the following formula

$$(1) \quad \text{Ad}(xy) = r_x \circ r_y$$

holds for any two unit vectors  $x, y \in V$ . The groups  $\text{Pin}(V)$  and  $\text{Spin}(V)$  are Lie groups whose Lie algebra is the Lie subalgebra  $\mathfrak{spin}(V) \subset \mathcal{C}\ell^0(V)$  generated by the commutators  $[x, y] = xy - yx$  of all elements  $x, y \in V$ . It is canonically isomorphic to the orthogonal Lie algebra  $\mathfrak{o}(V)$  via the adjoint representation

$$(2) \quad \text{ad} = \text{Ad}_* : \mathfrak{spin}(V) \xrightarrow{\sim} \mathfrak{o}(V), \quad \text{ad}(x)y = [x, y], \quad x \in \mathfrak{spin}(V), \quad y \in V.$$

In fact, if we identify  $\wedge^2 V = \mathfrak{o}(V)$  by

$$(3) \quad (x \wedge y)(z) := \langle y, z \rangle x - \langle x, z \rangle y, \quad x, y, z \in V$$

then  $\text{ad}^{-1} : \wedge^2 V = \mathfrak{o}(V) \xrightarrow{\sim} \mathfrak{spin}(V)$  is given by the following equation:

$$(4) \quad \text{ad}^{-1}(x \wedge y) = -\frac{1}{4}[x, y], \quad x, y \in V.$$

In particular,  $\text{ad}(xy) = -2x \wedge y$  if  $x$  and  $y$  are orthogonal.

Any  $\mathcal{C}\ell(V)$ -module  $W$  can be decomposed into irreducible submodules. Depending on the signature  $(p, q)$  of  $V$ , there exist one or two irreducible  $\mathcal{C}\ell(V)$ -modules up to equivalence. In case there are two, they are related by the unique automorphism of  $\mathcal{C}\ell(V)$  which preserves  $V$  and acts on  $V$  as  $-\text{Id}$ . The restriction of an irreducible  $\mathcal{C}\ell(V)$ -module  $S$  to  $\mathcal{C}\ell^0(V)$  (respectively, to  $\text{Spin}(V)$  and  $\mathfrak{spin}(V)$ ) is, up to equivalence, independent of  $W$  and is called the **spinor module** of  $\mathcal{C}\ell^0(V)$  (respectively, of  $\text{Spin}(V)$  and  $\mathfrak{spin}(V)$ ). The spinor module  $S$  is either irreducible or  $S = S^+ \oplus S^-$  is the sum of two irreducible **semispinor modules**  $S^\pm$ , which may be equivalent or not, depending on the signature of  $V$ . If  $S^+$  and  $S^-$  are not equivalent, they are related by an automorphism of  $\mathcal{C}\ell^0(V)$  which preserves  $V$  and acts as an isometry on  $V$ . In the following we will freely use standard notations such as  $\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q})$ ,  $\text{Spin}(p, q) = \text{Spin}(\mathbb{R}^{p,q})$ ,  $\text{Spin}(p) = \text{Spin}(p, 0)$  etc., cf. [L-M].

Now let  $W$  be a module of the even Clifford algebra  $\mathcal{C}\ell^0(V)$  and  $\Pi : \wedge^2 W \rightarrow V$  a  $\mathfrak{spin}(V)$ -equivariant linear map. Given these data we extend the Lie bracket on  $\mathfrak{p}(V)$  to a Lie bracket  $[\cdot, \cdot]$  on the vector space  $\mathfrak{p}(V) + W$  by the following requirements:

- 1)  $W$  is a  $\mathfrak{p}(V)$ -submodule with trivial action of  $V$  and action of  $\mathfrak{o}(V)$  defined by  $\text{ad}^{-1} : \mathfrak{o}(V) \rightarrow \mathfrak{spin}(V) \subset \mathcal{C}\ell^0(V)$ , see equation (2),
- 2)  $[s, t] = \Pi(s \wedge t)$  for all  $s, t \in W$ .

The reader may observe that the Jacobi identity follows from 1) and 2). The resulting Lie algebra will be denoted by  $\mathfrak{p}(\Pi)$ . Note that  $\mathfrak{p} = \mathfrak{p}(\Pi)$  has a  $\mathbb{Z}_2$ -grading

$$\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1, \quad \mathfrak{p}_0 = \mathfrak{p}(V), \quad \mathfrak{p}_1 = W,$$

compatible with the Lie bracket, i.e.  $[\mathfrak{p}_a, \mathfrak{p}_b] \subset \mathfrak{p}_{a+b}$ ,  $a, b \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ . In other words,  $\mathfrak{p}(\Pi)$  is a  $\mathbb{Z}_2$ -graded Lie algebra.

**Definition 2.** Any  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  as above is called an **extended Poincaré algebra** (of signature  $(p, q)$ ) if  $V \cong \mathbb{R}^{p,q}$ .  $\mathfrak{p}(\Pi)$  is called **nondegenerate** if  $\Pi$  is nondegenerate, i.e. if the map  $W \ni s \mapsto \Pi(s \wedge \cdot) \in W^* \otimes V$  is injective.

The structure of extended Poincaré algebra on the vector space  $\mathfrak{p}(V) + W$  is completely determined by the  $\mathfrak{o}(V)$ -equivariant map  $\Pi : \wedge^2 W \rightarrow V$  ( $\mathfrak{o}(V)$  acts on  $W$  via  $\text{ad}^{-1} : \mathfrak{o}(V) \rightarrow \mathfrak{spin}(V) \subset \mathcal{C}\ell^0(V)$ ). The set of all  $\mathfrak{o}(V)$ -equivariant linear maps  $\wedge^2 W \rightarrow V$  is naturally a vector space. In fact, it is the subspace  $(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)}$  of  $\mathfrak{o}(V)$ -invariant elements of the vector space  $\wedge^2 W^* \otimes V$  of all linear maps  $\wedge^2 W \rightarrow V$ . In the classification [A-C2] an explicit basis for the vector space  $(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)}$  of extended Poincaré algebra structures on  $\mathfrak{p}(V) + W$  is constructed for all possible signatures  $(p, q)$  of  $V$  and any  $\mathcal{C}\ell^0(V)$ -module  $W$ .

**1.2. The canonical symmetric bilinear form  $b$ .** Let  $V = \mathbb{R}^{p,q}$  be the standard pseudo-Euclidean vector space with scalar product  $\langle \cdot, \cdot \rangle$  of signature  $(p, q)$ . From now on we fix a decomposition  $p = p' + p''$  and assume that  $p' \equiv 3 \pmod{4}$ , see Remark 2 below.

**Remark 1:** Notice that  $p$  and  $q$  are on equal footing, since any extended Poincaré algebra of signature  $(q, p)$  is isomorphic to an extended Poincaré algebra of signature  $(p, q)$ . In fact, the canonical antiisometry which maps the standard orthonormal basis of  $\mathbb{R}^{q,p}$  to that of  $\mathbb{R}^{p,q}$  induces an isomorphism of the corresponding Poincaré algebras which is trivially extended to an isomorphism of extended Poincaré algebras.

We denote by  $(e_i) = (e_1, \dots, e_{p'})$  the first  $p'$  basis vectors of the standard basis of  $V$  and by  $(e'_i) = (e'_1, \dots, e'_{p''+q})$  the remaining ones. The two complementary orthogonal subspaces of  $V$  spanned by these bases are denoted by  $E = \mathbb{R}^{p'} = \mathbb{R}^{p',0}$  and  $E' = E^\perp = \mathbb{R}^{p'',q}$  respectively. The vector spaces  $V$ ,  $E$  and  $E'$  are oriented by their standard orthonormal bases. E.g, the orientation of Euclidean  $p'$ -space  $E$  defined by the basis  $(e_i)$  is  $e_1^* \wedge \dots \wedge e_{p'}^* \in \wedge^{p'} E^*$ . Here  $(e_i^*)$  denotes the basis of  $E^*$  dual to  $(e_i)$ . Now let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be an extended Poincaré algebra of signature  $(p, q)$  and  $(\tilde{e}_i)$  any orthonormal basis of  $E$ . Then we define a  $\mathbb{R}$ -bilinear form  $b_{\Pi, (\tilde{e}_i)}$  on the  $\mathcal{C}\ell^0(V)$ -module  $W$  by:

$$(5) \quad b_{\Pi, (\tilde{e}_i)}(s, t) = \langle \tilde{e}_1, [\tilde{e}_2 \dots \tilde{e}_{p'} s, t] \rangle = \langle \tilde{e}_1, \Pi(\tilde{e}_2 \dots \tilde{e}_{p'} s \wedge t) \rangle, \quad s, t \in W.$$

We put  $b = b(\Pi) := b_{\Pi, (e_i)}$  for the standard basis  $(e_i)$  of  $E$ .

**Remark 2:** Equation (5) defines a skew symmetric bilinear form on  $W$  if  $p' \equiv 1 \pmod{4}$ . For even  $p'$  the above formula does not make sense, unless one assumes that  $W$  is a  $\mathcal{C}\ell(V)$ -module rather than a  $\mathcal{C}\ell^0(V)$ -module. Here we are only interested in the case  $p' \equiv 3 \pmod{4}$ . Moreover, later on, for the construction of homogeneous quaternionic manifolds we will put  $p' = 3$ .

**Theorem 1.** *The bilinear form  $b$  has the following properties:*



- 1)  $b_{\Pi,(\tilde{e}_i)} = \pm b$  if  $\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{p'} = \pm e_1 \wedge \cdots \wedge e_{p'}$ . In particular,  $b_{\Pi,(\tilde{e}_i)} = b$  for any positively oriented orthonormal basis  $(\tilde{e}_i)$  of  $E$ .
- 2)  $b$  is symmetric.
- 3)  $b$  is invariant under the maximal connected subgroup  $K(p', p'') = \text{Spin}(p') \cdot \text{Spin}_0(p'', q) \subset \text{Spin}(p, q)$  which preserves the orthogonal decomposition  $V = E + E'$  (and is not  $\text{Spin}_0(p, q)$ -invariant, unless  $p'' + q = 0$ ).
- 4) Under the identification  $\mathfrak{o}(V) = \Lambda^2 V = \Lambda^2 E + \Lambda^2 E' + E \wedge E'$ , see equation (3), the subspace  $E \wedge E'$  acts on  $W$  by  $b$ -symmetric endomorphisms and the subalgebra  $\Lambda^2 E \oplus \Lambda^2 E' \cong \mathfrak{o}(p') \oplus \mathfrak{o}(p'', q)$  acts on  $W$  by  $b$ -skew symmetric endomorphisms.

**Proof:** Obviously 4)  $\Rightarrow$  3). We show first that 3)  $\Rightarrow$  1). If  $(\tilde{e}_i)$  is a positively oriented orthonormal basis then there exists  $\varphi \in \text{Spin}(p')$  such that  $\text{Ad}(\varphi)e_i = \tilde{e}_i$ ,  $i = 1, \dots, p'$ . Now  $\Pi = [\cdot, \cdot] \wedge^2 W : \Lambda^2 W \rightarrow V$  is  $\mathfrak{spin}(V)$ -equivariant and hence equivariant under the connected group  $\text{Spin}_0(V) \subset \text{Spin}(V)$ . In particular,  $\Pi$  is  $\text{Spin}(p')$ -equivariant. Under the condition 3), this implies that

$$\begin{aligned}
 b(s, t) &= b(\varphi^{-1}s, \varphi^{-1}t) = \langle e_1, [e_2 \dots e_{p'} \varphi^{-1}s, \varphi^{-1}t] \rangle \\
 &= \langle e_1, [\varphi^{-1} \text{Ad}_\varphi e_2 \dots \text{Ad}_\varphi e_{p'} s, \varphi^{-1}t] \rangle \\
 &= \langle e_1, \text{Ad}(\varphi^{-1})[\tilde{e}_2 \dots \tilde{e}_{p'} s, t] \rangle = \langle \tilde{e}_1, [\tilde{e}_2 \dots \tilde{e}_{p'} s, t] \rangle \\
 &= b_{\Pi,(\tilde{e}_i)}(s, t), \quad s, t \in W.
 \end{aligned}$$

Here we have used the notation  $\text{Ad}_\varphi = \text{Ad}(\varphi)$ . The case of negatively oriented orthonormal basis  $(\tilde{e}_i)$  follows now from the Clifford relation  $\tilde{e}_i \tilde{e}_j = -\tilde{e}_j \tilde{e}_i$ ,  $i \neq j$ .

Next we prove 2) using first the  $\mathfrak{spin}(V)$ -equivariance of  $\Pi$  then equation (4) and eventually  $p' \equiv 3 \pmod{4}$ :

$$\begin{aligned}
 b(t, s) &= \langle e_1, [e_2 \dots e_{p'} t, s] \rangle \\
 &= -\langle e_1, [e_4 \dots e_{p'} t, e_2 e_3 s] \rangle + \langle e_1, \text{ad}(e_2 e_3)[e_4 \dots e_{p'} t, s] \rangle \\
 &= -\langle e_1, [e_4 \dots e_{p'} t, e_2 e_3 s] \rangle = \cdots = -\langle e_1, [t, e_2 \dots e_{p'} s] \rangle \\
 &= \langle e_1, [e_2 \dots e_{p'} s, t] \rangle = b(s, t).
 \end{aligned}$$

Finally we prove 4). By equation (4) we have to check that  $e_i e_j, e'_k e'_l \in \mathfrak{spin}(V)$  ( $i \neq j$  and  $k \neq l$ ) act by  $b$ -skew symmetric endomorphisms and  $e_i e'_k$  by a  $b$ -symmetric endomorphism on  $W$ . This is done in the next computation, in which we use again the equivariance of  $\Pi$  and equations (1) and (4) to express the adjoint representations

Ad and ad respectively:

$$\begin{aligned}
b(e_1 e_2 s, t) &= \langle e_1, [e_2 \dots e_{p'} e_1 e_2 s, t] \rangle = \langle e_1, [e_1 e_2 e_2 \dots e_{p'} s, e_1 e_2 e_1 e_2 t] \rangle \\
&= \langle e_1, \text{Ad}(e_1 e_2)[e_2 \dots e_{p'} s, e_1 e_2 t] \rangle = -\langle e_1, [e_2 \dots e_{p'} s, e_1 e_2 t] \rangle \\
&= -b(s, e_1 e_2 t), \\
b(e_2 e_3 s, t) &= \langle e_1, [e_2 \dots e_{p'} e_2 e_3 s, t] \rangle = \langle e_1, [e_2 e_3 e_2 \dots e_{p'} s, t] \rangle \\
&= \langle e_1, \text{ad}(e_2 e_3)[e_2 \dots e_{p'} s, t] \rangle - \langle e_1, [e_2 \dots e_{p'} s, e_2 e_3 t] \rangle \\
&= -b(s, e_2 e_3 t), \\
b(e'_k e'_l s, t) &= \langle e_1, [e_2 \dots e_{p'} e'_k e'_l s, t] \rangle = \langle e_1, [e'_k e'_l e_2 \dots e_{p'} s, t] \rangle \\
&= -\langle e_1, [e_2 \dots e_{p'} s, e'_k e'_l t] \rangle + \langle e_1, \text{ad}(e'_k e'_l)[e_2 \dots e_{p'} s, t] \rangle \\
&= -b(s, e'_k e'_l t) + 0 = -b(s, e'_k e'_l t).
\end{aligned}$$

This already proves 3) and hence 1); in particular we have:

$$b_{\Pi, (e_1, e_2, \dots, e_{p'})} = b_{\Pi, (e_2, \dots, e_{p'}, e_1)}.$$

Due to this symmetry, it is sufficient to check that  $e_2 e'_k$  acts as  $b$ -symmetric endomorphism on  $W$ :

$$\begin{aligned}
b(e_2 e'_k s, t) &= \langle e_1, [e_2 \dots e_{p'} e_2 e'_k s, t] \rangle = \langle e_1, [e_3 \dots e_{p'} e'_k s, t] \rangle \\
&\stackrel{(*)}{=} -\langle e_1, [s, e_3 \dots e_{p'} e'_k t] \rangle = -\langle e_1, [s, e_2 \dots e_{p'} e_2 e'_k t] \rangle \\
&= \langle e_1, [e_2 \dots e_{p'} e_2 e'_k t, s] \rangle = b(e_2 e'_k t, s) \\
&= b(s, e_2 e'_k t).
\end{aligned}$$

At (\*) we have used  $(p' - 1)/2$  times the  $\mathfrak{spin}(V)$ -equivariance of  $\Pi$  and the fact that  $(p' - 1)/2$  is odd if  $p' \equiv 3 \pmod{4}$ .  $\square$

**Definition 3.** The bilinear form  $b = b(\Pi) = b_{\Pi, (e_1, \dots, e_{p'})}$  defined above is called the **canonical symmetric bilinear form** on  $W$  associated to the  $\mathfrak{spin}(V)$ -equivariant map  $\Pi : \wedge^2 W \rightarrow V = \mathbb{R}^{p, q}$  and the decomposition  $p = p' + p''$ .

**Proposition 1.** The kernels of the linear maps  $\Pi : W \rightarrow W^* \otimes V$  and  $b = b(\Pi) : W \rightarrow W^*$  coincide:  $\ker \Pi = \ker b$ .

**Proof:** It follows from Thm. 1, 4) that  $W_0 := \ker b \subset W$  is  $\mathfrak{o}(V)$ -invariant. This implies that  $\Pi(W_0 \wedge W) \subset V$  is an  $\mathfrak{o}(V)$ -submodule. The definition of  $W_0$  implies that  $\Pi(W_0 \wedge W) \subset E' = E^\perp$  and hence by Schur's lemma  $\Pi(W_0 \wedge W) = 0$ . This proves that  $\ker b \subset \ker \Pi$ . On the other hand, we have the obvious inclusion  $\ker \Pi \subset \ker (b \circ e_2 \dots e_{p'}) = \ker b$ . Here the equation follows from the  $\text{Spin}(p')$ -invariance of  $b$ , see Thm. 1, 3).  $\square$

**Corollary 1.**  $\mathfrak{p}(\Pi)$  is nondegenerate (see Def. 2) if and only if  $b(\Pi)$  is nondegenerate.

**Theorem 2.** Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be any extended Poincaré algebra of signature  $(p, q)$ ,  $p = p' + p''$ ,  $p' \equiv 3 \pmod{4}$ , and  $b$  the canonical symmetric bilinear form

associated to these data. Then there exists a  $b$ -orthogonal decomposition  $W = \bigoplus_{i=0}^{l+m} W_i$  into  $C\ell^0(V)$ -submodules with the following properties

- 1)  $[W_0, W] = 0$ ,  $[W_i, W_j] = 0$  if  $i \neq j$  and  $[W_i, W_i] = V$  for all  $i = 1, 2, \dots, l$ .
- 2)  $W_0 = \ker b$  and  $W_i$  is  $b$ -nondegenerate for all  $i \geq 1$ .
- 3) For  $i = 1, \dots, l$  the  $C\ell^0(V)$ -submodule  $W_i$  is irreducible and for  $j = l+1, \dots, l+m$  the  $C\ell^0(V)$ -submodule  $W_j = X_j \oplus X'_j$  is the direct sum of two irreducible  $b$ -isotropic  $C\ell^0(V)$ -submodules.
- 4) The restriction of  $b$  to a bilinear form on any irreducible  $C\ell^0(V)$ -submodule of  $X = \bigoplus_{j=l+1}^{l+m} W_j$  vanishes.

**Proof:** By Prop. 1,  $W_0 := \ker b = \ker \Pi$  satisfies  $[W_0, W] = 0$ . As kernel of the  $\mathfrak{o}(V)$ -equivariant map  $\Pi$  the subspace  $W_0$  is  $\mathfrak{o}(V)$ -invariant and hence a  $C\ell^0(V)$ -submodule. We denote by  $W'$  a complementary  $C\ell^0(V)$ -submodule. Every such submodule is  $b$ -nondegenerate. Let  $W_i \subset W'$  be any irreducible  $C\ell^0(V)$ -submodule. By Thm. 1, 4),  $\ker(b|_{W_i \times W_i})$  is  $\mathfrak{o}(V)$ -invariant and hence a  $C\ell^0(V)$ -submodule. Now by Schur's lemma we conclude that either  $b|_{W_i \times W_i} = 0$  or  $b$  is nondegenerate on  $W_i$ . In particular, we can decompose  $W' = \bigoplus_{i=1}^l W_i \oplus X$  as direct  $b$ -orthogonal sum of  $b$ -nondegenerate  $C\ell^0(V)$ -submodules such that  $W_i$  is irreducible and the restriction of  $b$  to a bilinear form on any irreducible  $C\ell^0(V)$ -submodule of  $X$  vanishes. Let  $Y, Z \subset X$  be two such submodules,  $Y \neq Z$ . The bilinear form  $b$  induces a linear map  $Y \rightarrow Z^*$ . By Thm. 1, 4), the kernel of this map is  $\mathfrak{o}(V)$ -invariant and hence a  $C\ell^0(V)$ -submodule. Now Schur's lemma implies that either the kernel is  $Y$  and hence the restriction of  $b$  to a bilinear form on  $Y \oplus Z$  vanishes or the kernel is trivial and  $b$  is nondegenerate on  $Y \oplus Z$ . In the second case  $X$  splits as direct  $b$ -orthogonal sum:  $X = (Y \oplus Z) \oplus (Y \oplus Z)^\perp$ . This shows that  $X = \bigoplus_{i=l+1}^{l+m} W_i$  is the direct orthogonal sum of  $b$ -nondegenerate  $C\ell^0(V)$ -submodules  $W_i$  such that  $W_i = X_i \oplus X'_i$  is the direct sum of two  $b$ -isotropic irreducible submodules. This proves 2), 3) and 4). Now 1) is established applying Schur's lemma to the  $\mathfrak{o}(V)$ -equivariant map  $\Pi$ . In fact,  $b(W_i, W_j) = 0$  (respectively,  $b(W_i, W_i) \neq 0$ ) implies  $\Pi(W_i \wedge W_j) \subset E'$  (respectively,  $\Pi(\wedge^2 W_i) \neq 0$ ) and thus  $[W_i, W_j] = \Pi(W_i \wedge W_j) = 0$  (respectively,  $[W_i, W_j] = \Pi(\wedge^2 W_i) = V$ ).  $\square$

Next we will construct the subgroup  $\hat{K}(p', p'') \subset \text{Spin}(p, q)$  which consists of all elements preserving the orthogonal decomposition  $V = E + E'$  and the canonical symmetric bilinear form  $b$  on  $W$ . Its identity component is the group  $K(p', p'') \subset \text{Spin}_0(p, q)$  introduced above. We will see that if  $p'' = 0$  then  $\hat{K}(p', p'') = \hat{K}(p, 0)$  is a maximal compact subgroup of  $\text{Spin}(p, q)$ , which together with the element  $1 \in C\ell_{p,q}^0$  generates the even Clifford algebra  $C\ell_{p,q}^0$ . This property will be very useful in the next section.

We denote by  $x \mapsto x'$  the linear map  $\mathbb{R}^q = \mathbb{R}^{q,0} \rightarrow \mathbb{R}^{0,q}$  which maps the standard orthonormal basis  $(e_1, \dots, e_q)$  of  $\mathbb{R}^q$  to the standard orthonormal basis  $(e'_1, \dots, e'_q)$  of  $\mathbb{R}^{0,q}$ . It is an antiisometry:  $\langle x', x' \rangle = -\langle x, x \rangle$ . Let  $\omega_{p'} = e_1 \dots e_{p'}$  be the volume element of  $C\ell_{p'} = C\ell_{p',0}$ ,  $(e_1, \dots, e_{p'})$  the standard orthonormal basis of  $\mathbb{R}^{p'} = \mathbb{R}^{p',0} \subset$

$\mathbb{R}^{p,q}$ . Note that, since  $p'$  is odd, the volume element  $\omega_{p'}$  commutes with  $\mathbb{R}^{p'}$  and anticommutes with  $\mathbb{R}^{p'',q} \supset \mathbb{R}^{0,q}$ . Moreover, it satisfies  $\omega_{p'}^2 = 1$ , due to  $p' \equiv 3 \pmod{4}$ .

**Lemma 1.** *The map*

$$\mathbb{R}^q \ni x \mapsto \omega_{p'} x' \in \mathcal{C}\ell_{p,q}^0$$

*extends to an embedding  $\iota : \mathcal{C}\ell_q \hookrightarrow \mathcal{C}\ell_{p,q}^0$  of algebras, which restricts to an embedding  $\iota|_{\text{Pin}(q)} : \text{Pin}(q) \hookrightarrow \text{Spin}(p, q)$  of groups.*

**Proof:** It follows from  $(\omega_{p'} x')^2 = -\omega_{p'}^2 x'^2 = -x'^2 = \langle x', x' \rangle = -\langle x, x \rangle$  that the map  $x \mapsto \omega_{p'} x'$  extends to a homomorphism  $\iota$  of Clifford algebras. Recall that  $\mathcal{C}\ell_q$  is either simple or the sum of two simple ideals. In the first case, we can immediately conclude that  $\ker \iota$  is trivial and hence  $\iota$  an embedding. In the second case, the two simple ideals of  $\mathcal{C}\ell_q$  are  $\mathcal{C}\ell_q^\pm := (1 \pm \omega_q)\mathcal{C}\ell_q$ , where  $\omega_q = e_1 \dots e_q$  is the volume element of  $\mathcal{C}\ell_q$ . Now it is sufficient to check that  $\iota(1 \pm \omega_q) \neq 0$ . Using the fact that  $q$  is odd if  $\mathcal{C}\ell_q$  is not simple we compute

$$\iota(\omega_q) = \pm \omega_{p'}^q e'_1 e'_2 \dots e'_q = \pm e_1 e_2 \dots e_{p'} e'_1 e'_2 \dots e'_q.$$

This shows that  $\iota(1 \pm \omega_q) = 1 \pm \iota(\omega_q) \neq 0$ .  $\square$

We denote by  $\hat{K}(p', p'') \subset \text{Spin}(p, q)$  the subgroup generated by the subgroups  $\text{Spin}(p') \cdot \text{Spin}_0(p'', q) \subset \text{Spin}(p, q)$  and  $\iota(\text{Pin}(q)) \subset \text{Spin}(p, q)$ .

**Theorem 3.** *The group  $\hat{K}(p', p'')$  has the following properties:*

- 1)  $\hat{K}(p', p'') \subset \text{Spin}(p, q)$  consists of all elements preserving the orthogonal decomposition  $V = E + E'$  and the canonical symmetric bilinear form  $b$  on  $W$ . Its identity component is the group  $K(p', p'') = \text{Spin}(p') \cdot \text{Spin}_0(p'', q) = \hat{K}(p', p'') \cap \text{Spin}_0(p, q)$ .
- 2) The homogeneous space  $\text{Spin}(p, q) / \hat{K}(p', p'')$  is connected.
- 3)  $\hat{K}(p', p'')$  is compact if and only if  $q = 0$  or  $p'' = 0$  (and hence  $p = p'$ ). In the latter case  $\hat{K}(p', p'') = \hat{K}(p, 0) = \text{Spin}(p) \cdot \iota(\text{Pin}(q)) \subset \text{Spin}(p, q)$  is a maximal compact subgroup and  $\hat{K}(p, 0) \cong (\text{Spin}(p) \times \text{Pin}(q)) / \{\pm 1\}$ . Finally, in this case, the even Clifford algebra  $\mathcal{C}\ell_{p,q}^0$  is generated by 1 and  $\hat{K}(p, 0)$ .

**Proof:** The first part of 1) can be checked using Thm. 1, 4) and implies the second part of 1). To prove 2) it is sufficient to observe that  $\iota(\text{Pin}(q)) \cong \text{Pin}(q)$  has nontrivial intersection with all connected components of  $\text{Spin}(p, q)$  (due to our assumption  $p \geq 3$  there are two such components if  $q \neq 0$ ). The first part of 3) now follows simply from the fact that  $\text{Spin}_0(p'', q)$  is compact if and only if  $p'' = 0$  or  $q = 0$ . The compact group  $\hat{K}(p, 0) \subset \text{Spin}(p, q)$  is maximal compact, because it has the same number of connected components as  $\text{Spin}(p, q)$ , and from  $\text{Spin}(p) \cap \iota(\text{Pin}(q)) = \{\pm 1\}$  we obtain the isomorphism  $\hat{K}(p, 0) = \text{Spin}(p) \cdot \iota(\text{Pin}(q)) \cong (\text{Spin}(p) \times \text{Pin}(q)) / \{\pm 1\}$ . Finally, to prove the last statement, one easily checks that  $\hat{K}(p, 0)$  contains all quadratic monomials  $xy$  in unit vectors  $x, y \in \mathbb{R}^{p,0} \cup \mathbb{R}^{0,q}$ .  $\square$

**Corollary 2.** *The correspondence  $\Pi \mapsto b(\Pi)$  defines an injective linear map  $(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)} \rightarrow (\vee^2 W^*)^{\hat{K}(p', p'')}$ .*

**Proof:** The existence of the map follows from Thm. 3. We prove the injectivity. From  $b(\Pi) = 0$  it follows that  $\Pi(\wedge^2 W) \subset E'$  and hence by Schur's lemma  $\Pi = 0$ .  $\square$

**1.3. The set of isomorphism classes of extended Poincaré algebras.** Starting from the decomposition proven in Thm. 2 we will derive the classification of extended Poincaré algebras of signature  $(p, q)$ ,  $p \equiv 3 \pmod{4}$ , up to isomorphism. It will turn out that the space of isomorphism classes is naturally parametrized by a finite number of integers. We fix the decomposition  $p = p' + p''$ ,  $p' = p$ ,  $p'' = 0$ , and for any extended Poincaré algebra  $\mathfrak{p}(\Pi)$  of signature  $(p, q)$  as above we consider the canonical symmetric bilinear form  $b = b_{\Pi, (e_1, \dots, e_p)}$ .

**Theorem 4.** *Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be any extended Poincaré algebra of signature  $(p, q)$ ,  $p \equiv 3 \pmod{4}$ . Then there exists a  $b$ -orthogonal decomposition  $W = \bigoplus_{i=0}^l W_i$  into  $\mathcal{C}l^0(V)$ -submodules with the following properties*

- 1)  $[W_0, W] = 0$ ,  $[W_i, W_j] = 0$  if  $i \neq j$  and  $[W_i, W_i] = V$  for all  $i = 1, 2, \dots, l$ .
- 2)  $W_0 = \ker b$  and  $W_i$  is  $b$ -nondegenerate for all  $i \geq 1$ .
- 3)  $W_i$ ,  $i \geq 1$ , is an irreducible  $\mathcal{C}l^0(V)$ -submodule on which  $b$  is (positive or negative) definite.

**Proof:** Let  $W = \bigoplus_{i=1}^{l+m} W_i$  be a decomposition as in Thm. 2. It only remains to prove that  $b$  is definite on  $W_i$  for  $i = 1, \dots, l$  and that  $X = \bigoplus_{i=l+1}^{l+m} W_i = 0$ . This follows from Lemma 3 and Lemma 4 below.  $\square$

**Lemma 2.** *The restriction of an irreducible  $\mathcal{C}l_{p,q}^0$ -module  $\Sigma$  to a module of the maximal compact subgroup  $\hat{K} = \hat{K}(p, 0) = \text{Spin}(p) \cdot \iota(\text{Pin}(q)) \subset \text{Spin}(p, q)$  is irreducible. Here  $\iota : \text{Pin}(p) \hookrightarrow \text{Spin}(p, q)$  is the embedding of Lemma 1. Moreover,  $\Sigma$  is irreducible as module of the connected group  $K = K(p, 0) = \text{Spin}(p) \cdot \iota(\text{Spin}(q)) = \text{Spin}(p) \cdot \text{Spin}(q)$  if and only if  $n = p + q \equiv 2, 4, 5$  or  $6 \pmod{8}$ . If  $n \equiv 0, 1, 3$  or  $7 \pmod{8}$  then  $\Sigma$  is the sum of two irreducible  $K$ -submodules.*

**Proof:** Recall that  $\mathcal{C}l_{p,q} = \mathcal{C}l_p \hat{\otimes} \mathcal{C}l_{0,q}$  is (identified with) the  $\mathbb{Z}_2$ -graded tensor product of the Clifford algebras  $\mathcal{C}l_p = \mathcal{C}l_{p,0}$  and  $\mathcal{C}l_{0,q}$ . It is easily checked, using the classification of Clifford algebras and their modules, see [L-M], that any irreducible  $\mathcal{C}l_{p,q}^0$ -module  $\Sigma$  is irreducible as module of the subalgebra  $\mathcal{C}l_p^0 \otimes \mathcal{C}l_{0,q}^0 \subset \mathcal{C}l_{p,q}^0 = \mathcal{C}l_p^0 \otimes \mathcal{C}l_{0,q}^0 + \mathcal{C}l_p^1 \otimes \mathcal{C}l_{0,q}^1$  if  $n \equiv 2, 4, 5$  or  $6 \pmod{8}$  and is the sum of two irreducible submodules if  $n \equiv 0, 1, 3$  or  $7 \pmod{8}$ . Now Lemma 2 follows from the fact that  $\mathcal{C}l_p^0 \otimes \mathcal{C}l_{0,q}^0$  (respectively,  $\mathcal{C}l_{p,q}^0$ ) is the subalgebra of  $\mathcal{C}l_{p,q}$  generated by 1 and  $K = \text{Spin}(p) \cdot \text{Spin}(0, q)$  (respectively, by 1 and  $\hat{K} = \text{Spin}(p) \cdot \iota(\text{Pin}(q))$ ).  $\square$

**Lemma 3.** *A  $\mathcal{C}l_{p,q}^0$ -module  $W$  is irreducible if and only if it is irreducible as module of the maximal compact subgroup  $\hat{K} \subset \text{Spin}(p, q)$ . In this case  $(\vee^2 W^*)^{\hat{K}}$  is one-dimensional and is spanned by a positive definite scalar product on  $W$ . Let  $W$  be*

an irreducible  $\mathcal{C}\ell_{p,q}^0$ -module and  $\Pi : \wedge^2 W \rightarrow V$  be any  $\mathfrak{o}(V)$ -equivariant linear map. Then either  $\Pi = 0$  or  $b(\Pi)$  is a definite  $\hat{K}$ -invariant symmetric bilinear form.

**Proof:** The first statement follows from Lemma 2. Since  $\hat{K}$  is compact there exists a positive definite  $\hat{K}$ -invariant symmetric bilinear form on  $W$ . From the irreducibility of  $W$  we conclude by Schur's lemma that  $(\vee^2 W^*)^{\hat{K}}$  is spanned by this form. Now the last statement is an immediate consequence of Cor. 2.  $\square$

**Lemma 4.** *Let  $W$  be a  $\mathcal{C}\ell^0(V)$ -module and  $\Pi : \wedge^2 W \rightarrow V$  an  $\mathfrak{o}(V)$ -equivariant linear map such that  $b = b(\Pi)$  is nondegenerate. Suppose that  $W = \Sigma \oplus \Sigma'$  is the direct sum of two irreducible submodules  $\Sigma$  and  $\Sigma'$ . Then there exists a  $b$ -orthogonal decomposition  $W = \Sigma_1 \oplus \Sigma_2$  into two  $b$ -nondegenerate (and hence  $b$ -definite by Lemma 3) irreducible submodules  $\Sigma_1$  and  $\Sigma_2$ .*

**Proof:** It is sufficient to show that  $W$  contains a  $b$ -nondegenerate irreducible  $\mathcal{C}\ell^0(V)$ -submodule  $\Sigma_1$ . Then  $\Sigma_2 := \Sigma_1^\perp$  is a  $b$ -nondegenerate  $\hat{K}$ -submodule. It is also a  $\mathcal{C}\ell^0(V)$ -submodule, because the algebra  $\mathcal{C}\ell^0(V)$  is generated by 1 and  $\hat{K}$ , and it is irreducible since the  $\mathcal{C}\ell^0(V)$ -module  $W$  is the direct sum of only two irreducible submodules. If  $W$  does not contain any  $b$ -nondegenerate irreducible  $\mathcal{C}\ell^0(V)$ -submodule  $\Sigma_1$  then, by Schur's lemma, the restriction of  $b$  to a bilinear form on any irreducible  $\mathcal{C}\ell^0(V)$ -submodule vanishes. In the following we derive a contradiction from this assumption. Since the bilinear form  $b$  is nondegenerate it defines a nondegenerate pairing between the  $b$ -isotropic subspaces  $\Sigma$  and  $\Sigma'$ . Due to the  $\hat{K}$ -invariance of  $b$  (Thm. 3)  $b : \Sigma' \xrightarrow{\sim} \Sigma^*$  is a  $\hat{K}$ -equivariant isomorphism. On the other hand  $\Sigma^* \cong \Sigma$  as irreducible modules of the compact group  $\hat{K}$ . This shows that  $\Sigma$  and  $\Sigma'$  are equivalent as  $\hat{K}$ -modules and thus as  $\mathcal{C}\ell^0(V)$ -modules, because the algebra  $\mathcal{C}\ell^0(V)$  is generated by 1 and  $\hat{K}$ . Hence there exists a  $\mathcal{C}\ell^0(V)$ -equivariant isomorphism  $\varphi : \Sigma \xrightarrow{\sim} \Sigma'$ . We define two  $\hat{K}$ -invariant bilinear forms  $\beta_\pm$  on  $\Sigma$  by:

$$\beta_\pm(s, t) := b(\varphi(s), t) \pm b(\varphi(t), s), \quad s, t \in \Sigma.$$

$\beta_+$  is symmetric and  $\beta_-$  is skew symmetric. If  $\beta_+ \neq 0$  then it is a definite  $\hat{K}$ -invariant scalar product on  $\Sigma$ , since  $\Sigma$  is an irreducible module of the compact group  $\hat{K}$ . So for  $s \in \Sigma - \{0\}$  we obtain

$$0 \neq \beta_+(s, s) = b(\varphi(s), s) + b(s, \varphi(s)) = b(s + \varphi(s), s + \varphi(s)).$$

This implies that the irreducible  $\mathcal{C}\ell^0(V)$ -submodule  $\Sigma_\varphi := \{s + \varphi(s) | s \in \Sigma\} \subset W$  is  $b$ -definite, which contradicts our assumption. We conclude that  $\beta_+ = 0$  and hence  $\beta_- = 2b \circ \varphi$  is a  $\hat{K}$ -invariant symplectic form on  $\Sigma$ . Let  $\beta$  be a  $\hat{K}$ -invariant positive definite scalar product on  $\Sigma$  (such scalar products exist since  $\hat{K}$  is compact). We define a  $\hat{K}$ -equivariant isomorphism  $\chi : \Sigma \rightarrow \Sigma$  by the equation

$$\beta(s, t) = \beta_-(\chi(s), t), \quad s, t \in \Sigma.$$

Then  $\psi := \varphi \circ \chi : \Sigma \rightarrow \Sigma'$  is a  $\hat{K}$ -equivariant and hence  $\mathcal{C}^\ell(V)$ -equivariant isomorphism. Using  $\beta_+ = 0$ , we compute for  $s \in \Sigma - \{0\}$ :

$$\begin{aligned} b(s + \psi(s), s + \psi(s)) &= b(\psi(s), s) + b(s, \psi(s)) = b(\psi(s), s) - b(\varphi(s), \chi(s)) \\ &= \beta_-(\chi(s), s) = \beta(s, s) \neq 0. \end{aligned}$$

As above, this implies that  $W$  contains a  $b$ -definite irreducible  $\mathcal{C}^\ell(V)$ -submodule  $\Sigma_\psi \cong \Sigma$  contradicting our assumption.  $\square$

**Theorem 5.** [A-C2] *Let  $V = \mathbb{R}^{p,q}$ ,  $p \equiv 3 \pmod{4}$ . If  $W$  is an irreducible  $\mathcal{C}^\ell(V)$ -module then*

$$\dim(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)} = 1.$$

**Proof:** Cor. 2 and Lemma. 3 show that  $\dim(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)} \leq 1$ . On the other hand, there exists a nontrivial  $\mathfrak{o}(V)$ -equivariant linear map  $\wedge^2 W \rightarrow V$ , see [A-C2], and hence  $\dim(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)} \geq 1$ . This proves that  $\dim(\wedge^2 W^* \otimes V)^{\mathfrak{o}(V)} = 1$ .  $\square$

Next, in order to parametrize the isomorphism classes of extended Poincaré algebras  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  we associate a certain number of nonnegative integers to  $\mathfrak{p}(\Pi)$ . Let us first consider the case when there is only one irreducible  $\mathcal{C}^\ell(V)$ -module  $\Sigma$  up to equivalence. Then  $W$  is necessarily isotypical and, due to Thm. 4, there exists a  $b$ -orthogonal decomposition  $W = W_0 \oplus \bigoplus_{i=1}^l W_i$  with  $W_0 = \ker b = \ker \Pi \cong l_0 \Sigma$  and irreducible  $\mathcal{C}^\ell(V)$ -submodules  $W_i \cong \Sigma$  for  $i = 1, \dots, l$  on which  $b$  is definite. We denote by  $l_+$  (repectively  $l_-$ ) the number of summands  $W_i$  on which  $b$  is positive (respectively, negative) definite. Note that the triple  $(l_0, l_+, l_-)$  does not depend on the choice of decomposition.

**Theorem 6.** *Let  $p \equiv 3 \pmod{4}$  and  $q \not\equiv 3 \pmod{4}$ . Then the isomorphism class  $[\mathfrak{p}(\Pi)]$  of an extended Poincaré algebra  $\mathfrak{p}(\Pi)$  of signature  $(p, q)$  is completely determined by the triple  $(l_0, l_+, l_-)$  introduced above. We put  $\mathfrak{p}(p, q, l_0, l_+, l_-) := [\mathfrak{p}(\Pi)]$ . Then  $\mathfrak{p}(p, q, l_0, l_+, l_-) = \mathfrak{p}(p, q, l'_0, l'_+, l'_-)$  if and only if  $l_0 = l'_0$  and  $\{l_+, l_-\} = \{l'_+, l'_-\}$ .*

**Proof:** If  $p \equiv 3 \pmod{4}$  then there is only one irreducible  $\mathcal{C}^{\ell_{p,q}}$ -module  $\Sigma$  up to equivalence if and only if  $q \not\equiv 3 \pmod{4}$ . Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  and  $\mathfrak{p}(\Pi') = \mathfrak{p}(V) + W'$  be two extended Poincaré algebras of signature  $(p, q)$  with the same integers  $l_0 = l_0(\Pi) = l_0(\Pi')$ ,  $l_+ = l_+(\Pi) = l_+(\Pi')$  and  $l_- = l_-(\Pi) = l_-(\Pi')$ . Then the modules  $W$  and  $W'$  are equivalent and we can assume that  $W = W' = W_0 \oplus \bigoplus_{i=1}^{l_+ + l_-} W_i$  is a decomposition as above. In particular, it is  $b(\Pi)$ - and  $b(\Pi')$ -orthogonal,  $b(\Pi)$  and  $b(\Pi')$  are both positive definite or both negative definite on  $W_i$  for  $i \geq 1$ ,  $\Pi(W_0 \wedge W) = \Pi'(W_0 \wedge W) = 0$ ,  $\Pi(W_i \wedge W_j) = \Pi'(W_i \wedge W_j) = 0$  if  $i \neq j$  and  $\Pi(\wedge^2 W_i) = \Pi'(\wedge^2 W_i) = V$  if  $i \geq 1$ . So the maps  $\Pi$  and  $\Pi'$  are completely determined by their restrictions  $\Pi_i := \Pi|_{\wedge^2 W_i} \neq 0$  and  $\Pi'_i := \Pi'|_{\wedge^2 W_i} \neq 0$  ( $i \geq 1$ ) respectively. By Thm. 5  $\Pi'_i = \lambda_i \Pi_i$  ( $i \geq 1$ ) for some constant  $\lambda_i \in \mathbb{R}^*$ . Now  $b = b(\Pi)$  and  $b' = b(\Pi')$  are both positive definite or both negative definite on  $W_i$  and hence  $\lambda_i = \mu_i^2 > 0$ . Now we can define an isomorphism  $\varphi : \mathfrak{p}(\Pi) \rightarrow \mathfrak{p}(\Pi')$  by  $\varphi|_{\mathfrak{p}(V) + W_0} = \text{Id}$  and  $\varphi|_{W_i} = \mu_i \text{Id}$ . This

shows that the integers  $(l_0, l_+, l_-)$  determine the extended Poincaré algebra  $\mathfrak{p}(\Pi)$  of signature  $(p, q)$  up to isomorphism. The  $\mathbb{Z}_2$ -graded Lie algebras  $\mathfrak{p}(\Pi)$  and  $\mathfrak{p}(-\Pi)$  are isomorphic via  $\alpha : \mathfrak{p}(\Pi) \rightarrow \mathfrak{p}(-\Pi)$  defined by:  $\alpha|_{\mathfrak{o}(V)} + W = \text{Id}$  and  $\alpha|_V = -\text{Id}$ . This proves  $\mathfrak{p}(p, q, l_0, l_+, l_-) = \mathfrak{p}(p, q, l_0, l_-, l_+)$ . It remains to show that  $\mathfrak{p}(p, q, l_0, l_+, l_-) = \mathfrak{p}(p, q, l'_0, l'_+, l'_-)$  implies  $l_0 = l'_0$  and  $\{l_+, l_-\} = \{l'_+, l'_-\}$ . Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W \in \mathfrak{p}(p, q, l_0, l_+, l_-)$  and  $\mathfrak{p}(\Pi') = \mathfrak{p}(V) + W' \in \mathfrak{p}(p, q, l'_0, l'_+, l'_-)$  be representative extended Poincaré algebras and  $W = W_0 \oplus \bigoplus_{i=1}^l W_i$  ( $l = l_+ + l_-$ ) a decomposition as above. We assume that there exists an isomorphism  $\varphi : \mathfrak{p}(\Pi) \rightarrow \mathfrak{p}(\Pi')$  of  $\mathbb{Z}_2$ -graded Lie algebras, i.e.  $\varphi\mathfrak{p}(V) = \mathfrak{p}(V)$  and  $\varphi W = W'$ . The automorphism  $\varphi|_{\mathfrak{p}(V)}$  preserves the radical  $V$  and maps the Levi subalgebra  $\mathfrak{o}(V)$  to an other Levi subalgebra of  $\mathfrak{p}(V)$ . Now by Malcev's theorem any two Levi subalgebras are conjugated by an inner automorphism, see [O-V]. So, using an inner automorphism of  $\mathfrak{p}(\Pi)$ , we can assume that  $\varphi\mathfrak{o}(V) = \mathfrak{o}(V)$ . The subalgebra  $\varphi(\mathfrak{o}(p) \oplus \mathfrak{o}(q)) \subset \mathfrak{o}(V)$  is maximal compact (i.e. the Lie algebra of a maximal compact subgroup of  $O(V) = O(p, q)$ ) and hence conjugated by an inner automorphism to the maximal compact subalgebra  $\mathfrak{o}(p) \oplus \mathfrak{o}(q) \subset \mathfrak{o}(V)$ . So, again, we can assume that  $\varphi$  preserves  $\mathfrak{o}(p) \oplus \mathfrak{o}(q)$ . Moreover, since  $p \neq q$  any automorphism of  $\mathfrak{o}(p) \oplus \mathfrak{o}(q)$  is inner and we can assume that  $\varphi|_{\mathfrak{o}(p) \oplus \mathfrak{o}(q)} = \text{Id}$ . From the fact that  $\varphi$  is an automorphism of  $\mathfrak{p}(V)$  we obtain that  $\phi := \varphi|_V \in \text{GL}(V)$  normalizes  $\mathfrak{o}(V)$  and  $\mathfrak{o}(p) \oplus \mathfrak{o}(q)$ . This implies  $\phi \in O(p) \times O(q)$ , in particular,  $\varphi\mathbb{R}^p = \mathbb{R}^p$  and  $\varphi\mathbb{R}^{0,q} = \mathbb{R}^{0,q}$ . Using an inner automorphism of  $\mathfrak{p}(\Pi)$  we can further assume that  $\varphi|_{\mathbb{R}^p} = \epsilon(\varphi)\text{Id}$ ,  $\epsilon(\varphi) \in \{+1, -1\}$ , and hence  $\varphi|_{\mathfrak{o}(p)} = \text{Ad}_\phi|_{\mathfrak{o}(p)} = \text{Id}$ . This implies that  $\varphi|_W$  is  $\text{Spin}(p)$ -equivariant. Now we can compute

$$\begin{aligned} b'(\varphi s, \varphi t) &= \langle e_1, [e_2 \dots e_p \varphi s, \varphi t] \rangle = \langle e_1, [\varphi e_2 \dots e_p s, \varphi t] \rangle \\ &= \langle e_1, \varphi [e_2 \dots e_p s, t] \rangle = \epsilon(\varphi) \langle e_1, [e_2 \dots e_p s, t] \rangle \\ &= \epsilon(\varphi) b(s, t), \quad s, t \in W. \end{aligned}$$

Put  $W'_i := \varphi W_i$ . Then, since  $\varphi^* b' = \epsilon(\varphi) b$ , we obtain a  $b'$ -orthogonal decomposition  $W' = W'_0 \oplus \bigoplus_{i=1}^{l'} W'_i$  ( $l' = l'_+ + l'_-$ ) as above;  $W'_0 = \ker \Pi' \cong l'_0 \Sigma$ ,  $W'_i \cong \Sigma$  ( $i \geq 1$ ) etc. This shows that  $l'_0 = l_0$ ,  $l'_\pm = l_\pm$  if  $\epsilon(\varphi) = +1$  and  $l'_\pm = l_\mp$  if  $\epsilon(\varphi) = -1$ .  $\square$

Now we discuss the complementary case  $p \equiv q \equiv 3 \pmod{4}$ . In this case, the spinor module  $S$  of  $\mathcal{C}\ell^0(V)$  is the sum  $S^+ \oplus S^-$  of two irreducible inequivalent semispinor modules  $S^+$  and  $S^-$  and any irreducible  $\mathcal{C}\ell^0(V)$ -module is equivalent to  $S^+$  or  $S^-$ . As  $\text{Spin}_0(V)$ -modules,  $S^+$  and  $S^-$  are dual:  $S^- \cong (S^+)^*$ . Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be an extended Poincaré algebra of signature  $(p, q)$  as above. Thanks to Thm. 4 there exists a  $b$ -orthogonal decomposition  $W = W_0 \oplus \bigoplus_{i=1}^l W_i$  as above with the following  $b$ -orthogonal refinements:  $W_0 = W_0^+ \oplus W_0^-$ ,  $W_0^\pm \cong l_0^\pm S^\pm$ , and  $\bigoplus_{i=1}^l W_i = \bigoplus_{i=1}^{l^+} W_i^+ \oplus \bigoplus_{i=1}^{l^-} W_i^-$ ,  $W_i^\pm \cong S^\pm$ . We denote by  $l_+^\epsilon$  (respectively,  $l_-^\epsilon$ ) the number of submodules  $W_i^\epsilon$ ,  $i = 1, \dots, l^\epsilon$ , on which  $b$  is positive (respectively, negative) definite,  $\epsilon \in \{+, -\}$ . So to  $\mathfrak{p}(\Pi)$  we have associated the nonnegative integers  $(l_0^+, l_0^-, l_+^+, l_+^-, l_-^+, l_-^-)$ .

**Theorem 7.** *Let  $p \equiv q \equiv 3 \pmod{4}$ . Then the isomorphism class  $[\mathfrak{p}(\Pi)]$  of an extended Poincaré algebra  $\mathfrak{p}(\Pi)$  of signature  $(p, q)$  is completely determined by the*



tuple  $(l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-)$  introduced above. We put  $\mathfrak{p}(p, q, l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-) := [\mathfrak{p}(\Pi)]$ . Then  $\mathfrak{p}(p, q, l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-) = \mathfrak{p}(p, q, \tilde{l}_0^+, \tilde{l}_+^+, \tilde{l}_-^+, \tilde{l}_0^-, \tilde{l}_+^-, \tilde{l}_-^-)$  if and only if  $(\tilde{l}_0^+, \tilde{l}_+^+, \tilde{l}_-^+, \tilde{l}_0^-, \tilde{l}_+^-, \tilde{l}_-^-) \in \Gamma(l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-)$ , where  $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  is the group generated by the following two involutions:

$$(l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-) \mapsto (l_0^+, l_-^+, l_+^+, l_0^-, l_-^-, l_+^-)$$

and

$$(l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-) \mapsto (l_0^-, l_+^-, l_-^-, l_0^+, l_+^+, l_-^+).$$

**Proof:** The proof uses again Thm. 4 and Thm. 5 and is similar to that of Thm. 6. Therefore, we explain only the reason for the appearance of the second involution. In terms of the standard basis  $(e_1, \dots, e_p, e'_1, \dots, e'_q)$  of  $V = \mathbb{R}^{p,q}$  we define an isometry  $\phi \in \text{SO}(p) \times \text{O}(q)$  by:  $\phi e_1 := e_1$ ,  $\phi e_i := -e_i$  ( $i \geq 2$ ) and  $\phi e'_j = -e'_j$  ( $j \geq 1$ ). Then  $\text{Ad}_\phi \in \text{Aut } \mathfrak{p}(V)$  induces an (outer) automorphism of  $\mathfrak{o}(V)$  interchanging the two semispinor modules. Let  $(\mathfrak{p}(V) + W, [\cdot, \cdot]) \in \mathfrak{p}(p, q, l_0^+, l_+^+, l_-^+, l_0^-, l_+^-, l_-^-)$  be an extended Poincaré algebra of signature  $(p, q)$ . Then we define a new extended Poincaré algebra  $(\mathfrak{p}(V) + W, [\cdot, \cdot]') \in \mathfrak{p}(p, q, l_0^-, l_+^-, l_-^-, l_0^+, l_+^+, l_-^+)$  by:

$$[\cdot, \cdot]' := [\cdot, \cdot] \quad \text{on} \quad \wedge^2 \mathfrak{p}(V) \oplus \wedge^2 W \oplus V \wedge W$$

and

$$[A, s]' := [\text{Ad}_\phi(A), s] \quad \text{for} \quad A \in \mathfrak{o}(V), s \in W.$$

The two  $\mathbb{Z}_2$ -graded Lie algebras are isomorphic via  $\varphi : (\mathfrak{p}(V) + W, [\cdot, \cdot]) \xrightarrow{\sim} (\mathfrak{p}(V) + W, [\cdot, \cdot]')$  defined by:  $\varphi|_{\mathfrak{p}(V)} = \text{Ad}_\phi$  and  $\varphi|_W = \text{Id}$ .  $\square$

## 2. THE HOMOGENEOUS QUATERNIONIC MANIFOLD $(M, Q)$ ASSOCIATED TO AN EXTENDED POINCARÉ ALGEBRA

### 2.1. Homogeneous manifolds associated to extended Poincaré algebras.

Any extended Poincaré algebra  $\mathfrak{p} = \mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  has an even derivation  $D$  with eigenspace decomposition  $\mathfrak{p} = \mathfrak{o}(V) + V + W$  and corresponding eigenvalues  $(0, 1, 1/2)$ . Therefore, the  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 = \mathfrak{p}(V) + W$  is canonically extended to a  $\mathbb{Z}_2$ -graded Lie algebra  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathbb{R}D + \mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$ , where  $\mathfrak{g}_0 = \mathbb{R}D + \mathfrak{p}_0 = \mathbb{R}D + \mathfrak{p}(V) =: \mathfrak{g}(V)$  and  $\mathfrak{g}_1 = \mathfrak{p}_1 = W$ . The next proposition describes the basic structure of the Lie algebra  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathfrak{g}(V) + W$ . To avoid trivial exceptions, in the following we assume that  $n = \dim V > 2$  and hence that  $\mathfrak{o}(V)$  is semisimple (for the construction of homogeneous quaternionic manifolds we will put  $V = \mathbb{R}^{p,q}$  and  $p \geq 3$ ).

**Proposition 2.** *The Lie algebra  $\mathfrak{g} = \mathfrak{g}(\Pi)$  has the Levi decomposition*

$$(6) \quad \mathfrak{g} = \mathfrak{o}(V) + \mathfrak{r}$$

into the radical  $\mathfrak{r} = \mathbb{R}D + V + W$  and the complementary (maximal semisimple) Levi subalgebra  $\mathfrak{o}(V)$ . The nilradical  $\mathfrak{n} = [\mathfrak{r}, \mathfrak{r}] = V + W$  is two-step nilpotent if  $\Pi \neq 0$  and Abelian otherwise.

For any Lie algebra  $\mathfrak{l}$  we denote by  $\text{der}(\mathfrak{l})$  the Lie algebra of its derivations.

**Proposition 3.** *The adjoint representation  $\mathfrak{g} \rightarrow \text{der}(\mathfrak{r})$  of  $\mathfrak{g} = \mathfrak{o}(V) + \mathfrak{r}$  on its ideal  $\mathfrak{r} = \mathbb{R}D + V + W$  is faithful.*

**Proof:** Let  $x \in \mathfrak{g}$ . We show that  $[x, \mathfrak{r}] = 0$  implies  $x = 0$ . First  $[x, D] = 0$  implies  $x \in \mathfrak{co}(V) = \mathbb{R}D + \mathfrak{o}(V)$ . Then  $[x, V] = 0$  implies  $x = 0$ , because the conformal Lie algebra  $\mathfrak{co}(V)$  acts faithfully on  $V$ .  $\square$

By Prop. 3 we can consider  $\mathfrak{g}(V)$  and  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathfrak{g}(V) + W$  as linear Lie algebras via the embedding  $\mathfrak{g}(V) \subset \mathfrak{g} \hookrightarrow \text{der}(\mathfrak{r}) \subset \mathfrak{gl}(\mathfrak{r})$ . The corresponding connected Lie groups of  $\text{Aut}(\mathfrak{r}) \subset \text{GL}(\mathfrak{r})$  will be denoted by  $G(V)$  and  $G = G(\Pi)$  respectively:  $\text{Lie}G(V) = \mathfrak{g}(V) \subset \text{Lie}G = \mathfrak{g}$ . Now let  $V = \mathbb{R}^{p,q}$  and fix a decomposition  $p = p' + p''$ . The subalgebra of  $\mathfrak{o}(V)$  preserving the corresponding orthogonal splitting  $V = E + E' = \mathbb{R}^{p',0} + \mathbb{R}^{p'',q}$  is  $\mathfrak{k} = \mathfrak{k}(p', p'') = \mathfrak{o}(p') \oplus \mathfrak{o}(p'', q) \subset \mathfrak{o}(p, q) = \mathfrak{o}(V)$ . We consider  $\mathfrak{k} \subset \mathfrak{o}(V) \subset \mathfrak{g}$  as a subalgebra of the linear Lie algebra  $\mathfrak{g} \hookrightarrow \text{der}(\mathfrak{r})$  and denote by  $K = K(p', p'') \subset G(V) \subset G \subset \text{Aut}(\mathfrak{r})$  the corresponding connected linear Lie group.  $K$  is a closed Lie subgroup, see Cor. 3 below. We are interested in the homogeneous spaces

$$M(V) := G(V)/K \subset M = M(\Pi) := G/K = G(\Pi)/K.$$

**Proposition 4.** *The Lie subalgebras  $\mathfrak{k}$ ,  $\mathfrak{n}$ ,  $\mathfrak{r}$ ,  $\mathfrak{o}(V)$ ,  $\mathfrak{p}(V)$ ,  $\mathfrak{p}$ ,  $\mathfrak{g}(V)$ ,  $\mathfrak{g} \subset \text{der}(\mathfrak{r})$  are algebraic subalgebras of the (real) algebraic Lie algebra  $\text{der}(\mathfrak{r})$ .*

**Proof:** We use the following sufficient conditions for algebraicity, see [O-V] Ch. 3 §3 8°:

- a) A linear Lie algebra coinciding with its derived algebra is algebraic.
- b) The radical of an algebraic linear Lie algebra is algebraic.
- c) A linear Lie algebra generated by algebraic subalgebras is algebraic.

The subalgebras  $\mathfrak{o}(V) \subset \mathfrak{p}(V) \subset \mathfrak{p} \subset \text{der}(\mathfrak{r})$  are algebraic by a). By c), to prove that  $\mathfrak{g} = \mathbb{R}D + \mathfrak{p}$  and  $\mathfrak{g}(V)$  are algebraic it is now sufficient to show that  $\mathbb{R}D \subset \text{der}(\mathfrak{r})$  is algebraic.  $D$  preserves  $\mathfrak{n} = V + W$  and acts trivially on the complement  $\mathbb{R}D + \mathfrak{o}(V)$ . The Lie algebra  $\mathbb{R}D \hookrightarrow \text{der}(\mathfrak{n})$  is the Lie algebra of the algebraic group

$$\{\lambda \text{Id}_V \oplus \mu \text{Id}_W \mid \lambda = \mu^2 \neq 0\} \subset \text{GL}(V \oplus W).$$

This shows that  $\mathbb{R}D$ ,  $\mathfrak{g}(V)$ ,  $\mathfrak{g} \subset \text{der}(\mathfrak{r})$  are algebraic. Now  $\mathfrak{n}$ , the radical of  $\mathfrak{p}$ , and  $\mathfrak{r}$ , the radical of  $\mathfrak{g}$ , are algebraic by b). Finally,  $\mathfrak{k} \subset \mathfrak{o}(V)$  is the subalgebra which preserves the orthogonal splitting  $V = E + E'$  and is hence algebraic.  $\square$

**Corollary 3.** *The connected linear Lie groups  $K$ ,  $S \cong \text{Spin}_0(V)$ ,  $R = \exp \mathfrak{r}$ ,  $G(V)$ ,  $G \subset \text{Aut}(\mathfrak{r})$  with Lie algebras  $\mathfrak{k}$ ,  $\mathfrak{s} := \mathfrak{o}(V)$ ,  $\mathfrak{r}$ ,  $\mathfrak{g}(V)$ ,  $\mathfrak{g} \subset \text{der}(\mathfrak{r})$  are closed.*

**Proof:** This follows from Prop. 4 and the fact that the identity component of a real algebraic linear group is a closed linear group.  $\square$

**Proposition 5.** *The Lie group  $G(\Pi)$  has the following Levi decomposition:*

$$(7) \quad G(\Pi) = S \ltimes R.$$

**Proof:** This follows from the corresponding Levi decomposition (6) of Lie algebras since  $S \cap R = \{e\}$ .  $\square$

Next we show that  $M(V)$  can be naturally endowed with a  $G(V)$ -invariant structure of pseudo-Riemannian locally symmetric space. With this in mind, we consider the pseudo-Euclidean vector space  $\tilde{V} = \mathbb{R}^{p+1, q+1}$  with scalar product  $\langle \cdot, \cdot \rangle$  and orthogonal decomposition  $\tilde{V} = \mathbb{R}e_0 + V + \mathbb{R}e'_0$ ,  $\langle e_0, e_0 \rangle = -\langle e'_0, e'_0 \rangle = 1$ . Recall that  $\mathfrak{o}(\tilde{V})$  is identified with  $\wedge^2 \tilde{V}$  via the pseudo-Euclidean scalar product  $\langle \cdot, \cdot \rangle$ , see (3).

**Proposition 6.** *The subspace*

$$\mathbb{R}e_0 \wedge e'_0 + \wedge^2 V + (e_0 - e'_0) \wedge V \subset \wedge^2 \tilde{V} = \mathfrak{o}(\tilde{V})$$

*is a subalgebra isomorphic to  $\mathfrak{g}(V)$ .*

**Proof:** The canonical embedding  $\mathfrak{o}(V) = \wedge^2 V \hookrightarrow \wedge^2 \tilde{V} = \mathfrak{o}(\tilde{V})$  is extended to an embedding  $\mathfrak{g}(V) \hookrightarrow \mathfrak{o}(\tilde{V})$  via

$$D \mapsto e_0 \wedge e'_0, \quad V \ni v \mapsto (e_0 - e'_0) \wedge v. \square$$

It is easy to see that the embedding  $\mathfrak{g}(V) \hookrightarrow \mathfrak{o}(\tilde{V})$  lifts to a homomorphism of connected Lie groups  $G(V) \rightarrow \text{SO}_0(\tilde{V})$  with finite kernel. In particular, we have a natural isometric action of  $G(V)$  on the pseudo-Riemannian symmetric space  $\tilde{M}(V) := \text{SO}_0(p+1, q+1)/(\text{SO}(p'+1) \times \text{SO}_0(p'', q+1))$ . We denote by  $[e] = e\tilde{K} \in \tilde{M}(V) := \text{SO}_0(\tilde{V})/\tilde{K}$ ,  $\tilde{K} := \text{SO}(p'+1) \times \text{SO}_0(p'', q+1)$ , the canonical base point and by  $G(V)[e]$  its  $G(V)$ -orbit.

**Proposition 7.** *The action of  $G(V)$  on  $\tilde{M}(V)$  induces a  $G(V)$ -equivariant open embedding*

$$M(V) = G(V)/K \xrightarrow{\sim} G(V)[e] \subset \tilde{M}(V).$$

*$G(V)$  acts transitively on  $\tilde{M}(V)$  if and only if  $\tilde{M}(V)$  is Riemannian, i.e. if and only if  $p'' = 0$ . In that case  $M(V) \cong \tilde{M}(V)$  is the noncompact dual of the Grassmannian  $\text{SO}(p+q+2)/(\text{SO}(p+1) \times \text{SO}(q+1))$  and admits a simply transitive splittable solvable subgroup  $I(S) \ltimes \exp(\mathbb{R}D + V) \subset G(V)$ . Here  $I(S)$  denotes the (solvable) Iwasawa subgroup of  $S \cong \text{Spin}_0(V)$ .*

**Proof:** The stabilizer  $G(V)_{[e]}$  of  $[e]$  in  $G(V)$  coincides with  $K$  and hence  $G(V)[e] \cong G(V)/G(V)_{[e]} = G(V)/K = M(V)$ . Now a simple dimension count shows that the orbit  $G(V)[e] \subset \tilde{M}(V)$  is open. If  $\tilde{M}(V)$  is Riemannian then it is a Riemannian symmetric space of noncompact type and, by the Iwasawa decomposition theorem (see [H]) the Iwasawa subgroup  $I(\text{SO}_0(p+1, q+1)) \subset \text{SO}_0(p+1, q+1)$  is a splittable solvable

subgroup which acts simply transitively on  $\tilde{M}(V)$ . Now let  $G(V) = S \ltimes R(V)$  be the Levi decomposition associated to the Levi decomposition  $\mathfrak{g}(V) = \mathfrak{o}(V) + (\mathbb{R}D + V)$ ,  $S \cong \text{Spin}_0(V)$ ,  $R(V) = \exp(\mathbb{R}D + V)$  (cf. Prop. 2). We denote by  $I(S) \subset S$  the Iwasawa subgroup of  $S$ . Then  $I(S) \ltimes R(V) \subset G(V)$  is mapped isomorphically onto  $I(\text{SO}_0(p+1, q+1)) \subset \text{SO}_0(\tilde{V})$  by the homomorphism  $G(V) \rightarrow \text{SO}_0(\tilde{V})$  introduced above. This shows that  $I(S) \ltimes R(V)$  and hence  $G(V)$  acts transitively on  $\tilde{M}(V)$  in the Riemannian case.

If  $\tilde{M}(V)$  is not Riemannian then the homogeneous spaces  $\tilde{M}(V)$  and  $M(V)$  are not homotopy equivalent and hence  $G(V)$  does not act transitively on  $\tilde{M}(V)$ . In fact,  $\tilde{M}(V)$  (respectively,  $M(V)$ ) has the homotopy type of the Grassmanian  $\text{SO}(p+1)/(\text{SO}(p'+1) \times \text{SO}(p''))$  (respectively,  $\text{SO}(p)/(\text{SO}(p') \times \text{SO}(p''))$ ). Now it is sufficient to observe that the Stiefel manifolds  $\text{SO}(p+1)/\text{SO}(p'+1)$  and  $\text{SO}(p)/\text{SO}(p')$  are homotopy equivalent only if  $p = p'$  and hence  $p'' = 0$ , see [O3].  $\square$

Before we can formulate the main result of the present paper in section 2.4 we need to recall the notions of quaternionic manifold and of (pseudo-) quaternionic Kähler manifold, cf. [A-M2]. The reader familiar with these concepts should skip the next section.

**2.2. Quaternionic structures.** It is instructive to introduce the basic concepts of quaternionic geometry as analogues of the more familiar concepts of complex geometry.

**Definition 4.** Let  $E$  be a (finite dimensional) real vector space. A **complex structure** on  $E$  is an endomorphism  $J \in \text{End}(E)$  such that  $J^2 = -\text{Id}$ . A **hypercomplex structure** on  $E$  is a triple  $(J_\alpha) = (J_1, J_2, J_3)$  of complex structures on  $E$  satisfying  $J_1 J_2 = J_3$ . A **quaternionic structure** on  $E$  is the three-dimensional subspace  $Q \subset \text{End}(E)$  spanned by a hypercomplex structure  $(J_\alpha): Q = \text{span}\{J_1, J_2, J_3\}$ . In that case, we say that the hypercomplex structure  $(J_\alpha)$  is **subordinate** to the quaternionic structure  $Q$ .

Note, first, that  $Q \subset \mathfrak{gl}(E)$  is a Lie subalgebra isomorphic to  $\mathfrak{sp}(1) \cong \text{Im}\mathbb{H} = \text{span}\{i, j, k\}$  the Lie algebra of the group  $Sp(1) = S^3 \subset \mathbb{H} = \text{span}\{1, i, j, k\}$  of unit quaternions and, second, that the real associative subalgebra of  $\text{End}(E)$  generated by  $\text{Id}$  and a quaternionic structure  $Q$  on  $E$  is isomorphic to the algebra of quaternions  $\mathbb{H}$ . In both cases, the choice of such isomorphism is equivalent to the choice of a hypercomplex structure  $(J_\alpha) = (J_1, J_2, J_3)$  subordinate to  $Q$ . In fact, given  $(J_\alpha)$  we can define an isomorphism of associative algebras by  $(\text{Id}, J_1, J_2, J_3) \mapsto (1, i, j, k)$  this induces also an isomorphism of Lie algebras  $Q \xrightarrow{\sim} \mathfrak{sp}(1)$ .

**Definition 5.** Let  $M$  be a (smooth) manifold. An **almost complex structure**  $J$  (respectively, **almost hypercomplex structure**  $(J_1, J_2, J_3)$ , **almost quaternionic structure**  $Q$ ) on  $M$  is a (smooth) field  $M \ni m \mapsto J_m \in \text{End}(T_m M)$  of complex structures (respectively,  $m \mapsto (J_1, J_2, J_3)_m$  of hypercomplex structures,  $m \mapsto Q_m$  of quaternionic structures). The pair  $(M, J)$  (respectively,  $(M, (J_\alpha))$ ,  $(M, Q)$ ) is called

an almost complex manifold (respectively, almost hypercomplex manifold, almost quaternionic manifold).

We recall that a connection on a manifold  $M$  (i.e. a covariant derivative  $\nabla$  on its tangent bundle  $TM$ ) induces a covariant derivative  $\nabla$  on the full tensor algebra over  $TM$  and, in particular, on  $\text{End}(TM) \cong TM \otimes T^*M$ . We will say that  $\nabla$  preserves a subbundle  $B \subset \text{End}(TM)$  if it maps (smooth) sections of  $B$  into sections of  $T^*M \otimes B$ .

**Definition 6.** A connection  $\nabla$  on an almost complex manifold  $(M, J)$  (respectively, almost hypercomplex manifold  $(M, (J_\alpha))$ , almost quaternionic manifold  $(M, Q)$ ) is called **almost complex** (respectively, **almost hypercomplex**, **almost quaternionic**) connection if  $\nabla J = 0$  (respectively, if  $\nabla J_1 = \nabla J_2 = \nabla J_3 = 0$ , if  $\nabla$  preserves the rank 3 subbundle  $Q \subset \text{End}(TM)$ ). A **complex** (respectively, **hypercomplex**, **quaternionic**) connection is a torsionfree almost complex (respectively, almost hypercomplex, almost quaternionic) connection.

Note that the equation  $\nabla J = 0$  is equivalent to the condition that  $\nabla$  preserves the rank 1 subbundle of  $\text{End}(TM)$  spanned by  $J$ , i.e.  $\nabla J = \theta \otimes J$  for some 1-form  $\theta$  on  $M$ . Therefore, the notion of almost quaternionic connection (as well as that of almost hypercomplex connection) is a direct quaternionic analogue of the notion of almost complex connection.

**Definition 7.** Let  $M$  be a manifold. An almost complex structure (respectively, almost hypercomplex structure, almost quaternionic structure) on  $M$  is called **1-integrable** if there exists a complex (respectively, hypercomplex, quaternionic) connection on  $M$ . A **complex structure** (respectively, **hypercomplex structure**, **quaternionic structure**) on  $M$  is a 1-integrable almost complex structure (respectively, almost hypercomplex structure, almost quaternionic structure) on  $M$ .

**Remark 3:** It is well known, see [N-N] and [K-NII], that an almost complex manifold  $(M, J)$  is integrable, i.e. admits an atlas with holomorphic transition maps, if and only if it is 1-integrable. This justifies the following definition of complex manifold.

**Definition 8.** A **complex manifold** (respectively, **hypercomplex manifold**) is a manifold  $M$  together with a complex structure  $J$  (respectively, hypercomplex structure  $(J_\alpha)$ ) on  $M$ . A **quaternionic manifold** of dimension  $d > 4$  is a manifold  $M$  of dimension  $d$  together with a quaternionic structure  $Q$  on  $M$ . Finally, a **quaternionic manifold** of dimension  $d = 4$  is a 4-dimensional manifold  $M$  together with an almost quaternionic structure  $Q$  which annihilates the Weyl tensor of the conformal structure defined by  $Q$ , see Remark 4 below.

For examples of hypercomplex and quaternionic manifolds (without metric condition) see [J1], [J2] and [B-D].

**Remark 4:** Notice that an almost quaternionic structure on a 4-manifold induces an oriented conformal structure. This follows from the fact that the normalizer in

$GL(4, \mathbb{R})$  of the standard quaternionic structure(s) on  $\mathbb{R}^4 = \mathbb{H}$  is the special conformal group  $CO_0(4)$ . The definition of quaternionic 4-manifold  $(M, Q)$  implies that this conformal structure is half-flat. More precisely, if the orientation of  $M$  is chosen such that  $Q$  is locally generated by positively oriented almost complex structures, then the self-dual half of the Weyl tensor vanishes. The special treatment of the 4-dimensional case in Def. 8 and also in Def. 11 below has the advantage that with these definitions many important properties of quaternionic (and also of quaternionic Kähler) manifolds of dimension  $> 4$  remain true in dimension 4, cf. Remark 5. In particular, all future statements about quaternionic manifolds and quaternionic Kähler manifolds in the present paper, such as the integrability of the canonical almost complex structure on the twistor space (see section 3), are valid also in the 4-dimensional case.

Next we discuss (almost) complex, hypercomplex and quaternionic structures on a pseudo-Riemannian manifold  $(M, g)$ .

**Definition 9.** *A pseudo-Riemannian metric  $g$  on an almost complex manifold  $(M, J)$  (respectively, almost hypercomplex manifold  $(M, (J_\alpha))$ , almost quaternionic manifold  $(M, Q)$ ) is called **Hermitian** if  $J$  is skew symmetric (respectively, the  $J_\alpha$  are skew symmetric,  $Q$  consists of skew symmetric endomorphisms).*

Note that, due to  $J^2 = -\text{Id}$ , an almost complex structure  $J$  on a pseudo-Riemannian manifold  $(M, g)$  is skew symmetric if and only if  $J$  is orthogonal, i.e. if and only if  $g(JX, JY) = g(X, Y)$  for all vector fields  $X, Y$  on  $M$ . Similarly, an almost quaternionic structure  $Q$  on a pseudo-Riemannian manifold  $(M, g)$  consists of skew symmetric endomorphisms if and only if  $Z := \{A \in Q \mid A^2 = -\text{Id}\}$  consists of orthogonal endomorphisms (here the equation  $A^2 = -\text{Id}$  is on  $T_{\pi A}M$ ,  $\pi : Q \rightarrow M$  the bundle projection).

**Definition 10.** *A complex (pseudo-) Hermitian manifold  $(M, J, g)$  (respectively, hypercomplex (pseudo-) Hermitian manifold  $(M, (J_\alpha), g)$ , quaternionic (pseudo-) Hermitian manifold  $(M, Q, g)$ , almost complex (pseudo-) Hermitian manifold  $(M, J, g)$  etc.) is a complex manifold  $(M, J)$  (respectively, hypercomplex manifold  $(M, (J_\alpha))$ , quaternionic manifold  $(M, Q)$ , almost complex manifold  $(M, J)$  etc.) with a Hermitian (pseudo-) Riemannian metric  $g$ .*

Next we define the hypercomplex and quaternionic analogues of (pseudo-) Kähler manifolds.

**Definition 11.** *A (pseudo-) Kähler manifold (respectively, (pseudo-) hyper-Kähler manifold, quaternionic (pseudo-) Kähler manifold of dimension  $d > 4$ ) is an almost complex (pseudo-) Hermitian manifold  $(M, J, g)$  (respectively, almost hypercomplex (pseudo-) Hermitian manifold  $(M, (J_\alpha), g)$ , almost quaternionic (pseudo-) Hermitian manifold  $(M, Q, g)$  of dimension  $d > 4$ ) with the property that the Levi-Civita connection  $\nabla^g$  of the (pseudo-) Riemannian metric  $g$  is complex (respectively, hypercomplex, quaternionic). An almost quaternionic Hermitian 4-manifold  $(M, Q, g)$*

is called **quaternionic Kähler manifold** if  $Q$  annihilates the curvature tensor  $R$  of  $\nabla^g$ .

See [Bes], [H-K-L-R], [C-F-G], [Sw1], [D-S1], [D-S2], [Bi1], [Bi2], [K-S2] and [C4] for examples of hyper-Kähler manifolds and [W1], [A3], [Bes], [G1], [G-L], [L2], [L3], [F-S], [dW-VP2], [A-G], [K-S1], [A-P], [D-S3] and [C2] for examples of quaternionic Kähler manifolds.

**Remark 5:** As explained in Remark 4, an almost quaternionic structure  $Q$  on a 4-manifold  $M$  defines a conformal structure. It is clear that a pseudo-Riemannian metric  $g$  on  $M$  defines the same conformal structure as  $Q$  if and only if it is  $Q$ -Hermitian and that any such metric is definite. Moreover, the Levi-Civita connection  $\nabla^g$  of an almost quaternionic Hermitian 4-manifold  $(M, Q, g)$  automatically preserves  $Q$ . In fact, its holonomy group at  $m \in M$  is a subgroup of  $SO(T_m M, g_m)$  because  $M$  is oriented (by  $Q$ ) and the latter group normalizes  $Q$  because  $g$  is  $Q$ -Hermitian. Now let  $(M, Q, g)$  be a quaternionic Kähler 4-manifold. Since  $Q$  annihilates the curvature tensor  $R$  and the metric  $g$  it must also annihilate the Ricci tensor  $Ric$  and the Weyl tensor of  $(M, g)$ . This shows first that  $Ric$  is  $Q$ -Hermitian and hence proportional to  $g$ :  $Ric = cg$ . In other words,  $(M, g)$  is an Einstein manifold. Second,  $(M, Q)$  is a quaternionic manifold because  $Q$  annihilates the Weyl tensor of the conformal structure defined by  $g$ , which coincides with the conformal structure defined by  $Q$ . For any quaternionic pseudo-Kähler manifold  $(M, g)$  (of arbitrary dimension) it is known, see e.g. [A-M2], that  $(M, g)$  is Einstein and that  $Q$  annihilates  $R$ .

As next, we introduce the appropriate notions in order to discuss transitive group actions on manifolds with the special geometric structures defined above.

### 2.3. Invariant connections on homogeneous manifolds and 1-integrability of homogeneous almost quaternionic manifolds.

**Definition 12.** *The automorphism group of an almost complex manifold  $(M, J)$  (respectively, almost hypercomplex manifold  $(M, (J_\alpha))$ , almost quaternionic manifold  $(M, Q)$ , almost complex Hermitian manifold  $(M, J, g)$ , etc.) is the group of diffeomorphisms of  $M$  which preserves  $J$  (respectively,  $(J_\alpha)$ ,  $Q$ ,  $(J, g)$ , etc.). An almost complex manifold (respectively, almost hypercomplex manifold, almost quaternionic manifold, almost complex Hermitian manifold, etc.) is called **homogeneous** if it has a transitive automorphism group.*

In the next section, we are going to construct an almost quaternionic structure  $Q$  on certain homogeneous manifolds  $M = G/K$  ( $G$  is a Lie group and  $K$  a closed subgroup). The almost quaternionic structure  $Q$  will be  $G$ -invariant by construction and hence  $(M, Q)$  will be a homogeneous almost quaternionic manifold. Similarly, we will construct homogeneous almost quaternionic (pseudo-) Hermitian manifolds  $(M = G/K, Q, g)$ . In order to prove that  $(M = G/K, Q, g)$  is a quaternionic (pseudo-) Kähler manifold, or simply to establish the 1-integrability of the almost quaternionic

structure  $Q$  it is useful to have an appropriate description of the affine space of  $G$ -invariant connections on the homogeneous manifold  $M = G/K$ . This is provided by the notion of Nomizu map, which we now recall, see [A-V-L], [V1]. Let  $\nabla$  be a connection on a manifold  $M$ . For any vector field  $X$  on  $M$  one defines the operator

$$(8) \quad L_X := \mathcal{L}_X - \nabla_X,$$

where  $\mathcal{L}_X$  is the Lie derivative (i.e.  $\mathcal{L}_X Y = [X, Y]$  for any vector field  $Y$  on  $M$ ).  $L_X$  is a  $C^\infty(M)$ -linear map on the  $C^\infty(M)$ -module  $\Gamma(TM)$  of vector fields on  $M$ , so it can be identified with a section  $L_X \in \Gamma(\text{End}(TM))$ . In particular,  $L_X|_m \in \text{End}(T_m M)$  for all  $m \in M$ .

Now let  $M = G/K$  be a homogeneous space and suppose that  $\nabla$  is  $G$ -invariant. The action of  $G$  on  $M$  defines an antihomomorphism of Lie algebras  $\alpha : \mathfrak{g} = \text{Lie } G \rightarrow \Gamma(TM)$  from the Lie algebra  $\mathfrak{g}$  of *left*-invariant vector fields on  $G$  to the Lie algebra  $\Gamma(TM)$  of vector fields on  $M$ . This antihomomorphism maps an element  $x \in \mathfrak{g}$  to the **fundamental vector field**  $\alpha(x) := X \in \Gamma(TM)$  defined by  $X(m) := \frac{d}{dt}|_{t=0} \exp(tx)m$ . The statement that  $\alpha$  is an antihomomorphism means that  $[\alpha(x), \alpha(y)] = -\alpha([x, y])$  for all  $x, y \in \mathfrak{g}$ . Note that  $\alpha$  becomes a homomorphism if we replace  $\mathfrak{g}$  by the Lie algebra of *right*-invariant vector fields on  $G$ . Without restriction of generality, we can assume that the action is almost effective, i.e.  $\mathfrak{g} \rightarrow \Gamma(TM)$  is injective. Then we can identify  $\mathfrak{g}$  with its faithful image in  $\Gamma(TM)$ . The isotropy subalgebra  $\mathfrak{k} = \text{Lie } K$  is mapped (anti)isomorphically onto a subalgebra of vector fields vanishing at the base point  $[e] = eK \in M = G/K$ . In this situation we define the **Nomizu map**  $L = L(\nabla) : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$ ,  $x \mapsto L_x$ , by the equation

$$L_x := L_X|_{[e]},$$

where again  $X$  is the fundamental vector field on  $M$  associated to  $x \in \mathfrak{g}$ . The operators  $L_x \in \text{End}(T_{[e]}M)$  will be called **Nomizu operators**. They have the following properties:

$$(9) \quad L_x = d\rho(x) \quad \text{for all } x \in \mathfrak{k}$$

and

$$(10) \quad L_{\text{Ad}_k x} = \rho(k)L_x\rho(k)^{-1} \quad \text{for all } x \in \mathfrak{g}, \quad k \in K,$$

where  $\rho : K \rightarrow \text{GL}(T_{[e]}M)$  is the isotropy representation. The first equation follows directly from equation (8) since  $(\nabla_X Y)_m = 0$  if  $X(m) = 0$ . The second equation expresses the  $G$ -invariance of  $\nabla$ . Conversely, any linear map  $L : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$  satisfying (9) and (10) is the Nomizu map of a uniquely defined  $G$ -invariant connection  $\nabla = \nabla(L)$  on  $M$ . Its torsion tensor  $T$  and curvature tensor  $R$  are expressed at  $[e]$  by:

$$T(\pi x, \pi y) = -(L_x \pi y - L_y \pi x + \pi[x, y])$$

and

$$R(\pi x, \pi y) = [L_x, L_y] + L_{[x, y]}, \quad x, y \in \mathfrak{g}$$



where  $\pi : \mathfrak{g} \rightarrow T_{[e]}M$  is the canonical projection  $x \mapsto \pi x = X([e]) = \frac{d}{dt}|_{t=0} \exp(tx)K$ .

**Remark 6:** The difference between our formulas for torsion and curvature and those of [A-V-L] is due to the fact that in [A-V-L] everything is expressed in terms of right-invariant vector fields on the Lie group  $G$  whereas we use left-invariant vector fields. To obtain the corresponding expressions in terms of right-invariant vector fields from the expressions in terms of left-invariant vector fields and vice versa it is sufficient to replace  $[x, y]$  by  $-[x, y]$  in all formulas ( $x, y \in \mathfrak{g}$ ). The same remark applies to formula (11) below, which expresses the Levi-Civita connection on a pseudo-Riemannian homogeneous manifold.

Suppose now that we are given a  $G$ -invariant geometric structure  $S$  on  $M$  (e.g. a  $G$ -invariant almost quaternionic structure  $Q$ ) defined by a corresponding  $K$ -invariant geometric structure  $S_{[e]}$  on  $T_{[e]}M$ . Then a  $G$ -invariant connection  $\nabla$  preserves  $S$  if and only if the corresponding Nomizu operators  $L_x$ ,  $x \in \mathfrak{g}$ , preserve  $S_{[e]}$ . So to construct a  $G$ -invariant connection preserving  $S$  it is sufficient to find a Nomizu map  $L : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$  such that  $L_x$  preserves  $S_{[e]}$  for all  $x \in \mathfrak{g}$ . We observe that, due to the  $K$ -invariance of  $S_{[e]}$ , the Nomizu operators  $L_x$  preserve  $S_{[e]}$  already for  $x \in \mathfrak{k}$ . The above considerations can be specialized as follows:

**Proposition 8.** *Let  $Q$  be a  $G$ -invariant almost quaternionic structure on a homogeneous manifold  $M = G/K$ . There is a natural one-to-one correspondence between  $G$ -invariant almost quaternionic connections on  $(M, Q)$  and Nomizu maps  $L : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$ , whose image normalizes  $Q_{[e]}$ , i.e. whose Nomizu operators  $L_x$ ,  $x \in \mathfrak{g}$ , belong to the normalizer  $\mathfrak{n}(Q) \cong \mathfrak{sp}(1) \oplus \mathfrak{gl}(d, \mathbb{H})$  ( $d = \dim M/4$ ) of the quaternionic structure  $Q_{[e]}$  in the Lie algebra  $\mathfrak{gl}(T_{[e]}M)$ .*

**Corollary 4.** *Let  $(M = G/K, Q)$  be a homogeneous almost quaternionic manifold and  $L : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$  a Nomizu map such that*

- (1)  $L_x \pi y - L_y \pi x = -\pi[x, y]$  for all  $x, y \in \mathfrak{g}$  (i.e.  $T = 0$ ) and
- (2)  $L_x$  normalizes  $Q_{[e]} \subset \text{End}(T_{[e]}M)$ .

*Then the connection  $\nabla(L)$  associated to the Nomizu map  $L$  is a  $G$ -invariant quaternionic connection on  $(M, Q)$  and hence  $Q$  is 1-integrable.*

For future use we give the well known formula for the Nomizu map  $L^g$  associated to the Levi-Civita connection  $\nabla^g$  of a  $G$ -invariant pseudo-Riemannian metric  $g$  on a homogeneous space  $M = G/K$ . Let  $\langle \cdot, \cdot \rangle = g_{[e]}$  be the  $K$ -invariant scalar product on  $T_{[e]}M$  induced by  $g$ . Then  $L_x^g \in \text{End}(T_{[e]}M)$ ,  $x \in \mathfrak{g}$ , is given by the following Koszul type formula:

$$(11) \quad -2\langle L_x^g \pi y, \pi z \rangle = \langle \pi[x, y], \pi z \rangle - \langle \pi x, \pi[y, z] \rangle - \langle \pi y, \pi[x, z] \rangle, \quad x, y, z \in \mathfrak{g}.$$

**Corollary 5.** *Let  $(M = G/K, Q, g)$  be a homogeneous almost quaternionic (pseudo-) Hermitian manifold and assume that  $L_x^g$  normalizes  $Q_{[e]}$  for all  $x \in \mathfrak{g}$ . Then the Levi-Civita connection  $\nabla^g = \nabla(L^g)$  is a  $G$  invariant quaternionic connection on  $(M, Q, g)$  and hence  $(M, Q, g)$  is a quaternionic (pseudo-) Kähler manifold if  $\dim M > 4$ .*

**2.4. The main theorem.** Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be an extended Poincaré algebra of signature  $(p, q)$  and  $p \geq 3$ . We fix the decomposition  $p = p' + p''$ , where  $p' = 3$ . For notational convenience we put  $r := p'' = p - 3$ . Then we consider the linear groups  $K = K(p', p'') = K(3, r) \subset G(V) \subset G = G(\Pi) \subset \text{Aut}(\mathfrak{r})$  introduced in section 2.1. As before  $\mathfrak{r}$  denotes the radical of  $\mathfrak{g} = \text{Lie } G$ .

**Theorem 8.** 1) *There exists a  $G$ -invariant quaternionic structure  $Q$  on  $M = M(\Pi) = G(\Pi)/K$ .*

2) *If  $\Pi$  is nondegenerate (see Def. 2) then there exists a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$  such that  $(M, Q, g)$  is a quaternionic pseudo-Kähler manifold.*

**Proof:** The main idea of the proof, which we will carry out in detail, is first to observe that the submanifold  $M(V) = G(V)/K \subset G/K = M$  has a natural  $G(V)$ -invariant structure of quaternionic pseudo-Kähler manifold and then to study the problem of extending this structure to the manifold  $M$ . As shown in section 2.1, we can embed  $M(V)$  as open  $G(V)$ -orbit into the pseudo-Riemannian symmetric space  $\tilde{M}(V) = \text{SO}_0(\tilde{V})/\tilde{K}$ ,  $\tilde{V} = \mathbb{R}^{p+1, q+1}$ ,  $\tilde{K} = \text{SO}(p' + 1) \times \text{SO}_0(p'', q + 1) = \text{SO}(4) \times \text{SO}_0(r, q + 1)$ . Now we claim that  $\tilde{M}(V)$  carries an  $\text{SO}_0(\tilde{V})$ -invariant (and hence  $G(V)$ -invariant) almost quaternionic structure  $Q$ . It is sufficient to specify the corresponding  $\tilde{K}$ -invariant quaternionic structure  $Q_{[e]} \subset \text{End}(T_{[e]}\tilde{M}(V))$  at the canonical base point  $[e] = e\tilde{K} \in \tilde{M}(V)$ . We first define a hypercomplex structure  $(J_\alpha)$  on the isotropy module  $T_{[e]}\tilde{M}(V) \cong \mathbb{R}^4 \otimes \mathbb{R}^{r, q+1}$ :

$$(12) \quad J_1 := \mu_l(i) \otimes \text{Id}, \quad J_2 := \mu_l(j) \otimes \text{Id}, \quad J_3 := \mu_l(k) \otimes \text{Id}.$$

Here  $\mu_l(x) \in \text{End}(\mathbb{R}^4)$  stands for left-multiplication by the quaternion  $x \in \mathbb{H} = \text{span}\{1, i, j, k\}$ :  $\mu_l(x)y = xy$  for all  $x, y \in \mathbb{H} = \mathbb{R}^4$ . Right-multiplication by  $x$  will be denoted by  $\mu_r(x)$ . Now it is easy to check that the quaternionic structure  $Q_{[e]} := \text{span}\{J_1, J_2, J_3\}$  is normalized under the isotropy representation of  $\tilde{K}$  and hence extends to an  $\text{SO}_0(\tilde{V})$ -invariant almost quaternionic structure  $Q$  on  $\tilde{M}(V)$ . Note that the complex structures  $J_\alpha$  are not invariant under the isotropy representation and hence do not extend to  $\text{SO}_0(\tilde{V})$ -invariant almost complex structures on  $\tilde{M}(V)$ . Next we observe that the tensor product of the canonical pseudo-Euclidean scalar products on  $\mathbb{R}^4$  and  $\mathbb{R}^{r, q+1}$  defines a pseudo-Euclidean scalar product  $-g_{[e]}$  on  $T_{[e]}\tilde{M}(V) \cong \mathbb{R}^4 \otimes \mathbb{R}^{r, q+1}$ . Notice that  $g_{[e]}$  is *positive* definite if  $r = 0$  and *indefinite* otherwise. It is invariant under the isotropy representation and hence extends to a ( $Q$ -Hermitian)  $\text{SO}_0(\tilde{V})$ -invariant pseudo-Riemannian metric on  $\tilde{M}(V)$ , which is unique, up to scaling, and defines on  $\tilde{M}(V)$  the well known structure of pseudo-Riemannian symmetric space. In fact, it is the symmetric space associated to the symmetric pair  $(\mathfrak{o}(\tilde{V}), \tilde{\mathfrak{k}} := \text{Lie } \tilde{K})$ . Let  $\tilde{\mathfrak{m}} \subset \mathfrak{o}(\tilde{V})$  be the  $\tilde{\mathfrak{k}}$ -invariant complement to the isotropy algebra  $\tilde{\mathfrak{k}}$ . Then  $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{k}}$  and hence the Nomizu operators  $L_x^g \in \text{End}(T_{[e]}\tilde{M}(V))$  associated to the Levi-Civita connection  $\nabla^g$  of  $g$  vanish for  $x \in \tilde{\mathfrak{m}}$ , see (11). On

the other hand if  $x \in \tilde{\mathfrak{k}}$  then the Nomizu operator  $L_x^g$  on  $T_{[e]} \tilde{M}(V) \cong \tilde{\mathfrak{m}}$  coincides with the image of  $x$  under the isotropy representation:  $L_x^g = \text{ad}_x \tilde{\mathfrak{m}}$ . This shows that  $L_x^g$  normalizes the quaternionic structure  $Q_{[e]}$  for all  $x \in \mathfrak{o}(\tilde{V})$  and by Cor. 5 we conclude that  $(\tilde{M}(V), Q, g)$  is a homogeneous quaternionic pseudo-Kähler manifold if  $\dim \tilde{M}(V) > 4$ . The manifold  $\tilde{M}(V)$  is 4-dimensional only if  $r = q = 0$  and in this case  $\tilde{M}(V) = \text{SO}_0(4, 1)/\text{SO}(4)$  reduces to real hyperbolic 4-space  $H_{\mathbb{R}}^4$ , i.e. to the quaternionic hyperbolic line  $H_{\mathbb{H}}^1 = \text{Sp}(1, 1)/\text{Sp}(1) \cdot \text{Sp}(1)$ , which is a standard example of (conformally half-flat Einstein) quaternionic Kähler 4-manifold. Of course, the pair  $(Q, g)$  defined on the manifold  $\tilde{M}(V)$  (for all  $r$  and  $q$ ) restricts to a  $G(V)$ -invariant quaternionic pseudo-Kähler structure on the open  $G(V)$ -orbit  $M(V) \hookrightarrow \tilde{M}(V)$ . So we have proven that  $(M(V) = G(V)/K, Q, g)$  is a homogeneous quaternionic pseudo-Kähler manifold.

Our strategy is now to extend the geometric structures  $Q$  and  $g$  from  $M(V)$  to  $G$ -invariant structures on  $M = G/K \supset G(V)/K = M(V)$ . First we will extend  $Q$  to a  $G$ -invariant almost quaternionic structure on  $M$ . Using the infinitesimal action of  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathfrak{g}(V) + W$  on  $M$  we can identify  $T_{[e]} M = (\mathfrak{g}(V)/\mathfrak{k}) \oplus W = T_{[e]} M(V) \oplus W$ . Note that the isotropy representation of  $K$  on  $T_{[e]} M$  preserves this decomposition and acts on  $T_{[e]} M(V) = T_{eK} M(V) \cong T_{e\tilde{K}} \tilde{M}(V)$  as restriction of the isotropy representation of  $\tilde{K}$  on  $T_{e\tilde{K}} \tilde{M}(V)$  to the subgroup  $K \subset \tilde{K}$ . We extend the hypercomplex structure  $(J_\alpha)$  defined above on  $T_{e\tilde{K}} \tilde{M}(V) \cong T_{eK} M(V)$  to a hypercomplex structure on  $T_{eK} M = T_{eK} M(V) \oplus W$  as follows:

$$(13) \quad J_\alpha s := e_\beta e_\gamma s, \quad s \in W,$$

where  $(e_\alpha, e_\beta, e_\gamma)$  is a cyclic permutation of the standard orthonormal basis of  $E = \mathbb{R}^{3,0} \subset V = E + E'$  (the product  $e_\beta e_\gamma$  is in the Clifford algebra). Now we can extend the quaternionic structure  $Q_{[e]}$  on  $T_{[e]} \tilde{M}(V)$  to a quaternionic structure on  $T_{[e]} M$  such that  $Q_{[e]} = \text{span}\{J_1, J_2, J_3\}$ . To prove that  $Q_{[e]} \subset \text{End}(T_{[e]} M)$  extends to a  $G$ -invariant almost quaternionic structure on  $M$  it is sufficient to check that the isotropy algebra  $\mathfrak{k}$  normalizes  $Q_{[e]}$ . It is obvious that the subalgebra  $\mathfrak{o}(E') = \mathfrak{o}(r, q) \subset \mathfrak{k} = \mathfrak{o}(E) \oplus \mathfrak{o}(E')$  centralizes  $Q_{[e]}$ . We have to show that  $\mathfrak{o}(E) = \mathfrak{o}(3)$  normalizes  $Q_{[e]}$ . In terms of the standard basis  $(e_1, e_2, e_3) = (i, j, k)$  of  $E = \mathbb{R}^3 = \text{Im} \mathbb{H}$  a basis of  $\mathfrak{o}(3) = \wedge^2 E$  is given by  $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$ . For any cyclic permutation  $(\alpha, \beta, \gamma)$  of  $(1, 2, 3)$  we have the following easy formulas, cf. (4):

$$(14) \quad -2d\rho(e_\beta \wedge e_\gamma) = (\mu_l(e_\alpha) - \mu_r(e_\alpha)) \otimes \text{Id}$$

on  $T_{eK} M(V) \cong T_{e\tilde{K}} \tilde{M}(V) \cong \mathbb{R}^4 \otimes \mathbb{R}^{r,q+1}$  and

$$(15) \quad -2d\rho(e_\beta \wedge e_\gamma) = e_\beta e_\gamma = J_\alpha$$

on  $W$ . As before  $d\rho : \mathfrak{k} \rightarrow \mathfrak{gl}(T_{[e]} M)$  denotes the isotropy representation. From the equations (12)-(15) we immediately obtain that

$$[d\rho(e_\beta \wedge e_\gamma), J_\alpha] = 0 \quad \text{and} \quad [d\rho(e_\beta \wedge e_\gamma), J_\beta] = -J_\gamma.$$

This shows that  $Q_{[e]} \subset \text{End}(T_{[e]}M)$  is invariant under  $\mathfrak{k}$  and hence extends to a  $G$ -invariant almost quaternionic structure  $Q$  on  $M$ . The 1-integrability of  $Q$  will be proven in the sequel.

Let us first treat the case where  $\Pi$  is nondegenerate. First of all, we extend the  $G(V)$ -invariant  $Q$ -Hermitian pseudo-Riemannian metric  $g$  on  $M(V)$  to a  $G$ -invariant  $Q$ -Hermitian pseudo-Riemannian metric on  $M$ . It is sufficient to extend the  $K$ -invariant pseudo-Euclidean scalar product  $g_{[e]}$  on  $T_{[e]}M(V)$  to a  $K$ -invariant and  $J_\alpha$ -invariant ( $\alpha = 1, 2, 3$ ) scalar product on  $T_{[e]} = T_{[e]}M(V) \oplus W$ . We do this in such a way that the above decomposition is orthogonal for the extended scalar product  $g_{[e]}$  on  $T_{[e]}M$  and define

$$g_{[e]}|W \times W := -b,$$

where  $b = b_{\Pi, (e_1, e_2, e_3)}$  is the canonical symmetric bilinear form associated to  $\Pi$  and to the decomposition  $p = p' + p'' = 3 + r$ . By Prop. 1 the nondegeneracy of  $\Pi$  implies the nondegeneracy of  $b$ . This shows that the symmetric bilinear form  $g_{[e]}$  on  $T_{[e]}M$  defined above is indeed a pseudo-Euclidean scalar product. This scalar product is invariant under the isotropy group  $K = K(p', p'') = K(3, r)$ , by virtue of Thm. 1, 3), and hence extends to a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$ . Moreover, the metric  $g$  is  $Q$ -Hermitian. In fact, it is sufficient to observe that  $b$  is  $J_\alpha$ -invariant. This property follows from the  $K$ -invariance of  $b$ , since  $J_\alpha|W \in d\rho(\mathfrak{k})|W$ , see (15). Summarizing, we obtain:  $(M, Q, g)$  is a homogeneous pseudo-Hermitian almost quaternionic manifold. The next step is to compute the Nomizu map  $L^g : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$  associated to the Levi-Civita connection  $\nabla^g$  of  $g$ . It is convenient to identify  $T_{[e]}M$  with a  $\mathfrak{k}$ -invariant complement  $\mathfrak{m}$  to  $\mathfrak{k}$  in  $\mathfrak{g}$ . Such complement is easily described in terms of the  $K$ -invariant orthogonal decomposition  $V = E + E' = \mathbb{R}^{3,0} + \mathbb{R}^{r,q}$ . Indeed, using the canonical identification  $\mathfrak{o}(V) = \wedge^2 V$  we have the following  $\mathfrak{k}$ -invariant decompositions:

$$\mathfrak{o}(V) = \mathfrak{k} + E \wedge E',$$

$$\mathfrak{g}(V) = \mathfrak{o}(V) + \mathbb{R}D + V = \mathfrak{k} + \mathfrak{m}(V), \quad \mathfrak{m}(V) = E \wedge E' + \mathbb{R}D + V,$$

$$\mathfrak{g} = \mathfrak{g}(V) + W = \mathfrak{k} + \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{m}(\Pi) := \mathfrak{m}(V) + W.$$

In the following, we will make the identifications  $T_{[e]}M(V) = \mathfrak{g}(V)/\mathfrak{k} = \mathfrak{m}(V) \subset T_{[e]}M = \mathfrak{g}/\mathfrak{k} = \mathfrak{m}$ . Let  $(e'_a)$ ,  $a = 1, \dots, r + q$ , be the standard orthonormal basis of  $E' = \mathbb{R}^{r,q}$ . Then we have the following orthonormal basis of  $(\mathfrak{m}(V), g_{[e]})$ :

$$(e_\alpha \wedge e'_a, D, e_\alpha, e'_a),$$

where  $\alpha = 1, 2, 3$  and  $a = 1, \dots, r + q$ . For notational convenience we denote this basis by  $(e_{i\alpha}), i = 0, \dots, 3, i = 0, \dots, r + q$ , where

$$\begin{aligned} e_{00} &:= e'_0 := e_0 := D \\ e_{0a} &:= e'_a \quad (a = 1, \dots, r + q) \\ e_{\alpha 0} &:= e_\alpha \quad (\alpha = 1, 2, 3) \\ e_{\alpha a} &:= e_\alpha \wedge e'_a. \end{aligned}$$

In terms of this basis the hypercomplex structure  $J_\alpha$  is expressed on  $\mathfrak{m}(V)$  simply by:

$$J_\alpha e_{0i} = J_\alpha e'_i = e_{\alpha i} \quad \text{and} \quad J_\alpha e_{\beta i} = e_{\gamma i},$$

where  $i = 0, \dots, r + q$  and  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ . The scalar product  $g_{[e]}$  is completely determined on  $\mathfrak{m}(V)$  by the condition that  $(e_{i\alpha})$  is orthonormal and that

$$\epsilon_i := g(e_{i\alpha}, e_{i\alpha}) = \begin{cases} -1 & \text{if } i = 1, \dots, r \\ +1 & \text{if } i = 0, r + 1, \dots, r + q. \end{cases}$$

The scalar product  $g_{[e]}$  on  $\mathfrak{m}$  induces an identification  $x \wedge y \mapsto x \wedge_g y$  of the exterior square  $\wedge^2 \mathfrak{m}$  with the vector space of  $g_{[e]}$ -skew symmetric endomorphisms on  $\mathfrak{m}$ , where  $x \wedge_g y(z) := g(y, z)x - g(x, z)y, x, y, z \in \mathfrak{m}$ . We denote by  $\mathfrak{n}(Q)$  (respectively,  $\mathfrak{z}(Q)$ ) the normalizer (respectively, centralizer) of the quaternionic structure  $Q_{[e]}$  in the Lie algebra  $\mathfrak{gl}(\mathfrak{m})$ .

**Lemma 5.** *The Nomizu map  $L^g = L(\nabla^g)$  associated to the Levi-Civita connection  $\nabla^g$  of the homogeneous pseudo-Riemannian manifold  $(M = G/K, g)$  is given by the following formulas:*

$$\begin{aligned} L^g_{e_0} &= L^g_{e_{\alpha\alpha}} = 0, \\ L^g_{e_\alpha} &= \frac{1}{2}J_\alpha + \bar{L}^g_{e_\alpha}, \end{aligned}$$

where

$$\bar{L}^g_{e_\alpha} = \frac{1}{2} \sum_{i=0}^{r+q} \epsilon_i e_{\alpha i} \wedge_g e'_i - \frac{1}{2} \sum_{i=0}^{r+q} \epsilon_i e_{\gamma i} \wedge_g e_{\beta i} \in \mathfrak{z}(Q),$$

$L^g_{e'_a} \in \mathfrak{z}(Q)$  is given by:

$$\begin{aligned} L^g_{e'_a} | \mathfrak{m}(V) &= \sum_{i=0}^3 e_{i\alpha} \wedge_g e_i, \\ L^g_{e'_a} | W &= \frac{1}{2} e_1 e_2 e_3 e'_a. \end{aligned}$$

For all  $s \in W$  the Nomizu operator  $L^g_s \in \mathfrak{z}(Q)$  maps the subspace  $\mathfrak{m}(V) \subset \mathfrak{m} = \mathfrak{m}(V) + W$  into  $W$  and  $W$  into  $\mathfrak{m}(V)$ . The restriction  $L^g_s | W$  ( $s \in W$ ) is completely

determined by  $L_s^g|_{\mathfrak{m}(V)}$  (and vice versa) according to the relation

$$g(L_s^g t, x) = -g(t, L_s^g x), \quad s, t \in W, \quad x \in \mathfrak{m}(V).$$

Finally,  $L_s^g|_{\mathfrak{m}(V)}$  ( $s \in W$ ) is completely determined by its values on the quaternionic basis  $(e'_i)$ ,  $i = 0, \dots, r+q$ , which are as follows:

$$\begin{aligned} L_s^g e'_0 &= \frac{1}{2}s, \\ L_s^g e'_\alpha &= \frac{1}{2}e_1 e_2 e_3 e'_\alpha s. \end{aligned}$$

(It is understood that  $L_x^g = d\rho(x) = \text{ad}_x|_{\mathfrak{m}}$  for all  $x \in \mathfrak{k}$ , cf. equation (9).) In the above formulas  $a = 1, \dots, r+q$ ,  $\alpha = 1, 2, 3$  and  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ .

**Proof:** This follows from equation (11) by a straightforward computation.  $\square$

**Corollary 6.** *The Levi-Civita connection  $\nabla^g$  of the homogeneous almost quaternionic pseudo-Hermitian manifold  $(M, Q, g)$  preserves  $Q$  and hence  $(M, Q, g)$  is a quaternionic pseudo-Kähler manifold.*

**Proof:** From the  $G$ -invariance of  $Q$ , we know already that  $L_x^g \in \mathfrak{n}(Q)$  for all  $x \in \mathfrak{k}$  and the formulas of Lemma 5 show that  $L_x^g \in \mathfrak{n}(Q)$  for all  $x \in \mathfrak{m}$ . Now the corollary follows from Cor. 5.  $\square$

By Cor. 6 we have already established part 2) of Thm. 8. Part 1) is a consequence of 2) provided that  $\Pi$  is nondegenerate. It remains to discuss the case of degenerate  $\Pi$ .

By Thm. 2, we have a direct decomposition  $W = W_0 + W'$  of  $C\ell^0(V)$ -modules, where  $W_0 = \ker \Pi$ . We put  $\Pi' := \Pi|_{\wedge^2 W'}$  and denote by  $L' : \mathfrak{g}(\Pi') \rightarrow \text{End}(\mathfrak{m}(\Pi'))$  the Nomizu map associated to the Levi-Civita connection of the quaternionic pseudo-Kähler manifold  $(M' := M(\Pi'), Q, g)$ . Note that the quaternionic structure  $Q$  of  $M'$  coincides with the almost quaternionic structure induced by the obvious  $G(\Pi')$ -equivariant embedding  $M' = G(\Pi')/K \subset G(\Pi)/K = M$ . By the next lemma, we can extend the map  $L'$  to a torsionfree Nomizu map  $L : \mathfrak{g}(\Pi) \rightarrow \text{End}(\mathfrak{m}(\Pi))$ , whose image normalizes  $Q_{[e]}$ . This proves the 1-integrability of  $Q$  (by Cor. 4), completing the proof of Thm. 8.  $\square$

By Cor. 6 we can decompose  $L'_x = \sum_{\alpha=1}^3 \omega'_\alpha(x) J_\alpha + \bar{L}'_x$ , where the  $\omega'_\alpha$  are 1-forms on  $\mathfrak{m}(\Pi')$  and  $\bar{L}'_x \in \mathfrak{z}(Q)$  belongs to the centralizer of the quaternionic structure on  $\mathfrak{m}(\Pi')$ .

**Lemma 6.** *The Nomizu map  $L' : \mathfrak{g}(\Pi') \rightarrow \text{End}(\mathfrak{m}(\Pi'))$  associated to the Levi-Civita connection of  $(M(\Pi') = G(\Pi')/K, g)$  can be extended to the Nomizu map  $L : \mathfrak{g}(\Pi) \rightarrow \text{End}(\mathfrak{m}(\Pi))$  of a  $G(\Pi)$ -invariant quaternionic connection  $\nabla$  on the homogeneous almost quaternionic manifold  $(M(\Pi) = G(\Pi)/K, Q)$ . The extension is*

defined as follows:

$$L_x := \sum_{\alpha=1}^3 \omega_\alpha(x) J_\alpha + \bar{L}_x, \quad x \in \mathfrak{m}(\Pi),$$

where  $\bar{L}_x \in \mathfrak{z}(\mathcal{Q})$ , with centralizer taken in  $\mathfrak{gl}(\mathfrak{m}(\Pi))$ , is defined below and the 1-forms  $\omega_\alpha$  on  $\mathfrak{m}(\Pi) := \mathfrak{m}(\Pi') + W_0$  satisfy  $\omega_\alpha|_{\mathfrak{m}(\Pi')} := \omega'_\alpha$  and  $\omega_\alpha|_{W_0} := 0$ . The operators  $\bar{L}_x$  are given by:

$$\bar{L}_x|_{\mathfrak{m}(\Pi')} := \bar{L}'_x \quad \text{if } x \in \mathfrak{m}(\Pi'),$$

$$\bar{L}_{e_0}|_{W_0} := \bar{L}_{e_{\alpha a}}|_{W_0} := \bar{L}_{e_a}|_{W_0} := 0,$$

$$\bar{L}_{e'_a}|_{W_0} := \frac{1}{2} e_1 e_2 e_3 e'_a,$$

$$\bar{L}_s|_{W_0} := 0 \quad \text{if } s \in W,$$

$$\bar{L}_s x := L_x s - [s, x] \quad \text{if } s \in W_0, \quad x \in \mathfrak{m}(\Pi').$$

(It is understood that  $L_x = \text{ad}_x|_{\mathfrak{m}(\Pi)}$  for all  $x \in \mathfrak{k}$ . In the above formulas, as usual,  $a = 1, \dots, r+q$  and  $\alpha = 1, 2, 3$ .)

**Proof:** To check that  $L : \mathfrak{g}(\Pi) \rightarrow \text{End}(\mathfrak{m}(\Pi))$  is a Nomizu map it is sufficient to check (10); (9) is satisfied by definition. The equation (10) expresses the  $K$ -invariance of  $L \in \mathfrak{g}(\Pi)^* \otimes \text{End}(\mathfrak{m}(\Pi))$  which is equivalent to the invariance of  $L$  under the Lie algebra  $\mathfrak{k}$ , since  $K$  is a connected Lie group. So (10) is equivalent to the following equation:

$$(16) \quad L_{[x,y]} z = [\text{ad}_x, L_y] z \quad x \in \mathfrak{k}, \quad y, z \in \mathfrak{m}(\Pi).$$

We first check this equation. Let always  $x \in \mathfrak{k}$ . Due to  $L_W W_0 = 0$  and  $[\mathfrak{k}, W_0] \subset W_0$  we have

$$L_{[x,y]} z = [\text{ad}_x, L_y] z = 0$$

if  $y \in W$  and  $z \in W_0$ . Also (16) is satisfied if  $y, z \in \mathfrak{m}(\Pi')$  because  $L_x|_{\mathfrak{m}(\Pi')} = L'_x$  is the Nomizu operator associated to the Nomizu map  $L'$ . Now let  $y \in W_0$  and  $z \in \mathfrak{m}(\Pi')$ . Then we compute:

$$\begin{aligned} L_{[x,y]} z - [\text{ad}_x, L_y] z &= (L_z[x, y] - [[x, y], z]) - [x, L_y z] + L_y[x, z] \\ &= L_z[x, y] - [[x, y], z] - [x, L_z y] + [x, [y, z]] + L_{[x,z]} y - [y, [x, z]] \\ &= L_{[x,z]} y - [x, L_z y] + L_z[x, y] = L_{[x,z]} y - [\text{ad}_x, L_z] y. \end{aligned}$$

This shows that it is sufficient to check (16) for  $y \in \mathfrak{m}(\Pi')$  and  $z \in W_0$ ; the case  $y \in W_0$  and  $z \in \mathfrak{m}(\Pi')$  then follows from the above computation. In the following let

$x \in \mathfrak{k}$  and  $z \in W_0$ . We check (16) for all  $y \in \mathfrak{m}(\Pi')$ . From  $L_{e_0} = 0$  and  $[\mathfrak{k}, e_0] = 0$  it follows that

$$L_{[x, e_0]}z - [x, L_{e_0}z] + L_{e_0}[x, z] = 0.$$

Next we check (16) for  $y = e_\alpha$ :

$$L_{[x, e_\alpha]}z - [x, L_{e_\alpha}z] + L_{e_\alpha}[x, z] = L_{[x, e_\alpha]}z - \frac{1}{2}[x, J_\alpha z] + \frac{1}{2}J_\alpha[x, z].$$

It is clear that the first summand and the sum of the second and third summands vanish if  $x \in \mathfrak{o}(E') \subset \mathfrak{k} = \mathfrak{o}(E) \oplus \mathfrak{o}(E')$ . For  $x = e_\alpha \wedge e_\beta \in \mathfrak{o}(E) = \mathfrak{o}(\mathfrak{3})$  ( $(\alpha, \beta, \gamma)$  cyclic) we compute:

$$\begin{aligned} & L_{[e_\alpha \wedge e_\beta, e_\alpha]}z - \frac{1}{2}[e_\alpha \wedge e_\beta, J_\alpha z] + \frac{1}{2}J_\alpha[e_\alpha \wedge e_\beta, z] \\ &= -L_{e_\beta}z + \frac{1}{4}e_\alpha e_\beta e_\beta e_\gamma z - \frac{1}{4}e_\beta e_\gamma e_\alpha e_\beta z \\ &= -\frac{1}{2}e_\gamma e_\alpha z + \frac{1}{2}e_\gamma e_\alpha z = 0 \end{aligned}$$

and for  $x = e_\beta \wedge e_\gamma$  we obtain:

$$\begin{aligned} & L_{[e_\beta \wedge e_\gamma, e_\alpha]}z - \frac{1}{2}[e_\beta \wedge e_\gamma, J_\alpha z] + \frac{1}{2}J_\alpha[e_\beta \wedge e_\gamma, z] \\ &= 0 + \frac{1}{4}e_\beta e_\gamma e_\beta e_\gamma z - \frac{1}{4}e_\beta e_\gamma e_\beta e_\gamma z = 0. \end{aligned}$$

Next we check (16) for  $y = e'_\alpha$ :

$$\begin{aligned} & L_{[x, e'_\alpha]}z - [x, L_{e'_\alpha}z] + L_{e'_\alpha}[x, z] = \\ & \frac{1}{2}e_1 e_2 e_3 [x, e'_\alpha]z - \frac{1}{2}[x, e_1 e_2 e_3 e'_\alpha z] + \frac{1}{2}e_1 e_2 e_3 e'_\alpha [x, z]. \end{aligned}$$

It is easy to see that this is zero if  $[x, e'_\alpha] = 0$ . So we can put  $x = e'_b \wedge e'_a$  obtaining:

$$\frac{1}{2}\langle e'_a, e'_a \rangle e_1 e_2 e_3 e'_b z + \frac{1}{4}e'_b e'_a e_1 e_2 e_3 e'_a z - \frac{1}{4}e_1 e_2 e_3 e'_a e'_b e'_a z = 0.$$

Finally, for  $y \in E \wedge E'$  we immediately obtain:

$$L_{[x, y]}z - [x, L_y z] + L_y[x, z] = L_{[x, y]}z = 0,$$

since  $L_y = 0$  for  $y \in E \wedge E'$  and  $[\mathfrak{k}, E \wedge E'] \subset E \wedge E'$ . So we have proven that  $L$  is a Nomizu map. It is easily checked that  $L_x y - L_y x = -\pi[x, y]$  for all  $x, y \in \mathfrak{m}(\Pi)$ , where  $\pi : \mathfrak{g}(\Pi) \rightarrow \mathfrak{m}(\Pi) \cong T_{[e]}M(\Pi)$  is the projection along  $\mathfrak{k}$ . This shows that  $\nabla = \nabla(L)$  has zero torsion. It only remains to check that  $L_x \in \mathfrak{n}(Q)$  for all  $x \in \mathfrak{g}(\Pi)$ . This is



easy to see for  $x \in \mathfrak{g}(\Pi')$  from the definition of the map  $L$  as extension of the Nomizu map  $L'$ . We present the calculation only for  $L_s$ ,  $s \in W_0$ :

$$L_s e_0 = L_{e_0} s - [s, e_0] = 0 + \frac{1}{2} s = \frac{1}{2} s,$$

$$L_s J_\alpha e_0 = L_s e_\alpha = L_{e_\alpha} s - [s, e_\alpha] = \frac{1}{2} J_\alpha s - 0 = J_\alpha L_s e_0,$$

$$L_s e'_a = L_{e'_a} s - [s, e'_a] = \frac{1}{2} e_1 e_2 e_3 e'_a s - 0 = \frac{1}{2} e_1 e_2 e_3 e'_a s,$$

$$\begin{aligned} L_s J_\alpha e'_a &= L_s e_{\alpha a} = L_{e_{\alpha a}} s - [s, e_{\alpha a}] = 0 - \frac{1}{2} e_\alpha e'_a s \\ &= e_\beta e_\gamma \left( \frac{1}{2} e_\alpha e_\beta e_\gamma e'_a s \right) = J_\alpha L_s e'_a, \end{aligned}$$

$(\alpha, \beta, \gamma)$  cyclic,

$$L_s t = L_t s - [s, t] = 0,$$

$$L_s J_\alpha t = L_{J_\alpha t} s - [s, J_\alpha t] = 0 = J_\alpha L_t s$$

if  $t \in W$ . We have used that  $L_W W_0 = 0$  and  $[W_0, W] = 0$ . This shows that  $L_s \in \mathfrak{z}(Q)$  for all  $s \in W_0$  finishing the proof of the lemma.  $\square$

## 2.5. The Riemannian case.

**Proposition 9.** *Let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be a nondegenerate extended Poincaré algebra of signature  $(p, q)$ ,  $p \geq 3$ , and  $(M(\Pi), Q, g)$  the corresponding homogeneous quaternionic pseudo-Kähler manifold, see Thm. 8. Then the pseudo-Riemannian metric  $g$  is positive definite, and hence a Riemannian metric, if and only if  $-b$  is positive definite and  $p = 3$ . In all other cases  $g$  is indefinite. Here  $b = b_{\Pi, (e_1, e_2, e_3)}$  is the canonical symmetric bilinear form associated to  $\Pi$  and to the decomposition  $p = 3 + r$ .*

**Proof:** By construction, the restriction of  $g$  to the submanifold  $M(V) \subset M(\Pi)$  is a (positive definite) Riemannian metric if and only if  $p = 3$  and is indefinite otherwise. Now the proposition follows from the fact that  $-b$  is precisely the restriction of the scalar product  $g_{[e]}$  on  $T_{[e]}M(\Pi) \cong T_{[e]}M(V) \oplus W$  to the subspace  $W$ .  $\square$

Next we will use the classification of extended Poincaré algebras of signature  $(3, q)$  up to isomorphism (see 1.3) to derive the classification of the quaternionic Kähler manifolds  $(M(\Pi), Q, g)$  up to isometry. We recall that  $\mathfrak{p}(3, q, 0, 0, l) = \mathfrak{p}(p = 3, q, l_0 = 0, l_+ = 0, l_- = l)$  (respectively,  $\mathfrak{p}(3, q, 0, 0, l^+) = \mathfrak{p}(p = 3, q, l_0^+ = 0, l_+^+ = l^+, l_0^- = 0, l_+^- = 0, l_-^- = l^-)$ ) is the set of isomorphism classes of extended Poincaré algebras for which  $-b$  is positive definite if  $q \not\equiv 3 \pmod{4}$  (respectively, if  $q \equiv 3 \pmod{4}$ ). We denote by  $M(q, l)$  (respectively,  $M(q, l^+, l^-)$ ) the homogeneous quaternionic Kähler manifold  $(M(\Pi), Q, g)$  associated to  $\Pi \in \mathfrak{p}(3, q, 0, 0, l)$  (respectively,  $\Pi \in \mathfrak{p}(3, q, 0, 0, l^+, 0, 0, l^-)$ ).

**Theorem 9.** *Every homogeneous quaternionic pseudo-Kähler manifold of the form  $(M(\Pi), Q, g)$  for which  $g$  is positive definite is isometric to one of the homogeneous quaternionic Kähler manifolds  $M(q, l)$  ( $q \not\equiv 3 \pmod{4}$ ) or  $M(q, l^+, l^-) \cong M(q, l^-, l^+)$  ( $q \equiv 3 \pmod{4}$ ). In particular, there are only countably many such Riemannian manifolds up to isometry.*

**Proof:** This is a direct consequence of Prop. 9, Thm. 6 and Thm. 7.  $\square$

Any real vector space  $E$  admitting a quaternionic structure has  $\dim E \equiv 0 \pmod{4}$ . Therefore, we can define its **quaternionic dimension**  $\dim_{\mathbb{H}} E := \dim E/4$ . Similarly, the quaternionic dimension of a quaternionic manifold  $(M, Q)$  is  $\dim_{\mathbb{H}} M := \dim M/4$ . We denote by  $N(q)$  the quaternionic dimension of an irreducible  $\mathcal{C}\ell_{3,q}^0$ -module.

**Proposition 10.** 1)  $N(0) = N(1) = N(2) = N(3) = 1$ ,  $N(4) = 2$ ,  $N(5) = 4$ ,  $N(6) = N(7) = 8$  and  $N(q+8) = 16N(q)$  for all  $q \geq 0$ . In particular,  $N(q)$  coincides with the dimension of an irreducible  $\mathbb{Z}_2$ -graded  $\mathcal{C}\ell_{q-3}$ -module if  $q \geq 3$ .  
2) The quaternionic dimension of the homogeneous quaternionic Kähler manifolds  $M(q, l)$  and  $M(q, l^+, l^-)$  is given by:

$$\dim_{\mathbb{H}} M(q, l) = q + 1 + lN(q)$$

and

$$\dim_{\mathbb{H}} M(q, l^+, l^-) = q + 1 + (l^+ + l^-)N(q).$$

**Proof:** The first part follows from the classification of Clifford algebras. The second part follows from  $\dim M = \dim M(V) + \dim W$ .  $\square$

The next theorem identifies the spaces  $M(q, l)$  and  $M(q, l^+, l^-)$  with Alekseevsky's quaternionic Kähler manifolds. We recall that an **Alekseevsky space** is a quaternionic Kähler manifold which admits a simply transitive non Abelian splittable solvable group of isometries, see [A3] and [C2]. Due to Iwasawa's decomposition theorem any symmetric quaternionic Kähler manifold of noncompact type is an Alekseevsky space. These are precisely the noncompact duals of the Wolf spaces. We recall that a **Wolf space** is a symmetric quaternionic Kähler manifold of compact type and that such manifolds are in 1-1-correspondence with the complex simple Lie algebras [W1]. The nonsymmetric Alekseevsky spaces are grouped into 3 series:  $\mathcal{V}$ -spaces,  $\mathcal{W}$ -spaces and  $\mathcal{T}$ -spaces, see [A3], [dW-VP2] and [C2]. These 3 series contain also all symmetric Alekseevsky spaces of rank  $> 2$  and no symmetric spaces of smaller rank. By definition an Alekseevsky space can be presented as metric Lie group, i.e. as homogeneous Riemannian manifold of the form  $(L, g)$ , where  $L$  is a Lie group and  $g$  a left-invariant Riemannian metric on  $L$ .

**Theorem 10.** *Let  $(M = M(\Pi) = G(\Pi)/K, Q, g)$  be a homogeneous quaternionic Kähler manifold as in Prop. 9,*

$$G(\Pi) = S \ltimes R \cong \text{Spin}_0(V) \ltimes R$$

the Levi decomposition (7) and  $I(S)$  the Iwasawa subgroup of  $S$ . Then  $L := L(\Pi) := I(S) \rtimes R \subset G(\Pi)$  is a (non Abelian) splittable solvable Lie subgroup which acts simply transitively on  $M$ . In particular,  $(M, Q, g)$  is an Alekseevsky space. More precisely, we have the following identifications with the  $\mathcal{V}$ -spaces,  $\mathcal{W}$ -spaces,  $\mathcal{T}$ -spaces and symmetric Alekseevsky spaces:

- 1)  $M(q, l) = \mathcal{V}(l, q - 3)$  and  $M(q, l^+, l^-) = \mathcal{V}(l^+, l^-, q - 3)$  if  $q \geq 4$ ,
- 2)  $M(3, l^+, l^-) = \mathcal{W}(l^+, l^-)$ ,
- 3)  $M(2, l) = \mathcal{T}(l)$ ,
- 4)  $M(1, l) = \text{SU}(l + 2, 2)/\text{S}(\text{U}(l + 2) \times \text{U}(2))$ ,
- 5)  $M(0, l) = \text{Sp}(l + 1, 1)/\text{Sp}(l + 1)\text{Sp}(1) = \mathbb{H}\mathbb{H}^{l+1}$  (quaternionic hyperbolic  $(l + 1)$ -space).

The above Riemannian manifolds  $M(q, l)$  and  $M(q, l^+, l^-) \cong M(q, l^-, l^+)$  are pairwise nonisometric and exhaust all Alekseevsky spaces with two symmetric exceptions:  $\mathbb{C}\mathbb{H}^2 = \text{SU}(1, 2)/\text{U}(2) =: M(1, -1)$  (complex hyperbolic plane) and  $\text{G}_2^{(2)}/\text{SO}(4)$ . (Note that these two symmetric spaces have rank  $\leq 2$  and so do not belong to any of the 3 series  $\mathcal{V}$ ,  $\mathcal{W}$  and  $\mathcal{T}$  of Alekseevsky spaces.)

**Proof:** The fact that  $L$  acts simply transitively on  $M$  follows from Prop. 7. This shows that  $(M, Q, g)$  is an Alekseevsky space. The Riemannian metric  $g$  induces a left-invariant metric  $g_L$  on the Lie group  $L$ . To establish the identifications given in the theorem it is sufficient to check that  $(L, g_L)$  is isomorphic (as metric Lie group) to one of the metric Lie groups which occur in the classification of Alekseevsky spaces, see [A3] and [C2]. (The quaternionic structure can be reconstructed from the holonomy of the Levi-Civita connection, up to an automorphism of the full isometry group which preserves the isotropy group.) Finally, to prove that  $M(q, l)$  and  $M(q, l^+, l^-) \cong M(q, l^-, l^+)$  are pairwise nonisometric it is, by [A4], sufficient to check that the corresponding metric Lie groups (which occur in the classification of Alekseevsky spaces) are pairwise nonisomorphic. This was done in [C2].  $\square$

**2.6. A class of noncompact homogeneous quaternionic Hermitian manifolds with no transitive solvable group of isometries.** There is a widely known conjecture by D.V. Alekseevsky which says that any noncompact homogeneous quaternionic Kähler manifold admits a transitive solvable group of isometries [A3]. The next theorem shows that this conjecture becomes false if we replace “Kähler” by “Hermitian”.

**Theorem 11.** *Let  $\mathfrak{p}(\Pi)$  be any extended Poincaré algebra of signature  $(p, q) = (3 + r, 0)$ ,  $r \geq 0$ , and  $(M = G/K, Q)$  the corresponding homogeneous quaternionic manifold, see Thm. 8, 1). Then there exists a  $G$ -invariant  $Q$ -Hermitian Riemannian metric  $h$  on  $M$ . Moreover, the noncompact homogeneous quaternionic Hermitian manifold  $(M, Q, h)$  does not admit any transitive solvable Lie group of isometries if  $r > 0$ .*

**Proof:** Since  $K = K(3, r) \cong \text{Spin}(3) \cdot \text{Spin}(r)$  is compact, one can easily construct a  $K$ -invariant  $Q_{[e]}$ -Hermitian Euclidean scalar product  $h_{[e]}$  on  $T_{[e]}M$  by the standard averaging procedure and extend it to a  $G$ -invariant  $Q$ -Hermitian Riemannian metric  $h$  on  $M$ . More explicitly, we can construct such a scalar product  $h_{[e]}$  on  $T_{[e]}M \cong T_{[e]}M(V) \oplus W$  as orthogonal sum of  $K$ -invariant Euclidean scalar products on  $T_{[e]}M(V)$  and  $W$  as follows. Using the open embedding  $M(V) \hookrightarrow \tilde{M}(V)$  we can identify  $T_{eK}M(V) \cong T_{e\tilde{K}}\tilde{M}(V) \cong \mathbb{R}^4 \otimes \mathbb{R}^{r,1} \cong \mathbb{R}^4 \otimes \mathbb{R}^{r+1}$  and choose the standard  $O(4) \times O(r+1)$ -invariant Euclidean scalar product on  $\mathbb{R}^4 \otimes \mathbb{R}^{r+1}$ . This scalar product is automatically  $K$ -invariant and  $Q_{[e]}$ -Hermitian. On  $W$  we choose any  $K$ -invariant Euclidean scalar product (which exists by compactness of  $K$ ). It is automatically  $Q_{[e]}$ -Hermitian because  $Q_{[e]} \subset \text{End}(W)$  is precisely the image of  $\mathfrak{o}(3) \subset \mathfrak{k} = \text{Lie } K$  under the isotropy representation of  $\mathfrak{k}$  on the  $\mathfrak{k}$ -invariant subspace  $W \subset T_{[e]}M \cong T_{[e]}M(V) \oplus W$ . It remains to show that  $\text{Isom}(M, h)$  does not contain any transitive solvable Lie subgroup if  $r > 0$ . In fact,  $M$  is homotopy equivalent to the simply connected real Grassmannian  $\text{SO}(3+r)/\text{SO}(3) \times \text{SO}(r)$  of oriented 3-planes in  $\mathbb{R}^{3+r}$  ( $r > 0$ ). On the other hand, if  $(M, h)$  admits a transitive solvable group of isometries then  $M$  must be homotopy equivalent to a (possibly trivial) torus, which contradicts the fact that  $M$  is simply connected (and not contractible).  $\square$

The last argument proves, in fact, the following theorem.

**Theorem 12.** *Let  $\mathfrak{p}(\Pi)$  be any extended Poincaré algebra of signature  $(p, q)$ ,  $p > 3$  and  $M = M(\Pi)$  the manifold constructed in Thm. 8. Then  $M$  does not admit any transitive (topological) action by a solvable Lie group.*

### 3. BUNDLES ASSOCIATED TO THE QUATERNIONIC MANIFOLD $(M, Q)$

To any almost quaternionic manifold  $(M, Q)$  one can canonically associate the following bundles over  $M$ : the twistor bundle  $Z(M)$ , the canonical  $\text{SO}(3)$ -principal bundle  $S(M)$  and the Swann bundle  $U(M)$ . The **twistor bundle** (or **twistor space**)  $Z(M) \rightarrow M$  is the subbundle of  $Q$  whose fibre  $Z(M)_m$  at  $m \in M$  consists of all complex structures subordinate to the quaternionic structure  $Q_m$ , i.e.

$$Z(M)_m = \{A \in Q_m \mid A^2 = -\text{Id}\}.$$

So  $Z(M)$  is a bundle of 2-spheres. The fibre  $S(M)_m$  of the  $\text{SO}(3)$ -principal bundle  $S(M)$  at  $m \in M$  consists of all hypercomplex structures  $(J_1, J_2, J_3)$  subordinate to  $Q_m$ . Finally,

$$U(M) = S(M) \times_{\text{SO}(3)} (\mathbb{H}^*/\{\pm 1\})$$

is associated to the action of  $\text{SO}(3) \cong \text{Sp}(1)/\{\pm 1\}$  on  $\mathbb{H}^*/\{\pm 1\}$  induced by left-multiplication of unit quaternions on  $\mathbb{H}$ . The total space  $Z(M)$  carries a canonical almost complex structure  $\mathbb{J}$ , which is integrable if  $Q$  is quaternionic, see [A-M-P]. Similarly, one can define an almost hypercomplex structure  $(\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3)$  on  $U(M)$ , which is integrable if  $Q$  is quaternionic, cf. [P-P-S]. We recall the definition of the

complex structure  $\mathbb{J}$  on the twistor space  $Z = Z(M)$  of a quaternionic manifold  $(M, Q)$ . Since  $Q$  is 1-integrable, there exists a quaternionic connection  $\nabla$  on  $M$ , see Def. 6 and Def. 7. The holonomy of  $\nabla$  preserves not only  $Q \subset \text{End}(TM)$  but also its sphere subbundle  $Z \subset Q$ , simply because  $\text{Id}$  is a parallel section of  $\text{End}(TM)$ . Let

$$(17) \quad TZ = TZ^{ver} \oplus TZ^{hor}$$

be the corresponding decomposition into the vertical space  $TZ^{ver}$  tangent to the fibres of the twistor bundle  $Z \rightarrow M$  and its  $\nabla$ -horizontal complement  $TZ^{hor}$ . The complex structure  $\mathbb{J}$  preserves the decomposition (17). Let  $m \in M$  be a point in  $M$  and  $z := J \in Z_m \subset Q_m$  a complex structure on  $T_mM$  subordinate to  $Q_m$ . Then  $\mathbb{J}_z \in \text{End}(T_zZ)$  is defined by:

$$\mathbb{J}A := JA \quad \text{and} \quad \mathbb{J}\tilde{X} = \widetilde{JX}$$

for all  $A \in T_zZ^{ver} = T_zZ_m = \{A \in Q_m \mid AJ = -JA\}$  and all  $X \in T_mM$ , where  $\tilde{X} \in T_zZ^{hor}$  denotes the  $\nabla$ -horizontal lift of  $X$ . It was proven in [A-M-P] that  $\mathbb{J}$  does not depend on the choice of quaternionic connection  $\nabla$ .

If  $(M, Q)$  admits a quaternionic pseudo-Kähler metric  $g$  (of nonzero scalar curvature) then it is known that  $(Z(M), \mathbb{J})$  admits a complex contact structure and a pseudo-Kähler-Einstein metric [S1], that  $S(M)$  admits a pseudo-3-Sasakian structure [Ko] and that  $(U(M), \mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3)$  admits a pseudo-hyper-Kähler metric [Sw1]. Moreover, all these special geometric structures are canonically associated to the data  $(M, Q, g)$ . We recall that a **complex contact structure** on a complex manifold  $Z$  is a holomorphic distribution  $\mathcal{D}$  of codimension one whose Frobenius form  $[\cdot, \cdot] : \wedge^2 \mathcal{D} \rightarrow TZ/\mathcal{D}$  is (pointwise) nondegenerate; for the definition of 3-Sasakian structure see [I-K] and [T]. If a Lie group  $G$  acts (smoothly) on an almost quaternionic manifold  $(M, Q)$  preserving  $Q$  then there is an induced  $\mathbb{J}$ -holomorphic action on  $Z$ . Similarly, if a Lie group  $G$  acts on a quaternionic pseudo-Kähler manifold  $(M, Q, g)$  preserving the data  $(Q, g)$  then it acts on any of the bundles  $Z(M)$ ,  $S(M)$  and  $U(M)$  preserving all the special geometric structures mentioned above.

**Theorem 13.** *Let  $(M(\Pi) = G(\Pi)/K, Q)$  be the homogeneous quaternionic manifold associated to an extended Poincaré algebra of signature  $(p, q)$ ,  $p \geq 3$ ,  $Z(\Pi) := Z(M(\Pi))$  its twistor space,  $S(\Pi) := S(M(\Pi))$  its canonical  $\text{SO}(3)$ -principal bundle and  $U(\Pi) := U(M(\Pi))$  its Swann bundle. Then  $G(\Pi)$  acts transitively on the manifolds  $Z(\Pi)$  and  $S(\Pi)$  and acts on  $U(\Pi)$  with an orbit of codimension one.*

**Proof:** Since  $G = G(\Pi)$  acts transitively on the base  $M = M(\Pi)$  of any of the bundles  $Z(\Pi) \rightarrow M$ ,  $S(\Pi) \rightarrow M$  and  $U(\Pi) \rightarrow M$ , it is sufficient to consider the action of the stabilizer  $K = \text{Spin}(3) \cdot \text{Spin}_0(r, q)$  on the fibres  $Z(\Pi)_{[e]}$ ,  $S(\Pi)_{[e]}$  and  $U(\Pi)_{[e]}$ ,  $[e] = eK \in M = G/K$ . The subgroup  $\text{Spin}_0(r, q) \subset K$  acts trivially on  $S(\Pi)_{[e]}$  and hence also on  $Z(\Pi)_{[e]}$  and  $U(\Pi)_{[e]}$ , whereas  $\text{Spin}(3)$  acts transitively on the set  $S(\Pi)_{[e]}$  of hypercomplex structures subordinate to  $Q_{[e]}$  and hence also on the

set  $Z(\Pi)_{[e]}$  of complex structures subordinate to  $Q_{[e]}$ . From this it follows that  $G$  acts transitively on  $Z(\Pi)$  and  $S(\Pi)$  and with an orbit of codimension one on  $U(\Pi)$ .  $\square$

From now on we denote by  $m_0 := [e] = eK \in M = G/K$  the canonical base point of  $M$  and fix the complex structure  $J_1 \in Q_{m_0}$  as base point  $z_0 := J_1 \in Z_{m_0}$  in  $Z = Z(\Pi)$ .

**Corollary 7.** *( $Z = Gz_0 \cong G/G_{z_0}, \mathbb{J}$ ) is a homogeneous complex manifold of the group  $G$ . The stabilizer of the point  $z_0 \in Z$  in  $G$  is the centralizer of  $J_1$  in  $K$ :  $G_{z_0} = Z_K(J_1) = Z_{\text{Spin}(3)}(J_1) \cdot \text{Spin}_0(r, q)$ ,  $Z_{\text{Spin}(3)}(J_1) \cong U(1)$ .*

Now we are going to construct a natural holomorphic immersion  $Z \rightarrow \bar{Z} = \bar{Z}(\Pi) = G^{\mathbb{C}}/H$  of  $Z$  into a homogeneous complex manifold of the complexified linear group  $G^{\mathbb{C}} \subset \text{Aut}(\mathfrak{t}^{\mathbb{C}})$ , where  $(G_{z_0})^{\mathbb{C}} \subset H \subset G^{\mathbb{C}}$  are closed complex Lie subgroups.

First of all, we give an explicit description of the complex structure  $\mathbb{J}$  on the twistor space  $Z$ . The choice of base point  $z_0 \in Z$  determines a  $G$ -equivariant diffeomorphism  $Z = Gz_0 \xrightarrow{\sim} G/G_{z_0}$ , which maps  $z_0$  to the canonical base point  $eG_{z_0} \in G/G_{z_0}$ . From now on we will identify  $Z$  and  $G/G_{z_0}$  via this map. The complex structure  $\mathbb{J}$  being  $G$ -invariant, it is completely determined by the  $G_{z_0}$ -invariant complex structure  $\mathbb{J}_{z_0}$  on  $T_{z_0}Z$ . In order to describe  $\mathbb{J}_{z_0}$  we introduce the following  $G_{z_0}$ -invariant complement  $\mathfrak{z} = \mathfrak{z}(\Pi)$  to  $\mathfrak{g}_{z_0} = \text{Lie } G_{z_0} = \mathbb{R}e_2 \wedge e_3 \oplus \mathfrak{o}(r, q)$  in  $\mathfrak{g}$ :

$$\mathfrak{z} = \mathbb{R}e_1 \wedge e_2 + \mathbb{R}e_1 \wedge e_3 + \mathfrak{m}.$$

The  $G_{z_0}$ -invariant decomposition  $\mathfrak{g} = \mathfrak{g}_{z_0} + \mathfrak{z}$  determines a  $G_{z_0}$ -equivariant isomorphism  $T_{z_0}Z \cong \mathfrak{g}/\mathfrak{g}_{z_0} \xrightarrow{\sim} \mathfrak{z}$ . Using it we can consider the  $G_{z_0}$ -invariant complex structure  $\mathbb{J}_{z_0}$  as a  $G_{z_0}$ -invariant complex structure on  $\mathfrak{z}$ .

**Proposition 11.** *The  $G$ -invariant complex structure  $\mathbb{J}$  on the twistor space  $Z = G/G_{z_0}$  is given on  $T_{z_0}Z \cong \mathfrak{z} = \mathbb{R}e_1 \wedge e_2 + \mathbb{R}e_1 \wedge e_3 + \mathfrak{m}$  by:*

$$\mathbb{J}_{z_0}e_1 \wedge e_2 = e_1 \wedge e_3, \quad \mathbb{J}_{z_0}|_{\mathfrak{m}} = J_1.$$

**Proof:** Let  $\nabla$  be the  $G$ -invariant quaternionic connection on  $(M, Q)$  constructed in Lemma 6 and  $L_x = \sum_{\alpha=1}^3 \omega_\alpha(x)J_\alpha + \bar{L}_x$ ,  $x \in \mathfrak{m}$ , its Nomizu operators, where  $\bar{L}_x \in \mathfrak{z}(Q) \cong \mathfrak{gl}(d, \mathbb{H})$  ( $d = \dim_{\mathbb{H}} M$ ). The connection  $\nabla$  induces the decomposition  $T_{z_0}Z = T_{z_0}Z^{ver} \oplus T_{z_0}Z^{hor} \cong \mathfrak{z} = \mathfrak{z}^{ver} \oplus \mathfrak{z}^{hor}$  into vertical space and horizontal space. The vertical space is  $T_{z_0}Z^{ver} = T_{z_0}Z_{z_0} = \mathbb{R}J_2 \oplus \mathbb{R}J_3 \subset Q_{z_0}$  and  $\mathfrak{z}^{ver} = \mathbb{R}e_1 \wedge e_2 \oplus \mathbb{R}e_1 \wedge e_3$  respectively, the identification being  $J_2 \mapsto -e_1 \wedge e_2$ ,  $J_3 \mapsto -e_1 \wedge e_3$ . For any vector  $x \in \mathfrak{m}$  we consider the curve  $c(t) = \exp txK \in G/K = M$  ( $t \in \mathbb{R}$ ) and define a lift  $s(t) \in Z_{c(t)}$  by the differential equation  $\mathcal{L}_X s = 0$  with initial condition  $s(0) = z_0 = J_1$ . Here  $X = \alpha(x)$  is the fundamental vector field on  $M$  associated to  $x$  (as defined on p. 56) and  $\mathcal{L}_X$  is the Lie derivative with respect to  $X$ . Then  $s(t) = (\exp tx)z_0$  is precisely the orbit of  $z_0 \in Z = G/G_{z_0}$  under the 1-parameter subgroup of  $G$  generated by  $x$ .

The vector

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}(s - t\nabla_X s) &= s'(0) - \nabla_X s\Big|_{t=0} \\ &= s'(0) + [L_x, J_1] = s'(0) - 2\omega_2(x)J_3 + 2\omega_3(x)J_2 \in T_{z_0}Z \end{aligned}$$

is horizontal. It is precisely the horizontal lift of  $X(m_0) \in T_{m_0}M$  and corresponds to

$$(18) \quad \tilde{x} := x + 2\omega_2(x)e_1 \wedge e_3 - 2\omega_3(x)e_1 \wedge e_2 \in \mathfrak{z}$$

under the identification  $T_{z_0}Z \cong \mathfrak{z}$ . This shows that

$$\begin{aligned} \mathfrak{z}^{hor} &= \{\tilde{x} | x \in \mathfrak{m}\} \\ &= \mathbb{R}(e_2 + e_1 \wedge e_3) + \mathbb{R}(e_3 - e_1 \wedge e_2) + \\ &\quad \mathbb{R}e_0 + \mathbb{R}e_1 + \text{span}\{e_{ia} | i = 0, \dots, 3, a = 1, \dots, r + q\} + W, \end{aligned}$$

see Lemma 6. Now the formulas for  $\mathbb{J}_{z_0}$  follow easily. In fact, it is clear that  $\mathbb{J}_{z_0}$  coincides with  $J_1$  on the  $J_1$ -invariant subspace  $\mathfrak{m} \cap \mathfrak{z}^{hor} = \mathbb{R}e_0 + \mathbb{R}e_1 + \text{span}\{e_{ia} | i = 0, \dots, 3, a = 1, \dots, r + q\} + W = e_2^\perp \cap e_3^\perp \subset \mathfrak{m}$ . The equation  $\mathbb{J}_{z_0}e_1 \wedge e_2 = e_1 \wedge e_3$  follows immediately from  $J_1J_2 = J_3$ , since  $e_1 \wedge e_2, e_1 \wedge e_3 \in \mathfrak{z}^{ver}$  are identified with the vertical vectors  $-J_2, -J_3 \in T_{z_0}Z^{ver}$ . It is now sufficient to check that  $\mathbb{J}_{z_0}e_2 = e_3$ . This is done in the next computation:

$$\mathbb{J}_{z_0}e_2 = \mathbb{J}_{z_0}(e_2 + e_1 \wedge e_3) - \mathbb{J}_{z_0}(e_1 \wedge e_3) = \mathbb{J}_{z_0}\tilde{e}_2 + e_1 \wedge e_2 = \widetilde{J_1}e_2 + e_1 \wedge e_2 = \tilde{e}_3 + e_1 \wedge e_2 = e_3$$

We denote by  $\mathfrak{z}^{1,0}$  (respectively,  $\mathfrak{z}^{0,1}$ ) the eigenspace of  $\mathbb{J}_{z_0} \in \text{End}(\mathfrak{z})$  for the eigenvalue  $i$  (respectively,  $-i$ ). The integrability of the complex structure  $\mathbb{J}$  implies that  $\mathfrak{h} := \mathfrak{h}(\Pi) := (\mathfrak{g}_{z_0})^{\mathbb{C}} + \mathfrak{z}^{0,1} \subset \mathfrak{g}^{\mathbb{C}}$  is a (complex) Lie subalgebra. Let  $H = H(\Pi) \subset G^{\mathbb{C}} \subset \text{Aut}(\mathfrak{t}^{\mathbb{C}})$  be the corresponding connected linear Lie group. Let us also consider the Lie algebra  $\mathfrak{h}(V) := \mathfrak{h} \cap \mathfrak{g}(V)^{\mathbb{C}} = (\mathfrak{g}_{z_0})^{\mathbb{C}} + \mathfrak{z}(V)^{0,1} \subset \mathfrak{g}(V)^{\mathbb{C}}$ , where  $\mathfrak{z}(V) = \mathfrak{z} \cap \mathfrak{g}(V)$  and  $\mathfrak{z}(V)^{0,1} := \mathfrak{z}^{0,1} \cap \mathfrak{g}(V)^{\mathbb{C}}$ .

**Proposition 12.**  $H = H(\Pi) \subset G^{\mathbb{C}} \subset \text{Aut}(\mathfrak{t}^{\mathbb{C}})$  are complex algebraic subgroups.

**Proof:** It follows from Prop. 4 that  $\mathfrak{g}^{\mathbb{C}} \subset \text{der}(\mathfrak{t}^{\mathbb{C}})$  is a complex algebraic subalgebra and hence  $G^{\mathbb{C}} \subset \text{Aut}(\mathfrak{t}^{\mathbb{C}})$  a complex algebraic subgroup. It only remains to show that  $\mathfrak{h}$  is a complex algebraic subalgebra. Let us consider the decomposition  $\mathfrak{h} = (\mathfrak{g}_{z_0})^{\mathbb{C}} + \mathfrak{z}^{0,1}$ . The subalgebra  $(\mathfrak{g}_{z_0})^{\mathbb{C}}$  is algebraic. In fact,  $\mathfrak{g}_{z_0} = \mathfrak{z}_{\mathfrak{k}}(e_2 \wedge e_3)$  is a centralizer in the real algebraic subalgebra  $\mathfrak{k} \subset \mathfrak{g}$ . If the subalgebra  $\langle \mathfrak{z}^{0,1} \rangle \subset \mathfrak{h}$  generated by the subspace  $\mathfrak{z}^{0,1}$  is an algebraic subalgebra of  $\text{der}(\mathfrak{t}^{\mathbb{C}})$ , then  $\mathfrak{h}$  is generated by algebraic linear Lie algebras and hence is itself algebraic, see [O-V]. The algebraicity of  $\langle \mathfrak{z}^{0,1} \rangle$  is proven in the next lemma.  $\square$

**Lemma 7.**  $\mathfrak{z}^{0,1}$  generates the algebraic subalgebra  $\langle \mathfrak{z}^{0,1} \rangle = \mathfrak{o}(E')^{\mathbb{C}} + \mathfrak{z}^{0,1} \subset \mathfrak{g}^{\mathbb{C}}$ .

**Proof:** First we compute the subalgebra  $\langle \mathfrak{z}^{0,1} \rangle$  of  $\mathfrak{g}^{\mathbb{C}}$  generated by  $\mathfrak{z}^{0,1} = \mathfrak{z}^{0,1}(V) + W^{0,1}$ . Note that

$$\begin{aligned} \mathfrak{z}^{0,1}(V) &= \text{span}\{e_1 \wedge e_2 + ie_1 \wedge e_3, e_0 + ie_1, e_2 + ie_3\} \\ &\quad + \text{span}\{e_{0a} + ie_{1a}, e_{2a} + ie_{3a} \mid a = 1, \dots, r+q\}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} [\mathfrak{z}^{0,1}(V), \mathfrak{z}^{0,1}(V)] &= \mathfrak{o}(E')^{\mathbb{C}} + \text{span}\{e_1 \wedge e_2 + ie_1 \wedge e_3, e_2 + ie_3\} \\ &\quad + \text{span}\{e_{2a} + ie_{3a} \mid a = 1, \dots, r+q\} \end{aligned}$$

and

$$[\mathfrak{z}^{0,1}(V), W^{0,1}] = W^{0,1}.$$

We show that  $[W^{0,1}, W^{0,1}] \subset \mathbb{C}(e_2 + ie_3)$ . This shows that  $\langle \mathfrak{z}^{0,1} \rangle \supset \mathfrak{o}(E')^{\mathbb{C}} + \mathfrak{z}^{0,1}$ . Since the right-hand side is closed under Lie brackets, we conclude that  $\langle \mathfrak{z}^{0,1} \rangle = \mathfrak{o}(E')^{\mathbb{C}} + \mathfrak{z}^{0,1}$ . It is easy to check that the subalgebra

$\mathfrak{o}(E')^{\mathbb{C}} + \text{span}\{e_1 \wedge e_2 + ie_1 \wedge e_3, e_2 + ie_3\} + \text{span}\{e_{0a} + ie_{1a}, e_{2a} + ie_{3a} \mid a = 1, \dots, r+q\} + W^{0,1}$  coincides with its derived Lie algebra. Therefore, it is an algebraic subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ , see [O-V]. Now, to prove that  $\langle \mathfrak{z}^{0,1} \rangle$  is algebraic, it is sufficient to check that  $\mathbb{C}(e_0 + ie_1)$  is algebraic. The element  $e_0 + ie_1 \in \mathfrak{g}^{\mathbb{C}}$  is conjugated to the algebraic element  $e_0 \in \mathfrak{g}^{\mathbb{C}}$  via  $\exp(-ie_1) \in G^{\mathbb{C}}$ . Consequently, it is algebraic.  $\square$

**Theorem 14.** *For the twistor space  $Z$  of any of the homogeneous quaternionic manifolds  $(M = G/K, Q)$  constructed in Thm. 8 there is a natural open  $G$ -equivariant holomorphic immersion*

$$Z \cong G/G_{z_0} \rightarrow \bar{Z} = \bar{Z}(\Pi) := G^{\mathbb{C}}/H$$

*into a homogeneous complex (Hausdorff) manifold of the complex algebraic group  $G^{\mathbb{C}}$ . This immersion is a (universal) finite covering over its image, which is an open  $G$ -orbit. (The group  $H$  was defined on p. 71.)*

**Proof:** It follows from Prop. 12 that  $H \subset G^{\mathbb{C}}$  is closed. This shows that  $G^{\mathbb{C}}/H$  is a homogeneous complex Hausdorff manifold. The inclusions  $G \subset G^{\mathbb{C}}$  and  $G_{z_0} \subset H$  define a  $G$ -equivariant map  $G/G_{z_0} \rightarrow G^{\mathbb{C}}/H$ , which is an immersion since  $\mathfrak{g} \cap \mathfrak{h} = \mathfrak{g}_{z_0}$ . The differential of this immersion at  $eG_{z_0}$  is canonically identified with the restriction  $\phi : \mathfrak{z} \rightarrow \mathfrak{g}^{\mathbb{C}}/\mathfrak{h}$  of the canonical projection  $\mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}/\mathfrak{h}$  to  $\mathfrak{z} \subset \mathfrak{g} = \mathfrak{g}_{z_0} + \mathfrak{z}$ . Obviously, the complex linear extension  $\phi^{\mathbb{C}}$  maps  $\mathfrak{z}^{1,0}$  isomorphically to  $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}$  and  $\mathfrak{z}^{0,1}$  to zero. This shows that  $Z \cong G/G_{z_0} \rightarrow G^{\mathbb{C}}/H$  is open and holomorphic with respect to the  $G$ -invariant complex structure  $\mathbb{J}$  on  $Z$  and the canonical complex structure on  $G^{\mathbb{C}}/H$ .  $\square$



**Theorem 15.** *The homogeneous complex manifold  $\bar{Z} = G^{\mathbb{C}}/H$  carries a  $G^{\mathbb{C}}$ -invariant holomorphic hyperplane distribution  $\mathcal{D} \subset T\bar{Z}$ . The hyperplane  $\mathcal{D}_{z_0} = T_{z_0}Z^{hor} \subset T_{z_0}Z = T_{z_0}\bar{Z}$  is the horizontal space associated to the  $G$ -invariant quaternionic connection  $\nabla$  on  $M$  constructed in Lemma 6. Moreover,  $\mathcal{D}$  defines a complex contact structure on  $\bar{Z} = \bar{Z}(\Pi)$  if and only if  $\Pi$  is nondegenerate. In this case the restriction  $\mathcal{D}|_Z$  coincides with the canonical complex contact structure on the twistor space  $Z$  of the quaternionic pseudo-Kähler manifold  $M$ .*

**Proof:** Recall that we identify  $T_{z_0}Z$  with the  $\mathfrak{g}_{z_0}$ -invariant subspace  $\mathfrak{z} \subset \mathfrak{g} = \mathfrak{g}_{z_0} + \mathfrak{z}$  complementary to  $\mathfrak{g}_{z_0}$ . Hereby the subspace  $\mathcal{D}_{z_0} = T_{z_0}Z^{hor}$  is identified with  $\mathfrak{z}^{hor} = \{\tilde{x} | x \in \mathfrak{m}\} \subset \mathfrak{z}$ , where  $\tilde{x} = x + 2\omega_2(x)e_1 \wedge e_3 - 2\omega_3(x)e_1 \wedge e_2$  is the  $\nabla$ -horizontal lift of  $x$ , see equation (18). The subspace  $\mathfrak{z}^{hor} \subset \mathfrak{z}$  is  $\mathbb{J}_{z_0}$ -invariant by the very definition of the complex structure  $\mathbb{J}_{z_0}$  and  $\mathbb{J}_{z_0}|_{\mathfrak{z}^{hor}}$  is given by  $\mathbb{J}_{z_0}\tilde{x} = \widetilde{J_1x}$  for all  $x \in \mathfrak{m}$ . The subspace  $(\mathfrak{z}^{hor})^{1,0} = (\mathfrak{z}^{hor})^{\mathbb{C}} \cap \mathfrak{z}^{1,0} \subset \mathfrak{z}^{1,0}$  is identified with the  $i$ -eigenspace  $\mathcal{D}_{z_0}^{1,0} \subset T_{z_0}^{1,0}Z = T_{z_0}^{1,0}\bar{Z}$  of  $\mathbb{J}_{z_0}$  on  $\mathcal{D}_{z_0}$ . In order to prove that  $\mathcal{D}_{z_0}^{1,0}$  extends to a  $G^{\mathbb{C}}$ -invariant holomorphic distribution  $\mathcal{D}^{1,0} \subset T^{1,0}\bar{Z}$  it is sufficient to check the following lemma.

**Lemma 8.** *The complex Lie algebra  $\mathfrak{h} = \mathfrak{g}_{z_0}^{\mathbb{C}} + \mathfrak{z}^{0,1}$  preserves the projection of  $(\mathfrak{z}^{hor})^{1,0} = \{\tilde{x} - iJ_1x | x \in \mathfrak{m}\} = \text{span}_{\mathbb{C}}\{e_2 + e_1 \wedge e_3 - i(e_3 - e_1 \wedge e_2), e_0 - ie_1, e_{0a} - ie_{1a}, e_{2a} - ie_{3a} | a = 1, \dots, r + q\} + W^{1,0}$  into  $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}$ , i.e.*

$$[\mathfrak{h}, (\mathfrak{z}^{hor})^{1,0}] \subset \mathfrak{h} + (\mathfrak{z}^{hor})^{1,0}.$$

Now  $\mathcal{D}^{1,0}$  defines a complex contact structure if and only if the Frobenius form  $\wedge^2 \mathcal{D}^{1,0} \xrightarrow{[\cdot, \cdot]} T^{1,0}\bar{Z}/\mathcal{D}^{1,0}$  is nondegenerate, which is equivalent to the nondegeneracy of the skew symmetric complex bilinear form

$$\omega : \wedge^2 (\mathfrak{z}^{hor})^{1,0} \xrightarrow{[\cdot, \cdot]} \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}/(\mathfrak{h} + (\mathfrak{z}^{hor})^{1,0}) \cong \mathbb{C}.$$

**Lemma 9.** *Let  $(\mathfrak{z}^{hor})^{1,0} = (\mathfrak{z}(V)^{hor})^{1,0} + W^{1,0}$  be the decomposition of  $(\mathfrak{z}^{hor})^{1,0}$  induced by the decomposition  $\mathfrak{z} = \mathfrak{z}(V) + W$ . Then  $\omega((\mathfrak{z}(V)^{hor})^{1,0}, W^{1,0}) = 0$ ,  $\omega|_{\wedge^2 (\mathfrak{z}(V)^{hor})^{1,0}}$  is nondegenerate and  $\omega|_{\wedge^2 W^{1,0}}$  is given by*

(19)

$$\omega(s - iJ_1s, t - iJ_1t) = 2(\langle e_2, [s, t] \rangle + i\langle e_3, [s, t] \rangle)(e_2 - ie_3) \pmod{\mathfrak{h} + (\mathfrak{z}^{hor})^{1,0}}.$$

From the lemma it follows that  $\omega$  is nondegenerate if and only if  $\omega|_{\wedge^2 W^{1,0}}$  is nondegenerate. The explicit formula for  $\omega|_{\wedge^2 W^{1,0}}$  given in equation (19) now shows that  $\omega$  is nondegenerate if and only if  $b(\Pi) = b_{\Pi, (e_1, e_2, e_3)}$  is nondegenerate, which is in turn equivalent to the nondegeneracy of  $\Pi$  by Cor. 1. This proves that  $\mathcal{D}$  is a complex contact structure if and only if  $\Pi$  is nondegenerate. If  $\Pi$  is nondegenerate then  $\mathcal{D}_{z_0} = T_{z_0}Z^{hor}$  is precisely the horizontal space associated to the Levi-Civita connection  $\nabla^g$  of the quaternionic pseudo-Kähler manifold  $(M, Q, g)$  and  $\mathcal{D}$  is the canonical complex contact structure on its twistor space  $Z$ , as defined by Salamon [S1].  $\square$

#### 4. HOMOGENEOUS QUATERNIONIC SUPERMANIFOLDS ASSOCIATED TO SUPEREXTENDED POINCARÉ ALGEBRAS

In this section we will show that our main result, Thm. 8, has a natural supergeometric analogue, Thm. 17. The fundamental idea is to replace the map  $\Pi : \wedge^2 W \rightarrow V$  defined on the exterior square  $\wedge^2 W$  by an  $\mathfrak{o}(V)$ -equivariant linear map  $\Pi : \vee^2 W \rightarrow V$  defined on the symmetric square  $\vee^2 W = \text{Sym}^2 W$ . We will freely use the language of supergeometry. The necessary background is outlined in the appendix.

**4.1. Superextended Poincaré algebras.** Let  $(V, \langle \cdot, \cdot \rangle)$  be a pseudo-Euclidean vector space,  $W$  a  $C^\infty(V)$ -module and  $\Pi : \vee^2 W \rightarrow V$  an  $\mathfrak{o}(V)$ -equivariant linear map. We recall that  $\mathfrak{o}(V)$  acts on  $W$  via  $\text{ad}^{-1} : \mathfrak{o}(V) \rightarrow \mathfrak{spin}(V) \subset C^\infty(V)$ , see equation (2).

Given these data we extend the Lie bracket on  $\mathfrak{p}_0 := \mathfrak{p}(V)$  to a super Lie bracket (see Def. 16)  $[\cdot, \cdot]$  on the  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{p}_0 + \mathfrak{p}_1$ ,  $\mathfrak{p}_1 = W$ , by the following requirements:

- 1) The adjoint representation (see Def. 28) of  $\mathfrak{o}(V)$  on  $\mathfrak{p}_1$  coincides with the natural representation of  $\mathfrak{o}(V) \cong \mathfrak{spin}(V)$  on  $W = \mathfrak{p}_1$  and  $[V, \mathfrak{p}_1] = 0$ .
- 2)  $[s, t] = \Pi(s \vee t)$  for all  $s, t \in W$ .

The super Jacobi identity follows from 1) and 2). The resulting super Lie algebra will be denoted by  $\mathfrak{p}(\Pi)$ .

**Definition 13.** Any super Lie algebra  $\mathfrak{p}(\Pi)$  as above is called a **superextended Poincaré algebra (of signature  $(p, q)$ )** if  $V \cong \mathbb{R}^{p,q}$ .  $\mathfrak{p}(\Pi)$  is called **nondegenerate** if  $\Pi$  is nondegenerate, i.e. if the map  $W \ni s \mapsto \Pi(s \vee \cdot) \in W^* \otimes V$  is injective.

The structure of superextended Poincaré algebra on the vector space  $\mathfrak{p}(V) + W$  is completely determined by the map  $\Pi : \vee^2 W \rightarrow V$ . An explicit basis for the vector space  $(\vee^2 W^* \otimes V)^{\mathfrak{o}(V)}$  of such  $\mathfrak{o}(V)$ -equivariant linear maps was constructed in [A-C2] for all  $V$  and  $W$ .

**4.2. The canonical supersymmetric bilinear form  $b$ .** Let  $V = \mathbb{R}^{p,q}$  be the standard pseudo-Euclidean vector space with scalar product  $\langle \cdot, \cdot \rangle$  of signature  $(p, q)$ . From now on we fix a decomposition  $p = p' + p''$  and assume that  $p' \equiv 3 \pmod{4}$ , see Remark 7 below. We denote by  $(e_i) = (e_1, \dots, e_{p'})$  the first  $p'$  basis vectors of the standard basis of  $V$  and by  $(e'_i) = (e'_1, \dots, e'_{p''+q})$  the remaining ones. The two complementary orthogonal subspaces of  $V$  spanned by these bases are denoted by  $E = \mathbb{R}^{p'} = \mathbb{R}^{p',0}$  and  $E' = \mathbb{R}^{p'',q}$  respectively. The vector spaces  $V$ ,  $E$  and  $E'$  are oriented by their standard orthonormal bases. E.g. the orientation of Euclidean  $p'$ -space  $E$  defined by the basis  $(e_i)$  is  $e_1^* \wedge \dots \wedge e_{p'}^* \in \wedge^{p'} E^*$ . Here  $(e_i^*)$  denotes the basis of  $E^*$  dual to  $(e_i)$ . Now let  $\mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  be a superextended Poincaré algebra of signature  $(p, q)$  and  $(\tilde{e}_i)$  any orthonormal basis of  $E$ . Then we define a  $\mathbb{R}$ -bilinear

form  $b_{\Pi,(\tilde{e}_i)}$  on the  $\mathcal{C}^\ell(V)$ -module  $W$  by:

$$(20) \quad b_{\Pi,(\tilde{e}_i)}(s, t) = \langle \tilde{e}_1, [\tilde{e}_2 \dots \tilde{e}_{p'} s, t] \rangle = \langle \tilde{e}_1, \Pi(\tilde{e}_2 \dots \tilde{e}_{p'} s \vee t) \rangle, \quad s, t \in W.$$

We put  $b = b(\Pi) := b_{\Pi,(e_i)}$  for the standard basis  $(e_i)$  of  $E$ . As in 4.1 we consider  $W = \mathfrak{p}_1$  as  $\mathbb{Z}_2$ -graded vector space of purely odd degree and recall that, by Def. 22, an even supersymmetric (respectively, super skew symmetric) bilinear form on  $W$  is simply an ordinary skew symmetric (respectively, symmetric) bilinear form on  $W$ .

**Remark 7:** Equation (20) defines an even super skew symmetric bilinear form on  $W$  if  $p' \equiv 1 \pmod{4}$ . For even  $p'$  the above formula does not make sense, unless one assumes that  $W$  is a  $\mathcal{C}^\ell(V)$ -module rather than a  $\mathcal{C}^{\ell^0}(V)$ -module. Here we are only interested in the case  $p' \equiv 3 \pmod{4}$ . Moreover, later on, for the construction of homogeneous quaternionic supermanifolds we will put  $p' = 3$ .

**Theorem 16.** *The bilinear form  $b$  has the following properties:*

- 1)  $b_{\Pi,(\tilde{e}_i)} = \pm b$  if  $\tilde{e}_1 \wedge \dots \wedge \tilde{e}_{p'} = \pm e_1 \wedge \dots \wedge e_{p'}$ . In particular, our definition of  $b$  does not depend on the choice of positively oriented orthonormal basis of  $E$ .
- 2)  $b$  is an even supersymmetric bilinear form.
- 3)  $b$  is invariant under the connected subgroup  $K(p', p'') = \text{Spin}(p') \cdot \text{Spin}_0(p'', q) \subset \text{Spin}_0(p, q)$  (and is not  $\text{Spin}_0(p, q)$ -invariant, unless  $p'' + q = 0$ ).
- 4) Under the identification  $\mathfrak{o}(V) = \wedge^2 V = \wedge^2 E + \wedge^2 E' + E \wedge E'$ , see equation (3), the subspace  $E \wedge E'$  acts on  $W$  by  $b$ -symmetric endomorphisms and the subalgebra  $\wedge^2 E \oplus \wedge^2 E' \cong \mathfrak{o}(p') \oplus \mathfrak{o}(p'', q)$  acts on  $W$  by  $b$ -skew symmetric endomorphisms.

**Proof:** The proof is the same as for Thm. 1, up to the modifications caused by the fact that  $\Pi$  is now symmetric instead of skew symmetric. Part 2) e.g. follows from the next computation, in which we use that  $p' \equiv 3 \pmod{4}$ :

$$\begin{aligned} b(t, s) &= \langle e_1, [e_2 \dots e_{p'} t, s] \rangle \\ &= -\langle e_1, [e_4 \dots e_{p'} t, e_2 e_3 s] \rangle + \langle e_1, \text{ad}(e_2 e_3)[e_4 \dots e_{p'} t, s] \rangle \\ &= -\langle e_1, [e_4 \dots e_{p'} t, e_2 e_3 s] \rangle = \dots = -\langle e_1, [t, e_2 \dots e_{p'} s] \rangle \\ &= -\langle e_1, [e_2 \dots e_{p'} s, t] \rangle = -b(s, t). \quad \square \end{aligned}$$

**Definition 14.** *The bilinear form  $b = b(\Pi) = b_{\Pi,(e_1, \dots, e_{p'})}$  defined above is called the canonical supersymmetric bilinear form on  $W$  associated to the  $\mathfrak{o}(V)$ -equivariant map  $\Pi : \vee^2 W \rightarrow V = \mathbb{R}^{p,q}$  and the decomposition  $p = p' + p''$ .*

**Proposition 13.** *The kernels of the linear maps  $\Pi : W \rightarrow W^* \otimes V$  and  $b = b(\Pi) : W \rightarrow W^*$  coincide:  $\ker \Pi = \ker b$ .*

**Proof:** See the proof of Prop. 1.  $\square$

**Corollary 8.**  *$\mathfrak{p}(\Pi)$  is nondegenerate (see Def. 13) if and only if  $b(\Pi)$  is nondegenerate.*

**4.3. The main theorem in the super case.** Any superextended Poincaré algebra  $\mathfrak{p} = \mathfrak{p}(\Pi) = \mathfrak{p}(V) + W$  has an even derivation  $D$  with eigenspace decomposition  $\mathfrak{p} = \mathfrak{o}(V) + V + W$  and corresponding eigenvalues  $(0, 1, 1/2)$ . Therefore, the super Lie algebra  $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 = \mathfrak{p}(V) + W$  is canonically extended to a super Lie algebra  $\mathfrak{g} = \mathfrak{g}(\Pi) = \mathbb{R}D + \mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$ , where  $\mathfrak{g}_0 = \mathbb{R}D + \mathfrak{p}_0 = \mathbb{R}D + \mathfrak{p}(V) = \mathfrak{g}(V)$  and  $\mathfrak{g}_1 = \mathfrak{p}_1 = W$ .

**Proposition 14.** *The adjoint representation (see Def. 28) of  $\mathfrak{g}$  is faithful and moreover it induces a faithful representation on its ideal  $\mathfrak{r} = \mathbb{R}D + V + W \subset \mathfrak{g} = \mathfrak{o}(V) + \mathfrak{r}$ .*

By Prop. 14 we can consider  $\mathfrak{g}$  as subalgebra of the super Lie algebra  $\mathfrak{gl}(\mathfrak{r})$  (defined in A.3), i.e.  $\mathfrak{g}$  is a linear super Lie algebra (see Def. 28). We denote by  $G = G(\Pi)$  the corresponding linear Lie supergroup, see A.3. Its underlying Lie group is  $G_0 = G(V)$  and has  $\mathfrak{g}_0 = \mathfrak{g}(V)$  as Lie algebra. For the construction of homogeneous quaternionic supermanifolds we will assume that  $V = \mathbb{R}^{p,q}$  with  $p \geq 3$ . Then we fix the decomposition  $p = p' + p''$ , where now  $p' = 3$  and  $p'' = p - 3 =: r$ . As before, we have a corresponding orthogonal decomposition  $V = E + E'$ , the subalgebra  $\mathfrak{k} = \mathfrak{k}(p', p'') = \mathfrak{k}(3, r) = \mathfrak{o}(3) \oplus \mathfrak{o}(r, q) \subset \mathfrak{o}(p, q) = \mathfrak{o}(V)$  preserving this decomposition and the corresponding linear Lie subgroup  $K = K(3, r) \subset G_0$ . We are interested in the homogeneous supermanifold (see A.4):

$$M = M(\Pi) := G/K = G(\Pi)/K.$$

Its underlying manifold is the homogeneous manifold

$$M_0 := G_0/K = G(V)/K = M(V).$$

**Theorem 17.** 1) *There exists a  $G$ -invariant quaternionic structure  $Q$  on  $M = G/K$ .*

2) *If  $\Pi$  is nondegenerate (see Def. 13) then there exists a  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M$  such that  $(M, Q, g)$  is a quaternionic pseudo-Kähler supermanifold.*

**Proof:** First let  $(Q_0, g_0)$  denote the  $G_0$ -invariant quaternionic pseudo-Kähler structure on  $M_0 = G_0/K$  which was introduced in the proof of Thm. 8 (previously it was denoted simply by  $(Q, g)$ ). As in that proof, see equation (13), the corresponding  $K$ -invariant quaternionic structure on  $T_{eK}M_0 \cong \mathfrak{g}_0/\mathfrak{k}$  is extended to a  $K$ -invariant quaternionic structure  $Q_{eK}$  on  $T_{eK}M \cong \mathfrak{g}/\mathfrak{k}$  defining a  $G$ -invariant almost quaternionic structure  $Q$  on  $G/K$  (see A.2 and A.4). Similarly, the  $G_0$ -invariant pseudo-Riemannian metric  $g_0$  on  $M_0$  corresponds to a  $K$ -invariant pseudo-Euclidean scalar product on  $\mathfrak{g}_0/\mathfrak{k}$ . This scalar product is extended by  $b = b_{\Pi, (e_1, e_2, e_3)}$ , in the obvious way, to a  $K$ -invariant supersymmetric bilinear form  $g_{eK}$  on  $\mathfrak{g}/\mathfrak{k}$ , which is nondegenerate if  $\Pi$  is nondegenerate. Let us first treat the case where  $\Pi$  is nondegenerate. In this case  $g_{eK}$  defines a  $G$ -invariant  $Q$ -Hermitian pseudo-Riemannian metric  $g$  on  $G/K$ . The Levi-Civita connection of the homogeneous pseudo-Riemannian supermanifold  $(M, g)$  is computed in Lemma 10 below. As on p. 60,  $\mathfrak{g}/\mathfrak{k}$  is identified with

the complement  $\mathfrak{m} = \mathfrak{m}(V) + W$  to  $\mathfrak{k}$  in  $\mathfrak{g}$ . We use the  $g_0$ -orthonormal basis  $(e_{ii})$  of  $\mathfrak{m}(V)$  introduced on p. 61 and recall that  $\epsilon_i = g_0(e_{ii}, e_{ii})$ . Also we will continue to write  $x \wedge_g y$  for the  $g_{eK}$ -skew symmetric endomorphism of  $\mathfrak{m}$  defined for  $x, y \in \mathfrak{m}(V)$  by:  $x \wedge_g y(z) := g_{eK}(y, z)x - g_{eK}(x, z)y$ ,  $z \in \mathfrak{m}$ .

**Lemma 10.** *The Nomizu map  $L^g = L(\nabla^g)$  associated to the Levi-Civita connection  $\nabla^g$  of the homogeneous pseudo-Riemannian supermanifold  $(M = G/K, g)$  is given by the following formulas:*

$$\begin{aligned} L_{e_0}^g &= L_{e_{\alpha\alpha}}^g = 0, \\ L_{e_\alpha}^g &= \frac{1}{2}J_\alpha + \bar{L}_{e_\alpha}^g, \end{aligned}$$

where

$$\bar{L}_{e_\alpha}^g = \frac{1}{2} \sum_{i=0}^{r+q} \epsilon_i e_{\alpha i} \wedge_g e'_i - \frac{1}{2} \sum_{i=0}^{r+q} \epsilon_i e_{\gamma i} \wedge_g e_{\beta i} \in \mathfrak{z}(Q_{eK}),$$

$L_{e'_a}^g \in \mathfrak{z}(Q_{eK})$  is given by:

$$\begin{aligned} L_{e'_a}^g | \mathfrak{m}(V) &= \sum_{i=0}^3 e_{ia} \wedge_g e_i, \\ L_{e'_a}^g | W &= \frac{1}{2} e_1 e_2 e_3 e'_a. \end{aligned}$$

For all  $s \in W$  the Nomizu operator  $L_s^g \in \mathfrak{z}(Q_{eK})$  maps the subspace  $\mathfrak{m}(V) \subset \mathfrak{m} = \mathfrak{m}(V) + W$  into  $W$  and  $W$  into  $\mathfrak{m}(V)$ . The restriction  $L_s^g | W$  ( $s \in W$ ) is completely determined by  $L_s^g | \mathfrak{m}(V)$  (and vice versa) according to the relation

$$g_{eK}(L_s^g t, x) = g_{eK}(t, L_s^g x), \quad s, t \in W, \quad x \in \mathfrak{m}(V).$$

Finally,  $L_s^g | \mathfrak{m}(V)$  ( $s \in W$ ) is completely determined by its values on the quaternionic basis  $(e'_i)$ ,  $i = 0, \dots, r+q$ , which are as follows:

$$\begin{aligned} L_s^g e'_0 &= \frac{1}{2}s, \\ L_s^g e'_a &= \frac{1}{2}e_1 e_2 e_3 e'_a s. \end{aligned}$$

(It is understood that  $L_x^g = \text{ad}_x | \mathfrak{m}$  for all  $x \in \mathfrak{k}$ , cf. equation (24).) In the above formulas  $a = 1, \dots, r+q$ ,  $\alpha = 1, 2, 3$  and  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ .

**Proof:** This follows from equation (26) by a straightforward computation.  $\square$

**Corollary 9.** *The Levi-Civita connection  $\nabla^g$  of the homogeneous almost quaternionic pseudo-Hermitian supermanifold  $(M, Q, g)$  preserves  $Q$  and hence  $(M, Q, g)$  is a quaternionic pseudo-Kähler supermanifold.*

By Cor. 9 we have already established part 2) of Thm. 17. Part 1) is a consequence of 2) provided that  $\Pi$  is nondegenerate. It remains to discuss the case of degenerate  $\Pi$ . From the  $\mathfrak{o}(V)$ -equivariance of  $\Pi$  it follows that  $W_0 = \ker \Pi \subset W$  is an  $\mathfrak{o}(V)$ -submodule. Let  $W'$  be a complementary  $\mathfrak{o}(V)$ -submodule. Then  $W_0$  and  $W'$  are  $C^\infty(V)$ -submodules; we put  $\Pi' := \Pi|_{\sqrt{2}W'}$ . We denote by  $(M' := M(\Pi'), Q', g')$  the corresponding quaternionic pseudo-Kähler supermanifold and by  $L' := L^{g'} : \mathfrak{g}(\Pi') \rightarrow \text{End}(\mathfrak{m}(\Pi'))$  the Nomizu map associated to its Levi-Civita connection. By the next lemma, we can extend the map  $L'$  to a torsionfree Nomizu map  $L : \mathfrak{g}(\Pi) \rightarrow \text{End}(\mathfrak{m}(\Pi))$ , whose image normalizes  $Q_{eK}$ . This proves the 1-integrability of  $Q$  (by Cor. 10), completing the proof of Thm. 17.  $\square$

By Cor. 9 we can decompose  $L'_x = \sum_{\alpha=1}^3 \omega'_\alpha(x)J_\alpha + \bar{L}'_x$ , where the  $\omega'_\alpha$  are 1-forms on  $\mathfrak{m}(\Pi')$  and  $\bar{L}'_x \in \mathfrak{z}(Q'_{eK})$  belongs to the centralizer of the quaternionic structure  $Q'_{eK} = Q_{eK}|_{\mathfrak{m}(\Pi')}$  on  $\mathfrak{m}(\Pi')$ .

**Lemma 11.** *The Nomizu map  $L' : \mathfrak{g}(\Pi') \rightarrow \text{End}(\mathfrak{m}(\Pi'))$  associated to the Levi-Civita connection of  $(M(\Pi') = G(\Pi')/K, g')$  can be extended to the Nomizu map  $L : \mathfrak{g}(\Pi) \rightarrow \text{End}(\mathfrak{m}(\Pi))$  of a  $G(\Pi)$ -invariant quaternionic connection  $\nabla$  on the homogeneous almost quaternionic supermanifold  $(M(\Pi) = G(\Pi)/K, Q)$ . The extension is defined as follows:*

$$L_x := \sum_{\alpha=1}^3 \omega_\alpha(x)J_\alpha + \bar{L}_x, \quad x \in \mathfrak{m}(\Pi),$$

where  $\bar{L}_x \in \mathfrak{z}(Q)$  (the centralizer is taken in  $\mathfrak{gl}(\mathfrak{m}(\Pi))$ ) is defined below and the 1-forms  $\omega_\alpha$  on  $\mathfrak{m}(\Pi) := \mathfrak{m}(\Pi') + W_0$  satisfy  $\omega_\alpha|_{\mathfrak{m}(\Pi')} := \omega'_\alpha$  and  $\omega_\alpha|_{W_0} := 0$ . The operators  $\bar{L}_x$  are given by:

$$\bar{L}_x|_{\mathfrak{m}(\Pi')} := \bar{L}'_x \quad \text{if } x \in \mathfrak{m}(\Pi'),$$

$$\bar{L}_{e_0}|_{W_0} := \bar{L}_{e_{\alpha\alpha}}|_{W_0} := \bar{L}_{e_\alpha}|_{W_0} := 0,$$

$$\bar{L}_{e'_\alpha}|_{W_0} := \frac{1}{2}e_1e_2e_3e'_\alpha,$$

$$\bar{L}_s|_{W_0} := 0 \quad \text{if } s \in W,$$

$$\bar{L}_s x := (-1)^{\tilde{x}} L_x s - [s, x] \quad \text{if } s \in W_0, \quad x \in \mathfrak{m}(\Pi').$$

(It is understood that  $L_x = \text{ad}_x|_{\mathfrak{m}(\Pi)}$  for all  $x \in \mathfrak{k}$ .) In the above formulas, as usual,  $a = 1, \dots, r+q$ ,  $\alpha = 1, 2, 3$  and  $\tilde{x} \in \mathbb{Z}_2 = \{0, 1\}$  stands for the  $\mathbb{Z}_2$ -degree of  $x \in \mathfrak{m}(\Pi') = \mathfrak{m}(\Pi')_0 + \mathfrak{m}(\Pi')_1 = \mathfrak{m}(V) + W'$ .

**Proof:** The proof is similar to that of Lemma 6.  $\square$

APPENDIX A. SUPERGEOMETRY

In this appendix we summarize the supergeometric material needed in 4. Standard references on supergeometry are [M], [L] and [K], see also [B-B-H], [Ba], [Be], [Bern], [B-O], [DW], [F], [O1], [O2], [Sch] and [S-W]. (D.A. Leites has informed us that he will soon publish a monograph on supergeometry.)

**A.1. Supermanifolds.** Let  $V = V_0 + V_1$  be a  $\mathbb{Z}_2$ -graded vector space. We recall that an element  $x \in V$  is called **homogeneous** (or **of pure degree**) if  $x \in V_{\tilde{x}}$ . The **degree** of a homogeneous element  $x \in V$  is the number  $\tilde{x} \in \mathbb{Z}_2 = \{0, 1\}$  such that  $x \in V_{\tilde{x}}$ . The element  $x \in V$  is said to be even if  $\tilde{x} = 0$  and odd if  $\tilde{x} = 1$ . For  $V_0$  and  $V_1$  of finite dimension, the **dimension** of  $V$  is defined as  $\dim V := \dim V_0 \mid \dim V_1 = m \mid n$  and a **basis** of  $V$  is by definition a tuple  $(x_1, \dots, x_m, \xi_1, \dots, \xi_n)$  such that  $(x_1, \dots, x_m)$  is a basis of  $V_0$  and  $(\xi_1, \dots, \xi_n)$  is a basis of  $V_1$ .

**Definition 15.** Let  $\mathbf{A}$  be a  $\mathbb{Z}_2$ -graded algebra. The **supercommutator** is the bilinear map  $[\cdot, \cdot] : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$  defined by:

$$[a, b] := ab - (-1)^{\tilde{a}\tilde{b}}ba$$

for all homogeneous elements  $a, b \in \mathbf{A}$ . The algebra  $\mathbf{A}$  is called **supercommutative** if  $[a, b] = 0$  for all  $a, b \in \mathbf{A}$ . A  $\mathbb{Z}_2$ -graded supercommutative associative (real) algebra  $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$  will simply be called a **superalgebra**.

**Example 1:** The exterior algebra  $\wedge E = \wedge^{\text{even}} E + \wedge^{\text{odd}} E$  over a finite dimensional vector space  $E$  is a superalgebra.

**Definition 16.** A **super Lie bracket** on a  $\mathbb{Z}_2$ -graded vector space  $V = V_0 + V_1$  is a bilinear map  $[\cdot, \cdot] : V \times V \rightarrow V$  such that for all  $x, y, z \in V_0 \cup V_1$  we have:

- i)  $\widetilde{[x, y]} = \tilde{x} + \tilde{y}$ ,
- ii)  $[x, y] = -(-1)^{\tilde{x}\tilde{y}}[y, x]$  and
- iii)  $[x, [y, z]] = [[x, y], z] + (-1)^{\tilde{x}\tilde{y}}[y, [x, z]]$  (“super Jacobi identity”).

The  $\mathbb{Z}_2$ -graded algebra with underlying  $\mathbb{Z}_2$ -graded vector space  $V$  and product defined by the super Lie bracket  $[\cdot, \cdot]$  is called a **super Lie algebra**.

**Example 2:** The supercommutator of any associative  $\mathbb{Z}_2$ -graded algebra  $\mathbf{A}$  is a super Lie bracket and hence defines on it the structure of super Lie algebra. For example, we may take  $\mathbf{A} = \text{End}(V)$  with the obvious structure of  $\mathbb{Z}_2$ -graded associative algebra (with unit). The corresponding super Lie algebra is denoted by  $\mathfrak{gl}(V)$  and is called the **general linear super Lie algebra**.

Let  $M_0$  be a (differentiable) manifold of dimension  $m$ . We denote by  $C_{M_0}^\infty$  its sheaf of functions. Sections of the sheaf  $C_{M_0}^\infty$  over an open set  $U \subset M_0$  are simply smooth functions on  $U$ :  $C_{M_0}^\infty(U) = C^\infty(U)$ . Now let  $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$  be a sheaf of superalgebras over  $M_0$ .

**Definition 17.** The pair  $M = (M_0, \mathcal{A})$  is called a (differentiable) **supermanifold of dimension**  $\dim M = m|n$  over  $M_0$  if for all  $p \in M_0$  there exists an open neighborhood  $U \ni p$  and a rank  $n$  free sheaf  $\mathcal{E}_U$  of  $\mathcal{C}_U^\infty$ -modules over  $U$  such that  $\mathcal{A}|_U \cong \wedge \mathcal{E}_U$  (as sheaves of superalgebras). A **function** on  $M$  (over an open set  $U \subset M_0$ ) is by definition a section of  $\mathcal{A}$  (over  $U$ ). The sheaf  $\mathcal{A} = \mathcal{A}_M$  is called the **sheaf of functions** on  $M$  and  $M_0$  is called the **manifold underlying the supermanifold**  $M$ . Let  $M = (M_0, \mathcal{A}_M)$  and  $N = (N_0, \mathcal{A}_N)$  be supermanifolds. A **morphism**  $\varphi : M \rightarrow N$  is a pair  $\varphi = (\varphi_0, \varphi^*)$ , where  $\varphi_0 : M_0 \rightarrow N_0$  is a smooth map and  $\varphi^* : \mathcal{A}_N \rightarrow (\varphi_0)_* \mathcal{A}_M$  is a morphism of sheaves of superalgebras. It is called an **isomorphism** if  $\varphi_0$  is a diffeomorphism and  $\varphi^*$  an isomorphism. An isomorphism  $\varphi : M \rightarrow M$  is called an **automorphism** of  $M$ . The set of all morphisms  $\varphi : M \rightarrow N$  (respectively, automorphisms  $\varphi : M \rightarrow M$ ) is denoted by  $\text{Mor}(M, N)$  (respectively,  $\text{Aut}(M)$ ).

From Def. 17 it follows that there exists a canonical epimorphism of sheaves  $\epsilon^* : \mathcal{A} \rightarrow \mathcal{C}_{M_0}^\infty$ , which is called the **evaluation map**. Its kernel is the ideal generated by  $\mathcal{A}_1$ :  $\ker \epsilon^* = \langle \mathcal{A}_1 \rangle = \mathcal{A}_1 + \mathcal{A}_1^2$ .

Given supermanifolds  $L, M, N$  and morphisms  $\psi \in \text{Mor}(L, M)$  and  $\varphi \in \text{Mor}(M, N)$ , there is a **composition**  $\varphi \circ \psi \in \text{Mor}(L, N)$  defined by:

$$(\varphi \circ \psi)_0 = \varphi_0 \circ \psi_0 \quad \text{and} \quad (\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

Here we have used the same symbol  $\psi^*$  for the map  $(\varphi_0)_* \mathcal{A}_M \rightarrow (\varphi_0)_*(\psi_0)_* \mathcal{A}_L = (\varphi_0 \circ \psi_0)_* \mathcal{A}_L$  induced by  $\psi^* : \mathcal{A}_M \rightarrow (\psi_0)_* \mathcal{A}_L$ . Similarly, if  $\varphi : M \rightarrow N$  is an isomorphism, then we can define it **inverse isomorphism** by:

$$\varphi^{-1} := (\varphi_0^{-1}, (\varphi^*)^{-1}) : N \rightarrow M.$$

Here, again, we have used the same notation  $(\varphi^*)^{-1}$  for the map  $\mathcal{A}_M \rightarrow (\varphi_0^{-1})_* \mathcal{A}_N$  induced by  $(\varphi^*)^{-1} : (\varphi_0)_* \mathcal{A}_M \rightarrow \mathcal{A}_N$ . Finally, for every supermanifold  $M = (M_0, \mathcal{A})$ , there is the **identity** automorphism  $\text{Id}_M := (\text{Id}_{M_0}, \text{Id}_{\mathcal{A}})$ . The above operations turn the set  $\text{Aut}(M)$  into a group.

**Example 3:** Let  $E \rightarrow M_0$  be a (smooth) vector bundle of rank  $n$  over the  $m$ -dimensional manifold  $M_0$  and  $\mathcal{E}$  its sheaf of sections. It is a locally free sheaf of  $\mathcal{C}_{M_0}^\infty$ -modules and  $SM(E) := (M_0, \wedge \mathcal{E})$  is a supermanifold of dimension  $m|n$ . Its evaluation map is the canonical projection onto 0-forms  $\wedge \mathcal{E} = \mathcal{C}_{M_0}^\infty + \sum_{j=1}^n \wedge^j \mathcal{E} \rightarrow \mathcal{C}_{M_0}^\infty$ . It is well known, see [Ba], that any supermanifold is isomorphic to a supermanifold of the form  $SM(E)$ . However, the isomorphism is not canonical, unless  $n = 0$ .

**Example 4:** Any manifold  $(M_0, \mathcal{C}_{M_0}^\infty)$  of dimension  $m$  can be considered as a supermanifold of dimension  $m|0$ . In fact, it is associated to the vector bundle of rank 0 over  $M_0$  via the construction of Example 3. For any supermanifold  $M = (M_0, \mathcal{A})$  the pair  $(\text{Id}_{M_0}, \epsilon^*)$  defines a canonical morphism  $\epsilon : M_0 \rightarrow M$ . The composition of  $\epsilon$  with the canonical constant map  $p : \{p\} \rightarrow M_0$  ( $p \in M_0$ ) defines a morphism  $\epsilon_p = (p, \epsilon_p^*) : \{p\} \rightarrow M$ . The epimorphism  $\epsilon_p^* : \mathcal{A} \rightarrow \mathbb{R}$  onto the constant sheaf is



called the **evaluation at  $p$** :

$$\epsilon_p^* f = (\epsilon^* f)(p).$$

$f(p) := \epsilon_p^* f \in \mathbb{R}$  is called the **value of  $f$  at the point  $p$** .

**Example 5:** Let  $V = V_0 + V_1$  be a  $\mathbb{Z}_2$ -graded vector space of dimension  $m|n$  and  $E_V = V_1 \times V_0 \rightarrow V_0$  the trivial vector bundle over  $V_0$  with fibre  $V_1$ . Then to  $V$  we can canonically associate the supermanifold  $SM(V) := SM(E_V)$ , see Example 3.

Let  $(x_0^i) = (x_0^1, \dots, x_0^m)$  be local coordinates for  $M_0$  defined on an open set  $U \subset M_0$ ,  $\mathcal{E}_U$  a rank  $n$  free sheaf of  $\mathcal{C}_U^\infty$ -modules over  $U$  and  $\phi : \wedge \mathcal{E}_U \rightarrow \mathcal{A}|_U$  an isomorphism. We can choose sections  $(\xi_0^j) = (\xi_0^1, \dots, \xi_0^n)$  of  $\mathcal{E}_U$  which generate  $\mathcal{E}_U$  freely over  $\mathcal{C}_U^\infty$ . Then any section of  $\wedge \mathcal{E}_U$  is of the form:

$$(21) \quad f = \sum_{\alpha \in \mathbb{Z}_2^n} f_\alpha(x_0^1, \dots, x_0^m) \xi_0^\alpha, \quad f_\alpha(x_0^1, \dots, x_0^m) \in C^\infty(U),$$

where  $\xi_0^\alpha := (\xi_0^1)^{\alpha_1} \wedge \dots \wedge (\xi_0^n)^{\alpha_n}$  for  $\alpha = (\alpha_1, \dots, \alpha_n)$ . The tuple  $(\phi, x_0^i, \xi_0^j)$  is called a **local coordinate system** for  $M$  over  $U$ . The open set  $U \subset M_0$  is called a **coordinate neighborhood** for  $M$ . Any function on  $M$  over  $U$  is of the form  $\phi(f)$  for some section  $f$  as in (21). The functions  $x^i := \phi(x_0^i) \in \mathcal{A}(U)_0$ ,  $\xi^j := \phi(\xi_0^j) \in \mathcal{A}_1$  are called **local coordinates** for  $M$  over  $U$ . The evaluation map  $\epsilon^* : \mathcal{A} \rightarrow \mathcal{C}_{M_0}^\infty$  is expressed in a local coordinate system simply by putting  $\xi_0^1 = \dots = \xi_0^n = 0$  in (21):

$$\epsilon^*(\phi(f)) = f(x_0^1, \dots, x_0^m, 0, \dots, 0) := f_{(0, \dots, 0)}(x_0^1, \dots, x_0^m).$$

In particular, we have  $\epsilon^* x^i = x_0^i$  and  $\epsilon^* \xi^j = 0$ .

**Example 6:** Let  $V = V_0 + V_1$  be a  $\mathbb{Z}_2$ -graded vector space and  $(x^i, \xi^j) = (x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  a basis of the  $\mathbb{Z}_2$ -graded vector space  $V^* = \text{Hom}(V, \mathbb{R})$ . Then  $(x^i, \xi^j)$  can be considered as global coordinates on the supermanifold  $SM(V)$ , i.e. as local coordinates for  $SM(V)$  over  $V_0 = SM(V)_0$ , see Example 5.

Let  $M$  and  $N$  be supermanifolds of dimension  $m|n$  and  $p|q$  respectively. In local coordinates  $(x^i, \xi^j)$  for  $M$  and  $(y^k, \eta^l)$  for  $N$  a morphism  $\varphi$  is expressed by  $p$  even functions  $y^k(x^1, \dots, x^m, \xi^1, \dots, \xi^n) := \varphi^* y^k$  and  $q$  odd functions  $\eta^l(x^1, \dots, x^m, \xi^1, \dots, \xi^n) := \varphi^* \eta^l$ .

There exists a supermanifold  $M \times N = (M_0 \times N_0, \mathcal{A}_{M \times N})$  called the **product** of the supermanifolds  $M$  and  $N$  and morphisms  $\pi_M : M \times N \rightarrow M$ ,  $\pi_N : M \times N \rightarrow N$  such that  $(\pi_M^* x^i, \pi_N^* y^k, \pi_M^* \xi^j, \pi_N^* \eta^l)$  are local coordinates for  $M \times N$  over  $U \times V$  if  $(x^i, \xi^j)$  are local coordinates for  $M$  over  $U$  and  $(y^k, \eta^l)$  are local coordinates for  $N$  over  $V$ . The morphism  $\pi_1 = \pi_M$  (respectively,  $\pi_2 = \pi_N$ ) is called the **projection** of  $M \times N$  onto the first (respectively, second) factor. Given morphisms  $\varphi_i : M_i \rightarrow N_i$ ,  $i = 1, 2$ , there is a corresponding morphism  $\varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow N_1 \times N_2$  such that  $\pi_{N_1} \circ (\varphi_1 \times \varphi_2) = \varphi_1 \circ \pi_{M_1}$  and  $\pi_{N_2} \circ (\varphi_1 \times \varphi_2) = \varphi_2 \circ \pi_{M_2}$ .

As next, we will discuss the notion for tangency on supermanifolds. For this purpose, we recall the following definition.

**Definition 18.** An endomorphism  $X = X_0 + X_1 \in \text{End}(\mathbf{A}) = \text{End}_{\mathbb{R}}(\mathbf{A})$  (here  $\tilde{X}_\alpha = \alpha$ ,  $\alpha = 0, 1$ ) of a  $\mathbb{Z}_2$ -graded algebra  $\mathbf{A}$  is called a **derivation** if it satisfies the Leibniz-rule

$$X_\alpha(ab) = X_\alpha(a)b + (-1)^{\alpha\bar{a}}aX_\alpha(b)$$

for all homogeneous  $a, b \in \mathbf{A}$  and  $\alpha \in \mathbb{Z}_2$ . The  $\mathbb{Z}_2$ -graded vector space of all derivations of  $\mathbf{A}$  is denoted by  $\text{Der } \mathbf{A} = (\text{Der } \mathbf{A})_0 + (\text{Der } \mathbf{A})_1$ .

Notice that the supercommutator on  $\text{End}(\mathbf{A})$  restricts to a super Lie bracket on  $\text{Der } \mathbf{A}$ .

**Definition 19.** Let  $M = (M_0, \mathcal{A})$  be a supermanifold. The **tangent sheaf** of  $M$  is the sheaf of derivations of  $\mathcal{A}$  and is denoted by  $\mathcal{T}_M = (\mathcal{T}_M)_0 + (\mathcal{T}_M)_1$ . A **vector field** on  $M$  is a section of  $\mathcal{T}_M$ . The **cotangent sheaf** is the sheaf  $\mathcal{T}_M^* = \text{Hom}_{\mathcal{A}}(\mathcal{T}_M, \mathcal{A})$ . The **full tensor superalgebra** over  $\mathcal{T}_M$  is the sheaf of superalgebras generated by tensor products ( $\mathbb{Z}_2$ -graded over  $\mathcal{A}$ ) of  $\mathcal{T}_M$  and  $\mathcal{T}_M^*$ . It is denoted by  $\otimes_{\mathcal{A}}\langle \mathcal{T}_M, \mathcal{T}_M^* \rangle$ . A **tensor field** on  $M$  is a section of  $\otimes_{\mathcal{A}}\langle \mathcal{T}_M, \mathcal{T}_M^* \rangle$ .

Explicitly, a section  $X \in \mathcal{T}_M(U)$  ( $U \subset M_0$  open) associates to any open subset  $V \subset U$  a derivation  $X|_V \in \text{Der } \mathcal{A}(V)$  such that

$$X|_V(f|_V) = X|_U(f)|_V \quad \text{for all } f \in \mathcal{A}(U).$$

Given local coordinates  $(x^i, \xi^j)$  on  $M$  over  $U$  there exist unique vector fields

$$\frac{\partial}{\partial x^i} \in \mathcal{T}_M(U)_0 \quad \text{and} \quad \frac{\partial}{\partial \xi^j} \in \mathcal{T}_M(U)_1$$

such that

$$\frac{\partial x^k}{\partial x^i} = \delta_i^k, \quad \frac{\partial \xi^l}{\partial x^i} = 0, \quad \frac{\partial x^k}{\partial \xi^j} = 0, \quad \frac{\partial \xi^l}{\partial \xi^j} = \delta_j^l.$$

(Here, of course,  $\delta_i^k \in \mathcal{A}(U)$  is the unit element in the algebra  $\mathcal{A}(U)$  if  $i = k$  and is zero otherwise.) Moreover, the vector fields  $(\partial/\partial x^i, \partial/\partial \xi^j)$  freely generate  $\mathcal{T}_M(U) \cong \text{Der } \mathcal{A}(U)$  over  $\mathcal{A}(U)$ . This shows that  $\mathcal{T}_M$  is a locally free sheaf of rank  $m|n = \dim M$  over  $\mathcal{A}$ . It is also a sheaf of super Lie algebras; simply because  $\text{Der } \mathcal{A}(U)$  is a subalgebra of the super Lie algebra  $\text{End}_{\mathbb{R}} \mathcal{A}(U)$  for all open  $U \subset M_0$ . As in the case of ordinary manifolds, given a vector field  $X$  there exists a unique derivation  $\mathcal{L}_X$  of the full tensor superalgebra  $\otimes_{\mathcal{A}}\langle \mathcal{T}_M, \mathcal{T}_M^* \rangle$  over  $\mathcal{T}_M$  compatible with contractions such that

$$\mathcal{L}_X f = X(f) \quad \text{and} \quad \mathcal{L}_X Y = [X, Y]$$

for all functions  $f$  and vector fields  $Y$  on  $M$  ( $[X, Y]$  is the *supercommutator* of vector fields).

**Definition 20.** Let  $M = (M_0, \mathcal{A})$  be a supermanifold. A **tangent vector** to  $M$  at  $p \in M_0$  is an  $\mathbb{R}$ -linear map  $v = v_0 + v_1 : \mathcal{A}_p \rightarrow \mathbb{R}$  such that

$$v_\alpha(fg) = v_\alpha(f)\epsilon_p^*(g) + (-1)^{\alpha\bar{f}}\epsilon_p^*(f)v_\alpha(g), \quad \alpha = 0, 1,$$

for all germs of functions  $f, g \in \mathcal{A}_p$  of pure degree. ( $\mathcal{A}_p$  denotes the stalk of  $\mathcal{A}$  at  $p$ .) The  $\mathbb{Z}_2$ -graded vector space of all tangent vectors to  $M$  at  $p$  is denoted by  $T_pM = (T_pM)_0 + (T_pM)_1$  and is called the **tangent space** to  $M$  at  $p$ .

Let  $X$  be a vector field defined on some open set  $U \subset M_0$  and  $p \in U$ . Then we can define the **value**  $X(p) \in T_pM$  of  $X$  at  $p$ :

$$X(p)(f) := \epsilon_p^*(X(f)), \quad f \in \mathcal{A}_p.$$

However, unless  $\dim M = m|n = m|0$ , a vector field is not determined by its values at all  $p \in M_0$ . The above definition of value at a point  $p$  is naturally extended to arbitrary tensor fields  $S$ ; the value of  $S$  at  $p$  is denoted by  $S(p)$ . Again, unless  $\dim M = m|n = m|0$ , a tensor field on  $M$  is not determined by its values at all  $p \in M_0$ .

Given a morphism  $\varphi : M \rightarrow N$ , to any local vector field  $X \in \mathcal{T}_M(U)$  on  $M$  we can associate a vector field  $d\varphi X \in (\varphi^*\mathcal{T}_N)(U)$  on  $N$  with values in  $\mathcal{A}_M$  which is defined by:

$$(d\varphi X)(f) = X(\varphi^*f), \quad f \in \mathcal{A}_N(V),$$

where  $V \subset N_0$  is an open set such that  $\varphi_0^{-1}(V) \supset U$ . We recall that  $\varphi^*\mathcal{T}_N$  is the sheaf of  $\mathcal{A}_M$ -modules over  $M_0$  defined by:

$$\varphi^*\mathcal{T}_N := \mathcal{A}_M \otimes_{\varphi_0^{-1}\mathcal{A}_N} \varphi_0^{-1}\mathcal{T}_N.$$

Here the action of  $\varphi_0^{-1}\mathcal{A}_N$  on  $\mathcal{A}_M$  is defined by the map

$$\varphi_0^{-1}\mathcal{A}_N \rightarrow \varphi_0^{-1}\varphi_*\mathcal{A}_M \rightarrow \mathcal{A}_M$$

induced by  $\varphi^* : \mathcal{A}_N \rightarrow \varphi_*\mathcal{A}_M$ . By the above construction, we obtain a section  $d\varphi$  of the sheaf  $\text{Hom}_{\mathcal{A}}(\mathcal{T}_M, \varphi^*\mathcal{T}_N)$ , which is expressed with respect to local coordinates  $(u^1, \dots, u^{m+n}) = (x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  on  $M$  and  $(v^1, \dots, v^{p+q}) = (y^1, \dots, y^p, \eta^1, \dots, \eta^q)$  on  $N$  by the Jacobian matrix  $(\frac{\partial \varphi^* v^i}{\partial u^j})$ . The value  $d\varphi(p) \in \text{Hom}(T_pM, T_{\varphi_0(p)}N)$  of the differential at a point  $p \in M_0$  is defined by:

$$d\varphi(p)X(p) = (d\varphi X)(p)$$

for all vector fields  $X$  on  $M$ . The differential  $d\epsilon$  of the canonical morphism  $\epsilon : M_0 \rightarrow M$  provides the canonical isomorphism  $d\epsilon(p) : T_pM_0 \xrightarrow{\sim} (T_pM)_0$  for all  $p \in M$ .

**Definition 21.** A morphism  $\varphi : M \rightarrow N$  is called an **immersion** (respectively, a **submersion**) if  $d\varphi$  has constant rank  $m|n = \dim M$  (respectively,  $p|q = \dim N$ ), i.e. if for all local coordinates as above the matrix  $(\epsilon^*(\partial\varphi^*y^i/\partial x^j))$  has constant rank  $m$  (respectively,  $p$ ) and  $(\epsilon^*(\partial\varphi^*\eta^i/\partial \xi^j))$  has constant rank  $n$  (respectively,  $q$ ). An immersion  $\varphi : M \rightarrow N$  is called **injective** (respectively, an **embedding**, a **closed**

**embedding**) and is denoted by  $\varphi : M \hookrightarrow N$  if  $\varphi_0 : M_0 \rightarrow N_0$  is injective (respectively, an embedding, a closed embedding). Two immersions  $\varphi : M \rightarrow N$  and  $\varphi' : M' \rightarrow N$  are called **equivalent** if there exists an isomorphism  $\psi : M \rightarrow M'$  such that  $\varphi = \varphi' \circ \psi$ . A **submanifold** (respectively, an **embedded submanifold**, a **closed submanifold**) is an equivalence class of injective immersions (respectively, embeddings, closed embeddings).

Notice that for any supermanifold the canonical morphism  $\epsilon = (\text{Id}_{M_0}, \epsilon^*) : M_0 \hookrightarrow M$  is a closed embedding.

As for ordinary manifolds, immersions and submersions admit adapted coordinates:

**Proposition 15.** (see [L], [K]) *Let  $M$  and  $N$  be supermanifolds of dimension  $\dim M = m|n$  and  $\dim N = p|q$ . A morphism  $\varphi : M \rightarrow N$  is an immersion (respectively, submersion) if and only if for all  $(x, y = \varphi_0(x)) \in M_0 \times \varphi_0(M_0) \subset M_0 \times N_0$  there exists local coordinates  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  for  $M$  defined near  $x$  and  $(y^1, \dots, y^p, \eta^1, \dots, \eta^q)$  for  $N$  defined near  $y$  such that*

$$\varphi^* y^i = \begin{cases} x^i & \text{for } i = 1, \dots, m \leq p \\ 0 & \text{for } m < i \leq p \end{cases} \quad \text{and} \quad \varphi^* \eta^j = \begin{cases} \xi^j & \text{for } j = 1, \dots, n \leq q \\ 0 & \text{for } n < j \leq q \end{cases}$$

(respectively,

$$\varphi^* y^i = x^i \quad \text{for } i = 1, \dots, p \leq m \quad \text{and} \quad \varphi^* \eta^j = \xi^j \quad \text{for } j = 1, \dots, q \leq n).$$

For any submanifold  $\varphi : M \hookrightarrow N$  we define its **vanishing ideal**  $\mathcal{J}_y \subset (\mathcal{A}_N)_y$  at  $y \in N_0$  as follows:  $\mathcal{J}_y := (\mathcal{A}_N)_y$  if  $y \notin \varphi_0(M_0)$  and  $\mathcal{J}_y := \ker(r \circ \varphi^*)$  if  $y = \varphi_0(x)$ ,  $x \in M_0$ , where  $r : ((\varphi_0)_* \mathcal{A}_M)_y \rightarrow (\mathcal{A}_M)_x$  is the natural restriction homomorphism. The union  $\mathcal{J} := \cup_{y \in N_0} \mathcal{J}_y \subset \mathcal{A}_N$  is called the **vanishing ideal** of the submanifold  $\varphi : M \hookrightarrow N$ . By Prop. 15, it has the following property (P): *If  $y \in \varphi_0(M_0)$  then there exists  $p - m$  even functions  $(y^i)$  and  $q - n$  odd functions  $\eta^j$  on  $N$  vanishing at  $y$  whose germs at  $y$  generate  $\mathcal{J}_y$  and which can be complemented to local coordinates for  $N$  defined near  $y$ . (Here  $m|n = \dim M$  and  $p|q = \dim N$ .) We denote by  $Z_U \subset N_0$  the submanifold defined by the equations  $\epsilon^* y^i = 0$  on some sufficiently small open neighborhood  $U \subset N_0$  of  $y$ . Then  $U$  can be chosen such that the germs of the functions  $(y^i, \eta^j)$  at  $z$  generate  $\mathcal{J}_z$  for all  $z \in Z_U$ .*

Conversely, let  $\varphi_0 : M_0 \hookrightarrow N_0$  be any submanifold and  $\mathcal{J}_y \subset (\mathcal{A}_N)_y$ ,  $y \in N$ , a collection of ideals with the above property (P) and such that  $\mathcal{J}_y = (\mathcal{A}_N)_y$  for  $y \notin \varphi_0(M_0)$ , then there exists a supermanifold  $M = (M_0, \mathcal{A}_M)$  and an injective immersion  $\varphi = (\varphi_0, \varphi^*) : M \hookrightarrow N$  with  $\mathcal{J} = \cup_{y \in N} \mathcal{J}_y$  as its vanishing ideal.

Notice that if  $M \hookrightarrow N$  is a *closed* submanifold then its vanishing ideal can be defined directly as the sheaf of ideals  $\mathcal{J} := \ker \varphi^*$ .

**A.2. Pseudo-Riemannian metrics, connections and quaternionic structures on supermanifolds.** Let  $\mathbf{A}$  be a superalgebra and  $T$  a free  $\mathbf{A}$ -module of rank  $m|n$ .

**Definition 22.** An even (respectively, odd) bilinear form on  $T$  is a biadditive map  $g : T \times T \rightarrow A$  such that

$$g(aX, bY) = (-1)^{\tilde{b}\tilde{X}} abg(X, Y)$$

$$(respectively, \quad g(aX, bY) = (-1)^{\tilde{b}\tilde{X} + \tilde{a} + \tilde{b}} abg(X, Y)),$$

for all homogeneous  $a, b \in \mathbf{A}$  and  $X, Y \in T$ . A bilinear form  $g$  on  $T$  is called **supersymmetric** (respectively, **super skew symmetric**) if

$$g(X, Y) = (-1)^{\tilde{X}\tilde{Y}} g(Y, X)$$

$$(respectively, \quad g(X, Y) = -(-1)^{\tilde{X}\tilde{Y}} g(Y, X)),$$

for all homogeneous  $X, Y \in T$ . It is called **nondegenerate** if

$$T \ni X \mapsto g(X, \cdot) \in T^* = \text{Hom}_{\mathbf{A}}(T, \mathbf{A})$$

is an isomorphism of  $\mathbf{A}$ -modules. The  $\mathbf{A}$ -module of bilinear forms on  $T$  is denoted by  $\text{Bil}_{\mathbf{A}}(T) = \text{Bil}_{\mathbf{A}}(T)_0 + \text{Bil}_{\mathbf{A}}(T)_1$ .

Notice that  $\text{Bil}_{\mathbf{A}}(T) \cong \text{Hom}_{\mathbf{A}}(T, T^*) \cong T^* \otimes_{\mathbf{A}} T^*$ , where  $T^* = \text{Hom}_{\mathbf{A}}(T, \mathbf{A})$ .

For a supermanifold  $M = (M_0, \mathcal{A})$  the sheaf  $\text{Bil}_{\mathbf{A}} \mathcal{T}_M$  of bilinear forms on  $\mathcal{T}_M$  is defined in the obvious way such that  $(\text{Bil}_{\mathbf{A}} \mathcal{T}_M)(U) = \text{Bil}_{\mathbf{A}(U)}(\mathcal{T}_M(U))$  for every open subset  $U \subset M_0$  with the property that  $\mathcal{T}_M(U)$  is a free  $\mathcal{A}(U)$ -module. We have obvious isomorphisms of sheaves of  $\mathcal{A}$ -modules:  $\text{Bil}_{\mathbf{A}} \mathcal{T}_M \cong \text{Hom}_{\mathbf{A}}(\mathcal{T}_M, \mathcal{T}_M^*) \cong \mathcal{T}_M^* \otimes_{\mathbf{A}} \mathcal{T}_M^*$ . Since any section  $g$  of  $\text{Bil}_{\mathbf{A}} \mathcal{T}_M$  can be considered as a tensor field on  $M$ , it has a well defined value  $g(p) \in \text{Bil}_{\mathbb{R}}(T_p M)$  for all  $p \in M_0$ . The restriction  $g(p)|_{(T_p M)_0 \times (T_p M)_0}$  defines a section  $g_0$  of  $\text{Bil}_{C_{M_0}^{\infty}} \mathcal{T}_{M_0}$  via the canonical identification  $d\epsilon(p) : T_p M_0 \xrightarrow{\sim} (T_p M)_0$ .

**Definition 23.** A pseudo-Riemannian metric on a supermanifold  $M = (M_0, \mathcal{A})$ ,  $M_0$  connected, is an even nondegenerate supersymmetric section  $g$  of  $\text{Bil}_{\mathbf{A}} \mathcal{T}_M$ . The **signature**  $(k, l)$  of  $g$  is the signature of the pseudo-Riemannian metric  $g_0$  on  $M_0$ . The pseudo-Riemannian metric  $g$  is said to be a **Riemannian metric** if  $g_0$  is Riemannian. Let  $M = (M, \mathcal{A})$  be a supermanifold and  $\mathcal{E}$  a locally free sheaf of  $\mathcal{A}$ -modules. A **connection** on  $\mathcal{E}$  is an even section  $\nabla$  of the sheaf  $\text{Hom}_{\mathbf{A}}(\mathcal{T}_M, \text{End}_{\mathbb{R}} \mathcal{E})$ , which to any vector field  $X$  on  $M$  associates a section  $\nabla_X$  of  $\text{End}_{\mathbb{R}} \mathcal{E}$  such that

$$\nabla_X f s = X(f)s + (-1)^{\tilde{X}\tilde{f}} f \nabla_X s$$

for all vector fields  $X$  on  $M$ , functions  $f$  on  $M$  and sections  $s$  of  $\mathcal{E}$  of pure degree. The **curvature**  $R$  of  $\nabla$  is the even super skew symmetric section of  $\text{End}_{\mathbf{A}} \mathcal{E} \otimes_{\mathbf{A}} \text{Bil}_{\mathbf{A}} \mathcal{T}_M$  defined by:

$$R(X, Y) := \nabla_X \nabla_Y - (-1)^{\tilde{X}\tilde{Y}} \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

for all vector fields  $X, Y$  on  $M$  of pure degree. A connection on a supermanifold  $M$  is by definition a connection on its tangent sheaf  $\mathcal{T}_M$ . Its torsion is the even super skew symmetric section of  $\mathcal{T}_M \otimes_{\mathcal{A}} \text{Bil}_{\mathcal{A}} \mathcal{T}_M$  defined by:

$$T(X, Y) := \nabla_X Y - (-1)^{\tilde{X}\tilde{Y}} \nabla_Y X - [X, Y]$$

for all vector fields  $X, Y$  on  $M$  of pure degree.

As for ordinary manifolds, a connection  $\nabla$  on a supermanifold  $M$  induces a connection on the full tensor superalgebra  $\otimes_{\mathcal{A}} \langle \mathcal{T}_M, \mathcal{T}_M^* \rangle$ . In particular, if  $g$  is e.g. an even section of  $\text{Bil}_{\mathcal{A}} \mathcal{T}_M \cong \mathcal{T}_M^* \otimes \mathcal{T}_M^*$  then we have

$$(\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - (-1)^{\tilde{X}\tilde{Y}} g(Y, \nabla_X Z)$$

for all vector fields  $X, Y$  and  $Z$  on  $M$  of pure degree. As in the case of ordinary manifolds, see e.g. [O'N], one can prove that a pseudo-Riemannian supermanifold  $(M, g)$  has a unique torsionfree connection  $\nabla = \nabla^g$  such that  $\nabla g = 0$ . We will call this connection the **Levi-Civita connection** of  $(M, g)$ . It is computable from the following superversion of the Koszul-formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + (-1)^{\tilde{X}(\tilde{Y}+\tilde{Z})} Yg(Z, X) \\ &\quad - (-1)^{\tilde{Z}(\tilde{X}+\tilde{Y})} Zg(X, Y) - g(X, [Y, Z]) \\ (22) \quad &\quad + (-1)^{\tilde{X}(\tilde{Y}+\tilde{Z})} g(Y, [Z, X]) + (-1)^{\tilde{Z}(\tilde{X}+\tilde{Y})} g(Z, [X, Y]) \end{aligned}$$

for all vector fields  $X, Y$  and  $Z$  on  $M$  of pure degree.

Next we are going to define quaternionic supermanifolds and quaternionic Kähler supermanifolds. First we define the notion of almost quaternionic structure.

**Definition 24.** Let  $M = (M_0, \mathcal{A})$  be a supermanifold. An **almost complex structure** on  $M$  is an even global section  $J \in (\text{End}_{\mathcal{A}} \mathcal{T}_M)(M_0)$  such that  $J^2 = -\text{Id}$ . An **almost hypercomplex structure** on  $M$  is a triple  $(J_\alpha) = (J_1, J_2, J_3)$  of almost complex structures on  $M$  satisfying  $J_1 J_2 = J_3$ . An **almost quaternionic structure** on  $M$  is a subsheaf  $Q \subset \text{End}_{\mathcal{A}} \mathcal{T}_M$  with the following property: for every  $p \in M_0$  there exists an open neighborhood  $U \subset M_0$  and an almost hypercomplex structure  $(J_\alpha)$  on  $M|_U$  such that  $Q(U)$  is a free  $\mathcal{A}(U)$ -module of rank  $3|0$  with basis  $(J_1, J_2, J_3)$ . A pair  $(M, J)$  (respectively,  $(M, (J_\alpha)), (M, Q)$ ) as above is called an **almost complex supermanifold** (respectively, **almost hypercomplex supermanifold**, **almost quaternionic supermanifold**).

Second we introduce the basic compatibility conditions between almost quaternionic structures, connections and pseudo-Riemannian metrics.

**Definition 25.** Let  $(M, Q)$  be an almost quaternionic supermanifold of dimension  $\dim M = m|n$ . A connection  $\nabla$  on  $(M, Q)$  is called an **almost quaternionic connection** if  $\nabla$  preserves  $Q$ , i.e. if  $\nabla_X S$  is a section of  $Q$  for any vector field  $X$  on  $M$  and any section  $S$  of  $Q$ . A **quaternionic connection** on  $(M, Q)$  is a torsionfree almost quaternionic connection. If the almost quaternionic structure  $Q$  on  $M$  admits a

quaternionic connection, then it is called **1-integrable** or **quaternionic structure**. In this case the pair  $(M, Q)$  is called a **quaternionic supermanifold**, provided that  $m = \dim M_0 > 4$ .

**Definition 26.** A (pseudo-) Riemannian metric  $g$  on an almost quaternionic supermanifold  $(M, Q)$  is called **Hermitian** if sections of  $Q$  are  $g$ -skew symmetric, i.e. if  $g(SX, Y) = -g(X, SY)$  for all sections  $S$  of  $Q$  and vector fields  $X, Y$  on  $M$ . In this case the triple  $(M, Q, g)$  is called an **almost quaternionic (pseudo-) Hermitian supermanifold**. If, moreover, the Levi-Civita connection  $\nabla^g$  of the  $Q$ -Hermitian metric  $g$  is quaternionic, then  $(M, Q, g)$  is called a **quaternionic (pseudo-) Kähler supermanifold**, provided that  $m = \dim M_0 > 4$ .

We recall that to any (pseudo-) Riemannian metric  $g$  on a supermanifold  $M$  we have associated the (pseudo-) Riemannian metric  $g_0$  on the manifold  $M_0$ , see Def. 23. Now we will associate an almost quaternionic structure  $Q_0$  on  $M_0$  to any almost quaternionic structure  $Q$  on  $M$ . To any even section  $S$  of  $\text{End}_{\mathcal{A}} \mathcal{T}_M$  we associate a section  $S_0$  of  $\text{End}_{\mathcal{C}_{M_0}^\infty} \mathcal{T}_{M_0}$  defined by  $S_0(p) = S(p)|_{T_p M_0}$ , where  $S(p) \in (\text{End}_{\mathbb{R}}(T_p M))_0$  is the value of the tensor field  $S$  at  $p \in M_0$  and, as usual,  $T_p M_0$  is canonically identified with  $(T_p M)_0$ . Let  $Q_0 \subset \text{End}_{\mathcal{C}_{M_0}^\infty} \mathcal{T}_{M_0}$  be the subsheaf of  $\mathcal{C}_{M_0}^\infty$ -modules spanned by the local sections of the form  $S_0$ , where  $S$  is a local section of  $Q$ . Then  $Q_0$  is a locally free sheaf of rank 3 and hence it defines a rank 3 subbundle of the vectorbundle  $\text{End } TM_0$ , more precisely, an almost quaternionic structure on  $M_0$  in the usual sense, see Def. 5.

Finally we give the definition of quaternionic supermanifolds and of quaternionic Kähler supermanifolds of dimension  $4|n$ .

**Definition 27.** An almost quaternionic supermanifold  $(M, Q)$  of dimension  $\dim M = 4|n$  is called a **quaternionic supermanifold** if

- i) there exists a quaternionic connection  $\nabla$  on  $(M, Q)$ ,
- ii)  $(M_0, Q_0)$  is a quaternionic manifold in the sense of Def. 8.

An almost quaternionic Hermitian supermanifold  $(M, Q, g)$  of dimension  $\dim M = 4|n$  is called a **quaternionic Kähler supermanifold** if

- i) the Levi-Civita connection  $\nabla^g$  preserves  $g$  and
- ii)  $(M_0, Q_0)$  is a quaternionic Kähler manifold in the sense of Def. 11.

**A.3. Supergroups.** Let  $V = V_0 + V_1$  be a  $\mathbb{Z}_2$ -graded vector space. We recall that  $\mathfrak{gl}(V)$  denotes the general linear super Lie algebra. Its super Lie bracket is the supercommutator on  $\text{End}_{\mathbb{R}} V$ , see Example 2.

**Definition 28.** A **representation** of a super Lie algebra  $\mathfrak{g}$  on a  $\mathbb{Z}_2$ -graded vector space  $V$  is a homomorphism of super Lie algebras  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . The **adjoint representation** of  $\mathfrak{g}$  is the representation  $\text{ad} : \mathfrak{g} \ni x \mapsto \text{ad}_x \in \mathfrak{gl}(\mathfrak{g})$  defined by:

$$\text{ad}_x y = [x, y], \quad x, y \in \mathfrak{g}.$$

Here  $[\cdot, \cdot]$  denotes the super Lie bracket in  $\mathfrak{g}$ . A linear super Lie algebra is a subalgebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$ .

Notice that, by the super Jacobi identity, see Def. 16,  $\text{ad}_x$  is a derivation of  $\mathfrak{g}$  for all  $x \in \mathfrak{g}$ .

Now let  $E$  be a module over a superalgebra  $\mathbf{A}$ ; say  $E = V \otimes \mathbf{A}$ , where  $V$  is a  $\mathbb{Z}_2$ -graded vector space (and  $\otimes$  stands for the  $\mathbb{Z}_2$ -graded tensor product over  $\mathbb{R}$ ). The invertible elements of  $(\text{End}_{\mathbf{A}} E)_0$  form a group, which is denoted by  $\text{GL}_{\mathbf{A}}(E)$ . Given a  $\mathbb{Z}_2$ -graded vector space  $V$ , the correspondence

$$\mathbf{A} \mapsto \text{GL}_{\mathbf{A}}(V \otimes \mathbf{A})$$

defines a covariant functor from the category of superalgebras into the category of groups. We can compose this functor with the contravariant functor

$$M = (M_0, \mathcal{A}) \mapsto \mathcal{A}(M_0)$$

from the category of supermanifolds into that of superalgebras. The resulting contravariant functor

$$M \mapsto \text{GL}(V)[M] := \text{GL}_{\mathcal{A}(M_0)}(V \otimes \mathcal{A}(M_0))$$

from the category of supermanifolds into that of groups is denoted by  $\text{GL}(V)[\cdot]$ .

**Definition 29.** A **supergroup**  $G[\cdot]$  is a contravariant functor from the category of supermanifolds into that of groups. The supergroup  $\text{GL}(V)[\cdot]$ , defined above, is called the **general linear supergroup** over the  $\mathbb{Z}_2$ -graded vector space  $V$ . A **subgroup** of a supergroup  $G[\cdot]$  is a supergroup  $H[\cdot]$  such that  $H[M]$  is a subgroup of  $G[M]$  for all supermanifolds  $M$  and  $H[\varphi] = G[\varphi]H[N]$  for all morphisms  $\varphi : M \rightarrow N$ . We will write  $H[\cdot] \subset G[\cdot]$  if  $H[\cdot]$  is a subgroup of the supergroup  $G[\cdot]$ . A **linear supergroup** is a subgroup  $H[\cdot] \subset \text{GL}(V)[\cdot]$ .

**Example 7:** To any linear super Lie algebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$  we can associate the linear supergroup  $G[\cdot] \subset \text{GL}(V)[\cdot]$  defined by:

$$G[M] := \langle \exp(\mathfrak{g} \otimes \mathcal{A}(M_0)) \rangle$$

for any supermanifold  $M = (M_0, \mathcal{A})$ . The right-hand side is the subgroup of  $\text{GL}_{\mathcal{A}(M_0)}(V \otimes \mathcal{A}(M_0))$  generated by the exponential image of the Lie algebra  $(\mathfrak{g} \otimes \mathcal{A}(M_0))_0 \subset (\mathfrak{gl}(V) \otimes \mathcal{A}(M_0))_0 \cong (\text{End}_{\mathcal{A}(M_0)}(V \otimes \mathcal{A}(M_0)))_0$ . The convergence of the exponential series, which is locally uniform in all derivatives of arbitrary order, follows from the analyticity of the exponential map by a (finite) Taylor expansion with respect to the odd coordinates.  $G[\cdot]$  is called the **linear supergroup associated to the linear super Lie algebra**  $\mathfrak{g}$ .

For any supermanifold  $M$  we denote by  $\Delta_M = (\Delta_{M_0}, \Delta_M^*)$  the **diagonal embedding** defined by:  $\Delta_{M_0}(x) = (x, x)$  ( $x \in M_0$ ) and  $\Delta_M^* \pi_i^* f = f$  for all functions  $f$  on  $M$ . Here  $\pi_i : M \times M \rightarrow M$  denotes the projection onto the  $i$ -th factor of  $M \times M$  ( $i = 1, 2$ ).

**Definition 30.** A **Lie supergroup** is a supermanifold  $G = (G_0, \mathcal{A}_G)$  whose underlying manifold is a Lie group  $G_0$  (which, for convenience, we will always assume to be connected) with neutral element  $e \in G_0$ , multiplication  $\mu_0 : G_0 \times G_0 \rightarrow G_0$  and



inversion  $\iota_0 : G_0 \rightarrow G_0$ , together with morphisms  $\mu = (\mu_0, \mu^*) : G \times G \rightarrow G$  and  $\iota = (\iota_0, \iota^*) : G \rightarrow G$  such that

- i)  $\mu \circ (\text{Id}_G \times \mu) = \mu \circ (\mu \times \text{Id}_G) \in \text{Mor}(G \times G \times G, G)$ ,
- ii)  $\mu \circ (\text{Id}_G \times \epsilon_e) = \pi_1 : G \times \{e\} \xrightarrow{\sim} G$ ,  $\mu \circ (\epsilon_e \times \text{Id}_G) = \pi_2 : \{e\} \times G \xrightarrow{\sim} G$  and
- iii)  $\mu \circ (\text{Id}_G \times \iota) \circ \Delta_G = \mu \circ (\iota \times \text{Id}_G) \circ \Delta_G = \text{Id}_G$ .

Here  $\epsilon_e = (e, \epsilon_e^*) : \{e\} \hookrightarrow G_0 \xrightarrow{\hookrightarrow} G$  is the canonical embedding. The morphisms  $\mu$  and  $\iota$  are called, respectively, **multiplication** and **inversion** in the Lie supergroup  $G$ . A **Lie subgroup** of a Lie supergroup  $G = (G_0, \mathcal{A}_G)$  is a submanifold  $H = (H_0, \mathcal{A}_H) \hookrightarrow G$  such that the immersion  $\varphi_0 : H_0 \hookrightarrow G_0$  induces on  $H_0$  the structure of Lie subgroup of  $G_0$  and  $\varphi$  induces on  $H$  the structure of Lie supergroup with underlying Lie group  $H_0$ . We will write  $H \subset G$  if  $H$  is a Lie subgroup of  $G$ . An **action** of a Lie supergroup  $G$  on a supermanifold  $M$  is a morphism  $\alpha : G \times M \rightarrow M$  such that

- i)  $\alpha \circ (\text{Id}_G \times \alpha) = \alpha \circ (\mu \times \text{Id}_M)$  and
- ii)  $\alpha \circ (\epsilon_e \times \text{Id}_M) = \pi_2 : \{e\} \times M \xrightarrow{\sim} M$ .

Given an action  $\alpha : G \times M \rightarrow M$  we have the notion of **fundamental vector field**  $X$  associated to  $x \in T_e G$ . It is defined by  $X(f) := x(\alpha^* f)$ . The correspondence  $x \mapsto X$  defines an even  $\mathbb{R}$ -linear map  $T_e G \rightarrow \mathcal{T}_M(M_0)$  from the  $\mathbb{Z}_2$ -graded vector space  $T_e G$  to the free  $\mathcal{A}_M(M_0)$ -module of global vector fields on  $M$ .

**Definition 31.** Let an action  $\alpha$  of a Lie supergroup  $G$  on a supermanifold  $M$  be given. A tensor field  $S$  on  $M$  is called  **$G$ -invariant** if the Lie derivative  $\mathcal{L}_X S = 0$  for all fundamental vector fields  $X$ . (Notice that, in particular, this defines the notion of  $G$ -invariant pseudo-Riemannian metric on  $M$ .) An almost quaternionic structure  $Q$  on  $M$  is called  **$G$ -invariant** if  $\mathcal{L}_X S$  is a section of  $Q$  for all sections  $S$  of  $Q$  and fundamental vector fields  $X$  on  $M$ .

We can specialize the above definition to the (left-) action  $\mu : G \times G \rightarrow G$  given by the multiplication in the Lie supergroup  $G$ . The  $G$ -invariant tensor fields with respect to that action are called **left-invariant**. A tensor field  $S$  is called **right-invariant** if  $L_X S = 0$  for all left-invariant vector fields  $X$  on  $G$ . The right-invariant vector fields are precisely the fundamental vector fields for the action  $\mu$ .

For any Lie supergroup  $G$ , we can define a group homomorphism  $\mu_l : G_0 \rightarrow \text{Aut}(G)$  by:

$$\mu_l(g) := \mu \circ (\epsilon_g \times \text{Id}_G) : \{g\} \times G \cong G \rightarrow G, \quad g \in G_0.$$

Here  $\epsilon_g : \{g\} \hookrightarrow G$  is the canonical embedding and  $\{g\} \times G$  is canonically identified with  $G$  via the projection  $\pi_2 : \{g\} \times G \xrightarrow{\sim} G$  onto the second factor. Similarly, we can define a group antihomomorphism  $\mu_r : G_0 \rightarrow \text{Aut}(G)$  by:

$$\mu_r(g) := \mu \circ (\text{Id}_G \times \epsilon_g) : G \times \{g\} \cong G \rightarrow G, \quad g \in G_0.$$

Notice that the canonical embedding  $\epsilon : G_0 \hookrightarrow G$  is  $G_0$ -equivariant with respect to the usual left- (respectively, right-) action on  $G_0$  and the action on  $G$  defined by  $\mu_l$  (respectively,  $\mu_r$ ).

Given a Lie supergroup  $G$  and a supermanifold  $M$  there is a natural group structure on  $\text{Mor}(M, G)$  with multiplication defined by:

$$\varphi \cdot \psi := \mu \circ (\varphi \times \psi) \circ \Delta_M, \quad \varphi, \psi \in \text{Mor}(M, G).$$

The correspondence  $M \mapsto \text{Mor}(M, G)$  defines a supergroup.

**Definition 32.** *The supergroup  $G[\cdot] := \text{Mor}(\cdot, G)$  is called the supergroup subordinate to the Lie supergroup  $G$ .*

**Example 8:** Let  $V$  be a  $\mathbb{Z}_2$ -graded vector space. Then  $\text{GL}_{\mathbb{R}}(V)$  is, by definition, the group of invertible elements of  $(\text{End}_{\mathbb{R}} V)_0$ . The Lie group  $\text{GL}_{\mathbb{R}}(V) \cong \text{GL}_{\mathbb{R}}(V_0) \times \text{GL}_{\mathbb{R}}(V_1)$  is an open submanifold of the vector space  $(\text{End}_{\mathbb{R}} V)_0 = \text{SM}(\text{End}_{\mathbb{R}} V)_0$ , see Example 5. We define the submanifold

$$\text{GL}(V) := \text{SM}(\text{End}_{\mathbb{R}} V)|_{\text{GL}_{\mathbb{R}}(V)} \hookrightarrow \text{SM}(\text{End}_{\mathbb{R}} V).$$

The manifold underlying the supermanifold  $\text{GL}(V) = (\text{GL}(V)_0, \mathcal{A}_{\text{GL}(V)})$  is the above Lie group:  $\text{GL}(V)_0 = \text{GL}_{\mathbb{R}}(V)$ . From the definition of the supermanifold  $\text{GL}(V)$  it is clear that  $\text{Mor}(M, \text{GL}(V))$  is canonically identified with the set  $\text{GL}_{\mathcal{A}(M_0)}(V \otimes \mathcal{A}(M_0)) = \text{GL}(V)[M]$  (cf. Def. 29) for any supermanifold  $M = (M_0, \mathcal{A})$ . Moreover,  $\text{GL}(V)$  has a unique structure of Lie supergroup inducing the canonical group structure on  $\text{GL}(V)[M]$  for any supermanifold  $M$ . In other words, the general linear supergroup  $\text{GL}(V)[\cdot]$  is the supergroup subordinate to the Lie supergroup  $\text{GL}(V)$ , see Def. 32.

**Definition 33.** *The Lie supergroup  $\text{GL}(V)$  is called the **general linear Lie supergroup**. A **linear Lie supergroup** is a Lie subgroup of  $\text{GL}(V)$ .*

There exists a unique morphism

$$\text{Exp} : \text{SM}(\mathfrak{gl}(V)) \rightarrow \text{GL}(V)$$

such that

$$\text{Exp} \circ \varphi = \exp(\varphi)$$

for all supermanifolds  $M = (M_0, \mathcal{A})$  and  $\varphi \in \text{Mor}(M, \text{SM}(\mathfrak{gl}(V))) = (\mathfrak{gl}(V) \otimes \mathcal{A}(M_0))_0 = (\text{End}_{\mathcal{A}(M_0)} V \otimes \mathcal{A}(M_0))_0$ , where, on the right-hand side,  $\exp : (\text{End}_{\mathcal{A}(M_0)} V \otimes \mathcal{A}(M_0))_0 \rightarrow \text{GL}_{\mathcal{A}(M_0)}(V \otimes \mathcal{A}(M_0)) = \text{Mor}(M, \text{GL}(V))$  is the exponential map for even endomorphisms of  $V \otimes \mathcal{A}(M_0)$  (for the definition of the supermanifold  $\text{SM}(\mathfrak{gl}(V))$  see Example 5). The underlying map  $\text{Exp}_0 : \mathfrak{gl}(V)_0 = \text{SM}(\mathfrak{gl}(V))_0 \rightarrow \text{GL}(V)_0$  is the ordinary exponential map for the Lie group  $\text{GL}(V)_0$ :  $\text{Exp}_0 = \exp$ . The morphism  $\text{Exp}$  is called the **exponential morphism** of  $\text{gl}(V)$ .

**Example 9:** Let  $\mathfrak{g} \subset \mathfrak{gl}(V)$  be a linear super Lie algebra and  $G[\cdot] \subset \text{GL}(V)[\cdot]$  the corresponding linear supergroup, see Example 7. We denote by  $G_0 \subset \text{GL}(V)_0$

the connected linear Lie group with Lie algebra  $\mathfrak{g}_0$ . It is the immersed Lie subgroup generated by the exponential image of  $\mathfrak{g}_0 = \mathfrak{gl}(V)_0$ . We define an ideal  $\mathcal{J}_g \subset (\mathcal{A}_{\mathrm{GL}(V)})_g$  as follows: A germ of function  $f \in (\mathcal{A}_{\mathrm{GL}(V)})_g$  ( $g \in G_0$ ) belongs to  $\mathcal{J}_g$  if and only if  $r(\varphi^* f) = 0$  for all injective immersions  $\varphi \in G[M] \subset \mathrm{Mor}(M, \mathrm{GL}(V))$ , where  $r : ((\varphi_0)_* \mathcal{A}_M)_g \rightarrow (\mathcal{A}_M)_{\varphi_0^{-1}(g)}$  is the natural restriction map. We claim that  $\mathcal{J}_G := \bigcup_{g \in G_0} \mathcal{J}_g$  is the vanishing ideal of a submanifold  $G \hookrightarrow \mathrm{GL}(V)$ . Due to the invariance of  $\mathcal{J}_G$  under the group  $\mu_l(G_0) \subset \mathrm{Aut}(G)$  it is sufficient to prove the claim locally over an open set of the form  $\exp U \subset G_0$ ,  $U \subset \mathfrak{g}_0$  an open neighborhood of  $0 \in \mathfrak{g}_0$ . The local statement follows from the fact that the morphism  $SM(\mathfrak{g}) \hookrightarrow SM(\mathfrak{gl}(V)) \xrightarrow{\mathrm{Exp}} \mathrm{GL}(V)$  has maximal rank at  $0 \in \mathfrak{g}_0 = SM(\mathfrak{g})_0$  and hence defines a submanifold  $SM(\mathfrak{g})|_U \hookrightarrow \mathrm{GL}(V)$  for some open neighborhood of  $U$  of  $0 \in \mathfrak{g}_0$ . The vanishing ideal of this submanifold coincides with  $\mathcal{J}_G$  over  $\exp U$  by the definition of the exponential morphism. Next we claim that multiplication  $\mu : \mathrm{GL}(V) \times \mathrm{GL}(V) \rightarrow \mathrm{GL}(V)$  and inversion  $\iota : \mathrm{GL}(V) \rightarrow \mathrm{GL}(V)$  have the property that  $\mu^* \mathcal{J}_G \subset \mathcal{J}_{G \times G}$  and  $\iota^* \mathcal{J}_G = \mathcal{J}_G$ . Here  $\mathcal{J}_{G \times G}$  is the vanishing ideal of the submanifold  $G \times G \hookrightarrow \mathrm{GL}(V) \times \mathrm{GL}(V)$ . The claim follows from the fact that  $G[\cdot]$  is a subgroup of  $\mathrm{GL}(V)[\cdot]$  and implies that the morphisms  $G \times G \hookrightarrow \mathrm{GL}(V) \times \mathrm{GL}(V) \xrightarrow{\mu} \mathrm{GL}(V)$  and  $G \hookrightarrow \mathrm{GL}(V) \xrightarrow{\iota} \mathrm{GL}(V)$  induce morphisms  $G \times G \rightarrow G$  and  $G \rightarrow G$ , which induce on  $G$  the structure of Lie supergroup. In other words,  $G$  is a Lie subgroup of the general linear Lie supergroup  $\mathrm{GL}(V)$ . It is called the **linear Lie supergroup associated to the linear Lie superalgebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$** . Notice that  $\mathrm{Exp}^*$  maps  $\mathcal{J}_G$  into the vanishing ideal of  $SM(\mathfrak{g}) \hookrightarrow SM(\mathfrak{gl}(V))$  and hence the restriction  $SM(\mathfrak{g}) \hookrightarrow SM(\mathfrak{gl}(V)) \xrightarrow{\mathrm{Exp}} \mathrm{GL}(V)$  of the exponential morphism induces a morphism  $SM(\mathfrak{g}) \rightarrow G$ , which is called the **exponential morphism** of  $\mathfrak{g}$  and is again denoted by  $\mathrm{Exp}$ . Its differential at  $0 \in \mathfrak{g}_0$  yields an isomorphism  $\mathfrak{g} \cong T_0 SM(\mathfrak{g}) \xrightarrow{\sim} T_e G$ .

**A.4. Homogeneous supermanifolds.** Let  $G = (G_0, \mathcal{A}_G)$  be a Lie supergroup,  $K \subset G_0$  a closed subgroup and  $\pi_0 : G_0 \rightarrow G_0/K$  the canonical projection. Then the subsheaf  $\mathcal{A}_G^K := \mathcal{A}_G^{\mu_r(K)} \subset \mathcal{A}_G$  of functions on  $G$  invariant under the subgroup  $\mu_r(K) \subset \mathrm{Aut}(G)$  is again a sheaf of superalgebras on  $G_0$ . Explicitly, a function  $f \in \mathcal{A}_G(U)$  ( $U \subset G_0$  open) belongs to  $\mathcal{A}_G^K(U)$  if it can be extended to a  $\mu_r(K)$ -invariant function over  $UK \subset G_0$ . Its pushed forward sheaf  $\mathcal{A}_{G/K} := (\pi_0)_* \mathcal{A}_G^K$  is a sheaf of superalgebras on the homogeneous manifold  $G_0/K$ .

**Theorem 18.** (cf. [K]) *Let  $G \hookrightarrow \mathrm{GL}(V)$  be a closed linear Lie supergroup and  $K \subset G_0$  a closed subgroup. Then  $G/K = (G_0/K, \mathcal{A}_{G/K})$  is a supermanifold with a canonical submersion  $\pi : G \rightarrow G/K$  and a canonical action  $\alpha : G \times G/K \rightarrow G/K$ .*

**Proof:** Since  $\mu_l(g) \in \mathrm{Aut}(G)$  induces isomorphisms

$$\mathcal{A}_{G/K}(gU) \xrightarrow{\sim} \mathcal{A}_{G/K}(U)$$

for all  $g \in G_0$  and  $U \subset G_0/K$  open, it is sufficient to check that  $G/K|_U = (U, \mathcal{A}_{G/K}|_U)$  is a supermanifold for some open neighborhood  $U$  of  $eK \in G_0/K$ .

**Lemma 12.** *Under the assumptions of Thm. 18, there exists local coordinates  $(x, y, \xi)$  for  $\mathrm{GL}(V)$  over some neighborhood  $U = UK \subset \mathrm{GL}(V)_0$  of  $e \in \mathrm{GL}(V)_0$  such that*

- 1)  $x = (x^i)$  and  $y = (y^j)$  consist of even functions and  $\xi = (\xi^k)$  of odd functions,
- 2)  $\mathcal{A}_{\mathrm{GL}(V)}^K(U)$  is the subalgebra of  $\mathcal{A}_{\mathrm{GL}(V)}(U)$  which consists of functions  $f(x, \xi)$  only of  $(x, \xi)$ , more precisely,

$$f(x, \xi) = \sum_{\alpha} f_{\alpha}(x) \xi^{\alpha},$$

where the  $f_{\alpha}(x) \in \mathcal{A}_{\mathrm{GL}(V)}(U)$  are functions only of  $x$ , i.e. if  $f_{\alpha}(x) \neq 0$  then  $f_{\alpha}(x)$  does not belong to the ideal generated by  $\xi$  and  $\epsilon^* f_{\alpha}(x) \in C^{\infty}(U)$  are functions only of  $\epsilon^* x$  (independent of  $\epsilon^* y$ ). Here we used the multiindex notation  $\alpha = (\alpha_1, \dots, \alpha_{2mn}) \in \mathbb{Z}_2^{2mn}$  ( $m|n = \dim V$ ) and  $\xi^{\alpha} = \prod_{j=1}^{2mn} (\xi^j)^{\alpha_j}$ .

**Proof:** The natural (global) coordinates on the supermanifold  $\mathrm{GL}(V)$  are the matrix coefficients with respect to some basis of  $V$ . We will denote them simply by  $(z^i, \zeta^j)$  instead of using matrix notation. For these coordinates it is clear that  $\mu_r(g)^* z^k$  is a linear combination (over the real numbers) of the even coordinates  $z := (z^i)$  and  $\mu_r(g)^* \zeta^l$  is a linear combination (over the real numbers) of the odd coordinates  $\zeta := (\zeta^j)$  for all  $g \in \mathrm{GL}(V)_0$ . In particular, we obtain a representation  $\rho$  of  $K \subset \mathrm{GL}(V)_0$  on the vector space spanned by the odd coordinates. Let  $E_{\rho} \rightarrow G_0/K$  denote the vector bundle associated to this representation. Any  $\mu_r(K)$ -invariant function on  $\mathrm{GL}(V)$  linear in  $\zeta$  defines a section of the dual vector bundle  $E_{\rho}^*$  and vice versa. Now, since  $E_{\rho}^*$  is locally trivial (like any vector bundle), we can find  $\mu_r(K)$ -invariant local functions  $\xi = (\xi^j)$  linear in  $\zeta$  such that  $(z, \xi)$  are local coordinates for  $\mathrm{GL}(V)$  over some open neighborhood  $U = UK \subset \mathrm{GL}(V)_0$  of  $e$ . Next, by way of a local diffeomorphism in the even coordinates  $z$ , we can arrange that  $z = (x, y)$ , where the  $x$  are  $\mu_r(K)$ -invariant functions on  $\mathrm{GL}(V)$  such that  $\epsilon^* x \in C^{\infty}(U)$  induce local coordinates on  $G_0/K$ . Now any function  $f \in \mathcal{A}_{\mathrm{GL}(V)}(U)$  has a unique expression of the form

$$(23) \quad \sum_{\alpha} f_{\alpha}(x, y) \xi^{\alpha},$$

where the  $f_{\alpha}(x, y)$  are functions only of  $(x, y)$ . From the  $\mu_r(K)$ -invariance of the  $\xi^{\alpha}$  it follows that  $f$  is  $\mu_r(K)$ -invariant if and only if the  $f_{\alpha}(x, y)$  are  $\mu_r(K)$ -invariant. The function  $f_{\alpha}(x, y)$  is  $\mu_r(K)$ -invariant if and only if  $\epsilon^* f_{\alpha}(x, y) \in C^{\infty}(U)$  is invariant under the right-action of  $K$  on  $\mathrm{GL}_0$ , i.e. if and only if  $\epsilon^* f_{\alpha}(x, y)$  is a function only of  $\epsilon^* x$ . This shows that  $f \in \mathcal{A}_{\mathrm{GL}(V)}(U)^K$  if and only if the coefficients  $f_{\alpha}(x, y)$  in the expansion (23) are functions only of  $x$   $\square$

We continue the proof of Thm. 18. From Lemma 12 it follows that  $\mathrm{GL}(V)/K$  is a supermanifold. In fact, the  $K$ -invariant local functions  $(x, \xi)$  on  $\mathrm{GL}(V)$  constructed in that lemma induce local coordinates on  $\mathrm{GL}(V)/K$ . Next we restrict the coordinates  $(x, y, \xi)$  to the submanifold  $G \hookrightarrow \mathrm{GL}(V)$ . We can decompose  $x = (x', x'')$ ,  $\xi = (\xi', \xi'')$  such that  $(x', y, \xi')$  restrict to local coordinates on  $G$  over some open

neighborhood (again denoted by)  $U$  of  $e \in G_0$  (notice that, by construction,  $y$  restrict to local coordinates on  $K$ ). Now, as in the proof of the corresponding statement for  $GL(V)$  (see Lemma 12), it follows that  $\mathcal{A}_G^K(U)$  consists precisely of all functions of the form  $f(x', \xi') = \sum f_\alpha(x')(\xi')^\alpha$ . This proves that  $G/K$  is a supermanifold with local coordinates  $(x', \xi')$  over  $U$ .

The inclusion  $\mathcal{A}_G^K \subset \mathcal{A}_G$  defines the canonical submersion  $\pi : G \rightarrow G/K$ . Finally, if  $f$  is a  $\mu_r(K)$ -invariant (local) function on  $G$  then  $\mu^*f$  is a (local) function on  $G \times G$  invariant under the group  $\text{Id}_G \times \mu_r(K) \subset \text{Aut}(G \times G)$ . This shows that the composition  $G \times G \xrightarrow{\mu} G \xrightarrow{\pi} G/K$  factorizes to a morphism  $\alpha : G \times G/K \rightarrow G/K$ , which defines an action of  $G$  on  $G/K$ .  $\square$

**Definition 34.** *The supermanifold  $M = G/K$  is called the **homogeneous supermanifold** associated to the pair  $(G, K)$ .*

For the rest of the paper let  $\mathfrak{g} \subset \mathfrak{gl}(V)$  be a linear super Lie algebra,  $\mathfrak{k} \subset \mathfrak{g}_0$  a subalgebra and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  a  $\mathfrak{k}$ -invariant direct decomposition compatible with the  $\mathbb{Z}_2$ -grading. We denote by  $K \subset G_0 \subset G \subset GL(V)$  the corresponding linear Lie supergroups (see Example 9) and assume that the (connected) Lie subgroups  $K \subset G_0 \subset GL(V)_0$  are closed. Then, by Thm. 18,  $M = G/K$  is a supermanifold with a canonical action of  $G$ . We have the canonical identification  $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{k} \cong T_{eK}M$  given by  $x \mapsto X(eK)$ , where  $X(eK)$  is the value of the fundamental vector field  $X$  on  $M$  associated to  $x \in \mathfrak{m}$  at the base point  $eK \in G_0/K = M_0$ . We claim that any  $\text{ad}_{\mathfrak{k}}$ -invariant tensor  $S_{eK}$  over  $\mathfrak{m}$  defines a corresponding  $G$ -invariant (see Def. 31) tensor field on  $M$  such that  $S(eK) = S_{eK}$ . Here by a tensor over  $\mathfrak{m}$  we mean an element of the full tensor superalgebra  $\otimes(\mathfrak{m}, \mathfrak{m}^*)$  over  $\mathfrak{m}$ . In fact, for any tensor  $S_{eK}$  over  $\mathfrak{m}$  there exists a corresponding left-invariant tensor field  $S$  on  $G$  such that  $S(eK) = S_{eK}$ . In order for  $S$  to define a tensor field on  $G/K$  it is necessary and sufficient that  $S$  is  $\mu_r(K)$ -invariant, or, equivalently, that  $S_{eK}$  is  $\text{ad}_{\mathfrak{k}}$ -invariant. In particular, we have the following proposition:

**Proposition 16.** *Let  $g_{eK}$  be an  $\text{ad}_{\mathfrak{k}}$ -invariant nondegenerate supersymmetric bilinear form on  $\mathfrak{m}$ . Then there exists a unique  $G$ -invariant pseudo-Riemannian metric  $g$  on  $M = G/K$  (see Def. 23 and Def. 31) such that  $g(eK) = g_{eK}$ . Let  $Q_{eK}$  be an  $\text{ad}_{\mathfrak{k}}$ -invariant quaternionic structure on  $\mathfrak{m}$  (i.e.  $\text{ad} : \mathfrak{k} \rightarrow \mathfrak{gl}(\mathfrak{m})$  normalizes  $Q_{eK}$ ). Then there exists a unique  $G$ -invariant almost quaternionic structure  $Q$  on  $M$  (see Def. 24 and Def. 31) such that  $Q(eK) = Q_{eK}$ .*

Finally, we need to discuss  $G$ -invariant connections on  $M = G/K$ .

**Definition 35.** *A connection  $\nabla$  on a homogeneous supermanifold  $M = G/K$  is called  **$G$ -invariant** if*

$$\mathcal{L}_X(\nabla_Y S) = \nabla_{[X, Y]} S + (-1)^{\tilde{X}\tilde{Y}} \nabla_Y(\mathcal{L}_X S)$$

for all vector fields  $X$  and  $Y$  on  $M$ .

Let  $\nabla$  be a connection on a supermanifold  $M$ . For any vector field  $X$  on  $M$  one defines the  $\mathcal{A}_M(M_0)$ -linear operator

$$L_X := \mathcal{L}_X - \nabla_X.$$

We denote by  $L_X(p) \in \text{End}_{\mathbb{R}} T_p M$  its value at  $p \in M_0$ ; it is defined by  $L_X(p)Y(p) = (L_X Y)(p)$  for all vector fields  $Y$  on  $M$ .

For a  $G$ -invariant connection  $\nabla$  on a homogeneous supermanifold  $M = G/K$  as above we define the **Nomizu map**  $L = L(\nabla) : \mathfrak{g} \rightarrow \text{End}(T_{eK}M)$ ,  $x \mapsto L_x$ , by the equation

$$L_x := L_X(eK),$$

where  $X$  is the fundamental vector field on  $M$  associated to  $x \in \mathfrak{g}$ . The operators  $L_x \in \text{End}(T_{eK}M)$  will be called **Nomizu operators**. They have the following properties:

$$(24) \quad L_x = d\rho(x) \quad \text{for all } x \in \mathfrak{k}$$

and

$$(25) \quad L_{\Lambda_k x} = \rho(k)L_x\rho(k)^{-1} \quad \text{for all } x \in \mathfrak{g}, \quad k \in K,$$

where  $\rho : K \rightarrow \text{GL}(T_{eK}M)$  is the isotropy representation (under the identification  $T_{eK}M \cong \mathfrak{m}$  the representation  $\rho$  is identified with adjoint representation of  $K$  on  $\mathfrak{m}$ ).

Conversely, any even linear map  $L : \mathfrak{g} \rightarrow \text{End}(T_{[e]}M)$  satisfying (24) and (25) is the Nomizu map of a uniquely defined  $G$ -invariant connection  $\nabla = \nabla(L)$  on  $M$ . Its torsion tensor  $T$  and curvature tensor  $R$  are expressed at  $eK$  by:

$$T(\pi x, \pi y) = -(L_x \pi y - (-1)^{\bar{x}\bar{y}} L_y \pi x + \pi[x, y])$$

and

$$R(\pi x, \pi y) = [L_x, L_y] + L_{[x, y]}, \quad x, y \in \mathfrak{g}$$

where  $\pi : \mathfrak{g} \rightarrow T_{eK}M$  is the canonical projection  $x \mapsto \pi x = X(eK) = \left. \frac{d}{dt} \right|_{t=0} \exp(tx)K$ .

Suppose now that we are given a  $G$ -invariant geometric structure  $S$  on  $M$  (e.g. a  $G$ -invariant almost quaternionic structure  $Q$ ) defined by a corresponding  $K$ -invariant geometric structure  $S_{eK}$  on  $T_{eK}M$ . Then a  $G$ -invariant connection  $\nabla$  preserves  $S$  if and only if the corresponding Nomizu operators  $L_x$ ,  $x \in \mathfrak{g}$ , preserve  $S_{eK}$ . So to construct a  $G$ -invariant connection preserving  $S$  it is sufficient to find a Nomizu map  $L : \mathfrak{g} \rightarrow \text{End}(T_{eK}M)$  such that  $L_x$  preserves  $S_e$  for all  $x \in \mathfrak{g}$ . We observe that, due to the  $K$ -invariance of  $S_e$ , the Nomizu operators  $L_x$  preserve  $S_e$  already for  $x \in \mathfrak{k}$ . The above considerations can be specialized as follows:

**Proposition 17.** *Let  $Q$  be a  $G$ -invariant almost quaternionic structure on a homogeneous supermanifold  $M = G/K$ . There is a natural one-to-one correspondence between  $G$ -invariant almost quaternionic connections on  $(M, Q)$  and Nomizu maps  $L : \mathfrak{g} \rightarrow \text{End}(T_{eK}M)$ , whose image normalizes  $Q(eK)$ , i.e. whose Nomizu operators  $L_x$ ,  $x \in \mathfrak{g}$ , belong to the normalizer  $\mathfrak{n}(Q) \cong \mathfrak{sp}(1) \oplus \mathfrak{gl}(d, \mathbb{H})$  ( $d = (m+n)/4$ ) of the quaternionic structure  $Q(eK)$  in the super Lie algebra  $\mathfrak{gl}(T_{eK}M)$ .*

**Corollary 10.** Let  $(M = G/K, Q)$  be a homogeneous almost quaternionic supermanifold and  $L : \mathfrak{g} \rightarrow \text{End}(T_{eK}M)$  a Nomizu map such that

- (1)  $L_x \pi y - (-1)^{\tilde{x}\tilde{y}} L_y \pi x = -\pi[x, y]$  for all  $x, y \in \mathfrak{g}$  (i.e.  $T = 0$ ) and
- (2)  $L_x$  normalizes  $Q(eK) \subset \text{End}(T_{eK}M)$ .

Then  $\nabla(L)$  is a  $G$ -invariant quaternionic connection on  $(M, Q)$  and hence  $Q$  is 1-integrable.

For use in 4, we give the formula for the Nomizu map  $L^g$  associated to the Levi-Civita connection  $\nabla^g$  of a  $G$ -invariant pseudo-Riemannian metric  $g$  on a homogeneous supermanifold  $M = G/K$ . Let  $\langle \cdot, \cdot \rangle = g(eK)$  be the  $K$ -invariant nondegenerate supersymmetric bilinear form on  $T_{eK}M$  induced by  $g$  (the value of  $g$  at  $eK$ ). Then  $L_x^g \in \text{End}(T_{eK}M)$ ,  $x \in \mathfrak{g}$ , is given by the following Koszul type formula:

$$(26) \quad -2\langle L_x^g \pi y, \pi z \rangle = \langle \pi[x, y], \pi z \rangle - \langle \pi x, \pi[y, z] \rangle - (-1)^{\tilde{x}\tilde{y}} \langle \pi y, \pi[x, z] \rangle, \quad x, y, z \in \mathfrak{g}.$$

**Corollary 11.** Let  $(M = G/K, Q, g)$  be a homogeneous almost quaternionic (pseudo-) Hermitian supermanifold and assume that  $L_x^g$  normalizes  $Q(eK)$  for all  $x \in \mathfrak{g}$ . Then the Levi-Civita connection  $\nabla^g = \nabla(L^g)$  is a  $G$  invariant quaternionic connection on  $(M, Q, g)$  and hence  $(M, Q, g)$  is a quaternionic (pseudo-) Kähler supermanifold if  $\dim M_0 > 4$ .

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**SPENCER MANIFOLDS**

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ABSTRACT. Almost-complex and hyper-complex manifolds are considered in this paper from the point of view of complex analysis and potential theory. The idea of holomorphic coordinates on an almost-complex manifold  $(M, \mathbf{J})$  is suggested by D. Spencer [Sp]. For hypercomplex manifolds we introduce the notion of hyper-holomorphic function and develop some analogous statements. Elliptic equations are developed in a different way than D. Spencer. In general here we describe only the formal aspect of the developed theory.

1. INTRODUCTION.

Differentiable manifolds are described locally by smooth real coordinates. This is typical in differential geometry. Complex-analytic manifolds are equipped locally by complex-analytic coordinates. This give rise to the possibility of applying the theory of holomorphic functions of many complex variables in the local geometry of complex-analytic manifolds. In the case of almost complex manifolds  $(M, \mathbf{J})$  one use ordinary real coordinates  $(x^1, \dots, x^{2n})$ . Here we shall consider complex self-conjugate coordinates  $(z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n)$ , where  $z^k = x^{2k-1} + ix^{2k}$  and  $\bar{z}^k = x^{2k-1} - ix^{2k}$ . We denote by  $\mathbf{J}^*$  the action of  $\mathbf{J}$  on differential forms of  $M$ , i. e. by definition  $(\mathbf{J}^*(\omega)X \stackrel{def}{=} \omega(\mathbf{J}X))$ , where  $X$  is a vector field, and  $\omega$  is a differential form on  $M$ . For a fixed index  $k$ , we say that  $z^k$  is a "holomorphic" coordinate if  $\mathbf{J}^*dz^k = idz^k$  and  $\mathbf{J}^*d\bar{z}^k = -id\bar{z}^k$ . For non-holomorphic coordinates  $z^q$  we have

$$\mathbf{J}^*dz^q = J_q^1 dz^1 + \dots + J_q^n dz^n + J_q^{n+1} d\bar{z}^{n+1} + \dots + J_q^{2n} d\bar{z}^{2n}$$

In the case  $z^k$  is a holomorphic coordinate for each  $k = 1, \dots, n$ , the almost complex structure  $\mathbf{J}$  is an integrable one. The interest of the existence of holomorphic coordinates  $z^k$  when the index  $k$  takes not all values  $1, \dots, n$  is suggested by Donald Spencer [Sp].

By  $\mathbb{H} = \mathbb{H}(\mathbf{1}, i, j, k), ij=k$ , we will denote the 4-dimensional quaternionic vector space, i. e.  $q \in \mathbb{H}$  means that  $q = x^0 + ix^1 + jx^2 + kx^3$ , where  $x^0, x^1, x^2, x^3 \in \mathbb{R}$ . We will use different complex number representation for quaternions  $q$ , namely  $q = z + \zeta j$ , where  $z = x^0 + ix^1$  and  $\zeta = x^2 + ix^3$ . So we obtain the right  $j$ -complex splitting of  $\mathbb{H}$ , denoted

by  $\mathbb{H}^j$ , i. e.  $\mathbb{H}^j = (\mathbb{R} \oplus i \mathbb{R}) \oplus (\mathbb{R} \oplus i \mathbb{R})j$ . By  $\mathbb{R} \oplus i \mathbb{R}$  is denoted the tensor product of  $\mathbb{R}$  with itself under the basis  $(1, 0)$  and  $(0, i)$ . Identifying  $\mathbb{R} \oplus i \mathbb{R}$  with  $\mathbb{C}$  we have that  $\mathbb{H}^j$  is isomorphic to  $\mathbb{C} \times \mathbb{C}$ . Analogously, we will consider the right  $i$ -complex splitting of  $\mathbb{H}$ , namely  $\mathbb{H}^i = (\mathbb{R} \oplus j \mathbb{R}) \oplus (\mathbb{R} \oplus j \mathbb{R})i$ , i.e.  $q = x^0 + jx^3 + (x^2 - jx^4)i$ .  $\mathbb{H}^i$  is isomorphic to  $\mathbb{C} \times \mathbb{C}$  too.

By  $\mathbb{H}^n$  is denoted the  $n$ -dimensional quaternionic vector space (real  $4n$ -dimensional)

$$\mathbb{H}^n = \{(q^1, \dots, q^n) : q^\alpha \in \mathbb{H}, \alpha = 1, \dots, n\}$$

According to the above accepted notation we have  $q^\alpha = z^\alpha + \zeta^\alpha j$ ,  $\alpha = 1, \dots, n$  or

$$\mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n j, \quad \mathbb{C}^n = \{z^1, \dots, z^n : z^\alpha \in \mathbb{C}\}$$

This representation is with respect of the right  $j$ -complex splitting  $\mathbb{H}^j$ . A similar representation of  $\mathbb{H}^n$  can be written with respect to the right  $i$ -complex splitting  $\mathbb{H}^i : \mathbb{H}^i = \mathbb{C}^n + \mathbb{C}^n i$ ,  $\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n i$ ,  $\mathbb{C}^n = \mathbb{R}^n \ominus \mathbb{R}^n i$ .

Let  $(M, \mathbf{J}, \mathbf{K})$  be a hyper-complex manifold,  $\mathbf{JK} + \mathbf{KJ} = 0$ ,  $\dim M_{\mathbb{R}} = 4n$ . A pair of complex coordinates  $(z, \zeta)$  is called *hyper-holomorphic pair* if  $z$  is holomorphic with respect to the almost-complex manifold  $(M, \mathbf{J})$  and  $\zeta$  is holomorphic with respect to  $(M, \mathbf{K})$ .

## 2. HOLOMORPHIC COORDINATES

2.0.1. *Almost-holomorphic functions.* By definition a function  $f : U \rightarrow \mathbb{C}$ , where  $U$  is an open subset of  $M$ , is called *almost holomorphic* or almost complex if  $\bar{\partial}f = 0$ . The above definition can be reformulated in the following equivalent form:

$$f \text{ is almost holomorphic iff } \mathbf{J}^*df = idf$$

Respectively,  $f$  is *almost-antiholomorphic* iff  $\mathbf{J}^*df = -idf$ . For the proof of the equivalence it is enough to take in view that the exterior derivative  $d$  is decomposed as  $d = \partial + \bar{\partial}$  over the space of smooth functions on  $M$ . Another form of this definition is obtained taking the real and imaginary parts of  $f$ , i.e.  $f = u + iv$ . In view of  $df = du + idv$  we receive  $\mathbf{J}^*du + i\mathbf{J}^*dv = idu - dv$ . This means that  $\mathbf{J}^*du = -dv$  and  $\mathbf{J}^*dv = du$ . As the obtained two equations are not independent, we can state the following Cauchy- Riemann type form of the definition

$$f = u + iv \text{ is almost-holomorphic iff } \mathbf{J}^*dv = du \text{ or equivalently } \mathbf{J}^*du = -dv.$$

Respectively:  $f = u + iv$  is *almost-anti holomorphic* iff  $\mathbf{J}^*dv = -du$  or equivalently  $\mathbf{J}^*du = dv$

*Remark:* For an almost complex manifold  $(M, \mathbf{J})$  with non-integrable  $\mathbf{J}$ , the decomposition  $d = \partial + \bar{\partial}$  is not valid over differential  $(p, q)$ -forms on  $(M, \mathbf{J})$ .

The following proposition is well-known:

**Proposition 1.** The almost complex structure  $\mathbf{J}$  of the almost complex manifold  $(M, \mathbf{J})$ ,  $\dim_{\mathbb{R}}M = 2n$ , is an integrable almost complex structure if and only if for every point  $p \in M$ , there is a neighborhood  $U$  of  $p$  and almost holomorphic functions

$f_j : U \rightarrow \mathbb{C}, j = 1, \dots, n$ , which differentials at  $p$ , i.e.  $d_p f_j, j = 1, \dots, n$ , are  $\mathbb{C}$ -linear independent.

*Remark:* Taking  $(U; f_1, \dots, f_n)$  as local coordinate system (as  $f_j$  are functionally independent on a neighborhood of  $p$ ), we obtain a local complex-analytic coordinate system  $(U; z_1, \dots, z_n)$ , where  $z^k = f_k$ .

### 3. SPENCER COORDINATES

We say that a local Spencer coordinate system of type  $m$  is defined on an almost complex manifold  $(M, \mathbf{J})$  if the following conditions hold:

1.) There exist an open subset  $U$  of  $M$  and  $m$  different functionally independent almost holomorphic functions  $f_j : U \rightarrow \mathbb{C}, j = 1, \dots, m$ , such that

2.) The sequence  $(f_1, \dots, f_m)$  is a maximal sequence of functionally independent on  $U$  almost-holomorphic functions. 3.) The sequence

$$(U, w^1, \dots, w^m, z^{m+1}, \dots, z^n, \bar{w}^{n+1}, \dots, \bar{w}^{n+m}, \bar{z}^{n+m+1}, \dots, \bar{z}^{2n})$$

where  $w^j = f_j, j = 1, \dots, m$ , determines a local self-conjugate system on  $(M, \mathbf{J})$ .

An almost complex manifold which is equipped with an atlas of local Spencer coordinate systems is by definition an almost-complex manifold of Spencer type  $m$ . It is to remark that the notion of Spencer type is correctly defined in the category of almost complex manifolds. This follows by the fact that each composition of almost-holomorphic mappings and each inverse of almost-holomorphic diffeomorphism are almost-holomorphic too.

**Lemma 1:** The matrix representation of  $\mathbf{J}^*$  in each local Spencer coordinate system

$$(U, w^1, \dots, w^m, z^{m+1}, \dots, z^n, \bar{w}^{n+1}, \dots, \bar{w}^{n+m}, \bar{z}^{n+m+1}, \dots, \bar{z}^{2n})$$

where  $w^j = f_j, j = 1, \dots, m$ , are functionally independent almost holomorphic functions, seems as follows

$$\begin{pmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{pmatrix}$$

$E_m$  being the unit  $m \times m$  matrix.

*Proof.* It is enough to take in view that:

$$(dw^1, \dots, dw^m, dz^{m+1}, \dots, dz^n, d\bar{w}^{n+1}, \dots, d\bar{w}^{n+m}, d\bar{z}^{n+m+1}, \dots, d\bar{z}^{2n})$$

is basis of the cotangent space and

$$\mathbf{J}^* dw^j = \mathbf{J}^* df_j = idf_j = idw^j, j = 1, \dots, m \blacksquare$$

*Consequences:* The first  $m$  equations of the system  $\mathbf{J}^* df = idf$  are just the conditions  $\partial f / \partial \bar{z}_j = 0, j = 1, \dots, m$

We shall consider the mapping from  $U$  to  $\mathbb{C}^m$  defined by  $f_1, \dots, f_m$ . This mapping is a smooth submersion as it can be considered as a composition of the diffeomorphism

defined by Spencer coordinates of  $U$  in  $\mathbb{C}^n \times \bar{\mathbb{C}}^n$  and the projection of  $\mathbb{C}^n \times \bar{\mathbb{C}}^n$  on  $\mathbb{C}^m$ ,  $m < n$ . This mapping will be denoted by  $f_U$ , and the image of  $U$  by  $f_U$  will be denoted  $U_m^c$ . It is an open subset of  $\mathbb{C}^m$ , which will be called a naturally associated  $m$ -dimensional open set to the considered local Spencer coordinate system.

**Lemma 2:** Each almost holomorphic function  $h$ , defined on a local Spencer coordinate system  $U$  is represented as a superposition of a holomorphic function  $H$  defined on  $U_m^c$  and the almost holomorphic functions  $f_1, \dots, f_m$  defined on  $U$ , i.e.

$$h = H \circ (f_1, \dots, f_m) = H(f_1, \dots, f_m)$$

*Proof:* As  $w_j = f_j$ ,  $j = 1, \dots, m$ , is a system of smooth functionally independent on  $U$  functions, we have  $h = H(w^1, \dots, w^m)$  with  $H \in \mathcal{C}^\infty(U)$ . But

$$\bar{\partial}H = (\bar{\partial}H/\partial\bar{w}^1)d\bar{w}^1 + \dots + (\partial H/\partial\bar{w}^m)d\bar{w}^m$$

and in view of  $\bar{\partial}H = \bar{\partial}h = 0$ , we get that the above written  $(0,1)$ -form is a zero-form, or  $\partial H/\partial\bar{w}_j = 0$ ,  $j = 1, \dots, m$ . ■

**Lemma 3:** Let  $(w^1, \dots, w^m)$  and  $(v^1, \dots, v^m)$  be two systems of holomorphic coordinates on  $U_m^c$  defined by two different systems of almost holomorphic on  $U$  systems  $(f_1, \dots, f_m)$  and  $(h_1, \dots, h_m)$ . Then there exists a bijective holomorphic transition mapping between the mentioned two coordinate systems.

*Proof.* According to Lemma 2 we have  $v_j = H_j(w^1, \dots, w^m)$ ,  $j = 1, \dots, m$ , where  $H_j$  are holomorphic functions of  $(w_1, \dots, w_m)$ . The system  $H = (H_1, \dots, H_m)$  defines the mentioned transition mapping as the differentials  $dH_j$  are  $\mathbb{C}$ -linear independent. ■

Recapitulating we obtain the following

**Proposition 2:** On each paracompact almost complex manifold  $(M, \mathbf{J})$  of constant Spencer type  $m$  there exists a locally finite covering  $U_j$  by self-conjugated Spencer's coordinate system  $(U_j, z_j^1, \dots, z_j^m, \dots)$  such that in every intersection  $U_j \cap U_k$  the holomorphic coordinates  $z_j^1, \dots, z_j^m$  change holomorphically in the other holomorphic coordinates  $z_k^1, \dots, z_k^m$ .

#### 4. LOCAL SUBMERSIONS AND LOCAL FOLIATIONS

As it was remarked above the mapping  $f_U : U \rightarrow \mathbb{C}^m$ , defined by the almost holomorphic functions  $(f_1, \dots, f_m)$  is a local submersion. According to the introduced notations

$$f_U(U) = U_m^c \subset \mathbb{C}^m$$

The leaves of this submersion are defined as the stalks of the mapping  $f_U$ . Each leaf is a smooth  $(2n - 2m)$ -dimensional submanifold of  $U$  on which all functions  $f_j$  have constant value. Transversal leaves are defined as univalent inverse images of  $U_m^c$ , i.e. as sections of  $U$  over  $U_m^c$ .

We shall consider the set of all open subsets  $U_m^c \subset \mathbb{C}^m$ , corresponding to different mappings  $f_U$ ,  $U$  open subset of  $M$ . This set together with the transition mappings



described in *Lemma 3* defines a pseudo-group of holomorphic transition mappings between open subsets of  $\mathbb{C}^m$  denoted as follows

$$\Gamma\{U_m^c, V_m^c, \dots; H : U_m^c \rightarrow V_m^c, \dots\}$$

We shall denote by  $\mathbb{C}^m/\Gamma$  the set of equivalent points of  $\mathbb{C}^m$  with respect to the natural equivalence defined by the holomorphic transition mappings. With this in mind we consider the family  $\{f_U : U \rightarrow M\}$  and will define a glued mapping

$$f : M \rightarrow \mathbb{C}^m/\Gamma$$

as follows: if  $p \in M$  we take an open subset  $U$  such that  $p \in U$  and we set

$$f(p) = \{\text{the equivalence class of the point } f_U(p).\}$$

Under the assumption that  $\mathbb{C}^m/\Gamma$  is equipped with the standard complex structure  $\mathbf{i}$  defined by holomorphic coordinates  $(w^1, \dots, w^m)$  we can formulate the following

**Lemma 4.** The glued mapping  $f : M \rightarrow \mathbb{C}^m/\Gamma$  is an almost holomorphic mapping between  $(M, \mathbf{J})$  and  $(\mathbb{C}^m/\Gamma, \mathbf{i})$ .

*Proof.* As the glued mapping  $f$  coincides locally with some  $f_U$  we have:

$$\mathbf{J}^*df_U = \mathbf{J}^*d(f_1, \dots, f_m) = \mathbf{J}^*(df_1, \dots, df_m) = (\mathbf{J}^*df_1, \dots, \mathbf{J}^*df_m) = \mathbf{i}(df_1, \dots, df_m) = id f_U. \text{ So each } f_U \text{ is an almost holomorphic mapping } \blacksquare$$

**Lemma 5.** The sheaf of almost holomorphic functions on  $M$  is the inverse image of the sheaf of holomorphic functions on  $\mathbb{C}^m/\Gamma$ .

*Proof.* The mentioned sheaf on  $M$  is defined by the presheaf  $\{U, \mathcal{O}_M(U)\}$  where  $U$  varies in the set of all open subsets of  $M$  and  $\mathcal{O}_M(U)$  is defined as follows:

$$\mathcal{O}_M(U) = \{h \circ f_U \mid h \in \mathcal{O}_{\mathbb{C}^m/\Gamma}(f_U(U))\} .$$

**4.1. Hypercomplex manifolds and hyperholomorphic functions.** Let  $M$  be a  $4n$ -dimensional ( $C^\infty$ ) smooth manifold. A hypercomplex structure on  $M$  is defined by a pair of two almost complex structures  $\mathbf{J}$  and  $\mathbf{K}$  such that  $\mathbf{JK} + \mathbf{KJ} = 0$ . It is easy to see that the composition  $\mathbf{JK}$  is an almost-complex structure too. Moreover, for each triple of real numbers  $b, c, d$ , such that  $b^2 + c^2 + d^2 = 1$ , the linear combination  $b\mathbf{J} + c\mathbf{K} + d(\mathbf{JK})$  is an almost-complex structure on  $M$ . So there is a family of almost complex structures on  $M$  parametrized by the points of sphere  $\Sigma^2$ . (See for instance [AM], [ABM]).

We shall consider almost-holomorphic functions on hypercomplex manifolds. The definition remains the same as in the above considered case, for instance on  $(M, \mathbf{J}, \mathbf{K})$  we have  $\mathbf{J}$ -almost- holomorphic function which are complex-valued function  $f$  on  $(M, \mathbf{J})$  such that  $\mathbf{J}^*df = idf$  using the right-side  $j$ -complex splitting of  $\mathbb{H}$ . Respectively  $\mathbf{K}$ -almost- holomorphic functions  $g$  on  $(M, \mathbf{J}, \mathbf{K})$  are the almost-holomorphic with respect to  $(M, \mathbf{K})$  such that  $\mathbf{K}^*dg = jdg$  using an  $i$ -complex splitting of  $\mathbb{H}$ . Let  $(M, \mathbf{J}, \mathbf{K})$  be a hypercomplex manifolds and  $\mathbb{H}$  be 4-dimensional quaternionic vector space. According to Sommese [So] the right-side multiplication by  $i$  and  $j$  are given respectively by the matrices  $S$  and  $T$ , called standard quaternionic structures.

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

In the paper of Sommese the matrix  $T$  is denoted by  $K$ .

As we have  $S^2 = -\mathbf{1}, T^2 = -\mathbf{1}, (ST)^2 = -\mathbf{1}$ , and  $ST + TS = 0$ , we can consider  $(\mathbb{H}, S, T)$  as a special hypercomplex manifold. (See [So]). A function  $F$  defined on an open subset  $U \subset M$  with valued in  $\mathbb{H}$  is called **J**-hyper-holomorphic function on  $U$  if  $dF \circ \mathbf{J} = S \circ dF$ , or  $\mathbf{J}^*dF = SdF$ . Using the right-side  $j$ -complex splitting  $\mathbb{H}^j$  we take the compositions of  $F$  with the projections of  $\mathbb{H}$  on the first and the second components of  $\mathbb{H}^j$ . So  $F$  is represented by a pair of complex valued functions denoted respectively by  $f$  and  $\varphi$ . If we set  $F = u + iv + j\zeta + k\eta$ , where  $u, v, \zeta, \eta$  are real-valued functions on  $U$ , we can write  $\varphi = u + iv + (\zeta + i\eta)j$ , with  $f = u + iv$ ,  $\varphi = \zeta + i\eta$ . Complexifying the matrix  $S$ , i.e. setting

$$S = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i} \end{bmatrix}, \quad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and taking  $dF = df + d\varphi j$ , we calculate that

$$J^*df + J^*d\varphi j = i df - i d\varphi j.$$

Having in mind the splitting  $\mathbb{H}^j$ , we get  $J^*df = i df$  and  $J^*d\varphi = -i d\varphi$ , which means that  $f$  is **J**-almost-holomorphic function on  $U$  and  $\varphi$  is **J**-almost-antiholomorphic.

For the definition of **K**-hyper-holomorphic function on  $U$  we shall use the other complex splitting of  $\mathbb{H}$ , namely  $\mathbb{H}^i$ . A function  $G : M \rightarrow \mathbb{H}^i$ , i.e.  $G = g + \psi i$ ,  $g = u' + j\zeta'$ ,  $\psi = v' - j\eta'$ , will be called **K**-hyper-holomorphic function on  $U$  if  $dG \circ K = T \circ dG$  or  $K^*dG = TdG$ . Taking a  $(2 \times 2)$ -representation of the matrix  $T$ , i.e.

$$T = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

after a short calculation we get

$$K^*dg + i K^*d\psi = d\psi - i dg$$

It follows that  $K^*dg = d\psi$  and  $K^*d\psi = -dg$ . This result is in terms of  $\mathbb{H}^i$ .

Now we will translate the obtained result in terms of  $\mathbb{H}^j$ . From

$$K^*(du' + d\zeta'j) = dv' - d\eta'j \text{ we get}$$

$$K^*du' = dv' \quad \text{and} \quad K^*d\zeta' = -d\eta'.$$

Analogously, from  $K^*(dv' - d\eta'j) = -(du' + d\zeta'j)$  we get

$$K^*dv' = -du' \quad \text{and} \quad K^*d\eta' = -d\zeta'.$$

But the system  $K^*du' = dv'$ ,  $K^*dv' = -du'$  is just the Cauchy-Riemann system, which says that the function  $u' + iv'$  is **J**-almost-antiholomorphic, i. e.  $J^*d(u' + iv') = -id(u' + iv')$ . The function  $\zeta' + i\eta'$  is **J**-almost-holomorphic.

**4.2. Hyper-Spencer coordinates.** Hyper-holomorphic coordinates on a hyper-complex manifold  $(M, \mathbf{J}, \mathbf{K})$  can be introduced by functionally independent quaternionic-valued functions  $f_\alpha + \varphi_\alpha j$ ,  $\alpha = 1, \dots, m$ ,  $m = (1/2)\dim M_{\mathbb{R}}$ , or by the complex-valued function  $(f_\alpha, \varphi_\alpha)$ . We are interested of the possibility to have  $m < (1/2)\dim M_{\mathbb{R}}$ . More precisely, a  $\mathbf{J}$ -hyper- Spencer coordinate system is defined locally on  $M$  as a maximal system of  $m$  functionally independent  $\mathbf{J}$ -hyper-holomorphic functions. A hypercomplex manifold equipped with an atlas of local  $\mathbf{J}$ -hyper-Spencer coordinate systems is called a hypercomplex manifold of Spencer type  $m$ .

Having in mind the interconnection between  $\mathbf{J}$ -hyper-holomorphic functions and  $\mathbf{J}$ -almost-holomorphic ones we derive the analogues of the *Lemmas 1,2* and *3* of the previous paragraphs. Let us remark that in view that  $f_\alpha$  are  $\mathbf{J}$ -almost- holomorphic, and  $\varphi_\alpha$  are  $\mathbf{J}$ -almost-antiholomorphic, the corresponding matrix representations of  $J^*$  is as follows (according to *Lemma 1*)

$$\begin{bmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{bmatrix} \times \begin{bmatrix} -iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & iE_m & * \\ 0 & * & 0 & * \end{bmatrix}$$

Analogously,  $\mathbf{K}$ -hyper-Spencer coordinates can be introduced with the help of  $\mathbf{K}$ -hyper holomorphic mappings. The *Proposition 2* remains valid for  $\mathbf{J}$ -holomorphic transition functions and  $\mathbf{K}$ -holomorphic transition functions. When the transition transformations are simultaneously  $\mathbf{J}$ - and  $\mathbf{K}$ -holomorphic it follows that they are affine.

Full coordinate systems defined by  $m = (1/2)\dim M_{\mathbb{R}}$  functions which are both  $\mathbf{J}$  and  $\mathbf{K}$  hyper-holomorphic lead to quaternionic manifolds.

### 5. ELLIPTIC EQUATIONS

**5.1. Potential structures on almost-complex manifolds.** Let  $(M, \mathbf{J})$  be an almost complex manifold. We shall consider the following globally defined on  $M$  Pfaffian form:  $\omega = J^* du$ , where  $u = u(p)$ ,  $p \in M$ , is a real-valued smooth (at least of class  $\mathcal{C}^2$ ) function. In the case the 1-form  $\omega$  is closed, we will say that  $\omega$  defines a *potential structure* on the almost complex manifold  $(M, \mathbf{J})$ . On each local real coordinate system  $(U, x = (x^k))$ ,  $x^k \in \mathbb{R}$ ,  $k = 1, \dots, 2n$ , we have a matrix representation of  $\mathbf{J}$ , i.e.  $J = \| J_j^k(x) \|$ , where  $J_j^k(x)$  are smooth real functions on  $U$ . By  $J_j$  is denoted the  $j$ -row of the mentioned matrix and  $\nabla u$  is the gradient of  $u$ . It is easy to see

$$J du = \sum_{q=1}^{2n} (J_q \cdot \nabla u) dx^q$$

where

$$J_q \cdot \nabla u = \sum_{p=1}^{2n} J_q^p \frac{\partial u}{\partial x^p},$$

For each potential structure on  $(M, \mathbf{J})$  the following two statements hold.

**Consequence 1.** On every simply connected domain  $\Omega \subset M$  it holds that

$$\int_{\gamma} J^* du = 0$$

for each closed curve  $\gamma$  in  $\Omega$ .

**Consequence 2.** The following system

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

$s, q = 1, \dots, 2n$ , is satisfied locally.

**5.2. Almost pluri-harmonic functions.** By  $(M, \mathbf{J}, \omega)$  is denoted an almost-complex manifold  $(M, \mathbf{J})$  equipped with potential structure  $\omega$ . Then the 1-form  $\omega = J^* du$  is close, and we have  $dJ^* du = 0$ . In this case we will say that the function  $u$  is an almost-pluriharmonic function. The interconnection between almost-pluriharmonic functions and almost-holomorphic ones (with respect to  $\mathbf{J}$ ) is like to this one between pluriharmonic functions and holomorphic ones. This follows directly from the Cauchy-Riemann equations  $J^* du = -dv$ ,  $J^* dv = du$ . Clearly the real part  $u$  and the imaginary part  $v$  of the almost-holomorphic function  $f = u + iv$  are almost-pluriharmonic functions.

**5.3. Elliptic equations on almost-complex manifolds.** We denote by  $\Delta_{\mathbf{J}}$  the following differential operator of second order (in terms of coordinates)

$$\Delta_{\mathbf{J}} = \sum_{s,p=1}^{2n} A_{sp} \frac{\partial^2}{\partial x^s \partial x^p} + \sum_{p=1}^{2n} B_p \frac{\partial}{\partial x^p}$$

where

$$A_{sp} = \sum_{q=1}^n (J_q^s J_q^p + \delta_q^s \delta_q^p),$$

and

$$B_p = \sum_{s,q=1}^{2n} J_q^s \left( \frac{\partial J_q^p}{\partial x^s} - \frac{\partial J_s^p}{\partial x^q} \right),$$

$\delta_q^s, \delta_q^p$  are the Kronecker symbols. Setting  $A_J = \|A_{sp}\|$ , we obtain

$$A_J = JJ^* + E_{2n}$$

where  $J^*$  is the transpose of  $J$  and  $E_{2n}$  is the unity  $2n \times 2n$  matrix.

We emphasize here that now we work with real coordinates, but not with complex self-conjugate ones. However this corresponds to the Spencer type 0. In the other extreme case of Spencer type  $n$  we have complex-analytic (holomorphic) coordinates.

This is the case of complex analytic manifold with the standard almost-complex structure denoted by  $S^0$  (it is different from  $S$  in the previous paragraph).

$$-S^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \dots \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (n \text{ times})$$

As  $S^0(S^0)^* = E_{2n}$  we get  $A_{S^0} = 2E_{2n}$  and  $\Delta_{S^0} = 2\Delta$ , where  $\Delta$  is the Laplace operator in  $2n$  real variables.

**Proposition 3:**  $\Delta_J$  is an elliptic differential operator.

*Proof:* It is sufficient to consider the following inequality

$$\sum_{s=1}^{2n} \sum_{p=1}^{2n} A_{sp} \xi_s \xi_p = \sum_{q=1}^{2n} \left( \sum_{s=1}^{2n} J_q^s \xi_s \right)^2 + \sum_{q=1}^{2n} \left( \sum_{s=1}^{2n} \delta_q^s \xi_s \right)^2 \geq \sum_{q=1}^{2n} \xi_q^2. \quad \blacksquare$$

Considering the PDE

$$\Delta_J u = 0,$$

we can state the following

**Theorem :** Each almost pluriharmonic function  $u$  satisfies locally the equation  $\Delta_J u = 0$

*Proof:* Let  $u$  be almost pluriharmonic, i.e.  $dJdu = 0$ , or the 1-form  $J^*du$  is closed. According to the previous paragraph  $u$  satisfies locally the following system of PDEs

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

$s, q = 1, \dots, 2n$ . Now replacing

$$J_q \bullet \nabla u = \sum_{p=1}^{2n} J_q^p \frac{\partial u}{\partial x^p} \quad \text{and} \quad J_s \bullet \nabla u = \sum_{p=1}^{2n} J_s^p \frac{\partial u}{\partial x^p}$$

in (4) we obtain the system

$$\sum_{p=1}^{2n} \left( \frac{\partial(J_q^p \frac{\partial u}{\partial x^p})}{\partial x^s} - \frac{\partial(J_s^p \frac{\partial u}{\partial x^p})}{\partial x^k} \right) = 0,$$

$k, s = 1, \dots, 2n$ . Multiplying each of the above written equations by  $J_q^s$  and summing with respect to  $s$  we obtain

$$\sum_{p=1}^{2n} \sum_{s=1}^{2n} \left( J_q^p J_q^s \frac{\partial^2 u}{\partial x^s \partial x^p} - J_q^s J_s^p \frac{\partial^2 u}{\partial x^k \partial x^p} \right) = \sum_{p=1}^{2n} \sum_{s=1}^{2n} J_q^s \left( \frac{\partial J_s^p}{\partial x^k} - \frac{\partial J_k^p}{\partial x^s} \right) \frac{\partial u}{\partial x^p}.$$

As we have

$$\sum_{s=1}^{2n} J_q^s J_s^p = -\delta_q^p$$

and

$$\frac{\partial^2 u}{\partial x^k \partial x^p} = \sum_{s=1}^{2n} \delta_k^s \frac{\partial^2 u}{\partial x^s \partial x^p},$$

we obtain

$$\sum_{p=1}^{2n} \sum_{s=1}^{2n} (J_k^p J_q^s + \delta_q^p \delta_k^s) \frac{\partial^2 u}{\partial x^s \partial x^p} = \sum_{p=1}^{2n} \sum_{s=1}^{2n} J_q^s \left( \frac{\partial J_s^p}{\partial x^s} - \frac{\partial J_k^p}{\partial x^s} \right) \frac{\partial u}{\partial x^p}.$$

Now taking  $q = k$  and summing with respect to  $k$  we get exactly

$$\Delta_{\mathbf{J}} u = 0. \quad \blacksquare$$

In the case  $\mathbf{J} = \mathbf{S}^0$  the above written equation is just the classical Cauchy-Riemann system.

**Consequences:**

1. Each almost pluriharmonic function and respectively every almost holomorphic function of class  $\mathcal{C}^2$  on a smooth manifold are of class  $\mathcal{C}^\infty$  too.
2. For connected smooth manifolds the maximum principle holds.
3. In the case of real analytic manifold  $M$ , equipped with real-analytic structure  $\mathbf{J}$ , each  $\mathbf{J}$ -pluriharmonic and each  $\mathbf{J}$ -almost-holomorphic function is real analytic.
4. In the case of connected real analytic manifold  $M$  with real-analytic structure  $\mathbf{J}$  the principle of unicity of the analytic continuation holds.

*Remark:* This theorem is inspired from the paper [BKW]. The first announcement is in [DM]

**5.4. The equation  $dJ^*du = 0$  in terms of vector fields - commutators and anti-commutators.** Applying the well known formula

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \omega \text{ is 1-form, } X, Y \text{ are vector fields}$$

to the 1-form  $\omega = \mathbf{J}du$  we present the equation (2) in terms of expressions of vector fields, namely

$$[X, Y]_{\mathbf{J}}(u) = \mathbf{J}[X, Y](u)$$

where  $[X, Y]_{\mathbf{J}} \stackrel{def}{=} X \circ \mathbf{J}Y - Y \circ \mathbf{J}X$ . It is to remark that  $[X, Y]_{\mathbf{J}}$  is not a vector field. For instance:

$$[X, Y]_{\mathbf{J}}(fh) = [X, Y]_{\mathbf{J}}(f)h + f [X, Y]_{\mathbf{J}}(h) + X(f)(\mathbf{J}Y)(h) - (\mathbf{J}X)(f)Y(h) + X(h)(\mathbf{J}Y)(f) - (\mathbf{J}X)(h)Y(f)$$

*Some properties of  $[X, Y]_{\mathbf{J}}$*

Considering the natural splitting

$$CTM = T^{1,0}M \oplus T^{0,1}M$$

we can take the restriction of  $[X, Y]_{\mathbf{J}}$  on  $T^{1,0}M$ . This means that

$$\mathbf{J}X = iX \text{ and } \mathbf{J}Y = iY$$

where  $X, Y \in T^{1,0}M$ . So we have

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (iX) = i[X, Y]$$

Analogously

$$[X, Y]_{\mathbf{J}} = (-i)[X, Y] \text{ on } T^{0,1}M$$

Now we take  $X \in T^{1,0}M$  and  $Y \in T^{0,1}M$

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (-iX) = i(X \circ Y + Y \circ X) = i\{X, Y\}$$

Here  $\{X, Y\}$  denotes the anticommutator of  $X$  and  $Y$ . Analogously, if  $X \in T^{0,1}M$  and  $Y \in T^{1,0}M$ :

$$[X, Y]_{\mathbf{J}} = -i\{X, Y\}$$

**5.5. Potential structures on hypercomplex manifolds.** On a hypercomplex manifold  $(M, \mathbf{J}, \mathbf{K})$  we can consider two separate potential structures, namely

$$\omega_1 = \mathbf{J}^* du \quad \text{and} \quad \omega_2 = \mathbf{K}^* d\zeta$$

or the sum

$$\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$$

The corresponding almost-pluriharmonic functions  $u, v, \zeta, \eta$  satisfy the equations:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \quad \text{and} \quad d\mathbf{J}^* d\zeta = d\mathbf{J}^* d\eta = 0$$

We have also the natural defined elliptic operators  $\Delta_{\mathbf{J}}$  and  $\Delta_{\mathbf{K}}$ . According to the proved theorem:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \quad \Rightarrow \quad \Delta_{\mathbf{J}} u = \Delta_{\mathbf{J}} v = 0$$

and

$$d\mathbf{K}^* d\zeta = d\mathbf{K}^* d\eta = 0 \quad \Rightarrow \quad \Delta_{\mathbf{K}} \zeta = \Delta_{\mathbf{K}} \eta = 0$$

For the sum  $\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$  a pair of functions  $(u, \zeta)$  appears, namely the solutions of the following second order equation:

$$d\mathbf{J}^* du + d\mathbf{K}^* d\zeta = 0$$

In terms of vector fields the above written equations seem as follows

$$[X, Y]_{\mathbf{J}} u = \mathbf{J}[X, Y](u) \quad \text{and} \quad [X, Y]_{\mathbf{K}} \zeta = \mathbf{K}[X, Y](\zeta)$$

## 6. GENERATION OF ALMOST-COMPLEX STRUCTURES

**6.1. Remarks on the local equation of almost-holomorphic functions.** Let  $(M, \mathbf{J})$  be an almost-complex manifold,  $\dim M = 2n$ . Having in mind the question of the local integration of the equation  $\mathbf{J}^*df = idf$ , we shall examine how "far away" a non-integrable almost complex structure  $\mathbf{J}$  is from the classical complex structure related with the standard almost-complex structure  $\mathbf{S}$ .

Let  $p$  be a point of  $M$ . Taking an open neighborhood  $U$  of the point  $p$ , small enough, we can accept that  $U$  is a neighborhood of the origin in  $\mathbb{R}^{2n}$  ( $p$  to be the origin). Now we shall replace  $\mathbf{J}$  by its matrix representation  $J$  on  $U$  and  $J^*$  will denote the transposed matrix. We will use general real coordinates  $x = (x^1, \dots, x^{2n}) \in \mathbb{R}^{2n}$ . Let  $G$  denote a non-degenerate  $(2n \times 2n)$  matrix, such that  $G^{-1}J^*(0)G = S^*$ , where  $S^*$  is the transposed matrix of  $S$ ,

$$S = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, \quad E_n \text{ being the unit } n \times n \text{ matrix.}$$

For  $x \in U$  we set:

$$G^{-1}J(x)G = \begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix}$$

$A(x), B(x), C(x), D(x)$  are  $n \times n$  matrices.

Clearly we have for  $x = 0$ :

$$\begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix} = S^* \text{ and } A(0) = B(0) = C(0) = D(0) = 0$$

Moreover, we have  $(G^{-1}J(x)G)^2 = -E_{2n}$ , which implies the following identities:

$$\begin{aligned} A^2(x) + (B(x) + E_n)(C(x) - E_n) &= -E_n \\ A(x)(B(x) + E_n) + (B(x) + E_n)D(x) &= 0_n \\ (C(x) - E_n)A(x) + D(x)(C(x) - E_n) &= 0_n \\ (C(x) - E_n)(B(x) + E_n) + D^2(x) &= -E_{2n} \end{aligned}$$

From the last system it follows that locally is valid:

$$\begin{aligned} A(x) &= -(C(x) - E_n)^{-1}D(x)(C(x) - E_n) \\ B(x) + E_n &= -(C(x) - E_n)^{-1}(D^2(x) + E_n) \end{aligned}$$

Indeed, as

$$\det(C(0) - E_n) = (-1)^n \neq 0$$

the inverse matrix  $(C(x) - E_n)^{-1}$  exists in some neighborhood of the origin  $0 \in \mathbb{R}^n$ .

Now let's consider the equation  $(J^* - iE_{2n})df = 0$ . It follows that

$$(G^{-1}J^*G - iE_{2n})df = 0$$

and also

$$\begin{bmatrix} A(x) - iE_n & B(x) + E_n \\ C(x) - E_n & D(x) - iE_n \end{bmatrix} df = 0$$

*Proposition:* The following block matrix identity is valid:



$$[ A(x) - iE_n \quad B(x) + E_n ] = (A(x) - iE_n)(C(x) - E_n)^{-1} [ C(x) - E_n \quad D(x) - iE_n ]$$

*Proof:* Let consider the right side of the identity:

$$\begin{aligned} & (A(x) - iE_n)(C(x) - E_n)^{-1} [ C(x) - E_n \quad D(x) - iE_n ] = \\ & = [ A(x) - iE_n \quad (A(x) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) ] \end{aligned}$$

But:

$$(A(x) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = B(x) + E_n, \text{ as } A(x) = -(C(x) - E_n)^{-1}D(x)(C(x) - E_n).$$

The last equality becomes:

$$\begin{aligned} & (-(C(x) - E_n)^{-1}D(x)(C(x) - E_n) - iE_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = \\ & = (C(x) - E_n)^{-1}(-D(x) - iE_n)(C(x) - E_n)(C(x) - E_n)^{-1}(D(x) - iE_n) = \\ & = -(C(x) - E_n)^{-1}(D(x) + iE_n)(D(x) - iE_n) = \\ & = -(C(x) - E_n)^{-1}(D^2(x) + iE_n) = B(x) + E_n. \blacksquare \end{aligned}$$

*Corollary:* The first  $n$  equations of the considered system

$$(J^* - iE_{2n})df = 0$$

follow from the last  $n$  ones. So we obtain that locally this system is equivalent to the next one:

$$[ C(x) - E_n \quad D(x) - iE_n ] df = 0$$

or:

$$[ E_n \quad (C(x) - E_n)^{-1}(D(x) - iE_n) ] df = 0$$

Setting  $P(x) \stackrel{def}{=} (C(x) - E_n)^{-1}D(x)$  and  $Q(x) \stackrel{def}{=} (C(x) - E_n)^{-1}$ , we receive the following block matrix form of the considered equation of almost holomorphic functions:

$$[ E_n \quad P(x) + iQ(x) ] df = 0.$$

**6.2. Local reconstruction of  $J$  by the matrices  $P$  and  $Q$ .** We will use the following equalities:

$$\begin{aligned} C - E_n &= Q^{-1}; \quad D = Q^{-1}P; \quad A = -QQ^{-1}PQ = -PQ^{-1}; \\ B + E_n &= -Q((Q^{-1}P)^2 + E_n) = -PQ^{-1}P - E_n. \end{aligned}$$

The matrix  $J$  can be reconstructed as follows:

$$J = \begin{bmatrix} -PQ^{-1} & -PQ^{-1}P - Q \\ Q^{-1} & Q^{-1}P \end{bmatrix} \quad (*)$$

The mentioned reconstruction (\*) can be considered as a generation of the matrix representation of  $J$  on the open set  $U$  by the pair of matrices  $(P, Q)$ . Denoting by  $\mathcal{M}(U, n)$  the algebra of all  $(n \times n)$ -matrices equipped with the topology of coordinate convergence, we can consider the Cartesian product  $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$  with the product topology as a continuous family which generates the set  $\mathcal{J}(U, 2n)$  of all  $(2n \times 2n)$ -matrices  $J$ , which verify the matrix equation

$$J^2 + E_{2n} = 0,$$

as a kind of moduli space (locally). More precisely, the following proposition holds

**Proposition 4:** For each  $J \in \mathcal{J}(U, 2n)$  there is a pair  $(P, Q) \in \mathcal{M}(U, n) \times \mathcal{M}(U, n)$  such that  $J$  is generated by  $(P, Q)$  in the sense of the rule (\*). Conversely, each pair  $(P, Q)$  defines a  $J$  according to the rule (\*). Each sequence  $(P_n, Q_n)$  of elements of  $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$  determines a sequence of elements of  $\mathcal{J}(U, 2n)$ , and the limit of the second sequence corresponds by the rule (\*) to the limit of the first sequence.

The proof is clear.

**6.3. Global reconstruction of  $J$ .** The problem of global reconstruction of almost complex structures on a smooth manifold by an appropriate algebraic objects is much more difficult. It seems that an approach can be developed on real-analytic almost complex manifold  $(M, J)$  having local matrix representation for  $J$  with real-analytic coefficients. Now we shall consider the sheaf of germs of almost complex structures, denoted by  $\mathcal{J}(M)$ , and the sheaf of germs of pairs of matrices  $(P, Q)$ . Supposing that each  $J$  can be considered as a global section of the sheaf  $\mathcal{J}(M)$ , we can develop the rule (\*) for germs of  $\mathcal{J}(M)$  and germs of pairs  $(P, Q)$  at each point  $p \in M$ . The set of global sections of  $\mathcal{J}(M)$  must be generated by the sections of the sheaf of germs of pairs  $(P, Q)$ .

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## QUATERNION KÄHLER FLAT MANIFOLDS

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This note is a revised version of the talk given by the author at the meeting *Quaternionic structures in Mathematics and Physics* at Rome in September, 1999. The results presented here are part of [4], a joint work with R. Miatello.

### 1. INTRODUCTION

A Riemannian manifold is *quaternion Kähler* if its holonomy group is contained in  $Sp(n)Sp(1)$ . It is known that quaternion Kähler manifolds are Einstein, so the scalar curvature  $s$  splits these manifolds according to whether  $s > 0$ ,  $s = 0$  or  $s < 0$ . Ricci flat quaternion Kähler manifolds include hyperkähler manifolds, that is, those with full holonomy group contained in  $Sp(n)$ . Such a manifold can be characterized by the existence of a pair of integrable anticommuting complex structures, compatible with respect to the Riemannian metric, and parallel with respect to the Levi-Civita connection.

It is the main purpose of this lecture to indicate a rather general method to construct quaternion-Kähler compact flat manifolds. This construction will give many families of quaternion Kähler manifolds of dimensions  $n \geq 8$ , which admit no Kähler structure (see Section 3). This will follow from the explicit calculation of the Betti numbers of the manifolds involved.

The simplest model of hyperkähler manifolds (and in particular, of quaternion Kähler manifolds) is provided by  $\mathbb{R}^{4n}$  with the standard flat metric and a pair  $J, K$  of orthogonal anticommuting complex structures. This hyperkähler structure descends to the  $4n$ -torus  $T_\Lambda := \Lambda \backslash \mathbb{R}^{4n}$ , for any lattice  $\Lambda$  in  $\mathbb{R}^{4n}$ . The main idea in the construction consists of finding finite groups  $F$  acting freely on the torus, endowed with the standard hyperkähler structure, in such a way that  $F \backslash T^{4n}$  becomes quaternion Kähler but its cohomology changes in such a way that the resulting manifold will not admit any Kähler structure.

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## 2. CONSTRUCTION OF QUATERNION KÄHLER FLAT MANIFOLDS.

One way of constructing free actions of finite groups on tori is via Bieberbach groups. A Bieberbach group  $\Gamma$  is a crystallographic group (i.e. a discrete cocompact subgroup of  $I(\mathbb{R}^n)$ ) which is torsion-free. The quotient  $M_\Gamma := \Gamma \backslash \mathbb{R}^n$  is a compact flat Riemannian manifold with fundamental group  $\Gamma$ . If  $v \in \mathbb{R}^n$ , let  $L_v$  denote translation by  $v$ . By Bieberbach's first theorem, if  $\Gamma$  is a crystallographic group then  $\Lambda = \{v : L_v \in \Gamma\}$  is a lattice in  $\mathbb{R}^n$ . The translation lattice  $L_\Lambda = \{L_v : v \in \Lambda\}$  is a normal and maximal abelian subgroup of  $\Gamma$  and the quotient  $F := L_\Lambda \backslash \Gamma$  is a finite group acting freely on  $\Lambda \backslash \mathbb{R}^n$ ; it represents the linear holonomy group of the flat Riemannian manifold  $M_\Gamma$  and is called the holonomy group of  $\Gamma$ . We will usually write  $\Lambda$  in place of  $L_\Lambda$ .

Any element  $\gamma \in I(\mathbb{R}^n)$  decomposes uniquely  $\gamma = BL_b$ , with  $B \in O(n)$  and  $b \in \mathbb{R}^n$  and the lattice  $\Lambda$  is  $B$ -stable for each  $BL_b \in \Gamma$ . The restriction to  $\Gamma$  of the canonical projection from  $I(\mathbb{R}^n)$  to  $O(n)$ , mapping  $BL_b$  to  $B$ , has kernel  $\Lambda$  and the image is a finite subgroup of  $O(n)$ , called the point group of  $\Gamma$ . We shall often identify the holonomy group  $F$  with the point group of  $\Gamma$ . The action of  $F$  on  $\Lambda$  defines an integral representation of  $F$ , usually called the holonomy representation.

If  $M_\Gamma = \Gamma \backslash \mathbb{R}^{4n}$  is a compact flat manifold such that the holonomy action of  $F = \Lambda \backslash \Gamma$  centralizes (resp. normalizes) the algebra generated by  $J, K$ , then  $M_\Gamma$  inherits a hyperkähler (resp. quaternion Kähler) structure. To produce Bieberbach groups having the previous property we introduced in [1] a "doubling" procedure for Bieberbach groups which allows to produce many flat hyperkähler (even Clifford Kähler) manifolds. In particular, we showed that any finite group is the holonomy group of a hyperkähler flat manifold. The main goal will be to give a variant of this construction which produces quaternion Kähler manifolds which are generically not Kähler.

Let  $\Gamma$  be a Bieberbach group with holonomy group  $F$  and translation lattice  $\Lambda \subset \mathbb{R}^n$ . Let  $\phi : F \rightarrow \mathbb{R}^n$  be a 1-cocycle modulo  $\Lambda$ , that is,  $\phi$  satisfies  $\phi(B_1 B_2) = B_2^{-1} \phi(B_1) + \phi(B_2)$ , modulo  $\Lambda$ , for each  $B_1, B_2 \in F$ . Then  $\phi$  defines a cohomology class in  $H^1(F; \mathbb{R}^n / \Lambda) \simeq H^2(F; \Lambda)$  and one may associate to  $\phi$  a crystallographic group with holonomy group  $F$  and translation lattice  $\Lambda$ . Furthermore, this group is torsion-free if and only if the class of  $\phi$  is a special class (see [2]).

**Definition 2.1.** Let  $\Gamma$  be a Bieberbach group with holonomy group  $F$  and translation lattice  $\Lambda \subset \mathbb{R}^n$ . Let  $\phi : F \rightarrow \mathbb{R}^n$  be any 1-cocycle modulo  $\Lambda$ . We let  $d_\phi \Gamma$  be the subgroup of  $I(\mathbb{R}^{2n})$  generated by elements of the form  $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} L_{(\phi(B), b)}$  and  $L_{(\lambda, \mu)}$ , for  $\gamma = BL_b \in \Gamma$  and  $(\lambda, \mu) \in \Lambda \oplus \Lambda$ .

**Proposition 2.2.** (compare with [1], Theorem 3.1) Let  $\Gamma, \phi$  and  $d_\phi \Gamma$  be as in Definition 2.1 Then

(i)  $d_\phi \Gamma$  is a Bieberbach group with holonomy group  $F$ , translation lattice  $\Lambda \oplus \Lambda$  and  $d_\phi \Gamma \backslash \mathbb{R}^{2n}$  is a Kähler compact flat manifold.

(ii) If  $\Gamma \backslash \mathbb{R}^n$  has a locally invariant Kähler structure, then  $d_\phi \Gamma \backslash \mathbb{R}^{2n}$  is hyperkähler. In particular, if  $\phi' : F \rightarrow \mathbb{R}^{2n}$  is any 1-cocycle modulo  $\Lambda \oplus \Lambda$ , then  $d_\phi d_\phi \Gamma \backslash \mathbb{R}^{4n}$  is hyperkähler. Any finite group is the holonomy group of a hyperkähler compact flat manifold.

We shall work mostly with the choice  $\phi = 0$  and we shall then write  $d_0 \Gamma$ . Other natural choice is to let  $\phi$  be the 1-cocycle associated to  $\Gamma$ , as in [1]; we denote  $d_\phi \Gamma$  by  $d\Gamma$  in this case.

It is clear that the procedure in (ii) of Proposition 2.2 can be iterated. If we assume that  $\phi = 0$ , for simplicity, and we set  $d_0^m \Gamma = d_0 d_0^{m-1} \Gamma$ , we get that  $d_0^m \Gamma$  is a Bieberbach subgroup of  $I(\mathbb{R}^{2^m n})$  with holonomy group  $F$ , diagonal holonomy representation and translation lattice  $\Lambda^{2^m}$ . Furthermore the holonomy representation commutes with  $m$  anticommuting complex structures on  $\mathbb{R}^{2^m n}$ , hence  $d_0^m \Gamma \backslash \mathbb{R}^{2^m n}$  has a Clifford structure of order  $m$  (compare [1], 3.1).

We wish to enlarge  $d_\phi \Gamma$  into a Bieberbach group  $d_{q,\phi} \Gamma$  in such a way that some element in the holonomy group of  $d_{q,\phi} \Gamma$  anticommutes with the complex structure  $J_{2n}$  in  $\mathbb{R}^{2n}$ . Once this is done, then by repeating the procedure twice, we shall get a Bieberbach group such that any element in the holonomy group will either commute or anticommute with each one of a pair of anticommuting complex structures, hence the quotient manifold will be a quaternion Kähler flat manifold which in general, will not be Kähler.

In order for this second construction to work we will restrict to Bieberbach groups with holonomy group  $\mathbb{Z}_2^k$ . We will make use of the following result from [3], Proposition 2.1 (see also [5], Proposition 1.1).

**Proposition 2.3.** *Assume that  $\Gamma = \langle \gamma_1, \dots, \gamma_k, \Lambda \rangle$  is a subgroup of  $Aff(\mathbb{R}^n)$ , with  $\gamma_i = B_i L_{b_i}$ ,  $b_i \in \mathbb{R}^n$ ,  $B_i \in Gl(n, \mathbb{R})$  such that  $\langle B_1, \dots, B_r \rangle$  is isomorphic to  $\mathbb{Z}_2^k$  and  $\Lambda$  is a lattice in  $\mathbb{R}^n$  stable by the  $B_i$ 's. Then  $\Gamma$  is torsion-free with translation lattice  $\Lambda$  if and only if the following two conditions hold:*

- (i) For each pair  $i, j$ ,  $1 \leq i, j \leq k$ ,  $(B_i - Id)b_j - (B_j - Id)b_i \in \Lambda$ .
- (ii) For each  $I = (i_1, \dots, i_s)$  with  $1 \leq i_1 \leq i_2 \leq \dots \leq i_s \leq k$ , let  $B_{i_1} L_{b_{i_1}} \dots B_{i_s} L_{b_{i_s}} = B_I L_{b(I)} \in \Gamma$ , with  $B_I := B_{i_1} \dots B_{i_s}$  and  $b(I) = B_{i_s} \dots B_{i_2} b_{i_1} + B_{i_s} \dots B_{i_3} b_{i_2} + \dots + B_{i_2} b_{i_{s-1}} + b_{i_s}$ . Then

$$(B_I + Id) b(I) \in \Lambda \setminus (B_I + Id) \Lambda.$$

Finally, if  $\Gamma$  satisfies conditions (i) and (ii), then  $\Gamma$  is isomorphic to a Bieberbach group with holonomy group  $F \simeq \mathbb{Z}_2^k$ .

In what follows we state the definitions and main results used to construct quaternion Kähler compact flat manifolds.

**Definition 2.4.** Let  $\Gamma$  be a Bieberbach group with holonomy group  $F \simeq \mathbb{Z}_2^k$ , with translation lattice  $\Lambda$  and such that  $b \in \frac{1}{2}\Lambda$  for any  $\gamma = BL_b \in \Gamma$ . Let  $\phi : F \rightarrow \mathbb{R}^n$  be a 1-cocycle modulo  $\Lambda$ . Set  $E_n = \begin{bmatrix} Id & \\ & -Id \end{bmatrix} \in I(\mathbb{R}^{2n})$ . Set  $d_{q,\phi}(\Gamma, v) = \langle d_\phi \Gamma, E_n L_{(v,0)} \rangle$ , where  $v \in \mathbb{R}^n$ .

As we shall see, under rather general conditions,  $d_{q,\phi}(\Gamma, v)$  contains  $d_\phi\Gamma$  as a normal subgroup of index 2, hence if  $v \in \mathbb{R}^n$  can be chosen so that  $d_{q,\phi}(\Gamma, v)$  is torsion free,  $M_{d_{q,\phi}(\Gamma, v)}$  will be a compact flat manifold with holonomy group  $F \times \mathbb{Z}_2$  having as a double cover the Kähler manifold  $M_{d_\phi\Gamma}$  (see 2.1). Furthermore  $F$  commutes with  $J$ , but  $E_n$  only anticommutes with  $J$ . If we use this construction twice we will get a Bieberbach group  $d_q^2(\Gamma, v, u) := d_{q,\phi'}(d_{q,\phi}(\Gamma, v), u) \subset I(\mathbb{R}^{4n})$  such that the holonomy group normalizes two anticommuting complex structures,  $J_1, J_2$ , on  $\mathbb{R}^{4n}$ , hence  $d_q^2(\Gamma, v, u) \backslash \mathbb{R}^{4n}$  will be a quaternion Kähler manifold. Thus, our main goal will be to give conditions on  $v \in \mathbb{R}^n$  that ensure that  $d_{q,\phi}(\Gamma, v)$  is torsion free. We also note that if  $n$  is even,  $M_{d_{q,\phi}(\Gamma, v)}$  will always be orientable. We will show that this can be done for a family  $\mathcal{F}$  of Bieberbach groups with holonomy group  $\mathbb{Z}_2^k$  (for a description of  $\mathcal{F}$ , which is technical, see [4]).

**Theorem 2.5.** *Let  $\Gamma, \phi$  be as in 2.4. Then*

- (i) *If  $v \in \mathbb{R}^n$  is such that  $2v \in \Lambda$  and satisfies*

$$(B - \text{Id})v \in \Lambda \text{ for each } \gamma = BL_b \in \Gamma,$$

*then  $d_{q,\phi}\Gamma$  is a crystallographic group with translation lattice  $\Lambda \oplus \Lambda$  and holonomy group  $\mathbb{Z}_2^{k+1}$ . Furthermore,  $d_{q,\phi}\Gamma$  is torsion-free if and only if  $v \notin \Lambda$  and for each  $\gamma = BL_b \in \Gamma$  we have:*

$$(B + \text{Id})(\phi(B) + v) \in \Lambda \setminus (B + \text{Id})\Lambda, \text{ or } (B - \text{Id})b \notin (B - \text{Id})\Lambda.$$

- (ii) *If every element in the holonomy group  $F$  commutes or anticommutes with a translation invariant complex structure and  $v$  satisfies the conditions in (i), then  $d_{q,\phi}(\Gamma, v) \backslash \mathbb{R}^{2n}$  is quaternion Kähler.*
- (iii) *If  $v$  satisfies the conditions in (i) we have that  $\beta_1(d_{q,\phi}(\Gamma, v) \backslash \mathbb{R}^{2n}) = \beta_1(\Gamma \backslash \mathbb{R}^n)$  and  $\beta_2(d_{q,\phi}(\Gamma, v) \backslash \mathbb{R}^{2n}) = 2\beta_2(d_{q,\phi}(\Gamma, v) \backslash \mathbb{R}^{2n})$ . Hence, if  $\beta_1(\Gamma \backslash \mathbb{R}^n)$  is odd, or if  $\beta_2(\Gamma \backslash \mathbb{R}^n) = 0$  and if  $F$  satisfies the condition in (ii), then  $d_{q,\phi}(\Gamma, v) \backslash \mathbb{R}^{2n}$  is quaternion Kähler and not Kähler.*
- (iv) *Assume  $\phi = 0$  and  $\Gamma \in \mathcal{F}$ . Then the vector  $v = \frac{1}{2} \sum_{i=1}^n e_i$  satisfies the conditions in (i), hence  $d_{q,0}(\Gamma, v)$  is a Bieberbach group. Furthermore,  $d_{q,0}(\Gamma, v) \in \mathcal{F}$ .*

**Corollary 2.6.** *In the notation of Theorem 2.5, assume  $v \in \mathbb{R}^n$  is such that  $d_{q,\phi}(\Gamma, v)$  is a Bieberbach group. Let  $\phi'$  be a cocycle on  $F$  modulo  $\Lambda \oplus \Lambda$ . If  $u \in \mathbb{R}^{2n}$  can be chosen so that  $d_{q,\phi,\phi'}^2(\Gamma, v, u) := d_{q,\phi'}(d_{q,\phi}(\Gamma, v), u)$  is torsion-free, then the quotient of  $\mathbb{R}^{4n}$  by  $d_{q,\phi,\phi'}^2(\Gamma, v, u)$  is a quaternion Kähler manifold. In particular, if  $\Gamma$  is a Bieberbach group in  $\mathcal{F}$  and we take  $\phi = 0$ ,  $v = \sum_{i=1}^n e_i$  and  $u = \sum_{i=2n+1}^{3n} e_i$ , then  $d_{q,0}(\Gamma, v) \in \mathcal{F}$  and  $d_{q,0,0}^2(\Gamma, v, u) \backslash \mathbb{R}^{4n}$  is a quaternion Kähler manifold.*

As it will be seen in the examples of the next section the vector  $v$  satisfying the conditions in the theorem is by no means unique, in general.



3. QUATERNION KÄHLER FLAT MANIFOLDS OF LOW DIMENSIONS

We will now illustrate the construction and results in the previous section by looking at particular Bieberbach groups in low dimensions. For more, and different examples we refer to [4]. In the examples below we will use  $\phi = 0$  and we will write  $d_{q,0}^2(\Gamma, v, u)$  in place of  $d_{q,0}(d_{q,0}(\Gamma, v), u)$ . Furthermore it will be convenient, for any  $C$  in  $O(n)$ , to denote by  $C' \in O(2n)$  the matrix  $C' = \begin{bmatrix} C & \\ & C \end{bmatrix}$ . Also,  $C'' \in O(4n)$  will have a similar meaning and  $\Lambda_n$  will denote the canonical lattice in  $\mathbb{R}^n$ .

**Examples** We let  $\Gamma$  be the Klein bottle Bieberbach group, for  $n = 2$ . By applying  $d_{q,0}$  twice to  $\Gamma$ , we shall obtain several 8-dimensional compact flat manifolds with holonomy group  $\mathbb{Z}_2^3$  which are quaternion Kähler and not Kähler. This will follow from the explicit computation of the real cohomology.

We take  $\Gamma = \langle BL_b, \Lambda_2 \rangle$ , where  $B = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ ,  $b = \frac{e_1}{2}$ . Then  $\Gamma \backslash \mathbb{R}^2$  is a Klein bottle. If  $v = \frac{1}{2}(m_1 e_1 + m_2 e_2)$ ,  $m_1, m_2 \in \mathbb{Z}$ , then

$$d_{q,0}(\Gamma, v) = \langle B'L_{b'}, E_2 L_{(v,0)}, \Lambda_4 \rangle,$$

with  $B' = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ ,  $E_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$  and  $b' = \frac{e_3}{2}$ .

We wish to find all  $m_1, m_2 \in \mathbb{Z}$  such that the conditions in (i) of Theorem 2.5 are satisfied, so that  $d_{q,0}(\Gamma, v)$  is torsion-free.

The first condition in (i) of 2.5 clearly holds for any choice of  $v$  since  $(B - Id)v = -m_2 e_2 \in \Lambda_2$ . Furthermore,  $v \in \frac{1}{2}\Lambda \setminus \Lambda$  if and only if at least one of the  $m_i$ 's is odd. We also need that  $(B + Id)v = m_1 e_1 \notin (B + Id)\Lambda_2 = \mathbb{Z}2e_1$ , hence  $m_1$  must be odd. Thus, the possible solutions, modulo  $\Lambda_2$  are  $v_1 = \frac{e_1}{2}$  and  $v_2 = \frac{e_1 + e_2}{2}$ . By computing the first integral homology groups in both cases, one can show that these solutions lead to flat manifolds non homeomorphic to each other.

We now form  $d_{q,0}^2(\Gamma, v_i, u)$  with  $i = 1, 2$  and  $u = \frac{1}{2} \sum_{j=1}^4 m_j e_j$ , with  $m_j \in \mathbb{Z}$  to be determined. Again we need that at least one of the  $m_j$ 's be odd. We now consider the second condition in (i) of 2.5 for each choice of  $v$ .

We have that

$$d_{q,0}^2 \left( \Gamma, \frac{e_1}{2}, u \right) = \langle B'' L_{\frac{e_7}{2}}, E_2' L_{\frac{e_8}{2}}, E_4 L_{(u,0)}, \Lambda_8 \rangle$$

$$d_{q,0}^2 \left( \Gamma, \frac{e_1 + e_2}{2}, u \right) = \langle B'' L_{\frac{e_7}{2}}, E_2' L_{\frac{e_8 + e_9}{2}}, E_4 L_{(u,0)}, \Lambda_8 \rangle$$

where

$$B'' = \begin{bmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{bmatrix} \quad E_2' = \begin{bmatrix} Id_2 & & & \\ & -Id_2 & & \\ & & Id_2 & \\ & & & -Id_2 \end{bmatrix} \quad E_4 = \begin{bmatrix} Id_4 & \\ & -Id_4 \end{bmatrix}.$$

The first condition in (i) of 2.5 is clearly satisfied in both cases, for any choice of  $u \in \frac{1}{2}\Lambda$ , since the matrices  $B', E_2$  are diagonal. For the second condition we also need:

$$\begin{aligned} (B' + Id)u &= m_1e_1 + m_3e_3 \notin (B' + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_3, \\ (E_2 + Id)u &= m_1e_1 + m_2e_2 \notin (E_2 + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_2, \\ (B'E_2 + Id)u &= m_1e_1 + m_4e_4 \notin (B'E_2 + Id)\Lambda_4 = \mathbb{Z}2e_1 \oplus \mathbb{Z}2e_4. \end{aligned}$$

These conditions are satisfied if and only if, either  $m_1$  is odd, or if each one of  $m_2, m_3$  and  $m_4$  are odd. This yields the following solutions modulo  $\Lambda_4$ : either  $u = u_Q = \frac{e_1 + e_Q}{2}$ , where  $e_Q = \sum_{j \in Q} e_j$  and  $Q$  runs through all subsets of  $\{e_2, e_3, e_4\}$ , or  $u = u' := \frac{e_2 + e_3 + e_4}{2}$ . We get 9 distinct solutions, the same set for both choices  $v = v_1, v = v_2$ . It will be convenient to order the subsets  $Q$  as follows:

$$\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}$$

and then to set  $u_j = u_Q$ , for  $j = 1, \dots, 8$  according to this ordering, letting  $u_9 = u'$ .

In this way we obtain 18 Bieberbach groups  $\Gamma_{i,j} := d_{g,0}^2(\Gamma, v_i, u_j)$  with  $1 \leq i \leq 2, 1 \leq j \leq 9$ , so that the quotients  $\Gamma_{i,j} \backslash \mathbb{R}^8$  are quaternion Kähler manifolds. We note that none of these manifolds is Kähler, since for all  $i, j$ ,  $\beta_1(\Gamma_{i,j} \backslash \mathbb{R}^8) = \beta_1(\Gamma \backslash \mathbb{R}^2) = 1$  and  $\beta_2(\Gamma_{i,j} \backslash \mathbb{R}^8) = 2\beta_2(\Gamma \backslash \mathbb{R}^2) = 0$ , by (iii) in 2.5. We also note that some of the groups may possibly be isomorphic to each other, however we see in [4] that many of them are pairwise non isomorphic, by computing  $\Gamma_{i,j}/[\Gamma_{i,j}, \Gamma_{i,j}]$  in each case.

We shall first determine all Betti numbers, by giving generators of  $\Lambda^h \mathbb{R}^{8F}$ , for  $1 \leq h \leq 8$ .

It is clear that the space of  $F$ -invariants in  $\mathbb{R}^8$  is spanned by  $e_1$  and furthermore  $\Lambda^2 \mathbb{R}^{8F} = 0$ . If  $h = 3$ , it is easy to see that a basis for the  $F$ -invariants is given by  $e_3 \wedge e_5 \wedge e_7, e_2 \wedge e_3 \wedge e_4, e_3 \wedge e_6 \wedge e_8, e_2 \wedge e_5 \wedge e_6, e_2 \wedge e_7 \wedge e_8, e_4 \wedge e_6 \wedge e_7, e_4 \wedge e_5 \wedge e_8$ , hence  $\beta_3 = \beta_5 = 7$ .

By Poincaré duality we have that  $\chi(\Gamma_{i,j} \backslash \mathbb{R}^8) = 2 - 2\beta_1 + 2\beta_2 - 2\beta_3 + \beta_4 = 0$ , hence (since  $\beta_1 = 1, \beta_2 = 0, \beta_3 = 7$ ) we get  $\beta_4 = 2\beta_3 = 14$ . We may check this value by finding a basis for the  $F$ -invariants in  $\Lambda^4 \mathbb{R}^8$ . This is given by vectors of the form  $e_i \wedge e_j \wedge e_k \wedge e_l$ , with  $\{i, j, k, l\}$  running through the sets

$$\begin{aligned} \{1, 3, 5, 7\}, \{2, 4, 6, 8\}, \{1, 2, 5, 6\}, \{3, 4, 7, 8\}, \{2, 3, 5, 8\}, \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \\ \{1, 3, 6, 8\}, \{1, 2, 7, 8\}, \{2, 4, 5, 7\}, \{1, 4, 6, 7\}, \{2, 3, 6, 7\}, \{2, 4, 5, 7\}, \{1, 4, 5, 8\} \end{aligned}$$

Summing up, we get that the Poincaré polynomial of each one of the flat manifolds  $\Gamma_{i,j} \backslash \mathbb{R}^8$  is  $p(t) = 1 + t + 7t^3 + 14t^4 + 7t^5 + t^7 + t^8$ .

We thus have 2-fold coverings  $M_{d_{g,0}^2\Gamma} \rightarrow M_{\Gamma_{i,j}}$ , where  $M_{d_{g,0}^2\Gamma}$  is hyperkähler, by Proposition 3.2, and  $M_{\Gamma_{i,j}}$  does not admit any Kähler structure, since  $\beta_1(M_{\Gamma_{i,j}}) = 1$ , for all  $i, j$ .

To conclude this example, one can show (see [4]) that many of the manifolds  $M_{\Gamma_{i,j}}$  are non homeomorphic to each other, by computing the first integral homology groups,  $H_1(M_{\Gamma_{i,j}}, \mathbb{Z}) \simeq \Gamma_{i,j}/[\Gamma_{i,j}, \Gamma_{i,j}]$ .

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HYPERHOLOMORPHIC FUNCTIONS IN  $\mathbb{R}^4$

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ABSTRACT. Let  $\mathbb{H}$  be the algebra of quaternions generated by  $e_1, e_2$  and  $e_{12}$  satisfying  $e_1e_2 = e_{12}$  and  $e_i e_j + e_j e_i = -2\delta_{ij}$  for  $i = 1, 2, 12$ . Any element  $x$  in  $\mathbb{H} \oplus \mathbb{H}$  may be decomposed as  $x = Px + Qxe_3$  for quaternions  $Px$  and  $Qx$ . The generalized Cauchy-Riemann operator in  $\mathbb{R}^4$  is defined by  $D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3$ . Leutwiler noticed that the power function  $(x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3)^m$  is the solution of the generalized Cauchy-Riemann system  $x_3 Df + 2f_3 = 0$  which has connections to the hyperbolic metric. We study solutions of the equation  $x_3 Df + 2Q'(f) = 0$  (the prime  $'$  is the main involution) called hyperholomorphic functions. If  $f = f_0 + f_1 e_1 + f_2 e_2 + f_3 e_3$  for some real functions  $f_0, f_1, f_2, f_3$  then  $f$  is the solution of the generalized Cauchy-Riemann system stated earlier.

1. INTRODUCTION

Let  $\mathbb{H}$  be the associative algebra of quaternions generated by  $e_1, e_2$  and  $e_{12}$  satisfying the usual relations

$$e_1^2 = e_2^2 = e_{12}^2 = -1$$

$$e_1 e_2 = e_{12} = -e_{12} e_1.$$

Set  $e_1 e_2 = e_{12}$ . The conjugation  $\bar{q}$  of the quaternion  $q = t + x e_1 + y e_2 + z e_{12}$  is defined by

$$\bar{q} = t - x e_1 - y e_2 - z e_{12}.$$

The involution  $' : \mathbb{H} \rightarrow \mathbb{H}$  is the isomorphism defined by

$$q' = t - x e_1 - y e_2 + z e_{12}.$$

The second involution  $* : \mathbb{H} \rightarrow \mathbb{H}$ , called **reversion**, is the anti-isomorphism defined by

$$q^* = t + x e_1 + y e_2 - z e_{12}.$$

We consider the set  $\mathbb{H} \oplus \mathbb{H}$  with the usual addition and the multiplication defined by

$$(1) \quad (u_1, u_2) (v_1, v_2) = (u_1 v_1 - u_2 v_2', u_1 v_2 + u_2 v_1').$$

It is known that

$$\mathbb{H} \oplus \mathbb{H} \cong Cl_{0,3},$$

where  $Cl_{0,3}$  is the Clifford algebra generated by the elements  $e_1, e_2$  and  $e_3$  satisfying the relation  $e_i e_j + e_j e_i = -2\delta_{ij}$  for the usual Kronecker delta  $\delta_{ij}$ . The isomorphism  $\varphi : \mathbb{H} \oplus \mathbb{H} \rightarrow Cl_{0,3}$  is given by  $\varphi(q_1, q_2) = q_1 + q_2 e_3$ . We identify the space  $\mathbb{H} \oplus \mathbb{H}$  with  $Cl_{0,3}$ .

The elements  $x = t + x e_1 + y e_2 + w e_3$  for  $t, x, y, w \in \mathbb{R}$  are called *paravectors* in  $\mathbb{H} \oplus \mathbb{H}$ . The space  $\mathbb{R}^4$  is identified with the set of paravectors. We also denote  $e_0 = 1$ .

The involution  $()'$  is extended to an isomorphism in  $\mathbb{H} \oplus \mathbb{H}$  by

$$(2) \quad (q_1 + q_2 e_3)' = q_1' - q_2' e_3 \quad (q_i \in \mathbb{H}).$$

Note that

$$(3) \quad e_3 q = q' e_3$$

for any  $q \in \mathbb{H}$ .

The involution  $*$  is extended to  $\mathbb{H} \oplus \mathbb{H}$  as follows

$$\begin{aligned} e_3^* &= e_3, (ab)^* = b^* a^* \\ (a + b)^* &= a^* + b^* \end{aligned}$$

and the conjugation by  $\bar{a} = (a^*)' = (a')^*$ . Note that  $\overline{ab} = \bar{b}\bar{a}$ .

Paravectors can be characterized as follows.

**Lemma 1.** *An element  $x \in \mathbb{H} \oplus \mathbb{H}$  is a paravector if and only if*

$$(4) \quad \sum_{i=0}^3 e_i x e_i = -2x'.$$

*Proof.* It is easy to see that the equation (4) holds for all  $e_i$  with  $i = 0, \dots, 3$ . Using the linearity we infer that it holds for all paravectors. Conversely, calculate first

$$\begin{aligned} \sum_{i=0}^3 e_i e_j e_k e_i &= 2e_j e_k \quad \text{for } 0 < j < k \leq 3 \\ \sum_{i=0}^3 e_i e_1 e_2 e_3 e_i &= -2e_1 e_2 e_3. \end{aligned}$$

Hence comparing the components of the left and right side of the equality (4) we note that the equality (4) implies that  $x$  has to be a paravector.  $\square$

The projection operators  $P : \mathbb{H} \oplus \mathbb{H} \rightarrow \mathbb{H}$  and  $Q : \mathbb{H} \oplus \mathbb{H} \rightarrow \mathbb{H}$  are defined by equations  $P(q_1 + q_2 e_3) = q_1$  and  $Q(q_1 + q_2 e_3) = q_2$  for  $q_i \in \mathbb{H}$ . Applying (2) we note that  $P(a') = (Pa)'$  and  $Q(a') = -(Qa)'$ . By virtue of (1) we obtain

$$(5) \quad P(ab) = PaPb - Qa(Qb)',$$

$$(6) \quad Q(ab) = PaQb + Qa(Pb)'.$$

The notation  $f \in \mathcal{C}^k(\Omega)$  for a function  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  means that the real coordinate functions of  $f$  are  $k$ -times continuously differentiable on  $\Omega \subset \mathbb{R}^4$ .

## 2. HYPERHOLOMORPHIC FUNCTIONS

The Cauchy Riemann operator is defined by

$$Df = \sum_{i=0}^3 e_i \frac{\partial f}{\partial x_i}$$

for a mapping  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$ ,  $\Omega \subset \mathbb{R}^4$ , whose components are partially differentiable. An operator  $\bar{D}$  is defined by

$$\bar{D}f = \frac{\partial f}{\partial x_0} - \sum_{j=1}^3 e_j \frac{\partial f}{\partial x_j}.$$

Note that  $D\bar{D} = \bar{D}D = \Delta$ , where  $\Delta$  is the Laplace operator in  $\mathbb{R}^4$ . If  $Df = 0$  the function  $f$  is called (left) *monogenic*. For the reference on properties of monogenic functions in the general case see [1] and for quaternions [21].

Using (5) we obtain

$$(7) \quad P(Df) = \sum_{i=0}^2 e_i \frac{\partial Pf}{\partial x_i} - \frac{\partial Q'f}{\partial x_3}.$$

Similarly, the property (6) implies that

$$(8) \quad Q(Df) = \sum_{i=0}^2 e_i \frac{\partial Qf}{\partial x_i} + \frac{\partial P'f}{\partial x_3}.$$

The **modified Cauchy-Riemann operator**  $M$  is defined by

$$(Mf)(x) = (Df)(x) + \frac{2}{x_3} Q'f$$

and the operator  $\bar{M}$  by

$$(\bar{M}f)(x) = (\bar{D}f)(x) - \frac{2}{x_3} Q'f.$$

Note that

$$(Mf)(x) + (\overline{Mf})(x) = (Df)(x) + (\overline{Df})(x) = 2 \frac{\partial f}{\partial x_0}$$

and

$$(9) \quad \overline{Df} = (D(f'))', \quad \overline{Mf} = (M(f'))'.$$

Let  $\Omega \subset \mathbb{R}^4$  be open. If  $f \in C^2(\Omega)$  and  $Mf(x) = 0$  for any  $x \in \Omega \setminus \{x \mid x_3 = 0\}$ , the function  $f$  is called **hyperholomorphic** in  $\Omega$ . If  $f$  is hyperholomorphic in  $\Omega$  and  $f = \sum_{i=0}^3 f_i e_i$  for some real functions  $f_i$  the function  $f$  is called an  **$H$ -solution**.

The  $H$ -solutions in  $\mathbb{R}^3$  were introduced by H. Leutwiler ([17]). They are notably studied in  $\mathbb{R}^n$  by H. Leutwiler ([18], [19], [20]), H. Hempfling ([13], [14], [15]), J. Cnops ([4]), P. Cerejeiras ([3]) and S.-L. Eriksson-Bique ([6], [11], [8], [9]). In  $\mathbb{R}^3$  the hyperholomorphic functions are researched by W. Hengartner and H. Leutwiler ([16]) and in  $\mathbb{R}^n$  by H. Leutwiler and S.-L. Eriksson-Bique ([11]).

Applying (7) and (8) for  $x_3 Mf = 0$  we obtain the following system.

**Proposition 2.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$  and  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  be a mapping with continuous partial derivatives. The equation  $Mf = 0$  is equivalent with the system of equations*

$$\begin{aligned} x_3 \left( D_2(Pf) - \frac{\partial(Qf)'}{\partial x_3} \right) + 2Q'f &= 0, \\ D_2(Qf) + \frac{\partial P'(f)}{\partial x_3} &= 0, \end{aligned}$$

where  $D_2 = \sum_{i=0}^2 e_i \frac{\partial f}{\partial x_i}$ .

Using the preceding result we infer the relation between monogenic and hyperholomorphic functions.

**Proposition 3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$ . Then if  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  is monogenic and  $Qf = 0$ , then  $f$  is hyperholomorphic.*

**Proposition 4.** *Let  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  be twice continuously differentiable on an open subset  $\Omega$  of  $\mathbb{R}^4$ . Then*

$$\begin{aligned} \overline{M}Mf = M\overline{M}f &= \Delta(Pf) - \frac{2}{x_3} \frac{\partial Pf}{\partial x_n} \\ &+ \left( \Delta(Qf) - \frac{2}{x_3} \frac{\partial Qf}{\partial x_3} + \frac{2}{x_3^2} Qf \right) e_3, \end{aligned}$$

where  $\Delta$  is the Laplacian in  $\mathbb{R}^4$ .

*Proof.* The property (8) implies that

$$QMf = QDf + \frac{2}{x_3} Q(Q'f) = D_2Qf + \frac{\partial P'f}{\partial x_3}.$$



Hence we have

$$\overline{M}Mf = \overline{D}Df + 2\overline{D}\left(\frac{Q'f}{x_3}\right) - \frac{2}{x_3}\left((D_2Qf)' + \frac{\partial Pf}{\partial x_3}\right).$$

Since by (9) and (3)

$$\begin{aligned} \overline{D}\left(\frac{Q'f}{x_3}\right) &= \frac{\overline{D}Q'f}{x_3} + \frac{e_3Q'f}{x_3^2} = \frac{(DQf)'}{x_3} + \\ &= \frac{(D_2Qf)'}{x_3} - \frac{1}{x_3}\frac{\partial Qf}{\partial x_3}e_3 + \frac{Qf}{x_3^2}e_3, \end{aligned}$$

we obtain

$$\overline{M}Mf = \Delta Pf - \frac{2}{x_3}\frac{\partial Pf}{\partial x_3} + \left(\Delta Qf - \frac{2}{x_3}\frac{\partial Qf}{\partial x_3} + 2\frac{Qf}{x_3^2}\right)e_3.$$

From the definitions we note that  $Mf + \overline{M}f = 2\frac{\partial f}{\partial x_0}$ . Hence we obtain

$$\begin{aligned} M\overline{M}f + M^2f &= 2M\left(\frac{\partial f}{\partial x_0}\right) = 2\frac{\partial Mf}{\partial x_0} \\ &= (M + \overline{M})(Mf) = \overline{M}Mf + M^2f, \end{aligned}$$

which implies  $M\overline{M}f = \overline{M}Mf$ , completing the proof. □

**Corollary 5.** *If  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  is hyperholomorphic then*

$$(10) \quad x_3\Delta Pf - 2\frac{\partial Pf}{\partial x_3} = 0$$

and

$$(11) \quad -2Qf = x_3^2\Delta(Qf) - 2x_3\frac{\partial Qf}{\partial x_3}.$$

*Conversely, if the equations (10) and (11) hold, then  $\overline{M}f$  is hyperholomorphic.*

The equation (10) is the Laplace-Beltrami equation associated with the hyperbolic metric

$$ds^2 = x_3^{-2}(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2).$$

The second equation (11) presents the eigenfunctions of the Laplace-Beltrami operator corresponding to the eigenvalue  $-2$ .

Using standard methods we obtain the following observation.

**Lemma 6.** *Let  $u : \Omega \rightarrow \mathbb{R}$  be twice continuously differentiable on an open subset  $\Omega$  of  $\mathbb{R}^4$ . If  $u$  satisfies the equation*

$$(12) \quad -2u(x) = x_3^2\Delta u(x) - 2x_3\frac{\partial u}{\partial x_3}(x), \quad x = (x_0, x_1, x_2, x_3)$$

then the mapping  $g : \Omega \rightarrow \mathbb{R}$  defined by

$$g(x) = \begin{cases} \frac{u(x)}{x_3}, & \text{if } x_3 \neq 0, \\ \frac{\partial u}{\partial x_3}(x), & \text{if } x_3 = 0, \end{cases}$$

is harmonic on  $\Omega \setminus \{x_3 = 0\}$ . Moreover, if  $u \in C^3(\Omega)$ , then  $g$  is harmonic on  $\Omega$ .

*Proof.* If  $x_3 \neq 0$ , then

$$\Delta g = \frac{\Delta u}{x_3} - \frac{2\partial u}{x_3^2 \partial x_3} + \frac{2u}{x_3^3}.$$

Using (12) we note that  $\Delta g = 0$ . Since by (12)  $\frac{\partial g}{\partial x_3} = \frac{1}{2} \Delta u$ , we see that  $g \in C^2(\Omega)$  and therefore  $\Delta g = 0$  on  $\Omega$  provided that  $u \in C^3(\Omega)$ .  $\square$

**Proposition 7.** *If  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  is hyperholomorphic and  $f \in C^3(\Omega)$ , then  $\Delta f$  is monogenic and therefore also harmonic.*

*Proof.* Assume that  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  is hyperholomorphic. Using preceding Lemma and  $\Delta = D\bar{D}$  we obtain

$$0 = D\bar{D}Mf = D\bar{D}Df + 2\Delta \left( \frac{Q'f}{x_3} \right) = D(\Delta f).$$

Hence  $\Delta f$  is monogenic and furthermore harmonic.  $\square$

**Example 8.** *Let  $f = u + iv$  be holomorphic in an open set  $\Omega \subset \mathbb{C}$ . We define the mapping  $\pi : \mathbb{R}^4 \rightarrow \mathbb{C}$  by*

$$\pi(x_0, x_1, x_2, x_3) = x_0 + x_1 i$$

and the mapping  $\rho : \mathbb{R}^4 \rightarrow \mathbb{C}$  by

$$\rho(x_0, x_1, x_2, x_3) = x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then the function  $f \circ \pi$  is hyperholomorphic in the set  $\{x \mid \pi(x) \in \Omega\}$ . Moreover the function  $\tilde{f}$  defined by

$$\tilde{f}(x) = u \circ \rho(x) + \frac{x_1 e_1 + x_2 e_2 + x_3 e_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} v \circ \rho(x)$$

is hyperholomorphic in the set  $\{x \mid \rho(x) \in \Omega\}$  (see [17, p.157]).

**Proposition 9.** *The space of hyperholomorphic functions in an open subset  $\Omega$  of  $\mathbb{R}^4$  forms a right quaternionic vector space.*

*Proof.* Let  $q \in \mathbb{H}$ . Using (6) we notice that

$$M(fq) = (Df)q + 2\frac{Q'(fq)}{x_3} = (Mf)q,$$

implying the assertion.  $\square$

The following result is easy to see.

**Lemma 10.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$  and  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  be hyperholomorphic. Then  $\frac{\partial f}{\partial x_i}$  is hyperholomorphic for  $i = 0, 1, 2$ .*

Product of hyperholomorphic functions is not necessarily hyperholomorphic.

**Theorem 11.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$  and  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  be hyperholomorphic. Then the product  $f(x)x$  is hyperholomorphic if and only if  $f$  is an  $H$ -solution.*

*Proof.* Assume that  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  and  $f(x)x$  are hyperholomorphic. Then by (6) we have

$$\begin{aligned} 0 &= M(fx) = (Df)x + \sum_{i=0}^3 e_i f e_i + 2 \frac{Q'(fx)}{x_3} \\ &= (Df)x + 2 \frac{Q'f(Px)}{x_3} + 2 \frac{P'(f)Q'x}{x_3} + \sum_{i=0}^3 e_i f e_i. \end{aligned}$$

Since  $Px = x - x_3 e_3$  and  $Qx = x_3$  we obtain

$$0 = (Mf)x + \sum_{i=0}^3 e_i f e_i - 2Q'f e_3 + 2P'(f) = \sum_{i=0}^3 e_i f e_i + 2f'.$$

By virtue of Lemma 1 we find out that  $f$  is para vector valued. The converse statement is proved similarly. □

**Corollary 12.** *The function  $x^m$  is an  $H$ -solution.*

**Theorem 13.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$  and  $F : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  be hyperholomorphic. Then the function  $f(x) = F(x)x^{-1}$  is hyperholomorphic in  $\Omega \setminus \{0\}$  if and only if it is paravector valued.*

*Proof.* Assume that  $F : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  and  $f(x) = F(x)x^{-1}$  are hyperholomorphic. Since  $F(x) = (F(x)x^{-1})x$  we obtain from the preceding theorem that  $(F(x)x^{-1})$  is para vector valued. On the other hand, assume that  $f$  is vector valued and  $F$  is hyperholomorphic. Then by (6) and Lemma 1, we conclude

$$0 = MF = (Mf)x + \sum_{j=0}^n e_j f e_j + 2f' = (Mf)x$$

and therefore  $f$  is also hyperholomorphic. □

**Corollary 14.** *The function  $x^{-m}$  is an  $H$ -solution.*

**Theorem 15.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^4$  and  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  twice continuously differentiable. Then  $f$  is hyperholomorphic if and only if for any  $a \in \Omega$  and any ball*

$B(a, r)$  with  $B(a, r) \subset \Omega$  there exists a continuous differentiable mapping  $H$  from  $B(a, r)$  into  $\mathbb{H}$  satisfying the equations

$$(13) \quad f = \overline{D}H$$

and

$$(14) \quad x_n \Delta H - 2 \frac{\partial H}{\partial x_n} = 0$$

on  $B(a, r)$ .

*Proof.* Assume that a mapping  $H : B(x, r) \rightarrow \mathbb{H}$  satisfies (14) and  $f = \overline{D}H$ . Since  $DD = \Delta$  we have  $x_3 Df = x_3 \Delta H$ . The equality  $-2 \frac{\partial H}{\partial x_n} = Q'f$  follows from

$$Qfe_3 = Q\overline{D}H = -e_3 \frac{\partial H}{\partial x_n}.$$

Hence  $Mf = 0$  and therefore  $f$  is hyperholomorphic.

Conversely assume that  $f : \Omega \rightarrow \mathbb{H} \oplus \mathbb{H}$  is hyperholomorphic. Set  $Pf = f_0 + f_1e_1 + f_2e_2 + f_{12}e_{12}$ . Let  $B(a, r)$  be a ball in  $\mathbb{R}^4$  centered at  $a = (a_0, \dots, a_n)$  satisfying  $B(a, r) \subset \Omega$ . Let  $s_i : (B(a, r) \cap \{x \mid x_3 = a_3\})^\sim \rightarrow \mathbb{R}$  be a twice continuously differentiable solution of the Poisson equation (which exists for example by [2, p.171])

$$\Delta s_i(\tilde{x}) = f_i(\tilde{x}, a_3).$$

for  $i = 0, 1, 2, 12$  and a ball  $B((a_0, a_1, a_2), r)$ . Set  $s = \sum_{i \in \{0, 1, 2, 12\}} s_i e_i$  and define a mapping  $H : B(a, r) \rightarrow \mathbb{H}$  by

$$H(x) = - \int_{a_3}^{x_3} Q'f(\tilde{x}, t) dt + D_2s.$$

Then we have by Proposition 2

$$\begin{aligned} \overline{D}H(x) &= e_3 2Q'f(x) - \int_{a_n}^{x_n} \overline{D}_2 Q'f(\tilde{x}, t) dt + \overline{D}_2 D_2 s \\ &= Q_2 f(x) e_n + \int_{a_n}^{x_n} \frac{\partial Pf}{\partial x_n}(\tilde{x}, t) dt + (Pf)(\tilde{x}, a_n) \\ &= f(x). \end{aligned}$$

Since the image space of  $H$  is  $\mathbb{H}$  we note that

$$Qfe_3 = -e_3 \frac{\partial H}{\partial x_3}.$$

Using the assumption  $f$  is hyperholomorphic we obtain

$$0 = x_3 Df + 2Q'f = x_3 D\overline{D}H - 2 \frac{\partial H}{\partial x_3}.$$

Hence the mapping  $H$  satisfies (14), completing the proof. □

Hyperholomorphic functions may be obtained from  $H$ -solutions as follows.

**Theorem 16.** *A mapping  $f$  is hyperholomorphic on a ball  $B(a, r) \subset \mathbb{R}^4$  if and only if there exist  $H$ -solutions  $g_i$  such that*

$$f = g_0 + g_1e_1 + g_2e_2 + g_{12}e_{12}.$$

*Proof.* Assume that  $f$  is hyperholomorphic on a ball  $B(a, r) \subset \mathbb{R}^4$ . Applying Theorem 15 we find a mapping  $H$  from  $B(a, r)$  into  $\mathbb{H}$  satisfying the equations (14) and  $f = \overline{D}H$ . Denote  $H = h_0 + h_1e_1 + h_2e_2 + h_{12}e_{12}$  for real functions  $h_i$ . Then the mapping  $g_i = \overline{D}h_i$  is vector valued and therefore an  $H$ -solution. Clearly we have  $f = g_0 + g_1e_1 + g_2e_2 + g_{12}e_{12}$ .  $\square$

**Corollary 17.** *If  $f$  is hyperholomorphic on  $\Omega$ , then it is real analytic on  $\Omega \setminus \{x_3 = 0\}$ . Moreover, if  $f \in C^3(\Omega)$  is hyperholomorphic on  $\Omega$ , then it is real analytic on  $\Omega$ .*

*Proof.* The  $H$ -solutions are real-analytic by [8, Theorem 4]. Hence the preceding theorem implies the statement.  $\square$

The fundamental homogeneous polynomial  $H$ -solutions are defined as follows.

**Definition 18.** *The homogeneous polynomials  $L_m^\alpha$  are defined for any multi-index  $\alpha \in \mathbb{N}_0^3$  and a non-negative integer  $m$  by*

$$L_m^\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} x^{m+|\alpha|}}{\partial x^\alpha}.$$

*The homogeneous polynomial  $H$ -solution  $T_m^\alpha$  of degree  $m$  is defined by*

$$\frac{\partial T_m^\alpha}{\partial x^\alpha} = \alpha! x_3^2 e_3$$

*and  $T_m^\alpha(x) = 0$  for any  $x$  with  $x_3 = 0$ .*

**Theorem 19.** *The set  $\{L_m^\alpha \mid |\alpha| \leq m\} \cup \{T_m^\alpha \mid |\alpha| = m - n + 1\}$  is a basis of the right  $\mathbb{H}$ -module of homogeneous hyperholomorphic polynomials of degree  $m$ .*

*Proof.* Using [8, Theorem 4] we obtain that the set in question is a basis of the right  $\mathbb{H}$ -module generated by the homogeneous polynomial  $H$ -solutions of degree  $m$ . If  $f$  is a homogenous hyperholomorphic polynomial of degree  $m$ , then by Theorem 16 there exists homogeneous polynomial  $H$ -solutions  $p_i$  of degree  $m$  satisfying  $f = \sum_{i \in \{0,1,2,12\}} p_i e_i$ . Hence it is also a basis of the right  $\mathbb{H}$ -module of homogeneous hyperholomorphic polynomials of degree  $m$ .  $\square$

**Theorem 20.** *Let  $f \in C^3(\Omega)$  be hyperholomorphic in a neighborhood of a point  $x = (x_0, x_1, x_2, 0)$ . Then there exist constants  $b_k(\alpha), c(\alpha) \in \mathbb{H}$  such that*

$$f = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=0}^k L_k^\alpha b_k(\alpha) + \sum_{|\alpha|=k-2} T_k^\alpha c_k(\alpha) \right)$$

*Proof.* Assume that  $f \in \mathcal{C}^3(\Omega)$  is hyperholomorphic in a neighborhood of a point  $x = (x_0, x_1, x_2, 0)$ . If  $T(y) = y + a$  for  $a \in \mathbb{R}^4$  with the last coordinate  $a_3 = 0$  then  $f \circ T$  is also hyperholomorphic. Hence we may assume that  $x = 0$ . Since  $f$  is hyperholomorphic,  $f$  is real analytic and therefore admits the presentation

$$f(y) = \sum_{\beta \in \mathbb{N}_0^{3+1}} a(\beta) y^\beta.$$

in some neighborhood  $B_r(0)$ . Applying  $M$  we obtain

$$0 = Mf(y) = \sum_{k=0}^{\infty} M \left( \sum_{|\beta|=k} a(\beta) y^\beta \right).$$

Since  $M \left( \sum_{|\beta|=k} a(\beta) y^\beta \right)$  is a homogeneous polynomial of degree  $k$  we infer

$$M \left( \sum_{|\beta|=k} a(\beta) y^\beta \right) = 0.$$

This implies that  $\sum_{|\beta|=k} a(\beta) y^\beta$  is hyperholomorphic. Applying Theorem 19 we obtain the result.  $\square$

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**A NOTE ON THE REDUCTION OF SASAKIAN MANIFOLDS**

GUEO GRANTCHAROV AND LIVIU ORNEA

**ABSTRACT.** This is a report on work in process. We show that the contact reduction can be specialized to Sasakian manifolds. We link this Sasakian reduction to Kähler reduction by considering the Kähler cone over a Sasakian manifold. Finally, we present an example of Sasakian manifold obtained by  $SU(2)$  reduction of a standard Sasakian sphere.

1. INTRODUCTION

Reduction technique was naturally extended from symplectic to contact structures by H. Geiges in [6]. Even earlier, Ch. Boyer, K. Galicki and B. Mann defined in [3] a moment map for 3-Sasakian manifolds, thus extending the reduction procedure for nested metric contact structures. Quite surprisingly, a reduction scheme for Sasakian manifolds (contact manifolds endowed with a compatible Riemannian metric satisfying a curvature condition), was still missing.

In this note - presenting work in progress - we fill the gap by defining a Sasakian moment map and constructing the associated reduced space. We then relate Sasakian reduction to Kähler reduction *via* the Kähler cone over a Sasakian manifold.

In a forthcoming paper we shall discuss the compatibility between the Einstein property and the reduction scheme.

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2. DEFINITIONS OF SASAKIAN MANIFOLDS

We recall here the notion of a Sasakian manifold, referring to [2] and [4] for details and examples.

**Definition 2.1.** *A Sasakian manifold is a  $(2n+1)$ -dimensional Riemannian manifold  $(N, g)$  endowed with a unitary Killing vector field  $\xi$  such that the curvature tensor of  $g$  satisfies the equation:*

$$(1) \quad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi$$

where  $\eta$  is the metric dual 1-form of  $\xi$ :  $\eta(X) = g(\xi, X)$ .

Let  $\varphi = \nabla\xi$ , where  $\nabla$  is the Levi-Civita connection of  $g$ . The following formulae are then easily deduced:

$$(2) \quad \varphi\xi = 0, \quad g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

It can be seen that  $\eta$  is a contact form on  $N$ , whose Reeb field is  $\xi$  (it is also called the characteristic vector field). Moreover, the restriction of  $\varphi$  to the contact distribution  $\eta = 0$  is a complex structure.

The simplest example is the standard sphere  $S^{2n+1} \subset \mathbb{C}^{n+1}$ , with the metric induced by the flat one of  $\mathbb{C}^{n+1}$ . The characteristic Killing vector field is  $\xi_p = -i\vec{p}$ ,  $i$  being the imaginary unit. Other Sasakian structures on the sphere can be obtained by  $D$ -homothetic transformations (cf. [7]). Also, the unit sphere bundle of any space form is Sasakian. A large class of examples is obtained *via* the converse construction of the Boothby-Wang fibration. Moreover, the join of two Sasakian Einstein manifolds is Sasakian Einstein.

The following equivalent definition puts Sasakian geometry in the framework of holonomy groups. Let  $C(N) = N \times \mathbb{R}_+$  be the cone over  $(N, g)$ . Endow it with the warped-product cone metric  $C(g) = r^2g + dr^2$ . Let  $R_0 = r\partial r$  and define on  $C(N)$  the complex structure  $J$  acting like this (with obvious identifications):  $JY = \varphi Y - \eta(Y)R_0$ ,  $JR_0 = \xi$ . We have:

**Theorem 2.1.** [4]  *$(N, g, \xi)$  is Sasakian if and only if the cone over  $N$   $(C(N), C(g), J)$  is Kählerian.*

3. THE RESULTS

**Theorem 3.1.** *Let  $(N, g, \xi)$  be a compact  $2n + 1$  dimensional Sasakian manifold and  $G$  a compact  $d$ -dimensional Lie group acting on  $N$  by contact isometries. Suppose  $0 \in \mathfrak{g}^*$  is a regular value of the associated moment map  $\mu$ . Then the reduced space  $M = N//G := \mu^{-1}(0)/G$  is a Sasakian manifold of dimension  $2(n - d) + 1$ .*

*Proof.* (A sketch.) By [6], the contact moment map  $\mu : N \rightarrow \mathfrak{g}^*$  is defined by

$$\langle \mu(x), \underline{X} \rangle = \eta(X)$$

for any  $\underline{X} \in \mathfrak{g}$  and  $X$  the corresponding field on  $N$ . We know that the reduced space is a contact manifold, *loc. cit.* Hence we only need to check that (1) the Riemannian metric is projected on  $M$  and (2) the field  $\xi$  projects to a unitary Killing field on  $M$  such that the curvature tensor of the projected metric satisfies formula (1).

To this end, we first describe the metric geometry of the Riemannian submanifold  $\mu^{-1}(0)$ . One proves that the restriction of the Reeb field to  $\mu^{-1}(0)$  is Killing with respect to the induced metric. Moreover, using the Gauss equation we obtain

$$\begin{aligned}
 (3) \quad & g(R^{\mu^{-1}(0)}(X, \xi)Y, Z) - g(R^N(X, \xi)Y, Z) = \\
 & = - \sum_{i=1}^d \|X_i\|^{-2} \{h_i(X, Y)h_i(\xi, Z) - h_i(X, Z)h_i(\xi, Y)\} \\
 & = - \sum_{i=1}^d \|X_i\|^{-2} \{g(X_i, Z)g(\nabla_X^N X_i, \varphi Y) - g(X_i, Y)g(\nabla_X^N X_i, \varphi Z)\}
 \end{aligned}$$

where  $\{\underline{X}_1, \dots, \underline{X}_d\}$  is a basis of  $\mathfrak{g}$  and let  $\{X_1, \dots, X_d\}$  is the corresponding vector fields on  $N$ . (Note that  $\nu_i = \|X_i\|^{-1}\varphi X_{i_p}$  are chosen to be orthonormal in  $p$ ; this is always possible pointwise by appropriate choice of the initial  $\underline{X}_i$ ).

Let now  $\pi : \mu^{-1}(0) \rightarrow M$  and endow  $M$  with the projection  $g^M$  of the metric  $g$  such that  $\pi$  becomes a Riemannian submersion. This is possible because  $G$  acts by isometries. In this setting, the vector fields  $X_i$  span the vertical distribution of the submersion, whilst  $\xi$  is horizontal and projectable (because  $\mathcal{L}_{X_i}\xi = 0$ ). Denote with  $\zeta$  its projection on  $M$ .  $\zeta$  is obviously unitary. To prove that  $\zeta$  is Killing on  $M$ , we just observe that  $\mathcal{L}_\zeta g(Y, Z) = \mathcal{L}_\xi g(Y^h, Z^h)$ , where  $Y^h$  denotes the horizontal lift of  $Y$ . Finally, to compute the values  $R^M(X, \zeta)Y$  of the curvature tensor of  $g^M$ , we use O'Neill formula (cf. [1], (9.28f)) and find

$$R^M(X, \zeta)Y = g(\xi, Y^h)X^h - g(X^h, y^h)\xi = g^M(\zeta, Y)X - g^M(X, Y)\zeta$$

which proves that  $(M, g^M, \zeta)$  is a Sasakian manifold. □

**Remark 3.1.**  $\mu^{-1}(0)$  is a natural example of a contact CR-submanifold (in the terminology of K. Yano and M. Kon [9], a semi-invariant submanifold in the terminology of A. Bejancu). In general, this means that the tangent space of the submanifold decomposes in three mutually orthogonal distributions:  $\mathbb{R}\xi$ , a distribution  $\mathcal{D}$  on which  $\varphi$  restricts to an endomorphism and a distribution  $\mathcal{D}^\perp$  which is mapped by  $\varphi$  in the normal space of the submanifold. It is known that on a contact CR-submanifold the distribution  $\mathcal{D}^\perp$  is always integrable. Here the integrability of this distribution expresses the fact that it is generated by fundamental vector fields corresponding to a basis of the Lie algebra of the group defining the moment map. In general, the invariant distribution is not integrable. In our case, one can show that its integrability is equivalent with strong restrictions on the geometry of the quotient.

In the following we relate Sasakian reduction to Kähler reduction by using the cone construction. Roughly speaking, we prove that reduction and taking the cone are commuting operations.

Let  $\omega = dr^2 \wedge \eta + r^2 d\eta$  be the Kähler form of the cone  $C(N)$  over a Sasakian manifold  $(N, g, \xi)$ . If  $\rho_t$  are the translations acting on  $C(N)$  by  $(x, r) \mapsto (x, tr)$ , then the vector field  $R_0 = r\partial r$  is the one generated by  $\{\rho_t\}$ . Moreover, the following two relations are useful:

$$(4) \quad \mathcal{L}_{R_0}\omega = \omega, \quad \rho_t^*\omega = t\omega.$$

If a compact Lie group  $G$  acts on  $C(N)$  by holomorphic isometries, commuting with  $\rho_t$ , we obtain a corresponding action of  $G$  on  $N$ . In fact, we can consider  $G \cong G \times \{Id\}$  acting as  $(g, (x, r)) \times (gx, r)$ .

Suppose that a moment map  $\Phi : C(N) \rightarrow \mathfrak{g}$  exists.

As above notations, let  $\{\underline{X}_1, \dots, \underline{X}_d\}$  be a basis of  $\mathfrak{g}$  and let  $\{X_1, \dots, X_d\}$  be the corresponding vector fields on  $C(N)$ . We see that  $X_i$  are independent on  $r$ , hence can be considered as vector fields on  $N$ . Furthermore, the commutation of  $G$  with  $\rho_t$  implies

$$(5) \quad \Phi(\rho_t(p)) = t\Phi(p).$$

Now embed  $N$  in the cone as  $N \times \{1\}$  and let  $\mu := \Phi|_{N \times \{1\}}$ . We can prove that this is precisely the moment map of the action of  $G$  on  $N$ .

Let  $P = \Phi^{-1}(0)/G$  be the reduced Kähler manifold. The key remark is that because of (5),  $\Phi^{-1}(0)$  is the cone  $N' \times \mathbb{R}_+$  over  $N' = \{x \in N ; (x, 1) \in \Phi^{-1}(0)\}$ . Moreover, since the actions of  $G$  and  $\rho_t$  commute, one has an induced action of  $G$  on  $N'$ . Then

$$\Phi^{-1}(0)/G \cong (N' \times \mathbb{R}_+)/G \cong N'/G \times \mathbb{R}_+$$

The manifold  $N'//G \times \mathbb{R}_+$  is Kähler, as reduction of a Kähler manifold, but we still have to check that this Kähler structure is a cone one. For the more general, symplectic case, this was done in [5]. Let  $g_0$  be the reduced Kähler metric and  $g'$  be the Sasakian reduced metric on  $N'//G$ . It is easily seen that the lift of  $g_0$  to  $\Phi^{-1}(0)$  coincides with the lift of the cone metric  $r^2 g' + dr^2$  on horizontal fields. This implies that the cone metric coincides with  $g_0$ .

Summing up we have proved:

**Theorem 3.2.** *Let  $(N, g, \xi)$  be a Sasakian manifold and let  $(C(N), C(g), J)$  be the Kähler cone over it. Let a compact Lie group  $G$  act by holomorphic isometries on  $C(N)$  and commuting with the action of the 1-parameter group generated by the field  $R_0$ . If a moment map with regular value 0 exists for this action, then a moment map with regular value 0 exists also for the induced action of  $G$  on  $N$ . Moreover, the reduced space  $C(N)//G$  is the Kähler cone over the reduced Sasakian manifold  $N//G$ .*

The advantage of defining the Sasakian reduction *via* Kähler reduction, as done in [3] for 3-Sasakian manifolds, is the avoiding of curvature computations.

4. EXAMPLES:  $SU(2)$  ACTIONS ON SASAKIAN SPHERES

**Example 4.1.** In a future note we are planning to consider with details  $S^1$  actions on Sasakian manifolds but now we concentrate to the actions of  $SU(2)$  with homogeneous reduced spaces. Consider the standard Sasakian structure on  $S^{4n-1} \subset \mathbb{C}^{2n}$  given by the "round metric" and vector field  $\zeta$  generated by the left action of  $S^1 = e^{it}$ . Then the right action of the unit quaternions on  $S^{4n-1} \subset \mathbb{H}^n$  by:

$$(q, (q_0, \dots, q_{n-1})) \mapsto (q_0q, \dots, q_{n-1}q).$$

satisfies the conditions of Theorem 3.1. The associated moment map is the same as the 3-Sasakian moment map of the  $S^1$  action given in [3]:

$$\mu(q) = \sum q_a i \bar{q}_a$$

The reason is that in both cases the coordinates  $(\mu_1, \mu_2, \mu_3)$  of  $\mu$  are given by a scalar product of the vector fields generated by the left actions of  $i, j$  and  $k$  with  $\zeta$ . So using the result from [3] we have:

$$\mu^{-1}(0) \cong SU(n+1)/SU(n-1)$$

. The reduced space is diffeomorphic to the homogeneous space  $SU(n+1)/SU(2) \times SU(n-1)$  which is a  $S^1$  bundle over  $SU(n+1)/S(U(2) \times U(n-1))$ , a Hermitian symmetric space. Note also that the latter space is a quaternionic Kähler manifold and is the base for the 3-Sasakian fibration with  $S^3$  fiber, obtained as a reduced space after the 3-Sasakian reduction mentioned above. One can also check that the reduced metric is the homogeneous Einstein metric arising from the Wang and Ziller's construction, [8].

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A THEORY OF QUATERNIONIC ALGEBRA,  
WITH APPLICATIONS TO HYPERCOMPLEX GEOMETRY

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1. INTRODUCTION

*Mathematicians are like Frenchmen: whenever you say something to them, they translate it into their own language, and at once it is something entirely different.*

Goethe, Maxims and Reflections (1829)

The subject of this paper is an algebraic device, a means to construct algebraic structures over the quaternions  $\mathbb{H}$  as though  $\mathbb{H}$  were a commutative field. As far as the author can tell, the idea seems to be new. We shall provide the reader with a dictionary, giving the equivalents of simple concepts such as commutative field, vector space, tensor products of vector spaces, symmetric and antisymmetric products, dimensions of vector spaces, and so on. The reader will then be able to translate her own favourite algebraic objects into this new quaternionic language. The results often turn out to be surprisingly, entirely different.

The basic building blocks of the theory are the quaternionic analogues of vector space and the tensor product of vector spaces over a commutative field. A vector space is replaced by an *A $\mathbb{H}$ -module*  $(U, U')$ , which is a left  $\mathbb{H}$ -module  $U$  with a given real vector subspace  $U'$ . The tensor product  $\otimes$  is replaced by the *quaternionic tensor product*  $\otimes_{\mathbb{H}}$ , which has a complex definition given in §1.1. It shares some important properties of  $\otimes$  (e.g. it is commutative and associative), but also has important differences (e.g. the dimension of a quaternionic tensor product behaves strangely).

The theory arose out of my attempts to understand the algebraic structure of noncompact hypercomplex manifolds. Let  $M$  be a  $4n$ -manifold. A *hypercomplex structure* on  $M$  is a triple  $(I_1, I_2, I_3)$  of complex structures satisfying  $I_1 I_2 = I_3$ . These induce an  $\mathbb{H}$ -action on the tangent bundle  $TM$ , and  $M$  is called a hypercomplex manifold. Hypercomplex manifolds are the subject of the second half of the paper.

The best quaternionic analogue of a holomorphic function on a complex manifold is a *q-holomorphic function* on a hypercomplex manifold  $M$ , defined in §3.1. This is an  $\mathbb{H}$ -valued function on  $M$  satisfying an equation analogous to the Cauchy-Riemann equations, which was introduced by Fueter in 1935 for the case  $M = \mathbb{H}$ . Affine

algebraic geometry is the study of complex manifolds using algebras of holomorphic functions upon them. Seeking to generalize this to hypercomplex manifolds using  $q$ -holomorphic functions, I was led to this quaternionic theory of algebra as the best language to describe quaternionic algebraic geometry.

Although the applications are all to hypercomplex manifolds, I hope that much of the paper will be of interest to those who study algebra rather than geometry, and maybe to others who, like myself, are fascinated by the quaternions. The paper has been laid out with this in mind. There are four chapters. Chapter 1 explains  $A\mathbb{H}$ -modules and the quaternionic tensor product. It is quite long and wholly algebraic, without references to geometry. Chapter 2 gives quaternionic analogues of various algebraic structures. It is short and is mostly definitions. The most important idea is that of an  $H$ -algebra, the quaternionic version of a commutative algebra.

Chapter 3 is about hypercomplex geometry.  $Q$ -holomorphic functions are defined, their properties explored, and it is shown that the  $q$ -holomorphic functions on a hypercomplex manifold form an  $H$ -algebra. A similar result is proved for hyperkähler manifolds. The problem of reconstructing a hypercomplex manifold from its  $H$ -algebra is considered, and  $H$ -algebras are used to study a special class of noncompact, complete hyperkähler manifolds, called asymptotically conical manifolds.

Chapter 4 is a collection of examples and applications of the theory. Two interesting topics covered here are an algebraic treatment of the ‘coadjoint orbit’ hyperkähler manifolds, and some new types of singularities of hypercomplex manifolds that have remarkable properties. To control the length of the paper I have kept the list of examples and applications short, and I have omitted a number of proofs. However, I believe that there is much interesting work still to be done on these ideas, and in §4.7 I shall indicate some directions in which I would like to see the subject develop.

Much of the material in this paper has already been published by the author in [14]; this paper is mostly an expanded version of [14], with rather more detail, and some new material on the geometry of noncompact hypercomplex and hyperkähler manifolds. However, the results of [14, §12] on coadjoint orbits are not included in this paper, and instead we approach the subject from a slightly different direction.

Two other references on the subject of this paper are Quillen [25], who reinterprets the  $A\mathbb{H}$ -modules and the quaternionic tensor product  $\otimes_{\mathbb{H}}$  in terms of sheaves on  $\mathbb{C}P^1$ , and Widdows [28], who explores a number of issues in hypercomplex algebraic geometry, including the classification of finite-dimensional  $A\mathbb{H}$ -modules up to isomorphism, and quaternionic analogues of the Dolbeault double complex on complex manifolds.

**1.1. Quaternionic tensor products.** In this section we give some notation, and define a quaternionic analogue of the tensor product of two vector spaces. This idea is central to the whole paper. First, a remark about the real tensor product. Let  $U, V$  be real vector spaces. If  $U, V$  are infinite-dimensional, there is more than one possible definition for the real tensor product  $U \otimes V$ , if  $U, V$  are equipped with topologies. In this paper we choose the simplest definition: for us, every element of  $U \otimes V$  is a *finite*



sum  $\sum_j u_j \otimes v_j$  where  $u_j \in U, v_j \in V$ . Also, in this paper the dual  $U^*$  of  $U$  means the vector space of *all* linear maps  $U \rightarrow \mathbb{R}$ , whether continuous in some topology or not.

The quaternions are

$$\mathbb{H} = \{r_0 + r_1i_1 + r_2i_2 + r_3i_3 : r_0, \dots, r_3 \in \mathbb{R}\},$$

and quaternion multiplication is given by

$$i_1i_2 = -i_2i_1 = i_3, \quad i_2i_3 = -i_3i_2 = i_1, \quad i_3i_1 = -i_1i_3 = i_2, \quad i_1^2 = i_2^2 = i_3^2 = -1.$$

The quaternions are an associative, noncommutative algebra. The imaginary quaternions are  $\mathbb{I} = \langle i_1, i_2, i_3 \rangle$ . This notation  $\mathbb{I}$  is not standard, but we will use it throughout the paper. If  $q = r_0 + r_1i_1 + r_2i_2 + r_3i_3$  then we define the *conjugate*  $\bar{q}$  of  $q$  by  $\bar{q} = r_0 - r_1i_1 - r_2i_2 - r_3i_3$ . Then  $(pq) = \bar{q}\bar{p}$  for  $p, q \in \mathbb{H}$ .

A (*left*)  $\mathbb{H}$ -*module* is a real vector space  $U$  with an action of  $\mathbb{H}$  on the left. We write this action  $(q, u) \mapsto q \cdot u$  or  $qu$ , for  $q \in \mathbb{H}$  and  $u \in U$ . The action is a bilinear map  $\mathbb{H} \times U \rightarrow U$ , and satisfies  $p \cdot (q \cdot u) = (pq) \cdot u$  for  $p, q \in \mathbb{H}$  and  $u \in U$ . In this paper, all  $\mathbb{H}$ -modules will be left  $\mathbb{H}$ -modules.

Let  $U$  be an  $\mathbb{H}$ -module. We define the *dual  $\mathbb{H}$ -module*  $U^\times$  to be the vector space of linear maps  $\alpha : U \rightarrow \mathbb{H}$  that satisfy  $\alpha(qu) = q\alpha(u)$  for all  $q \in \mathbb{H}$  and  $u \in U$ . If  $q \in \mathbb{H}$  and  $\alpha \in U^\times$  we may define  $q \cdot \alpha$  by  $(q \cdot \alpha)(u) = \alpha(u)\bar{q}$  for  $u \in U$ . Then  $q \cdot \alpha \in U^\times$ , and  $U^\times$  is a (*left*)  $\mathbb{H}$ -module. If  $V$  is a real vector space, we write  $V^*$  for the dual of  $V$  as a real vector space. It is important to distinguish between the dual vector spaces and dual  $\mathbb{H}$ -modules. Dual  $\mathbb{H}$ -modules behave just like dual vector spaces. In particular, there is a canonical map  $U \rightarrow (U^\times)^\times$ , that is an isomorphism when  $U$  is finite-dimensional.

**Definition 1.1.1** Let  $U$  be an  $\mathbb{H}$ -module. Let  $U'$  be a real vector subspace of  $U$ , that need not be closed under the  $\mathbb{H}$ -action. Define a real vector subspace  $U^\dagger$  of  $U^\times$  by

$$U^\dagger = \{\alpha \in U^\times : \alpha(u) \in \mathbb{I} \text{ for all } u \in U'\}.$$

We define an *augmented  $\mathbb{H}$ -module*, or *A $\mathbb{H}$ -module*, to be a pair  $(U, U')$ , such that if  $u \in U$  and  $\alpha(u) = 0$  for all  $\alpha \in U^\dagger$ , then  $u = 0$ . Usually we will refer to  $U$  as an A $\mathbb{H}$ -module, implicitly assuming that  $U'$  is also given. We consider  $\mathbb{H}$  to be an A $\mathbb{H}$ -module, with  $\mathbb{H}' = \mathbb{I}$ .

A $\mathbb{H}$ -modules should be thought of as the quaternionic analogues of real vector spaces. It is easy to define the dual of an A $\mathbb{H}$ -module, but we are not going to do this, as it seems not to be a fruitful idea. We can interpret  $U^\times$  as the dual of  $U$  as a real vector space, and then  $U^\dagger$  is the annihilator of  $U'$ . Thus if  $U$  is finite-dimensional,  $\dim U' + \dim U^\dagger = \dim U = \dim U^\times$ . The letters  $A, U, V, \dots, Z$  will usually denote A $\mathbb{H}$ -modules. Here are the natural concepts of linear map between A $\mathbb{H}$ -modules, and A $\mathbb{H}$ -submodules.

**Definition 1.1.2** Let  $U, V$  be A $\mathbb{H}$ -modules. Let  $\phi : U \rightarrow V$  be a linear map satisfying  $\phi(qu) = q\phi(u)$  for each  $q \in \mathbb{H}$  and  $u \in U$ . Such a map is called *quaternion linear*, or  $\mathbb{H}$ -linear. We say that  $\phi$  is a *morphism of A $\mathbb{H}$ -modules*, or *A $\mathbb{H}$ -morphism*, if  $\phi : U \rightarrow V$  is  $\mathbb{H}$ -linear and satisfies  $\phi(U') \subset V'$ . Define a linear map  $\phi^\times : V^\times \rightarrow U^\times$  by  $\phi^\times(\beta)(u) = \beta(\phi(u))$  for  $\beta \in V^\times$  and  $u \in U$ . Then  $\phi(U') \subset V'$  implies that  $\phi^\times(V^\dagger) \subset U^\dagger$ . If  $\phi$  is an isomorphism of  $\mathbb{H}$ -modules and  $\phi(U') = V'$ , we say that  $\phi$  is an *A $\mathbb{H}$ -isomorphism*.

Clearly, if  $\phi : U \rightarrow V$  and  $\psi : V \rightarrow W$  are A $\mathbb{H}$ -morphisms, then  $\psi \circ \phi : U \rightarrow W$  is an A $\mathbb{H}$ -morphism. If  $V$  is an A $\mathbb{H}$ -module, we say that  $U$  is an *A $\mathbb{H}$ -submodule of  $V$*  if  $U$  is an  $\mathbb{H}$ -submodule of  $V$  and  $U' = U \cap V'$ . This implies that  $U^\dagger$  is the restriction of  $V^\dagger$  to  $U$ , so the condition that  $u = 0$  if  $\alpha(u) = 0$  for all  $\alpha \in U^\dagger$  holds automatically, and  $U$  is an A $\mathbb{H}$ -module.

In this paper ‘Im’ always means the image of a map, and we will write ‘id’ for the identity map on any vector space. If  $U$  is an A $\mathbb{H}$ -module, then  $\text{id} : U \rightarrow U$  is an A $\mathbb{H}$ -morphism. Next we will define a sort of tensor product of A $\mathbb{H}$ -modules, which is the key to the whole paper.

**Definition 1.1.3** Let  $U$  be an A $\mathbb{H}$ -module. Then  $\mathbb{H} \otimes (U^\dagger)^*$  is an  $\mathbb{H}$ -module, with  $\mathbb{H}$ -action  $p \cdot (q \otimes x) = (pq) \otimes x$ . Define a map  $\iota_U : U \rightarrow \mathbb{H} \otimes (U^\dagger)^*$  by  $\iota_U(u) \cdot \alpha = \alpha(u)$ , for  $u \in U$  and  $\alpha \in U^\dagger$ . Then  $\iota_U(q \cdot u) = q \cdot \iota_U(u)$  for  $u \in U$ . Thus  $\iota_U(U)$  is an  $\mathbb{H}$ -submodule of  $\mathbb{H} \otimes (U^\dagger)^*$ . Suppose  $u \in \text{Ker } \iota_U$ . Then  $\alpha(u) = 0$  for all  $\alpha \in U^\dagger$ , so that  $u = 0$  as  $U$  is an A $\mathbb{H}$ -module. Thus  $\iota_U$  is injective, and  $\iota_U(U) \cong U$ .

**Definition 1.1.4** Let  $U, V$  be A $\mathbb{H}$ -modules. Then  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$  is an  $\mathbb{H}$ -module, with  $\mathbb{H}$ -action  $p \cdot (q \otimes x \otimes y) = (pq) \otimes x \otimes y$ . Exchanging the factors of  $\mathbb{H}$  and  $(U^\dagger)^*$ , we may regard  $(U^\dagger)^* \otimes \iota_V(V)$  as a subspace of  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ . Thus  $\iota_U(U) \otimes (V^\dagger)^*$  and  $(U^\dagger)^* \otimes \iota_V(V)$  are  $\mathbb{H}$ -submodules of  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ . Define an  $\mathbb{H}$ -module  $U \otimes_{\mathbb{H}} V$  by

$$(1) \quad U \otimes_{\mathbb{H}} V = (\iota_U(U) \otimes (V^\dagger)^*) \cap ((U^\dagger)^* \otimes \iota_V(V)) \subset \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*.$$

Define a vector subspace  $(U \otimes_{\mathbb{H}} V)'$  by  $(U \otimes_{\mathbb{H}} V)' = (U \otimes_{\mathbb{H}} V) \cap (\mathbb{I} \otimes (U^\dagger)^* \otimes (V^\dagger)^*)$ . Define a linear map  $\lambda_{U,V} : U^\dagger \otimes V^\dagger \rightarrow (U \otimes_{\mathbb{H}} V)^\times$  by  $\lambda_{U,V}(x)(y) = y \cdot x \in \mathbb{H}$ , for  $x \in U^\dagger \otimes V^\dagger$ ,  $y \in U \otimes_{\mathbb{H}} V$ , where ‘ $\cdot$ ’ contracts together the factors of  $U^\dagger \otimes V^\dagger$  and  $(U^\dagger)^* \otimes (V^\dagger)^*$ .

Clearly, if  $x \in U^\dagger \otimes V^\dagger$  and  $y \in (U \otimes_{\mathbb{H}} V)'$ , then  $\lambda_{U,V}(x)(y) \in \mathbb{I}$ . As this holds for all  $y \in (U \otimes_{\mathbb{H}} V)'$ ,  $\lambda_{U,V}(x) \in (U \otimes_{\mathbb{H}} V)^\dagger$ , so that  $\lambda_{U,V}$  maps  $U^\dagger \otimes V^\dagger \rightarrow (U \otimes_{\mathbb{H}} V)^\dagger$ . If  $y \in U \otimes_{\mathbb{H}} V$ , then  $\lambda_{U,V}(x)(y) = 0$  for all  $x \in U^\dagger \otimes V^\dagger$  if and only if  $y = 0$ . Thus  $U \otimes_{\mathbb{H}} V$  is an A $\mathbb{H}$ -module, by Definition 1.1.1. This A $\mathbb{H}$ -module will be called the *quaternionic tensor product of  $U$  and  $V$* , and the operation  $\otimes_{\mathbb{H}}$  will be called the *quaternionic tensor product*. When  $U, V$  are finite-dimensional,  $\lambda_{U,V}$  is surjective, so that  $(U \otimes_{\mathbb{H}} V)^\dagger = \lambda_{U,V}(U^\dagger \otimes V^\dagger)$ .

Here are some basic properties of the operation  $\otimes_{\mathbb{H}}$ .

**Lemma 1.1.5.** *Let  $U, V, W$  be Aℍ-modules. Then there are canonical Aℍ-isomorphisms*

$$\mathbb{H} \otimes_{\mathbb{H}} U \cong U, \quad U \otimes_{\mathbb{H}} V \cong V \otimes_{\mathbb{H}} U, \quad \text{and} \quad (U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W).$$

*Proof.* As  $\mathbb{H}^\dagger \cong \mathbb{R}$ , we may identify  $\mathbb{H} \otimes (\mathbb{H}^\dagger)^* \otimes (U^\dagger)^*$  and  $\mathbb{H} \otimes (U^\dagger)^*$ . Under this identification, it is easy to see that  $\mathbb{H} \otimes_{\mathbb{H}} U$  and  $\iota_U(U)$  are identified. Since  $\iota_U(U) \cong U$  as in Definition 1.1.3, this gives an isomorphism  $\mathbb{H} \otimes_{\mathbb{H}} U \cong U$ , which is an Aℍ-isomorphism. The Aℍ-isomorphism  $U \otimes_{\mathbb{H}} V \cong V \otimes_{\mathbb{H}} U$  is trivial, because the definition of  $U \otimes_{\mathbb{H}} V$  is symmetric in  $U$  and  $V$ .

It remains to show that  $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$ . The maps  $\lambda_{U,V} : U^\dagger \otimes V^\dagger \rightarrow (U \otimes_{\mathbb{H}} V)^\dagger$  and  $\lambda_{U \otimes_{\mathbb{H}} V, W} : (U \otimes_{\mathbb{H}} V)^\dagger \otimes W^\dagger \rightarrow ((U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W)^\dagger$  compose to give a linear map  $\lambda_{UV,W} : U^\dagger \otimes V^\dagger \otimes W^\dagger \rightarrow ((U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W)^\dagger$ , defined in the obvious way. Define a linear map  $\iota_{UV,W} : (U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \rightarrow \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \otimes (W^\dagger)^*$  by  $\iota_{UV,W}(y) \cdot x = \lambda_{UV,W}(x)(y) \in \mathbb{H}$ , for each  $x \in U^\dagger \otimes V^\dagger \otimes W^\dagger$ .

Suppose that  $y \in (U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W$ , and  $\iota_{UV,W}(y) = 0$ . Then  $\lambda_{UV,W}(x)(y) = 0$  for each  $x \in U^\dagger \otimes V^\dagger \otimes W^\dagger$ . It can be shown that this implies that  $y = 0$ . Thus  $\iota_{UV,W}$  is injective. Now from (1) and the definitions, it is easy to show that

$$(2) \quad \begin{aligned} \iota_{UV,W}((U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W) &= (\iota_U(U) \otimes (V^\dagger)^* \otimes (W^\dagger)^*) \\ &\cap ((U^\dagger)^* \otimes \iota_V(V) \otimes (W^\dagger)^*) \cap ((U^\dagger)^* \otimes (V^\dagger)^* \otimes \iota_W(W)), \end{aligned}$$

interpreting this equation as we did (1). As  $\iota_{UV,W}$  is injective,  $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W$  is isomorphic to the r.h.s. of (2). By the same argument,  $U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$  is also isomorphic to the r.h.s. of (2). This gives a canonical isomorphism  $(U \otimes_{\mathbb{H}} V) \otimes_{\mathbb{H}} W \cong U \otimes_{\mathbb{H}} (V \otimes_{\mathbb{H}} W)$ . It turns out to be an Aℍ-isomorphism, and the lemma is complete.  $\square$

Lemma 1.1.5 tells us that  $\otimes_{\mathbb{H}}$  is commutative and associative, and that  $\mathbb{H}$  acts as an identity element for  $\otimes_{\mathbb{H}}$ . Since  $\otimes_{\mathbb{H}}$  is associative, we shall not bother to put brackets in multiple products such as  $U \otimes_{\mathbb{H}} V \otimes_{\mathbb{H}} W$ . Also, the commutativity and associativity of  $\otimes_{\mathbb{H}}$  enable us to define symmetric and antisymmetric products, analogous to  $S^k V$  and  $\Lambda^k V$  for  $V$  a real vector space.

**Definition 1.1.6** Let  $U$  be an Aℍ-module. Write  $\bigotimes_{\mathbb{H}}^k U$  for the product  $U \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{H}} U$  of  $k$  copies of  $U$ . Then the  $k^{\text{th}}$  symmetric group  $S_k$  acts on  $\bigotimes_{\mathbb{H}}^k U$  by permutation of the  $U$  factors in the obvious way. Define  $S_{\mathbb{H}}^k U$  to be the Aℍ-submodule of  $\bigotimes_{\mathbb{H}}^k U$  that is symmetric under these permutations, and  $\Lambda_{\mathbb{H}}^k U$  to be the Aℍ-submodule of  $\bigotimes_{\mathbb{H}}^k U$  that is antisymmetric under these permutations. Define  $\bigotimes_{\mathbb{H}}^0 U$ ,  $S_{\mathbb{H}}^0 U$  and  $\Lambda_{\mathbb{H}}^0 U$  to be the Aℍ-module  $\mathbb{H}$ . The *symmetrization operator*  $\sigma_{\mathbb{H}}$ , defined in the obvious way, is a projection  $\sigma_{\mathbb{H}} : \bigotimes_{\mathbb{H}}^k U \rightarrow S_{\mathbb{H}}^k U$ . Clearly,  $\sigma_{\mathbb{H}}$  is an Aℍ-morphism. Similarly, there is an *antisymmetrization operator*, that is an Aℍ-morphism projection from  $\bigotimes_{\mathbb{H}}^k U$  to  $\Lambda_{\mathbb{H}}^k U$ .

Here is the definition of the tensor product of two  $\mathbb{A}\mathbb{H}$ -morphisms.

**Definition 1.1.7** Let  $U, V, W, X$  be  $\mathbb{A}\mathbb{H}$ -modules, and let  $\phi : U \rightarrow W$  and  $\psi : V \rightarrow X$  be  $\mathbb{A}\mathbb{H}$ -morphisms. Then  $\phi^\times(W^\dagger) \subset U^\dagger$  and  $\psi^\times(X^\dagger) \subset V^\dagger$ , by definition. Taking the duals gives maps  $(\phi^\times)^* : (U^\dagger)^* \rightarrow (W^\dagger)^*$  and  $(\psi^\times)^* : (V^\dagger)^* \rightarrow (X^\dagger)^*$ . Combining these, we have a map

$$(3) \quad \text{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^* : \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \rightarrow \mathbb{H} \otimes (W^\dagger)^* \otimes (X^\dagger)^*.$$

Now  $U \otimes_{\mathbb{H}} V \subset \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$  and  $W \otimes_{\mathbb{H}} X \subset \mathbb{H} \otimes (W^\dagger)^* \otimes (X^\dagger)^*$ . It is easy to show that  $(\text{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*)(U \otimes_{\mathbb{H}} V) \subset W \otimes_{\mathbb{H}} X$ . Define  $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \rightarrow W \otimes_{\mathbb{H}} X$  to be the restriction of  $\text{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$  to  $U \otimes_{\mathbb{H}} V$ . It follows trivially from the definitions that  $\phi \otimes_{\mathbb{H}} \psi$  is  $\mathbb{H}$ -linear and satisfies  $(\phi \otimes_{\mathbb{H}} \psi)((U \otimes_{\mathbb{H}} V)^\dagger) \subset (W \otimes_{\mathbb{H}} X)^\dagger$ . Thus  $\phi \otimes_{\mathbb{H}} \psi$  is an  $\mathbb{A}\mathbb{H}$ -morphism from  $U \otimes_{\mathbb{H}} V$  to  $W \otimes_{\mathbb{H}} X$ . This is the *quaternionic tensor product* of  $\phi$  and  $\psi$ .

**Lemma 1.1.8.** *Suppose that  $\phi : U \rightarrow W$  and  $\psi : V \rightarrow X$  are injective  $\mathbb{A}\mathbb{H}$ -morphisms. Then  $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \rightarrow W \otimes_{\mathbb{H}} X$  is an injective  $\mathbb{A}\mathbb{H}$ -morphism.*

*Proof.* Consider the map  $\text{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$  of (3). Clearly this maps  $\iota_U(U) \otimes (V^\dagger)^*$  to  $\iota_W(W) \otimes (X^\dagger)^*$ . As  $\iota_U(U) \cong U$  and  $\iota_W(W) \cong W$  and the map  $\phi : U \rightarrow W$  is injective, we see that the kernel of  $\text{id} \otimes (\phi^\times)^* \otimes (\psi^\times)^*$  on  $\iota_U(U) \otimes (V^\dagger)^*$  is  $\iota_U(U) \otimes \text{Ker}(\psi^\times)^*$ . Similarly, the kernel on  $(U^\dagger)^* \otimes \iota_V(V)$  is  $\text{Ker}(\phi^\times)^* \otimes \iota_V(V)$ . Thus the kernel of  $\phi \otimes_{\mathbb{H}} \psi$  is

$$\text{Ker}(\phi \otimes_{\mathbb{H}} \psi) = (\iota_U(U) \otimes \text{Ker}(\psi^\times)^*) \cap (\text{Ker}(\phi^\times)^* \otimes \iota_V(V)).$$

But this is contained in  $(\iota_U(U) \cap (\mathbb{H} \otimes \text{Ker}(\phi^\times)^*)) \otimes (V^\dagger)^*$ . Now  $\iota_U(U) \cap (\mathbb{H} \otimes \text{Ker}(\phi^\times)^*) = 0$ , since if  $\iota(u)$  lies in  $\mathbb{H} \otimes \text{Ker}(\phi^\times)^*$  then  $\phi(u) = 0$  in  $W$ , so  $u = 0$  as  $\phi$  is injective. Thus  $\text{Ker}(\phi \otimes_{\mathbb{H}} \psi) = 0$ , and  $\phi \otimes_{\mathbb{H}} \psi$  is injective.  $\square$

The following lemma is trivial to prove.

**Lemma 1.1.9.** *Let  $U, V$  be  $\mathbb{A}\mathbb{H}$ -modules, and let  $u \in U$  and  $v \in V$  be nonzero. Suppose that  $\alpha(u)\beta(v) = \beta(v)\alpha(u) \in \mathbb{H}$  for every  $\alpha \in U^\dagger$  and  $\beta \in V^\dagger$ . Define an element  $u \otimes_{\mathbb{H}} v$  of  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$  by  $(u \otimes_{\mathbb{H}} v) \cdot (\alpha \otimes \beta) = \alpha(u)\beta(v) \in \mathbb{H}$ . Then  $u \otimes_{\mathbb{H}} v$  is a nonzero element of  $U \otimes_{\mathbb{H}} V$ .*

The philosophy of the algebraic side of this paper is that much algebra that works over a commutative field such as  $\mathbb{R}$  or  $\mathbb{C}$  also has a close analogue over  $\mathbb{H}$  (or some other noncommutative algebra), when we replace vector spaces over  $\mathbb{R}$  or  $\mathbb{C}$  by  $\mathbb{A}\mathbb{H}$ -modules, and tensor products of vector spaces by the quaternionic tensor product  $\otimes_{\mathbb{H}}$ . Lemmas 1.1.5, 1.1.8 and 1.1.9 are examples of this philosophy, as they show that essential properties of the usual tensor product also hold for the quaternionic tensor product.

However, the quaternionic tensor product also has properties that are very unlike the usual tensor product, which come from the noncommutativity of the quaternions.

For example, let  $U, V$  be  $\mathbb{H}$ -modules, and let  $U' = V' = \{0\}$ . Then  $U^\dagger = U^\times$  and  $V^\dagger = V^\times$ . Suppose that  $x \in U \otimes_{\mathbb{H}} V$ , and let  $p, q \in \mathbb{H}$ ,  $\alpha \in U^\dagger$  and  $\beta \in V^\dagger$ . Then

$$(4) \quad \begin{aligned} x(\alpha \otimes \beta)\bar{p}\bar{q} &= x((p \cdot \alpha) \otimes \beta)\bar{q} = x((p \cdot \alpha) \otimes (q \cdot \beta)) \\ &= x(\alpha \otimes (q \cdot \beta))\bar{p} = x(\alpha \otimes \beta)\bar{q}\bar{p}, \end{aligned}$$

where we use the facts that  $U^\dagger, V^\dagger$  are closed under the  $\mathbb{H}$ -action on  $U^\times$ , and the definition (1) of  $U \otimes_{\mathbb{H}} V$ . Choosing  $p$  and  $q$  such that  $\bar{p}\bar{q} \neq \bar{q}\bar{p}$ , equation (4) shows that  $x(\alpha \otimes \beta) = 0$ . Since this holds for all  $\alpha, \beta, x = 0$ . Thus  $U \otimes_{\mathbb{H}} V = \{0\}$ . We have shown that the quaternionic tensor product of two nonzero A $\mathbb{H}$ -modules can be zero.

Suppose that  $U \cong \mathbb{H}^k$ . Then the condition in Definition 1.1.1 implies that the dimension  $\dim(U^\dagger) \geq k$ . But  $\dim(U') + \dim(U^\dagger) = 4k$ , so  $\dim(U') \leq 3k$ . The example above illustrates the general principle that if  $\dim(U')$  is small, then quaternionic tensor products involving  $U$  tend to be small or zero. A good rule is that the most interesting A $\mathbb{H}$ -modules  $U$  are those in the range  $2k \leq \dim(U') \leq 3k$ .

Here are some differences between the quaternionic and ordinary tensor products.

- In contrast to Lemma 1.1.9, if  $u \in U$  and  $v \in V$ , there is in general no element ‘ $u \otimes_{\mathbb{H}} v$ ’ in  $U \otimes_{\mathbb{H}} V$ . At best, there is a linear map from some vector subspace of  $U \otimes V$  to  $U \otimes_{\mathbb{H}} V$ .
- Suppose that  $U, V$  are finite-dimensional A $\mathbb{H}$ -modules, with  $U \cong \mathbb{H}^k, V \cong \mathbb{H}^l$ . It is easy to show that  $U \otimes_{\mathbb{H}} V \cong \mathbb{H}^n$ , for some integer  $n$  with  $0 \leq n \leq kl$ . However,  $n$  can vary discontinuously under smooth variations of  $U^\dagger, V^\dagger$ .
- If we wish, we can make  $U^\times, V^\times$  into A $\mathbb{H}$ -modules. However, it is not in general true that  $U^\times \otimes_{\mathbb{H}} V^\times \cong (U \otimes_{\mathbb{H}} V)^\times$ , and there is no reason for  $U^\times \otimes_{\mathbb{H}} V^\times$  and  $(U \otimes_{\mathbb{H}} V)^\times$  even to have the same dimension. For this reason, dual A $\mathbb{H}$ -modules seem not to be a very powerful tool.
- In contrast to Lemma 1.1.8, if  $\phi : U \rightarrow W$  and  $\psi : V \rightarrow X$  are *surjective* A $\mathbb{H}$ -morphisms, then  $\phi \otimes_{\mathbb{H}} \psi : U \otimes_{\mathbb{H}} V \rightarrow W \otimes_{\mathbb{H}} X$  does not have to be surjective. In particular,  $U \otimes_{\mathbb{H}} V$  may be zero, but  $W \otimes_{\mathbb{H}} X$  nonzero.

**1.2. Stable and semistable A $\mathbb{H}$ -modules.**

Now two special sorts of A $\mathbb{H}$ -modules will be defined, called *stable* and *semistable* A $\mathbb{H}$ -modules. Our aim in this paper has been to develop a strong analogy between the theories of A $\mathbb{H}$ -modules and vector spaces over a field. For stable A $\mathbb{H}$ -modules it turns out that this analogy is more complete than in the general case, because various important properties of the vector space theory hold for stable but not for general A $\mathbb{H}$ -modules. Therefore, in applications of the theory it will often be useful to restrict to stable A $\mathbb{H}$ -modules, to exploit their better behaviour. We begin with a definition.

**Definition 1.2.1** Let  $q \in \mathbb{I}$  be nonzero. Define an A $\mathbb{H}$ -module  $X_q$  by  $X_q = \mathbb{H}$ , and  $X'_q = \{p \in \mathbb{H} : pq = -qp\}$ . Then  $X'_q \subset \mathbb{I}$  and  $\dim X'_q = 2$ . Let  $\chi_q : X_q \rightarrow \mathbb{H}$  be the identity. Then  $\chi_q(X'_q) \subset \mathbb{I} = \mathbb{H}'$ , so  $\chi_q$  is an A $\mathbb{H}$ -morphism. Suppose  $U$  is any finite-dimensional A $\mathbb{H}$ -module. Then  $U \otimes_{\mathbb{H}} X_q$  is an A $\mathbb{H}$ -module, and  $U \otimes_{\mathbb{H}} \mathbb{H} \cong U$ ,

so there is a canonical AHH-morphism  $\text{id} \otimes_{\mathbb{H}} \chi_q : U \otimes_{\mathbb{H}} X_q \rightarrow U$ . We say that  $U$  is a semistable AHH-module if

$$U = \langle (\text{id} \otimes_{\mathbb{H}} \chi_q)(U \otimes_{\mathbb{H}} X_q) : 0 \neq q \in \mathbb{I} \rangle,$$

that is,  $U$  is generated as an  $\mathbb{H}$ -module by the images  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_q)$  for nonzero  $q \in \mathbb{I}$ . Clearly, every AHH-module contains a unique maximal semistable AHH-submodule, the submodule generated by the images  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_q)$ .

**Proposition 1.2.2.** *Let  $q \in \mathbb{I}$  be nonzero. Then  $X_q \otimes_{\mathbb{H}} X_q$  is AHH-isomorphic to  $X_q$ . Let  $U$  be a finite-dimensional AHH-module. Then  $U \otimes_{\mathbb{H}} X_q$  is AHH-isomorphic to  $nX_q$ , the direct sum of  $n$  copies of  $X_q$ , for some  $n \geq 0$ . Now suppose that  $U$  is semistable, with  $\dim U = 4j$  and  $\dim U' = 2j + r$ , for integers  $j, r$ . Then  $U \otimes_{\mathbb{H}} X_q \cong nX_q$ , where  $n \geq r$  for all nonzero  $q \in \mathbb{I}$ , and  $n = r$  for generic  $q \in \mathbb{I}$ . Thus  $r \geq 0$ .*

*Proof.* A short calculation shows that  $X_q \otimes_{\mathbb{H}} X_q \cong X_q$  as AHH-modules. Let  $U$  be a finite-dimensional AHH-module. Then  $\text{id} \otimes_{\mathbb{H}} \chi_q : U \otimes_{\mathbb{H}} X_q \rightarrow U$  is an injective AHH-morphism, by Lemma 1.1.8. It is easy to show that  $(U \otimes_{\mathbb{H}} X_q)'$  is identified with  $U' \cap (q \cdot U')$  by  $\text{id} \otimes_{\mathbb{H}} \chi_q$ . Now  $\langle 1, q \rangle \subset \mathbb{H}$  is a subalgebra  $\mathbb{C}_q$  of  $\mathbb{H}$  isomorphic to  $\mathbb{C}$ , and clearly  $(U \otimes_{\mathbb{H}} X_q)'$  is closed under  $\mathbb{C}_q$ . Choose a basis  $u_1, \dots, u_n$  of  $(U \otimes_{\mathbb{H}} X_q)'$  over the field  $\mathbb{C}_q$ . Suppose that  $\sum q_j u_j = 0$  in  $U \otimes_{\mathbb{H}} X_q$ , for  $q_1, \dots, q_n \in \mathbb{H}$ . Let  $p \in \mathbb{H}$  be nonzero, and such that  $pq = -qp$ . Then  $\mathbb{H}$  splits as  $\mathbb{H} = \mathbb{C}_q \oplus p\mathbb{C}_q$ . Using this splitting, write  $q_j = a_j + pb_j$ , with  $a_j, b_j \in \mathbb{C}_q$ .

If  $\alpha \in (U \otimes_{\mathbb{H}} X_q)^\dagger$ , then  $\alpha(u_j) \in p\mathbb{C}_q$ , so  $\alpha(\sum q_j u_j) = 0$  implies that  $\alpha(\sum a_j u_j) = \alpha(\sum b_j u_j) = 0$ . As this holds for all  $\alpha \in (U \otimes_{\mathbb{H}} X_q)^\dagger$ , we have  $\sum a_j u_j = 0$  and  $\sum b_j u_j = 0$ . But  $\{u_j\}$  is a basis over  $\mathbb{C}_q$  and  $a_j, b_j \in \mathbb{C}_q$ . Thus  $a_j = b_j = 0$ , and  $q_j = 0$ . We have proved that  $u_1, \dots, u_n$  are linearly independent over  $\mathbb{H}$ . It is now easily shown that  $\mathbb{H} \cdot u_j \cong X_q$ , and that  $U \otimes_{\mathbb{H}} X_q = \mathbb{H} \cdot u_1 \oplus \dots \oplus \mathbb{H} \cdot u_n \cong nX_q$ , as we have to prove.

Now let  $U$  be semistable, with  $\dim U = 4j$  and  $\dim U' = 2j + r$ . From above,  $U' \cap (q \cdot U') = \mathbb{C}_q^n$ . But we have

$$(5) \quad \begin{aligned} \dim(U' \cap (q \cdot U')) + \dim(U' + q \cdot U') \\ = \dim U' + \dim(q \cdot U') = 4j + 2r. \end{aligned}$$

Since  $U' + q \cdot U' \subset U$ ,  $\dim(U' + q \cdot U') \leq 4j$ , and so (5) shows that  $2n \geq 2r$ , with equality if and only if  $U = U' + q \cdot U'$ . Thus  $n \geq r$  for all nonzero  $q$ , as we have to prove. To complete the proposition, it is enough to show that  $U = U' + q \cdot U'$  for generic  $q \in \mathbb{I}$ .

As  $U$  is semistable, it is generated by the images  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_q)$ . So suppose  $U$  is generated by  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_{q_j})$  for  $j = 1, \dots, k$ , where  $0 \neq q_j \in \mathbb{I}$ . Let  $q \in \mathbb{I}$ , and suppose that  $qq_j \neq q_j q$  for  $j = 1, \dots, k$ . This is true for generic  $q$ . Clearly  $X'_{q_j} + q \cdot X'_{q_j} = X_{q_j}$ , as  $qq_j \neq q_j q$ . Thus  $(U \otimes_{\mathbb{H}} X_{q_j})' + q \cdot (U \otimes_{\mathbb{H}} X_{q_j})' = U \otimes_{\mathbb{H}} X_{q_j}$ , as  $U \otimes_{\mathbb{H}} X_{q_j} \cong nX_{q_j}$ . We deduce that  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_{q_j})$  is contained in  $U' + q \cdot U'$ . But  $U$  is generated by the spaces  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_{q_j})$ , so  $U = U' + q \cdot U'$ . This completes the proof.  $\square$

Now we can define stable Aℍ-modules.

**Definition 1.2.3** Let  $U$  be a finite-dimensional Aℍ-module. Then  $\dim U = 4j$  and  $\dim U' = 2j + r$ , for some integers  $j, r$ . Define the *virtual dimension* of  $U$  to be  $r$ . We say that  $U$  is a *stable Aℍ-module* if  $U$  is a semistable Aℍ-module, so that  $r \geq 0$ , and  $U \otimes_{\mathbb{H}} X_q \cong rX_q$  for each nonzero  $q \in \mathbb{I}$ .

The point of this definition will become clear soon. Here are two propositions about stable and semistable Aℍ-modules.

**Proposition 1.2.4.** *Let  $U$  be a stable Aℍ-module with  $\dim U = 4j$ , and  $\dim U' = 2j + r$  for integers  $j, r$ . Let  $V$  be a semistable Aℍ-module with  $\dim V = 4k$  and  $\dim V' = 2k + s$  for integers  $k, s$ . Then  $\dim(U \otimes_{\mathbb{H}} V) = 4l$ , where  $l = js + rk - rs$ .*

*Proof.* Regard  $\mathbb{H} \otimes U^\dagger \otimes V^\dagger$  and  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$  as  $\mathbb{H}$ -modules in the obvious way. Define a bilinear map  $\Theta : \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* \times \mathbb{H} \otimes U^\dagger \otimes V^\dagger \rightarrow \mathbb{H}$  by  $\Theta(p \otimes \alpha \otimes \beta, q \otimes x \otimes y) = \alpha(x)\beta(y)p\bar{q}$  for  $p, q \in \mathbb{H}$ ,  $x \in U^\dagger$ ,  $y \in V^\dagger$ ,  $\alpha \in (U^\dagger)^*$  and  $\beta \in (V^\dagger)^*$ . Recall that  $\iota_U(U) \otimes (V^\dagger)^*$  and  $(U^\dagger)^* \otimes \iota_V(V)$  are  $\mathbb{H}$ -submodules of  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ . Define a subspace  $K_{U,V}$  of  $\mathbb{H} \otimes U^\dagger \otimes V^\dagger$  by  $z \in K_{U,V}$  if  $\Theta(\zeta, z) = 0$  whenever  $\zeta \in \iota_U(U) \otimes (V^\dagger)^*$  or  $\zeta \in (U^\dagger)^* \otimes \iota_V(V)$ . Then  $K_{U,V}$  is an  $\mathbb{H}$ -submodule of  $\mathbb{H} \otimes U^\dagger \otimes V^\dagger$ .

Now  $\iota_U(U) \otimes (V^\dagger)^* + (U^\dagger)^* \otimes \iota_V(V)$  is an  $\mathbb{H}$ -submodule of  $\mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ , and clearly

$$\begin{aligned} & \dim K_{U,V} + \dim(\iota_U(U) \otimes (V^\dagger)^* + (U^\dagger)^* \otimes \iota_V(V)) \\ &= \dim \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^* = 4(2j - r)(2k - s). \end{aligned}$$

But  $U \otimes_{\mathbb{H}} V = (\iota_U(U) \otimes (V^\dagger)^*) \cap ((U^\dagger)^* \otimes \iota_V(V))$ , and thus

$$\begin{aligned} \dim(U \otimes_{\mathbb{H}} V) &= \dim(\iota_U(U) \otimes (V^\dagger)^*) + \dim((U^\dagger)^* \otimes \iota_V(V)) \\ &\quad - \dim(\iota_U(U) \otimes (V^\dagger)^* + (U^\dagger)^* \otimes \iota_V(V)), \end{aligned}$$

so that  $\dim(U \otimes_{\mathbb{H}} V) = 4j(2k - s) + (2j - r)4k - \{4(2j - r)(2k - s) - \dim K_{U,V}\} = 4l + \dim K_{U,V}$ , where  $l = js + rk - rs$ . Therefore  $\dim(U \otimes_{\mathbb{H}} V) = 4l$  if and only if  $K_{U,V} = \{0\}$ .

Suppose that  $W$  is an Aℍ-module, and  $\phi : W \rightarrow V$  is an Aℍ-morphism. Then  $\phi^\times : V^\dagger \rightarrow W^\dagger$ , so that  $\text{id} \otimes \phi^\times : \mathbb{H} \otimes U^\dagger \otimes V^\dagger \rightarrow \mathbb{H} \otimes U^\dagger \otimes W^\dagger$ . We have  $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes V^\dagger$  and  $K_{U,W} \subset \mathbb{H} \otimes U^\dagger \otimes W^\dagger$ . It is easy to show that  $(\text{id} \otimes \phi^\times)(K_{U,V}) \subset K_{U,W}$ . Let  $0 \neq q \in \mathbb{I}$ , and put  $W = V \otimes_{\mathbb{H}} X_q$ , and  $\phi = \text{id} \otimes_{\mathbb{H}} \chi_q$ . In this case  $W \cong nX_q$ . The argument above shows that  $K_{U,X_q} = \{0\}$  if and only if  $\dim(U \otimes_{\mathbb{H}} X_q) = 4r$ . But by this holds by Definition 1.2.3, as  $U$  is stable.

Thus  $K_{U,X_q} = \{0\}$ , and  $K_{U,W} = \{0\}$  as  $W \cong nX_q$ . It follows that  $(\text{id} \otimes \phi^\times)(K_{U,V}) = \{0\}$ , so  $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes \text{Ker } \phi^\times$ . Now  $V$  is semistable. Therefore  $V$  is generated by submodules  $\phi(W)$  of the above type, and the intersection of the subspaces  $\text{Ker } \phi^\times \subset V^\dagger$  for all nonzero  $q$ , must be zero. So  $K_{U,V} \subset \mathbb{H} \otimes U^\dagger \otimes \{0\}$ , giving  $K_{U,V} = \{0\}$ , and  $\dim(U \otimes_{\mathbb{H}} V) = 4l$  from above, which completes the proof.  $\square$

**Proposition 1.2.5.** *Let  $U$  be a stable  $\mathbb{A}\mathbb{H}$ -module, and  $V$  a semistable  $\mathbb{A}\mathbb{H}$ -module. Then  $U \otimes_{\mathbb{H}} V$  is semistable.*

*Proof.* Let  $\dim U = 4j$ ,  $\dim V = 4k$ ,  $\dim U' = 2j + r$  and  $\dim V' = 2k + s$ . Then Proposition 1.2.4 shows that  $\dim(U \otimes_{\mathbb{H}} V) = 4l$ , where  $l = js + rk - rs$ . Let  $W \subset U \otimes_{\mathbb{H}} V$  be the  $\mathbb{A}\mathbb{H}$ -submodule of  $U \otimes_{\mathbb{H}} V$  generated by the images  $\text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_q)$ , where  $\text{id} \otimes_{\mathbb{H}} \chi_q : U \otimes_{\mathbb{H}} V \otimes_{\mathbb{H}} X_q \rightarrow U \otimes_{\mathbb{H}} V$  and  $0 \neq q \in \mathbb{I}$ . Then  $W$  is the maximal semistable  $\mathbb{A}\mathbb{H}$ -submodule of  $U \otimes_{\mathbb{H}} V$ . We shall prove the proposition by explicitly constructing  $l$  elements of  $W$ , that are linearly independent over  $\mathbb{H}$ . This will imply that  $\dim W \geq 4l$ . Since  $W \subset U \otimes_{\mathbb{H}} V$  and  $\dim(U \otimes_{\mathbb{H}} V) = 4l$ , we see that  $W = U \otimes_{\mathbb{H}} V$ , so  $U \otimes_{\mathbb{H}} V$  is semistable.

Here is some new notation. Let  $0 \neq q \in \mathbb{I}$ , and define  $U_q = \{u \in U : \alpha(u) \in \langle 1, q \rangle \text{ for all } \alpha \in U^\dagger\}$ . Similarly define  $V_q, (U \otimes_{\mathbb{H}} V)_q$ . It can be shown that  $\mathbb{H} \cdot U_q = \text{Im}(\text{id} \otimes_{\mathbb{H}} \chi_q)$ , and similarly for  $V_q, (U \otimes_{\mathbb{H}} V)_q$ . Thus  $(U \otimes_{\mathbb{H}} V)_q \subset W$ . Since  $V$  is semistable, we can choose nonzero elements  $q_1, \dots, q_k \in \mathbb{I}$  and  $v_1, \dots, v_k \in V$ , such that  $v_a \in V_{q_a}$  and  $(v_1, \dots, v_k)$  is a basis for  $V$  over  $\mathbb{H}$ . Since  $U$  is stable,  $U_q \cong \mathbb{C}^r$  for each  $0 \neq q \in \mathbb{I}$ . Therefore for each  $a = 1, \dots, k$  we may choose elements  $u_{a1}, \dots, u_{ar}$  of  $U$ , such that  $u_{ab} \in U_{q_a}$  and  $u_{a1}, \dots, u_{ar}$  are linearly independent over  $\mathbb{H}$  in  $U$ .

As  $U$  is stable, it is not difficult to see that there are nonzero elements  $p_1, \dots, p_{j-r} \in \mathbb{I}$  and  $u_1, \dots, u_{j-r} \in U$ , such that  $u_c \in U_{p_c}$ , and for each  $a = 1, \dots, k$ , the set  $u_{a1}, \dots, u_{ar}, u_1, \dots, u_{j-r}$  is linearly independent over  $\mathbb{H}$ . This is just a matter of picking *generic* elements  $p_c$  and  $u_c$ , and showing that generically, linear independence holds.

We shall also need another property. Define  $F \subset U^\dagger$  by  $F = \{\alpha \in U^\dagger : \alpha(u_c) = 0 \text{ for } c = 1, \dots, j-r\}$ . Now as  $\alpha(u_c) \in \langle 1, q_c \rangle$  for each  $\alpha \in U^\dagger$ , the codimension of  $F$  in  $U^\dagger$  is at most  $2(j-r)$ . Since  $\dim U^\dagger = 2j-r$ , this gives  $\dim F \geq r$ . The second property we need is that for each  $a = 1, \dots, k$ , if  $u \in \langle u_{a1}, \dots, u_{ar} \rangle_{\mathbb{H}}$  and  $\alpha(u) = 0$  for all  $\alpha \in F$ , then  $u = 0$ . Here  $\langle \cdot \rangle_{\mathbb{H}}$  means the linear span over  $\mathbb{H}$ . Again, it can be shown that for generic choice of  $p_c, v_c$ , this property holds.

Now for  $a = 1, \dots, k$  and  $b = 1, \dots, r$ ,  $u_{ab} \in U_{q_a}$  and  $v_a \in V_{q_a}$ . Lemma 1.1.9 gives an element  $u_{ab} \otimes_{\mathbb{H}} v_a$  in  $U \otimes_{\mathbb{H}} V$ . Moreover  $u_{ab} \otimes_{\mathbb{H}} v_a \in (U \otimes_{\mathbb{H}} V)_{q_a}$ . Thus  $u_{ab} \otimes_{\mathbb{H}} v_a \in W$ . Therefore, we have made  $kr$  elements  $u_{ab} \otimes_{\mathbb{H}} v_a$  of  $W$ . Similarly, for  $c = 1, \dots, j-r$  and  $d = 1, \dots, s$ , the elements  $u_c \otimes_{\mathbb{H}} v_{cd}$  exist in  $W$ , which gives a further  $(j-r)s$  elements of  $W$ . Since  $kr + (j-r)s = l$ , we have constructed  $l$  explicit elements  $u_{ab} \otimes_{\mathbb{H}} v_a$  and  $u_c \otimes_{\mathbb{H}} v_{cd}$  of  $W$ .

Suppose that  $\sum_{a,b} x_{ab} u_{ab} \otimes_{\mathbb{H}} v_a + \sum_{c,d} y_{cd} u_c \otimes_{\mathbb{H}} v_{cd} = 0$  in  $U \otimes_{\mathbb{H}} V$ , where  $x_{ab} \in \mathbb{H}$  for  $a = 1, \dots, k$ ,  $b = 1, \dots, r$  and  $y_{cd} \in \mathbb{H}$  for  $c = 1, \dots, j-r$  and  $d = 1, \dots, s$ . We shall show that  $x_{ab} = y_{cd} = 0$ . Now this equation implies that

$$(6) \quad \sum_{a,b} x_{ab} \alpha(u_{ab}) \beta(v_a) + \sum_{c,d} y_{cd} \alpha(u_c) \beta(v_{cd}) = 0$$



for all  $\alpha \in U^\dagger$  and  $\beta \in V^\dagger$ , by Lemma 1.1.9. Let  $\alpha \in F$ . Then  $\alpha(u_c) = 0$ , so  $\sum_{a,b} x_{ab} \alpha(u_{ab}) \beta(v_a) = 0$ . As this holds for all  $\beta \in V^\dagger$  and the  $v_a$  are linearly independent over  $\mathbb{H}$ , it follows that  $\sum_b x_{ab} \alpha(u_{ab}) = 0$  for all  $a$  and all  $\alpha \in F$ .

By the property of  $F$  assumed above, this implies that  $x_{ab} = 0$  for all  $a, b$ . It is now easy to show that  $y_{cd} = 0$  for all  $c, d$ . Thus the  $l$  elements  $u_{ab} \otimes_{\mathbb{H}} v_a$  and  $u_c \otimes_{\mathbb{H}} v_{cd}$  of  $W$  are linearly independent over  $\mathbb{H}$ , so  $\dim W \geq 4l$ . But  $\dim(U \otimes_{\mathbb{H}} V) = 4l$  by Proposition 1.2.4. So  $U \otimes_{\mathbb{H}} V = W$ , and  $U \otimes_{\mathbb{H}} V$  is semistable. This finishes the proof.  $\square$

We can now prove the main result of this section.

**Theorem 1.2.6.** *Let  $U$  be a stable  $\text{A}\mathbb{H}$ -module and  $V$  be a semistable  $\text{A}\mathbb{H}$ -module with*

$$\dim U = 4j, \dim U' = 2j + r, \dim V = 4k \text{ and } \dim V' = 2k + s$$

*for integers  $j, k, r$  and  $s$ . Then  $U \otimes_{\mathbb{H}} V$  is a semistable  $\text{A}\mathbb{H}$ -module with  $\dim(U \otimes_{\mathbb{H}} V) = 4l$  and  $\dim(U \otimes_{\mathbb{H}} V)' = 2l + t$ , where  $l = js + rk - rs$  and  $t = rs$ . If  $V$  is stable, then  $U \otimes_{\mathbb{H}} V$  is stable.*

*Proof.* Let  $l = js + rk - rs$  and  $t = rs$ . Proposition 1.2.4 shows that  $\dim(U \otimes_{\mathbb{H}} V) = 4l$ . As  $U$  is stable,  $U \otimes_{\mathbb{H}} X_q \cong rX_q$  for nonzero  $q \in \mathbb{I}$ . As  $V$  is semistable, Proposition 1.2.2 shows that  $V \otimes_{\mathbb{H}} X_q \cong sX_q$  for generic  $q \in \mathbb{I}$ . Thus  $U \otimes_{\mathbb{H}} V \otimes_{\mathbb{H}} X_q \cong rsX_q = tX_q$  for generic  $q \in \mathbb{I}$ . Also, Proposition 1.2.5 shows that  $U \otimes_{\mathbb{H}} V$  is semistable. Combining these two facts with Proposition 1.2.2, we see that  $\dim(U \otimes_{\mathbb{H}} V)' = 2l + t$ , as we have to prove.

It remains to show that if  $V$  is stable, then  $U \otimes_{\mathbb{H}} V$  is stable. Suppose  $V$  is stable. Then  $V \otimes_{\mathbb{H}} X_q \cong sX_q$  for all nonzero  $q \in \mathbb{I}$ , so  $U \otimes_{\mathbb{H}} V \otimes_{\mathbb{H}} X_q = tX_q$  for all nonzero  $q \in \mathbb{I}$ . As  $U \otimes_{\mathbb{H}} V$  is semistable, it is stable, by Definition 1.2.3. This completes the proof.  $\square$

Theorem 1.2.6 shows that if  $U$  is stable and  $V$  semistable, then the virtual dimension of  $U \otimes_{\mathbb{H}} V$  is the product of the virtual dimensions of  $U$  and  $V$ . Thus the virtual dimension is a good analogue of the dimension of a vector space, as it multiplies under  $\otimes_{\mathbb{H}}$ . Note also that  $(j - r)/r + (k - s)/s = (l - t)/t$ , so that the nonnegative function  $U \mapsto (j - r)/r$  behaves additively under  $\otimes_{\mathbb{H}}$ .

Next we will show that generic  $\text{A}\mathbb{H}$ -modules  $(U, U')$  with positive virtual dimension are stable. Thus there are many stable  $\text{A}\mathbb{H}$ -modules.

**Lemma 1.2.7.** *Let  $j, r$  be integers with  $0 < r \leq j$ . Let  $U = \mathbb{H}^j$ , and let  $U'$  be a real vector subspace of  $U$  with  $\dim U' = 2j + r$ . Then for generic subspaces  $U'$ ,  $(U, U')$  is a stable  $\text{A}\mathbb{H}$ -module.*

*Proof.* Let  $G$  be the Grassmannian of real  $(2j + r)$ -planes in  $U \cong \mathbb{R}^{4j}$ . Then  $U' \in G$ , and  $\dim G = 4j^2 - r^2$ . The condition for  $(U, U')$  to be an  $\text{A}\mathbb{H}$ -module is that  $\mathbb{H} \cdot U^\dagger = U^\times$ . A calculation shows that this fails for a subset of  $G$  of codimension

$4(j - r + 1)$ . Thus for generic  $U' \in G$ ,  $(U, U')$  is an  $\text{A}\mathbb{H}$ -module. Suppose  $(U, U')$  is an  $\text{A}\mathbb{H}$ -module. Let  $W$  be the maximal semistable  $\text{A}\mathbb{H}$ -submodule in  $U$ . Then  $W \cong \mathbb{H}^k$ , for some  $k$  with  $r \leq k \leq j$ . A calculation shows that for given  $k$ , the subset of  $G$  with  $W \cong \mathbb{H}^k$  is of codimension  $2(j - k)r$ . Thus for generic  $U' \in G$ ,  $W = \mathbb{H}^j = U$ , and  $U$  is semistable.

Suppose  $(U, U')$  is semistable. Let  $0 \neq q \in \mathbb{I}$ . Then  $U \otimes_{\mathbb{H}} X_q \cong rX_q$  if and only if  $U' \cap q \cdot U' = \mathbb{R}^{2r}$ . A computation shows that this fails for a subset of  $G$  of codimension  $2r + 2$ . Thus the condition  $U' \cap q \cdot U' = \mathbb{R}^{2r}$  for all nonzero  $q \in \mathbb{I}$  fails for a subset of  $G$  of codimension at most  $2r$ , since this subset is the union of a 2-dimensional family of  $2r + 2$ -codimensional subsets, the 2-dimensional family being  $\mathcal{S}^2$ , the unit sphere in  $\mathbb{I}$ . Therefore for generic  $U' \in G$ ,  $U \otimes_{\mathbb{H}} X_q \cong rX_q$  for all nonzero  $q \in \mathbb{I}$ , and  $U$  is stable.  $\square$

We leave the proof of this proposition to the reader, as a (difficult) exercise.

**Proposition 1.2.8.** *Let  $U$  be a stable  $\text{A}\mathbb{H}$ -module, with  $\dim U = 4j$  and  $\dim U' = 2j + r$ . Let  $n$  be a positive integer. Then  $S_{\mathbb{H}}^n U$  and  $\Lambda_{\mathbb{H}}^n U$  are stable  $\text{A}\mathbb{H}$ -modules, with  $\dim(S_{\mathbb{H}}^n U) = 4k$ ,  $\dim(S_{\mathbb{H}}^n U)' = 2k + s$ ,  $\dim(\Lambda_{\mathbb{H}}^n U) = 4l$  and  $\dim(\Lambda_{\mathbb{H}}^n U)' = 2l + t$ , where*

$$k = (j - r) \binom{r + n - 1}{n - 1} + \binom{r + n - 1}{n}, \quad s = \binom{r + n - 1}{n},$$

$$l = (j - r) \binom{r - 1}{n - 1} + \binom{r}{n} \quad \text{and} \quad t = \binom{r}{n}.$$

**1.3. Stable  $\text{A}\mathbb{H}$ -modules and exact sequences.** Recall that if  $U, V, W$  are vector spaces and  $\phi : U \rightarrow V$ ,  $\psi : V \rightarrow W$  are linear maps, then we say that the sequence  $U \xrightarrow{\phi} V \xrightarrow{\psi} W$  is *exact at  $V$*  if  $\text{Im } \phi = \text{Ker } \psi$ . Here is the analogue of this for  $\text{A}\mathbb{H}$ -modules.

**Definition 1.3.1** Let  $U, V, W$  be  $\text{A}\mathbb{H}$ -modules, and let  $\phi : U \rightarrow V$  and  $\psi : V \rightarrow W$  be  $\text{A}\mathbb{H}$ -morphisms. We say that *the sequence  $U \xrightarrow{\phi} V \xrightarrow{\psi} W$  is  $\text{A}\mathbb{H}$ -exact at  $V$*  if the sequence  $U \xrightarrow{\phi} V \xrightarrow{\psi} W$  is exact at  $V$ , and the sequence  $U' \xrightarrow{\phi} V' \xrightarrow{\psi} W'$  is exact at  $V'$ . We say that a sequence of  $\text{A}\mathbb{H}$ -morphisms is  $\text{A}\mathbb{H}$ -exact if it is  $\text{A}\mathbb{H}$ -exact at every term.

Here is an example of some bad behaviour of the theory.

**Example 1.3.2** Define  $U = \mathbb{H}$  and  $U' = \{0\}$ . Define  $V = \mathbb{H}^2$  and  $V' = \langle (1, i_1), (1, i_2) \rangle$ . Define  $W = \mathbb{H}$  and  $W' = \langle i_1, i_2 \rangle$ . Then  $U, V, W$  are  $\text{A}\mathbb{H}$ -modules. Define linear maps  $\phi : U \rightarrow V$  by  $\phi(q) = (q, 0)$ , and  $\psi : V \rightarrow W$  by  $\psi((p, q)) = q$ . Then  $\phi, \psi$  are  $\text{A}\mathbb{H}$ -morphisms, and the sequence  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is  $\text{A}\mathbb{H}$ -exact.

Now set  $Z$  to be the  $\text{A}\mathbb{H}$ -module  $W$ . A short calculation shows that  $U \otimes_{\mathbb{H}} Z = \{0\}$ ,  $V \otimes_{\mathbb{H}} Z = \{0\}$ , but  $W \otimes_{\mathbb{H}} Z \cong W$ . It follows that the sequence

$$0 \rightarrow U \otimes_{\mathbb{H}} Z \xrightarrow{\phi \otimes_{\mathbb{H}} \text{id}} V \otimes_{\mathbb{H}} Z \xrightarrow{\psi \otimes_{\mathbb{H}} \text{id}} W \otimes_{\mathbb{H}} Z \rightarrow 0$$

is not  $\text{AH}$ -exact at  $W \otimes_{\mathbb{H}} Z$ . This contrasts with the behaviour of exact sequences of real vector spaces under the tensor product.

The following proposition gives a clearer idea of what is happening.

**Proposition 1.3.3.** *Suppose that  $U, V, W$  and  $Z$  are finite-dimensional  $\text{AH}$ -modules, that  $\phi : U \rightarrow V$  and  $\psi : V \rightarrow W$  are  $\text{AH}$ -morphisms, and that the sequence  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is  $\text{AH}$ -exact. Then the sequence*

$$(7) \quad 0 \rightarrow W^\dagger \xrightarrow{\psi^\times} V^\dagger \xrightarrow{\phi^\times} U^\dagger \rightarrow 0$$

is exact, and the sequence

$$(8) \quad 0 \rightarrow U \otimes_{\mathbb{H}} Z \xrightarrow{\phi \otimes_{\mathbb{H}} \text{id}} V \otimes_{\mathbb{H}} Z \xrightarrow{\psi \otimes_{\mathbb{H}} \text{id}} W \otimes_{\mathbb{H}} Z \rightarrow 0$$

is  $\text{AH}$ -exact at  $U \otimes_{\mathbb{H}} Z$  and  $V \otimes_{\mathbb{H}} Z$ , but it need not be  $\text{AH}$ -exact at  $W \otimes_{\mathbb{H}} Z$ .

*Proof.* Since  $\psi : V \rightarrow W$  is surjective,  $\psi^\times : W^\times \rightarrow V^\times$  is injective, and thus  $\psi^\times : W^\dagger \rightarrow V^\dagger$  is injective. Now  $\phi$  induces a map  $\phi : U/U' \rightarrow V/V'$ . Suppose that  $u + U'$  lies in the kernel of this map. Then  $\phi(u) \in V'$ . By exactness,  $\psi(\phi(u)) = 0$ . As the sequence  $U' \rightarrow V' \rightarrow W'$  is also exact,  $\phi(u) = \phi(u')$  for some  $u' \in U'$ . But  $\phi$  is injective, so  $u \in U'$ . Thus the map  $\phi : U/U' \rightarrow V/V'$  is injective. It follows that the map  $\phi^* : (V/V')^* \rightarrow (U/U')^*$  is surjective.

Since  $U, V$  are finite-dimensional, it follows that  $U^\dagger \cong (U/U')^*$  and  $V^\dagger \cong (V/V')^*$ . Under these isomorphisms  $\phi^*$  is identified with  $\phi^\times$ . Thus  $\phi^\times : V^\dagger \rightarrow U^\dagger$  is surjective. Using exactness we see that  $\dim V = \dim U + \dim W$  and  $\dim V' = \dim U' + \dim W'$ . By subtraction we find that  $\dim V^\dagger = \dim U^\dagger + \dim W^\dagger$ . But we have already shown that  $\psi^\times : W^\dagger \rightarrow V^\dagger$  is injective, and  $\phi^\times : V^\dagger \rightarrow U^\dagger$  is surjective. Using these facts, we see that the sequence (7) is exact, as we have to prove.

Now  $\phi : U \rightarrow V$  is injective, and clearly  $\text{id} : Z \rightarrow Z$  is injective. Thus Lemma 1.1.8 shows that  $\phi \otimes_{\mathbb{H}} \text{id} : U \otimes_{\mathbb{H}} Z \rightarrow V \otimes_{\mathbb{H}} Z$  is injective. So the sequence (8) is  $\text{AH}$ -exact at  $U \otimes_{\mathbb{H}} Z$ , as we have to prove. Suppose that  $x \in \text{Ker}(\psi \otimes_{\mathbb{H}} \text{id})$ . It is easy to show, using exactness, injectivity, and the definition of  $\otimes_{\mathbb{H}}$ , that  $x \in \text{Im}(\phi \otimes_{\mathbb{H}} \text{id})$ . Thus  $\text{Ker}(\psi \otimes_{\mathbb{H}} \text{id}) = \text{Im}(\phi \otimes_{\mathbb{H}} \text{id})$ , and the sequence (8) is exact at  $V \otimes_{\mathbb{H}} Z$ , as we have to prove.

Recall from Definition 1.1.4 that as  $U, Z$  are finite-dimensional,  $\lambda_{U,Z}$  is surjective. Suppose that  $a \in (U \otimes_{\mathbb{H}} Z)^\dagger$ . Then  $a = \lambda_{U,Z}(b)$  for some  $b \in U^\dagger \otimes Z^\dagger$ . The map  $\phi^\times : V^\dagger \rightarrow U^\dagger$  is surjective from above, and thus  $b = (\phi^\times \otimes \text{id})(c)$  for some  $c \in V^\dagger \otimes Z^\dagger$ . Thus  $a = \lambda_{U,Z} \circ (\phi^\times \otimes \text{id})(c)$ . But it is easy to see that  $\lambda_{U,Z} \circ (\phi^\times \otimes \text{id}) = (\phi \otimes_{\mathbb{H}} \text{id})^\times \circ \lambda_{V,Z}$ , as maps  $V^\dagger \otimes Z^\dagger \rightarrow (U \otimes_{\mathbb{H}} Z)^\dagger$ . Thus if  $a \in (U \otimes_{\mathbb{H}} Z)^\dagger$ , then  $a = (\phi \otimes_{\mathbb{H}} \text{id})^\times(d)$ , for  $d = \lambda_{V,Z}(c) \in (V \otimes_{\mathbb{H}} Z)^\dagger$ . Therefore  $(\phi \otimes_{\mathbb{H}} \text{id})^\times : (V \otimes_{\mathbb{H}} Z)^\dagger \rightarrow (U \otimes_{\mathbb{H}} Z)^\dagger$  is surjective.

Let  $x \in (V \otimes_{\mathbb{H}} Z)'$ , and suppose  $(\psi \otimes_{\mathbb{H}} \text{id})(x) = 0$ . As (8) is exact,  $x = (\phi \otimes_{\mathbb{H}} \text{id})(y)$ , and as  $\phi \otimes_{\mathbb{H}} \text{id}$  is injective,  $y$  is unique. We shall show that  $y \in (U \otimes_{\mathbb{H}} Z)'$ . It is enough to show that for each  $\alpha \in (U \otimes_{\mathbb{H}} Z)^\dagger$ ,  $\alpha(y) \in \mathbb{I}$ . Since  $(\phi \otimes_{\mathbb{H}} \text{id})^\times : (V \otimes_{\mathbb{H}} Z)^\dagger \rightarrow (U \otimes_{\mathbb{H}} Z)^\dagger$  is surjective,  $\alpha = (\phi \otimes_{\mathbb{H}} \text{id})^\times(\beta)$  for  $\beta \in (V \otimes_{\mathbb{H}} Z)^\dagger$ . Then  $\alpha(y) = \beta(x)$ . But

$x \in (V \otimes_{\mathbb{H}} Z)'$  and  $\beta \in (V \otimes_{\mathbb{H}} Z)^\dagger$ , so  $\beta(x) \in \mathbb{I}$ . Thus  $\alpha(y) \in \mathbb{I}$ , and  $y \in (U \otimes_{\mathbb{H}} Z)'$ . It follows that the sequence  $(U \otimes_{\mathbb{H}} Z)' \xrightarrow{\phi \otimes_{\mathbb{H}} \text{id}} (V \otimes_{\mathbb{H}} Z)' \xrightarrow{\psi \otimes_{\mathbb{H}} \text{id}} (W \otimes_{\mathbb{H}} Z)'$  is exact at  $(V \otimes_{\mathbb{H}} Z)'$ . Therefore (8) is Aℍ-exact at  $V \otimes_{\mathbb{H}} Z$ , as we have to prove. The example above shows that the sequence need not be Aℍ-exact at  $W \otimes_{\mathbb{H}} Z$ , and the proposition is finished.  $\square$

In the language of category theory, Proposition 1.3.3 shows that when  $Z$  is a finite-dimensional Aℍ-module, the operation  $\otimes_{\mathbb{H}} Z$  is a *left-exact functor*, but may not be a *right-exact functor*. However, the next proposition shows that right-exactness does hold for stable and semistable Aℍ-modules.

**Proposition 1.3.4.** *Suppose that  $U, V, W$  are Aℍ-modules, that  $U$  and  $W$  are stable, that  $\phi : U \rightarrow V$  and  $\psi : V \rightarrow W$  are Aℍ-morphisms, and that the sequence  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is Aℍ-exact. Let  $Z$  be a semistable Aℍ-module. Then the following sequence is Aℍ-exact:*

$$(9) \quad 0 \rightarrow U \otimes_{\mathbb{H}} Z \xrightarrow{\phi \otimes_{\mathbb{H}} \text{id}} V \otimes_{\mathbb{H}} Z \xrightarrow{\psi \otimes_{\mathbb{H}} \text{id}} W \otimes_{\mathbb{H}} Z \rightarrow 0.$$

**Proof.** Let  $\dim U = 4j$ ,  $\dim V = 4k$ ,  $\dim W = 4l$ ,  $\dim U' = 2j + r$ ,  $\dim V' = 2k + s$ ,  $\dim W' = 2l + t$ ,  $\dim Z = 4a$  and  $\dim Z' = 2a + b$ . Then by Theorem 1.2.6 we have

$$(10) \quad \begin{aligned} \dim(U \otimes_{\mathbb{H}} Z) &= 4(jb + ra - rb), & \dim(W \otimes_{\mathbb{H}} Z) &= 4(lb + ta - tb), \\ \dim(U \otimes_{\mathbb{H}} Z)' &= 2jb + 2ra - rb, & \dim(W \otimes_{\mathbb{H}} Z)' &= 2lb + 2ta - tb. \end{aligned}$$

Theorem 1.2.6 calculates the dimensions of a quaternionic tensor product of stable and semistable Aℍ-modules. Examining the proof, it is easy to see that these dimensions are actually *lower bounds* for the dimensions of a quaternionic tensor product of general Aℍ-modules. Therefore

$$(11) \quad \dim(V \otimes_{\mathbb{H}} Z) \geq 4(kb + sa - sb) \text{ and } \dim(V \otimes_{\mathbb{H}} Z)' \geq 2kb + 2sa - sb.$$

By Proposition 1.3.3, the sequence (9) is Aℍ-exact at  $U \otimes_{\mathbb{H}} Z$  and  $V \otimes_{\mathbb{H}} Z$ . The only way Aℍ-exactness at  $W \otimes_{\mathbb{H}} Z$  can fail is for  $\psi \otimes_{\mathbb{H}} \text{id} : V \otimes_{\mathbb{H}} Z \rightarrow W \otimes_{\mathbb{H}} Z$  or  $\psi \otimes_{\mathbb{H}} \text{id} : (V \otimes_{\mathbb{H}} Z)' \rightarrow (W \otimes_{\mathbb{H}} Z)'$  not to be surjective. We deduce that

$$(12) \quad \begin{aligned} \dim(U \otimes_{\mathbb{H}} Z) + \dim(W \otimes_{\mathbb{H}} Z) &\geq \dim(V \otimes_{\mathbb{H}} Z), \\ \dim(U \otimes_{\mathbb{H}} Z)' + \dim(W \otimes_{\mathbb{H}} Z)' &\geq \dim(V \otimes_{\mathbb{H}} Z)'. \end{aligned}$$

But  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is Aℍ-exact, so that  $k = j + l$  and  $s = r + t$ . Combining (10), (11) and (12), we see that equality holds in (11) and (12), because the inequalities go opposite ways. Counting dimensions,  $\psi \otimes_{\mathbb{H}} \text{id} : V \otimes_{\mathbb{H}} Z \rightarrow W \otimes_{\mathbb{H}} Z$  and  $\psi \otimes_{\mathbb{H}} \text{id} : (V \otimes_{\mathbb{H}} Z)' \rightarrow (W \otimes_{\mathbb{H}} Z)'$  must be surjective. Thus by definition, (9) is Aℍ-exact at  $W \otimes_{\mathbb{H}} Z$ , and the proposition is proved.  $\square$

The proposition gives one reason why it is convenient, in many situations, to work with stable Aℍ-modules rather than general Aℍ-modules.

**Proposition 1.3.5.** *Suppose  $U, V, W$  are  $\mathbb{A}\mathbb{H}$ -modules,  $\phi : U \rightarrow V$  and  $\psi : V \rightarrow W$  are  $\mathbb{A}\mathbb{H}$ -morphisms, and that the sequence  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is  $\mathbb{A}\mathbb{H}$ -exact. If  $U$  and  $W$  are stable  $\mathbb{A}\mathbb{H}$ -modules, then  $V$  is a stable  $\mathbb{A}\mathbb{H}$ -module.*

*Proof.* Let  $q \in \mathbb{I}$  be nonzero, and apply Proposition 1.3.4 with  $Z = X_q$ , the semistable  $\mathbb{A}\mathbb{H}$ -module defined in §1.2. Let  $\dim V = 4k$  and  $\dim V' = 2k + s$  as in the proof of the proposition. Since equality holds in (11),  $\dim(V \otimes_{\mathbb{H}} X_q) = 4s$ , so that  $V \otimes_{\mathbb{H}} X_q \cong sX_q$ . But this is the main condition for  $V$  to be stable. Thus, it remains only to show that  $V$  is semistable.

Let  $S$  be the maximal semistable  $\mathbb{A}\mathbb{H}$ -submodule of  $V$ . As  $U$  is semistable,  $\phi(U) \subset S$ . Also, from above the map  $\psi \otimes_{\mathbb{H}} \text{id} : V \otimes_{\mathbb{H}} X_q \rightarrow W \otimes_{\mathbb{H}} X_q$  is surjective. As  $W$  is semistable, we deduce that  $\psi(S) = W$ . But  $0 \rightarrow U \xrightarrow{\phi} V \xrightarrow{\psi} W \rightarrow 0$  is  $\mathbb{A}\mathbb{H}$ -exact, so  $\phi(U) \subset S$  and  $\psi(S) = W$  imply that  $S = V$ . Therefore  $V$  is semistable, so  $V$  is stable.  $\square$

## 2. ALGEBRAIC STRUCTURES OVER THE QUATERNIONS

In this chapter, the machinery of Chapter 1 will be used to define quaternionic analogues of various algebraic concepts. We shall only discuss those structures we shall need for our study of hypercomplex geometry, but the reader will soon see how the process works in general. First, in §2.1 we define  $\mathbb{H}$ -algebras, the quaternionic version of commutative algebras, and modules over  $\mathbb{H}$ -algebras. Then in §2.2 we define  $\mathbb{H}\mathbb{L}$ -algebras and  $\mathbb{H}\mathbb{P}$ -algebras, the analogues of Lie algebras and Poisson algebras. Section 2.3 is about filtered and graded  $\mathbb{H}$ -algebras, and §2.4 considers free and finitely-generated  $\mathbb{H}$ -algebras.

**2.1.  $\mathbb{H}$ -algebras and modules.** Now we will define the quaternionic version of a commutative algebra, which we shall call an  $\mathbb{H}$ -algebra, and also modules over  $\mathbb{H}$ -algebras. In Chapter 3 we shall see that the  $q$ -holomorphic functions on a hypercomplex manifold form an  $\mathbb{H}$ -algebra. Here are two axioms.

**Axiom A.** (i)  $A$  is an  $\mathbb{A}\mathbb{H}$ -module.

(ii) There is an  $\mathbb{A}\mathbb{H}$ -morphism  $\mu_A : A \otimes_{\mathbb{H}} A \rightarrow A$ , called the *multiplication map*.

(iii)  $\Lambda_{\mathbb{H}}^2 A \subset \text{Ker } \mu_A$ . Thus  $\mu_A$  is *commutative*.

(iv) The  $\mathbb{A}\mathbb{H}$ -morphisms  $\mu_A : A \otimes_{\mathbb{H}} A \rightarrow A$  and  $\text{id} : A \rightarrow A$  combine to give  $\mathbb{A}\mathbb{H}$ -morphisms  $\mu_A \otimes_{\mathbb{H}} \text{id}$  and  $\text{id} \otimes_{\mathbb{H}} \mu_A : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A$ . Composing with  $\mu_A$  gives  $\mathbb{A}\mathbb{H}$ -morphisms  $\mu_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id})$  and  $\mu_A \circ (\text{id} \otimes_{\mathbb{H}} \mu_A) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A$ . Then  $\mu_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id}) = \mu_A \circ (\text{id} \otimes_{\mathbb{H}} \mu_A)$ . This is *associativity of multiplication*.

(v) An element  $1 \in A$  called the *identity* is given, with  $1 \notin A'$  and  $\mathbb{I} \cdot 1 \subset A'$ .

(vi) Part (v) implies that if  $\alpha \in A^\dagger$  then  $\alpha(1) \in \mathbb{R}$ . Thus for each  $a \in A$ ,  $1 \otimes_{\mathbb{H}} a$  and  $a \otimes_{\mathbb{H}} 1 \in A \otimes_{\mathbb{H}} A$  by Lemma 1.1.9. Then  $\mu_A(1 \otimes_{\mathbb{H}} a) = \mu_A(a \otimes_{\mathbb{H}} 1) = a$  for each  $a \in A$ . Thus  $1$  is a *multiplicative identity*.

**Axiom M.** (i) Let  $U$  be an  $\mathbb{A}\mathbb{H}$ -module.

- (ii) There is an  $\mathbb{A}\mathbb{H}$ -morphism  $\mu_U : A \otimes_{\mathbb{H}} U \rightarrow U$  called the *module multiplication map*.
- (iii) The maps  $\mu_A$  and  $\mu_U$  define  $\mathbb{A}\mathbb{H}$ -morphisms  $\mu_A \otimes_{\mathbb{H}} \text{id}$  and  $\text{id} \otimes_{\mathbb{H}} \mu_U : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} U \rightarrow A \otimes_{\mathbb{H}} U$ . Composing with  $\mu_U$  gives  $\mathbb{A}\mathbb{H}$ -morphisms  $\mu_U \circ (\mu_A \otimes_{\mathbb{H}} \text{id})$  and  $\mu_U \circ (\text{id} \otimes_{\mathbb{H}} \mu_U) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} U \rightarrow U$ . Then  $\mu_U \circ (\mu_A \otimes_{\mathbb{H}} \text{id}) = \mu_U \circ (\text{id} \otimes_{\mathbb{H}} \mu_U)$ . This is *associativity of module multiplication*.
- (iv) For  $u \in U$ ,  $1 \otimes_{\mathbb{H}} u \in A \otimes_{\mathbb{H}} U$  by Lemma 1.1.9. Then  $\mu_U(1 \otimes_{\mathbb{H}} u) = u$  for all  $u \in U$ . Thus 1 acts as an identity on  $U$ .

Now we can define H-algebras and modules over them. Here H-algebra stands for *Hamilton algebra* (but my wife calls them Happy algebras).

### Definition 2.1.1

- An *H-algebra* satisfies Axiom A.
- A *noncommutative H-algebra* satisfies Axiom A, except part (iii).
- Let  $A$  be an H-algebra. A *module  $U$  over  $A$*  satisfies Axiom M.

An H-algebra is basically a commutative algebra over the skew field  $\mathbb{H}$ . This is a strange idea: how can the algebra commute when the field does not? The obvious answer is that the algebra is only a partial algebra, and multiplication is only allowed when the elements commute. I'm not sure if this is the full story, though. In this paper our principal interest is in commutative H-algebras, but noncommutative H-algebras also exist.

The associative axiom A(iv) gives a good example of the issues involved in finding quaternionic analogues of algebraic structures. The usual formulation is that  $(ab)c = a(bc)$  for all  $a, b, c \in A$ . This is not suitable for the quaternionic case, as not all elements in  $A$  can be multiplied, so we rewrite the axiom in terms of linear maps of tensor products, and the quaternionic analogue becomes clear. Finally we define morphisms of H-algebras.

**Definition 2.1.2** Let  $A, B$  be H-algebras, and let  $\phi : A \rightarrow B$  be an  $\mathbb{A}\mathbb{H}$ -morphism. Write  $1_A, 1_B$  for the identities in  $A, B$  respectively. We say  $\phi$  is a *morphism of H-algebras* or an *H-algebra morphism* if  $\phi(1_A) = 1_B$  and  $\mu_B \circ (\phi \otimes_{\mathbb{H}} \phi) = \phi \circ \mu_A$  as  $\mathbb{A}\mathbb{H}$ -morphisms  $A \otimes_{\mathbb{H}} A \rightarrow B$ .

**2.2. H-algebras and Poisson brackets.** Recall the idea of a *Lie bracket* on a vector space. A real algebra may be equipped with a Lie bracket satisfying certain conditions, and in this case the Lie bracket is called a *Poisson bracket*, and the algebra is called a *Poisson algebra*. Poisson algebras are studied in [7]. In this section we define one possible analogue of these concepts in our theory of quaternionic algebra. The analogue of a Poisson algebra will be called an *HP-algebra*. We begin by defining a special  $\mathbb{A}\mathbb{H}$ -module.

**Definition 2.2.1** Define  $Y \subset \mathbb{H}^3$  by  $Y = \{(q_1, q_2, q_3) : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\}$ . Then  $Y \cong \mathbb{H}^2$  is an  $\mathbb{H}$ -module. Define  $Y' \subset Y$  by  $Y' = \{(q_1, q_2, q_3) \in Y : q_j \in \mathbb{I}\}$ . Then

$\dim Y' = 5$  and  $\dim Y^\dagger = 3$ . Thus  $\dim Y = 4j$  and  $\dim Y' = 2j + r$ , where  $j = 2$  and  $r = 1$ . Define a map  $\nu : Y \rightarrow \mathbb{H}$  by  $\nu((q_1, q_2, q_3)) = i_1 q_1 + i_2 q_2 + i_3 q_3$ . Then  $\text{Im } \nu = \mathbb{I}$ , and  $\text{Ker } \nu = Y'$ . But  $Y/Y' \cong (Y^\dagger)^*$ , so that  $\nu$  induces an isomorphism  $\nu : (Y^\dagger)^* \rightarrow \mathbb{I}$ . Since  $\mathbb{I} \cong \mathbb{I}^*$ , we have  $(Y^\dagger)^* \cong \mathbb{I} \cong Y^\dagger$ . It is easy to see that  $Y$  is a stable AH-module. Thus  $Y \otimes_{\mathbb{H}} Y$  satisfies  $\dim(Y \otimes_{\mathbb{H}} Y) = 12$  and  $\dim(Y \otimes_{\mathbb{H}} Y)' = 7$ , by Theorem 1.2.6. Proposition 1.2.8 then shows that  $S_{\mathbb{H}}^2 Y = Y \otimes_{\mathbb{H}} Y$  and  $\Lambda_{\mathbb{H}}^2 Y = \{0\}$ .

Let  $A$  be an AH-module. Here is the axiom for a Lie bracket on  $A$ .

**Axiom P1.** (i) There is an AH-morphism  $\xi_A : A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$  called the *Lie bracket* or *Poisson bracket*, where  $Y$  is the AH-module of Definition 2.2.1.

(ii)  $S_{\mathbb{H}}^2 A \subset \text{Ker } \xi_A$ . Thus  $\xi_A$  is *antisymmetric*.

(iii) There are AH-morphisms  $\text{id} \otimes_{\mathbb{H}} \xi_A : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y$  and  $\xi_A \otimes_{\mathbb{H}} \text{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y \rightarrow A \otimes_{\mathbb{H}} Y \otimes_{\mathbb{H}} Y$ . Composing gives an AH-morphism  $(\xi_A \otimes_{\mathbb{H}} \text{id}) \circ (\text{id} \otimes_{\mathbb{H}} \xi_A) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y \otimes_{\mathbb{H}} Y$ . Then  $\Lambda_{\mathbb{H}}^3 A \subset \text{Ker}((\xi_A \otimes_{\mathbb{H}} \text{id}) \circ (\text{id} \otimes_{\mathbb{H}} \xi_A))$ . This is *the Jacobi identity* for  $\xi_A$ .

For the next axiom, let  $A$  be an H-algebra.

**Axiom P2.** (i) If  $a \in A$ , we have  $1 \otimes_{\mathbb{H}} a \in A \otimes_{\mathbb{H}} A$ . Then  $\xi_A(1 \otimes_{\mathbb{H}} a) = 0$ .

(ii) There are AH-morphisms  $\text{id} \otimes_{\mathbb{H}} \xi_A : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y$  and  $\mu_A \otimes_{\mathbb{H}} \text{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y \rightarrow A \otimes_{\mathbb{H}} Y$ . Composing gives an AH-morphism  $(\mu_A \otimes_{\mathbb{H}} \text{id}) \circ (\text{id} \otimes_{\mathbb{H}} \xi_A) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$ . Similarly, there are AH-morphisms  $\mu_A \otimes_{\mathbb{H}} \text{id} : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A$  and  $\xi_A : A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$ . Composing gives an AH-morphism  $\xi_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id}) : A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} Y$ . Then  $\xi_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id}) = 2(\mu_A \otimes_{\mathbb{H}} \text{id}) \circ (\text{id} \otimes_{\mathbb{H}} \xi_A)$  on  $S_{\mathbb{H}}^2 A \otimes_{\mathbb{H}} A$ . This is *the derivation property*.

Now we can define HL-algebras and HP-algebras. Here HL-algebra stands for *Hamilton-Lie algebra*, and HP-algebra stands for *Hamilton-Poisson algebra* (but my wife calls these Happy Fish algebras).

**Definition 2.2.2**

- an *HL-algebra* is an AH-module  $A$  satisfying Axiom P1.
- an *HP-algebra* satisfies Axioms A, P1 and P2.

Here is a little motivation for the definitions above. If  $M$  is a symplectic manifold, then the algebra of smooth functions on  $M$  acquires a Poisson bracket. Since a hyperkähler manifold  $M$  has 3 symplectic structures, the algebra of smooth functions on  $M$  has 3 Poisson brackets, and these interact with the H-algebra  $A$  of q-holomorphic functions on  $M$ , that will be defined in Chapter 3.

Our definition of HP-algebra is an attempt to capture the essential algebraic properties of this interaction. We may regard  $A \otimes_{\mathbb{H}} Y$  as a subspace of  $A \otimes (Y^\dagger)^*$ , and  $(Y^\dagger)^* \cong \mathbb{I}$  by Definition 2.2.1. Thus  $\xi_A$  is an antisymmetric map from  $A \otimes_{\mathbb{H}} A$  to  $A \otimes \mathbb{I}$ , i.e. a triple of antisymmetric maps from  $A \otimes_{\mathbb{H}} A$  to  $A$ . These 3 antisymmetric maps should be interpreted as the 3 Poisson brackets on the hyperkähler manifold.

**2.3. Filtered and graded H-algebras.** We begin by defining filtered and graded Aℍ-modules.

**Definition 2.3.1** Let  $U$  be an Aℍ-module. A *filtration* of  $U$  is a sequence  $U_0, U_1, \dots$  of Aℍ-submodules of  $U$ , such that  $U_j \subset U_k$  whenever  $j \leq k$ , and  $U = \bigcup_{k=0}^{\infty} U_k$ . We call  $U$  a *filtered Aℍ-module* if it has a filtration  $U_0, U_1, \dots$ .

Let  $U$  be an Aℍ-module. A *grading* of  $U$  is a sequence  $U^0, U^1, \dots$  of Aℍ-submodules of  $U$ , such that  $U = \bigoplus_{k=0}^{\infty} U^k$ . We call  $U$  a *graded Aℍ-module* if it has a grading  $U^0, U^1, \dots$ . If  $U$  is a graded Aℍ-module, define  $U_k = \bigoplus_{j=0}^k U^j$ . Then  $U_0, U_1, \dots$  is a filtration of  $U$ , so every graded Aℍ-module is also a filtered Aℍ-module.

Let  $U, V$  be filtered Aℍ-modules, and  $\phi : U \rightarrow V$  be an Aℍ-morphism. We say that  $\phi$  is a *filtered Aℍ-morphism* if  $\phi(U_k) \subset V_k$  for each  $k \geq 0$ . Graded Aℍ-morphisms are also defined in the obvious way.

Here are axioms for filtered and graded H- and HP-algebras.

**Axiom AF.** (i)  $A$  is a filtered Aℍ-module.

(ii)  $\mathbb{H} \cdot 1 \subset A_0$ .

(iii) For each  $j, k$ ,  $\mu_A(A_j \otimes_{\mathbb{H}} A_k) \subset A_{j+k}$ .

**Axiom AG.** (i)  $A$  is a graded Aℍ-module.

(ii)  $\mathbb{H} \cdot 1 \subset A^0$ .

(iii) For each  $j, k$ ,  $\mu_A(A^j \otimes_{\mathbb{H}} A^k) \subset A^{j+k}$ .

**Axiom PF.** For each  $j, k$ ,  $\xi_A(A_j \otimes_{\mathbb{H}} A_k) \subset A_{j+k-1} \otimes_{\mathbb{H}} Y$ .

**Axiom PG.** For each  $j, k$ ,  $\xi_A(A^j \otimes_{\mathbb{H}} A^k) \subset A^{j+k-1} \otimes_{\mathbb{H}} Y$ .

**Definition 2.3.2**

- A *filtered H-algebra* satisfies Axioms A and AF.
- A *graded H-algebra* satisfies Axioms A and AG.
- a *filtered HP-algebra* satisfies Axioms A, AF, P1, P2, and PF.
- a *graded HP-algebra* satisfies Axioms A, AG, P1, P2, and PG.

Morphisms of filtered and graded H-algebras are defined in the obvious way, following Definition 2.1.2. The choice of the grading  $j+k-1$  in Axioms PF and PG is not always appropriate, but depends on the situation. For some of our applications, the grading  $j+k-2$  is better. Now let  $U$  be an Aℍ-module, and  $V$  an Aℍ-submodule of  $U$ . Then  $U/V$  is naturally an ℍ-module. As  $V' = U' \cap V$ , we may interpret  $U'/V'$  as a real vector subspace of  $U/V$ . Put  $(U/V)' = U'/V'$ . Then  $U/V$  is an ℍ-module with a real vector subspace  $(U/V)'$ . Note that  $U/V$  may or may not be an Aℍ-module, because it may not satisfy the condition of Definition 1.1.1.

**Definition 2.3.3** Let  $A$  be a filtered H-algebra. Define  $A_k = \{0\}$  for  $k < 0$  in  $\mathbb{Z}$ . We say that  $A$  is a *stable filtered H-algebra*, or *SFH-algebra*, if for each  $k \geq 0$ ,  $A_k/A_{k-1}$  is a stable Aℍ-module. Let  $B$  be a graded H-algebra. We say  $B$  is a *stable graded H-algebra*, or *SGH-algebra*, if  $B^k$  is stable for each  $k \geq 0$ .



**Lemma 2.3.4.** *Let  $A$  be an SFH-algebra. Then for each  $j, k \geq 0$ ,  $A_k$  and  $A_j/A_{j-k}$  are stable AHH-modules. Let  $j, k \geq 0$  and  $l > 0$  be integers. Then the multiplication map  $\mu_A : A_j \otimes_{\mathbb{H}} A_k \rightarrow A_{j+k}$  induces a natural AHH-morphism*

$$\mu_{jkl}^A : (A_j/A_{j-l}) \otimes_{\mathbb{H}} (A_k/A_{k-l}) \rightarrow A_{j+k}/A_{j+k-l}.$$

*Proof.* We shall prove that  $A_k$  is stable, by induction on  $k$ . Firstly,  $A_0 = A_0/A_{-1}$  is stable, by definition. Suppose by induction that  $A_{k-1}$  is stable. The sequence  $0 \rightarrow A_{k-1} \rightarrow A_k \xrightarrow{\pi_k} A_k/A_{k-1} \rightarrow 0$  is AHH-exact, and  $A_{k-1}$  and  $A_k/A_{k-1}$  are stable. Therefore, Proposition 1.3.5 shows that  $A_k$  is stable, so all  $A_k$  are stable, by induction. By a similar argument involving induction on  $k$ ,  $A_j/A_{j-k}$  is stable.

Now let  $j, k, l$  be as given, and let  $\pi_m : A_m \rightarrow A_m/A_{m-l}$  be the natural projection, for  $m \geq 0$ . Because  $0 \rightarrow A_{j-l} \rightarrow A_j \xrightarrow{\pi_j} A_j/A_{j-l} \rightarrow 0$  and  $0 \rightarrow A_{k-l} \rightarrow A_k \xrightarrow{\pi_k} A_k/A_{k-l} \rightarrow 0$  are AHH-exact sequences of stable AHH-modules, two applications of Proposition 1.3.4 show that the sequence

$$(13) \quad A_j \otimes_{\mathbb{H}} A_k \xrightarrow{\pi_j \otimes_{\mathbb{H}} \pi_k} (A_j/A_{j-l}) \otimes_{\mathbb{H}} (A_k/A_{k-l}) \rightarrow 0$$

is AHH-exact at the middle term. Now  $\mu_A$  maps  $A_j \otimes_{\mathbb{H}} A_k$  to  $A_{j+k}$ . By identifying the kernels of  $\pi_j \otimes_{\mathbb{H}} \pi_k$  and  $\pi_{j+k} \circ \mu_A$ , it can be seen that there exists a linear map  $\mu_{jkl}^A$  as in the lemma, such that  $\mu_{jkl}^A \circ (\pi_j \otimes_{\mathbb{H}} \pi_k) = \pi_{j+k} \circ \mu_A$  as maps  $A_j \otimes_{\mathbb{H}} A_k \rightarrow A_{j+k}/A_{j+k-l}$ . Using the AHH-exactness of (13) and that  $\pi_{j+k} \circ \mu_A$  is an AHH-morphism, we deduce that  $\mu_{jkl}^A$  is unique and is an AHH-morphism.  $\square$

Using this lemma, we make a definition.

**Definition 2.3.5** Let  $A, B$  be SFH-algebras and let  $l > 0$  be an integer. By Lemma 2.3.4,  $A_j/A_{j-l}$  and  $B_j/B_{j-l}$  are stable AHH-modules for each  $j \geq 0$ . We say that  $A$  and  $B$  are isomorphic to order  $l$  if the following holds. For  $j \geq 0$  there are AHH-isomorphisms  $\phi_j : A_j/A_{j-l} \rightarrow B_j/B_{j-l}$ . These satisfy  $\phi_j(A_k/A_{j-l}) \subset B_k/B_{j-l}$  when  $j-l \leq k \leq j$ . Therefore  $\phi_j$  projects to a map  $A_j/A_{j-l+1} \rightarrow B_j/B_{j-l+1}$ , and  $\phi_{j+1}$  restricts to a map  $A_j/A_{j-l+1} \rightarrow B_j/B_{j-l+1}$ . Then  $\phi_j = \phi_{j+1}$  on  $A_j/A_{j-l+1}$ . Also, for all  $j, k \geq 0$ ,  $\mu_{jkl}^B \circ (\phi_j \otimes_{\mathbb{H}} \phi_k) = \phi_{j+k} \circ \mu_{jkl}^A$  as AHH-morphisms from  $(A_j/A_{j-l}) \otimes_{\mathbb{H}} (A_k/A_{k-l})$  to  $B_{j+k}/B_{j+k-l}$ . Here  $\mu_{jkl}^A$  and  $\mu_{jkl}^B$  are defined in Lemma 2.3.4.

There is a well-known way to construct a graded algebra from a filtered algebra (for instance [7, p. 35-37] gives the associated graded Poisson algebra of a filtered Poisson algebra). Here is the analogue of this for H-algebras, which will be applied in §3.5.

**Proposition 2.3.6.** *Let  $A$  be an SFH-algebra. Define  $B^k = A_k/A_{k-1}$  for  $k \geq 0$ . Define  $B = \bigoplus_{k=0}^{\infty} B^k$ . Then  $B$  has the structure of an SGH-algebra, in a natural way.*

*Proof.* As  $A$  is an SFH-algebra,  $B^k$  is a stable AHH-module, by definition. Let  $j, k \geq 0$  be integers. Putting  $l = 1$ , Lemma 2.3.4 defines an AHH-morphism  $\mu_{jk1}^A : B^j \otimes_{\mathbb{H}} B^k \rightarrow$

$B^{j+k}$ . Let  $\mu_B : B \otimes_{\mathbb{H}} B \rightarrow B$  be the unique AHH-morphism, such that the restriction of  $\mu_B$  to  $B^j \otimes_{\mathbb{H}} B^k$  is  $\mu_{j+k}^A$ . It is elementary to show that because  $A$  is an H-algebra,  $\mu_B$  makes  $B$  into an H-algebra, and we leave this to the reader. As  $B$  satisfies Axiom AG,  $B$  is graded, so  $B$  is an SGH-algebra, and the proposition is complete.  $\square$

We call the SGH-algebra  $B$  defined in Proposition 2.3.6 the *associated graded H-algebra of  $A$* . Note that  $A$  is isomorphic to  $B$  to order 1, in the sense of Lemma 2.3.5. One might ask if the construction would work even if  $A$  were only a filtered H-algebra. There are two problems here: firstly,  $B^k$  might not be an AHH-module, and secondly, even if  $B^j$  and  $B^k$  were AHH-modules, the map  $\pi_j \otimes_{\mathbb{H}} \pi_k : A_j \otimes_{\mathbb{H}} A_k \rightarrow B^j \otimes_{\mathbb{H}} B^k$  might not be surjective. If it were not, we could only define  $\mu_{j+k}^A$  uniquely on part of  $B^j \otimes_{\mathbb{H}} B^k$ . For these reasons we prefer SFH-algebras.

**2.4. Free and finitely-generated H-algebras.** First we define ideals in SFH-algebras.

**Definition 2.4.1** Let  $A$  be an SFH-algebra, and  $I$  an AHH-submodule of  $A$ . Set  $I_k = I \cap A_k$  for  $k \geq 0$ . We say that  $I$  is a *stable filtered ideal in  $A$*  if  $1 \notin I$ ,  $\mu_A(I \otimes_{\mathbb{H}} A) \subset I$ , and for each  $k \geq 0$ ,  $I_k$  and  $A_k/I_k$  are stable AHH-modules. Suppose that  $J$  is an AHH-submodule of  $A$ , and  $I = \mu_A(J \otimes_{\mathbb{H}} A)$ . Then we say that  $I$  is *generated by  $J$* .

The proof of the next lemma is similar to that of Proposition 2.3.6, so we omit it.

**Lemma 2.4.2.** *Let  $A$  be an SFH-algebra, and  $I$  a stable filtered ideal in  $A$ . Then there exists a unique SFH-algebra  $B$ , with a filtered H-algebra morphism  $\pi : A \rightarrow B$ , such that  $0 \rightarrow I_k \xrightarrow{\iota} A_k \xrightarrow{\pi} B_k \rightarrow 0$  is an AHH-exact sequence for each  $k \geq 0$ . Here  $\iota : I \rightarrow A$  is the inclusion map.*

We shall call this new H-algebra  $B$  the *quotient of  $A$  by  $I$* . Next, here is the definition of a free H-algebra.

**Definition 2.4.3** Let  $Q$  be an AHH-module. Define the *free H-algebra  $F^Q$  generated by  $Q$*  as follows. Put  $F^Q = \bigoplus_{k=0}^{\infty} S_{\mathbb{H}}^k Q$ . Then  $F^Q$  is an AHH-module. Now  $(S_{\mathbb{H}}^k Q) \otimes_{\mathbb{H}} (S_{\mathbb{H}}^l Q) \subset \bigotimes_{\mathbb{H}}^{k+l} Q$ , and from Definition 1.1.6 there is an AHH-module projection  $\sigma_{\mathbb{H}} : \bigotimes_{\mathbb{H}}^{k+l} Q \rightarrow S_{\mathbb{H}}^{k+l} Q$ . Define  $\mu_{k,l} : (S_{\mathbb{H}}^k Q) \otimes_{\mathbb{H}} (S_{\mathbb{H}}^l Q) \rightarrow S_{\mathbb{H}}^{k+l} Q$  to be the restriction of  $\sigma_{\mathbb{H}}$  to  $(S_{\mathbb{H}}^k Q) \otimes_{\mathbb{H}} (S_{\mathbb{H}}^l Q)$ .

Define  $\mu_{F^Q} : F^Q \otimes_{\mathbb{H}} F^Q \rightarrow F^Q$  to be the unique linear map such that the restriction of  $\mu_{F^Q}$  to  $(S_{\mathbb{H}}^k Q) \otimes_{\mathbb{H}} (S_{\mathbb{H}}^l Q)$  is  $\mu_{k,l}$ . Recall that  $S_{\mathbb{H}}^0 Q = \mathbb{H}$ , and define  $1 \in F^Q$  to be  $1 \in \mathbb{H} = S_{\mathbb{H}}^0 Q$ . It is easy to show that with these definitions,  $F^Q$  is an H-algebra. The *natural grading on  $F^Q$*  is  $(F^Q)^k = S_{\mathbb{H}}^k Q$  for  $k \geq 0$ . The *natural filtration on  $F^Q$*  is  $F_k^Q = \bigoplus_{j=0}^k (F^Q)^j$  for  $k \geq 0$ .

If  $Q$  is stable, then  $S_{\mathbb{H}}^k Q$  is stable by Proposition 1.2.8. Thus, if  $Q$  is stable then  $F^Q$  is an SFH-algebra (SGH-algebra) with the natural filtration (grading). The proof of the next lemma is trivial, and we omit it.

**Lemma 2.4.4.** *Let  $A$  be an H-algebra, and let  $Q \subset A$  be an AH-submodule. Let  $\iota_Q : Q \rightarrow F^Q$  be the inclusion map. Then there is a unique H-algebra morphism  $\phi_Q : F^Q \rightarrow A$  such that  $\phi_Q \circ \iota_Q : Q \rightarrow A$  is the identity on  $Q$ .*

**Definition 2.4.5** A free SFH-algebra is an SFH-algebra  $A$  that is isomorphic, as an H-algebra, to some  $F^Q$ , for finite-dimensional  $Q$ . Note that the filtration on  $A$  need not be the natural filtration on  $F^Q$ . A finitely-generated SFH-algebra, or FGH-algebra is the quotient  $B$  of a free SFH-algebra  $A$  by a stable filtered ideal  $I$  in  $A$ . Suppose that  $Q$  is an AH-submodule of  $B$ , and that there exists an H-algebra isomorphism  $A \cong F^Q$  identifying the maps  $\pi : A \rightarrow B$  and  $\phi_Q : F^Q \rightarrow B$ . Then we say that  $Q$  generates the FGH-algebra  $B$ .

The purpose of this definition is as follows. The polynomials on an affine algebraic variety form a finitely-generated, filtered algebra, and in algebraic geometry one studies this algebra to learn about the variety. In the opinion of the author, FGH-algebras are the best quaternionic analogue of finitely-generated, filtered algebras. Clearly, they are finitely-generated, filtered H-algebras, and the extra stability conditions we impose enable us to exploit the ‘right-exactness’ results of §1.3.

Moreover, the author believes that there is a wide class of noncompact hypercomplex manifolds, to which one can naturally associate an FGH-algebra. The study of FGH-algebras should be interpreted as the ‘quaternionic algebraic geometry’ of these hypercomplex manifolds. Therefore, the author proposes that the study of FGH-algebras, from the algebraic point of view, may be interesting and worthwhile. More will be said on these ideas in Chapter 4.

We leave the proof of this final result as an exercise. It will be useful later.

**Proposition 2.4.6.** *Let  $A$  be an SFH-algebra, and  $B$  the associated graded algebra, as in §2.3. Then  $B$  is also an SFH-algebra. Suppose that  $B$  is an FGH-algebra, generated by  $B_k$ . Then  $A$  is an FGH-algebra, generated by  $A_k$ .*

### 3. HYPERCOMPLEX GEOMETRY

We begin in §3.1 by defining hypercomplex manifolds, hyperkähler manifolds and q-holomorphic functions on hypercomplex manifolds, and some elementary properties of q-holomorphic functions are given. Section 3.2 proves that the vector space  $A$  of q-holomorphic functions on a hypercomplex manifold  $M$  forms an H-algebra, and §3.3 shows that if  $M$  is hyperkähler, then  $A$  is an HP-algebra. Section 3.4 discusses the possibility of reconstructing a hypercomplex manifold from an H-algebra of q-holomorphic functions upon it. Finally, §3.5 discusses hyperkähler manifolds that are asymptotic to a conical metric.

**3.1. Q-holomorphic functions on hypercomplex manifolds.** We begin by defining hypercomplex manifolds ([26, p. 137-139]) and hyperkähler manifolds ([26, p. 114-123]). Let  $M$  be a manifold of dimension  $4n$ . A hypercomplex structure on  $M$  is a

triple  $(I_1, I_2, I_3)$  on  $M$ , where  $I_j$  is a complex structure on  $M$ , and  $I_1 I_2 = I_3$ . A *hyperkähler structure* on  $M$  is a quadruple  $(g, I_1, I_2, I_3)$ , where  $g$  is a Riemannian metric on  $M$ ,  $(I_1, I_2, I_3)$  is a hypercomplex structure on  $M$ , and  $g$  is Kähler w.r.t. each  $I_j$ . If  $M$  has a hypercomplex (hyperkähler) structure, then  $M$  is called a *hypercomplex (hyperkähler) manifold*.

If  $M$  is a hypercomplex manifold, then  $I_1, I_2, I_3$  satisfy the quaternion relations, so that each tangent space  $T_m M$  is an  $\mathbb{H}$ -module isomorphic to  $\mathbb{H}^n$ . Also, if  $r_1, r_2, r_3 \in \mathbb{R}$  with  $r_1^2 + r_2^2 + r_3^2 = 1$ , then  $r_1 I_1 + r_2 I_2 + r_3 I_3$  is a complex structure. Thus a hypercomplex manifold possesses a 2-dimensional family of integrable complex structures, parametrized by  $\mathcal{S}^2$ . We will often use  $\mathcal{S}^2$  to denote this family of complex structures.

Let  $M$  be a hypercomplex manifold. For  $k \geq 0$ , define  $\Omega^k = C^\infty(\Lambda^k T^* M)$ , and  $\Omega^k(\mathbb{H}) = C^\infty(\mathbb{H} \otimes \Lambda^k T^* M)$ . Then  $\Omega^1$  is the vector space of smooth 1-forms on  $M$ , and  $\Omega^0(\mathbb{H})$  is the vector space of smooth, quaternion-valued functions on  $M$ . Define an operator  $D : \Omega^0(\mathbb{H}) \rightarrow \Omega^1$  by

$$(14) \quad D(a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3) = da_0 + I_1(da_1) + I_2(da_2) + I_3(da_3),$$

where  $a_0, \dots, a_3$  are smooth real functions on  $M$ .

We define a *q-holomorphic function on  $M$*  to be an element  $a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3$  of  $\Omega^0(\mathbb{H})$  for which  $D(a) = 0$ . The term q-holomorphic is short for *quaternion-holomorphic*, and it is intended to indicate that a q-holomorphic function on a hypercomplex manifold is the appropriate quaternionic analogue of a holomorphic function on a complex manifold. The operator  $D$  of (14) should be thought of as the quaternionic analogue of the  $\bar{\partial}$  operator on a complex manifold. It can be seen that  $D(a) = 0$  is equivalent to the equation

$$(15) \quad da - I_1(da)i_1 - I_2(da)i_2 - I_3(da)i_3 = 0,$$

where each term is an  $\mathbb{H}$ -valued 1-form,  $I_j$  acts on 1-forms and  $i_j$  acts on  $\mathbb{H}$  by multiplication.

Now in 1935, Fueter defined a class of ‘regular’  $\mathbb{H}$ -valued functions on  $\mathbb{H}$ , using an analogue of the Cauchy-Riemann equations, and Fueter and his co-workers went on to develop the theory of *quaternionic analysis*, by analogy with complex analysis. An account of this theory, with references, is given by Sudbery in [27]. On the hypercomplex manifold  $\mathbb{H}$ , Fueter’s definition of regular function coincides with that of q-holomorphic function, given above. We shall make little reference to Fueter’s theory, because we are interested in rather different questions. However, in Chapter 4 we will use our theory to give an elegant construction of the spaces of homogeneous q-holomorphic functions on  $\mathbb{H}$ , which are important in quaternionic analysis.

Suppose that  $M$  is hyperkähler. Then using the metric  $g$  on  $M$  we construct the operator  $D^* : \Omega^1 \rightarrow \Omega^0(\mathbb{H})$ , which is given by

$$(16) \quad D^*(\alpha) = d^* \alpha - d^*(I_1 \alpha) i_1 - d^*(I_2 \alpha) i_2 - d^*(I_3 \alpha) i_3.$$

Now for a smooth real function  $f$  on a Kähler manifold,  $d^*(Idf) = 0$ . Using this we can show that  $D^* D(a) = \Delta a$ , where  $\Delta$  is the usual Laplacian. Thus q-holomorphic

functions on a hyperkähler manifold are harmonic. When  $n = 1$ ,  $D$  is elliptic, and is the Dirac operator  $D_+$ . When  $n > 1$ ,  $D$  is overdetermined elliptic.

Here are two basic properties of  $q$ -holomorphic functions.

**Lemma 3.1.1.** *Suppose that  $a$  is  $q$ -holomorphic on  $M$ , and that  $q \in \mathbb{H}$ . Then  $qa$  is  $q$ -holomorphic.*

*Suppose that  $i = r_1i_1 + r_2i_2 + r_3i_3 \in \mathbb{I}$  satisfies  $i^2 = -1$ , and that  $I = r_1I_1 + r_2I_2 + r_3I_3$  is the corresponding complex structure on  $M$ . Suppose that  $y + zi$  is a complex function on  $M$  that is holomorphic w.r.t.  $I$ . Then  $y + zi$  is  $q$ -holomorphic on  $M$ , regarding  $y + zi = y + zr_1i_1 + zr_2i_2 + zr_3i_3$  as an element of  $\Omega^0(\mathbb{H})$ .*

*Proof.* Let  $q = q_0 + q_1i_1 + q_2i_2 + q_3i_3 \in \mathbb{H}$ , and define  $Q = q_0 + q_1I_1 + q_2I_2 + q_3I_3$ , regarding  $Q$  as an endomorphism of  $T^*M$ . Let  $a \in \Omega^0(\mathbb{H})$ . It is easy to verify that  $D(qa) = Q \cdot D(a)$ . Thus  $D(qa) = 0$  if  $D(a) = 0$ , and  $qa$  is  $q$ -holomorphic whenever  $a$  is  $q$ -holomorphic. This proves the first part.

If  $y + zi$  is holomorphic w.r.t.  $I$ , then  $dy + I(dz) = 0$  by the Cauchy-Riemann equations. But

$$\begin{aligned} 0 &= dy + I(dz) = dy + (r_1I_1 + r_2I_2 + r_3I_3)(dz) \\ &= D(y + zr_1i_1 + zr_2i_2 + zr_3i_3), \end{aligned}$$

so that  $y + zr_1i_1 + zr_2i_2 + zr_3i_3$  is  $q$ -holomorphic. This completes the lemma.  $\square$

The operator  $D$  of (14) was also studied by Baston [5], upon quaternionic manifolds rather than hypercomplex manifolds. He calls  $D$  the *Dirac-Fueter operator*, and uses the Penrose transform to interpret  $D$  as a holomorphic object on the twistor space  $Z$  of  $M$ . Baston [5, p. 44-45] shows that  $\text{Ker } D$  on a quaternionic manifold  $M$  can be identified with the sheaf cohomology group  $H^1(Z, \mathcal{O}_Z(-3))$ , giving a twistor interpretation of  $q$ -holomorphic functions. He also constructs an exact complex of operators resolving  $D$ , [5, p. 43-44].

**3.2. Hypercomplex manifolds and H-algebras.** Let  $M$  be a hypercomplex manifold, and let  $A$  be the vector space of  $q$ -holomorphic functions on  $M$ . In this section we will prove that  $A$  is an  $\mathbb{H}$ -algebra.

**Definition 3.2.1** Let  $M$  be a hypercomplex manifold. Define  $A \subset \Omega^0(\mathbb{H})$  to be the vector space of  $q$ -holomorphic functions on  $M$ . Let  $a \in A$ , and define  $(q \cdot a)(m) = q(a(m))$  for  $m \in M$ . Then  $q \cdot a \in A$  by Lemma 3.1.1, and this gives an  $\mathbb{H}$ -action on  $A$ , so  $A$  is an  $\mathbb{H}$ -module. Define a subspace  $A'$  in  $A$  by  $A' = \{a \in A : a(m) \in \mathbb{I} \text{ for all } m \in M\}$ . For each  $m \in M$ , define  $\theta_m : A \rightarrow \mathbb{H}$  by  $\theta_m(a) = a(m)$ . Then  $\theta_m \in A^\times$ , and if  $a \in A'$  then  $\theta_m(a) \in \mathbb{I}$ , so that  $\theta_m \in A^\dagger$ .

Suppose  $a \in A$ , and  $\alpha(a) = 0$  for all  $\alpha \in A^\dagger$ . Since  $\theta_m \in A^\dagger$ ,  $a(m) = 0$  for each  $m \in M$ , and so  $a = 0$ . Thus  $A$  is an  $\mathbb{A}\mathbb{H}$ -module, by Definition 1.1.1. Define the element  $1 \in A$  to be the constant function on  $M$  with value 1. Then  $1 \notin A'$ , but  $\mathbb{I} \cdot 1 \subset A'$ . We will also write  $A_M$  for  $A$ , when we wish to specify the manifold  $M$ .

The following proposition gives us a greater understanding of the quaternionic tensor product.

**Proposition 3.2.2.** *Let  $M$  and  $N$  be hypercomplex manifolds, and let  $U, V$  be  $\mathbb{A}\mathbb{H}$ -submodules of the  $\mathbb{A}\mathbb{H}$ -modules  $A_M, A_N$  of  $q$ -holomorphic functions on  $M, N$  respectively. Define  $W$  to be the vector space of smooth,  $\mathbb{H}$ -valued functions  $w$  on  $M \times N$ , such that for each  $m \in M$ , the function  $n \mapsto w(m, n)$  lies in  $V$ , and for each  $n \in N$ , the function  $m \mapsto w(m, n)$  lies in  $U$ . Then each such  $w$  is a  $q$ -holomorphic function on  $M \times N$ , and  $W$  is an  $\mathbb{A}\mathbb{H}$ -submodule of  $A_{M \times N}$ . Also, there is a canonical injective  $\mathbb{A}\mathbb{H}$ -morphism  $\phi : U \otimes_{\mathbb{H}} V \rightarrow W$ . If  $U, V$  are finite-dimensional,  $\phi$  is an  $\mathbb{A}\mathbb{H}$ -isomorphism.*

*Proof.* Suppose  $w : M \times N \rightarrow \mathbb{H}$  is a smooth function, such that for each  $m \in M$ , the function  $n \mapsto w(m, n)$  lies in  $V$ , and for each  $n \in N$ , the function  $m \mapsto w(m, n)$  lies in  $U$ . The condition for  $w$  to be  $q$ -holomorphic is  $D(w) = 0$ . But  $D(w) = D_M(w) + D_N(w)$ , where  $D_M$  involves only derivatives in the  $M$  directions, and  $D_N$  only derivatives in the  $N$  directions.

Let  $n \in N$ . Then the function  $m \mapsto w(m, n)$  is equal to some  $u \in U$ . Thus  $D_M(w)(m, n) = D(u)(m)$ . But the functions in  $U$  are  $q$ -holomorphic, so  $D(u) = 0$ . Therefore  $D_M(w) = 0$ , and similarly  $D_N(w) = 0$ . So  $D(w) = 0$ , and  $w$  is  $q$ -holomorphic, as we have to prove. It is clear that the space  $W$  of such functions  $w$  is closed under addition and multiplication by  $\mathbb{H}$ . Thus  $W$  is an  $\mathbb{H}$ -submodule of  $A_{M \times N}$ , so  $W$  is an  $\mathbb{A}\mathbb{H}$ -submodule of  $A_{M \times N}$ .

Now let  $\epsilon \in U \otimes_{\mathbb{H}} V$ . Then  $\epsilon \in \mathbb{H} \otimes (U^\dagger)^* \otimes (V^\dagger)^*$ , so  $\epsilon$  defines a linear map  $\epsilon : U^\dagger \otimes V^\dagger \rightarrow \mathbb{H}$ . Define a map  $w : M \times N \rightarrow \mathbb{H}$  by  $w(m, n) = \epsilon(\theta_m \otimes \theta_n)$ . For  $m \in M$ , define  $w_m : N \rightarrow \mathbb{H}$  by  $w_m(n) = w(m, n)$ . Since  $\epsilon \in U \otimes_{\mathbb{H}} V$ ,  $\epsilon \in (U^\dagger)^* \otimes \iota_V(V)$ , so  $w_m \in \iota(V)$ , regarding  $w_m$  as an element of  $\mathbb{H} \otimes (V^\dagger)^*$ . Thus  $w_m \in V$ . Similarly, defining  $w_n(m) = w(m, n)$  for  $n \in N$ , we find  $w_n \in U$  for each  $n \in N$ .

To show that  $w \in W$ , we only need to show that  $w$  is smooth. In general, if  $f$  is a function on  $M \times N$ , such that for each  $m \in M$ , the function  $n \mapsto f(m, n)$  is smooth, and for each  $n \in N$ , the function  $m \mapsto f(m, n)$  is smooth, it does not follow that  $f$  is smooth. However, because real tensor products involve only finite sums as in §1.1,  $\epsilon \in (U^\dagger)^* \otimes \iota_V(V)$  implies that  $w_m$  lies in some finite-dimensional subspace of  $V$  for all  $m \in M$ , and similarly  $w_n$  lies in a finite-dimensional subspace of  $U$  for all  $n \in N$ . These imply that  $w$  is smooth. Thus  $w \in W$ .

Define  $\phi(\epsilon) = w$ . In this way we define a map  $\phi : U \otimes_{\mathbb{H}} V \rightarrow W$ . It is easy to show that  $\phi$  is an  $\mathbb{A}\mathbb{H}$ -morphism. Also, if  $w = 0$  it is easily seen that  $\epsilon = 0$ , so  $\phi$  is injective. Thus  $\phi$  is an injective  $\mathbb{A}\mathbb{H}$ -morphism, as we have to prove. Suppose  $U, V$  are finite-dimensional, and let  $w \in W$ . We must find  $\epsilon$  in  $U \otimes_{\mathbb{H}} V$  such that  $\phi(\epsilon) = w$ .

Choose bases of the form  $\{\theta_{m_b} : b = 1, \dots, k\}$  for  $U^\dagger$  and  $\{\theta_{n_c} : c = 1, \dots, l\}$  for  $V^\dagger$ . It can be shown that such bases exist. Let  $\epsilon : U^\dagger \otimes V^\dagger \rightarrow \mathbb{H}$  be the unique linear map satisfying  $\epsilon(\theta_{m_b} \otimes \theta_{n_c}) = w(m_b, n_c)$  for  $b = 1, \dots, k, c = 1, \dots, l$ . One may prove

that  $\epsilon \in U \otimes_{\mathbb{H}} V$ , and  $\phi(\epsilon) = w$ . Thus  $\phi$  is an injective and surjective AHH-morphism, and clearly is an AHH-isomorphism. This completes the proof.  $\square$

The following lemma is trivial, and the proof will be omitted.

**Lemma 3.2.3.** *Suppose  $M$  is a hypercomplex manifold, and  $N$  is a hypercomplex submanifold of  $M$ . If  $a$  is a  $q$ -holomorphic function on  $M$ , then  $a|_N$  is  $q$ -holomorphic on  $N$ . Let  $\rho : A_M \rightarrow A_N$  be the restriction map. Then  $\rho$  is an AHH-morphism.*

Now we can define the multiplication map  $\mu_A$  on  $A$ .

**Definition 3.2.4** Let  $M$  be a hypercomplex manifold, and  $A$  the AHH-module of  $q$ -holomorphic functions on  $M$ . By Proposition 3.2.2 there is a canonical AHH-morphism  $\phi : A \otimes_{\mathbb{H}} A \rightarrow A_{M \times M}$ . Now  $M$  is embedded in  $M \times M$  as the diagonal submanifold  $\{(m, m) : m \in M\}$ , and this is a hypercomplex submanifold of  $M \times M$ , isomorphic to  $M$  as a hypercomplex manifold. Therefore Lemma 3.2.3 gives an AHH-morphism  $\rho : A_{M \times M} \rightarrow A$ . Define an AHH-morphism  $\mu_A : A \otimes_{\mathbb{H}} A \rightarrow A$  by  $\mu_A = \rho \circ \phi$ .

Here is the main result of this section.

**Theorem 3.2.5.** *Let  $M$  be a hypercomplex manifold. Then Definition 3.2.1 defines an AHH-module  $A$  and an element  $1 \in A$ , and Definition 3.2.4 defines an AHH-morphism  $\mu_A : A \otimes_{\mathbb{H}} A \rightarrow A$ . With these definitions,  $A$  is an H-algebra in the sense of §2.1.*

*Proof.* We must show that Axiom A is satisfied. Parts (i) and (ii) are trivial. For part (iii), observe that the permutation map  $A \otimes_{\mathbb{H}} A \rightarrow A \otimes_{\mathbb{H}} A$  that swaps round the factors, is induced by the map  $M \times M \rightarrow M \times M$  given by  $(m_1, m_2) \mapsto (m_2, m_1)$ . Since the diagonal submanifold is invariant under this, it follows that  $\mu_A$  is invariant under permutation, and so  $\Lambda_{\mathbb{H}}^2 A \subset \text{Ker } \mu_A$ .

Let  $\Delta_M^2$  be the ‘diagonal’ submanifold in  $M \times M$ , and let  $\Delta_M^3$  be the ‘diagonal’ submanifold in  $M \times M \times M$ . We interpret part (iv) as follows.  $A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A$  is a space of  $q$ -holomorphic functions on  $M \times M \times M$ . The maps  $\mu_A \otimes_{\mathbb{H}} \text{id}$  and  $\text{id} \otimes_{\mathbb{H}} \mu_A$  are the maps restricting to  $\Delta_M^2 \times M$  and  $M \times \Delta_M^2$  respectively. Thus  $\mu_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id})$  is the result of first restricting to  $\Delta_M^2 \times M$  and then to  $\Delta_M^3$ , and  $\mu_A \circ (\text{id} \otimes_{\mathbb{H}} \mu_A)$  is the result of first restricting to  $M \times \Delta_M^2$  and then to  $\Delta_M^3$ . Clearly  $\mu_A \circ (\mu_A \otimes_{\mathbb{H}} \text{id}) = \mu_A \circ (\text{id} \otimes_{\mathbb{H}} \mu_A)$ , proving part (iv).

Interestingly, the proof of part (iv) does not use the associativity of quaternion multiplication. This raises the possibility of generalizing the definition to give ‘associative algebras over a nonassociative field’. Part (v) is given in Definition 3.2.1. Finally, part (vi) follows easily from the fact that  $1$  is the identity in  $\mathbb{H}$ . Thus all of Axiom A of §2.1 applies, and  $A$  is an H-algebra.  $\square$

One problem with the H-algebra of all  $q$ -holomorphic functions on a hypercomplex manifold is that it is too large to work with – it is not in general finitely-generated, for instance. Therefore, it is convenient to restrict to H-subalgebras of functions satisfying some condition. The condition we shall use is that of *polynomial growth*.

**Definition 3.2.6** Let  $M$  be a hypercomplex manifold, and let  $r : M \rightarrow [0, \infty)$  be a given continuous function. Suppose  $f \in \Omega^0(\mathbb{H})$  on  $M$ , and let  $k \geq 0$  be an integer. We say that  $f$  has polynomial growth of order  $k$ , written  $f = O(r^k)$ , if there exist positive constants  $C_1, C_2$  such that  $|f| \leq C_1 + C_2 r^k$  on  $M$ .

The H-algebra of q-holomorphic functions on  $M$  is  $A$ . For integers  $k \geq 0$ , define  $P_k = \{a \in A : a = O(r^k)\}$ , and define  $P = \bigcup_{k=0}^\infty P_k$ . Then  $P$  is a filtered AH-module. It is easy to see that  $P$  is an H-subalgebra of  $A$ , and satisfies Axiom AF of §2.3. Thus,  $P$  is a filtered H-algebra. We call  $P$  the filtered H-algebra of q-holomorphic functions of polynomial growth on  $M$ , and we write  $P_M$  for  $P$  when we wish to specify the manifold  $M$ .

The main example we have in mind in making this definition, is the case that  $M$  is a complete, noncompact hyperkähler manifold, and  $r : M \rightarrow [0, \infty)$  is the distance function from some point  $m_0 \in M$ . Then  $P_k$  is independent of the choice of base point  $m_0$ . In good cases, such as those discussed in Chapter 4,  $P$  is an FGH-algebra. Again, in a good case,  $P$  determines the hypercomplex structure of  $M$  explicitly and uniquely. Thus we can define the hypercomplex structure of  $M$  completely using only a finite-dimensional amount of algebraic data.

**3.3. Hyperkähler manifolds and HP-algebras.** Let  $M$  be a hyperkähler manifold, and  $A$  the vector space of q-holomorphic functions on  $M$ . Since  $M$  is hypercomplex,  $A$  is an H-algebra by Theorem 3.2.5. In this section we will see that  $A$  is also an HP-algebra in the sense of §2.2. To save space, and because we have wandered from the main subject of the paper, we shall omit the proofs of Proposition 3.3.2 and Theorem 3.3.4. The proofs are elementary calculations, though not especially easy, and the author can supply them to the interested reader on request.

**Definition 3.3.1** Let  $M$  be a hyperkähler manifold. Then  $M \times M$  is also a hyperkähler manifold. Let  $\Delta_M^2 = \{(m, m) : m \in M\}$ . Then  $\Delta_M^2$  is a hyperkähler submanifold of  $M \times M$ . We shall write  $M \times M = M^1 \times M^2$ , using the superscripts <sup>1</sup> and <sup>2</sup> to distinguish the two factors. Let  $\nabla$  be the Levi-Civita connection on  $M$ . Define  $\nabla^1, \nabla^2$  to be the lift of  $\nabla$  to the first and second factors of  $M$  in  $M \times M$  respectively. Then  $\nabla^1$  and  $\nabla^2$  commute. Let  $\nabla^{12}$  be the Levi-Civita connection on  $M \times M$ . Then  $\nabla^{12} = \nabla^1 + \nabla^2$ .

Let  $x \in A_{M \times M}$ . Then  $\nabla^1 \nabla^2 x \in C^\infty(\mathbb{H} \otimes T^* M^1 \otimes T^* M^2)$  over  $M \times M$ . Restrict  $\nabla^1 \nabla^2 x$  to  $\Delta_M^2$ . Then  $\Delta_M^2 \cong M$  and  $T^* M^1|_{\Delta_M^2} \cong T^* M^2|_{\Delta_M^2} \cong T^* M$ . Thus  $\nabla^1 \nabla^2 x|_{\Delta_M^2} \in C^\infty(\mathbb{H} \otimes T^* M \otimes T^* M)$  over  $M$ . Define a linear map  $\Theta : A_{M \times M} \rightarrow \Omega^0(\mathbb{H}) \otimes \mathbb{I}$  by

$$(17) \quad \Theta(x) = \{g^{ab}(I_1)_a^c \nabla_b^1 \nabla_c^2 x|_{\Delta_M^2}\} \otimes i_1 + \{g^{ab}(I_2)_a^c \nabla_b^1 \nabla_c^2 x|_{\Delta_M^2}\} \otimes i_2 + \{g^{ab}(I_3)_a^c \nabla_b^1 \nabla_c^2 x|_{\Delta_M^2}\} \otimes i_3,$$

using index notation for tensors on  $M$  in the obvious way. Here  $g$  is the hyperkähler metric on  $M$ , and  $I_1, I_2, I_3$  the complex structures.



Here are some properties of  $\Theta$ .

**Proposition 3.3.2.** *This map satisfies  $\Theta(x) \in A \otimes \mathbb{I}$ . Also,  $\Theta : A_{M \times M} \rightarrow A \otimes \mathbb{I}$  is an A $\mathbb{H}$ -morphism, and if  $\Theta(x) = x_1 \otimes i_1 + x_2 \otimes i_2 + x_3 \otimes i_3$  and  $m \in M$ , then  $x_1(m)i_1 + x_2(m)i_2 + x_3(m)i_3 = 0 \in \mathbb{H}$ .*

Now we can define the map  $\xi_A$  of §2.2.

**Definition 3.3.3** Proposition 3.2.2 defines an A $\mathbb{H}$ -morphism  $\phi : A \otimes_{\mathbb{H}} A \rightarrow A_{M \times M}$ . Definition 3.3.1 and Proposition 3.3.2 define an A $\mathbb{H}$ -morphism  $\Theta : A_{M \times M} \rightarrow A \otimes \mathbb{I}$ . Let  $Y$  be the A $\mathbb{H}$ -module of Definition 2.2.1. Then  $(Y^\dagger)^* \cong \mathbb{I}$ . Recall that  $A \cong \iota_A(A)$ , so we may identify  $A \otimes \mathbb{I} \cong \iota_A(A) \otimes (Y^\dagger)^* \subset \mathbb{H} \otimes (A^\dagger)^* \otimes (Y^\dagger)^*$ . Define  $\xi_A : A \otimes_{\mathbb{H}} A \rightarrow \iota_A(A) \otimes (Y^\dagger)^*$  to be the composition  $\xi_A = \Theta \circ \phi$ .

Here is the main result of this section.

**Theorem 3.3.4.** *This  $\xi_A$  maps  $A \otimes_{\mathbb{H}} A$  to  $A \otimes_{\mathbb{H}} Y$ . It is an A $\mathbb{H}$ -morphism, and satisfies Axioms P1 and P2 of §2.2. Thus, by Theorem 3.2.5 and Definition 2.2.2, if  $M$  is a hyperkähler manifold, then the vector space  $A$  of  $q$ -holomorphic functions on  $M$  is an HP-algebra.*

Given a continuous function  $r : M \rightarrow [0, \infty)$ , Definition 3.2.6 defines the filtered H-algebra  $P$  of  $q$ -holomorphic functions on  $M$  of polynomial growth. It is natural to ask whether  $P$  is a filtered HP-algebra. One must show that Axiom PF of §2.3 holds, which relates the Poisson bracket and the filtration. This is not automatic, but depends on the asymptotic properties of the hyperkähler structure and the function  $r$  on  $M$ , and must be verified for each case. These properties also determine whether  $P$  is closed under  $\xi_A$  at all.

**3.4. Reconstructing a hypercomplex manifold from its H-algebra.** In §3.2 we saw that the vector space of  $q$ -holomorphic functions on a hypercomplex manifold is an H-algebra, providing a transform from geometric to algebraic objects. Now we shall consider whether this transform can be reversed. It turns out that under certain circumstances, an H-algebra does explicitly determine a unique hypercomplex manifold. Therefore, it should be possible to construct new hypercomplex manifolds by writing down their H-algebras.

Throughout this section, let  $M$  be a hypercomplex manifold, let  $A$  be the H-algebra of  $q$ -holomorphic functions on  $M$ , let  $P$  be an H-subalgebra of  $A$ , and let  $Q$  be an A $\mathbb{H}$ -submodule of  $P$  that generates  $P$  as an H-algebra, in the sense of §2.4. From §3.2, if  $m \in M$ , then  $m$  defines an  $\mathbb{H}$ -linear map  $\theta_m : A \rightarrow \mathbb{H}$  such that  $\theta_m \in A^\dagger$ . Now  $\mathbb{H}$  is itself an H-algebra in the obvious way, and it is easy to see that in fact  $\theta_m$  is an H-algebra morphism. Therefore  $\theta_m|_P : P \rightarrow \mathbb{H}$  is also an H-algebra morphism, and  $\theta_m|_P \in P^\dagger$ .

Conversely, suppose  $\theta \in P^\dagger$ , so that  $\theta : P \rightarrow \mathbb{H}$  is an  $\mathbb{H}$ -linear map. From Definition 2.1.2 we calculate that  $\theta$  is an H-algebra morphism if and only if  $\theta$  satisfies the quadratic equation  $\mu_P^\dagger(\theta) = \lambda_{P,P}(\theta \otimes \theta)$  in  $(P \otimes_{\mathbb{H}} P)^\dagger$ , where  $\mu_P^\dagger : P^\dagger \rightarrow (P \otimes_{\mathbb{H}} P)^\dagger$  is

the dual of the multiplication map  $\mu_P$ , and  $\lambda_{P,P} : P^\dagger \otimes P^\dagger \rightarrow (P \otimes_{\mathbb{H}} P)^\dagger$  is defined in §1.1.

Suppose  $\theta_1, \theta_2 : P \rightarrow \mathbb{H}$  are H-algebra morphisms, and that  $\theta_1|_Q = \theta_2|_Q$ . Because  $Q$  generates  $P$ , it is easy to see that  $\theta_1 = \theta_2$ . Hence, H-algebra morphisms from  $P$  are determined by their restrictions to  $Q$ . As  $\theta_1 \in P^\dagger$ , we have  $\theta_1|_Q \in Q^\dagger$ . The principal case we have in mind is that  $P$  is a ‘polynomial growth’ H-subalgebra as in Definition 3.2.6, and  $Q$  is finite-dimensional. By restricting to  $Q$  we can work in a finite-dimensional situation. Motivated by the above, we make the following definition.

**Definition 3.4.1** Let  $P$  be an H-algebra, and  $Q$  an AH-submodule of  $P$  that generates  $P$ . Define  $M_{P,Q}$  by

$$M_{P,Q} = \{ \theta|_Q : \theta \in P^\dagger, \mu_P^\dagger(\theta) = \lambda_{P,P}(\theta \otimes \theta) \},$$

so that  $M_{P,Q}$  is a closed subset of  $Q^\dagger$ . Now suppose that  $M$  is a hypercomplex manifold, and  $P$  an H-subalgebra of the H-algebra  $A$  of q-holomorphic functions on  $M$ . For each  $m \in M$ ,  $\theta_m : A \rightarrow \mathbb{H}$  is an H-algebra morphism, and so  $\theta_m|_Q$  lies in  $M_{P,Q}$ . Define a map  $\pi_{P,Q} : M \rightarrow M_{P,Q}$  by  $\pi_{P,Q}(m) = \theta_m|_Q$ .

**Lemma 3.4.2.** *Suppose  $Q$  is finite-dimensional. Then  $M_{P,Q}$  is an affine real algebraic subvariety of  $Q^\dagger$ , that is, it is the zeros of a finite collection of polynomials on  $Q^\dagger$ .*

*Proof.* By Lemma 2.4.4 there is an H-algebra morphism  $\phi_Q : F^Q \rightarrow P$ , and as  $Q$  generates  $P$ ,  $\phi_Q$  is surjective. Let  $I \subset F^Q$  be the kernel of  $\phi_Q$ . As  $F^Q$  is a free H-algebra, each element  $x$  of  $Q^\dagger$  defines a unique H-algebra morphism  $\theta_x : F^Q \rightarrow \mathbb{H}$ , that restricts to  $x$  on  $Q$ . Clearly, if  $x \in Q^\dagger$ , then  $x \in M_{P,Q}$  if and only if  $I \subset \text{Ker } \theta_x$ .

For each  $y \in F^Q$ , define a function  $\psi_y : Q^\dagger \rightarrow \mathbb{H}$  by  $\psi_y(x) = \theta_x(y)$  for  $x \in Q^\dagger$ . It is easy to see that if  $y \in F_k^Q$ , then  $\psi_y$  is an  $\mathbb{H}$ -valued polynomial on  $Q^\dagger$  of degree at most  $k$ . But  $x \in M_{P,Q}$  if and only if  $\psi_y(x) = 0$  for each  $y \in I$ . Thus  $M_{P,Q}$  is the zeros of a collection of polynomials on  $Q^\dagger$ . By Hilbert’s Basis Theorem, we can choose a finite number of polynomials, which define  $M_{P,Q}$ .  $\square$

Our aim is to recover  $M$  and its hypercomplex structure from  $P$  and  $Q$ . If we are lucky,  $\pi_{P,Q}$  will be (locally) a bijection, so that  $M_{P,Q}$  gives us the manifold  $M$ , at least as a set. Then we can try and use  $P$  to define a hypercomplex structure on  $M_{P,Q}$ . However, there are a number of ways in which this process could fail.

- $M_{P,Q}$  might not be a submanifold of  $Q^\dagger$ .
- $\pi_{P,Q}$  might not be (locally) injective.
- $\pi_{P,Q}$  might not be (locally) surjective.
- Even if  $M_{P,Q}$  is a submanifold of  $Q^\dagger$  and  $\pi_{P,Q}$  is a diffeomorphism, we may be unable to define the hypercomplex structure on  $M_{P,Q}$ , because  $P$  may contain only partial information about the structure.

Unfortunately, all of these possibilities do occur, and examples will be given in §4.2. Sometimes the hypercomplex manifold  $M$  can be reconstructed, and sometimes not; for polynomial growth  $H$ -algebras  $P$  and complete  $M$ , this seems to depend only on the asymptotic behaviour of  $M$  at infinity. Next we shall explain how, in good cases, the hypercomplex structure of  $M$  may be recovered from  $P$ .

**Lemma 3.4.3.** *Let  $M$  be a hypercomplex manifold of dimension  $4k$ ,  $A$  the  $H$ -algebra of  $q$ -holomorphic functions on  $M$ ,  $P$  an  $H$ -subalgebra of  $A$ , and  $Q$  an  $A\mathbb{H}$ -submodule of  $P$  generating  $P$ . Let  $m \in M$ . Then the derivative of  $\pi_{P,Q}$  at  $m$  gives a linear map  $d_m\pi_{P,Q} : T_mM \rightarrow Q^\dagger$ .*

*Regard  $Q$  as a vector space of  $\mathbb{H}$ -valued functions on  $Q^\dagger$ . Pulling these functions back to  $T_mM$  using  $d_m\pi_{P,Q}$  gives a linear map  $(d_m\pi_{P,Q})^* : Q \rightarrow T_m^*M \otimes \mathbb{H}$ . Define  $V_m = \text{Im}((d_m\pi_{P,Q})^*)$ , so that  $V_m$  is a linear subspace of  $T_m^*M \otimes \mathbb{H}$ . Then  $\dim V_m \leq 12k$ , and  $V_m$  determines the hypercomplex structure on  $T_mM$  if and only if  $\dim V_m = 12k$ .*

*Proof.* Let  $v \in V_m$ , and write  $v = v_0 \otimes 1 + v_1 \otimes i_1 + v_2 \otimes i_2 + v_3 \otimes i_3$ . Now  $v$  is the first derivative at  $m$  of some element of  $Q$ , which is a  $q$ -holomorphic function on  $M$ . By definition of  $q$ -holomorphic function, it follows that  $v_0 + I_1v_1 + I_2v_2 + I_3v_3 = 0$ , where  $I_1, I_2, I_3$  are the complex structures on  $T_m^*M$ . Let  $W_m \subset T_m^*M \otimes \mathbb{H}$  be the subspace of elements  $w_0 \otimes 1 + w_1 \otimes i_1 + w_2 \otimes i_2 + w_3 \otimes i_3$  satisfying  $w_0 + I_1w_1 + I_2w_2 + I_3w_3 = 0$ . Then  $W_m$  has dimension  $12k$ , and  $V_m \subset W_m$ , so  $\dim V_m \leq 12k$ , as we have to prove.

If  $\dim V_m = 12k$ , then  $V_m = W_m$ . But  $W_m$  determines  $I_1, I_2$  and  $I_3$ . (For instance,  $w_1 = I_1w_0$  if and only if  $w_0 \otimes 1 + w_1 \otimes i_1 \in W_m$ , so  $W_m$  determines  $I_1$ .) Thus if  $\dim V_m = 12k$ , then  $V_m$  determines the hypercomplex structure on  $T_mM$ . Now  $W_m$  is an  $\mathbb{H}$ -submodule of  $T_m^*M \otimes \mathbb{H}$ , and it can be shown that if  $W$  is an  $\mathbb{H}$ -submodule of  $T^*M \otimes \mathbb{H}$  such that  $W \neq W_m$ ,  $\dim W = 12k$ , and  $W$  is close to  $W_m$ , then  $W$  determines a different hypercomplex structure. It is easy to see that if  $\dim V_m < 12k$ , then we may choose such a  $W$  with  $V_m \subset W$ . Thus  $V_m$  is consistent with two different hypercomplex structures. Therefore  $V_m$  determines the hypercomplex structure on  $T_mM$  if and only if  $\dim V_m = 12k$ .  $\square$

**Corollary 3.4.4.** *In the situation of Lemma 3.4.3, suppose that  $\dim V_m = 12k$ , and that  $\pi_{P,Q}$  is surjective near  $m$ . Then  $P$  determines the hypercomplex structure of  $M$  near  $m$ .*

*Proof.* As  $\dim V_m = 12k$ , we deduce that  $d_m\pi_{P,Q}$  is injective. Together with the surjectivity assumption, this implies that  $\pi_{P,Q}$  is a diffeomorphism near  $m$ . Also,  $\dim V_n = 12k$  for  $n$  near  $m$ . The conclusion follows from the lemma.  $\square$

In the special case  $\dim M = 4$ , we can give a more convenient condition for  $P$  to determine the hypercomplex structure of  $M$ .

**Proposition 3.4.5.** *Let  $M$  be a hypercomplex manifold of dimension 4,  $A$  the  $H$ -algebra of  $q$ -holomorphic functions on  $M$ ,  $P$  an  $H$ -subalgebra of  $A$ , and  $Q$  an  $A\mathbb{H}$ -submodule of  $P$  generating  $P$ . Suppose that  $Q$  is stable, that  $M_{P,Q}$  is a submanifold*

of  $Q^\dagger$ , and that  $\pi_{P,Q}$  is a (local) diffeomorphism. Then  $P$  determines the (local) hypercomplex structure of  $M$ .

*Proof.* Because of Corollary 3.4.4, it is sufficient to show that  $\dim V_m = 12$  for each  $m \in M$ . Suppose for a contradiction that  $m \in M$  and  $\dim V_m < 12$ . As  $\pi_{P,Q}$  is a diffeomorphism,  $d_m \pi_{P,Q}$  is injective. Let  $R = \text{Ker}((d_m \pi_{P,Q})^*)$ , so that  $R$  is a proper  $\text{A}\mathbb{H}$ -submodule of  $Q$ , and  $V_m \cong Q/R$ . Since  $d_m \pi_{P,Q}$  is injective and  $\dim M = 4$ , we have  $\dim Q^\dagger = \dim R^\dagger + 4$ .

Now  $V_m$  is an  $\mathbb{H}$ -module with  $\dim V_m < 12$ , so  $\dim V_m \leq 8$ . But  $\dim Q = \dim R + \dim V_m$ , so  $\dim Q \leq \dim R + 8$ . By Definition 1.2.3, using  $\dim Q^\dagger = \dim R^\dagger + 4$ , we see that the virtual dimension of  $R$  is greater or equal to that of  $Q$ . Using the stability of  $Q$ , we then prove that for each nonzero  $q \in \mathbb{I}$ , the image of  $\text{id}_{\otimes_{\mathbb{H}} \chi_q} : Q \otimes_{\mathbb{H}} X_q \rightarrow Q$  lies in  $R$ , and as  $R$  is a proper  $\text{A}\mathbb{H}$ -submodule, this contradicts the semistability of  $Q$ . Thus  $\dim V_m = 12$  for all  $m \in M$ , and the proposition is complete.  $\square$

The author has not found a satisfactory analogue of this proposition for higher dimensions. One can also consider the problem of reconstructing a hyperkähler manifold from an HP-algebra. It can be solved easily using a similar approach.

**3.5. Asymptotically conical hyperkähler manifolds.** In this section we shall define an interesting class of hyperkähler manifolds, and conjecture some theory about them.

**Definition 3.5.1** Let  $N$  be a compact manifold of dimension  $4n - 1$ , and set  $C = N \times (0, \infty)$ . Let  $t : C \rightarrow (0, \infty)$  be the projection to the second factor. Let  $v$  be the vector field  $t \partial/\partial t$  on  $C$ . Suppose  $C$  has a hyperkähler structure, with metric  $g$  and complex structures  $I_1, I_2, I_3$ . We say that  $C$  is a hyperkähler cone if  $v$  is a Killing vector of  $I_1, I_2, I_3$ , and  $g = t^2 h + dt^2$ , where  $h$  is a Riemannian metric on  $N$  that is independent of  $t$ .

**Definition 3.5.2** Let  $C$  be a hyperkähler cone. Let  $f$  be a  $q$ -holomorphic function on  $C$ , and let  $k \geq 0$  be an integer. We say  $f$  is homogeneous of degree  $k$  if  $f = t^k f_N$ , where  $f_N$  is an  $\mathbb{H}$ -valued function on  $N$ , independent of  $t$ . Define  $B^k$  to be the  $\text{A}\mathbb{H}$ -module of  $q$ -holomorphic functions on  $C$  that are homogeneous of degree  $k$ . Define  $B = \bigoplus_{k=0}^\infty B^k$ . Then  $B$  is an  $\text{A}\mathbb{H}$ -submodule of the  $\text{H}$ -algebra  $A_C$  of  $q$ -holomorphic functions on  $C$ . Clearly,  $B$  is a graded  $\text{H}$ -algebra.

Here are some basic properties of hyperkähler cones. The proof is left to the reader.

**Lemma 3.5.3.** *Let  $C$  be a hyperkähler cone. Then the vector fields  $v, I_1 v, I_2 v, I_3 v$  generate an action of the Lie algebra  $\mathbb{R} \oplus \mathfrak{su}(2)$  on  $C$ , and exponentiating them gives an action of the Lie group  $\mathbb{R} \times SU(2)$  on  $C$ . Let  $c \in C$ , and let  $\mathcal{O}_c$  be the orbit of  $c$  under  $\mathbb{R} \times SU(2)$ . Then  $\mathcal{O}_c$  is a hyperkähler submanifold of  $C$ , and is isomorphic as a hyperkähler manifold to  $(\mathbb{H} \setminus \{0\})/\Gamma$ , for some finite subgroup  $\Gamma \subset SU(2)$ .*

**Lemma 3.5.4.** *Let  $C$  be a hyperkähler cone. Then the graded  $H$ -algebra  $B$  of Definition 3.5.2 is equal to the filtered  $H$ -algebra  $P_C$  of  $q$ -holomorphic functions of polynomial growth on  $C$ .*

*Proof.* Let  $f$  be a  $q$ -holomorphic function on  $C$ , of polynomial growth of degree  $k$ . Let  $c \in C$ . By Lemma 3.5.3,  $\mathcal{O}_c$  is isomorphic to  $(\mathbb{H} \setminus \{0\})/\Gamma$ , so that the universal cover  $\tilde{\mathcal{O}}_c$  is isomorphic to  $\mathbb{H} \setminus \{0\}$ . Restricting  $f$  to  $\mathcal{O}_c$  and lifting to  $\tilde{\mathcal{O}}_c$ , the result is a  $q$ -holomorphic function on  $\mathbb{H} \setminus \{0\}$ , of polynomial growth of degree  $k$ .

Now all such functions are in fact polynomials of degree  $k$  on  $\mathbb{H}$ , by a classical result about harmonic functions of polynomial growth on  $\mathbb{R}^n$ . Therefore, we may decompose  $f$  on  $C$  into a sum of homogeneous polynomials on each  $\mathcal{O}_c$ . Clearly, these homogeneous pieces lie in  $B^j$  for  $j \leq k$ , so  $f \in \bigoplus_{j=0}^k B^j$ . We have shown that if  $f \in P_C$ , then  $f \in B$ , and  $P_C \subset B$ . But the inclusion  $B \subset P_C$  is immediate, so  $B = P_C$ . Clearly, the filtrations on  $B$  and  $P_C$  agree, and the lemma is complete.  $\square$

Next, we shall define asymptotically conical hyperkähler manifolds.

**Definition 3.5.5** Let  $M$  be a complete hyperkähler manifold of dimension  $4n$ , with metric  $g^M$  and complex structures  $I_1^M, I_2^M, I_3^M$ . Let  $C = N \times (0, \infty)$  be a hyperkähler cone of dimension  $4n$ , with metric  $g^C$  and complex structures  $I_1^C, I_2^C, I_3^C$ . Let  $K$  be a compact subset of  $M$ . Then  $M \setminus K$  and  $N \times (1, \infty)$  are submanifolds of  $M$  and  $C$ . Suppose that  $\Phi : M \setminus K \rightarrow N \times (1, \infty)$  is a diffeomorphism. Let  $\nabla$  be the Levi-Civita connection on  $C$ , and  $l$  be a positive integer.

We say that  $M$  is *asymptotically conical*, or *AC* to order  $l$ , if

$$(18) \quad \begin{aligned} \Phi_*(g^M) &= g^C + O(t^{-l}), & \nabla(\Phi_*(g^M)) &= O(t^{-l-1}), \\ \nabla^2(\Phi_*(g^M)) &= O(t^{-l-2}), & \text{and} \end{aligned}$$

$$(19) \quad \begin{aligned} \Phi_*(I_j^M) &= I_j^C + O(t^{-l}), & \nabla(\Phi_*(I_j^M)) &= O(t^{-l-1}), \\ \nabla^2(\Phi_*(I_j^M)) &= O(t^{-l-2}), & \text{for } j &= 1, 2, 3. \end{aligned}$$

These equations should be interpreted as follows. Let  $T$  be a tensor field on  $N \times (1, \infty)$ . Then  $T = O(t^{-k})$  means that  $|T| \leq \kappa t^{-k}$  on  $N \times (1, \infty)$ , where  $\kappa$  is a positive constant, and  $|\cdot|$  is taken w.r.t. the metric  $g^C$ . We call  $C$  the *asymptotic cone* of  $M$ .

Our aim in the remainder of this section is to explore the structure of the  $H$ -algebra of  $q$ -holomorphic functions on an AC hyperkähler manifold. We shall now state – but not prove, and I do not know a complete proof – a powerful result relating the  $q$ -holomorphic functions on the AC manifold and its asymptotic cone. An *incomplete* proof will be given shortly, that depends on conjectures to be stated below.

**Theorem 3.5.6.** *Let  $M$  be a hyperkähler manifold that is AC of order  $l$ , with asymptotic cone  $C$ . Let  $B$  be the graded  $H$ -algebra of  $q$ -holomorphic functions on  $C$ , defined in Definition 3.5.2. Let  $P$  be the filtered  $H$ -algebra of  $q$ -holomorphic functions on  $M$  with polynomial growth. Then  $B$  is an SGH-algebra,  $P$  is an SFH-algebra, and*

$B$  is isomorphic to  $P$  to order  $l$ , in the sense of Definition 2.3.5. This implies that  $B$  is the associated graded  $H$ -algebra of  $P$ , as in Proposition 2.3.6.

We shall state two conjectures, and then prove the theorem assuming these. Before making the conjectures, we shall define *tensor fields of polynomial growth* on AC hyperkähler manifolds.

**Definition 3.5.7** Let  $M$  be an AC hyperkähler manifold, with asymptotic cone  $C$ . Let  $T$  be a smooth tensor field or function on  $M$ , and  $k$  be an integer. Then the expression  $|T| = O(t^k)$  means that  $|T| \leq \kappa \Phi^*(t)^k$  on  $M \setminus K$  for some positive constant  $\kappa$ , where  $|\cdot|$  is taken w.r.t. the metric  $g^M$ , and  $\Phi, K, t$  and  $g^M$  are as in Definition 3.5.5.

With this definition we can state the first conjecture.

**Conjecture 3.5.8** Let  $M$  be a hyperkähler manifold that is AC of order  $l \geq 1$ , and let  $k \geq -1$  be an integer. Let  $x : M \rightarrow \mathbb{H}$  be a smooth function, so that  $D(x)$  is a 1-form on  $M$ , where  $D$  is the operator of §3.1. Suppose that  $\nabla^a D(x) = O(t^{k-a-1})$  for  $a = 0, 1, 2$ , where  $\nabla$  is the Levi-Civita connection on  $M$ . Then there exists a smooth function  $y : M \rightarrow \mathbb{H}$  such that  $D(x) = D(y)$ , and  $y = O(t^k)$ . Moreover, if  $x$  takes values in  $\mathbb{I}$ , then  $y$  can be chosen to take values in  $\mathbb{I}$ .

This conjecture is a result in analysis, and I believe that it can be proved using existing mathematical ideas and techniques. Much work has been done on a similar problem, that of studying the *harmonic* functions of polynomial growth on an *asymptotically flat* manifold. Some relevant examples are [22, Th. 9.2, p. 76], [4, Th. 1.17, p. 674], and in particular the proof of Theorem 3.1 in [4, p. 678]. See also [23], in which Li and Yau study *holomorphic* functions of subquadratic growth on an asymptotically flat Kähler manifold.

These results give strong relations between the harmonic functions of polynomial growth on a Riemannian manifold  $M$  asymptotic to  $\mathbb{R}^n$ , and the harmonic polynomials on  $\mathbb{R}^n$ . Conjecture 3.5.8 is modelled on them. I believe that generalizing from asymptotically flat to asymptotically conical manifolds should be easy. The problems will come from dealing with the operator  $D$  rather than  $\Delta$ . One possible tool to use here is Baston's elliptic complex of operators [5, p. 43-44] resolving  $D$ .

**Conjecture 3.5.9** Let  $C$  be a hyperkähler cone. Then the graded  $H$ -algebra  $B$  of Definition 3.5.2 is an SGH-algebra.

In fact, the author's calculations suggest that hyperkähler cones  $C$  always have  $B^{2k} \cong \mathbb{R}^a \otimes S_{\mathbb{H}}^k Y$ , where  $a \geq 0$  is an integer depending on  $k$  and  $Y$  is the  $\mathbb{A}\mathbb{H}$ -module of §2.2, and that most  $C$  also have  $B^{2k+1} = \{0\}$ . Conjecture 3.5.9 would follow immediately from this. Assuming Conjectures 3.5.8 and 3.5.9, we will now prove Theorem 3.5.6.

*Sketch proof of Theorem 3.5.6.* Let  $j \geq 0$  be an integer. We shall construct a linear map  $\phi_j : B_j/B_{j-l} \rightarrow P_j/P_{j-l}$ . Let  $b \in B_j$ . Then  $b$  is a  $q$ -holomorphic function on  $C$ ,

with polynomial growth of degree  $j$ . With a partition of unity, one may construct a smooth,  $\mathbb{H}$ -valued function  $x$  on  $M$ , such that  $\Phi_*(x) = b$  on  $N \times (2, \infty) \subset C$ . Using equations (18) and (19) and the fact that  $\nabla^a b = O(t^{j-a})$  on  $C$  for large  $t$  (this is easily proved), it can be shown that  $\nabla^a D(x) = O(t^{j-l-a-1})$  for  $a = 0, 1, 2$  on  $M$ .

Putting  $k = j - l$ , or  $k = -1$  if  $l > j + 1$ , we may apply Conjecture 3.5.8. It shows that there exists a smooth,  $\mathbb{H}$ -valued function  $y$  on  $M$  with  $y = O(t^{j-l})$  or  $y = O(t^{-1})$  respectively, such that  $D(x) = D(y)$  on  $M$ . Therefore,  $D(x - y) = 0$ , so  $x - y$  is  $q$ -holomorphic on  $M$ . Clearly,  $x - y$  has polynomial growth of order  $j$ , so  $x - y \in P_j$ . Now  $y$  may not be unique, but since  $y = O(t^{j-l})$  or  $O(t^{-1})$ , any two solutions  $y$  differ by an element of  $P_{j-l}$  or  $P_{-1} = \{0\}$ . Therefore,  $x - y + P_{j-l}$  is a well-defined element of  $P_j/P_{j-l}$ , depending only on  $b$ .

Define a map  $\phi_j : B_j/B_{j-l} \rightarrow P_j/P_{j-l}$  by  $\phi_j(b + B_{j-l}) = x - y + P_{j-l}$ . Then  $\phi_j$  is a well-defined  $\mathbb{H}$ -linear map. Suppose that  $b \in B'_j$ . Then  $b$  takes values in  $\mathbb{I}$ . So  $x$  takes values in  $\mathbb{I}$ , and by Conjecture 3.5.8 we may choose  $y$  to take values in  $\mathbb{I}$ . Thus  $x - y \in P'_j$ , and  $\phi_j$  maps  $B'_j/B'_{j-l}$  to  $P'_j/P'_{j-l}$ . Therefore  $\phi_j$  is an  $\text{A}\mathbb{H}$ -morphism, provided  $B_j/B_{j-l}$  and  $P_j/P_{j-l}$  are  $\text{A}\mathbb{H}$ -modules. With a little more work, one shows that  $\phi_j$  is an  $\text{A}\mathbb{H}$ -isomorphism.

It remains to verify the conditions of Definition 2.3.5. From the definition of  $\phi_j$  it immediately follows that  $\phi_j$  takes  $B_k/B_{j-l}$  to  $P_k/P_{j-l}$  for  $j - l \leq k \leq j$ , and also that  $\phi_j = \phi_{j+1}$  on  $B_j/B_{j-l+1}$ . By Conjecture 3.5.9,  $B_j/B_{j-l}$  is a stable  $\text{A}\mathbb{H}$ -module, so  $P_j/P_{j-l}$  is also a stable  $\text{A}\mathbb{H}$ -module. The equation  $\mu_{jkl}^P \circ (\phi_j \otimes_{\mathbb{H}} \phi_k) = \phi_{j+k} \circ \mu_{jkl}^B$  comes naturally out of the construction of  $\phi_j$ . Thus  $B$  is isomorphic to  $P$  to order  $l$ , by definition. Finally, it follows easily that  $B$  is the associated graded  $\text{H}$ -algebra of  $P$ . ■

#### 4. EXAMPLES, APPLICATIONS AND CONCLUSIONS

This chapter was difficult to write, because of the many examples, little bits of theory, and quaternionic versions of this and that which begged to be included. For reasons of time and space I have been ruthless, discussing a few topics only, and not in great depth. However, I think that one could easily fill another paper the length of this one with interesting material.

Section 4.1 finds the  $\text{HP}$ -algebra of  $q$ -holomorphic functions of polynomial growth on  $\mathbb{H}$ , in a series of simple steps, as an example. In §4.2 we give examples of how the programme of §3.4 (to recover a hypercomplex manifold from its  $\text{H}$ -algebra) may fail. Then §4.3 shows how to make  $\text{HL}$ -algebras and  $\text{HP}$ -algebras out of ordinary Lie algebras. This device is applied in §4.4, which is about hyperkähler manifolds with symmetries. The high point of §4.4 is a (conjectural) algebraic method to explicitly construct ‘coadjoint orbit’ hyperkähler manifolds, using  $\text{HP}$ -algebras.

In §4.5 we look at at the simplest nontrivial hyperkähler manifold — the Eguchi-Hanson space. It fits into our theory both as an  $\text{AC}$  hyperkähler manifold, and as a ‘coadjoint orbit’. A careful investigation of the  $\text{H}$ -algebra and its deformations

reveals a surprise: an unexpected family of singular hypercomplex structures with remarkable properties. Section 4.6 interprets self-dual connections, or ‘instantons’, over a hypercomplex manifold, as modules over its H-algebra. Finally, §4.7 concludes the paper with some research problems.

**4.1. Q-holomorphic functions on  $\mathbb{H}$ .** Let  $\mathbb{H}$  have real coordinates  $(x_0, \dots, x_3)$ , so that  $(x_0, \dots, x_3)$  represents  $x_0 + x_1i_1 + x_2i_2 + x_3i_3$ . Now  $\mathbb{H}$  is naturally a hypercomplex manifold with complex structures given by  $I_1dx_2 = dx_3$ ,  $I_2dx_3 = dx_1$ ,  $I_3dx_1 = dx_2$  and  $I_jdx_0 = dx_j$ , for  $j = 1, 2, 3$ . The study of q-holomorphic functions on  $\mathbb{H}$  is called *quaternionic analysis*, and is surveyed in [27].

**Example 4.1.1** First we shall determine the  $\text{A}\mathbb{H}$ -module  $U$  of all linear q-holomorphic functions on  $\mathbb{H}$ . Let  $q_0, \dots, q_3 \in \mathbb{H}$ , and define  $u = q_0x_0 + \dots + q_3x_3$  as an  $\mathbb{H}$ -valued function on  $\mathbb{H}$ . A calculation shows that  $u$  is q-holomorphic if and only if  $q_0 + q_1i_1 + q_2i_2 + q_3i_3 = 0$ . It follows that  $U \cong \mathbb{H}^3$ . Also,  $U'$  is the vector subspace of  $U$  with  $q_j \in \mathbb{I}$  for  $j = 0, \dots, 3$ . Let us identify  $U$  with  $\mathbb{H}^3$  explicitly by taking  $(q_1, q_2, q_3)$  as quaternionic coordinates. Then

$$U' = \{(q_1, q_2, q_3) \in \mathbb{H}^3 : q_j \in \mathbb{I} \text{ for } j = 1, 2, 3 \text{ and } q_1i_1 + q_2i_2 + q_3i_3 \in \mathbb{I}\}.$$

Thus  $U' \cong \mathbb{R}^8$ , and  $\dim U = 4j$ ,  $\dim U' = 2j + r$  with  $j = 3$  and  $r = 2$ , so the virtual dimension of  $U$  is 2. This is because  $\mathbb{H} \cong \mathbb{C}^2$ , so the complex dimension of  $\mathbb{H}$  is 2. It is easy to see that  $U$  is a stable  $\text{A}\mathbb{H}$ -module.

**Example 4.1.2** Let  $k \geq 0$  be an integer, and let  $U^{(k)}$  be the  $\text{A}\mathbb{H}$ -module of q-holomorphic functions on  $\mathbb{H}$  that are homogeneous polynomials of degree  $k$ . We shall determine  $U^{(k)}$ . Write  $A$  for the H-algebra of q-holomorphic functions on  $\mathbb{H}$ , and  $\mu_A : A \otimes_{\mathbb{H}} A \rightarrow A$  for the multiplication map. By Example 4.1.1,  $U^{(1)} = U \subset A$ . Thus  $\mu_A$  induces an  $\text{A}\mathbb{H}$ -morphism  $\mu_A : U \otimes_{\mathbb{H}} U \rightarrow A$ , and composing  $\mu_A$   $k-1$  times gives an  $\text{A}\mathbb{H}$ -morphism  $\mu_A^{k-1} : \otimes_{\mathbb{H}}^k U \rightarrow A$ . Clearly,  $\text{Im } \mu_A^{k-1} \subset U^{(k)}$ . Also,  $\mu_A^{k-1}$  is symmetric in the  $k$  factors of  $U$ , so it makes sense to restrict to  $S_{\mathbb{H}}^k U$ .

Thus we have constructed an  $\text{A}\mathbb{H}$ -morphism  $\mu_A^{k-1} : S_{\mathbb{H}}^k U \rightarrow U^{(k)}$ . It is easy to show that  $\mu_A^{k-1}$  is injective on  $S_{\mathbb{H}}^k U$ . By Example 4.1.1,  $U$  is stable with  $j = 3$  and  $r = 2$ . Thus Proposition 1.2.8 shows that  $\dim S_{\mathbb{H}}^k U = 2(k+1)(k+2)$ . But Sudbery [27, Th. 7, p. 217] shows that  $\dim U^{(k)} = 2(k+1)(k+2)$ . It follows that  $\mu_A^{k-1}$  is an isomorphism, and  $U^{(k)} \cong S_{\mathbb{H}}^k U$ .

The interpretation of Example 4.1.2 is simple. If  $V$  is the linear polynomials on some vector space, then  $S^k V$  is the homogeneous polynomials of degree  $k$ . Here we have a quaternionic analogue of this, replacing  $S^k$  by  $S_{\mathbb{H}}^k$ . We have found an elegant construction of the spaces  $U^{(k)}$ , important in quaternionic analysis, that gives insight into their algebraic structure and dimension.

**Example 4.1.3** Let us consider the filtered H-algebra  $P$  of q-holomorphic functions of polynomial growth on  $\mathbb{H}$ , as in §3.2. Clearly, the functions in  $U^{(k)}$  have polynomial



growth of order  $k$ , so that  $U^{(k)} \subset P_k \subset P$ . Thus  $\bigoplus_{j=0}^k U^{(j)} \subset P_k$ , and  $\bigoplus_{j=0}^\infty U^{(j)} \subset P$ . Now it is a well-known result in complex analysis that all holomorphic functions on  $\mathbb{C}$  of polynomial growth, are polynomials. The obvious analogue of this is that all q-holomorphic functions on  $\mathbb{H}$  of polynomial growth are sums of elements of  $U^{(k)}$ .

This is in fact true, and can be proved using the theory in [27]. Therefore  $P_k = \bigoplus_{j=0}^k U^{(j)}$ , and  $P = \bigoplus_{j=0}^\infty U^{(j)}$ . But from Example 4.1.2,  $U^{(j)} = S_{\mathbb{H}}^j U$ . Thus  $P = \bigoplus_{j=0}^\infty S_{\mathbb{H}}^j U$ . So, by Definition 2.4.3,  $P$  is isomorphic to the free algebra  $F^U$  generated by  $U$ , with its natural filtration. As  $U$  is finite-dimensional,  $P$  is finitely-generated, and in particular,  $P$  is an FGH-algebra in the sense of §2.4.

The full H-algebra  $A$  of q-holomorphic functions on  $\mathbb{H}$  is obtained by completing  $P$ , by adding in convergent power series. The analytic details are beyond the scope of this paper. Note, however, that because  $A$  contains all holomorphic functions on  $\mathbb{C}^2$ ,  $A$  is certainly not finitely-generated, so that  $P$  has a much simpler structure than  $A$ . We may generalize this example to the hypercomplex manifold  $\mathbb{H}^n$ . It is easy to see that the H-algebra of q-holomorphic functions on  $\mathbb{H}^n$  of polynomial growth is  $F^{nU}$ , the free H-algebra generated by  $n$  copies of  $U$ .

Now  $\mathbb{H}$  is a hyperkähler manifold, so by Theorem 3.3.4,  $A$  and  $P$  should be HP-algebras. We shall define the HP-algebra structure on  $P$ .

**Example 4.1.4** We must construct an AH-morphism  $\xi_P : P \otimes_{\mathbb{H}} P \rightarrow P \otimes_{\mathbb{H}} Y$ . From above  $U = U^{(1)} \subset P$ , so consider  $\xi_P : U \otimes_{\mathbb{H}} U \rightarrow P \otimes_{\mathbb{H}} Y$ . Since  $\xi_P$  is antisymmetric, we may restrict to  $\Lambda_{\mathbb{H}}^2 U$ . Now  $U$  is stable and has  $j = 3, r = 2$ , so by Proposition 1.2.8, we have  $\dim \Lambda_{\mathbb{H}}^2 U = 8$  and  $\dim (\Lambda_{\mathbb{H}}^2 U)' = 5$ . But these are the same dimensions as those of the AH-module  $Y$  of Definition 2.2.1. In fact there is a natural isomorphism  $\Lambda_{\mathbb{H}}^2 U \cong Y$ . Now  $U^{(0)} \cong \mathbb{H}$ , so that  $U^{(0)} \otimes_{\mathbb{H}} Y \cong Y$ . Thus we have AH-isomorphisms  $\Lambda_{\mathbb{H}}^2 U^{(1)} \cong U^{(0)} \otimes_{\mathbb{H}} Y \cong Y$ .

It is easy to show that the restriction of  $\xi_P$  to  $\Lambda_{\mathbb{H}}^2 U^{(1)}$  gives exactly this isomorphism  $\Lambda_{\mathbb{H}}^2 U^{(1)} \cong U^{(0)} \otimes_{\mathbb{H}} Y$ . Thus we have defined  $\xi_P$  on a generating subspace  $U^{(1)}$  for  $P$ . Using Axiom P2, we may extend  $\xi_P$  *uniquely* to all of  $P$ , because the action of  $\xi_P$  on the generators defines the whole action. Now  $P$  is a filtered H-algebra, and  $\xi_P$  satisfies  $\xi_P(P_j \otimes_{\mathbb{H}} P_k) \subset P_{j+k-2}$ . Thus Axiom PF of §2.3 holds, and  $P$  is a filtered HP-algebra.

**4.2. Hypercomplex manifolds undetermined by their H-algebras.** In §3.4 we explained how, under good conditions, it is possible to reconstruct a hypercomplex manifold from an H-algebra of q-holomorphic functions upon it. Here are three examples where this cannot be done, illustrating different ways in which the reconstruction can fail.

**Example 4.2.1** Since  $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{H}$ ,  $\mathbb{Z}$  acts on  $\mathbb{H}$  by translation, and so  $\mathbb{H}/\mathbb{Z} \cong \mathbb{R}^3 \times \mathbb{S}^1$  is a hyperkähler manifold. We shall determine the filtered H-algebra  $P$  of q-holomorphic functions on  $M = \mathbb{H}/\mathbb{Z}$  of polynomial growth. But  $\mathbb{H}$  covers  $M$ , so any

q-holomorphic function of polynomial growth on  $M$  lifts to a q-holomorphic function of polynomial growth on  $\mathbb{H}$ . So by Example 4.1.3,  $P$  is the H-subalgebra of  $F^U$  that is invariant under  $\mathbb{Z}$ .

Clearly, any polynomial invariant under  $\mathbb{Z}$  is also invariant under  $\mathbb{R} \subset \mathbb{H}$ . Referring to Example 4.1.3, the elements of  $U$  invariant under  $\mathbb{R}$  are those with  $q_0 = 0$ . Therefore, the  $\mathbb{R}$ -invariant polynomials in  $U$  are  $\{(q_1, q_2, q_3) \in \mathbb{H}^3 : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\} \subset U$ . But this is the AH-module  $Y$  of §2.2. So, one may show that the H-algebra  $P$  of q-holomorphic functions on  $M$  of polynomial growth is the free H-algebra  $F^Y$ .

Now consider reconstructing  $M$  from  $P$ , as in §3.4. We have  $P = F^Y$ , so we put  $Q = Y$ . Then  $Q^\dagger = \mathbb{I} = \mathbb{R}^3$ . As  $P = F^Q$ ,  $M_{P,Q}$  is the whole of  $Q^\dagger = \mathbb{R}^3$ . But  $M \cong \mathbb{R}^3 \times \mathcal{S}^1$ , and the map  $\pi_{P,Q} : M \rightarrow M_{P,Q}$  is simply the projection to the first factor  $\mathbb{R}^3 \times \mathcal{S}^1 \rightarrow \mathbb{R}^3$ . Therefore, in this case,  $M_{P,Q}$  is a manifold, and  $\pi_{P,Q}$  is surjective, but not (even locally) injective. Thus, we cannot recover the manifold  $M$  from  $P$ .

**Example 4.2.2** Much of Atiyah and Hitchin’s book [2] is an in-depth study of a particular complete, noncompact hyperkähler 4-manifold, the 2-monopole moduli space  $M_2^0$ . As a manifold,  $M_2^0$  is diffeomorphic to a complex line bundle over  $\mathbb{R}P^2$ , and the zero section gives a submanifold  $\mathbb{R}P^2 \subset M_2^0$ . The isometry group of the metric  $g$  on  $M_2^0$  is  $SO(3)$ . This isometry group acts in a *nontrivial* way on the hyperkähler structure. There is a vector space isomorphism  $\langle I_1, I_2, I_3 \rangle \cong \mathfrak{so}(3)$ , that identifies the action of  $SO(3)$  on  $\langle I_1, I_2, I_3 \rangle$  with the adjoint action on  $\mathfrak{so}(3)$ .

Because  $SO(3)$  does not fix the hyperkähler structure, there are no hyperkähler moment maps (see §4.4). However, there are some Kähler moment maps. Let  $v_1, v_2, v_3 \in \mathfrak{so}(3)$  be identified with  $I_1, I_2, I_3$  under the isomorphism above. Then  $v_1, v_2, v_3$  are Killing vectors of  $g$  on  $M$ . Let  $\alpha_1, \alpha_2, \alpha_3$  be the 1-forms on  $M_2^0$  dual to  $v_1, v_2, v_3$  under  $g$ . Then a brief calculation shows that the 1-form  $I_j \alpha_k + I_k \alpha_j$  is closed, for  $j, k = 1, \dots, 3$ . Since  $b^1(M_2^0) = 0$ , there exists a unique real function  $f_{jk}$  with  $df_{jk} = I_j \alpha_k + I_k \alpha_j$ , and such that  $\int_{\mathbb{R}P^2} f_{jk} dA = 0$ .

The  $f_{jk}$  form a vector space  $S^2 \mathbb{R}^3 \cong \mathbb{R}^6$  of real functions on  $M_2^0$ . For  $j, k = 1, 2, 3$ , let  $q_{jk} \in \mathbb{H}$  with  $q_{jk} = q_{kj}$ , and define  $z = \sum_{j,k=1}^3 q_{jk} f_{jk}$ . When is  $z$  a q-holomorphic function on  $M_2^0$ ? Calculation shows that  $z$  is q-holomorphic if and only if  $\sum_{j=1}^3 q_{jk} i_j = 0$  for  $k = 1, 2, 3$ . Therefore, the AH-module of q-holomorphic  $z$  is

$$(20) \quad Z = \langle f_{22} - f_{33} + 2i_1 f_{23}, f_{33} - f_{11} + 2i_2 f_{31}, f_{11} - f_{22} + 2i_3 f_{12} \rangle.$$

Note that only the trace-free part  $S_0^2 \mathfrak{so}(3) \cong \mathbb{R}^5$  appears here. From (20) we find that  $Z \cong \mathbb{H}^3$  and  $Z' \cong \mathbb{R}^7$ , and in fact there is a canonical isomorphism  $Z \cong Y \otimes_{\mathbb{H}} Y$ , where  $Y$  is the AH-module defined in §2.2.

We have found an AH-module  $Z \cong Y \otimes_{\mathbb{H}} Y$  of q-holomorphic functions on  $M_2^0$ . By Lemma 2.4.4 there is an H-algebra morphism  $\phi_Z : F^Z \rightarrow A$ , where  $A$  is the H-algebra of q-holomorphic functions on  $M_2^0$ . Now Atiyah and Hitchin [2] define the metric on  $M_2^0$  explicitly, using an elliptic integral. Their construction uses transcendental

functions, not just algebraic functions. Because of this, it can be shown that the 5 functions  $f_{jk}$  used in (20) are algebraically independent. Therefore, there can be no polynomial relations between them, so that  $\phi_Z$  is injective.

Thus  $F^Z$  is an H-algebra of q-holomorphic functions on  $M_2^0$ . Now  $Z^\dagger \cong \mathbb{R}^5$ , and since  $F^Z$  is free,  $M_{F^Z, Z}$  is the whole of  $Z^\dagger$ . Thus  $\pi_{F^Z, Z}$  maps  $M_2^0$  to  $\mathbb{R}^5$ . It can be shown that  $\pi_{F^Z, Z}$  is generically injective, but because of the dimensions  $\pi_{F^Z, Z}$  cannot be surjective. Thus we cannot recover the manifold  $M_2^0$  from  $F^Z$ . I claim that  $F^Z$  is in fact the whole H-algebra of q-holomorphic functions of polynomial growth on  $M_2^0$ .

**Example 4.2.3** Let  $k \geq 2$ , and let  $M = \mathbb{H}^k$ . Then by Example 4.1.3, the H-algebra of q-holomorphic functions of polynomial growth on  $M$  is  $F^{kU}$ . Now  $U \cong \mathbb{H}^3$ , so  $kU \cong \mathbb{H}^{3k}$ . It can be shown that if  $k \geq 2$ , then the generic  $\mathbb{H}$ -submodule  $Q \cong \mathbb{H}^{3k-1}$  of  $kU$  is a stable A $\mathbb{H}$ -module with  $Q' \cong \mathbb{R}^{8k-4}$  and  $Q^\dagger = \mathbb{R}^{4k}$ . Let  $Q$  be such an A $\mathbb{H}$ -submodule, and let  $P = F^Q \subset F^U$ . Then  $P$  is an H-algebra of q-holomorphic functions on  $M = \mathbb{H}^k$ .

Consider reconstructing  $M$  and its hypercomplex structure from  $P$ , as in §3.4. Since  $P$  is free,  $M_{P, Q}$  is the whole of  $Q^\dagger$ , which is  $\mathbb{R}^{4k}$ . But  $M \cong \mathbb{R}^{4k}$ , and in fact  $\pi_{P, Q} : M \rightarrow M_{P, Q}$  is a diffeomorphism, the identity. However, in Lemma 3.4.3 we have  $V_m \cong Q$  for each  $m \in M$ , so that  $\dim V_m = 12k - 4$ . Thus, the lemma shows that the hypercomplex structure of  $M$  cannot be recovered from  $P$ . This means that  $P$  gives full information about the manifold  $M$ , but only partial information about the hypercomplex structure.

**4.3. HL-algebras and HP-algebras.** Here is a simple construction of HL-algebras.

**Example 4.3.1** Let  $\mathfrak{g}$  be a Lie algebra. Then the Lie bracket  $[\cdot, \cdot]$  on  $\mathfrak{g}$  gives a linear map  $\lambda : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ , such that  $\lambda(x \otimes y) = [x, y]$  for  $x, y \in \mathfrak{g}$ . Let  $Y$  be the A $\mathbb{H}$ -module defined in §2.2, and define  $A_{\mathfrak{g}}$  to be the A $\mathbb{H}$ -module  $\mathfrak{g} \otimes Y$ . Then  $A_{\mathfrak{g}} \otimes_{\mathbb{H}} A_{\mathfrak{g}} \cong (\mathfrak{g} \otimes \mathfrak{g}) \otimes (Y \otimes_{\mathbb{H}} Y)$  and  $A_{\mathfrak{g}} \otimes_{\mathbb{H}} Y \cong \mathfrak{g} \otimes (Y \otimes_{\mathbb{H}} Y)$ . Define a linear map  $\xi_{A_{\mathfrak{g}}} : A_{\mathfrak{g}} \otimes_{\mathbb{H}} A_{\mathfrak{g}} \rightarrow A_{\mathfrak{g}} \otimes_{\mathbb{H}} Y$  by  $\xi_{A_{\mathfrak{g}}} = \lambda \otimes \text{id}$ , as a map from  $(\mathfrak{g} \otimes \mathfrak{g}) \otimes (Y \otimes_{\mathbb{H}} Y)$  to  $\mathfrak{g} \otimes (Y \otimes_{\mathbb{H}} Y)$ . It is easy to show that  $\xi_{A_{\mathfrak{g}}}$  satisfies Axiom P1 of §2.2, using the Jacobi identity for  $\mathfrak{g}$  to prove part (iii). Therefore,  $A_{\mathfrak{g}}$  is an HL-algebra, by Definition 2.2.2.

**Example 4.3.2** Example 4.1.4 constructed a filtered HP-algebra  $P$ . Because  $\xi_P(P_j \otimes_{\mathbb{H}} P_k) \subset P_{j+k-2}$ , each of  $P_0, P_1$  and  $P_2$  are closed under  $\xi_P$ . Therefore, restricting  $\xi_P$  to  $P_0, P_1$  and  $P_2$  gives them the structure of HL-algebras. Now  $P_0 \cong \mathbb{H}$  and  $\xi_P$  is zero on  $P_0$ , but  $P_1 \cong \mathbb{H} \oplus U$ ,  $P_2 \cong \mathbb{H} \oplus U \oplus S_{\mathbb{H}}^2 U$ , which are both finite-dimensional, and  $\xi_P$  is nontrivial on both. It is easy to see that none of these is of the form  $\mathfrak{g} \otimes Y$ . Thus, there exist nontrivial, finite-dimensional HL-algebras that do not arise from Example 4.3.1.

The next example is an aside about Lie and Poisson algebras.

**Example 4.3.3** Let  $\mathfrak{g}$  be a Lie algebra. The *symmetric algebra*  $S(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} S^k \mathfrak{g}$  of  $\mathfrak{g}$  is a free, commutative algebra generated by  $\mathfrak{g}$ . Define a bracket  $\{, \} : S(\mathfrak{g}) \times S(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  as follows. Let  $k, l \geq 0$  be integers. If  $x, y \in \mathfrak{g}$ , then  $x^k \in S^k \mathfrak{g}$  and  $y^l \in S^l \mathfrak{g}$ , and  $S^k \mathfrak{g}, S^l \mathfrak{g}$  are generated by such elements. When  $k = 0$  or  $l = 0$  define  $\{, \} = 0$  on  $S^k \mathfrak{g} \times S^l \mathfrak{g}$ , and when  $k, l > 0$  define  $\{x^k, y^l\} = kl \sigma(x^{k-1} \otimes [x, y] \otimes y^{l-1})$ . Here  $[, ]$  is the Lie bracket on  $\mathfrak{g}$  and  $\sigma : S^{k-1} \mathfrak{g} \otimes \mathfrak{g} \otimes S^{l-1} \mathfrak{g} \rightarrow S^{k+l-1} \mathfrak{g}$  is the symmetrization operator, a projection.

This definition extends uniquely to give a bilinear operator  $\{, \} : S^k \mathfrak{g} \times S^l \mathfrak{g} \rightarrow S^{k+l-1} \mathfrak{g}$ , so we have found a bilinear bracket  $\{, \}$  on  $S(\mathfrak{g}) \times S(\mathfrak{g})$ . It is shown in [7, §1.4] that  $\{, \}$  is a Poisson bracket on  $S(\mathfrak{g})$ , so that  $S(\mathfrak{g})$  is a Poisson algebra, the Poisson algebra of the Lie algebra  $\mathfrak{g}$ .

Using Example 4.3.3 as a model, here is a construction of HP-algebras.

**Example 4.3.4** Let  $A$  be an HL-algebra. Then §2.4 defines the free H-algebra  $F^A$  generated by  $A$ . To make  $F^A$  into an HP-algebra, we must give a Poisson bracket  $\xi_{F^A}$  on  $F^A$ . Let  $k, l$  be positive integers, and define a map  $\xi_{k,l} : S_{\mathbb{H}}^k A \otimes_{\mathbb{H}} S_{\mathbb{H}}^l A \rightarrow S_{\mathbb{H}}^{k+l-1} A$  by  $\xi_{k,l} = kl \sigma_{\mathbb{H}} \circ (\text{id} \otimes_{\mathbb{H}} \xi_A \otimes_{\mathbb{H}} \text{id}) \circ \iota$ , using the sequence of maps

$$\begin{aligned} S_{\mathbb{H}}^k A \otimes_{\mathbb{H}} S_{\mathbb{H}}^l A &\xrightarrow{\iota} S_{\mathbb{H}}^{k-1} A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} S_{\mathbb{H}}^{l-1} A \xrightarrow{\text{id} \otimes_{\mathbb{H}} \xi_A \otimes_{\mathbb{H}} \text{id}} \\ &S_{\mathbb{H}}^{k-1} A \otimes_{\mathbb{H}} A \otimes_{\mathbb{H}} Y \otimes_{\mathbb{H}} S_{\mathbb{H}}^{l-1} A \xrightarrow{\sigma_{\mathbb{H}}} S_{\mathbb{H}}^{k+l-1} A \otimes_{\mathbb{H}} Y. \end{aligned}$$

Here  $\iota$  is the inclusion, and  $\sigma_{\mathbb{H}}$  the symmetrization operator of §1.1. For  $k = 0$  or  $l = 0$ , let  $\xi_{k,l} = 0$ . Define  $\xi_{F^A} : F^A \otimes_{\mathbb{H}} F^A \rightarrow F^A \otimes_{\mathbb{H}} Y$  to be the unique linear map such that the restriction of  $\xi_{F^A}$  to  $S_{\mathbb{H}}^k A \otimes_{\mathbb{H}} S_{\mathbb{H}}^l A$  is  $\xi_{k,l}$ , for all  $k, l \geq 0$ . Then  $\xi_{F^A}$  is a well-defined AH-morphism. A calculation following those in [7, §1.4] shows that  $\xi_{F^A}$  satisfies Axioms P1 and P2 of §2.2. Thus, by Definition 2.2.2,  $F^A$  is an HP-algebra. Moreover,  $F^A$  is a filtered H-algebra with the natural filtration, and it is easy to show that Axiom PF holds. So  $F^A$  is a filtered HP-algebra.

Combining Examples 4.3.1 and 4.3.4, we see that if  $\mathfrak{g}$  is a Lie algebra, then  $F^{A\mathfrak{g}}$  is a filtered HP-algebra.

**4.4. Hyperkähler manifolds with symmetries.** Let  $M$  be a hyperkähler manifold, and suppose  $v$  is a Killing vector of the hyperkähler structure on  $M$ . A *hyperkähler moment map for  $v$*  is a triple  $(f_1, f_2, f_3)$  of smooth real functions on  $M$  such that  $\alpha = I_1 df_1 = I_2 df_2 = I_3 df_3$ , where  $\alpha$  is the 1-form dual to  $v$  under the metric  $g$ . Moment maps always exist if  $b^1(M) = 0$ , and are unique up to additive constants.

More generally, let  $M$  be a hyperkähler manifold, let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ , and suppose  $\Phi : G \rightarrow \text{Aut}(M)$  is a homomorphism from  $G$  to the group of automorphisms of the hyperkähler structure on  $M$ . Let  $\phi : \mathfrak{g} \rightarrow \text{Vect}(M)$  be the induced map from  $\mathfrak{g}$  to the Killing vectors. Then a *hyperkähler moment map for the action  $\Phi$  of  $G$*  is a triple  $(f_1, f_2, f_3)$  of smooth functions from  $M$  to  $\mathfrak{g}^*$ , such that for each  $x \in \mathfrak{g}$ ,  $(x \cdot f_1, x \cdot f_2, x \cdot f_3)$  is a hyperkähler moment map for the vector field

$\phi(x)$ , and in addition,  $(f_1, f_2, f_3)$  is equivariant under the action  $\Phi$  of  $G$  on  $M$  and the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

Moment maps are a familiar part of symplectic geometry, and hyperkähler moment maps were introduced by Hitchin et al. as part of a quotient construction for hyperkähler manifolds [12], [26, p. 118-122]. Hyperkähler moment maps will exist under quite mild conditions on  $M$  and  $G$ , for instance if  $b^1(M) = 0$  and  $G$  is compact. We shall use them to construct  $q$ -holomorphic functions on hyperkähler manifolds with symmetries.

**Example 4.4.1** Let  $M$  be a hyperkähler manifold and  $v$  a nonzero Killing vector of the hyperkähler structure on  $M$ . Suppose  $(f_1, f_2, f_3)$  is a hyperkähler moment map for  $v$ . We shall make  $q$ -holomorphic functions on  $M$  out of the real functions  $f_j$ . Let  $q_1, q_2, q_3 \in \mathbb{H}$ , and consider the  $\mathbb{H}$ -valued function  $y = q_1 f_1 + q_2 f_2 + q_3 f_3$  on  $M$ . By construction the  $f_j$  satisfy  $I_1 df_1 = I_2 df_2 = I_3 df_3$ . Using this equation, the fact that  $v$  is nonzero, and the definition of  $q$ -holomorphic in §3.1, it is easy to show that  $y$  is  $q$ -holomorphic if and only if  $q_1 i_1 + q_2 i_2 + q_3 i_3 = 0$ . Thus we have constructed an  $\text{A}\mathbb{H}$ -module  $Y \cong \{(q_1, q_2, q_3) \in \mathbb{H}^3 : q_1 i_1 + q_2 i_2 + q_3 i_3 = 0\}$  of  $q$ -holomorphic functions on  $M$ . It is isomorphic to the  $\text{A}\mathbb{H}$ -module  $Y$  of Definition 2.2.1.

**Example 4.4.2** Now let  $M$  be a hyperkähler manifold,  $G$  a Lie group,  $\Phi : G \rightarrow \text{Aut}(M)$  an action of  $G$  on  $M$  preserving the hyperkähler structure, and  $\phi : \mathfrak{g} \rightarrow \text{Vect}(M)$  the induced map. Suppose  $\phi$  is injective, and that  $(f_1, f_2, f_3)$  is a hyperkähler moment map for  $\Phi$ . By Example 4.4.1, each nonzero  $x \in \mathfrak{g}$  gives us an  $\text{A}\mathbb{H}$ -module  $Y$  of  $q$ -holomorphic functions on  $M$ . Clearly, these fit together to form a canonical  $\text{A}\mathbb{H}$ -module  $\mathfrak{g} \otimes Y$  of  $q$ -holomorphic functions on  $M$ .

We have already met  $\mathfrak{g} \otimes Y = A_{\mathfrak{g}}$  as an  $\text{HL}$ -algebra in Example 4.3.1, where we defined a Poisson bracket  $\xi_{A_{\mathfrak{g}}}$  on it. In the context of this example,  $\mathfrak{g} \otimes Y$  is an  $\text{A}\mathbb{H}$ -submodule of the  $\text{HP}$ -algebra  $A_M$  of  $q$ -holomorphic functions on  $M$ , which derives its own Poisson bracket  $\xi_{A_M}$  from the hyperkähler structure of  $M$ . A computation shows that  $\mathfrak{g} \otimes Y$  is closed under  $\xi_{A_M}$ , and that  $\xi_{A_M} = \xi_{A_{\mathfrak{g}}}$ .

Example 4.4.2 shows that the  $\text{HL}$ -algebras of Example 4.3.1 are related to the  $\text{HP}$ -algebras of hyperkähler manifolds with symmetry groups. In the following lemma we extend this to the associated  $\text{HP}$ -algebras defined by Example 4.3.4.

**Lemma 4.4.3.** *Let  $M$  be a hyperkähler manifold, and  $A_M$  the  $\text{HP}$ -algebra of  $q$ -holomorphic functions on  $M$ . Let  $G$  be a Lie group,  $\Phi : G \rightarrow \text{Aut}(M)$  be an action of  $G$  on  $M$  preserving the hyperkähler structure, and  $\phi : \mathfrak{g} \rightarrow \text{Vect}(M)$  be the induced map. Suppose  $(f_1, f_2, f_3)$  is a hyperkähler moment map for  $\Phi$ . Then there is a canonical  $\text{HP}$ -algebra morphism  $\Phi_* : F^{A_{\mathfrak{g}}} \rightarrow A_M$ , where  $F^{A_{\mathfrak{g}}}$  is defined by Examples 4.3.1 and 4.3.4.*

*Proof.* In Example 4.4.2 we constructed an  $\text{A}\mathbb{H}$ -submodule  $A_{\mathfrak{g}} = \mathfrak{g} \otimes Y$  of  $A_M$ . By Lemma 2.4.4, there is a unique  $\text{H}$ -algebra morphism  $\Phi_* = \phi_{A_{\mathfrak{g}}} : F^{A_{\mathfrak{g}}} \rightarrow A_M$ . To

complete the proof we must show that  $\Phi_*$  is an HP-algebra morphism, where the HP-algebra structure on  $F^{A_{\mathfrak{g}}}$  is defined by Example 4.3.4, so we must show that  $\Phi_*$  identifies the Poisson brackets on  $F^{A_{\mathfrak{g}}}$  and  $A_M$ .

In Example 4.3.4 we remarked that  $\xi_{A_M} = \xi_{A_{\mathfrak{g}}}$  on  $A_{\mathfrak{g}}$ . Thus  $\Phi_*$  identifies the Poisson brackets on this subspace. Because  $\Phi_*$  is an H-algebra morphism and  $A_{\mathfrak{g}}$  generates  $F^{A_{\mathfrak{g}}}$ , we can deduce from Axiom P2 that if  $\xi_{A_{\mathfrak{g}}}$  and the pullback of  $\xi_{A_M}$  agree on  $A_{\mathfrak{g}}$ , they must agree on the whole of  $F^{A_{\mathfrak{g}}}$ . Thus  $\Phi_*$  identifies the Poisson brackets of  $F^{A_{\mathfrak{g}}}$  and  $A_M$ , and  $\Phi_*$  is an HP-algebra morphism.  $\square$

In the situation of the lemma,  $\Phi_*(F^{A_{\mathfrak{g}}})$  is an HP-algebra containing information about  $M$  and  $G$ . This suggests that to understand hyperkähler manifolds with symmetries better, it may be helpful to study HP-algebras that are images of  $F^{A_{\mathfrak{g}}}$ . The next two examples construct such images.

**Example 4.4.4** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Then  $A_{\mathfrak{g}} = \mathfrak{g} \otimes Y$  and  $Y^\dagger = \mathbb{I}$  by definition, so  $A_{\mathfrak{g}}^\dagger = \mathfrak{g}^* \otimes \mathbb{I}$ . As in the proof of Lemma 3.4.2, each element  $f$  of  $F^{A_{\mathfrak{g}}}$  induces an  $\mathbb{H}$ -valued polynomial  $\psi_f$  on  $A_{\mathfrak{g}}^\dagger = \mathfrak{g}^* \otimes \mathbb{I}$ . Now  $G$  acts on  $\mathfrak{g}^*$  by the coadjoint action, so  $G$  acts on  $\mathfrak{g}^* \otimes \mathbb{I}$ , with trivial action on  $\mathbb{I}$ . Let  $\Omega \subset \mathfrak{g}^* \otimes \mathbb{I}$  be an orbit of  $G$ , that is contained in no proper vector subspace of  $\mathfrak{g}^* \otimes \mathbb{I}$ . Define an  $\mathbb{A}\mathbb{H}$ -submodule  $I^\Omega$  in  $F^{A_{\mathfrak{g}}}$  by

$$(21) \quad I^\Omega = \{f \in F^{A_{\mathfrak{g}}} : \psi_f \equiv 0 \text{ on } \Omega\}.$$

Clearly,  $\mu_{F^{A_{\mathfrak{g}}}}(I^\Omega \otimes_{\mathbb{H}} F^{A_{\mathfrak{g}}}) \subset I^\Omega$ , so  $I^\Omega$  is an *ideal*.

Define a subset  $M^\Omega \subset \mathfrak{g}^* \otimes \mathbb{I}$  by

$$(22) \quad M^\Omega = \{x \in \mathfrak{g}^* \otimes \mathbb{I} : \psi_f(x) = 0 \text{ for all } f \in I^\Omega\}.$$

Then  $M^\Omega$  is an affine real algebraic variety in  $\mathfrak{g}^* \otimes \mathbb{I}$ , as in Lemma 3.4.2. Also,  $\Omega \subset M^\Omega$  by (21), and  $M^\Omega$  is invariant under the action of  $G$  on  $\mathfrak{g}^* \otimes \mathbb{I}$ . Define  $A^\Omega = \{\psi_f|_{M^\Omega} : f \in F^{A_{\mathfrak{g}}}\}$ . Thus  $A^\Omega$  is a vector space of  $\mathbb{H}$ -valued functions on  $M^\Omega$ .

Now  $I^\Omega$ ,  $F^{A_{\mathfrak{g}}}$  and  $A^\Omega$  fit into a natural,  $\mathbb{A}\mathbb{H}$ -exact sequence

$$(23) \quad 0 \rightarrow I^\Omega \xrightarrow{\iota} F^{A_{\mathfrak{g}}} \xrightarrow{\rho} A^\Omega \rightarrow 0,$$

where  $\iota$  is the inclusion map, and  $\rho$  is the restriction map from  $\mathfrak{g}^* \otimes \mathbb{I}$  to  $M^\Omega$ . Since  $I^\Omega$  is an ideal, intuitively  $A^\Omega$  should be an H-algebra.

In §2.4 we saw that the best sort of ideal is a *stable filtered ideal*. Therefore, let us assume that  $I^\Omega$  is a stable filtered ideal. Then Lemma 2.4.2 shows that  $A^\Omega$  is an SFH-algebra, and  $\rho$  a filtered H-algebra morphism. Also, because we suppose that  $\Omega$  is not contained in any proper subspace of  $\mathfrak{g}^* \otimes \mathbb{I}$ , and  $\Omega \subset M^\Omega$ , we see that  $\rho|_{\mathfrak{g} \otimes Y}$  is injective, and  $\mathfrak{g} \otimes Y$  is an  $\mathbb{A}\mathbb{H}$ -submodule of  $A^\Omega$  that generates  $A^\Omega$ . It can be shown that  $M_{A^\Omega, \mathfrak{g} \otimes Y} = M^\Omega$ . Observe that  $A^\Omega$  is an FGH-algebra in the sense of §2.4.

**Example 4.4.5** In the situation of the previous example, we shall show that  $A^\Omega$  is an HP-algebra. The linear map  $\xi_{F^{A_{\mathfrak{g}}}} : F^{A_{\mathfrak{g}}} \otimes_{\mathbb{H}} A_{\mathfrak{g}} \rightarrow F^{A_{\mathfrak{g}}} \otimes_{\mathbb{H}} Y$  is easy to understand,

because  $A_{\mathfrak{g}} = \mathfrak{g} \otimes Y$ , so we may write the map as  $\xi_{F^{A_{\mathfrak{g}}}} : \mathfrak{g} \otimes (F^{A_{\mathfrak{g}}} \otimes_{\mathbb{H}} Y) \rightarrow F^{A_{\mathfrak{g}}} \otimes_{\mathbb{H}} Y$ , which just gives the Lie algebra action of  $\mathfrak{g}$  on  $F^{A_{\mathfrak{g}}}$ . Since  $I^{\Omega}$  is  $G$ -invariant, it follows that  $\xi_{F^{A_{\mathfrak{g}}}}$  maps  $I^{\Omega} \otimes_{\mathbb{H}} A_{\mathfrak{g}}$  to  $I^{\Omega} \otimes_{\mathbb{H}} Y$ . But  $A_{\mathfrak{g}}$  generates  $F^{A_{\mathfrak{g}}}$ , and so using Axiom P2 and the fact that  $\mu_{F^{A_{\mathfrak{g}}}}(I^{\Omega} \otimes_{\mathbb{H}} F^{A_{\mathfrak{g}}}) \subset I^{\Omega}$ , it follows that  $\xi_{F^{A_{\mathfrak{g}}}}(I^{\Omega} \otimes_{\mathbb{H}} F^{A_{\mathfrak{g}}}) \subset I^{\Omega} \otimes_{\mathbb{H}} Y$ .

Using the assumption in Example 4.4.4, this inclusion is just what is needed to prove that the Poisson bracket  $\xi_{F^{A_{\mathfrak{g}}}}$  can be pushed down to  $A^{\Omega}$  using  $\rho$ , inducing a Poisson bracket  $\xi_{A^{\Omega}}$  on  $A^{\Omega}$ , so that  $A^{\Omega}$  is an HP-algebra. As  $\rho$  is a filtered H-algebra morphism,  $A^{\Omega}$  is a filtered HP-algebra, and  $\rho$  a filtered HP-algebra morphism.

Examples 4.4.4 and 4.4.5 construct a large family of HP-algebras  $A^{\Omega}$  associated to Lie groups. As in §3.4, we can try to use  $A^{\Omega}$  to construct a hyperkähler structure on  $M^{\Omega}$ . This suggests that associated to each Lie group  $G$ , there is a natural family of hyperkähler manifolds. Now Kronheimer [19, 20], Biquard [8] and Kovalev [16] have also constructed hyperkähler manifolds associated to Lie groups, from a completely different point of view.

Let  $G$  be a compact Lie group with Lie algebra  $\mathfrak{g}$ , and let the complexification of  $G$  be  $G^c$  with Lie algebra  $\mathfrak{g}^c$ . Kronheimer found that certain moduli spaces of singular  $G$ -instantons on  $\mathbb{R}^4$  are hyperkähler manifolds. These moduli spaces can be identified with coadjoint orbits of  $G^c$  in  $(\mathfrak{g}^c)^*$ , and have hyperkähler metrics invariant under  $G$ . Kronheimer’s construction worked only for certain special coadjoint orbits, and more general cases were handled by Biquard and Kovalev.

Although these metrics look very algebraic, their construction is in fact analytic, and the algebraic description of these metrics is not well understood. I propose that Examples 4.4.4 and 4.4.5 provide this algebraic description. This was proved in [14, §11-12] for Kronheimer’s metrics [19, 20], which are the simplest case, but I have not yet proved it for the metrics of Biquard and Kovalev. Here is a conjecture about this.

**Conjecture 4.4.6** We conjecture the following relations between Examples 4.4.4 and 4.4.5, and Kronheimer, Biquard and Kovalev’s ‘coadjoint orbit’ metrics.

- The assumption made in Example 4.4.4 always holds.
- Using the techniques of §3.4, the HP-algebra  $A^{\Omega}$  determines a hyperkähler structure on a dense open set of  $M^{\Omega}$ .
- These hyperkähler structures on subsets of  $M^{\Omega}$  include all those of [19], [20], [8] and [16] as special cases. However, generically the structures on  $M^{\Omega}$  are new, and do not coincide with those of Kronheimer, Biquard and Kovalev.
- For some  $\Omega$ ,  $M^{\Omega}$  is a cone in  $\mathfrak{g}^* \otimes \mathbb{I}$ , and  $A^{\Omega}$  is an SGH-algebra. Then  $M^{\Omega}$  is a hyperkähler cone, as in §3.5. These are the *nilpotent orbits* of [20].
- For each  $\Omega$  there is an  $\tilde{\Omega}$ , such that  $M^{\tilde{\Omega}}$  is a ‘nilpotent orbit’, and the associated graded H-algebra of  $A^{\Omega}$  is  $A^{\tilde{\Omega}}$ . Then the (singular) hyperkähler manifold  $M^{\Omega}$  has an AC end, as in Definition 3.5.5, with asymptotic cone  $M^{\tilde{\Omega}}$ .

Further study of these spaces from the HP-algebra point of view will probably lead to a much clearer understanding of the algebra and geometry underlying Kronheimer’s metrics. We will look at an example in detail in the next section.

**4.5. The Eguchi-Hanson space.** The Eguchi-Hanson space  $M$  is a noncompact hyperkähler manifold of dimension 4, with a metric written down by Eguchi and Hanson [9]. It is of interest to us as an example for two reasons. Firstly, it is the simplest interesting example of a ‘coadjoint orbit’ metric, as it fits into the theory of §4.4 with  $G = SU(2)$  or  $SO(3)$ . Secondly, it is the simplest hyperkähler *asymptotically locally Euclidean* or *ALE space*. An ALE space is an AC hyperkähler manifold of dimension 4, with asymptotic cone  $\mathbb{H}/\Gamma$ , for some finite subgroup  $\Gamma \subset SU(2)$ .

The ALE spaces for cyclic  $\Gamma$  were described explicitly by Gibbons and Hawking [10] and Hitchin [11], and a complete construction and classification of ALE spaces was given by Kronheimer [17], [18]. The Eguchi-Hanson space has asymptotic cone  $\mathbb{H}/\{\pm 1\}$ , and as a manifold it is diffeomorphic to the total space of  $T^*\mathbb{C}P^1$ . In this section we will treat the Eguchi-Hanson space from our H-algebra point of view. It gives us our first explicit example of the theories of §3.5 and §4.4. On the way, we will determine the deformations of the H-algebra of the Eguchi-Hanson space. The result and its implications are quite surprising.

Let  $M$  be the Eguchi-Hanson space. Then the hyperkähler cone of  $M$ , in the sense of §3.5, is  $\mathbb{H}/\{\pm 1\}$ . Our first step is to find the SGH-algebra  $B$  of  $q$ -holomorphic functions on  $\mathbb{H}/\{\pm 1\}$  with polynomial growth.

**Example 4.5.1** By Example 4.1.3, the graded H-algebra of  $q$ -holomorphic functions on  $\mathbb{H}$  is  $F^U$ , where  $U$  is defined in the example. Define  $\sigma : \mathbb{H} \rightarrow \mathbb{H}$  by  $\sigma(q) = -q$ . Then  $\sigma$  acts on  $F^U$ , and as the functions in  $U$  are linear,  $\sigma$  acts as  $-1$  on  $U$ . Therefore  $\sigma$  acts as  $(-1)^j$  on  $S_{\mathbb{H}}^j U$ . Now the SGH-algebra  $B$  of  $q$ -holomorphic functions on  $\mathbb{H}/\{\pm 1\}$  is just the  $\sigma$ -invariant part of  $F^U$ . Therefore  $B = \bigoplus_{j=0}^{\infty} S_{\mathbb{H}}^{2j} U$ . For compatibility with §3.5 we adopt the grading  $B^{2j} = S_{\mathbb{H}}^{2j} U$  and  $B^{2j+1} = \{0\}$ , and the corresponding filtration.

A calculation using Proposition 1.2.8 and the definitions of  $Y$  and  $U$  in §§2.2 and 4.1 shows that  $\dim(S_{\mathbb{H}}^j Y) = 4k$ ,  $\dim(S_{\mathbb{H}}^j Y)' = 2k + s$  with  $k = j + 1$  and  $s = 1$ , and  $\dim(S_{\mathbb{H}}^{2j} U) = 4l$ ,  $\dim(S_{\mathbb{H}}^{2j} U)' = 2l + t$  with  $l = (2j + 1)(j + 1)$  and  $t = 2j + 1$ . Thus  $l = (2j + 1)k$  and  $t = (2j + 1)s$ . In fact, by carefully investigating the geometry it can be shown that  $S_{\mathbb{H}}^{2j} U \cong \mathbb{R}^{2j+1} \otimes S_{\mathbb{H}}^j Y$ .

Here is one way to see this. It is well-known that  $SO(4)$  acts irreducibly on  $\mathbb{R}^4$ , but that  $\mathbb{R}^4 \otimes \mathbb{C}$  splits as  $\mathbb{C}^2 \otimes \mathbb{C}^2$ . In the same way,  $U$  is irreducible, but  $U \otimes \mathbb{C}$  may be written as  $\mathbb{C}^2 \otimes W$ , where  $W$  is a complex AH-module with  $W \otimes_{\mathbb{H}} W = S_{\mathbb{H}}^2 W \cong Y \otimes \mathbb{C}$  as complex AH-modules. Thus we may write  $B^{2j} \cong \mathbb{R}^{2j+1} \otimes S_{\mathbb{H}}^j Y$ , where  $\mathbb{R}^{2j+1}$  is the real part of  $\mathbb{C}^{2j+1} = S^j(\mathbb{C}^2)$ , and  $B = \bigoplus_{j=0}^{\infty} \mathbb{R}^{2j+1} \otimes S_{\mathbb{H}}^j Y$ .

Next, we present  $B$  as an FGH-algebra generated by  $Q \subset B$ , and determine the algebraic variety  $M_{B,Q}$ .



**Example 4.5.2** To write  $B$  as an FGH-algebra, the quotient of a free  $\mathbb{H}$ -algebra  $F^Q$  by an ideal  $I \subset F^Q$ , one must choose a subspace  $Q$  of  $B$  that generates  $B$ . Define  $Q = B^2 = \mathbb{R}^3 \otimes Y$ , where  $\mathbb{R}^3$  is equipped with a Euclidean metric and inner product. Then  $\phi_Q : F^Q \rightarrow B$  is defined in Lemma 2.4.4, and we set  $I = \text{Ker } \phi_Q$ . Since  $B^{2j} = \mathbb{R}^{2j+1} \otimes S_{\mathbb{H}}^j Y$  and  $S_{\mathbb{H}}^j Q = \mathbb{R}^{(j+1)(j+2)/2} \otimes S_{\mathbb{H}}^j Y$ , it can be shown that  $\phi_Q : S_{\mathbb{H}}^j Q \rightarrow B^{2j}$  is surjective for all  $j \geq 0$ , and has kernel  $I^{2j} \subset S_{\mathbb{H}}^j Q$  with  $I^{2j} \cong \mathbb{R}^{j(j-1)/2} \otimes S_{\mathbb{H}}^j Y$ .

Now  $S_{\mathbb{H}}^j Y$  is stable by Proposition 1.2.8, as  $Y$  is stable. Thus  $I$  is a stable filtered ideal in the sense of Definition 2.4.1. Therefore  $B$  is generated by  $Q$ , and  $B$  is an FGH-algebra by Definition 2.4.5. Set  $J = I^4 \cong S_{\mathbb{H}}^2 Y$ . Another calculation shows that  $I$  is generated by  $J$  in the sense of Definition 2.4.1. Moreover, if  $h$  is the Euclidean metric on  $\mathbb{R}^3$ , then the  $\mathbb{A}\mathbb{H}$ -module  $J \subset S_{\mathbb{H}}^2 Q$  is  $J = \langle h \rangle \otimes S_{\mathbb{H}}^2 Y \subset S^2 \mathbb{R}^3 \otimes S_{\mathbb{H}}^2 Y = S_{\mathbb{H}}^2 Q$ .

Thus  $B$  is the quotient of the free SFH-algebra  $F^Q$ , where  $Q = \mathbb{R}^3 \otimes Y$ , by the stable filtered ideal  $I$  in  $F^Q$ , where  $I$  is generated by the  $\mathbb{A}\mathbb{H}$ -module  $J = S_{\mathbb{H}}^2 Y \subset S_{\mathbb{H}}^2 Q$ . This is an explicit description of  $B$  as an FGH-algebra. To finish the example, we will find and describe the subset  $M_{B,Q}$  of  $Q^\dagger$ , defined in §3.4. As  $Q = \mathbb{R}^3 \otimes Y$ ,  $Q^\dagger = (\mathbb{R}^3)^* \otimes \mathbb{I}$ , as  $Y^\dagger \cong \mathbb{I}$ . Identify  $(\mathbb{R}^3)^*$  with  $\mathbb{R}^3$  using the metric. Then a point  $\gamma$  in  $Q^\dagger$  may be written  $\gamma = \sum_{k=1}^3 v_k \otimes i_k$ , where  $v_1, v_2, v_3$  are vectors in  $\mathbb{R}^3$ .

The proof of Lemma 3.4.2 showed that each element  $y$  of  $F^Q$  defines an  $\mathbb{H}$ -valued polynomial  $\psi_y$  on  $Q^\dagger$ . In our case, as  $J$  generates  $I$ ,  $M_{B,Q}$  is the zeros of the polynomials  $\psi_y$  for  $y \in J$ . Moreover, as  $J \subset S_{\mathbb{H}}^2 Q$ , these polynomials are homogeneous quadratics on  $Q^\dagger$ . A computation using the definitions of  $Y$ ,  $Q$  and  $J$  shows that  $\gamma \in M_{B,Q}$  if and only if

$$(24) \quad v_1 \cdot v_1 = v_2 \cdot v_2 = v_3 \cdot v_3, \quad v_1 \cdot v_2 = 0, \quad v_2 \cdot v_3 = 0, \quad v_3 \cdot v_1 = 0,$$

where ‘ $\cdot$ ’ is the inner product on  $\mathbb{R}^3$ . So  $v_1, v_2, v_3$  must be orthogonal in  $\mathbb{R}^3$ , and of equal length.

Now we look at  $M_{B,Q}$  more closely, and interpret it as a ‘coadjoint orbit’.

**Example 4.5.3** The previous example identified the algebraic variety  $M_{B,Q}$  with the set of triples  $(v_1, v_2, v_3)$  of vectors in  $\mathbb{R}^3$  that are orthogonal and of equal length. For each  $r > 0$ , define  $S_{r,+}$  to be the set of orthogonal triples  $(v_1, v_2, v_3)$  that form a positively oriented basis of  $\mathbb{R}^3$ , and for which  $|v_1| = |v_2| = |v_3| = r$ . Let  $S_{r,-}$  be defined in the same way, but with negative orientation. Then  $S_{r,+}$  and  $S_{r,-}$  are subsets of  $M_{B,Q}$ .

Now  $SO(3)$  acts on  $\mathbb{R}^3$  preserving the Euclidean metric, and the action preserves the equations (24). Thus  $M_{B,Q}$  decomposes into orbits of the  $SO(3)$ -action. It is easy to show that the orbits are the single point  $(0, 0, 0)$  and the sets  $S_{r,\pm}$  for  $r > 0$ . Moreover,  $SO(3)$  acts freely on the set  $S_{r,\pm}$ , so that  $S_{r,\pm} \cong SO(3) \cong \mathbb{R}\mathbb{P}^3$ . Therefore  $M_{B,Q}$  is the disjoint union of  $\{0\}$  and 2 copies of  $(0, \infty) \times \mathbb{R}\mathbb{P}^3$ .

However,  $\mathbb{H}/\{\pm 1\}$  is the disjoint union of  $\{0\}$  and 1 copy of  $(0, \infty) \times \mathbb{R}\mathbb{P}^3$ . Therefore,  $M_{B,Q}$  is actually the union of two distinct copies of  $\mathbb{H}/\{\pm 1\}$ , which meet at 0, and the map  $\pi_{B,Q} : \mathbb{H}/\{\pm 1\} \rightarrow M_{B,Q}$  is an isomorphism with one of these copies. This

is something of a surprise. The 2 copies are not separate algebraic components of the variety  $M_{B,Q}$ , as the complexification of  $M_{B,Q}$  in  $Q^\dagger \otimes \mathbb{C}$  has only one component containing both copies.

Let us look at  $B$  from the point of view of §4.4. Above we gave an action of  $SO(3)$  on  $M_{B,Q}$ . This action induces an isomorphism between the  $\mathbb{R}^3$  in the equation  $Q = \mathbb{R}^3 \otimes Y$ , with the Lie algebra  $\mathfrak{so}(3)$  of  $SO(3)$ . Thus  $Q \cong A_{\mathfrak{so}(3)}$ , where  $A_{\mathfrak{so}(3)}$  is defined in Example 4.3.1. So  $B$  is the quotient of  $F^{A_{\mathfrak{so}(3)}}$  by a stable filtered ideal. Therefore,  $B$  fits into framework of Example 4.4.4.

In fact,  $S_{r\pm}$  are orbits in  $\mathfrak{so}(3)^* \otimes \mathbb{I}$ , and if we put  $\Omega = S_{r,+}$  or  $S_{r,-}$  for any  $r > 0$ , then calculation shows that  $A^\Omega = B$ , so that  $M^\Omega = M_{B,Q}$ . This is the simplest example of an HP-algebra of this form, and it is one of the ‘nilpotent orbits’ mentioned in Conjecture 4.4.6.

Our goal, remember, is to find the filtered H-algebra  $P$  of q-holomorphic functions of polynomial growth on the Eguchi-Hanson space  $M$ . From the definition [9] of the Eguchi-Hanson metric one finds that  $M$  is AC to order 4, with asymptotic cone  $\mathbb{H}/\{\pm 1\}$ . Therefore, Theorem 3.5.6 applies to show that  $P$  is an SFH-algebra and  $B$  and  $P$  are isomorphic to order 4, in the sense of Definition 2.3.5. In the next example we will determine all such SFH-algebras.

**Example 4.5.4** Suppose  $P$  is an SFH-algebra isomorphic to order 4 with the H-algebra  $B$  of the previous examples. As  $B_{-2}, P_{-2}$  are zero,  $\phi_2 : B_2 \rightarrow P_2$  is an Aℍ-isomorphism. Thus  $P_2 \cong \mathbb{H} \oplus Q$ . Since  $Q$  generates  $B$ , Proposition 2.4.6 shows that  $P_2$  generates  $P$ . But the  $\mathbb{H}$  in  $P_2$  is the multiples of 1, so it does not generate anything. Therefore  $P$  is generated by  $Q$ , and  $P$  is the quotient of  $F^Q$  by a stable filtered ideal  $I^P$ .

Define  $J^P = I_4^P$ , so that  $J^P$  is an Aℍ-submodule of  $\mathbb{H} \oplus Q \oplus S_{\mathbb{H}}^2 Q$ . By a similar argument to Proposition 2.4.6, we find that because  $J$  generates  $I$ ,  $J^P$  generates  $I^P$ . Thus  $P$  is determined by the Aℍ-submodule  $J^P$ . Using the maps  $\phi_2, \phi_4$  to compare  $I^P$  and  $I$ , we see that  $I_2^P = \{0\}$ , that  $J^P \subset \mathbb{H} \oplus J$ . Let  $\pi_1, \pi_2$  be the projections from  $\mathbb{H} \oplus J$  to the  $\mathbb{H}$  and  $J$ . Then  $\pi_2 : J^P \rightarrow J$  is an Aℍ-isomorphism. Let  $\iota : J \rightarrow J^P$  be its inverse, so that  $\iota : J \rightarrow \mathbb{H} \oplus J$  is an Aℍ-morphism with  $J^P = \iota(J)$ .

As  $\pi_2 \circ \iota$  is the identity on  $J$ ,  $J^P$  is determined by  $\pi_1 \circ \iota$ , i.e. by an Aℍ-morphism from  $J$  to  $\mathbb{H}$ . Let  $\lambda \in J^\dagger$ . Then  $\lambda \in J^\times$ , so  $\lambda : J \rightarrow \mathbb{H}$  is an  $\mathbb{H}$ -linear map. By definition of  $J^\dagger$ ,  $\lambda$  maps  $J$  to  $\mathbb{I}$  which is  $\mathbb{H}'$ . Thus  $\lambda$  is an Aℍ-morphism from  $J$  to  $\mathbb{H}$ , and it is easy to see that all such Aℍ-morphisms are elements of  $J^\dagger$ .

Therefore, we have shown that all SFH-algebras  $P$  isomorphic with  $B$  to order 4 (in the sense of Definition 2.3.5) are constructed in the following manner. We are given  $Q = \mathbb{R}^3 \otimes Y$ , and a fixed Aℍ-module  $J \subset S_{\mathbb{H}}^2 Q$ , that has  $J \cong S_{\mathbb{H}}^2 Y$ , so that  $J^\dagger \cong \mathbb{R}^5$ . Choose an element  $\lambda \in J^\dagger$ . Then  $\lambda : J \rightarrow \mathbb{H}$  is an Aℍ-morphism. Define  $J^\lambda$  to be the image of the Aℍ-morphism  $\lambda \oplus \text{id} : J \rightarrow \mathbb{H} \oplus J$ .

As  $\mathbb{H} = S_{\mathbb{H}}^0 Q$  and  $J \subset S_{\mathbb{H}}^2 Q$ , we may regard  $\mathbb{H} \oplus J$  as an Aℍ-submodule of  $F^Q$ . Since  $J^\lambda$  is an Aℍ-submodule of  $\mathbb{H} \oplus J$ , it is an Aℍ-submodule of  $F^Q$ . Define  $I^\lambda$

by  $I^\lambda = \mu_{F^Q}(J^\lambda \otimes_{\mathbb{H}} F^Q)$ . Then  $I^\lambda$  is a stable filtered ideal of  $F^Q$ . Define  $P^\lambda$  to be the quotient of  $F^Q$  by  $I^\lambda$ , following Lemma 2.4.2. By Definition 2.4.5,  $P^\lambda$  is an FGH-algebra.

Next we identify the variety  $M_{P^\lambda, Q}$ .

**Example 4.5.5** Let  $\lambda \in J^\dagger$  be given. Then the previous example defines an FGH-algebra  $P^\lambda$ , generated by  $Q \subset P^\lambda$ . We shall describe the real algebraic variety  $M_{P^\lambda, Q} \subset Q^\dagger$ . Our treatment follows Example 4.5.2 closely, and uses the same notation. Now  $J^\lambda$  may be interpreted as a vector space of  $\mathbb{H}$ -valued polynomials on  $Q^\dagger$ , and  $M_{P^\lambda, Q}$  is the zeros of these polynomials.

There is a natural identification between  $(S_{\mathbb{H}}^2 Y)^\dagger$  and  $S_0^2 \mathbb{R}^3$ , which is the space of trace-free  $3 \times 3$  symmetric matrices. Using this identification we may write  $\lambda$  in coordinates as  $\lambda = (a_{kl})$ , where  $(a_{kl})$  is a  $3 \times 3$  matrix with  $a_{jk} = a_{kj}$  and  $\sum_j a_{jj} = 0$ . Using the vectors  $v_1, v_2, v_3$  as coordinates on  $Q^\dagger$  as before, the equations defining  $M_{P^\lambda, Q}$  turn out to be

$$(25) \quad \begin{aligned} v_1 \cdot v_1 - a_{11} &= v_2 \cdot v_2 - a_{22} = v_3 \cdot v_3 - a_{33}, \\ v_1 \cdot v_2 &= a_{12}, \quad v_2 \cdot v_3 = a_{23}, \quad v_3 \cdot v_1 = a_{31}. \end{aligned}$$

These are 5 real equations, because  $(J^\lambda)^\dagger \cong \mathbb{R}^5$ . When  $\lambda = 0$ ,  $a_{kl} = 0$  and we recover the equations (24).

As in Example 4.5.3, the equations (25) are invariant under the action of  $SO(3)$  on  $\mathbb{R}^3$ . Thus  $M_{P^\lambda, Q}$  is invariant under the  $SO(3)$ -action, and is a union of  $SO(3)$  orbits. Choose any point  $m = (v_1, v_2, v_3)$  in  $M_{P^\lambda, Q}$  such that  $v_1, v_2, v_3$  are linearly independent (this is true for a generic point), and let  $\Omega$  be the orbit of  $m$ . Then it can be shown that  $A^\Omega = P^\lambda$  and  $M^\Omega = M_{P^\lambda, Q}$ , as in Example 4.5.3. Thus  $P^\lambda$  is one of the algebras  $A^\Omega$  of Example 4.4.4, with Lie group  $G = SO(3)$ . Therefore, by Example 4.4.5,  $P^\lambda$  is a filtered HP-algebra.

Finally, we interpret  $M_{P^\lambda, Q}$ , and describe its hyperkähler structure.

**Example 4.5.6** In Example 4.5.5, the element  $\lambda \in J^\dagger$  determines a matrix  $(a_{jk})$ . Now  $SO(3)$  acts on  $\mathbb{I}$ , inducing automorphisms of  $\mathbb{H}$ . But automorphisms of  $\mathbb{H}$  act on HP-algebras in a natural way. Thus  $SO(3)$  acts on the category of HP-algebras. This action of  $SO(3)$  on the HP-algebras  $P^\lambda$  turns out to be conjugation of the matrix  $(a_{jk})$  by elements of  $SO(3)$ . So, up to automorphisms of  $\mathbb{H}$ , we may suppose that  $P^\lambda$  is defined by a matrix  $(a_{jk})$  that is diagonal, and for which  $a_{11} \geq a_{22} \geq a_{33}$ .

Let the matrix  $(a_{jk})$  be of this form. Then the equations (25) can be rewritten

$$(26) \quad v_1 \cdot v_1 - a = v_2 \cdot v_2 - b = v_3 \cdot v_3, \quad v_1 \cdot v_2 = 0, \quad v_2 \cdot v_3 = 0, \quad v_3 \cdot v_1 = 0,$$

where  $a, b$  are constants with  $a \geq b \geq 0$ . We shall consider the following three cases separately. Case 1 is  $a = b = 0$ , case 2 is  $a > 0, b = 0$ , and case 3 is  $a \geq b > 0$ . In

case 1, we have  $\lambda = 0$ , so  $P^\lambda = B$  and  $M_{P^\lambda, Q}$  is 2 copies of  $\mathbb{H}/\{\pm 1\}$  meeting at 0, as in Example 4.5.3.

In case 2, there is one special orbit of  $SO(3)$ , the orbit  $\{(v_1, 0, 0) : v_1 \cdot v_1 = a\}$ . Clearly this is a 2-sphere  $S^2$ . It is easy to see that  $M_{P^\lambda, Q}$  is the union of  $S^2$  with 2 copies of  $(0, \infty) \times \mathbb{R}P^3$ . Also,  $M_{P^\lambda, Q}$  is singular at  $S^2$  but nonsingular elsewhere. A more careful investigation shows that  $M_{P^\lambda, Q}$  is actually the union of 2 nonsingular, embedded submanifolds  $M_1, M_2$  of  $Q^\dagger$ , that meet in a common  $S^2$ . In fact, both  $M_1$  and  $M_2$  are the Eguchi-Hanson space. This case is the HP-algebra of the Eguchi-Hanson space  $M$ , and the map  $\pi_{P^\lambda, Q} : M \rightarrow M_{P^\lambda, Q}$  is a diffeomorphism from  $M$  to  $M_1$ , say.

Thus, we have found the HP-algebra  $P^\lambda$  of q-holomorphic functions of polynomial growth on the Eguchi-Hanson space  $M$ . When we attempt to reconstruct  $M$  from  $P^\lambda$  we find that  $M_{P^\lambda, Q}$  contains not one but two distinct copies of  $M$ , that intersect in an  $S^2$ .

In case 3,  $SO(3)$  acts freely on  $M_{P^\lambda, Q}$ , and  $M_{P^\lambda, Q}$  is a *nonsingular* submanifold of  $Q^\dagger$  diffeomorphic to  $\mathbb{R} \times \mathbb{R}P^3$ . This 4-manifold has two ends, each modelled on that of  $\mathbb{H}/\{\pm 1\}$ . In cases 1 and 2, we saw that  $M_{P^\lambda, Q}$  was the singular union of 2 copies of  $\mathbb{H}/\{\pm 1\}$  or the Eguchi-Hanson space; in case 3 this singular union is resolved into one nonsingular 4-manifold.

Let us apply the programme of §3.4, to construct a hypercomplex structure on  $M_{P^\lambda, Q}$ . This can be done, and as  $Q$  is stable, Proposition 3.4.5 shows that the hypercomplex structure is determined by  $P^\lambda$  wherever it exists. At first sight, therefore, it seems that  $M_{P^\lambda, Q}$  carries a nonsingular hypercomplex structure with two ends, both asymptotic to  $\mathbb{H}/\{\pm 1\}$ . However, explicit calculation reveals that although  $M_{P^\lambda, Q}$  is nonsingular, its hypercomplex structure has a singularity on the hypersurface  $v_3 = 0$  in  $M_{P^\lambda, Q}$ , which is diffeomorphic to  $\mathbb{R}P^3$ .

What is the nature of this hypersurface singularity? Consider the involution  $(v_1, v_2, v_3) \mapsto (-v_1, -v_2, -v_3)$  of  $M_{P^\lambda, Q}$ . This preserves the hypercomplex structure, but it is orientation-reversing on  $M_{P^\lambda, Q}$ . It also preserves the hypersurface  $v_3 = 0$ . Now a hypercomplex structure has its own natural orientation. Therefore, the hypercomplex structure changes orientation over its singular hypersurface  $v_3 = 0$ .

This singular hypercomplex manifold has a remarkable property. Each element of  $P^\lambda$  is a q-holomorphic function on  $M_{P^\lambda, Q}$ , which is smooth, as  $M_{P^\lambda, Q}$  is a submanifold of  $Q^\dagger$ . Thus, the hypercomplex manifold has a full complement of q-holomorphic functions that *extend smoothly over the singularity*. I shall christen singularities with this property *invisible singularities*, because you cannot see them using q-holomorphic functions. I think this phenomenon is worth further study.

As  $P^\lambda$  is an HP-algebra,  $M_{P^\lambda, Q}$  is actually hyperkähler. In Example 4.5.5 we showed that this hyperkähler structure is  $SO(3)$ -invariant. Now in [6], Belinskii et al. explicitly determine all hyperkähler metrics with an  $SO(3)$ -action of this form, by solving an ODE. Thus, the metric on  $M_{P^\lambda, Q}$  is given in [6]. Belinskii et al. show that

the singularity is a curvature singularity – that is, the Riemann curvature becomes infinite upon it.

The metrics can also be seen from the twistor point of view. In [11], Hitchin constructs the twistor spaces of some ALE spaces, including the Eguchi-Hanson space. He uses a polynomial  $z^k + a_1 z^{k-1} + \dots + a_k$ , introduced on [11, p. 467]. On p. 468 he assumes this polynomial has a certain sort of factorization, to avoid singularities. If this assumption is dropped, then in the case  $k=2$ , Hitchin’s construction yields the twistor spaces of our singular hyperkähler manifolds  $M_{P\lambda, Q}$ .

**4.6. Self-dual vector bundles and H-algebra modules.** First we define q-holomorphic sections of a vector bundle.

**Definition 4.6.1** Let  $M$  be a hypercomplex manifold. Let  $E$  be a vector bundle over  $M$ , and  $\nabla_E$  a connection on  $E$ . Define an operator  $D_E : \mathbb{H} \otimes C^\infty(E) \rightarrow C^\infty(E \otimes T^*M)$  by

$$D_E(1 \otimes e_0 + i_1 \otimes e_1 + i_2 \otimes e_2 + i_3 \otimes e_3) = \nabla_E e_0 + I_1(\nabla_E e_1) + I_2(\nabla_E e_2) + I_3(\nabla_E e_3),$$

where  $I_1, I_2, I_3$  act on the  $T^*M$  factor of  $E \otimes T^*M$ , and  $e_0, \dots, e_3 \in C^\infty(E)$ . We call an element  $e$  of  $\mathbb{H} \otimes C^\infty(E)$  a *q-holomorphic section* if  $D_E(e) = 0$ .

Define  $Q_{M,E}$  to be the vector space of q-holomorphic sections in  $\mathbb{H} \otimes C^\infty(E)$ . Then  $Q_{M,E}$  is closed under the  $\mathbb{H}$ -action  $p \cdot (q \otimes e) = (pq) \cdot e$  on  $\mathbb{H} \otimes C^\infty(E)$ , so  $Q_{M,E}$  is an  $\mathbb{H}$ -module. Define a real vector subspace  $Q'_{M,E}$  by  $Q'_{M,E} = Q_{M,E} \cap \mathbb{I} \otimes C^\infty(E)$ . As in Definition 3.2.1, one can show that  $Q_{M,E}$  is an  $\text{AH}$ -module.

The point of this definition is the following theorem. The proof follows that of Theorem 3.2.5 in §3.2 very closely, so we leave it as an exercise.

**Theorem 4.6.2.** *Let  $M$  be a hypercomplex manifold, and  $A_M$  the H-algebra of q-holomorphic functions on  $M$ . Let  $E$  be a vector bundle over  $M$ , with connection  $\nabla_E$ . Then the  $\text{AH}$ -module  $Q_{M,E}$  of Definition 4.6.1 is a module over the H-algebra  $A_M$ , in a natural way.*

Recall that modules over H-algebras were defined in Definition 2.1.1. To see the link between Theorems 3.2.5 and 4.6.2, put  $E = \mathbb{R}$  with the flat connection. Then a q-holomorphic section of  $E$  is a q-holomorphic function, so  $Q_{M,E} = A_M$ , and Theorem 4.6.2 states that  $A_M$  is a module over itself. But this follows trivially from Theorem 3.2.5.

Now the equation  $D_E(e) = 0$  is in general overdetermined, and for a *generic* connection  $\nabla_E$  we find that  $Q_{M,E} = \{0\}$  for  $\dim M > 4$ , and  $Q'_{M,E} = \{0\}$  for  $\dim M = 4$ . In these cases, Theorem 4.6.2 is trivial. To make the situation interesting, we need  $\nabla_E$  to satisfy a curvature condition (integrability condition), ensuring that  $D_E(e) = 0$  has many solutions locally. We will give this condition. The following proposition is a collection of results from [24, §2].

**Proposition 4.6.3.** *Let  $M$  be a hypercomplex manifold of dimension  $4n$ . Then  $I_1, I_2, I_3$  act as maps  $T^*M \rightarrow T^*M$ , so we may consider the map  $\delta : \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M$  defined by  $\delta = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$ . Then  $\delta^2 = 2\delta + 3$ , so the eigenvalues of  $\delta$  are 3 and  $-1$ . This induces a splitting  $\Lambda^2 T^*M = \Lambda_+ \oplus \Lambda_-$ , where  $\Lambda_+$  is the eigenspace of  $\delta$  with eigenvalue 3, and  $\Lambda_-$  is the eigenspace of  $\delta$  with eigenvalue  $-1$ .*

*The fibre dimensions are  $\dim \Lambda_+ = 2n^2 + n$  and  $\dim \Lambda_- = 6n^2 - 3n$ . Also,  $\Lambda_+$  is the subbundle of  $\Lambda^2 T^*M$  of 2-forms that are of type  $(1, 1)$  w.r.t. every complex structure  $r_1 I_1 + r_2 I_2 + r_3 I_3$ , for  $r_1, r_2, r_3 \in \mathbb{R}$  and  $r_1^2 + r_2^2 + r_3^2 = 1$ . When  $n = 1$ , the hypercomplex structure induces a conformal structure, and the splitting  $\Lambda^2 T^*M = \Lambda_+ \oplus \Lambda_-$  is the usual splitting into self-dual and anti-self-dual 2-forms.*

Following Mamone Capria and Salamon [24, p. 520], we make the following definition.

**Definition 4.6.4** Let  $M$  be a hypercomplex manifold, let  $E$  be a vector bundle over  $M$ , and let  $\nabla_E$  be a connection on  $E$ . Let  $F_E$  be the curvature of  $\nabla_E$ , so that  $F_E \in C^\infty(E \otimes E^* \otimes \Lambda^2 T^*M)$ . We say that  $\nabla_E$  is a *self-dual connection* if  $F_E \in C^\infty(E \otimes E^* \otimes \Lambda_+)$ , that is, if the component of  $F_E$  in  $E \otimes E^* \otimes \Lambda_-$  is zero.

We use the notation  $\Lambda_+$ ,  $\Lambda_-$  and self-dual to stress the analogy with the four-dimensional case, where this notation is already standard. In four dimensions, self-dual connections (also called *instantons*) are a very important tool in differential topology, and have been much studied.

The point of the definition is this. If  $\nabla_E$  is self-dual, then  $F_E$  is of type  $(1, 1)$  w.r.t. the complex structure  $r_1 I_1 + r_2 I_2 + r_3 I_3$  by Proposition 4.6.3, and so  $\nabla_E$  makes  $\mathbb{C} \otimes E$  into a holomorphic bundle w.r.t.  $r_1 I_1 + r_2 I_2 + r_3 I_3$ . Therefore  $\mathbb{C} \otimes E$  has many local holomorphic sections w.r.t.  $r_1 I_1 + r_2 I_2 + r_3 I_3$ . But these are just special sections  $e$  of  $\mathbb{H} \otimes E$  (or  $\mathbb{I} \otimes E$ ) satisfying  $D_E(e) = 0$ .

We deduce that locally, if  $F_E$  is self-dual, then the equation  $D_E(e) = 0$  admits many solutions. Thus, it is clear that vector bundles with self-dual connections are the appropriate quaternionic analogue of *holomorphic vector bundles* in complex geometry.

Suppose that  $M$  has a notion of q-holomorphic functions of polynomial growth, as in Definition 3.2.6, and that  $E$  is equipped with a metric. Sections of  $E$  with polynomial growth can then be defined in the obvious way. Let  $P_{M,E} \subset Q_{M,E}$  be the filtered AH-submodule of q-holomorphic sections of  $E$  of polynomial growth, and let  $P_M$  be the filtered H-algebra of q-holomorphic functions on  $M$  with polynomial growth. Then  $P_{M,E}$  is a filtered module over the H-algebra  $P_M$ .

The ideas of §3.4 can also be applied to the problem of reconstructing a self-dual connection from its module  $P_{M,E}$ . Under suitable conditions,  $P_{M,E}$  entirely determines the bundle  $E$  and its connection  $\nabla_E$ . This suggests that H-algebra techniques could be used to construct explicit self-dual connections over hypercomplex manifolds of interest.

Now the *ADHM construction* [1] is an explicit construction of self-dual connections over the hypercomplex manifold  $\mathbb{H}$ . By Example 4.1.3, each such connection yields a module  $P_{M,E}$  over the  $\mathbb{H}$ -algebra  $F^U$ . The author hopes to study these modules in a later paper, and hence to provide a new treatment and proof of the ADHM construction. The approach generalizes to instantons over other hypercomplex manifolds such as the hyperkähler ALE spaces, for which an ADHM-type construction is given in [21].

**4.7. Directions for future research.** We have developed an extensive and detailed comparison between vector spaces, tensor products and linear maps, and their quaternionic analogues. As a result, many pieces of algebra that use vector spaces, tensor products and linear maps as their building blocks have a quaternionic version. We shall discuss the quaternionic version of algebraic geometry.

Complex algebraic geometry is the study of complex manifolds using algebras of holomorphic functions upon them. In the same way, let ‘quaternionic algebraic geometry’ be the study of hypercomplex manifolds using  $\mathbb{H}$ -algebras of  $q$ -holomorphic functions upon them. I believe that quaternionic algebraic geometry and its generalizations may be interesting enough to develop a small field of algebraic geometry devoted to them.

I have tried to take the first steps in this direction in Chapters 2-4. These methods seem to have no application to compact hypercomplex manifolds, unfortunately, so instead one should study noncompact hypercomplex manifolds satisfying some restriction, such as AC hypercomplex manifolds. The best category of  $\mathbb{H}$ -algebras to use appears to be FGH-algebras. Here are a number of questions I think are worth further study. Almost all are problems in quaternionic algebraic geometry, and should clarify what I mean by it.

### Research problems about hypercomplex manifolds

- Study the theory of ‘coadjoint orbit’ hyperkähler manifolds and HP-algebras, begun in §4.4, in much greater depth.
- In Example 4.5.6 we saw that hypercomplex manifolds derived from FGH-algebras can have ‘invisible singularities’ with interesting properties. Study these singularities. Develop a theory of the singularities possible in ‘algebraic’ hypercomplex manifolds.
- Develop a *deformation theory* for FGH-algebras. That is, given a fixed FGH-algebra, describe the family of ‘nearby’ FGH-algebras. One expects to find the usual machinery of versal and universal deformations, infinitesimal deformations and obstructions, cohomology groups. However, the author’s calculations suggest that the  $\mathbb{H}$ -algebra setting makes the theory rather complex and difficult. The application is to deformations of ‘algebraic’ hypercomplex manifolds.
- Use this deformation theory to construct hyperkähler deformations of singular hyperkähler manifolds. Can you find new explicit examples of complete, non-singular hyperkähler manifolds of dimension at least 8?

- In particular, try and understand the deformations of  $\mathbb{H}^n/\Gamma$ , for  $\Gamma$  a finite subgroup of  $Sp(n)$ . In [15] the author used analysis to construct a special class of hyperkähler metrics on crepant resolutions of  $\mathbb{C}^{2n}/\Gamma$ . They are called *Quasi-ALE metrics*, and satisfy complicated asymptotic conditions at infinity.

It seems likely that these Quasi-ALE metrics are natural solutions to the deformation problem for the FGH-algebra of  $\mathbb{H}^n/\Gamma$ , and thus that one could use hypercomplex algebraic geometry to study them, and even construct them explicitly.

- Study the H-algebras of ALE spaces, mentioned in §4.5. Use H-algebras to give a second proof of Kronheimer’s classification of ALE spaces [17], [18].
- Rewrite the ADHM construction [1] for self-dual connections on  $\mathbb{H}$  in H-algebra language, as suggested in §4.6. Interpreted this way it becomes a beautiful algebraic construction for modules of the H-algebra  $F^U$  of Example 4.1.3.
- In a similar way, rewrite the ADHM construction on ALE spaces [21] in terms of H-algebras. Consider the possibility of a ‘general ADHM construction’ for FGH-algebras and algebraic hypercomplex manifolds, that captures the algebraic essence of the constructions of [1] and [21].

### Other research problems

- Understand and classify finite-dimensional HL-algebras, by analogy with Lie algebras.
- Apply quaternionic algebra in other areas of mathematics, to produce quaternionic analogues of existing pieces of mathematics. These need have no connection at all with hypercomplex manifolds. For instance, one can look at a quaternionic version of the Quantum Yang-Baxter Equation, and try and produce quaternionic knot invariants.
- Generalize the quaternionic algebra idea by replacing  $\mathbb{H}$  by some noncommutative ring, or more general algebraic object. Can you find any interesting algebraic applications for this? One particularly interesting case is when  $\mathbb{H}$  is replaced by a division ring over an algebraic number field, since these rings have a strong analogy with  $\mathbb{H}$ .
- We have seen that one may associate an H-algebra to a hypercomplex manifold. However, the converse is false, and one cannot in general associate a hypercomplex manifold to an H-algebra. Following §3.4, given an H-algebra  $P$  generated by finite-dimensional  $Q$ , one constructs a real algebraic variety  $M_{P,Q} \subset Q^\dagger$ . What geometric structure does  $P$  induce on  $M_{P,Q}$ , for general  $P$ ? Study these structures.
- In [13] the author constructed nontrivial examples of manifolds with  $n$  anticommuting complex structures, for arbitrary  $n > 1$ . Such structures have also been found by Barberis et al. [3], who call them *Clifford structures*. The programme of this paper may be generalized to Clifford structures. One simply replaces  $\mathbb{H}$



by the Clifford algebra  $C_n$  in the definition of H-algebra, and then to a manifold with a Clifford structure one may associate a ' $C_n$ -algebra', the analogue of H-algebra. Use this theory to study and find new examples of manifolds with Clifford structures.

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## A CANONICAL HYPERKÄHLER METRIC ON THE TOTAL SPACE OF A COTANGENT BUNDLE

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ABSTRACT. A canonical hyperkähler metric on the total space  $T^*M$  of a cotangent bundle to a complex manifold  $M$  has been constructed recently by the author in [K]. This paper presents the results of [K] in a streamlined and simplified form. The only new result is an explicit formula obtained for the case when  $M$  is an Hermitian symmetric space.

### INTRODUCTION.

Constructing a hyperkähler metric on the total space  $T^*M$  of the cotangent bundle to a Kähler manifold  $M$  is an old problem, dating back to the very first examples of hyperkähler metrics given by E. Calabi in [C]. Since then, many people have obtained a lot of important results valid for manifolds  $M$  in this or that particular class (see, for example, the papers [DS], [BG], [Kr2], [Kr2], [N]). Finally, the general problem has been more or less solved a couple of years ago, independently by B. Feix [F] and by the author [K].

The metrics constructed in [F] and [K] are the same. In fact, this metric satisfies an additional condition which makes it essentially unique – which justifies the use of the term “canonical metric”. But the approaches in [F] and [K] are very different. Feix’s method is very geometric in nature; it is based on a direct description of the associated twistor space. The approach in [K] is much farther from geometric intuition. However, it seems to be more likely to lead to explicit formulas.

Unfortunately, the paper [K] is 100 pages long and the exposition is not very good. Sometimes, it is sometimes canonical to the point of obscurity. Some of the proofs are not at all easy to understand and to check.

Recently the author has been invited to give a talk on the results of [K] at the Second Quaternionic Meeting in Rome. It was a good opportunity to revisit the subject and to streamline and simplify some of the proofs. This paper, written for the Proceedings volume of the Rome conference, is an attempt to present the results of [K] in a concrete and readable form. The exposition is parallel to [K] but the paper is mostly independent.

Compared to [K], the emphasis in the present paper has been shifted from canonical and conceptually correct abstract constructions to things more explicit and down-to-earth. The number of definitions is reduced to the necessary minimum. I have also tried to give a concrete geometric interpretation to everything that admits such an interpretation. In a sense, the exposition intentionally goes to the other extreme. Thus the present paper is not so much a replacement for [K] but rather a companion paper – the same story told in a different way.

The words “more or less” in the first paragraph of this Introduction refer to one important defect of the canonical hyperkähler metric on  $T^*M$  – namely, it is defined only in an open neighborhood  $U \subset T^*M$  of the zero section  $M \subset T^*M$ . This raises a very interesting and difficult “convergence problem”. One would like to describe the maximal open subset  $U \subset T^*M$  where the canonical metric is defined, and to say when  $U$  is the whole total space  $T^*M$ . Unfortunately, very little is known about this. In fact, in the present paper we even restrict ourselves to giving the formal germ of the canonical metric near  $M \subset T^*M$ . The fact that this formal germ converges to an actual metric at least on an open subset  $U \subset T^*M$  is proved in the last Section of [K]. The proof is long and tedious but completely straightforward. Since I don’t know how to improve it, I have decided to omit it altogether to save space.

This reader will find the precise statements of all the results in Section 1. The last part of that Section contains a brief description of the rest of the paper, and indicates the parallel places in [K], the differences in notation and terminology and so on. The only thing in this paper which is completely new is the last Section 8. It contains an explicit formula for the canonical metric (or rather, for the canonical hypercomplex structure) in the case when  $M$  is an Hermitian symmetric space. The formula is similar to the general formula for symmetric  $M$  obtained by O. Biquard and P. Gauduchon in [BG].

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## 1. STATEMENTS AND DEFINITIONS.

To save space, we will assume some familiarity with hyperkähler and hypercomplex geometry. We only give a brief reminder. The reader will find excellent expositions of the subject in [B], [HKLR], [Sal1], [Sal2].

Let  $\mathbb{H}$  be the algebra of quaternions. A smooth manifold  $X$  is called *almost quaternionic* if it is equipped with a smooth action of the algebra  $\mathbb{H}$  on the tangent bundle  $TX$ . Equivalently, one can consider a smooth action on the cotangent bundle  $\Lambda^1(M)$ . To fix terminology, we will assume that  $\mathbb{H}$  acts on  $\Lambda^1(M)$  on the left.

An almost quaternionic manifold  $X$  is called *hypercomplex* if it admits a torsion-free connection preserving the  $\mathbb{H}$ -module structure on  $\Lambda^1(M)$ . If such a connection exists, it is unique and called the *Obata connection* of the hypercomplex manifold  $X$ .

A Riemannian almost quaternionic manifold  $X$  is called *hyperhermitian* if the Riemannian pairing satisfies

$$(h\alpha_1, \alpha_2) = (\alpha_1, \bar{h}\alpha_2), \quad \alpha_1, \alpha_2 \in \Lambda^1(M), h \in \mathbb{H},$$

where  $\bar{h}$  is the quaternion conjugate to  $h$ . A hyperhermitian almost quaternionic manifold  $X$  is called *hyperkähler* if the  $\mathbb{H}$ -action is parallel with respect to the Levi-Civita connection  $\nabla_{LC}$ . In other words,  $X$  must be hypercomplex, and the Obata connection must be  $\nabla_{LC}$ .

Let  $X$  be a almost quaternionic manifold. Every embedding  $\mathbb{C} \hookrightarrow \mathbb{H}$  from the field of complex numbers to the algebra of the quaternions induces an almost complex structure on  $X$ . If  $X$  is hypercomplex, then all these induced almost complex structures are integrable. If  $X$  is also hyperkähler, then all these complex structures are Kähler with respect to the metric.

Throughout this paper it will be convenient to choose an embedding  $I : \mathbb{C} \rightarrow \mathbb{H}$  and an additional element  $j \in \mathbb{H}$  such that

$$j^2 = -1 \\ j \cdot I(z) = I(\bar{z}) \cdot j, \quad z \in \mathbb{C}$$

With these choices, every left  $\mathbb{H}$ -module  $V_{\mathbb{R}}$  defines a complex vector space  $V = V_I$  and a map  $j : V \rightarrow \bar{V}$  which satisfies

$$(1) \quad j \circ \bar{j} = -\text{id}$$

We will call  $V_I$  the *main complex structure* on the real vector space  $V_{\mathbb{R}}$ . Conversely, every pair  $\langle V, j \rangle$  of a complex vector space  $V$  and a map  $j : V \rightarrow \bar{V}$  which satisfies (1) defines an  $\mathbb{H}$ -module structure on the real vector space  $V_{\mathbb{R}}$  underlying  $V$ .

The map  $j : V \rightarrow \bar{V}$  can be considered as an automorphisms  $J : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$  of the underlying real vector space. This map induces a second complex structure on  $V_{\mathbb{R}}$ . We will call it the *complementary* complex structure and denote the resulting complex vector space by  $V_j$

Applying this to vector bundles, we see that an almost quaternionic manifold  $X$  is the same as an almost complex manifold  $X$  equipped with a smooth complex bundle map

$$(2) \quad j : \mathcal{T}(X) \rightarrow \bar{\mathcal{T}}(X)$$

from the tangent bundle  $\mathcal{T}(X)$  to its complex-conjugate bundle  $\overline{\mathcal{T}}(X)$  which satisfies (1).

It would be very convenient to have some way to know whether an almost quaternionic manifold  $X$  is hypercomplex or hyperkähler without working explicitly with torsion-free connections. For hypercomplex manifolds, the integrability condition is very simple. An almost quaternionic manifold  $X$  is hypercomplex if and only if both the main complex structure on  $X$  and the complementary almost complex structure  $X_J$  are integrable.

The simplest way to determine whether a hyperhermitian almost quaternionic manifold  $X$  is hyperkähler is to consider the complex-bilinear form  $\Omega$  on the tangent bundle  $\mathcal{T}(X)$  given by

$$(3) \quad \Omega(\xi_1, \xi_2) = h(\xi_1, j(\xi_2)), \quad \xi_1, \xi_2 \in \mathcal{T}(X),$$

where  $h$  is the Hermitian metric on  $X$ . Then (1) insures that the form  $\Omega$  is skew-symmetric. The manifold  $X$  is hyperkähler if and only if both the  $(2, 0)$ -form  $\Omega$  and the Kähler  $\omega$  are closed. Alternatively, one can define a  $(2, 0)$ -form  $\Omega_J$  for the complementary almost complex structure  $X_J$  instead of the  $(2, 0)$ -form  $\Omega$  for the main almost complex structure. Then  $X$  is hyperkähler if and only if both  $\Omega$  and  $\Omega_J$  are closed. Moreover, it suffices to require that these  $(2, 0)$ -forms are holomorphic, each in its respective almost complex structure on  $X$  ("holomorphic" here means that  $d\Omega$  is a form of type  $(3, 0)$ ). Finally, if we already know that the manifold  $X$  is hypercomplex, then it suffices to require that only one of the forms  $\Omega, \Omega_J$  is a holomorphic 2-form.

If  $X$  is a Kähler manifold equipped with a closed  $(2, 0)$ -form  $\Omega$ , one can define a map  $j : \mathcal{T}(X) \rightarrow \overline{\mathcal{T}}(X)$  by (3). Then  $X$  is hyperkähler if and only if this map  $j$  satisfies (1).

Consider now the case when  $X = T^*M$  is the total space of the cotangent bundle to a Kähler manifold  $M$  with the Kähler metric  $h$ . Then  $X$  carries a canonical holomorphic 2-form  $\Omega$ . Moreover, the unitary group  $U(1)$  acts on  $X$  by homoteties along the fibers of the projection  $X \rightarrow M$ , so that we have

$$(4) \quad z^*\Omega = z\Omega$$

for every  $z \in U(1) \subset \mathbb{C}$ . Using these data, we can formulate the main result of [K] as follows.

**Theorem 1.1.** *There exists a unique, up to fiberwise automorphisms of  $X/M$ ,  $U(1)$ -invariant Kähler metric  $h$  on  $X = T^*M$ , defined in the formal neighborhood of the zero section  $M \subset T^*M = X$ , such that*

1.  $h$  restricts to the given Kähler metric on the zero section  $M \subset X$ , and
2. the pair  $\langle \Omega, h \rangle$  defines a hyperkähler structure on  $X$  near  $M \subset X$ .

*Moreover, if the Kähler metric  $h$  on  $M$  is real-analytic, then the formal canonical metric on  $X$  comes from a real-analytic hyperkähler metric defined in an open neighborhood of  $M \subset X$ .*

The  $U(1)$ -action on  $T^*M$  is very important for this theorem. In fact,  $T^*M$  with this action belongs to a general class of hyperkähler manifolds equipped with a  $U(1)$ -action, introduced in [H], [HKLR].

**Definition 1.2.** *We will say that a holomorphic  $U(1)$ -action on a hyperkähler manifold  $X$  is compatible with the hyperkähler structure if and only if*

1. *the metric  $h$  on  $X$  is  $U(1)$ -invariant,*
2. *the holomorphic 2-form  $\Omega$  satisfies (4),*
3. *the map  $j : \mathcal{T}(X) \rightarrow \overline{\mathcal{T}}(X)$  satisfies*

$$(5) \quad j(z^*\xi) = z \cdot z^*(j(\xi)), \quad \xi \in \mathcal{T}(X), z \in U(1) \subset \mathbb{C}.$$

It is easy to check using (3) that every two of these conditions imply the third.

Theorem 1.1 can be generalized to the following statement, somewhat analogous to the Darboux-Weinstein Theorem in symplectic geometry.

**Theorem 1.3.** *Let  $X$  be a hyperkähler manifold equipped with a regular compatible holomorphic  $U(1)$ -action. Then there exists an open neighborhood  $U \subset X$  of the  $U(1)$ -fixed point subset  $M = X^{U(1)} \subset X$  and a canonical embedding  $\mathcal{L} : U \rightarrow T^*M$  such that the hyperkähler structure on  $U$  is induced by means of the map  $\mathcal{L}$  from the canonical hyperkähler structure on  $T^*M$ .*

Here *regular* is a certain condition on the  $U(1)$ -action near the fixed points subset  $X^{U(1)} \subset X$  which is formulated precisely in Definition 2.1 (roughly speaking, weights of the action on the tangent space  $T_m X$  at every point  $m \in X^{U(1)} \subset X$  should be 0 and 1). The map  $\mathcal{L}$  will be called the *normalization map*. Note that this Theorem allows one to reformulate Theorem 1.1 so that the metric is indeed unique – not just unique up to a fiberwise automorphism of  $X/M$ . To fix the metric, it suffices to require that the associated normalization map  $\mathcal{L} : X \rightarrow X = \overline{TM}$  is identical. Metrics with this property will be called *normalized*.

There is also a form of Theorem 1.1 for hypercomplex manifolds (and it is this form which is the most important for [K] – all the other statements are obtained as its corollaries). To formulate it, we note that out of the three conditions of Definition 1.2, the third one makes sense for almost quaternionic (in particular, hypercomplex) manifolds. We will say that a holomorphic  $U(1)$ -action on an almost quaternionic manifold  $X$  is *compatible with the quaternionic action* if Definition 1.2 (iii) is satisfied.

Let  $\overline{TM}$  be the total space of the bundle  $\overline{\mathcal{T}}(M)$  complex-conjugate to the tangent bundle of the manifold  $M$ . Then  $\overline{TM}$  is a smooth manifold, and we have the canonical projection  $\rho : \overline{TM} \rightarrow M$  and the zero section  $i : M \rightarrow \overline{TM}$ . The group  $U(1)$  acts on  $\overline{TM}$  by homoteties along the fibers of the projection  $\rho$ . Moreover, for any compatible hypercomplex structure on the  $U(1)$ -manifold  $\overline{TM}$  the corresponding Obata connection  $\nabla_O$  induces a torsion-free connection  $\nabla$  on  $M$  by the following rule

$$\nabla(\alpha) = i^*(\nabla_O \rho^* \alpha), \quad \alpha \in \Lambda^1(M).$$

The hypercomplex version of Theorem 1.1 is the following.

**Theorem 1.4.** *Let  $M$  be a complex manifold  $M$  equipped with a holomorphic connection  $\nabla$  on the tangent bundle  $\mathcal{T}(M)$  such that*

1.  $\nabla$  has no torsion, and
2. the curvature of the connection  $\nabla$  is of type  $(1, 1)$ .

*Let  $X = \overline{TM}$  be the total space of the complex-conjugate to the tangent bundle  $TM$ . Let the group  $U(1)$  act on  $X$  by homoteties along the fibers of the projection  $\rho : X \rightarrow M$ .*

*Then there exists a unique, up to a fiberwise automorphism of  $X/M$ , hypercomplex structure on  $X$ , defined in the formal neighborhood of the zero section  $M \subset X$ , such that the embedding  $i : M \hookrightarrow X$  and the projection  $\rho : X \rightarrow M$  are holomorphic and the Obata connection on  $X$  induces the given connection  $\nabla$  on  $M$ .*

*Moreover, if the connection  $\nabla$  is real-analytic, then the hypercomplex structure on  $X$  is real-analytic in an open neighborhood  $U \subset X$  of  $M \subset X$ .*

Note that *a priori* there is no natural complex structure on the space  $X = \overline{TM}$  (we write  $\overline{T}$  instead of  $T$  just to indicate the correct  $U(1)$ -action). Therefore Theorem 1.4 in fact claims two things: firstly, there exists an intergrable almost complex structure on  $X$ , and secondly, there exists a map  $j : \mathcal{T}(X) \rightarrow \overline{\mathcal{T}}(X)$  which extends it to a hypercomplex structure.

Connections  $\nabla$  that satisfy the conditions of this Theorem we called *kählerian* in [K] (see [K, 8.1.2]). I would like to thank D. Joyce for attracting my attention to his paper [J], where he uses the same class of connections to construct *commuting* almost complex structures on the total space  $TM$  of the tangent bundle to a complex manifold  $M$ . Joyce calls these connection *Kähler-flat*.

Theorem 1.4 also admits a generalized Darboux-like version in the spirit of Theorem 1.3; we do not formulate it here to save space.

We will now give a brief outline of the remaining part of the paper. In Section 2 we consider an arbitrary  $U(1)$ -equivariant hypercomplex manifold  $X$  and construct the normalization map  $\mathcal{L} : X \rightarrow \overline{TM}$ , thus proving Theorem 1.3. This corresponds to [K, Section 4]. What we call *normalization* here was called *linearization* in [K]; *normalized* corresponds to *linear*. The terminology of [K] has been changed because it was misleading: connections on the fibration  $\overline{TM} \rightarrow M$  that were called linear in [K] are not linear in the usual sense of the word.

Section 3 introduces  $\mathbb{R}$ -Hodge structures and the so-called Hodge bundles (Definition 3.1) which are the basis of our approach to  $U(1)$ -equivariant quaternionic manifolds  $X$ . This corresponds to [K, Sections 2,3]. Proposition 3.3 is a version of [K, Proposition 3.1].

In Section 4 we turn to the case  $X = \overline{TM}$  and introduce the so-called Hodge connections (Definition 4.2). This corresponds to [K, Section 5]. The proofs have been considerably shortened. There are also some new facts on the relation between



our formalism and the objects one usually associates with hypercomplex manifolds. In particular, Lemma 4.3 is new.

Section 5 and Section 6 introduce the Weil algebra  $\mathcal{B}^*(M)$  of a complex manifold  $M$  (Definition 5.1) and then use it to prove Theorem 1.4. Section 5 contains the preliminaries; Section 6 gives the proof itself. This material corresponds to [K, Sections 6-8]. The approach has been changed in the following way. Keeping track of various gradings and bigradings and on the Weil algebra presents considerable difficulties: when the proof of Theorem 1.4 is written down, the number of indices becomes overwhelming. In [K] we have tried to handle this by an auxiliary technical device called *the total Weil algebra* ([K, 7.2.4]). It was quite a natural thing to do from the conceptual point of view. Unfortunately, the proof became more abstract than one would like. Here we have opted for the direct approach. To make things comprehensible, we rely on pictures (Figure 1, Figure 2) which graphically represent the Hodge diamonds of the relevant pieces of the Weil algebra  $\mathcal{B}^*(M)$ .

Finally, Section 7 deals with things hyperkähler: we deduce Theorem 1.1 from Theorem 1.4. This corresponds to [K, Section 9]. We believe that the exposition has also been simplified, and the proofs are easier to check.

The last Section 8 of this paper is new. We try to illustrate our constructions by a concrete example of an Hermitian symmetric space  $M$ . We obtain a formula similar to [BG]. The last section of [K] contains the proof of convergence of our formal series in the case when the Kähler manifold  $M$  is real-analytic. In this paper, this proof is entirely omitted.

## 2. NORMALIZATION.

Of all the statement formulated in the last Section, the most straightforward one is the Darboux-like Theorem 1.3 and its hypercomplex version. In this Section, we explain how to construct the normalization map  $\mathcal{L}$ . Most of the proofs are replaced with references to [K].

We begin with some generalities. Assume that the group  $U(1)$  acts smoothly on a smooth manifold  $X$ . For any point  $x \in X$  fixed by the action, we have an action of  $U(1)$  on the tangent space  $T_x X$ . Equivalently, we have the weight decomposition

$$T_x X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_k (T_x X)^k,$$

where  $z \in U(1) \subset \mathbb{C}$  acts on  $(T_x X)^k$  by multiplication by  $z^k$ . We will say that the fixed point  $x$  is *regular* if the only non-trivial weight components  $(T_x X)^k$  correspond to weights  $k = 0, 1$ . The subset  $X^{U(1)} \subset X$  of fixed points is a disjoint union of smooth submanifolds of different dimensions. Regular fixed points form a connected component of this set. Denote this component by  $M \subset X$ .

Let  $\phi$  be the differential of the  $U(1)$ -action – that is, the vector field on  $X$  which gives the infinitesimal action of the generator  $\frac{\partial}{\partial \theta}$  of the Lie algebra of the group  $U(1)$ . Assume further that  $X$  is a complex manifold and that  $U(1)$  preserves the complex

structure. Say that a point  $x \in X$  is *stable* if for any  $t \in \mathbb{R}$ ,  $t \geq 0$  there exists  $\exp(\sqrt{-1}t\phi)x$ , and moreover, the limit

$$x_0 = \lim_{t \rightarrow +\infty} \exp(\sqrt{-1}t\phi)x$$

also exist. When the limit point  $x_0$  does exist, it is obviously fixed by  $U(1)$ . Say that a stable point  $x \in X$  is *regular stable* if the limit point is a regular fixed point,  $x_\infty \in M \subset X$ . Regular stable points form an open subset  $X^{reg} \subset X$ .

**Definition 2.1.** *A complex  $U(1)$ -manifold  $X$  is called regular if every point  $x \in X$  is regular stable,  $X^{reg} = X$ .*

For example, the total space of an arbitrary complex vector bundle on an arbitrary complex manifold is regular (if  $U(1)$  acts by homoteties along the fibers). The submanifold of regular fixed points in this example is the zero section.

When the  $U(1)$ -manifold  $X$  is hypercomplex, we will say that it is regular if it is regular in the main complex structure. In this case, the subset  $M \subset X$  of regular fixed points is a complex submanifold. Setting

$$\rho(x) = x_0 = \lim_{t \rightarrow +\infty} \exp(\sqrt{-1}t\phi)x$$

defines a  $U(1)$ -invariant projection  $\rho : X \rightarrow M$ .

**Lemma 2.2** ([K, 4.2.1-3]). *The projection  $\rho : X \rightarrow M$  is a holomorphic submersion.* □

We can now define the normalization map. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}(X/M) \longrightarrow \mathcal{T}(X) \longrightarrow \rho^*\mathcal{T}(M) \xrightarrow{d\rho} 0$$

of tangent bundles associated to the submersion  $\rho : X \rightarrow M$ . The differential  $\phi$  of the  $U(1)$ -action is a vertical holomorphic vector field on  $X$ ,  $\phi \in \mathcal{T}(X/M)$ . Applying the operator  $j : \mathcal{T}(X) \rightarrow \overline{\mathcal{T}}(X)$  to  $\phi$  gives a section of the bundle  $\overline{\mathcal{T}}(X)$ . We can project this section to obtain a section

$$d\rho(j(\phi)) \in \rho^*\overline{\mathcal{T}}(M)$$

of the pullback bundle  $\rho^*\overline{\mathcal{T}}(M)$ . But such a section tautologically defines a map  $\mathcal{L} : X \rightarrow \overline{\mathcal{T}}M$  from  $X$  to the total space of the complex bundle  $\overline{\mathcal{T}}$  on the manifold  $M$ . Since  $\phi$  is  $U(1)$ -invariant, and  $j$  is of weight 1 with respect to the  $U(1)$ -action, the section  $d\rho(j(\phi))$  is also of weight one. This means that the associated map

$$\mathcal{L} : X \rightarrow \overline{\mathcal{T}}M$$

is  $U(1)$ -equivariant. We will call it the *normalization map* for the regular hypercomplex  $U(1)$ -manifold  $X$ .

**Lemma 2.3** ([K, Proposition 4.1]). *The normalization map  $\mathcal{L} : X \rightarrow \overline{\mathcal{T}}M$  is an open embedding.* □

This Lemma essentially reduces Theorem 1.3 to Theorem 1.1.

A particular case occurs when  $X = \overline{TM}$  is itself the total space of the complex-conjugate to the tangent bundle on a complex manifold  $M$  – or, more generally, an open  $U(1)$ -invariant neighborhood  $U \subset \overline{TM}$  of the zero section  $M \subset \overline{TM}$ . As noted above, in this case the zero section  $M \subset \overline{T}(M)$  coincides with the subset of fixed points. Therefore the normalization map  $\mathcal{L} : U \rightarrow \overline{TM}$  is an open embedding from  $U$  into  $\overline{TM}$ , possibly different from the given one.

**Definition 2.4.** *The hypercomplex structure on  $U \subset \overline{TM}$  is called normalized if the normalization map  $\mathcal{L} : U \rightarrow \overline{TM}$  coincides with the given embedding.*

(In particular, when  $U = \overline{TM}$  is the whole total space, the normalization map must be identical.)

To prove Theorem 1.1 and Theorem 1.4, it is sufficient to be able to classify all normalized hypercomplex structures on the  $U(1)$ -manifold  $\overline{T}(M)$  and germs of such structures near the zero section  $M \subset \overline{TM}$ . It will be convenient to slightly rewrite the normalization condition. Namely, the identity map  $\text{id} : \overline{TM} \rightarrow \overline{TM}$  defines a section on  $X = \overline{T}(M)$  of the pullback bundle  $\rho^*\overline{T}(M)$ . We will denote this section by  $\tau$  and call it the *tautological section*. Then a hypercomplex structure on  $U \subset X$  is normalized if and only if we have

$$(6) \quad j(\phi) = \tau \in \rho^*\overline{T}(M).$$

### 3. HODGE BUNDLES.

The first step in the proof of Theorem 1.4 is to give a workable description of hypercomplex structures on the total space  $X = \overline{TM}$ . For this we use the language of  $\mathbb{R}$ -Hodge structures.

Recall that an  $\mathbb{R}$ -Hodge structure  $V$  of weight  $k$  is by definition a real vector space  $V_{\mathbb{R}}$  equipped with a grading

$$(7) \quad V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_p V^{p,k-p}$$

such that

$$(8) \quad \overline{V^{p,q}} = V^{q,p}, \quad p, q \in \mathbb{Z}, p + q = k.$$

Equivalently, instead of the grading (7) one can consider a  $U(1)$  action on  $V$  defined by

$$z \cdot v = z^p v, \quad v \in V^{p,q} \subset V, z \in U(1) \subset \mathbb{C}.$$

Then (8) becomes

$$\overline{z \cdot v} = z^k \overline{z \cdot \overline{v}}, \quad v \in V, z \in U(1) \subset \mathbb{C}.$$

When the weight  $k$  is equal to 1, this equation on the complex conjugation map becomes precisely (5). The difference between the complex conjugation map and the map  $j$  used to define quaternionic structures is that the first one is an involution,

$\bar{v} = v$ , while for the second one we have  $j(j(v)) = -v$ . Nevertheless, we will exploit the similarity between them to describe quaternionic actions via Hodge structures. To do this, we use the following trick. Let  $V$  be an  $\mathbb{R}$ -Hodge structure of weight 1, and consider the map

$$\iota : V \rightarrow V$$

given by the action of  $-1 \in U(1) \subset \mathbb{C}$  – in other words, let

$$\iota(v) = (-1)^p, \quad v \in V^{p,1-p} \subset V.$$

Then the map

$$(9) \quad j = \iota \circ -: V \rightarrow \bar{V}$$

still satisfies (5), and (1) also holds. This turns  $V$  into a left  $\mathbb{H}$ -module. Conversely, every left  $\mathbb{H}$  module  $\langle V, j \rangle$  equipped with a  $U(1)$ -action on  $V$  such that  $j$  satisfies (5) defines an  $\mathbb{R}$ -Hodge structure of weight 1.

To use this for a description of hypercomplex structures on manifolds, we introduce the following.

**Definition 3.1.** *Let  $X$  be a smooth manifold equipped with an action of the group  $U(1)$ , and let  $\iota : X \rightarrow X$  be the action of the element  $-1 \in U(1) \subset \mathbb{C}$ . A Hodge bundle  $\mathcal{E}$  of weight  $k$  on  $X$  is by a  $U(1)$ -equivariant complex vector bundle  $\mathcal{E}$  equipped with a  $U(1)$ -equivariant isomorphism*

$$- : \mathcal{E} \rightarrow \bar{\mathcal{E}}(k)$$

such that  $- \circ - = \text{id}$ .

Here  $\bar{\mathcal{E}}(k)$  is the bundle complex conjugate to  $\mathcal{E}$ , whose  $U(1)$ -equivariant structure is twisted by tensoring with the 1-dimensional representation  $\mathbb{C}(k)$  of the group  $U(1)$  of weight  $k$ ,

$$z \cdot x = z^k x, \quad x \in \mathbb{C}(k), z \in U(1) \subset \mathbb{C}.$$

When the  $U(1)$ -action on the manifold  $X$  is trivial, a weight- $k$  Hodge bundle  $\mathcal{E}$  on  $X$  is just the bundle of  $\mathbb{R}$ -Hodge structures of weight  $k$  in the obvious sense. In particular, if  $X$  is an almost complex manifold, then the bundle  $\Lambda^k(X)$  of all complex-valued  $k$ -forms on  $X$  is a Hodge bundle of weight  $k$ .

When the  $U(1)$ -action on  $X$  is no longer trivial, every bundle  $\mathcal{E}$  of  $\mathbb{R}$ -Hodge structures on  $X$  still defines a Hodge bundle. Thus  $\Lambda^k(X)$  is still a weight- $k$  Hodge bundle. But this Hodge bundle structure is not interesting, since it does not take into account the natural  $U(1)$ -action on  $\Lambda^k(X)$ . Assuming that the  $U(1)$ -action preserves the almost complex structure on  $X$ , we define instead a Hodge bundle structure on  $\Lambda^1(X, \mathbb{C})$  by keeping the usual complex conjugation map and twisting the  $U(1)$ -action so that

$$\Lambda^1(X, \mathbb{C}) \cong \Lambda^{1,0}(X)(1) \oplus \Lambda^{0,1}(X)$$

as a  $U(1)$ -equivariant vector bundle. It is easy to check that this indeed defines on  $\Lambda^1(X, \mathbb{C})$  a Hodge bundle structure of weight 1.

Assume now that the almost complex manifold  $X$  is equipped with an almost quaternionic structure which is compatible with the  $U(1)$ -action. Then the complex vector bundle  $\Lambda^{0,1}(X)$  of  $(0, 1)$ -forms on  $X$  already has a structure of a Hodge bundle of weight 1. The  $U(1)$ -action for this structure is the natural one, and the complex conjugation map is induced by the map  $j : \mathcal{T}(X) \rightarrow \overline{\mathcal{T}}X$  via (9). This Hodge bundle structure completely determines the quaternionic action. More precisely, for every smooth  $U(1)$ -manifold  $X$ , every Hodge bundle  $\mathcal{E}$  of weight 1 whose underlying real vector bundle  $\mathcal{E}_{\mathbb{R}}$  is identified with the cotangent bundle  $\Lambda^1(X, \mathbb{R})$  comes from a unique compatible almost quaternionic structure  $X$ .

The natural embedding  $\Lambda^{0,1}(X) \subset \Lambda^1(X, \mathbb{C})$  is not a map of Hodge bundles – it is  $U(1)$ -equivariant, but it obviously does not commute with the complex conjugation map. This can be corrected. To do this, one has to look at the picture in a different way (which will turn out to be very useful). Return for a moment to linear algebra. Let  $V_{\mathbb{R}}$  be a left  $\mathbb{H}$ -module, and let  $V, V_J$  be the complex vector spaces obtained from  $V_{\mathbb{R}}$  by the main and the complementary complex structures. Consider the complex vector space  $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ . This vector space does not depend on the  $\mathbb{H}$ -action on  $V_{\mathbb{R}}$ . Given an  $\mathbb{H}$ -action, we have the main and the complementary complex structure operators  $I = I(\sqrt{-1}) : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$  and  $J : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$  and the associated eigenspace decompositions

$$(10) \quad V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V},$$

$$(11) \quad V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V_J \oplus \overline{V_J}$$

Since the operators  $I$  and  $J$  anti-commute, these decompositions are distinct: we have

$$V \cap V_J = V \cap \overline{V_J} = \overline{V} \cap V_J = \overline{V} \cap \overline{V_J} = 0.$$

In particular, the composition

$$(12) \quad H : \overline{V} \rightarrow V_{\mathbb{R}} \otimes \mathbb{C} \rightarrow \overline{V_J}$$

of the canonical embedding in (10) and the canonical projection in (11) is an isomorphism. We will call it the *canonical isomorphism between the main and the complementary complex structures*. On the level of the real vector space  $V_{\mathbb{R}}$ , the map  $H$  is induced by a non-trivial automorphism  $H : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$  (in fact it is the action of the element  $I(\sqrt{-1}) + j \in \mathbb{H}$ ). Conjugation with this map interchanges the operators  $I$  and  $J$ .

Return now to the case of an almost quaternionic manifold  $X$ . Then we claim that the complementary almost complex structure operator  $J : \Lambda^1(X, \mathbb{C}) \rightarrow \Lambda^1(X, \mathbb{C})$  is a map of Hodge bundles. Indeed, it commutes with the complex conjugation map by definition. Therefore it suffices to show that it is  $U(1)$ -equivariant on  $\Lambda^{0,1}(X) \subset \Lambda^1(X, \mathbb{C})$ . But for every  $v \in \Lambda^{1,0}(X)$  we have  $J(v) = j(v) \in \Lambda^{0,1}(X)$ , and the map  $j$  is of weight 1. Thus the operator  $J$  is indeed  $U(1)$ -equivariant (recall the twisting of the  $U(1)$ -action in the definition of the Hodge bundle structure on  $\Lambda^1(X, \mathbb{C})$ ).

Since the endomorphism  $J : \Lambda^1(X, \mathbb{C}) \rightarrow \Lambda^1(X, \mathbb{C})$  is a map of Hodge bundles, its eigenbundles

$$\Lambda_J^{1,0}(X), \Lambda_J^{0,1}(X) \subset \Lambda^1(X, \mathbb{C})$$

are Hodge subbundles. Therefore we obtain a canonical weight-1 Hodge bundle structure on the bundle  $\Lambda_J^{0,1}(X)$  of  $(0, 1)$ -forms for the *complementary* almost complex structure on  $X$ .

This Hodge bundle is not a new one. Indeed, the canonical isomorphism

$$H : \Lambda^{0,1}(X) \rightarrow \Lambda_J^{0,1}(X)$$

defined in (12) is  $U(1)$ -equivariant – it is obtained as a composition of  $U(1)$ -equivariant maps. Moreover, it is very easy to check that  $H$  commutes with the complex conjugation. Thus  $\Lambda^{0,1}(X) \cong \Lambda_J^{0,1}(X)$  as Hodge bundles. But the projections

$$(13) \quad \Lambda^1(X, \mathbb{C}) \rightarrow \Lambda^{0,1}(X),$$

$$(14) \quad \Lambda^1(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X) \cong \Lambda^{0,1}(X),$$

are different. Only the second one is a Hodge bundle map.

All this linear algebra is somewhat tautological, but it becomes useful when we consider the integrability conditions on an almost quaternionic quaternionic structure. The real advantage of  $\mathbb{R}$ -Hodge structures over  $\mathbb{H}$ -modules is the presence of higher weights. Namely, the category of  $\mathbb{H}$ -modules admits no natural tensor product. On the other hand, the category of  $\mathbb{R}$ -Hodge structures and the category of Hodge bundles are obviously tensor categories. Thus, for example, the weight-1 Hodge bundle structure on the cotangent bundle  $\Lambda^1(X, \mathbb{C})$  induces a weight- $k$  Hodge bundle structure on the bundle  $\Lambda^k(X, \mathbb{C})$  of  $k$ -forms.

To make use of these higher-weight Hodge bundles, we need a convenient notion of maps between Hodge bundles of different weights.

**Definition 3.2.** A bundle map (or, more generally, a differential operator)  $f : \mathcal{E} \rightarrow \mathcal{F}$  between Hodge bundles  $\mathcal{E}, \mathcal{F}$  of weights  $m, n$  is called weakly Hodge if it commutes with the complex conjugation map and admits a decomposition

$$(15) \quad f = \sum_{0 \leq p \leq n-m} f^p,$$

where  $f^p : \mathcal{E} \rightarrow \mathcal{F}$  is of weight  $p$  with respect to the  $U(1)$ -action – in other words,  $f^p$  is  $U(1)$ -equivariant when considered as a map

$$f^p : \mathcal{E} \rightarrow \mathcal{F}(p).$$

We see that non-trivial weakly Hodge maps between Hodge bundles  $\mathcal{E}, \mathcal{F}$  exist only when their weights satisfy  $\text{wt } \mathcal{F} \geq \text{wt } \mathcal{E}$ .

When the  $U(1)$ -action on the manifold  $X$  is trivial, the Hodge bundles  $\mathcal{E}$  and  $\mathcal{F}$  come from bundles of  $\mathbb{R}$ -Hodge structures on  $X$ , and the decomposition  $f = \sum_p f^p$  of a weakly Hodge map  $f : \mathcal{E} \rightarrow \mathcal{F}$  is simply the Hodge type decomposition,

$$f^p = f^{p,q}, \quad p + q = \text{wt } \mathcal{F} - \text{wt } \mathcal{E}.$$

If the  $U(1)$ -action is not trivial, but preserves an almost complex structure on  $X$ , the de Rham differential  $d = \partial + \bar{\partial} = d^{1,0} + d^{0,1} : \Lambda^0(X) \rightarrow \Lambda^1(X)$  is weakly Hodge. If the almost complex structure is integrable, then the same is true for the de Rham differential  $d : \Lambda^k(X, \mathbb{C}) \rightarrow \Lambda^{k+1}(X, \mathbb{C})$  for every  $k \geq 0$ .

When the  $U(1)$ -manifold is almost quaternionic, we have a Hodge bundle structure of weight 1 on the bundle  $\Lambda_J^{0,1}(X)$ . Then the Dolbeault differential

$$D = \bar{\partial}_J : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X)$$

is weakly Hodge. Indeed, it is the composition of the weakly Hodge de Rham differential and the projection (14) which is a Hodge bundle map. Denote by

$$(16) \quad D = D^0 + D^1$$

be the decomposition (15) for the weakly Hodge map  $D$ . Looking at the definition of the canonical isomorphism  $H : \Lambda^{0,1}(X) \rightarrow \Lambda_J^{0,1}(X)$ , we see that the Dolbeault differential  $\partial$  for the main complex structure considered as a derivation

$$\partial : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda^{0,1}(X) \cong \Lambda_J^{0,1}(X)$$

coincides with the component  $D^0 : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X)$  in the decomposition (16).

Assume now that the almost quaternionic  $U(1)$ -manifold  $X$  is hypercomplex. The bundle  $\Lambda_J^{0,k}(x)$  of  $(0, k)$ -forms on  $X_J$  is a Hodge bundle of weight  $k$  for every  $k \geq 0$ , and we have the Dolbeault differential

$$D = \bar{\partial}_J : \Lambda_J^{0,k}(x) \rightarrow \Lambda_J^{0,k+1}(x).$$

Since the projections  $\Lambda^k(X, \mathbb{C}) \rightarrow \Lambda_J^{0,k}(X)$  are Hodge bundle maps for every  $k \geq 0$ , this Dolbeault differential is weakly Hodge. It turns out that this is a sufficient integrability condition for an almost quaternionic manifold equipped with a compatible  $U(1)$ -action.

**Proposition 3.3.** *Let  $X$  be an almost quaternionic manifold equipped with a compatible  $U(1)$ -action. Assume that the Dolbeault differential*

$$D : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X)$$

*extends to a weakly Hodge derivation  $D : \Lambda_J^{0,*}(x) \rightarrow \Lambda_J^{0,*+1}(X)$  of the algebra  $\Lambda_J^{0,*}(X)$  satisfying  $D \circ D = 0$ . Then the manifold  $X$  is hypercomplex.*

*Proof.* It suffices to prove that both the main and the complementary almost complex structures on  $X$  are integrable. For this, it is enough to prove that the Dolbeault differentials

$$\begin{aligned} \bar{\partial}_J &= D : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X), \\ \bar{\partial} &= D^0 : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_J^{0,1}(X) \cong \Lambda^{0,1}(X), \end{aligned}$$

extend to square-zero derivations of the exterior algebra  $\Lambda_J^{0,*}(X)$ . The differential  $D$  extends by assumption. To extend  $D^0$ , take the component  $D^0 : \Lambda_J^{0,*}(x) \rightarrow \Lambda_J^{0,*+1}(X)$

of the weakly Hodge map  $D : \Lambda_J^{0,*}(x) \rightarrow \Lambda_J^{0,*+1}(X)$ . Then  $D^0 \circ D^0$  is a component in the decomposition (15) of the weakly Hodge map  $D \circ D : \Lambda_J^{0,*}(x) \rightarrow \Lambda_J^{0,*+2}(X)$ . Since  $D \circ D = 0$ , we also have  $D^0 \circ D^0 = 0$ .  $\square$

We will now say a couple of words about hyperkähler manifolds and Hodge bundles. Let  $X$  be an almost quaternionic  $U(1)$ -manifold. Then every Riemannian metric on  $X$  defines a  $(2, 0)$ -form  $\Omega_J \in \Lambda^{2,0}(X)$ . It turns out that if the metric is hyperhermitian and  $U(1)$ -invariant, then in terminology of [K], the form  $\Omega_J$  of  $H$ -type  $(1, 1)$ . This means the following. Consider the form  $\Omega_J$  as a bundle map

$$(17) \quad \Omega_J : \mathbb{R}(-1) \rightarrow \Lambda^{2,0}(X),$$

where  $\mathbb{R}(-1)$  is the trivial bundle on  $X$  equipped with the so-called Hodge-Tate  $\mathbb{R}$ -Hodge structure of weight  $-1$  – that is, the complex conjugation map on  $\mathbb{R}(-1)$  is minus the complex conjugation map on  $\mathbb{R}$ , and the  $U(1)$ -equivariant structure is twisted by 1. The form  $\Omega_J$  is said to be of  $H$ -type  $(1, 1)$  if the map (17) is a Hodge bundle map.

Conversely, every  $(2, 0)$ -form  $\Omega_J \in \Lambda^{2,0}(X)$  of  $H$ -type  $(1, 1)$  on an almost quaternionic  $U(1)$ -manifold  $X$  which satisfies a positivity condition

$$(18) \quad \Omega(\xi_1, I(\xi_2)) > 0, \quad \xi_1, \xi_2 \in T(X, \mathbb{R})$$

defines a  $U(1)$ -invariant hyperhermitian metric on  $X$ . (See [K, 1.5.4], but the proof is almost trivial.)

If  $X$  is hypercomplex, then, as indicated in Section 1, the metric corresponding to such a form  $\Omega_J$  is hyperkähler if and only if the form  $\Omega_J$  is holomorphic,  $D\Omega_J = 0 \in \Lambda_J^{2,1}(X)$ .

**Remark 3.4.** *In fact, using the  $U(1)$ -action on  $X$ , one can even drop the integrability condition. Indeed, if a form  $\Omega_J$  of  $H$ -type  $(1, 1)$  on an almost quaternionic  $U(1)$ -manifold  $X$  satisfies  $D\Omega_J = 0$ , then it also must satisfy*

$$D^0\Omega_J = D^1\Omega_J = 0.$$

*The canonical endomorphism  $H : \Lambda^1(X, \mathbb{R}) \rightarrow \Lambda^1(X, \mathbb{R})$ , being the conjugation with a quaternion, preserves up to a coefficient the metric associated to  $\Omega_J$  and interchanges the almost complex structure operators  $I$  and  $J$ . Therefore it sends  $\Omega_J$  to a form proportional to  $\Omega$ . Then  $D^0\Omega_J = 0$  implies that not only  $\Omega_J$  is holomorphic, but that  $\Omega$  is holomorphic as well. This proves that  $X$  is hyperkähler (a posteriori, also hypercomplex). We will never need nor use this argument. An interested reader will find details in [K, 3.3.4].*

#### 4. HODGE CONNECTIONS.

We will now restrict our attention to the case when the  $U(1)$ -manifold  $X$  is the total space  $\overline{TM}$  of the complex-conjugate to the tangent bundle of a complex manifold  $M$ . In this case, Proposition 3.3 is really useful, because it turns out that the Hodge



bundle algebra  $\Lambda_j^{0,k}(X)$  does not depend on an almost quaternionic structure on  $X$ . To see this, denote by  $\rho : X = \overline{T}M \rightarrow M$  the canonical projection, and let

$$\delta\rho : \rho^*\Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(X, \mathbb{C})$$

be the codifferential of the map  $\rho$ . Then for every compatible hypercomplex structure on  $X$ , we have canonical bundle maps

$$\rho^*\Lambda^1(M, \mathbb{C}) \xrightarrow{\delta\rho} \Lambda^1(X, \mathbb{C}) \longrightarrow \Lambda_j^{0,1}(X)$$

Assume that the manifold  $X$  is equipped with a hypercomplex structure satisfying the conditions of Theorem 1.4 – namely, assume that the projection  $\rho : X \rightarrow M$  and the zero section  $i : M \rightarrow X$  are holomorphic maps. Then the codifferential  $\delta\rho$  is obviously a map of Hodge bundles. However, we also have the following.

**Lemma 4.1** ([K, 5.1.9-10]). *The composition map*

$$\rho^*\Lambda^1(M, \mathbb{C}) \rightarrow \Lambda_j^{0,1}(X)$$

*is an isomorphism of Hodge bundles in an open neighborhood of the zero section  $M \subset X$ .* □

From this point on, it will be convenient to only consider germs of hypercomplex structures defined near the zero section  $M \subset X$ . In other words, we replace  $X = \overline{T}M$  with an unspecified and shrinkable  $U(1)$ -invariant open neighborhood of the zero section. Since we are only interested in hypercomplex structures on  $X$  that satisfy the conditions of Theorem 1.4, Lemma 4.1 shows that no matter what the particular hypercomplex structure on  $X$  is, we can *a priori* canonically identify the Hodge bundle  $\Lambda_j^{0,1}(X)$  with the pullback bundle  $\rho^*\Lambda^1(M, \mathbb{C})$ . The only thing that depends on the hypercomplex structure is the derivation  $D : \Lambda^0(X, \mathbb{C}) \rightarrow \Lambda_j^{0,1}(X) \cong \rho^*\Lambda^1(M, \mathbb{C})$ .

To formalize the situation, we introduce the following.

**Definition 4.2.**

1. A  $\mathbb{C}$ -valued connection  $\Theta$  on  $X/M$  is a bundle map

$$\Theta : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$$

*which splits the codifferential  $\delta\rho : \rho^*\Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(X, \mathbb{C})$  of the projection  $\rho : X \rightarrow M$ .*

2. *The derivation  $D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  associated to a  $\mathbb{C}$ -valued connection  $\Theta$  is the composition  $D = \Theta \circ d$  of  $\Theta$  with the de Rham differential  $d$ .*
3. *A Hodge connection  $\Theta$  on  $X/M$  is a  $\mathbb{C}$ -valued connection such that the associated derivation  $D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  is weakly Hodge.*
4. *A Hodge connection  $\Theta$  on  $X/M$  is called flat if the associated derivation  $D$  extends to a weakly Hodge derivation*

$$D : \rho^*\Lambda^*(M, \mathbb{C}) \rightarrow \rho^*\Lambda^{*+1}(M, \mathbb{C})$$

*of the pullback of the de Rham algebra  $\Lambda^*(M, \mathbb{C})$ , and the extended map  $D$  satisfies  $D \circ D = 0$ .*

Of course, a Hodge connection  $\Theta$  is completely defined by the associated derivation  $D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$ . Conversely, an arbitrary weakly Hodge derivation

$$D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$$

comes from a Hodge connection if and only if we have

$$D\rho^*f = \rho^*df \in \rho^*\Lambda^1(M, \mathbb{C})$$

for every smooth function  $f \in \Lambda^0(M, \mathbb{C})$ . Say that a derivation

$$D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$$

is *holonomic* if the associated Hodge connection  $\Theta$  induces an isomorphism

$$(19) \quad \Theta : \Lambda^1(X, \mathbb{R}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$$

between the real cotangent bundle  $\Lambda^1(X, \mathbb{R})$  and the real bundle underlying the complex vector bundle  $\rho^*\Lambda^1(M, \mathbb{C})$ . Then Proposition 3.3 and Lemma 4.1 show that hypercomplex structures on  $X$  satisfying the conditions of Theorem 1.4 are in one-to-one correspondence with Hodge connections on  $X/M$  whose associated derivations are holonomic. Indeed, the isomorphism (19) induces a Hodge bundle structure of weight 1 on the cotangent bundle  $\Lambda^1(X, \mathbb{R})$ , hence an almost quaternionic structure on  $X$ . Applying Proposition 3.3, we see that flatness of the Hodge connection is equivalent to the integrability of this almost quaternionic structure. All derivations  $D$  that we will work with will be automatically holonomic – this will turn out to be a consequence of the normalization condition (6) (see Lemma 4.5).

The name ‘‘Hodge connection’’ invokes the notion of a connection on a smooth fibration. This is somewhat misleading. The problem is that a Hodge connection  $\Theta : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  is only defined over  $\mathbb{C}$ . So it has a real part  $\Theta_{Re}$  and an imaginary part  $\Theta_{Im}$ . The real part

$$\Theta_{Re} : \Lambda^1(X, \mathbb{R}) \rightarrow \rho^*\Lambda^1(M, \mathbb{R})$$

is indeed a connection on the fibration  $\rho : X \rightarrow M$  in the usual sense – that is, it defines a smooth splitting

$$\Lambda^1(X, \mathbb{R}) \cong \rho^*\Lambda^1(M, \mathbb{R}) \oplus \text{Ker } \Theta_{Re}$$

connection on the cotangent of the real cotangent bundle  $\Lambda^1(X, \mathbb{R})$  into a horizontal and a vertical part. The vertical part  $\text{Ker } \Theta_{Re}$  is canonically isomorphic to the relative cotangent bundle  $\Lambda^1(X/M, \mathbb{R})$ .

The imaginary part  $\Theta_{Im}$ , on the other hand, vanishes on the subbundle  $\rho^*\Lambda^1(M, \mathbb{R}) \subset \Lambda^1(X, \mathbb{R})$  and defines therefore a certain map

$$(20) \quad R_J : \Lambda^1(X/M, \mathbb{R}) \rightarrow \rho^*\Lambda^1(M, \mathbb{R})$$

from the relative cotangent bundle  $\Lambda^1(X/M, \mathbb{R})$  to the pullback bundle  $\rho^*\Lambda^1(M)$ .

Since  $X$  is an open subset in  $\overline{TM}$ , we can canonically identify the bundle  $\Lambda^1(X/M)$  with the pullback bundle  $\rho^*\Lambda^1(M)$ . Under this identification, the map  $R_J$  becomes an endomorphism of the bundle  $\rho^*\Lambda^1(M)$ .

Typically, when a Hodge connection  $\Theta$  comes from a hypercomplex structure on  $X$ , the associated real connection  $\Theta_{Re}$  on  $X/M$  is *not* flat. It is only the sum

$$\Theta = \Theta_{Re} + \sqrt{-1}\Theta_{Im}$$

which is flat – but it is no longer a real connection. This situation is somewhat similar to what happens in C. Simpson’s theory of Higgs bundles and harmonic metrics ([S]).

The presence of a non-trivial imaginary part  $\Theta_{Im}$  seems to imply a contradiction. Indeed, a Hodge connection  $\Theta : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  must by definition be compatible with the Hodge bundle structures – in particular, it must commute with the complex conjugation map. But this is different from “real”. The reason for this is the twist by the involution  $\iota : X \rightarrow X$  that we have introduced in the definition of a Hodge bundle. This can be seen clearly if instead of the connection  $\Theta$  one considers the associated derivation  $D$ . The derivation  $D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  satisfies

$$(21) \quad D\iota^*\bar{f} = \iota^*\overline{Df}$$

for any  $\mathbb{C}$ -valued smooth function  $f \in \Lambda^0(X, \mathbb{C})$ . Take the decomposition  $D = D_- + D_+$  into the odd and the even part with respect to the involution  $\iota$ , so that every function  $f \in \Lambda^0(X, \mathbb{C})$  we have

$$D_-(\iota^*f) = -\iota^*D_-(f), \quad D_+(\iota^*f) = \iota^*D_+(f),$$

There is no reason for either one of these parts to vanish. But (21) shows that

$$(22) \quad D_- = \Theta_{Re} \circ d,$$

$$(23) \quad D_+ = \sqrt{-1}\Theta_{Im} \circ d,$$

where  $d : \Lambda^0(X) \rightarrow \Lambda^1(X)$  is the de Rham differential.

The imaginary part  $\Theta_{Im}$  of a Hodge connection  $\Theta$  on  $X/M$  – or rather, the associated map  $R_J$  – by itself has a very direct geometric meaning in terms of the hypercomplex structure on  $X$  given by  $\Theta$ . To describe it, consider the splitting

$$(24) \quad \Lambda^1(X, \mathbb{R}) = \Lambda^1(X/M, \mathbb{R}) \oplus \rho^*\Lambda^1(M, \mathbb{R})$$

given by the real part  $\Theta_{Re}$  and identify  $\rho^*\Lambda^1(M, \mathbb{R}) \cong \Lambda^1(X/M, \mathbb{R})$ .

**Lemma 4.3.** *The operator  $j : \Lambda^1(X, \mathbb{R}) \rightarrow \Lambda^1(X, \mathbb{R})$  of the hypercomplex structure given by  $\Theta$  can be written with respect to the decomposition (24) as the matrix*

$$\begin{pmatrix} 0 & -R_J^{-1} \\ R_J & 0 \end{pmatrix},$$

where  $R_J : \rho^*\Lambda^1(M, \mathbb{R}) \rightarrow \rho^*\Lambda^1(M, \mathbb{R})$  is the bundle endomorphism (20).

*Proof.* Since  $j^2 = -\text{id}$ , it suffices to prove that for every horizontal 1-form  $\alpha \in \rho^*\Lambda^1(M, \mathbb{R})$  the 1-form  $j(\alpha)$  is vertical, – that is,

$$(25) \quad \Theta_{Re}(j(\alpha)) = 0,$$

and moreover, that we have

$$(26) \quad \Theta_{Im}(j(\alpha)) = -\alpha.$$

Let  $\alpha$  be such a form. By definition, the kernel  $\text{Ker } \Theta \subset \Lambda^1(X, \mathbb{C})$  of the projection  $\Theta : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  is the subbundle  $\Lambda_J^{1,0}(X)$  of  $(1, 0)$ -forms for the complementary complex structure on  $X$ . Therefore we have

$$\alpha - \sqrt{-1}j(\alpha) \in \text{Ker } \Theta,$$

which means that

$$\Theta(\alpha) = \sqrt{-1}\Theta(j(\alpha)).$$

Since  $\alpha = \Theta(\alpha)$ , equations (25) and (26) are the real and the imaginary parts of this equality. □

In keeping with the general philosophy of this section, we will use the formula for  $j$  given by Lemma 4.4 to express the bundle endomorphism  $R_J : \rho^*\Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(M, \mathbb{C})$  entirely in terms of operators on the algebra  $\rho^*\Lambda^*(M, \mathbb{C})$ . To do this, consider the tautological section of the pullback tangent bundle  $\rho^*T(M)$ , and let

$$\tau : \rho^*\Lambda^{*+1}(M, \mathbb{C}) \rightarrow \rho^*\Lambda^*(M, \mathbb{C})$$

be the operator given by contraction with this tautological section. Thus  $\tau$  vanishes on functions, and for every 1-form  $\alpha \in \Lambda^1(M, \mathbb{R})$  the function  $\tau(\rho^*\alpha)$  is just  $\alpha$  considered as a fiberwise-linear function on the total space  $\overline{T}(M)$ .

**Lemma 4.4.** *For any 1-form  $\alpha \in \Lambda^1(M, \mathbb{C})$  we have*

$$R_J(\alpha) = -\sqrt{-1}D_+\tau(\alpha).$$

*Proof.* By (23), the right-hand side is equal to

$$\Theta_{Im}(d\tau(\alpha)) \in \rho^*\Lambda^1(M, \mathbb{C}).$$

Since the projection  $\Theta_{Im} : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$  vanishes on the subbundle  $\rho^*\Lambda^1(M, \mathbb{C}) \subset \Lambda^1(X, \mathbb{C})$ , this expression depends only the relative 1-form  $P(d\tau(\alpha)) \in \Lambda^1(X/M, \mathbb{C})$  obtained from the 1-form  $d\tau(\alpha) \in \Lambda^1(X, \mathbb{C})$  by the projection  $P : \Lambda^1(X, \mathbb{C}) \rightarrow \Lambda^1(X/M, \mathbb{C})$ . But  $P(d\tau(\alpha))$  is precisely the image of the form  $\alpha \in \rho^*\Lambda^1(M, \mathbb{C})$  under the canonical isomorphism  $\rho^*\Lambda^1(M, \mathbb{C}) \cong \Lambda^1(X/M, \mathbb{C})$ . □

We will now rewrite in the same spirit the normalization condition (6) on the hypercomplex structure on  $X$  associated to  $D$ . For this we need to extend the canonical isomorphism  $\Lambda^1(X/M, \mathbb{C}) \cong \rho^*\Lambda^1(M, \mathbb{C})$  to an algebra isomorphism  $\Lambda^*(X/M, \mathbb{C}) \cong \rho^*\Lambda^*(M, \mathbb{C})$ . Then the map

$$\tau : \rho^*\Lambda^{*+1}(M, \mathbb{C}) \rightarrow \rho^*\Lambda^*(M, \mathbb{C})$$

becomes the contraction with the relative Euler vector field (that is, the differential of the  $\mathbb{R}^*$ -action by homoteties along the fibers of the projection  $\rho : X \rightarrow M$ ).

The normalization condition (6) involves a different vector field – namely, the differential  $\phi$  of the action by homoteties of the group  $U(1)$ . It will be more convenient

now to multiply it by  $\sqrt{-1}$  (or, equivalently, to change the generator of the Lie algebra of the circle  $U(1)$  from  $\frac{\partial}{\partial \theta}$  to  $z \frac{\partial}{\partial z}$ ). Denote by

$$\sigma : \rho^* \Lambda^{*+1}(M, \mathbb{C}) \cong \Lambda^{*+1}(X/M, \mathbb{C}) \longrightarrow \rho^* \Lambda^*(M, \mathbb{C}) \cong \Lambda^*(X/M, \mathbb{C})$$

the contraction with the vertical vector field  $\sqrt{-1}\phi$ . The operators  $\sigma$  and  $\tau$  are related by

$$\sigma(\alpha) = \sqrt{-1}\tau(I\alpha), \quad \alpha \in \Lambda^1(M, \mathbb{C}),$$

where  $I : \Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(M, \mathbb{C})$  is the complex structure operator – in other words,

$$\sqrt{-1}I = \begin{cases} -\text{id} & \text{on } \Lambda^{1,0}(M). \\ \text{id} & \text{on } \Lambda^{0,1}(M). \end{cases}$$

**Lemma 4.5.** *Let  $\Theta$  be a Hodge connection on  $X/M$ , and let  $D_+$  be the even component of the associated derivation*

$$D : \Lambda^0(X, \mathbb{C}) \rightarrow \rho^* \Lambda^1(M, \mathbb{C}).$$

*The hypercomplex structure on  $X$  given by  $\Theta$  satisfies the normalization condition (6) if and only for every 1-form  $\alpha \in \Lambda^1(M, \mathbb{C})$  we have*

$$\sigma \circ D_+(f) = f.$$

*where  $f = \tau(\rho^* \alpha) \in \Lambda^0(X, \mathbb{C})$ . Moreover, if this is the case, then the Hodge connection  $\Theta$  is holonomic.*

*Proof.* It suffices to check (6) by evaluating both sides on every 1-form  $\alpha \in \rho^* \Lambda^1(M, \mathbb{C})$ . Moreover, it is even enough to check it for forms of the type  $\rho^* \alpha$ , where  $\alpha \in \Lambda^1(M, \mathbb{C})$  is a 1-form on  $M$ . Let  $\alpha$  be such a form. We have to check that

$$j(\rho^* \alpha) \lrcorner \phi = \tau(\rho^* \alpha).$$

By Lemma 4.3 this is equivalent to

$$\sigma(R_J(\rho^* \alpha)) = -\sqrt{-1}\tau(\rho^* \alpha),$$

and by Lemma 4.4 this can be further rewritten as

$$-\sqrt{-1}\sigma(D_+\tau(\rho^* \alpha)) = -\sqrt{-1}\tau(\rho^* \alpha).$$

Replacing  $\tau(\rho^* \alpha)$  with  $f$  gives precisely the first claim of the lemma.

To prove the second claim, we have to show that the map

$$\Theta : \Lambda^1(X, \mathbb{R}) \rightarrow \rho^* \Lambda^1(M, \mathbb{C})$$

is surjective. By definition, on the second term  $\rho^* \Lambda^1(M, \mathbb{R}) \subset \Lambda^1(X, \mathbb{R})$  in the splitting (24) we have  $\Theta = \Theta_{Re} = \text{id}$ . Therefore it suffices to prove that

$$\Theta_{Im} : \Lambda^1(X/M, \mathbb{R}) \rightarrow \sqrt{-1}\rho^* \Lambda^1(M, \mathbb{R})$$

is surjective. But by (23) and the first claim of the lemma, this is the inverse map to

$$\sigma : \sqrt{-1}\rho^* \Lambda^1(M, \mathbb{R}) \rightarrow S^1(M, \mathbb{R}) \cong \Lambda^1(X/M, \mathbb{R}). \quad \square$$

5. THE WEIL ALGEBRA.

The final preliminary step in the proof of Theorem 1.4 is to reduce it from a question about the total space  $X = \overline{TM}$  of the complex-conjugate to the tangent bundle on  $M$  to a question about the manifold  $M$ . To do this, we introduce the following.

**Definition 5.1.** *The Weil algebra  $\mathcal{B}^*(M)$  of a complex manifold  $M$  is the algebra on  $M$  defined by*

$$\mathcal{B}^*(M) = \rho_* \rho^* \Lambda^*(M, \mathbb{C}),$$

where  $\rho : \overline{TM} \rightarrow M$  is the canonical projection.

This requires an explanation – indeed, for a vector bundle  $\mathcal{E}$  on  $\overline{TM}$ , the direct image sheaf  $\rho_* \mathcal{E}$  a priori is not a sheaf of sections of any vector bundle on  $M$ . We have to consider a smaller subsheaf. From now on and until Theorem 1.4 is proved, we will be interested not in hypercomplex structures on the total space  $\overline{TM}$  but in their formal Taylor decompositions in the neighborhood of the zero section  $M \subset \overline{TM}$ . Therefore it will be sufficient for our purposes to define the direct image  $\rho_* \mathcal{E}$  as the sheaf of sections of the bundle  $\mathcal{E}$  on  $\overline{TM}$  which are polynomial along the fibers of the projection  $\rho : \overline{TM} \rightarrow M$ . Formal germs of bundle maps on  $\mathcal{E}$  will give formal series of maps between the corresponding direct image bundles.

Having said this, we can explicitly describe the Weil algebra  $\mathcal{B}^*(M)$ . Our first remark is that  $\mathcal{B}^k(M)$  is canonically a Hodge bundle on  $M$  of weight  $k$ . Moreover, since the  $U(1)$ -action on  $M$  is trivial, Hodge bundles on  $M$  are just bundles of  $\mathbb{R}$ -Hodge structures in the usual sense. Thus we have a Hodge type bigrading

$$\mathcal{B}^k(M) = \bigoplus_{p+q=k} \mathcal{B}^{p,q}(M)$$

and a canonical real structure on every one of the complex vector bundles  $\mathcal{B}^k(M)$ .

The projection formula show that for every  $k$  we have a canonical isomorphism

$$\mathcal{B}^k(M) \cong \mathcal{B}^0(M) \otimes \Lambda^k(M, \mathbb{C}).$$

These isomorphisms are compatible with the Hodge structures and with multiplication. The degree-0 Hodge bundle  $\mathcal{B}^0(M)$  is a symmetric algebra freely generated by the bundle  $S^1(M, \mathbb{C})$  of functions on  $\overline{TM}$  linear along the fibers of  $\rho : \overline{TM} \rightarrow M$ . The complex vector bundle  $S^1(M, \mathbb{C})$  is canonically isomorphic to the bundle  $\Lambda^1(M, \mathbb{C})$  of 1-forms on  $M$ . However, the Hodge structures on these bundles are different. The Hodge type grading on  $S^1(M, \mathbb{C})$  is given by

$$S^1(M, \mathbb{C}) = S^{1,-1}(M) \oplus S^{-1,1}(M, \mathbb{C}),$$

where  $S^{1,-1}(M) \cong \Lambda^{1,0}(M)$  and  $S^{-1,1}(M) \cong \Lambda^{0,1}(M)$  – the grading is the same as on  $\Lambda^1(M, \mathbb{C})$  but graded pieces are assigned different weights. Moreover, the complex conjugation map on  $S^1(M, \mathbb{C})$  is *minus* the complex conjugation map on  $\Lambda^1(M, \mathbb{C})$ . This is the last vestige of the twist by the involution  $\iota : \overline{TM} \rightarrow \overline{TM}$  in Definition 3.1.

To simplify notation, denote by  $S^k(M, \mathbb{C})$  the  $k$ -th symmetric power of the Hodge bundle  $S^1(M, \mathbb{C})$ . Then we have

$$\mathcal{B}^0(M) = \bigoplus_{k \geq 0} S^k(M, \mathbb{C}),$$

and the Weil algebra  $\mathcal{B}^*(M) = \mathcal{B}^0(M) \otimes \Lambda^*(M, \mathbb{C})$  is the free graded-commutative algebra generated by  $S^1(M, \mathbb{C})$  and  $\Lambda^1(M, \mathbb{C})$  (where, contrary to notation,  $S^1(M, \mathbb{C})$  is placed in degree 0).

It will be convenient to introduce another grading on the Weil algebra  $\mathcal{B}^*(M)$  by assigning to both of the generator bundles  $S^1(M, \mathbb{C})$ ,  $\Lambda^1(M, \mathbb{C})$  degree 1. We will call it *augmentation grading* and denote by lower indices, so that we have

$$\begin{aligned} S^1(M, \mathbb{C}) &= \mathcal{B}_1^0(M) \subset \mathcal{B}^0(M), \\ \Lambda^1(M, \mathbb{C}) &= \mathcal{B}_1^1(M) \subset \mathcal{B}^1(M). \end{aligned}$$

The augmentation grading corresponds to the Taylor decomposition near the zero section  $M \subset \overline{TM}$ . Namely, every formal germ near  $M \subset \overline{TM}$  of a flat Hodge connection on  $\overline{TM}/M$  induces a formal series

$$(27) \quad D = \sum_{k \geq 0} D_k$$

of algebra bundle derivations

$$D_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M),$$

where each of the derivations  $D_k$  is weakly Hodge and has augmentation degree  $k$ . Their (formal) sum satisfies

$$D \circ D = 0.$$

Conversely, every formal series (27) induces a (formal germ of a) weakly Hodge derivation  $D : \rho^* \Lambda^*(M, \mathbb{C}) \rightarrow \rho^* \Lambda^{*+1}(M, \mathbb{C})$  on the total space  $\overline{TM}$ . This derivation comes from a flat Hodge connection if and only if we have

$$D(f) = df$$

for every function  $f \in \mathcal{B}_0^0(M) \cong \Lambda^0(M, \mathbb{C})$ . Since we have  $D \circ D = 0$ , this immediately implies that  $D$  coincides with the de Rham differential  $d$  on the whole subalgebra

$$\Lambda^*(M, \mathbb{C}) \subset \mathcal{B}^*(M) \cong \mathcal{B}^0(M) \otimes \Lambda^*(M, \mathbb{C}).$$

More precisely, we must have  $D_1 = d$  on  $\Lambda^*(M, \mathbb{C})$ , and all the other components  $D_k, k \neq 1$  must vanish on this subalgebra. Since  $D$  is a derivation, this in turn implies that all the components  $D_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$  for  $k \neq 1$  are not differential operators but bundle maps. Moreover, since the algebra  $\mathcal{B}^*(M)$  is freely generated by  $S^1(M, \mathbb{C})$  and  $\Lambda^1(M, \mathbb{C})$ , and we know *a priori* the derivation  $D$  on the generator subbundle  $\Lambda^1(M, \mathbb{C})$ , it always suffices to specify the restriction  $D : S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M) \rightarrow \mathcal{B}^1(M)$ .

The decomposition  $D = D_- + D_+$  of a Hodge connection into an even and an odd part is quite transparent on the level of the Weil algebra – we simply have

$$D_- = \sum_{k \geq 0} D_{2k+1} \qquad D_+ = \sum_{k \geq 0} D_{2k}.$$

We will now rewrite the normalization condition (6) in terms of the Weil algebra. To do this, note that the map  $\sigma : \rho^* \Lambda^{\bullet+1}(M, \mathbb{C}) \rightarrow \Lambda^\bullet(M, \mathbb{C})$  induces a bundle map  $\sigma : \mathcal{B}^{\bullet+1}(M) \rightarrow \mathcal{B}^\bullet(M)$ . This map is in fact a derivation of the Weil algebra. It vanishes on the generator bundle  $S^1(M, \mathbb{C})$ , while on the generator bundle  $\Lambda^1(M, \mathbb{C})$  it is given by

$$\sigma = \begin{cases} \text{id} : \Lambda^{1,0}(M) \rightarrow S^{1,-1}(M) \cong \Lambda^{1,0}(M), \\ -\text{id} : \Lambda^{0,1}(M) \rightarrow S^{-1,1}(M) \cong \Lambda^{0,1}(M). \end{cases}$$

Then Lemma 4.5 immediately shows that the (formal germ of the) hypercomplex structure on  $\overline{T}M$  induced by a derivation  $D : \mathcal{B}^\bullet(M) \rightarrow \mathcal{B}^{\bullet+1}(M)$  is normalized if and only if we have

$$(28) \qquad \qquad \qquad \sigma \circ D_+ = \text{id}$$

on the generator subbundle  $S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M)$ . It is convenient to modify this in the following way. Let  $C : S^1(M, \mathbb{C}) \rightarrow \Lambda^1(M, \mathbb{C})$  be the isomorphism inverse to  $\sigma : \Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(M, \mathbb{C})$ . Set  $C = 0$  on the generator subbundle  $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$  and extend it a derivation  $C : \mathcal{B}^\bullet(M) \rightarrow \mathcal{B}^{\bullet+1}(M)$  of the Weil algebra. Both derivations  $C$  and  $\sigma$  are real. Moreover, the derivation  $C$  is weakly Hodge (the derivation  $\sigma$  is not – simply because it decreases the weight). Then the normalization condition is equivalent to

$$\begin{cases} D_0 = C, \\ \sigma \circ D_k = 0 \text{ on } S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M) \text{ for every even } k = 2p \geq 1. \end{cases}$$

To sum up, formal germs near  $M \subset \overline{T}M$  of normalized flat Hodge connections on  $\overline{T}M$  are in a natural one-to-one correspondence with derivations

$$D = \sum_{k \geq 0} D_k : \mathcal{B}^\bullet(M) \rightarrow \mathcal{B}^{\bullet+1}(M)$$

of the Weil algebra  $\mathcal{B}^\bullet(M)$  which satisfy the following conditions.

1.  $D \circ D = 0$ .
2.  $D_k : \mathcal{B}^\bullet(M) \rightarrow \mathcal{B}^{\bullet+1}(M)$  is a weakly Hodge algebra derivation of augmentation degree  $k$ .
3.  $D_0 = C$ .
4.  $D_1 = d$  and  $D_k = 0, k \neq 0$  on the subalgebra  $\Lambda^\bullet(M, \mathbb{C}) \subset \mathcal{B}^\bullet(M)$ .
5.  $\sigma \circ D_{2k} = 0$  on  $S^1(M, \mathbb{C}) = \mathcal{B}_1^0(M) \subset \mathcal{B}^0(M)$  for every  $k \geq 1$ .



For every such derivation, the differential operator

$$D_1 : S^1(M, \mathbb{C}) = \mathcal{B}_1^0(M) \rightarrow \mathcal{B}_2^1(M) \cong S^1(M, \mathbb{C}) \otimes \Lambda^1(M, \mathbb{C})$$

satisfies the Leibnitz rule

$$D_1(fa) = fD_1(a) + adf, \quad a \in S^1(M, \mathbb{C}), f \in \Lambda^0(M, \mathbb{C}).$$

Therefore it is a connection on the bundle  $S^1(M, \mathbb{C})$ . We postpone the proof of the following Lemma till the end of Section 7.

**Lemma 5.2.** *The connection  $D_1$  on the bundle  $S^1(M, \mathbb{C}) \cong \Lambda^1(M, \mathbb{C})$  coincides with the connection on  $M$  induced by the Obata connection for the hypercomplex structure on  $\bar{T}M$  defined by the derivation  $D$ .*

With Lemma 5.2 in mind, we see that Theorem 1.4 is reduced to the following statement.

**Proposition 5.3.** *Suppose that  $\nabla$  is a torsion-free connection on the cotangent bundle  $\Lambda^1(M, \mathbb{C})$  of a complex manifold  $M$  such that the curvature of the connection  $\nabla$  is of type  $(1, 1)$ .*

*Then there exists a unique derivation*

$$D = \sum_{k \geq 0} D_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$$

*of the Weil algebra  $\mathcal{B}^*(M)$  of the manifold  $M$  such that  $D$  satisfies the conditions (i)-(v) above and we have*

$$D_1 = \nabla$$

*on  $S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M)$ .*

This ends the preliminaries. We now begin the proof of Proposition 5.3.

### 6. THE PROOF OF PROPOSITION 5.3.

The proof proceeds by induction on the augmentation degree. Denote

$$D_{\leq k} = D_0 + D_1 + \dots + D_k.$$

To base the induction, consider the derivation  $D_{\leq 1}$ . By assumptions it is equal to

$$D_{\leq 1} = C + D_1.$$

Let  $R : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$  be the composition

$$R = D_{\leq 1} \circ D_{\leq 1}.$$

Since the derivation  $D_{\leq 1}$  of the graded-commutative algebra  $\mathcal{B}^*(M)$  is of odd degree, up to a coefficient the composition  $R$  coincides with the supercommutator  $\{D_{\leq 1}, D_{\leq 1}\}$ . In particular, it is also an algebra derivation.

The derivation  $R : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+2}(M)$  *a priori* has components  $R_0, R_1, R_2$  of augmentation degrees 0, 1 and 2. However,

$$R_0 = \{C, C\} = 0.$$

Moreover,

$$R_1 = \{C, D_1\} : S^1(M, \mathbb{C}) \rightarrow \mathcal{B}_2^2(M) \cong \Lambda^2(M, \mathbb{C})$$

is precisely the torsion of the connection  $\nabla = D_1$ . Thus it vanishes by assumption. Since  $C = 0$  on  $\Lambda^*(M, \mathbb{C}) \subset \mathcal{B}^*(M)$ , we also have  $R_1 = 0$  on  $\Lambda^1(M, \mathbb{C})$ , which implies that  $R_1 = 0$  everywhere. What remains is  $R_2$ . In general, it does *not* vanish, and to kill it we have to add new terms  $D_k$ .

We now turn to the induction step. Assume that for some  $k \geq 2$  we are already given the derivation  $D_{\leq k-1}$  which satisfies the conditions (ii)-(v) on page 222, and assume that the composition  $D_{\leq k-1} \circ D_{\leq k-1}$  has no non-trivial components of augmentation degrees  $< k$ . Denote by  $R_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$  its component of augmentation degree  $k$ . We have to find a derivation  $D_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}$  which also satisfies (ii)-(v) and such that

$$(D_{\leq k-1} + D_k) \circ (D_{\leq k-1} + D_k) = 0$$

in augmentation degree  $k$ . This is equivalent to

$$(29) \quad \{D_0, D_k\} = \{C, D_k\} = -R_k.$$

The conditions (ii) and (iii) mean that  $D_k : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$  must be a weakly Hodge derivation which vanishes on  $\Lambda^1(M, \mathbb{C}) = \mathcal{B}_1^1(M) \subset \mathcal{B}^1(M)$ . The condition (iv) is only relevant for  $D_1$ . Finally, the condition (v) is relevant for all even  $k$  and means that

$$\sigma \circ D_k = 0$$

on  $S^1(M, \mathbb{C}) \subset \mathcal{B}^0(M)$ .

Because of (iii), it suffices to define  $D_k$  on the generator subbundle  $S^1(M, \mathbb{C})$ . Since  $D_k$  commutes with the complex conjugation map, it even suffices to consider only  $S^{1,-1}(M) \subset S^1(M, \mathbb{C})$ . Moreover, (iii) implies that it also suffices to check (29) only on  $S^{1,-1}(M) = \mathcal{B}_1^{1,-1}(M) \subset \mathcal{B}^0(M)$ . Note that on this subbundle we have  $\{C, D_k\} = C \circ D_k$ .

There will be two slightly different cases. The first is one when  $k = 2p + 1$  is odd, the second one is when  $k = 2p$  is even.

In both cases, the weakly Hodge map  $R_k : \mathcal{B}_1^0(M) \rightarrow \mathcal{B}_{k+1}^2(M)$  only has non-trivial pieces of Hodge bidegrees  $(2, 0)$ ,  $(1, 1)$  and  $(0, 2)$ . Moreover, for any map  $\Theta$  of odd degree we tautologically have  $[\{\Theta, \Theta\}, \Theta] = 0$ . Applying this to  $\Theta = D_{\leq k}$  and collecting terms of augmentation degree  $k$ , we see that

$$C \circ R_k = 0.$$

To track various components of the map  $R_k : \mathcal{B}_1^0(M) \rightarrow \mathcal{B}_{k+1}^2(M)$  it is convenient to refine the augmentation grading on the Weil algebra  $\mathcal{B}^*(M)$  to an *augmentation*

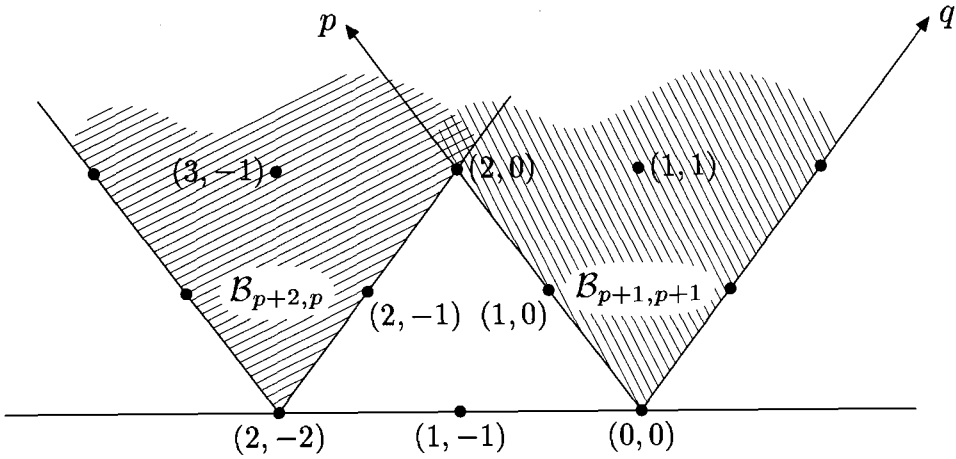


FIGURE 1. The augmentation bigrading on  $\mathcal{B}_{k+1}^*$  for an odd  $k$ ,  $k = 2p + 1$ .

bigrading by setting

$$\deg S^{1,-1}(M) = \deg \Lambda^{1,0}(M) = (1, 0),$$

$$\deg S^{-1,1}(M) = \deg \Lambda^{0,1}(M) = (0, 1),$$

on the generator bundles

$$S^1(M, \mathbb{C}) = S^{1,-1}(M) \oplus S^{-1,1}(M)$$

and

$$\Lambda^1(M, \mathbb{C}) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M).$$

The augmentation bigrading will be denoted by lower indices, so that we have

$$\mathcal{B}_k^* = \bigoplus_{p+q=k} \mathcal{B}_{p,q}^*.$$

The relevant pieces of the augmentation bigrading on the bundle  $\mathcal{B}_{k+1}^*(M)$  are shown on Figure 1 for  $k = 2p + 1$  odd, and on Figure 2 for  $k = 2p$  even. The axes on the figures correspond to the grading by Hodge type. A Hodge bidegree component  $\mathcal{B}_{m,n}^{p,q}$  can be non-trivial only when  $p \geq m - n$  and  $q \geq n - m$ . Thus the component  $\mathcal{B}_{m,n}^{*,*}$  is represented by an upward-looking angle with vertex  $(m - n, n - m)$ : a graded piece  $\mathcal{B}_{m,n}^{p,q}$  can be non-trivial only if the point  $(p, q)$  lies in the interior (or on the boundary) of this angle.

Consider the Hodge bigrading decompositions

$$C = C^{1,0} + C^{0,1} \quad \sigma = \sigma^{-1,0} + \sigma^{0,-1}$$

of the derivations  $C, \sigma$  of the Weil algebra  $\mathcal{B}^*(M)$ . Then the augmentation bigrading is essentially the eigenvalue decomposition for the commutators  $\{C^{1,0}, \sigma^{-1,0}\}$  and

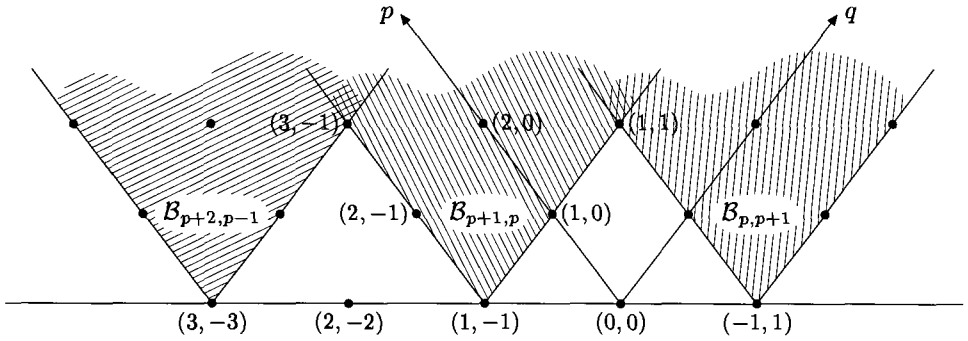


FIGURE 2. The augmentation bigrading on  $\mathcal{B}_{k+1}^*$  for an even  $k$ ,  $k = 2p$ .

$\{C^{0,1}, \sigma^{0,-1}\}$ . More precisely, we have

$$\begin{aligned}
 \{C^{1,0}, \sigma^{0,-1}\} &= \{C^{0,1}, \sigma^{-1,0}\} = 0, \\
 \{C^{1,0}, \sigma^{-1,0}\} &= m \text{ id on } \mathcal{B}_{m,n}^*, \\
 \{C^{0,1}, \sigma^{0,-1}\} &= n \text{ id on } \mathcal{B}_{m,n}^*
 \end{aligned}
 \tag{30}$$

Indeed, since all these commutators are derivations of the Weil algebra, it suffices to check this on the generator bundles  $S^1(M, \mathbb{C})$ ,  $\Lambda^1(M, \mathbb{C})$ , which is elementary. In particular, we see that both  $C$  and  $\sigma$  preserve the augmentation bidegree. The equalities (30) also immediately imply that

$$\{C, \sigma\} = k \text{ id on } \mathcal{B}_k^*.$$

One further corollary of (30) will be very important (we leave the proof to the reader as an easy exercise).

**Lemma 6.1.** *If  $m, n \geq 1$ , then the map  $C^{1,0}$  is injective on every graded piece  $\mathcal{B}_{m,n}^{p,q}$  with  $q = n - m$ , while  $C^{0,1}$  is injective on  $\mathcal{B}_{m,n}^{p,q}$  with  $p = m - n$ .*

Graphically, this means that  $C^{1,0}$  is injective on  $\mathcal{B}_{m,n}^{p,q}$  when the point  $(p, q)$  lies on the right-hand boundary of the angle representing  $\mathcal{B}_{m,n}^*$ , and  $C^{0,1}$  is injective in this graded piece when the point  $(p, q)$  lies on the left-hand boundary of the same angle. We will call this *the boundary rule*.

We can now proceed with the proof of the induction step.

*Case when  $k = 2p + 1$  is odd.* Looking at Figure 1, we see that the only non-trivial augmentation-bidegree components of the map  $R_k : \mathcal{B}_1^0(M) \rightarrow \mathcal{B}_{k+1}^2(M)$  are  $R_{p,p+1}$  and  $R_{p+1,p}$ ,

$$R_k = R_{p,p+1} + R_{p+1,p},$$

and the same is true for any weakly Hodge map  $D_k : S^{1,-1}(M) \rightarrow \mathcal{B}_{k+1}^1(M)$ ,

$$D_k = D_{p,p+1} + D_{p+1,p}.$$

Moreover, on  $S^{1,-1}(M) \subset \mathcal{B}^0(M)$  we have

$$\begin{aligned} D_k^{0,1} &= D_{p,p+1} & D_k^{1,0} &= D_{p+1,p}, \\ R_k^{0,2} &= R_{p,p+1}^{0,2} & D_k^{2,0} &= R_{p+1,p}^{2,0}, \end{aligned}$$

while  $R_k^{1,1}$  further decomposes as  $R_{p+1,p}^{1,1} + R_{p,p+1}^{1,1}$ .

We have to find  $D_k$  which satisfies (29). In particular, we must have

$$(31) \quad R_k^{2,0} = -C^{1,0} \circ D_k^{1,0} \quad R_k^{0,2} = -C^{0,1} \circ D_k^{0,1}.$$

But by the boundary rule the map  $C^{1,0}$  is injective on  $\mathcal{B}_{p+2,p}^{2,-1}$ , while the map  $C^{0,1}$  is injective on  $\mathcal{B}_{p+1,p+1}^{1,0}$ . Therefore there exists at most one weakly Hodge map  $D_k : \mathcal{B}_{1,0}^0(M) \rightarrow \mathcal{B}_{k+1}^1$  which satisfies (31). Setting (again on  $S^{1,-1}(M)$ )

$$\begin{aligned} D_k^{0,1} &= D_{p,p+1}^{0,1} = -\frac{1}{p+1} \sigma^{0,-1} \circ R_k^{0,2}, \\ D_k^{1,0} &= D_{p+1,p}^{1,0} = -\frac{1}{p+2} \sigma^{-1,0} \circ R_k^{2,0}. \end{aligned}$$

gives this unique solution to (31). Indeed, we have

$$\begin{aligned} C^{1,0} \circ D_k^{1,0} &= -\frac{1}{p+2} C^{1,0} \circ \sigma^{-1,0} \circ R_k^{2,0} \\ &= -\frac{1}{p+2} (\{C^{1,0}, \sigma^{-1,0}\} \circ R_{p+1,p}^{2,0} + \sigma^{-1,0} \circ C^{1,0} \circ R_k^{2,0}). \end{aligned}$$

The second summand in the brackets vanishes since  $C \circ R_k = 0$ , while the first is equal to  $(p+2)R_k^{2,0}$  by (30). This proves the first equation in (31). The second one is proved in exactly the same way.

It remains to prove that this map  $D_k$  satisfies not only (31) but also the stronger condition (29). To do this, note that

$$C \circ (C \circ D_k + R_k) = (C \circ C) \circ D_k + C \circ R_k = 0.$$

But from (31) we see that  $C \circ D_k + R_k$  is of Hodge bidegree  $(1, 1)$ . Therefore this implies that

$$C^{1,0} \circ (C \circ D_k + R_k) = C^{0,1} \circ (C \circ D_k + R_k) = 0.$$

The only possible non-trivial components of  $C \circ D_k + R_k$  with respect to the augmentation bigrading have bidegrees  $(p+1, p)$  and  $(p, p+1)$ , and by the boundary rule  $C^{1,0}$  is injective on  $\mathcal{B}_{p+1,p}^{2,0}$ , while  $C^{0,1}$  is injective on  $\mathcal{B}_{p,p+1}^{2,0}$ . Thus  $C \circ D_k + R_k = 0$ .

*Case when  $k = 2p$  is odd.* Assume for the moment that  $k \geq 4$ , thus  $p \geq 2$ .

Looking at Figure 2, we see that *a priori* the map  $R_k$  can have three non-trivial augmentation-bidegree components, namely,

$$R_k = R_{p-1,p+1}^{0,2} + R_{p,p} + R_{p+1,p-1}^{2,0}.$$

However, since  $C \circ R_k = 0$ , and the map  $C^{1,0}$  is injective on  $\mathcal{B}_{p+2,p-1}^{3,-1}(M)$  by the boundary rule, we see that  $R_{p+1,p-1}^{0,2} = 0$ .

Analogously,  $R_{p-1,p+1} = 0$ . Therefore  $R_k$  is of pure augmentation bidegree  $(p, p)$ .

Since the map  $D_k : S^1(M) \rightarrow \mathcal{B}_{k+1}^1(M)$  is weakly Hodge, it must also be of augmentation bidegree  $(p, p)$ . Conversely, looking at the angle representing  $\mathcal{B}_{p+1,p-1}^{1,*}$ , we see that every real map  $D_k : S^1(M, \mathbb{C}) \rightarrow \mathcal{B}_{k+1}^1$  of pure augmentation bidegree  $(p, p)$  is necessarily weakly Hodge. In particular, setting

$$(32) \quad D_k = -\frac{1}{k+1} \sigma \circ R_k$$

defines a weakly Hodge map. This map is a solution to (29):

$$C \circ D_k = -\frac{1}{k+1} C \circ \sigma \circ R_k = -\frac{1}{k+1} \{C, \sigma\} \circ R_k - \sigma \circ C \circ R_k = -R_k.$$

This solution is not unique. However, since  $k$  is even, we have the additional normalization condition  $\sigma \circ D_k = 0$ . This condition (automatically satisfied by the solution (32)) ensures uniqueness. Indeed, the difference  $P = D_k - D'_k$  between two solutions  $D_k, D'_k$  must satisfy  $C \circ P = \sigma \circ P = 0$ , which implies

$$P = \frac{1}{k+1} \{C, \sigma\} \circ P = 0.$$

Finally, it remains to consider the case  $k = 2$ . The general argument works in this case just as well, with a single exception. Since  $p - 1 = 0$  is no longer strictly positive, the boundary rule does not apply: it is not true that  $C^{1,0}$  is injective on  $\mathcal{B}_{p+1,p-1}^{3,-1} = \mathcal{B}_{3,0}^{3,-1}$  (in fact, on this graded piece  $C^{1,0}$  is equal to zero). Therefore the component  $R_{p+1,p-1}$  does not vanish automatically. However, this component

$$R_{2,0}^{2,0} : S^{1,-1}(M) \rightarrow \mathcal{B}_{3,0}^{3,-1}(M) \cong \Lambda^{2,0}(M) \otimes S^{1,-1}(M)$$

is precisely the  $(2, 0)$ -curvature of the connection  $\nabla$  on  $M$ . It vanishes by the second assumption on this connection. □

### 7. METRICS.

The last Section essentially finishes the proof of the hypercomplex Theorem 1.4 (it remains to prove Lemma 5.2). We will now sketch a proof of the hyperkähler Theorem 1.1.

As we have already noted, Theorem 1.1 will be a corollary of Theorem 1.4. Namely, given a Kähler manifold  $M$  we proceed in the following way. First we note that the Levi-Civita connection  $\nabla_{LC}$  on  $M$  has no torsion and no  $(2, 0)$ -curvature. Therefore Theorem 1.4 applies to  $\nabla_{LC}$  and provides a hypercomplex structure on the total space  $X = \overline{T}M$ . Then we show that every Hermitian metric on  $M$  which is preserved by  $\nabla_{LC}$  (in particular, the given Kähler metric) extends uniquely from the zero section  $M \subset \overline{T}M$  to a (formal germ of a) hyperhermitian metric on the hypercomplex manifold  $X = \overline{T}M$  which is compatible with the hypercomplex structure. After this,

we finish the proof by identifying the holomorphically symplectic manifolds  $\overline{T}M$  and  $T^*M$ .

We will go through these steps in reverse order, starting with the last one.

**Lemma 7.1.** *Assume given a hypercomplex structure on the total space  $X = \overline{T}M$  which satisfies the conditions of Theorem 1.4. Let  $h$  be a  $U(1)$ -invariant hyperkähler metric on  $X$  compatible with this hypercomplex structure, and let  $\Omega_X \in \Lambda^{2,0}(X)$  be the associated holomorphic 2-form. Let  $T^*M$  be the total space of the cotangent bundle to  $M$  equipped with the standard holomorphic 2-form  $\Omega$ .*

*Then there exists a unique  $U(1)$ -equivariant biholomorphic map  $\eta : X \rightarrow T^*M$  such that  $\Omega_X = \eta^*\Omega$ .*

*Proof.* Since the map  $\eta$  must be  $U(1)$ -equivariant, it must commute with the canonical projections  $\rho : X, T^*M \rightarrow M$  and send the zero section  $M \subset X$  to the zero section  $M \subset T^*M$ . Denote by  $\phi$  the differential of the  $U(1)$ -action. Then we also must have

$$\Omega_X \lrcorner \phi = \eta^*\Omega \lrcorner \phi = \eta^*(\Omega \lrcorner \phi).$$

But the 1-form  $\alpha = \Omega \lrcorner \phi$  is the tautological 1-form  $\alpha \in \rho^*(\Lambda^1(M)) \subset \Lambda^1(T^*M)$ . Therefore the 1-form  $\rho^*\alpha$  on  $X$  completely defines the map  $\eta$ .

Conversely, the form  $\alpha_X = \Omega_X \lrcorner \phi$  satisfies  $\alpha_X = \eta^*\alpha$  for a unique map  $\eta : X \rightarrow T^*M$ . Since the metric  $h$  is  $U(1)$ -invariant, the forms  $\Omega_X$  and  $\alpha_X$  are of weight 1. Therefore the map  $\eta : X \rightarrow T^*M$  is  $U(1)$ -equivariant. By the Cartan homotopy formula, we have

$$\Omega_X = d\alpha_X = d\eta^*\alpha = \eta^*d\alpha = d\Omega. \quad \square$$

We will now explain how to construct the metric  $h$  – or, equivalently, the associated holomorphic 2-form  $\Omega_J \in \Lambda^{2,0}(X)$ .

Keep the notation of last two Sections. Let  $\omega \in \Lambda^{1,1}(M, \mathbb{C})$  be the Kähler form (more generally, any  $(1, 1)$ -form preserved by the connection). We have to prove that there exists a unique (formal germ of a) holomorphic  $(2, 0)$ -form  $\Omega \in \Lambda_J^{2,0}(X)$  which is of  $H$ -type  $(1, 1)$  and whose restriction to the zero section  $M \subset X$  coincides with  $\omega$  (since the positivity (18) is an open condition, it is satisfied automatically in a neighborhood of the zero section  $M \subset X$ ).

To reformulate this in terms of  $M$ , consider the complex  $\Lambda_J^{2,*}(X)$  of Hodge bundles on  $\overline{T}M$  with the Dolbeault differential  $D = \bar{\partial}_J : \Lambda_J^{2,*}(X) \rightarrow \Lambda^{2,*+1}(X)$ . Denote by

$$\mathcal{C}^{*+2}(M) = \rho_*\Lambda_J^{2,*}(X)$$

its direct image on  $M$  (the grading is shifted by 2 to make it compatible with the Hodge degrees). We are given a section  $\omega \in \Lambda^{1,1}(M)$  of Hodge type  $(1, 1)$ . We have to prove that there exists a section  $\Omega = \Omega_J \in \mathcal{C}^{1,1}(M)$  such that  $\Omega = \omega$  on the zero section  $M \subset X$  and  $D\Omega = 0$ .

Since  $\Lambda_J^{2,*}(X) \cong \Lambda_J^{2,0}(X) \otimes \Lambda_J^{0,*}(X)$  and  $\Lambda_J^{2,0}(X) \cong \rho^*\Lambda^2(M, \mathbb{C})$ , the complex  $\mathcal{C}^*(M)$  is a free module

$$\mathcal{C}^*(M) \cong L^2(M) \otimes \mathcal{B}^*(M)$$

over the Weil algebra  $\mathcal{B}^*(M) = \rho_* \Lambda_j^{0,*}(X)$  generated by some subbundle

$$L^2(M) \subset \mathcal{C}^2(M)$$

which is isomorphic to  $\Lambda^2(M, \mathbb{C})$ . We introduce the augmentation grading on the  $\mathcal{B}^*(M)$ -module  $\mathcal{C}^*(M)$  by setting  $\text{deg } L^2(M) = 2$ . Just as in Proposition 5.3, the proof will proceed by induction on the augmentation degree – namely, we will construct the form  $\Omega \in \mathcal{C}^{1,1}(M)$  as a sum

$$\Omega = \Omega_0 + \dots + \Omega_k + \dots$$

with  $\Omega_k \in \mathcal{C}_k^{1,1}(M)$  of augmentation degree  $k + 2$ . We begin with the induction step. It is completely parallel to Proposition 5.3, so we give only a sketch.

*Induction step – a sketch.* We can assume that we already have

$$\Omega_{<k} = \Omega_0 + \dots + \Omega_{k-1}$$

such that  $\Phi = D\Omega_{<k}$  is of augmentation degree  $\geq k + 2$ . Denote by  $\Phi_k = \Phi_k^{2,1} + \Phi_k^{1,2}$  the component of augmentation degree exactly  $k + 2$ . We have to show that there exists a unique  $\Omega_k \in \mathcal{C}_{k+2}^{1,1}$  such that  $\Phi_k = D_0\Omega_k$ .

The derivations  $C$  and  $\sigma$  of the Weil algebra  $\mathcal{B}^*(M)$  extend to endomorphisms of the free module  $L^2 \otimes \mathcal{B}^*(M)$  by setting  $C = \sigma = 0$  on  $L^2(M)$ . Just as on  $\mathcal{B}^*(M)$ , we have  $D_0 = C$ . For every  $k \geq 0$ , we have  $\{C, \sigma\} = k \text{ id}$  on  $\mathcal{C}_{k+2}^*$ . This immediately implies that  $C = D_0$  is injective on  $\mathcal{C}_{k+2}^{1,1}(M)$  for  $k \geq 1$ , which proves the uniqueness of  $\Omega_k$ .

The space  $\mathcal{C}_{k+2}^*$  splits into the sum of parts of the form

$$L^{p,q} \otimes \mathcal{B}_{m,n}^*, \quad p + q = 2; \quad m + n = k; \quad p, q, m, n \geq 0.$$

Such a part can have a non-trivial piece of Hodge bidegree  $p_1, q_1$  only if  $p_1 \geq p + m - n$  and  $q_1 \geq q + n - m$ . Having in mind the graphical representation as in Figure 1 and Figure 2, we will say that the part  $L^{p,q} \otimes \mathcal{B}_{m,n}^* \subset \mathcal{C}_{k+2}^*$  is an *angle based at*  $(p + m - n, q + n - m)$ . Each angle is preserved by the maps  $C$  and  $\sigma$ . In this terminology,  $\mathcal{C}_{k+2}^{2,1}$  can intersect non-trivially with various angles based at  $(2, 0)$  and  $(1, 1)$ , while  $\mathcal{C}_{k+2}^{1,2}$  can intersect non-trivially with angles based at  $(1, 1)$  and  $(0, 2)$ . But by induction we have  $C\Phi_k = 0$ , which implies that  $C^{1,0}\Phi_k^{2,1} = C^{0,1}\Phi_k^{1,2} = 0$ . Applying the boundary rule Lemma 6.1, we see that both  $\Phi_k^{2,1} \in \mathcal{C}_{k+2}^{2,1}$  and  $\Phi_k^{1,2} \in \mathcal{C}_{k+2}^{1,2}$  must lie entirely within angles based at  $(1, 1)$ . Therefore  $\sigma\Phi_k$  must also lie within angles based in  $(1, 1)$ . This means that  $\sigma\Phi_k \in \mathcal{C}_{k+2}^{1,1}$  is of Hodge type  $(1, 1)$ , and we can set  $\Omega_k = \frac{1}{k}\sigma\Phi_k$ . □

We now have to base the induction – namely, to find the section  $\Omega_0 \in \mathcal{C}_2^{1,1}(M)$  with correct restriction to the zero section  $M \subset X$ , and to handle those angles  $L^{p,q}(M) \otimes \mathcal{B}_{m,n}^*$  to which the boundary rule does not apply – which means the angles



with  $m = 0$  or  $n = 0$ . There are three such angles. We denote the corresponding components of the derivation  $D : L^{1,1}(M) \rightarrow L^2(M) \otimes \mathcal{B}^1(M)$  by

$$(33) \quad D_1 : L^{1,1}(M) \rightarrow L^{1,1}(M) \otimes \mathcal{B}_1^1(M),$$

$$(34) \quad D_2^l : L^{1,1}(M) \rightarrow L^{2,0}(M) \otimes \mathcal{B}_{0,2}^{-1,2}(M),$$

$$(35) \quad D_2^r : L^{1,1}(M) \rightarrow L^{0,2}(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M).$$

We have to choose  $\Omega_0$  so that  $D_1\Omega_0 = D_2^l\Omega_0 = D_2^r\Omega_0 = 0$ . We note that  $D_2^l\Omega_0 = 0$  implies  $D_2^r\Omega_0 = 0$  by complex conjugation.

Moreover, we note that we have one more degree of freedom. So far nothing depended on the choice of the generator subbundle  $L^2(M) \subset \mathcal{C}^2(M)$  – all we needed was to know that it exists. We will now make this choice. It will not be the most obvious one, but the one which will make computations as easy as possible. We consider the splitting

$$(36) \quad \Lambda^1(X, \mathbb{C}) = \Lambda_J^{1,0}(X) \oplus \rho^*\Lambda^1(M, \mathbb{C})$$

given by the Hodge connection  $\Theta : \Lambda^1(X, \mathbb{C}) \rightarrow \rho^*\Lambda^1(M, \mathbb{C})$ . This splitting induces a bigrading on the de Rham algebra  $\Lambda^*(X, \mathbb{C})$ . The complex  $\Lambda^*(X, \mathbb{C})$  with this bigrading is a bicomplex which we will denote by  $\Lambda_\Theta^{*,*}(X)$ . More precisely, the de Rham differential  $d$  is a sum of two anticommuting differentials

$$\tilde{d} : \Lambda_\Theta^{*,*}(X) \rightarrow \Lambda_\Theta^{*+1,*}(X) \quad D : \Lambda_\Theta^{*,*}(X) \rightarrow \Lambda_\Theta^{*,*+1}(X)$$

(this is essentially equivalent to the flatness of the Hodge connection  $\Theta$ ).

Since the first term in the splitting (36) is  $\Lambda_J^{1,0}(X)$ , the subcomplexes  $\Lambda_\Theta^{\geq k,*}(X)$  and  $\Lambda_J^{\geq k,*}(X)$  of the de Rham complex  $\Lambda^*(X, \mathbb{C})$  are the same for every  $k$ . Therefore the associated graded quotients  $\Lambda_\Theta^{k,*}(X)$  and  $\Lambda_J^{k,*}(X)$  are also isomorphic for every  $k$ . In particular, we have

$$\mathcal{C}^{*+2}(M) = \rho_*\Lambda_J^{2,0}(X) \cong \rho_*\Lambda_\Theta^{2,*}(X).$$

On the other hand, since

$$\Lambda_\Theta^{1,0}(X) = \Lambda^1(X, \mathbb{C})/\rho^*\Lambda^1(M, \mathbb{C}) \cong \Lambda^1(X/M, \mathbb{C})$$

is the bundle of relative 1-forms on  $X/M$ , the quotient complex  $\Lambda_\Theta^{*,0}(X)$  with the differential  $\tilde{d}$  is canonically isomorphic

$$(37) \quad \Lambda_\Theta^{*,0}(X) \cong \Lambda^*(X/M, \mathbb{C})$$

to the relative de Rham complex  $\Lambda^*(X/M, \mathbb{C})$ . We use this identification and choose as

$$L^k(M) \subset \rho_*\Lambda^*(X/M, \mathbb{C}) \cong \rho_*\Lambda_\Theta^{*,0}(X)$$

the subbundle of  $k$ -forms which are constant along the fibers of the projection  $\rho : X \rightarrow M$  (by a constant  $k$ -form on a vector space  $V$  with a basis  $e_1, \dots, e_n$  we mean a linear combination of forms  $e_{a_1} \wedge \dots \wedge e_{a_k}$  with constant coefficients).

This choice guarantees that the relative de Rham differential  $\tilde{d} : L^\bullet(M) \otimes \mathcal{B}^0(M) \rightarrow L^{\bullet+1}(M) \otimes \mathcal{B}^0(M)$  takes a very simple form. Namely, it vanishes on  $L^k(M)$ , and induces an isomorphism

$$(38) \quad \tilde{d} : S^1(M) \cong L^1(M)$$

between the generator subbundles  $S^1(M) \subset \mathcal{B}^0(M)$  and  $L^1(M) \subset L^1(M) \otimes \mathcal{B}^1(M)$ . This is important because by construction  $\tilde{d}$  anticommutes with  $D$ . Moreover, since  $\tilde{d}$  obviously preserves the augmentation degrees, it anticommutes separately with each of the components  $D_k$ . Since we already know the derivations  $D_k$  on  $\mathcal{B}^\bullet(M)$ , the isomorphism (38) will allow us to compute individual components  $D_k : L^\bullet(M) \rightarrow L^\bullet(M) \otimes \mathcal{B}_k^1(M)$ .

The first result is the following: the map  $D_1 = D_1^{1,1}$  in (33) is minus the connection

$$\nabla : L^{1,1}(M) \rightarrow L^{1,1}(M) \otimes \Lambda^1(M, \mathbb{C})$$

on the bundle  $L^{1,1}(M) \cong \Lambda^{1,1}(M, \mathbb{C})$ . Indeed,  $D_1 : L^\bullet(M) \rightarrow L^\bullet(M) \otimes \Lambda^1(M, \mathbb{C})$  is a derivation of the exterior algebra  $L^\bullet(M)$ . Thus it suffices to prove that  $D_1 = -\nabla$  on  $L^1(M)$ . Since  $\tilde{d}D_1 = -D_1\tilde{d}$ , this follows from (38) and the construction of the map  $D : \mathcal{B}^\bullet(M) \rightarrow \mathcal{B}^{\bullet+1}(M)$  given in Section 6. This shows that taking

$$(39) \quad \Omega_0 = \omega \in L^{1,1}(M) \cong \Lambda^{1,1}(M)$$

guarantees that  $D_1\Omega_0 = 0$ .

At this point we will also choose the isomorphism  $L^k(M) \cong \Lambda^1(M, \mathbb{C})$  – namely, we take the composition of the embedding

$$L^k(M) \subset \Lambda^k(X/M, \mathbb{C}) \cong \Lambda_{\Theta}^{k,0}(X)$$

and the restriction  $i^*\Lambda^{k,0}(X) \rightarrow \Lambda^k(M, \mathbb{C})$  to the zero section  $i : M \hookrightarrow X$ . Then the form  $\Omega_0$  defined by (39) automatically restricts to  $\omega$ .

It remains to prove that  $D_2^l\Omega_0 = 0$ . This is a corollary of the following claim.

**Lemma 7.2.** *The map  $D_2^l$  defined in (35) is the composition of the curvature*

$$R : L^{1,1}(M) \rightarrow L^{1,1}(M) \otimes \Lambda^{1,1}(M).$$

*of the connection  $\nabla = D_1 : L^{1,1}(M) \rightarrow L^{1,1}(M) \otimes \Lambda^1(M, \mathbb{C})$  and a certain bundle map*

$$Q : L^{1,1}(M) \otimes \Lambda^{1,1}(M) \rightarrow L^{0,2}(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M).$$

*Proof.* Extend the map  $D_2^l$  to an algebra derivation

$$D_2^l : L^\bullet(M) \rightarrow L^\bullet(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M) \cong L^\bullet(M) \otimes S^{1,-1}(M) \otimes \Lambda^{1,0}(M)$$

by setting  $D_2^l = 0$  on  $L^{0,1}(M)$  and taking as

$$D_2^l : L^{1,0}(M) \rightarrow L^{0,1}(M) \otimes \mathcal{B}_{2,0}^{2,-1}(M)$$

the corresponding component of the map  $D_2 : L^{1,0}(M) \rightarrow L^1(M) \otimes \mathcal{B}_2^1(M)$ . Since  $D_2^l$  is a derivation of the algebra  $L^*(M)$  which vanishes on  $L^{0,1}(M)$ , on  $L^{1,1}(M)$  it is equal to the composition

$$L^{1,0}(M) \otimes L^{0,1}(M) \xrightarrow{D_2^l \otimes \text{id}} \mathcal{B}_{2,0}^{2,-1}(M) \otimes L^{0,1}(M) \otimes L^{0,1}(M) \longrightarrow \mathcal{B}_{2,0}^{2,-1}(M) \otimes L^{0,2}(M)$$

$\xrightarrow{\text{id} \otimes \text{Alt}}$

(here  $\text{Alt} : L^{0,1}(M) \otimes L^{0,1}(M) \rightarrow \Lambda^{0,2}(M)$  is the alternation map). Therefore it suffices to prove that on  $L^{1,0}(M)$  we have

$$D_2^l = P \circ R : L^{1,0}(M) \rightarrow L^{1,0}(M) \otimes \Lambda^{1,1}(M) \rightarrow L^{0,1} \otimes \mathcal{B}_{2,0}^{2,-1}(M)$$

for a certain bundle map  $P : L^{1,0}(M) \otimes \Lambda^{1,1}(M) \rightarrow L^{0,1} \otimes \mathcal{B}_{2,0}^{2,-1}(M)$ . Since  $D_2 \tilde{D} = -\tilde{d}D_2$ , this follows directly from (38) and (32) with  $k = 2$ . □

This Lemma implies that  $D_2^l(\Omega_0) = Q(R(\omega)) = Q(\nabla(\nabla(\omega))) = 0$ . This finishes the proof of Theorem 1.1.

The last application of the formalism that we have developed in this Section is the proof of Lemma 5.2.

*Proof of Lemma 5.2.* The Obata connection  $\nabla_O$  on a hypercomplex manifold  $X$  is defined as follows: for every  $(0, 1)$ -form  $\alpha \in \Lambda^{1,0}(X)$  the  $(1, 0)$ -part  $\nabla_O^{1,0}$  is equal to

$$\nabla_O^{1,0} \alpha = \partial \alpha \in \Lambda^{0,1}(M) \otimes \Lambda^{1,0}(M),$$

while the  $(0, 1)$ -part satisfies

$$\nabla_O^{0,1} \alpha = -(\text{id} \otimes j)(\partial(j(\alpha))) \in \Lambda^{0,1}(M) \otimes \Lambda^{0,1}(M).$$

The first condition in fact automatically follows from the absence of torsion.

Consider a hypercomplex structure on  $X = \bar{T}M$  corresponding to a torsion-free connection  $\nabla$  on  $M$ , and let  $\alpha \in \Lambda^{0,1}(M)$  be a  $(0, 1)$ -form on  $M$ . We have to prove that  $\nabla_O^{0,1}(\rho^* \alpha) = \rho^* \nabla^{0,1} \alpha$  on the zero section  $i : M \hookrightarrow X$ . It suffices to prove that

$$(40) \quad i^* \bar{\partial} j(\rho^* \alpha) = i^*(\text{id} \otimes j)(\nabla^{0,1} \alpha)$$

as section of the bundle  $\Lambda^{0,1}(M) \otimes i^* \Lambda^1(X, \mathbb{C})$  on the zero section  $M \subset X$ . The canonical isomorphism  $H : \Lambda^{*,*}(X) \cong \Lambda_j^{*,*}(X)$  sends the Dolbeault differential  $\bar{\partial}$  to the component  $D^{1,0}$  of the Dolbeault differential  $D = \bar{\partial}_J : \Lambda_j^{*,*}(X) \rightarrow \Lambda_j^{*,*+1}(X)$ . Moreover, the composition of the map  $H \circ j : \rho^* \Lambda^{0,1}(M) \rightarrow \Lambda_j^{1,0}(X)$  and the restriction to the zero section  $M \subset X$  induces a bundle map

$$P : \Lambda^{0,1}(M) \rightarrow L^1(M) = i^* \Lambda_j^{1,0}(X) \subset i^* \Lambda^1(X, \mathbb{C}).$$

It is easy to check that this map is proportional to the canonical embedding  $\Lambda^{0,1}(M) \cong L^{0,1}(M) \hookrightarrow L^1(M)$ . Therefore it commutes with the connection  $\nabla^{1,0}$  – namely, we have  $(\text{id} \otimes P) \circ \nabla^{1,0} = \nabla^{1,0} \circ P$ . The equation (40) becomes

$$D_1^{1,0} P(\alpha) = \nabla^{1,0} P(\alpha).$$

But we have already proved that on  $L^1(M)$  we have  $\nabla = D_1$ . □

### 8. SYMMETRIC SPACES.

To illustrate our rather abstract methods by a concrete example, we would like now to derive a formula for the canonical hypercomplex structure on  $X = \overline{T}M$  in the case when  $M$  is a symmetric space. In this case the classic equality  $\nabla R = 0$  bring many simplifications, so that the constructions of Section 6 can be seen through to a reasonably explicit final result. The formula that we obtain is similar to the one obtained by O. Biquard and P. Gauduchon in [BG].

Let us introduce some notations. Let  $M$  be a symmetric space with Levi-Civita connection  $\nabla$  and curvature  $R$ . Consider the total space  $X = \overline{T}M$  with the canonical projection  $\rho : \overline{T}M \rightarrow M$ . Let  $A : \rho^*\Lambda^1(M) \rightarrow \rho^*\Lambda^1(M)$  be the endomorphism of the pullback bundle  $\rho^*\Lambda^1(M)$  given by

$$(41) \quad A(\xi)(\alpha) = -\frac{1}{3}R_m(\alpha) \lrcorner (\xi \otimes \xi), \quad \alpha \in \rho^*\Lambda^1(M).$$

Here  $(\xi, m)$ ,  $m \in M$ ,  $\xi \in T_mM$  is a point in  $\overline{T}M$ ,  $R_m$  is the curvature evaluated at the point  $m \in M$ , and  $R(\alpha)$  is interpreted as a section of the bundle  $\rho^*\Lambda^1(M) \otimes \rho^*\Lambda^{1,1}(M)$ . Let also

$$f(z) = \sum_{p \geq 1} f_p z^p$$

be the generating function for the recurrence relation

$$(42) \quad f_p = -\frac{1}{2p+1} \sum_{1 \leq i \leq p-1} f_i f_{p-i}$$

with the initial condition  $f_1 = 1$ . In other words,  $f(z)$  is the solution of the ODE

$$2zf'(z) + f(z) + f^2(z) = 3z$$

with the initial condition  $f(0) = 0$ . With these notations, we can formulate our result.

**Proposition 8.1.** *Let  $I : \Lambda^1(M) \rightarrow \Lambda^1(M)$  be the complex structure operator on  $M$ . Then the map  $J : \Lambda^1(X) \rightarrow \Lambda^1(X)$  for the canonical normalized hypercomplex structure on  $X$  is given by the matrix*

$$J = \begin{pmatrix} 0 & f(A)I \\ I(f(A))^{-1} & 0 \end{pmatrix}$$

with respect to the decomposition

$$\Lambda^1(X) = \Lambda^1(X/M) \oplus \rho^*\Lambda^1(M) \cong \rho^*\Lambda^1(M) \oplus \rho^*\Lambda^1(M)$$

associated to the Levi-Civita connection on  $M$ .

We have already noted that this result is very similar to the formula obtained in [BG]. However, it is not the same. The reason is the following: Biquard and Gauduchon work with the cotangent bundle  $T^*M$ , and they use the normalization natural to the cotangent bundle. As the result, their analog of the map  $A$  is slightly different, and the function  $f(z)$  is also different – in particular, it is given by an explicit expression. To compare our results with those of [BG], one should either compute the normalization map  $\mathcal{L} : T^*M \rightarrow \overline{T}M$  for the Biquard-Gauduchon hypercomplex structure, or go the other way around and compute the map  $\eta : \overline{T}M \rightarrow T^*M$  provided by Lemma 7.1. Unfortunately, I haven't been able to do either.

*Proof of Proposition 8.1.* Throughout the proof, we will freely use the notation of preceding Sections.

The main simplification in the case of symmetric spaces is the well-known equality  $\nabla R = 0$ . For our construction it immediately implies that the odd augmentation degree components  $D_{2p+1}$  of the weakly Hodge derivation  $D : \mathcal{B}^*(M) \rightarrow \mathcal{B}^{*+1}(M)$  vanish for  $p \geq 1$ . The only non-trivial component is  $D_1$ . By (22), this immediately shows that the connection  $\Theta_{Re}$  on the fibration  $\rho : X \rightarrow M$  is simply the linear connection associated to the Levi-Civita connection  $\nabla$ . Applying Lemma 4.3, we see that all we have to prove is the equality  $R_J = f(A)$ . By Lemma 4.4 this can be rewritten in terms of the Weil algebra  $\mathcal{B}^*(M)$  as

$$D_+ \circ \sigma = f(A) : \Lambda^1(M, \mathbb{C}) \rightarrow \mathcal{B}^1(M).$$

Moreover, by (32) with  $k = 2$  the endomorphism  $A$  of the  $\mathcal{B}^0(M)$ -module  $\mathcal{B}^1(M) \cong \rho_*\rho^*\Lambda^1(M, \mathbb{C})$  satisfies

$$A = D_2 \circ \sigma$$

on  $\Lambda^1(M, \mathbb{C})$ . In fact, this should be taken as the definition – I apologize to the reader for any possible mistakes in writing down the explicit formula (41).

Next we note that since  $D_2$  vanishes on  $\Lambda^1(M, \mathbb{C})$ , the endomorphism  $A$  is in fact equal

$$A = \{D_2, \sigma\}$$

to the commutator  $\{D_2, \sigma\}$ . Moreover, this formula holds not only on the generator subbundle  $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$ , but on the whole  $\mathcal{B}^0(M)$ -module  $\mathcal{B}^1(M)$ . Indeed, by the normalization condition (28) this commutator vanishes on the generator subbundle  $S^1(M) \subset \mathcal{B}^0(M)$ . Since it is a derivation of the Weil algebra, it vanishes on the whole  $\mathcal{B}^0(M)$ . Therefore it restricts to a map of  $\mathcal{B}^0(M)$ -modules on  $\mathcal{B}^1(M) \subset \mathcal{B}^1(M)$ .

We now trace one-by-one the induction steps in the proof of Proposition 5.3. Since all the odd terms vanish, we only need to consider the even terms  $D_{2p}$ . We have to prove that

$$\{D_{2p}, \sigma\} = f_k A^k : \mathcal{B}^1(M) \rightarrow \mathcal{B}^1(M).$$

Since both sides are maps of  $\mathcal{B}^0(M)$ -modules, it suffices to prove this on  $\Lambda^1(M, \mathbb{C}) \subset \mathcal{B}^1(M)$ , where we can replace  $\{D_{2p}, \sigma\}$  with  $D_p \circ \sigma$ . By (32), on  $S^1(M) \subset \mathcal{B}^0(M)$  we

have

$$D_{2p} = -\frac{1}{2p+1} \sum_{1 \leq l \leq p-1} \sigma \circ D_{2l} \circ D_{2(p-l)}.$$

Comparing this to (42), we see that it suffices to prove that

$$\sigma \circ D_m \circ D_n \circ \sigma = \{D_m, \sigma\} \circ \{D_n, \sigma\} : \Lambda^1(M) \rightarrow \mathcal{B}^1(M)$$

for all even  $m, n \geq 2$ . Writing out the commutators on the right-hand side, we see that the difference is equal to

$$D_m \circ \sigma \circ D_n \circ \sigma + D_m \circ \sigma \circ \sigma \circ D_n + \sigma \circ D_m \circ \sigma \circ D_n.$$

The first summand vanishes since by the normalization condition (28) we have  $\sigma \circ D_n = 0$  on  $S^1(M) = \sigma(\Lambda^1(M, \mathbb{C}))$ . The last two summands vanish since  $D_n = 0$  on  $\Lambda^1(M, \mathbb{C})$ .  $\square$

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## EQUIVARIANT COHOMOLOGY RINGS OF TORIC HYPERKÄHLER MANIFOLDS

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ABSTRACT. A smooth hyper Kähler quotient of a quaternionic vector space  $\mathbf{H}^N$  by a subtorus of  $T^N$  is called a toric hyperKähler manifold. We determine the ring structure of the integral equivariant cohomology of a toric hyperKähler manifold.

### 1. INTRODUCTION

The topology of symplectic quotients has been studied intensively in the last two decades, using Morse theory and equivariant cohomology theory [Ki][JK]. However, the topology of hyperKähler quotients has not been studied well. In this note we study the topology of smooth hyperKähler quotients of a quaternionic vector space  $\mathbf{H}^N$  by subtori of  $T^N$ , which we call toric hyperKähler manifolds. Originally Bielawski and Dancer introduced and studied toric hyperKähler manifolds in [BD], being influenced by [D], [A] and [Gu]. Especially they computed their Betti numbers. In [Ko] we determined the ring structure of the integral cohomology of toric hyperKähler manifolds. In this note we show that the method in [Ko] is enough to compute the integral equivariant cohomology rings. Since the topology of toric hyperKähler manifolds depends only on the subtori of  $T^N$ , we describe the ring structures of their equivariant cohomology in terms of the subtori (Theorem 2.4). We also describe them in terms of the arrangement of hyperplanes which were associated to toric hyperKähler manifolds (Theorem 2.6).

In section 2 we review some basic properties of toric hyperKähler manifolds and state our main results. In section 3 we prove them.

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2. MAIN RESULTS

First let us recall the hyperKähler structure on a quaternionic vector space  $\mathbf{H}^N$ . Let  $\{1, I_1, I_2, I_3\}$  be the standard basis of  $\mathbf{H}$ . We define three complex structures on  $\mathbf{H}^N$  by the multiplication of  $I_1, I_2, I_3$  from the left respectively. We fix the identification  $i: \mathbf{H}^N \rightarrow \mathbf{C}^N \times \mathbf{C}^N$  by  $i(\xi) = (z, w)$ , where  $\xi = (\xi_1, \dots, \xi_N) \in \mathbf{H}^N$ ,  $z = (z_1, \dots, z_N), w = (w_1, \dots, w_N) \in \mathbf{C}^N$  and  $\xi_j = z_j + w_j I_2$  for  $j = 1, \dots, N$ .

The real torus  $T^N = \{\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{C}^N \mid |\alpha_i| = 1\}$  acts on  $\mathbf{H}^N$  by

$$(z, w)\alpha = (z\alpha, w\alpha^{-1}).$$

This action preserves the hyperKähler structure. Let  $\text{Exp}_{T^N}: t^N \rightarrow T^N$  be the exponential map and  $\{X_1, \dots, X_N\} \subset t^N$  be the basis satisfying  $\text{Exp}_{T^N}(\sum_{i=1}^N a_i X_i) = (e^{2\pi\sqrt{-1}a_1}, \dots, e^{2\pi\sqrt{-1}a_N}) \in T^N$ . We define  $\{u_1, \dots, u_N\} \subset (t^N)^*$  to be the dual basis of  $\{X_1, \dots, X_N\} \subset t^N$ . Then the hyperKähler moment map

$$\mu_{T^N} = (\mu_{T^N,1}, \mu_{T^N,2}, \mu_{T^N,3}): \mathbf{H}^N \rightarrow (t^N)^* \otimes \mathbf{R}^3$$

is given by

$$\begin{aligned} \mu_{T^N,1}(z, w) &= \pi \sum_{i=1}^N (|z_i|^2 - |w_i|^2) u_i, \\ (\mu_{T^N,2} + \sqrt{-1}\mu_{T^N,3})(z, w) &= -2\pi\sqrt{-1} \sum_{i=1}^N z_i w_i u_i. \end{aligned}$$

Let  $K$  be a subtorus of  $T^N$  with the Lie algebra  $k \subset t^N$ . Then we have the torus  $T^n = T^N/K$  and its Lie algebra  $t^n = t^N/k$ . We also have the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0, \\ 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0. \end{array}$$

We remark that some  $\pi(X_i)$  may be zero and some  $\iota^* u_j$  may be zero. Since the torus  $K$  acts on  $\mathbf{H}^N$  preserving its hyperKähler structure, we obtain the hyperKähler moment map

$$\mu_K = (\iota^* \otimes id) \circ \mu_{T^N}: \mathbf{H}^N \rightarrow k^* \otimes \mathbf{R}^3.$$

Now we define a toric hyperKähler manifold.

**Definition.** If  $\nu \in k^* \otimes \mathbf{R}^3$  is a regular value of the hyperKähler moment map  $\mu_K$  and if the action of  $K$  on  $\mu_K^{-1}(\nu)$  is free, we call the hyperKähler quotient

$$X(\nu) = \mu_K^{-1}(\nu)/K$$

a toric hyperKähler manifold.  $\square$

In fact  $X(\nu)$  is a  $4n$  dimensional hyperKähler manifold. The torus  $T^n = T^N/K$  acts on  $X(\nu)$ , preserving its hyperKähler structure.



In [Ko] we discussed when the hyperKähler quotient becomes a toric hyperKähler manifold as follows.

**Lemma 2.1.** Fix an element  $\nu = (\nu_1, \nu_2, \nu_3) \in k^* \otimes \mathbf{R}^3$ . Then the following (1) and (2) are equivalent.

- (1)  $\nu \in k^* \otimes \mathbf{R}^3$  is a regular value of the hyperKähler moment map  $\mu_K$ .
- (2) For any  $J \subset \{1, \dots, N\}$ , whose number  $\#J$  is less than  $\dim k = N - n$ ,  $\nu_1, \nu_2$  and  $\nu_3$  are not simultaneously contained in the subspace of  $k^*$  which is spanned by  $\{\iota^*u_j | j \in J\}$ .

**Lemma 2.2.** Let  $\nu \in k^* \otimes \mathbf{R}^3$  be a regular value of the hyperKähler moment map  $\mu_K$ . Then the following (1) and (2) are equivalent.

- (1) The action of  $K$  on  $\mu_K^{-1}(\nu)$  is free.
- (2) For any  $J \subset \{1, \dots, N\}$  such that  $\{\iota^*u_j | j \in J\}$  forms a basis of  $k^*$

$$t_{\mathbf{Z}}^N = k_{\mathbf{Z}} + \sum_{j \in J^c} \mathbf{Z}X_j \quad \text{as a } \mathbf{Z}\text{-module,}$$

where  $t_{\mathbf{Z}}^N$  is the standard lattice in  $t^N$  and  $k_{\mathbf{Z}} = k \cap t_{\mathbf{Z}}^N$ .

These lemmas lead the following property of a toric hyperKähler manifold, which was proved in [BD].

**Proposition 2.3.** Let  $X(\nu)$  be a toric hyperKähler manifold. Then its diffeomorphism type is independent of the choice of  $\nu$ .

Next we construct line bundles  $L_i$  on a toric hyperKähler manifold  $X(\nu)$  for  $i = 1, \dots, N$ . Let  $\chi_i: T^N \rightarrow S^1$  be a character defined by  $\chi_i(\alpha) = \alpha_i$ . Define the action of  $T^N$  on  $\mu_K^{-1}(\nu) \times \mathbf{C}$  by

$$((z, w), v)\alpha = ((z\alpha, w\alpha^{-1}), v\chi_i(\alpha)).$$

This action induces a  $T^n$ -equivariant line bundle  $L_i = (\mu_K^{-1}(\nu) \times \mathbf{C})/K$  on  $X(\nu)$ .

Let  $ET^n \rightarrow BT^n$  be a universal  $T^n$ -bundle. Then we define the homotopy quotient of a  $T^n$ -space  $M$  by  $M_{T^n} = (ET^n \times M)/T^n$ . The equivariant cohomology  $H_{T^n}^*(M; \mathbf{Z})$  is by definition  $H^*(M_{T^n}; \mathbf{Z})$ . The  $T^n$ -equivariant line bundle  $L_i$  on  $X(\nu)$  induces a line bundle  $\mathcal{L}_i$  on  $X(\nu)_{T^n}$ . The equivariant first Chern class of  $L_i$  is defined by  $c_1(\mathcal{L}_i) \in H_{T^n}^*(X(\nu); \mathbf{Z})$ .

Now we can state main result of this paper, which determines the ring structure of the integral equivariant cohomology of  $X(\nu)$ .

**Theorem 2.4.** Let  $X(\nu)$  be a toric hyperKähler manifold and

$$\Psi: \mathbf{Z}[u_1, \dots, u_N] \rightarrow H_{T^n}^*(X(\nu); \mathbf{Z})$$

be a ring homomorphism, which is defined by  $\Psi(u_i) = c_1(\mathcal{L}_i)$ . Then the following holds.

- (1) The map  $\Psi$  is a surjective ring homomorphism.

(2) Let  $I$  be the ideal generated by all  $\prod_{a_i \neq 0} u_i$  for  $\sum_{i=1}^N a_i X_i \in k \setminus \{0\}$ . Then  $I = \ker \Psi$ .

**Example.** Let  $\pi: t^5 \rightarrow t^3$  be a map such that

1.  $\{\pi(X_1), \pi(X_2), \pi(X_3)\}$  forms a basis of  $t^3$ ,
2.  $\pi(X_4) = -\pi(X_1) - \pi(X_2)$ ,
3.  $\pi(X_5) = -\pi(X_1) - \pi(X_3)$ .

Then we have a toric hyperKähler manifold  $X(\nu)$  for  $\nu \in k^* \otimes \mathbf{R}^3$  satisfying the condition mentioned above. Since  $k$  is spanned by

$$\{X_1 + X_2 + X_4, X_1 + X_3 + X_5\},$$

there are 4 types of elements in  $k$  as follows:

$$X_1 + X_2 + X_4, \quad X_1 + X_3 + X_5, \quad X_2 - X_3 + X_4 - X_5, \\ \sum_{i=1}^5 a_i X_i \quad \text{where } a_i \neq 0 \text{ for } i = 1, \dots, 5.$$

Therefore Theorem 2.4 implies

$$H_{T^3}^*(X(\nu); \mathbf{Z}) \cong \mathbf{Z}[u_1, \dots, u_5]/I,$$

where the ideal  $I$  is generated by  $\{u_1 u_2 u_4, u_1 u_3 u_5, u_2 u_3 u_4 u_5\}$ .  $\square$

It is worth mentioning the results in [Ko] here. Let  $p: X(\nu)_{T^n} \rightarrow BT^n$  be the natural projection. If we fix  $x \in BT^n$ , then we can identify the fiber  $F_x = p^{-1}(x)$  with  $X(\nu)$  and the restriction of  $\mathcal{L}_i$  to  $F_x$  is isomorphic to  $L_i$ . If we denote the embedding of  $F_x$  into  $X(\nu)_{T^n}$  by  $i: F_x \rightarrow X(\nu)_{T^n}$ , then we have the map

$$\Phi = i^* \circ \Psi: \mathbf{Z}[u_1, \dots, u_N] \rightarrow H^*(X(\nu); \mathbf{Z}).$$

Then we proved the following theorem in [Ko], which determines the structure of the integral cohomology ring of  $X(\nu)$ .

**Theorem 2.5.** *The map  $\Phi$  is a surjective ring homomorphism. Moreover  $\ker \Phi$  is generated by  $\ker \Psi$  and  $\pi^*((t^n)^*) \cap \sum_{i=1}^N \mathbf{Z}u_i$ .*

Next we give another description of the ideal  $I$  in Theorem 2.4. We assume that  $X(\nu)$  is a toric hyperKähler manifold with  $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$ . We fix an element  $h \in (t^N)^*$  such that  $\iota^* h = \nu_1$ . Since every toric hyperKähler manifold  $X(\nu')$  can be deformed to  $X(\nu)$  with  $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$ , this assumption does not lose any generality.

We associate an arrangement of hyperplanes  $F_1, \dots, F_N$  in  $(t^n)^*$  to a toric hyperKähler manifold  $X(\nu)$  with

$$F_i = \{p \in (t^n)^* \mid \langle \pi^* p + h, X_i \rangle = 0\} \quad \text{for } i = 1, \dots, N.$$

We note that  $F_i = \emptyset$  in this case  $\pi(X_i) = 0$ , because  $\nu = (\nu_1, 0, 0)$  is a regular value of  $\mu_K$ . We also note that the above arrangement of hyperplanes is determined by  $\nu = (\nu_1, 0, 0)$  up to parallel translation.

Then we have the following theorem.

**Theorem 2.6.** *Let  $X(\nu)$  be a toric hyperKähler manifold with  $\nu = (\nu_1, 0, 0)$ . Let  $I \subset \mathbf{Z}[u_1, \dots, u_N]$  be the ideal in Theorem 2.4 and  $I' \subset \mathbf{Z}[u_1, \dots, u_N]$  be the ideal generated by all  $\prod_{j \in S} u_j$  for  $\emptyset \neq S \subset \{1, \dots, N\}$  such that  $\bigcap_{j \in S} F_j = \emptyset$ . Then we have  $I = I'$ .*

Since this theorem is essentially proved in [Ko], we omit it.

### 3. PROOF OF THEOREM 2.4

In this section we prove Theorem 2.4. We prove it by induction on  $N$ .

First we prove the theorem for  $N = 1$ . In this case we have  $k = \{0\}$  or  $k = t^1$ .

Suppose  $k = \{0\}$ . In this case  $X(\nu)$  is  $\mathbf{H}$  with  $S^1$ -action. So we have  $H_{S^1}^*(X(\nu); \mathbf{Z}) \cong H_{S^1}^*(point; \mathbf{Z})$ , which is generated by  $\Psi(u_1)$ . Therefore  $\Psi$  is surjective and  $\ker \Psi = \{0\}$ . On the other hand, since  $k = \{0\}$ , we have  $I = \{0\}$ . So in this case Theorem 2.4 is true.

Suppose  $k = t^1$ . It is easy to see that the hyperKähler quotient  $X(\nu)$  is a point without torus action. So we have  $\ker \Psi = (u_1)$ . On the other hand, since  $X_1 \in k$ , we have  $I = (u_1)$ . So in this case Theorem 2.4 is true. Thus we proved Theorem 2.4 for  $N = 1$ .

From now on we assume that Theorem 2.4 is true up to  $N - 1$ . So we prove the theorem for  $N$ .

We begin with a few remarks.

Suppose that  $\iota^*u_N = 0$ , that is,  $k \subset t^{N-1} = \sum_{i=1}^{N-1} \mathbf{R}X_i$ . In this case the action of  $K$  preserves the product structure  $\mathbf{H}^{N-1} \times \mathbf{H}$ , where  $\mathbf{H} = \{(z_N, w_N)\}$ . Moreover  $K$  acts on  $\mathbf{H}$  trivially. Therefore the hyperKähler quotient  $X(\nu)$  of  $\mathbf{H}^N$  by  $K$  is a product of the hyperKähler quotient  $X_{(1)}(\nu^{(1)})$  of  $\mathbf{H}^{N-1}$  by  $K$  and  $\mathbf{H}$  itself. Here we note that  $X(\nu)$  is a  $T^n$ -space,  $X_{(1)}(\nu^{(1)})$  is a  $T^{n-1} = T^{N-1}/K$ -space and  $\mathbf{H}$  is a  $S^1$ -space, where  $S^1$  is the group with the Lie algebra  $\mathbf{R}\pi(X_N)$ . Let  $I_1$  be the ideal and  $\Psi_1: \mathbf{Z}[u_1, \dots, u_{N-1}] \rightarrow H_{T^{n-1}}^*(X_{(1)}(\nu^{(1)}); \mathbf{Z})$  be the map in Theorem 2.4 for  $X_{(1)}(\nu^{(1)})$ . Since  $X(\nu)_{T^n}$  is homotopy equivalent to  $X_{(1)}(\nu^{(1)})_{T^{n-1}} \times \mathbf{H}_{S^1}$ ,  $H_{S^1}^*(\mathbf{H}; \mathbf{Z}) \cong \mathbf{Z}[u_N]$  and  $\Psi_1$  is surjective, we see that  $\Psi$  is surjective and  $\ker \Psi$  is generated by  $\ker \Psi_1$ . On the other hand, since  $k \subset t^{N-1}$ , we see that  $I$  is generated by  $I_1$ . By the assumption of the induction we have  $\ker \Psi = I$ . That is, Theorem 2.4 is true in this case.

Suppose that  $\pi(X_N) = 0$ , that is,  $X_N \in k$ . In this case the Lie algebra  $k$  is decomposed into the direct sum  $k = k_2 \oplus \mathbf{R}X_N$ , where  $k_2 = k \cap \sum_{i=1}^{N-1} \mathbf{R}X_i$ . Let  $K_2$  be the corresponding Lie group to  $k_2$ . Then the hyperKähler quotient  $X(\nu)$  of  $\mathbf{H}^N$  by  $K$  is just the hyperKähler quotient  $X_{(2)}(\nu^{(2)})$  of  $\mathbf{H}^{N-1}$  by  $K_2$ . Here we note that both  $X(\nu)$  and  $X_{(2)}(\nu^{(2)})$  are  $T^m$ -spaces. Let  $I_2$  be the ideal and

$\Psi_2: \mathbf{Z}[u_1, \dots, u_{N-1}] \rightarrow H^*(X_{(2)}(\nu^{(2)}); \mathbf{Z})$  be the map in Theorem 2.4 for  $X_{(2)}(\nu^{(2)})$ . Since  $X(\nu)_{T^n} = X_{(2)}(\nu^{(2)})_{T^n}$  and  $\Psi_2$  is surjective, we see that  $\Psi$  is surjective and  $\ker \Psi$  is generated by  $\ker \Psi_2$  and  $u_N$ . On the other hand, since  $k = k_2 \oplus \mathbf{R}X_N$ , we see that the ideal  $I$  is generated by  $I_2$  and  $u_N$ . By the assumption of the induction we have  $\ker \Psi = I$ . That is, Theorem 2.4 is true in this case.

From now on we assume that we have a fixed toric hyperKähler manifold  $X(\nu)$  with  $\pi(X_i) \neq 0, \iota^*u_i \neq 0$  for any  $i = 1, \dots, N$ . We may also assume that  $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$ . We fix  $h \in (t^N)^*$  such that  $\iota^*h = \nu_1$ . Then we have an arrangement of hyperplanes  $F_1, \dots, F_N$  in  $(t^n)^*$ .

To proceed the induction argument, we will recover the equivariant cohomology ring of  $X(\nu)$  from the equivariant cohomology rings of  $X_{(1)}(\nu^{(1)})$  and  $X_{(2)}(\nu^{(2)})$ , whose associated arrangements of hyperplanes are  $F_1 \cap F_N, \dots, F_{N-1} \cap F_N$  in  $F_N$  and  $F_1, \dots, F_{N-1}$  in  $(t^n)^*$  respectively. In [BD] Bielawski and Dancer computed the Betti numbers of  $X(\nu)$  from the information of  $X_{(1)}(\nu^{(1)})$  and  $X_{(2)}(\nu^{(2)})$  by Mayer-Vietoris argument. Since we study the ring structure of the equivariant cohomology, we need further argument as we explain below.

We begin with constructing  $X_{(1)}(\nu^{(1)})$ . Let  $\rho: t^N \rightarrow t^{N-1}$  be the projection such that  $\rho(X_i) = X_i$  for  $i = 1, \dots, N - 1$  and  $\rho(X_N) = 0$ . Since  $\pi(X_N) \neq 0$ , we have an isomorphism  $\rho|_k: k \rightarrow k_1$ , where  $k_1 = \rho(k)$ . Then we have the following diagrams:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n & \longrightarrow & 0 \\
 & & \rho|_k \downarrow & & \rho \downarrow & & \bar{\rho} \downarrow & & \\
 0 & \longrightarrow & k_1 & \xrightarrow{\iota_1} & t^{N-1} & \xrightarrow{\pi_1} & t^{n-1} & \longrightarrow & 0, \\
 \\
 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* & \longleftarrow & 0 \\
 & & (\rho|_k)^* \uparrow & & \rho^* \uparrow & & \bar{\rho}^* \uparrow & & \\
 0 & \longleftarrow & k_1^* & \xleftarrow{\iota_1^*} & (t^{N-1})^* & \xleftarrow{\pi_1^*} & (t^{n-1})^* & \longleftarrow & 0.
 \end{array}$$

Since  $(\rho|_k)^*: k_1^* \rightarrow k^*$  is an isomorphism, there exists  $\nu_1^{(1)} \in k_1^*$  uniquely such that  $(\rho|_k)^*(\nu_1^{(1)}) = \nu_1$ . Let  $K_1$  be the torus corresponding to  $k_1$ . Then the action of  $K_1$  on  $\mathbf{H}^{N-1}$  gives the hyperKähler moment map

$$\mu_{K_1}: \mathbf{H}^{N-1} \rightarrow k_1^* \otimes \mathbf{R}^3.$$

We set  $\nu^{(1)} = (\nu_1^{(1)}, 0, 0)$ . In [Ko] we showed the following.

**Claim 1.** (1)  $X_{(1)}(\nu^{(1)}) = \mu_{K_1}^{-1}(\nu^{(1)})/K_1$  is a toric hyperKähler manifold with  $T^{n-1}$ -action.

(2)  $X_{(1)}(\nu^{(1)})$  is a hyperKähler submanifold of  $X(\nu)$ , which is preserved by  $T^n$ -action.

(3) We can identify  $(t^{n-1})^*$  with  $F_N \subset (t^n)^*$  naturally. Moreover under this identification the associated arrangement of hyperplanes for  $X_{(1)}(\nu^{(1)})$  is  $F_1 \cap F_N, \dots, F_{N-1} \cap F_N$  in  $F_N$ .

Next we construct  $X_{(2)}(\nu^{(2)})$ . Let  $t^{N-1} = \sum_{i=1}^{N-1} \mathbf{R}X_i$  and  $j: t^{N-1} \rightarrow t^N$  be the inclusion. If we set  $k_2 = k \cap t^{N-1}$ , then we have the following diagrams:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & k_2 & \xrightarrow{\iota_2} & t^{N-1} & & \\
 & & j|_{k_2} \downarrow & & j \downarrow & \searrow \pi_2 & \\
 0 & \longrightarrow & k & \xrightarrow{\iota} & t^N & \xrightarrow{\pi} & t^n \longrightarrow 0, \\
 \\ 
 0 & \longleftarrow & k_2^* & \xleftarrow{\iota_2^*} & (t^{N-1})^* & & \\
 & & (j|_{k_2})^* \uparrow & & j^* \uparrow & \swarrow \pi_2^* & \\
 0 & \longleftarrow & k^* & \xleftarrow{\iota^*} & (t^N)^* & \xleftarrow{\pi^*} & (t^n)^* \longleftarrow 0.
 \end{array}$$

We remark that  $\pi_2$  is surjective, because  $\iota^*u_N \neq 0$ , that is,  $k_2 \subsetneq k$ . Let  $K_2$  be the torus corresponding to  $k_2$ . Then the action of  $K_2$  on  $\mathbf{H}^{N-1}$  gives the hyperKähler moment map

$$\mu_{K_2}: \mathbf{H}^{N-1} \rightarrow k_2^* \otimes \mathbf{R}^3.$$

We set  $\nu_1^{(2)} = (j|_{k_2})^*\nu_1 \in k_2^*$  and  $\nu^{(2)} = (\nu_1^{(2)}, 0, 0)$ . In [Ko] we showed the following.  
**Claim 2.** (1)  $X_{(2)}(\nu^{(2)}) = \mu_{K_2}^{-1}(\nu^{(2)})/K_2$  is a toric hyperKähler manifold with  $T^n$ -action.  
 (2) The associated arrangement of hyperplanes for  $X_{(2)}(\nu^{(2)})$  is  $F_1, \dots, F_{N-1}$  in  $(t^n)^*$ .  
 Let  $I_1, I_2$  be the ideal in Theorem 2.4 and

$$\begin{aligned}
 \Psi_1: \mathbf{Z}[u_1, \dots, u_{N-1}] &\rightarrow H_{T^{n-1}}^*(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \\
 \Psi_2: \mathbf{Z}[u_1, \dots, u_{N-1}] &\rightarrow H_{T^n}^*(X_{(2)}(\nu^{(2)}); \mathbf{Z})
 \end{aligned}$$

be the map in Theorem 2.4 for  $X_{(1)}(\nu^{(1)})$  and  $X_{(2)}(\nu^{(2)})$  respectively. Then by the assumption of the induction  $\Psi_i$  is surjective and  $\ker \Psi_i = I_i$ . Moreover  $X_{(1)}(\nu^{(1)})_{T^n}$  is homotopy equivalent to  $X_{(1)}(\nu^{(1)})_{T^{n-1}} \times BS^1$ , where  $S^1$  is the group with the Lie algebra  $\mathbf{R}\pi(X_N)$ . Therefore we have

$$\tilde{\Psi}_1: \mathbf{Z}[u_1, \dots, u_N] \rightarrow H_{T^n}^*(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H_{T^{n-1}}^*(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \otimes H_{S^1}^*(point; \mathbf{Z})$$

and  $\tilde{\Psi}_1$  is surjective and  $\ker \tilde{\Psi}_1$  is generated by  $\ker \Psi_1 = I_1$ .

The following claim shows how we can recover the equivariant cohomology ring of  $X(\nu)$  from the equivariant cohomology rings of  $X_{(1)}(\nu^{(1)})$  and  $X_{(2)}(\nu^{(2)})$ .

**Claim 3.** (1)  $H_{T^n}^{2k+1}(X(\nu); \mathbf{Z}) = 0$  for each  $k \in \mathbf{Z}$ .  
 (2) There exists the following short exact sequence for each  $k \in \mathbf{Z}$ .

$$0 \longrightarrow H_{T^n}^{2k-2}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \xrightarrow{e} H_{T^n}^{2k}(X(\nu); \mathbf{Z}) \xrightarrow{r} H_{T^n}^{2k}(X_{(2)}(\nu^{(2)}); \mathbf{Z}) \longrightarrow 0.$$

- (3)  $e(\tilde{\Psi}_1(f(u_1, \dots, u_N))) = \Psi(f(u_1, \dots, u_N)u_N)$  for  $f \in \mathbf{Z}[u_1, \dots, u_N]$ .
- (4)  $r(\Psi(g(u_1, \dots, u_N))) = \Psi_2(g(u_1, \dots, u_{N-1}, 0))$  for  $g \in \mathbf{Z}[u_1, \dots, u_N]$ .

**Proof.** We have fixed  $h \in (t^N)^*$  and  $\nu_1 = \iota^*h$ . In Claim 1 and Claim 2 we only assumed that  $\nu = (\nu_1, 0, 0) \in k^* \otimes \mathbf{R}^3$  is a regular value of  $\mu_K$ . Now we can choose

$h$  and  $\nu_1 = \iota^*h$  such that all vertices  $\bigcap_{i \in S} F_i$  for  $S \subset \{1, \dots, N\}$  with  $\#S = n$  are contained in  $\{p \in (t^n)^* | \langle \pi^*p + h, X_N \rangle \geq 0\}$ . We set

$$\begin{aligned} U_0 &= \{p \in (t^n)^* | \langle \pi^*p + h, X_N \rangle = 0\} = F_N, \\ U_1 &= \{p \in (t^n)^* | \langle \pi^*p + h, X_N \rangle \geq 0\}, \\ U_2 &= \{p \in (t^n)^* | \langle \pi^*p + h, X_N \rangle > 0\}, \\ \tilde{V}_i &= \mu_{T^n}^{-1}(U_i, (t^n)^*, (t^n)^*) \quad \text{for } i = 0, 1, 2, \\ V_i &= \mu_{T^n}^{-1}(U_i, 0, 0) \quad \text{for } i = 0, 1, 2, \end{aligned}$$

where  $\mu_{T^n}: X(\nu) \rightarrow (t^n)^* \otimes \mathbf{R}^3$  is a hyperKähler moment map for the action of  $T^n$  on  $X(\nu)$ .

Now we consider the cohomology exact sequence for  $(X(\nu), \tilde{V}_2)$ :

$$\longrightarrow H_{T^n}^l(X(\nu), \tilde{V}_2; \mathbf{Z}) \longrightarrow H_{T^n}^l(X(\nu); \mathbf{Z}) \longrightarrow H_{T^n}^l(\tilde{V}_2; \mathbf{Z}) \longrightarrow H_{T^n}^{l+1}(X(\nu), \tilde{V}_2; \mathbf{Z}) \longrightarrow$$

In [BD] it was shown that  $\tilde{V}_2$  is  $T^n$ -equivariant homotopy equivalent to  $X_{(2)}(\nu^{(2)})$ . Therefore we have

$$H_{T^n}^l(\tilde{V}_2; \mathbf{Z}) \cong H_{T^n}^l(X_{(2)}(\nu^{(2)}); \mathbf{Z}).$$

By the same argument we also showed in [Ko] that  $(V_1, V_2)$  is a  $T^n$ -equivariant deformation retract of  $(X(\nu), \tilde{V}_2)$ . Moreover the neighborhood  $W$  of  $V_0$  in  $V_1$  can be identified with the neighborhood of  $V_0$  in  $E = L_N|_{V_0}$  by the  $T^n$ -equivariant map  $i: W \rightarrow E$ , which is defined by

$$i([z_1, w_1, \dots, z_{N-1}, w_{N-1}, z_N, 0]) = [(z_1, w_1, \dots, z_{N-1}, w_{N-1}, 0, 0), z_N],$$

where  $[\dots]$  denotes equivalence class. If  $[z, w] \in V_1$ , then  $z_i w_i = 0$  for  $i = 1, \dots, N-1$  and  $w_N = 0$ . Moreover  $V_0$  is defined by the equation  $z_N = 0$  in  $V_1$ . So we have

$$H_{T^n}^l(X(\nu), \tilde{V}_2; \mathbf{Z}) \cong H_{T^n}^l(V_1, V_2; \mathbf{Z}) \cong H_{T^n}^l(E, E \setminus V_0; \mathbf{Z}).$$

$V_0$  is not smooth, but it is a  $T^n$ -equivariant deformation retract of  $X_{(1)}(\nu^{(1)})$ . Moreover  $E = L_N|_{V_0}$  is the restriction of  $\tilde{E} = L_N|_{X_{(1)}(\nu^{(1)})}$ . Therefore we have

$$H_{T^n}^l(E, E \setminus V_0; \mathbf{Z}) \cong H_{T^n}^l(\tilde{E}, \tilde{E} \setminus X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H_{T^n}^{l-2}(X_{(1)}(\nu^{(1)}); \mathbf{Z}),$$

where the second isomorphism is the Thom isomorphism. Thus we have

$$H_{T^n}^l(X(\nu), \tilde{V}_2; \mathbf{Z}) \cong H_{T^n}^{l-2}(X_{(1)}(\nu^{(1)}); \mathbf{Z}).$$

Moreover by the assumption of the induction we have

$$H_{T^n}^{2k+1}(X_{(1)}(\nu^{(1)}); \mathbf{Z}) \cong H_{T^n}^{2k+1}(X_{(2)}(\nu^{(2)}); \mathbf{Z}) \cong 0.$$

Therefore we proved the claim.  $\square$

Now we prove the first part of Theorem 2.4.

**Claim 4.**  $\Psi$  is surjective.

**Proof.** Fix any  $a \in H_{T^n}^{2k}(X(\nu); \mathbf{Z})$ . Since  $\Psi_2$  is surjective, there exists  $f \in \mathbf{Z}[u_1, \dots, u_{N-1}]$  such that

$$r(a) = \Psi_2(f(u_1, \dots, u_{N-1})) = r(\Psi(f(u_1, \dots, u_{N-1}))).$$

Since  $a - \Psi(f(u_1, \dots, u_{N-1})) \in \ker r$ , there exists  $b \in H_{T^n}^{2k-2}(X_{(1)}(\nu^{(1)}); \mathbf{Z})$  such that

$$e(b) = a - \Psi(f(u_1, \dots, u_{N-1})).$$

Since  $\tilde{\Psi}_1$  is surjective, there exists  $g \in \mathbf{Z}[u_1, \dots, u_N]$  such that

$$b = \tilde{\Psi}_1(g(u_1, \dots, u_N)).$$

Thus we have

$$\begin{aligned} a &= \Psi(f(u_1, \dots, u_{N-1})) + e(b) \\ &= \Psi(f(u_1, \dots, u_{N-1})) + e(\tilde{\Psi}_1(g(u_1, \dots, u_N))) \\ &= \Psi(f(u_1, \dots, u_{N-1})) + g(u_1, \dots, u_N)u_N. \quad \square \end{aligned}$$

Finally we show the second part of Theorem 2.4.

**Claim 5.**  $I = \ker \Psi$ .

**Proof.** First we show  $I \subset \ker \Psi$ . To do this, we show that all generators of  $I$  belong to  $\ker \Psi$ . Take  $\emptyset \neq S \subset \{1, \dots, N\}$  such that  $\bigcap_{j \in S} F_j = \emptyset$ . According to Theorem 2.6, it is sufficient to show  $\prod_{j \in S} u_j \in \ker \Psi$ .

Suppose  $N \in S$ . Since  $\bigcap_{j \in S \setminus \{N\}} (F_j \cap F_N) = \bigcap_{j \in S} F_j = \emptyset$ , according to Claim 1 and Theorem 2.6, we have

$$\prod_{j \in S \setminus \{N\}} u_j \in I_1 = \ker \Psi_1 \subset \ker \tilde{\Psi}_1.$$

Therefore, according to Claim 3, we have

$$\Psi\left(\prod_{j \in S} u_j\right) = e(\tilde{\Psi}_1\left(\prod_{j \in S \setminus \{N\}} u_j\right)) = 0.$$

Thus we proved  $\prod_{j \in S} u_j \in \ker \Psi$  in the case  $N \in S$ . Here we used the assumption  $\pi(X_N) \neq 0$ . However we have assumed  $\pi(X_i) \neq 0$  for any  $i$ . So we use the same argument in the case  $i \in S$ . Thus we proved  $I \subset \ker \Psi$ .

Next we show  $\ker \Psi \subset I$ . For  $f(u_1, \dots, u_N) \in \ker \Psi$  we have to show  $f \in I$ . First we rewrite

$$f(u_1, \dots, u_N) = g_1(u_1, \dots, u_{N-1}) + g_2(u_1, \dots, u_N)u_N.$$

Since

$$0 = r(\Psi(f(u_1, \dots, u_N))) = \Psi_2(g_1(u_1, \dots, u_{N-1})),$$

we have that  $g_1(u_1, \dots, u_{N-1}) \in \ker \Psi_2 = I_2$ . Since  $I_2 \subset I \subset \ker \Psi$ , we have  $g_2(u_1, \dots, u_N)u_N \in \ker \Psi$ . We have to show  $g_2(u_1, \dots, u_N)u_N \in I$ .

Since

$$0 = \Psi(g_2(u_1, \dots, u_N)u_N) = e(\tilde{\Psi}_1(g_2(u_1, \dots, u_N)))$$

and  $e$  is injective, we have  $g_2(u_1, \dots, u_N) \in \ker \tilde{\Psi}_1 = I_1\mathbf{Z}[u_N]$ . According to Claim 1 and Theorem 2.6, we have  $I_1u_N \subset I$ . So we have  $g_2(u_1, \dots, u_N)u_N \in I$ . Thus we proved  $\ker \Psi \subset I$ . So we finish the proof of Claim 5.  $\square$

Thus we finish the proof of Theorem 2.4.

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## AN INTRODUCTION TO PSEUDOTWISTORS BASIC CONSTRUCTIONS

JULIAN LAWRYNOWICZ AND OSAMU SUZUKI

R. Penrose has observed in 1976 [11] that the points of the Minkowski space-time can be represented by two-dimensional linear subspaces of a complex four-dimensional vector space on which an hermitian form of signature  $(+, +, -, -)$  is defined. He called this *flat twistor space*, and the method of investigating deformation of complex structures, yielded from there, the *twistor programme*. This initiated a series of papers and monographs by various authors. In the present research we are dealing with dynamical systems generated by the Hermitian Hurwitz pairs of the signature  $(\sigma, s)$ ,  $\sigma + s = 5 + 4\mu$ ,  $|\sigma + 1 - s| = 2 + 4m$ ;  $\mu, m = 0, 1, \dots$ . In particular, for  $(3, 2)$  and its dual  $(1, 4)$  the role of entropy is indicated as well as the relationship between Hurwitz and Penrose twistors; *Hurwitz twistors* being objects introduced by us. The signatures  $(1, 8)$  and  $(7, 6)$  give rise for introducing *pseudotwistors* and *bitwistors*, respectively; for pseudotwistors we can prove [9] a counterpart of the original fundamental Penrose theorem in the local version (on real analytic solutions of the related spinor equations vs. harmonic forms) and in the semi-global version (on holomorphic solutions of those equations vs. Dolbeault cohomology groups). This has to be preceded by basic constructions (which is the core topic of this paper), a study of the related pseudotwistors and spinor equations as well as complex structures on spinors. This will allow us to prove a theorem (which we call the atomization theorem) saying that there exist complex structures on isometric embeddings for the Hermitian Hurwitz pairs concerned so that the embeddings are real parts of holomorphic mappings.

### 1. INTRODUCTION

In 1985 Lawrynowicz and Rembieliński [5] initiated a geometrization of the problem of A. Hurwitz concerned with the composition of quadratic forms and a geometrical study of the related differential operators of the Cauchy-Riemann-, Dirac- and Fueter-types by introducing the so-called Hurwitz pairs, also in the hyperbolic case [17] discussed by Penrose [11]. The results obtained quickly appeared in text-books [12]

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and enabled [7, 8] to formulate and prove *counterparts of two Penrose’s fundamental theorems* within the theory of Hurwitz pairs.

Consider the Hurwitz pair consisting of the Hermitian space  $\mathbb{C}^4(\kappa) := (\mathbb{C}^4, \kappa)$ , equipped with the metric

$$\kappa \equiv I_{2,2} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the real space  $\mathbb{R}^5(\eta) := (\mathbb{R}^5, \eta)$ , equipped with the metric

$$\eta \equiv I_{2,3} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let  $(e_1, \dots, e_4)$  be the canonical basis of  $\mathbb{C}^4(\kappa)$ . We consider a pair

$$H = (\mathbb{C}^4(\kappa), \mathbb{R}^5(\eta)).$$

If there exists a bilinear mapping  $\circ : \mathbb{R}^5(\eta) \times \mathbb{C}^4(\kappa) \rightarrow \mathbb{C}^4(\kappa)$  called *multiplication* of elements of  $\mathbb{R}^5(\eta)$  by elements of  $\mathbb{C}^4(\kappa)$  such that, for  $f \in \mathbb{C}^4(\kappa)$  and  $\alpha \in \mathbb{R}^5(\eta)$ , we have

$$\langle a, \alpha \rangle_\eta \langle \langle f, f \rangle \rangle_\kappa = \langle \langle a \circ f, a \circ f \rangle \rangle_\kappa,$$

where

$$\langle \langle f, g \rangle \rangle_\kappa := f^* \kappa g, \quad f, g \in \mathbb{C}^n; \quad \langle a, b \rangle_\eta := a^T \eta b, \quad a, b \in \mathbb{R}^p$$

and  $*$  denotes the hermitian conjugation, and, moreover,  $H$  is *irreducible*, i.e. there exists no subspace  $V$  of  $\mathbb{C}^4$ ,  $V \neq \{0\}$ ,  $\mathbb{C}^4$ , such that  $\circ |_{\mathbb{R}^5(\eta) \times V} : \mathbb{R}^5(\eta) \times V \rightarrow V$ , then  $H$  is called an *Hermitian Hurwitz pair* (cf. e.g. [7, 8]). This is of course a particular case of the general definition.

Further, let

$$\epsilon_\alpha \circ e_k = C_{\alpha k}^1 e_1 + \dots + C_{\alpha k}^n e_n,$$

where  $(\epsilon_1, \dots, \epsilon_5)$  is the canonical basis of  $\mathbb{R}^5(\eta)$  and let  $C_\alpha = (C_{\alpha k}^j)$ ,  $\alpha = 1, \dots, 5$ . We define the algebra  $\mathcal{A}_{2,3}$  which is generated by  $\{C_\alpha^\# C_\beta : \alpha \leq \beta\}$ , where  $C_\alpha^\# = \kappa C_\alpha^* \kappa^{-1}$ . An element  $x \in \mathcal{A}_{2,3}$  is called *Hurwitz twistor* [7, 8] whenever  $x$  has the form

$$(1.1) \quad x = \sum_{\alpha < \beta} \xi_{\alpha\beta} C_\alpha^\# C_\beta, \quad \xi_{\alpha\beta} \in \mathbb{C}$$

and  $\text{im } x^2 = 0$ , where  $\text{im } x$ ,  $x \in \mathcal{A}_{2,3}$  is defined in the following manner:  $x \in \mathcal{A}_{2,3}$  can be written uniquely as

$$x = \sum_{k=0}^4 x_k, \quad x_k = \sum_{\alpha_1 < \beta_1 < \dots < \alpha_k < \beta_k} \xi_{\alpha_1 \beta_1 \dots \alpha_k \beta_k} C_{\alpha_1}^\# C_{\beta_1} \dots C_{\alpha_k}^\# C_{\beta_k},$$

with  $x_0 = \xi_0 I_4$ . We define  $\text{im } x := x - x_0$  and denote the collection of Hurwitz twistors by  $P^1 = \mathcal{J}_H$ :

$$\mathcal{J}_H := \{x = \sum_{\alpha < \beta} \xi_{\alpha\beta} C_\alpha^\# C_\beta : \text{im } x^2 = 0\}.$$

2. DYNAMICAL SYSTEMS GENERATED BY THE HERMITIAN HURWITZ PAIRS OF SIGNATURES (3, 2) AND (1, 4)

Following [1] we are looking in our case for a dynamical system  $(X, \tilde{\mu}, T)$  in the sense of ergodic theory. Here  $X$  is a measure space,  $\tilde{\mu}$  is a measure on the space and  $T$  an invertible, measurable map  $X \rightarrow X$  that preserves  $\tilde{\mu}$ , i.e., for any measurable set  $A \subset X$  we have  $\tilde{\mu}[A] = \tilde{\mu} \circ T^{-1}[A]$ , and  $\tilde{\mu}[X] = 1$ . Of course it is natural to take

$$X = \mathbb{C}^4, \tilde{\mu} = \langle\langle \cdot, \cdot \rangle\rangle_{I_{2,2}} \quad \text{resp.} \quad X = \mathbb{C}^{16}, \tilde{\mu} = \langle\langle \cdot, \cdot \rangle\rangle_{I_{8,8}}.$$

If  $\xi$  is a finite partitioning of  $X$  in measurable sets  $C_1, \dots, C_{N(\xi)}$ , i.e.,

$$C_j \subset X, \quad C_j \cap C_k = \emptyset, \text{ for } j \neq k, \text{ and } C_1 \cup \dots \cup C_{N(\xi)} = X,$$

the *entropy*  $H_{\tilde{\mu}}(\xi)$  of the partition  $\xi$  is the quantity

$$H_{\tilde{\mu}}(\xi) := -\sum_{j=1}^{N(\xi)} \tilde{\mu}(C_j) \log_2 \tilde{\mu}(C_j),$$

where  $\tilde{\mu}(C_j) \log_2 \tilde{\mu}(C_j) = 0$  whenever  $\tilde{\mu}(C_j) = 0$ .

If  $\xi = \{C_j\}, 1 \leq j \leq N(\xi)$ , and  $\tilde{\xi} = \{C_k\}, 1 \leq k \leq N(\tilde{\xi})$  are two finite partitions of  $X$ , we shall denote by  $\xi \vee \tilde{\xi}$  the partition of  $X$  into  $C_j \cap C_k$ , where the indices  $j$  and  $k$  run independently from 1 to  $N(\xi)$  and from 1 to  $N(\tilde{\xi})$ , respectively. For an arbitrary partition  $\xi = \{C_j\}, 1 \leq j \leq N(\xi)$ , we denote by  $T^{-1}\xi$  the partition of  $X$  into the sets  $T^{-1}C_1, \dots, T^{-1}C_{N(\xi)}$ . For all positive integers  $n$  we form  $\xi^n = T^{-0}\xi \vee \dots \vee T^{-n+1}\xi$  and consider the limit of  $H_{\tilde{\mu}}(\xi^n)/n$  as  $n \rightarrow \infty$ . The limit exists and is called the *entropy* of the partition  $\xi$  for *unit time*. We denote it by  $b_{\tilde{\mu}}(T|X; \xi)$ . For the mapping  $T : X \rightarrow X$  of the dynamical system  $(X, \tilde{\mu}, T)$ , the *metric entropy* in the sense of Ja. G. Sinai is the quantity  $b_{\tilde{\mu}}(T|X) := \sup_{\xi} b_{\tilde{\mu}}(T|X; \xi)$ , where the upper bound is taken over all finite partitions of  $X$ .

In particular, we may take as  $b_{\tilde{\mu}}(T|X)$  the entropy in the physical sense (cf., e.g. E. Fermi's book [2]); treating it as a stochastic instant  $\tau$  [10, 4], analogous to a time instant  $t$ , when considering a relativistic particle at  $(x, y, z) \in \mathbb{R}^3$  within the spaces  $\mathbb{R}^5(I_{1,4})$  and  $\mathbb{R}^5(I_{3,2})$  of space-time elements

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - d\tau^2 \quad \text{and} \quad -ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 - d\tau^2,$$

where  $c$  is a positive constant interpreted as the light velocity. Already in the Newtonian dynamics an additional dimension corresponding to time is needed because of the necessity of introducing an inertial frame and absolute time in connection with the Newtonian laws of dynamics. This means that, in contrast to the Aristotelian physics, the space  $\mathbb{E}^3 = \{(x, y, z) \in \mathbb{R}^3, ds^2 = dx^2 + dy^2 + dz^2\}$  together with time  $t \in \mathbb{T} = \mathbb{R}$  are no more absolute: there is a projection mapping  $\pi : \mathbb{E}^3 \times \mathbb{T} \rightarrow \mathbb{T}$  which

associates to any element  $\tilde{p} \in \mathbb{E}^3 \times \mathbb{R}$  the corresponding instant of time  $t = \pi(\tilde{p})$ ;  $\mathbb{T}$  is called the base space. The inverse image of  $t$ ,  $\pi^{-1}(\tilde{p})$  is called a fibre. Each fibre is isomorphic to the Euclidean space  $\mathbb{E}^3$ , which is therefore called a typical fibre. Such a triple  $(\mathbb{E}^3 \times \mathbb{T}, \mathbb{T}, \pi)$  with  $\pi$  being a surjective projection map is called a bundle with the base space  $\mathbb{T}$  and bundle space  $\mathbb{E}^3 \times \mathbb{T}$ . The bundle approach is very convenient and naturally extendable to higher dimensions and curved spaces [12].

In the case of the space-time elements in question, the usual variational procedure with respect to the action integral leads to discussion of the Lorentz transformation

$$\begin{aligned} dx &= \rho_1(dx' + v_t^x dt' + v_\tau^x d\tau'), & dy &= dy', & dz &= dz', \\ dt &= \rho_2(v_x^t dx' + dt' + v_\tau^t d\tau'), & \rho_1, \rho_2, \rho_3 &\in \mathbb{R}^+, \\ d\tau &= \rho_3(v_x^\tau dx' + v_t^\tau dt' + d\tau'), & v_t^x \text{ etc.} &\in \mathbb{R}, \end{aligned}$$

satisfying the condition  $A^T \rho A = \rho$ , where  $A$  is the matrix of the transformation and  $\rho$  the metric in question. Hence [10]:

$$\begin{aligned} \rho_1 &= [1 - (\frac{1}{c}v_t^x)^2 \pm (v_\tau^x)^2]^{-1/2}, & \rho_2 &= [1 - (\frac{1}{c}v_x^t)^2 \mp (cv_\tau^t)^2]^{-1/2}, \\ \rho_3 &= [1 \pm (v_x^\tau)^2 \mp (\frac{1}{c}v_t^\tau)^2]^{-1/2} \end{aligned}$$

with various restrictions for  $v_t^x$  etc. The corresponding Euler-Lagrange equations, interpreted as the equations of motion, include the stochastic force (cf. [10, 3]):

$$\vec{F} = \pm m v_t^\tau \nabla_{\vec{r}} v_t^\tau, \quad \vec{r} = (x, y, z)$$

( $m$  denoting the mass), being now an intrinsic part of the geometry. More generally, if we consider a one-parameter family of symplectic transformations and the parameter value 0 corresponds to the identity transformation, a Hamiltonian operator is defined as the derivative of the transformations of the family with respect to the parameter (at 0). By differentiating the condition for symplecticity of a transformation, we may find the condition for an operator  $H$  to be Hamiltonian:  $\omega(H\mathbf{x}, \mathbf{y}) + \omega(\mathbf{x}, H\mathbf{y}) = 0$  for all  $\mathbf{x}, \mathbf{y}$  belonging to a symplectic space in question endowed with a skew-scalar product  $\omega$ ; the Hamiltonian is supposed to be related in a standard way with the Lagrangian density appearing in the action integral.

### 3. BASIC CONSTRUCTIONS FOR THE HURWITZ PAIRS

$$(\mathbb{C}^{16}(I_{8,8}), \mathbb{R}^9(I_{\sigma,s}), \sigma + s = 9)$$

In this section, we recall basic constructions of Hurwitz algebras and give generators of the Hurwitz algebras  $\mathcal{H}_{\sigma,s}$ ,  $\sigma + s = 9$ . They are counterparts of  $\mathcal{A}_{2,3}$ . Generally, under a Hurwitz algebra we understand a central Clifford algebra whose generators  $S_\alpha$  satisfy the condition  $S_\alpha^\# = S_\alpha$ .

The basic constructions are methods of giving explicit forms for generators of Hurwitz algebras. These constructions involve three different methods. (I)  $\mathcal{H}_{\sigma,0} \implies \mathcal{H}_{\sigma+2,0}$ ,  $\sigma \equiv 1 \pmod{2}$ . Let  $S_1, S_2, \dots, S_\sigma$  be generators of  $\mathcal{H}_{\sigma,0}$ . Then

$$\begin{aligned} \tilde{S}_1 &= \begin{pmatrix} S_1 & 0 \\ 0 & -S_1 \end{pmatrix}, \quad \tilde{S}_2 = \begin{pmatrix} S_2 & 0 \\ 0 & -S_2 \end{pmatrix}, \dots, \quad \tilde{S}_\sigma = \begin{pmatrix} S_\sigma & 0 \\ 0 & -S_\sigma \end{pmatrix}, \\ \tilde{S}_{\sigma+1} &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \tilde{S}_{\sigma+2} = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}, \quad n = 2^{\lfloor \frac{1}{2}(\sigma-1) \rfloor} \end{aligned}$$

become generators of  $\mathcal{H}_{\sigma+2,0}$ .

(II)  $\mathcal{H}_{\sigma,0} \implies \mathcal{H}_{\sigma,2}$ ,  $\sigma \equiv 1 \pmod{2}$ . Let  $S_1, \dots, S_\sigma$  be generators of  $\mathcal{H}_{\sigma,0}$ . Then

$$\begin{aligned} \tilde{S}_1 &= \begin{pmatrix} S_1 & 0 \\ 0 & -S_1 \end{pmatrix}, \dots, \quad \tilde{S}_\sigma = \begin{pmatrix} S_\sigma & 0 \\ 0 & -S_\sigma \end{pmatrix}, \\ \tilde{S}_{\sigma+1} &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \tilde{S}_{\sigma+2} = \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \quad n = 2^{\lfloor \frac{1}{2}(\sigma-1) \rfloor} \end{aligned}$$

become generators of  $\mathcal{H}_{\sigma,2}$ . (III)  $\mathcal{H}_{\sigma,s} \implies \mathcal{H}_{\sigma,s+2}$ ,  $\sigma + s \equiv 1 \pmod{2}$ ,  $s > 0$ . Let

$S_1, \dots, S_{\sigma+s}$  be generators of  $\mathcal{H}_{\sigma,s}$  of the form

$$S_\alpha = \begin{pmatrix} A_\alpha & iB_\alpha \\ iB_\alpha^* & -D_\alpha \end{pmatrix} \quad \alpha = 1, 2, \dots, \sigma + s,$$

$$A_\alpha^* = A_\alpha, \quad D_\alpha^* = D_\alpha,$$

$$A_\alpha, B_\alpha, D_\alpha \in M_n(\mathbb{C}) \quad n = 2^{\lfloor \frac{1}{2}(\sigma+s)-\frac{1}{2} \rfloor}.$$

Then the generators of  $\mathcal{H}_{\sigma,s+2}$  are given by

$$\tilde{S}_\alpha = \begin{pmatrix} A_\alpha & 0 & 0 & iB_\alpha \\ 0 & D_\alpha & iB_\alpha^* & 0 \\ 0 & iB_\alpha & -A_\alpha & 0 \\ iB_\alpha^* & 0 & 0 & -D_\alpha \end{pmatrix}, \quad \alpha = 1, 2, \dots, \sigma + s,$$

$$\tilde{S}_{\sigma+s+1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \tilde{S}_{\sigma+s+2} = \begin{pmatrix} 0 & iI_{\frac{1}{2}n} \otimes \sigma_3 \\ iI_{\frac{1}{2}n} \otimes \sigma_3 & 0 \end{pmatrix},$$

where  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Applying the above construction methods, we can find generators of the Hurwitz algebras of  $(\mathbb{C}^4(I_{2,2}), \mathbb{R}^5(I_{\sigma,s}))$ ,  $\sigma + s = 5$ , and  $(\mathbb{C}^{16}(I_{8,8}), \mathbb{R}^9(I_{\sigma,s}))$ ,  $\sigma + s = 9$ . At first we notice that

$$\begin{array}{ccc} (\mathbb{C}^n(I_{\frac{1}{2}n, \frac{1}{2}n}), \mathbb{R}^{\sigma+s}(I_{\sigma,s})) & & \\ \swarrow & & \searrow \\ \mathcal{H}_{\sigma-1,s} & & \mathcal{H}_{s-1,\sigma} \end{array}$$

which implies that a Hurwitz pair gives rise to two Hurwitz algebras  $\mathcal{H}_{\sigma-1,s}$  and  $\mathcal{H}_{s-1,\sigma}$ . Explicitly, in the case of  $\sigma + s = 5$ ,

for $\sigma=1$	and $s=4$	we get	$\mathcal{H}_{0,4}$ , $\mathcal{H}_{3,1}$
2	3		$\mathcal{H}_{1,3}$ , $\mathcal{H}_{2,2}$
3	2		$\mathcal{H}_{2,2}$ , $\mathcal{H}_{1,3}$
4	1		$\mathcal{H}_{3,1}$ , $\mathcal{H}_{0,4}$

In the case of  $\sigma + s = 9$ ,

for $\sigma=1$	and $s=8$	we get	$\mathcal{H}_{0,8}$ , $\mathcal{H}_{7,1}$
2	7		$\mathcal{H}_{1,7}$ , $\mathcal{H}_{6,2}$
3	6		$\mathcal{H}_{2,6}$ , $\mathcal{H}_{5,3}$
4	5		$\mathcal{H}_{3,5}$ , $\mathcal{H}_{4,4}$
5	4		$\mathcal{H}_{4,4}$ , $\mathcal{H}_{3,5}$
6	3		$\mathcal{H}_{5,3}$ , $\mathcal{H}_{2,6}$
7	2		$\mathcal{H}_{6,2}$ , $\mathcal{H}_{1,7}$
8	1		$\mathcal{H}_{7,1}$ , $\mathcal{H}_{0,8}$

We give the generators of Hurwitz algebras involved explicitly as well as the type of the basic construction. We take the Pauli matrices as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(IV)  $\mathcal{H}_{\sigma,s}$  with  $\sigma + s = 3$  and  $\sigma + s = 5$ .

For  $\mathcal{H}_{3,0}$  the generators can be chosen as  $S_\alpha = \sigma_\alpha$ ,  $\alpha = 1, 2, 3$ . Subsequently, we take

$$\text{for } \mathcal{H}_{3,2} \quad \begin{aligned} S_1 &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad S_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \\ S_4 &= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix} \end{aligned}$$

with the type of basing construction  $\mathcal{H}_{3,0} \xrightarrow{\text{(II)}} \mathcal{H}_{3,2}$ ;

for  $\mathcal{H}_{1,2}$   $S_1 = i\sigma_1$ ,  $S_2 = i\sigma_2$ ,  $S_3 = \sigma_3$ ;

$$\text{for } \mathcal{H}_{1,4} \quad \begin{aligned} S_1 &= \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \\ S_4 &= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \end{aligned}$$

with the type of basing construction  $\mathcal{H}_{1,2} \xrightarrow{\text{(III)}} \mathcal{H}_{1,4}$ ; (V)  $\mathcal{H}_{\sigma,s}$  with  $\sigma + s = 7$ . Let

$$\text{diag}(A, B) := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ etc. and } \text{diag}^*(A, B) := \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \text{ etc.,}$$

where  $A, B$ , etc. denote square matrices. For  $\mathcal{H}_{1,6}$  the generators can be chosen as

$$S_\alpha = \text{diag}^*(i\sigma_\alpha, i\sigma_\alpha, i\sigma_\alpha, i\sigma_\alpha), \quad \alpha = 1, 2, 3,$$

$$\begin{aligned} S_4 &= \text{diag}^*(-I_2, I_2, -I_2, I_2), & S_5 &= \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}, & S_6 &= \begin{pmatrix} 0 & 0 & iI_2 & 0 \\ 0 & 0 & 0 & -iI_2 \\ iI_2 & 0 & 0 & 0 \\ 0 & -iI_2 & 0 & 0 \end{pmatrix} \\ S_7 &= \text{diag}(\sigma_3, -\sigma_3, -\sigma_3, \sigma_3), \end{aligned}$$

with the type of construction  $\mathcal{H}_{1,2} \xrightarrow{\text{(III)}} \mathcal{H}_{1,4} \xrightarrow{\text{(III)}} \mathcal{H}_{1,6} \xrightarrow{\text{(III)}} \mathcal{H}_{1,8}$ ;

$$\begin{aligned} \text{for } \mathcal{H}_{5,2} \quad S_\alpha &= \text{diag}(\sigma_\alpha, -\sigma_\alpha, -\sigma_\alpha, \sigma_\alpha), \quad \alpha = 1, 2, 3, \\ S_4 &= \begin{pmatrix} 0 & I_2 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_2 \\ 0 & 0 & -I_2 & 0 \end{pmatrix}, & S_5 &= \begin{pmatrix} 0 & iI_2 & 0 & 0 \\ -iI_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -iI_2 \\ 0 & 0 & iI_2 & 0 \end{pmatrix}, \\ S_6 &= \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}, & S_7 &= \begin{pmatrix} 0 & -iI_4 \\ -iI_4 & 0 \end{pmatrix} \end{aligned}$$

with the type of construction  $\mathcal{H}_{3,0} \xrightarrow{\text{(I)}} \mathcal{H}_{5,0} \xrightarrow{\text{(II)}} \mathcal{H}_{5,2}$ ;

$$\begin{aligned} \text{for } \mathcal{H}_{3,4} \quad S_\alpha &= \text{diag}(\sigma_\alpha, \sigma_\alpha, -\sigma_\alpha, -\sigma_\alpha), \quad \alpha = 1, 2, 3, \\ S_4 &= \text{diag}^*(-I_2, I_2, -I_2, I_2), \quad S_6 \text{ resp. } S_7 \text{ as } S_5 \text{ resp. } S_6 \text{ for } \mathcal{H}_{1,6}, \\ S_5 &= \text{diag}^*(iI_2, iI_2, iI_2, iI_2) \end{aligned}$$

with the type of construction  $\mathcal{H}_{3,0} \xrightarrow{\text{(II)}} \mathcal{H}_{3,2} \xrightarrow{\text{(II)}} \mathcal{H}_{3,4}$ ;

$$\begin{aligned} \text{for } \mathcal{H}_{7,0} \quad S_\alpha &= \text{diag}(\sigma_\alpha, -\sigma_\alpha, \sigma_\alpha, -\sigma_\alpha), \quad \alpha = 1, 2, 3, \\ S_4, S_5 &\text{ as for } \mathcal{H}_{1,6}, \quad S_6 = \begin{pmatrix} 0 & iI_4 \\ iI_4 & 0 \end{pmatrix}, \quad S_7 = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix} \end{aligned}$$

with the type of construction  $\mathcal{H}_{3,0} \xrightarrow{\text{(I)}} \mathcal{H}_{5,0} \xrightarrow{\text{(I)}} \mathcal{H}_{7,0}$ .

#### 4. PSEUDOTWISTORS RELATED TO HERMITIAN HURWITZ PAIRS

In this section we define the pseudotwistors for the Hermitian Hurwitz pairs

$$(\mathbb{C}^{16}(I_{8,8}), \quad \mathbb{R}^9(I_{\sigma,s})),$$

$\sigma + s = 9$ , and discuss the duality of them and the Penrose diagrams. These are the counterparts of  $(\mathbb{C}^4(I_{2,2}), \mathbb{R}^5(I_{2,3}))$  in [7, 8].

Let  $(\mathbb{C}^{16}(I_{8,8}), \mathbb{R}^9(I_{\sigma,s}))$ ,  $\sigma + s = 9$  be one of the Hurwitz pairs and let  $C_\alpha$ ,  $\alpha = 1, 2, \dots, 9$ , be the corresponding Hurwitz matrices. Then we have the Hurwitz algebras:

$$\mathcal{A} = \bigoplus_{k=0}^4 \mathcal{A}_{2k},$$

$$\mathcal{A}_{2k} = \{ \sum_{1 \leq \alpha_1 < \beta_1 < \dots < \alpha_k < \beta_k \leq 9} \xi_{\alpha_1 < \beta_1 \dots \alpha_k \beta_k} C_{\alpha_1}^\# C_{\beta_1} \dots C_{\alpha_k}^\# C_{\beta_k} \}.$$

**Definition.** An element  $\xi \in \mathcal{A}$  is called a *pseudotwistor* of  $\mathcal{A}$ , if  $\text{im } \xi^2 = 0$ . If  $\xi \in \mathcal{A}_{2k}$ , it is said to be of *degree*  $k$ . Here we denote the non-scalar part of  $\xi^2$  is denoted by  $\text{im } \xi^2$ .

*Example 1.* Scalar elements are pseudotwistors.

*Example 2.* Monomials

$$C_{\alpha_1}^\# C_{\beta_1} \dots C_{\alpha_k}^\# C_{\beta_k}, \quad \alpha_1 < \beta_1 < \alpha_2 < \dots < \beta_k,$$

are pseudotwistors. Hence we see that any element of  $\mathcal{A}$  can be written as a linear combination of pseudotwistors.

*Example 3.* There are elements of  $\mathcal{A}$  which are not pseudotwistors, e.g,  $\xi = C_1^\# C_2 + C_3^\# C_4$ . We note that  $\text{im } \xi^2 = 2C_1^\# C_2 C_3^\# C_4$ .

Now we introduce an analogue of Wick's theorem in the fermionic algebras to Hurwitz algebras. We describe it with the use of a simple example: For

$$\xi_1 = C_1^\# C_3 + C_1^\# C_5, \quad \xi_2 = C_2^\# C_5,$$

we have

$$\xi_1 \xi_2 = -C_1^\# C_2 C_3^\# C_5 + \eta_{55} C_1^\# C_5.$$

Because of contractions, we have a term of lower degrees. We denote this term by

$$C_1^\# C_2 = \eta_{55} C_1^\# C_5 C_2^\# C_5,$$

where underlining indicates the way of contraction. The first term is called *the normal product* of  $\xi_1$  and  $\xi_2$  and is denoted by

$$: \xi_1 \xi_2 := -C_1^\# C_2 C_3^\# C_5.$$

Hence we can write

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} :$$

where  $\overline{\quad}$  denotes contraction. In general, for  $\xi_1, \xi_2 \in \mathcal{A}$ , we have

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} : + : \overline{\xi_1 \xi_2}^{(2)} : + \dots,$$

where  $\overline{\xi_1 \xi_2}^{(2)} := \overline{\overline{\xi_1 \xi_2}}$  etc., i.e.  $\overline{\quad}^{(\ell)}$  denotes the  $\ell$ -time contraction.

*Example 4.* For  $\xi_1 = C_1^\# C_3 C_5^\# C_7$ ,  $\xi_2 = C_4^\# C_8 + C_5^\# C_8 + C_1^\# C_3$ , we get

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} : + : \overline{\overline{\xi_1 \xi_2}} :,$$

$$: \xi_1 \xi_2 := C_1^\# C_3 C_5^\# C_7 \cdot C_4^\# C_8,$$



$$\begin{aligned} &: \overline{\xi_1 \xi_2} := -C_1^\# C_3 C_7^\# C_8, \\ &: \overline{\overline{\xi_1 \xi_2}} := C_5^\# C_7. \end{aligned}$$

We have the following

*Lemma 4.1.* An element  $\xi \in \mathcal{A}$  is a pseudotwistor if and only if

$$: \xi \xi := 0, : \overline{\xi \xi} := 0, \dots, \overline{\xi \xi}^{(k-1)} := 0.$$

We are going to prove

*Theorem 4.2.* The pseudotwistor space of degree  $k$

$$\mathcal{J}^{(k)} = \{\xi \in \mathcal{A}^{(2k)} : \text{im } \xi^2 = 0\}, \quad k = 1, 2, 3, 4,$$

has a decomposition

$$\mathcal{J}^{(k)} = \mathcal{J}_-^{(k)} + \mathcal{J}_+^{(k)}$$

and introduces a flag structure, i.e.,

- (i)  $\mathcal{J}_-^{(k)} \subset G(2k - 1, 8)$  and  $\mathcal{J}_+^{(k)} \subset G(2k, 8)$  and
- (ii)  $\xi = \xi_- + \xi_+$  implies  $\xi_- \subset \xi_+$ .

*Proof.* The proof is divided into five steps.

**S t e p A.** We are going to choose a special basis of the algebra  $\mathcal{A}$  in question. We take

$$x_2 = C_1^\# C_2, \quad x_3 = C_1^\# C_3, \dots, \quad x_9 = C_1^\# C_9.$$

Then we arrive at the following basis  $\{x_\alpha x_\beta : 2 \leq \alpha \leq \beta \leq 9\}$  of  $\mathcal{A}$ :

$$x_2 x_3 = -\eta_{1,1} C_2^\# C_3, \quad x_2 x_4 = -\eta_{1,1} C_2^\# C_4, \quad \dots, \quad x_8 x_9 = -\eta_{1,1} C_8^\# C_9,$$

where  $\eta$  is the metric of  $\mathbb{R}^9(I_{\sigma,s})$ . An element  $x_k \in \mathcal{A}_{2k}$ ,  $k = 1, 2, 3, 4$ , can be written as

$$\begin{aligned} x_1 &= \sum_{\beta_1=2}^9 \xi_{1\beta_1} x_{\beta_1} + \sum_{2 \leq \alpha_1 < \beta_1} \xi_{\alpha_1 \beta_1} x_{\alpha_1} x_{\beta_1}, \\ x_2 &= \sum_{2 \leq \beta_1 < \alpha_1 < \beta_2} \xi_{1\beta_1 \alpha_2 \beta_2} x_{\beta_1} x_{\alpha_2} x_{\beta_2} + \sum_{2 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2} \xi_{\alpha_1 \beta_1 \alpha_2 \beta_2} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2}, \\ x_3 &= \sum_{2 \leq \beta_1 < \alpha_2 < \dots < \beta_3} \xi_{1\beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_3} x_{\beta_3} \\ &\quad + \sum_{2 \leq \alpha_1 < \beta_1 < \dots < \alpha_3 < \beta_3} \xi_{\alpha_1 \beta_1 \dots \alpha_3 \beta_3} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_3} x_{\beta_3}, \\ x_4 &= \xi_{12345678} x_2 x_3 \dots x_8 + \xi_{23\dots 9} x_2 x_3 \dots x_9. \end{aligned}$$

Let  $x_k = x_1^{(k)} + x_2^{(k)}$ ,  $k = 1, 2, 3, 4$ . Since the degree  $k$  is indicated, we denote it simply by  $x = x_1 + x_2$ .

Step B. From the above formulae we calculate directly:

$$x^2 =: x_1x_2 : + x_2x_2 :, \quad k = 1,$$

where

$$: x_1x_2 := \sum_{\beta_1, \alpha'_1, \beta'_1 \geq 2} \xi_{1\beta_1} \xi_{\alpha'_1\beta'_1} x_{\beta_1} x_{\alpha'_1} x_{\beta'_1} \quad (\text{as.} = \text{antisymmetric}),$$

$$: x_1x_2 := \sum_{\alpha_1, \beta_1, \alpha'_1, \beta'_1 \geq 2} \xi_{\alpha_1\beta_1} \xi_{\alpha'_1\beta'_1} x_{\alpha_1} x_{\beta_1} x_{\alpha'_1} x_{\beta'_1} \quad (\text{as.});$$

$$\text{im } x^2 =: \overline{x_1x_1} : + : \overline{x_2x_2} : + : \overline{x_2x_2} : + : x_1x_2 : + : x_2x_2 :,$$

where

$$: \overline{x_1x_1} := \sum_{\alpha_2, \beta_1, \beta_2, \alpha'_2, \beta'_2 \geq 2} \xi_{1\beta_1\alpha_2\beta_2} \xi_{1\beta'_1\alpha'_2\beta'_2} x_{\alpha_2} x_{\beta_2} x_{\alpha'_2} x_{\beta'_2} \quad (\text{as.}),$$

$$: \overline{x_1x_2} := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{1\beta_1\alpha_2\beta_2} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2} x_{\beta_2} x_{\alpha'_2} x_{\beta'_2} \quad (\text{as.}),$$

$$: \overline{x_2x_2} := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{\alpha_1\beta_1\alpha_2\beta_2} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2} x_{\alpha_2} x_{\beta_2} x_{\alpha'_2} x_{\beta'_2} \quad (\text{as.}),$$

$$: x_2x_2 := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{\alpha_1\beta_1\alpha_2\beta_2} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha'_1} x_{\beta'_1} x_{\alpha'_2} x_{\beta'_2} \quad (\text{as.}),$$

$$: x_1x_2 := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{1\beta_1\alpha_2\beta_2} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha'_1} x_{\beta'_1} x_{\alpha'_2} x_{\beta'_2} \quad (\text{as.});$$

$$\text{im } x^2 =: \overline{x_1x_1}^{(3)} : + : \overline{x_1x_2}^{(4)} : + : \overline{x_2x_2}^{(4)} :,$$

where

$$: \overline{x_1x_1}^{(3)} := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \xi_{1\beta'_1\alpha'_2\beta'_2\alpha'_3\beta'_3} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha'_2} x_{\beta'_2} x_{\alpha'_3} x_{\beta'_3} \quad (\text{as.}),$$

$$: \overline{x_1x_2}^{(4)} := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2\alpha_3\beta_3} x_{\beta_1} x_{\alpha'_1} x_{\beta'_1} \quad (\text{as.}),$$

$$: \overline{x_2x_2}^{(4)} := \sum_{\alpha_j, \alpha'_j; \beta_j, \beta'_j \geq 2} \xi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \xi_{\alpha'_1\beta'_1\alpha'_2\beta'_2\alpha_3\beta_3} x_{\alpha_1} x_{\beta_1} x_{\alpha'_1} x_{\beta'_1} \quad (\text{as.});$$

$$\text{im } x^2 = 0, \quad k = 4,$$

where

$$x = \xi_{12345678} x_2x_3 \dots x_8 + \xi_{23\dots 9} x_2x_3 \dots x_9.$$

Step C. We observe that  $\mathcal{J}^{(1)}$  determines a flag structure

$$M_1 = \{(L_1, L_2) : L_1 \subset L_2 \subset \mathbb{C}^8, \dim L_1 = 1, \dim L_2 = 2\}.$$

Indeed, let us write down the system of equations. We fix the indices  $\beta_1, \alpha'_1, \beta'_1$  and write down :  $x_1x_2 := 0$ . If we take  $(2, 3, 4)$ , for example, we obtain

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0.$$

The second equation :  $x_1x_2 := 0$  for  $(\alpha_1, \beta_1, \alpha'_1, \beta'_1) = (2, 3, 4, 5)$  implies

$$\xi_{23}\xi_{45} - \xi_{24}\xi_{35} + \xi_{25}\xi_{34} = 0.$$

Hence we see that  $(x_1)_{234} \subset (x_2)_{2345}$ . From the same discussion in the general case, we conclude that

$$\xi_- \subset \xi_+,$$

where  $\xi_- = \sum_{\beta' \geq 2} \xi_{1\beta'}x_{\beta'}$ ,  $\xi_+ = \sum_{2 \leq \alpha'_i, \beta'_i} \xi_{\alpha'_i\beta'_i}x_{\alpha'_i}x_{\beta'_i}$ , which proves the assertion of Step C.

S t e p D.  $\mathcal{J}^{(2)}$  determines a flag-structure

$$M_2 = \{(L_3, L_4) : L_3 \subset L_4 \subset \mathbb{C}^8, \dim L_3 = 3, \dim L_4 = 4\}.$$

Indeed, let

$$L_k(\alpha_1, \dots, \beta_l) = L_k \cap \{\alpha_1 = \dots = \beta_l = 0\}.$$

Then :  $x_1x_1 := 0$  implies that  $L_3(\beta_1)$  determines a 2-dimensional subspace in  $\mathbb{C}^8$  for a fixed  $\beta_1$ . Hence we have a family of 2-dimensional subspaces. By this we infer that :  $x_1x_1 := 0$  determines a 3-dimensional subspace  $L_3$  of  $\mathbb{C}^8$ . Hence :  $x_2x_2 := 0$  is the system of Plücker relations for 4-dimensional subspaces, so we have  $L_4 \subset \mathbb{C}^8$ . From :  $x_1x_2^{(2)} := 0$ , we conclude that

$$L_3(\beta_1, \alpha_2) \subset L_4(\alpha'_1, \beta'_1), \quad \beta_1 = \alpha'_1, \quad \alpha_2 = \beta'_1,$$

and hence  $L_3 \subset L_4$ . By this we obtain the desired correspondence.

S t e p E. A similar observation leads to the conclusion that  $\mathcal{J}^{(3)}$  determines a flag-structure

$$M_3 = \{(L_5, L_6) : L_5 \subset L_6 \subset \mathbb{C}^8, \dim L_5 = 5, \dim L_6 = 6\}.$$

Conclusions of Steps A-E suffice to complete the proof of the theorem.

*Remark.* We have not used in full the conditions to be pseudotwistors. In fact, we shall see more subtle structures of pseudotwistors, which will lead us to the "quaternary analysis" for the Hermitian Hurwitz pairs [9].

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## DIFFERENTIAL GEOMETRY OF CIRCLES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. It is well-known that all geodesics in a Riemannian symmetric space of rank one are congruent each other under the action of isometry group. In this paper we are interested in *circles* in a quaternionic projective space  $\mathbb{Q}P^n$ . Recently we have known that each circle in  $\mathbb{Q}P^n$  is congruent to a circle in  $\mathbb{C}P^n$  which is a totally geodesic submanifold of  $\mathbb{Q}P^n$ . This fact leads us to the study about circles in  $\mathbb{C}P^n$ .

### 1. INTRODUCTION

A smooth curve  $\gamma : \mathbb{R} \rightarrow M$  parametrized by its arclength  $s$  in a complete Riemannian manifold  $M$  is called a *circle* of curvature  $\kappa (\geq 0)$ , if there exists a field of unit vectors  $Y_s$  along the curve satisfying the following equations:  $\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa Y_s$  and  $\nabla_{\dot{\gamma}} Y_s = -\kappa \dot{\gamma}$ , where  $\kappa$  is a non-negative constant and  $\nabla_{\dot{\gamma}}$  denotes the covariant differentiation along  $\gamma$  with respect to the Riemannian connection  $\nabla$  of  $M$ . A circle of null curvature is nothing but a geodesic. For given a point  $x \in M$ , orthonormal pair of vectors  $u, v \in T_x M$  and for given each positive constant  $\kappa$ , we have a unique circle  $\gamma = \gamma(s)$  of curvature  $\kappa$  satisfying the initial condition that  $\gamma(0) = x$ ,  $\dot{\gamma}(0) = u$  and  $(\nabla_{\dot{\gamma}} \dot{\gamma})(0) = \kappa v$ . It is known that in a complete Riemannian manifold every circle can be defined for  $-\infty < s < \infty$  (cf. [N]).

In general, a circle in a Riemannian manifold is not closed. Here, a curve  $\gamma = \gamma(s)$  is said to be *closed* if there exists a positive  $s_0$  with  $\gamma(s + s_0) = \gamma(s)$  for every  $s$ . For a circle  $\gamma$ , the definition of closedness of  $\gamma$  can be rewritten as follows: A circle  $\gamma$  is said to be closed if there exists a positive  $s_0$  with

$$\gamma(s_0) = \gamma(0), \quad \dot{\gamma}(s_0) = \dot{\gamma}(0) \quad \text{and} \quad (\nabla_{\dot{\gamma}} \dot{\gamma})(s_0) = (\nabla_{\dot{\gamma}} \dot{\gamma})(0).$$

Of course, any circles of positive curvature in Euclidean  $n$ -space  $\mathbb{R}^n$  are closed. And also any circles in Euclidean  $n$ -sphere  $S^n(c)$  are closed. But in a real hyperbolic space  $n$ -space  $H^n(c)$ , there exist many open circles. In fact, a circle of curvature  $\kappa$  is closed if and only if  $\kappa > \sqrt{|c|}$  (see [C]).

In this paper we make mention of length of circles. For a closed curve  $\gamma$ , we call the minimum positive constant  $s_0$  with the condition  $\gamma(s + s_0) = \gamma(s)$  for every  $s$  its

length, and denote by  $\text{Length}(\gamma)$ . For an open circle, a circle which is not closed, we put its length as  $\text{Length}(\gamma) = \infty$ . In order to get rid of the influence of the action of the full isometry group, we shall consider the moduli space of circles under the action of isometries. We say that two circles  $\gamma_1$  and  $\gamma_2$  are congruent each other if there exist an isometry  $\varphi$  and a constant  $s_1$  with  $\gamma_2(s) = \varphi \circ \gamma_1(s + s_1)$  for each  $s$ . The moduli space  $\text{Cir}(M)$  of circles is the quotient space of the set of all circles in  $M$  under this congruence relation. The *length spectrum* of circles in  $M$  is the map  $\mathfrak{L} : \text{Cir}(M) \rightarrow \mathbb{R} \cup \{\infty\}$  defined by  $\mathfrak{L}([\gamma]) = \text{Length}(\gamma)$ . Sometimes we also call the image  $\text{LSpec}(M) = \mathfrak{L}(\text{Cir}(M)) \cap \mathbb{R}$  in the real line the length spectrum of circles on  $M$ .

In a real space form  $M^n(c)(= S^n(c), \mathbb{R}^n$  or  $H^n(c))$  of constant sectional curvature  $c$ , circles are well-understood. In these spaces, two circles are congruent each other if and only if they have the same curvature. If the curvature of a circle is  $\kappa$ , then its length is  $\frac{2\pi}{\sqrt{\kappa^2+c}}$  in  $S^n(c)$ ,  $\frac{2\pi}{\kappa}$  in  $\mathbb{R}^n$  and  $\frac{2\pi}{\sqrt{\kappa^2+c}}$  when  $\kappa > \sqrt{|c|}$  in  $H^n(c)$ . Therefore length spectrum of these spaces are  $\text{LSpec}(S^n(c)) = \left(0, \frac{2\pi}{\sqrt{c}}\right]$ ,  $\text{LSpec}(\mathbb{R}^n) = \text{LSpec}(H^n(c)) = (0, \infty)$ . So we treat an  $n$ -dimensional complex projective space  $\mathbb{C}P^n(c)$  of constant holomorphic sectional curvature  $c$  and a quaternionic projective space  $\mathbb{Q}P^n(c)$  of constant quaternionic sectional curvature  $c$  as model spaces. We are particularly interested in the following problem:

**Problem** In a complex projective space  $\mathbb{C}P^n(c)$  (resp. a quaternionic projective space  $\mathbb{Q}P^n(c)$ ), for each positive  $\ell$  does there exist a unique closed circle  $\gamma$  whose length is  $\ell$  up to an isometry of  $\mathbb{C}P^n(c)$  (resp.  $\mathbb{Q}P^n(c)$ )?

In order to give an answer to this problem, we shall study the length spectrum of circles in  $\mathbb{C}P^n(c)$  in detail (see section 4).

## 2. CONGRUENCE THEOREM FOR CIRCLES

In order to state the congruence theorem for circles in a complex projective space, we introduce an important invariant for circles in a Kähler manifold. Let  $(M, J)$  be a Kähler manifold with complex structure  $J$ . For a circle  $\gamma = \gamma(s)$  in  $M$  satisfying the equations  $\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa Y_s$  and  $\nabla_{\dot{\gamma}}Y_s = -\kappa\dot{\gamma}$ , we call  $\tau = \langle \dot{\gamma}, JY_s \rangle$  its *complex torsion*. The complex torsion  $\tau$  is constant along  $\gamma$ . Indeed,

$$\begin{aligned} \nabla_{\dot{\gamma}}\langle \dot{\gamma}, JY_s \rangle &= \langle \nabla_{\dot{\gamma}}\dot{\gamma}, JY_s \rangle + \langle \dot{\gamma}, J\nabla_{\dot{\gamma}}Y_s \rangle \\ &= \kappa \cdot \langle Y_s, JY_s \rangle - \kappa \cdot \langle \dot{\gamma}, J\dot{\gamma} \rangle = 0. \end{aligned}$$

Clearly it satisfies  $|\tau| \leq 1$ . We denote by  $M_n(c)$  an  $n$ -dimensional complete simply connected complex space form of constant holomorphic sectional curvature  $c$ . It is well-known that any isometry  $\varphi$  of a non-flat complex space form  $M_n(c)(= \mathbb{C}P^n(c)$  or  $\mathbb{C}H^n(c))$  is holomorphic or anti-holomorphic. The congruence theorem for circles in  $M_n(c)$ ,  $c \neq 0$  is stated as follows (see Theorem 5.1 in [MO]):

*Proposition 2.1.* Two circles in a non-flat complex space form  $M_n(c)$  are congruent if and only if they have the same curvatures and the same absolute values of complex torsions.

For a circle  $\gamma$  in a quaternionic Kähler manifold  $(M, \{I, J, K\})$  with quaternionic Kähler structure  $\{I, J, K\}$ , the corresponding invariant *structure torsion*  $\tau$  is defined by

$$\tau = \sqrt{\langle \dot{\gamma}, IY_s \rangle^2 + \langle \dot{\gamma}, JY_s \rangle^2 + \langle \dot{\gamma}, KY_s \rangle^2}.$$

On a quaternionic projective space and on a quaternionic hyperbolic space, this invariant can be interpreted in terms of the sectional curvature  $\text{Riem}(\dot{\gamma}, Y)$  of the plane spanned by  $\dot{\gamma}$  and  $Y$ :  $\text{Riem}(\dot{\gamma}, Y) = \frac{c}{4}(1 + 3\tau^2)$ , where  $c$  is the quaternionic sectional curvature of the base manifold.

*Proposition 2.2.* Two circles in a quaternionic projective space or in a quaternionic hyperbolic space are congruent if and only if they have the same curvatures and the same structure torsions.

Since a quaternionic projective (resp. hyperbolic) space contains a complex projective (resp. hyperbolic) space as a totally geodesic submanifold, we are enough to study circles in a complex space form (c.f. [A1]). For a circle  $\gamma$  in a Cayley plane and in a Cayley hyperbolic plane we can define its invariant by  $\text{Riem}(\dot{\gamma}, Y)$  and obtain congruence theorem of the same type (see [MT]). In the following, we only study on a complex projective space  $\mathbb{C}P^n(c)$ . But all the results similarly hold for a quaternionic projective space  $\mathbb{Q}P^n(c)$  of constant quaternionic sectional curvature  $c$  and for a Cayley plane of maximal sectional curvature  $c$ .

### 3. When is a circle closed in $\mathbb{C}P^n(c)$ ?

We first suppose that a complex projective space  $\mathbb{C}P^n$  is furnished with the standard metric of constant holomorphic sectional curvature 4. First of all we are devoted to the study about circles of curvature  $\frac{1}{\sqrt{2}}$ .

Our main tool is the following parallel isometric imbedding  $h$  of  $S^1 \times S^{n-1}/\phi$  into  $\mathbb{C}P^n(4)$ . Here the identification  $\phi$  is defined by

$$\phi((e^{i\theta}, a_1, \dots, a_n)) = (-e^{i\theta}, -a_1, \dots, -a_n),$$

where  $\sum a_j^2 = 1$ . The isometric imbedding  $h : S^1 \times S^{n-1}/\phi \rightarrow \mathbb{C}P^n(4)$  is defined by

$$h(e^{i\theta}; a_1, \dots, a_n) = \pi \left( \begin{pmatrix} \frac{1}{3}(e^{-2i\theta/3} + 2a_1e^{i\theta/3}) \\ \frac{\sqrt{2}}{3}(e^{-2i\theta/3} - a_1e^{i\theta/3}) \\ \frac{2}{\sqrt{6}}ia_2e^{i\theta/3} \\ \vdots \\ \frac{2}{\sqrt{6}}ia_n e^{i\theta/3} \end{pmatrix} \right),$$

where  $\pi : S^{2n+1}(1) \rightarrow \mathbb{C}P^n(4)$  is the Hopf fibration.

We recall that the map  $h$  is injective and that for each geodesic  $\gamma$  on  $M = S^1 \times S^{n-1}/\phi$  the curve  $h \circ \gamma$  is a circle of  $\frac{1}{\sqrt{2}}$  in  $\mathbb{C}P^n(4)$  (for details, see [N]). Hence, investigating all geodesics on  $M$ , we obtain the following theorem which gives us information about all circles of curvature  $\frac{1}{\sqrt{2}}$  in  $\mathbb{C}P^n$ .

*Theorem 2.3.* For any unit vector  $X = \alpha u + v \in T_x(S^1 \times S^{n-1}/\phi) \simeq T_{x_1}S^1 \oplus T_{x_2}S^{n-1}$  at a point  $x$ , we denote by  $\gamma_X$  the geodesic along  $X$  on  $S^1 \times S^{n-1}/\phi$ . Then the circle  $h \circ \gamma_X$  on  $\mathbb{C}P^n(4)$  satisfies the following properties:

1. The curvature of  $h \circ \gamma_X$  is  $\frac{1}{\sqrt{2}}$ .
2. The complex torsion of  $h \circ \gamma_X$  is  $4\alpha^3 - 3\alpha$  for  $-1 \leq \alpha \leq 1$ .
3. The circle  $h \circ \gamma_X$  is closed if and only if either  $\alpha = 0$  or  $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$  is rational.
4. When  $\alpha = 0$ , the length of the closed circle is  $\frac{2\sqrt{6}}{3}\pi$ .
5. When  $\alpha \neq 0$  and  $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$  is rational, we denote by  $\frac{p}{q}$  the irreducible fraction defined by  $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$ . Then the length  $\ell$  of a closed circle  $h \circ \gamma_X$  is as follows;
6. When  $pq$  is even,  $\ell$  is the least common multiple of  $\frac{2\sqrt{2}}{3|\alpha|}\pi$  and  $\frac{2\sqrt{2}}{\sqrt{3(1-\alpha^2)}}\pi$ . In particular, when  $\alpha = \pm 1$ , then  $\ell = \frac{2\sqrt{2}}{3}\pi$ .
7. When  $pq$  is odd,  $\ell$  is the least common multiple of  $\frac{\sqrt{2}}{3|\alpha|}\pi$  and  $\frac{\sqrt{2}}{\sqrt{3(1-\alpha^2)}}\pi$ .

Next, we prepare the following in order to consider circles of arbitrary positive curvature. Let  $N$  be the outward unit normal on  $S^{2n+1}(1) (\subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1})$ . We here mix the complex structures of  $\mathbb{C}^{n+1}$  and  $\mathbb{C}P^n(4)$ . We shall study circles in  $\mathbb{C}P^n(4)$  by making use of the Hopf fibration  $\pi : S^{2n+1}(1) \rightarrow \mathbb{C}P^n(4)$ . For the sake of simplicity we identify a vector field  $X$  on  $\mathbb{C}P^n(4)$  with its horizontal lift  $X^*$  on  $S^{2n+1}(1)$ . Then the relation between the Riemannian connection  $\nabla$  of  $\mathbb{C}P^n(4)$  and the Riemannian connection  $\tilde{\nabla}$  of  $S^{2n+1}(1)$  is as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle X, JY \rangle JN$$

for any vector fields  $X$  and  $Y$  on  $\mathbb{C}P^n(4)$ , where  $\langle \cdot, \cdot \rangle$  is the natural metric of  $\mathbb{C}^{n+1}$ . By using this relation we can see that for each circle  $\gamma$  of positive curvature any horizontal lift  $\tilde{\gamma}$  of  $\gamma$  in  $S^{2n+1}(1)$  is a helix in  $S^{2n+1}(1)$ .

*Proposition 2.4.* Let  $\gamma$  denote a circle with curvature  $\kappa (> 0)$  and complex torsion  $\tau$  in  $\mathbb{C}P^n(4)$  satisfying that  $\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa Y_s$  and  $\nabla_{\dot{\gamma}} Y_s = -\kappa \dot{\gamma}$ . Then a horizontal lift  $\tilde{\gamma}$  of  $\gamma$  in  $S^{2n+1}(1)$  is a helix of order 2, 3 or 5 corresponding to  $\tau = 0, \tau = \pm 1$  or  $\tau \neq 0, \pm 1$ ,



respectively. Moreover, it satisfies the following differential equations:

$$\begin{cases} \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa Y_s, \\ \tilde{\nabla}_{\dot{\gamma}}Y_s = -\kappa\dot{\gamma} + \tau JN, \\ \tilde{\nabla}_{\dot{\gamma}}(JN) = -\tau Y_s + \sqrt{1-\tau^2}Z_s, \\ \tilde{\nabla}_{\dot{\gamma}}Z_s = -\sqrt{1-\tau^2}JN + \kappa W_s, \\ \tilde{\nabla}_{\dot{\gamma}}W_s = -\kappa Z_s, \end{cases}$$

where  $Z_s = \frac{1}{\sqrt{1-\tau^2}}(J\dot{\gamma} + \tau Y_s)$ ,  $W_s = \frac{1}{\sqrt{1-\tau^2}}(JY_s - \tau\dot{\gamma})$ .

Note that a curve  $\gamma = \gamma(s)$  in  $\mathbb{C}P^n(4)$  is closed if and only if there exists a positive constant  $s_*$  such that a horizontal lift  $\tilde{\gamma} = \tilde{\gamma}(s)$  of  $\gamma$  in  $S^{2n+1}(1)$  satisfies  $\tilde{\gamma}(s + s_*) = e^{i\theta_s}\tilde{\gamma}(s)$  with some  $\theta_s \in [0, 2\pi)$  for every  $s$ . Then by studying a horizontal lift  $\tilde{\gamma}$  of a circle  $\gamma$  in  $\mathbb{C}P^n(4)$  we establish the following.

*Theorem 2.5.* Let  $\gamma$  be a circle of curvature  $\kappa(> 0)$  and of complex torsion  $\tau$  in a complex projective space  $\mathbb{C}P^n(4)$ . Then the following hold:

1. When  $\tau = 0$ , a circle  $\gamma$  is a simple closed curve with length  $\frac{2\pi}{\sqrt{\kappa^2+1}}$ .
2. When  $\tau = \pm 1$ , a circle  $\gamma$  is a simple closed curve with length  $\frac{2\pi}{\sqrt{\kappa^2+4}}$ .
3. When  $\tau \neq 0, \pm 1$ , we denote by  $a, b$  and  $d$  ( $a < b < d$ ) the nonzero solutions for  $\lambda^3 - (\kappa^2 + 1)\lambda + \kappa\tau = 0$ .

Then we find the following:

4. If one of (hence all of) the three ratios  $\frac{a}{b}$ ,  $\frac{b}{d}$  and  $\frac{d}{a}$  is rational, then  $\gamma$  is a simple closed curve. Its length is the least common multiple of  $\frac{2\pi}{b-a}$  and  $\frac{2\pi}{d-a}$ .
5. If each of the three ratios  $\frac{a}{b}$ ,  $\frac{b}{d}$  and  $\frac{d}{a}$  is irrational, then  $\gamma$  is a simple open curve.

Let  $\gamma$  be a circle of curvature  $\kappa$  in a Riemannian manifold  $(M, g)$ . When we change the metric  $g$  homothetically to  $m^2 \cdot g$  for some positive constant  $m$ , the curve  $\sigma(s) = \gamma(\frac{s}{m})$  is a circle of curvature  $\frac{\kappa}{m}$  in  $(M, m^2 \cdot g)$ . Under the operation  $g \rightarrow m^2 \cdot g$ , the length of a closed curve changes to  $m$ -times of the original length. Hence, by virtue of Theorem 3.3 we can conclude the following which is the main result in this section.

*Theorem 2.6.* Let  $\gamma$  be a circle with curvature  $\kappa(> 0)$  and with complex torsion  $\tau$  in a complex projective space  $\mathbb{C}P^n(c)$  of constant holomorphic sectional curvature  $c$ . Then the following hold:

1. When  $\tau = 0$ , a circle  $\gamma$  is a simple closed curve with length  $\frac{4\pi}{\sqrt{4\kappa^2+c}}$ .
2. When  $\tau = \pm 1$ , a circle  $\gamma$  is a simple closed curve with length  $\frac{2\pi}{\sqrt{\kappa^2+c}}$ .
3. When  $\tau \neq 0, \pm 1$ , we denote by  $a, b$  and  $d$  ( $a < b < d$ ) the nonzero solutions for  $c\lambda^3 - (4\kappa^2 + c)\lambda + 2\sqrt{c}\kappa\tau = 0$ .

Then we find the following:

4. If one of (hence all of) the three ratios  $\frac{a}{b}$ ,  $\frac{b}{d}$  and  $\frac{d}{a}$  is rational,  $\gamma$  is a simple closed curve. Its length is the least common multiple of  $\frac{4\pi}{\sqrt{c(b-a)}}$  and  $\frac{4\pi}{\sqrt{c(d-a)}}$ .
5. If each of the three ratios  $\frac{a}{b}$ ,  $\frac{b}{d}$  and  $\frac{d}{a}$  is irrational,  $\gamma$  is a simple open curve.

**Remarks.** A circle  $\gamma = \gamma(s)$  with complex torsion  $\tau$  is a plane curve in  $\mathbb{C}P^n(c)$  (that is,  $\gamma$  is locally contained on some real 2-dimensional totally geodesic submanifold of  $\mathbb{C}P^n(c)$ ) if and only if  $\tau = 0$  or  $\tau = \pm 1$ .

1. When  $\tau = 0$ , the circle  $\gamma$  lies on  $\mathbb{R}P^2(\frac{c}{4})$  which is a totally real totally geodesic submanifold of  $\mathbb{C}P^n(c)$ .
2. When  $\tau = 1$  or  $-1$ , the circle  $\gamma$  lies on  $\mathbb{C}P^1(c)$  which is a holomorphic totally geodesic submanifold of  $\mathbb{C}P^n(c)$ .

Circles of complex torsion  $\pm 1$  are called *holomorphic circles*, and circles of null complex torsion are called *totally real circles*.

### 3. LENGTH SPECTRUM OF CIRCLES IN $\mathbb{C}P^n(c)$

In this section, we study the length spectrum of circles in  $\mathbb{C}P^n(c)$ . For a spectrum  $\lambda \in \text{LSpec}(M)$  the cardinality  $m_M(\lambda)$  of the set  $\mathfrak{L}^{-1}(\lambda)$  is called the *multiplicity* of the length spectrum  $\mathfrak{L}$  at  $\lambda$ . When  $m_M(\lambda) = 1$ , we say that  $\lambda$  is *simple*. For example, every length spectrum of circles in a real space form is simple. When the multiplicity of  $\mathfrak{L}$  is greater than one at some point  $\lambda$ , this means that we can find circles which are not congruent each other but have the same length  $\lambda$ .

Rewriting Theorem 3.1, we find the following which is our main tool in this section.

*Proposition 3.1.* In  $\mathbb{C}P^n(c)$  a circle  $\gamma$  of curvature  $\frac{\sqrt{2c}}{4}$  and complex torsion  $\tau = 3\alpha - 4\alpha^3$  ( $0 < |\alpha| < \frac{1}{2}$ ) is closed if and only if  $\sqrt{\frac{1-\alpha^2}{3\alpha^2}}$  is rational. In this case if we denote  $\sqrt{\frac{1-\alpha^2}{3\alpha^2}} = \frac{p}{q}$  by relatively prime positive integers  $p$  and  $q$ , then its length is

$$\text{Length}(\gamma) = \begin{cases} \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2 + q^2)}, & \text{if } pq \text{ is even,} \\ \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2 + q^2)}, & \text{if } pq \text{ is odd.} \end{cases}$$

We denote by  $[\gamma_{\kappa,\tau}]$  the congruency class of circles of curvature  $\kappa$  and complex torsion  $\tau (\geq 0)$  in  $\mathbb{C}P^n(c)$ . The moduli space of circles have a natural stratification by their curvatures. We denote by  $\text{Cir}_\kappa(M)$  the moduli space of circles of curvature  $\kappa$  in  $M$  and by  $\mathfrak{L}_\kappa$  the restriction of  $\mathfrak{L}$  on this space.

For a positive constant  $\kappa$  we define a canonical transformation

$$\Phi_\kappa : \text{Cir}_\kappa(\mathbb{C}P^n(c)) \setminus \{[\gamma_{\kappa,1}]\} \rightarrow \text{Cir}_{\sqrt{2c}/4}(\mathbb{C}P^n(c)) \setminus \{[\gamma_{\sqrt{2c}/4,1}]\}$$

by

$$\Phi_\kappa([\gamma_{\kappa,\tau}]) = [\gamma_{\sqrt{2c}/4, 3\sqrt{3c}\kappa\tau(4\kappa^2+c)^{-3/2}}].$$

The following lemma guarantees that the structure of the length spectrum  $\mathfrak{L}_\kappa$  of circles of curvature  $\kappa$  essentially does not depend on  $\kappa$ .

*Lemma 3.2.* The canonical transformation  $\Phi_\kappa$  satisfies

$$\mathfrak{L}([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3c}{2(4\kappa^2 + c)}} \cdot \mathfrak{L}(\Phi_\kappa([\gamma_{\kappa,\tau}]))$$

for every  $\tau$  ( $0 \leq \tau < 1$ ).

We denote by  $\text{LSpec}_\kappa(M) = \mathfrak{L}(\text{Cir}_\kappa(M)) \cap \mathbb{R}$  the length spectrum of circles of curvature  $\kappa$  in  $M$ . This lemma yields that

$$\begin{aligned} \text{LSpec}_\kappa(\mathbb{C}P^n(c)) = & \left\{ \frac{2\pi}{\sqrt{\kappa^2 + c}}, \frac{4\pi}{\sqrt{4\kappa^2 + c}} \right\} \\ & \cup \left\{ \frac{4\pi\sqrt{3p^2 + q^2}}{3(4\kappa^2 + c)} \mid \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is even and } p > \alpha_\kappa q > 0 \end{array} \right\}, \\ & \cup \left\{ \frac{2\pi\sqrt{3p^2 + q^2}}{3(4\kappa^2 + c)} \mid \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is odd and } p > \alpha_\kappa q > 0 \end{array} \right\}, \end{aligned}$$

where  $\alpha_\kappa$  ( $\geq 1$ ) denotes the number with

$$\frac{3\sqrt{3c}\kappa}{(4\kappa^2 + c)^{3/2}} = \frac{9\alpha_\kappa^2 - 1}{(3\alpha_\kappa^2 + 1)^{3/2}}.$$

Note that the constant  $\alpha_\kappa$  satisfies

1.  $\alpha_{\sqrt{2c}/4} = 1$ ,
2. monotone decreasing when  $0 < \kappa \leq \frac{\sqrt{2c}}{4}$ , and monotone increasing when  $\kappa \geq \frac{\sqrt{2c}}{4}$ ,
3.  $\lim_{\kappa \rightarrow 0} \alpha_\kappa = \lim_{\kappa \rightarrow \infty} \alpha_\kappa = \infty$ .

Lemma 4.2 also guarantees that

$$\text{LSpec}(\mathbb{C}P^n(c)) = \left(0, \frac{4\pi}{\sqrt{c}}\right) \cup \bigcup \left\{ I_{p,q} \mid \begin{array}{l} p > q, p \text{ and } q \text{ are relatively} \\ \text{prime positive integers} \end{array} \right\},$$

where

$$I_{p,q} = \begin{cases} \left( \frac{4\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{4\pi}{3\sqrt{c}}\sqrt{9p^2 - q^2} \right), & \text{if } pq \text{ is even,} \\ \left( \frac{2\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{2\pi}{3\sqrt{c}}\sqrt{9p^2 - q^2} \right), & \text{if } pq \text{ is odd.} \end{cases}$$

We denote by  $\text{Cir}^\tau(M)$  the moduli space of circles with complex torsion  $\tau$  in a Kähler manifold  $M$  by  $\mathfrak{L}^\tau$  the restriction of  $\mathfrak{L}$  onto this space. From these expressions on length spectrum of circles we establish the following main result.

*Theorem 3.3.* For a complex projective space  $\mathbb{C}P^n(c)$  ( $n \geq 2$ ) of constant holomorphic sectional curvature  $c$ , the length spectrum of circles has the following properties.

1. Both the sets

$$\text{LSpec}_\kappa(\mathbb{C}P^n(c)) = \mathcal{L}(\text{Cir}_\kappa(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

and

$$\text{LSpec}^\tau(\mathbb{C}P^n(c)) = \mathcal{L}(\text{Cir}^\tau(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

are unbounded discrete subsets of  $\mathbb{R}$  for each  $\kappa(> 0)$  and  $0 < \tau < 1$ .

2. The length spectrum  $\text{LSpec}(\mathbb{C}P^n(c))$  of circles coincides with the real positive line  $(0, \infty)$ .
3. For  $\kappa > 0$  the bottom of  $\text{LSpec}_\kappa(\mathbb{C}P^n(c))$  is  $\frac{2\pi}{\sqrt{\kappa^2+c}}$ , which is the length of the holomorphic circle of curvature  $\kappa$ . The second lowest spectrum of  $\text{LSpec}_\kappa(\mathbb{C}P^n(c))$  is  $\frac{4\pi}{\sqrt{4\kappa^2+c}}$ , which is the length of the totally real circle of curvature  $\kappa$ . They are simple for  $\mathcal{L}_\kappa$ .
4. The multiplicity of  $\mathcal{L}$  is finite at each point  $\lambda \in \mathbb{R}$ .
5.  $\lambda(\in \mathbb{R})$  is simple for  $\mathcal{L}$  if and only if  $\lambda \in \left( \frac{2}{\sqrt{c}}\pi, \frac{4}{3}\sqrt{\frac{5}{c}}\pi \right]$ .
6. The multiplicity of  $\mathcal{L}_\kappa$  ( $\kappa > 0$ ) is not uniformly bounded;

$$\limsup_{\lambda \rightarrow \infty} \#(\mathcal{L}_\kappa^{-1}(\lambda)) = \infty.$$

The growth order of the multiplicity with respect to  $\lambda$  is not so rapid. It satisfies  $\lim_{\lambda \rightarrow \infty} \lambda^{-\delta} \#(\mathcal{L}_\kappa^{-1}(\lambda)) = 0$  for arbitrary positive  $\delta$ .

The statements (2) and (5) in our theorem give the complete answer to the problem in the introduction.

**Remark** It follows from Proposition 4.1 that a circle of curvature  $\frac{\sqrt{2c}}{4}$  and complex torsion  $\tau$  in  $\mathbb{C}P^n(c)$  is closed if and only if

$$\tau = \tau(p, q) = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2},$$

for some relatively prime positive integers  $p$  and  $q$  with  $p > q$ . We find that the length spectrum  $\mathcal{L}_{\frac{\sqrt{2c}}{4}}$  is not simple at the following points for examples.

1. Let  $\gamma_1$  be a circle of curvature  $\frac{\sqrt{2c}}{4}$  and complex torsion  $\tau = \tau(27, 7) = \frac{5698}{559\sqrt{559}}$  and  $\gamma_2$  be a circle of curvature  $\frac{\sqrt{2c}}{4}$  and complex torsion  $\tau = \tau(25, 19) = \frac{12502}{559\sqrt{559}}$ . These two closed circles have the same curvature and the same length  $\frac{4\sqrt{1118}}{3\sqrt{c}}\pi$ . But they are not congruent.
2. Let  $\gamma_i$  be a circle of the same curvature  $\frac{\sqrt{2c}}{4}$  and complex torsion  $\tau_i = \tau(p_i, q_i)$ ,  $i = 1, 2, 3$ . Here we set  $(p_1, q_1) = (129, 71)$ ,  $(p_2, q_2) = (131, 59)$  and  $(p_3, q_3) = (135, 17)$ . Note that  $3p_i^2 + q_i^2 = 54964$  for  $i = 1, 2, 3$ . Then these three circles have the same curvature and the same length. But these three circles are not congruent each other.

Finally we investigate the asymptotic behaviour of the number of congruency classes of closed circles of curvature  $\kappa$ . Let  $n_M(\lambda; \kappa)$  denotes the number of congruency classes of closed circles of curvature  $\kappa$  in  $M$  with length not greater than  $\lambda$ .

*Theorem 3.4.* For a complex projective space  $\mathbb{C}P^n(c)$  ( $n \geq 2$ ) of constant holomorphic sectional curvature  $c$ , we have for  $\kappa > 0$

$$\lim_{\lambda \rightarrow \infty} \frac{n_{\mathbb{C}P^n(c)}(\lambda; \kappa)}{\lambda^2} = \frac{3\sqrt{3}(4\kappa^2 + c)}{8\pi^4} \tan^{-1} \left( \frac{1}{\sqrt{3}\alpha_\kappa} \right),$$

where  $\alpha_\kappa (\geq 1)$  denotes the number with

$$\frac{3\sqrt{3}\kappa}{(4\kappa^2 + c)^{3/2}} = \frac{9\alpha_\kappa^2 - 1}{(3\alpha_\kappa^2 + 1)^{3/2}}.$$

In particular,

$$\lim_{\lambda \rightarrow \infty} \frac{n_{\mathbb{C}P^n(c)}(\lambda; \sqrt{2c}/4)}{\lambda^2} = \frac{3\sqrt{3}c}{32\pi^3}.$$

**Sketch of the proof.** For a positive integer  $d$ , we put  $n_\alpha(\lambda)$  and  $k_\alpha(\lambda; d)$  the cardinalities of the sets

$$\left\{ (p, q) \in \mathbb{Z} \times \mathbb{Z} \mid \begin{array}{l} p \text{ and } q \text{ are relatively prime integers with} \\ 3p^2 + q^2 \leq \lambda^2 \text{ and } p > \alpha q > 0 \end{array} \right\}$$

and

$$K_\alpha(\lambda; d) = \{ (p, q) \in d\mathbb{Z} \times d\mathbb{Z} \mid 3p^2 + q^2 \leq \lambda^2, p > \alpha q > 0 \},$$

respectively. Here  $d\mathbb{Z}$  denotes the set  $\{dj \mid j \in \mathbb{Z}\}$ . Since the correspondence  $(p, q) \mapsto (dp, dq)$  of  $K_\alpha(\lambda/d; 1)$  to  $K_\alpha(\lambda; d)$  is bijective, we find the following relation between  $n_\alpha(\lambda)$  and  $k_\alpha(\lambda; 1)$  by using the Möbius function  $\mu$ ;

$$n_\alpha(\lambda) = \sum_{d \geq 1} \mu(d) k_\alpha(\lambda; d) = \sum_{1 \leq d \leq [\lambda/2]} \mu(d) k_\alpha(\lambda/d; 1),$$

where  $[\delta]$  denotes the integer part of a real number  $\delta$ . Put

$$C = \frac{1}{2\sqrt{3}} \tan^{-1} \left( \frac{1}{\sqrt{3}\alpha} \right),$$

which is the area of the set  $\{(x, y) \in \mathbb{R}^2 \mid 3x^2 + y^2 \leq \lambda^2, x \geq \alpha y \geq 0\}$ . One can easily find positive constants  $C_1, C_2$  with  $|k_\alpha(\lambda; 1) - C\lambda^2| < C_1\lambda + C_2$ . Thus we obtain

$$\lim_{\lambda \rightarrow \infty} \frac{n_\alpha(\lambda)}{\lambda^2} = C \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{C}{\zeta(2)} = \frac{6C}{\pi^2},$$

where  $\zeta$  denotes the Riemann zeta function.

We now put  $n_\alpha^o(\lambda)$  and  $n_\alpha^e(\lambda)$  the cardinalities of the sets

$$\left\{ (p, q) \in \mathbb{Z} \times \mathbb{Z} \mid \begin{array}{l} p \text{ and } q \text{ are relatively prime integers which satisfy} \\ pq \text{ is odd, } 3p^2 + q^2 \leq \lambda^2 \text{ and } p > \alpha q > 0 \end{array} \right\}$$

and

$$\left\{ (p, q) \in \mathbb{Z} \times \mathbb{Z} \mid \begin{array}{l} p \text{ and } q \text{ are relatively prime integers which satisfy} \\ pq \text{ is even, } \alpha p^2 + \beta q^2 \leq \lambda^2 \text{ and } p > \alpha q > 0 \end{array} \right\},$$

respectively. By similar argument we obtain

$$\lim_{\lambda \rightarrow \infty} \frac{n_{\alpha}^o(\lambda)}{\lambda^2} = \frac{C}{4} \sum_{\substack{1 \leq d < \infty, \\ d \text{ is odd}}} \frac{\mu(d)}{d^2} = \frac{2C}{\pi^2},$$

and

$$\lim_{\lambda \rightarrow \infty} \frac{n_{\alpha}^e(\lambda)}{\lambda^2} = \lim_{\lambda \rightarrow \infty} \left( \frac{n_{\alpha, \beta}(\lambda)}{\lambda^2} - \frac{n_{\alpha, \beta}^o(\lambda)}{\lambda^2} \right) = \frac{4C}{\pi^2}.$$

Since we have

$$n_{\mathbb{C}P^n(c)}(\lambda; \kappa) = 2 + n_{\alpha_{\kappa}}^o \left( \frac{\sqrt{4\kappa^2 + c}}{2\sqrt{3}\pi} \lambda \right) + n_{\alpha_{\kappa}}^e \left( \frac{\sqrt{4\kappa^2 + c}}{4\sqrt{3}\pi} \lambda \right)$$

for  $\lambda > \frac{4\pi}{\sqrt{4\kappa^2 + c}}$ , we obtain the conclusion. □

**Remark.** The constant  $c(\kappa) = \lim_{\lambda \rightarrow \infty} \lambda^{-2} n_{\mathbb{C}P^n(c)}(\lambda; \kappa)$  satisfies

$$\lim_{\kappa \rightarrow 0} c(\kappa) = 0 \quad \text{and} \quad \lim_{\kappa \rightarrow \infty} c(\kappa) = \frac{9c}{16\pi^4}.$$

We finally pose some problems on length spectrum of circles.

**Problems.**

1. Are there non-simple spectrum for  $\mathfrak{L}^{\tau}$  ( $0 < \tau < 1$ )?
2. Whether is the multiplicity of  $\mathfrak{L}^{\tau}$  ( $0 < \tau < 1$ ) uniformly bounded or not?
3. Give an explicit formula of the first spectrum for  $\mathfrak{L}^{\tau}$  ( $0 < \tau < 1$ ).
4. Study the asymptotic behaviour of the number of congruency classes of closed circles of complex torsion  $\tau (\neq 0, 1)$  with respect to length.
5. Find nice properties for the multiplicity  $m_{\mathbb{C}P^n(c)}(\lambda)$  of the full length spectrum  $\mathfrak{L}$ . For a complex hyperbolic space, it is monotone increasing left continuous function with polynomial growth and its jumping step is not uniformly bounded (see [A2]).
6. Study the behaviour of  $c(\kappa)$ . What is the maximum value of this function  $c(\kappa)$ ?
7. Study the geometric meaning of the constant  $\lim_{\kappa \rightarrow \infty} c(\kappa)$ .

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## ON SPECIAL 4-PLANAR MAPPINGS OF ALMOST HERMITIAN QUATERNIONIC SPACES

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ABSTRACT. In the paper special 4-planar mappings of almost Hermitian quaternionic spaces are studied. Fundamental equations of these mappings are expressed in linear Cauchy form. Our results improve results of I.N. Kurbatova [9].

4-quasiplanar mappings of an almost quaternionic space have been studied in [5], [9] and [14]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces, see [4], [12], [13], [15], [17], [18], [19]. Similar problems are studied on complex manifolds in [3]. Anti-quaternionic spaces which were studied e.g. in [11], [16] have some properties similar to those of quaternions [1]. This fact can be used in the study of 4-planar mappings.

I.N. Kurbatova studied a special kind of 4-planar mappings (called 4-quasiplanar, see [9]) from a Riemannian space  $V_n$  onto another Riemannian space  $\bar{V}_n$  where an almost quaternionic structure on  $V_n$  is Hermitian and it satisfies additional conditions so that  $V_n$  and  $\bar{V}_n$  are APT spaces.

Analyzing the results of [9] (theorems 2 – 6) we noticed that the space  $\bar{V}_n$  is implicitly supposed to be Hermitian and this assumption is essential. Hermitian structure of  $\bar{V}_n$  is more important than Hermitian structure of  $V_n$  and, moreover, it simplifies fundamental equations of 4-planar mappings. In this paper we do not assume  $V_n$  to be Hermitian.

1. A well-known definition says that an *almost quaternionic* space is a differentiable manifold  $M_n$  with almost complex structures  $\overset{1}{F}$  and  $\overset{2}{F}$  satisfying

$$\overset{1}{F}_\alpha^h \overset{1}{F}_i^\alpha = -\delta_i^h; \quad \overset{2}{F}_\alpha^h \overset{2}{F}_i^\alpha = -\delta_i^h; \quad \overset{1}{F}_\alpha^h \overset{2}{F}_i^\alpha + \overset{2}{F}_\alpha^h \overset{1}{F}_i^\alpha = 0, \quad (1)$$

where  $\delta_i^h$  is the Kronecker symbol, see e.g. [1], [4].

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The tensor  $F_i^h \equiv F_i^\alpha F_\alpha^h$  defines an almost complex structure, too. The relations among the tensors  $F, \bar{F}, \bar{F}$  are the following

$$F_i^h = F_i^\alpha F_\alpha^h = -F_i^\alpha F_\alpha^h, \quad \bar{F}_i^h = \bar{F}_i^\alpha F_\alpha^h = -\bar{F}_i^\alpha F_\alpha^h, \quad \bar{F}_i^h = \bar{F}_i^\alpha F_\alpha^h = -\bar{F}_i^\alpha F_\alpha^h. \quad (2)$$

Any two of the above three structures  $F, \bar{F}, \bar{F}$  define the same almost quaternionic structure.

Let  $A_n \equiv (M_n, \Gamma, F, \bar{F}, \bar{F})$  be an almost quaternionic space with a torsion-free affine connection  $\Gamma$ .

**Definition 1** A curve  $\ell: x^h = x^h(t)$  in  $A_n$  is called *4-planar* if the tangent vector  $\lambda^h = dx^h/dt$  being parallelly transported along this curve, remains in the linear 4-dimensional space generated by the tangent vector  $\lambda^h$  and the corresponding vectors  $F_\alpha^h \lambda^\alpha, \bar{F}_\alpha^h \lambda^\alpha$  and  $\bar{F}_\alpha^h \lambda^\alpha$ .

A curve is 4-planar if and only if the equations

$$\frac{d\lambda^h}{dt} + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \sum_{s=0}^3 \rho_s F_\alpha^s \lambda^\alpha$$

hold, where  $F_i^h \equiv \delta_i^h$  is the Kronecker symbol,  $\Gamma_{\alpha\beta}^h$  are components of the affine connection on  $A_n$  and  $\rho_s = \rho_s(t)$  ( $s = 0, \dots, 3$ ) denote functions of the parameter  $t$ .

Any geodesic curve is a special case of a 4-planar curve where  $\rho_1 \equiv \rho_2 \equiv \rho_3 \equiv 0$ .

Consider two spaces  $A_n$  and  $\bar{A}_n$  with the same underlying manifold  $M_n$  and the same almost quaternionic structure  $(F, \bar{F}, \bar{F})$  but with two different torsion-free affine connection  $\Gamma$  and  $\bar{\Gamma}$ , respectively.

**Definition 2** A diffeomorphism  $f: A_n \rightarrow \bar{A}_n$  is called a *4-planar mapping*, if it maps any geodesic of  $A_n$  to a 4-planar curve of  $\bar{A}_n$ .

*Remark.* In the following we shall attach to each local map  $\varphi$  around a point  $p \in A_n$  the local map  $\varphi \circ f^{-1}$  around the point  $f(p) \in \bar{A}_n$ . This means that any point  $x \in A_n$  and the corresponding point  $f(x) \in \bar{A}_n$  will have the same local coordinates.

The following theorem holds [5]:

**Theorem 1.** A diffeomorphism of  $A_n$  onto  $\bar{A}_n$  is a 4-planar mapping if and only if in every local coordinate system  $x = (x^1, x^2, \dots, x^n)$  the conditions

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \sum_{s=0}^3 \psi_{(i}^s F_{j)}^h \quad (3)$$

hold, where  $\Gamma_{ij}^h$  and  $\bar{\Gamma}_{ij}^h$  are components of the affine connections  $\Gamma$  and  $\bar{\Gamma}$ , respectively,  $\psi_s^i(x)$ ,  $s = 0, \dots, 3$ , are covectors, and  $(ij)$  denotes a symmetrization of indices.

Using Theorem 1 one can prove the all 4-planar curves of  $A_n$  are mapped onto 4-planar curves of  $\bar{A}_n$  (I.N. Kurbatova [9] defined 4-quasiplanar mappings preserving almost-quaternionic structure by the conditions (3)).

Finally, we will consider a special case of  $\bar{A}_n$ , namely an almost quaternionic Riemannian space  $\bar{V}_n \equiv (M_n, \bar{g}, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$  in which  $\bar{\Gamma}$  denote the Levi-Civita connection of  $\bar{g}$ .

The following theorem holds (see [5]).

**Theorem 2.** A diffeomorphism  $f: A_n \rightarrow \bar{V}_n$  is a 4-planar mapping if and only if the metric tensor  $\bar{g}_{ij}(x)$  satisfies the following equations:

$$\bar{g}_{ij,k} = \sum_{s=0}^3 \left( \psi_s^k \bar{g}_{\alpha(i} \overset{s}{F}_{j)}^\alpha + \psi_s^i \bar{g}_{j)\alpha} \overset{s}{F}_k^\alpha \right) \tag{4}$$

where comma denotes the covariant derivative in  $A_n$ .

Recall that the covariant derivative of  $\bar{g}$  in  $\bar{A}_n$  is zero.

The proof follows from the fact that formulas (3) and (4) are equivalent in our special case.

**2.** Now we shall prove the following two lemmas.

Consider the spaces  $A_n, \bar{A}_n$  and let " , " or " | " before an index denote a covariant derivative w.r. to the corresponding local variable on  $A_n$  and  $\bar{V}_n$ , respectively.

**Lemma 1.** Let a 4-planar mapping  $A_n \rightarrow \bar{A}_n$  be given and let  $\psi_s^i$  denote the corresponding covectors from (3). Then

$$\overset{s}{F}_{i,\alpha}^\alpha = \overset{s}{F}_{i|\alpha}^\alpha \quad s = 1, 2, 3. \tag{5}$$

holds if and only if the covectors  $\psi_s^i$  are expressed by formulas

$$\psi_s^i = -\frac{n}{n-4} \psi_\alpha \overset{s}{F}_i^\alpha, \quad s = 1, 2, 3, \quad \psi_i \equiv \psi_0^i. \tag{6}$$

The proof of the above Lemma 1 is a consequence of (5) and fundamental equations of 4-planar mappings (3). We use also algebraic properties of quaternionic structures (1) and (2).

A manifold with an affine connection  $\Gamma$  and an almost complex structure  $F$  is said to be an *Apt space* (see [2], [4], [9], or *nearly Kählerian space* or *Tachibana space* [4], [6],

[7], [8], [10], [20]) if its structure  $F$  satisfies  $F_{i,\alpha}^\alpha = 0$ . A space  $A_n = (M_n, \Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$  to be an *almost quaternionic Apt space* if

$$\overset{s}{F}_{i,\alpha}^\alpha = 0, \quad s = 1, 2, 3.$$

Lemma 1 implies that an Apt spaces  $A_n$  is 4-planarly mapped on an Apt space  $\bar{A}_n$  iff (6) holds. Evidently Kählerian spaces are Apt spaces and also quaternionic Kählerian spaces are Apt spaces.

Contracting (3) with respect to  $h$  and  $j$  we got the lemma

**Lemma 2.** *If for a 4-planar mapping  $A_n \rightarrow \bar{A}_n$  the formulae (6) hold and the spaces  $A_n$  and  $\bar{A}_n$  are equiaffine, then the vector  $\psi_i$  is a gradient, i.e. there exists a function  $\psi$  such that  $\psi_i = \psi_{,i}$ .*

**3.** Now we shall show that if a 4-planar mapping from  $A_n$  onto a Riemannian space  $\bar{V}_n$  is given, then the formulae (3) and (4) are both equivalent to the following formula:

$$\bar{g}^{ij}_{,k} = - \sum_{s=0}^3 \left( \psi_{,k} \bar{g}^{\alpha(i} \overset{s}{F}_{\alpha}^{j)} + \psi_{\alpha} \bar{g}^{\alpha(i} \overset{s}{F}_{k}^{j)} \right) \tag{7}$$

where  $\bar{g}^{ij}$  is the inverse matrix of metric tensor  $\bar{g}_{ij}$ . In fact, (7) is a consequence of the identity  $\bar{g}^{ij}_{,k} = -\bar{g}_{\alpha\beta,k} \bar{g}^{\alpha i} \bar{g}^{\beta j}$ .

In what follows we shall assume a quaternionic structure on  $\bar{V}_n$  which is Hermitian, i.e. we have

$$\bar{g}_{i\alpha} \overset{s}{F}_j^\alpha + \bar{g}_{j\alpha} \overset{s}{F}_i^\alpha = 0, \quad s = 1, 2, 3. \tag{8}$$

(8) is equivalent with

$$\bar{g}^{i\alpha} \overset{s}{F}_\alpha^j + \bar{g}^{j\alpha} \overset{s}{F}_\alpha^i = 0, \quad s = 1, 2, 3, \tag{9}$$

or with

$$\bar{g}^{\alpha\beta} \overset{s}{F}_\alpha^i \overset{s}{F}_\alpha^j = \bar{g}^{ij}, \quad s = 1, 2, 3. \tag{10}$$

Using (9) the equations of 4-planar mappings are simplified to

$$\bar{g}^{ij}_{,k} = -2\psi_{,k} \bar{g}^{ij} - \sum_{s=0}^3 \psi_{\alpha} \bar{g}^{\alpha(i} \overset{s}{F}_k^{j)} \tag{11}$$

Suppose now that the covector  $\psi_i$  is a gradient, i.e.  $\psi_i \equiv \psi_{,i} \equiv \psi_{,i}$  where  $\psi$  is a function. We define the tensor

$$a^{ij} \equiv e^{2\psi} \bar{g}^{ij}.$$

Then (11) can be rewritten in the form

$$a^{ij}_{,k} = \sum_{s=0}^3 \lambda_s^{(i} \overset{s}{F}_k^{j)}, \tag{12}$$

where

$$\lambda_s^i \equiv -\psi_s \alpha \bar{g}^{\alpha i} . \tag{13}$$

By the definition of the tensor  $a^{ij}$  (10) is equivalent with

$$a^{\alpha\beta} \overset{s}{F}_\alpha^i \overset{s}{F}_\alpha^j = a^{ij}, \quad s = 1, 2, 3. \tag{14}$$

Due to the fact that  $\bar{V}_n$  is Hermitian and using (13) we see that the formula (6) is equivalent with

$$\lambda_s^i = \frac{n}{n-4} \lambda^\alpha \overset{s}{F}_\alpha^i, \quad s = 1, 2, 3, \quad \lambda^i \equiv \lambda_0^i . \tag{15}$$

Now we come back to the affine case. Let a space  $A_n$  be given as before and let the system of equations (12), (14) and (15) has a solution for a regular matrix function  $a^{ij}$  and a vector function  $\lambda^i$ . Then one can prove that the inverse matrix  $||\tilde{g}_{ij}|| = ||a^{ij}||^{-1}$  defines a Riemannian metric  $\tilde{g}$  on  $M_n$  and the covector  $\lambda^\alpha \tilde{g}_{\alpha i}$  is a gradient  $\text{grad}\psi$ . By the conformal change  $\tilde{g}_{ij} = e^{2\psi} \bar{g}_{ij}$  we obtain a new metric  $\bar{g}$  for which  $A_n$  becomes a Hermitian almost quaternionic space  $\bar{V}_n$ . Moreover, there exists a 4-planar mapping  $A_n \rightarrow \bar{V}_n$ .

This results coincides with the result by N.S. Sinyukov for geodetic mappings and the results by V.V. Domashev and J. Mikeš for holomorphically projective mappings of Kählerian spaces etc., see [12], [13], [18], [19]. Now we can conclude the above results with

**Theorem 3.** *Under the condition (5) an equiaffine space  $A_n$  admits a 4-planar mapping on a Hermitian quaternionic space  $\bar{V}_n$  if and only if there exists a regular tensor  $a^{ij}$  on  $A_n$  satisfying (12), (14), and (15).*

The result analogous to Theorem 3 was proved by I.N. Kurbatova [9] under the assumption that  $A_n$  is Hermitian and from the proof it is evident that also  $\bar{V}_n$  is supposed the be Hermitian.

4. Analysing the equation of I.N. Kurbatova [9] analogous to (12) we can modify this equation as a system of linear differential equations of Cauchy type. In what follows we give more simple modification which uses also conditions (14).

We consider covariant derivatives of (14) in  $A_n$ , i.e.

$$a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^i \overset{r}{F}_\beta^j + a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^i \overset{r}{F}_\beta^j + a^{\alpha\beta} \overset{r}{F}_\alpha^i \overset{r}{F}_{\beta,k}^j = a^{ij}_{,k}, \quad r = 1, 2, 3.$$

Putting (12) into the above equation we get

$$\sum_{s=0}^3 \left( \lambda_s^{(i} \overset{s}{F}_k^j) - \lambda_s^\alpha \overset{r}{F}_\alpha^{(i} \overset{r}{F}_\beta^j) \overset{s}{F}_k^\beta \right) = a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^{(i} \overset{r}{F}_\beta^j) . \tag{16}$$

For  $r = 1$ , using (1), (2) and (15) we have

$$\lambda^{(i}\delta_k^{j)} - \lambda^\alpha \frac{1}{F_\alpha^i} F_k^{(i} F^{j)} = \frac{n-4}{4} a^{\alpha\beta} \frac{1}{F_{\alpha,k}^i} F_\beta^{(i} F^{j)} \tag{17}$$

and contracting (17) with respect to  $j$  and  $k$  we have the following expression of the vector  $\lambda^i$ :

$$\lambda^i = \frac{n-4}{n(2n+1)} a^{\alpha\beta} \left( \frac{1}{F_{\alpha,\gamma}^i} F_\beta^\gamma + \frac{1}{F_\alpha^i} F_{\beta,\gamma}^\gamma \right). \tag{18}$$

It implies that  $\lambda^i$  can be expressed as a linear functions in  $a^{ij}$ . It implies

**Theorem 4.** *Under the condition (5) an equiaffine space  $A_n$  admits a 4-planar mapping onto a Hermitian almost quaternionic space  $\tilde{V}_n$  if and only if the following system of differential equations of Cauchy type is solvable with respect to the unknown functions  $a^{ij}$ :*

$$a^{ij}_{,k} = \sum_{s=0}^3 \lambda_s^{(i} F_k^{s)j)}, \tag{19}$$

where

$$\lambda_s^i = \frac{n}{n-4} \lambda^\alpha F_\alpha^i, \quad s = 1, 2, 3, \quad \lambda^i = \frac{n-4}{n(2n+1)} a^{\alpha\beta} \left( \frac{1}{F_{\alpha,\gamma}^i} F_\beta^\gamma + \frac{1}{F_\alpha^i} F_{\beta,\gamma}^\gamma \right)$$

and the matrix  $(a^{ij})$  should satisfying addition  $|a^{ij}| \neq 0$  and the algebraic condition

$$a^{\alpha\beta} F_\alpha^i F_\alpha^j = a^{ij}, \quad s = 1, 2, 3.$$

The system (19) does not have more than one solution for the initial Cauchy conditions  $a^{ij}(x_o) = a_o^{ij}$  under the conditions (20). Therefore the general solution of (19) does not depend on more than  $N_o = (n/2)^2$  parameters. The question of existence of a solution of (19) leads to the studium of integrability conditions, which are linear equations w.r. to the unknowns  $a^{ij}(x)$  with coefficients from the space  $A_n$ .

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## SPECIAL SPINORS AND CONTACT GEOMETRY

ANDREI MOROIANU

### 1. INTRODUCTION

The aim of this note is to outline some new results obtained in contact geometry by means of spinorial methods and in particular to exhibit some interesting relations between (complex) contact structures and (Kählerian) Killing spinors.

While the notion of a contact structure is well-known to most differential geometers, that of a Killing spinor (though intensively studied by physicists under the name of supersymmetry) remained, for a quite long time, neglected by mathematicians. Killing spinors came to be studied only after 1980, when Th. Friedrich [3] proved that they arise as the eigenspinors corresponding to the least possible eigenvalue of the Dirac operator on compact spin manifolds with positive scalar curvature. More precisely, we have the

*Theorem 1.1. (Friedrich, 1980)* Any eigenvalue  $\lambda$  of the Dirac operator on a compact spin manifold  $M^n$  with positive scalar curvature  $S$  satisfies the inequality

$$(1.1) \quad \lambda^2 \geq \frac{n}{4(n-1)} \inf_M S.$$

Moreover, if the equality holds, then every eigenspinor  $\psi$  corresponding to  $\lambda$  is a real Killing spinor, i.e. satisfies the equation

$$(1.2) \quad \nabla_X \psi = \alpha X \cdot \psi, \quad \forall X \in TM (\alpha = -\frac{\lambda}{n}).$$

After several steps were made towards their classification by H. Baum, Th. Friedrich, R. Grunewald and I. Kath (these are presented in a unified manner in [2]), Killing spinors (or, properly speaking, manifolds carrying them) were finally classified by C. Bär [1], who made a very elegant use of the so-called cone construction. This is where contact structures come into the play, since Bär shows that, with some low-dimensional exceptions, all simply connected manifolds carrying Killing spinors are contact manifolds (or round spheres, in even dimensions). More precisely, if  $M^{2m+1}$  ( $m > 3$ ) carries Killing spinors, then  $M$  is either Einstein-Sasakian or 3-Sasakian (for the definitions see [1] for example).

Using the explicit relations between Sasakian structures and Killing spinors given by Th. Friedrich and I. Kath [2], we gave a description in [11] of the splitting of the algebra of infinitesimal isometries of Einstein-Sasakian and 3-Sasakian manifolds, and furthermore proved the following rigidity result:

*Theorem 1.2.* The only simply connected 3-Sasakian manifold  $(M^7, g, \xi_i)$  possessing an infinitesimal isometry of unit length, other than the Sasakian vector fields, is the unit sphere  $S^7$ .

Let us now turn our attention to the complex case, and recall the following

*Definition 1.1.* (cf. [7]) Let  $M^{2m}$  be a complex manifold of complex dimension  $m = 2k + 1$ . A *complex contact structure* is a family  $\mathcal{C} = \{(U_i, \omega_i)\}$  satisfying the following conditions:

- (i)  $\{U_i\}$  is an open covering of  $M$ .
- (ii)  $\omega_i$  is a holomorphic 1-form on  $U_i$ .
- (iii)  $\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0}M)$  is non vanishing at every point of  $U_i$ .
- (iv)  $\omega_i = f_{ij}\omega_j$  in  $U_i \cap U_j$ , where  $f_{ij}$  is a holomorphic function on  $U_i \cap U_j$ .

Our main result will be the classification of all Kähler-Einstein manifolds of positive scalar curvature admitting a complex contact structure. This goes roughly as follows: first of all, if  $M^{4k+2}$  is a Kähler manifold admitting a complex contact structure, then we construct for  $k$  odd a canonical spinor on  $M$  and for  $k$  even a canonical section of the spinor bundle associated to a suitable  $\text{Spin}^c$  structure of  $M$  (this idea - for  $k$  odd - stems from K.-D. Kirchberg and U. Semmelmann, see [6]). We then prove that the constructed spinor is a *Kählerian Killing spinor* (we have to define this notion in the  $\text{Spin}^c$  case) if the given Kähler metric on the manifold  $M$  is also Einstein. The next step is to construct a canonical  $S^1$  bundle  $N$  over  $M$ , which is endowed with a Riemannian metric and a spin structure, such that the above constructed Kählerian Killing spinor on  $M$  induces a Killing spinor on  $N$ . Finally, using Bär's classification of such manifolds and some further algebraic properties of the Killing spinor, we conclude that  $N$  has to be 3-Sasakian and furthermore, by the naturality of the construction of  $N$ , we are also able to characterise  $M$  geometrically.

## 2. A SHORT REVIEW ON SPIN AND $\text{Spin}^c$ GEOMETRY

We will first recall some basic facts about spin and  $\text{Spin}^c$  structures. Consider an oriented Riemannian manifold  $(M^n, g)$ . Let  $P_{SO(n)}M$  denote the bundle of oriented orthonormal frames on  $M$ .

*Definition 2.1.* The manifold  $M$  is called spin if there exists a 2-fold covering  $P_{\text{Spin}_n}M$  of  $P_{SO(n)}M$  with projection  $\theta : P_{\text{Spin}_n}M \rightarrow P_{SO(n)}M$  satisfying the following conditions :

- i)  $P_{\text{Spin}_n}M$  is a principal bundle over  $M$  with structure group  $\text{Spin}_n$ ;

ii) If we denote by  $\phi$  the canonical projection of  $\text{Spin}_n$  over  $\text{SO}(n)$ , then for every  $u \in P_{\text{Spin}_n}M$  and  $a \in \text{Spin}_n$  we have

$$\theta(ua) = \theta(u)\phi(a).$$

The bundle  $P_{\text{Spin}_n}M$  is called a spin structure. Representation theory shows that the complex Clifford algebra  $Cl(n)$  has (up to equivalence) exactly one irreducible complex representation  $\Sigma_n$  for  $n$  even and two irreducible complex representations  $\Sigma_n^\pm$  for  $n$  odd. In the last case, these two representations are equivalent when restricted to  $\text{Spin}_n$ , and this restriction is denoted by  $\Sigma_n$ . For  $n$  even, there is a splitting of  $\Sigma M$  with respect to the action of the volume element in  $\Sigma_n := \Sigma_n^+ \oplus \Sigma_n^-$  and one usually calls elements of  $\Sigma_n^+$  (resp.  $\Sigma_n^-$ ) positive (resp. negative) half-spinors. For arbitrary  $n$ ,  $\Sigma_n$  is called the complex spin representation, and its associated vector bundle  $\Sigma M$  is called the complex spinor bundle. Sections of  $\Sigma M$  are called spinors. If  $M$  is even-dimensional we denote by  $\Sigma^\pm M$  the subbundles of  $\Sigma M$  corresponding to  $\Sigma_n^\pm$ . If, with respect to the decomposition  $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ , a spinor  $\psi$  is written as  $\psi = \psi_+ + \psi_-$ , then its conjugate  $\bar{\psi}$  is defined to be  $\bar{\psi}_+ - \bar{\psi}_-$ .

*Definition 2.2.* A  $\text{Spin}^c$  structure on an oriented Riemannian manifold  $(M^n, g)$  is given by a  $U(1)$  principal bundle  $P_{U(1)}M$  and a  $\text{Spin}_n^c$  principal bundle  $P_{\text{Spin}_n^c}M$  together with a projection  $\theta : P_{\text{Spin}_n^c}M \rightarrow P_{\text{SO}(n)}M \times P_{U(1)}M$  satisfying

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

for every  $\tilde{u} \in P_{\text{Spin}_n^c}M$  and  $\tilde{a} \in \text{Spin}_n^c$ , where  $\xi$  is the canonical 2-fold covering of  $\text{Spin}_n^c$  over  $\text{SO}(n) \times U(1)$ . The complex line bundle associated to  $P_{U(1)}M$  is called the auxiliary bundle of the given  $\text{Spin}^c$  structure.

Recall that  $\text{Spin}_n^c = \text{Spin}_n \times_{\mathbb{Z}_2} U(1)$ , and that  $\xi$  is given by  $\xi([u, a]) = (\phi(u), a^2)$ , where  $\phi : \text{Spin}_n \rightarrow \text{SO}(n)$  is the canonical 2-fold covering. The complex representations of  $\text{Spin}_n^c$  are obviously the same as those of  $\text{Spin}_n$ ; thus to every  $\text{Spin}^c$  manifold is associated a spinor bundle just as is the case for spin manifolds.

If  $M$  is spin, the Levi-Civita connection on  $P_{\text{SO}(n)}M$  induces a connection on the spin structure  $P_{\text{Spin}_n}M$ , and thus a covariant derivative on  $\Sigma M$  denoted by  $\nabla$ . Similarly, if  $M$  has a  $\text{Spin}^c$  structure, then every connection form  $A$  on  $P_{U(1)}M$  defines (together with the Levi-Civita connection of  $M$ ) a covariant derivative on  $\Sigma M$  denoted by  $\nabla^A$ .

Spin structures are special case of  $\text{Spin}^c$  structures, because of the following

*Lemma 2.1.* A  $\text{Spin}^c$  structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection  $A$  of the auxiliary bundle  $L$  is flat, then under this identification  $\nabla^A$  corresponds to  $\nabla$  on the spinor bundles.

**Proof.** Notice that the triviality of the auxiliary bundle implies that we can exhibit a global section of  $U(1)$  that we shall call  $\sigma$ . Denote by  $P_{\text{Spin}_n}M$  the inverse image by  $\theta$  of  $P_{\text{SO}(n)}M \times \sigma$ . It is straightforward to check that this defines a spin structure

on  $M$  and that the connection on  $P_{Spin^c} M$  restricts to the Levi-Civita connection on  $P_{Spin} M$  if  $\sigma$  can be chosen to be parallel, i.e. if  $A$  defines a flat connection.

Q.E.D.

Let  $M$  be a  $Spin^c$  manifold with auxiliary connection  $A$ . On  $\Sigma M$  there is a canonical hermitian product  $(\cdot, \cdot)$ , with respect to which the Clifford multiplication by vectors (which arises via the Clifford representation) is skew-Hermitian:

$$(2.1) \quad (X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM, \psi, \varphi \in \Sigma M.$$

We now define the Dirac operator as the composition  $\gamma \circ \nabla^A$ , where  $\gamma$  denotes the Clifford contraction. The Dirac operator can be expressed using a local orthonormal frame  $\{e_1, \dots, e_n\}$  as

$$D = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^A.$$

Suppose now that  $(M^{2m}, g, J)$  is a Kähler manifold. We define the twisted Dirac operator  $\tilde{D}$  as

$$\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla_{e_i}^A = - \sum_{i=1}^{2m} e_i \cdot \nabla_{J(e_i)}^A.$$

It satisfies

$$(2.2) \quad \tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0.$$

We also define the complex Dirac operators  $D_{\pm} := \frac{1}{2}(D \mp i\tilde{D})$ , and (2.2) becomes

$$(2.3) \quad D_-^2 = D_+^2 = 0 \quad \text{and} \quad D^2 = D_+D_- + D_-D_+.$$

Consider a local orthonormal frame  $\{X_{\alpha}, Y_{\alpha}\}$  such that  $Y_{\alpha} = J(X_{\alpha})$ . Then  $Z_{\alpha} = \frac{1}{2}(X_{\alpha} - iY_{\alpha})$  and  $Z_{\bar{\alpha}} = \frac{1}{2}(X_{\alpha} + iY_{\alpha})$  are local frames of  $T^{1,0}(M)$  and  $T^{0,1}(M)$ , and  $D_{\pm}$  can be expressed as

$$(2.4) \quad D_+ = 2 \sum_{\alpha=1}^m Z_{\alpha} \cdot \nabla_{Z_{\bar{\alpha}}}^A, \quad D_- = 2 \sum_{\alpha=1}^m Z_{\bar{\alpha}} \cdot \nabla_{Z_{\alpha}}^A.$$

A  $k$ -form  $\omega$  acts on  $\Sigma M$  by

$$\omega \cdot \Psi := \sum_{i_1 < \dots < i_k} \omega(e_{i_1}, \dots, e_{i_k}) e_{i_1} \cdot \dots \cdot e_{i_k} \cdot \Psi.$$

With respect to this action, the Kähler form  $\Omega$  (defined by  $\Omega(X, Y) = g(X, JY)$ ) satisfies

$$(2.5) \quad \Omega = \frac{1}{2} \sum_{i=1}^{2m} J(e_i) \cdot e_i = -\frac{1}{2} \sum_{i=1}^{2m} e_i \cdot J(e_i).$$

For later use let us note that

$$(2.6) \quad \sum_{\alpha=1}^m Z_\alpha \cdot Z_{\bar{\alpha}} = -\frac{i}{2}\Omega - \frac{m}{2}, \quad \sum_{\alpha=1}^m Z_{\bar{\alpha}} \cdot Z_\alpha = \frac{i}{2}\Omega - \frac{m}{2},$$

where  $Z_\alpha$  and  $Z_{\bar{\alpha}}$  are local frames of  $T^{1,0}(M)$  and  $T^{0,1}(M)$  as before.

The action of  $\Omega$  on  $\Sigma M$  yields an orthogonal decomposition

$$\Sigma M = \bigoplus_{r=0}^m \Sigma_r M,$$

where  $\Sigma_r M$  is the eigenbundle associated to the eigenvalue  $i\mu_r = i(m - 2r)$  of  $\Omega$ . If we define  $\Sigma_{-1}M = \Sigma_{m+1}M = \{0\}$ , then

$$(2.7) \quad D_\pm \Gamma(\Sigma_r M) \subset \Gamma(\Sigma_{r\pm 1} M).$$

### 3. RELATIONS BETWEEN COMPLEX CONTACT STRUCTURES AND SPINORS

Let  $\mathcal{C} = \{(U_i, \omega_i)\}$  be a complex contact structure. Then there exists an associated holomorphic line subbundle  $L_C \subset \Lambda^{1,0}(M)$  with transition functions  $\{f_{ij}^{-1}\}$  and local sections  $\omega_i$ . It is easy to see that

$$\mathcal{D} := \{Z \in T^{1,0}M \mid \omega(Z) = 0, \forall \omega \in L_C\}$$

is a codimension 1 maximally non-integrable holomorphic sub-bundle of  $T^{1,0}M$ , and conversely, every such bundle defines a complex contact structure. Condition (iii) in Definition 1.1 entails that  $L_C^{k+1}$  is isomorphic to  $K$ , where  $K = \Lambda^{m,0}(M)$  denotes the canonical bundle of  $M$ .

Suppose for a while that  $k$  is even, say  $k = 2l$ . The collection  $(U_i, \omega_i \wedge (\partial\omega_i)^l)$  defines a holomorphic line bundle  $L_l \subset \Lambda^{2l+1,0}M$ , and from the definition of  $\mathcal{C}$  we easily obtain

$$(3.1) \quad L_l \cong L_C^{l+1}.$$

We now fix some  $(U, \omega) \in \mathcal{C}$  and define a local section  $\psi_C$  of  $\Lambda^{0,2l+1}M \otimes L_C^{l+1}$  by

$$(3.2) \quad \psi_C|_U := |\xi_\tau|^{-2} \bar{\tau} \otimes \xi_\tau,$$

where  $\tau := \omega \wedge (\partial\omega)^l$  and  $\xi_\tau$  is the element corresponding to  $\tau$  through the isomorphism (3.1). The fact that  $\psi_C$  does not depend on the element  $(U, \omega) \in \mathcal{C}$  shows that it actually defines a global section  $\psi_C$  of  $\Lambda^{0,2l+1}M \otimes L_C^{l+1}$ .

We now recall ([8], Appendix D) that  $\Lambda^{0,*}M$  is just the spinor bundle associated to the canonical  $\text{Spin}^c$  structure on  $M$ , whose auxiliary line bundle is  $K^{-1}$ , so that  $\Lambda^{0,*}M \otimes L_C^{l+1}$  is actually the spinor bundle associated to the  $\text{Spin}^c$  structure on  $M$  with auxiliary bundle  $L = K^{-1} \otimes L_C^{2(l+1)} \cong L_C^{-(2l+1)} \otimes L_C^{2(l+1)} \cong L_C$ . The section  $\psi_C$  is thus a spinor lying in  $\Lambda^{0,2l+1}M \otimes L_C^{l+1} \cong \Sigma_{2l+1}M$ , which shows that

$$(3.3) \quad \Omega \cdot \psi_C = -i\psi_C.$$

The case  $k = 2l + 1$  is similar: the section  $\psi_C$  is defined by the same formulae as before, and it lies in  $\Lambda^{0,2l+1}M \otimes L_C^{l+1} \cong \Lambda^{0,2l+1}M \otimes K^{\frac{1}{2}}$ . Thus in this case  $\psi_C$  is an usual spinor on  $M$  (see[4]).

We suppose from now on that  $M$  is Kähler-Einstein with positive scalar curvature. The manifold  $M$  is compact, by Myers' Theorem. By rescaling the metric on  $M$  if necessary, we can suppose that the scalar curvature of  $M$  is equal to  $2m(2m + 2)$ , and thus the Ricci form  $\rho$  and the Kähler form  $\Omega$  are related by the equality  $\rho = (2m + 2)\Omega$ .

*Proposition 3.1.* For  $k$  even the spinor field  $\psi_C$  satisfies

$$(3.4) \quad \nabla_Z \psi_C = 0, \quad \forall Z \in T^{1,0}M$$

and

$$(3.5) \quad D^2 \psi_C = D_- D_+ \psi_C = \left(\frac{1}{4}R \psi_C - \frac{i}{2} \rho \cdot \psi_C\right),$$

and for  $k$  odd, say  $k = 2l + 1$

$$(3.6) \quad D^2 \psi_C = D_- D_+ \psi_C = \frac{l + 1}{2l + 1} \left(\frac{1}{2}R \psi_C - i \rho \cdot \psi_C\right),$$

where  $R$  is the scalar curvature of  $M$ . In particular (3.4) shows that  $D_- \psi_C = 0$ .

The proof of the first two assertions can be found in [6]. The proof of (3.6) is analogous to that of (3.5) (one only has to replace some  $\frac{1}{2}$  coefficients by  $\frac{l+1}{2l+1}$  coefficients. Using (3.3), (3.6) and the fact that  $\rho = \frac{1}{8l+2}R\Omega = (8l + 4)\Omega$  for  $k = 2l$  and  $\rho = \frac{1}{8l+6}R\Omega = (8l + 8)\Omega$  for  $k = 2l + 1$ , we obtain

*Corollary 3.1.* The spinor field  $\psi_C$  is an eigenspinor of  $D^2$  with eigenvalue  $16l(l + 1)$  for  $k = 2l$  and with eigenvalue  $16(l + 1)^2$  for  $k = 2l + 1$ .

It is now easy to see that for  $k = 2l + 1$   $\psi_C$  is a Kählerian Killing spinor. Indeed, it is enough to use the above corollary and the fact that the scalar curvature of  $M$  is (due to our normalisation)  $S = 2m(2m + 2) = (8l + 6)(8l + 8)$ , together with the following result from [5]

*Theorem 3.1. (Kirchberg, 1986)* Any eigenvalue  $\lambda$  of the Dirac operator on a compact Kähler spin manifold  $(M^{2m}, g, J)$  ( $m$  odd) with positive scalar curvature  $S$  satisfies the inequality

$$(3.7) \quad \lambda^2 \geq \frac{m + 1}{4m} \inf_M S.$$

Moreover, if the equality holds, then every eigenspinor  $\psi$  corresponding to  $\lambda$  is a Kählerian Killing spinor, i.e. satisfies the equation

$$(3.8) \quad \nabla_X \psi = \alpha X \cdot \psi + \alpha J(X) \cdot \bar{\psi}, \quad \forall X \in TM \quad \left(\alpha = -\frac{\lambda}{2m + 2}\right).$$

We thus have the

*Corollary 3.2.* For  $k$  odd, the spinor  $\psi_C$  is a Kählerian Killing spinor.

The case  $k = 2l$  is somewhat harder, since no analogue of Kirchberg’s Theorem is known for  $\text{Spin}^c$  manifolds and we have to resort to an "ad-hoc" argument. Let us first introduce some notations:

$$(3.9) \quad \psi_- := \psi_C \in \Gamma(\Sigma_{2l+1}M) \quad , \quad \psi_+ := \frac{1}{4l+4}D\psi_C \in \Gamma(\Sigma_{2l+2}M) \quad .$$

Integrating over  $M$  we immediately obtain from Corollary 3.1

$$(3.10) \quad |\psi_-|_{L^2}^2 = \frac{l+1}{l}|\psi_+|_{L^2}^2.$$

*Proposition 3.2.* The following relations hold

$$(3.11) \quad \nabla_Z\psi_- = 0, \quad \forall Z \in T^{1,0}M,$$

$$(3.12) \quad \nabla_{\bar{Z}}\psi_- + \bar{Z} \cdot \psi_+ = 0, \quad \forall \bar{Z} \in T^{0,1}M,$$

$$(3.13) \quad \nabla_Z\psi_+ = 0, \quad \forall Z \in T^{0,1}M,$$

$$(3.14) \quad \nabla_Z\psi_+ + Z \cdot \psi_- = 0, \quad \forall Z \in T^{1,0}M.$$

**Proof.** The first relation is part of Proposition 3.1. In order to prove (3.12), let us consider the local frames of  $T^{1,0}(M)$  and  $T^{0,1}(M)$  introduced in Section 2:  $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$  and  $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$ , where  $Y_\alpha = J(X_\alpha)$ , and  $\{X_\alpha, Y_\alpha\}$  is a local orthonormal frame of  $TM$ . From (3.11) we find  $\nabla_{Z_{\bar{\alpha}}}\psi_- = \nabla_{X_\alpha}\psi_- = i\nabla_{Y_\alpha}\psi_-$ , so using (2.6) and (3.9) gives

$$\begin{aligned} 0 &\leq \sum_{\alpha=1}^m |\nabla_{Z_{\bar{\alpha}}}\psi_- + Z_{\bar{\alpha}} \cdot \psi_+|^2 \\ &= \sum_{\alpha=1}^m |\nabla_{X_\alpha}\psi_-|^2 - 2\Re \sum_{\alpha=1}^m (\psi_+, Z_\alpha \cdot \nabla_{Z_{\bar{\alpha}}}\psi_-) - \sum_{\alpha=1}^m (\psi_+, Z_\alpha \cdot Z_{\bar{\alpha}} \cdot \psi_+) \\ &= \frac{1}{2}|\nabla\psi_-|^2 - \Re(\psi_+, D_+\psi_-) - \frac{1}{2}(\psi_+, (-i\Omega - m)\psi_+) \\ &= \frac{1}{2}|\nabla\psi_-|^2 - (4l+4)|\psi_+|^2 + \frac{1}{2}(4l+4)|\psi_+|^2. \end{aligned}$$

The last expression is by construction a positive function on  $M$ , say  $|F|^2$ . Integrating over  $M$  and using the generalised Lichnerowicz formula ([8], Appendix D),

Corollary 3.1 and (3.10), we obtain

$$\begin{aligned} |F|_{L^2}^2 &= \frac{1}{2}(\nabla^* \nabla \psi_-, \psi_-)_{L^2} - (4l+4)|\psi_+|_{L^2}^2 + \frac{1}{2}(4l+4)|\psi_+|_{L^2}^2 \\ &= \frac{1}{2}(D^2 \psi_- - \frac{1}{4}R\psi_- + \frac{i}{2} \frac{1}{2l+1} \rho \cdot \psi_-, \psi_-)_{L^2} - (2l+2)|\psi_+|_{L^2}^2 \\ &= |\psi_-|_{L^2}^2 \left( 8l(l+1) - \frac{(8l+2)(8l+4)}{8} + \frac{i-i(8l+4)}{4} \frac{1}{2l+1} - 2l \right) = 0, \end{aligned}$$

thus proving that  $F = 0$  and consequently (3.12). To check the last two equations one has to make use of the operator  $\tilde{D}$ . From  $D_- \psi_- = 0$  we find

$$(3.15) \quad 0 = \frac{1}{4l+4} D_+^2 \psi_- = D_+ \psi_+,$$

and so

$$(3.16) \quad \tilde{D} \psi_+ = -iD \psi_+.$$

Let us choose a local orthonormal frame  $e_j$ ; using (2.1), (2.5), (3.9) and (3.16) we compute

$$\begin{aligned} 0 &\leq \sum_{j=1}^n |\nabla_{e_j} \psi_+ + \frac{1}{2}(e_j - iJ(e_j)) \cdot \psi_-|^2 \\ &= |\nabla \psi_+|^2 - \Re e((D + i\tilde{D})\psi_+, \psi_-) \\ &\quad - \frac{1}{4} \sum_{j=1}^n ((e_j + iJ(e_j)) \cdot (e_j - iJ(e_j)) \cdot \psi_-, \psi_-) \\ &= |\nabla \psi_+|^2 - 2\Re e(D\psi_+, \psi_-) + ((m - i\Omega) \cdot \psi_-, \psi_-) \\ &= |\nabla \psi_+|^2 - 8l|\psi_-|^2 + 4l|\psi_-|^2 := |G|^2 \end{aligned}$$

Just as before, we compute the integral over  $M$  of the positive function  $|G|^2$ , namely

$$\begin{aligned} |G|_{L^2}^2 &= |\nabla \psi_+|_{L^2}^2 - 4l|\psi_-|_{L^2}^2 \\ &= (\nabla^* \nabla \psi_+, \psi_+)_{L^2} - 4l|\psi_-|_{L^2}^2 \\ &= (D^2 \psi_+ - \frac{1}{4}R\psi_+ + \frac{i}{2} \frac{1}{2l+1} \rho \cdot \psi_+, \psi_+)_{L^2} - 4l|\psi_-|_{L^2}^2 \\ &= |\psi_+|_{L^2}^2 \left( 16l(l+1) - \frac{(8l+2)(8l+4)}{4} + \frac{i-3i(8l+4)}{2} \frac{1}{2l+1} - 4(l+1) \right) = 0, \end{aligned}$$

thus proving  $G = 0$ . Consequently  $\nabla_X \psi_+ + \frac{1}{2}(X - iJ(X)) \cdot \psi_- = 0$  for all  $X \in TM$ , which is equivalent to (3.13) and (3.14).

Q.E.D.

The above proposition motivates the following



*Definition 3.1.* A section  $\psi$  of the spinor bundle of a given  $\text{Spin}^c$  structure on a Kähler manifold  $(M^{8l+2}, g, J)$  satisfying

$$(3.17) \quad \nabla_X^A \psi = \frac{1}{2} X \cdot \psi + \frac{i}{2} JX \cdot \bar{\psi}, \quad \forall X \in TM$$

is called a Kählerian Killing spinor.

Defining  $\psi := \psi_+ - \psi_-$  we immediately obtain the

*Corollary 3.3.* Let  $\mathcal{C}$  be a complex contact structure on a Kähler–Einstein manifold  $(M^{8l+2}, g, J)$ . Then the  $\text{Spin}^c$  structure on  $M$  with auxiliary bundle  $L_{\mathcal{C}}$  carries a Kählerian Killing spinor  $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ .

#### 4. THE CLASSIFICATION OF POSITIVE KÄHLER-EINSTEIN CONTACT MANIFOLDS

Let us first recall the following results from the theory of projectable spinors:

*Theorem 4.1.* ([10]) Let  $M$  be a compact Kähler manifold of positive scalar curvature and complex dimension  $4l + 3$ . If  $\Sigma M$  carries a Kählerian Killing spinor, then the principal  $U(1)$  bundle  $\bar{M}$  associated to any maximal root of the canonical bundle of  $M$  admits a canonical spin structure carrying Killing spinors.

*Theorem 4.2.* ([12]) Let  $M$  be a compact Kähler manifold of positive scalar curvature and complex dimension  $4l + 1$  such that there exists a  $\text{Spin}^c$  structure on  $M$  with auxiliary bundle  $L$  and spinor bundle  $\Sigma M$  satisfying  $L^{\otimes(2l+1)} \cong \Lambda^{4l+1,0}M$ . If  $\Sigma M$  carries a Kählerian Killing spinor  $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ , then the principal  $U(1)$  bundle  $\bar{M}$  associated to any maximal root of the canonical bundle of  $M$  admits a canonical spin structure carrying Killing spinors.

We are now able to give the classification of positive Kähler-Einstein contact manifolds:

*Theorem 4.3.* The only Kähler-Einstein manifolds of positive scalar curvature admitting a complex contact structure are the twistor spaces of quaternionic Kähler manifolds of positive scalar curvature.

The notion of the twistor space over a quaternionic Kähler manifold was introduced by S. Salamon in [13], where he proves that these twistor spaces all admit Kähler-Einstein metrics and complex contact structures. Our Theorem 4.3 is thus a converse of Salamon’s result, and it should be noted that it was also recently proved by C. LeBrun [9] using rather different methods.

**Proof of Theorem 4.3.** Let  $M^{4k+2}$  be a positive Kähler-Einstein contact manifold and let  $\bar{M}$  be the principal  $U(1)$  bundle associated to any maximal root of the canonical bundle of  $M$ . From Corollaries 3.2 and 3.3 and Theorems 4.1 and 4.2 we deduce that  $\bar{M}$  carries a *projectable* Killing spinor  $\psi$ . This spinor then induces a parallel spinor  $\Psi$  on the cone  $CM$  over  $\bar{M}$ , which is a Kähler manifold (cf. [1], [10], [12]). Moreover, using the projectability of  $\psi$  we can compute the action of the Kähler form

of  $C\bar{M}$  on  $\Psi$  (see [10]) and obtain that  $\Psi \in \Sigma_{k+1}C\bar{M}$ . From C. Bär's classification [1] we know that the restricted holonomy group of  $C\bar{M}$  is one of the following:  $SU(2k+2)$ ,  $Sp(k+1)$  or 0. The fixed points of the spin representation of  $SU(2k+2)$  lie in  $\Sigma_0$  and  $\Sigma_{2k+2}$ , so since  $\Psi$  is a parallel spinor in  $\Sigma_{k+1}C\bar{M}$ , the restricted holonomy group of  $C\bar{M}$  cannot be equal to  $SU(2k+2)$ . This implies that the universal covering of  $C\bar{M}$  is hyperkähler, and thus that the universal covering of  $\bar{M}$  is 3-Sasakian (see [1]). Actually, using the Gysin exact sequence we can easily deduce that  $\bar{M}$  is simply connected (see [2], p.85). On the other hand, the unit vertical vector field  $V$  on  $\bar{M}$  defines a Sasakian structure (see [2]) and it is well known that any Sasakian structure on a 3-Sasakian manifold  $P^{4k-1}$  of non-constant sectional curvature belongs to the 2-sphere of Sasakian structures. Indeed, the cone  $CP$  over  $P$  has restricted holonomy  $Sp(k)$ , and since the centralizer of  $Sp(k)$  in  $U(2k)$  is just  $Sp(1)$ , every Kähler structure on  $CP$  must belong to the 2-sphere of Kähler structures of  $CP$ , which is equivalent to our statement.

Now,  $\bar{M}$  is regular in the direction of  $V$ , so an old result of Tanno implies that it is actually a regular 3-Sasakian manifold (cf. [14]). It is then well known that the quotient of  $\bar{M}$  by the corresponding  $SO(3)$  action is a quaternionic Kähler manifold of positive scalar curvature, say  $N$ , and that the twistor space over  $N$  is biholomorphic to the quotient of  $\bar{M}$  by each of the  $S^1$  actions given by the Sasakian vector fields, so in particular to  $M$ , which is the quotient of  $\bar{M}$  by the  $S^1$  action generated by  $V$ .

Q.E.D.

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## GENERALIZED ADHM-CONSTRUCTION ON WOLF SPACES

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### 1. Introduction

On  $4n$ -dimensional quaternion-Kähler manifolds, self-dual (SD) connections can be defined, which is the same as self-dual connections in 4-dimensional Riemannian geometry in the case  $n = 1$ .

However the situation in higher dimensional case is quite different. For example, there exists “rank 2” vector bundles with an ASD connection on 4-dimensional sphere  $S^4 \cong \mathbb{H}P^1$ , which is one of conclusions from ADHM-construction. On the contrary, algebraic geometers believe that there do *not* exist any indecomposable “rank 2” holomorphic vector bundles on 5-dimensional complex projective sapce  $\mathbb{C}P^5$  which is the Salamon twistor space of  $\mathbb{H}P^2$ . Hence it is conjectured that we have no “rank 2” ASD bundle on  $\mathbb{H}P^2$ . In general, the lack of low-rank holomorphic vector bundles on higher dimensional Kähler manifolds prevented us from finding concrete examples of ASD bundles on higher dimensional quaternion-Kähler manifolds via the twistor theory.

It is natural that we try to generalize ADHM-construction, when intending to construct some ASD bundles on higher-dimensional quaternionic projective spaces, because so called ADHM-data is comprised of finite dimensional vector spaces and linear maps between them with some conditions [4, p.97]. Indeed, this approach is adopted by Mamone Capria and Salamon [10]. (These “ $k$ -instantons” in higher dimensional case can be *classified* via vanishing theorems ([11] and [9].)) As examples independent of ASD bundles on 4-dimensional manifolds, Mamone Capria and Salamon found that the well known Horrocks bundle (rank “3”!) on  $\mathbb{C}P^5$  can be obtained as the pull-back bundle of an anti-self-dual bundle on  $\mathbb{H}P^2$  [10]. They also showed that there exists a rank 3 ASD homogeneous vector bundle on  $G_2/SO(4)$ . These were the only known concrete examples of anti-self-dual bundles on higher-dimensional quaternion-Kähler manifolds until 1990.

In my talk, these ASD bundles are reinterpreted from **representation theory** of compact Lie groups (and complexified Lie groups of them). The method of **monad**

(or ADHM-construction) is generalized to the Wolf spaces from the viewpoint of representation theory. The purpose of my talk is to give classification of irreducible homogeneous bundles with ASD canonical connections and construct non-homogeneous ASD connections on the Wolf spaces. We will obtain many examples of ASD bundles systematically, which include all the examples provided by Mamone Capria and Salamon. Secondly, the moduli spaces of such connections are described via the theory of monad and the Bott-Borel-Weil theorem. Finally, we focus attention on the boundary of the moduli spaces. Such a boundary point represents an ASD-connection “with a singular set”. The relation between such a singular set and a vector bundle on which a singular ASD connection is defined will be understood through the Poincaré duality.

2. PRELIMINARIES

• The Wolf spaces and the Salamon twistor spaces

*Theorem 2.1.* [19] For every complex simple Lie group, there exists only one compact quaternion symmetric space. (These compact quaternion symmetric spaces are called the Wolf spaces.)

*Example.* type  $A_{n+1} = Gr_2(\mathbb{C}^{n+2})$ , type  $C_{n+1} = \mathbb{H}P^n$ .

In particular, we have only two **4-dimensional** compact quaternion symmetric space :

type  $A_2 = Gr_2(\mathbb{C}^3) = \mathbb{C}P^2$ , type  $C_2 = \mathbb{H}P^1 = S^4$

*Theorem 2.2.* [7] A compact 4-dimensional manifold of which the twistor space admits a Kähler metric is conformally equivalent to  $\mathbb{C}P^2$  or  $S^4$  with a standard metric.

Let  $Z$  be the Salamon twistor space. (In the 4-dimensional case,  $Z$  is the Penrose twistor space.)

*Theorem 2.3.* [18] The total space of the twistor space  $Z$  has a natural complex structure and so,  $Z$  is a complex manifold whose dimension is  $2n + 1$ . The fibre of  $Z$  is a complex submanifold and is holomorphically isomorphic to  $\mathbb{C}P^1$ .

*Example.*

$$\begin{array}{ccc}
 \mathbb{C}P^{2n+1} & F^{2n+1} & \\
 \downarrow_{\mathbb{C}P^1} & \downarrow_{\mathbb{C}P^1} & \\
 \mathbb{H}P^n & Gr_2(\mathbb{C}^{n+2}) & \\
 & & F^{2n+1} = \frac{SU(n+2)}{S(U(1) \times U(n) \times U(1))}
 \end{array}$$

• ASD-connection

We shall treat metric connections on a complex vector bundle  $E$  equipped with a Hermitian metric  $h$  over a quaternion-Kähler manifold  $M$ .

*Definition.* [6, 10, 16] A connection  $\nabla$  is called **anti-self-dual(ASD)**

$$\stackrel{\text{def}}{\iff} R^\nabla(IX, IY) = R^\nabla(JX, JY) = R^\nabla(KX, KY) = R^\nabla(X, Y),$$

for all  $x \in M$  and all  $X, Y \in T_xM$ ,

where  $R^\nabla$  is the curvature of  $\nabla$ , which is regarded as  $\text{End } E$  valued 2-form on  $M$ .

A vector bundle with an ASD connection is called **ASD bundle** or **instanton** (bundle).

*Theorem 2.4.* [6, 10, 16] Any ASD connection is a Yang-Mills connection.

*Remark.* Moreover, if  $M$  is compact, then ASD connection minimizes the Yang-Mills functional [6, 10].

*Example. 4-dimensional case*

- **ADHM**-construction (Atiyah-Drinfeld-Hitchin-Manin)

All instanton bundles on  $S^4$  are classified by the twistor method (for example, see [1, 5]).

- All instanton bundles on  $\mathbb{C}P^2$  are also classified in a similar way by Buchdahl [3].

*Remark.* Before 1990, in the **higher-dimensional** case, concrete examples of vector bundles with ASD connections had not known except examples presented by Mamone Capria and Salamon [10].

The twistor method in the examples is explained in the next theorem, originated with Atiyah, Hitchin and Singer.

*Theorem 2.5.* [2, 10, 16] The pull-back connection of an ASD connection induces a holomorphic structure on the pull-back bundle on the twistor space  $Z$ , and so the pull-back bundle is a holomorphic vector bundle on  $Z$ .

### 3. HOMOGENEOUS ASD BUNDLES

By ADHM-construction [1, 5] and Buchdahl [3], the standard ASD bundles with  $c_2 = 1$  on  $S^4$  and  $\mathbb{C}P^2$  are homogeneous bundles with canonical connections. In this section, we determine irreducible homogeneous vector bundles with ASD canonical connections in terms of weights.

*Definition.*

$\mathfrak{g}^{\mathbb{C}}$  : complex simple Lie algebra     $B$  : the Killing form of  $\mathfrak{g}^{\mathbb{C}}$   
 $\theta$  : maximal root     $I$  : the set of integral weights

*Definition.*  $f : I \rightarrow \mathbb{Z}$

$$f(\lambda) = \frac{B(\lambda, \theta^\vee)}{B(\theta, \theta)} \quad (\lambda \in I)$$

where,  $\theta^\vee$  is the co-root of  $\theta$ . ( $\theta^\vee = 2\theta/B(\theta, \theta)$ .)

**Notation**

- $E(\lambda)$  :=the irreducible representation space of  $\mathfrak{g}^{\mathbb{C}}$   
has  $(-\lambda)$  as an extremal weight
- $E_{\mathfrak{p}}(\lambda)$  :=the irreducible representation space of  $\mathfrak{p}$   
has  $(-\lambda)$  as an extremal weight

where,  $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ : parabolic subalgebra.

$G^{\mathbb{C}}$  : simply connected Lie group whose Lie algebra is  $\mathfrak{g}^{\mathbb{C}}$

$\mathfrak{g}$  : a compact real form of  $\mathfrak{g}^{\mathbb{C}}$

$G$  : the corresponding compact simply connected Lie group to  $\mathfrak{g}$

$G/K_4$  : compact quaternion symmetric space

$G/K_Z$  : the twistor space

*Remark.* Since the twistor space  $G/K_Z$  is a compact simply connected homogeneous Kähler manifold, we can also express the twistor space using a complex simply connected Lie group  $G^{\mathbb{C}}$ . Then the twistor space is denoted by  $G^{\mathbb{C}}/P$ , where  $P$  is the corresponding parabolic subgroup of  $G^{\mathbb{C}}$ .

*Definition.*  $\mathcal{O}_{\mathfrak{p}}(\lambda) = G^{\mathbb{C}} \times_P E_{\mathfrak{p}}(\lambda)$  : irreducible homogeneous holomorphic vector bundle on the twistor space  $G^{\mathbb{C}}/P$ .

We have the classification theorem for ASD irreducible homogeneous vector bundles.

*Theorem 3.1.* [12] Let  $E$  be an irreducible homogeneous bundle over  $G/K_4$  of which the canonical connection is ASD. Then, there exists an integral weight  $\lambda$  with  $f(\lambda) = 0$  such that  $\mathcal{O}_{\mathfrak{p}}(\lambda)$  is the pull-back bundle of  $E$  on the twistor space  $G/K_Z$ . Conversely, if an integral weight  $\lambda$  satisfies  $f(\lambda) = 0$ , an irreducible homogeneous holomorphic bundle  $\mathcal{O}_{\mathfrak{p}}(\lambda)$  on  $G/K_Z$  is the pull-back of an ASD homogeneous bundle on  $G/K_4$ .

4. MONAD AND REPRESENTATION THEORY

In this section, we show that a dominant integral weight induces a monad of vector bundles on the twistor space of which the cohomology bundle is the pull-back of an ASD bundle.

*Definition.* (cf.[17]) A “**monad**” is a complex of vector bundles

$$A \xrightarrow{a} B \xrightarrow{b} C,$$

with homomorphisms  $a$  and  $b$  between them, such that  $a$  is injective and  $b$  is surjective. The quotient bundle

$$E = \text{Ker } b / \text{Im } a$$



is called the **cohomology of the monad**.

Let  $W$  be the Weyl group of  $\mathfrak{g}^{\mathbb{C}}$  and  $w^0$  be the longest element of  $W$ .

□ **The unified construction of monad**

1. We take an irreducible representation space  $E(\lambda)$  of  $G^{\mathbb{C}}$ , where  $\lambda$  is a dominant integral weight.
2. Restrict the homomorphism  $G^{\mathbb{C}} \rightarrow \text{End } E(\lambda)$  to  $P$ , then we have a complex of representation spaces:

$$E_{\mathfrak{p}}(w^0\lambda) \xrightarrow{i} E(\lambda) \xrightarrow{\pi} E_{\mathfrak{p}}(\lambda)$$

where,  $i$ :**injection**,  $\pi$ :**surjection**,  $\pi \circ i = 0$  and  $i, \pi$ :  $P$ -**equivariant homomorphism**.

We call this complex a *monad of representation*.

3. A complex of representation spaces yields a monad of vector bundles.

$$\mathcal{O}_{\mathfrak{p}}(w^0\lambda) \xrightarrow{\alpha} G^{\mathbb{C}}/P \times E(\lambda) \xrightarrow{\beta} \mathcal{O}_{\mathfrak{p}}(\lambda),$$

where,

$$\alpha([g, e]) = ([g, gi(e)] \quad \text{and} \quad \beta([g, u]) = [g, \pi(g^{-1}u)],$$

$$g \in G^{\mathbb{C}}, e \in E_{\mathfrak{p}}(w^0\lambda) \text{ and } u \in E(\lambda).$$

*Example.* • If we take the dominant integral weight  $\varpi_1$  of  $C_{n+1}$  ( $\mathbb{H}P^n$ ), then we have

$$\mathcal{O}_{\mathfrak{p}}(w^0\varpi_1) \xrightarrow{\alpha} \underline{E(\varpi_1)} \xrightarrow{\beta} \mathcal{O}_{\mathfrak{p}}(\varpi_1),$$

where,  $\underline{E(\varpi_1)} = G^{\mathbb{C}}/P \times E(\varpi_1)$ . In the case  $n = 1$ , this is nothing but a monad for 1-instanton bundle which is presented by ADHM-construction [1]: (In higher dimensional case, see [10] and [9].)

$$\mathcal{O}(-1) \xrightarrow{\alpha} \underline{E(\varpi_1)} \xrightarrow{\beta} \mathcal{O}(1).$$

• If we take two dominant integral weights  $\varpi_1$  and  $\varpi_n$  of  $A_{n+1}$  ( $Gr_2(\mathbb{C}^{n+2})$ ), then we have

$$\mathcal{O}_{\mathfrak{p}}(w^0\varpi_1) \oplus \mathcal{O}_{\mathfrak{p}}(w^0\varpi_n) \xrightarrow{\alpha} \underline{E(\varpi_1)} \oplus \underline{E(\varpi_n)} \xrightarrow{\beta} \mathcal{O}_{\mathfrak{p}}(\varpi_1) \oplus \mathcal{O}_{\mathfrak{p}}(\varpi_n).$$

In the case  $n = 1$ , this is nothing but a monad for 1-instanton bundle which is presented by Buchdahl [3]. (In higher dimensional case, see [15].)

$$\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0) \xrightarrow{\alpha} \underline{E(\varpi_1)} \oplus \underline{E(\varpi_n)} \xrightarrow{\beta} \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1),$$

*Definition.* A monad of vector bundles on  $G^{\mathbb{C}}/P$  obtained in the above way is called the *standard monad induced by  $\lambda$* .

*Theorem 4.1.* For an integral dominant weight  $\lambda$ , the following two conditions are equivalent:

1.  $f(\lambda) = 1$ .

- The cohomology bundle of the standard monad induced by  $\lambda$  is the pull-back of an ASD bundle on  $G/K_4$ .

In the next section, we apply this method on the Wolf spaces of type  $B, D, E, F$  and  $G$ .

### 5. MODULI SPACES

From now on, we pick up the dominant integral weights (or the corresponding irreducible representation spaces) of  $G^{\mathbb{C}}$ :

$$\begin{aligned} A_n &: E(\varpi_1) \oplus E(\varpi_n), & B_n &: E(\varpi_n), & C_n &: E(\varpi_1), \\ D_n &: E(\varpi_{n-1}) \oplus E(\varpi_n), & E(\varpi_1) \oplus E(\varpi_{n-1}), & & E(\varpi_1) \oplus E(\varpi_n), \\ E_6 &: E(\varpi_1) \oplus E(\varpi_6), & F_4 &: E(\varpi_4), & G_2 &: E(\varpi_1). \end{aligned}$$

These dominant integral weights (say,  $\lambda$ ) which we choose satisfy  $f(\lambda) = 1$ . As in the previous section, we obtain the standard monad of vector bundles. We describe moduli spaces of ASD bundles, which are obtained by deforming vector bundle homomorphisms  $\alpha$  and  $\beta$  of the standard monad.

#### □ Description of moduli

For simplicity, we restrict ourselves to the case that the weight which we pick up is  $\varpi_1$  of type  $C_n$ .

$$\mathcal{O}_{\mathbb{P}}(w^0\varpi_1) \xrightarrow{\alpha} E(\varpi_1) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}}(\varpi_1),$$

- Applying the Bott-Borel-Weil theorem:  
we have the identification as the  $G$ -representation spaces such that

$$\begin{aligned} H^0(\text{Hom}(\mathcal{O}_{\mathbb{P}}(w^0\varpi_1), E(\varpi_1))) &\cong \text{End } E(\varpi_1), \\ H^0(\text{Hom}(E(\varpi_1), \mathcal{O}_{\mathbb{P}}(\varpi_1))) &\cong \text{End } E(\varpi_1). \end{aligned}$$

Hence  $\alpha$  and  $\beta$  are identified with  $A \in \text{End } E(\varpi_1)$  and  $B \in \text{End } E(\varpi_1)$ , respectively.

- $\beta \circ \alpha = 0 \Leftrightarrow BA \in \mathbb{C} \oplus E(\varpi_2) \subset \text{End } E(\varpi_1)$
- $\alpha$ :injection,  $\beta$ :surjection(non-degeneracy condition)  $\Leftrightarrow \det BA \neq 0$
- the cohomology bundles are the pull-back of some ASD bundles (reality condition)  $\Leftrightarrow B = A^*$

As a result, we obtain the following.

*Theorem 5.1.* [12] The moduli spaces are identified with the following spaces, respectively.

- Table 5.1(The moduli spaces of “1-instanton bundles”)

base spaces	rep. spaces	moduli space
$A_n$	$E(\varpi_1) \oplus E(\varpi_n)$	an open cone over $\mathbb{P}(E(\varpi_2))$
$B_n$	$E(\varpi_n)$	an open ball in $E(\varpi_1)^\mathbb{R}$
$C_n$	$E(\varpi_1)$	an open ball in $E(\varpi_2)^\mathbb{R}$
$D_n$	$E(\varpi_{n-1}) \oplus E(\varpi_n)$	an open cone over $\mathbb{P}(E(\varpi_1))$
	$E(\varpi_1) \oplus E(\varpi_{n-1})$	an open cone over $\mathbb{P}(E(\varpi_{n-1}))$
	$E(\varpi_1) \oplus E(\varpi_n)$	an open cone over $\mathbb{P}(E(\varpi_n))$
$E_6$	$E(\varpi_1) \oplus E(\varpi_6)$	an open cone over $\mathbb{P}(E(\varpi_1))$
$F_4$	$E(\varpi_4)$	an open ball in $E(\varpi_4)^\mathbb{R}$
$G_2$	$E(\varpi_1)$	an open ball in $E(\varpi_1)^\mathbb{R}$

where, for example,  $E(\varpi_1)^\mathbb{R}$  denotes the real representation space of  $G$  in  $E(\varpi_1)$ .

*Remark.* In the case of  $A_2$  ( $Gr_2(\mathbb{C}^3) \cong \mathbb{C}P^2$ ), the moduli space is an open cone over  $\mathbb{P}(E(\varpi_2)) \cong \mathbb{P}(\mathbb{C}^3) \cong \mathbb{C}P^2$ . In the case of  $C_2$  ( $\mathbb{H}P^1 \cong S^4$ ), the moduli space is an open ball in  $E(\varpi_2)^\mathbb{R} \cong \mathbb{R}^5$ . These are well known moduli spaces of 1-instantons.

*Remark.* In the case of type  $G_2$  ( $G_2/SO(4)$ ), “the center” of the moduli space represents the canonical ASD connection which is found by Mamone Capria and Salamon [10].

On the complex Grassmannian manifold  $Gr_2(\mathbb{C}^{n+2})$  (the Wolf space of type  $A_{n+1}$ ), we obtain another type of ASD bundles in a slightly different way. However these ASD bundles are also 1-instantons in the case  $n = 1$  ( $\mathbb{C}P^2$ ).

*Theorem 5.2.* [13] The moduli space is identified with an open cone over  $\mathbb{P}(E(\varpi_1)) \cong \mathbb{C}P^{n+1}$ .

Finally, we introduce generalized Horrocks bundles on odd-dimensional complex projective spaces.

*Theorem 5.3.* [12] On  $\mathbb{C}P^{2n+1}$  ( $n \geq 2$ ), we have a monad of the following type:

$$\mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}(-\varpi_1 + \varpi_{n+1}) \longrightarrow \mathcal{O}(1),$$

and the cohomology bundle of this monad is the pull-back of an anti-self-dual bundle on  $\mathbb{H}P^n$ . In particular, in the case of  $n = 2$ , this cohomology bundle is the well known Horrocks bundle on  $\mathbb{C}P^5$  [8, 10].

6. SINGULAR SETS

In our geometric description in §5, obvious compactifications are suggested, (though we do not explicitly refer to the topology of the moduli spaces.)

For simplicity, we explain our theorem in the case of 1-instanton bundle  $E$  on the Wolf space of type  $B_3$  ( $Gr_4(\mathbb{R}^7)^\sim$ ). In our description of moduli by monad, the boundary point of the moduli space represents bundle homomorphisms  $a : \mathcal{O}_p(w_0\varpi_3) \rightarrow \underline{E}(\varpi_3)$  and  $b : \underline{E}(\varpi_3) \rightarrow \mathcal{O}_p(\varpi_3)$  which are *not* injective and surjective, respectively. We fix  $G$ -invariant Hermitian metrics on the homogeneous bundles  $\mathcal{O}_p(w_0\varpi_3)$  and  $\mathcal{O}_p(\varpi_3)$ . Using Hermitian metrics and bundle homomorphisms  $a$  and  $b$ , we obtain a bundle homomorphism

$$B := a^* \oplus b : \underline{E}(\varpi_3) \rightarrow \mathcal{O}_p(w_0\varpi_3) \oplus \mathcal{O}_p(\varpi_3).$$

Because of the reality condition ( $B = A^*$ ) in §5, a bundle homomorphism  $B$  pushed down to  $Gr_4(\mathbb{R}^7)^\sim$ . The subset  $S$  in  $Gr_4(\mathbb{R}^7)^\sim$  is defined:

$$S := \left\{ x \in Gr_4(\mathbb{R}^7)^\sim \mid B_x : \underline{E}(\varpi_3)_x \rightarrow \mathcal{O}_p(w_0\varpi_3)_x \oplus \mathcal{O}_p(\varpi_3)_x \text{ is not surjective} \right\}.$$

The subset  $S$  is called singular set.

*Theorem 6.1.* • The restricted bundle  $\text{Ker}B$  to  $Gr_4(\mathbb{R}^7)^\sim \setminus S$  is still an ASD bundle.  
 • The singular set  $S$  is a quaternion submanifold  $Gr_2(\mathbb{C}^4) \subset Gr_4(\mathbb{R}^7)^\sim$ .  
 • The Poincaré dual of  $S$  is the second Chern class  $c_2(E)$ .  
 • In some sense, on the singular set  $S$ ,  $E|_S$  is identified with the standard 1-instanton bundle on  $Gr_2(\mathbb{C}^4)$  which corresponds to the vertex of the moduli space (see Table 5.1).

• Table 6.1 (Singular set)

base spaces	singular set	Poincaré dual
(1) $Gr_2(\mathbb{C}^{n+2})$	1point, $\mathbb{H}P^1, \dots,$ $\mathbb{H}P^{\lfloor \frac{n}{2} \rfloor}$	$c_{2n}(E), c_{2n-2}(E), \dots,$ $c_n(E)(n:\text{even}), c_{n+1}(E)(n:\text{odd})$
(2) $Gr_2(\mathbb{C}^{n+2})$	$Gr_2(\mathbb{C}^{n+1})$	$c_2(E)$
$Gr_4(\mathbb{R}^7)^\sim$	$Gr_2(\mathbb{C}^4)$	$c_2(E)$
$\mathbb{H}P^n$	1point, $\mathbb{H}P^1, \dots, \mathbb{H}P^{n-1}$	$c_{2n}(E), c_{2n-2}(E), \dots, c_2(E)$
$G_2/SO(4)$	$\mathbb{C}P^2$	$c_2(E)$

where  $[m]$  is the greatest integer not greater than  $m$ .

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$Sp(1)^n$ -INVARIANT QUATERNIONIC KÄHLER METRIC

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We study  $Sp(1)^n$ -invariant hyperKähler or quaternionic Kähler manifolds of real dimension  $4n$ . In the case of  $n = 1$ , Hitchin classified these kinds of metrics associated with special functions. They are written as

$$g = dt^2 + \sum_{i=1}^3 f_i(t)\sigma_i^2 \quad \text{on } \mathbb{R} \times Sp(1),$$

where  $\sigma_1, \sigma_2, \sigma_3$  are canonical 1-forms associated with  $i, j, k \in \mathfrak{sp}(1)$ . We obtain a generalization of the Hitchin's result ([2]).

*Theorem 0.1.* Let  $\mathbb{H}$  be the Hamilton's quaternion number field  $\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ . Then  $\mathbb{H}^n$  has a natural quaternionic structure  $I, J, K$  induced by the action of  $i, j, k$ . Since  $\mathbb{H} \setminus \{0\}$  is diffeomorphic to  $\mathbb{R} \times Sp(1)$  canonically,  $(\mathbb{H} \setminus \{0\})^n$  is diffeomorphic to  $\mathbb{R}^n \times (Sp(1))^n$ . We denote the coordinate of  $\mathbb{R}^n$  by  $(t_1, t_2, \dots, t_n)$ . Let a Riemannian metric  $g$  be written as

$$g = \sum_{i=1}^n (dt_i^2 + \sum_{j=1}^3 f_{ij}(t_1, t_2, \dots, t_n)\sigma_{ij}^2),$$

where  $\sigma_{i1}, \sigma_{i2}, \sigma_{i3}$  are canonical 1-forms associated with  $i, j, k \in \mathfrak{sp}(1)$ . Then we obtain the following:

(i) If  $g$  is hyperKählerian with respect to the quaternionic structure  $I, J, K$ , then each  $f_{ij}(t_1, t_2, \dots, t_n)$  depend only on  $t_i$ . Hence the Riemannian metric is an  $n$ -times product of hyperKähler metric obtained by Hitchin.

(ii) If  $g$  is quaternionic Kählerian with respect to the quaternionic structure  $\mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ , then  $g$  is hyperKählerian.

By Hitchin, the coefficient functions  $f_{ij}$  satisfy

$$\begin{cases} \frac{d}{dt_i} f_{i1} = 2f_{i2}f_{i3}, \\ \frac{d}{dt_i} f_{i2} = 2f_{i3}f_{i1}, \\ \frac{d}{dt_i} f_{i3} = 2f_{i1}f_{i2}. \end{cases}$$

These equations imply the first integral

$$\begin{cases} f_{i1} - f_{i2} = a_i, \\ f_{i1} - f_{i3} = b_i, \end{cases}$$

where  $a_i, b_i$  are constant. Associated to  $(a_i \neq 0, b_i \neq 0)$ ,  $(a_i = 0, b_i \neq 0)$  and  $(a_i = 0, b_i = 0)$ , the metric is the type of Belinski-Gibbons-Page-Pope metric, Eguchi-Hanson metric and conformally flat metric.

One of our backgrounds is a natural metric on a moduli space of self-dual connections on  $\mathbb{H}$ . It coincides to a framed moduli space of self-dual connections on  $S^4$ . The quaternionic Kähler manifold  $\mathbb{H}$  has an isometry  $Sp(1) \cdot Sp(1)$ , that acts on the framed moduli space  $\mathcal{M}_k$  on a Hermitian vector bundle  $V$  of rank 2 with the second Chern class  $k$ .

$$\mathcal{M}_k = \{\nabla : \text{self-dual connection on } V, c_2(V) = k\} / \text{gauge group}.$$

The tangent space of  $\mathcal{M}_k$  is represented as the first cohomology of the following elliptic complex:

$$0 \longrightarrow \text{End}(V) \xrightarrow{\nabla} \text{End}(V) \otimes T^*\mathbb{R}^4 \xrightarrow{pr_- \circ \alpha^\nabla} \text{End}(V) \otimes \wedge_- \longrightarrow 0$$

where  $\wedge^2 T^*\mathbb{R}^4$  is decomposed into the self-dual part  $\wedge_+$  and the anti-self-dual part  $\wedge_-$ ,  $pr_- : \wedge^2 T^*\mathbb{R}^4 \rightarrow \wedge_-$  is the natural projection. The tangent space of the moduli space is represented as a subset of  $\text{End}(V)$ -valued 1-forms. The  $L_2$ -metric of  $\text{End}(V)$ -valued 1-forms induces a Riemannian metric on the moduli space  $\mathcal{M}_k$

$$\langle \alpha, \beta \rangle = \int_{\mathbb{R}^4} \text{tr}(\alpha \wedge \beta).$$

Furthermore the quaternionic structure  $I, J, K$  induces a hyperKählerian structure with respect to the Riemannian metric. It is known that the dimension of  $\mathcal{M}_k$  is  $8k$ . These are represented as elements of

$$M_{k,k+1}(\mathbb{H}) = \{(A, B) | A \in M_{k,1}(\mathbb{H}), B \in M_{k,k}(\mathbb{H})\}$$

by the A.D.H.M. construction. We denote

$$M_{k,k+1}^0(\mathbb{H}) = \{(A, B) | (A, B) \in M_{k,k+1}(\mathbb{H}), \text{tr}(B) = 0\}.$$

It corresponds to a hyperKähler submanifold in  $\mathcal{M}_k$ , whose dimension is equal to  $8k - 4$ . We denote it by  $\mathcal{M}_k^0$ . The conformal group  $(Sp(1) \times Sp(1))/\mathbb{Z}_2 \times \mathbb{R}^+ \times \mathbb{H}$  on  $\mathbb{H}$  and the gauge group  $Sp(1)/\mathbb{Z}_2$  at the infinity act on  $\mathcal{M}_k^0$

- i.  $(q, p) \in (Sp(1) \times Sp(1))/\mathbb{Z}_2, x \mapsto qxp^{-1} \quad (A, B) \mapsto (Ap, qBp),$
- ii.  $\lambda \in \mathbb{R}^+, x \mapsto \frac{1}{\lambda}x \quad (A, B) \mapsto (\lambda A, \lambda B),$
- iii.  $a \in \mathbb{H}, x \mapsto x - a \quad (A, B) \mapsto (A, B + aid),$
- iv.  $r \in Sp(1)/\mathbb{Z}_2, (A, B) \mapsto (rA, B).$

We denote vector fields generated from the action i, ii by  $V_1(\lambda), V_2(a)$ . Then the norms of  $V_1(\lambda), V_2(a)$  are constant on each orbit.



*Proposition .*

$$\|V_1(\lambda)\|^2 = \lambda^2 C_1$$

$$\|V_2(a)\|^2 = \sum_{i,j=0}^3 C_{2ij} a_i a_j,$$

where  $C_1, C_2$  are constant,  $a = a_0 + ia_1 + ja_2 + ka_3$ .

The  $Sp(1) \times \mathbb{R}^+$  acts on  $\mathcal{M}_k^0$ . The reduced space  $\mathbb{P}(\mathcal{M}_k^0)$  is known to be quaternionic Kählerian ([1]). These are not smooth manifolds, they have singularities. Now in the case  $k = 2$ ,  $\mathcal{M}_2^0$  and  $\mathbb{P}(\mathcal{M}_2^0)$  are examples that are hyperKähler or quaternionic Kähler space of dimension  $4n$  with  $Sp(1)^n$ -symmetry. In fact  $\mathcal{M}_2^0$  is a hyperKähler space of dimension  $3 \times 4$  with  $(Sp(1) \times Sp(1))/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2$ -symmetry and  $\mathbb{P}(\mathcal{M}_2^0)$  is a quaternionic Kähler space of dimension  $2 \times 4$  with  $Sp(1)/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2$ -symmetry.

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## BRANE SOLITONS AND HYPERCOMPLEX STRUCTURES

G. PAPADOPOULOS

**ABSTRACT.** The investigation of strings and M-theory involves the understanding of various BPS solitons which in a certain approximation can be thought of as solutions of ten- and eleven-dimensional supergravity theories. These solitons have a brane or an intersecting brane interpretation, saturate a bound and are associated with parallel spinors with respect to a connection of the spin bundle of spacetime. A class of intersecting brane configurations is examined and it is shown that the geometry of spacetime is hyper-Kähler with torsion. A relation between these hyper-Kähler geometries with torsion and quaternionic calibrations is also demonstrated.

### 1. INTRODUCTION

The main achievement in theoretical physics the past few years is the realization that all five string theories [1] are related amongst themselves and that are limits of another theory which has been called M-theory [2, 3]. The precise nature of M-theory remains a mystery but by now an impressive amount of evidence has been gathered which points to its existence. Most of these arise from investigating the low energy approximation of strings and M-theory which are described by ten- and eleven-dimensional supergravities, respectively; see however [4]. The use of supergravity theories in this context is two-fold. First, the conjectured duality symmetries of string theories are discrete subgroups of the continuous duality symmetries of the field equations of supergravity theories [5]. Second, the supergravity theories admit solutions which have the interpretation of extended objects embedded in the spacetime, called branes, which are the BPS solitons of strings and M-theory. A consequence of their BPS property is that branes are stable under deformations of the various parameters of the theories, like for example coupling constants. Because of this, they can be used to compare the various limits of M-theory and thus establish the relations amongst the various string theories. Extrapolating from the properties of eleven-dimensional and ten-dimensional supergravities, M-theory is thought to have the following essential ingredients:

- A low energy description in terms of the eleven-dimensional supergravity,
- M-2- and M-5-branes, and

- limits that describe all five string theories in ten dimensions.

The solutions of supergravity theories with a brane interpretation have some common properties. The associate spacetime of a  $p$ -brane has an asymptotic region isometric to  $\mathbb{R}^{(1,p)} \times \mathbb{R}^n$ , where, viewing the  $p$ -brane as a  $(p+1)$ -dimensional submanifold of spacetime,  $\mathbb{R}^{(1,p)}$  and  $\mathbb{R}^n$  are identified with the worldvolume and transverse directions of the  $p$ -brane, respectively;  $(p+n)$  is equal either to nine (string theory) or to ten (M-theory). In the above asymptotic region a mass  $m$  and a charge  $q$  per unit  $\mathbb{R}^p \subset \mathbb{R}^{(1,p)}$  volume is defined, i.e  $m$  and  $q$  are energy and charge densities, respectively. Then using the properties of supergravity theory, a bound can be established [6] as

$$m \geq \alpha |q| ,$$

where for string theory branes  $\alpha$  depends on the string coupling constant  $\lambda$ . The precise dependence of  $\alpha$  distinguishes between the various types of branes as follows:  $\alpha \sim \lambda^0$  for fundamental strings,  $\alpha \sim \lambda^{-1}$  for Dirichlet branes or D-branes for short and  $\alpha \sim \lambda^{-2}$  for Neveu-Schwarz 5-branes or NS-5-branes for short. The solutions that are of interest are those that saturate the above bound leading to BPS type configurations. BPS configurations are associated with parallel spinors with respect to a connection which occurs naturally in supergravity theories. The BPS branes of strings and M-theory admit sixteen parallel spinors. Apart from the stability of these BPS solutions that has already been mentioned above, superposition rules have been found that allow to combine two or more such solutions and construct new ones [7]. The solutions that arise from superpositions of BPS branes also admit parallel spinors which typically are less than those of the branes involved in the superposition.

In this paper, the geometry of a class of BPS brane solutions of supergravity theories and that of their superpositions will be described. I shall begin with a description of BPS M-2-brane [8] and M-5-brane [9] solutions of eleven-dimensional supergravity [10]. Then I shall explain the connection between BPS solutions and parallel spinors. I shall continue with the NS-5-brane solution of type II ten-dimensional supergravity theories [11] and show that the geometry of this solution is hyper-Kähler with torsion (HKT). Then I shall explore the various superpositions of NS-5-branes and I shall demonstrate that these superpositions are related to the quaternionic calibrations in  $\mathbb{R}^8$  [12]. I shall interpret these superpositions as intersecting NS-5-branes and I shall show that the geometry of these solutions is again HKT. Most of these results have appeared in [13, 14]. Finally, I shall state my conclusions.

## 2. ELEVEN-DIMENSIONAL SUPERGRAVITY

I shall not attempt to give a full description of eleven-dimensional supergravity. This can be found in the original paper of Cremmer, Julia and Scherk who constructed the theory [10]. Here I shall only emphasize some aspects of the geometric structure of the theory. In field theoretic terms, the theory describes the dynamics of the graviton  $g$ , a three-form gauge potential  $A$  and a gravitino  $\psi$ . The latter is a spinor-valued

one-form which does not enter in the analysis below and so it will be neglected in what follows. Geometrically, let  $(N; g, F, \nabla)$  be an eleven-dimensional spin manifold  $N$  of signature  $(-, +, \dots, +)$  equipped with a metric  $g$ , a closed four-form  $F$ , locally  $F = dA$ , and a connection  $\nabla$ . In the physics literature  $\nabla$  is called superconnection and

$$\nabla : C^\infty(S) \rightarrow \Omega^1(N) \otimes C^\infty(S) ,$$

where  $S$  is the spin bundle over  $N$  and  $\text{rank}(S) = 32$ . This connection can be written as

$$\nabla = D + T(F) ,$$

$D$  is the connection of  $S$  induced from the Levi-Civita connection of the metric  $g$  and

$$T_M(F)dx^M = -\frac{1}{144}F_{NPQR}(\Gamma_M^{NPQR} - 8\delta^N_M\Gamma^{PQR})dx^M ,$$

where  $\{\Gamma^M; M = 0, \dots, 10\}$  is a basis in the Clifford algebra  $\text{Clif}(1, 10)$  and  $\Gamma^{M_1M_2\dots M_n} = \Gamma^{[M_1}\Gamma^{M_2}\dots\Gamma^{M_n]}$ . The dynamics of the theory is described<sup>1</sup> by the action

$$S = \int d^{11}x \sqrt{|\det g|} (R(g) - 2|F|^2) - \frac{4}{3}A \wedge F \wedge F ,$$

where  $R(g)$  is the Ricci scalar of the metric  $g$  and the norm of  $F$  is taken with respect to  $g$ . The above action consists from the Einstein-Hilbert term, the standard kinetic term for  $F$  and a Chern-Simons term, respectively. The equations of the fields  $g$  and  $A$  can be derived by varying the above action.

There are two classes of solutions to the field equations depending on whether or not  $F = 0$ . If  $F = 0$ , then the field equations imply that the Ricci tensor of  $g$  vanishes. Therefore a large class of solutions is  $\mathbb{R}^{(1,10-n)} \times M^n$ , where  $M^n$  is a manifold of appropriate special holonomy, i.e  $SU(k)$ ,  $k = 2, 3, 4$  ( $n = 2k$ );  $Sp(2)$  ( $n = 8$ );  $G_2$  ( $n = 7$ );  $Spin(7)$  ( $n = 8$ ). Such solutions admit parallel spinors and have found application in the various compactifications of M-theory [15]. The other class of solutions is that for which  $F \neq 0$ . For such solutions to have a brane interpretation, it is required that they have an asymptotic region which is isometric to either  $\mathbb{R}^{(1,2)} \times \mathbb{R}^8$  or  $\mathbb{R}^{(1,5)} \times \mathbb{R}^5$ . The former asymptotic behaviour is that of M-2-brane while the latter is that of M-5-brane. Then after imposing appropriate decaying conditions on the fields as they approach these asymptotic regions, the charges per unit volume of the M-2- and M-5-branes can be defined as follows:

$$q_2 = \int_{S^7} (*F - A \wedge F) ,$$

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<sup>1</sup>The conventions for forms are as follows:  $\omega = \frac{1}{p!}\omega_{a_1,\dots,a_p}dx^{a_1} \wedge \dots \wedge dx^{a_p}$ ,  $|\omega|^2 = \frac{1}{p!}\omega_{a_1,\dots,a_p}\omega^{a_1,\dots,a_p}$  and  $**\omega = -(-1)^{p(n-p)}\omega$ , where  $*$  is the Hodge star operation and  $n = 11$  in the present case.

where  $S^7 \subset \mathbb{R}^8$ , and

$$q_5 = \int_{S^4} F ,$$

where  $S^4 \subset \mathbb{R}^5$ , respectively. Then adapting the positive mass theorem of general relativity to this case, the bounds can be established,

$$m_2 \geq \alpha_2 |q_2|$$

or

$$m_5 \geq \alpha_5 |q_5| ,$$

where  $m_2$  and  $m_5$  are the M-2-brane and M-5-brane masses per unit volume, respectively. The manifolds that saturate those bounds admit sixteen parallel spinors with respect to the connection  $\nabla$ .

To be specific, the BPS M-2-brane solution [8] is

$$\begin{aligned} ds^2 &= h^{\frac{1}{3}} (h^{-1} ds^2(\mathbb{R}^{(1,2)}) + ds^2(\mathbb{R}^8)) \\ *F &= \mp \frac{1}{2} \star dh , \end{aligned}$$

where

$$h = 1 + \frac{q_2}{|y|^6}$$

is a harmonic function on  $\mathbb{R}^8$ ,  $y \in \mathbb{R}^8$ , the Hodge star operation on  $F$  is with respect to the metric  $g$  on  $N$  and the Hodge star operation on  $dh$  is with respect to the flat metric on  $\mathbb{R}^8$ . The M-2-brane is located at  $y = 0$ . There are two asymptotic regions. One is as  $|y| \rightarrow \infty$ , where the spacetime  $N$  becomes isometric to  $\mathbb{R}^{(1,2)} \times \mathbb{R}^8$  as expected. The other is as  $|y| \rightarrow 0$  in which case  $N$  becomes isometric to  $AdS_4 \times S^7$ ;  $AdS_4$  is a Minkowski signature analogue of the standard hyperbolic four-manifold. It turns out though that the BPS membrane solution develops a singularity behind the  $AdS_4 \times S^7$  region.

The BPS M-5-brane solution [9] is

$$\begin{aligned} ds^2 &= h^{\frac{2}{3}} (h^{-1} ds^2(\mathbb{R}^{(1,2)}) + ds^2(\mathbb{R}^5)) \\ F &= \mp \frac{1}{2} \star dh , \end{aligned}$$

where

$$h = 1 + \frac{q_5}{|y|^3}$$

is a harmonic function on  $\mathbb{R}^5$ ,  $y \in \mathbb{R}^5$  and the Hodge star operation on  $dh$  is with respect to the flat metric on  $\mathbb{R}^5$ . The M-5-brane is located at  $y = 0$ . There are also two asymptotic regions in this case. One is as  $|y| \rightarrow \infty$  where the spacetime  $N$  becomes isometric to  $\mathbb{R}^{(1,5)} \times \mathbb{R}^5$  as expected. The other is as  $|y| \rightarrow 0$  in which case  $N$  becomes isometric to  $AdS_7 \times S^4$ ;  $AdS_7$  is a Minkowski signature analogue of the standard hyperbolic seven-manifold. The BPS M-5-brane solution is not singular.

Despite much progress in constructing solutions of eleven-dimensional supergravity which admit parallel spinors, there is no systematic way to tackle the problem of constructing solutions of the theory with any number of parallel spinors. Of course

this is related to understanding the properties of the connection  $\nabla$ . To give an example of the subtleties involved, let us consider the case of solutions with thirty two parallel spinors which is the maximal number possible. A straightforward example of a such spacetime is the Minkowski  $N = \mathbb{R}^{(1,10)}$  space with vanishing  $F$ . However, this is not all. There are two more cases,  $N = AdS_4 \times S^7$  with  $F$  the volume form of  $AdS_4$  and  $N = AdS_7 \times S^4$  with  $F$  the volume form of  $S^4$ .

A related and still unresolved problem is to construct *localized* solutions of eleven-dimensional supergravity theory with the interpretation of intersecting M-2- and M-5-branes. In particular consistency of the M-theory picture suggests that there should be a solution that has the interpretation of a M-2-brane ending orthogonally on a M-5-brane associated with eight parallel spinors. No such solution has been found so far; For a recent discussion of this see [16]. Other BPS solutions involving M-5-branes and M-2-branes have been found though, like a solution which has the interpretation of a M-2-brane ‘passing through’ a M-5-brane [17].

### 3. TYPE IIA STRINGS

A geometric insight into the properties of the connection  $\nabla$  of eleven-dimensional supergravity can be given after reducing M-theory to type IIA string theory. It turns out that in a sector of the supergravity theory associated with type IIA strings, IIA supergravity, the connection  $\nabla$  of the spin bundle is induced from certain connections with torsion of the tangent bundle of spacetime.

For this consider the eleven-dimensional spacetime  $N = S^1 \times M$  with

$$(3.1) \quad \begin{aligned} ds^2(N) &= e^{\frac{4}{3}\phi} d\theta^2 + e^{-\frac{2}{3}\phi} ds^2(M) \\ F &= d\theta \wedge H, \end{aligned}$$

where  $\theta$  is the angle which parameterizes the circle of radius  $r$ . It is assumed that the vector field  $X = \frac{\partial}{\partial\theta}$  is an isometry of  $N$  which leaves in addition  $F$  invariant. In field theoretic terms, the metric  $\gamma$  in  $ds^2(M)$  describes the graviton in ten dimensions, the closed three-form  $H$  is the  $NS \otimes NS$  form field strength and  $\phi$  is the dilaton;  $\gamma$ ,  $H$  and  $\phi$  are the so called common sector fields of string theory. The IIA supergravity has additional fields which can also be derived from eleven dimensions but the above sector will suffice for our purpose. The type IIA string coupling constant is related to the radius of the circle  $S^1$  as  $\lambda = r^{\frac{3}{2}}$  [3]. So for small radius, the string coupling constant is small and M-theory reduces to IIA strings.

The dynamics of the common sector fields in ten dimensions can be described by the action

$$S = \int d^{10}x e^{-2\phi} \sqrt{|\det \gamma|} (R(\gamma) - 2|H|^2 + 4|d\phi|^2),$$

where the norms are taken with respect to the metric  $\gamma$  on  $M$ .

As it has been mentioned, a simplification occurs in the description of the connection  $\nabla$  using  $N = S^1 \times M$  and 3.1 as above. For this, first observe that the spin bundle  $S$  decomposes as  $S = S^+ \otimes S^-$ , where  $S^+$  and  $S^-$  are spin bundles over  $M$

with  $\text{rank } S^+ = \text{rank } S^- = 16$ . This is due to the decomposition of the spinor representation of  $\text{spin}(1, 10)$  into the sum of the two irreducible spinor representations of  $\text{spin}(1, 9)$ . Next it turns out that the connection  $\nabla$  decomposes into two connections one on  $S^+$  and one on  $S^-$  which are induced by the connections

$$\nabla^\pm = D \pm H$$

of the tangent bundle, respectively, where  $D$  is the Levi-Civita connection of the metric  $\gamma$ . The connections  $\nabla^\pm$  are metric connections with torsion the closed three-forms  $\pm H$ , respectively. There are two more conditions that arise from reducing the connection  $\nabla$  of eleven-dimensional supergravity to IIA supergravity which involve the dilaton  $\phi$ . However, these two conditions do not give rise to additional restrictions on the parallel spinors of  $\nabla^\pm$  connections in the examples that we shall investigate below. So we shall not consider them further.

The above simplification in the structure of the connection  $\nabla$  has some profound consequences. One of them is that the existence of parallel spinors with respect to the connection  $\nabla$  of  $S$  depends on the holonomy of the connections  $\nabla^\pm$  of the tangent bundle of  $M$ .

The NS-5-brane solution of IIA supergravity [11] is

$$\begin{aligned} ds^2(M) &= ds^2(\mathbb{R}^{(1,5)}) + h ds^2(\mathbb{Q}) \\ H &= \mp \frac{1}{2} \star dh \\ e^{2\phi} &= h, \end{aligned}$$

where the Hodge star operation on  $dh$  is with respect to the flat metric on  $\mathbb{R}^4$  and  $h$  is a harmonic function on  $\mathbb{R}^4 = \mathbb{Q}$ ,  $\mathbb{Q}$  is the quaternionic line,

$$h = 1 + \frac{1}{|q|^2},$$

$q \in \mathbb{Q}$ . The NS-5-brane is located at  $q = 0$ . The spacetime  $M$  is diffeomorphic to  $\mathbb{R}^{(1,5)} \times (\mathbb{Q} - \{0\})$  and it has two asymptotic regions,  $\mathbb{R}^{(1,5)} \times \mathbb{R}^4$  as  $|q| \rightarrow \infty$  and  $\mathbb{R}^{(1,5)} \times \mathbb{R} \times S^3$  as  $|q| \rightarrow 0$ . In what follows we shall choose  $H = -\frac{1}{2} \star dh$ .

The non-trivial part of the metric of  $M$  is that on  $\mathbb{Q} - \{0\}$ . To investigate the geometry on  $(\mathbb{Q} - \{0\})$ , we introduce two hypercomplex structures  $\mathbf{I} = \{I_1, I_2, I_3\}$  and  $\mathbf{J} = \{J_1, J_2, J_3\}$  as follows:

$$\begin{aligned} I_1(dq) &= i dq, & I_2(dq) &= j dq, & I_3(dq) &= k dq, \\ J_1(dq) &= -dq i, & J_2(dq) &= -dq j, & J_3(dq) &= -dq k, \end{aligned}$$

where  $i, j, k$  are the imaginary unit quaternions. Observe that the two hypercomplex structures commute,  $[\mathbf{I}, \mathbf{J}] = 0$ . The metric on  $\mathbb{Q} - \{0\}$  is hermitian with respect to both hypercomplex structures. In addition, the hypercomplex structure  $\mathbf{I}$  is compatible with the  $\nabla^-$  connection ( $\nabla^- \mathbf{I} = 0$ ) and the hypercomplex structure  $\mathbf{J}$  is compatible with the  $\nabla^+$  connection ( $\nabla^+ \mathbf{J} = 0$ ); Note that the torsion  $H$  has support on  $\mathbb{Q} - \{0\}$ . Therefore the holonomy of  $\nabla^\pm$  is in  $SU(2)$ . In fact the holonomy of  $\nabla^\pm$  is  $SU(2)$  and so the NS-5-brane admits sixteen parallel spinors. This fact follows



from representation theory. As it will be demonstrated shortly, the geometry of the NS-5-brane can be summarized by saying that it admits two commuting strong HKT structures.

#### 4. HYPER-KÄHLER MANIFOLDS WITH TORSION

Let  $(M, g, \mathbf{J})$  be a Riemannian hyper-complex manifold with metric  $g$  and hyper-complex structure  $\mathbf{J}$ ;  $\dim M = 4k$ . The manifold  $(M, g, \mathbf{J})$  admits a HKT structure [18] if

- The metric  $g$  is hermitian with respect to all three complex structures.
- There is a compatible connection  $\nabla$  with both the metric  $g$  and the hypercomplex structure  $\mathbf{J}$  which has torsion a three form  $H$ .

There are two types of HKT structures on manifolds, the *strong* and the *weak*, depending on whether or not the torsion three-form is a closed, respectively.

Torsion has appeared in the physics literature since the early attempts to incorporate it in a relativistic theory of gravity. In supersymmetry, metric connections with torsion a closed three-form have appeared in the context of IIA and IIB supergravities but the relation to HKT geometry was not established. Connections with torsion a closed three-form appeared next in the investigation of two-dimensional supersymmetric sigma models [19, 20]. For a class of models, the sigma model manifold satisfies conditions which can be organized in one or two copies of what it is now called *strong* HKT structure. The general case of connections with torsion *any* three-form were found in the investigation of one-dimensional supersymmetric sigma models [21, 22]. For a class of models, the sigma model manifold satisfies conditions which can be organized as one or two copies of what it is now called *weak* HKT structure. The definition of the HKT structure as a new structure on manifolds was given in [18]. In the same paper, the strong and weak HKT structures were introduced, a formulation of a HKT structure in terms of conditions on Kähler forms was given, and a twistor construction for the HKT manifolds was proposed. The latter two properties of HKT manifolds are similar to those of hyper-Kähler manifolds [23]. There is also a generalization of the Quaternionic Kähler structure on manifolds to include torsion. The Quaternionic Kähler manifolds with torsion (QKT) have been introduced in [24] and further investigated in [25]. QKT manifolds admit a twistor construction similar to that of Quaternionic Kähler ones [26].

A straightforward consequence of the definition of HKT manifolds is that the holonomy of the connection  $\nabla$  is in  $Sp(k)$ . Some other developments related to these connections with three-form torsion are the vanishing theorems of [27, 28] for certain cohomology groups. Many examples of HKT manifolds have been constructed. These include a class of group manifolds with strong HKT structures in [29]. Specifically, the Hopf surface  $S^1 \times S^3$  admits two strong HKT structures. Homogeneous weak HKT manifolds have been constructed in [30] using the hypercomplex structures of [31]. Inhomogeneous weak HKT structures have been given on  $S^1 \times S^{4k-1}$  in [32].

In physics, the NS-5-brane solution constructed in the previous section clearly admits two strong HKT structures each associated with the hypercomplex structures on  $\mathbb{Q} - \{0\}$  defined by left and right quaternionic multiplication, respectively. Other examples are certain (strong and weak) HKT structures that appear on the moduli spaces of five-dimensional black holes [22, 33, 34].

The HKT structure has many properties some of them found in [18] and more have been derived in [32]. One of them is the following: Let  $M$  be hypercomplex manifold with respect to  $\mathbf{J}$  and equipped with a three-form  $H$ .  $M$  admits a HKT structure if

$$(4.1) \quad d\omega_{\mathbf{J}} - 2i_{\mathbf{J}}H = 0 ,$$

where  $\omega_{\mathbf{J}}$  are the three Kähler forms associated with the hyper-complex structure and  $i_{\mathbf{J}}$  are the inner derivations with respect to the three complex structures. This equation will be used later to construct new HKT manifolds. In fact if two of the above conditions are satisfied, they imply the third. Observe also that the torsion of an HKT manifold can be specified from the metric and the complex structures [18]. So in what follows, we shall not give the expression for the torsion.

### 5. QUATERNIONIC CALIBRATIONS

Calibrations have been introduced by Harvey and Lawson [35] to construct a large class of minimal submanifolds. Here, I shall use calibrations to find a new class of solutions of IIA supergravity that has the interpretation of intersecting branes. This new class of solutions admits a strong HKT structure.

A calibration of degree  $k$  is a  $k$ -form  $\omega$  such that for every  $k$ -plane  $\eta$  in  $\mathbb{R}^n$

$$\omega(\vec{\eta}) \leq 1 ,$$

where  $\vec{\eta}$  is the co-volume form of  $\eta$ .

The contact set  $G_\omega$  of a calibration is the subset of  $\text{Gr}(k, \mathbb{R}^n)$  of  $k$ -planes that saturate the above bound. Usually  $G_\omega$  is a homogeneous space. There are many examples of calibrations, like Kähler and Special Lagrangian, which have been extensively investigated both in mathematics and physics. In the present case, the relevant class of calibrations are the quaternionic calibrations that have been described by Daroc, Harvey and Morgan in [12]. For this, we identify  $\mathbb{R}^8 = \mathbb{Q}^2$  and introduce the hypercomplex structures  $\mathbb{I} = \{I_1, I_2, I_3\}$  and  $\mathbb{J} = \{J_1, J_2, J_3\}$  on  $\mathbb{Q}^2$  induced by left and right quaternionic multiplication, respectively, i.e.

$$\begin{aligned} I_1(du) &= i du , & I_2(du) &= j du , & I_3(du) &= k du , \\ J_1(du) &= -du i , & J_2(du) &= -du j , & J_3(du) &= -du k , \end{aligned}$$

where  $i, j, k$  are the imaginary unit quaternions. In addition, we define

$$\phi_{\mathbf{J}} = \frac{1}{3} \sum_{r=1}^3 \omega_{J_r} \wedge \omega_{J_r} ,$$

where  $\omega_{J_r}$  is the Kähler form of the  $J_r$  complex structure with respect to the standard metric on  $\mathbb{Q}^2$ , and similarly  $\phi_{\mathbb{I}}$  for  $\mathbb{I}$ . There are several calibration forms that can be constructed using the Kähler forms above. The calibration forms of quaternionic calibrations are found by averaging the calibration form  $\phi_{\mathbb{J}}$  with various calibration forms constructed using the  $\mathbb{I}$  hypercomplex structure. These calibration forms and their contact sets are summarized in the following table:

<u>Quaternionic Calibrations in <math>\mathbb{R}^8</math></u>	
Calibration Form $\omega$	Contact Set $G_\omega$
$\frac{1}{2}(\phi_{\mathbb{I}} + \phi_{\mathbb{J}})$	$S^1$
$\frac{1}{5}\omega_{I_1} \wedge \omega_{I_1} + \frac{3}{5}\phi_{\mathbb{J}}$	$S^2$
$\frac{1}{4}\Omega + \frac{3}{4}\phi_{\mathbb{J}}$	$S^3$
$\phi_{\mathbb{I}}$	$S^4$

In the second case above, instead of  $I_1$  any other of the complex structures of  $\mathbb{I}$  can be used. In the third case, the form  $\Omega$  is the *Spin*(7) invariant form associated with  $\mathbb{I}$ . In the last case, the contact set is the grassmannian of quaternionic lines in  $\mathbb{Q}^2$ ,  $Gr(1; \mathbb{Q}^2) = S^4$ . The contact sets of the quaternionic calibrations are computed by observing that the groups that leave invariant the above forms act transitively on the calibrated planes. For more details see [12].

### 6. HKT GEOMETRIES AND BRANES

The strategy of constructing HKT geometries in eight dimensions is to superpose a HKT geometry on  $\mathbb{Q} - \{0\}$  along four-planes in  $\mathbb{R}^8$ . The four-planes which will be used are in the contact sets of quaternionic calibrations that has been described in the previous section. The construction [13, 14] involves the following steps:

(i) Consider the HKT metric

$$(6.1) \quad d\sigma^2 = \frac{1}{|q|^2} dqd\bar{q}$$

on  $\mathbb{Q} - \{0\}$ , where  $q \in \mathbb{Q}$ . The torsion of this HKT geometry is that of the NS-5-brane that we have described.

(ii) Introduce the maps

$$\begin{aligned} \tau : \quad \mathbb{Q}^2 &\rightarrow \mathbb{Q} \\ u &\rightarrow \tau(u) = p_1u^1 + p_2u^2 - a, \end{aligned}$$

where  $p_1, p_2, a$  are quaternions.

(iii) Define the metric

$$ds^2 = \sum_{\tau} r^2(\tau)\tau^*d\sigma^2 + ds^2(\mathbb{Q}^2),$$

where the sum is over a finite number of maps  $\tau$ ,  $r(\tau) \in \mathbb{R}$  and  $ds^2(\mathbb{Q}^2)$  is the standard flat metric on  $\mathbb{Q}^2$ .

The manifold  $K = \mathbb{Q}^2 - \cup_{\tau} \{\tau^{-1}(0)\}$  admits a HKT structure associated with the hypercomplex structure  $\mathbf{J}$  induced by the hypercomplex structure  $\mathbb{J}$  on  $\mathbb{Q}^2$ . To show this, the torsion of the HKT structure on  $K$  is given by pulling back the torsion of the NS-5-brane with respect to the maps  $\tau$  and then summing up over the various maps  $\tau$ . The key observation is that the differential  $d\tau$  commutes with the hypercomplex structures on  $\mathbb{Q} - \{0\}$  and  $K$  defined by *right* quaternionic multiplication. Using this, the condition 4.1 for  $K$  can be written as

$$\sum_{\tau} \tau^*(d\omega_{\mathbf{J}} - 2i_{\mathbf{J}}H) = 0 ,$$

where the expression inside the brackets is that of the condition 4.1 for the HKT structure (i) on  $\mathbb{Q} - \{0\}$  and so vanishes identically. Thus  $K$  admits a HKT structure with respect to the  $\nabla^+$  connection and  $\mathbf{J}$  hypercomplex structure.

Observe that  $d\tau$  for generic parameters  $\{p_1, p_2\}$  does not commute with the hypercomplex structure induced by left quaternionic multiplication on  $\mathbb{Q} - \{0\}$  and  $K$ . So  $K$  does not admit a HKT structure with respect to this hypercomplex structure.

To make connection with the calibrations of the previous sections as promised, we consider a HKT geometry constructed using several maps  $\tau$  with generic parameters  $(p_1, p_2; a)$ . Such a HKT geometry is independent from the parameterization of the maps  $\tau$  and depends only on the arrangements of quaternionic lines  $\tau^{-1}(0)$  in  $\mathbb{Q}^2$ . To see this, observe that two maps  $\tau$  and  $\tau'$  give the same HKT structure if their parameters are related as

$$(p'_1, p'_2; a') = (up_1, up_2; ua)$$

for some  $u \in \mathbb{Q}, u \neq 0$ . So the inequivalent HKT structures associated with each map  $\tau$  are parameterized by the bundle space of the canonical quaternionic line bundle over the Grassmannian  $\text{Gr}(1; \mathbb{Q}^2)$ . In turn the quaternionic lines defined by  $\text{Ker } d\tau$  are in  $\text{Gr}(1; \mathbb{Q}^2)$  which is precisely the contact set of last calibration in the table given in the previous section. Observe that the calibration form and the HKT connection  $\nabla^+$  are compatible with the same hypercomplex structure  $\mathbf{J}$ .

The HKT geometries that we are considering are complete provided that the subspaces  $\tau^{-1}(0)$  are in general position. Near the intersection of two such subspaces, the HKT metric is isometric to that of  $(\mathbb{Q} - \{0\}) \times (\mathbb{Q} - \{0\})$ , where the metric on  $\mathbb{Q} - \{0\}$  is given as in 6.1.

Finally, the above HKT geometries can be used to construct new solutions of IIA supergravity as

$$\begin{aligned} ds^2(M) &= ds^2(\mathbb{R}^{(1,1)}) + ds^2(K) \\ e^{2\phi} &= (\det\gamma)^{\frac{1}{4}} . \end{aligned}$$

The IIA supergravity three-form field strength  $H$  is given in terms of the torsion of the HKT manifold  $K$ . The brane interpretation of such solution is that of NS-5-branes intersecting on a string. The NS-5-brane associated with the map  $\tau$  is located at  $\tau^{-1}(0)$ .

7. SPECIAL CASES

As we have seen for a generic choice of maps  $\tau$  the HKT geometries in eight dimensions found in the previous section were associated with quaternionic lines in  $\mathbb{Q}^2$  given by  $\text{Ker } d\tau$ . This establishes a correspondence between HKT geometries and quaternionic calibrations with calibration form  $\phi_J$ . This correspondence can be extended to the rest of the quaternionic calibrations. For this, instead of considering generic maps  $\tau$  with parameters  $(p_1, p_2; a)$  to construct the HKT geometries, we restrict them in an appropriate way. There are four cases to consider, including the HKT geometry on  $K$  that has been mentioned above, as illustrated in the following table:

HKT Geometries in Eight Dimensions

$\bar{p}_1 p_2$	$\text{Ker } d\tau$
$\mathbb{R}$	$S^1$
$\mathbb{C}$	$S^2$
$\text{Im}\mathbb{Q}$	$S^3$
$\mathbb{Q}$	$S^4$

In the first column we denote the restriction on the parameters of the map and in the second column the set that  $\text{ker } d\tau$  lies as we vary the map  $\tau$  in the same class. Comparing the above table with that which contains the contact sets of quaternionic calibrations in section five, we observe that  $\text{ker } d\tau$  lies in a contact set in all four cases.

The holonomy groups of the connections  $\nabla^\pm$  in each of the above cases are given in the following table:

$\bar{p}_1 p_2$ :	$\mathbb{R}$	$\mathbb{C}$	$\text{Im}\mathbb{Q}$	$\mathbb{Q}$
$\nabla^-$ :	$Sp(2)$	$SU(4)$	$Spin(7)$	$SO(8)$
$\nabla^+$ :	$Sp(2)$	$Sp(2)$	$Sp(2)$	$Sp(2)$

The holonomy of  $\nabla^+$  is in  $Sp(2)$  in all cases because it is compatible with the  $J$  hypercomplex structure. Now if  $\bar{p}_1 p_2$  is real for all  $\tau$  involved in the construction of HKT geometry, then  $d\tau$  commutes with the hypercomplex structures of  $K$  and  $\mathbb{Q} - \{0\}$  induced by *left* quaternionic multiplication. This leads to another HKT structure on  $M$  compatible with the  $\nabla^-$  connection. So the holonomy of  $\nabla^-$  is in  $Sp(2)$  as well. This HKT geometry was found in [36] and a special case in [37]. If  $\bar{p}_1 p_2$  is complex, say, with respect to the  $I_1$  complex structure, then  $d\tau$  commutes with the complex structures of  $K$  and  $\mathbb{Q} - \{0\}$  induced by left quaternionic multiplication with the quaternionic unit  $i$ . This makes the  $\nabla^-$  connection compatible with the  $I_1$  complex structure which implies that the holonomy of  $\nabla^-$  is in  $U(4)$ . In fact it turns out that the holonomy of  $\nabla^-$  is in  $SU(4)$ . Observe that the Kähler form of  $I_1$  appears in the construction of the calibration form in this case. A similar analysis can be done for the remaining case.

Some of the above HKT geometries can be related to toric hyper-Kähler geometries [23, 36]. In particular, the HKT geometries associated with maps  $\tau$  such that  $\bar{p}_1 p_2 \in \mathbb{R}$  are T-dual (mirror symmetry) to toric hyper-Kähler manifolds [22]. In this case mirror

symmetry transforms manifolds of one class, HKT manifolds, to manifolds of another class, hyper-Kähler manifolds. This is because the T-duality above is performed as many times as the number of tri-holomorphic vector fields of the toric hyper-Kähler manifold which is *less* than the middle dimension of the manifold. This is unlike the case of mirror symmetry for Calabi-Yau spaces where T-duality is performed in as many directions as the middle dimension of the manifold [38]. The HKT geometries with  $\bar{p}_1 p_2 \in \mathbb{R}$  also appear in the context moduli spaces of a class of black holes in five dimensions [22].

It is of interest to ask the question whether it is possible to construct supergravity solutions which have the interpretation of intersecting branes using other calibrations from those employed above. To be more specific instead of the quaternionic calibrations, one may also use Kähler or special Lagrangian calibrations to do the superposition. Unfortunately, in both these cases, a superposition similar to that employed for quaternionic calibrations does not lead to solutions of supergravity field equations. This may be due to the fact that the resulting geometries depend on the particular parameterization of the maps  $\tau$ . On the other hand string perturbation theory considerations seem to suggest that superpositions of the kind employed above lead to BPS brane configurations [39, 40, 41]. However, it is not known how to construct in a systematic way the corresponding supergravity solution from a BPS brane configuration of string theory.

## 8. CONCLUDING REMARKS

The understanding of the non-perturbative properties of string theory requires the investigation of various solitons. In the low energy approximation these solitons have an interpretation as branes or as intersecting branes and are solutions of various supergravity theories. A class of such solutions was presented and their construction was related to quaternionic calibrations.

The problem of finding the intersecting brane solutions of supergravity theories has not been tackled in complete generality. Although many examples of such solutions are known, there does not seem to be a systematic way to find a solution for each BPS brane configuration of string theory. The resolution of this will require a better understanding of the supercovariant derivative of supergravity theories. The method of calibrations that I have presented led to the construction of a large class of these solutions but it has limitations some of which have already been mentioned. However, the solutions that we are seeking for which the form field strengths do not vanish ( $F \neq 0$ ) are in the same universality class as hyper-Kähler, Calabi-Yau and other special holonomy manifolds as far as the holonomy of the supercovariant connections of the supergravity theories is concerned. So it may be that powerful methods of algebraic and differential geometry that have been developed to construct examples of the latter may be extended to find examples of the former.

After the end of the conference, the moduli space of a class of five-dimensional black holes was determined and it was found to be a weak HKT manifold [33]. This

result was further generalized in [34] to a larger class of four- and five-dimensional black holes. The moduli spaces of all five-dimensional black holes that admit at least four parallel spinors are HKT manifolds.

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## HYPERCOMPLEX GEOMETRY

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### 1. INTRODUCTION

A manifold  $M$  is said to be *hypercomplex* if there exist three integrable complex structures  $I_1, I_2, I_3$  on  $M$  satisfying the quaternion identities:  $I_1 I_2 = -I_2 I_1 = I_3$ .

**Example 1.** Let  $\mathbb{H}$  denote the quaternion numbers and consider  $(\mathbb{H} \setminus 0)^n = (S^3 \times \mathbb{R})^n$ . Define a hypercomplex structure by

$$I_\lambda(\vec{q}, \vec{x}) = (\vec{q}\lambda, \vec{x})$$

for  $(\vec{q}, \vec{x}) \in (S^3)^n \times (\mathbb{R})^n$  and  $\lambda \in \{i, j, k\}$ . Note that this structure is left invariant. We get compact examples on  $(S^3 \times S^1)^n$  via  $(\mathbb{Z})^n$ -quotients of  $(S^3 \times \mathbb{R})^n$ .

Thus, the Hopf surface  $S^3 \times S^1$  is a simple example of a compact hypercomplex manifold. In the following we shall generalize this example in three directions. The Hopf surface together with the projection  $S^3 \times S^1 \rightarrow S^3$  is an example of a special Kähler-Weyl 4-manifold  $M^4$  with symmetry, fibering over an Einstein-Weyl 3-space  $M^4 \rightarrow B^3$ . This point of view leads to a construction of hypercomplex 4-manifolds via Abelian monopoles and geodesic congruences on Einstein-Weyl 3-manifolds [6].

We may also think of the Hopf surface as the Lie group  $SU(2) \times S^1$  with a homogeneous hypercomplex structure. Spindel et al. [20] and independently Joyce [12] showed how such homogeneous structures may be constructed on  $G \times T^k$  for  $G$  a compact Lie group. Using the twistorial description of hypercomplex geometry [16], we may bring complex deformations to bear on these examples and obtain non-homogeneous structures on  $G \times T^k$  [17].

The third theme we shall address is the following: to any quaternionic  $4n$ -manifold  $M$  we may associate a hypercomplex  $(4n+4)$ -space  $\mathcal{V}(M)$  [18] generalizing the Swann bundle of a quaternionic Kähler manifold [21]. Joyce [12] showed how to twist this construction with an instanton  $P \rightarrow M$  to obtain a hypercomplex manifold  $\mathcal{V}_P(M)$  fibering over  $M$  with fiber the Hopf surface  $S^3 \times S^1$ . Again such structures may be deformed using twistor theory [16].

2. KÄHLER-WEYL 4-MANIFOLDS

Consider a hypercomplex 4-manifold  $M$ . On  $M$  we may define a conformal structure  $[g]$ : to each non-zero vector  $X$  we declare  $(X, I_1X, I_2X, I_3X)$  to be orthonormal. Any hypercomplex manifold has a unique torsion-free connection preserving each of the complex structures, the Obata connection  $D$  [14]. This connection clearly preserves the conformal structure, so we have a Weyl manifold  $(M, [g], D)$  [6]. A Weyl manifold with vanishing trace-free-symmetric part of the Ricci curvature  $S_0r^D$  is called *Einstein-Weyl* [5]. In the following we shall see how Einstein-Weyl geometry in 3 and 4 dimensions interacts with hypercomplex geometry.

Let  $V_{\pm}$  be the spin bundles and let  $L$  be the bundle coming from the representation  $A \mapsto |\det(A)|^{\frac{1}{4}}$ . Then the complexified tangent bundle  $T_cM$  is equal to  $V_+ \otimes V_- \otimes L$  and the curvature

$$R^D = W_+ + W_- + S_0r^D + F_+^D + F_-^D + s^D$$

of the Weyl connection  $D$  is contained in

$$L^{-2} \otimes (S^4V_+ \oplus S^4V_- \oplus (S^2V_+ \otimes S^2V_-) \oplus S^2V_+ \oplus S^2V_- \oplus \mathbb{R}).$$

For a hypercomplex manifold, the structure is reduced to  $\mathbb{R}_{>0} \times \text{SU}(2)_+$ , so the curvature is contained in  $L^{-2} \otimes (S^4V \oplus S^2V_+)$ . Therefore, half of the Weyl curvature vanishes,  $W_- = 0$ , the trace-free-symmetric part of the Ricci curvature vanishes,  $S_0r^D = 0$ , half of the Faraday curvature vanishes,  $F_-^D = 0$ , and the scalar curvature  $s^D$  vanishes. In particular, a hypercomplex manifold is an example of a special selfdual 4-manifold (which is also Einstein-Weyl).

Via the Penrose correspondence, a selfdual conformal 4-manifold  $M$  with a conformal Killing vector  $K$  corresponds to a 3-dimensional complex twistor space  $Z$  with a complex holomorphic vector field  $K_c$  [1]. The quotient  $M/K$  is an Einstein-Weyl 3-space  $B$  with a monopole  $(w, A)$  consisting of a section  $w$  of  $L^{-1}$  and a 1-form  $A$  such that  $*D^B w = dA$  [10]. The quotient  $Z/K_c$  is the minitwistor space  $S$  of  $B$  [8].

A conformal 4-manifold  $(M, [g])$  with compatible complex structure  $I$  has a natural weight-less anti-selfdual 2-form  $\Omega$  ( $\Omega \in L^{-2} \otimes \Lambda^2$ ) and a unique Weyl connection  $D$  (i.e. a torsion-free connection preserving the conformal structure) such that  $d^D\Omega = 0$  [6]. We called such a structure  $(M, [g], I, D)$  a *Kähler-Weyl manifold*.

For a selfdual Kähler-Weyl manifold the twistor space  $Z$  contains degree one divisors  $\mathcal{D}, \overline{\mathcal{D}}$  corresponding to the complex structures  $\pm I$ . The line bundle  $\mathcal{L}_\tau = [\mathcal{D} - \overline{\mathcal{D}}]$  over  $Z$  is clearly trivial on twistor lines. Via the Ward correspondence such a degree zero bundle gives an instanton [1], which in this case is the Ricci form  $\rho^D$ . Therefore, the 4-manifold is hypercomplex iff  $\mathcal{L}_\tau$  is trivial. When  $\mathcal{L}_\tau$  is trivial the meromorphic function defining the divisor  $\mathcal{D} - \overline{\mathcal{D}}$  gives a map from  $Z$  to  $\mathbb{C}P^1$ .

If a selfdual Kähler-Weyl manifold has a conformal Killing vector  $K$ , preserving the complex structure, then  $\mathcal{D}, \overline{\mathcal{D}}$  project to divisors  $\mathcal{C}, \overline{\mathcal{C}}$  contained in the minitwistor space  $S$ . The space  $B$  parameterizes degree two rational curves in  $S$  and points in  $S$  correspond to oriented geodesics in  $B$ . The rational curve in  $S$  corresponding to

a point  $x$  in  $B$  intersects  $C, \bar{C}$  in a pair of points defining a geodesic in  $B$  through  $x$  with two orientations. In this way we obtain a shear-free geodesic congruence which may be formulated as a section  $\chi$  of the bundle  $L^{-1} \otimes TB$  satisfying

$$D^B \chi = \tau(id - \chi \otimes \chi) + \kappa * \chi$$

where shear-free means that the conformal structure normal to  $\chi$  is preserved. The sections  $\tau, \kappa$  of  $L^{-1}$  are monopoles representing the divergence and twist respectively of the congruence [6].

Conversely, from an Einstein Weyl space  $(B^3, [h], D^B)$  with a monopole  $(w, A)$  we may construct a selfdual 4-metric

$$g = w^2 h + (dt + A)^2.$$

The twistor space  $Z$  is the total space of the monopole line bundle over the minitwistor space  $S$  of  $B$ . Choose a shear-free geodesic congruence  $\chi$ . This corresponds to a divisor in  $S$  which lifts to a divisor in  $Z$  defining a compatible complex structure on the 4-manifold. In fact this conformal 4-space is hypercomplex iff the divergence of  $\chi$  is proportional to the monopole  $w$  used to construct  $g$ . This can be seen as follows: the twistor space is the total space of  $\mathcal{L}_\tau \xrightarrow{p} S$  and the pull back  $p^* \mathcal{L}_\tau$  is trivial over  $Z$ , so the Ricci form vanishes. As an example we could take the Einstein-Weyl space given by the round 3-sphere and let  $\chi$  be a left or right invariant congruence. Since these congruences have vanishing  $\tau$  any sum  $w$  of fundamental solutions to the Laplace equations would give a hypercomplex 4-space. The solution  $w = 1$  (in the gauge given by the round sphere) gives the Hopf surface  $S^3 \times S^1$ .

### 3. LIE GROUPS AND HYPERCOMPLEX GEOMETRY

The hypercomplex structure of the Hopf surface defined in the example in the introduction may be considered as a left invariant structure on the Lie group  $S^1 \times \text{SU}(2)$ . Consider the Lie group  $\text{SU}(3)$ . The Lie algebra  $\mathfrak{g} = \mathfrak{su}(3)$  decomposes as  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}_1 \oplus \mathfrak{f}_1$  where

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{a}_1 & \mathfrak{f}_1 \\ \mathfrak{f}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{su}(2) & \mathbb{C}^2 \\ \mathbb{C}^2 & \mathfrak{u}(1) \end{pmatrix} = \mathfrak{su}(3).$$

Think of  $\mathfrak{b} \oplus \mathfrak{a}_1$  as  $\mathbb{H}$  and think of  $\mathfrak{a}_1$  as the imaginary quaternions acting on  $\mathfrak{f}_1$  via the adjoint representation. Applying left translations we obtain in this way a hypercomplex structure on  $\text{SU}(3)$ . Now, let  $G$  be a compact semi-simple Lie group. The Lie algebra  $\mathfrak{g}$  decomposes as follows

$$\mathfrak{g} = \mathfrak{b} \oplus_{j=1}^n \mathfrak{a}_j \oplus_{j=1}^n \mathfrak{f}_j,$$

where  $\mathfrak{b}$  is Abelian,  $\mathfrak{a}_j$  is isomorphic to  $\mathfrak{su}(2)$  and  $[\mathfrak{a}_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$ . The rank  $r$  of  $G$  is equal to  $n + \dim \mathfrak{b}$  and if we add  $2n - r$  Abelian factors we can think of  $(2n - r)\mathfrak{u}(1) \oplus \mathfrak{b} \oplus_{j=1}^n \mathfrak{a}_j$  as  $\mathbb{H}^n$ . Since  $[\mathfrak{a}_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$  we can proceed as with  $\text{SU}(3)$  above to get a left

invariant hypercomplex structure on  $T^{2n-r} \times G$  [12]. In this way we get homogeneous hypercomplex structures on for example

$$\text{SU}(2\ell + 1), T^1 \times \text{SU}(2\ell), T^\ell \times \text{SO}(2\ell + 1), T^\ell \times \text{Sp}(\ell), T^{2\ell} \times \text{SO}(4\ell), \\ T^{2\ell-1} \times \text{SO}(4\ell + 2), T^2 \times E_6, T^7 \times E_7, T^8 \times E_8, T^4 \times F_4 \text{ and } T^2 \times G_2.$$

The issue is now how to get more than these homogeneous examples. For a general hypercomplex manifold  $(M^{4n}, I_1, I_2, I_3)$  we note that we have a 2-sphere of complex structures  $I_{\mathbf{v}} = v_1 I_1 + v_2 I_2 + v_3 I_3$  for  $\mathbf{v} = (v_1, v_2, v_3) \in S^2$ . The *twistor space* of  $M$  is the space  $W = M \times S^2$  of these compatible complex structures [15, 16]. This space is a complex manifold of dimension  $2n + 1$ : the complex structure  $\mathcal{I}$  at  $(x, \mathbf{v}) \in M \times S^2$  is standard along the 2-sphere and it is equal to  $I_{\mathbf{v}}(x)$  along  $T_x M$ . The integrability of  $\mathcal{I}$  is a consequence of  $M$  being hypercomplex. The holomorphic projection  $W \xrightarrow{p} S^2 = \mathbb{C}\mathbb{P}^1$  has fiber  $p^{-1}(z)$  which is  $M$  together with the complex structure determined by the point  $z \in \mathbb{C}\mathbb{P}^1$ . The non-holomorphic projection  $W \xrightarrow{\pi} M$  has as fibers, rational curves of normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$ .

The idea is to deform the hypercomplex structure on  $M$  by deforming the map  $W \xrightarrow{p} \mathbb{C}\mathbb{P}^1$  [17]. Consider the sheaf  $\mathcal{D}$  defined by the exact sequence

$$0 \rightarrow \mathcal{D} \rightarrow \Theta_W \xrightarrow{dp} p^* \Theta_{\mathbb{C}\mathbb{P}^1} \rightarrow 0.$$

where  $\Theta$  is the tangent sheaf. The deformations of the map  $p$  (and therefore the deformations of the hypercomplex geometry on  $M$ ) is measured by the cohomology groups of the sheaf  $\mathcal{D}$  [9]:  $H^0(W, \mathcal{D})$  is the space of hypercomplex symmetries,  $H^1(W, \mathcal{D})$  is the parameter space of deformations and  $H^2(W, \mathcal{D})$  is the obstruction space.

For  $M = T^k \times G$  the twistor space  $W$  is a homogeneous complex manifold and one may expect that  $H^j(W, \mathcal{D})$  is computable via Bott-Borel-Weil-Hirzebruch theory for representations and cohomology. Consider the natural map  $\Phi$  from  $W$  to  $G/U$  where  $U$  is a maximal torus in  $G$ . The spaces  $Z = G/U$  is a complex manifold and is called the Borel flag [2, 7]. The cohomology of the Borel flag has indeed been studied using representation theory and this will help us getting information about the cohomology on  $W$ : let  $X$  be  $M$  with a complex structure  $X = p^{-1}(z)$ . The restriction of  $\Phi$  to  $X$  has fiber  $E$  which is a product of elliptic curves. We may compute  $H^j(X, \mathcal{O}_X)$ , say, using a Leray spectral sequence

$$E_2^{p,q} = H^p(Z, R^q \Phi_* \mathcal{O}_X), E_\infty^{p,q} = H^{p+q}(X, \mathcal{O}_X).$$

We find  $R^q \Phi_* \mathcal{O}_X = \mathcal{O}_Z \otimes H^q(E, \mathcal{O}_E)$  and since  $H^p(Z, \mathcal{O}_Z)$  vanishes for  $p \geq 1$  [3], the spectral sequence is easy to handle and we get

$$H^q(X, \mathcal{O}_X) = E_\infty^{0,q} = E_2^{0,q} = H^q(E, \mathcal{O}_E) \cong \Lambda^q \mathbb{C}^n.$$

In much the same way we can compute the cohomology  $H^j(W, \mathcal{O}_W), H^j(W, \Phi^* \Theta_Z)$  etc. via vanishing results of Bott [4]. Then using the sequences

$$0 \rightarrow \mathcal{O}_W \rightarrow p^* \Theta_{\mathbb{C}P^1} \rightarrow \mathcal{O}_{X_1 \cup X_2} \rightarrow 0$$

$$0 \rightarrow \mathcal{D} \rightarrow \Theta_W \xrightarrow{dp} p^* \Theta_{\mathbb{C}P^1} \rightarrow 0,$$

we are able to find  $H^j(W, \mathcal{D})$ .

It turns out that the obstruction space  $H^2(W, \mathcal{D})$  is non-trivial. Therefore we study the possible obstructions using Kuranishi theory [13]. However, we can prove that for the  $U$ -invariant part of  $H^1(W, \mathcal{D})$  the obstruction vanishes and we obtain (see [17] for a more precise formulation of the theorem):

**Theorem 1.** *Suppose  $G$  is a compact semi-simple Lie group of rank  $r$  and containing  $n$  factors of  $\mathfrak{sp}(1)$ . Then the local moduli at a generic deformation of left-invariant hypercomplex structures on  $T^{2n-r} \times G$  is a smooth manifold of dimension  $n(n+r)$ . The identity component of the group of hypercomplex symmetries of a generic deformation is the Abelian group  $T^{2n}$ .*

In the introduction we defined one hypercomplex structure on  $(S^3 \times S^1)^n$ . Inspired by the theory of Abelian varieties, we shall now construct a family of hypercomplex structures on  $(S^3 \times S^1)^n$  and use the theorem above to secure completeness. Let  $(q_1, \dots, q_n; x_1, \dots, x_n) = (\mathbf{q}; \mathbf{x})$  be coordinates for  $(S^3)^n \times \mathbb{R}^n$ . Here the  $q_j$  are unit quaternions. Choose a hypercomplex structure on  $\mathbb{H}^n$  by right multiplication of unit quaternions. Then we define a hypercomplex structure on  $(S^3 \times \mathbb{R})^n$  through the embedding into  $\mathbb{H}^n$ .

For  $1 \leq j \leq n$ , define an action generated by

$$\gamma_j(\mathbf{q}; \mathbf{x}) = (e^{2\pi i \theta_{1j}} q_1, \dots, e^{2\pi i \theta_{nj}} q_n; \mathbf{x} + \mathbf{v}_j).$$

The action of  $\gamma_j$  is represented by the column vectors  $\mathbf{v}_j$  and  $\Theta_j = (\theta_{1j}, \dots, \theta_{nj})^T$ , where  $\theta_{ij}$  are in  $\mathbb{R}/\mathbb{Z}$ .

Assume that the vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  are linearly independent. Let  $\Gamma \cong \mathbb{Z}^n$  be the group generated by  $\{\gamma_1, \dots, \gamma_n\}$ . We call

$$(\Theta|V) = (\Theta_1, \dots, \Theta_n | \mathbf{v}_1, \dots, \mathbf{v}_n)$$

the *period matrix* of the manifold  $(S^3 \times \mathbb{R})^n / \Gamma$ . Thus the groups  $\Gamma$  are parameterized by the space  $(\mathbb{R}/\mathbb{Z})^{n^2} \times GL(n, \mathbb{R})$ . However, different period matrices may generate the same group. In fact, the period matrices  $(\Theta|V)$  and  $(\hat{\Theta}|\hat{V})$  generate the same group if and only if there is a matrix  $M = (m_{ij})$  in  $GL(n, \mathbb{Z})$  such that

$$(\hat{\Theta}|\hat{V}) = (\Theta M | V M).$$

The quotient space  $(S^3 \times \mathbb{R})^n / \Gamma$  is a hypercomplex manifold because the actions of  $\Gamma$  commute with the right multiplications of the quaternions on  $(q_1, \dots, q_n)$ . The quotient space is clearly diffeomorphic to  $(S^3 \times S^1)^n$ . Using the fact that symmetries

lifts to holomorphic maps of the twistor space (which is built out of a complex projective space), it is seen that hypercomplex manifolds  $(S^3 \times \mathbb{R})^n / \Gamma$  and  $(S^3 \times \mathbb{R})^n / \Gamma'$  are equivalent if and only if there exist period matrices  $(\Theta|V)$  and  $(\Theta'|V')$  for  $\Gamma$  and  $\Gamma'$  respectively such that  $V = V'$ , and  $\Theta_j = \pm\Theta'_j$ . Thus we obtain

**Theorem 2.** *The quotient space  $((\mathbb{R}/\mathbb{Z})^{n^2} \times \text{GL}(n, \mathbb{R})) / (\mathbb{Z}_2^n \times \text{GL}(n, \mathbb{Z}))$  is a complete moduli space for hypercomplex structures on the product manifold  $(S^3 \times S^1)^n$ .*

The constructions above are currently being modified to work for the case of nilpotent automorphisms and for combinations of the semi-simple and the nilpotent situation in joint work with Grantcharov and Poon.

#### 4. THE SWANN BUNDLE

Now we turn to the third theme where  $S^3 \times S^1$  appears as the fiber of a bundle. The definition of a hypercomplex manifold is equivalent to requiring that the holonomy group lies in  $\text{GL}(n, \mathbb{H})$ . More generally for a quaternionic manifold  $M$  the frame bundle has a torsion free connection with holonomy in

$$\text{GL}(n, \mathbb{H}) \text{GL}(1, \mathbb{H}) = (\mathbb{R}_{>0} \times \text{SL}(n, \mathbb{H}) \times \text{Sp}(1)) / \{\pm 1\}.$$

This group acts on  $\mathbb{H}/\mathbb{Z}_2$  by

$$\rho(\lambda, A, q)(\eta) = \lambda^{\frac{n}{n+1}} \eta q^{-1}.$$

The associated bundle is denoted by  $\mathcal{U}(M)$  and was studied by Swann for  $M$  a quaternionic Kähler manifold [21]. For  $M$  quaternionic  $\mathcal{U}(M)$  is hypercomplex [18]. The group  $\mathbb{H}^*$  acts from the left on  $\mathcal{U}(M)$  and the center  $\mathbb{Z}$  preserves the hypercomplex structure. The quotient  $\mathcal{U}(M)/\mathbb{Z}$  is denoted  $\mathcal{V}(M)$  and is a compact hypercomplex manifold which we call the *Swann bundle* [18], [19]. Now, let  $P$  be an  $S^1$ -instanton on  $M$ . Then Joyce [12] introduces the twisted bundle  $\mathcal{V}_P(M) = P \times_{S^1} \mathcal{V}(M)$  which again provides us with an example of a compact hypercomplex manifold. The fiber from  $\mathcal{V}_P(M)$  to  $M$  is  $S^3 \times S^1$ .

**Example 2.** Let  $M$  be the complex projective plane and let  $P \rightarrow M$  be the instanton given by the Hopf fibration  $S^5 \rightarrow \mathbb{C}P^2$ . Then in this case the hypercomplex manifold  $\mathcal{V}_P(M)$  is equal to  $\text{SU}(3)/\mathbb{Z}_2$ .

We may now apply complex deformation theory to these twisted Swann bundles. The twistor space  $W$  of  $\mathcal{V}_P(M)$  fibers over the twistor space  $Z$  of  $M$  and via the Leray spectral sequence we are able to compute the cohomology  $H^j(W, \mathcal{D})$  in terms of the cohomology on  $Z$  [16].

**Example 3.** Let  $M$  be the connected sum  $2\mathbb{C}P^2$  equipped with a Poon conformal structure  $c_\lambda$ ,  $\lambda \in (0, 1)$ . Then the deformation theory gives a 4-parameter space of  $T^3$ -symmetric hypercomplex structures on the 8-manifold  $\mathcal{V}(2\mathbb{C}P^2)$ . Furthermore, we can integrate and find these hypercomplex manifolds locally as a (Joyce-) hypercomplex

quotient [11] of  $\mathbb{H}^4$  with a  $T^2$  action. The space is realized as a subspace of  $\mathbb{C}^6 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  given by simple equations [16].

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## EXAMPLES OF HYPER-KÄHLER CONNECTIONS WITH TORSION

YAT SUN POON

This article is a revised version of the author's lecture given at the Second Meeting on Quaternionic Structures in Mathematics and Physics at Roma, September 8, 1999. It will appear in the proceeding of this conference. The work presented here are jointly done with Gueo Grantcharov. This article is composed while the author visits the ESI.

1. It has been known for a fairly long time that when the Wess-Zumino term is present in the  $N = 4$  supersymmetric one-dimensional sigma model, the internal space has a torsionful linear connection with holonomy in  $\text{Sp}(n)$  [3]. Such geometry also arises when one considers T-duality of toric hyper-Kähler manifolds [5]. In this conference, G. Papadopoulos explains the role of HKT-geometry in M-theory, or more specifically in IIA Superstring theory [10].

In this lecture, we discuss HKT-geometry entirely from a mathematical point of view, and present several methods to produce series of examples that may interest mathematicians.

2. The background object of HKT-geometry is a hyper-Hermitian manifold. Three complex structures  $I_1, I_2$  and  $I_3$  on a smooth manifold  $M$  form a hypercomplex structure if

$$(0.1) \quad I_1^2 = I_2^2 = I_3^2 = -1, \quad \text{and} \quad I_1 I_2 = I_3 = -I_2 I_1.$$

A triple of such complex structures is equivalent to the existence of a 2-sphere worth of integrable complex structures:  $\mathcal{I} = \{a_1 I_1 + a_2 I_2 + a_3 I_3 : a_1^2 + a_2^2 + a_3^2 = 1\}$ . When  $g$  is a Riemannian metric on the manifold  $M$  such that it is Hermitian with respect to every complex structure in the hypercomplex structure,  $(M, \mathcal{I}, g)$  is called a hyper-Hermitian manifold.

On a hyper-Hermitian manifold, there are two natural torsion-free connections, namely the Levi-Civita connection and the Obata connection. However, in general the Levi-Civita connection does not preserve the hypercomplex structure and the

Obata connection does not preserve the metric. We are interested in the following type of connections.

*Definition 1.* A linear connection  $\nabla$  on a hyper-Hermitian manifold  $(M, \mathcal{I}, g)$  is hyper-Hermitian if  $\nabla g = 0$ , and  $\nabla I_1 = \nabla I_2 = \nabla I_3 = 0$ .

Although a general hyper-Hermitian connection has torsion, physical requirement limits our discussion to a special type of hyper-Hermitian connections. We follow physicists' conventions of definitions. Recall that when  $T^\nabla$  is the torsion tensor for a connection  $\nabla$ , we can construct the  $(3, 0)$ -tensor  $c(X, Y, Z) = g(X, T^\nabla(Y, Z))$ .

*Definition 2.* A linear connection  $\nabla$  on a hyper-Hermitian manifold  $(M, \mathcal{I}, g)$  is hyper-Kähler with torsion (HKT) if it is hyper-Hermitian and its torsion  $(3, 0)$ -tensor is totally skew-symmetric. It is a strong HKT-connection if its torsion 3-form is closed.

*Example 1.* Let  $q$  be the quaternion coordinate for the one-dimensional quaternion module  $\mathbf{H}$ . Through left multiplications by the unit quaterions  $i, j$  and  $k$ , one obtains a hypercomplex structure  $\mathcal{I}$  on  $\mathbf{H}$ . The Euclidean metric  $g = dqd\bar{q}$  is hyper-Kähler. The Levi-Civita connection is a HKT-connection.

A less obvious and more relevant example for us is to consider the following metric on  $\mathbf{H} \setminus \{0\}$ .

$$(0.2) \quad \hat{g} = \frac{dqd\bar{q}}{|q\bar{q}|}.$$

Considering the diffeomorphis  $\mathbf{H} \setminus \{0\} = \mathbf{R}^+ \times S^3$ , we choose a spherical coordinate  $(r, \theta, \phi, \psi)$ . Let  $g_S$  be the metric on the round unit-sphere. Then

$$(0.3) \quad \hat{g} = \frac{dr^2}{r^2} + g_S.$$

Now,  $(\mathbf{H} \setminus \{0\}, \mathcal{I}, \hat{g})$  is a HKT-structure. The torsion form  $c$  is the volume form of the sphere  $S^3$ . It is also a closed 3-form.

If one chooses to study the Hermitian geometry for one of the complex structures  $J$  in the hypercomplex structure, one should note that Gauduchon found a collection of canonical Hermitian connections on any Hermitian manifold. The collection forms an affine subspace of the space of linear connections [4]. This collection of Hermitian connections include Chern connection and Lichnerowicz's first canonical connection. Within this family, there exists exactly one connection whose torsion  $(3, 0)$ -tensor is a 3-form. To describe it, we recall the following definitions and convention. For any  $n$ -form  $\omega$ ,  $d^c\omega := (-1)^n JdJ\omega$  where  $(J\omega)(X_1, \dots, X_n) := (-1)^n \omega(JX_1, \dots, JX_n)$ . Then  $\partial = \frac{1}{2}(d + id^c)$  and  $\bar{\partial} = \frac{1}{2}(d - id^c)$ . By [4], the Hermitian connection with totally skew-symmetric torsion  $(3, 0)$ -tensor  $c$  is uniquely determined by the following identity.

$$(0.4) \quad c(X, Y, Z) = -\frac{1}{2}d^cF(X, Y, Z),$$

where  $F(X, Y) = g(JX, Y)$  is the Kähler form for the complex structure  $J$ .

Now the HKT-connection serves as such a unique connection for each complex structure in the hypercomplex structure. Therefore, if we use  $F_a$  and  $d_a$  to represent the Kähler form and complex exterior differential for the complex structure  $J_a$ ,  $a = 1, 2, 3$ , we have the following observation.

*Proposition 1.* A hyper-Hermitian manifold  $(M, \mathcal{I}, g)$  admits a HKT-connection if and only if  $d_1F_1 = d_2F_2 = d_3F_3$ . If it exists, it is unique.

In view of the uniqueness, we say that  $(M, \mathcal{I}, g)$  is a HKT-structure if it admits a HKT-connection.

*Example 2.* A non-trivial class of HKT-structures can be found on semi-simple Lie groups and homogeneous spaces [13] [9]. For instance, the Killing-Cartan form  $-B$  on the Lie group  $SU(2n + 1)$  defines a bi-invariant metric  $g = -B$ . This group has a left-invariant hypercomplex structure  $\mathcal{I}$  so that with the bi-invariant metric  $g$ , it forms a HKT-structure. The HKT-connection is the left-invariant connection defined by having all left-invariant vector fields to be parallel. The torsion of this connection is the Lie bracket, and the torsion tensor  $c(X, Y, Z) = -B(X, [Y, Z])$  is totally skew-symmetric. Similar constructions can be applied to  $U(1) \times SU(2n)$  and other homogeneous spaces.

**3.** To further our analysis of HKT-geometry, we note a holomorphic characterization of HKT-structures.

*Proposition 2.* Let  $(M, \mathcal{I}, g)$  be a hyper-Hermitian manifold and  $F_a$  be the Kähler form for  $(J_a, g)$ . Then  $(M, \mathcal{I}, g)$  is a HKT-structure if and only if  $\partial_1(F_2 + iF_3) = 0$ ; or equivalently  $\bar{\partial}_1(F_2 - iF_3) = 0$ .

Applying this proposition to any complex structure in the given hypercomplex structure, one obtains a section of twisted 2-form on the twistor space of the hypercomplex structures. However, this 2-form is only  $J_2$ -holomorphic in the sense of Eells-Salamon [2]. Since the almost complex structure  $J_2$  is never integrable [2], we shall concentrate on the holomorphic characterization given above and ignore the twistor characterization.

Due to the absence of type  $(3, 0)$ -form with respect to any complex structure on any real four-dimensional manifold, it is now apparent that any four-dimensional hyper-Hermitian manifold is a HKT-structure.

The holomorphic characterization also yields new examples of HKT-structures.

*Example 3.* Let  $\{X_1, \dots, X_{2n}, Y_1, \dots, Y_{2n}, Z\}$  be a basis for  $\mathbf{R}^{4n+1}$ . Define commutators by  $[X_i, Y_i] = 4Z$ , and all others are zero. These commutators define on  $\mathbf{R}^{4n+1}$  the structure of the Heisenberg Lie algebra  $\mathfrak{h}$ . Let  $\mathbf{R}^3$  be the 3-dimensional Abelian algebra. The direct sum  $\mathfrak{n} = \mathfrak{h} \oplus \mathbf{R}^3$  is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis  $\{E_1, E_2, E_3\}$  for  $\mathbf{R}^3$ . Consider the endomorphisms  $I_1,$

$I_2$  and  $I_3$  of  $\mathfrak{n}$  defined by left multiplications of the quaternions  $i, j$  and  $k$  on the module of quaternions  $\mathbf{H}$ , and the identifications

$$(0.5) \quad \begin{aligned} x_0X_{2a-1} + x_1X_{2a-1} + x_2Y_{2a-1} + x_3Y_{2a} &\rightarrow x_0 + x_1i + x_2j + x_3k; \\ x_0Z + x_1E_1 + x_2E_2 + x_3E_3 &\rightarrow x_0 + x_1i + x_2j + x_3k. \end{aligned}$$

Through left translations, these endomorphisms define almost complex structures on the product of the Heisenberg group and the Abelian group  $N = H \times \mathbf{R}^3$ . It is clear from the definition that these almost complex structures satisfy the algebra (0.1). Moreover, for  $a = 1, 2, 3$  and  $X, Y \in \mathfrak{n}$ ,

$$(0.6) \quad [I_aX, I_aY] = [X, Y]$$

so  $I_a$  are Abelian complex structures on  $\mathfrak{n}$  in the sense of [1]. In particular, they are integrable. It implies that  $\{I_a : a = 1, 2, 3\}$  is a left-invariant hypercomplex structure on the Lie group  $N$ . It is known [12] that the complex structures  $I_a$  on  $\mathfrak{n}$  satisfy  $d(\Lambda_{I_a}^{1,0}\mathfrak{n}^*) \subset \Lambda_{I_a}^{1,1}\mathfrak{n}^*$  where  $\mathfrak{n}^*$  is the space of left-invariant 1-forms on  $N$  and  $\Lambda_{I_a}^{i,j}\mathfrak{n}^*$  is the  $(i, j)$ -component of  $\mathfrak{n}^* \otimes \mathbf{C}$  with respect to  $I_a$ . But then we have  $d(\Lambda_{I_a}^{2,0}\mathfrak{n}^*) \subset \Lambda_{I_a}^{2,1}\mathfrak{n}^*$  and any left-invariant  $(2,0)$ -form is  $\partial_1$ -closed. Now consider the invariant metric on  $N$  for which the basis  $\{X_i, Y_i, Z, E_a\}$  is orthonormal. Since it is compatible with the complex structures  $I_a$ , in view of the holomorphic characterization of HKT-geometry, we obtain a left-invariant HKT-structure on  $N$ .

*Example 4.* Based on the above computation, we could also see that there is a left-invariant HKT-structure on the product of the  $4n + 1$ -dimensional Heisenberg group and the compact simple Lie group  $SU(2)$ , an interesting mixture of the last example and Example 2.

Recall that the underlying manifold of the Heisenberg group  $H_{4n+1}$  is the manifold  $\mathbf{R}^{4n+1}$ . Consider it as the product space  $\mathbf{R}^{2n} \times \mathbf{R}^{2n} \times \mathbf{R}$ , the group law for the Heisenberg group is

$$(0.7) \quad (\vec{x}, \vec{y}, z) * (\vec{x}', \vec{y}', z') = (\vec{x} + \vec{x}', \vec{y} + \vec{y}', z + z' + 2 \sum_{i,j=1}^{2n} (x_i y'_j - y_i x'_j)).$$

The 1-forms  $\alpha_j = dx_j, \beta_j = dy_j, \gamma = dz + 2 \sum (y_j dx_i - x_j dy_i)$  are left-invariant. Let  $\{X_j, Y_j, Z\}$  be the dual left-invariant vector fields.

On  $SU(2)$ , choose left-invariant vector fields  $A_1, A_2$  and  $A_3$  such that  $[A_1, A_2] = 2A_3$ , etc., then the dual left-invariant 1-forms  $\sigma_1, \sigma_2$  and  $\sigma_3$  satisfy the identities

$$(0.8) \quad d\sigma_1 = 2\sigma_2 \wedge \sigma_3, d\sigma_2 = 2\sigma_3 \wedge \sigma_1, d\sigma_3 = 2\sigma_1 \wedge \sigma_2.$$

Now, using  $\{A_1, A_2, A_3\}$  instead of  $\{E_1, E_2, E_3\}$ , we define endomorphisms  $I_1, I_2$  and  $I_3$  on  $\mathfrak{h} \oplus \mathfrak{su}(2)$  as in (0.5). Through left translation, we define three almost complex structures on the product group  $H \times SU(2)$  satisfying the identities (0.1). To prove that these almost complex structures are integrable, one first notes that when  $\mathfrak{c}$  is the center of the Heisenberg algebra, then the vector space  $\mathfrak{h} \oplus \mathfrak{su}(2)$

has a direct sum decomposition  $\mathfrak{t}_{4n} \oplus \mathfrak{c} \oplus \mathfrak{su}(2)$  where  $\mathfrak{t}_{4n}$  is the linear span of all the  $X_j$  and  $Y_j$ . On  $\mathfrak{t}_{4n}$ , the almost complex structures satisfy the identity (0.6). Therefore, the Nijenhuis tensor vanishes on  $\mathfrak{t}_{4n}$ . On  $\mathfrak{c} \oplus \mathfrak{su}(2)$ , the almost complex structures are the standard ones for  $\mathbf{H} \setminus \{0\}$ . Therefore, the Nijenhuis tensor vanishes on this summand. Since  $\mathfrak{c} \oplus \mathfrak{su}(2)$  commutes with  $\mathfrak{t}_{4n}$ , and both  $\mathfrak{t}_{4n}$  and  $\mathfrak{c} \oplus \mathfrak{su}(2)$  are invariant of the endomorphisms  $I_1, I_2$  and  $I_3$ , the Nijenhuis tensor vanishes completely. Therefore, the left-invariant almost complex structures  $I_1, I_2$  and  $I_3$  define a left-invariant hypercomplex structure on the product group  $H \times \text{SU}(2)$ . However the hypercomplex structure is no longer Abelian.

We define a left-invariant metric  $g$  on the product group by requiring the left-invariant vector fields  $\{X_1, \dots, X_{2n}, Y_1, \dots, Y_{2n}, Z, E_1, E_2, E_3\}$  to be an orthonormal frame. Equivalently,

$$g = \sum_{a=1}^n (\alpha_{2a-1}^2 + \alpha_{2a}^2 + \beta_{2a-1}^2 + \beta_{2a}^2) + \gamma^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$$

This metric, along with the left-invariant hypercomplex structure, is a HKT-structure on the product group  $H \times \text{SU}(2)$ . Indeed, when  $F_1, F_2$  and  $F_3$  are the three Kähler forms for the complex structures  $I_1, I_2$  and  $I_3$ ,  $dF_a = 2i(d\gamma \wedge \sigma_a - \gamma \wedge \sigma_b \wedge \sigma_c)$ , where  $(abc)$  is any even permutation of  $(123)$ . Since  $d\gamma = -4 \sum_{a=1}^n (\alpha_{2a-1} \wedge \beta_{2a-1} + \alpha_{2a} \wedge \beta_{2a})$ ,

$$(0.9) \quad I_1 dF_1 = I_2 dF_2 = I_3 dF_3 = -2i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3).$$

Therefore, we have a HKT-structure. Since the torsion 3-form  $c = i(\gamma \wedge d\gamma + \sigma_1 \wedge \sigma_2 \wedge \sigma_3)$

$$(0.10) \quad dc = id\gamma \wedge d\gamma.$$

This is not a closed 3-form, the corresponding HKT-structure is weak.

**4.** The holomorphic characterization shows that the form  $F_2 + iF_3$  has a locally defined  $(1, 0)$ -form  $\beta$  as its potential. Although the  $(0, 1)$ -form  $I_2\beta$  is not a priori  $\partial_1$ -closed, we consider the case when it is. From this observation, we extract the following definition.

*Definition 3.* Let  $(M, \mathcal{I}, g)$  be a HKT-structure with Kähler forms  $F_1, F_2$  and  $F_3$ . A possibly locally defined function  $\mu$  is a potential function for the HKT-structure if

$$(0.11) \quad F_1 = \frac{1}{2}(dd_1 + d_2d_3)\mu, \quad F_2 = \frac{1}{2}(dd_2 + d_3d_1)\mu, \quad F_3 = \frac{1}{2}(dd_3 + d_1d_2)\mu.$$

Referring to the holomorphic characterization of HKT-geometry, we reformulate the definition of HKT-potential in the following way.

*Proposition 3.* Let  $(M, \mathcal{I}, g)$  be a HKT-structure with Kähler form  $F_1, F_2$  and  $F_3$ . A possibly locally defined function  $\mu$  is a potential function for the HKT-structure if

$$(0.12) \quad F_2 + iF_3 = 2\partial_1 I_2 \bar{\partial}_1 \mu.$$

*Example 5.* On the complex vector space  $(\mathbf{C}^n \oplus \mathbf{C}^n) \setminus \{0\} \cong \mathbf{H}^n \setminus \{0\}$ , let  $(z_\alpha, w_\alpha)$ ,  $1 \leq \alpha \leq n$ , be its coordinates. Define a hypercomplex structure  $\mathcal{I}$  by right multiplication of the pure quaternions  $i$ ,  $j$  and  $k$ . Let  $g$  be the flat metric. It is a hyper-Kähler metric with hyper-Kähler potential  $\mu = \frac{1}{2}(|z|^2 + |w|^2)$ . Consider a new metric

$$\hat{g} = \frac{1}{\mu}g - \frac{1}{4\mu^2}(d\mu \otimes d\mu + I_1 d\mu \otimes I_1 d\mu + I_2 d\mu \otimes I_2 d\mu + I_3 d\mu \otimes I_3 d\mu).$$

Then the hyper-Hermitian structure  $(\mathcal{I}, \hat{g})$  is a HKT-structure. Moreover, the function  $\ln(\mu)$  is its potential.

Now, for any real number  $r$ , with  $0 < r < 1$ , and  $\theta_1, \dots, \theta_n$  modulo  $2\pi$ , we consider the integer group  $\langle \gamma \rangle$  generated by the following action on  $(\mathbf{C}^n \oplus \mathbf{C}^n) \setminus \{0\}$ .

$$(0.13) \quad \gamma(z_\alpha, w_\alpha) = (r e^{i\theta_\alpha} z_\alpha, r e^{-i\theta_\alpha} w_\alpha).$$

Since  $\gamma$  is a hyper-holomorphic isometry, the HKT-structure on  $(\mathbf{C}^n \oplus \mathbf{C}^n) \setminus \{0\}$  descends to a HKT-structure on the quotient space with respect to the group  $\langle \gamma \rangle$ . As this quotient space is diffeomorphic to  $S^1 \times S^{4n-1}$  [11], and the quotient hypercomplex structure is not homogeneous, we obtain a family of inhomogeneous HKT-structures on the manifold  $S^1 \times S^{4n-1}$ .

It should be noted that this method of generating HKT-geometry through a transformation from HKT-potentials to HKT-potentials can easily generate large classes of inhomogeneous HKT-structures on homogeneous manifolds especially when we start from well known hyper-Kähler metrics with hyper-Kähler potentials.

**Remark** To produce more examples, one may develop a reduction theory along the line of hyper-Kähler reduction [7]. One can also prove that Joyce's twist construction of hypercomplex manifolds [8] carries HKT-manifolds to HKT-manifolds. We do not present details of these theories here. Details of our work can be found in [6].

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THEOREMS OF EXISTENCE  
 OF LOCAL AND GLOBAL SOLUTIONS OF PDEs  
 IN THE CATEGORY OF  
 NONCOMMUTATIVE QUATERNIONIC MANIFOLDS

AGOSTINO PRÁSTARO

ABSTRACT - In this paper we apply our recent geometric theory of noncommutative (quantum) manifolds and noncommutative (quantum) PDEs [7,8,12] to the category of quantum quaternionic manifolds. These are manifolds modelled on spaces built starting from quaternionic algebras. For PDEs considered in such category we determine theorems of existence of local and global quantum quaternionic solutions. We show also that such a category of quantum quaternionic manifolds properly contains that of manifolds with (almost) quaternionic structure. So our theorems of existence of quantum quaternionic manifolds for PDEs produce a cascade of new solutions with nontrivial topology.

1 - QUANTUM MANIFOLDS AND QUANTUM PDEs

In order to give a geometrical model for quantum physics we introduced in some recent works [7,8,12] a new category of noncommutative manifolds, (**quantum manifolds**), where the "first brick" used to build them is a suitable structured noncommutative Fréchet algebra, (**quantum algebra**). The aim of this paper is to show that the category of quantum manifolds contains subcategories of great interest whose noncommutative manifolds have the quaternionic algebra as a fundamental structure element. (We call such manifolds **noncommutative quaternionic manifolds**). Then for PDEs built in such subcategories we shall apply our geometric theory of QPDEs and obtain theorems of existence of local and global solutions for noncommutative quaternionic PDEs.

Set  $\mathbf{K} = \mathbf{R}, \mathbf{C}$ . Let us recall some fundamental definitions and results on quantum manifolds as given by us in refs.[7,8,12]. A **quantum algebra** is a triplet  $(A, \epsilon, c)$ , where: (i)  $A$  is a metrizable, complete, Hausdorff, locally convex topological  $\mathbf{K}$ -vector space, that is also a ring with unit; (ii)  $\epsilon : \mathbf{K} \rightarrow A_0 \subset A$  is a ring homomorphism, where  $A_0$  is the centre of  $A$ ; (iii)  $c : A \rightarrow \mathbf{K}$  is a  $\mathbf{K}$ -linear morphism, with  $c(e) = 1$ ,  $e = \text{unit of } A$ . For any  $a \in A$  we call  $a_C \equiv c(a) \in \mathbf{K}$  the **classic limit** of  $a$ ; (iv)  $A$  is an associative  $\mathbf{K}$ -algebra. A **quantum vector space** of dimension  $(m_1, \dots, m_s) \in \mathbf{N}^s$ , built on the quantum algebra  $A \equiv A_1 \times \dots \times A_s$ , is a locally convex topological  $\mathbf{K}$ -vector space  $E$  isomorphic to  $A_1^{m_1} \times \dots \times A_s^{m_s}$ . A **quantum manifold** of dimension  $(m_1, \dots, m_s)$  over a quantum algebra  $A \equiv A_1 \times \dots \times A_s$  of class  $Q_w^k$ ,  $0 \leq k \leq \infty, \omega$ , is a locally convex manifold  $M$  modelled on  $E$  and with a  $Q_w^k$ -atlas of local coordinate mappings, i.e., the transition functions  $f : U \subset E \rightarrow U' \subset E$  define a pseudogroup of local  $Q_w^k$ -homeomorphisms on  $E$ , where  $Q_w^k$  means  $C_w^k$ , i.e., weak differentiability [5], and derivatives  $A_0$ -lineaires. So for each open coordinate set  $U \subset M$  we have a set of  $m_1 + \dots + m_s$  coordinate functions  $x^A : U \rightarrow A$ , (**quantum coordinates**). The **tangent space**  $T_p M$  at  $p \in M$ , is the vector space of the equivalence classes  $v \equiv [f]$  of  $C_w^1$  (or equivalently  $C^1$ ) curves  $f : I \rightarrow M$ ,  $I \equiv$  open neighborhood of  $0 \in \mathbf{R}$ ,  $f(0) = p$ ; two curves  $f, f'$

are equivalent if for each (equivalently, for some) coordinate system  $\mu$  around  $p$  the functions  $\mu \circ f, \mu \circ f' : I \rightarrow A_1^{m_1} \times \dots \times A_s^{m_s}$  have the same derivative at  $0 \in \mathbf{R}$ . Then, derived tangent spaces associated to a quantum manifold  $M$  can be naturally defined. (For details see refs.[7,8,12].) We say that a quantum manifold of dimension  $(m_1, \dots, m_s)$  is **classic regular** if it admits a projection  $c : M \rightarrow M_C$  on a  $n$ -dimensional manifold  $M_C$ . We will call  $M_C$  the **classic limit** of  $M$  and in order to emphasize this structure we say that the dimension of  $M$  is  $(n \downarrow m_1, \dots, m_s)$ . A **quantum PDE** (QPDE) of order  $k$  on the fibre bundle  $\pi : W \rightarrow M$ , defined in the category of quantum manifolds, is a subfibrebundle  $\hat{E}_k \subset J\hat{D}^k(W)$  of the jet-quantum derivative space  $J\hat{D}^k(W)$  over  $M$ . ( $J\hat{D}^k(W)$  is, in the category of quantum manifolds, the corresponding of the jet-derivative space for usual manifolds. For more details see refs.[7,8,12].) In refs.[8,12] we have formulated also a geometric theory for quantum PDEs that generalizes the theory of PDEs for usual manifolds. In particular in the following we shall emphasize some important definitions and results about. A QPDE  $\hat{E}_k$  is **quantum regular** if the  $r$ -quantum prolongations  $\hat{E}_{k+r} \equiv J\hat{D}^r(\hat{E}_k) \cap J\hat{D}^{k+r}(W)$  are subbundles of  $\pi_{k+r, k+r-1} : J\hat{D}^{k+r}(W) \rightarrow J\hat{D}^{k+r-1}(W), \forall r \geq 0$ . Furthermore, we say that  $\hat{E}_k$  is **formally quantumintegrable** if  $\hat{E}_k$  is quantum regular and if the mappings  $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}, \forall r \geq 0$ , and  $\pi_{k,0} : \hat{E}_k \rightarrow W$  are surjective. In the following we shall consider QPDEs on a fiber bundle  $\pi : W \rightarrow M$ , where  $M$  is a quantum manifold of dimension  $m$  on the quantum algebra  $A$  and  $W$  is a quantum manifold of dimension  $(m, s)$  on the quantum algebra  $B \equiv A \times E$ , where  $E$  is also an  $A_0$ -algebra. The **quantum symbol**  $\hat{g}_{k+r}$  of  $\hat{E}_{k+r}$  is a family of  $A_0$ -modules over  $\hat{E}_k$  characterized by means of the following short exact sequence of  $A_0$ -modules:  $0 \rightarrow \pi_{k+r, k+r-1}^* \hat{g}_{k+r} \rightarrow vT\hat{E}_{k+r} \rightarrow \pi_{k+r, k+r-1}^* vT\hat{E}_{k+r-1}$ . Then one has the following complex of  $A_0$ -modules over  $\hat{E}_k$  ( **$\delta$ -quantum complex**):

$$0 \rightarrow \hat{g}_m \xrightarrow{\delta} TMO\hat{g}_{m-1} \xrightarrow{\delta} \hat{\Lambda}_0^2 MO\hat{g}_{m-2} \xrightarrow{\delta} \dots \xrightarrow{\delta} \hat{\Lambda}_0^{m-k} MO\hat{g}_k \xrightarrow{\delta} \hat{\Lambda}_0^{m-k} MO\hat{g}_k \rightarrow 0$$

where  $\hat{\Lambda}_0^r M$  is the skewsymmetric subbundle of  $\hat{T}_0^r M \equiv TM \otimes_{A_0} \dots \otimes_{A_0} TM$ . We call **Spencer quantumcohomology** of  $\hat{E}_k$  the homology of such complex. We denote by  $\{H_q^{m-j,j}\}_{q \in \hat{E}_k}$  the homology at  $(\hat{\Lambda}_0^j MO\hat{g}_{m-j})_q$ . We say that  $\hat{E}_k$  is  **$r$ -quantumacyclic** if  $H_q^{m,j} = 0, m \geq k, 0 \leq j \leq r, \forall q \in \hat{E}_k$ . We say that  $\hat{E}_k$  is **quantuminvolutive** if  $H_q^{m,j} = 0, m \geq k, j \geq 0$ . We say that  $\hat{E}_k$  is  **$\delta$ -regular** if there exists an integer  $\kappa_0 \geq \kappa$ , such that  $\hat{g}_{\kappa_0}$  is quantum involutive or 2-quantumacyclic. **THEOREM 1.1 - ( $\delta$ -POINCARÉ LEMMA FOR QUANTUM PDES)[12].** *Let  $\hat{E}_k \subset J\hat{D}^k(W)$  be a quantum regular QPDE. If  $A_0$  is a Noetherian  $\mathbf{K}$ -algebra, then  $\hat{E}_k$  is a  $\delta$ -regular QPDE.*

**THEOREM 1.2 - (CRITERION OF FORMAL QUANTUM INTEGRABILITY)[12].** *Let  $\hat{E}_k \subset J\hat{D}^k(W)$  be a quantum regular,  $\delta$ -regular QPDE. Then if  $\hat{g}_{k+r+1}$  is a bundle of  $A_0$ -modules over  $\hat{E}_k$ , and  $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}$  is surjective for  $0 \leq r \leq m$ , then  $\hat{E}_k$  is formally quantumintegrable.*

An **initial condition** for QPDE  $\hat{E}_k \subset J\hat{D}^k(W)$  is a point  $q \in \hat{E}_k$ . A **solution** of  $\hat{E}_k$  passing for the initial condition  $q$  is a  $m$ -dimensional quantum manifold  $N \subset \hat{E}_k$  such that  $q \in N$  and such that  $N$  can be represented in a neighborhood of any of its points  $q' \in N$ , except for a nowhere dense subset  $\Sigma(N) \subset N$  of dimension  $\leq m - 1$ , as image of the  $k$ -derivative  $D^k s$  of some  $Q_w^k$ -section  $s$  of  $\pi : W \rightarrow M$ . We call  $\Sigma(N)$  the set of **singular points** (of Thom-Bordman type) of  $N$ . If  $\Sigma(N) \neq \emptyset$  we say that  $N$  is a **regular solution** of  $\hat{E}_k \subset J\hat{D}^k(W)$ . Furthermore, let us denote by  $\hat{J}_m^k(W)$  the  $k$ -jet of  $m$ -dimensional quantum manifolds (over  $A$ ) contained into  $W$ . One has the natural embeddings  $\hat{E}_k \subset J\hat{D}^k(W) \subset \hat{J}_m^k(W)$ . Then, with respect to the embedding  $\hat{E}_k \subset \hat{J}_m^k(W)$  we can consider solutions of  $\hat{E}_k$  as  $m$ -dimensional (over  $A$ ) quantum manifolds  $V \subset \hat{E}_k$  such that  $V$  can be representable in the neighborhood of any of its points  $q' \in V$ , except for a nowhere dense subset  $\Sigma(V) \subset V$ , of dimension  $\leq m - 1$ , as  $N^{(k)}$ , where  $N^{(k)}$  is the  $k$ -quantum prolongation of a  $m$ -dimensional (over  $A$ ) quantum manifold  $N \subset W$ . In the case that  $\Sigma(V) = \emptyset$ , we say that  $V$  is a **regular solution** of  $\hat{E}_k \subset \hat{J}_m^k(W)$ . Of course, solutions  $V$  of  $\hat{E}_k \subset \hat{J}_m^k(W)$ , even if regular ones, are

not, in general diffeomorphic to their projections  $\pi_k(V) \subset M$ , hence are not representable by means of sections of  $\pi : W \rightarrow M$ .

Therefore, above theorem allows us to obtain existence theorems of local solutions. Now, in order to study the structure of global solutions it is necessary to consider the integral bordism groups of QPDEs. In refs.[7,8,12] we extended to QPDEs our previous results on the determination of integral bordism groups of PDEs [6-11,13]. Let us denote by  $\Omega_p^{\hat{E}_k}$ ,  $0 \leq p \leq m-1$ , the integral bordism groups of a QPDE  $\hat{E}_k \subset \hat{J}_m^k(W)$  for closed integral quantum submanifolds of dimension  $p$ , over a quantum algebra  $A$ , of  $\hat{E}_k$ . The structure of smooth global solutions of  $\hat{E}_k$  are described by the integral bordism group  $\Omega_{m-1}^{\hat{E}_\infty}$  corresponding to the  $\infty$ -quantumprolongation  $\hat{E}_\infty$  of  $\hat{E}_k$ . Beside the groups  $\Omega_p^{\hat{E}_k}$ ,  $0 \leq p \leq m-1$ , we can also introduce the **integral singular  $p$ -bordism groups**  ${}^B\Omega_{p,s}^{\hat{E}_k}$ ,  $0 \leq p \leq m-1$ , where  $B$  is a quantum algebra. Then one can prove [12] that  ${}^B\Omega_{p,s}^{\hat{E}_k} \cong \Omega_{p,s}^{\hat{E}_k} \otimes_K B$ , where  $\Omega_{p,s}^{\hat{E}_k}$  are the integral singular bordism groups for  $B = \mathbf{K}$ . Furthermore, the equivalence classes in the groups  ${}^B\Omega_{p,s}^{\hat{E}_k}$  are characterized by means of suitable characteristic numbers (belonging to  $B$ ), similarly to what happens for PDEs [7-11]. In ref.[12] we given also a general method to explicitly calculate such bordism groups for quantum PDEs.

## 2 - THE CATEGORY OF QUANTUM QUATERNIONIC MANIFOLDS

Let us first recall some fundamental definitions and results on quaternionic algebra [2]. Let  $\mathbf{K} = \mathbf{R}, \mathbf{C}$ . Let  $R$  be a commutative ring. Let  $\alpha, \beta \in R$ ,  $(e_1, e_2)$  the canonical basis of the  $R$ -module  $R^2$ . We say **quadratic algebra of type  $(\alpha, \beta)$**  over  $R$  the  $R$ -module  $R^2$  endowed with the structure of algebra defined by means of the following multiplication: ( $\clubsuit$ )  $e_1^2 = e_1, e_1 e_2 = e_2 e_1 = e_2, e_2^2 = \alpha e_1 + \beta e_2$ . Any  $R$ -algebra  $E$ , isomorphic to a quadratic algebra is called a quadratic algebra too. (Any  $R$ -algebra  $E$  that admits a basis of two elements (one being the identity) is a quadratic algebra.) Then the basis is called a **basis of type  $(\alpha, \beta)$** . A quadratic algebra  $E$  is associative and commutative. Let  $E$  be a quadratic  $R$ -algebra,  $e$  its unit. Let  $u \in E$  and  $T(u)$  the trace of the endomorphism  $x \mapsto ux$  of the free  $R$ -module  $E$ . Then the application  $s : E \rightarrow E$ ,  $s(u) = T(u) \cdot e - u$ , is an endomorphism of the  $R$ -algebra  $E$  and one has  $s^2(u) = u, \forall u \in E$ . A **Cayley algebra** on  $R$  is a couple  $(E, s)$ , where  $E$  is a  $R$ -algebra, with unit  $e \in E$ , and  $s$  is a skewendomorphism of  $E$  such that: (a)  $u + \bar{u} \in Re$ , (b)  $u \cdot \bar{u} \in Re$ , with  $\bar{u} \equiv s(u), \forall u \in E$ .  $s$  is called **conjugation** of the Cayley algebra  $E$  and  $s(u) \equiv \bar{u}$  is the **conjugated** of  $u$ . From the condition (a) it follows that  $u\bar{u} = \bar{u}u$ . One defines **Cayley trace** and **Cayley norm** respectively the following maps:  $T : E \rightarrow Re$ ,  $u \mapsto T(u) = u + \bar{u}$ ;  $N : E \rightarrow Re$ ,  $u \mapsto N(u) = u \cdot \bar{u}$ . One has the following properties: (1)  $\bar{\bar{e}} = e$ ; (2)  $s(u + s(u)) = u + s(u) \Rightarrow s(u) + s^2(u) = u + s(u) \Rightarrow s^2(u) = u \Rightarrow s^2 = id_E$ ; (3)  $T(\bar{u}) = T(u)$ ; (4)  $N(\bar{u}) = N(u)$ ; (5)  $(u - u)(u - \bar{u}) = 0 \Rightarrow u^2 - T(u) \cdot u + N(u) = 0$ ; (6) Let  $E$  be a  $R$ -algebra and let  $s, s'$  be skewendomorphisms of  $E$  such that  $(E, s)$  and  $(E, s')$  are Cayley algebras. If  $E$  admits a basis containing  $E$ , one has  $s = s'$ ; (7)  $\overline{u + v} = \bar{u} + \bar{v}$ ;  $\overline{\alpha u} = \alpha \bar{u}$ ;  $\overline{u \cdot v} = \bar{v} \cdot \bar{u}, \forall \alpha \in R, \forall u, v \in E$ ; (8)  $T(e) = 2e$ ;  $N(e) = e$ ; (9)  $T(uv) = T(vu)$ ; (10)  $T(v\bar{u}) = T(u\bar{v}) = N(u + v) - N(u) - N(v) = T(u)T(v) - T(uv)$ ; (11)  $N(\alpha u) = \alpha^2 N(u)$ ; (12)  $(T(u))^2 - T(u^2) = 2N(u)$ ; (13)  $T$  is a linear form on  $E$  and  $N$  is a quadratic form on  $E$ .

**EXAMPLE 2.1 - (CAYLEY EXTENSION OF A CAYLEY ALGEBRA  $(E, s)$  DEFINED BY AN ELEMENT  $\gamma \in R$ ).** 1) Let  $(E, s)$  be a Cayley algebra and let  $\gamma \in R$ . Let  $F$  be the  $R$ -algebra with underlying module  $E \times E$  and with multiplication  $(x, y)(x', y') = (xx' + \gamma y' y, yx' + y'y)$ . Then  $(e, 0)$  is the unit of  $F$  and  $E \times \{0\}$  is a subalgebra of  $F$  isomorphic to  $E$  that can be identified with  $E$ . Let  $\iota$  be the permutation of  $F$  defined by  $\iota(x, y) = (\bar{x}, -y), \forall x, y \in E$ . Then the couple  $(F, \iota)$  is a Cayley algebra over  $R$ . Set  $j = (0, e)$ . So we can write  $(x, y) = (x, 0)(e, 0) + (0, y)(0, e) = xe + yj$ . One has  $yj = j\bar{y}$ ,  $x(yj) = (yx)j - (xj)y = (x\bar{y})j$ ,  $(xj)(yj) = \bar{y}xe$ ,  $j^2 = e$ . Furthermore, one has  $T_F(xe + yj) = T(x)$ ,  $N_F(xe + yj) = N(x) - \gamma N(y)$ .  $F$  is associative iff  $E$  is associative and

commutative.

2) As a particular case one has: If  $E=R$  (hence  $s=id_R$ ), the Cayley extension of  $(R, id_R)$  by an element  $\gamma \in R$  is a **quadratic  $R$ -algebra** with basis  $(e, j)$  with  $j^2 = \gamma e$ .

3) Another particular case is the following. Let  $E$  be a quadratic algebra of type  $(\alpha, \beta)$  such that the underlying module is  $R^2$  with multiplication rule given by means of  $(\clubsuit)$  for the canonical basis. Let the conjugation  $s$  be the conjugation in  $E$ . Then for any  $\gamma \in R$ , the Cayley extension  $F$  of  $(E, s)$  by means of  $\gamma$  is called **quaternionic algebra of type  $(\alpha, \beta, \gamma)$** . (This is an associative algebra.) The underlying module is  $R^4$ . Let us denote by  $(0, i, j, k)$  the canonical basis of  $R^4$ . Then the corresponding multiplication rule is given by the following table. (In the same table it are also reported the trace and norm formulas.)

TAB.2.2 - Multiplication table and trace and norm formulas.

	$i$	$j$	$k$	trace and norm formulas
$i$	$\alpha e + \beta i$	$k$	$\alpha j + \beta k$	$T_F(u) = 2\rho + \beta\xi$
$j$	$\beta j - \kappa$	$\gamma e$	$\beta\gamma - \gamma i$	$N_F(u) = \rho^2 + \beta\rho\xi - \alpha\xi^2 - \gamma(\eta^2 + \beta\eta\xi - \alpha\zeta^2)$
$k$	$-\alpha j$	$\gamma i$	$-\alpha\gamma e$	$N_F(uv) = N_F(u)N_F(v)$

$$u = \rho e + \xi i + \eta j + \zeta k, \quad \rho, \xi, \eta, \zeta \in \mathbf{K}; \quad \bar{u} = (\rho + \beta\xi)e - \xi i - \eta j - \zeta k.$$

4) An  $A$ -algebra isomorphic to a quaternionic algebra is called a quaternionic algebra; if a basis of such an algebra has the multiplication to be  $(0)$  then it is called of type  $(\alpha, \beta, \gamma)$ .

5) If  $\beta=0$  we say that the quaternionic algebra is of type  $(\alpha, \gamma)$ . One has:

TAB.2.3 - Multiplication table and trace and norm formulas.

	$i$	$j$	$k$	trace and norm formulas
$i$	$\alpha e$	$k$	$\alpha j$	$T_F(u) = 2\rho$
$j$	$-\kappa$	$\gamma e$	$-\gamma i$	$N_F(u) = \rho^2 - \alpha\xi^2 - \gamma\eta^2 + \alpha\gamma\zeta^2$
$k$	$-\alpha j$	$\gamma i$	$-\alpha\gamma e$	

$$u = \rho e + \xi i + \eta j + \zeta k, \quad \rho, \xi, \eta, \zeta \in \mathbf{K}; \quad \bar{u} = \rho e - \xi i - \eta j - \zeta k.$$

(Of course as  $-1 \neq 1$  this algebra is not commutative.)

□ In particular if  $A=K=R$ ,  $\alpha=\gamma=-1$ ,  $\beta=0$ ,  $F$  is called the **Hamiltonian quaternionic algebra** and is denoted by  $\mathbf{H}$ . In this case  $N(u) \neq 0$ , hence  $u$  admits an inverse  $u^{-1} = N(u)^{-1} \bar{u}$  in  $\mathbf{H}$ , therefore  $\mathbf{H}$  is a noncommutative corp. Any finite  $R$ -algebra that is also a corp (noncommutative) is isomorphic to  $\mathbf{H}$ . Any quaternion  $q \in \mathbf{H}$  can be represented by  $q = \rho e + \xi i + \eta j + \zeta k$ , where  $i, j, k$  are linearly independent symbols that satisfy the following multiplication rules:  $ij = k = -ji$ ,  $jk = i = -kj$ ,  $ki = j = -ik$ ,  $i^2 = j^2 = k^2 = -1$ . One has the following  $R$ -algebras homomorphism:  $A: \mathbf{H} \rightarrow M(2; \mathbf{C})$ ,  $q \rightarrow \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ , where  $i$  is the imaginary unity of  $\mathbf{C}$ . The matrices  $\sigma_x \equiv -iA(k)$ ,  $\sigma_y \equiv -iA(j)$ ,  $\sigma_z \equiv -iA(i)$ , where  $A(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ ,  $A(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $A(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ , are called **Pauli matrices** and satisfy  $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ ,  $\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$ . The set  $\mathbf{H}_1 \equiv N_{\mathbf{H}}^{-1}(1)$  of quaternions of norm 1 is isomorphic to the group  $SU(2)$ :  $\mathbf{H}_1 \cong SU(2)$ . The  $n$ -dimensional quaternionic space  $\mathbf{H}^n$  has a canonical basis  $\{e_k\}_{1 \leq k \leq n}$ ,  $e_k \in \mathbf{H}$ , and any  $v \in \mathbf{H}^n$  can be represented in the form  $v = \sum_{1 \leq k \leq n} q^k e_k$ ,  $q^k \in \mathbf{H}$ , ( $q^k \equiv$  quaternionic components). As any quaternionic number  $q$  admits the following representation  $q = x + yj = x + j\bar{y}$ , with  $x = \rho e + \xi i$ ,  $y = \eta + \zeta i$ , where  $x$  and  $y$  can be considered complex

numbers, then one has the following isomorphism  $\mathbf{H}^n \cong \mathbf{C}^{2n}$ ,  $(q^k) \rightarrow (x^k, y^k)$ , where  $\mathbf{C}^{2n}$  has the following basis  $(e_1, \dots, e_n, j e_1, \dots, j e_n)$ . We write  $\dim_{\mathbf{H}} \mathbf{H}^n = n$ ,  $\dim_{\mathbf{C}} \mathbf{H}^n = 2n$ . By using different quaternionic bases in  $\mathbf{H}^n$  one has that the quaternionic components of any vector  $v \in \mathbf{H}^n$  transform by means of the following rule  $q^k = \sum_{1 \leq i \leq n} q^i \lambda_i^k$ ,  $(\lambda_i^k) \in GL(n, \mathbf{H})$ . Furthermore, the corresponding complex components transform in the following way:

$$\{x'^k = x^i a_i^k - y^i b_i^k, \quad y'^k = x^i b_i^k + y^i a_i^k\}, \quad \lambda_i^k = a_i^k + b_i^k j.$$

Then one has a group-homomorphism  $GL(n, \mathbf{H}) \xrightarrow{\sim} GL(2n, \mathbf{C})$ , such that if  $\Lambda = A + B j \in GL(n, \mathbf{H})$ , then  $c(\Lambda) = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ . On  $\mathbf{H}^n$  there is a canonical quadratic form  $|v|^2 = \sum_{1 \leq k \leq n} |q^k|^2 = \sum_{1 \leq k \leq n} q^k \bar{q}^k = \sum_{1 \leq k \leq n} (|x^k|^2 + |y^k|^2) \in \mathbf{R}$ , where  $v = q^k e_k$ ,  $q^k \in \mathbf{H}$ ,  $q^k = x^k + y^k j$ ,  $x^k, y^k \in \mathbf{C}$ . So such a quadratic form coincides with the ordinary norm of the vector space  $\mathbf{C}^{2n}$ . Furthermore one has on  $\mathbf{H}^n$  the following form  $\langle v_1, v_2 \rangle_{\mathbf{H}} = \sum_{1 \leq k \leq n} q_i^k \bar{q}_2^k \in \mathbf{H}$ ,  $v_i = \sum_{1 \leq k \leq n} q_i^k e_k$ ,  $i=1,2$ . The quaternionic transformations of  $\mathbf{H}^n$ , that conserve above form form a group  $Sp(n) \subset GL(n, \mathbf{H})$ . As we can write  $\langle v_1, v_2 \rangle_{\mathbf{H}}$  in the following way:

$$\langle v_1, v_2 \rangle_{\mathbf{H}} = \left\{ \begin{array}{l} \sum_{1 \leq k \leq n} (x_1^k x_2^k + y_1^k y_2^k) = \langle v_1, v_2 \rangle_{\mathbf{C}} = \text{hermitian form in } \mathbf{C}^{2n} \\ \sum_{1 \leq k \leq n} (y_1^k x_2^k - x_1^k y_2^k) j = \sigma(v_1, v_2)_{\mathbf{C}} = \text{skewsymmetric form in } \mathbf{C}^{2n} \end{array} \right\}$$

we see that  $\Lambda \in Sp(n)$  preserves the hermitian form and the skewsymmetric form. Therefore  $c(Sp(n)) \subset U(2n)$  and it is formed by the unitary transformations of  $\mathbf{C}^{2n}$  that preserves the antisymmetric form  $\sigma(v_1, v_2)_{\mathbf{C}}$ .

EXAMPLE 2.2 -  $Sp(1) \cong SU(2) \subset U(2)$ . So all the transformations contained in  $c(Sp(1))$  are unimodular. ■

DEFINITION 2.1 - Let  $B$  be a quantum algebra. We define **Cayley  $B$ -quantum algebra** any quantum algebra  $C$  that is obtained from a Cayley  $\mathbf{K}$ -algebra  $A$ , by "extending the scalars" from  $\mathbf{K}$  to  $B$ , i.e.,  $C \cong B \otimes_{\mathbf{K}} A$ .

EXAMPLE 2.3 - The noncommutative  $B$ -quaternionic algebra  $B \otimes_{\mathbf{K}} \mathbf{H}$  is a Cayley  $B$ -quantum algebra over  $\mathbf{K} = \mathbf{R}$  endowed with the natural topology of Banach space and considering the  $\mathbf{K}$ -linear morphism  $c = c_B \otimes \frac{1}{2} T : B \otimes_{\mathbf{K}} \mathbf{H} \rightarrow \mathbf{K}$ , where  $T$  is the trace of  $\mathbf{H}$ . (Another possibility is to take  $c = c_B \otimes N$ , where  $N$  is the norm of  $\mathbf{H}$ . In this last case, whether  $B$  is an augmented quantum algebra, then  $B \otimes_{\mathbf{K}} \mathbf{H}$  becomes an augmented quantum algebra too.) ■

DEFINITION 2.2 - A **quantum  $B$ -quaternionic manifold** of dimension  $n$  and class  $Q_w^k$ ,  $0 \leq k \leq \infty, \omega$ , is a quantum manifold  $M$  of dimension  $n$  and class  $Q_w^k$  over the  $B$ -quantum algebra  $C \equiv B \otimes_{\mathbf{K}} \mathbf{H}$ . Then the quantum coordinates in an open coordinate subset  $U \subset M$  are called  **$B$ -quaternionic coordinates**,  $\{q^k\}_{1 \leq k \leq n}$ ,  $q^k : U \rightarrow C^1$ .

DEFINITION 2.3 - The category  $\mathcal{C}_{\mathbf{H}}^B$ , of **quantum  $B$ -quaternionic manifolds** of class  $Q_w^k$ , is defined by considering as morphisms maps of class  $Q_w^k$ , between quantum  $B$ -quaternionic manifolds.

EXAMPLE 2.4 - **Quantum quaternionic Möbius strip**. Let us denote  $I \equiv [-\pi, \pi] \subset \mathbf{R}$ ,  $N \equiv I \times \mathbf{H}$ . Let us introduce the following equivalence relation in  $N$ :  $(x, y) \sim (x, y)$  if  $x \neq -\pi, \pi$ ,  $(-\pi, -y) \sim (\pi, y)$ . Then  $N / \sim \equiv M$  is called noncommutative quaternionic Möbius strip. One has a natural projection  $p : M \rightarrow S^1$ , given by  $p([x, y]) = x \in S^1$  if  $x \neq -\pi, \pi$ , and  $p([\pi, y]) = * \in S^1$ , where  $*$  is the point of  $S^1 \equiv I / \{-\pi, \pi\}$ , corresponding to  $\{-\pi, \pi\}$ . One can recover  $M$  with two open sets:

$$\{\Omega_1 \equiv p^{-1}(U_1), \quad U_1 \equiv ]-\pi, \pi[; \quad \Omega_2 \equiv p^{-1}(U_2), \quad U_2 \equiv S^1 \setminus \{0\}\}.$$

We put quaternionic coordinates on  $\Omega_i$ ,  $i = 1, 2$ , in the following way. On  $\Omega_1$ ,  $\{x^1[x, y] = p[x, y] = x \in \mathbf{R}, x^2[x, y] = y \in \mathbf{H}\}$ . On  $\Omega_2$ , if  $x \neq -\pi, \pi$ ,  $\bar{x}^1[x, y] = -\pi + x \in \mathbf{R}$ ,  $\bar{x}^1[-x, y] = \pi - x \in \mathbf{R}$ ,

<sup>1</sup> As a particular case we can take  $B = \mathbf{K}$ . In this case  $C = \mathbf{H}$  and we call such quantum  $\mathbf{K}$ -quaternionic manifolds simply **quantum quaternionic manifolds**.

$\bar{x}^2[\pm x, y] = y \in \mathbf{H}$ ;  $\bar{x}^1[-\pi, -y] = \bar{x}^1[\pi, y] = 0$ ,  $\bar{x}^2[-\pi, -y] = \bar{x}^2[\pi, y] = |y| \in c^{-1}(\mathbf{R}^+) \subset \mathbf{H}$ , with  $c = \frac{1}{2}T$ . The change of coordinates is given by:

$$\forall q \in \Omega_1 \cap \Omega_2, \quad (p(q) \in S^1 \setminus \{*, 0\} \equiv U_+ \cup U_-), \quad \left\{ \begin{array}{l} \bar{x}^1|_{U_-} = \pi - x^1 \in \mathbf{R} \\ \bar{x}^1|_{U_+} = -\pi + x^1 \in \mathbf{R} \\ \bar{x}^2 = x^2 \in \mathbf{H} \end{array} \right\}.$$

The structure group, i.e. the group of the jacobian matrix, is isomorphic to  $\mathbf{Z}_2$ . In fact one has:

$$\left\{ (\partial x_j, \bar{x}^k) = \begin{pmatrix} (\partial x_1, \bar{x}^1) & (\partial x_1, \bar{x}^2) \\ (\partial x_2, \bar{x}^1) & (\partial x_2, \bar{x}^2) \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \subset GL(2; \mathbf{R}) \right\}.$$

Therefore,  $M$  is a quantum quaternionic manifold modelled on  $\mathbf{R} \times \mathbf{H} \subset \mathbf{H}^2$ , hence  $\dim_{\mathbf{H}} M = 2$ . Furthermore one has the canonical projection  $M \rightarrow S^1$ , therefore  $M$  is a regular quantum manifold of dimension  $(1 \downarrow 2)$ . Finally remark that as  $M$  is not covered by a global chart, it is a non trivial example of quantum quaternionic manifold. Of course this can be also seen by means of homological arguments. In fact one has  $H_1(M; \mathbf{R}) \cong H_1(M_{\mathcal{C}}; \mathbf{R}) \cong H_1(S^1; \mathbf{R}) \cong \mathbf{R}$ . Therefore,  $M$  is not homotopy equivalent to  $\mathbf{R}^5$ , as  $H_1(\mathbf{R}^5; \mathbf{R}) = 0$ . ■

**EXAMPLE 2.5 - Quaternionic manifolds** [3,15,16]. The category  $\mathcal{C}_{\mathbf{H}}$  of quaternionic manifolds is a subcategory of  $\mathcal{C}_{\mathbf{H}}^{B \equiv \mathbf{R}}$ , where the morphisms are quaternionic affine maps [15]. Therefore any of such morphisms  $f \in Hom_{\mathcal{C}_{\mathbf{H}}}(M, N)$ , where  $\dim M = 4m$ ,  $\dim N = 4n$ , are locally represented by formulas like the following:  $f^k = A_j^k q^j + r^k$ ,  $A_j^k, q^j, r^k \in \mathbf{H}$ ,  $1 \leq k \leq n$ ,  $1 \leq j \leq m$ .  $A_j^k$  identify  $m \times n$  matrices with entries in  $\mathbf{H}$ , or equivalently, real matrices of the form

$$(A_j^k) = \begin{pmatrix} \hat{A}_1^1 & \dots & \hat{A}_m^1 \\ \dots & \dots & \dots \\ \hat{A}_1^n & \dots & \hat{A}_m^n \end{pmatrix}, \quad \hat{A}_i^j = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & b & a \end{pmatrix}, \quad a, b, c, d \in \mathbf{R}.$$

The set of such matrices is denoted by  $M(n, m; \mathbf{H})$ . The structure group of a  $4n$ -dimensional quaternionic manifold is  $GL(n; \mathbf{H})$ , (that is the subset of  $M(n, m; \mathbf{H})$  of invertible matrices). Therefore, quaternionic manifolds are quantum quaternionic manifolds where the local maps  $f : U \subset \mathbf{H}^n \rightarrow \bar{U} \subset \mathbf{H}^n$ , change of coordinates, are  $\mathbf{H}$ -linear. Hence  $Df(p) \in \mathbf{H}^{n^2}$ ,  $\forall p \in U$ . In fact, one has the following commutative diagram:

$$\begin{array}{ccccc} Hom_{\mathbf{H}}(\mathbf{H}^n; \mathbf{H}^n) & \hookrightarrow & Hom_{\mathbf{R}}(\mathbf{H}^n; \mathbf{H}^n) & \cong & Hom_{\mathbf{R}}(\mathbf{H}; \mathbf{H})^{n^2} \\ \parallel \wr & & & & \parallel \wr \\ Hom_{\mathbf{H}}(\mathbf{H}; \mathbf{H})^{n^2} & & & & Hom_{\mathbf{R}}(\mathbf{R}^4; \mathbf{R}^4)^{n^2} \\ \parallel \wr & & & & \parallel \wr \\ \mathbf{H}^{n^2} & \cong & \mathbf{R}^{4n^2} & \hookrightarrow & \mathbf{R}^{4^2 n^2} \end{array}$$

On the other hand the tangent space  $T_p M$  has a natural structure of  $\mathbf{H}$ -module iff  $M$  is an affine manifold. (As in this case the action of  $\mathbf{H}$  on  $T_p M \cong \mathbf{H}^n$  does not depend on the coordinates used to obtain the identification of  $T_p M$  with  $\mathbf{H}^n$ .) Hence the category  $\mathcal{C}_{\mathbf{H}}$  is the subcategory of  $\mathcal{C}_{\mathbf{H}}^R$  of affine quantum quaternionic manifolds. A trivial example of quaternionic manifold is  $\mathbf{R}^{4n} \cong \mathbf{H}^n$ . If  $\{x^i, y^i, u^i, v^i\}_{1 \leq i \leq n}$  are real coordinates on  $\mathbf{R}^{4n}$ , then the almost quaternionic structure given by

$$\left\{ \begin{array}{l} J(\partial x_i) = \partial y_i, \quad J(\partial y_i) = -\partial x_i, \quad J(\partial u_i) = -\partial v_i, \quad J(\partial v_i) = \partial u_i \\ K(\partial x_i) = \partial u_i, \quad K(\partial y_i) = \partial v_i, \quad K(\partial u_i) = -\partial x_i, \quad J(\partial v_i) = -\partial y_i \end{array} \right\}$$

is called the **standard right quaternionic structure** on  $\mathbf{R}^{4n}$ .<sup>2</sup> A non trivial example of quaternionic manifold is  $\mathbf{R}^{4n}$  with the standard quaternionic structure quotiented by a discrete translation group that gives a torus. ■

**EXAMPLE 2.6 - Almost quaternionic manifolds.** The category  $\widetilde{\mathcal{C}}_{\mathbf{H}}$  of almost quaternionic manifolds is a subcategory of  $\mathcal{C}_{\mathbf{H}}^{\mathbb{B}=\mathbf{R}}$ , where the structure group of a  $4n$ -dimensional quantum quaternionic manifold is  $GL(n; \mathbf{H})Sp(1) \subset GL(4n; \mathbf{R})$ . The category  $\widetilde{\mathcal{C}}_{\mathbf{H}}$  properly contains  $\mathcal{C}_{\mathbf{H}}$ . An example of almost quaternionic manifold, that is not contained into  $\mathcal{C}_{\mathbf{H}}$ , is the quaternionic projective space  $\mathbf{H}P^1$ . This cannot be a quaternionic manifold, since it does not admit a structure of complex manifold. ■

**REMARK 2.1 -** As the centre  $C_0$  of  $C \equiv B \otimes_{\mathbf{K}} \mathbf{H}$  is isomorphic to  $B_0$ , that is the centre of  $B$ , we get that, whether  $B_0$  is Noetherian, one can apply above Theorem 1.1 and Theorem 1.2 for QPDEs, in order to state the formal quantum-integrability for quantum  $B$ -quaternionic PDEs. Note that in such a way we obtain as solutions submanifolds that have natural structures of quantum  $B$ -quaternionic manifolds. Then applying our theorems on the integral bordism groups for quantum PDEs [12], we can also calculate theorem of existence of global solutions for quantum  $B$ -quaternionic PDEs.

**EXAMPLE 2.7 - Quantum  $B$ -quaternionic heat equation.** Let us consider the fiber bundle  $\pi : W \equiv C^3 \rightarrow C^2 \equiv M$  with coordinates  $(t, x, u) \mapsto (t, x)$ . The quantum  $B$ -quaternionic heat equation is the following QPDE:  $(\widehat{HE})_C \subset \widehat{J}\mathcal{D}^2(W) \subset \widehat{J}_2^2(W) : u_{xx} - u_t = 0$ . This is a formally (quantum)integrable QPDE. Hence, for  $(\widehat{HE})_C$  we have the existence of local solutions for any initial condition. This means that in the neighborhood of any point  $q \in (\widehat{HE})_C$  we can built an integral quantum  $B$ -quaternionic manifold of dimension 2 over  $C$ ,  $V \subset (\widehat{HE})_C$ , such that  $V \cong \pi_2(V) \subset M$ , where  $\pi_2$  is the canonical projection  $\pi_2 : \widehat{J}\mathcal{D}^2(W) \rightarrow M$ . Then by using a Theorem 5.6 given in ref.[12] we have that the first integral bordism groups of  $(\widehat{HE})_C$  is:  $\Omega_1^{(\widehat{HE})_C} \cong H_1(W; \mathbf{K}) \otimes_{\mathbf{K}} C \cong 0$ . Hence we get that any admissible closed integral 1-dimensional quantum  $B$ -quaternionic manifold,  $N \subset (\widehat{HE})_C$  is the boundary of an integral 2-dimensional quantum  $B$ -quaternionic manifold  $V$ ,  $\partial V \subset N$ ,  $V \subset (\widehat{HE})_C$ , such that  $V$  is diffeomorphic to its projection into  $W$  by means of the canonical projection  $\pi_{2,0} : \widehat{J}_2^2(W) \rightarrow W$ . ■

**EXAMPLE 2.8 - Quantum quaternionic heat equation.** As a particular case of above equation one can take  $B \equiv \mathbf{R}$ . Then one has:

$$\left\{ \begin{array}{l} \pi : W \equiv \mathbf{H}^4 \rightarrow \mathbf{H}^2 \equiv M; \quad (t, x, u) \mapsto (t, x) \\ (\widehat{HE})_{\mathbf{H}} \subset J\mathcal{D}^2(W) \subset J_2^2(W) : \quad u_{xx} - u_t = 0 \end{array} \right\}, \quad \Omega_1^{(\widehat{HE})_{\mathbf{H}}} \cong H_1(W; \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{H} \cong 0.$$

We can see that the set  $Sol((\widehat{HE})_{\mathbf{H}})$  of solutions of  $(\widehat{HE})_{\mathbf{H}}$  contains also quaternionic manifolds, i.e., affine quantum quaternionic solutions. For example a torus<sup>3</sup>  $X \subset \mathbf{H}^2 \equiv M$  can be embedded into

<sup>2</sup> An **almost complex structure** on a  $C^\infty$  manifold  $M$  is a fiberwise endomorphism  $J$  of the tangent bundle  $TM$  such that  $J^2 = -1$ . A **complex analytic map** between almost complex manifolds  $(X, J_1)$  and  $(Y, J_2)$  is a  $C^\infty$  map  $\phi: X \rightarrow Y$ , such that  $T(\phi) \circ J_1 = J_2 \circ T(\phi)$ . An **almost quaternionic structure** on a  $C^\infty$  manifold  $M$  is a pair of two almost complex structures  $J$  and  $K$  such that  $JK + KJ = 0$ . A **quaternionic map**  $\phi$  between two almost quaternionic manifolds  $(X, J_1, K_1)$  and  $(Y, J_2, K_2)$  is a map  $\phi: X \rightarrow Y$  that is complex analytic from  $(X, J_1)$  to  $(Y, J_2)$  and from  $(X, K_1)$  to  $(Y, K_2)$ . A **quaternionic manifold** is a  $C^\infty$  manifold  $M$  endowed with an atlas  $\{\phi_i: U_i \rightarrow \mathbf{R}^{4n}\}$ , for some  $n$ , such that  $\phi_j \circ \phi_i^{-1}: \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$  is a quaternionic function with respect to the standard structure on  $\mathbf{R}^{4n}$ . (See also ref.[1].)

<sup>3</sup> Recall [15] that if  $(X, J, K)$  is a quaternionic manifold, then  $X$  with the complex structure  $aJ + bK + cJK$ ,  $a, b, c \in \mathbf{R}$ , is an affine complex manifold, hence has zero rational Pontryagin classes. Furthermore, if  $X$  is compact has zero index and Euler characteristic. Moreover, if  $\dim_{\mathbf{H}} X = 1$  and, for some  $a, b, c$ ,  $X$  is Kähler, then it is a torus.

$(\widehat{HE})_{\mathbf{H}}$  by means of the second holonomic prolongation of the zero section  $u \equiv 0 : M \rightarrow W$ . In fact,  $(\widehat{HE})_{\mathbf{H}}$  is a linear equation. Therefore,  $X^{(2)} \equiv D^2u(X)$  is a 1-dimensional smooth closed compact admissible integral manifold contained into  $(\widehat{HE})_{\mathbf{H}}$ , that is the boundary of a 2-dimensional integral admissible manifold contained into  $(\widehat{HE})_{\mathbf{H}}$  too. This last is also a quaternionic manifold. Moreover, all the regular solutions of  $(\widehat{HE})_{\mathbf{H}} \subset JD^2(W)$  are quaternionic manifolds, as they are diffeomorphic to  $\mathbf{H}^2$ . However, no all the regular solutions of  $(\widehat{HE})_{\mathbf{H}} \subset J_2^2(W)$  are necessarily quaternionic manifolds too. ■

We are ready now to state the main results of this paper.

**THEOREM 2.3** - *Let  $B$  be a quantum algebra such that its centre  $B_0$  is a Noetherian  $\mathbf{R}$ -algebra. Let  $\hat{E}_k \subset J\hat{D}^k(W)$  be a quantum regular QPDE in the category  $C_{\mathbf{H}}^{\mathbf{R}}$ , where  $\pi : W \rightarrow M$  is a fibre bundle with  $\dim_C M = m$ ,  $C \equiv B \otimes_{\mathbf{R}} \mathbf{H}$ . If  $\hat{g}_{k+r+1}$  is a bundle of  $C_0$ -modules over  $\hat{E}_k$ , and  $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}$  is surjective for  $0 \leq r \leq m$ , then  $\hat{E}_k$  is formally quantumintegrable. In such a case, and further assuming that  $W$  is  $p$ -connected,  $p \in \{0, \dots, m-1\}$ , then the integral bordism groups of  $\hat{E}_k \subset \hat{J}_m^k(W)$  are given by:*

$$\Omega_p^{\hat{E}_k} \cong H_p(W; \mathbf{R}) \otimes_{\mathbf{R}} C, \quad 0 \leq p \leq m-1.$$

All the regular solutions of  $\hat{E}_k \subset \hat{J}_m^k(W)$  are quantum  $B$ -quaternionic submanifolds of  $\hat{E}_k$  of dimension  $m$ , over  $C$ , identified with  $m$ -dimensional quantum  $B$ -quaternionic submanifolds of  $W$ .

**PROOF.** It follows directly from above definitions and remarks by specializing Theorem 1.1, Theorem 1.2 and our results in ref.[12], about integral bordism groups in QPDEs, to the category  $C_{\mathbf{H}}^{\mathbf{R}}$ . □

**COROLLARY 2.1** - *Let  $\hat{E}_k \subset JD^k(W)$  be a quantum regular QPDE in the category  $C_{\mathbf{H}}^{\mathbf{R}}$ , (resp.  $\tilde{C}_{\mathbf{H}}$ ), where  $\pi : W \rightarrow M$  is a fibre bundle with  $\dim_{\mathbf{H}} M = m$ . If  $\hat{g}_{k+r+1}$  is a bundle of  $\mathbf{R}$ -modules over  $\hat{E}_k$ , and  $\hat{E}_{k+r+1} \rightarrow \hat{E}_{k+r}$  is surjective for  $0 \leq r \leq m$ , then  $\hat{E}_k$  is formally quantumintegrable. In such a case, further assuming that  $W$  is  $p$ -connected,  $p \in \{0, \dots, m-1\}$ , then the integral bordism groups of  $\hat{E}_k \subset J_m^k(W)$  are given by:*

$$\Omega_p^{\hat{E}_k} \cong H_p(W; \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{H}, \quad 0 \leq p \leq m-1.$$

All the regular solutions of  $\hat{E}_k \subset J_m^k(W)$  are quantum quaternionic, (resp. almost quaternionic), submanifolds of  $\hat{E}_k$  of dimension  $m$ , identified with  $m$ -dimensional quantum quaternionic, (resp. almost quaternionic), submanifolds of  $W$ .

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## OPTIMAL CONTROL PROBLEMS ON THE LIE GROUP $SP(1)$

MIRCEA PUTA

ABSTRACT. An optimal control problem on the Lie group  $SP(1)$  is discussed and some of its dynamical and geometrical properties are pointed out.

### 1. INTRODUCTION

Recent work in nonlinear control has drawn attention to drift-free, left invariant control systems on matrix Lie groups. We can remind here the case of the matrix Lie group  $SO(3)$  studied in connection with the spacecraft dynamics [4], [8], the case of the matrix Lie group  $SE(3)$  studied in connection with the control tower problem [6], the case of the matrix Lie group  $SO(n)$  studied in connection with the electrical circuits [11] and the case of the matrix Lie group  $U(n)$  studied in connection with the molecular dynamics [1]. The goal of our paper is to make a similar study for the matrix Lie group  $SP(1)$ .

### 2. THE LIE GROUP $SP(1)$

Let  $\mathbf{H}$  be the noncommutative field of quaternions, i.e.,

$$\mathbf{H} = \{q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbf{R}\}.$$

Then we have the usual identification:

$$\mathbf{H} \simeq \mathbf{R}^4.$$

We denote by  $GL_1(\mathbf{H})$  the group of automorphisms of  $\mathbf{H}$ , that is the group of transformations  $t : \mathbf{H} \rightarrow \mathbf{H}$ , of the form:

$$\xi' = a\xi, \tag{2.1}$$

where  $\xi, \xi' \in \mathbf{H} \simeq \mathbf{R}^4$  and  $a \in \mathbf{H} \setminus \{0\}$ , and by  $SP(1)$  the subgroup of  $GL_1(\mathbf{H})$  consisting of unitary automorphisms of  $\mathbf{H}$  with respect to the canonical Hermitian product:

$$\xi \cdot \eta = \xi\bar{\eta}, \quad \forall \xi, \eta \in \mathbf{H} \simeq \mathbf{R}^4,$$

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that is,  $t \in SP(1)$  iff it is of the form (2.1) with

$$a \cdot \bar{a} = 1.$$

On the other hand, the canonical scalar product on  $\mathbf{R}^4$  is expressed in quaternionic form by:

$$\langle \xi, \eta \rangle = Re(\xi, \bar{\eta}),$$

where  $\xi, \eta \in \mathbf{H} \simeq \mathbf{R}^4$ . It follows that

$$a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

and then one get the identification:

$$SP(1) \simeq S^3 \subset \mathbf{R}^4.$$

Let  $Im \mathbf{H} \simeq \mathbf{R}^3$  be the space of imaginary quaternions. Since each element of  $SO(3)$  can be expressed in quaternionic form by:

$$\xi' = q\xi\bar{q},$$

where  $q \in \mathbf{H}$ ,  $q\bar{q} = 1$ ,  $\xi, \xi' \in Im \mathbf{H}$ , one has the isomorphism:

$$SO(3) \simeq SP(1)/\mathbf{Z}_2,$$

where

$$\mathbf{Z}_2 = \{1, -1\}.$$

Correspondingly we have an identification of the Lie algebra  $so(3)$  of  $SO(3)$  with the Lie algebra  $sp(1)$  of  $SP(1)$ . In terms of matrices one can get also the following identification:

$$sp(1) \cong \left\{ \left[ \begin{array}{cccc} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{array} \right] \mid a_1, a_2, a_3 \in \mathbf{R} \right\}.$$

Let  $\{A_1, A_2, A_3\}$  be the canonical basis of  $sp(1)$  given by:

$$A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the Lie algebra structure of  $sp(1)$  is given by the following table:

$[\cdot, \cdot]$	$A_1$	$A_2$	$A_3$
$A_1$	0	$2A_3$	$-2A_2$
$A_2$	$-2A_3$	0	$2A_1$
$A_3$	$2A_2$	$-2A_1$	0

Let us consider now on  $SP(1)$  the left invariant system:

$$\dot{g} = g(A_1u_1 + A_2u_2). \tag{2.2}$$

**THEOREM 2.1** *The system (2.2) is controllable and it is a single bracket one.*

**Proof.** Indeed, the proof is a consequence of the fact that  $\mathfrak{sp}(1)$  is generated by  $A_1, A_2, [A_1, A_2]$ . ■

**REMARK 2.1** It is not hard to see that the controlled system (2.2) can be put in the equivalent form:

$$\begin{cases} \dot{x} = zu_2 \\ \dot{y} = zu_1 \\ \dot{z} = xu_2 - yu_1 \\ xu_1 + yu_2 = 0. \end{cases} \quad (2.3)$$

■

### 3. AN OPTIMAL CONTROL PROBLEM

Let  $\mathcal{J}$  be the cost function given by:

$$\mathcal{J}(u_1, u_2) = \frac{1}{2} \int_0^{t_f} [c_1 u_1^2(t) + c_2 u_2^2(t)] dt; \quad c_1 > 0, \quad c_2 > 0. \quad (3.1)$$

Then we can prove:

**THEOREM 3.1** *The controls that minimize  $\mathcal{J}$  and steer the system (2.2) from  $X = X_0$  at  $t = 0$  to  $X = X_f$  at  $t = t_f$  are given by:*

$$u_1 = \frac{1}{c_1} P_1; \quad u_2 = \frac{1}{c_2} P_2,$$

where the functions  $P_i$  are solutions of:

$$\begin{cases} \dot{P}_1 = -\frac{2}{c_2} P_2 P_3 \\ \dot{P}_2 = \frac{2}{c_1} P_1 P_3 \\ \dot{P}_3 = \left( \frac{2}{c_2} - \frac{2}{c_1} \right) P_1 P_2. \end{cases} \quad (3.2)$$

**Proof.** Simply apply Krishnaprasad's theorem, [3]. It follows that the optimal Hamiltonian is given by:

$$H_o = \frac{1}{2} \left( \frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} \right). \quad (3.3)$$

It is in fact the controlled Hamiltonian  $H$  given by

$$H = P_1 u_1 + P_2 u_2 - \frac{1}{2} (c_1 u_1^2 + c_2 u_2^2),$$

which is reduced to  $(\mathfrak{sp}(1))_-^*$  via the Poisson reduction. Here  $(\mathfrak{sp}(1))_-^*$  is  $(\mathfrak{sp}(1))^* \simeq \mathbf{R}^3$  together with the minus Lie-Poisson structure given by the matrix:

$$\mathbb{I} = \begin{bmatrix} 0 & -2P_3 & 2P_2 \\ 2P_3 & 0 & -2P_1 \\ -2P_2 & 2P_2 & 0 \end{bmatrix}. \tag{3.4}$$

Then the optimal controls are given by:

$$u_1 = \frac{1}{c_1} P_1; \quad u_2 = \frac{1}{c_2} P_2,$$

where the functions  $P_i$  are solutions of the reduced Hamilton's equations (or momentum equations) given by:

$$[\dot{P}_1, \dot{P}_2, \dot{P}_3]^t = \mathbb{I} \cdot \nabla H_o,$$

which are nothing else but the required equations. ■

**REMARK 3.1** The function  $C$  given by

$$C = P_1^2 + P_2^2 + P_3^2 \tag{3.5}$$

is a Casimir of our configuration  $((\mathfrak{sp}(1))^*, \mathbb{I}) \simeq (\mathbf{R}^3, \mathbb{I})$ , i.e.,

$$(\nabla C)^t \cdot \mathbb{I} = 0.$$

**REMARK 3.2** The phase curves of our system (3.2) are the intersections of the elliptic cylinders ■

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = 2H_o$$

with the spheres

$$P_1^2 + P_2^2 + P_3^2 = C.$$

**THEOREM 3.2** *The dynamics (3.2) is equivalent to the pendulum dynamics.* ■

**Proof.** Indeed,  $H_o$  is a constant of motion, so

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = l^2.$$

Let us take now

$$\begin{cases} P_1 = l\sqrt{c_1} \cos \theta \\ P_2 = l\sqrt{c_2} \sin \theta. \end{cases}$$

Then

$$\begin{aligned} \dot{P}_1 &= -\sqrt{c_1} \dot{\theta} \sin \theta \\ &= -\sqrt{\frac{c_1}{c_2}} \sqrt{c_2} \dot{\theta} \sin \theta \\ &= -\sqrt{\frac{c_1}{c_2}} \dot{\theta} P_2 \end{aligned}$$

or equivalently,

$$\begin{aligned}\dot{\theta} &= -\sqrt{\frac{c_1}{c_2}} \frac{\dot{P}_1}{P_2} \\ &= -\sqrt{\frac{c_1}{c_2}} \left( -\frac{P_2 P_3}{P_2} \right) \frac{2}{c_2} \\ &= \frac{2}{\sqrt{c_1 c_2}} P_3.\end{aligned}$$

Differentiating again, we get

$$\ddot{\theta} = 2l^2 \sqrt{c_1 c_2} (c_1 - c_2) \sin 2\theta.$$

Thus, pendulum mechanics as required.  $\blacksquare$

**THEOREM 3.4** *The system (3.2) may be realized as a Hamilton-Poisson system in an infinite number of different ways, i.e., there exists infinitely many different, in general nonisomorphic Poisson structures on  $\mathbf{R}^3$  such that the system (3.2) is induced by an appropriate Hamiltonian.*

**Proof.** Indeed, to begin with, let us observe that our system can be put in an equivalent form:

$$\dot{P} = \nabla C \times \nabla H_o,$$

where  $P = [P_1, P_2, P_3]^t$  and  $C$  and  $H_o$  are respectively given by (3.5) and (3.3). Now, an easy computation shows us that the system (3.3) may be realized as a Hamilton-Poisson system with the phase space  $\mathbf{R}^3$ , the Poisson bracket  $\{\cdot, \cdot\}_{ab}$  given by

$$\{f, g\}_{ab} = -\nabla C' \cdot (\nabla f \times \nabla g),$$

where  $a, b \in \mathbf{R}$ ,

$$C' = aC + bH_o,$$

and the Hamiltonian  $H'$  defined by

$$H' = cC + dH_o,$$

where  $c, d \in \mathbf{R}$ ,  $ad - bc = 1$ .  $\blacksquare$

**THEOREM 3.5** *The system (3.2) has a Lax formulation.*

**Proof.** Let us take:

$$L = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{2}{c_1} P_3 & \left( \frac{2}{c_2} - \frac{2}{c_1} \right) P_2 \\ -\frac{2}{c_1} P_3 & 0 & 0 \\ \left( \frac{2}{c_1} - \frac{2}{c_2} \right) P_2 & 0 & 0 \end{bmatrix}.$$

Then a direct computation shows us that the system (3.2) can be put in the equivalent form

$$\dot{L} = [L, B],$$

as required. ■

**REMARK 3.3** As a consequence of the above theorem it follows that the flow of the system (3.2) is isospectral. ■

**THEOREM 3.6** *The system (3.2) may be explicitly integrated by elliptic functions.*

**Proof.** It is known that:

$$P_1^2 c_2 + P_2^2 c_1 = 2H_o c_1 c_2 = l,$$

and

$$P_1^2 + P_2^2 + P_3^2 = C,$$

are constants of motion. Then an easy computation shows us that:

$$P_2^2 = \frac{c_2}{c_2 - c_1} \left[ \frac{C c_2 - l}{c_2} - P_3^2 \right]$$

and

$$P_1^2 = \frac{c_1}{c_1 - c_2} \left[ \frac{C c_1 - l}{c_1} - P_3^2 \right].$$

Using now the third equation from (3.2) we get

$$(\dot{P}_3)^2 = \frac{4}{c_1 c_2} \left( P_3^2 - \frac{C c_2 - l}{c_2} \right) \left( \frac{C c_1 - l}{c_1} - P_3^2 \right)$$

that is:

$$t = \int_{P_3(0)}^{P_3} \frac{dt}{\sqrt{\frac{4}{c_1 c_2} \left( P_3^2 - \frac{C c_2 - l}{c_2} \right) \left( \frac{C c_1 - l}{c_1} - P_3^2 \right)}}$$

which shows that  $P_3$ , and hence  $P_1, P_2$  are elliptic functions of time. ■

#### 4. NUMERICAL INTEGRATION OF THE SYSTEM (3.2)

In this section we shall discuss the numerical integration of the system (3.2) via the Lie-Trotter integrator and we shall point out some of their geometrical properties. To begin with, let us observe that the Hamiltonian vector field  $X_{H_o}$  splits as follows:

$$X_{H_o} = X_{H_1} + X_{H_2},$$

where

$$H_1 = \frac{1}{2} \frac{P_1^2}{c_1}; \quad H_2 = \frac{1}{2} \frac{P_2^2}{c_2}.$$

The integral curves of  $X_{H_1}$  and  $X_{H_2}$  are given by:

$$P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_1(t, P(0))$$



and respectively

$$P(t) = \exp(tX_{H_2}) \cdot P(0) = \phi_2(t, P(0)).$$

Now, following [10] (see also [5] and [9]), the Lie-Trotter formula gives rise to an explicit integrator of the equation (3.2) namely:

$$P^{k+1} = \phi_1(t, \phi_2(t, P^{(k)}),$$

or explicitly:

$$\left\{ \begin{array}{l} P_1^{k+1} = P_1^k \cos \frac{2P_2(0)}{c_2} t - P_3^k \sin \frac{2P_2(0)}{c_2} t \\ P_2^{k+1} = P_1^k \sin \frac{2P_1(0)}{c_1} t \sin \frac{2P_2(0)}{c_2} t + P_2^k \cos \frac{2P_1(0)}{c_1} t \\ \quad + P_3^k \sin \frac{2P_1(0)}{c_1} t \cos \frac{2P_1(0)}{c_1} t \\ P_3^{k+1} = P_1^k \cos \frac{2P_1(0)}{c_1} t \sin \frac{2P_2(0)}{c_2} t - P_2^k \sin \frac{2P_1(0)}{c_1} t \\ \quad + P_3^k \cos \frac{2P_1(0)}{c_1} t \cos \frac{2P_2(0)}{c_2} t. \end{array} \right. \quad (4.1)$$

Some of its properties are sketched in the following theorem:

**THEOREM 4.1** *The numerical integrator (4.1) has the following properties: (i) The numerical integrator (4.1) preserves the Poisson structure (3.4). (ii) The numerical integrator (4.1) preserves the Casimirs of our configuration  $(\mathbf{R}^3, \prod)$ . (iii) Its restriction to each coadjoint orbit  $(P_1^2 + P_2^2 + P_3^2 = k, \omega_k = \frac{1}{k}(P_2 dP_1 \wedge dP_3 - P_3 dP_1 \wedge dP_2 - P_1 dP_2 \wedge dP_3))$  gives rise to a symplectic integrator. (iv) The numerical integrator (4.1) does not preserve the Hamiltonian  $H_o$  given by (3.3).*

**Proof.** The items (i)–(iii) hold because  $\phi_1$  and  $\phi_2$  are flows of some Hamiltonian vector fields, hence they are Poisson maps. Item (iv) is essentially due to the fact that

$$\{H_1, H_2\} \neq 0.$$

■

## 5. STABILITY

It is not hard to see that the equilibrium states of our system (3.2) are:

$$e_1 = (M, 0, 0); \quad e_2 = (0, M, 0); \quad e_3 = (0, 0, M),$$

where  $M \in \mathbf{R}$ . Now we shall discuss their nonlinear stability. Recall that an equilibrium point  $p$  is nonlinear stable if trajectories starting close to  $p$  stay close to  $p$ . In other words, a neighborhood of  $p$  must be flow invariant.

**THEOREM 5.1** *The equilibrium state  $e_1$  is: (i) unstable, if  $c_1 > c_2$ ; (ii) nonlinear stable if  $c_1 < c_2$ .*

**Proof.** First consider the system linearized about  $e_1$ . Its eigenvalues are given by solutions of the equation:

$$\lambda \left( \lambda^2 - 4M^2 \frac{c_1 - c_2}{c_1^2 c_2} \right) = 0.$$

(i) If  $c_1 > c_2$  then a root of the characteristic polynomial has positive real part, thus  $e_1$  is unstable as required. (ii) If  $c_1 < c_2$ , then the characteristic polynomial has two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is, via the energy-Casimir method, [2], [7]. Consider the energy-Casimir function:

$$H_\varphi = \frac{1}{2} \left( \frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} \right) + \varphi \left( \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) \right),$$

where  $\varphi : \mathbf{R} \rightarrow \mathbf{R}$  is an arbitrary smooth real valued function defined on  $\mathbf{R}$ . Let  $\varphi', \varphi''$  denote its first and second derivatives. Now, the first variation of  $H_\varphi$  is given by:

$$\delta H_\varphi = \frac{P_1}{c_1} \delta P_1 + \frac{P_2}{c_2} \delta P_2 + \varphi'(\cdot)(P_1 \delta P_1 + P_2 \delta P_2 + P_3 \delta P_3).$$

This equals zero at the equilibrium of interest if and only if:

$$\varphi' \left( \frac{1}{2} M^2 \right) = -\frac{1}{c_1}. \tag{5.1}$$

The second variation of  $H_\varphi$  at the equilibrium of interest is given via (5.1) by:

$$\delta^2 H_\varphi(e_1) = \varphi'' \left( \frac{1}{2} M^2 \right) M^2 (\delta P_1)^2 + \frac{c_1 - c_2}{c_1 c_2} (\delta P_2)^2 - \frac{1}{c_1} (\delta P_3)^2.$$

Since  $c_1 < c_2$ ;  $c_1, c_2 > 0$  and having chosen  $\varphi$  such that:

$$\varphi'' \left( \frac{1}{2} M^2 \right) < 0,$$

we can conclude that the second variation at the equilibrium of interest is negative definite thus  $e_1$  is nonlinear stable. ■

Similar arguments lead us to:

**THEOREM 5.2** *The equilibrium state  $e_2$  is: (i) unstable, if  $c_1 < c_2$ ; (ii) nonlinear stable, if  $c_1 > c_2$ .*

**THEOREM 5.3** *The equilibrium state  $e_3$  is always nonlinear stable.*

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## A NEW WEIGHT SYSTEM ON CHORD DIAGRAMS VIA HYPERKÄHLER GEOMETRY

JUSTIN SAWON

ABSTRACT. A weight system on graph homology was constructed by Rozansky and Witten using a compact hyperkähler manifold. A variation of this construction utilizing holomorphic vector bundles over the manifold gives a weight system on chord diagrams. We investigate these weights from the hyperkähler geometry point of view.

### 1. INTRODUCTION

New invariants of hyperkähler manifolds were introduced by Rozansky and Witten in [10]. They occur as the weights in a Feynman diagram expansion of the partition function

$$Z^{\text{RW}}(M) = \sum b_{\Gamma}(X) I_{\Gamma}^{\text{RW}}(M)$$

of a three-dimensional physical theory. In this expansion the terms  $I_{\Gamma}^{\text{RW}}(M)$  depend on the three-manifold  $M$  but not on the compact hyperkähler manifold  $X$ , whereas the weights  $b_{\Gamma}(X)$  depend on  $X$  but not on  $M$ . Both terms are indexed by the trivalent graph  $\Gamma$ , though  $b_{\Gamma}(X)$  actually only depends on the *graph homology* class which  $\Gamma$  represents. There are many similarities with Chern-Simons theory, for which a Feynman diagram expansion of the partition function

$$Z^{\text{CS}}(M) = \sum c_{\Gamma}(\mathfrak{g}) I_{\Gamma}^{\text{CS}}(M)$$

gives us the more ‘familiar’ weights  $c_{\Gamma}(\mathfrak{g})$  on graph homology constructed from a Lie algebra  $\mathfrak{g}$  (in this case, the Lie algebra of the gauge group). We wish to further exploit the analogies.

For example, in Chern-Simons theory we can introduce Wilson lines, ie. a link embedded in the three-manifold. This leads to correlation functions which are invariants of the link, depending on representations  $V_a$  of the Lie algebra  $\mathfrak{g}$  which are attached to the components of the link (the Wilson lines). Perturbatively we get

$$Z^{\text{CS}}(M; \mathcal{L}) = \sum c_D(\mathfrak{g}; V_a) Z_D^{\text{Kont}}(M; \mathcal{L})$$

where we sum over all *chord diagrams*  $D$  (unitrivalent graphs whose univalent vertices lie on a collection of oriented circles), the weights  $c_D(\mathfrak{g}; V_a)$  depend on the Lie algebra  $\mathfrak{g}$  and its representations  $V_a$ , but not on the three-manifold  $M$  or the link  $\mathcal{L}$ , and

$$Z^{\text{Kont}}(M; \mathcal{L}) = \sum Z_D^{\text{Kont}}(M; \mathcal{L})D$$

is the Kontsevich integral of the link  $\mathcal{L}$  in  $M$ . We would like to imitate this construction in Rozansky-Witten theory, but although Rozansky and Witten give a construction using spinor bundles, it is not clear in general how to associate observables to Wilson lines using arbitrary holomorphic vector bundles over  $X$ . However, in this article we show that ‘perturbatively’ this is possible. In other words, we construct explicitly a weight system  $b_D(X; E_a)$  on chord diagrams from a collection of holomorphic vector bundles  $E_a$  over a compact hyperkähler manifold  $X$ . This leads to potentially new invariants of links

$$Z^{\text{RW}}(M; \mathcal{L}) = \sum b_D(X; E_a)Z_D^{\text{Kont}}(M; \mathcal{L}).$$

Rather than investigate these invariants of links, our main purpose in this paper is to use these ideas to obtain new results in hyperkähler geometry. For example, the weights  $b_\Gamma(X)$  are invariant under deformations of the hyperkähler metric, and for particular choices of  $\Gamma$  give characteristic numbers. We can use the formalism of graph homology to relate certain invariants, in particular arriving at a formula for the norm of the curvature of  $X$  in terms of characteristic numbers and the volume of  $X$ . This is our most fruitful application of this theory to hyperkähler geometry, though the result should extend to the invariants of holomorphic vector bundles over  $X$ .

Some of the ideas presented in this paper have already been described in more detail in Hitchin and Sawon [7], and the entire work is a continuation of the research first presented in [11]. A complete account may be found in the author’s PhD thesis [12].

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## 2. HYPERKÄHLER GEOMETRY AND DEFINITIONS

Let  $X$  be a compact hyperkähler manifold of real-dimension  $4k$ . This means there is a metric on  $X$  whose Levi-Civita connection has holonomy contained in  $\text{Sp}(k)$ . Such a manifold admits the following structures:

- complex structures  $I, J$ , and  $K$  acting like the quaternions on the tangent bundle  $T$ ,
- a *hyperkähler* metric  $g$  Kählerian wrt  $I, J$ , and  $K$ ,
- corresponding Kähler forms  $\omega_1, \omega_2$ , and  $\omega_3$ .

Although there is a whole sphere of complex structures compatible with the hyperkähler metric, there is no natural way to choose one of them. However, since we wish to use the techniques of complex geometry we shall choose to regard  $X$  as a complex manifold with respect to  $I$ . Then we can construct a holomorphic symplectic form

$$\omega = \omega_2 + i\omega_3 \in H^0(X, \Lambda^2 T^*)$$

on  $X$ , whose dual is

$$\tilde{\omega} \in H^0(X, \Lambda^2 T).$$

Note that in local complex coordinates  $\tilde{\omega}$  has matrix  $\omega^{ij}$  which is minus the inverse of the matrix  $\omega_{ij}$  of  $\omega$ . The Riemann curvature tensor of the Levi-Civita connection of  $g$  is

$$K \in \Omega^{1,1}(\text{End}T)$$

which has components  $K^i_{j\bar{k}\bar{l}}$  with respect to local complex coordinates. Using  $\omega$  to identify  $T$  and  $T^*$ , we get

$$\Phi_{ijk\bar{l}} = \sum_m \omega_{im} K^m_{j\bar{k}\bar{l}}.$$

This tensor is symmetric in  $j$  and  $k$  as the Levi-Civita connection is torsion-free and complex structure preserving. It is also symmetric in  $i$  and  $j$  due to the  $\text{Sp}(2k, \mathbb{C})$  reduction of the frame bundle which accompanies the hyperkähler structure. Therefore

$$\Phi \in \Omega^{0,1}(\text{Sym}^3 T^*).$$

Let  $E$  be a holomorphic vector bundle over  $X$  of complex-rank  $r$ , and choose a Hermitian structure  $h$  on  $E$ . The unique connection  $\nabla$  on  $E$  which is compatible with both the Hermitian and holomorphic structures is called the *Hermitian connection*. The curvature

$$R \in \Omega^{1,1}(\text{End}E)$$

of this connection is of pure Hodge type and has components  $R^I_{j\bar{k}\bar{l}}$  with respect to local complex coordinates on  $X$  and a local basis of sections of  $E$ .

Let  $\Gamma$  be an oriented trivalent graph with  $2k$  vertices. The orientation means an equivalence class of orientations of the edges and an ordering of the vertices; if two such differ by a permutation  $\pi$  of the vertices and a reversal of the orientation on  $n$  edges then they are equivalent if  $\text{sign}\pi = (-1)^n$ . Due to an argument of Kapranov [8] this notion of orientation is equivalent to the usual one given by an equivalence class of cyclic orderings of the outgoing edges at each vertex, with two such equivalent if they differ at an even number of vertices. Hence any trivalent graph drawn in the plane has a canonical orientation given by taking the anticlockwise cyclic ordering at each vertex. Note that  $\Gamma$  need not be connected, but we do not allow connected components which simply consist of closed circles.

Place a copy of  $\Phi$  at each vertex of  $\Gamma$  and attach the holomorphic indices  $i, j$ , and  $k$  to the outgoing edges in any way. Place a copy of  $\tilde{\omega}$  on each edge of  $\Gamma$  and attach the holomorphic indices  $i$  and  $j$  to the ends of the edges in a way compatible with the

orientations of the edges. The ends of each edge will then have two indices attached to them, one coming from  $\Phi$  and one coming from  $\tilde{\omega}$ . Now multiply all these copies of  $\Phi$  and  $\tilde{\omega}$ , with the  $\Phi$ s multiplied in a way compatible with the ordering of the vertices, and then contract the indices at the ends of each edge. Finally, project to the exterior product to get an element

$$\Gamma(\Phi) \in \Omega^{0,2k}(X).$$

For example, suppose that  $\Gamma$  is the two-vertex graph



which we denote by the Greek letter  $\Theta$  and call *theta*. The canonical orientation of this graph corresponds to ordering the vertices 1 and 2 with the three edges all oriented from 1 to 2 (or any equivalent arrangement). Therefore in local complex coordinates  $\Theta(\Phi) \in \Omega^{0,2}(X)$  looks like

$$\Theta(\Phi)_{\bar{i}_1 \bar{i}_2} = \Phi_{i_1 j_1 k_1 \bar{i}_1} \Phi_{i_2 j_2 k_2 \bar{i}_2} \omega^{i_1 i_2} \omega^{j_1 j_2} \omega^{k_1 k_2}.$$

Note that  $X$  must be four real-dimensional in this example, ie. either a K3 surface  $S$  or a torus.

Returning to the general case, we multiply  $\Gamma(\Phi)$  by  $\omega^k$  which is a trivializing section of  $\Lambda^{2k} T^*$ . This gives us an element of  $\Omega^{2k,2k}(X)$  which we can integrate to get a number.

**Definition** *The Rozansky-Witten invariant of  $X$  corresponding to the oriented trivalent graph  $\Gamma$  is*

$$(1) \quad b_\Gamma(X) = \frac{1}{(8\pi^2)^k k!} \int_X \Gamma(\Phi) \omega^k.$$

Now let  $D$  be a *chord diagram*, which consists of an oriented univalent graph whose univalent (or *external*) vertices lie on a collection of oriented circles which we call the *skeleton* of the diagram. The orientation is given by an equivalence class of cyclic orderings of the outgoing edges at each trivalent (or *internal*) vertex, with two such equivalent if they differ at an even number of vertices. Including the skeleton, we can regard the entire diagram as being a trivalent graph with some extra information. Since the skeleton consists of *oriented* circles, it induces a canonical cyclic ordering of the outgoing edges at the external vertices, so this trivalent graph is also oriented. The corresponding ordering of the vertices and orientations of the edges can be chosen in a way compatible with the orientation of the skeleton since we are working in an equivalence class. Note that  $D$  may be disconnected; we even allow circles in the skeleton with no external vertex on them. We assume that  $D$  has  $2k$  vertices (internal and external).

Let  $E_1, \dots, E_m$  be a collection of holomorphic vector bundles over  $X$ , one for each circle in the skeleton of  $D$ . Choose Hermitian structures on these bundles and denote



the curvatures of the corresponding Hermitian connections by

$$R_a \in \Omega^{1,1}(\text{End}E_a).$$

As before, we place a copy of  $\Phi$  at each internal vertex of  $D$  and a copy of  $\tilde{\omega}$  at each edge, and attach indices as before. Each circle in the skeleton will have a vector bundle  $E_a$  associated with it, and we place a copy of the curvature  $R_a$  of that vector bundle at each external vertex lying on that circle. Recall that in local complex coordinates, and with respect to a local basis of sections of  $E_a$ , this curvature has components  $(R_a)^{I_a}_{J_a k \bar{l}}$ . Then  $k$  should be attached to the outgoing edge,  $I_a$  to the *incoming* part of the skeleton, and  $J_a$  to the *outgoing* part of the skeleton (recall that the skeleton consists of *oriented* circles). Now multiply all these copies of  $\Phi$ ,  $\tilde{\omega}$ , and  $R_1, \dots, R_m$ , with the  $\Phi$ s and  $R_a$ s multiplied in a way compatible with the ordering of the vertices, and then contract the indices as before. For the curvatures attached to the external vertices, we contract indices like

$$\dots (R_a)^{I_a}_{J_a k \bar{l}} (R_a)^{J_a}_{K_a m \bar{n}} \dots$$

in an order which is compatible with the orientations of the circles making up the skeleton. If one of the circles has no external vertices lying on it, then we simply include a factor given by minus the rank of the vector bundle attached to that circle. Finally, projecting to the exterior product we get an element

$$D(\Phi; R_a) \in \Omega^{0,2k}(X).$$

As before, multiplying by  $\omega^k$  gives us an element of  $\Omega^{2k,2k}(X)$  which we can integrate.

**Definition** *The weight on the chord diagram  $D$  given by the vector bundles  $E_a$  over  $X$  is*

$$(2) \quad b_D(X; E_a) = \frac{1}{(8\pi^2)^k k!} \int_X D(\Phi; R_a) \omega^k.$$

For example, let  $S$  be a K3 surface,  $E$  a vector bundle over  $S$ , and  $D$  the chord diagram which is like the trivalent graph  $\Theta$ , but with the outer circle being the skeleton. We usually break the skeleton at some (arbitrary) point and draw it as a directed line, and hence  $D$  looks like

In local complex coordinates  $D(\Phi; R) \in \Omega^{0,2}(S)$  looks like

$$D(\Phi; R)_{\bar{l}_1 \bar{l}_2} = R^I_{J k_1 \bar{l}_1} R^J_{I k_2 \bar{l}_2} \omega^{k_1 k_2}$$

and

$$b_D(S; E) = \frac{1}{8\pi^2} \int_S D(\Phi; R) \omega.$$

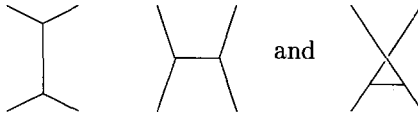
This construction may be varied by replacing the curvatures  $R_1, \dots, R_m$ , and  $K$  by the Dolbeault cohomology classes which they represent; these are the Atiyah classes

of the bundles  $E_1, \dots, E_m$ , and  $T$ , and are independent of the choices of Hermitian structures and hyperkähler metric (respectively) on these bundles. We can then calculate using cohomology instead of differential forms. In fact, we can define the Rozansky-Witten invariants  $b_\Gamma(X)$  and the weights  $b_D(X; E_a)$  for any holomorphic symplectic manifold  $X$  (ie. not necessarily Kähler). This approach is due to Kapranov [8].

### 3. PROPERTIES OF $b_\Gamma(X)$ AND $b_D(X; E_a)$

Let us first mention some of the basic properties of  $b_\Gamma(X)$ .

1. Recall that we chose to regard  $X$  as a complex manifold with respect to  $I$ . In fact,  $b_\Gamma(X)$  is independent of this choice of compatible complex structure. Furthermore, it is a real number.
2. If we deform the hyperkähler metric  $b_\Gamma(X)$  remains invariant. In other words,  $b_\Gamma(X)$  is constant on connected components of the moduli space of hyperkähler metrics on  $X$ . This essentially follows from the cohomological approach mentioned above.
3. The invariant  $b_\Gamma(X)$  depends on the trivalent graph  $\Gamma$  only through its *graph homology* class. Graph homology is the space of rational linear combinations of oriented trivalent graphs modulo the AS and IHX relations. The former says that reversing the orientation of a graph is equivalent to changing its sign, and the latter says that three graphs  $\Gamma_I, \Gamma_H$ , and  $\Gamma_X$  which are identical except for in a small ball where they look like



respectively, are related by

$$\Gamma_I \equiv \Gamma_H - \Gamma_X.$$

In the Rozansky-Witten invariant context, the AS relations follow easily from the definition whereas the IHX relations follow by integrating by parts.

4. The factor

$$\frac{1}{(8\pi^2)^k k!}$$

in Equation (1) has been carefully chosen. Firstly, dividing by  $k!$  ensures that the invariants satisfy the following multiplicative property

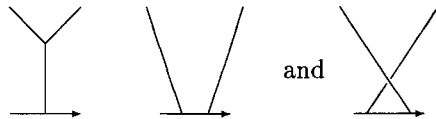
$$(3) \quad b_\Gamma(X \times Y) = \sum_{\gamma \sqcup \gamma' = \Gamma} b_\gamma(X) b_{\gamma'}(Y)$$

where  $X$  and  $Y$  are compact hyperkähler manifolds and the sum is over all ways of decomposing  $\Gamma$  into the disjoint union of two trivalent graphs  $\gamma$  and  $\gamma'$ . The

additional factors lead to a nice formula for characteristic numbers in terms of Rozansky-Witten invariants which we shall describe in the next section. More importantly, our overall normalization agrees with Rozansky and Witten's.

These properties were known to Rozansky and Witten (for a slightly different presentation see [7] or [12]). We can show that the weights  $b_D(X; E_a)$  satisfy similar properties.

1. Since the Atiyah class of  $E$  does not depend on the choice of Hermitian structure, we can show via the cohomological approach that neither does  $b_D(X; E_a)$  depend on the choices of Hermitian structures on the bundles  $E_1, \dots, E_m$ .
2. The weight  $b_D(X; E_a)$  depends on  $D$  only through its chord diagram equivalence class. In other words, we should consider rational linear combinations of chord diagrams modulo the AS, IHX, and STU relations. The AS and IHX relations are as before, applied to internal vertices, while the STU relations are essentially the IHX relations applied to external vertices. More precisely, let  $D_S$ ,  $D_T$ , and  $D_U$  be three chord diagrams which are identical except for in a small ball where they look like



respectively. Then they are related by

$$D_S \equiv D_T - D_U.$$

In fact, it can be shown that all of the AS and IHX relations follow from the STU relations. Once again, in the hyperkähler context the STU relations follow by integrating by parts.

3. In the case that all of the vector bundles  $E_1, \dots, E_m$  are trivial, we can choose flat connections and hence the curvatures  $R_a$  vanish. The only non-zero weights  $b_D(X; E_a)$  will come from chord diagrams which are given by a trivalent graph  $\Gamma$  plus a skeleton consisting of a collection of disjoint circles<sup>1</sup>. Up to the additional factors corresponding to these circles, we simply get  $b_\Gamma(X)$ , and this is why we have chosen the same factor

$$\frac{1}{(8\pi^2)^k k!}$$

in Equation (2). Note that another way to obtain the Rozansky-Witten invariants  $b_\Gamma(X)$  is by letting the vector bundles  $E_1, \dots, E_m$  be the tangent bundle  $T$ . In this case, the chord diagram  $D$  becomes a trivalent graph, with no distinction between the edges of the univalent graph and the skeleton.

<sup>1</sup>This is analogous to the vanishing of Wilson lines in Chern-Simons theory when we associate the trivial representation to them.

These properties are discussed at greater length in [12]. Whether or not the weights  $b_D(X; E_a)$  are also independent of the holomorphic structures on the vector bundles  $E_1, \dots, E_m$  is a question requiring further investigation. We expect that in the case of hyperholomorphic bundles (as described in Verbitsky’s talk at this meeting) the answer should be in the affirmative.

4. EXAMPLES

In this section we shall discuss some specific trivalent graphs and chord diagrams, and the Rozansky-Witten invariants and weights which they lead to. To begin with, suppose we have an *irreducible* hyperkähler manifold  $X$ . For such a manifold we know that

$$h^{0,q} = \begin{cases} 0 & \text{if } q \text{ is odd} \\ 1 & \text{if } q \text{ is even} \end{cases}$$

where  $h^{p,q}$  are the Hodge numbers of  $X$  (see [3] for example). Now suppose we have a trivalent graph  $\gamma$  with  $2m < 2k$  vertices. We can still construct

$$\gamma(\Phi) \in \Omega^{0,2m}(X)$$

as before. Furthermore, the Dolbeault cohomology class that this element represents lies in the one-dimensional cohomology group

$$H_{\bar{\partial}}^{2m}(X).$$

This group is generated by  $[\bar{\omega}^m]$  and hence

$$(4) \quad [\gamma(\Phi)] = c_{\gamma}[\bar{\omega}^m]$$

for some constant  $c_{\gamma}$ . Therefore if  $\Gamma$  is a trivalent graph with  $2k$  vertices which decomposes into the disjoint union of the trivalent graphs  $\gamma_1, \dots, \gamma_t$ , then

$$(5) \quad b_{\Gamma}(X) = \frac{1}{(8\pi^2)^k k!} c_{\gamma_1} \cdots c_{\gamma_t} \int_X \bar{\omega}^k \omega^k$$

for irreducible  $X$ . This formula clearly generalizes to the case that the  $\gamma_i$  may be chord diagrams instead of trivalent graphs, and we introduce a collection of holomorphic vector bundles over  $X$ .

For example, if we let  $\Theta_2$  denote the trivalent graph



then

$$(6) \quad b_{\Theta^4}(X) b_{\Theta_2^2}(X) = b_{\Theta^2 \Theta_2}(X) b_{\Theta^2 \Theta_2}(X)$$

for an irreducible sixteen-dimensional manifold  $X$ , as both sides equal

$$c_{\Theta^4}^4 c_{\Theta_2}^2 \left( \frac{1}{(8\pi^2)^4 4!} \int_X \bar{\omega}^4 \omega^4 \right)^2.$$

We can also calculate  $c_\Theta$  explicitly in terms of the  $\mathcal{L}^2$ -norm of the curvature  $\|R\|$  and the volume of  $X$ , and this leads to the formula

$$(7) \quad b_{\Theta^k}(X) = \frac{k!}{(4\pi^2 k)^k} \frac{\|R\|^{2k}}{(\text{vol} X)^{k-1}}$$

for an irreducible hyperkähler manifold of real-dimension  $4k$  (see [7]).

The other type of trivalent graphs (and chord diagrams) we shall be interested in are those constructed from *wheels*. Wheels are univalent graphs consisting of a circle with attached spokes. We use the notation  $w_{2\lambda}$  to denote a wheel with  $2\lambda$  spokes. In the case of chord diagrams, we use the notation  $\mathbf{w}_{2\lambda}$  to denote a wheel whose circle is oriented and part of the skeleton. Figure 1 shows some examples. Note

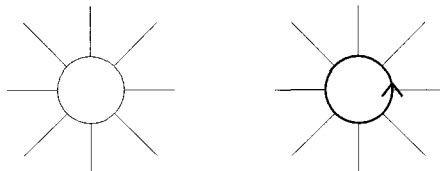


FIGURE 1. The wheels  $w_8$  and  $\mathbf{w}_8$

that we are primarily interested in wheels with an even number of spokes. A *polywheel* is obtained by taking the disjoint union of a collection of wheels  $w_{2\lambda_1}, \dots, w_{2\lambda_t}$  and then summing over all possible ways of joining their spokes pairwise, in order to obtain a trivalent graph. We denote this

$$\langle w_{2\lambda_1} \cdots w_{2\lambda_t} \rangle.$$

In the chord diagram case, some of the wheels  $w_{2\lambda}$  may be replaced by  $\mathbf{w}_{2\lambda}$ . Now suppose that

$$\lambda_1 + \dots + \lambda_t = k$$

so that the trivalent graphs in the polywheel all have  $2k$  vertices. Then for a hyperkähler manifold of real-dimension  $4k$

$$(8) \quad b_{\langle w_{2\lambda_1} \cdots w_{2\lambda_t} \rangle}(X) = (-1)^t (2\lambda_1)! \cdots (2\lambda_t)! \int_X \text{ch}_{2\lambda_1} \cdots \text{ch}_{2\lambda_t}$$

where  $\text{ch}_{2\lambda}$  is the  $2\lambda^{\text{th}}$  component of the Chern character of  $X$  (see [7]). If some of the wheels  $w_{2\lambda}$  are replaced by  $\mathbf{w}_{2\lambda}$  and we introduce a collection of holomorphic vector bundles over  $X$ , then in the above formula  $\text{ch}_{2\lambda}$  should be replaced by  $\text{ch}_{2\lambda}(E)$ , the  $2\lambda^{\text{th}}$  component of the Chern character of  $E$ , where  $E$  is the vector bundle associated to that particular oriented circle in the skeleton of the chord diagram.

Thus every characteristic number of  $X$  can be expressed as a Rozansky-Witten invariant for some choice of linear combination of trivalent graphs. A fundamental question in this theory is “to what extent is the converse true?”, i.e. can every

**k = 1**

$$\bigcirc = \langle w_2 \rangle$$

**k = 2**

$$\bigcirc^2 = \langle w_2^2 \rangle - \frac{4}{5} \langle w_4 \rangle$$

$$\bigcirc \bigcirc = \frac{2}{5} \langle w_4 \rangle$$

**k = 3**

$$\bigcirc^3 = \langle w_2^3 \rangle - \frac{12}{5} \langle w_2 w_4 \rangle + \frac{64}{35} \langle w_6 \rangle$$

$$\bigcirc \bigcirc \bigcirc = \frac{2}{5} \langle w_2 w_4 \rangle - \frac{16}{35} \langle w_6 \rangle$$

$$\triangle = \frac{4}{35} \langle w_6 \rangle$$

**k = 4**

$$\bigcirc^4 = \langle w_4^2 \rangle - \frac{24}{5} \langle w_2^2 w_4 \rangle + \frac{48}{25} \langle w_4^2 \rangle + \frac{256}{35} \langle w_2 w_6 \rangle - \frac{1152}{175} \langle w_8 \rangle$$

$$\bigcirc^2 \bigcirc \bigcirc = \frac{2}{5} \langle w_2^2 w_4 \rangle - \frac{8}{25} \langle w_4^2 \rangle - \frac{32}{35} \langle w_2 w_6 \rangle + \frac{192}{175} \langle w_8 \rangle$$

$$\bigcirc \bigcirc \triangle = -\frac{1}{2} \bigcirc \bigcirc + \frac{2}{25} \langle w_4^2 \rangle + \frac{4}{35} \langle w_2 w_6 \rangle - \frac{48}{175} \langle w_8 \rangle$$

$$\square = \frac{1}{12} \bigcirc \bigcirc - \frac{1}{75} \langle w_4^2 \rangle + \frac{8}{175} \langle w_8 \rangle$$

$$\square = -\frac{41}{96} \bigcirc \bigcirc + \frac{41}{600} \langle w_4^2 \rangle - \frac{16}{175} \langle w_8 \rangle$$

TABLE 1. Polywheels and graph homology

Rozansky-Witten invariant be expressed as a linear combination of Chern numbers? We will answer this in the negative in the next section, but first observe Table 1. For  $k = 1, 2,$  and  $3$  the graphs given on the left hand side span graph homology and can all be expressed as linear combinations of polywheels. Therefore the Rozansky-Witten invariants *are* all characteristic numbers for  $k = 1, 2,$  and  $3$ . The first trivalent graph which is not equivalent to a linear combination of polywheels in graph homology is  $\Theta_2^3$  which occurs in degree  $k = 4$ . It is precisely this graph which we will show leads to an invariant which is not a linear combination of Chern numbers.

5. SOME CALCULATIONS

There are two well-known families of irreducible compact hyperkähler manifolds, the Hilbert schemes  $S^{[k]}$  of  $k$  points on a K3 surface  $S$  and the generalized Kummer varieties  $K_k$  (see Beauville [3]). Apart from these, the only other known example of an irreducible compact hyperkähler manifold was constructed by O’Grady [9] in real-dimension 20 (as presented at this meeting). For the Hilbert schemes  $S^{[k]}$  and generalized Kummer varieties  $K_k$ , there are generating sequences for the Hirzebruch  $\chi_\gamma$ -genuses due to Cheah [5] and Göttsche and Soergel [6]. We can try to use the Riemann-Roch formula to determine the characteristic numbers from this information. For  $k = 1, 2,$  and  $3$  this gives us  $k$  independent equations in  $k$  unknowns (the Chern numbers) which we can invert. Then according to the relations in Table 1, all the Rozansky-Witten invariants may be determined from this information. When  $k = 4$  we get four independent equations in five unknowns, and hence we cannot determine all of the Chern numbers, let alone the Rozansky-Witten invariants, from what we know thus far.

Recall Equation (4) which says that

$$[\gamma(\Phi)] = c_\gamma[\bar{\omega}^m] \in H_{\mathbb{R}}^{2m}(X).$$

The number  $c_\gamma$  may be a constant, but it depends on the manifold  $X$ . If we let  $X$  run through the family  $S^{[k]}$  (respectively  $K_k$ ), this means a dependence on  $k$ . For  $\gamma = \Theta$ ,  $c_\Theta$  is a linear expression in  $k$  (as proved in [12]), and using our calculations for  $k = 1, 2,$  and  $3$  we can determine this expression precisely. Substituting into Equation (5) gives us the following results

$$\begin{aligned} (9) \quad & b_{\Theta^k}(S^{[k]}) = 12^k(k+3)^k \\ (10) \quad & b_{\Theta^k}(K_k) = 12^k(k+1)^{k+1}. \end{aligned}$$

From Table 1 we can see that  $b_{\Theta^k}$  is a characteristic number. Therefore when  $k = 4$  we get a fifth equation for the Chern numbers which we can combine with the four independent equations we already have, and this system can then be solved to give all of the Chern numbers. According to Table 1

$$b_{\Theta^2\Theta_2}(X)$$

may also be written in terms of Chern numbers, and hence can now be determined. Then

$$b_{\Theta_2^2}(X)$$

can be calculated from Equation (6), and this allows us to determine all the remaining Rozansky-Witten invariants for  $S^{[4]}$  and  $K_4$ . For *reducible* compact hyperkähler manifolds in real-dimension sixteen, we merely need to apply the product formula (3).

There is evidence to suggest that  $c_\gamma$  is also linear in  $k$  for graphs  $\gamma$  other than  $\Theta$  (possibly for all trivalent graphs). This would enable us to perform many more

calculations, ie. for  $k > 4$ , though we shall not need such results here. In fact, we already know enough to show that the invariant

$$b_{\mathbb{O}_2^2}(X)$$

in real-dimension sixteen is not a linear combination of Chern numbers. We simply take two (disconnected) compact hyperkähler manifolds

$$48K_4 + 294S \times S^{[3]} + 144S^{[2]} \times S^{[2]} + 63S^4$$

and

$$336S^{[4]} + 268S^2 \times S^{[2]}$$

where ‘+’ denotes disjoint union. The coefficients have been chosen so that both of these manifolds have the same Chern numbers. However, our calculations reveal that

$$b_{\mathbb{O}_2^2}(48K_4 + 294S \times S^{[3]} + 144S^{[2]} \times S^{[2]} + 63S^4) \neq b_{\mathbb{O}_2^2}(336S^{[4]} + 268S^2 \times S^{[2]})$$

and therefore the Rozansky-Witten invariant  $b_{\mathbb{O}_2^2}$  is *not* a characteristic number. On the other hand, although this Rozansky-Witten invariant cannot be written as a linear combination of Chern numbers, for  $X$  irreducible and connected Equation (6) implies that it can be written as a *rational* function of Chern numbers. Hence whether or not the Rozansky-Witten invariants are *really* more general than characteristic numbers is a fairly subtle question.

### 6. THE WHEELING THEOREM

The space of equivalence classes of chord diagrams admits two different product structures. The Wheeling Theorem is an isomorphism  $\hat{\Omega}$  between the two resulting algebras, which is constructed quite explicitly from a particular linear combination of disjoint unions of wheels

$$\begin{aligned} \Omega &= 1 + \frac{1}{48}w_2 + \frac{1}{2!48^2}(w_2^2 - \frac{4}{5}w_4) + \dots \\ &= \exp_{\cup} \sum_{m=1}^{\infty} b_{2m}w_{2m} \end{aligned}$$

where

$$\sum_{m=0}^{\infty} b_{2m}x^{2m} = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}$$

and  $\exp_{\cup}$  means we exponentiate using disjoint union of graphs as our product. This Theorem was recently proved by Bar-Natan, Le, and Thurston [2], and we refer to [1] for a detailed statement of the result. Of course, the isomorphism may be thought of as a family of relations among equivalence classes of chord diagrams

$$(11) \quad \hat{\Omega}(xy) = \hat{\Omega}(x)\hat{\Omega}(y)$$

where  $x$  and  $y$  are chord diagrams, and in this sense it is really a statement about the remarkable properties of  $\Omega$ . In the Rozansky-Witten context, we wish to investigate



the consequences of these relations for our invariants of hyperkähler manifolds and their vector bundles.

The particular relations we are interested in are a special case of (11) and look like

$$(12) \quad \left( \text{---} \overbrace{\hspace{1.5cm}}^{\curvearrowright} \text{---} + \frac{1}{24} \text{---} \overbrace{\hspace{1.5cm}}^{\bigcirc} \text{---} \right)^{\times k} = 2^k k! \left( \frac{1}{(2k)!} \langle \Omega_0 \mathbf{w}_{2k} \rangle + \frac{1}{(2k-2)!} \langle \Omega_2 \mathbf{w}_{2k-2} \rangle + \dots + \langle \Omega_{2k} \mathbf{w}_0 \rangle \right)$$

where  $\Omega_{2m}$  is the  $2m^{\text{th}}$  term of  $\Omega$ , which consists of wheels and their disjoint unions having  $2m$  external legs, and  $\times k$  means that we take the  $k^{\text{th}}$  power where multiplication is given by juxtaposition of skeletons (which are written as directed lines). Note that since we are quotienting by the STU relations, this multiplication is in fact commutative. For example, when  $k = 2$  the left hand side of Equation (12) looks like

$$\text{---} \overbrace{\hspace{1.5cm}}^{\curvearrowright} \text{---} \overbrace{\hspace{1.5cm}}^{\curvearrowright} \text{---} + \frac{2}{24} \text{---} \overbrace{\hspace{1.5cm}}^{\bigcirc} \text{---} + \frac{1}{24^2} \text{---} \overbrace{\hspace{1.5cm}}^{\bigcirc^2} \text{---}$$

Now suppose we have a compact hyperkähler manifold  $X$  of real-dimension  $4k$  with a holomorphic vector bundle  $E$  over it. Since polywheels give rise to Chern numbers, we expect the weight corresponding to the right hand side of Equation (12) to give us some characteristic number. In fact, the precise form of  $\Omega$  (in particular, the appearance of  $\frac{\sinh(x/2)}{x/2}$  in its generating function) means that we get

$$(13) \quad -2^k k! \int_X \text{Td}^{1/2}(T) \wedge \text{ch}(E)$$

where projection of the integrand to the space of top degree forms before integrating is assumed. The weight corresponding to the left hand side of Equation (12) is less easy to interpret. Let us look at the simplest possible case where  $E$  is a trivial vector bundle.

As mentioned earlier, the only weights which do not vanish in this case are those coming from chord diagrams consisting of a trivalent graph plus a skeleton consisting of a disjoint circle. The only such chord diagram in the left hand side of Equation (12) is

$$\frac{1}{24^k} \text{---} \overbrace{\hspace{1.5cm}}^{\bigcirc^k} \text{---}$$

and the corresponding weight is

$$\frac{-\text{rank} E}{24^k} b_{\Theta^k}(X).$$

On the other hand, the Chern character

$$\text{ch}(E) = \text{rank} E$$

for  $E$  trivial and hence (13) becomes

$$-2^k k! \operatorname{rank} E \int_X \operatorname{Td}^{1/2}(T).$$

Therefore

$$(14) \quad b_{\Theta^k}(X) = 48^k k! \int_X \operatorname{Td}^{1/2}(T).$$

We already have a formula for  $b_{\Theta^k}(X)$  when  $X$  is irreducible, namely Equation (7), and it follows that in this case the  $\mathcal{L}^2$ -norm of the curvature  $\|R\|$  of  $X$  can be expressed in terms of characteristic numbers and the volume of  $X$ . This is the main result of [7], where the precise formula may be found. Also, since  $\|R\|$  and the volume must be positive, we can conclude that

$$\int_X \operatorname{Td}^{1/2}(T) > 0$$

for irreducible manifolds  $X$ . For example, in eight real-dimensions this implies that the Euler characteristic

$$c_4(X) < 3024.$$

In fact, it follows from a result of Bogomolov and Verbitsky (see Beauville [4]) that the sharp upper bound in this case is 324. The author is grateful to Beauville for pointing this out.

Of course the holomorphic vector bundle  $E$  has disappeared entirely from Equation (14). To generalize this result to non-trivial vector bundles  $E$  we need a better understanding of the weights corresponding to particular chord diagrams, and their relations to standard invariants of vector bundles; we have already seen that characteristic numbers arise - perhaps certain norms of the curvatures of these bundles should also appear. Ultimately one would like a complete interpretation of Equation (11) in the Rozansky-Witten context.

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## QUATERNIONIC GROUP REPRESENTATIONS AND THEIR CLASSIFICATIONS

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ABSTRACT. We study quaternionic group representations of finite groups systematically and obtain some basic tools of the theory, such as orthogonality relations and the Clabsch-Gordan series for reducible representations. We also derive all irreducible inequivalent  $Q$ -representations of a group  $G$ , classifying them according to a suitable generalization of the Wigner and the Frobenius-Schur classification. Some applications to physical problems and to the time reversal symmetry are shown.

### 1. INTRODUCTION

In the first part of this communication we intend to inquire quaternionic group representations (QGR) directly (i.e. without the detour of transcribing the quaternion operators into complex ones via the symplectic representation) and systematically, going over the basic steps of the theory.

When dealing with this subject the main difficulties come from the non commutativity of  $Q$ , which complicates from the very beginning the basic problem of the invertibility of a linear mapping, and the usual form of the character of a representation must be abandoned in favor of a (seemingly) weaker characterization. Moreover the corollary of the Schur's lemma (which is a basic tool for the analysis of representations and for deriving orthogonality relations) fails to be true in its usual form. This notwithstanding, we obtain some orthogonality relations for linear representations and characters in QGR, that can be applied to analyze any reducible  $Q$ -representation; in particular we obtain all the ( inequivalent ) irreducible  $Q$ -representations ( $Q$ -irreps) of a (finite) group  $G$  and classify them according to a generalization of the well-known Frobenius-Schur classification [2,8] of  $C$ -representations.

The second part of this communication, is devoted to some applications, mainly regarding magnetic groups. The time-reversal symmetry, which is described in complex quantum mechanics by an antiunitary operator, brings out the necessity of introducing the more general concept of *corepresentations* (i.e., representations by unitary and antiunitary operators) whenever the symmetry group contains a time-inversion operator.

In the framework of Quaternionic Quantum Mechanics ( $QQM$ ) the time-inversion operator is still unitary, with the remarkable property that it anticommutes with the anti-self-adjoint operator which represents the Hamiltonian of the physical system [1]. It follows that one can study the symmetry groups including time-reversal by the same methods adopted in order to study symmetry groups containing spatial symmetries only.

We apply to the magnetic groups a further classification of groups, which in some sense replaces the Wigner classification of corepresentations [9,10] and can be crossed with the generalized Frobenius-Schur classification in order to get a more general classification of these groups. Some physical applications are briefly sketched in the conclusions, from which a suggestion arises to inquire into parity violation from a purely group theoretical point of view.

The main results of this communication have been exposed in ref.[14,15].

## 2. UNITARY $Q$ -REPRESENTATIONS

In a (right)  $n$ -dimensional vector space  $Q^n$  over  $Q$ , every linear operator is associated in a standard way [4] to a  $n \times n$  matrix acting on the left.

In analogy with the case of complex group representations, one can then define the hermitian conjugate  $A^\dagger = \overline{A}^T$  of a matrix  $A$  ( $A^T$  and  $\overline{A}$  denote, as usual, the transpose and the quaternionic conjugate of  $A$ , respectively), and introduce the concepts of unitarity, hermiticity and so on. The properties of hermitian and unitary matrices in  $Q^n$  have been widely investigated [6,7]; moreover if  $G$  is a finite (or a compact) group, one can always assume unitarity and complete reducibility of the quaternionic representations [13].

Finally we recall that for  $Q$ -irreps Schur's lemma still holds [7] and one obtains as a corollary that: "*If a Hermitian matrix  $H$  commutes with an irreducible set  $D$  of matrices, it is a (real) multiple of the unit matrix*" [7].

The above corollary allows one to prove the following proposition:

*"The equivalence between unitary  $Q$ -representations can always be effected by a unitary matrix".*

*Proof.* Let  $D_1$  and  $D_2$  be two equivalent unitary irreducible  $Q$ -representations, and let  $T$  be the matrix that effects the equivalence between them:

$$D_1 T = T D_2.$$

The conjugate of previous equation reads

$$T^\dagger D_1^\dagger = D_2^\dagger T^\dagger$$

or, recalling the unitarity of  $D_1$  and  $D_2$ ,

$$T^\dagger D_1 = D_2 T^\dagger.$$

Then,  $TT^\dagger D_1 = TD_2T^\dagger = D_1TT^\dagger$ , i.e., the hermitian matrix  $TT^\dagger$  commutes with  $D_1$  and by the Schur Lemma,  $TT^\dagger = r\mathbf{1}$ ,  $r \in R$ . Moreover,  $r \geq 0$  being trivially

$$\forall |\phi\rangle \in Q^n, \langle \phi | T^\dagger T | \phi \rangle = r \langle \phi | \phi \rangle = \|T | \phi \rangle\|^2.$$

Hence,  $T' = \frac{1}{\sqrt{r}}T$  is a unitary matrix such that

$$D_1 = T' D_2 T'^{-1}.$$

The proof for reducible representations follows at once, observing that  $D_1 \cong D_2$  if and only if the irreducible blocks in their decomposition are both equivalent (see sect.3).

### 3. ORTHOGONALITY RELATIONS AND ANALYSIS OF Q-REPRESENTATIONS

Let  $D(G)$  be an  $n$ -dimensional irreducible and unitary  $Q$ -representation of a finite group  $G$  and let us consider the matrix

$$(3.1) \quad A = \sum_{g \in G} D(g^{-1}) X D(g) = \sum_{g \in G} \overline{D}^T(g) X D(g)$$

with  $X$  hermitian; then, trivially,  $A = A^\dagger$ .

Indeed

$$(3.2) \quad A_{ij} = \sum_{g \in G} \sum_{k,l} \overline{D}_{ki}(g) X_{kl} D_{lj}(g) = \overline{A}_{ji};$$

moreover  $D(g) A = A D(g)$ ,  $\forall g \in G$ .

By using the corollary of Schur's lemma [7],

$$(3.3) \quad A = \lambda^{(X)} I_n$$

where  $\lambda^{(X)} \in R$  and  $I_n$  is the unit  $n \times n$  matrix.

Let us now choose in Eq. (3.1) a matrix  $X^{(r)}$  in such a way that  $X_{kl}^{(r)} = \delta_{kr} \delta_{lr}$ , with  $r$  fixed, and take the real trace of  $A$ . Recalling that the real trace satisfies the cyclic property  $ReTr BC = ReTr CB$  [7], we obtain

$$(3.4) \quad ReTr A = \sum_g ReTr X^{(r)} = [G] = \lambda^{(r)} n$$

where  $[G]$  is the order of  $G$ .

By substituting the explicit form of  $X_{kl}^{(r)}$  and  $\lambda^{(r)}$  in Eq. (3.2), we easily obtain

$$(3.5) \quad \sum_{g \in G} \overline{D}_{ri}(g) D_{rj}(g) = \frac{[G]}{n} \delta_{ij}.$$

Analogously, let  $D^{(\mu)}(G)$  and  $D^{(\nu)}(G)$  ( $\mu \neq \nu$ ) be two unitary inequivalent  $Q$ -irreps of  $G$  whose dimensions respectively are  $n_\mu$  and  $n_\nu$ ; then the matrix

$$(3.6) \quad A = \sum_{g \in G} D^{(\mu)}(g^{-1}) X D^{(\nu)}(g)$$

for every matrix  $X$ , satisfies the condition

$$D^{(\mu)}(h) A = A D^{(\nu)}(h) \quad \forall h \in G.$$

By using Schur' lemma [7], we conclude that  $A$  must vanish identically.

Choosing in Eq. (3.6) a matrix  $X^{(rs)}$  such that  $X_{ki}^{(rs)} = \delta_{kr} \delta_{is}$  with  $r, s$  fixed and writing down the explicit form of  $A_{ij}$ , we obtain

$$(3.7) \quad \sum_{g \in G} \overline{D_{ri}^{(\mu)}}(g) D_{sj}^{(\nu)}(g) = 0$$

and finally (expressing Eqs. (3.5) and (3.7) in a more compact form),

$$(3.8) \quad \sum_{g \in G} \overline{D_{ri}^{(\mu)}}(g) D_{rj}^{(\nu)}(g) = \frac{|G|}{n_\mu} \delta_{ij} \delta_{\mu\nu},$$

which is the (weaker) analogue for  $Q$ -irreps of the orthogonality relation for  $C$ -irreps.

Let us put now  $r = i$  and  $s = j$  in Eq. (3.8), and let us sum over  $i$  and  $j$ ; then,

$$(3.9) \quad \sum_g \chi^{(\mu)Q}(g) \chi^{(\nu)}(g) = 0$$

where  $\chi^{(\mu)}(g)$  denotes the (full) trace of  $D^{(\mu)}(g)$ . Eq. (3.9) express the orthogonality between (quaternionic) characters of two inequivalent  $Q$ -irreps of the group  $G$ .

On the other hand, the following identity holds:

$$\hat{\chi}^{(\mu)}(g) \equiv Re \chi^{(\mu)}(g) = \frac{1}{4} [\chi^{(\mu)}(g) - i \chi^{(\mu)}(g) i - j \chi^{(\mu)}(g) j - k \chi^{(\mu)}(g) k]$$

and each term in parentheses, say  $-i \chi^{(\mu)}(g) i$ , can be considered as the character of  $g$  in a  $Q$ -representation (in our case  $-i D^{(\mu)} i$ ), which is equivalent to the  $D^{(\mu)}$  but certainly inequivalent to the  $D^{(\nu)}$  [13]. For, we easily get the following relation from Eq. (3.9)

$$\frac{1}{\sum_g \hat{\chi}^{(\mu)2}(g)} \sum_g \hat{\chi}^{(\mu)}(g) \hat{\chi}^{(\nu)}(g) = \delta_{\mu\nu}$$

or also (remembering that conjugated elements of a group have the same real character)

$$(3.10) \quad \frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i^{(\mu)} \hat{\chi}_i^{(\nu)} = \delta_{\mu\nu},$$



where  $\hat{\chi}_i^{(\mu)}$  obviously indicates the (real) character of all elements belonging to the  $i$ -th conjugation class of  $G$ , and  $k_i$  is the number of the elements of such a class.

As usual in CGR theory, Eq. (3.10) can be read as an orthogonality relation between vectors in a  $\kappa$ -dimensional space (where  $\kappa$  is the number of the conjugation classes of  $G$ ), so that we finally obtain that the number  $r$  of inequivalent  $Q$ -irreps of  $G$  must satisfy the following inequality

$$r \leq \kappa$$

(and some cases occur in which strict inequality holds).

The possibility of decomposing any reducible  $Q$ -representation follows at once from these results. Indeed, let

$$D(G) = \sum_{\mu} a_{\mu} D^{(\mu)}(G)$$

be the Clebsh-Gordan series of a reducible  $Q$ -representation  $D(G)$ . Then, trivially,

$$\hat{\chi}(g) = \sum_{\mu} a_{\mu} \hat{\chi}^{(\mu)}(g) \quad \forall g \in G.$$

By using Eq. (3.10) we obtain

$$a_{\mu} = \frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i \hat{\chi}_i^{(\mu)}$$

and this decomposition is unique, so that we can finally assert that *two  $Q$ -representations are equivalent if and only if their (real) characters coincide.*

#### 4. Q-IRREPS AND THE GENERALIZED FROBENIUS-SCHUR CLASSIFICATION

In order to obtain all the  $Q$ -irreps, we recall that any  $C$ -irrep of a group  $G$  can obviously be considered as a (not necessary irreducible)  $Q$ -representation and an important theorem (*Main Reduction Theorem*) states that: *"A  $C$ -irrep  $D$  reduces over  $Q$  (into two equivalent  $Q$ -irreps  $D_1$  and  $D_2$ ) if and only if  $D$  is equivalent to its complex conjugate  $D^*$  by an antisymmetric matrix"* [7].

Moreover we can prove that: *"All the  $Q$ -representations found in the sense of Main Reduction Theorem are inequivalent to each other, with the exception of those generated by a pair of complex conjugated representations such that  $D \not\cong D^*$ "* [14],

Recalling that, in the realm of CGR, *"Two inequivalent  $C$ -irreps share the same real part of the character if and only if they are complex conjugate of each other"* [14], we can conclude that the choice of characterizing any  $Q$ -representation by means of the real part of the trace (due to the necessity of maintaining the cyclic property of this quantity) does not eliminate any relevant information.

Finally we prove that:

*"No  $Q$ -irrep exists besides those generated (in the sense of the Main Reduction Theorem) by the  $C$ -irreps"* [14].

*Proof.* (We give here a more direct proof of this proposition, with respect to ref. [14])

Let  $D = D_1 + jD_2$  be a purely quaternionic representation (i.e.,  $D_1$  and  $D_2$  are complex matrices and  $D_2 \neq \mathbf{0}$  in every basis); if we take the direct sum

$$\begin{pmatrix} D_1 + jD_2 & 0 \\ 0 & D_1 + jD_2 \end{pmatrix},$$

and perform the similarity transformation

$$\frac{1}{2} \begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -j\mathbf{1} & k\mathbf{1} \end{pmatrix} \begin{pmatrix} D_1 + jD_2 & 0 \\ 0 & D_1 + jD_2 \end{pmatrix} \begin{pmatrix} \mathbf{1} & j\mathbf{1} \\ i\mathbf{1} & -k\mathbf{1} \end{pmatrix} = \begin{pmatrix} D_1 & -D_2^* \\ D_2 & D_1^* \end{pmatrix},$$

we obtain a complex representation which is equivalent to its complex conjugate by an antisymmetric matrix:

$$\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} D_1 & -D_2^* \\ D_2 & D_1^* \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} D_1^* & -D_2 \\ D_2^* & D_1 \end{pmatrix}.$$

One can easily verify that the complex commutant of such representation is a complex multiple of identity operator, therefore, by the corollary of Schur's Lemma for C-irreducible representations, this representation is irreducible over the complex field. The Main Reduction Theorem, cited above, ensures on the other hand that to any C-representation with the previous properties is associated an purely quaternionic irreducible Q-representation and the theorem is proved.

We have shown elsewhere [14] that all irreducible linear (inequivalent) Q-representations of a finite group  $G$  fall into three classes: *potentially real or of type R*, *potentially complex or of type C*, *(purely) quaternionic or of type Q (generalized Frobenius-Schur classification)*. The generalized irreducibility criterion reads

$$(4.1) \quad \sum_g \hat{\chi}^{(\mu)2}(g) = \frac{|G|}{c^{(\mu)}}$$

$$\text{where } c^{(\mu)} = \begin{cases} 1 \\ 2 \\ 4 \end{cases} \text{ when the representation } D^{(\mu)} \text{ is of type } \begin{cases} R \\ C \\ Q \end{cases}.$$

Let us recall [14] that the following relation occur between the character  $\chi_C^{(\mu)}$  of a complex representation and the character  $\hat{\chi}^{(\mu)}$  of the corresponding quaternionic representation:

$$(4.2) \quad \hat{\chi}^{(\mu)} = \begin{cases} \chi_C^{(\mu)} \\ \text{Re}\chi_C^{(\mu)} \\ \frac{1}{2}\chi_C^{(\mu)} \end{cases} \text{ when } D^{(\mu)} \text{ of type } \begin{cases} R \\ C \\ Q \end{cases}.$$

Then, we also obtain, by using the classical Frobenius-Schur criterion:

$$(4.3) \quad \sum_g \hat{\chi}^{(\mu)}(g^2) = d^{(\mu)} [G]$$

where  $d^{(\mu)} = \begin{cases} +1 \\ 0 \\ -\frac{1}{2} \end{cases}$  if  $D^{(\mu)}$  is a Q-irrep of type  $\begin{cases} R \\ C \\ Q \end{cases}$ .

We conclude that the couple of values of  $\sum_g \hat{\chi}^{(\mu)^2}(g)$  and  $\sum_g \hat{\chi}^{(\mu)}(g^2)$  uniquely identifies all Q-irreps and their class.

5. MAGNETIC GROUPS AND THEIR CLASSIFICATION

Color groups are defined in the literature[12] by

$$(5.1) \quad G' = G + aG, \quad a \notin G,$$

where  $a$  is an operator which switches color (or, even, a product of such operator with a spatial symmetry which does not belong to  $G$ ) and  $G$  is a (normal) subgroup of  $G'$  of index 2, whose elements represent spatial symmetries. In the CGR theory, the same equation defines the *magnetic groups* [3,11], where the elements of the coset  $aG$  are antiunitary operators. We call magnetic group in the following any group defined by Eq. (5.1), without entering into the physical interpretation of the elements of  $aG$ . We only characterize algebraically these elements by requiring that *all elements in  $G$  commute with a given operator, say  $H$ , while all elements in  $aG$  anticommute with it.*

We are now ready to study and possibly classify the representations of magnetic groups in the spaces  $Q^n$ . Let  $X$  be a finite dimensional vector space and let  $D(G')$  be an irreducible (unitary) representation of a magnetic group  $G'$  in  $X$ . Whenever the restriction of  $D(G')$  to  $G$  is reducible, let  $X_1$  be an irreducible  $G$ -invariant subspace of  $X$  and let  $\{|e_i\rangle\}$  be a basis in it. Then,

$$(5.2) \quad \langle e_i | D(g) | e_j \rangle = D_{ij}(g) \doteq \Delta_{ij}(g) \quad \forall g \in G;$$

moreover, if  $|f_i\rangle \doteq D(a) |e_i\rangle$ , we get

$$(5.3) \quad \bar{\Delta}_{ij}(g) \doteq \langle f_i | D(g) | f_j \rangle = \langle e_i | D(a^{-1}) D(g) D(a) | e_j \rangle = \Delta_{ij}(a^{-1}ga).$$

Since  $a^{-1}ga \in G$ , the set of matrices  $\bar{\Delta}(G)$  coincides with  $\Delta(G)$  which is supposed irreducible in  $X_1$  (then, they share the same global properties); for,  $\bar{\Delta}(G)$  too is an irreducible representation of  $G$  in  $X_2 = D(a)X_1$ . Furthermore, we note that

$$D(a)X_2 = D(a^2)X_1 = X_1.$$

Now, let us observe that both the subspaces  $X_1 \cap X_2$  and  $X_1 \oplus X_2$  are  $G'$ -invariant; being by hypothesis  $D(G')$  irreducible, we easily obtain  $X_1 \cap X_2 = \emptyset$  and  $X_1 \oplus X_2 = X$ .

Choosing as a basis in  $X$  the set  $\{| e_i \rangle \} \cup \{| f_j \rangle \}$ , we get

$$(5.4) \quad D(G) = \begin{pmatrix} \Delta(G) & 0 \\ 0 & \bar{\Delta}(G) \end{pmatrix}, D(a) = \begin{pmatrix} 0 & \Delta(a^2) \\ 1 & 0 \end{pmatrix},$$

and two cases arise, according to whether  $\Delta$  is equivalent to  $\bar{\Delta} : \Delta \cong \bar{\Delta}$  or not.

We have thus obtained a threefold classification of the irreducible representations of magnetic groups :

- I- the restriction  $D(G)$  of  $D$  to the subgroup  $G$  is irreducible;
- II-  $D(G)$  is reducible and has the above form, with  $\Delta \not\cong \bar{\Delta}$ ;
- III-  $D(G)$  is reducible and has the above form, with  $\Delta \cong \bar{\Delta}$ .

This classification makes no reference to the scalar field of the vector space  $X$ , so that it generalizes the Wigner classification of corepresentations [18] in CGR theory and can replace it in the framework of QGR theory.

Thus, the idea arises to cross this new classification with the generalized FS classification discussed in Sect.(4), so as to obtain a more detailed description of Q-irreps of magnetic groups.

By using the orthogonality relations we can prove [15] that case I splits into five subcases:

- I-R  $D(G') \rightsquigarrow R, D(G) \rightsquigarrow R$  (i.e.,  $D(G)$  and  $D(G')$  both of type  $R$ )
- I-C/R  $D(G') \rightsquigarrow C, D(G) \rightsquigarrow R$
- I-C/C  $D(G') \rightsquigarrow C, D(G) \rightsquigarrow C$
- I-Q/C  $D(G') \rightsquigarrow Q, D(G) \rightsquigarrow C$
- I-Q/Q  $D(G') \rightsquigarrow Q, D(G) \rightsquigarrow Q$ ,

case II splits into three subcases [15]:

- II-R  $D(G') \rightsquigarrow R, D(G) \rightsquigarrow R + R$
- II-C  $D(G') \rightsquigarrow C, D(G) \rightsquigarrow C + C$
- II-Q  $D(G') \rightsquigarrow Q, D(G) \rightsquigarrow Q + Q$ .

(We denote by  $R + R, C + C, Q + Q$  a decomposition of  $D(G)$  in two inequivalent representations of type  $R, C, Q$  respectively.)

Finally, case III splits in two subcases [15] only:

- III-R  $D(G') \rightsquigarrow R, D(G) \rightsquigarrow 2C$ ;
- III-C  $D(G') \rightsquigarrow C, D(G) \rightsquigarrow 2Q$ .

(We denote by  $2C, 2Q$  here a decomposition of  $D(G)$  in two equivalent representations of type  $C, Q$  respectively.) We observe that the above crossed classification is not trivial, because some of the nine cases that one could in principle obtain split in subcases, whereas one of them cannot occur, so that it provides a valuable insight into the properties of magnetic groups and their Q-irreps.

A remarkable role is played among the magnetic groups by the *factorizable* groups, the physical interest of which has been widely outlined [5,10].

We recall that a magnetic group  $G' = G + aG$  is said to be factorizable if the automorphism

$$(5.5) \quad g \rightarrow g' = a^{-1}ga \quad \forall g \in G$$

is an inner automorphism, i.e., an element  $w \in G$  exists such that  $g' = w^{-1}gw, \forall g \in G$ . It is easy to see that  $G'$  is factorizable if and only if an element  $t = aw^{-1} \in aG$  exists which commutes with all elements in  $G$  (hence with  $a$ , that is with all elements in  $G'$ ).

In many physical applications, when such an operator  $t$  exists, it is interpreted as a *time-inversion operator*. Indeed Adler [ 1] has shown that in the realm of QQM all spatial symmetries commute with the Hamiltonian  $H$  of the system, whereas the time-inversion operator anticommutes with  $H$  and commutes with all spatial symmetries; thus, in this framework, every symmetry group containing the time-inversion operator is a factorizable group.

We therefore studied magnetic groups of the form

$$(5.6) \quad G' = G + tG, \quad [t, G] = 0,$$

and determined that only the cases *I-R, I-C/C, I-Q/Q, I-C/R, III-C* of the crossed classification actually occur for such groups. In the cases *I-R, I-C/C, I-Q/Q* results  $D(t^2) = 1$ , and in the cases *I-C/R, I-C/C, III-C* we obtain  $D(t^2) = -1$ .

If one now recalls that the squared time-inversion operator in QQM [1] is equal to the identity for fermionic systems, it has opposite sign for bosonic systems, we conclude that:

- i) whenever a fermionic system is considered, a magnetic factorizable group falls into one of the cases I-R, I-C/C, I-Q/Q of the previous classification and  $D(t^2) = 1$  ;*
- ii) whenever a bosonic system is considered, a magnetic factorizable group falls into one of the cases I-C/R, I-C/C, III-C of the previous classification and  $D(t^2) = -1$  .*

## 6. CONCLUSIONS

We conclude the discussion recalling that the mathematical methods and results developed in this communication have been applied to quantum physical problems, such as the study of degeneracy of energy levels in QQM whenever a time-reversal symmetry exists [15] (Kramers degeneracy). Kramers theorem applies in the context of CQM [17] and states that all energy levels of a fermionic system must be at least doubly degenerate, as really happens. Of course, Kramers degeneracy must appear in all attempts of modifying or generalizing ordinary quantum mechanics; our results perfectly agree with the experimental data.

Secondly we obtained the Q-representations of the quaternionic complete symmetry group [16] (obtained by extending the connected Poincare' group and the internal symmetry group by means of the CPT ( $\Theta_0$ ) and the generalized parity ( $\mathcal{P}$ ) operators), in order to classify the particle multiplets.

Further investigations are suggested by an examination of the explicit forms of the  $\mathbb{Q}$ -representations of the complete group. For instance, one of the possible forms of such extensions is, in a suitable basis:

$$(6.1) \quad \mathcal{D}(G) = \begin{pmatrix} \Delta^*(G) & \mathbf{0} \\ \mathbf{0} & \Delta(G) \end{pmatrix}, \mathcal{D}(\Theta_0) = \begin{pmatrix} \mathbf{0} & k\mathbf{1} \\ k\mathbf{1} & \mathbf{0} \end{pmatrix}, \mathcal{D}(\mathcal{P}) = \begin{pmatrix} \mathbf{0} & -S_1 \\ S_1^* & \mathbf{0} \end{pmatrix}$$

and the Hamiltonian is

$$(6.2) \quad H = ih_0\mathbf{1}, \quad h_0 \in R,$$

where  $\Delta(G)$  is a  $\mathbb{Q}$ -irrep of the internal symmetry group  $G$ , and  $\Theta_0$  and  $\mathcal{P}$  denote the CPT and the parity operators, respectively.

On the other hand, if we consider a physical theory which is not invariant with respect to the (generalized) parity operator and then study the extension of the same representation  $\Delta(G)$  of  $G$  obtained by means only of  $\Theta_0$ , the case I-C/C arises. Performing again a suitable change of basis, we obtain:

$$(6.3) \quad D(G) = \begin{pmatrix} \Delta^*(G) & \mathbf{0} \\ \mathbf{0} & \Delta(G) \end{pmatrix}, D(\Theta_0) = \begin{pmatrix} \mathbf{0} & k\mathbf{1} \\ k\mathbf{1} & \mathbf{0} \end{pmatrix}$$

and the Hamiltonian is:

$$(6.4) \quad H' = ih_0\mathbf{1} + jh_1 \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad h_0, h_1 \in R,$$

It follows at once that the representations of  $G$  and  $\Theta_0$  are identical in both cases: but the presence in the former case of a further symmetry, namely  $\mathcal{P}$ , forces the cancellation in the form of  $H$  of the genuinely quaternionic term in  $j$ , to which we can then ascribe the parity violation (in perfect accordance with some arguments due to Adler [1] in a very different context).

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## VANISHING THEOREMS FOR QUATERNIONIC KÄHLER MANIFOLDS

UWE SEMMELMANN & GREGOR WEINGART

ABSTRACT. In this article we discuss a peculiar interplay between the representation theory of the holonomy group of a Riemannian manifold, the Weitzenböck formula for the Hodge–Laplace operator on forms and the Lichnerowicz formula for twisted Dirac operators. For quaternionic Kähler manifolds this leads to simple proofs of eigenvalue estimates for Dirac and Laplace operators. We determine which representations may contribute to harmonic forms and prove the vanishing of certain odd Betti numbers on compact quaternionic Kähler manifolds of negative scalar curvature. We simplify the proofs of several related results in the positive case.

**AMS Subject Classification:** 53C25, 58J50

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### 1. INTRODUCTION

Since decades the Weitzenböck formulas for the Dirac operator on Clifford bundles have inspired intensive and important research. The full Weitzenböck machinery is now beginning to take its definite place in differential geometry incorporating recent

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ideas about Kato inequalities (cf. [CGH99]) and more and more representation theory. It is inevitable to get the impression that geometrically interesting second order operators like the Hodge–Laplace or the Dirac operator can be defined abstractly apart from their original setting. In particular it is thus possible to compare geometric differential operators defined on completely different vector bundles. In this article we will describe the impact of this idea and discuss potential applications for quaternionic Kähler manifolds in detail.

Studying manifolds of special holonomy may lead to new insights into underlying structures and concepts of differential geometry. In fact the primary feature of a manifold of special holonomy is its richness in geometric vector bundles  $\pi(M)$  corresponding to the representations  $\pi$  of the holonomy group. In this article we will use Meyer’s interpretation (cf. [Me71]) of the Weitzenböck formula for the Hodge–Laplacian  $\Delta$  to define an elliptic selfadjoint second order differential operator

$$\Delta_\pi : \Gamma \pi(M) \longrightarrow \Gamma \pi(M)$$

for every geometric vector bundle  $\pi(M)$ . For a homogeneous vector bundle on a symmetric space  $G/K$  of compact type the operator  $\Delta_\pi$  becomes the Casimir of  $G$ . Moreover  $\Delta_\pi$  agrees with the Hodge–Laplacian  $\Delta$  for all parallel subbundles  $\pi(M)$  of the differential forms. This immediately implies the generalized Lefschetz decomposition of the de Rham cohomology

$$H_{dR}^\bullet(M, \mathbb{C}) = \bigoplus_{\pi} \text{Hom}_{\text{Hol}}(\pi, \Lambda^\bullet(T^*M \otimes_{\mathbb{R}} \mathbb{C})) \otimes \ker \Delta_\pi$$

where the sum is over all irreducible representations  $\pi$  of the holonomy group  $\text{Hol}$ . Considering  $\Delta_\pi$  as a generalization of the Casimir of a symmetric space of compact type to arbitrary Riemannian manifolds it is only natural to derive formulas linking this operator to other second order differential operators. In particular we will generalize Parthasarathy’s formula which expresses the twisted Dirac operator on symmetric spaces in terms of the Casimir.

The general result for twisted spinor bundles can be applied to a very prominent family of twisted spinor bundles on quaternionic Kähler manifolds. The indices of the twisted Dirac operators in this family are of fundamental importance in studying quaternionic Kähler manifolds in general. Our main technical result is a general eigenvalue estimate for the Dirac operators in this family leading to an interpretation of their kernels in terms of eigenspaces of operators  $\Delta_\pi$  corresponding to the minimal eigenvalue. In the case of positive scalar curvature  $\kappa$  this can be used to give new proofs for results of S. Salamon on the Betti numbers (c.f. [Sal82]) and to prove the strong Lefschetz Theorem:

*Theorem 1.1. (Strong Lefschetz Theorem for Quaternionic Kähler Manifolds with  $\kappa > 0$ )*

Let  $(M^{4n}, g)$  be a quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ . The odd Betti numbers  $b_{2k+1}(M) = 0$  of  $M$  vanish. The wedge product with the

parallel Kraines form  $\Omega \in \Gamma(\Lambda^4 T^*M)$  descends to an injective map on the level of cohomology

$$\Omega \wedge : H^{2k}(M, \mathbb{R}) \longrightarrow H^{2k+4}(M, \mathbb{R})$$

for all  $k < n$ . In particular the Betti numbers of  $M$  satisfy the inequality:

$$b_{2k}(M) \leq b_{2k+4}(M)$$

for all  $k < n$ . Moreover the space of primitive forms of degree  $2k$  agrees with the kernel of the operator  $\Delta_{\pi_k}$  for the irreducible representation  $\pi_k = \Lambda_{\text{top}}^{k,k} E$  of  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ .

It is then rather surprising to see that the same techniques can be applied to obtain completely new results on the cohomology of quaternionic Kähler manifolds with negative scalar curvature. Here we can prove:

*Theorem 1.2. (Weak Lefschetz Theorem for Quaternionic Kähler Manifolds with  $\kappa < 0$ )*

Let  $(M^{4n}, g)$  be a quaternionic Kähler manifold of negative scalar curvature  $\kappa < 0$ . Its odd Betti numbers vanish  $b_{2k+1}(M) = 0$  for  $2k + 1 < n$ . In general the Betti numbers of  $M$  satisfy for all  $k < n$  the inequalities

$$b_{2k}(M) \leq b_{2k+4}(M) \quad \text{and} \quad b_{2k+1}(M) \leq b_{2k+3}(M)$$

For quaternionic Kähler manifolds of negative scalar curvature the wedge product with the parallel Kraines form  $\Omega$  still descends to an injective map on the level of cohomology in all degrees one could possibly hope for. Philosophically however this is not really the strong Lefschetz theorem, because the space of primitive forms decomposes non-trivially into different isotypical components with respect to the holonomy group.

## 2. HOLONOMY GROUPS AND WEITZENBÖCK FORMULAS

In this section we will discuss the classical Weitzenböck formula for the Hodge-Laplacian or more general for the Dirac operator on a Clifford bundle and introduce the Laplace operator  $\Delta_{\pi}$ . The basic example of a Clifford bundle is the bundle of exterior forms  $\Lambda^{\bullet} T^*M$  endowed with the scalar product induced by the metric on  $M$  and Clifford multiplication with tangent vectors

$$\star : T_p M \times \Lambda^{\bullet} T_p^* M \longrightarrow \Lambda^{\bullet} T_p^* M, \quad (X, \omega) \longmapsto X \star \omega$$

defined by  $X \star \omega := X \lrcorner \omega - \omega \lrcorner X$ . The Levi-Civita-connection induces a connection  $\nabla$  on  $\Lambda^{\bullet} T^*M$  and an associated second order elliptic differential operator  $\nabla^* \nabla := -\sum_i \nabla_{E_i}^2$  where  $\nabla_{X,Y}^2 := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$  and the sum is over a local orthonormal base  $\{E_i\}$ . On the other hand we have the exterior differential  $d$  and its formal adjoint  $d^*$  as natural first order differential operators on  $\Lambda^{\bullet} T^*M$  linked to  $\nabla^* \nabla$  by the classical Weitzenböck formula

$$(2.1) \quad \Delta := (d + d^*)^2 = \nabla^* \nabla + \frac{1}{2} \sum_{ij} E_i \star E_j \star R_{E_i, E_j}$$

where  $R_{X,Y}$  is the curvature endomorphism of  $\Lambda^*T_p^*M$ . However the connection on  $\Lambda^*T^*M$  is induced by a connection on  $TM$  and consequently the curvature endomorphism  $R_{X,Y}$  is just the curvature endomorphism of  $T_pM$  in a different representation, namely the representation

$$\bullet : \mathfrak{so}(T_pM) \times \Lambda^*T_p^*M \longrightarrow \Lambda^*T_p^*M, \quad (X, \omega) \longmapsto X \bullet \omega$$

of the Lie algebra  $\mathfrak{so}(T_pM)$  of  $\mathbf{SO}(T_pM)$  on the exterior algebra induced by its representation on  $T_pM$ . The canonical identification of  $\mathfrak{so}(T_pM)$  with the bivectors  $\Lambda^2T_pM$  characterized by  $\langle (X \wedge Y) \bullet A, B \rangle := \langle X \wedge Y, A \wedge B \rangle$  reads  $(X \wedge Y) \bullet A := \langle X, A \rangle Y - \langle Y, A \rangle X$  and defines a unique bivector  $R(X \wedge Y)$  via:

$$\langle R(X \wedge Y) \bullet Z, W \rangle := \langle R_{X,Y}Z, W \rangle \quad R(X \wedge Y) = \frac{1}{2} \sum_i E_i \wedge R_{X,Y}E_i$$

In the spirit of this identification the representation of  $\mathfrak{so}(T_pM)$  on  $\Lambda^*T_p^*M$  is given by  $(X \wedge Y) \bullet = Y^\sharp \wedge X \lrcorner - X^\sharp \wedge Y \lrcorner$ . In particular, the classical Weitzenböck formula becomes

$$\begin{aligned} \Delta &= \nabla^* \nabla + \frac{1}{2} \sum_{ij} (E_i^\sharp \wedge E_j^\sharp \wedge - E_i \lrcorner E_j^\sharp \wedge - E_i^\sharp \wedge E_j \lrcorner + E_i \lrcorner E_j \lrcorner) R(E_i \wedge E_j) \bullet \\ &= \nabla^* \nabla + \frac{1}{2} \sum_{ij} (E_i \wedge E_j) \bullet R(E_i \wedge E_j) \bullet \end{aligned}$$

because both potentially troublesome inhomogeneous terms cancel by the first Bianchi identity leaving us with a curvature term depending linearly on the curvature tensor:

$$R := \frac{1}{4} \sum_{ij} (E_i \wedge E_j) \cdot R(E_i \wedge E_j) \in \text{Sym}^2(\Lambda^2T_pM).$$

It will be convenient to compose the identification  $\Lambda^2T_pM \xrightarrow{\cong} \mathfrak{so}(T_pM)$  with the quantization map  $q : \text{Sym}^2\mathfrak{so}(T_pM) \longrightarrow \mathcal{U}\mathfrak{so}(T_pM)$ ,  $X^2 \longmapsto X^2$ , into the universal enveloping algebra of  $\mathfrak{so}(T_pM)$  to get an element  $q(R) \in \mathcal{U}\mathfrak{so}(T_pM)$  with:

$$(2.2) \quad \Delta = \nabla^* \nabla + 2q(R)$$

Writing the well known classical Weitzenböck formula (2.1) this way we can bring the holonomy group of the underlying manifold into play. Recall that the holonomy group  $\text{Hol}_pM \subset \mathbf{O}(T_pM)$  is the closure of the group of all parallel transports along piecewise smooth loops in  $p \in M$ . We will assume that  $M$  is connected so that the holonomy groups in different points  $p$  and  $\tilde{p}$  are conjugated by parallel transport  $T_pM \longrightarrow T_{\tilde{p}}M$ . Choosing a suitable representative  $\text{Hol} \subset \mathbf{O}_n\mathbb{R}$  with  $n := \dim M$  of their common conjugacy class acting on the abstract vector space  $\mathbb{R}^n$  we can define the holonomy bundle of  $M$ :

$$\text{Hol}(M) := \{ f : \mathbb{R}^n \longrightarrow T_pM \mid p \in M, f \text{ isometry with } f(\text{Hol}) = \text{Hol}_pM \}.$$

The holonomy bundle is a reduction of the orthonormal frame bundle  $\mathbf{O}(M)$  to a principal bundle with structure group  $\text{Hol}$ , which is stable under parallel transport. Consequently the Levi-Civita connection is tangent to  $\text{Hol}(M)$  and descends to a connection on  $\text{Hol}(M)$ .

The associated fibre bundle  $\text{Hol}(M) \times_{\text{Hol}} \mathbf{O}_n\mathbb{R}$  is canonically diffeomorphic to the full orthonormal frame bundle  $\mathbf{O}(M)$ . This construction provides an explicit foliation of  $\mathbf{O}(M)$  into mutually equivalent principal subbundles stable under parallel transport. Choosing a leaf different from the distinguished leaf  $\text{Hol}(M)$  amounts to choosing a different representative for the conjugacy class of  $\text{Hol} \subset \mathbf{O}_n\mathbb{R}$ . In particular every principal subbundle of  $\mathbf{O}(M)$  stable under parallel transport is a union of leaves and is characterized by a subgroup of  $\mathbf{O}_n\mathbb{R}$  containing a representative of the conjugacy class of the holonomy group  $\text{Hol}$ .

With the Levi-Civita connection being tangent to the holonomy bundle  $\text{Hol}(M)$  its curvature tensor  $R$  takes values in the holonomy algebra  $\mathfrak{hol}_p M$  at every point  $p \in M$ , so that  $R \in \text{Sym}^2 \mathfrak{hol}_p M \subset \text{Sym}^2 \Lambda^2 T_p M$  and  $q(R) \in \mathcal{U} \mathfrak{hol}_p M$ . However by definition every point  $f \in \text{Hol}(M)$  identifies  $\mathfrak{hol}_p M$  with  $\mathfrak{hol}$  making  $q(R)$  a  $\mathcal{U} \mathfrak{hol}$ -valued function on  $\text{Hol}(M)$ :

$$q(R) \in C^\infty(\text{Hol}(M), \mathcal{U} \mathfrak{hol})^{\text{Hol}} \cong \Gamma(\text{Hol}(M) \times_{\text{Hol}} \mathcal{U} \mathfrak{hol})$$

For an arbitrary irreducible complex representation  $\pi$  of  $\text{Hol}$  the associated vector bundle  $\pi(M) := \text{Hol}(M) \times_{\text{Hol}} \pi$  over  $M$  is endowed with the connection induced from the Levi-Civita connection. Moreover there is a canonical second order differential operator defined on sections of  $\pi(M)$ :

$$(2.3) \quad \Delta_\pi := \nabla^* \nabla + 2q(R)$$

It is evident from the Weitzenböck formula (2.1) written as in (2.2) that the diagram

$$\begin{array}{ccc} \pi(M) & \xrightarrow{\Delta_\pi} & \pi(M) \\ F \downarrow & & \downarrow F \\ \Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\Delta} & \Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

commutes for any  $F \in \text{Hom}_{\text{Hol}}(\pi, \Lambda^* \mathbb{C}^{n*})$  or equivalently for any globally parallel embedding  $F : \pi(M) \rightarrow \Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C}$ . Hence the pointwise decomposition of  $\Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C}$  into irreducible complex representations of  $\text{Hol}_p M$  becomes a global decomposition of any eigenspace of  $\Delta$ , e. g. we have for its kernel:

$$H_{dR}^*(M, \mathbb{C}) = \bigoplus_{\pi} \text{Hom}_{\text{Hol}}(\pi, \Lambda^* \mathbb{C}^{n*}) \otimes \ker \Delta_\pi$$

The same kind of reasoning is possible for the Dirac operator on spinors, assuming the manifold  $M$  to be spin and taking  $\text{Hol}_p M$  to be its spin holonomy group. Ignoring for the moment the Lichnerowicz result that the curvature term reduces to multiplication by the scalar curvature and employing the formula  $(X \wedge Y) \bullet :=$

$\frac{1}{2}(X \star Y \star + \langle X, Y \rangle)$  for the representation of  $\mathfrak{so}(T_p M)$  on the spinor bundle  $\mathbf{S}(M)$  we can proceed from (2.1) directly to:

$$(2.4) \quad D^2 = \nabla^* \nabla + 4q(R).$$

In particular, all eigenspaces of  $D^2$  decompose globally according to the pointwise decomposition of the spinor bundle under the spin holonomy group  $\text{Hol}_p M$ . From Lichnerowicz’s result we already know that  $q(R)$  acts by scalar multiplication with  $\frac{\kappa}{16}$  on  $\mathbf{S}(M)$ , where  $\kappa$  is the scalar curvature of  $M$ . Hence we can read equation (2.4) as

$$D^2 \Big|_{\pi} = \Delta_{\pi} + \frac{\kappa}{8}$$

where the restriction to  $\pi$  is a short hand notation for any globally parallel embedding  $F : \pi(M) \rightarrow \mathbf{S}(M)$  induced by some non-trivial  $F \in \text{Hom}_{\text{Hol}}(\pi, \mathbf{S})$ . Written in this way formula (2.4) is seen to be a generalization of the Parthasarathy formula for the Dirac square  $D^2$  on a symmetric space  $G/K$  of compact type, because in this case the operators  $\Delta_{\pi}$  defined above on sections of  $\pi(M)$  all become the Casimir of  $G$ .

Counterexamples to the idea that eigenspaces of intrinsically defined differential operators always decompose globally according to the pointwise decomposition under the holonomy group are easily found among twisted Dirac operators. Consider therefore a geometric vector bundle  $\mathcal{R}(M) := \text{Hol}(M) \times_{\text{Hol}} \mathcal{R}$  associated to the holonomy bundle via some not necessarily irreducible representation  $\mathcal{R}$  of the holonomy group. The Levi-Civita connection on  $\text{Hol}(M)$  defines a geometric connection on this vector bundle, whose curvature endomorphism is still given through the representation

$$\bullet : \mathfrak{hol}_p M \times \mathcal{R}_p(M) \rightarrow \mathcal{R}_p(M)$$

of the Lie algebra  $\mathfrak{hol}_p M$  on  $\mathcal{R}_p(M)$  by the formula  $R_{X,Y}^{\mathcal{R}} = R(X \wedge Y) \bullet$ . The twisted Dirac operator  $D_{\mathcal{R}}$  is a first order differential operator acting on sections of the vector bundle  $(\mathbf{S} \otimes \mathcal{R})(M)$ . It satisfies a twisted Weitzenböck formula derived from (2.1):

$$(2.5) \quad D_{\mathcal{R}}^2 = \nabla^* \nabla + \frac{1}{2} \sum_{ij} \left( E_i \star E_j \star R(E_i \wedge E_j) \bullet \otimes \text{id}_{\mathcal{R}} + E_i \star E_j \star \otimes R(E_i \wedge E_j) \bullet \right)$$

This formula has an apparent asymmetry between the spinor bundle and the twist. However, we still have the formula  $(X \wedge Y) \bullet = \frac{1}{2}(X \star Y \star + \langle X, Y \rangle)$  for the representation of  $\mathfrak{so}(T_p M)$  on the fibre  $\mathbf{S}_p(M)$  of the spinor bundle and we may try to balance this asymmetry to cast equation (2.5) into a form similar to (2.4). This is most easily achieved by rewriting the action of  $q(R)$  on the tensor product  $\mathbf{S} \otimes \mathcal{R}$  in the following asymmetric way:

$$q(R) = \frac{1}{2} \sum_{ij} \left( (E_i \wedge E_j) \bullet R(E_i \wedge E_j) \bullet \otimes \text{id}_{\mathcal{R}} + (E_i \wedge E_j) \bullet \otimes R(E_i \wedge E_j) \bullet \right) - q(R) \otimes \text{id}_{\mathcal{R}} + \text{id}_{\mathbf{S}} \otimes q(R)$$

With Lichnerowicz's result  $q(R) = \frac{\kappa}{16}$  for the spinor representation  $\mathbf{S}$  equation (2.5) becomes

$$(2.6) \quad D_{\mathcal{R}}^2 = \Delta_{\mathbf{S} \otimes \mathcal{R}} + \frac{\kappa}{8} \otimes \text{id}_{\mathcal{R}} - \text{id}_{\mathbf{S}} \otimes 2q(R)$$

In conclusion, the squares  $D_{\mathcal{R}}^2$  of twisted Dirac operators will in general not respect the decomposition of  $(\mathbf{S} \otimes \mathcal{R})(M)$  into parallel subbundles because of the critical summand  $\text{id}_{\mathbf{S}} \otimes 2q(R)$ . Nevertheless, if  $q(R)$  acts by scalar multiplication not only on  $\mathbf{S}$  but on  $\mathcal{R}$ , too, the global decomposition of the eigenspaces of  $D_{\mathcal{R}}^2$  according to the pointwise decomposition of  $\mathbf{S} \otimes \mathcal{R}$  is restored.

Equation (2.6) is the key relation of this article and forms the cornerstone and motivation of all statements and calculations to come. In fact, we can take advantage of equation (2.6) even if the manifold in question is not spin, because the twisted Dirac operator may be well defined on the vector bundle  $(\mathbf{S} \otimes \mathcal{R})(M)$  although  $M$  is neither spin nor  $\mathbf{S}(M)$  or  $\mathcal{R}(M)$  are well defined vector bundles. The only thing that really matters is whether the representation  $\mathbf{S} \otimes \mathcal{R}$  is defined for the holonomy group  $\text{Hol}$  itself or only for some covering group.

### 3. QUATERNIONIC KÄHLER HOLONOMY

In this section we introduce the main notions of quaternionic Kähler holonomy based on the group  $\text{Hol} = \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$  with  $n \geq 2$ . Very few examples of compact manifolds with this particular holonomy group are known, and it is a deep result that in every quaternionic dimension  $n$  there are up to isometry only finitely many of these manifolds with positive scalar curvature  $\kappa > 0$  ([LeBSa94]). In fact, the only known examples with  $\kappa > 0$  are symmetric spaces, the so-called Wolf spaces.

In order to introduce quaternionic Kähler holonomy we return for a moment to a point we glossed over in the definition of the holonomy bundle. There we had to choose a suitable representative  $\text{Hol} \subset \mathbf{O}_{4n}\mathbb{R}$  in the conjugacy class of the holonomy groups acting on an abstract vector space  $\mathbb{R}^{4n}$ . This abstract vector space has no meaning in itself but plays the role of the tangent representation of  $\text{Hol}$  just as  $T_pM$  is the tangent representation of  $\text{Hol}_pM$ . Instead of really choosing a representative  $\text{Hol} \subset \mathbf{O}_{4n}\mathbb{R}$  it is always better to start with specifying this tangent representation. Let us begin with an abstract complex vector space  $E \cong \mathbb{C}^{2n}$  endowed with a symplectic form  $\sigma \in \Lambda^2 E^*$  and an adapted, positive quaternionic structure  $J$ , i. e., a conjugate linear map  $J : E \rightarrow E$  satisfying

$$J^2 = -1 \quad \sigma(Je_1, Je_2) = \overline{\sigma(e_1, e_2)} \quad \sigma(e, Je) > 0$$

for all  $e_1, e_2 \in E$  and  $e \neq 0$ . Such a set of structures is consistent and can be defined on the underlying complex vector space of  $\mathbb{H}^n$ . One merit of this explicit construction is that the group of all symplectic transformations of  $E$  commuting with  $J$  agrees in this picture with the quaternionic unitary group  $\mathbf{Sp}(n) := \{A \in M_{n,n}\mathbb{H} \text{ such that } \overline{A^t}A = 1\}$ . The symplectic form  $\sigma$  induces mutually inverse isomorphisms  $\sharp : E \rightarrow E^*, e \mapsto \sigma(e, \cdot)$  and  $\flat : E^* \rightarrow E$ . Similar to the representation

of  $\Lambda^2 T_p M$  on  $T_p M$  considered in the first section there is an action

$$\bullet : \text{Sym}^2 E \times E \longrightarrow E, \quad (e_1 e_2, e) \longmapsto (e_1 e_2) \bullet e := \sigma(e_1, e) e_2 + \sigma(e_2, e) e_1$$

of the second symmetric power  $\text{Sym}^2 E$  on  $E$ . This action is skew symplectic and commutes with  $J$  for all real elements of  $\text{Sym}^2 E$ . It identifies this real subspace with the Lie algebra  $\mathfrak{sp}(n)$  of  $\mathbf{Sp}(n)$  and makes  $\bullet$  not only an action but a representation.

Let  $H \cong \mathbb{C}^2$  be another abstract vector space with the same structures, a symplectic form  $\sigma \in \Lambda^2 H^*$  and an adapted, positive quaternionic structure  $J$ . The tensor product  $H \otimes E$  of these two vector spaces carries a real structure  $J \otimes J$  and a complex bilinear symmetric form  $\langle , \rangle := \sigma \otimes \sigma$ , which is positive definite on the real subspace. In this way the group  $\mathbf{O}(H \otimes E)$  of all complex linear isometries of  $H \otimes E$  commuting with  $J \otimes J$  is isomorphic to  $\mathbf{O}_{4n} \mathbb{R}$  and has a distinguished subgroup  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) := \mathbf{Sp}(1) \times \mathbf{Sp}(n) / \mathbb{Z}_2$  preserving the tensor product structure of  $H \otimes E$ :

*Definition 3.1. (Quaternionic Kähler Manifolds)*

A quaternionic Kähler manifold  $M$  is a Riemannian manifold of dimension  $4n$ ,  $n \geq 2$ , endowed with a reduction of the frame bundle  $\mathbf{O}(M)$  to a principal  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ -bundle  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(M)$  stable under parallel transport. Such a reduction exists if and only if the holonomy group  $\text{Hol}$  of  $M$  is conjugated to a subgroup of  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) \subset \mathbf{O}(H \otimes E)$  and in case of equality it may be defined as:

$$\mathbf{Sp}(1) \cdot \mathbf{Sp}(M) := \{ f : H \otimes E \longrightarrow T_p M \otimes_{\mathbb{R}} \mathbb{C} \mid f \text{ isometry, } f(\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)) = \text{Hol}_p M \}$$

There are a few remarks to make on this definition. First of all we insist on  $n \geq 2$ , because taking this definition as it stands it applies to every oriented Riemannian manifold  $M$  of dimension 4. In addition a quaternionic Kähler manifold with vanishing scalar curvature  $\kappa = 0$  is locally hyperkähler, its universal cover thus hyperkähler, and we will usually exclude these manifolds from consideration. In general, however, a quaternionic Kähler manifold with non-vanishing scalar curvature is despite nomenclature not Kähler.

In order to justify terminology after all these negative remarks and to get into contact with a more common definition of quaternionic Kähler manifolds we recall that  $\text{Sym}^2 H$  acts via  $(h_1 h_2) \bullet h := \sigma(h_1, h) h_2 + \sigma(h_2, h) h_1$  on  $H$ . For a normed real element  $i h J h \in \text{Sym}^2 H$  with  $\sigma(h, J h) = 1$  the action on  $H$  commutes with  $J$  and satisfies:

$$(i h J h) \bullet (i h J h) \bullet = -\text{id} .$$

This follows from the fundamental identity  $\sigma(h_1, h) h_2 - \sigma(h_2, h) h_1 = \sigma(h_1, h_2) h$  for 2-dimensional symplectic vector spaces and hence does not work for  $E$ . Extending this action from  $H$  to the tangent representation  $H \otimes E$  we conclude that normed real local sections of the parallel subbundle  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(M) \times_{\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)} \text{Sym}^2 H$  of the complexified endomorphism bundle  $\text{End}(TM \otimes_{\mathbb{R}} \mathbb{C})$  act as local complex structures on the tangent bundle  $TM$ . Choosing in this way three local complex structures  $I, J$  and  $K$  satisfying  $IJ = K$  we define the canonical quaternionic orientation of  $M$  by



declaring every base of the form  $X_1, IX_1, JX_1, KX_1, \dots, X_n, IX_n, JX_n, KX_n$  to be positively oriented. Alternatively the canonical quaternionic orientation is induced by the  $n$ -th power of the parallel Kraines form  $\Omega \in \Lambda^4(H \otimes E)$  defined in ([Kra66]).

A rather subtle remark concerns the two representations  $H$  and  $E$ , which do not factor through the projection  $\mathbf{Sp}(1) \times \mathbf{Sp}(n) \rightarrow \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ . Although we may think of the complex tangent bundle as a tensor product of two complex vector bundles  $H$  and  $E$ , these vector bundles are not well defined and in general exist only locally. In passing from representation theory to geometry we always have to check, whether the representations factor through the projection  $\mathbf{Sp}(1) \times \mathbf{Sp}(n) \rightarrow \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ . Things get actually simpler in some respect, as the spinor representation  $\mathbf{S}$  of  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  factors through to a representation of  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$  whenever  $n$  is even. Thus all quaternionic Kähler manifolds of even quaternionic dimension  $n$  are spin:

*Proposition 3.2. (The Signed Spinor Representation ([BaS83], [Wan89]))*

The spinor representation  $\mathbf{S}$  of  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  decomposes into the direct sum

$$(3.1) \quad \mathbf{S} = \bigoplus_{r=0}^n \mathbf{S}_r := \bigoplus_{r=0}^n \text{Sym}^r H \otimes \Lambda_0^{n-r} E$$

where  $\Lambda_0^{n-r} E$  is the kernel of the contraction  $\sigma : \Lambda^{n-r} E \rightarrow \Lambda^{n-r-2} E$  with the symplectic form. For the canonical quaternionic orientation of  $H \otimes E$  the half spin representations are given by:

$$\mathbf{S}^+ := \bigoplus_{r \equiv n(2)} \mathbf{S}_r \quad \mathbf{S}^- := \bigoplus_{r \not\equiv n(2)} \mathbf{S}_r.$$

The delicate point in a constructive proof of this proposition is the choice of Clifford multiplication  $\star : (H \otimes E) \times \mathbf{S} \rightarrow \mathbf{S}$ . Besides the Clifford identity there is another crucial property of this multiplication, namely the compatibility condition with the action of the Lie algebra  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$  on  $\mathbf{S}$ . The representation  $\bullet$  of the complexified Lie algebra  $\text{Sym}^2 H \oplus \text{Sym}^2 E$  of the group  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  on  $\mathbf{S}$  has to agree with the representation implicitly defined by Clifford multiplication via  $(X \wedge Y) \bullet := \frac{1}{2}(X \star Y \star + \langle X, Y \rangle)$ . Choosing dual pairs of bases  $\{de_\mu\}, \{e_\nu\}$  for  $E^*, E$  with  $\langle de_\mu, e_\nu \rangle = \delta_{\mu\nu}$  and  $\{dh_\alpha\}, \{h_\beta\}$  for  $H^*, H$  we can check that

$$(3.2) \quad (e \tilde{e}) \mapsto \sum_{\alpha} (dh_{\alpha}^b \otimes e) \wedge (h_{\alpha} \otimes \tilde{e}) \quad (h \tilde{h}) \mapsto \sum_{\mu} (h \otimes de_{\mu}^b) \wedge (\tilde{h} \otimes e_{\mu})$$

is a Lie algebra homomorphism  $\text{Sym}^2 H \oplus \text{Sym}^2 E \rightarrow \Lambda^2(TM \otimes_{\mathbb{R}} \mathbb{C})$  intertwining the given representations of  $\text{Sym}^2 H, \text{Sym}^2 E$  and  $\Lambda^2(TM \otimes_{\mathbb{R}} \mathbb{C})$  on  $H \otimes E = TM \otimes_{\mathbb{R}} \mathbb{C}$ . Consequently the following two operator identities on the spinor representation  $\mathbf{S}$  are

at the heart of Proposition 3.2:

$$(3.3) \quad (e \tilde{e}) \bullet = \frac{1}{2} \sum_{\alpha} \left( (dh_{\alpha}^b \otimes e) \star (h_{\alpha} \otimes \tilde{e}) \star + \sigma(e, \tilde{e}) \right)$$

$$(3.4) \quad (h \tilde{h}) \bullet = \frac{1}{2} \sum_{\mu} \left( (h \otimes de_{\mu}^b) \star (\tilde{h} \otimes e_{\mu}) \star + \sigma(h, \tilde{h}) \right)$$

We will not go into the details of this construction given in [KSW97a], but will take Proposition 3.2 as the assertion that a Clifford multiplication  $\star : (H \otimes E) \times \mathbf{S} \rightarrow \mathbf{S}$  exists satisfying the Clifford identity together with the properties (3.3) and (3.4).

The most important point in our present discussion of quaternionic Kähler holonomy is of course the discussion of the curvature tensor of a quaternionic Kähler manifold and of the associated element  $q(R)$  in the universal enveloping algebra of the Lie algebra  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$  of the holonomy group  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ . In fact compared to other holonomy groups quaternionic Kähler holonomy is rather rigid. This is mainly due to the fact that the curvature tensor of a quaternionic Kähler manifold has to satisfy very stringent constraints and can be described completely by the scalar curvature  $\kappa$  and a section  $\mathfrak{R}$  of  $\text{Sym}^4 E$ . This decomposition was first derived by D. V. Alekseevskii (cf.: [Al68] or [Sal82]) and can be made explicit in the following way (cf.: [KSW97a]):

*Lemma 3.3. (The Curvature Tensor)*

A quaternionic Kähler manifold  $M$  is Einstein with constant scalar curvature  $\kappa$ . Its curvature tensor depends only on  $\kappa$  and a section  $\mathfrak{R}$  of  $\text{Sym}^4 E$ , this dependence reads

$$(3.5) \quad R = -\frac{\kappa}{8n(n+2)}(R^H + R^E) + R^{hyper}$$

where the endomorphism valued two forms  $R^H$ ,  $R^E$  and  $R^{hyper}$  are defined by:

$$(3.6) \quad \begin{aligned} R_{h_1 \otimes e_1, h_2 \otimes e_2}^H &= \sigma_E(e_1, e_2)(h_1 h_2 \bullet \otimes \text{id}_E) \\ R_{h_1 \otimes e_1, h_2 \otimes e_2}^E &= \sigma_H(h_1, h_2)(\text{id}_H \otimes e_1 e_2 \bullet) \\ R_{h_1 \otimes e_1, h_2 \otimes e_2}^{hyper} &= \sigma_H(h_1, h_2)(\text{id}_H \otimes (e_2^{\sharp} \lrcorner e_1^{\sharp} \lrcorner \mathfrak{R}) \bullet) \end{aligned}$$

At the end of this section we want to describe the action of the element  $q(R)$  of the universal enveloping algebra  $\mathcal{U}(\mathfrak{sp}(1) \oplus \mathfrak{sp}(n))$  on some representations. In particular we will see that for a large class of representations of  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  the element  $q(R)$  acts by scalar multiplication, because the contributions from the hyperkähler part  $R^{hyper}$  of the curvature tensor drop out. Observe first that  $q(R)$  depends linearly on  $R$ :

$$q(R) = -\frac{\kappa}{8n(n+2)} (q(R^H) + q(R^E)) + q(R^{hyper})$$

Using equation (3.2) we can write down the terms appearing in this sum more explicitly:

Lemma 3.4.

$$\begin{aligned}
 q(R^H) &= \frac{1}{4} \sum_{\alpha\beta} (dh_\alpha^b dh_\beta^b) \bullet (h_\alpha h_\beta) \bullet \\
 q(R^E) &= \frac{1}{4} \sum_{\mu\nu} (de_\mu^b de_\nu^b) \bullet (e_\mu e_\nu) \bullet \\
 q(R^{hyper}) &= \frac{1}{4} \sum_{\mu\nu} (de_\mu^b de_\nu^b) \bullet (e_\mu^\sharp \lrcorner e_\nu^\sharp \lrcorner \mathfrak{A}) \bullet
 \end{aligned}$$

**Proof:** Converting the sum over a local orthonormal base  $\{E_i\}$  into the sum

$$\sum_i E_i \otimes E_i = \sum_{\alpha\mu} (dh_\alpha^b \otimes de_\mu^b) \otimes (h_\alpha \otimes e_\mu)$$

over dual pairs  $\{de_\mu\}$ ,  $\{e_\mu\}$  and  $\{dh_\alpha\}$ ,  $\{h_\alpha\}$  of bases we calculate say for  $q(R^{hyper})$

$$\begin{aligned}
 &\frac{1}{4} \sum_{ij} (E_i \wedge E_j) \bullet R_{E_i, E_j}^{hyper} = \\
 &\frac{1}{4} \sum_{\alpha\beta\mu\nu} (dh_\alpha^b \otimes de_\mu^b \wedge dh_\beta^b \otimes de_\nu^b) \bullet \sigma(h_\alpha, h_\beta)(e_\mu^\sharp \lrcorner e_\nu^\sharp \lrcorner \mathfrak{A}) \bullet = \\
 &\frac{1}{4} \sum_{\alpha\mu\nu} (dh_\alpha^b \otimes de_\mu^b \wedge h_\alpha \otimes de_\nu^b) \bullet (e_\mu^\sharp \lrcorner e_\nu^\sharp \lrcorner \mathfrak{A}) \bullet
 \end{aligned}$$

which is equivalent to the stated equality in view of equation (3.2).  $\square$

Evidently  $2q(R^H)$  and  $2q(R^E)$  respectively are the Casimir operators for  $\mathfrak{sp}(1)$  and  $\mathfrak{sp}(n)$  in  $\sigma$ -normalization, i. e. the defining invariant symmetric form on the Lie algebra  $\text{Sym}^2 H$  or  $\text{Sym}^2 E$  is not the Killing form itself but the natural extension of  $\sigma$  to the second symmetric powers using Gram's permanent. Hence the Casimir eigenvalues of  $q(R^H)$  and  $q(R^E)$  are easily calculated directly for the simplest representations of  $\mathbf{Sp}(n)$ :

Lemma 3.5. (Casimir Eigenvalues)

For the irreducible representations  $\text{Sym}^l E$  and  $\Lambda^d E$  the Casimir eigenvalues for  $q(R^E)$  are:

$$q(R^E)_{\text{Sym}^l E} = -l \left( n + \frac{l}{2} \right) \qquad q(R^E)_{\Lambda^d E} = -d \left( n - \frac{d}{2} + 1 \right)$$

The eigenvalues of  $q(R^H)$  are given by the same formulas with  $n = 1$ . Setting  $l = 2$  we get the Casimir eigenvalues for  $q(R^E)$  and  $q(R^H)$  in the adjoint representations  $\text{Sym}^2 E$  and  $\text{Sym}^2 H$  of  $\mathfrak{sp}(n)$  and  $\mathfrak{sp}(1)$ . Since by definition the Casimir eigenvalue of the adjoint representation is always one for Casimirs in the Killing normalization we see in particular:

$$q(R^E) = -2(n + 1) \text{Cas}_{\mathfrak{sp}(n)} \qquad q(R^H) = -4 \text{Cas}_{\mathfrak{sp}(1)}$$

Now we claim that the hyperkähler contribution  $q(R^{hyper})$  to the element  $q(R)$  acts trivially on every irreducible representation occurring in the representation  $\Lambda E$ , i. e., on all representations  $\Lambda^d E$  with  $d = 0, \dots, n$ . Because  $q(R^{hyper})$  depends linearly on  $\mathfrak{R} \in \text{Sym}^4 E$  we are allowed to expand  $\mathfrak{R}$  into a sum of fourth powers  $\frac{1}{24}e^4$ ,  $e \in E$ , to calculate  $q(R^{hyper})$ . It is thus sufficient to prove that the action of  $q(\frac{1}{24}e^4)$  on  $\Lambda E$  is trivial for all  $e \in E$ . According to Lemma 3.4 the element  $q(\frac{1}{24}e^4)$  acts on  $\Lambda E$  as:

$$q(\frac{1}{24}e^4) = \frac{1}{2}(\frac{1}{2}e^2) \bullet (\frac{1}{2}e^2) \bullet = \frac{1}{2}(e \wedge e^\sharp \lrcorner)(e \wedge e^\sharp \lrcorner) = -\frac{1}{2}e \wedge e \wedge e^\sharp \lrcorner e^\sharp \lrcorner = 0$$

Consequently the curvature tensor  $q(R)$  will act by scalar multiplication on all representations  $\mathcal{R}^{l,d} := \text{Sym}^l H \otimes \Lambda^d E$ . From equation (2.6) we conclude that the squares  $D_{\mathcal{R}^{l,d}}^2$  of the twisted Dirac operators with these particular twists have properties similar to the Hodge–Laplacian  $\Delta$  and the square  $D^2$  of the untwisted Dirac operator:

*Proposition 3.6. (Global Decomposition Principle)*

The restriction  $D_{\mathcal{R}^{l,d}}^2|_{\pi}$  of the square of a twisted Dirac operator  $D_{\mathcal{R}^{l,d}}^2$  with twisting bundle  $\mathcal{R}^{l,d} := (\text{Sym}^l H \otimes \Lambda^d E)(M)$  to a parallel subbundle  $\pi(M) \subset (\mathbf{S} \otimes \mathcal{R}^{l,d})(M)$  does not depend on the specific embedding of this subbundle and equation (2.6) becomes in this case:

$$\Delta_{\pi} = D_{\mathcal{R}^{l,d}}^2|_{\pi} + \frac{\kappa}{8n(n+2)}(l+d-n)(l-d+n+2)$$

#### 4. CLASSIFICATION OF MINIMAL AND MAXIMAL TWISTS

In this section we will focus attention on the technicalities necessary to draw conclusions from Proposition 3.6. The irreducible representations occurring in the twisted spinor representations  $\mathbf{S} \otimes \mathcal{R}^{l,d}$  are all of the form  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$ , where  $\Lambda_{\text{top}}^{a,b} E$  is the irreducible representation of highest weight in the tensor product  $\Lambda^a E \otimes \Lambda^b E$ . Alternatively we see from Weyl’s construction of the irreducible representations of the classical matrix groups that  $\Lambda_{\text{top}}^{a,b} E$  is the common kernel of the diagonal contraction with the symplectic form  $\sigma : \Lambda^a E \otimes \Lambda^b E \rightarrow \Lambda^{a-1} E \otimes \Lambda^{b-1} E$  and the Plücker differential:

$$\sum_{\mu} e_{\mu} \wedge \otimes de_{\mu} \lrcorner : \Lambda^a E \otimes \Lambda^b E \rightarrow \Lambda^{a+1} E \otimes \Lambda^{b-1} E$$

In particular, we will characterize the twists  $\mathcal{R}^{l,d}$  with  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E \subset \mathbf{S} \otimes \mathcal{R}^{l,d}$ . Moreover, for each representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  in this class and will classify the special twists maximizing the curvature expression

$$-\frac{\kappa}{8n(n+2)}(l+d-n)(l-d+n+2)$$

of Proposition 3.6 for  $\kappa > 0$  and  $\kappa < 0$ . This classification is the most important step used in the applications of the ideas encoded in Proposition 3.6. Global questions are postponed to the next sections. Hence, we will deal with representations of  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  only.

*Theorem 4.1. (Characterization of Admissible Twists)*

A representation  $\mathcal{R}^{l,d} := \text{Sym}^l H \otimes \Lambda^d E$  with  $l \geq 0$  and  $n \geq d \geq 0$  is called an admissible twist for the irreducible representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$ , if there exists a non-trivial, equivariant homomorphism from  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  to the twisted spinor representation  $\mathbf{S} \otimes \mathcal{R}^{l,d}$ :

$$\text{Hom}_{\mathbf{Sp}(1) \times \mathbf{Sp}(n)}(\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E, \mathbf{S} \otimes \mathcal{R}^{l,d}) \neq \{0\}$$

A twist  $\mathcal{R}^{l,d}$  is admissible in this sense if and only if  $k + a + b \equiv n + l + d \pmod 2$  and:

- (4.1)  $b \leq d$
- (4.2)  $|k - l| + |a - d| \leq n - b$
- (4.3)  $|n - a + b - d| \leq k + l$

A simple consequence of Theorem 4.1 is that all the representations  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  occur in twisted spinor representations, e. g. in  $\mathbf{S} \otimes \mathcal{R}^{k+n-b,a}$  and  $\mathbf{S} \otimes \mathcal{R}^{|n-a-k|,b}$ . In fact for the twist  $\mathcal{R}^{k+n-b,a}$  inequality (4.2) is trivial and (4.3) needs  $|n - 2a + b| \leq |n - a| + |a - b|$ . For the second twist  $\mathcal{R}^{|n-a-k|,b}$  inequality (4.2) follows from the distance decreasing property  $||x| - |y|| \leq |x - y|$  of the absolute value via  $||-k| - |n - a - k|| \leq n - a$ , whereas (4.3) reduces to  $|n - a| \leq \max\{n - a, 2k - n + a\} = k + |n - a - k|$ . These two twists are the prototype examples of maximal and minimal twists to be defined below.

**Proof:** For the proof we recall a well-known fusion rule for the tensor product  $\Lambda^c E \otimes \Lambda^d E$  of two irreducible  $\mathbf{Sp}(n)$ -representations  $\Lambda^c E$  and  $\Lambda^d E$  (cf. [OnVi90])

$$\Lambda^c E \otimes \Lambda^d E = \bigoplus_{\substack{a+b \equiv c+d \pmod 2 \\ a+b \leq c+d \\ |c-d| \leq a-b \leq 2n-c-d}} \Lambda_{\text{top}}^{a,b} E$$

Note in particular that each irreducible representation  $\Lambda_{\text{top}}^{a,b} E$  occurs at most once in the tensor product  $\Lambda^c E \otimes \Lambda^d E$ . Using this fusion rule together with the Clebsch-Gordan formula for irreducible  $\mathbf{Sp}(1)$ -representations and the decomposition of the spinor representation  $\mathbf{S}$  under  $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$  given in Proposition 3.2 we can formally

write down the decomposition

(4.4)

$$\bigoplus_{c=0}^n (\text{Sym}^{n-c} H \otimes \Lambda^c_0 E) \otimes (\text{Sym}^l H \otimes \Lambda^d_0 E) = \sum_{\substack{k \geq 0 \\ n \geq a \geq b \geq 0}} \# \mathfrak{M}_{k,a,b}(l, d) \cdot \text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$$

of  $\mathbf{S} \otimes \mathcal{R}^{l,d}$ , where  $\mathfrak{M}_{k,a,b}(l, d)$  is the set of all  $n \geq c \geq 0$  satisfying the set of constraints:

$$(4.5) \quad \begin{array}{ll} k \equiv n + c + l \pmod{2} & a + b \equiv c + d \pmod{2} \\ k \leq n - c + l & a + b \leq c + d \\ k \geq |n - c - l| & a - b \geq |c - d| \\ & a - b \leq 2n - c - d \end{array}$$

It is clear from these constraints that  $\mathfrak{M}_{k,a,b}(l, d)$  is empty unless  $k + a + b \equiv n + l + d \pmod{2}$  reflecting in a way the consistency of the action of  $(-1, -1) \in \mathbf{Sp}(1) \times \mathbf{Sp}(n)$ . In particular,  $k + a + b \equiv n + l + d \pmod{2}$  is a necessary condition for the twist  $\mathcal{R}^{l,d}$  to be admissible.

In view of this congruence we can drop one of the two constraints  $a + b \equiv c + d \pmod{2}$  or  $k \equiv n + c + l \pmod{2}$  and solve the inequalities (4.5) for  $c$  to arrive after a little manipulation at an equivalent description of  $\mathfrak{M}_{k,a,b}(l, d)$  as the set of all  $c \equiv a + b + d \pmod{2}$  satisfying:

$$(4.6) \quad \max \{ b + |a - d|, n - k - l \} \leq c \leq n - \max \{ |k - l|, |n - a + b - d| \}$$

Under the standing hypothesis  $k + a + b \equiv n + l + d \pmod{2}$  we evidently have

$$\max \{ b + |a - d|, n - k - l \} \equiv a + b + d \equiv n - \max \{ |k - l|, |n - a + b - d| \} \pmod{2}$$

so that  $\mathfrak{M}_{k,a,b}(l, d)$  will be non-empty if and only if the inequality (4.6) is consistent, because the congruence  $c \equiv a + b + d \pmod{2}$  will be fulfilled by either end of the resulting interval. However, the consistency condition for (4.6) is given by four inequalities in  $l, d$  depending of course on  $k, a, b$ . The first  $n - k - l \leq n - |k - l|$  is trivial for  $k, l \geq 0$  and the next two become inequalities (4.2) and (4.3), whereas the last  $b + |a - d| \leq n - |n - a + b - d|$  is equivalent to inequality (4.1) for all  $b \leq a \leq n$  and  $d \leq n$ .  $\square$

Note that if the set  $\mathfrak{M}_{k,a,b}(l, d)$  is non-empty all its elements will have the same parity as  $a + b + d$ . Of course their number  $\# \mathfrak{M}_{k,a,b}(l, d)$  is just the multiplicity of the representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  in  $\mathbf{S} \otimes \mathcal{R}^{l,d}$ , which we will need below as index multiplicity:

*Definition 4.2. (The Index of an Admissible Twist)*

The index of an admissible twist  $\mathcal{R}^{l,d}$  for an irreducible representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  is the index multiplicity of  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  in the twisted spinor representation

$\mathbf{S}^\pm \otimes \mathcal{R}^{l,d}$ :

$$\begin{aligned} \text{index}(k, a, b; l, d) &:= \dim \text{Hom}_{\mathbf{Sp}(1) \times \mathbf{Sp}(n)}(\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E, \mathbf{S}^+ \otimes \mathcal{R}^{l,d}) \\ &\quad - \dim \text{Hom}_{\mathbf{Sp}(1) \times \mathbf{Sp}(n)}(\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E, \mathbf{S}^- \otimes \mathcal{R}^{l,d}) \end{aligned}$$

From the proof of Theorem 4.1 we can easily read off an explicit formula for this index:

$$\begin{aligned} &\text{index}(k, a, b; l, d) := \\ &\frac{(-1)^{a+b+d}}{2} \left( n + 2 - \max\{|k-l|, |n-a+b-d|\} - \max\{b+|a-d|, n-k-l\} \right) \end{aligned}$$

Although we have calculated the index multiplicity of the representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  for an arbitrary twisted spinor representation  $\mathbf{S} \otimes \mathcal{R}^{l,d}$ , it will turn out below that only very few representations actually contribute to the index of a particular twisted Dirac operator. These representations are characterized by the following extremality condition:

*Definition 4.3. (Minimal and Maximal Twists)*

An admissible twist  $\mathcal{R}^{l,d} := \text{Sym}^l H \otimes \Lambda_{\circ}^d E$  for the irreducible representation  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  is called a minimal or maximal twist, if the curvature term of Proposition 3.6, or equivalently the function  $\phi(\tilde{l}, \tilde{d}) := (\tilde{l} + \tilde{d} - n)(\tilde{l} - \tilde{d} + n + 2)$ , assumes its minimum or maximum among all admissible twists  $\mathcal{R}^{\tilde{l}, \tilde{d}}$  in the twist  $\mathcal{R}^{l,d}$ .

To determine the index of a twisted Dirac operator in terms of the dimension of the eigenspaces of the operators  $\Delta_\pi$ , all we will further need is a classification of all minimal twists for negative scalar curvature  $\kappa < 0$  and similarly of all maximal twists for  $\kappa > 0$ :

*Theorem 4.4. (Classification of Maximal Twists)*

All representations  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  with  $k > 0$  or  $a > b$  have unique maximal twists:

$$\mathcal{R}^{k+n-b,a} = \text{Sym}^{k+n-b} H \otimes \Lambda_{\circ}^a E \quad \text{index}(k, a, b; k+n-b, a) = (-1)^b$$

For the special representations  $\Lambda_{\text{top}}^{a,a} E$  with  $k = 0$  and  $a = b$  all admissible twists  $\mathcal{R}^{n-d,d}$  with  $d = a, \dots, n$  have  $\phi(n-d, d) = 0$  and are thus automatically maximal and minimal:

$$\mathcal{R}^{n-d,d} = \text{Sym}^{n-d} H \otimes \Lambda_{\circ}^d E \quad \text{index}(0, a, a; n-d, d) = (-1)^d$$

The classification of all minimal twists splits into more cases:

*Theorem 4.5. (Classification of Minimal Twists)*

According to their minimal twists the irreducible representations  $\text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  are divided into four classes. In the first class we have  $k > (n-a) + (n-b)$  and a unique minimal twist:

$$\mathcal{R}^{k-n+b,a} = \text{Sym}^{k-n+b} H \otimes \Lambda_{\circ}^a E \quad \text{index}(k, a, b; k-n+b, a) = (-1)^b$$

In the second class with  $k = (n - a) + (n - b)$  the minimal twist is no longer unique. All minimal twists for representations in this class are given by

$$\mathcal{R}^{n-d,d} = \text{Sym}^{n-d} H \otimes \Lambda_{\circ}^d E \quad \text{index}(k, a, b; n - d, d) = (-1)^{k+d}$$

with  $d = b, \dots, a$ . The special representations  $\Lambda_{\text{top}}^{a,a} E$  with  $k = 0$  and  $a = b$  form the third class overlapping in  $k = 0$  and  $a = b = n$  with the second. All admissible twists  $\mathcal{R}^{n-d,d}$  with  $d = a, \dots, n$  for these special representations are minimal and maximal at the same time:

$$\mathcal{R}^{n-d,d} = \text{Sym}^{n-d} H \otimes \Lambda_{\circ}^d E \quad \text{index}(0, a, a; n - d, d) = (-1)^d$$

The remaining representations are characterized by  $k < (n - a) + (n - b)$  and  $k + (a - b) > 0$ . The minimal twists of the representations in this fourth class are all unique:

$$\mathcal{R}^{|n-a-k|,b} = \text{Sym}^{|n-a-k|} H \otimes \Lambda_{\circ}^b E \quad \text{index}(k, a, b; |n - a - k|, b) = (-1)^a$$

Before proceeding to the actual proofs of Theorem 4.4 and Theorem 4.5 let us agree on some geometric terms in order to help intuition. The set of solutions to the inequality (4.2) in  $(l, d)$ -space is a ball in  $L^1$ -norm, i. e. a diamond, with center  $(k, a)$  and radius  $n - b$ . Its right and left corner are thus  $(k \pm (n - b), a)$  with  $(k, a \pm (n - b))$  being its top and bottom corner. On the other hand the set of solutions to the inequality (4.3) is the cone  $\{(l, d) \mid l + d \geq -k + n - a + b \text{ and } l - d \geq -k - n + a - b\}$  opening diagonally to the right from its vertex in the point  $(-k, n - a + b)$ .

In particular the set of solutions to both inequalities (4.2) and (4.3) is always a rectangle in  $(l, d)$ -space, which may degenerate into a straight line but always contains at least the points  $(k + n - b, a)$  and  $(|n - a - k|, b)$ . Note that all corners of the diamond as well as the vertex of the cone and the corners of the resulting intersection rectangle satisfy the congruence condition  $l + d \equiv n + k + a + b$ , which consequently will care for itself below.

Finally the level sets of the function  $\phi(l, d) = (l + d - n)(l - d + n + 2)$ , which we are going to extremize, are hyperbolas with two diagonal axes  $l + d = n$  and  $l - d = -n - 2$  dividing  $(l, d)$ -space into four quadrants. In the first quadrant with  $l + d \geq n, l - d \geq -n - 2$  the function  $\phi \geq 0$  is positive, whereas it is negative in the second  $l + d \leq n, l - d \geq -n - 2$ . Eventually we only care for points  $l \geq 0$  and  $n \geq d \geq 0$  in these two quadrants.

**Proof of Theorem 4.4:** We already know that the right corner  $(k + n - b, a)$  of the diamond always corresponds to the admissible twist  $\mathcal{R}^{k+n-b,a}$  since  $|n - 2a + b| \leq |n - a| + |a - b| = n - b$ . If this right corner of the diamond lies in the strict interior of the first quadrant, then it will be the unique point, where  $\phi$  assumes its maximum on the diamond, tacitly ignoring of course third and fourth quadrant. In particular the twist  $\mathcal{R}^{k+n-b,a}$  will be the unique maximal twist as soon as  $k + n - b + a > n$ , equivalently  $k > 0$  or  $a > b$ .

Assuming now  $k = 0$  and  $a = b$  we see that the top corner  $(0, n)$  of the diamond coincides with the vertex of the cone. Thus the intersection rectangle degenerates



into the face of the diamond running from its top corner  $(0, n)$  to its right corner  $(n - a, a)$ . Consequently the admissible twists are exactly the twists  $\mathcal{R}^{n-d,d}$  with  $d = a, \dots, n$  and  $\phi(n - d, d) = 0$ . The calculation of the index multiplicities is left to the reader.  $\square$

**Proof of Theorem 4.5:** We first concentrate on the case  $k > (n - a) + (n - b)$  or equivalently  $k - n + b + a > n$ , where the diamond lies completely in the strict interior of the first quadrant since its left corner does. With the axes of the level sets of  $\phi$  running parallel to the faces of the diamond  $\phi$  assumes a unique minimum on the diamond in its left corner. Consequently we are done once we have checked that  $\mathcal{R}^{k-n+b,a}$  is an admissible twist. However inequality (4.3) immediately follows from  $|n - 2a + b| \leq n - b < k$ , which is needed for calculating the index multiplicity, too.

Assuming next that  $k = (n - a) + (n - b)$  the left corner of the diamond is the point  $(n - a, a)$  in the first quadrant. Hence, all of the diamond lies in the first quadrant  $\phi \geq 0$  with  $\phi = 0$  only on the face from its left to its bottom corner  $(2n - a - b, a - n + b)$ . Note that the bottom corner fails to satisfy inequality (4.1) and that inequality (4.3) is satisfied by the left corner  $(n - a, a)$  due to  $|n - 2a + b| \leq n - b \leq k$ . Taking this into account the only admissible twists satisfying  $\phi = 0$  are exactly the twists  $\mathcal{R}^{n-d,d}$  with  $d = b, \dots, a$ .

The admissible twists for the special representations  $\Lambda_{\text{top}}^{a,a} E$  with  $k = 0$  and  $a = b$  are exactly the twists  $\mathcal{R}^{n-d,d}$  with  $d = a, \dots, n$ , because the top corner  $(0, n)$  of the diamond coincides with the vertex of the cone. As all these admissible twists have  $\phi(n - d, d) = 0$ , they are all both minimal and maximal.

Recall now that  $\mathcal{R}^{|n-a-k|,b}$  is an admissible twist, because  $|n - a| \leq k + |n - a - k|$  and  $|| - k| - |n - a - k|| \leq n - a$  by distance decrease. Turning to geometry we see that the bottom corner of the intersection rectangle of cone and diamond will be either  $(k, a - n + b)$  for  $k \geq n - a$  or  $(n - a, b - k)$  for  $k \leq n - a$ , i. e. whatever point has larger  $l$  and  $d$ -coordinate. In particular this bottom corner fails in general to satisfy inequality (4.1) chopping off a triangle from the rectangle. The resulting face runs from the point  $(|n - a - k|, b)$  to  $(n - a + k, b)$  independent of whether  $k \geq n - a$  or  $k \leq n - a$ . Note that the geometry may become even more complicated, but we already know that the twist  $\mathcal{R}^{|n-a-k|,b}$  is admissible, which fixes this problem as far as we need it.

In order to classify the minimal twists of the remaining representations characterized by  $k < (n - a) + (n - b)$  and  $k + (a - b) > 0$  we observe that these two assumptions together are equivalent to  $|n - a - k| + b < n$ , so that the point  $(|n - a - k|, b)$  will lie in the strict interior of the second quadrant. From the geometric discussion above we conclude that  $\phi$  assumes a unique minimum in this point, because the tangents to the level surfaces of  $\phi$  are never diagonal and horizontal only for  $l = -1 < |n - a - k|$ .  $\square$

5. EIGENVALUE ESTIMATES

The potential applications of Proposition 3.6 include eigenvalue estimates for the Laplace and for twisted Dirac operators. The general procedure is described in this section and carried out in some particularly interesting cases. Our first example are the irreducible  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ -representations  $\text{Sym}^r H \otimes \Lambda^r_0 E$  defining parallel subbundles in the bundle of  $r$ -forms (cf. [Sal86]). On these parallel subbundles we have the following lower bound for the spectrum of the Laplace operator.

*Proposition 5.1. (Eigenvalue Estimate on  $\text{Sym}^r H \otimes \Lambda^r_0 E$ )*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ . Then any eigenvalue  $\lambda$  of the Laplace operator restricted to  $\text{Sym}^r H \otimes \Lambda^r_0 E$  satisfies

$$\lambda \geq \frac{r(n+1)}{2n(n+2)} \kappa.$$

**Proof:** It follows from Theorem 4.4 that  $\text{Sym}^{n+r} H \otimes \Lambda^r_0 E$  is a maximal twist for the representation  $\text{Sym}^r H \otimes \Lambda^r_0 E$ . Using Proposition 3.6 with  $l = n + r$  and  $d = r$  we obtain:

$$\Delta_{\text{Sym}^r H \otimes \Lambda^r_0 E} = D^2_{\mathcal{R}^{n+r,r}} \Big|_{\text{Sym}^r H \otimes \Lambda^r_0 E} + \frac{r(n+1)}{2n(n+2)} \kappa \geq \frac{r(n+1)}{2n(n+2)} \kappa. \quad \square$$

An interesting special case is  $H \otimes E = TM \otimes_{\mathbb{R}} \mathbb{C}$  for  $r = 1$ , leading to an eigenvalue estimate for the Laplace operator on 1-forms. In particular, the first Betti number has to vanish. Since the differential of any eigenfunction of the Laplace operator is an eigenform for the same eigenvalue we also obtain an estimate on functions (cf. [AlMa95] and [LeB95]):

*Corollary 5.2. (Vanishing of the First Betti Number for Positive Scalar Curvature)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ . Any eigenvalue  $\lambda$  of the Laplace operator on non-constant functions or 1-forms satisfies

$$\lambda \geq \frac{n+1}{2n(n+2)} \kappa.$$

Replacing maximal by minimal twists to compensate the sign of the scalar curvature the same argument provides eigenvalue estimates on  $\text{Sym}^r H \otimes \Lambda^r_0 E$  on manifolds with  $\kappa < 0$ :

*Proposition 5.3. (Vanishing of the First Betti Number for Negative Scalar Curvature)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of negative scalar curvature  $\kappa < 0$ . Then any eigenvalue  $\lambda$  of the Laplace operator on 1-forms satisfies:

$$\lambda \geq \frac{|\kappa|}{2(n+2)}.$$

In particular the first Betti number has to vanish even in the case of negative scalar curvature.

**Proof:** Recall that we excluded the case  $n = 1$  from the very beginning in Definition 3.1. Since  $n \geq 2$  and  $r = 1$  we are in the fourth case of Theorem 4.5. The unique minimal twist for  $H \otimes E$  is thus  $\text{Sym}^{n-2}H$  and we can apply Proposition 3.6 with  $l = n - 2$  and  $d = 0$  to obtain:

$$\Delta_{H \otimes E} = D_{\mathcal{R}^{n-2,0}}^2 \Big|_{H \otimes E} - \frac{\kappa}{2(n+2)} \geq \frac{|\kappa|}{2(n+2)}. \quad \square$$

The vanishing of the first Betti number in the case of negative scalar curvature was also proved in [Ho96]. In Proposition 6.8 we will prove a stronger vanishing result for the odd Betti numbers.

As an other application we consider the Laplace operator on 2-forms  $\Lambda^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$ , which decompose into  $\text{Sym}^2 H \oplus (\text{Sym}^2 H \otimes \Lambda^2_0 E) \oplus \text{Sym}^2 E$ . In the next section we will see that the Laplace operator may have a kernel in the sections of the parallel subbundle  $\text{Sym}^2 E$ . Nevertheless we have a positive lower bound on the other two parallel subbundles:

*Proposition 5.4. (Eigenvalue Estimates on 2-forms)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa$ . Then all eigenvalues  $\lambda$  of the Laplace operator on 2-forms in  $\text{Sym}^2 H$  or  $\text{Sym}^2 H \otimes \Lambda^2_0 E$  satisfy

$$\lambda(\Delta_{\text{Sym}^2 H}) \geq \frac{\kappa}{2n} \quad \text{and} \quad \lambda(\Delta_{\text{Sym}^2 H \otimes \Lambda^2_0 E}) \geq \frac{n+1}{n(n+2)} \kappa.$$

The estimate for the Laplace operator on  $\text{Sym}^2 H \subset \Lambda^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}$  was proved for the first time in [AlMa98]. Again we have similar results in the case of negative curvature. In particular, the lower bound for  $\Delta_{\text{Sym}^2 H}$  is the same as in Proposition 5.3.

Our next aim is to derive properties of twisted Dirac operators. For doing so we make the following crucial observation. If  $\pi$  is any representation with admissible twists  $\mathcal{R}^{l,d}$  and  $\mathcal{R}^{\tilde{l},\tilde{d}}$  then we can apply Proposition 3.6 twice to obtain

$$(5.1) \quad D_{\mathcal{R}^{l,d}}^2 \Big|_{\pi} = D_{\mathcal{R}^{\tilde{l},\tilde{d}}}^2 \Big|_{\pi} + \frac{\kappa}{8n(n+2)} \left( \phi(\tilde{l}, \tilde{d}) - \phi(l, d) \right),$$

with  $\phi(l, d) = (l + d - n)(l - d + n + 2)$ . We first use this observation to give a short proof of the eigenvalue estimate for the untwisted Dirac operator:

*Proposition 5.5. (Eigenvalue Estimate for the Untwisted Dirac Operator [KSW97a])*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler spin manifold of positive scalar curvature  $\kappa$ . Then any eigenvalue  $\lambda$  of the untwisted Dirac operator satisfies

$$\lambda^2 \geq \frac{n+3}{n+2} \frac{\kappa}{4}.$$

**Proof:** According to Proposition 3.2 the spinor bundle decomposes into the parallel subbundles  $\mathbf{S} = \oplus_{r=0}^n \mathbf{S}_r$  with  $\mathbf{S}_r = \text{Sym}^r H \otimes \Lambda_0^{n-r} E$ . To estimate the square of

the Dirac operator on  $\text{Sym}^r H \otimes \Lambda^{\circ n-r} E$  we observe that the unique maximal twist for  $\text{Sym}^r H \otimes \Lambda^{\circ n-r} E$  is  $\mathcal{R}^{n+r, n-r}$  and for  $l = d = 0$  and  $\tilde{l} = n + r, \tilde{d} = n - r$  equation (5.1) reads:

$$D^2 \Big|_{\mathbf{S}_r} = D^2_{\mathcal{R}^{n+r, n-r}} \Big|_{\mathbf{S}_r} + \frac{\kappa}{8n(n+2)} \left( n(2r+n+2) + n(n+2) \right) \geq \frac{n+2+r}{n+2} \frac{\kappa}{4}.$$

Consequently some hypothetical eigenspinor  $\phi \in \Gamma(\mathbf{S})$  of  $D^2$  with eigenvalue  $\lambda^2 < \frac{n+3}{n+2} \frac{\kappa}{4}$  would have to be localized in the subbundle  $\mathbf{S}_0 \subset \mathbf{S}$ . But the Dirac operator on a manifold of positive scalar curvature has trivial kernel so that  $D\phi \in \Gamma(\mathbf{S}_1)$  would be a nontrivial eigenspinor for  $D^2$  again with eigenvalue  $\lambda^2$  in contradiction to the estimate for  $\mathbf{S}_1$ .  $\square$

We now use equation (5.1) for describing the kernels of twisted Dirac operators in the case of positive scalar curvature. If  $\pi$  is any representations which contributes to the kernel of  $D^2_{\mathcal{R}^{l,d}}$  then  $\mathcal{R}^{l,d}$  has to be a maximal twist for  $\pi$ . In fact equation (5.1) implies that  $D^2_{\mathcal{R}^{l,d}}$  is positive on  $\pi$  as soon as there is another admissible twist  $\mathcal{R}^{\tilde{l}, \tilde{d}}$  for  $\pi$  with  $\phi(\tilde{l}, \tilde{d}) > \phi(l, d)$ . From this remark and Proposition 3.6 we conclude in the case of positive scalar curvature

$$(5.2) \quad \ker(D^2_{\mathcal{R}^{l,d}}) = \bigoplus_{\pi} \ker \left( \Delta_{\pi} - \frac{\kappa}{8n(n+2)} \phi(l, d) \right)$$

where the sum is over all representations  $\pi$  for which  $\mathcal{R}^{l,d}$  is a maximal twist. Since  $\frac{\kappa}{8n(n+2)} \phi(l, d)$  is the smallest possible eigenvalue of the operator  $\Delta_{\pi}$  equation (5.2) is in essence a decomposition of  $\ker(D^2_{\mathcal{R}^{l,d}})$  into a sum of minimal eigenspaces for the operators  $\Delta_{\pi}$ .

If  $\mathcal{R}^{l,d}$  is a maximal twist for a representation  $\pi$  then Theorem 4.4 also provides us with the information whether  $\pi$  occurs in  $\mathbf{S}^+ \otimes \mathcal{R}^{l,d}$  or in  $\mathbf{S}^- \otimes \mathcal{R}^{l,d}$ . Hence a corollary of equation (5.2) is a formula for the index of the twisted Dirac operator  $D_{\mathcal{R}^{l,d}}$  in terms of dimensions of certain minimal eigenspaces. We will describe this in two examples:

*Proposition 5.6.*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ , then:

$$\ker(D^2_{\mathcal{R}^{l,d}}) = \{0\} \quad \text{for} \quad l + d < n.$$

**Proof:** All maximal twists  $\mathcal{R}^{l,d}$  satisfy  $l + d \geq n$  by Theorem 4.4.  $\square$

An immediate consequence of this proposition is the vanishing of the index  $\text{index}(D_{\mathcal{R}^{l,d}})$  for  $l + d < n$ . This was also proved in [LeBSa94] by using the Akizuki–Nakano vanishing theorem on the twistor space. For the second example we consider the twisted Dirac operator  $D_{\mathcal{R}^{n+2,0}}$ . It easily follows from Theorem 4.4 that  $\text{Sym}^2 H$  is the unique representation with maximal twist  $\mathcal{R}^{n+2,0}$ :

*Proposition 5.7. (Killing Vector Fields)*

On every compact quaternionic Kähler manifold  $(M^{4n}, g)$  of positive scalar curvature  $\kappa$  we have:

$$\ker(D_{\mathcal{R}^{n+2,0}}^2) = \ker\left(\Delta_{\text{Sym}^2 H} - \frac{\kappa}{2n}\right).$$

The index of  $D_{\mathcal{R}^{n+2,0}}$  equals the dimension of the isometry group of  $(M, g)$  (cf. [Sal82]). But since  $\text{Sym}^2 H$  is the only representation contributing to  $\ker(D_{\mathcal{R}^{n+2,0}}^2)$  the index is just the dimension of the minimal eigenspace of  $\Delta_{\text{Sym}^2 H}$ . In fact, there is an explicit isomorphism from the space of Killing vector fields to  $\text{Sym}^2 H$  (cf. [AlMa98]). It is given by projecting the covariant derivative of a Killing vector field onto its component in  $\text{Sym}^2 H \subset \Lambda^2 T^* M \otimes_{\mathbb{R}} \mathbb{C}$ .

6. HARMONIC FORMS AND BETTI NUMBERS

This section contains the most important application of Proposition 3.6. We will determine which parallel subbundles of the differential forms may carry harmonic forms and thus prove vanishing theorems for Betti numbers both for positive and negative scalar curvature. These results will lead to quaternionic Kähler analogues of the weak and strong Lefschetz theorem in Kähler geometry. Recall that the weak Lefschetz theorem for Kähler manifolds  $M$  states the inequality  $b_k \leq b_{k+2}$  of the Betti numbers for  $k < \frac{1}{2} \dim M$ , whereas the strong Lefschetz theorem asserts that the wedge product with the parallel 2-form descends to an injective map of the cohomology  $H^k(M, \mathbb{R}) \rightarrow H^{k+2}(M, \mathbb{R})$  in the same range.

*Proposition 6.1. (Representations and Harmonic Forms)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of scalar curvature  $\kappa \neq 0$  and let  $\pi$  be an irreducible representation of  $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$  occurring in the forms  $\Lambda^\bullet(H \otimes E)$ :

$$\text{Hom}_{\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)}(\pi, \Lambda^\bullet(H \otimes E)) \neq \{0\}$$

If the scalar curvature is positive then  $\ker \Delta_\pi = \{0\}$  unless  $\pi = \Lambda_{\text{top}}^{a,a} E$  for some  $a$  with  $n \geq a \geq 0$ . Similarly if the scalar curvature is negative then  $\ker(\Delta_\pi) = \{0\}$  unless either  $\pi = \Lambda_{\text{top}}^{a,a} E$  as before or  $\pi$  is a representation of the form  $\pi = \text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$  with  $n \geq a \geq b \geq 0$ .

Although the representations  $\text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$  form a larger class of representations they are still rather special among all the representations occurring in the forms. The appearance of these exceptional representations potentially carrying harmonic forms could have been foreseen from the difficulties encountered in the attempt to push Kraines original strong Lefschetz theorem ([Kra66]) for quaternionic Kähler manifolds beyond degree  $n$ . In higher degrees the given proofs fail precisely for these representations. It follows from Proposition 6.1 that this problem is absent in the positive scalar curvature case.

**Proof:** For any manifold of even dimension the bundle of exterior forms is the tensor product of the spinor bundle  $\mathbf{S}$  with its dual  $\mathbf{S}^*$ . In the quaternionic Kähler case  $\mathbf{S} \cong \mathbf{S}^*$  is real and the decomposition given in Proposition 3.2 implies:

$$\Lambda^*(H \otimes E) = \mathbf{S} \otimes \mathbf{S} = \bigoplus_{r=0}^n \mathbf{S} \otimes \mathcal{R}^{r, n-r}.$$

In particular, a representation  $\pi$  occurs in the forms if and only if it occurs in a twisted spinor bundle  $\mathbf{S} \otimes \mathcal{R}^{r, n-r}$  for some  $r$  with  $n \geq r \geq 0$ . It is consequently of the form  $\pi = \text{Sym}^k H \otimes \Lambda_{\text{top}}^{a,b} E$  for suitable  $k \geq 0$  and  $n \geq a \geq b \geq 0$ . In this situation Proposition 3.6 becomes:

$$\Delta \Big|_{\pi} = \Delta_{\pi} = D_{\mathcal{R}^{r, n-r}}^2 \Big|_{\pi}$$

A harmonic form in the parallel subbundle determined by  $\pi$  is thus identified with an harmonic twisted spinor for the twist  $\mathcal{R}^{r, n-r}$ . However, we have already expressed the kernel of the twisted Dirac operators  $D_{\mathcal{R}^{r, n-r}}^2$  in formula (5.2) at least for positive scalar curvature.

The point in this formula is of course that only those representations  $\pi$  may contribute to the kernel of the twisted Dirac operator  $D_{\mathcal{R}^{r, n-r}}^2$ , for which the twist  $\mathcal{R}^{r, n-r}$  is a maximal twist. Replacing maximal by minimal twists the same argument applies in the case of negative scalar curvature and we conclude that a representation  $\pi$  may carry harmonic forms in the case of negative or positive scalar curvature if and only if it has a minimal or maximal twist respectively of the form  $\mathcal{R}^{r, n-r}$  for some  $r$  with  $n \geq r \geq 0$ . A look at the classification of maximal and minimal twists in Theorems 4.4 and 4.5 completes the proof.  $\square$

We now want to point out a remarkable property of minimal and maximal twists: If a twist  $\mathcal{R}^{l,d}$  is minimal or maximal for a representation  $\pi$  then  $\pi$  always occurs with multiplicity one in the twisted spinor representation  $\mathbf{S} \otimes \mathcal{R}^{l,d}$ . Although this property seems very natural it is obtained only as a corollary of the calculation of the index multiplicities in Theorems 4.4 and 4.5 using all the rather technical calculations of that section. Surely it is tempting to search for a direct argument providing better insight into the nature of this property.

For us this property is very convenient counting the total multiplicity of those representations  $\pi$  in the differential forms, which may carry harmonic forms. In fact for any representation  $\pi$  this total multiplicity is given by:

(6.1)

$$\dim \text{Hom}_{\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)}(\pi, \Lambda^*(H \otimes E)) = \sum_{r=0}^n \dim \text{Hom}_{\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)}(\pi, \mathbf{S} \otimes \mathcal{R}^{r, n-r})$$

However, in the course of the proof of Theorem 6.1 we characterized the representations  $\pi$  potentially carrying harmonic forms in negative or positive scalar curvature by their property of having a minimal or maximal twist respectively of the form

$\mathcal{R}^{r,n-r}$ ,  $n \geq r \geq 0$ . For such a representation  $\pi$  a twist of the form  $\mathcal{R}^{\tilde{r},n-\tilde{r}}$  is minimal or maximal respectively if and only if it is admissible, because in this case  $\phi(r,n-r) = 0 = \phi(\tilde{r},n-\tilde{r})$ .

Consequently for any representation  $\pi$  which may carry harmonic forms the summands on the right hand side of equation (6.1) are all either 0 or 1 and the total multiplicity of  $\pi$  in the differential forms is just the number of different minimal or maximal twists respectively. This number is easily read off from Theorems 4.4 and 4.5 and is part of the following lemma:

*Lemma 6.2. (Embeddings of Harmonic Forms)*

The representation  $\pi = \Lambda_{\text{top}}^{a,a} E$ ,  $n \geq a \geq 0$ , occurs  $n - a + 1$  times in the forms: it occurs with multiplicity one in the forms of degree  $2a, 2a + 4, 2a + 8, \dots, 4n - 2a$ . Similarly the representation  $\pi = \text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$ ,  $n \geq a \geq b \geq 0$ , occurs in the forms of degree  $2n - a + b, 2n - a + b + 2, 2n - a + b + 4, \dots, 2n + a - b$  with multiplicity 1 and  $a - b + 1$  times in total.

**Proof:** We have already calculated the total multiplicity of the representations  $\Lambda_{\text{top}}^{a,a} E$  and  $\text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$  in the differential forms so that it is sufficient to prove the existence of embeddings of these representations into the forms of the claimed degrees. First let us recall the well known general decomposition of the exterior forms  $\Lambda^k(H \otimes E)$  into Schur functors

$$\Lambda^k(H \otimes E) = \bigoplus_{\mathfrak{Y}} \text{Schur}_{\mathfrak{Y}} H \otimes \text{Schur}_{\overline{\mathfrak{Y}}} E$$

where the sum is over all Young tableaux  $\mathfrak{Y}$  of size  $|\mathfrak{Y}| = k$  and  $\overline{\mathfrak{Y}}$  denotes the conjugated Young tableau ([FuHa]). All Schur functors have two preferred realizations as the images of Schur symmetrizers in iterated tensor products. Specifying the Young tableau  $\mathfrak{Y}$  either by the length of its rows  $(r_1, r_2, \dots, r_{c_1})$  or of its columns  $(c_1, c_2, \dots, c_{r_1})$  satisfying  $r_1 \geq r_2 \geq \dots \geq r_{c_1}$  and  $c_1 \geq c_2 \geq \dots \geq c_{r_1}$  these two preferred realizations of the Schur functors

$$\begin{aligned} \text{Schur}_{\mathfrak{Y}} H &\subset \Lambda^{c_1} H \otimes \Lambda^{c_2} H \otimes \dots \otimes \Lambda^{c_{r_1}} H \\ \text{Schur}_{\mathfrak{Y}} E &\subset \text{Sym}^{r_1} E \otimes \text{Sym}^{r_2} E \otimes \dots \otimes \text{Sym}^{r_{c_1}} E \end{aligned}$$

are given as the intersection of the kernels of all possible Plücker differentials. In our case all Schur functors in  $H$  corresponding to Young tableaux of more than two rows vanish and since  $\Lambda^2 H \cong \mathbb{C}$  is trivial the Schur functor in  $H$  for the Young tableau of size  $k$  with two rows  $(k - s, s)$  is equivalent to  $\text{Sym}^{k-2s} H$ :

$$\Lambda^k(H \otimes E) = \bigoplus_{s=0}^{\lfloor \frac{k}{2} \rfloor} \text{Sym}^{k-2s} H \otimes \text{Schur}_{\overline{(k-s,s)}} E.$$

Conjugation of Young tableaux is defined by exchanging rows and columns. Conjugated to the Young tableau with two rows  $(k - s, s)$  is the tableau with two columns

$(k - s, s)$ . Thus Schur  $\overline{(k-s, s)}E$  can be defined as the kernel of the Plücker differential:

$$\sum_{\mu} e_{\mu} \wedge \text{de}_{\mu} \lrcorner : \Lambda^{k-s}E \otimes \Lambda^sE \longrightarrow \Lambda^{k-s+1}E \otimes \Lambda^{s-1}E.$$

From Weyl’s construction of the representation  $\Lambda_{\text{top}}^{a,b}E$  as the intersection of the kernel of the Plücker differential  $\Lambda_{\circ}^aE \otimes \Lambda_{\circ}^bE \longrightarrow \Lambda^{a+1}E \otimes \Lambda_{\circ}^{b-1}E$  with the kernel of the diagonal contraction with the symplectic form we see that  $\Lambda_{\text{top}}^{a,a}E \subset \text{Schur}_{\overline{(a,a)}}E$ . Consider now the map

$$\Omega : \Lambda^aE \otimes \Lambda^bE \longrightarrow \Lambda^{a+2}E \otimes \Lambda^{b+2}E$$

defined by

$$\Omega := \sum_{\mu, \nu} (de_{\mu}^b \wedge de_{\nu}^b \wedge \otimes e_{\mu} \wedge e_{\nu} \wedge + de_{\mu}^b \wedge e_{\mu} \wedge \otimes de_{\nu}^b \wedge e_{\nu} \wedge),$$

which curiously enough commutes with the Plücker differential. Consequently we may extend the above embedding to a chain of  $\mathbf{Sp}(n)$ -equivariant linear maps:

$$\Lambda_{\text{top}}^{a,a}E \longrightarrow \text{Schur}_{\overline{(a,a)}}E \xrightarrow{\Omega} \text{Schur}_{\overline{(a+2, a+2)}}E \xrightarrow{\Omega} \dots \xrightarrow{\Omega} \text{Schur}_{\overline{(2n-a, 2n-a)}}E.$$

Explicit calculation shows that  $\Omega^{n-a} = (2n - 2a + 1)! (\star \otimes \star)$  on  $\Lambda_{\text{top}}^{a,a}E$ , where  $\star$  denotes the Hodge isomorphism  $\Lambda^aE \longrightarrow \Lambda^{2n-a}E$ . Hence  $\Lambda_{\text{top}}^{a,a}E$  embeds into all the Schur functors  $\text{Schur}_{\overline{(a+2s, a+2s)}}E$  with  $n - a \geq s \geq 0$  and further into the forms  $\Lambda^{2a+4s}(H \otimes E)$  of degree  $2a + 4s$  with  $n - a \geq s \geq 0$ . The appearance of the map  $\Omega$  is by no means an accident, it can be shown that it corresponds exactly to the wedge product with the parallel Kraines form  $\Omega$  on the level of forms.

The construction of the different embeddings of the representations  $\text{Sym}^{2n-a-b}H \otimes \Lambda_{\text{top}}^{a,b}E$  is simpler, although it is a dead end to start with the inclusion  $\Lambda_{\text{top}}^{a,b}E \subset \text{Schur}_{\overline{(a,b)}}E$ . Instead we have to use the Hodge isomorphism  $(\star \otimes 1) : \Lambda^aE \otimes \Lambda^bE \longrightarrow \Lambda^{2n-a}E \otimes \Lambda^bE$ , which interchanges in a sense the roles of the Plücker differential and the diagonal contraction with the symplectic form. The Hodge isomorphism can be extended to a chain of maps

$$\Lambda_{\text{top}}^{a,b}E \longrightarrow \Lambda^{2n-a}E \otimes \Lambda^bE \xrightarrow{\sigma} \Lambda^{2n-a+1}E \otimes \Lambda^{b+1}E \xrightarrow{\sigma} \dots \xrightarrow{\sigma} \Lambda^{2n-b}E \otimes \Lambda^aE$$

using the diagonal multiplication  $\sigma$  with the symplectic form. Since diagonal contraction and multiplication with the symplectic form generate an  $\mathfrak{sl}_2$ -algebra of operators the final map  $\Lambda_{\text{top}}^{a,b}E \longrightarrow \Lambda^{2n-b}E \otimes \Lambda^aE$  is injective and maps into the kernel of  $\sigma$ . In addition the commutator relations between the Plücker differential and  $\sigma$  imply that  $\Lambda_{\text{top}}^{a,b}E$  is mapped into the kernel  $\text{Schur}_{\overline{(2n-a+s, b+s)}}E$  of the Plücker differential at each step, so that

$$\text{Sym}^{2n-a-b}H \otimes \Lambda_{\text{top}}^{a,b}E \longrightarrow \text{Sym}^{2n-a-b}H \otimes \text{Schur}_{\overline{(2n-a+s, b+s)}}E \xrightarrow{\subset} \Lambda^{2n-a+b+2s}(H \otimes E)$$

embeds into the forms of degree  $2n - a + b + 2s$  for all  $a - b \geq s \geq 0$ .  $\square$



*Remark 6.3. (Strong Lefschetz Theorems)*

In the course of the proof of Lemma 6.2 we have sketched a proof of the strong Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature. The wedge product with the parallel Kraines form  $\Omega$  is injective on the forms of type  $\Lambda_{\text{top}}^{a,a}E$  in all degrees  $k < \frac{1}{2} \dim M$  and hence descends to an injective map of the cohomology  $H^k(M, \mathbb{R}) \rightarrow H^{k+4}(M, \mathbb{R})$ .

A completely different argument can be given to show that the wedge product with the Kraines form is injective on forms of type  $\text{Sym}^{2n-a-b}H \otimes \Lambda_{\text{top}}^{a,b}E$  in degrees  $k < \frac{1}{2} \dim M - 1$ , too. In contrast to the positive scalar curvature case however, the decomposition of the cohomology given in Proposition 6.1 for quaternionic manifolds of negative scalar curvature is finer than the decomposition into primitive cohomologies with respect to the Kraines form.

The weak Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature was proved by S. Salamon (cf. [Sal82]) by analyzing the cohomology of the twistor space. Applying Proposition 6.1 in combination with Lemma 6.2 we get a more explicit version of this result:

*Proposition 6.4. (Weak Lefschetz Theorem for Positive Scalar Curvature)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ . Its Betti numbers  $b_k$  satisfy for all  $0 \leq k \leq n$  the following relations:

$$\begin{aligned} b_{2k+1} &= 0, \\ b_{2k} &= \sum_{\nu=0}^{\lfloor \frac{k}{2} \rfloor} \dim(\ker \Delta_{\Lambda_{\text{top}}^{k-2\nu, k-2\nu}E}), \\ b_{2k} - b_{2k-4} &= \dim(\ker \Delta_{\Lambda_{\text{top}}^{k,k}E}) \geq 0. \end{aligned}$$

**Proof:** For a compact quaternionic Kähler manifold of positive scalar curvature it follows from Proposition 6.1 that the only representations potentially carrying harmonic forms are  $\Lambda_{\text{top}}^{a,a}E$  with  $n \geq a \geq 0$ . But according to Lemma 6.2 all these representations embed into forms of even degree, i. e. all odd Betti numbers necessarily vanish. Moreover the representations  $\Lambda_{\text{top}}^{a,a}E$  occur in the forms of degree  $2k$  if and only if  $a = k, k - 2, \dots$  and in this case they occur with multiplicity one.  $\square$

*Remark 6.5. (Associated Twistor Space and 3-Sasakian Manifold [GaSa96])*

Let  $\mathcal{S}$  be the 3-Sasakian manifold and  $\mathcal{Z}$  the twistor space associated with the quaternionic Kähler manifold  $M^{4n}$ . The dimension of  $\ker \Delta_{\Lambda_{\text{top}}^{k,k}E}$  can be reinterpreted as the dimension of the cohomology of  $\mathcal{S}$  and as the dimension of the primitive cohomology group of  $\mathcal{Z}$ :

$$\dim(\ker \Delta_{\Lambda_{\text{top}}^{k,k}E}) = b_{2k}(\mathcal{S}) = b_{2k}(\mathcal{Z}) - b_{2k-2}(\mathcal{Z}) \quad k \leq n .$$

As an immediate consequence of Proposition 6.4 we obtain a result of S. Salamon and C. LeBrun (cf. [LeBSa94]) on the index of the twisted Dirac operator  $D_{\mathcal{R}^l, a}$  with  $l + d = n$ :

*Corollary 6.6. (Index of Twisted Dirac Operators and Betti Numbers)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of positive scalar curvature  $\kappa > 0$ :

$$\begin{aligned} \ker(D_{\mathcal{R}^{n-d,d}}^2) &= \bigoplus_{a \leq d} \ker(\Delta_{\Lambda_{\text{top}}^{a,a} E}), \\ \dim \ker(D_{\mathcal{R}^{n-d,d}}^2) &= b_{2d} + b_{2d-2}, \\ \text{index}(D_{\mathcal{R}^{n-d,d}}) &= (-1)^d (b_{2d} + b_{2d-2}). \end{aligned}$$

**Proof:** We already observed in formula (5.2) that in the case of positive scalar curvature a representation  $\pi$  may contribute to the kernel of a twisted Dirac operator  $D_{\mathcal{R}^{l,d}}^2$  only if the twist  $\mathcal{R}^{l,d}$  is maximal for  $\pi$ . On the other hand the twisted spinor representation  $\mathbf{S} \otimes \mathcal{R}^{n-d,d}$  occurs in the forms so that a representation  $\pi$  contributes to the kernel of  $D_{\mathcal{R}^{n-d,d}}^2$  if and only if it carries harmonic forms, i. e.  $\pi$  must be one of the representations  $\Lambda_{\text{top}}^{a,a} E$  for some  $a$  with  $n \geq a \geq 0$ . From equation (4.1) of Theorem 4.1 it is evident that  $\pi = \Lambda_{\text{top}}^{a,a} E$  occurs in  $\mathbf{S} \otimes \mathcal{R}^{n-d,d}$  if and only if  $a \leq d$ . Consequently Proposition 6.4 provides the expression for the dimension of the kernel of  $D_{\mathcal{R}^{n-d,d}}$  in terms of Betti numbers.  $\square$

In dealing with quaternionic Kähler manifolds of negative scalar curvature it is convenient to decompose their cohomology into two direct summands with quite different behavior:

*Definition 6.7. ( $\mathfrak{sp}(1)$ -Invariant and Exceptional Cohomology)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of negative scalar curvature. According to Proposition 6.1 two different series of representations contribute to harmonic forms on  $M$ , namely  $\Lambda_{\text{top}}^{a,a} E$ ,  $n \geq a \geq 0$  and  $\text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$ ,  $n \geq a \geq b \geq 0$ . In particular the de Rham cohomology of  $M$  splits into the direct sum

$$H_{dR}^*(M, \mathbb{C}) = H_{\mathfrak{sp}(1)}^*(M, \mathbb{C}) \oplus H_{\text{expt}}^*(M, \mathbb{C})$$

of its  $\mathfrak{sp}(1)$ -invariant cohomology  $H_{\mathfrak{sp}(1)}^*(M, \mathbb{C})$ , which is the sum of all isotypical components corresponding to the representations  $\Lambda_{\text{top}}^{a,a} E$ ,  $n \geq a \geq 0$ , and its exceptional cohomology  $H_{\text{expt}}^*(M, \mathbb{C})$ , which is the direct sum of all isotypical components corresponding to the remaining representations  $\text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$ ,  $n \geq a \geq b \geq 0$ ,  $b \neq n$ .

Because the curvature tensor of  $M$  is  $\mathfrak{sp}(1)$ -invariant the same is true for all its characteristic classes. Moreover  $H_{\mathfrak{sp}(1)}^*(M, \mathbb{C})$  is closed under multiplication and the decomposition of the de Rham-cohomology into  $\mathfrak{sp}(1)$ -invariant and exceptional cohomology is respected by the induced modul structure. A deeper analysis of the ring structure of the cohomology ring of  $M$  will be given in a forthcoming paper (cf. [Wei00]).

As a final application of the ideas developed in this article we combine Proposition 6.1 and Lemma 6.2 to obtain new information on the Betti numbers of compact quaternionic Kähler manifolds of negative scalar curvature.

*Proposition 6.8. (Weak Lefschetz Theorem for Negative Scalar Curvature)*

Let  $(M^{4n}, g)$  be a compact quaternionic Kähler manifold of negative scalar curvature  $\kappa < 0$ . Its  $\mathfrak{sp}(1)$ -invariant and exceptional Betti numbers  $b_{\mathfrak{sp}(1),k}$  and  $b_{\text{expt},k}$  satisfy:

$$\begin{aligned} b_{\mathfrak{sp}(1),k} &= 0 && \text{for } k \text{ odd,} \\ b_{\text{expt},k} &= 0 && \text{for } k \leq n-1, \\ b_{\mathfrak{sp}(1),k} &\leq b_{\mathfrak{sp}(1),k+4} && \text{for } k \leq 2n-2, \\ b_{\text{expt},k} &\leq b_{\text{expt},k+2} && \text{for } k \leq 2n-1. \end{aligned}$$

In particular, its Betti numbers  $b_k = b_{\mathfrak{sp}(1),k} + b_{\text{expt},k}$  satisfy:

$$\begin{aligned} b_{2k+1} &= 0 && \text{for } 2k+1 \leq n-1, \\ b_k &\leq b_{k+2} && \text{for odd } k \leq 2n-1, \\ b_k &\leq b_{k+4} && \text{for } k \leq 2n-2. \end{aligned}$$

**Proof:** Since the  $\mathfrak{sp}(1)$ -invariant Betti numbers correspond by definition to the representations  $\Lambda_{\text{top}}^{a,a} E$ ,  $n \geq a \geq 0$ , they have the same properties as Betti numbers of a quaternionic Kähler manifolds of positive scalar curvature given in Proposition 6.4.

It follows from Lemma 6.2 that the remaining representations  $\text{Sym}^{2n-a-b} H \otimes \Lambda_{\text{top}}^{a,b} E$  with  $n \geq a \geq b \geq 0$  and  $b \neq n$  corresponding to the exceptional Betti numbers embed into forms of degree  $2n-a+b$ ,  $2n-a+b+2$ ,  $\dots$ ,  $2n+a-b$ . For  $a \not\equiv b \pmod{2}$  these embeddings give rise to harmonic forms of odd degree. Nevertheless the odd Betti numbers of degree less than  $n$  have to vanish because of  $2n-a+b \geq n$ .  $\square$

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**WEAKENING HOLONOMY**

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These notes are based on a talk I gave at the Erwin Schrödinger International Institute for Mathematical Physics, Vienna, on the 20th October, 1999. This is work in progress, partly based on joint work with F. M. Cabrera and M. D. Monar and partly results of my Ph. D. student Richard Cleyton. It is a pleasure to thank the Erwin Schrödinger Institute and the organisers of the program on Holonomy Groups in Differential Geometry for their kind hospitality.

1. INTRODUCTION

Suppose  $(M, g)$  is a Riemannian manifold. One fundamental piece of data determined by  $g$  is the restricted holonomy group  $Hol$ . If we assume that  $Hol$  acts irreducibly on  $TM$ , which is the case if  $M$  is complete and irreducible, then the main classification theorem implies that either  $(M, g)$  is locally isometric to a symmetric space  $K/Hol$  or  $Hol$  is one of  $SO(n)$ ,  $U(n)$ ,  $SU(n)$ ,  $Sp(n)$ ,  $Sp(n)Sp(1)$ ,  $G_2$  or  $Spin(7)$  (see [3]). Studying the geometries determined by these holonomy groups one finds that if  $M \neq SO(n)$  or  $U(n)$ , then  $g$  is automatically Einstein. This may be restated as follows.

*Theorem 1.1.* Suppose  $G$  is a proper connected subgroup of  $SO(n)$  that acts irreducibly on  $\mathbb{R}^n$  and if  $n$  is even suppose that  $G \neq U(n/2)$ . Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with structure group  $G$ . If  $M$  admits a torsion-free  $G$ -connection then  $g$  is Einstein.

A natural question is:

Are there weaker conditions than the existence of a torsion-free connection that imply useful restrictions on the curvature?

In 1971, Gray [9] provided one such notion which he called "weak holonomy". He studied this idea for the groups  $G$  that act transitively on the sphere. For the groups  $SO(n)$ ,  $Sp(n)$ ,  $Sp(n)U(1)$ ,  $Sp(n)Sp(1)$  and  $Spin(7)$ , the weak holonomy condition implies that the holonomy group reduces and we obtain no new geometries. However,

Group	Geometry
$U(n), SU(n)$	Nearly Kähler, i.e., $(\nabla_X J)(X) = 0$ Einstein if $n = 3$
$G_2$	$d\varphi = \lambda*\varphi, \lambda \neq 0$ Einstein with $s > 0$
$Spin(9)$	Not Einstein

TABLE 1. The geometries determined by weak holonomy groups acting transitively on a sphere when the holonomy does not reduce to that group.

for  $U(n), SU(n)$  and  $G_2$  Gray found that the weak holonomy condition does give new structures. Very recently Th. Friedrich has shown that the group  $Spin(9)$  also occurs as a weak holonomy group [7]. These results are summarised in Table 1.

As the table indicates, the only new examples of Einstein structures are provided by nearly Kähler six-manifolds and seven-dimensional manifolds with weak holonomy  $G_2$ . Many examples of the latter are known. For example each Aloff-Wallach space  $SU(3)/U(1)_{k,\ell}$ , given by embedding  $U(1)$  in  $SU(3)$  via

$$\exp(i\theta) \mapsto \text{diag}(\exp(ik\theta), \exp(i\ell\theta), \exp(-i(k + \ell)\theta)),$$

carries a homogeneous metric with weak holonomy  $G_2$  [1]. Also non-homogeneous examples can be constructed from non-homogeneous 3-Sasakian metrics in dimension 7 by using the results of [8]. In the case of nearly Kähler six-manifolds that are not Kähler, the only examples known are 3-symmetric spaces, so these are homogeneous. Moroianu & Semmelmann have a proof that there are no other homogeneous examples [10].

As far as I know Gray’s condition has not been studied for other  $G$ -structures. This may be because his definition is not particularly easy to work with. More in the spirit of Gray’s other work would be to look for  $G$ -structures which admit a connection whose torsion is ‘simple’. This is the approach I wish to take.

## 2. TORSION AND CURVATURE

Fix a closed connected Lie subgroup  $G$  of  $SO(n)$ . Suppose that  $M$  is an  $n$ -dimensional manifold with a reduction of its structure group to  $G$ . Let  $g$  be the corresponding Riemannian metric and write  $\nabla$  for the Levi-Civita connection.

If  $\nabla'$  is any  $G$ -connection on  $M$ , then the difference  $\nabla - \nabla'$  is tensorial and is a one-form with values in (the bundle associated to) the Lie algebra  $\mathfrak{so}(n)$  of  $SO(n)$ . If  $\eta$  is any one-form with values in the Lie algebra  $\mathfrak{g}$  of  $G$ , then  $\nabla' + \eta$  is also a  $G$ -connection. It is now easy to see:

*Lemma 2.1.* If  $G$  is a subgroup of  $SO(n)$  and  $M$  is a Riemannian manifold with a  $G$ -structure, then there is a unique  $G$ -connection  $\tilde{\nabla}$  such that the Levi-Civita connection satisfies

$$\nabla = \tilde{\nabla} + \xi$$

with  $\xi$  an element of  $T^*M \otimes \mathfrak{g}^\perp \subset T^*M \otimes \mathfrak{so}(n)$ .

*Definition 2.2.* We call the connection  $\tilde{\nabla}$  of Lemma 2.1 the *natural metric connection* of the  $G$ -structure.

The torsion of  $\tilde{\nabla}$  is given by  $T^{\tilde{\nabla}}(X, Y) = \xi_Y X - \xi_X Y$ . Moreover, this torsion determines  $\xi$  by

$$g(\xi_X Y, Z) = \frac{1}{2}(g(T^{\tilde{\nabla}}(X, Z), Y) - g(T^{\tilde{\nabla}}(X, Y), Z) + g(X, T^{\tilde{\nabla}}(Y, Z))).$$

We will therefore often abuse terminology and refer to  $\xi$  as the torsion of  $\tilde{\nabla}$ .

Write  $V$  for the representation of  $G$  on  $\mathbb{R}^n$ . Then  $\xi$  is an element of the bundle associated to the representation  $V \otimes \mathfrak{g}^\perp$ . Let us assume that  $\xi$  lies in a subrepresentation  $W \subset V \otimes \mathfrak{g}^\perp$ . It is now possible to deduce some restrictions on the Riemann curvature tensor  $R$  of  $M$ .

The curvature  $R$  of the Levi-Civita connection is an element of  $S^2(\mathfrak{so}(n))$ . Using  $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$  we obtain the decomposition

$$S^2(\mathfrak{so}(n)) = S^2(\mathfrak{g}) \oplus S(\mathfrak{g} \otimes \mathfrak{g}^\perp) \oplus S^2(\mathfrak{g}^\perp),$$

and a corresponding splitting of  $R$ :

$$R = R^{\mathfrak{g}} + R^m + R^\perp.$$

We can refine this decomposition further. Let  $b: S^2(\Lambda^2 V) \rightarrow \Lambda^4 V$  be the map defined by

$$b(\alpha)(X, Y, Z, W) = \alpha(X, Y, Z, W) + \alpha(X, Z, W, Y) + \alpha(X, W, Y, Z).$$

The space  $\mathcal{K}(\mathfrak{g}) := \ker b \cap S^2(\mathfrak{g})$  consists of elements in  $S^2(\mathfrak{g})$  that satisfy the Bianchi identity, and so is the space of algebraic curvature tensors whose holonomy lies in  $G$ .

We write  $S^2(\mathfrak{g}) = \mathcal{K}(\mathfrak{g}) \oplus \mathcal{K}(\mathfrak{g})^\perp$  and use this to make a splitting  $R^{\mathfrak{g}} = R_0^{\mathfrak{g}} + R_1^{\mathfrak{g}}$ . Note that  $b$  is injective on  $\mathcal{K}(\mathfrak{g})^\perp$ , so the fact that  $R$  satisfies the Bianchi identity  $b(R) = 0$  implies that  $R_1^{\mathfrak{g}}$  is uniquely determined by  $R^m + R^\perp$  via

$$b(R_1^{\mathfrak{g}}) = -b(R^m + R^\perp).$$

To obtain information on  $R^m$  and  $R^\perp$ , let us locally choose a tensor  $\varphi$  on  $M$  (not necessarily of pure type) such that

- (a)  $\text{Lie stab}_{SO(n)} \varphi = \mathfrak{g}$ , and
- (b)  $\tilde{\nabla} \varphi = 0$ ,

where  $\text{Listab}_{SO(n)} \varphi$  denotes the Lie algebra of the stabiliser of  $\varphi$  under the action of  $SO(n)$ . One can find such a  $\varphi_0$  satisfying condition (a) at a given point. Suppose  $\varphi_0 \in \bigoplus_{i=1}^s V^{\otimes r_i}$ , for a minimal set of integers  $r_1, \dots, r_s$ . Let  $U_i$  be the trivial submodule of  $V^{\otimes r_i}$ , for  $i = 1, \dots, s$ . Then  $U_i$  defines a subbundle of tensor algebra of  $TM$  and the restriction of  $\tilde{\nabla}$  to  $U_i$  is flat, so we may locally extend  $\varphi_0$  to a tensor  $\varphi$  satisfying (b).

Now consider the action of the curvature  $R$  on  $\varphi$ . We have

$$R.\varphi = R^m.\varphi + R^\perp.\varphi.$$

Moreover,  $R.\varphi$  determines  $R^m$  and  $R^\perp$ . On the other hand

$$\begin{aligned} R.\varphi &= \mathbf{a}(\nabla\nabla\varphi) = \mathbf{a}(\nabla(\tilde{\nabla}\varphi + \xi.\varphi)) = \mathbf{a}(\nabla(\xi.\varphi)) \\ &= \mathbf{a}((\tilde{\nabla}\xi).\varphi) + \mathbf{a}(\xi^2.\varphi), \end{aligned}$$

where  $\mathbf{a}$  denotes the alternation map. From this we see that if  $\xi$  lies in a subrepresentation  $W$  of  $V \otimes \mathfrak{g}^\perp$ , then  $R^m + R^\perp$  lies in the representation  $V \otimes W + W \otimes W$ . Note that  $R_1^\mathfrak{g} \in V \otimes W + W \otimes W$  too, since  $R_1^\mathfrak{g}$  is determined by  $R^m + R^\perp$ .

Write  $S_0^2V$  for the space of trace-free symmetric tensors on  $V$ . Then the trace-free Ricci tensor lies in  $S_0^2V$  and the vanishing of this component of  $R$  is exactly the Einstein condition. The above discussion now implies that  $R$  is Einstein provided  $R_0^\mathfrak{g}$  and  $R_1^\mathfrak{g} + R^m + R^\perp$  are both Einstein. We thus have:

*Theorem 2.3.* Suppose  $M$  is a manifold with structure group  $G \leq SO(n)$ . Let  $V$  denote the representation of  $G$  on  $TM$  and suppose that the torsion  $\xi$  lies in the subrepresentation  $W \subset V \otimes \mathfrak{g}^\perp$ . Then a sufficient condition for  $M$  to be Einstein is

- (a)  $(V \otimes W + W \otimes W) \cap S_0^2V = \{0\}$ , and
- (b) any element of  $\mathcal{K}(\mathfrak{g})$  is Einstein.

The simplest case is when the representation  $W$  is trivial. This occurs precisely when  $\xi$  is invariant under the action of  $G$ .

*Definition 2.4.* We say that the  $M$  is a Riemannian manifold with *invariant torsion* if the structure group of  $M$  reduces to a proper subgroup  $G$  of  $SO(n)$  and the torsion  $\xi$  of the natural metric connection for this  $G$ -structure is invariant under the action of  $G$ .

### 3. EXAMPLES

Let us consider some examples of manifolds with invariant torsion and see how they relate to Theorem 2.3. We begin with Gray's weak holonomy structures that are Einstein.



3.1. **Weak Holonomy  $G_2$ .** We have  $G = G_2$  and  $V$  is the irreducible representation on  $\mathbb{R}^7$ . Then

$$\mathfrak{so}(7) = \Lambda^2 V = \mathfrak{g}_2 \oplus V$$

so  $\mathfrak{g}^\perp = V$  and  $V \otimes \mathfrak{g}^\perp$  certainly contains a trivial representation. In fact

$$V \otimes \mathfrak{g}^\perp = V \otimes V = \mathbb{R} \oplus V \oplus \mathfrak{g}_2 \oplus S_0^2 V$$

as a sum of irreducible modules.

Taking  $\xi \in W = \mathbb{R}$ , we have  $V \otimes W + W \otimes W = V + \mathbb{R}$  which has no subrepresentation in common with  $S_0^2 V$ , which is irreducible. Therefore condition (a) of Theorem 2.3 is satisfied.

For condition (b), we have that  $\mathcal{K}(\mathfrak{g}_2)$  is the algebraic space of curvature tensors of metrics with holonomy  $G_2$ . But all such metrics are Ricci flat and condition (b) holds. (In fact,  $\mathcal{K}(\mathfrak{g}_2)$  is an irreducible representation of dimension 77.) Thus for these  $G_2$ -structures,  $\xi \in \mathbb{R}$  implies that  $M^7$  is Einstein.

The tensor  $\varphi$  in the proof of Theorem 2.3 may be taken to be the fundamental 3-form of the  $G_2$ -structure [4]. The condition that  $\xi$  lies in  $\mathbb{R}$  implies that  $\nabla\varphi = \xi \cdot \varphi$  is an invariant tensor in  $V \otimes \Lambda^3 V$ . But

$$\Lambda^3 V = \mathbb{R} \oplus V \oplus S_0^2 V$$

and so  $V \otimes \Lambda^3 V$  contains a unique invariant summand. This is spanned by the four-form  $\ast\varphi$ , so we have  $d\varphi = \mathbf{a}(\nabla\varphi) = \lambda\ast\varphi$  and the structure has weak holonomy  $G_2$ . Conversely, for  $\lambda \neq 0$ , a metric with weak holonomy  $G_2$  always has invariant torsion  $\xi = c\varphi$ .

3.2. **Nearly Kähler Six-Manifolds.** Let  $U(n)$  act irreducibly on  $V = \mathbb{R}^{2n}$ . Then  $V$  is the real representation underlying  $\Lambda^{1,0}$  and we write  $V = [\Lambda^{1,0}]$ . In this case  $\mathfrak{u}(n)^\perp = [\Lambda^{2,0}]$ , so in order for  $V \otimes \mathfrak{u}(n)^\perp$  to have a trivial summand we need an isomorphism of  $[\Lambda^{1,0}]$  with  $[\Lambda^{2,0}]$ . For the dimensions of these two representations to be equal we have to have  $n = 3$ . However, even in that case the centre of  $U(3)$  acts on these two representations with different weights. We therefore conclude that there is a trivial summand only with respect to the action of  $SU(3)$ .

So we must take  $G = SU(3)$  and  $V = [\Lambda^{1,0}]$ . We then have  $V \otimes \mathfrak{g}^\perp$  contains  $W = 2\mathbb{R}$ . For this choice of  $W$ , condition (a) of Theorem 2.3 is satisfied. Condition (b) is also satisfied, as metrics of holonomy  $SU(3)$  are Ricci-flat. Therefore, an  $SU(3)$ -structure with invariant torsion  $\xi \in 2\mathbb{R}$  is Einstein.

For  $\xi \neq 0$ , these are exactly the nearly Kähler six-manifolds that are not Kähler. Notice that the structure group of such a manifold always reduces from  $U(3)$  to  $SU(3)$  as  $dw + i\ast dw$  trivialises  $\Lambda^{3,0}$ .

3.3. **Holonomy Representations.** Suppose  $G$  acts irreducibly on  $V$  via the holonomy representation of a Riemannian metric. Assume that  $G \neq SO(\dim V)$ . Looking

at each individual case, one can see that the only times where  $\mathfrak{g}^\perp$  contains a copy of  $V$  are (i)  $G = SU(3)$ ,  $V = [\Lambda^{1,0}]$  and (ii)  $G = G_2$ ,  $V = \mathbb{R}^7$ . We thus have:

*Proposition 3.1.* Suppose  $M$  is a Riemannian manifold with non-zero invariant torsion. If the structure group  $G$  acts on  $TM$  via a holonomy representation, then  $M$  is either a six-dimensional nearly Kähler manifold or  $M$  is a seven-dimensional manifold with weak holonomy  $G_2$ .

**3.4. Representations of  $SU(2)$ .** Let us consider the case when  $G = SU(2)$  and  $V$  is an irreducible representation of  $G$ . This implies that  $V \otimes \mathbb{C} = S^k \mathbb{C}^2$ , the  $k$ th symmetric power of  $\mathbb{C}^2$ , for some integer  $k$ . This representation only admits an invariant metric if  $k$  is even. The condition that  $V \otimes \mathfrak{su}(2)^\perp$  contains a trivial representation then implies that  $k \equiv 2 \pmod{4}$  and that  $k \neq 2$ . One can now check that conditions (a) and (b) of Theorem 2.3 are satisfied. Therefore, if they exist, such structures will give an Einstein metric in dimensions  $4r + 3$  for  $r > 0$ .

Two examples can be easily found. For  $r = 1$ , the space  $M^7 = Sp(2)/Sp(1)$ , with  $Sp(1)$  embedded maximally in  $Sp(2)$  has complexified isotropy representation  $S^6 \mathbb{C}^2$  and the only invariant metric is a structure with invariant torsion  $SU(2)$ . Similarly, for  $r = 2$ ,  $M^{11} = G_2/SU(2)$ , again with  $SU(2)$  maximally embedded, is isotropy irreducible and carries an Einstein metric with invariant torsion. We will see later that these are the only examples that arise from this family of representations of  $SU(2)$ .

**3.5. Homogeneous Spaces.** Let  $M = K/G$  be a reductive homogeneous space with  $K$  and  $G$  semi-simple and compact. Write  $\mathfrak{k} = \mathfrak{g} + \mathfrak{p}$ , then  $T_e M = \mathfrak{p}$  and the negative  $\langle \cdot, \cdot \rangle$  of the Killing form induces a positive definite  $\mathfrak{g}$ -invariant inner product on  $\mathfrak{p}$  and hence a Riemannian metric on  $M$ . The canonical connection on  $M$  is a  $G$ -connection with torsion  $\xi(X, Y, Z) = \langle [X, Y], Z \rangle$ , for left-invariant vector fields  $X, Y$  and  $Z$ .

If  $\mathfrak{p}$  is an irreducible  $\mathfrak{g}$ -module then  $M = K/G$  is isotropy irreducible. These spaces have been classified by Wolf [13]. One can check directly that condition (b) is satisfied for all these spaces. However, with  $W = \mathbb{R}$ , it is not always the case that  $V = \mathfrak{p}$  satisfies condition (a), even though  $K/G$  is well-known to be Einstein.

**3.6. Three-Sasakian Manifolds.** 3-Sasakian manifolds give another class of Einstein manifolds with invariant torsion. However, in this neither condition (a) nor condition (b) is satisfied.

#### 4. GENERAL RESULTS

Some general results may be obtained by studying conditions (a) and (b) of Theorem 2.3 in more detail.

First we note that for any representation  $W$  in  $V \otimes \mathfrak{g}^\perp$  will have  $\mathbb{R}$  as a subrepresentation of  $W \otimes W$ . Thus condition (a) implies that  $S_0^2 V$  does not contain a trivial representation. It is straightforward to check that  $V$  is then forced to be irreducible.

For irreducible representations  $V$ , condition (b) is easy to satisfy given the current state of knowledge of the holonomy classification. Let us consider Berger's approach to the holonomy problem [2] as explained by Bryant [5] and Schwachhöfer [11].

*Definition 4.1.* Let  $G$  be a subgroup of  $SO(n)$ . Define the *Berger algebra*  $\underline{\mathfrak{g}}$  of  $G$  by

$$\underline{\mathfrak{g}} = \{ R(X, Y) : R \in \mathcal{K}(\mathfrak{g}), X, Y \in V \}$$

It is easy to show

*Lemma 4.2.* The Berger algebra  $\underline{\mathfrak{g}}$  is an ideal of  $\mathfrak{g}$ , i.e.,  $\underline{\mathfrak{g}} \triangleleft \mathfrak{g}$ , and  $\mathcal{K}(\underline{\mathfrak{g}}) = \mathcal{K}(\mathfrak{g})$ .

Berger two necessary conditions for  $\mathfrak{g}$  to be a holonomy algebra, if  $G$  acts irreducibly on  $\mathbb{R}^n$ . The first is that  $\mathcal{K}(\mathfrak{h})$  should be strictly smaller than  $\mathcal{K}(\mathfrak{g})$  for any proper subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . This may be rephrased as  $\mathfrak{g} = \underline{\mathfrak{g}}$ . The second criteria comes from consideration of the possible covariant derivatives of curvature tensors. It turns out that this second condition merely distinguishes holonomy groups which can only occur for symmetric spaces from the others. The work on existence of metrics with non-symmetric holonomies now implies that each algebra satisfying Berger's first criterion is the holonomy algebra of some torsion-free connection.

*Theorem 4.3.*  $\mathfrak{g}$  is a holonomy algebra of an irreducible Riemannian manifold if and only if  $\underline{\mathfrak{g}} = \mathfrak{g}$ .

We may thus calculate the space  $\mathcal{K}(\mathfrak{g})$  by considering all ideals of  $\mathfrak{g}$  and comparing them with holonomy representations.

*Corollary 4.4.* Suppose  $G$  is a proper subgroup of  $SO(n)$  acting irreducibly on  $V = \mathbb{R}^n$ . Then  $\mathcal{K}(\mathfrak{g})$  consists only of Einstein tensors unless  $n$  is even and  $\mathfrak{g} = \mathfrak{u}(n/2)$ .

*Proof.* If  $G$  is simple then either  $\mathfrak{g}$  is a holonomy algebra or  $\underline{\mathfrak{g}} = \{0\}$  and there is nothing to prove.

Suppose  $G$  is not simple and that  $\{0\} \neq \underline{\mathfrak{g}} \neq \mathfrak{g}$ . If  $\underline{\mathfrak{g}}$  acts irreducibly on  $V$  then the only possibility we have to rule out is  $\underline{\mathfrak{g}} = \mathfrak{u}(n/2)$ , if  $n$  is even. However  $\mathfrak{u}(n/2)$  is maximal in  $\mathfrak{so}(n)$ , and so the fact that the containments  $\underline{\mathfrak{g}} < \mathfrak{g} < \mathfrak{so}(n)$  are strict, rule out this case.

If the representation of  $\mathfrak{h}_1 := \underline{\mathfrak{g}}$  on  $V$  is reducible then  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ . Depending on the type of the representation  $V$ , we may decompose  $V \otimes \mathbb{C}$  as a sum of 1, 2 or 4 tensor products of irreducible  $\mathfrak{h}_i$ -modules over  $\mathbb{C}$ . If  $\mathfrak{h}_i$  is not Abelian, we find that  $V$  is the isotropy representation of a Grassmann symmetric space. The space of curvature tensors for such representations are known and the condition  $\underline{\mathfrak{g}} = \mathfrak{h}_1$  can not be satisfied. If one  $\mathfrak{h}_i$  is Abelian, then a direct calculation shows that there are no non-trivial curvature tensors with values in  $\mathfrak{h}_1$ .  $\square$

We now return to condition (a) of Theorem 2.3. When  $W$  is a trivial representation and  $V$  is irreducible, (a) is equivalent to  $S_0^2 V$  not containing a copy of  $V$ .

*Lemma 4.5.* If  $V$  is irreducible,  $\xi$  lies in a trivial submodule of  $V \otimes \mathfrak{g}^\perp$  and  $S_0^2V$  has no submodule isomorphic to  $V$ , then  $\xi$  is a three-form  $\xi \in \Lambda^3V$ .

*Proof.* If  $V \otimes \mathfrak{g}^\perp$  contains a trivial summand, then we have  $V \subset \mathfrak{g}^\perp \subset \Lambda^2V$ . Now, under the action of  $SO(n)$ , we have

$$(4.1) \quad V \otimes \Lambda^3V = V + \Lambda^3V + U,$$

where  $U$  is irreducible, and we also have

$$(4.2) \quad V \otimes S_0^2V = S^3V + U.$$

Our hypotheses imply that for the action of  $G$ , (4.2) contains no trivial submodules. In particular,  $U^G = \{0\}$ . Therefore, any trivial submodule of (4.1) lies in either  $V$  or  $\Lambda^3V$ . But  $V$  is irreducible, so any trivial module is in  $\Lambda^3V$ .  $\square$

Thus for  $V$  irreducible, condition (a) of Theorem 2.3 forces the torsion to be totally skew. This is interesting, as such a totally skew condition on torsion seems to be natural in physical consideration of for example hyperKähler geometries with torsion [6].

Interestingly, Lemma 4.5 has a converse.

*Proposition 4.6.* If  $M$  is a Riemannian manifold with a  $G$ -structure whose natural metric connection has torsion  $\xi$  and  $\xi$  is a three-form, then  $\tilde{\nabla}\xi$  does not contribute to the curvature of the Levi-Civita connection and condition (a) of Theorem 2.3 can be replaced by

$$(a') \quad W \otimes W \cap S_0^2V = \{0\}$$

*Proof.* The tensor  $\tilde{\nabla}\xi$  is a sum of four tensors  $\psi(X, Y, Z, W)$  which are totally skew in their last three entries. The corresponding element of  $S^2(\Lambda^2V)$  is

$$\begin{aligned} \tilde{\psi}(X, Y, Z, W) &= \psi(X, Y, Z, W) - \psi(Y, X, Z, W) \\ &\quad + \psi(Z, W, X, Y) - \psi(W, Z, X, Y). \end{aligned}$$

Now one can check directly that  $\tilde{\psi}$  is skew in its first two indices and  $\tilde{\psi}(Y, Z, W, X) = -\tilde{\psi}(X, Y, Z, W)$ . Therefore  $\tilde{\psi}$  is a four-form and so orthogonal to the kernel of the Bianchi map  $b: S^2(\Lambda^2V) \rightarrow \Lambda^4V$ .  $\square$

As we have already seen, for  $W$  trivial and  $V$  irreducible, condition (a') of Proposition 4.6 is satisfied. We therefore have:

*Theorem 4.7.* Let  $(M, g)$  be a Riemannian manifold with structure group  $G$  acting irreducibly and for which the natural torsion is invariant and totally-skew. Suppose that the structure group is not  $SO(n)$  or  $U(n/2)$ . Then  $g$  is Einstein.

In certain cases we can show uniqueness of the Einstein metric.

*Theorem 4.8.* Let  $(M, g)$  be a complete Riemannian manifold satisfying the hypotheses of Theorem 4.7 with structure group  $G$  and tangent representation  $V$ . Suppose that the space of invariant three-forms  $(\Lambda^3 V)^G$  on  $M$  is one-dimensional and that the scalar curvature of  $M$  is non-zero. If  $\mathfrak{g} \neq \mathfrak{g}_2$ , then  $M$  is homogeneous and isometric to an isotropy irreducible space.

*Sketch Proof.* If the Berger algebra  $\mathfrak{g}$  is non-trivial then it acts on  $V$  preserving  $\xi$ . Proposition 3.1 then implies that  $\mathfrak{g} = \{0\}$ , as we have specifically excluded the other cases apart from  $\mathfrak{su}(3)$ . But for  $\mathfrak{su}(3)$  the space of invariant three-forms has dimension 2 rather than 1, so this case does not occur.

Now we see that  $R_0^{\mathfrak{g}} = 0$  and by Proposition 4.6  $R^m = 0$ . Thus  $R$  is algebraically determined by the torsion  $\xi$ . Write  $R = R(\xi^2)$ .

As the space of invariant three-forms is one-dimensional, locally the  $\xi$  is proportional to a  $\tilde{\nabla}$ -parallel three-form  $\varphi$ . Write  $\xi = f\varphi$ . Then  $R(\xi^2) = f^2 R(\varphi^2)$ , and in particular the scalar curvature  $s(\xi^2) = f^2 s(\varphi^2)$ . But  $s(\xi^2)$  is constant, as  $g$  is Einstein, and  $s(\varphi^2)$  is constant, since it is parallel for  $\tilde{\nabla}$ . Therefore,  $f$  is constant under the hypothesis that  $s(\xi^2) \neq 0$ .

We thus have that  $\tilde{\nabla}\xi = 0$  and  $\tilde{\nabla}R = 0$ . By definition this means that  $\tilde{\nabla}$  is an Ambrose-Singer connection. Results of Tricerri & Vanhecke [12] imply that  $M$  is a homogeneous space with isotropy group  $\text{stab } R \cap \text{stab } \xi$ . However, this group contains  $G$ , and so  $M$  is isotropy irreducible.  $\square$

*Example 4.9.* One instructive example might be helpful at this point. As mentioned above, the Aloff-Wallach spaces  $M_{k,\ell} = SU(3)/U(1)_{k,\ell}$  carry invariant metrics of weak holonomy  $G_2$ . However, in dimension 7 we also have the isotropy irreducible space  $M^7 = Sp(2)/Sp(1)$  with isotropy representation  $S^6\mathbb{C}^2$ . Theorem 4.8 applies to the  $SU(2)$ -structure of  $M^7$  and shows that this is the only complete metric with invariant torsion.

Now  $G_2$  has a subgroup  $SU(2)$  that acts on the seven-dimensional representation of  $G_2$  as  $S^6\mathbb{C}^2$ . Remarkably, the space of invariant tensors in  $T \otimes \Lambda^2 T^*$  is the same for both groups.

If we look parameters  $k$  and  $\ell$  such that  $M_{k,\ell}$  carries such an  $SU(2)$ -structure we find that topologically the only solution is  $k = 1$  and  $\ell = 4$ . Thus  $M_{1,4}$  has an invariant metric with weak holonomy  $G_2$  and a reduction of the structure group to the seven-dimensional irreducible representation of  $SU(2)$ . Theorem 4.8 implies that with respect to the structure group  $SU(2)$ ,  $M_{1,4}$  can not be a manifold with invariant torsion, even though it has invariant torsion with respect to  $G_2$ . We can see that this is not a contradiction by considering the relations

$$\begin{aligned} \nabla &= \tilde{\nabla}^{\mathfrak{g}_2} + \xi^{\mathfrak{g}_2} \\ &= \tilde{\nabla}^{\mathfrak{su}(2)} + \xi^{\mathfrak{su}(2)} \end{aligned}$$

This implies  $\xi^{\mathfrak{su}(2)} = \xi^{\mathfrak{g}_2} + (\tilde{\nabla}^{\mathfrak{g}_2} - \tilde{\nabla}^{\mathfrak{su}(2)})$ . The last bracket takes values in  $\mathfrak{g}_2 \ominus \mathfrak{su}(2)$  and there is no particular reason for it to vanish. Thus, if  $\xi^{\mathfrak{g}_2}$  is invariant, this will not imply that  $\xi^{\mathfrak{su}(2)}$  is. However, the converse *is* true, and the  $SU(2)$ -structure on  $M^7$  is also a metric of weak holonomy  $G_2$ .

Giving this result it is therefore interesting to find representations  $V$  of  $G$  for which the dimension of  $(\Lambda^3 V)^G$  is at least 2, as these would give some hope of giving non-homogeneous Einstein structures. It is interesting to remark that there are isotropy irreducible spaces that satisfy this condition. For example, if  $G$  is a simple group with Lie algebra not equal to  $\mathfrak{su}(2)$  or  $\mathfrak{sp}(2)$  then the isotropy irreducible space

$$\frac{SO(\dim G)}{G}$$

has at each point a two-dimensional family of invariant three-forms if  $G$  is not of type  $A_n$ ,  $n \geq 3$ , and a four-dimensional family in these remaining cases. There therefore appears to be a second natural three-form for these representations and it would be interesting to determine that and to see whether non-homogeneous Einstein structures can be constructed.

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Second Meeting on  
Quaternionic Structures  
in Mathematics and Physics  
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## MAXWELL'S VISION: ELECTROMAGNETISM WITH HAMILTON'S QUATERNIONS

D. SWEETSER AND G. SANDRI

We present two results that we have not found in the literature and that we believe therefore to be new, and some of their consequences. First, the Maxwell equations and the Lorentz force are formulated with strict use of Hamilton's quaternions (two quaternion field equations and one quaternion force equation). Second, formulas for the Lorentz transformation, in fact for the 15 parameter conformal group, are presented, again with strict use of Hamilton's quaternions.

The first result was expected by Maxwell, but he did not complete this program. He presented the theory as eight field equations in Cartesian coordinates and, of course, did not include the three components of the Lorentz equation of motion. The task of reaching the first of our results has been discussed extensively with the use of biquaternions ("complex quaternions")[3]. While this direction is interesting in itself, we insist in the present work on the strict use of Hamilton's quaternions and prove that they are fully adequate for the task.

The second result of our paper, the formulas for the Lorentz transformations, was attempted by Dirac[2]. Dirac's analysis shows the existence of the subgroup of the algebraic field of quaternions that corresponds to Lorentz transformations in abstract terms, but does not reach explicit formulas. Biquaternions have also been used to characterize the Lorentz group from the early days[1][4].

We give below explicit quaternionic formulas for 3-space rotations, for the proper Lorentz transformations (boosts), and for proper conformal transformations (accelerations). We thus provide one explicit quaternion representation for the 15 parameter conformal group  $C_{15} \sim SO(2,4) \sim SU(2,2)$ . Two observations are in order: (1) our formula for space rotations is not identical but equivalent to the Rodriguez-Hamilton half-angle similarity formula. (2) One can see the proper conformal transformation in Dirac's paper but no explicit mention of the subject is made by him. In 1945, the  $C_{15}$  group was not as prominent in the mind of theoretical physicists as it is now.

The basic idea of our analysis is that any expression that involves three dimensional scalars and/or vectors (Gibbs' "scalar" and "vectors") can be written strictly in terms of Hamilton's quaternions. This "translation rule" extends to 4-scalars and 4-vectors as well as to tensors and interestingly to spinors using on the left-hand

part of the Rodriguez-Hamilton similarity. Imitating Hamilton's formulation of the complex numbers as ordered pairs of reals, we view quaternions as embedded pairs of a Gibbs scalar (= 3-scalar) and a Gibbs 3-vector (= 3 vector). The formulation of classical electrodynamics which is the current standard, for example in the text of Stratton and Jackson, is based on Gibbs' vector algebra, two 3-vector and two 3-scalar field equations plus one 3-vector equation of motion. Consequently, our solution to both problems, the formulation of electrodynamics and the formulation of the Lorentz transformations, in strict quaternion terms, consists of a strict "translation" of Gibbs' formulas into Hamilton's counterparts.

Immediate consequences of our results are quaternion formulas for charge and energy conservation. These two basic conservation laws are based on a quaternion current and on a Poynting's quaternion, respectively.

We now give the essential formulas.

### 1. Quaternions

A quaternion

$$q = (s, \vec{V})$$

The conjugate

$$q^* = (s, -\vec{V})$$

The 3-scalar

$$s = Sc(q)$$

The 3-vector

$$\vec{V} = Vect(q)$$

The anti-commutator

$$\{q_1, q_2\} = q_1 q_2 + q_2 q_1$$

The commutator

$$[q_1, q_2] = q_1 q_2 - q_2 q_1$$

### 2. Fields ( $c = 1$ )

The grad

$$\square = (\partial_t, \vec{\nabla})$$

The potential

$$A = (\phi, \vec{A})$$

The electric field

$$E = (0, \vec{E}) = \frac{1}{2} Vect(\{\square^*, A^*\})$$

The magnetic field

$$B = (0, \vec{B}) = \frac{1}{2} [\square^*, A^*]$$

The current density

$$J = (\rho, \vec{J})$$

### 3. The Maxwell Equations

The homogeneous Maxwell equations

$$\{\square, B\} + [\square, E] = 0$$

The inhomogeneous Maxwell equations

$$([\square, B] - \{\square, E\})/2 = 4\pi J$$

Electric charge conservation

$$Sc(\square^* \frac{1}{2}([\square, B] - \{\square, E\})) = Sc(\square^* 4\pi J)$$

$$0 = \partial_t \rho + \vec{\nabla} \cdot \vec{J}$$

Poynting's quaternion

$$Sc(E^* \frac{1}{2}([\square, B] - \{\square, E\})) = Sc(E^* 4\pi J)$$

$$\vec{\nabla} \cdot (\vec{B} \times \vec{E}) - \frac{1}{2} \partial_t^2 \vec{E} - \frac{1}{2} \partial_t^2 \vec{B} = 4\pi \vec{E} \cdot \vec{J}$$

The Lorentz force

let  $e$  = electric charge

$\beta$  = relativistic velocity,  $v/c$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$u = \gamma(1, \vec{\beta})$$

$$F = \frac{e}{2}(\{u^*, E\}^* + [u, B])$$

#### 4. Transformations

Space rotations

let  $\hat{e}$  = unit rotational 3-vector

$$R : q \rightarrow q' = q \cos \Theta + \frac{1}{4} \{ \{ \hat{e}, q \}, \hat{e} \} (\cos \Theta - 1) - \frac{1}{2} \{ \hat{e}, q \} \sin \Theta$$

Lorentz transformations

let  $\hat{v}$  = unit velocity 3-vector

$$\Lambda : q \rightarrow q' = q + \frac{1}{4}(\gamma - 1) \{ \{ \hat{v}, q \}, \hat{v} \} + \frac{1}{2} \gamma \{ \vec{\beta}^*, q^* \}$$

Note: this has an antilinear term

Proper conformal (accelerations)

Let  $b$  = the acceleration parameter

$$N = \sqrt{1 + 2Sc(bq) + Sc(q^2)Sc(b^2)}$$

$$C : q \rightarrow q' = N(q + bSc(q^2))$$

Among the many formulations of electrodynamics known, an obvious competitor to quaternions is the Minkowski tensors in terms of succinctness. Both formulations consist of two field equations and a single force equation. In addition, both utilize a single current and a single potential. Quaternions have a simpler version of Poynting's Theorem but Minkowski extends to  $n$ -dimensional manifolds. We do not wish to express a preference in applications, but we believe that the quaternion formulation opens the way to asking novel and interesting questions such as the meaning of the antilinear terms in the Lorentz transformation and in the conservation laws.

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## SPECIAL KÄHLER GEOMETRY

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ABSTRACT. The geometry that is defined by the scalars in couplings of Einstein–Maxwell theories in  $N = 2$  supergravity in 4 dimensions is denoted as special Kähler geometry. There are several equivalent definitions, the most elegant ones involve the symplectic duality group. The original construction used conformal symmetry, which immediately clarifies the symplectic structure and provides a way to make connections to quaternionic geometry and Sasakian manifolds.

### 1. INTRODUCTION

In the previous workshop in this series on quaternionic geometry, B. de Wit and me gave talks [1] on the classification of quaternionic homogeneous spaces [2]. Results in special geometry had lead to new homogeneous quaternionic spaces. We have discussed on this topic further with D. Alekseevsky and V. Cortés<sup>1</sup>, and realised that it would be useful to have a definition of special Kähler geometry that does not refer to the constructions of supersymmetric actions. The text of the proceedings was a first step in that direction. Meanwhile, in 1994, the second superstring revolution took place. The main issue was that theories which were previously thought as different, are recognized as perturbations around ‘vacua’ of a master theory. Essential for that are the duality relations which make the connections between the different descriptions. The first example was provided by Seiberg and Witten [4]. They used a model with  $N = 2$  supersymmetry in 4 dimensions with vector multiplets, being multiplets involving Maxwell fields. Special Kähler geometry [5] is defined by the couplings of the scalars in the locally supersymmetric theory, i.e. in the coupled Einstein–Maxwell theory. The model used by Seiberg–Witten thus involves a similar geometry, which has been called rigid special Kähler geometry [6], as it appears in rigid supersymmetry. The structure of that geometry was important for the obtained results. In particular the analyticity properties of fields in these theories allowed them to find exact solutions.

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<sup>1</sup>V. Cortés made our results more accessible to the mathematical audience [3]

The so-called vector multiplets in  $d = 4$ ,  $N = 2$  supersymmetry are multiplets with spins  $(0, 0, \frac{1}{2}, \frac{1}{2}, 1)$ , the latter being the vector providing the Maxwell theory. The scalars are moduli, whose values parametrize the different vacua. The two (real) scalars in a multiplet can be combined to a complex one, and the supersymmetry will indeed provide a complex structure. As will become clear below, the structure of special Kähler geometry implies holomorphicity of the resulting field equations. Then the result of Seiberg–Witten is based on the fact that singularities and the asymptotic behaviour determine exact answers. The singularities, see figure 1, are points around

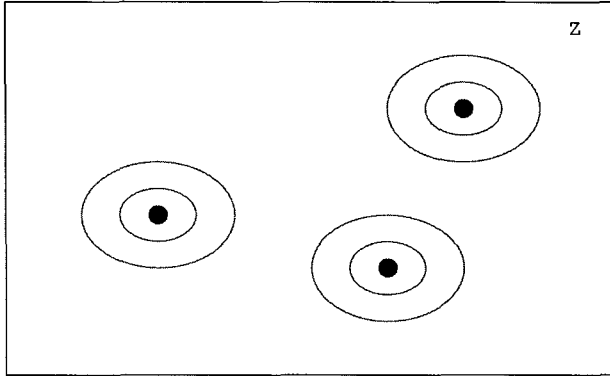


FIGURE 1. *The moduli space with 3 singularities.*

which a classical limit can be considered. The theory allows perturbation expansions around these points. Each one leads classically to a different theory, but there is only one full quantum theory. The singular points form a family of inequivalent vacua.

These developments motivated us to look for a definition of special geometry independent of supersymmetry. A first step in that direction had meanwhile been taken by Strominger [7]. He had in mind the moduli spaces of Calabi–Yau spaces. His definition is already based on the symplectic structure, which we also have emphasized. However, being already in the context of Calabi–Yau moduli spaces, his definition of special Kähler geometry omitted some ingredients that are automatically present in any Calabi–Yau moduli space, but have to be included as necessary ingredients in a generic definition. Another important step was obtained in [8]. Before, special geometry was connected to the existence of a holomorphic prepotential function  $F(z)$ . The special Kähler manifolds were recognized as those for which the Kähler potential can be determined by this prepotential, in a way to be described below. However, in [8] it was found that one can have  $N = 2$  supergravity models coupled to Maxwell multiplets such that there is no such prepotential. These models were constructed by applying a symplectic transformation to a model with prepotential. This fact raised

new questions: are all the models without prepotential symplectic dual to models with a prepotential? Can one still define special Kähler geometry starting from the models with a prepotential? Is there a more convenient definition which does not involve this prepotential? These questions have been answered in [9], and are reviewed here.

Section 2 introduces some ingredients. I give some elements of the algebraic context of  $N = 2$  supersymmetry, and how the geometric quantities are encoded in the action. Then I show the emergence of symplectic transformations in the actions with vector fields coupled to scalars. Rigid  $N = 2$  supersymmetry and the associated rigid special Kähler geometry is discussed in section 3. Section 4 will then discuss the supergravity case. For that, it is useful to look first at the conformal group, as a formulation from that perspective will show more structure, in particular it clarifies the role of the symplectic transformations, and gives the connection with Sasakian manifolds. This is the central section where the definitions, their equivalence and some examples are discussed. The special Kähler manifolds appear in moduli spaces of Riemann surfaces for the rigid version and in those of Calabi–Yau manifolds for the local version. That is illustrated in section 5. A summary is given in section 6. We briefly discuss there also the usage of the same construction methods for quaternionic geometry as recently applied in [10].

## 2. INGREDIENTS

For supersymmetry in 4-dimensional spacetime, the fermionic charges should belong to a spinor representation of  $SO(3, 1)$ . Therefore, in the minimal supersymmetric case, the supercharges have 4 real components. This minimal situation is called  $N = 1$ . Field theory allows realizations up to  $N = 8$  supersymmetry, i.e. with 32 real supercharges. Special Kähler geometry appears in the context of  $N = 2$  supersymmetry. The 8 real spinor supercharges are denoted as  $Q_\alpha^i$ , where  $\alpha = 1, \dots, 4$  and  $i = 1, 2$ . They satisfy the anticommutation rule

$$(2.1) \quad \{Q_\alpha^i, Q_\beta^j\} = \gamma_{\alpha\beta}^\mu P_\mu \delta^{ij},$$

thus involving the translation operator  $P_\mu$  in 4-dimensional spacetime. There are representations with spins

$$(2.2) \quad \begin{aligned} (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}) &: \text{ hypermultiplet } \quad \text{quaternionic scalars} \\ (0, 0, \frac{1}{2}, \frac{1}{2}, 1) &: \text{ vector multiplet } \quad \text{complex scalars} \\ (1, \frac{3}{2}, \frac{3}{2}, 2) &: \text{ supergravity } \quad , \end{aligned}$$

where I have indicated their names and the types of scalars. The quaternionic and complex structures are guaranteed by the supersymmetry.

The ingredients of the geometry are found in the action. In general, having scalars  $\phi^i(x)$ , vectors with field strength  $\mathcal{F}_{\mu\nu}^I(x)$ , and possibly a non-trivial spacetime metric

$g_{\mu\nu}(x)$ , the bosonic kinetic part of the action has the general form

$$\begin{aligned}
 S &= \int d^4x \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j G_{ij}(\phi) \\
 &\quad + \frac{1}{4} \sqrt{g} g^{\mu\rho} g^{\nu\sigma} (\Im \mathcal{N}_{IJ})(\phi) \mathcal{F}_{\mu\nu}^I \mathcal{F}_{\rho\sigma}^J - i \delta(\Re \mathcal{N}_{IJ})(\phi) \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu}^I \mathcal{F}_{\rho\sigma}^J \\
 (2.3) \quad &\quad + \dots
 \end{aligned}$$

$G_{ij}(\phi)$  is identified as the metric of the manifold of scalars. The complex symmetric matrix  $\mathcal{N}_{IJ}$  determines the kinetic terms of the vectors, and its meaning will be clarified below.

Supersymmetry relates bosons and fermions, e.g. for the scalars

$$(2.4) \quad \delta\phi^i(x) = \bar{\epsilon} \chi^i(x),$$

where  $\epsilon$  are the supersymmetry parameters and  $\chi^i(x)$  are the fermions. In the context of local supersymmetry the parameters depend on spacetime, and we thus have

$$(2.5) \quad \delta\phi^i(x) = \overline{\epsilon(x)} \chi^i(x).$$

In order to have an action invariant under these local symmetries, one needs connection fields, which are the gravitini for the supersymmetry. Due to the algebra (2.1) this should be related to local translations, i.e. general coordinate transformations, whose connection field is the (spin 2) graviton.

A prerequisite to understand the following development, is the understanding of the meaning of the symplectic transformations. These are the duality symmetries of 4 dimensions, the generalizations of the Maxwell dualities. They were first discussed in [11]. Consider the kinetic terms of the vector fields as in (2.3) with  $I = 1, \dots, m$ .  $\mathcal{N}_{IJ}$  are coupling constants or functions of scalars. One defines (anti)selfdual combinations as

$$(2.6) \quad \mathcal{F}_{\mu\nu}^\pm = \frac{1}{2} (\mathcal{F}_{\mu\nu} \pm \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}).$$

The conventions<sup>2</sup> are such that the complex conjugate of  $\mathcal{F}^+$  is  $\mathcal{F}^-$ . Defining

$$(2.7) \quad G_{+I}^{\mu\nu} \equiv 2i \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{+I}} = \mathcal{N}_{IJ} \mathcal{F}^{+J \mu\nu},$$

the Bianchi identities and field equations can be written as

$$\begin{aligned}
 (2.8) \quad \partial^\mu \Im \mathcal{F}_{\mu\nu}^{+I} &= 0 && \text{Bianchi identities} \\
 \partial_\mu \Im G_{+I}^{\mu\nu} &= 0 && \text{Equations of motion.}
 \end{aligned}$$

This set of equations is invariant under  $GL(2m, \mathbb{R})$ :

$$(2.9) \quad \begin{pmatrix} \tilde{\mathcal{F}}^+ \\ \tilde{G}_+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix}.$$

<sup>2</sup>The Levi-Civita symbol has  $\varepsilon_{0123} = i$ .



In order that this transformation be consistent with (2.7), we should have

$$\begin{aligned}
 \tilde{G}^+ &= (C + D\mathcal{N})\mathcal{F}^+ = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\tilde{\mathcal{F}}^+ \\
 (2.10) \quad &\rightarrow \boxed{\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}}
 \end{aligned}$$

However, this matrix should remain symmetric,  $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}^T$ , which implies that

$$(2.11) \quad \mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R}),$$

as the explicit condition is

$$(2.12) \quad \mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

Thus the remaining transformations are real symplectic ones in dimension  $2m$ , where  $m$  is the number of vector fields.

In the following we will denote by symplectic vectors, those vectors  $V$  such that its symplectic transformed is  $\tilde{V} = \mathcal{S}V$ . The prime example is thus  $V = \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix}$ . An invariant inner product of symplectic vectors is defined by

$$(2.13) \quad \langle V, W \rangle \equiv V^T \Omega W.$$

The important properties for the matrix  $\mathcal{N}$  is that it should be symmetric and  $\Im \mathcal{N} < 0$  in order to have positive kinetic terms. These properties are preserved under symplectic transformations defined by (2.10).

### 3. RIGID SPECIAL KÄHLER GEOMETRY

As mentioned in the introduction, the ‘rigid’ special Kähler geometry is the geometric structure encountered in rigid  $N = 2$  supersymmetry in 4 dimensions. This supersymmetry has as field representations multiplets with spins  $(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$ , the hypermultiplet, and multiplets with spin  $(0, 0, \frac{1}{2}, \frac{1}{2}, 1)$ , the vector multiplet. For the former, the scalar field geometry is based on quaternions, and is a hyper-Kähler structure. Here, we will consider the vector multiplets, for which the scalars combine to complex fields, whose geometry is Kählerian. A natural description for such multiplets uses  $N = 2$  superspace, that is an extension of usual spacetime (with points labelled by  $x$ ) by fermionic coordinates  $\theta$ , such that the superspace is a representation of the superalgebra. The vector multiplets are then described by superfields  $\Phi^A(x, \theta)$  that satisfy some constraints, restricting the way in which they depend on the  $\theta$ . The result is some superfield

$$(3.1) \quad \Phi^A(x, \theta) = X^A(x) + \bar{\theta} \chi_\alpha^A(x) + \bar{\theta} \gamma^{\mu\nu} \theta \mathcal{F}_{\mu\nu}(x) + \dots,$$

where the lowest components  $X^A$  are complex fields.  $A = 1, \dots, n$  labels different vector multiplets. To build an action, one integrates a general holomorphic function  $F$  over one half of the  $\theta$  variables (the chiral superspace). The above mentioned

constraints have, between other restrictions, restricted the superfields to depend only on this chiral superspace. With

$$(3.2) \quad S = \int d^4x \int d^4\theta F(\Phi) + c.c. ,$$

one obtains that the scalars have a metric of Kählerian type:

$$(3.3) \quad \begin{aligned} G_{A\bar{B}}(X, \bar{X}) &= \partial_A \partial_{\bar{B}} K(X, \bar{X}) \\ K(X, \bar{X}) &= i(\bar{F}_A(\bar{X})X^A - F_A(X)\bar{X}^A) \\ \mathcal{N}_{AB} &= F_{AB} , \end{aligned}$$

where the latter defines the kinetic term of the vectors as in (2.3). Further,  $F_A(X) = \frac{\partial}{\partial X^A} F(X)$  or  $\bar{F}_A(\bar{X}) = \frac{\partial}{\partial \bar{X}^A} \bar{F}(\bar{X})$ ,  $F_{AB}(X) = \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^B} F(X)$ .

The equations of motion turn out to be those equations that determine that  $F_A(\Phi)$  satisfy the same superfield constraints as  $\Phi^A$ . Comparing with (2.8), the superfield constraints on  $\Phi^A$  contain the first equations (Bianchi identities) while the same superfield equations on  $F_A$  contain the second line.

It is therefore appropriate to combine the superfield in a ‘symplectic vector’

$$(3.4) \quad \tilde{\Phi} = \begin{pmatrix} \Phi^A \\ F_A(\Phi) \end{pmatrix} \quad \begin{array}{l} \text{chiral superfields which} \\ \text{satisfy extra constraints.} \end{array}$$

The scalars, i.e. the  $\theta = 0$  part of this vector form also a symplectic vector

$$(3.5) \quad V \equiv \begin{pmatrix} X^A \\ F_A(X) \end{pmatrix} \quad \text{is a symplectic vector.}$$

A further improvement is to allow general coordinates. So far, we parametrize the scalars as  $X^A$ , which are special coordinates (occurring in the superfields). We can, however, allow arbitrary coordinates [12]  $z^\alpha$  with  $\alpha = 1, \dots, n$ . Then the special coordinates are holomorphic functions of the  $z^\alpha$ , i.e.  $X^A(z^\alpha)$ , such that  $e_\alpha^A \equiv \partial_\alpha X^A(z)$  is invertible.

Now we have all the ingredients to give definitions [9].

**Definition 1 of rigid special Kähler geometry.**

A rigid special Kähler manifold is an  $n$ -dimensional Kähler manifold with on any chart  $n$  holomorphic functions  $X^A(z)$  and a holomorphic function  $F(X)$  such that

$$(3.6) \quad K(z, \bar{z}) = i \left( X^A \frac{\partial}{\partial \bar{X}^A} \bar{F}(\bar{X}) - \bar{X}^A \frac{\partial}{\partial X^A} F(X) \right) .$$

On overlap of charts these functions should be related by (inhomogeneous) symplectic transformations  $ISp(2n, \mathbb{R})$ :

$$(3.7) \quad \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(i)} = e^{ic_{ij}} M_{ij} \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(j)} + b_{ij} ,$$

with

$$(3.8) \quad c_{ij} \in \mathbb{R} ; \quad M_{ij} \in Sp(2n, \mathbb{R}) ; \quad b_{ij} \in \mathbb{C}^{2n} ,$$

satisfying the cocycle condition on overlaps of 3 charts.

There is, however, a second definition of rigid special Kähler manifolds, which is based on the symplectic structure, rather than on the prepotential.

**Definition 2 of rigid special Kähler geometry.**

A Kähler manifold is the base manifold of a  $U(1) \times ISp(2n, \mathbb{R})$  bundle. A holomorphic section  $V(z)$  defines the Kähler potential by

$$(3.9) \quad K(z, \bar{z}) = i \langle V, \bar{V} \rangle ,$$

and it should satisfy the constraint

$$(3.10) \quad \langle \partial_\alpha V, \partial_\beta V \rangle = 0 .$$

One can show that the prepotential exists locally, but it is thus not essential for the definition. In rigid special geometry the choice of definition is rather a question of esthetics. However, in the local case, it will be important to have the analogue of the second definition available. The kinetic matrix for the vectors is

$$(3.11) \quad \mathcal{N}_{AB} = (\partial_\alpha F_A(z)) e_B^\alpha .$$

The condition (3.10) guarantees that this matrix is symmetric. Finally, let us remark that the symplectic metric  $\Omega$  should in general not assume the canonical form (2.12), but can be an arbitrary non-degenerate real antisymmetric matrix. However, in order to distinguish  $X^A$  and  $F_A$  components, and thus to write a prepotential, one should bring it to this canonical form.

4.  $N = 2$  SUPERGRAVITY AND SPECIAL KÄHLER GEOMETRY

In this section we introduce the ‘local’ special Kähler geometry, which is the one generally denoted as special Kähler geometry. It is this one which was found in [5], and has most interesting applications. It was introduced in the context of supergravity. To explain its structure, it is useful to consider again its origin.

To describe a supergravity theory, there are several methods. One of them is the introduction of a **superspace**. This formalism shows a lot of structure of the theory. It is very transparent for rigid supersymmetry. However, in its local version, necessary for supergravity, there appear a lot of extra superfield symmetries. These symmetries are an artifact of the formalism. They have to be gauge-fixed to obtain the physical theory.

**Superconformal tensor calculus** is in-between. Also here extra gauge symmetries occur, and these are in fact the symmetries of the superconformal group, the basic ingredient of the formalism. The experience tells us that these symmetries are the relevant ones to display the structure of the theory, but this formalism does not have the many other symmetries present in the superspace approach. It turns out that we just remain with those that are useful to get insight in complicated formulae. Also

for the calculation of the action, the superconformal symmetries are just appropriate to simplify the construction. This is particularly interesting in our case. The superconformal tensor calculus gives the proper setup for the symplectic (duality-adapted) formulation.

The idea is to start by constructing an action invariant under superconformal group. Then, one choose gauges for the extra gauge invariances of the superconformal group, such that the remaining theory has just the super-Poincaré symmetries.

The formalism can be used for theories in various dimensions and amount of supersymmetry. Let us review here the structure for 4 dimensions with 8 real supersymmetry generators ( $N = 2$ ). The superconformal group contains first of all the conformal group (translations, Lorentz rotations, dilatations and special conformal transformations). This group is  $S0(4, 2) = SU(2, 2)$ . The supersymmetries should sit in a spinor representation of this group. This singles out the supergroup  $SU(2, 2|2)$ , which means essentially that the group can be represented by matrices of the form

$$(4.1) \quad \begin{pmatrix} SU(2, 2) & SUSY \\ SUSY & SU(2) \times U(1) \end{pmatrix} .$$

The off-diagonal blocks are the fermionic symmetries. The diagonal blocks are the bosonic ones. They split up in the above-mentioned conformal group and an ‘R-symmetry group’,  $SU(2) \times U(1)$ . This extra group plays an important role:

- the gauge connection of  $U(1)$  will be the Kähler curvature. It acts on the manifold of scalars in vector multiplets,
- the gauge connection of  $SU(2)$  promotes the hyperKähler manifold of hypermultiplets to a quaternionic manifold.

As we neglect here the hypermultiplets, we have to consider the basic supergravity multiplet and the vector multiplets. The physical content that one should have (from representation theory of the super-Poincaré group) can be represented as follows:

$$(4.2) \quad \begin{array}{ccc} \text{SUGRA} & & \text{vectorm.} \\ & 2 & \\ & \frac{3}{2} & \frac{3}{2} \\ & 1 & \\ & & +n * \quad \begin{array}{cc} 1 & \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array} & \rightarrow n + 1 \end{array}$$

The supergravity sector contains the graviton, 2 gravitini and a so-called graviphoton. That spin-1 field gets, by coupling to  $n$  vector multiplets, part of a set of  $n + 1$  vectors, which will be uniformly described by the special Kähler geometry. The scalars appear as  $n$  complex ones  $z^\alpha$ , with  $\alpha = 1, \dots, n$ .

To describe this, we start with  $n + 1$  superconformal vector multiplets with scalars  $X^I$  with  $I = 0, \dots, n$ . The action is determined by a holomorphic function  $F(X)$ . Compared with the rigid case, there is one additional requirement. The conformal

invariance requires  $F(X)$  to be homogeneous of weight 2, where the  $X$  fields carry weight 1. These scalar fields transform also under a local  $U(1)$  symmetry.

The obtained metric is a cone [13, 10]. To see this, one splits the  $n + 1$  complex variables  $\{X\}$  in  $\{\rho, \theta, z^\alpha\}$

- $r$  is scale which is a gauge degree of freedom for translations
- $\theta$  is the  $U(1)$  degree of freedom;
- the  $n$  complex variables  $z^\alpha$ .

The metric now takes the form

$$(4.3) \quad ds^2 = dr^2 + \frac{1}{18}r^2 [A + d\theta + i(\partial_\alpha K(z, \bar{z}) dz^\alpha - \partial_{\bar{\alpha}} K(z, \bar{z}) d\bar{z}^{\bar{\alpha}})]^2 + r^2 \partial_\alpha \partial_{\bar{\alpha}} K(z, \bar{z}) dz^\alpha d\bar{z}^{\bar{\alpha}},$$

where  $A$  is the one-form gauging the  $U(1)$  group, and  $K(z, \bar{z})$  is a function of the holomorphic prepotential  $F(X)$ , to be explained below. With  $A = 0$ , this defines the cone over a Sasakian manifold. However, in supergravity, the field equation of  $A$  implies that it is a composite field, given by (minus) the other parts of the second term of (4.3). With fixed  $\rho$  (gauge fixing the superfluous dilatations), the remaining manifold is Kähler, with the Kähler potential determined by  $F(X)$ . That gives the special Kähler metric.

Let us explain this now in more detail, using at the same time more of the symplectic formalism. The dilatational gauge fixing (the fixing of  $r$  above), is done by the condition

$$(4.4) \quad X^I \bar{F}_I(\bar{X}) - \bar{X}^I F_I(X) = i.$$

This condition is chosen in order to decouple kinetic terms of the graviton from those of the scalars. Using again symplectic vectors

$$(4.5) \quad V = \begin{pmatrix} X^I \\ F_I \end{pmatrix},$$

this can be written as the condition on the symplectic inner product:

$$(4.6) \quad \langle V, \bar{V} \rangle = i.$$

To solve this condition, we define

$$(4.7) \quad V = e^{K(z, \bar{z})/2} v(z),$$

where  $v(z)$  is a holomorphic symplectic vector,

$$(4.8) \quad v(z) = \begin{pmatrix} Z^I(z) \\ \frac{\partial}{\partial Z^I} F(Z) \end{pmatrix}.$$

The upper components here are arbitrary functions (up to conditions for non-degeneracy), reflecting the freedom of choice of coordinates  $z^\alpha$ . The Kähler potential is

$$(4.9) \quad e^{-K(z, \bar{z})} = -i \langle v, \bar{v} \rangle.$$

The kinetic matrix for the vectors is given by

$$(4.10) \quad \mathcal{N}_{IJ} = (F_I \quad \mathcal{D}_{\bar{\alpha}} \bar{F}_I(\bar{X})) (X^J \quad \mathcal{D}_{\bar{\alpha}} \bar{X}^J)^{-1},$$

where the matrices are  $(n + 1) \times (n + 1)$  and

$$(4.11) \quad \mathcal{D}_{\bar{\alpha}} \bar{F}_I(\bar{X}) = \partial_{\bar{\alpha}} \bar{F}_I(\bar{X}) + \frac{1}{2}(\partial_{\bar{\alpha}} K) \bar{F}_I(\bar{X}), \quad \mathcal{D}_{\bar{\alpha}} \bar{X}^J = \partial_{\bar{\alpha}} \bar{X}^J + \frac{1}{2}(\partial_{\bar{\alpha}} K) \bar{X}^J.$$

Before continuing with general statements, it is time for an example. Consider the prepotential  $F = -iX^0X^1$ . This is a model with  $n = 1$ . There is thus just one coordinate  $z$ . One has to choose a parametrization to be used in the upper part of (4.8). Let us take a simple choice:  $Z^0 = 1$  and  $Z^1 = z$ . The full symplectic vector is then (as e.g.  $F_0(Z) = -iZ^1(z)$ )

$$(4.12) \quad v = \begin{pmatrix} Z^0 \\ Z^1 \\ F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix}.$$

The Kähler potential is then directly obtained from (4.9), determining the metric:

$$(4.13) \quad e^{-K} = 2(z + \bar{z}) ; \quad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K = (z + \bar{z})^{-2}.$$

The kinetic matrix for the vectors is diagonal. From (4.10) follows

$$(4.14) \quad \mathcal{N} = \begin{pmatrix} -iz & 0 \\ 0 & -i\frac{1}{z} \end{pmatrix}.$$

Therefore the action contains

$$(4.15) \quad e^{-1} \mathcal{L}_1 = -\frac{1}{2} \Re \left[ z (F_{\mu\nu}^{+0})^2 + z^{-1} (F_{\mu\nu}^{+1})^2 \right].$$

The domain of positivity for both metrics is  $\Re z > 0$ .

We formulate again two definitions, the first using the prepotential, and the second one using only the symplectic vectors.

**Definition 1 of (local) special Kähler geometry.**

A special Kähler manifold is an  $n$ -dimensional Hodge-Kähler manifold with on any chart  $n + 1$  holomorphic functions  $Z^I(z)$  and a holomorphic function  $F(Z)$ , homogeneous of second degree, such that, with (4.8), the Kähler potential is given by

$$(4.16) \quad e^{-K(z,\bar{z})} = -i \langle v, \bar{v} \rangle,$$

and on overlap of charts, the  $v(z)$  are connected by symplectic transformations  $Sp(2(n + 1), \mathbb{R})$  and/or Kähler transformations.

$$(4.17) \quad v(z) \rightarrow e^{f(z)} \mathcal{S}v(z).$$

**Definition 2 of (local) special Kähler geometry.**

A special Kähler manifold is an  $n$ -dimensional Kähler–Hodge manifold, that is the base manifold of a  $Sp(2(n + 1)) \times U(1)$  bundle. There should exist a holomorphic section  $v(z)$  such that the Kähler potential can be written as

$$(4.18) \quad e^{-K(z,\bar{z})} = -i\langle v, \bar{v} \rangle,$$

and it should satisfy the condition

$$(4.19) \quad \langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v \rangle = 0.$$

Note that the latter condition guarantees the symmetry of  $\mathcal{N}_{IJ}$ . This condition did not appear in [7], where the author had in mind Calabi–Yau manifolds. As we will see below, in those applications, this condition is automatically fulfilled. For  $n > 1$  the condition can be replaced by the equivalent condition

$$(4.20) \quad \langle \mathcal{D}_\alpha v, v \rangle = 0.$$

For  $n = 1$ , the condition (4.19) is empty, while (4.20) is not. In [14] it has been shown that models with  $n = 1$  not satisfying (4.20) can be formulated.

The appearance of ‘Hodge’ manifold in the definitions refers to a global requirement. The  $U(1)$  curvature should be of even integer cohomology. This has been considered first in [15], and for an explanation on the normalization, one can consult [9]. Note that in the mathematics literature ‘Hodge’ refers to integer cohomology. Here, however, the presence of fermions makes the condition stronger by a factor of two: one needs even integers.

Let us come back to the example, on which we will perform a symplectic mapping:

$$(4.21) \quad \tilde{v} = \mathcal{S}v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ i \\ -iz \\ z \end{pmatrix}.$$

After this mapping,  $z$  is not any more a good coordinate for  $(\tilde{Z}^0, \tilde{Z}^1)$ , the upper two components of the symplectic vector  $z$ . This means that the symplectic vector can not be obtained from a prepotential. We can not obtain the symplectic vector from a form (4.8). No function  $\tilde{F}(\tilde{Z}^0, \tilde{Z}^1)$  exists. Therefore, the first definition is not applicable. However, nothing prevents us from using the second definition. The Kähler metric is still the same, (4.13), and one can again compute the vector kinetic matrix, either directly from (4.10), as the denominator is still invertible, or from (2.10):

$$(4.22) \quad \tilde{\mathcal{N}} = (C + DN)(A + BN)^{-1} = -iX^1(X^0)^{-1}\mathbb{1} = -iz\mathbb{1}.$$

In this parametrization, the action is thus

$$(4.23) \quad e^{-1}\mathcal{L}_1 = -\frac{1}{2}\Re \left[ z (F_{\mu\nu}^{+0})^2 + z (F_{\mu\nu}^{+1})^2 \right].$$

This action is not the same as the one before, but is a ‘dual formulation’ of the same theory, being obtained from (4.15) by a duality transformation. The straightforward

construction in superspace or superconformal tensor calculus does not allow to construct actions without a superpotential. However, in [14] it has been shown that the field equations of these models can also be obtained from the superconformal tensor calculus. One just has to give up the concept of a superconformal invariant action.

It is thus legitimate to ask about the equivalence of the two definitions. Indeed, we saw that in some cases definition 2 is satisfied, but one can not obtain a prepotential  $F$ . However, that example, as others in [8], was obtained from performing a symplectic transformation from a formulation where the prepotential does exist. In [9] it was shown that this is true in general. If definition 2 is applicable, then there exists a symplectic transformation to a basis such that  $F(Z)$  exists. Note, however, that in the way physical problems are handled, the existence of formulations without prepotentials is important. Going to a dual formulation, one obtains a formulation with different symmetries in perturbation theory. The example that we used here appears in a reduction to  $N = 2$  of two versions of  $N = 4$  supergravity, known respectively as the ‘ $SO(4)$  formulation’ [16] and the ‘ $SU(4)$  formulation’ of pure  $N = 4$  supergravity [17].

Finally let us note that we still could apply (4.10) because the matrix

$$(4.24) \quad (X^I \quad \mathcal{D}_\alpha \bar{X}^I)$$

is always invertible if the metric  $g_{\alpha\bar{\alpha}} = \partial_\alpha \partial_{\bar{\alpha}} K(z, \bar{z})$  is positive definite. Therefore, the inverse exists, and  $\mathcal{N}_{IJ}$  can be constructed. However, the matrix

$$(4.25) \quad (X^I \quad \mathcal{D}_\alpha X^I)$$

is not invertible in the formulation (4.21). If that matrix is invertible, then a prepotential exists [9].

### 5. REALIZATIONS IN MODULI SPACES OF RIEMANN SURFACES AND CALABI–YAU MANIFOLDS

The realizations of special Kähler geometry that are mostly studied in physics these days, are the moduli spaces of Riemann surfaces for the rigid case, and those of Calabi–Yau 3-folds for the local case.

First, consider the Hodge diamond of Riemann surfaces, listing the number of non-trivial (anti)holomorphic  $(p, q)$  forms:

$$\begin{array}{ccc} & h^{00} = 1 & \\ h^{10} = g & & h^{01} = g \\ & h^{11} = 1 & \end{array}$$

Rigid special Kähler geometry is obtained for the moduli spaces of such Riemann surfaces when we consider

- with  $n$  complex moduli  $z^\alpha$
- $n \leq g$  holomorphic 1-forms  $\gamma_\alpha$  ( $\alpha = 1, \dots, n$ )
- $2n$  cycles  $c_A$  that form a complete basis for 1-cycles for which  $\int_c \gamma_\alpha \neq 0$ .



In this situation

$$(5.1) \quad \gamma_\alpha(z) = \partial_\alpha \lambda(z) + d\eta_\alpha(z),$$

where  $\lambda(z)$  is a meromorphic 1-form with zero residues. The symplectic formulation of rigid special Kähler geometry is obtained with as symplectic vector the vector of periods of  $\lambda$  over the chosen cycles:

$$(5.2) \quad V = \int_{c_\Lambda} \lambda.$$

The intersection matrix of the cycles plays the role of the symplectic metric. This type of realizations was used in Seiberg–Witten models. The general features have been discussed in [9].

To obtain local special Kähler manifolds, one considers the moduli space of Calabi–Yau 3-folds. In this case the Hodge diamond of the manifold is

$$\begin{array}{ccccccc}
 & & & h^{00} = 1 & & & \\
 & & & 0 & & 0 & \\
 & & 0 & h^{11} = m & & 0 & \\
 h^{30} = 1 & & h^{21} = n & & h^{12} = n & & h^{03} = 1 \\
 & & 0 & h^{22} = m & & 0 & \\
 & & 0 & & 0 & & \\
 & & & h^{33} = 1 & & & 
 \end{array}$$

These manifolds have  $h^{21} = n$  complex structure moduli, which play the role of the variables  $z^\alpha$  of the previous section. There are  $2(n + 1)$  3-cycles  $c_\Lambda$ , with intersection matrix  $Q_{\Lambda\Sigma} = c_\Lambda \cap c_\Sigma$ . The canonical form is obtained with so-called  $A$  and  $B$  cycles, and then  $Q$  takes the form of  $\Omega$  in (2.12). Symplectic vectors are identified again as vectors of integrals over the  $2(n + 1)$  3-cycles:

$$(5.3) \quad v = \int_{c_\Lambda} \Omega^{(3,0)}, \quad \mathcal{D}_\alpha v = \int_{c_\Lambda} \Omega_\alpha^{(2,1)}.$$

$\Omega^{(3,0)}$  is the unique  $(3, 0)$  form that characterizes the Calabi–Yau manifold.  $\Omega_\alpha^{(2,1)}$  is a basis of the  $(2, 1)$  forms, determined by the choice of basis for  $z^\alpha$ . That these moduli spaces give rise to special Kähler geometry became clear in [18]. Details on the relation between the geometric quantities and the fundamentals of special Kähler geometry have been discussed in [19, 9].

The defining equations of special Kähler geometry are automatically satisfied. E.g. one can easily see how the crucial equation (4.19) is realized:

$$\begin{aligned}
 (\mathcal{D}_\alpha v, \mathcal{D}_\beta v) &= \int_{c_\Lambda} \Omega_{(\alpha)}^{(2,1)} \cdot Q^{\Lambda\Sigma} \cdot \int_{c_\Sigma} \Omega_{(\beta)}^{(2,1)} \\
 (5.4) \quad &= \int_{CY} \Omega_{(\alpha)}^{(2,1)} \wedge \Omega_{(\beta)}^{(2,1)} = 0.
 \end{aligned}$$

The symplectic transformations correspond now to changes of the basis of the cycles used to construct the symplectic vectors. The statement that a formulation with a prepotential can always be obtained in special Kähler geometry by using a symplectic transformation, can now be translated to the statement that the geometry can be obtained from a prepotential for some choice of cycles.

Finally, it is interesting that singularities of Calabi–Yau manifolds may be used to obtain a ‘rigid limit’. Indeed, in [20] it is shown how Calabi–Yau manifolds that are  $K3$  fibrations can be reduced near the singularity to fibrations of ALE manifolds. Then the special geometry of the moduli space of the Calabi–Yau manifold reduces to the rigid special geometry with Kähler potential determined by the ALE manifold. This mechanism is considered further in [21]. There it has been shown how the Kähler potential of special geometry approaches the one of rigid special geometry, and how the periods of the local theory behave around the singular points and thus around the rigid limit. In the superstring theory this allows to get the gravity corrections to the rigid theory, which can be used for applications [22].

## 6. SUMMARY AND CONNECTION WITH QUATERNIONIC MANIFOLDS

Special Kähler geometry is defined by the couplings of  $N = 2$  supersymmetric theories (‘rigid’ special Kähler) or supergravity theories ((local) special Kähler) with vector multiplets. There are several ways to describe the geometry. We discussed two ways:

- by using a prepotential function
- by symplectic vectors and constraints

In *rigid* special Kähler geometry, these are completely equivalent. In the *local* theory, all special Kähler manifolds can be obtained from a prepotential, but in some cases that involves a duality transformation. Therefore not all actions can be described by the prepotential.

*Rigid* special Kähler geometry is realised by moduli spaces of certain Riemann manifold. That construction is not straightforward, and involves a choice of cohomology subspace and moduli. The *local* special Kähler geometry appears in the moduli space of Calabi–Yau threefolds. In this case the construction is straightforward. For a particular Calabi–Yau manifold one includes all the moduli. In this way a clear geometrical interpretation of the building blocks of special geometry is obtained. Duality transformations correspond then to a change of the basis of cycles. A prepotential does exist at least for a suitable choice of basis of the cycles.

Note, however, that not all special Kähler manifolds can be obtained as realizations in moduli spaces. E.g. the homogeneous manifolds, treated in [1, 2] are not obtained in this way.

In the First Meeting on Quaternionic Structures in Mathematics and Physics, 5 years ago, we have shown [1, 2] how homogeneous special Kähler spaces are related by the  $c$ -map to homogeneous quaternionic spaces and by the  $r$ -map to homogeneous

‘very special’ real spaces. The construction of special Kähler geometry that we have outlined here can be used as well for the quaternionic spaces (and for the real ones). In a recent work [10] it has been shown how the conformal tensor calculus can be applied to obtain the actions based on the quaternionic spaces (actions for ‘hypermultiplets’). The scalars are the lowest components of superfields (or superconformal multiplets)  $A_i^\alpha$ , with  $i = 1, 2$  and  $\alpha = 1, \dots, 2(r+1)$  with a reality condition. The  $A_i^\alpha$  can be considered as  $Sp(1) \times Sp(r+1)$  sections. Again the number of multiplets ( $r+1$ ) is one more than the number of physical multiplets ( $r$ ) that we will obtain. We thus start with  $4(r+1)$  scalars. One of those will be a scale degree of freedom<sup>3</sup>, three are  $SU(2)$  degrees of freedom, the second part of the  $R$ -symmetry as was mentioned after (4.1), and the remaining ones form  $r$  quaternions. As in the metric of the vector multiplets, there is a connection to Sasakian manifolds. Putting the gauge fields of the  $SU(2)$  invariance to zero, rather than using their field equations, one obtains a 3-Sasakian manifold. This is related to the talk of Galicki in the meeting 5 years ago [23].

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<sup>3</sup>When vector multiplets and hypermultiplets are simultaneously coupled, there is one overall dilatational gauge degree of freedom. An auxiliary field of the superconformal gauge multiplet gives a second relation, such that as well the compensating field of the vector multiplet as that of the hypermultiplet are fixed.

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**SINGULARITIES IN HYPERKÄHLER GEOMETRY**

MISHA VERBITSKY

**ABSTRACT.** This is a survey of some of the work done in 1993-99 on resolution of singularities in the context of hyperkähler geometry. We define a singular hypercomplex variety and its desingularization; similar methods are applied to desingularising coherent sheaves. We relate the singularities of reflexive sheaves over hyperkähler manifolds to quaternionic-Kähler geometry. Finally, we study holomorphic symplectic orbifolds and their resolutions.

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## 1. INTRODUCTION

This introduction is a quick summary of the works presented in this paper. The reader who is not sufficiently acquainted with the hyperkähler geometry is advised to start with Section 2.

Hyperkähler manifold is a Riemannian manifold  $M$  with three complex structures  $I, J, K$ ,  $I \circ J = -J \circ I = K$ , such that  $M$  is Kähler with respect to  $I, J$  and  $K$ . Clearly, the operators  $I, J, K$  define a quaternion action on the tangent space to  $M$ . Hyperkähler manifolds are quaternionic analogues of the usual Kähler manifolds.

The hyperkähler manifold is, by definition, smooth. However, there were attempts to introduce singularities in hyperkähler geometry, starting from [De], [S] (Deligne and Simpson; see Definition 3.13). More recently, D.Kaledin (unpublished Ph.D. thesis) and A.Dancer – A.Swann ([DS]) studied singular varieties, appearing as a result of hyperkähler reduction. Unfortunately, as Kaledin noticed, the hyperkähler reduction does not result in the Deligne-Simpson's type singular hyperkähler varieties.

There is an obvious source of examples of singular hyperkähler varieties. Let  $M$  be a hyperkähler manifold,  $I, J, K$  the standard complex structures on  $M$ , and  $X \subset M$  a closed subset. The subset  $X$  is called **trianalytic** if  $X$  is complex analytic with respect to  $I, J$  and  $K$ . As [V-h], Remark 4.4 implies, trianalytic subsets are singular hyperkähler, in the sense of Deligne and Simpson.

A group of unitarian quaternions is naturally isomorphic to  $SU(2)$ . This defines an  $SU(2)$ -action on the tangent space to a hyperkähler manifold  $M$ . If  $M$  is compact, this action defines a natural action of  $SU(2)$  on the cohomology of  $M$ .

Another source of examples of singular hyperkähler varieties is given by the theory of hyperholomorphic bundles. A hyperholomorphic bundle over a compact hyperkähler manifold is a stable holomorphic bundle with first and second Chern classes  $SU(2)$ -invariant. In [V1], it was shown that the moduli space of hyperholomorphic bundles is singular hyperkähler.

The definition of Deligne and Simpson was studied in [V-d], [V-d2] and [V-h]. It was found that the singularities of the singular hyperkähler varieties are remarkably simple. A canonical desingularization was constructed (Theorem 5.1); the desingularization is a smooth hyperkähler manifold.

This means that hyperkähler varieties (in the sense of Deligne and Simpson) are "almost" non-singular. Indeed, the canonical desingularization is provided by normalization.

It is possible that there is a more relaxed notion of a hyperkähler variety, which allows for more varied singularities. We study the singular structures in hyperkähler geometry, hoping to come across such a notion.



There is a notion of hyperholomorphic connection on a vector bundle  $B$  over a hyperkähler manifold  $M$  (Definition 4.3). It is a “hyperkähler analogue” of the usual  $(1,1)$  connections on holomorphic bundles. When  $M$  is compact, such connection exists if and only if  $B$  is a direct sum of stable bundles with first and second Chern classes  $SU(2)$ -invariant. In such a case, the hyperholomorphic connection is unique.

A similar result exists for stable sheaves (Theorem 6.11). If  $F$  is a reflexive stable coherent sheaf over a hyperkähler manifold, and the first and second Chern classes of  $F$  are  $SU(2)$ -invariant, then  $F$  admits a unique hyperholomorphic connection with admissible type of singularities (Definition 6.5). The study of such sheaves (called **hyperholomorphic sheaves**, Definition 6.9) is done by the same methods as the study of hyperkähler varieties. A version of desingularization theorem holds in this situation as well (Theorem 6.12).

It is easy to see that hyperkähler manifolds admit a holomorphic symplectic form (Subsection 2.1). Conversely, a compact holomorphic symplectic Kähler manifold admits a natural hyperkähler structure (this follows from Calabi conjecture, proven by S.-T. Yau (Theorem 2.8). Therefore, to study compact holomorphic symplectic Kähler manifold we need to learn about holomorphic symplectic geometry.

What is “a singular holomorphic symplectic variety”? This is not clear. However, the most natural generalization of holomorphic symplectic manifold is a holomorphic symplectic orbifold, that is, a variety which is locally isomorphic to a quotient of a holomorphic symplectic manifold by a finite group action.

Let  $M$  be a holomorphic symplectic manifold, and  $G$  a finite group acting on  $M$  preserving the symplectic structure. It is natural to consider the quotient  $M/G$  as a holomorphic symplectic orbifold. Suppose we have a resolution of singularities  $\widetilde{M} \rightarrow M/G$  with  $\widetilde{M}$  a smooth holomorphically symplectic manifold. Such a situation arises, for instance, when  $M = S^n$  is a product of  $n$  copies of a holomorphic symplectic surface  $S$ ,  $G = S_n$  the symmetric group and  $\widetilde{M}$  a Hilbert scheme of  $S$ . Another instance when such a situation arises is described in [KV2]. Suppose that  $T^{[n]}$  is a so-called “generalized Hilbert scheme” of a torus  $T$ , and  $X \subset T^{[n]}$  a complex subvariety which survives a generic deformation of  $T^{[n]}$  (that is, for any deformation of  $T^{[n]}$ , there exists a flat deformation of  $X \subset T^{[n]}$ ). From a definition of  $T^{[n]}$  (see e.g. [Bea]) it follows that  $T^{[n]}$  is equipped by a generically finite map  $\pi : T^{[n]} \rightarrow T^{n+1}$ . It was proven in [KV2] that in the above assumptions,  $\pi(X)$  is a quotient of a torus by a Coxeter group action on it, and that  $\pi : X \rightarrow \pi(X)$  is a holomorphic symplectic resolution of  $\pi(X)$ .

It is not clear how the holomorphically symplectic resolutions are related to the hyperkähler geometry. For instance, it is not clear, even in the most simple cases, whether a Hilbert scheme of a non-compact hyperkähler surface is hyperkähler. Still, the desingularizations of hyperkähler orbifolds is one of the most common ways of obtaining hyperkähler manifolds.

The work [KV2] goes some way explaining why the ubiquitous Coxeter groups appear in the study of subvarieties of generalized Kummer varieties. Given a holomorphic symplectic manifold  $M$  and a finite group  $G$  acting on  $M$  by symplectomorphisms, let  $\tilde{M}$  be a holomorphic symplectic resolution of  $M/G$ . Then,  $G$  is generated by **symplectic reflections**, that is, by automorphisms with fixed set of codimension 2 (see Section 7 for detail).

## 2. HYPERKÄHLER MANIFOLDS

**2.1. Hyperkähler manifolds.** This subsection contains a compression of the basic and best known results and definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Bea].

**Definition 2.1:** Let  $M$  be a smooth manifold, equipped with an action of quaternion algebra in  $TM$ . Then  $M$  is called an **almost hypercomplex manifold**.

Let  $M$  be an almost hypercomplex manifold and  $I, J \neq \pm I$  quaternions satisfying  $I^2 = J^2 = -1$ . Clearly,  $I, J$  define almost complex structures on  $M$ .

**Proposition 2.2:** [K1] In the above situation, assume that the almost complex structures  $I$  and  $J$  are integrable. Let  $K \in \mathbb{H}$  be a quaternion satisfying  $K^2 = -1$ . Then  $K$  defines an integrable complex structure on  $M$ .

■

**Definition 2.3:** Let  $M$  be a smooth manifold, and  $I, J, K$  almost complex structures satisfying  $I \circ J = -J \circ I = K$ . Assume that  $I$  and  $J$  are integrable. Then  $M$  is called a hypercomplex manifold.

**Remark 2.4:** *A posteriori*, we obtain that every quaternion satisfying  $K^2 = -1$  defines an integrable complex structure on a hypercomplex manifold (Proposition 2.2).

**Definition 2.5:** ([Bes]) A hyperkähler manifold is a hypercomplex manifold equipped with a Riemannian metric  $(\cdot, \cdot)$ , such that  $I, J, K$  are Kähler complex structures with respect to  $(\cdot, \cdot)$ .

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has a natural action of the quaternion algebra  $\mathbb{H}$  in its real tangent bundle  $TM$ . Therefore its complex dimension is even. For each quaternion  $L \in \mathbb{H}$ ,  $L^2 = -1$ , the corresponding automorphism of  $TM$  is an almost

complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

**Definition 2.6:** Let  $M$  be a hyperkähler (or hypercomplex) manifold, and  $L$  a quaternion satisfying  $L^2 = -1$ . The corresponding complex structure on  $M$  is called **an induced complex structure**. The  $M$ , considered as a complex manifold, is denoted by  $(M, L)$ .

**Definition 2.7:** Let  $M$  be a complex manifold and  $\Theta$  a closed holomorphic 2-form over  $M$  such that  $\Theta^n = \Theta \wedge \Theta \wedge \dots$ , is a nowhere degenerate section of a canonical class of  $M$  ( $2n = \dim_{\mathbb{C}}(M)$ ). Then  $M$  is called **holomorphically symplectic**.

Let  $M$  be a hyperkähler manifold; denote the Riemannian form on  $M$  by  $(\cdot, \cdot)$ . Let the form  $\omega_I := (I(\cdot), \cdot)$  be the usual Kähler form which is closed and parallel (with respect to the Levi-Civita connection). Analogously defined forms  $\omega_J$  and  $\omega_K$  are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form  $\Theta := \omega_J + \sqrt{-1}\omega_K$  is of type  $(2, 0)$  and, being closed, this form is also holomorphic. Also, the form  $\Theta$  is nowhere degenerate, as another linear algebraic argument shows. It is called **the canonical holomorphic symplectic form of a manifold  $M$** . Thus, for each hyperkähler manifold  $M$ , and an induced complex structure  $L$ , the underlying complex manifold  $(M, L)$  is holomorphically symplectic. The converse assertion is also true:

**Theorem 2.8:** ([Bea], [Bes]) Let  $M$  be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form  $\Theta$ , a Kähler class  $[\omega] \in H^{1,1}(M)$  and a complex structure  $I$ . Let  $n = \dim_{\mathbb{C}} M$ . Assume that  $\int_M \omega^n = \int_M (\operatorname{Re}\Theta)^n$ . Then there is a unique hyperkähler structure  $(I, J, K, (\cdot, \cdot))$  over  $M$  such that the cohomology class of the symplectic form  $\omega_I = (\cdot, I\cdot)$  is equal to  $[\omega]$  and the canonical symplectic form  $\omega_J + \sqrt{-1}\omega_K$  is equal to  $\Theta$ .

Theorem 2.8 follows from the conjecture of Calabi, proven by Yau ([Y]). ■

Let  $M$  be a hyperkähler manifold. We identify the group  $SU(2)$  with the group of unitary quaternions. This gives a canonical action of  $SU(2)$  on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of  $SU(2)$  on the bundle of differential forms.

**Lemma 2.9:** The action of  $SU(2)$  on differential forms commutes with the Laplacian.

**Proof:** This is Proposition 1.1 of [V2]. ■

Thus, for compact  $M$ , we may speak of the natural action of  $SU(2)$  in cohomology.

The following lemma is clear from the properties of the Hodge decomposition.

**Lemma 2.10:** Let  $\omega$  be a differential form over a hyperkähler manifold  $M$ . The form  $\omega$  is  $SU(2)$ -invariant if and only if it is of Hodge type  $(p, p)$  with respect to all induced complex structures on  $M$ .

**Proof:** This is [V1], Proposition 1.2. ■

**2.2. Trianalytic subvarieties in hyperkähler manifolds.** In this subsection, we give a definition and basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let  $M$  be a compact hyperkähler manifold,  $\dim_{\mathbb{R}} M = 2m$ .

**Definition 2.11:** Let  $N \subset M$  be a closed subset of  $M$ . Then  $N$  is called **trianalytic** if  $N$  is a complex analytic subset of  $(M, L)$  for any induced complex structure  $L$ .

Let  $I$  be an induced complex structure on  $M$ , and  $N \subset (M, I)$  be a closed analytic subvariety of  $(M, I)$ ,  $\dim_{\mathbb{C}} N = n$ . Consider the homology class represented by  $N$ . Let  $[N] \in H^{2m-2n}(M)$  denote the Poincaré dual cohomology class, so called **fundamental class** of  $N$ . Recall that the hyperkähler structure induces the action of the group  $SU(2)$  on the space  $H^{2m-2n}(M)$ .

**Theorem 2.12:** Assume that  $[N] \in H^{2m-2n}(M)$  is invariant with respect to the action of  $SU(2)$  on  $H^{2m-2n}(M)$ . Then  $N$  is trianalytic.

**Proof:** This is Theorem 4.1 of [V2]. ■

The following assertion is the key to the proof of Theorem 2.12 (see [V2] for details).

**Proposition 2.13:** (Wirtinger's inequality) Let  $M$  be a compact hyperkähler manifold,  $I$  an induced complex structure and  $X \subset (M, I)$  a closed complex subvariety for complex dimension  $k$ . Let  $J$  be an induced complex structure,  $J \neq \pm I$ , and  $\omega_I, \omega_J$  the associated Kähler forms. Consider the numbers

$$\deg_I X := \int_X \omega_I^k, \quad \deg_J X := \int_X \omega_J^k$$

Then  $\deg_I X \geq |\deg_J X|$ , and the inequality is strict unless  $X$  is trianalytic.

■

**Remark 2.14:** Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

**Definition 2.15:** Let  $M$  be a complex manifold admitting a hyperkähler structure  $\mathcal{H}$ . We say that  $M$  is **of general type** or **generic** with respect to  $\mathcal{H}$  if all elements of the group

$$\bigoplus_p H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z}) \subset H^*(M)$$

are  $SU(2)$ -invariant.

The following result is an elementary application of representation theory.

**Proposition 2.16:** Let  $M$  be a compact manifold,  $\mathcal{H}$  a hyperkähler structure on  $M$  and  $S$  be the set of induced complex structures over  $M$ . Denote by  $S_0 \subset S$  the set of  $L \in S$  such that  $(M, L)$  is generic with respect to  $\mathcal{H}$ . Then  $S_0$  is dense in  $S$ . Moreover, the complement  $S \setminus S_0$  is countable.

**Proof:** This is Proposition 2.2 from [V2] ■

Theorem 2.12 has the following immediate corollary:

**Corollary 2.17:** Let  $M$  be a compact holomorphically symplectic manifold. Assume that  $M$  is of general type with respect to a hyperkähler structure  $\mathcal{H}$ . Let  $S \subset M$  be closed complex analytic subvariety. Then  $S$  is trianalytic with respect to  $\mathcal{H}$ .

■

**2.3. Twistor spaces.** Let  $M$  be a hyperkähler manifold. Consider the product manifold  $X = M \times S^2$ . Embed the sphere  $S^2 \subset \mathbb{H}$  into the quaternion algebra  $\mathbb{H}$  as the subset of all quaternions  $J$  with  $J^2 = -1$ . For every point  $x = m \times J \in X = M \times S^2$  the tangent space  $T_x X$  is canonically decomposed  $T_x X = T_m M \oplus T_J S^2$ . Identify  $S^2 = \mathbb{C}P^1$  and let  $I_J : T_J S^2 \rightarrow T_J S^2$  be the complex structure operator. Let  $I_m : T_m M \rightarrow T_m M$  be the complex structure on  $M$  induced by  $J \in S^2 \subset \mathbb{H}$ .

The operator  $I_x = I_m \oplus I_J : T_x X \rightarrow T_x X$  satisfies  $I_x \circ I_x = -1$ . It depends smoothly on the point  $x$ , hence defines an almost complex structure on  $X$ . This almost complex structure is known to be integrable (see [Sal]).

**Definition 2.18:** The complex manifold  $(X, I_x)$  is called *the twistor space* for the hyperkähler manifold  $M$ , denoted by  $\text{Tw}(M)$ . This manifold is equipped with a real analytic projection  $\sigma : \text{Tw}(M) \rightarrow M$  and a complex analytic projection  $\pi : \text{Tw}(M) \rightarrow \mathbb{C}P^1$ .

The twistor space  $\text{Tw}(M)$  is not, generally speaking, a Kähler manifold. For  $M$  compact, it is easy to show that  $\text{Tw}(M)$  does not admit a Kähler metric.

### 3. HYPERCOMPLEX VARIETIES

This section is based on [V-h], Section 4 and 8. In this section, we shall state all results for hypercomplex varieties, instead of hyperkähler ones. However, everything we say can be stated (and proven) for hyperkähler varieties (this approach was chosen in [V-d] and [V-d2]).

**3.1. Real analytic varieties and complex structures.** In this subsection, we follow [GMT] and [V-h], Section 2.

Let  $I$  be an ideal sheaf in the ring of real analytic functions in an open ball  $B$  in  $\mathbb{R}^n$ . The set of common zeroes of  $I$  is equipped with a structure of ringed space, with  $\mathcal{O}(B)/I$  as the structure sheaf. We denote this ringed space by  $\text{Spec}(\mathcal{O}(B)/I)$ .

**Definition 3.1:** By a **weak real analytic space** we understand a ringed space which is locally isomorphic to  $\text{Spec}(\mathcal{O}(B)/I)$ , for some ideal  $I \subset \mathcal{O}(B)$ . A **real analytic space** is a weak real analytic space for which the structure sheaf is coherent (i. e., locally finitely generated and presentable).

For every real analytic variety  $X$ , there is a natural sheaf morphism of evaluation,  $\mathcal{O}(X) \xrightarrow{ev} C(X)$ , where  $C(X)$  is the sheaf of real analytic functions on  $X$ .

**Definition 3.2:** A **real analytic variety** is a weak real analytic space for which the natural sheaf morphism  $\mathcal{O}(X) \rightarrow C(X)$  is injective.

Let  $(X, \mathcal{O}(X))$  be a real analytic space and  $N(X) \subset \mathcal{O}(X)$  be the kernel of the natural sheaf morphism  $\mathcal{O}(X) \rightarrow C(X)$ . Clearly, the ringed space  $(X, \mathcal{O}(X)/N(X))$  is a real analytic variety. This variety is called a **reduction of  $X$** , denoted  $X_r$ . The structure sheaf of  $X_r$  is not necessarily coherent, for examples see [GMT], III.2.15.

For an ideal  $I \subset \mathcal{O}(B)$  we define the module of real analytic differentials on  $\mathcal{O}(B)/I$  by

$$\Omega^1(\mathcal{O}(B)/I) = \Omega^1(\mathcal{O}(B)) / \left( I \cdot \Omega^1(\mathcal{O}(B)) + dI \right),$$

where  $B$  is an open ball in  $\mathbb{R}^n$ , and  $\Omega^1(\mathcal{O}(B)) \cong \mathbb{R}^n \otimes \mathcal{O}(B)$  is the module of real analytic differentials on  $B$ . Patching this construction, we define the sheaf of real analytic differentials on any real analytic space. Likewise, one defines sheaves of analytic differentials for complex varieties and in other similar situations.

Let  $X$  be a complex analytic variety. The **real analytic space underlying  $X$**  (denoted by  $X_{\mathbb{R}}$ ) is the following object. By definition,  $X_{\mathbb{R}}$  is a ringed space with the same topology as  $X$ , but with a different structure sheaf, denoted by  $\mathcal{O}_{X_{\mathbb{R}}}$ . Let  $i : U \hookrightarrow B^n$  be a closed complex analytic embedding of an open subset  $U \subset X$  to an open ball  $B^n \subset \mathbb{C}^n$ , and  $I$  be an ideal defining  $i(U)$ . Then

$$\mathcal{O}_{X_{\mathbb{R}}}|_U := \mathcal{O}_{B_{\mathbb{R}}^n}/\text{Re}(I)$$

is a quotient sheaf of the sheaf of real analytic functions on  $B^n$  by the ideal  $\text{Re}(I)$  generated by the real parts of the functions  $f \in I$ .

Note that the real analytic space underlying  $X$  needs not be reduced, though it has no nilpotents in the structure sheaf.

Consider the sheaf  $\mathcal{O}_X$  of holomorphic functions on  $X$  as a subsheaf of the sheaf  $C(X, \mathbb{C})$  of continuous  $\mathbb{C}$ -valued functions on  $X$ . The sheaf  $C(X, \mathbb{C})$  has a natural automorphism  $f \rightarrow \bar{f}$ , where  $\bar{f}$  is complex conjugation. By definition, the section  $f$  of  $C(X, \mathbb{C})$  is called **antiholomorphic** if  $\bar{f}$  is holomorphic. Let  $\mathcal{O}_X$  be the sheaf of holomorphic functions, and  $\bar{\mathcal{O}}_X$  be the sheaf of antiholomorphic functions on  $X$ . Let  $\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X \xrightarrow{i} \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$  be the natural multiplication map.

**Claim 3.3:** Let  $X$  be a complex variety,  $X_{\mathbb{R}}$  the underlying real analytic space. Then the natural sheaf homomorphism  $i : \mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X \rightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$  is injective. For each point  $x \in X$ ,  $i$  induces an isomorphism on  $x$ -completions of  $\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X$  and  $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ .

**Proof:** Clear from the definition. ■

In the assumptions of Claim 3.3, let

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}), \quad \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X), \quad \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C})$$

be the sheaves of real analytic differentials associated with the corresponding sheaves of rings. There is a natural sheaf map

$$(3.1) \quad \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}) \rightarrow \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X),$$

corresponding to the monomorphism

$$\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X \hookrightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}.$$

**Claim 3.4:** Tensoring both sides of (3.1) by  $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$  produces an isomorphism

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \bar{\mathcal{O}}_X} (\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}) = \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}).$$

**Proof:** Clear. ■

According to the general results about differentials (see, for example, [H], Chapter II, Ex. 8.3), the sheaf  $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$  admits a canonical decomposition:

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) = \Omega^1(\mathcal{O}_X) \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \oplus \mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1(\overline{\mathcal{O}}_X).$$

Let  $\tilde{I}$  be an endomorphism of  $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$  which acts as a multiplication by  $\sqrt{-1}$  on

$$\Omega^1(\mathcal{O}_X) \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \subset \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$$

and as a multiplication by  $-\sqrt{-1}$  on

$$\mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1(\overline{\mathcal{O}}_X) \subset \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X).$$

Let  $\underline{I}$  be the corresponding  $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ -linear endomorphism of

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \otimes \mathbb{C} = \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X} (\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}).$$

A quick check shows that  $\underline{I}$  is *real*, that is, comes from the  $\mathcal{O}_{X_{\mathbb{R}}}$ -linear endomorphism of  $\Omega^1(\mathcal{O}_{X_{\mathbb{R}}})$ . Denote this  $\mathcal{O}_{X_{\mathbb{R}}}$ -linear endomorphism by

$$I : \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \longrightarrow \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}),$$

$I^2 = -1$ . The endomorphism  $I$  is called **the complex structure operator on the underlying real analytic space**. In the case when  $X$  is smooth,  $I$  coincides with the usual complex structure operator on the cotangent bundle.

**Definition 3.5:** Let  $M$  be a weak real analytic space, and

$$I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying  $I^2 = -1$ . Then  $I$  is called **an almost complex structure on  $M$** .

**3.2. Almost complex structures on real analytic varieties and integrability.** In this Subsection, we follow [V-h], Section 2.

From the definition (see [V-h], Lemma 2.6), it follows that a real analytic variety underlying a given complex variety is equipped with a natural almost complex structure. The corresponding operator is called **the complex structure operator in the underlying real analytic variety**.

The following theorem is quite easy to prove.



**Theorem 3.6:** Let  $X, Y$  be complex analytic varieties, and

$$f_{\mathbb{R}} : X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$$

be a morphism of underlying real analytic varieties which commutes with the complex structure. Then there exist a morphism  $f : X \longrightarrow Y$  of complex analytic varieties, such that  $f_{\mathbb{R}}$  is its underlying morphism.

**Proof:** This is [V-h], Theorem 2.10. ■

From Theorem 3.6, it follows that the complex structure on  $X$  is uniquely determined by the complex structure on the underlying real analytic variety.

**Definition 3.7:** Let  $M$  be a real analytic variety, and

$$I : \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$$

be an endomorphism satisfying  $I^2 = -1$ . Then  $I$  is called an **almost complex structure on  $M$** . If there exist a structure  $\mathfrak{C}$  of complex variety on  $M$  such that  $I$  appears as the complex structure operator associated with  $\mathfrak{C}$ , we say that  $I$  is **integrable**. Theorem 3.6 implies that this complex structure is unique if it exists.

**3.3. Hypercomplex varieties: the definition. Definition 3.8:** Let  $M$  be a real analytic variety equipped with almost complex structures  $I, J$  and  $K$ , such that  $I \circ J = -J \circ I = K$ . Then  $M$  is called an **almost hypercomplex variety**.

An almost hypercomplex variety is equipped with an action of quaternion algebra in its differential sheaf. Each quaternion  $L \in \mathbb{H}$ ,  $L^2 = -1$  defines an almost complex structure on  $M$ . Such an almost complex structure is called **induced by the hypercomplex structure**.

**Definition 3.9:** Let  $M$  be an almost hypercomplex variety. We say that  $M$  is **hypercomplex** if there exist a pair of induced complex structures  $I_1, I_2 \in \mathbb{H}$ ,  $I_1 \neq \pm I_2$ , such that  $I_1$  and  $I_2$  are integrable.

**Caution:** Not everything which looks hypercomplex satisfies the conditions of Definition 3.9. Take a quotient  $M/G$  of a hypercomplex manifold by an action of a finite group  $G$ , acting compatible with the hypercomplex structure. Then  $M/G$  is *not* hypercomplex, unless  $G$  acts freely.

**Claim 3.10:** Let  $M$  be a hypercomplex manifold. Then  $M$  is a hypercomplex variety in the sense of Definition 3.9.

**Proof:** Let  $I, J$  be induced complex structures. We need to identify  $(M, I)_{\mathbb{R}}$  and  $(M, J)_{\mathbb{R}}$  in a natural way. These varieties are canonically identified as  $C^{\infty}$ -manifolds;

we need only to show that this identification is real analytic. This is [V3], Proposition 6.5. ■

ll

**Remark 3.11:** Trianalytic subvarieties of hyperkähler manifolds are obviously hypercomplex. Define trianalytic subvarieties of hypercomplex varieties as subvarieties which are complex analytic with respect to all induced complex structures. Clearly, trianalytic subvarieties of hypercomplex varieties are equipped with a natural hypercomplex structure. Another example of a hypercomplex variety is given in Corollary 4.11. For additional examples, see [V3].

**3.4. Hypercomplex varieties and twistor spaces.** For a hypercomplex variety, it is clear how to define the twistor space, which is a complex variety (see [V-h], Section 7 for details). This definition coincides with the usual one for the hypercomplex manifolds in the smooth case.

Following [HKLR], Deligne and Simpson defined hypercomplex varieties in terms of their twistor spaces. This is done as follows.

Let  $M$  be a hypercomplex variety and  $\text{Tw}$  its twistor space. Consider the unique anticomplex involution  $\iota_0 : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$  with no fixed points. This involution is obtained by central symmetry with center in 0 if we identify  $\mathbb{C}P^1$  with a unit sphere in  $\mathbb{R}^3$ . Let  $\iota : \text{Tw} \rightarrow \text{Tw}$  be an involution of the twistor space mapping  $(s, m) \in S^2 \times M = \text{Tw}$  to  $(\iota_0(s), m)$ . Clearly,  $\iota$  is anticomplex.

**Definition 3.12:** Let  $s : \mathbb{C}P^1 \rightarrow \text{Tw}$  be a section of the natural holomorphic projection  $\pi : \text{Tw} \rightarrow \mathbb{C}P^1$ ,  $s \circ \pi = \text{Id}|_{\mathbb{C}P^1}$ . Then  $s$  is called **the twistor line**. The space  $\text{Sec}$  of twistor lines is finite-dimensional and equipped with a natural complex structure, as follows from deformation theory ([Do]).

Let  $\text{Sec}^\iota$  be the space of all lines  $s \in \text{Sec}$  which are fixed by  $\iota$ . The space  $\text{Sec}^\iota$  is equipped with a structure of a real analytic space. We have a natural map  $\tau : M_{\mathbb{R}} \rightarrow \text{Sec}^\iota$  associating to  $m \in M$  the line  $s : \mathbb{C}P^1 \rightarrow \text{Tw}$ ,  $s(x) = (x, m) \in S^2 \times M = \text{Tw}$ . Such twistor lines are called **horizontal twistor lines**. Denote the set of horizontal twistor lines by  $\text{Hor} \subset \text{Sec}$ .

The linear algebra of quaternions implies that the normal bundle of a horizontal twistor line  $s \cong \mathbb{C}P^1$  is a direct sum of several copies of  $O(1)$ . A section of  $O(1)$  is uniquely determined by its values in two distinct points. Therefore, (at least if  $M$  is smooth), through every two generic points in a neighbourhood of  $s$  passes a unique deformation of  $s$  (if this statement needs a justification, see [V-h], (7.2)).

This motivates the following definition, proposed by Delidne and Simpson ([De], [S]).

**Definition 3.13:** (Hypercomplex spaces) Let  $\text{Tw}$  be a complex analytic space,  $\pi : \text{Tw} \rightarrow \mathbb{C}P^1$  a holomorphic map, and  $\iota : \text{Tw} \rightarrow \text{Tw}$  an anticomplex automorphism, such that  $\iota \circ \pi = \pi \circ \iota_0$ . Let  $\text{Sec}$  be the space of sections of  $\pi$  equipped with a structure of a complex analytic space, and  $\text{Sec}^c$  be the real analytic space of sections  $s$  of  $\pi$  satisfying  $s \circ \iota_0 = \iota \circ s$ . Let  $\text{Hor}$  be a connected component of  $\text{Sec}^c$ . Then  $(\text{Tw}, \pi, \iota, \text{Hor})$  is called a **hypercomplex space** if

- (i): For each point  $x \in \text{Tw}^r$ , there exists a unique line  $s \in \text{Hor}^r$  passing through  $x$ , where  $\text{Tw}^r, \text{Hor}^r$  is a reduction of  $\text{Tw}, \text{Hor}$ .
- (ii): Let  $s \in \text{Hor}$ , and  $U \subset \text{Tw}$  be a neighbourhood of  $s$  such that an irreducible decomposition of  $U$  coincides with the irreducible decomposition of  $\text{Tw}$  in a neighbourhood of  $s \subset \text{Tw}^r$ . Let

$$\overline{X} := \pi^{-1}(I) \times \pi^{-1}(J) \cap U \times U,$$

where  $I, J$  distinct points of  $\mathbb{C}P^1$ . Let  $p_{IJ} : U \rightarrow \overline{X} \subset \pi^{-1}(I) \times \pi^{-1}(J)$  be the evaluation map,  $s \rightarrow (s(I), s(J))$ . Then there exist a closed subspace  $X \subset \overline{X}$ , obtained as a union of some of irreducible components of  $\overline{X}$ , and an open neighbourhood  $V \subset \text{Sec}$  of  $s \in \text{Sec}$ , such that  $p_{IJ}$  is an open embedding of  $V$  to  $X$ .

For varieties, this definition is equivalent to Definition 3.9 ([V-h], Theorem 8.1).

#### 4. HYPERHOLOMORPHIC BUNDLES

**4.1. Hyperholomorphic bundles: the definition.** This subsection contains several versions of a definition of hyperholomorphic connection in a complex vector bundle over a hyperkähler manifold. We follow [V1].

Let  $B$  be a holomorphic vector bundle over a complex manifold  $M$ ,  $\nabla$  a connection in  $B$  and  $\Theta \in \Lambda^2 \otimes \text{End}(B)$  be its curvature. This connection is called **compatible with a holomorphic structure** if  $\nabla_X(\zeta) = 0$  for any holomorphic section  $\zeta$  and any antiholomorphic tangent vector field  $X \in T^{0,1}(M)$ . If there exists a holomorphic structure compatible with the given Hermitian connection then this connection is called **integrable**.

One can define a **Hodge decomposition** in the space of differential forms with coefficients in any complex bundle, in particular,  $\text{End}(B)$ .

**Theorem 4.1:** Let  $\nabla$  be a Hermitian connection in a complex vector bundle  $B$  over a complex manifold. Then  $\nabla$  is integrable if and only if  $\Theta \in \Lambda^{1,1}(M, \text{End}(B))$ ,

where  $\Lambda^{1,1}(M, \text{End}(B))$  denotes the forms of Hodge type  $(1,1)$ . Also, the holomorphic structure compatible with  $\nabla$  is unique.

**Proof:** This is Proposition 4.17 of [Ko], Chapter I. ■

This result has the following more general version:

**Proposition 4.2:** Let  $\nabla$  be an arbitrary (not necessarily Hermitian) connection in a complex vector bundle  $B$ . Then  $\nabla$  is integrable if and only if its  $(0,1)$ -part has square zero.

■

This proposition is a version of Newlander-Nirenberg theorem. For vector bundles, it was proven by Atiyah and Bott.

**Definition 4.3:** Let  $B$  be a Hermitian vector bundle with a connection  $\nabla$  over a hyperkähler manifold  $M$ . Then  $\nabla$  is called **hyperholomorphic** if  $\nabla$  is integrable with respect to each of the complex structures induced by the hyperkähler structure.

As follows from Theorem 4.1,  $\nabla$  is hyperholomorphic if and only if its curvature  $\Theta$  is of Hodge type  $(1,1)$  with respect to any of complex structures induced by a hyperkähler structure.

As follows from Lemma 2.10,  $\nabla$  is hyperholomorphic if and only if  $\Theta$  is a  $SU(2)$ -invariant differential form.

**Example 4.4:** (Examples of hyperholomorphic bundles)

- (i): Let  $M$  be a hyperkähler manifold, and  $TM$  be its tangent bundle equipped with the Levi-Civita connection  $\nabla$ . Consider a complex structure on  $TM$  induced from the quaternion action. Then  $\nabla$  is a Hermitian connection which is integrable with respect to each induced complex structure, and hence, is Yang-Mills.
- (ii): For  $B$  a hyperholomorphic bundle, all its tensor powers are also hyperholomorphic.
- (iii): Thus, the bundles of differential forms on a hyperkähler manifold are also hyperholomorphic.

**4.2. Stable bundles and Yang-Mills connections.** This subsection is a compendium of the most basic results and definitions from the Yang-Mills theory over Kähler manifolds, concluding in the fundamental theorem of Uhlenbeck-Yau [UY].

**Definition 4.5:** Let  $F$  be a coherent sheaf over an  $n$ -dimensional compact Kähler manifold  $M$ . We define  $\text{deg}(F)$  as

$$\deg(F) = \int_M \frac{c_1(F) \wedge \omega^{n-1}}{\text{vol}(M)}$$

and  $\text{slope}(F)$  as

$$\text{slope}(F) = \frac{1}{\text{rank}(F)} \cdot \deg(F).$$

The number  $\text{slope}(F)$  depends only on a cohomology class of  $c_1(F)$ .

Let  $F$  be a coherent sheaf on  $M$  and  $F' \subset F$  its proper subsheaf. Then  $F'$  is called **destabilizing subsheaf** if  $\text{slope}(F') \geq \text{slope}(F)$

A coherent sheaf  $F$  is called **stable**<sup>1</sup> if it has no destabilizing subsheaves. A coherent sheaf  $F$  is called **semistable** if for all destabilizing subsheaves  $F' \subset F$ , we have  $\text{slope}(F') = \text{slope}(F)$ .

Later on, we usually consider the bundles  $B$  with  $\deg(B) = 0$ .

Let  $M$  be a Kähler manifold with a Kähler form  $\omega$ . For differential forms with coefficients in any vector bundle there is a Hodge operator  $L : \eta \rightarrow \omega \wedge \eta$ . There is also a fiberwise-adjoint Hodge operator  $\Lambda$  (see [GH]).

**Definition 4.6:** Let  $B$  be a holomorphic bundle over a Kähler manifold  $M$  with a holomorphic Hermitian connection  $\nabla$  and a curvature  $\Theta \in \Lambda^{1,1} \otimes \text{End}(B)$ . The Hermitian metric on  $B$  and the connection  $\nabla$  defined by this metric are called **Yang-Mills** if

$$\Lambda(\Theta) = \text{constant} \cdot \text{Id} \Big|_B,$$

where  $\Lambda$  is a Hodge operator and  $\text{Id} \Big|_B$  is the identity endomorphism which is a section of  $\text{End}(B)$ .

Further on, we consider only these Yang–Mills connections for which this constant is zero.

A holomorphic bundle is called **indecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

The following fundamental theorem provides examples of Yang–Mills bundles.

**Theorem 4.7:** (Uhlenbeck–Yau) Let  $B$  be an indecomposable holomorphic bundle over a compact Kähler manifold. Then  $B$  admits a Hermitian Yang–Mills connection if and only if it is stable, and this connection is unique.

<sup>1</sup>In the sense of Mumford–Takemoto

**Proof:** [UY]. ■

**4.3. Hyperholomorphic connections and Yang-Mills theory.** In this subsection, we apply Yang-Mills theory to hyperholomorphic connections. We follow [V1].

**Proposition 4.8:** Let  $M$  be a hyperkähler manifold,  $L$  an induced complex structure and  $B$  be a complex vector bundle over  $(M, L)$ . Then every hyperholomorphic connection  $\nabla$  in  $B$  is Yang-Mills and satisfies  $\Lambda(\Theta) = 0$  where  $\Theta$  is a curvature of  $\nabla$ .

**Proof:** We use the definition of a hyperholomorphic connection as one with  $SU(2)$ -invariant curvature. Then Proposition 4.8 follows from the

**Lemma 4.9:** Let  $\Theta \in \Lambda^2(M)$  be a  $SU(2)$ -invariant differential 2-form on  $M$ . Then  $\Lambda_L(\Theta) = 0$  for each induced complex structure  $L$ .<sup>2</sup>

**Proof:** This is Lemma 2.1 of [V1]. ■

Let  $M$  be a compact hyperkähler manifold,  $I$  an induced complex structure. For any stable holomorphic bundle on  $(M, I)$  there exists a unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. It is possible to tell when this happens.

**Theorem 4.10:** Let  $B$  be a stable holomorphic bundle over  $(M, I)$ , where  $M$  is a hyperkähler manifold and  $I$  is an induced complex structure over  $M$ . Then  $B$  admits a compatible hyperholomorphic connection if and only if the first two Chern classes  $c_1(B)$  and  $c_2(B)$  are  $SU(2)$ -invariant.<sup>3</sup>

**Proof:** This is Theorem 2.5 of [V1]. ■

From Theorem 4.10 it follows that hyperholomorphic bundles can be described in two ways: either as holomorphic objects over  $(M, I)$ , or as certain types of connections. The first definition implies that there exists a complex structure on the moduli of hyperholomorphic connections. The second implies that if we replace  $I$  by another induced complex structure, the  $C^\infty$ -structure of the moduli of hyperholomorphic connections remains the same. In other words, the real analytic variety underlying the moduli of hyperholomorphic connections admits a set of complex structures parametrized by  $CP^1$ . Using Kodaira relations, it is easy to check that these complex structures satisfy quaternionic relations. We obtain the following

**Corollary 4.11:** ([V-h], Subsection 10.2) The moduli space of hyperholomorphic bundles is singular hyperkähler.

<sup>2</sup>By  $\Lambda_L$  we understand the Hodge operator  $\Lambda$  associated with the Kähler complex structure  $L$ .

<sup>3</sup>We use Lemma 2.9 to speak of action of  $SU(2)$  in cohomology of  $M$ .

## 5. HYPERCOMPLEX VARIETIES: THE DESINGULARIZATION

The Desingularization Theorem is stated as follows.

**Theorem 5.1:** (Desingularization theorem) Let  $M$  be a hypercomplex variety  $I$  an integrable induced complex structure. Let

$$\widetilde{(M, I)} \xrightarrow{n} (M, I)$$

be the normalization of  $(M, I)$ . Then  $\widetilde{(M, I)}$  is smooth and has a natural hypercomplex structure  $\mathcal{H}$ , such that the associated map  $n : \widetilde{(M, I)} \rightarrow (M, I)$  agrees with  $\mathcal{H}$ . Moreover, the hypercomplex manifold  $\widetilde{M} := \widetilde{(M, I)}$  is independent from the choice of induced complex structure  $I$ .

**Proof:** This is [V-h], Theorem 6.2 ■

In this Section, we give a sketch of a proof of Theorem 5.1. This sketch assumes some background in commutative algebra. Some readers might prefer to read [V-h], where we don't skip these details.

The idea of the proof is following. First of all, we prove Theorem 5.1 under an addition assumption, called LHS (locally homogeneous singularities). Then, we prove that LHS always holds for hypercomplex varieties. LHS is the following beast.

**Definition 5.2:** (local rings with LHS) Let  $A$  be a local ring. Denote by  $\mathfrak{m}$  its maximal ideal. Let  $A_{gr}$  be the corresponding associated graded ring for the  $\mathfrak{m}$ -adic filtration. Let  $\hat{A}$ ,  $\widehat{A_{gr}}$  be the  $\mathfrak{m}$ -adic completion of  $A$ ,  $A_{gr}$ . Let  $(\hat{A})_{gr}$ ,  $(\widehat{A_{gr}})_{gr}$  be the associated graded rings, which are naturally isomorphic to  $A_{gr}$ . We say that  $A$  has **locally homogeneous singularities** (LHS) if there exists an isomorphism  $\rho : \hat{A} \rightarrow \widehat{A_{gr}}$  which induces the standard isomorphism  $i : (\hat{A})_{gr} \rightarrow (\widehat{A_{gr}})_{gr}$  on associated graded rings.

**Definition 5.3:** (SLHS) Let  $X$  be a complex or real analytic space. Then  $X$  is called a **space with locally homogeneous singularities** (SLHS) if for each  $x \in X$ , the local ring  $\mathcal{O}_x X$  has locally homogeneous singularities.

To say that a ring is graded is the same as to say that it is equipped with an action of  $\mathbb{C}^*$ . Therefore, a local ring is LHS if and only if its completion is equipped with an action  $\rho$  of  $\mathbb{C}^*$ , and  $\rho$  acts on its tangent space by dilatations. This is why we use the word the word "homogeneous".

The following proposition was the main result of [V-d].

**Proposition 5.4:** Let  $M$  be a hypercomplex variety, and  $I$  an induced complex structure. Assume that the complex variety  $(M, I)$  has locally homogeneous singularities (LHS). Then the normalization of  $(M, I)$  is smooth.

**Proof:** Let  $O_x$  be the adic completion of the localization of the structure sheaf  $\mathcal{O}_{(M,I)}$  in  $x \in M$ . The normalization is compatible with the adic completions ([M], Chapter 9, Proposition 24.E). Therefore to prove that the normalization of  $(M, I)$  is smooth we need only to show that the normalization of  $O_x$  is regular. Since  $(M, I)$  is LHS, the ring  $O_x$  is isomorphic to the completion of the coordinate ring  $\mathcal{O}(Z_x)$  of the Zariski tangent cone  $Z_x$  of  $(M, I)$ . Therefore, it suffices to show that the normalization of  $Z_x$  is smooth. On the other hand, by [V-h], Theorem 4.5, the Zariski tangent cone  $Z_x$  of  $(M, I)$  is hypercomplex (this is easy to see from the differential-geometric definition of the Zariski tangent cone). Moreover, the natural embedding of the Zariski tangent cone to the Zariski tangent space  $T_x M$  is compatible with the hypercomplex structure (the space  $T_x M$  is quaternionic, which follows immediately from the definition of a hypercomplex structure). The manifold  $T_x M$  is hyperkähler. It is well known (see, for instance, [V3]) that trianalytic subvarieties of hyperkähler manifolds are completely geodesic. Since the manifold  $T_x M$  is flat, a completely geodesic subvariety must be a union of planes. But the normalization of a union of planes is smooth. This finishes the proof of Proposition 5.4; for more details, please read [V-h] and [V-d2]. ■

To finish the sketch of the proof of Theorem 5.1, it remains to prove the following proposition, which is the main result of [V-d2].

**Proposition 5.5:** Let  $M$  be a hypercomplex variety. Then  $M$  is a space with locally homogeneous singularities (SLHS).

**Proof:** Let  $I$  be an induced complex structure,  $x \in M$  a point and  $O_x$  the adic completion of the localization  $\mathcal{O}_x(M, I)$  of the structure ring of  $(M, I)$ . To produce a grading on  $O_x$ , we need to construct an action  $\rho$  of  $\mathbb{C}^*$  on  $O_x$ , such that  $\rho$  acts by dilatations on the tangent space  $T_x(M, I)$ . This is done geometrically as follows.

Let  $x \in M$  be a point,  $\pi : \text{Tw} \rightarrow \mathbb{C}P^1$  a twistor space of  $M$ , and  $s_x : \mathbb{C}P^1 \rightarrow \text{Tw}$  the line corresponding to the set  $(i, x)$  where  $i$  runs through  $\mathbb{C}P^1$  (such lines are called **horizontal twistor lines**, see Definition 3.12). As we have mentioned before, for “generic” pair of points  $(\alpha, \beta)$  sufficiently close to  $s_x$ , there exists a unique twistor line  $s$  passing through  $\alpha$  and  $\beta$ . To be more precise, let  $I, I' \in \mathbb{C}P^1$  be distinct induced complex structures. Then  $s_x$  has a neighbourhood  $U$  such that for all  $\alpha \in \pi^{-1}(I) \cap U$ ,  $\beta \in \pi^{-1}(I') \cap U$ , there exists a unique twistor line  $s_{\alpha, \beta} : \mathbb{C}P^1 \rightarrow \text{Tw}$  passing through  $\alpha, \beta$ . Fix a point  $I'' \in \mathbb{C}P^1$  which is distinct from  $I$  and  $I'$ . Let  $\delta = (I'', x) \in (\mathbb{C}P^1, M) \cong \text{Tw}$  be the corresponding point of  $s_x$ . For each  $\alpha \in \pi^{-1}(I) \cap U$  there exists a unique twistor line  $s_{\alpha, \delta}$  passing through  $\alpha$  and  $\delta$ . Evaluating this map at  $I'$ ,



we obtain a point  $\beta$  in  $\pi^{-1}(I') \cap U$ . Consider this as an operation producing  $\beta$  from  $\alpha$ . Clearly, this way we obtain an isomorphism

$$\hat{\mathcal{O}}_x(M, I') \longrightarrow \hat{\mathcal{O}}_x(M, I)$$

where  $\hat{\mathcal{O}}_x(M, I')$ ,  $\hat{\mathcal{O}}_x(M, I)$  is an adic completion of a localization of a ring of regular functions on  $(M, I')$ ,  $(M, I)$ . This isomorphism depends from the parameter  $I''$ . Varying  $I''$ , we obtain different isomorphisms between  $\hat{\mathcal{O}}_x(M, I')$  and  $\hat{\mathcal{O}}_x(M, I)$ . A composition of two such isomorphisms is an automorphism of  $\hat{\mathcal{O}}_x(M, I)$ . A simple linear-algebraic argument shows that this automorphism acts as a dilatation on the tangent space  $T_x M$  (Lemma 5.13, [V-h]). This proves Proposition 5.5. ■

## 6. HYPERHOLOMORPHIC SHEAVES AND THEIR SINGULARITIES

**6.1. Stable sheaves and Yang-Mills connections.** In [BS], S. Bando and Y.-T. Siu developed the machinery allowing one to apply the methods of Yang-Mills theory to torsion-free coherent sheaves. In the course of this paper, we apply their work to generalise the results of [V1]. In this Subsection, we give a short exposition of their results.

**Definition 6.1:** Let  $X$  be a complex manifold, and  $F$  a coherent sheaf on  $X$ . Consider the sheaf  $F^* := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$ . There is a natural functorial map  $\rho_F : F \longrightarrow F^{**}$ . The sheaf  $F^{**}$  is called a **reflexive hull**, or **reflexization** of  $F$ . The sheaf  $F$  is called **reflexive** if the map  $\rho_F : F \longrightarrow F^{**}$  is an isomorphism.

**Remark 6.2:** For all coherent sheaves  $F$ , the map  $\rho_{F^*} : F^* \longrightarrow F^{***}$  is an isomorphism ([OSS], Ch. II, the proof of Lemma 1.1.12). Therefore, a reflexive hull of a sheaf is always reflexive.

**Claim 6.3:** Let  $X$  be a Kähler manifold, and  $F$  a torsion-free coherent sheaf over  $X$ . Then  $F$  (semi)stable if and only if  $F^{**}$  is (semi)stable.

**Proof:** This is [OSS], Ch. II, Lemma 1.2.4. ■

**Definition 6.4:** Let  $X$  be a Kähler manifold, and  $F$  a coherent sheaf over  $X$ . The sheaf  $F$  is called **polystable** if  $F$  is a direct sum of stable sheaves.

The admissible Hermitian metrics, introduced by Bando and Siu in [BS], play the role of the ordinary Hermitian metrics for vector bundles.

Let  $X$  be a Kähler manifold. In Hodge theory, one considers the operator  $\Lambda : \Lambda^{p,q}(X) \longrightarrow \Lambda^{p-1,q-1}(X)$  acting on differential forms on  $X$ , which is adjoint to the multiplication by the Kähler form. This operator is defined on differential forms with

coefficient in every bundle. Considering a curvature  $\Theta$  of a bundle  $B$  as a 2-form with coefficients in  $\text{End}(B)$ , we define the expression  $\Lambda\Theta$  which is a section of  $\text{End}(B)$ .

**Definition 6.5:** Let  $X$  be a Kähler manifold, and  $F$  a reflexive coherent sheaf over  $X$ . Let  $U \subset X$  be the set of all points at which  $F$  is locally trivial. By definition, the restriction  $F|_U$  of  $F$  to  $U$  is a bundle. An **admissible metric** on  $F$  is a Hermitian metric  $h$  on the bundle  $F|_U$  which satisfies the following assumptions

- (i): the curvature  $\Theta$  of  $(F, h)$  is square integrable, and
- (ii): the corresponding section  $\Lambda\Theta \in \text{End}(F|_U)$  is uniformly bounded.

**Definition 6.6:** Let  $X$  be a Kähler manifold,  $F$  a reflexive coherent sheaf over  $X$ , and  $h$  an admissible metric on  $F$ . Consider the corresponding Hermitian connection  $\nabla$  on  $F|_U$ . The metric  $h$  and the connection  $\nabla$  are called **Yang-Mills** if its curvature satisfies

$$\Lambda\Theta \in \text{End}(F|_U) = c \cdot \text{Id}$$

where  $c$  is a constant and  $\text{Id}$  the unit section  $\text{Id} \in \text{End}(F|_U)$ .

Further in this paper, we shall only consider Yang-Mills connections with  $\Lambda\Theta = 0$ .

**Remark 6.7:** By Gauss-Bonnet formule, the constant  $c$  is equal to  $\text{deg}(F)$ , where  $\text{deg}(F)$  is the degree of  $F$  (Definition 4.5).

One of the main results of [BS] is the following analogue of Uhlenbeck–Yau theorem (Theorem 4.7).

**Theorem 6.8:** Let  $M$  be a compact Kähler manifold, and  $F$  a coherent sheaf without torsion. Then  $F$  admits an admissible Yang–Mills metric is and only if  $F$  is polystable. Moreover, if  $F$  is stable, then this metric is unique, up to a constant multiplier.

**Proof:** In [BS], Theorem 6.8 is proved for Kähler  $M$  ([BS], Theorem 3). ■

**6.2. Stable sheaves over hyperkähler manifolds.** Let  $M$  be a compact hyperkähler manifold,  $I$  an induced complex structure,  $F$  a torsion-free coherent sheaf over  $(M, I)$  and  $F^{**}$  its reflexization. Recall that the cohomology of  $M$  are equipped with a natural  $SU(2)$ -action (Lemma 2.9). The motivation for the following definition is Theorem 4.10 and Theorem 6.8.

**Definition 6.9:** Assume that the first two Chern classes of the sheaves  $F, F^{**}$  are  $SU(2)$ -invariant. Then  $F$  is called **stable hyperholomorphic** if  $F$  is stable, and **semistable hyperholomorphic** if  $F$  can be obtained as a successive extension of stable hyperholomorphic sheaves.

Consider the natural  $SU(2)$ -action in the bundle  $\Lambda^i(M, B)$  of the differential  $i$ -forms with coefficients in a vector bundle  $B$ . Let  $\Lambda_{inv}^i(M, B) \subset \Lambda^i(M, B)$  be the bundle of  $SU(2)$ -invariant  $i$ -forms.

**Definition 6.10:** Let  $X \subset (M, I)$  be a complex subvariety of codimension at least 2, such that  $F|_{M \setminus X}$  is a bundle,  $h$  be an admissible metric on  $F|_{M \setminus X}$  and  $\nabla$  the associated connection. Then  $\nabla$  is called **hyperholomorphic** if its curvature

$$\Theta_{\nabla} = \nabla^2 \in \Lambda^2 \left( M, \text{End} \left( F|_{M \setminus X} \right) \right)$$

is  $SU(2)$ -invariant, i. e. belongs to  $\Lambda_{inv}^2 \left( M, \text{End} \left( F|_{M \setminus X} \right) \right)$ .

**Theorem 6.11:** Let  $M$  be a compact hyperkähler manifold,  $I$  an induced complex structure and  $F$  a reflexive sheaf on  $(M, I)$ . Then  $F$  admits a hyperholomorphic connection if and only if  $F$  is polystable hyperholomorphic in the sense of Definition 6.9. Moreover, such a connection is unique.

**Proof:** This is [V-c], Theorem 3.19. ■

The proof of Theorem 6.11 is based on an elementary linear algebra argument (see Lemma 2.10).

**6.3. Desingularization of hyperholomorphic sheaves.** Hyperholomorphic sheaves (at least ones with isolated singularities) can be desingularized in the same fashion as the hyperkähler varieties; in fact, almost the same argument applies to both cases.

**Theorem 6.12:** Let  $M$  be a hyperkähler manifold, not necessarily compact,  $I$  an induced complex structure, and  $F$  a reflexive coherent sheaf over  $(M, I)$  equipped with a hyperholomorphic connection (Definition 6.10). Assume that  $F$  has isolated singularities. Let  $\widetilde{M} \xrightarrow{\sigma} M$  be a blow-up of  $(M, I)$  in the singular set of  $F$ , and  $\sigma^*F$  the pullback of  $F$ . Then  $\sigma^*F$  is a locally trivial sheaf, that is, a holomorphic vector bundle.

**Proof:** This is [V-c], Theorem 6.1. ■

The idea of the proof is the following. We apply to  $F$  the methods used in the proof of Desingularization Theorem (Theorem 5.1). The main ingredient in the proof of Desingularization Theorem is the existence of a natural  $\mathbb{C}^*$ -action on the completion  $\hat{\mathcal{O}}_x(M, I)$  of the local ring  $\mathcal{O}_x(M, I)$ , for all  $x \in M$ . This  $\mathbb{C}^*$ -action identifies  $\hat{\mathcal{O}}_x(M, I)$  with a completion of a graded ring. Here we show that a sheaf  $F$  is  $\mathbb{C}^*$ -equivariant. Therefore, a germ of  $F$  at  $x$  has a grading, which is compatible with the natural

$\mathbb{C}^*$ -action on  $\hat{\mathcal{O}}_x(M, I)$ . Singularities of such reflexive sheaves can be resolved by a single blow-up.

**6.4. Quaternionic-Kähler geometry.** In this Subsection, we follow [V-c] (Section 7). We give here a number of preliminary constructions, which are used later on to describe the singularities of hyperholomorphic sheaves. These constructions are mostly due to A.Swann and T. Nitta ([Sw], [N1], [N2]).

**Definition 6.13:** ([Sal], [Bes]) Let  $M$  be a Riemannian manifold. Consider a bundle of algebras  $\text{End}(TM)$ , where  $TM$  is the tangent bundle to  $M$ . Assume that  $\text{End}(TM)$  contains a 4-dimensional bundle of subalgebras  $W \subset \text{End}(TM)$ , with fibers isomorphic to a quaternion algebra  $\mathbb{H}$ . Assume, moreover, that  $W$  is closed under the transposition map  $\perp : \text{End}(TM) \rightarrow \text{End}(TM)$  and is preserved by the Levi-Civita connection. Then  $M$  is called **quaternionic-Kähler**.

**Example 6.14:** Consider the quaternionic projective space

$$\mathbb{H}P^n = (\mathbb{H}^n \setminus 0) / \mathbb{H}^*$$

It is easy to see that  $\mathbb{H}P^n$  is a quaternionic-Kähler manifold. For more examples of quaternionic-Kähler manifolds, see [Bes].

A quaternionic-Kähler manifold is Einstein ([Bes]), i. e. its Ricci tensor is proportional to the metric:  $Ric(M) = c \cdot g$ , with  $c \in \mathbb{R}$ . When  $c = 0$ , the manifold  $M$  is hyperkähler, and its restricted holonomy group is  $Sp(n)$ ; otherwise, the restricted holonomy is  $Sp(n) \cdot Sp(1)$ . The number  $c$  is called **the scalar curvature** of  $M$ . Further on, we shall use the term *quaternionic-Kähler manifold* for manifolds with non-zero scalar curvature.

The quaternionic projective space  $\mathbb{H}P^n$  has positive scalar curvature.

Let  $M$  be a quaternionic-Kähler manifold, and  $W \subset \text{End}(TM)$  the corresponding 4-dimensional bundle. For  $x \in M$ , consider the set  $\mathcal{R}_x \subset W|_x$ , consisting of all  $I \in W|_x$  satisfying  $I^2 = -1$ . Consider  $\mathcal{R}_x$  as a Riemannian submanifold of the total space of  $W|_x$ . Clearly,  $\mathcal{R}_x$  is isomorphic to a 2-dimensional sphere. Let  $\mathcal{R} = \cup_x \mathcal{R}_x$  be the corresponding spherical fibration over  $M$ , and  $\text{Tw}(M)$  its total space. The manifold  $\text{Tw}(M)$  is equipped with an almost complex structure, which is defined in the same way as the almost complex structure for the twistor space of a hyperkähler manifold. This almost complex structure is known to be integrable (see [Sal]).

A role of  $SU(2)$ -invariant 2-forms is played by so-called  $B_2$ -forms.

**Definition 6.15:** Let  $SO(TM) \subset \text{End}(TM)$  be a group bundle of all orthogonal automorphisms of  $TM$ , and  $G_M := W \cap SO(TM)$ . Clearly, the fibers of  $G_M$  are

isomorphic to  $SU(2)$ . Consider the action of  $G_M$  on the bundle of 2-forms  $\Lambda^2(M)$ . Let  $\Lambda_{inv}^2(M) \subset \Lambda^2(M)$  be the bundle of  $G_M$ -invariants. The bundle  $\Lambda_{inv}^2(M)$  is called **the bundle of  $B_2$ -forms**. In a similar fashion we define  $B_2$ -forms with coefficients in a bundle.

**Definition 6.16:** In the above assumptions, let  $(B, \nabla)$  be a bundle with connection over  $M$ . The bundle  $B$  is called a  **$B_2$ -bundle**, and  $\nabla$  is called a  **$B_2$ -connection**, if its curvature is a  $B_2$ -form.

The  $B_2$ -bundles were introduced and studied by T. Nitta in a serie of papers ([N1], [N2] etc.)

Consider the natural projection  $\sigma : Tw(M) \rightarrow M$ . The following observation is clear (Claim 6.17 (ii) is, in fact, an immediate consequence of Claim 6.17 (i), which is proven by linear algebra).

**Claim 6.17:** ([V-c], Claim 7.13)

- (i): Let  $\omega$  be a 2-form on  $M$ . The pullback  $\sigma^*\omega$  is of type  $(1, 1)$  on  $Tw(M)$  if and only if  $\omega$  is a  $B_2$ -form on  $M$ .
- (ii): Let  $B$  be a complex vector bundle on  $M$  equipped with a connection  $\nabla$ , not necessarily Hermitian. The pullback  $\sigma^*B$  of  $B$  to  $Tw(M)$  is equipped with a pullback connection  $\sigma^*\nabla$ . Then,  $\nabla$  is a  $B_2$ -connection if and only if  $\sigma^*\nabla$  has curvature of Hodge type  $(1, 1)$ .

■

**Definition 6.18:** Let  $Tw(M)$  be the twistor space of a quaternioni-Kähler manifold  $M$ . A  $B_2$ -bundle  $F$  on  $M$  gives a holomorphic bundle  $F'$  on  $Tw(M)$ . We say that  $F'$  is a **twistor transform**, or **direct twistor transform** of  $F$ .

The  $B_2$ -bundle  $F$  can be recovered from  $F'$  ([V-c], Corollary 7.15). This procedure is called **the inverse twistor transform**.

In [Sw], A.Swann discovered a construction which relies a hyperkähler manifold with a special  $\mathbb{H}^*$ -action to every quaternionic-Kähler manifold of positive scalar curvature. This is done as follows.

Let  $\mathbb{H}^*$  be the group of non-zero quaternions. Consider an embedding  $SU(2) \hookrightarrow \mathbb{H}^*$ . Clearly, every quaternion  $h \in \mathbb{H}^*$  can be uniquely represented as  $h = |h| \cdot g_h$ , where  $g_h \in SU(2) \subset \mathbb{H}^*$ . This gives a natural decomposition  $\mathbb{H}^* = SU(2) \times \mathbb{R}^{>0}$ . Recall that  $SU(2)$  acts naturally on the set of induced complex structures on a hyperkähler manifold.

**Definition 6.19:** Let  $M$  be a hyperkähler manifold equipped with a free smooth action  $\rho$  of the group  $\mathbb{H}^* = SU(2) \times \mathbb{R}^{>0}$ . The action  $\rho$  is called **special** if the following conditions hold.

- (i): The subgroup  $SU(2) \subset \mathbb{H}^*$  acts on  $M$  by isometries.
- (ii): For  $\lambda \in \mathbb{R}^{>0}$ , the corresponding action  $\rho(\lambda) : M \rightarrow M$  is compatible with the hyperholomorphic structure (which is a fancy way of saying that  $\rho(\lambda)$  is holomorphic with respect to any of induced complex structures).
- (iii): Consider the smooth  $\mathbb{H}^*$ -action  $\rho_e : \mathbb{H}^* \times \text{End}(TM) \rightarrow \text{End}(TM)$  induced on  $\text{End}(TM)$  by  $\rho$ . For any  $x \in M$  and any induced complex structure  $I$ , the restriction  $I|_x$  can be considered as a point in the total space of  $\text{End}(TM)$ . Then, for all induced complex structures  $I$ , all  $g \in SU(2) \subset \mathbb{H}^*$ , and all  $x \in M$ , the map  $\rho_e(g)$  maps  $I|_x$  to  $g(I)|_{\rho_e(g)(x)}$ .

Speaking informally, this can be stated as “ $\mathbb{H}^*$ -action interchanges the induced complex structures”.

- (iv): Consider the automorphism of  $S^2T^*M$  induced by  $\rho(\lambda)$ , where  $\lambda \in \mathbb{R}^{>0}$ . Then  $\rho(\lambda)$  maps the Riemannian metric tensor  $s \in S^2T^*M$  to  $\lambda^2 s$ .

**Example 6.20:** Consider the flat hyperkähler manifold  $M_{fl} = \mathbb{H}^n \setminus 0$ . There is a natural action of  $\mathbb{H}^*$  on  $\mathbb{H}^n \setminus 0$ . This gives a special action of  $\mathbb{H}^*$  on  $M_{fl}$ .

The case of a flat manifold  $M_{fl} = \mathbb{H}^n \setminus 0$  is the only case where we apply the results of this section. However, the general statements are just as difficult to prove, and much easier to comprehend.

**Definition 6.21:** Let  $M$  be a hyperkähler manifold with a special action  $\rho$  of  $\mathbb{H}^*$ . Assume that  $\rho(-1)$  acts non-trivially on  $M$ . Then  $M/\rho(\pm 1)$  is also a hyperkähler manifold with a special action of  $\mathbb{H}^*$ . We say that the manifolds  $(M, \rho)$  and  $(M/\rho(\pm 1), \rho)$  are **hyperkähler manifolds with special action of  $\mathbb{H}^*$  which are special equivalent**. Denote by  $H_{sp}$  the category of hyperkähler manifolds with a special action of  $\mathbb{H}^*$  defined up to special equivalence.

A. Swann ([Sw]) developed an equivalence between the category of quaternionic-Kähler manifolds of positive scalar curvature and the category  $H_{sp}$ .

Let  $Q$  be a quaternionic-Kähler manifold. The restricted holonomy group of  $Q$  is  $Sp(n) \cdot Sp(1)$ , that is,  $(Sp(n) \times Sp(1))/\{\pm 1\}$ . Consider the principal bundle  $\mathcal{G}$  with the fiber  $Sp(1)/\{\pm 1\}$ , corresponding to the subgroup

$$Sp(1)/\{\pm 1\} \subset (Sp(n) \times Sp(1))/\{\pm 1\}.$$

of the holonomy. There is a natural  $Sp(1)/\{\pm 1\}$ -action on the space  $\mathbb{H}^*/\{\pm 1\}$ . Let

$$\mathcal{U}(Q) := \mathcal{G} \times_{Sp(1)/\{\pm 1\}} \mathbb{H}^*/\{\pm 1\}.$$

Clearly,  $\mathcal{U}(Q)$  is fibered over  $Q$ , with fibers which are isomorphic to  $\mathbb{H}^*/\{\pm 1\}$ . We are going to show that the manifold  $\mathcal{U}(Q)$  is equipped with a natural hypercomplex structure.

There is a natural smooth decomposition  $\mathcal{U}(Q) \cong \mathcal{G} \times \mathbb{R}^{>0}$  which comes from the isomorphism  $\mathbb{H}^* \cong Sp(1) \times \mathbb{R}^{>0}$ .

Consider the standard 4-dimensional bundle  $W$  on  $Q$ . Let  $x \in Q$  be a point. The fiber  $W|_q$  is isomorphic to  $\mathbb{H}$ , in a non-canonical way. The choices of isomorphism  $W|_q \cong \mathbb{H}$  are called **quaternion frames in  $q$** . The set of quaternion frames gives a fibration over  $Q$ , with a fiber  $\text{Aut}(\mathbb{H}) \cong Sp(1)/\{\pm 1\}$ . Clearly, this fibration coincides with the principal bundle  $\mathcal{G}$  constructed above. Since  $\mathcal{U}(Q) \cong \mathcal{G} \times \mathbb{R}^{>0}$ , a choice of  $u \in \mathcal{U}(Q)|_q$  determines an isomorphism  $W|_q \cong \mathbb{H}$ .

Let  $(q, u)$  be the point of  $\mathcal{U}(Q)$ , with  $q \in Q$ ,  $u \in \mathcal{U}(Q)|_q$ . The natural connection in  $\mathcal{U}(Q)$  gives a decomposition

$$T_{(q,u)}\mathcal{U}(Q) = T_u\left(\mathcal{U}(Q)|_q\right) \oplus T_qQ.$$

The space  $\mathcal{U}(Q)|_q \cong \mathbb{H}^*/\{\pm 1\}$  is equipped with a natural hypercomplex structure.

This gives a quaternion action on  $T_u\left(\mathcal{U}(Q)|_q\right)$ . The choice of  $u \in \mathcal{U}(Q)|_q$  determines a quaternion action on  $T_qQ$ , as we have seen above. We obtain that the total space of  $\mathcal{U}(Q)$  is an almost hypercomplex manifold.

**Proposition 6.22:** (A. Swann) Let  $Q$  be a quaternionic-Kähler manifold. Consider the manifold  $\mathcal{U}(Q)$  constructed as above, and equipped with a quaternion algebra action in its tangent space. Then  $\mathcal{U}(Q)$  is a hypercomplex manifold.

**Proof:** This is [V-c], Proposition 7.22. ■

Consider the action of  $\mathbb{H}^*$  on  $\mathcal{U}(M)$  defined in the proof of Proposition 6.22. This action satisfies the conditions (ii) and (iii) of Definition 6.19. The conditions (i) and (iv) of Definition 6.19 are easy to check (see [Sw] for details). This gives a functor from the category  $\mathcal{C}$  of quaternionic-Kähler manifolds of positive scalar curvature to

the category  $H_{sp}$  of Definition 6.21. This is an equivalence of categories, constructed by A.Swann ([Sw]).

The inverse functor from  $H_{sp}$  to  $C$  is constructed by taking a quotient of  $M$  by the action of  $\mathbb{H}^*$ . Using the technique of quaternionic-Kähler reduction and hyperkähler potentials ([Sw]), one can equip the quotient  $M/\mathbb{H}^*$  with a natural quaternionic-Kähler structure. We call this equivalence **Swann's formalism for quaternionic-Kähler manifolds**.

**6.5. Swann's formalism for vector bundles.** Here we use the correspondence constructed by A.Swann to construct a correspondence between  $B_2$ -bundles on a quaternionic-Kähler manifold and  $\mathbb{H}^*$ -invariant hyperholomorphic bundles on the corresponding  $\mathbb{H}^*$ -invariant hyperkähler manifold. We follow [V-c], Section 8.

For the duration of this Subsection, we fix a hyperkähler manifold  $M$ , equipped with a special  $\mathbb{H}^*$ -action  $\rho$ , and the corresponding quaternionic-Kähler manifold  $Q = M/\mathbb{H}^*$ . Denote the standard quotient map by  $\varphi : M \rightarrow Q$ .

**Lemma 6.23:** Let  $\omega$  be a 2-form over  $Q$ , and  $\varphi^*\omega$  its pullback to  $M$ . Then the following conditions are equivalent

- (i):  $\omega$  is a  $B_2$ -form
- (ii):  $\varphi^*\omega$  is of Hodge type (1,1) with respect to some induced complex structure  $I$  on  $M$
- (iii):  $\varphi^*\omega$  is  $SU(2)$ -invariant.

**Proof:** The proof is elementary linear algebra ([V-c], Lemma 8.1). ■

The following proposition is an immediate corollary of Lemma 6.23

**Proposition 6.24:** ([V-c], Proposition 8.2) Let  $(B, \nabla)$  be a Hermitian vector bundle with connection over  $Q$ , and  $(\varphi^*B, \varphi^*\nabla)$  its pullback to  $M$ . Then the following conditions are equivalent

- (i):  $(B, \nabla)$  is a  $B_2$ -bundle
- (ii): The curvature of  $(\varphi^*B, \varphi^*\nabla)$  is of Hodge type (1,1) with respect to some induced complex structure  $I$  on  $M$
- (iii): The bundle  $(\varphi^*B, \varphi^*\nabla)$  is hypercomplex.

**Proof:** Follows from Lemma 6.23 applied to  $\omega = \nabla^2$ . ■

Let  $N$  be a hyperkähler manifold  $\sigma : \text{Tw}(N) \rightarrow N$  its twistor space and  $B$  a hyperholomorphic bundle. It is easy to check that the lift  $\sigma^*B$  is a holomorphic bundle on  $\text{Tw}(N)$ . The holomorphic structure  $\sigma^*B$  defines the connection on  $B$  in a unique way. This is called **direct and inverse twistor transform for hyperholomorphic**



**bundles** ([KV1]). A holomorphic bundle on  $\text{Tw}(N)$  is called **compatible with the twistor transform** if it is obtained from some hyperholomorphic bundle on  $NN$ .

Proposition 6.24 has the following fundamental corollary

**Theorem 6.25:** In the above assumptions, let  $\mathcal{C}_{B_2}$  be the category of  $B_2$ -bundles on  $Q$ , and  $\mathcal{C}_{\text{Tw}, \mathbb{C}^*}$  the category of  $\mathbb{C}^*$ -equivariant holomorphic bundles on  $\text{Tw}(M)$  which are compatible with the twistor transform. Consider the functor

$$(\sigma^* \varphi^*)^{0,1} : \mathcal{C}_{B_2} \longrightarrow \mathcal{C}_{\text{Tw}, \mathbb{C}^*},$$

$(B, \nabla) \longrightarrow (\sigma^* \varphi^* B, (\sigma^* \varphi^* \nabla)^{0,1})$ , constructed above. Then  $(\sigma^* \varphi^*)^{0,1}$  establishes an equivalence of categories.

**Proof:** This is [V-c], Theorem 8.5. ■

Now, let  $x \in R$  be an isolated singularity of a reflexive hyperholomorphic sheaf  $F$  over a hyperkähler manifold  $R$ . Consider the twistor space  $\text{Tw}(N)$  and the horizontal twistor line  $s_x : \mathbb{C}P^1 \longrightarrow \text{Tw}(N)$  corresponding to  $x$ . Let  $I$  be an ideal sheaf of  $s_x$ . Denote the associated graded sheaf by  $\mathcal{O}(\text{Tw}(N))_{gr}$ . Then  $\text{Spec}(\mathcal{O}(\text{Tw}(N))_{gr})$  is isomorphic to the twistor space of  $T_x M$ . Taking an associate graded sheaf of  $F$ , we obtain a sheaf  $F_{gr}$  over  $\text{Spec}(\mathcal{O}(\text{Tw}(N))_{gr}) \cong \text{Tw}(T_x M)$ . We proved the Desingularization Theorem by establishing the natural  $\mathbb{C}^*$ -action on the fibers  $(N, I)$  of the twistor projection  $\pi : \text{Tw}(N) \longrightarrow \mathbb{C}P^1$ . As we have shown, the sheaf  $F$  is  $\mathbb{C}^*$ -equivariant with respect to this  $\mathbb{C}^*$ -action. Therefore, the associated graded sheaf  $F_{gr}$  has the same singularities as  $F$ .

This reasoning leads to the following theorem.

**Theorem 6.26:** ([V-c], Theorem 8.15) Let  $N$  be a hyperkähler manifold,  $I$  an induced complex structure and  $F$  a reflexive sheaf on  $(N, I)$  admitting a hyperholomorphic connection. Assume that  $F$  has an isolated singularity in  $x \in N$ , and is locally trivial outside of  $x$ . Let  $\pi : \tilde{N} \longrightarrow (N, \widetilde{I})$  be the blow-up of  $(N, I)$  in  $x$ . Consider the holomorphic vector bundle  $\pi^* F$  on  $\tilde{M}$  (Theorem 6.12). Let  $C \subset (N, I)$  be the blow-up divisor,  $C = \mathbb{C}P T_x M$ . Clearly, the manifold  $C$  is canonically isomorphic to the twistor space of the quaternionic-Kähler manifold  $\mathbb{H}P(T_x N)$ . Then  $\pi^* F$  is the twistor transform of a  $B_2$ -bundle on  $\mathbb{H}P(T_x N)$ .

■

T.Nitta ([N2]) has shown that a bundle, obtained by twistor transform, is Yang-Mills (hence, direct sum of stable bundles of the same degree). This leads to the following corollary.

**Corollary 6.27:** In the assumptions of Theorem 6.26, consider the natural connection  $\nabla$  on the bundle  $\pi^*F|_C$  obtained from the twistor transform. Then  $\nabla$  is Yang-Mills, with respect to the Fubini-Study metric on  $C = \mathbb{P}T_x N$ , the degree  $\deg c_1(\pi^*F|_C)$  vanishes, and the holomorphic vector bundle  $\pi^*F|_C$  is a direct sum of stable bundles of the same degree.

■

## 7. CREPANT RESOLUTIONS AND HOLOMORPHIC SYMPLECTIC GEOMETRY

Let  $V$  be a complex symplectic vector space, and  $G$  a finite group acting on  $V$  by linear transformations preserving symplectic structure. The variety  $X = V/G$  is usually singular. In this section, we are working with the resolutions of these singularities.

The **crepant** resolutions of  $X$  are resolutions  $\pi : \tilde{X} \rightarrow X$  such that the canonical class of  $\tilde{X}$  is obtained as a pullback of a canonical class of  $X$ .

Further on, we shall assume that  $\pi : \tilde{X} \rightarrow X = V/G$  is a crepant resolution. In such a case, the manifold  $\tilde{X}$  is also holomorphically symplectic, which is quite easy to see ([V-r], Theorem 2.5).

Here is an example of such situation.

**Example 7.1:** The Hilbert scheme of  $n$  points on  $\mathbb{C}^2$  provides a crepant resolution of the quotient  $(\mathbb{C}^2)^n/S_n$  of  $(\mathbb{C}^2)^n$  by the natural action of the symmetric group  $S_n$  (this is well known; see e. g. [N]).

A reason to study the symplectic desingularization comes from the hyperkähler geometry. Consider a compact complex torus  $T$ ,  $\dim_{\mathbb{C}} T = 2$ , and its  $n$ -th Hilbert scheme of points  $T^{[n]}$ . Let  $Alb : T^{[n]} \rightarrow T$  be the Albanese map. A **generalized Kummer variety**  $K^{[n-1]}$  is defined as

$$K^{[n-1]} \subset T^{[n]}, \quad K^{[n-1]} := Alb^{-1}(0).$$

The variety  $K^{[n-1]}$  is smooth and holomorphically symplectic ([Bea]). By Calabi-Yau theorem ([Y], [Bea]), the variety  $K^{[n]}$  is equipped with a set of hyperkähler structures, parametrized by the Kähler cone.

In [KV2] (Section 4), it was shown that all trianalytic subvarieties of generalized Kummer varieties (at least, for generic hyperkähler structures) are isomorphic to symplectic desingularizations of a quotient of a compact torus by an action of a Coxeter group. This establishes a very interesting relation between the Dynkin diagrams and hyperkähler geometry, and motivates the study of symplectic desingularization of quotient singularities.

In [V-r], this argument was carried a step further, to obtain information about the structure of finite groups  $G \subset Sp(V)$  such that  $V/G$  admits a symplectic desingularization. This is done as follows. Let  $g \in \text{End}(V)$  be a symplectomorphism of finite order. We say that  $g$  is a **symplectic reflection** if

$$\text{codim}_V \left( \{x \in V \mid g(x) = x\} \right) = 2,$$

that is, the dimension of the fixed set of  $g$  is maximal possible for non-trivial  $g$ . This definition parallels that of complex reflections – a complex reflection is an endomorphism of finite order with fixed point set of codimension 1. The main result of [V-r] is the following theorem.

**Theorem 7.2:** ([V-r], Theorem 3.2) Let  $V$  be a symplectic vector space over  $\mathbb{C}$ , and  $G \subset Sp(V)$  a finite group of symplectic transformations. Assume that  $V/G$  admits a symplectic resolution. Then  $G$  is generated by symplectic reflections.

The proof of Theorem 7.2 is modeled on the proof of well-known theorem about groups generated by complex reflections (that is, endomorphisms which fix a subspace of codimension 1).

**Proposition 7.3:** ([Bou], Ch. V, §5 Theorem 4) Let  $V$  be a complex vector space, and  $G \subset GL(V)$  a finite group acting on  $V$ . Assume that  $X := V/G$  is smooth. Then  $G$  is generated by complex reflections. Conversely, if  $G$  generated by complex reflections, the quotient  $V/G$  is smooth.

Another important ingredient is the following theorem, which is based on elementary arguments from linear algebra.

**Definition 7.4:** Let  $\pi : \tilde{X} \rightarrow X$  be a resolution of singularities. The map  $\pi$  is called **semismall** if  $X$  admits a stratification  $\mathfrak{S}$  with open strata  $U_i$ , such that

$$\forall x \in U_i \mid \dim \pi^{-1}(x) \leq \frac{1}{2} \text{codim } U_i$$

**Theorem 7.5:** Let  $\pi : \tilde{X} \rightarrow X$  be a crepant resolution of a quotient singularity  $X = V/G$ ,  $G \in Sp(V)$ . Then  $\pi$  is semismall.

**Proof:** This statement easily follows from Proposition 4.16 and Proposition 4.5 of [V4] (see also [K1], Proposition 4.4). ■

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