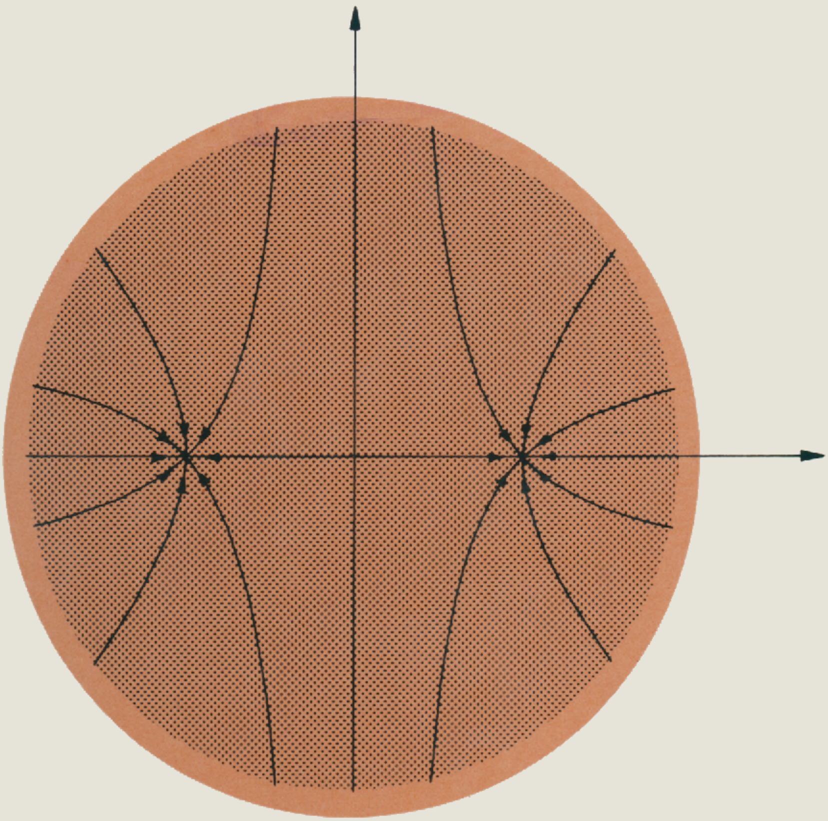
Second Edition

Groundwater Flow



A. Verruijt

Theory of Groundwater Flow

Consulting Editor: Professor E. M. Wilson, University of Salford

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Preface

This is the second edition of a book originally written in 1970 as a textbook for courses on the fundamentals of the theory of groundwater flow, including the most effective methods for solving problems in engineering practice involving groundwater flow.

In this second edition both steady and non-steady groundwater flow are considered, and the emphasis has shifted from analytical to numerical methods. The complex variable method is still treated rather extensively because it gives such simple and elegant solutions to a variety of problems, such as problems involving a free surface or an interface between fresh and salt groundwater.

The numerical methods discussed are the finite difference method and the finite element method. All details of these methods are given, in an elementary way, leading up to some simple computer programs. It is hoped that some readers will be encouraged to use these programs as a basis for more sophisticated computer programs.

Very little of the material presented in this book has been developed by me. Some care has been taken to mention the original sources. In addition to these references, however, I would like to thank my former teachers, in particular Professor G. De Josselin de Jong, several of my present colleagues, and many former students, for their stimulation and help. For typing the manuscripts I am indebted to my sister Alice, and to Mrs Ellen van der Salm.

A.V.

1

Introduction

It is the aim of this book to describe the most effective methods for solving problems of groundwater flow, as encountered in engineering practice. Among these problems are those connected with seepage through earth dams, underneath hydraulic structures, in natural or artificial infiltration, and flow towards systems of wells.

In general the problem is to determine the pressure distribution in the groundwater, and the flow rates, under certain conditions imposed on the boundaries of the soil mass. Mathematically speaking, the problem is in the class of boundary-value problems of mathematical physics. After some general considerations on the fundamental physical laws involved in the flow of groundwater, several methods for the solution of practical problems will be presented. These include elementary analytical solutions, more advanced techniques using conformal transformations or Laplace transforms, and various approximate methods. Special attention is paid to numerical methods, which are the most powerful of all.

In the first chapter some of the most important physical properties of soils and water will be discussed.

1.1 Properties of Soils

Natural soils consist of solid material, water and air. The water and air together fill the porespace between the solid grains. A measure of the amount of pores is provided by the porosity n, which is defined as the volume of the pores per unit total volume. For sandy soils the porosity is about 0.35 to 0.45. For natural clays and peat the porosity is usually in the range from 0.40 to 0.60, but in exceptional cases it may be even higher. The effective porosity ϵ denotes the pore volume which actually contributes to the flow of groundwater, which excludes dead-end pores, etc. In clays ϵ may be considerably smaller than n, but in sands the two quantities are almost equal.

The degree of saturation, S_r, is the volume of the water in the pores per unit total pore volume. It varies between 0, for a completely dry soil, and 1, for a completely saturated soil.

The coefficient of compressibility, α, is defined by the equation

$$\mathrm{d}h/\mathrm{d}\sigma' = -\alpha h \qquad (1.1)$$

where σ' is the vertical pressure on a soil sample of height h, the sample being confined horizontally, so that its cross-sectional area remains constant. Common values for the compressibility are 10^{-8} to 10^{-7} m²/N for sands, and 10^{-7} - 10^{-6} m²/N for clays. In soil mechanics literature the compressibility is usually denoted by m_v. Expressed in terms of the constants of the theory of elasticity, the compressibility is

$$\alpha = m_v = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} = \frac{1}{K+(4/3)G} \tag{1.2}$$

where E is Young's modulus, ν is Poisson's ratio, K is the bulk modulus and G is the shear modulus. It should be noted that natural soils are far from linearly elastic. The use of a unique compressibility is restricted to small stress changes, and it should be interpreted in the sense of a tangent modulus.

1.2 Properties of Water

The density of a material is defined as the mass per unit volume. For water the density, which will be denoted by ρ, is about 1000 kg/m³. The density may vary with pressure, temperature, and the concentration of dissolved material (for example, salt). An impression of the variation of ρ with temperature can be obtained from table 1.1.

TABLE 1.1 *Variation of Density and Viscosity of Water with Temperature*

Temperature (°C)	Density (kg/m³)	Dynamic viscosity (kg/m s)
0	999.868	1.79×10^{-3}
5	999.992	1.52×10^{-3}
10	999.727	1.31×10^{-3}
15	999.126	1.14×10^{-3}
20	998.230	1.01×10^{-3}

The unit weight, γ, is obtained by multiplying the density ρ by the gravity constant g (≈ 9.81 m/s²), $\gamma = \rho g$.

The dynamic viscosity, denoted by μ, is defined by the formula

$$\tau = \mu \, \mathrm{d}v_x/\mathrm{d}y \tag{1.3}$$

where τ is the shear stress producing a velocity gradient $\mathrm{d}v_x/\mathrm{d}y$ in a flow in the x-direction. For water the dynamic viscosity is about 10^{-3} kg/m s (see also table 1.1). The kinematic viscosity ν is defined by the relation $\nu = \mu/p$. Its value is about 10^{-6} m²/s for water.

The compressibility, β, describes the variation of the density of water with pressure. It is defined by the equation

$$\mathrm{d}\rho/\mathrm{d}p = \rho\beta \tag{1.4}$$

where p is the pressure in the fluid. For pure water the value of β is about

0.5×10^{-9} m^2/N. It should be noted, however, that the effective compressibility of the water in a porous material can be much larger because of entrapped air (see for example, Barends, 1980). As a first approximation one may use the following expression for the effective compressibility β'

$$\beta' = \beta + (1 - S_r)/p \tag{1.5}$$

where S_r is the degree of saturation and p is the fluid pressure.

2

Basic Equations for Steady Flow

In this chapter the basic equations of groundwater flow are presented, for the case of steady flow only. These basic equations are Darcy's law and the equation of continuity. The case of non-steady flow, with the possibility of storage of groundwater in the soil, is treated in chapter 7.

2.1 Darcy's Law

The science of groundwater flow originates from about 1856, in which year the city engineer of Dijon, Henri Darcy, published the results of the investigations that he had carried out for the design of a water supply system based on subsurface water carried to the valley in which Dijon is located, by permeable layers of soil, and supplied by rainfall on the surroundings. Since that time the basic law of groundwater movement carries his name, but the presentation has been developed, and the law has been generalised in several ways.

Before introducing Darcy's law it is useful to consider, as a point of reference, the hydrostatics in a porous medium, the pores of which are completely filled with a fluid of density ρ. If the pressure in the fluid is denoted by p the principles of hydrostatics teach that in the absence of flow the pressure increases with depth, and the local pressure gradient is equal to ρg, where g is the gravity constant. Thus, in a cartesian coordinate system, with the positive z-axis pointing upwards, if there is no flow

$$\begin{aligned} \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial p}{\partial z} + \rho g &= 0 \end{aligned} \qquad (2.1)$$

These equations express equilibrium of the pore fluid. They are independent of the actual pore geometry, provided that all the pores are interconnected.

In the case when the pore fluid moves with respect to the solid matrix a frictional resistance is generated, due to the viscosity of the fluid and the small dimensions of the pores. The essence of Darcy's experimental results, and those of later researchers, is that for relatively slow movements the frictional resistance is proportional to the flow rate. If inertia effects are disregarded, and if the porous medium is isotropic (that is, the geometry of the pore space is independent of the direction of flow), the equations of equilibrium can now be written as

$$\begin{aligned} \frac{\partial p}{\partial x} + \frac{\mu}{\kappa} q_x &= 0 \\ \frac{\partial p}{\partial y} + \frac{\mu}{\kappa} q_y &= 0 \\ \frac{\partial p}{\partial z} + \rho g + \frac{\mu}{\kappa} q_z &= 0 \end{aligned} \tag{2.2}$$

where μ is the viscosity of the fluid, and κ is a soil parameter indicating the so-called permeability of the porous medium. The quantities q_x, q_y and q_z are the three components of the specific discharge vector, where specific discharge denotes the discharge through a certain area of soil, divided by that area. Equations 2.2 are consistent with the formulae used in hydrodynamics for the movement of a viscous fluid through a thin tube, or for the flow between two parallel plates, at a small distance. In the first case, which was studied by Poiseuille and Hagen (see for example, Lamb, 1932, p. 331) the appropriate value of the constant κ appears to be $R^2/8$, where R is the radius of the tube. In the case of flow of a viscous fluid between two parallel plates, which was investigated by Hele Shaw (see for example, Lamb, 1932, p. 582), the value of κ to be used is $d^2/12$, where d is the distance between the two plates.

It is to be noted that equations 2.2 can be considered to be consistent with the Navier-Stokes equations of hydrodynamics, if in the latter all inertia effects are disregarded, and the terms due to viscous shear stresses are represented by some sort of average resistance, proportional to the flow rate, and inversely proportional to the characteristic parameter κ. Attempts to derive the forms of equation 2.2 in a more or less rigorous way from the Navier-Stokes equations, by an averaging process, do give some insight into the microscopical quantities involved, but they usually fail to give a quantitative expression for the permeability κ (see for example, Bear, 1972).

Equations 2.2 can also be written as follows

$$\begin{aligned} q_x &= -\frac{\kappa}{\mu}\frac{\partial p}{\partial x} \\ q_y &= -\frac{\kappa}{\mu}\frac{\partial p}{\partial y} \\ q_z &= -\frac{\kappa}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right) \end{aligned} \tag{2.3}$$

When the fluid density ρ is constant it is useful to introduce a new variable, the groundwater head ϕ, defined by

$$\phi = z + \frac{p}{\rho g} \tag{2.4}$$

In that case (that is, with ρ constant) one may write

$$q_x = -k \frac{\partial \phi}{\partial x}$$

$$q_y = -k \frac{\partial \phi}{\partial y} \tag{2.5}$$

$$q_z = -k \frac{\partial \phi}{\partial z}$$

where k is a new parameter, the so-called hydraulic conductivity

$$k = \kappa \rho g / \mu \tag{2.6}$$

In many publications the parameter k is also referred to as the coefficient of permeability. Written in the form of equations 2.5 the basic equations are equivalent to the original law discovered by Darcy, on the basis of experiments such as shown in figure 2.1. From his experiments Darcy found that the discharge through a sample of sand was directly proportional to the difference of the water levels at the two ends of the sample. It is obvious that the dis-

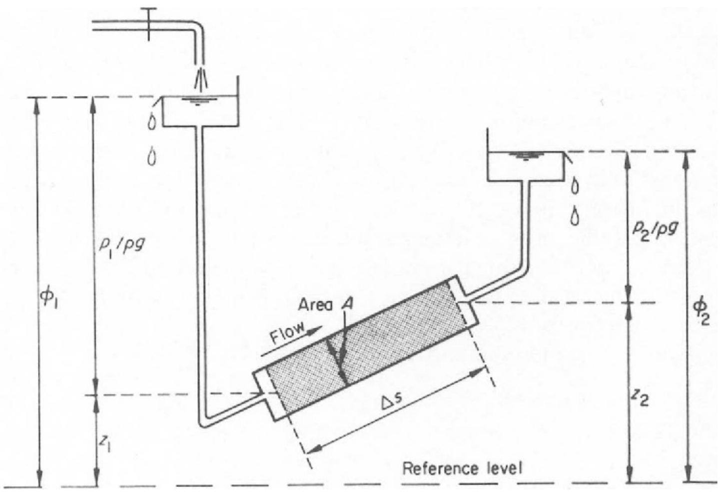

Figure 2.1 *Darcy's experiment*

charge is also proportional to the cross-sectional area of the sample, and the linear relationship between discharge and drop in water level implies that the discharge is inversely proportional to the length of the sample, as can be seen by imagining two samples of equal length to be placed one after the other. All this leads to the following formula

$$Q = kA \frac{\phi_1 - \phi_2}{\Delta s} = -kA \frac{\Delta\phi}{\Delta s} \tag{2.7}$$

If the specific discharge q is defined as Q/A one now obtains, when passing to the limit $\Delta s \to 0$

$$q = -k \frac{d\phi}{ds} \tag{2.8}$$

which is a special form of equations 2.5.

It should be noted that the average velocity of the water is Q/nA, which is greater than the specific discharge.

2.2 Permeability

In the previous section two parameters describing the permeability have been introduced, the permeability κ and the hydraulic conductivity k, related to each other by equation 2.6

$$k = \kappa \frac{\rho g}{\mu}$$

The most fundamental property is the permeability κ, which only depends upon the properties of the pore space. Because of the factors ρ and μ in this equation, the hydraulic conductivity also depends upon the fluid properties, in particular upon the viscosity. This means that it requires more effort to let a more viscous fluid flow through a porous medium, which is intuitively acceptable. It also means that the hydraulic conductivity, through the viscosity, depends upon the temperature. In areas with great fluctuations between the temperatures in summer and winter this may result in seasonal variations in the groundwater discharge.

Because of the analogy with the flow of a viscous fluid through a thin tube, already mentioned above, in which case the value of κ is $R^2/8$, it seems reasonable to expect that the hydraulic conductivity k will be proportional to the second power of some characteristic pore size. This is confirmed by experimental evidence. Both theoretical and experimental investigations have been carried out to establish a formula predicting the value of the permeability. The most familiar equation is that of Kozeny–Carman

$$\kappa = cd^2 \frac{n^3}{(1-n)^2} \tag{2.9}$$

where n is the porosity of the soil and d is some mean particle size. A convenient definition can be made in terms of the specific surface M (the total area of the wetted surface per unit volume of the solid material), namely

$$d = 6/M \tag{2.10}$$

The factor 6 has been introduced so that for a packing of equal spheres the value of d corresponds to the diameter of the spheres. The value of the constant c in equation 2.9, corresponding to the definition in equation 2.10 for d, and best fitting the experimental data is of the order of magnitude of $c \approx 1/180$. It should be realised that equations such as the one given above can at best give a rough idea of the value of the permeability. Because factors such as the angularity of the particles are ignored, the predicted value may differ considerably from the real value. It should also be noted that the experimental determination of the permeability, by letting the groundwater flow under controlled conditions, in the laboratory or in the field, is relatively easy. Thus the value of formulae such as equation 2.9 is actually only that they give an insight into the variation of permeability with porosity (which may be useful if one intends to densify the soil), or to give a first estimation of the permeability of a soil of which a grain size analysis, but no sample, is available.

The circumstance that the permeability κ is independant of the fluid, in contrast to the hydraulic conductivity, implies that for problems involving two fluids (such as oil and water, or fresh and salt water) Darcy's law should be formulated in terms of the permeability (see equation 2.3). In many applications in the field of groundwater flow only one single fluid is considered: fresh water. In such cases a formulation in terms of the hydraulic conductivity (see equation 2.5) is more convenient. In this book, which is mainly concerned with standard problems of the flow of fresh groundwater, this practice will be followed, using Darcy's law in the form of equation 2.5.

An indication of the numerical values of the permeability κ and the hydraulic conductivity k (for fresh groundwater) is given, in table 2.1, for certain soils frequently occurring in engineering practice.

TABLE 2.1 *The Order of Magnitude of the Permeability of Natural Soils*

	$\kappa\,(\mathrm{m}^2)$	$k\,(\mathrm{m/s})$
Clay	10^{-17} to 10^{-15}	10^{-10} to 10^{-8}
Silt	10^{-15} to 10^{-13}	10^{-8} to 10^{-6}
Sand	10^{-12} to 10^{-10}	10^{-5} to 10^{-3}
Gravel	10^{-9} to 10^{-8}	10^{-2} to 10^{-1}

The analogy of groundwater flow with the slow movement of a viscous fluid through a thin tube suggests that, just as in hydrodynamics, the flow must be laminar for the simple linear viscous relationships to be valid. This is needed the case, and it has been observed that for very rapid flows, as may occur in coarse materials such as gravel, deviations of Darcy's law occur due to turbulence. The region of applicability of Darcy's law can be defined in terms of a Reynolds number

$$Re = vd/\nu \tag{2.11}$$

where d is an average pore size (as defined above), v is the average flow rate, and ν is the kinematic viscosity ($\nu = \mu/\rho$). The critical value of Reynolds number for groundwater flow is of the order of magnitude of 10. Beyond that value a resistance due to turbulence, and proportional to the second power of the flow rate, plays a significant role. In this book restriction will be made to relatively slow movements, for which Darcy's law applies.

2.3 Anisotropy

In the preceding sections it has been assumed that the properties of the soil that are responsible for the resistance to flow are independent of the direction: such a material is said to be isotropic with regard to permeability. Not every soil possesses that property, however. In many soil deposits the resistance to flow in the vertical direction is considerably larger than the resistance to horizontal flow, due to the presence of a layered structure in the soil, generated by its geological history. For such anisotropic porous media Darcy's law has to be generalised. The proper generalisation is, in terms of hydraulic conductivity

$$\begin{aligned}
q_x &= -k_{xx}\frac{\partial\phi}{\partial x} - k_{xy}\frac{\partial\phi}{\partial y} - k_{xz}\frac{\partial\phi}{\partial z}\\
q_y &= -k_{yx}\frac{\partial\phi}{\partial x} - k_{yy}\frac{\partial\phi}{\partial y} - k_{yz}\frac{\partial\phi}{\partial z}\\
q_z &= -k_{zx}\frac{\partial\phi}{\partial x} - k_{zy}\frac{\partial\phi}{\partial y} - k_{zz}\frac{\partial\phi}{\partial z}
\end{aligned} \tag{2.12}$$

These equations express the most general linear relationship between the specific discharge vector and the gradient of the groundwater head. The coefficients $k_{xx}, \ldots, k_{zz}$ are said to be components of a second-order tensor. It is usually assumed, on the basis of thermodynamic considerations, that this is a symmetric tensor (that is, $k_{xy} = k_{yx}$, $k_{yz} = k_{zy}$, $k_{zx} = k_{xz}$). It can be shown that this means that there exist three mutually orthogonal directions, the so-called principal directions of permeability, in which the cross-components vanish. Physically speaking this means that a gradient of the groundwater head in one of these directions leads to a flow in that same direction. Although the considerations in this book will usually be restricted to isotropic soils it may be illuminating to consider here in some more detail the consequences of anisotropic permeability.

Consider the two-dimensional case of an anisotropic porous medium consisting of two orthogonal systems of channels of different cross-section, and hence of different resistance (see figure 2.2). In this case the principal directions coincide with the directions of the channels, and one may write

$$q_x = -k_{xx} \frac{\partial \phi}{\partial x}$$
$$q_y = -k_{yy} \frac{\partial \phi}{\partial y} \qquad (2.13)$$

If the channels in the x-direction are wider than those in the y-direction, the permeability k_{xx} will be greater than k_{yy}. Now consider the same situation being described with respect to coordinates ξ and η, which are obtained from x and y by a rotation through an angle α

$$\xi = x \cos \alpha + y \sin \alpha$$
$$\eta = y \cos \alpha - x \sin \alpha \qquad (2.14)$$

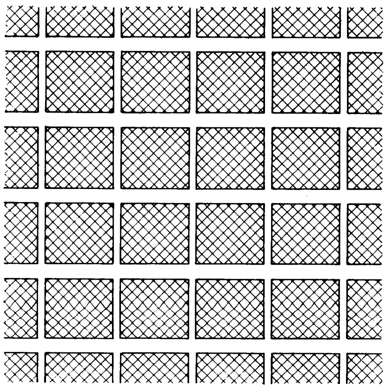

Figure 2.2 *Anisotropic porous medium*

A vector q with components q_x and q_y can also be decomposed into components q_ξ and q_η, where

$$q_\xi = q_x \cos \alpha + q_y \sin \alpha$$
$$q_\eta = q_y \cos \alpha - q_x \sin \alpha$$

or with equation 2.13

$$q_\xi = -k_{xx} \cos \alpha \frac{\partial \phi}{\partial x} - k_{yy} \sin \alpha \frac{\partial \phi}{\partial y}$$
$$q_\eta = -k_{yy} \cos \alpha \frac{\partial \phi}{\partial y} + k_{xx} \sin \alpha \frac{\partial \phi}{\partial x} \qquad (2.15)$$

The partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ can be related to $\partial\phi/\partial\xi$ and $\partial\phi/\partial\eta$ with the aid of equation 2.14. This gives

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial\xi}\cos\alpha - \frac{\partial\phi}{\partial\eta}\sin\alpha$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial y} = \frac{\partial\phi}{\partial\xi}\sin\alpha + \frac{\partial\phi}{\partial\eta}\cos\alpha$$

Using these expressions the equations 2.15 can be written as

$$q_\xi = -k_{\xi\xi}\frac{\partial\phi}{\partial\xi} - k_{\xi\eta}\frac{\partial\phi}{\partial\eta}$$
$$q_\eta = -k_{\eta\xi}\frac{\partial\phi}{\partial\xi} - k_{\eta\eta}\frac{\partial\phi}{\partial\eta} \qquad (2.16)$$

where

$$k_{\xi\xi} = k_{xx}\cos^2\alpha + k_{yy}\sin^2\alpha = \tfrac{1}{2}(k_{xx}+k_{yy}) - \tfrac{1}{2}(k_{yy}-k_{xx})\cos 2\alpha$$
$$k_{\eta\eta} = k_{yy}\cos^2\alpha + k_{xx}\sin^2\alpha = \tfrac{1}{2}(k_{xx}+k_{yy}) + \tfrac{1}{2}(k_{yy}-k_{xx})\cos\alpha$$
$$k_{\xi\eta} = k_{\eta\xi} = (k_{yy}-k_{xx})\sin\alpha\cos\alpha = \tfrac{1}{2}(k_{yy}-k_{xx})\sin 2\alpha \qquad (2.17)$$

Equations 2.16 are the description of a general flow, using the coordinates ξ and η. It is interesting to note the appearance of the cross-coefficients $k_{\xi\eta}$ and $k_{\eta\xi}$. These coefficients vanish only if $k_{xx} = k_{yy}$ (when the soil is isotropic) or if $\alpha = 0, \pi, \ldots$, etc. (when the ξ, η-coordinates coincide with x and y). It now follows that a gradient of the groundwater head in ξ-direction not only leads to a flow in that direction, but also to a flow in η-direction. This can be realised physically by noting (see figure 2.2) that in that case the channels in x-direction will transport much more water than the narrow channels in y-direction (because $k_{xx} > k_{yy}$). This means that the resultant flow will always have a tendency towards the most permeable direction. In the example this means a (negative) component of flow in the η-direction. It also means that in general anisotropy cannot be formulated by simply using different coefficients in the three directions in which Darcy's law is formulated. In general the anisotropy law should be of the form of equation 2.12, with six independent coefficients. Fortunately in engineering practice it is usually acceptable to distinguish only between the permeability in vertical direction and one in horizontal direction, assuming that this difference has been created during the geological process of deposition of the soil. Then it may be assumed that the x, y and z-directions are principal directions (if the z-axis is vertical), with $k_{xx} = k_{yy} = k_h$ and $k_{zz} = k_v$. Darcy's law can then be used in the form

$$q_x = -k_h\frac{\partial\phi}{\partial x}$$
$$q_y = -k_h\frac{\partial\phi}{\partial y} \qquad (2.18)$$
$$q_z = -k_v\frac{\partial\phi}{\partial z}$$

which involves only two coefficients. They must be measured by doing two independent tests. It is perhaps surprising that the horizontal permeability can conveniently be measured by a field test using a dipole arrangement along a vertical axis (Rietsema and Viergever, 1979).

2.4 Continuity

For the description of a groundwater movement Darcy's law alone is not enough, unless the distribution of the groundwater head ϕ were measured throughout the soil body. In general, especially in engineering, when a future situation must be predicted, the head ϕ is not known beforehand, and then Darcy's law only gives three relations between four unknown quantities: the three components of the specific discharge vector and the head. A fourth equation may be obtained by noting that the flow has to satisfy the fundamental physical law of conservation of mass. Whatever the pattern of flow, no mass can be gained or lost. In the present paragraph the considerations will be restricted to the case of a fully saturated and incompressible porous medium. Then, according to the principle of conservation of mass, there can be no net inward or outward flux to or from an elementary control volume in the soil (see figure 2.3). The mass flux through the left face of the elementary volume is

$$(\rho q_y)_1 \; \Delta x \; \Delta z$$

and this amount of water flows into the element. The net outward flux, taking into account the flow through all the six faces can be written as

$$[(\rho q_x)_2 - (\rho q_x)_1] \; \Delta y \; \Delta z + [(\rho q_y)_2 - (\rho q_y)_1] \; \Delta z \; \Delta x +$$
$$[(\rho q_z)_2 - (\rho q_z)_1] \; \Delta x \; \Delta y$$

This quantity must be zero according to the principle of conservation of mass (or the water balance principle, as it is often called in hydrology), there being no possibility for storage, because it has been assumed that the soil is incompressible and fully saturated. One now obtains, after dividing by $\Delta x \Delta y \Delta z$

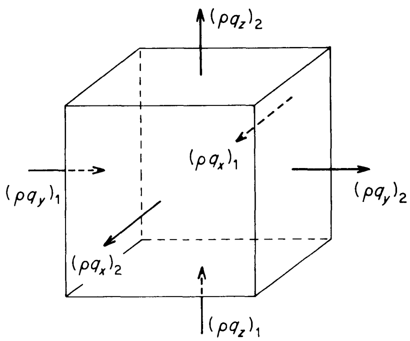

Figure 2.3 *Conservation of mass*

$$\frac{\partial (\rho q_x)}{\partial x} + \frac{\partial (\rho q_y)}{\partial y} + \frac{\partial (\rho q_z)}{\partial z} = 0 \tag{2.19}$$

This is the equation of conservation of mass, in the absence of storage (i.e. for steady flow conditions). In most cases the variations of the density ρ can be disregarded. Then it follows from equation 2.19, by assuming ρ to be constant

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0 \tag{2.20}$$

This is usually called the equation of continuity for steady flow. It is the fourth equation required to give a complete description, together with Darcy's law, of the groundwater flow phenomenon.

Substitution of Darcy's law (equation 2.5), into equation 2.20 gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \phi}{\partial z}\right) = 0 \tag{2.21}$$

which is a single equation in one variable only, the head ϕ. If it is assumed that the hydraulic conductivity is constant, thereby restricting the considerations to homogeneous porous media, equation 2.21 reduces to the so-called Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.22}$$

which is often written, in abbreviated form using the Laplace operator ∇^2, as

$$\nabla^2 \phi = 0 \tag{2.23}$$

The basic problem of steady groundwater flow is to find solutions to Laplace's equation, or, more generally, to equation 2.21, taking into account the appropriate boundary conditions. This is essentially a problem of applied mathematics. In most cases, however, simplifications must be made in order to solve a certain problem or a class of problems, and these simplifications involve a careful consideration of the physical nature of the groundwater flow in the problem under consideration, which may require a good engineering judgement or insight. In the following chapters several methods for solving certain classes of problems will be considered.

2.5 Problems

1. In a laboratory Darcy's experiment is used to determine the permeability of a sand sample. The length of the sample is 20 cm and its cross-sectional area is 10 cm^2. The difference in head between the two ends is 25 cm, and the amount of water flowing through the sample in 5 min is measured to be 75 cm^3. Calculate the value of the hydraulic conductivity k.
2. For a certain type of sand the hydraulic conductivity has been measured

to be $k = 2 \times 10^{-4}$ m/s, with water at a temperature of 20 °C (kinematic viscosity $\nu = 10^{-6}$ m^2/s). What would be the value at a temperature of 5 °C ($\nu = 1.5 \times 10^{-6}$ m^2/s)?

3. For a certain type of sand the hydraulic conductivity has been measured to be $k = 2 \times 10^{-4}$ m/s. The porosity of the sand was $n = 0.40$. What would be the hydraulic conductivity if the sand were compacted (for instance by vibration) so that the porosity is reduced to 0.35, without crushing the particles?
4. In engineering practice the hydraulic conductivity is often expressed in m/day, and in the United States in g pd/sq ft. Give the appropriate conversion factors to the SI unit of m/s.
5. In the petroleum industry the permeability κ is sometimes expressed in darcy's. The unit darcy is defined to be the permeability of a porous medium through which a fluid with a viscosity of 1 centipoise flows at a rate of 1 cm/s under a pressure gradient of 1 atmosphere per cm. Express the unit of 1 darcy in the SI unit of m^2.

3

The Aquifer Approach

In many cases of practical interest the groundwater flow occurs in a layered soil, consisting of permeable water-bearing layers, bounded at their bottom and/or their top surface by layers of very low permeability (clay layers), see figure 3.1. All these layers are more or less horizontal, so that it is reasonable to assume that in the permeable layers (denoted as aquifers) the flow is predominantly horizontal. The less permeable clay layers (sometimes called aquicludes) act as separation layers between the aquifers. When the bounding clay layers are completely impermeable the aquifer is said to be (completely) confined. An aquifer is considered to be semi-confined when the percolation through one or both of the bounding clay layers cannot be completely disregarded but is still small enough to justify the assumption that the flow in the aquifer is mainly horizontal. A third type of aquifer is the so-called unconfined aquifer, in which the upper boundary of the ground water is a free surface. The uppermost aquifer in figure 3.1 might be considered as unconfined. For the three types of aquifers mentioned above some elementary problems are considered in this chapter.

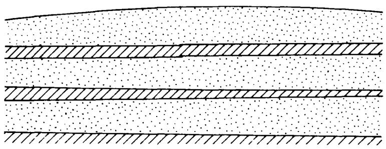

Figure 3.1 *Aquifers*

3.1 Confined Aquifers

In a completely confined aquifer which is practically horizontal, (see figure 3.2), there can be no flow across the upper and lower boundaries. It is then reasonable to disregard all flow perpendicular to the plane of the aquifer. This means that the head ϕ is assumed to be independent of z

$$\phi = \phi(x,y) \tag{3.1}$$

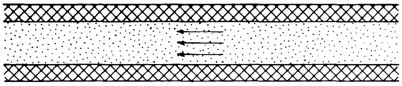

Figure 3.2 *Confined aquifer*

This is the main assumption in all aquifer schematisations. Only two relevant components of the specific discharge remain, namely

$$q_x = -k\frac{\partial\phi}{\partial x}$$
$$q_y = -k\frac{\partial\phi}{\partial y} \qquad (3.2)$$

If the aquifer is fully saturated and incompressible, so that no storage of groundwater can occur, the equation of continuity must now express the fact that the net outward flux from an elementary control volume of dimensions Δx, Δy and of thickness H, is zero, that is

$$\frac{\partial}{\partial x}(q_x H) + \frac{\partial}{\partial y}(q_y H) = 0 \qquad (3.3)$$

Substitution from equation 3.2 into equation 3.3 gives

$$\frac{\partial}{\partial x}\left(T\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial\phi}{\partial y}\right) = 0 \qquad (3.4)$$

which is the basic equation of steady groundwater flow in a confined aquifer, with $T = kH$, the so-called transmissivity of the aquifer. If the transmissivity T is constant the differential equation reduces to Laplace's equation in two dimensions

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (3.5)$$

3.1.1 *Radial Flow*

As an elementary example the case of flow towards a well in the centre of a circular island will be discussed (see figure 3.3). In this case it is convenient to introduce polar coordinates r and θ, because it can be expected that the head ϕ will be a function of the radius r only. Then one obtains, using the chain rule of differentiation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{d^2\phi}{dr^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{d\phi}{dr}\frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{d^2 \phi}{dr^2} \left(\frac{\partial r}{\partial y}\right)^2 + \frac{d\phi}{dr} \frac{\partial^2 r}{\partial y^2}$$

Because $r = (x^2 + y^2)^{\frac{1}{2}}$ this gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} \tag{3.6}$$

Thus the differential equation becomes

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} = 0 \tag{3.7}$$

which can also be written in the more compact form

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr}\right) = 0 \tag{3.8}$$

Successive integration now leads to the following general solution

$$\phi = A \ln r + B \tag{3.9}$$

where A and B are constants, to be determined from the boundary conditions.

As boundary conditions one could impose certain fixed values for the head for two values of r (an interior and exterior boundary of the soil mass). A more realistic set of boundary conditions for the case under consideration is to assume a constant head (ϕ_0) at the outer boundary, and to impose that at the inner boundary a certain given discharge (Q_0) is extracted from the soil. Then the boundary conditions are

$$\begin{aligned} r = R&: \ \phi = \phi_0 \\ r = r_w&: \ 2\pi r H q_r = -2\pi T r \frac{d\phi}{dr} = -Q_0 \end{aligned} \tag{3.10}$$

where Darcy's law has been used in the form $q_r = -k \ d\phi/dr$, and where the minus sign before Q_0 means that the discharge Q_0 flows in the negative r-direction, that is, towards the well. With these boundary conditions the constants A and B can easily be determined, and the final solution becomes

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r}{R}\right) \tag{3.11}$$

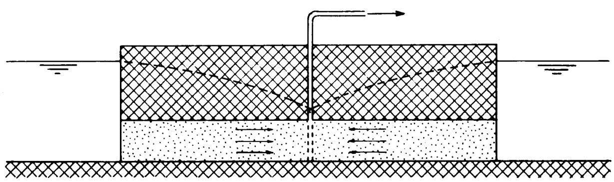

Figure 3.3 *Radial flow in confined aquifer*

Since in the region occupied by the soil body r is always smaller than R, the logarithm is always negative, and this indicates that the head ϕ is everywhere lower than its boundary value ϕ_0 (provided that $Q > 0$). Thus, if water is extracted from the soil, the head is lowered, as one would expect.

The water level in the well is obtained by putting $r = r_w$ in equation 3.11. This gives

$$\phi_w = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_w}{R}\right) \tag{3.12}$$

In engineering practice the discharge Q_0 is usually produced by a pump. Such a pump is characterised by a relationship between discharge and drawdown. This relationship may be such that over a certain range of values for the drawdown the discharge is practically constant. If that is the case the boundary conditions used in this example are relevant.

The solution of equation 3.11 for a well in a circular confined aquifer has certain interesting properties. First, it should be noted that the total discharge Q through the surface of a circular cylinder of radius r and height H is equal to $2\pi r H q_r$, and this is found to be equal to $-Q_0$. This quantity is independent of r, which is not very surprising since it is an immediate consequence of the principle of continuity. It may also be noted that the solution in equation 3.11 is independent of the radius of the well, r_w. This means that the influence of a well upon the head at a certain distance depends only upon the discharge of the well, and not upon its radius. One might be tempted to think that this suggests the use of a very thin pipe, which can be driven into the ground very easily, for the well. However, the price to be paid for this is, that the drawdown in the well itself may become very large (see equation 3.12), perhaps even to such an extent that the pump is unable to produce the discharge Q_0.

Another interesting property of equation 3.11 is that when R, the outer radius of the aquifer, becomes very large, the solution degenerates since $\ln(r/R)$ tends to $-\infty$ if R tends to $+\infty$. This means that it is impossible to extract water at an ever constant rate from a confined aquifer of infinite extent. There seems to be something fair in this statement: there can be no constant production of water without adequate supply from somewhere else. More light on this question will be shed in section 7.2, when considering the non-steady case.

3.2 Semi-confined Aquifers

A semi-confined aquifer is a water-bearing layer bounded by two layers of low premeability through one (or both) of which small amounts of water percolate, thus contributing to the discharge in the aquifer, (see figure 3.4). Even though there must be a certain vertical component of flow in this case, it may still be assumed that the flow is predominantly horizontal. For the major components of the specific discharge Darcy's law states that

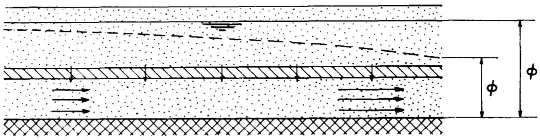

Figure 3.4 *Semi-confined aquifer*

$$q_x H = -T \frac{\partial \phi}{\partial x} \qquad q_y H = -T \frac{\partial \phi}{\partial y} \tag{3.13}$$

In the equation of continuity the supply of water to an elementary volume due to the leakage must now be taken into account, hence

$$\frac{\partial}{\partial x}(q_x H) + \frac{\partial}{\partial y}(q_y H) = L \tag{3.14}$$

where L is the leakage, expressed as a specific discharge. If the leakage occurs through a single clay layer of thickness d and hydraulic conductivity k' one may write, on the basis of Darcy's law for the vertical flow through the clay layer

$$L = k' \frac{\phi' - \phi}{d} \tag{3.15}$$

where ϕ' is the groundwater head in the aquifer on the other side of the clay layer. Because in equation 3.15 only the ratio of permeability and thickness appears, this is often written as

$$L = \frac{\phi' - \phi}{c} \tag{3.16}$$

where c is the so-called resistance of the clay layer, defined by $c = d/k'$. Substitution of equations 3.13 and 3.16 into equation 3.14 gives

$$\frac{\partial}{\partial x}\left(T \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T \frac{\partial \phi}{\partial y}\right) - \frac{\phi - \phi'}{c} = 0 \tag{3.17}$$

When the aquifer possesses a constant transmissivity T this equation can be written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{\phi - \phi'}{\lambda^2} = 0 \tag{3.18}$$

where λ is a new parameter, defined by

$$\lambda^2 = Tc$$

λ, the so-called leakage factor, has the dimension of length.

3.2.1 *One-dimensional Flow*

As a first example let us consider the case of a semi-confined aquifer bounded by a straight canal. Above the aquifer a constant water level ϕ' is maintained, which is lower than the water level ϕ_0 in the canal (see figure 3.5). Direct

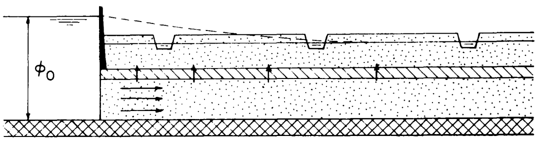

Figure 3.5 *One-dimensional flow in semi-confined aquifer*

flow from the canal into the upper aquifer is prevented by a dike of very small width, but there is of course some leakage into that aquifer through the lower aquifer and the clay layer. The groundwater head ϕ in the lower aquifer can be determined by solving the differential equation 3.18, which now reduces to

$$\frac{d^2\phi}{dx^2} - \frac{\phi - \phi'}{\lambda^2} = 0 \qquad (3.19)$$

because there is no flow in the y-direction in this case.

The general solution equation 3.19 is

$$\phi = \phi' + A \exp(x/\lambda) + B \exp(-x/\lambda) \qquad (3.20)$$

The constants A and B have to be determined from the boundary conditions, which should express the fact that at the left side boundary the head is equal to the water level in the canal, and that at great distances from the canal, say at infinity, the head in the aquifer is equal to the water level above it

$$x = 0: \quad \phi = \phi_0$$

$$x \to \infty: \quad \phi = \phi'$$

From these conditions the constants A and B can be determined as $A = 0$, and $B = \phi_0 - \phi'$. The solution now becomes

$$\phi = \phi' + (\phi_0 - \phi') \exp(-x/\lambda) \qquad (3.21)$$

In figure 3.5 the head in the aquifer is indicated by a dashed line.

An interesting quantity to be calculated is the total discharge. The specific discharge is first determined from equation 3.21 and Darcy's law

$$q_x = -k \frac{d\phi}{dx} = k \, [(\phi_0 - \phi')/\lambda] \exp(-x/\lambda)$$

The total discharge through a section of the canal of length L is obtained by multiplication of the specific discharge at $x = 0$ by the area HL

$$Q = kHL \, (\phi_0 - \phi')/\lambda \qquad (3.22)$$

It may be noted that this is the same amount of water as would flow under completely confined conditions through a soil body of length λ under the influence of a difference in head $\phi_0 - \phi'$. The leakage factor λ also determines how large an aquifer must be in order that it can be considered as infinite. Because exp(-3) = 0.0498 and exp(-4) = 0.0183 it follows that the head, as expressed by equation 3.21, is practically equal to ϕ', the head above the aquifer, at a distance of 3 or 4 times the value of λ. The value of λ appears to determine the magnitude of the zone of influence of the water level in the canal. An aquifer of which the dimensions are 3 or 4 times λ can be considered as infinite.

3.2.2 *Radial Flow*

As a second example of flow in a semi-confined aquifer, the case of a well in an aquifer of infinite extent is considered (see figure 3.6). Here the flow is purely radial, and the Laplace operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ can be replaced by equation 3.6. Thus the differential equation 3.18 reduces to

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi - \phi'}{\lambda^2} = 0 \tag{3.23}$$

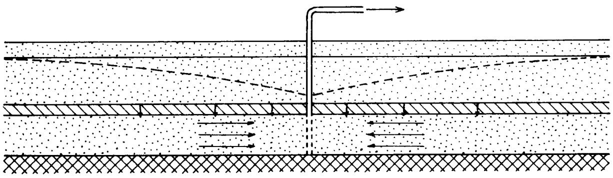

Figure 3.6 *Radial flow in semi-confined aquifer*

The general solution of this differential equation is

$$\phi = \phi' + AI_0\,(r/\lambda) + BK_0\,(r/\lambda) \tag{3.24}$$

where $I_0\,(x)$ and $K_0\,(x)$ are the so-called modified Bessel functions of the first and second kind, respectively, see appendix A. The boundary conditions are

$$r = r_w:\ r\frac{d\phi}{dr} = \frac{Q_0}{2\pi T}$$

$$r \to \infty:\ \phi = \phi'$$

It follows from the second boundary condition that $A = 0$, because $I_0\,(x)$ tends to infinity for $x \to \infty$, whereas $K_0\,(x)$ vanishes for $x \to \infty$. From the first boundary condition one obtains

$$B = -\frac{Q_0}{2\pi T}\,\frac{1}{(r_w/\lambda)\,K_1\,(r_w/\lambda)} \tag{3.25}$$

In engineering practice the value of λ is usually of the order of magnitude of several hundred metres, and r_w, the radius of the well, is much smaller. Thus

$r_w/\lambda \ll 1$. In that case the expression $(r_w/\lambda)\,K_1\,(r_w/\lambda)$ is very close to 1 (see appendix A) so that equation 3.25 reduces to

$$B = -\frac{Q_0}{2\pi T} \tag{3.26}$$

Substitution of the values of A and B just obtained into equation 3.24 finally gives as the solution of the problem

$$\phi = \phi' - \frac{Q_0}{2\pi T}\,K_0\,(r/\lambda) \tag{3.27}$$

Again the value of λ determines the zone of influence of the well. Because $K_0\,(4) = 0.0112$ the influence of the well becomes very small at a distance of about 4 times λ.

Close to the well equation 3.27 becomes impractical, because the tables usually do not give data for very small values of the argument. In that case the Bessel function can be approximated by a logarithm (see appendix A). Hence

$$r \ll \lambda: \ \phi = \phi' + \frac{Q_0}{2\pi T}\ln\left(\frac{r}{1.123\lambda}\right) \tag{3.28}$$

This means that in the vicinity of the well the groundwater head varies in the same manner as in the case of a completely confined aquifer (see equation 3.11). The apparent radius (or equivalent radius) is 1.123λ. The observation that in the vicinity of the pumping well the head is described by a logarithmic formula has a general validity. The drawdown s due to a well, in its vicinity, can always be written as

$$s = -\frac{Q_0}{2\pi T}\ln\left(\frac{r}{R_{eq}}\right) \tag{3.29}$$

where the actual value of the parameter R_{eq} depends upon the nature and the geometry of the problem.

3.3 Unconfined Aquifers

An unconfined aquifer is a water-bearing layer in which the upper boundary of the groundwater is an interface with the open air. In the aquifer approach it is usually assumed that there exists a sharp interface separating a completely saturated zone from a completely dry zone. In reality such a sharp interface (the groundwater table) does not exist, because there is always a certain transition zone in which the degree of saturation S (the volume of water per unit volume of the pore space) gradually varies from 1 to 0 (see figure 3.7). A well-defined surface is the so-called phreatic surface: the locus of points in which the pressure in the water is equal to the atmospheric pressure. Because of capil-

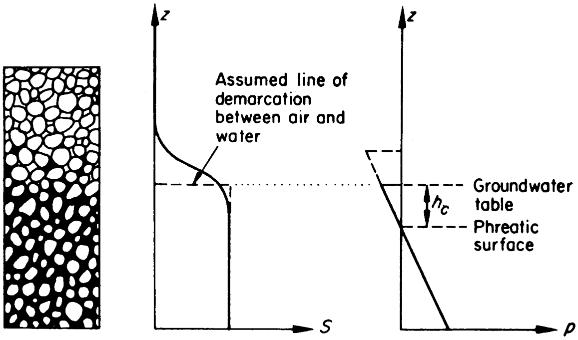

Figure 3.7 *Groundwater table and phreatic surface*

lary forces in the narrow channels in the soil the groundwater table is usually located somewhat higher than the phreatic surface. The difference is called the capillary height, and is denoted by h_c. In sands the value of h_c is usually about 10 or 20 cm, but in clays it can be as large as several metres. In this book the difference between groundwater table and phreatic surface will be disregarded, which is justified, as a first approximation, for fairly permeable aquifers of sufficient thickness. The error made in neglecting 10 cm of water over a total thickness of say 10 m, is insignificant.

A major difficulty in unconfined aquifers may seem to be that one cannot simply disregarded the vertical component of flow, because the water table need not be horizontal, and thus a water particle must be given the possibility to move from a higher position to a lower one. However, it may be argued that in many cases the horizontal flow components will be much greater than the vertical one; the slope of the phreatic surface will usually be in the order of magnitude of 1 : 100 or even smaller. This implies that the derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ in the horizontal plane will be much greater than $\partial\phi/\partial z$. By way of approximation the value of $\partial\phi/\partial z$, the variation of the groundwater head with depth, will now be disregarded entirely. Physically speaking this assumption (usually called Dupuit's assumption) means that the head ϕ along any vertical line is constant. Because the pressure is zero at the upper boundary the head there is equal to the elevation of the water table, and thus one may write Dupuit's assumption as

$$\phi = h(x, y) \tag{3.30}$$

where h denotes the height of the free surface, measured from the bottom of the aquifer, which is assumed to be impermeable. The basic differential equation for the flow of groundwater in an unconfined aquifer will be derived from Darcy's law for the flow in the horizontal directions, and from the principle of continuity. It should be noted that the use of Darcy's law for the vertical flow component is avoided, in order to prevent the statement that this is now exactly zero.

The components of the specific discharge in the x- and y-directions are, from Darcy's law

$$q_x = -k \frac{\partial h}{\partial x}, \qquad q_y = -k \frac{\partial h}{\partial y} \tag{3.31}$$

For steady flow conditions (which are assumed to apply in this chapter) the principle of continuity requires that no water is stored in an elementary volume, as indicated in figure 3.8. This can be expressed mathematically as

$$\frac{\partial}{\partial x}(q_x h) + \frac{\partial}{\partial y}(q_y h) - N = 0 \tag{3.32}$$

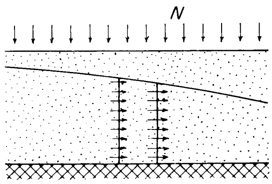

Figure 3.8 *Continuity in unconfined aquifer*

Substitution of equation 3.31 into equation 3.32 yields terms of the form

$$\frac{\partial}{\partial x}\left(k\,h\,\frac{\partial h}{\partial x}\right)$$

which are non-linear in h. Following Dupuit it is convenient to introduce h^2 as the variable, because one may write

$$h\,\frac{\partial h}{\partial x} = \tfrac{1}{2}\,\frac{\partial (h^2)}{\partial x}$$

If it is assumed that the hydraulic conductivity k is constant (homogeneous aquifer) the basic differential equation now becomes, from equations 3.31 and 3.32

$$\frac{\partial^2 (h^2)}{\partial x^2} + \frac{\partial^2 (h^2)}{\partial y^2} + \frac{2N}{k} = 0 \tag{3.33}$$

where N represents the infiltration rate from above (due to precipitation). Written in this form the differential equation is linear, in the variable h^2. This is a very valuable property, not only since many solutions of linear differential equations are available, but also because it enables the superposition of solutions (see chapter 4). Some simple examples of solutions of equation 3.33 will be discussed in this chapter.

3.3.1 *One-dimensional Flow*

The simplest example of unconfined flow is that of the flow through an aquifer bounded by two very long parallel canals, without infiltration (see figure 3.9). In this case, where h is a function of x only, the basic differential equation 3.33 reduces to

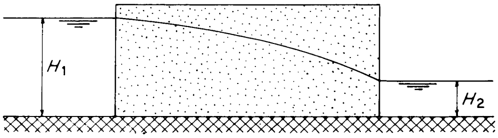

Figure 3.9 *One-dimensional flow in unconfined aquifer*

$$\frac{d^2 (h^2)}{dx^2} = 0 \tag{3.34}$$

with the general solution

$$h^2 = Ax + B \tag{3.35}$$

Again the two integration constants A and B have to be determined from the boundary conditions. In the case illustrated in figure 3.9 these are

$$x = 0: h = H_1$$

$$x = \mathrm{L}: h = H_2$$

With these boundary conditions the constants A and B can be evaluated. Substitution of the result into equation 3.35 then yields the following final solution

$$h^2 = H_1^2 + (H_2^2 - H_1^2)\, x/L \tag{3.36}$$

This equation expresses that the water table is of parabolic shape, (see figure 3.9). The greater slope at the downstream side indicates that there the flow rate is larger because the water has to flow through a smaller area.

The total discharge through a section of length B (perpendicular to the plane drawn in figure 3.9) can be obtained as follows

$$Q = Bh\, q_x = -k\, Bh \frac{dh}{dx} = -\frac{kB}{2} \frac{d(h^2)}{dx}$$

With equation 3.36 one now obtains

$$Q = \frac{k\, B\, (H_1^2 - H_2^2)}{2L} \tag{3.37}$$

This formula was first derived by Dupuit in 1863. Although it has been derived by disregarding the variation of the head with the vertical coordinate z, and although the so-called seepage surface at the downstream boundary was not taken into consideration, the formula has been found to give excellent results, even when the length L is very small and the head difference $H_1 - H_2$ is very large. An extensive discussion of this surprising precision has been given by Muskat (1937), who also presented results of other approximate calculations and investigations using electrical analogues, which all indicated that the error in Dupuit's formula must be extremely small. The matter was finally settled by Charny in 1951, who demonstrated, by a simple and ingenious argument, that equation 3.37 is an exact formula (see for example, Polubarinova-Kochina, 1962).

A very crude derivation of Dupuit's formula, which may serve as an aid to memorising it, is by simply writing the total discharge as the product of the specific discharge and the area

$$Q = qA$$

For the specific discharge one may estimate, on the basis of Darcy's law

$$q = k(H_1 - H_2)/L$$

and for the cross-sectional area a reasonable estimation seems to be the width B multiplied by the average thickness, $\frac{1}{2}(H_1 + H_2)$. Thus one obtains immediately

$$Q = \frac{kB(H_1^2 - H_2^2)}{2L}$$

which happens to be just Dupuit's formula.

3.3.2 *Radial Flow*

As a second example of unconfined flow consider the case of a well in the centre of a circular island, see figure 3.10. In this case the differential equation 3.33 reduces to

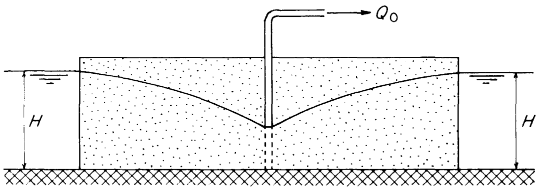

Figure 3.10 *Radial flow in unconfined aquifer*

$$\frac{d^2(h^2)}{dr^2} + \frac{1}{r}\frac{d(h^2)}{dr} = 0 \tag{3.38}$$

where use has been made of equation 3.6 to express the Laplace operator in polar coordinates for radial symmetry. The solution of this differential equation is, analogous to the solution for the confined case (see equations 3.7 and 3.9)

$$h^2 = A \ln r + B$$

The constants A and B can be determined from the boundary conditions

$$r = R:\ h = H$$

$$r = r_w:\ 2\pi r h q_r = -\pi k r \frac{d(h^2)}{dr} = -Q_0$$

where as before $h\,dh/dr$ has been written as $\frac{1}{2}\,d\,(h^2)/dr$. The final result is

$$h^2 = H^2 + \frac{Q_0}{\pi k} \ln\left(\frac{r}{R}\right) \tag{3.39}$$

This is the formula for the position of the phreatic surface in an unconfined aquifer of radius R, with a central well producing a discharge Q_0, in the absence of precipitation.

When the drawdown s is introduced as

$$s = H - h$$

then one may write $h^2 = (H - s)^2 = H^2 - 2Hs + s^2 = H^2 - 2Hs(1 - s/2H)$. Hence equation 3.39 can be written as

$$s(1 - s/2H) = -\frac{Q_0}{2\pi k H} \ln\left(\frac{r}{R}\right) \tag{3.40}$$

Written in this form the correspondence with the formula for a central well in a confined aquifer, equation 3.11, is striking (see also equation 3.29). In fact, when the drawdown is small compared to the original height of the water table, the term $(1 - s/2H)$ can be taken equal to 1, and the difference between an unconfined and a confined aquifer completely disappears. In engineering practice this is sometimes accurate enough, as a first approximation. It is to be noted, however, that with a little effort, a much better approximation for the flow in an unconfined aquifer can be obtained, by using Dupuit's approach, based upon the differential equation 3.33 in terms of h^2.

3.4 Problems

1. In the centre of a circular confined aquifer of thickness $H = 10$ m, hydraulic conductivity $k = 10^{-4}$ m/s and radius $R = 2000$ m, a well is constructed by installing a filter piper of radius $r_w = 0.20$ m, surrounded by a gravel pack of radius $r_a = 1$ m. The hydraulic conductivity of the gravel is 10^{-2} m/s. The production of the well is 10^{-3} m^3/s. Calculate the drawdown in the well, and the drawdown just outside the gravel pack.

2. Calculate the total leakage into a circular polder (see figure 3.11) of radius $R = 1000$ m, if the head in the polder is maintained at a level 5 m below the water level in the lake surrounding the polder, and using the values $k = 10^{-5}$ m/s, $H = 10$ m for the lower aquifer, and $d = 1$ m, $k' = 10^{-10}$ m/s for the confining layer.

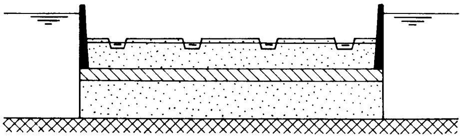

Figure 3.11 *Problem 2*

3. An unconfined aquifer is bounded by two very long parallel straight canals, 200 m apart, in which the water level is 10 m above the impermeable bedrock. Over the entire aquifer infiltration occurs at a rate $N = 10^{-8}$ m/s. The hydraulic conductivity of the aquifer is 10^{-5} m/s. Calculate the water level in the middle of the aquifer.

4. Extend the solution given in equation 3.39 for the water level in an unconfined aquifer in case of radial flow, by taking into account a uniform infiltration rate N.

4

Some General Aspects of Two-dimensional Problems

In this chapter some general aspects of two-dimensional problems, and some particular solutions, will be discussed. The considerations will be restricted to the flow in regions having a very simple geometrical shape, such as an aquifer of infinite extent, or a circular aquifer. The solution of problems for more general types of regions will be treated in subsequent chapters.

4.1 Superposition

The differential equation for two-dimensional steady flow in a homogeneous confined aquifer was found to be (see section 3.1)

$$\frac{\partial \phi^2}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4.1}$$

where ϕ is the groundwater head. Because this is a linear and homogeneous differential equation the principle of superposition applies, that is, if $\phi = \phi_1\ (x, y)$ is a solution and $\phi = \phi_2\ (x, y)$ is another solution then a linear combination $\phi = A\phi_1\ (x, y) + B\phi_2\ (x, y)$ is also a solution. This property can easily be verified by substitution into equation 4.1. The validity of the superposition principle then appears to be a consequence of the linearity of the differential operators, such as $\partial(A\phi_1 + B\phi_2)/\partial x = A\partial\phi_1/\partial x + B\partial\phi_2/\partial x$.

Superposition of solutions is especially valuable for constructing solutions for the flow generated by systems of isolated wells. By the appropriate addition of the elementary solutions for a single well it is possible to establish the solution for the case of flow due to a number of wells. The procedure can most conveniently be explained on the basis of some examples.

The first example refers to a case of two wells, of opposite discharge, in an infinite aquifer. This is usually called a pair of recharge and discharge wells, or a source and a sink (see figure 4.1). The solution is required to have the following properties.

(1) It should satisfy Laplace's equation 4.1 everywhere, except in the two points $x = \pm p, y = 0$, the points where the wells are located.

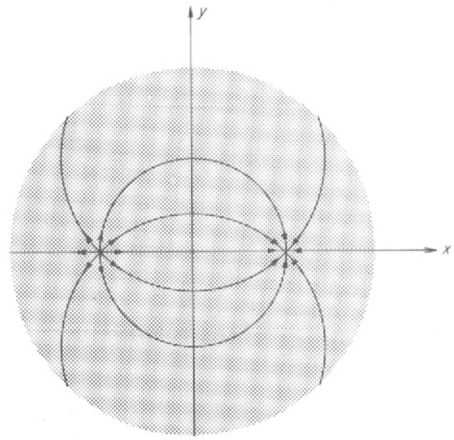

Figure 4.1 *Source and sink in confined flow*

(2) In one point, say at $x = -p, y = 0$, an amount of water Q_0 should be supplied to the aquifer.

(3) In the second singular point (that is, $x = p, y = 0$) an amount of water Q_0 should be extracted from the aquifer.

(4) At infinity the groundwater head must be constant, say $\phi = \phi_0$.

A solution satisfying all these requirements can be written down on the basis of the elementary solution in equation 3.11 for a well in a circular aquifer

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r}{R}\right)$$

The first term in the right-hand side provides for the value of the head along the outer boundary ($r = R$), and the second term describes the influence of the well. This suggests as a possible solution of the problem considered here, the equation

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1}{R}\right) - \frac{Q_0}{2\pi T} \ln\left(\frac{r_2}{R}\right) \tag{4.2}$$

or

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1}{r_2}\right) \tag{4.3}$$

It can immediately be verified that this solution satisfies requirements 2, 3 and 4, the boundary conditions. That it also satisfies requirement 1 is a consequence of the linearity of the differential equation, and needs no verification, although verification of course will show that equation 4.3 is indeed a solution of Laplace's equation. Hence equation 4.3 is the correct solution of the problem. It is interesting to note that the quantity R, the radius of the (very large) outer boundary no longer appears in the final expression, equation 4.3. This is due to the fact that the two composing terms in equation 4.2 have precisely the same coefficient, or, physically speaking, that the recharge in the source is just equal to the discharge in the sink. The pair of wells constitute a system of hydraulic equilibrium, which does not need any external supply of water. As a consequence the solution in equation 4.3 can be considered to apply in an aquifer of infinite extent.

Very close to the sink (that is, if $r_1 << p$) the value of r_2 can be set equal to $2p$, the distance between the source and the sink. The solution then reduces to

$$r_1 << p:\ \phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1}{2p}\right) \tag{4.4}$$

which is again of the form of equation 3.29, involving a logarithm of the form $\ln(r/R_{eq})$. In this case the equivalent radius R_{eq} appears to be $2p$.

An interesting property of the solution in equation 4.3 is that along the y-axis (that is, for $x = 0$) the drawdown is zero. This can be seen from figure 4.1. For any point on the y-axis $r_1 = r_2$ and hence the logarithm in equation 4.3 vanishes, because $\ln 1 = 0$. This property can be considered to be a consequence of the symmetry of the system with respect to the y-axis, together with the linearity of the problem. The drawdown due to the sink is exactly balanced by the rise of the head due to the source in points at equal distances from both.

A second example of the use of the superposition principle concerns the case of a system of two sinks, of equal discharge Q_0, again at a mutual distance $2p$, in a circular aquifer of radius R. This radius R is supposed to be very large compared to p, so that the sinks can each be considered, as a first approximation, to be operating very close to the centre of the aquifer. By addition of the influences of the two individual wells one now obtains as the probable solution

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1 r_2}{R^2}\right) \tag{4.5}$$

That this is indeed the correct solution will not be verified in detail here. It should be noted, however, that the condition along the outer boundary ($\phi = \phi_0$ if $r = R$) is satisfied only in an approximate way. Along that boundary both r_1 and r_2 become almost equal to R, so that $r_1 r_2/R^2$ approaches 1, but this is not exactly true. The error can be seen to be of the order $0\,(p/R)$, which means that the solution is applicable only if $p << R$.

In the vicinity of one of the sinks the head can again be written in the standard form, equation 3.29

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \left(\frac{r_1}{R^2/2p}\right) \tag{4.6}$$

In this case the equivalent radius is found to be $R^2/2p$.

Because of the symmetry of the system it can be expected that the solution

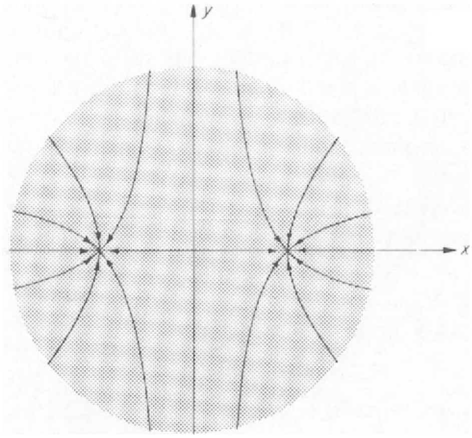

Figure 4.2 *Two sinks in confined flow*

possesses some special property along the axis of symmetry ($x = 0$). In fact the resultant flow in points of the y-axis (see figure 4.2) due to the influence of both wells is directed along the y-axis. This means that no water crosses the y-axis, which therefore might be considered as an impermeable screen. This property will be used in the so-called method of images (see section 4.2).

To conclude this section, the solution for a system of n sinks, numbered from $j = 1$ to $j = n$, at points with coordinates x_j, y_j, and of strength Q_j, will be given. A negative value of Q_j indicates a source. The solution for such a system, operating in a circular confined aquifer with a very large radius R, can be written down immediately by using superposition

$$\phi = \phi_0 + \frac{1}{2\pi T} \sum_{j=1}^{n} Q_j \ln\left(\frac{r_j}{R}\right) \tag{4.7}$$

or, in terms of cartesian coordinates

$$\phi = \phi_0 + \frac{1}{2\pi T} \sum_{j=1}^{n} Q_j \ln \frac{[(x - x_j)^2 + (y - y_j)^2]^{\frac{1}{2}}}{R} \tag{4.8}$$

When the sum of all discharges is zero the wells constitute a system of hydraulic equilibrium. Since in that case $\Sigma Q_j \ln R$ vanishes, equation 4.7 reduces to

$$\Sigma Q_j = 0: \ \phi = \phi_0 + \frac{1}{2\pi T} \sum_{j=1}^{n} \ln r_j \tag{4.9}$$

This solution is also applicable in case of an aquifer of infinite extent.

4.1.1 *Unconfined Aquifers*

It has been seen in chapter 3 that problems of unconfined flow can most conveniently be expressed in terms of a variable h^2. In the absence of infiltration the basic differential equation is again Laplace's equation, $\nabla^2 (h^2) = 0$, with basic solutions of the same form as in the case of a confined aquifer. Because the differential equation is a linear one, the superposition principle can be applied, provided that the solutions are expressed in terms of h^2. The solution for a system of n wells in an unconfined circular aquifer, having a very large radius R, is for instance

$$h^2 = H^2 + \frac{1}{\pi k} \sum_{j=1}^{n} Q_j \ln (r_j/R) \tag{4.10}$$

where use has been made of the elementary equation 3.39.

When applying the principle of superposition some care may be needed in adjusting the constant term, H^2 in equation 4.10. This constant should be chosen such that the boundary condition along the outer boundary is correctly satisfied. This condition should always be checked separately, which may result in the need to add another particular solution, namely $h^2 = C$, where C is a constant. In the present case all values r_j approach R along the outer boundary, so that all logarithms approach zero, and h^2 will be practically equal to H^2. Hence equation 4.10 indeed satisfies the condition that $h = H$ along the outer boundary.

4.1.2 *Semi-confined Aquifers*

The differential equation for flow in a semi-confined aquifer, equation 3.18

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{\phi - \phi'}{\lambda^2} = 0 \tag{4.11}$$

is linear, but not homogeneous, because of the appearance of the constant term ϕ'. This means that the superposition principle can be applied to this differential equation, provided that a particular solution is always added. A convenient particular solution is $\phi = \phi'$. As an example the solution for a system of n wells in a semi-confirmed aquifer of infinite extent will be given. This solution is

$$\phi = \phi' - \frac{1}{2\pi T} \sum_{j=1}^{n} Q_j \, K_0(r_j/\lambda) \tag{4.12}$$

where use has been made of the elementary equation 3.27.

4.2 Method of Images

Several problems involving the flow induced by sources and sinks in aquifers of relatively simple geometrical form, such as a half-plane, a quarter-plane, or an infinite strip, can be solved in a simple way by means of the so-called method of images. The method is most easily explained by considering some specific examples.

The first example concerns the problem of a well near a very long straight canal (or river, or lake) in which the water level is constant. The confined aquifer is assumed to occupy a semi-infinite region bounded by an infinite straight line, the half-plane $x > 0$ in figure 4.3. The solution should have the following properties.

(1) It should satisfy Laplace's equation 4.1 everywhere in the half-plane $x > 0$, except in the point $x = p, y = 0$.

(2) In the point $x = p$, $y = 0$ an amount of water Q_0 should be extracted from the aquifer.

(3) At infinity the groundwater head must be constant, say $\phi = \phi_0$.

(4) Along the y-axis (that is, for $x = 0$), the head must also remain constant, $\phi = \phi_0$.

It is now assumed that these four requirements are sufficient to ensure that there is one and only one solution. In fact we now touch upon such fundamental aspects of the theory as existence and uniqueness of solution. To prove mathematically that there exists one solution satisfying the four conditions listed above is not very easy, and is outside the scope of this book. The interested reader is referred to treatises on potential theory, such as the one by Kellogg (1929). For the present purpose it seems sufficient to assume, on physical grounds, that the four conditions given constitute a complete mathematical description of a unique situation that can be realised in the field. If

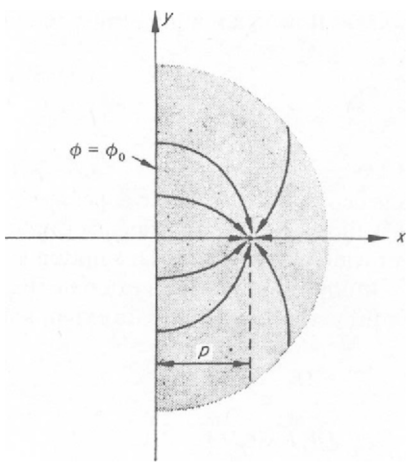

Figure 4.3 *Well near straight canal*

there is just one solution that satisfies all four conditions, then it does not matter in what way this solution is obtained, and what other properties this solution may have in addition to the four original requirements, for instance in the half-plane $x < 0$.

A function satisfying all four conditions listed above is given by equation 4.3, which represented the solution for a source and a sink in an infinite confined aquifer

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1}{r_2}\right) \tag{4.13}$$

As already noted in the preceding section this function certainly satisfies conditions 1 to 3, and it also satisfies the condition 4, because of symmetry. It is irrelevant that this equation also has a value for points in the half-plane $x < 0$. These points are located outside the boundary of the region considered.

The interesting feature of equation 4.13 is that this function has now been used as the solution for two apparently different problems. One might say that the present problem has been solved by replacing the semi-infinite aquifer by an infinite one, and locating an (imaginary) source in such a point (in fact, $x = -p$, $y = 0$) that the line $x = 0$ becomes a line of constant head. The point $x = -p$, $y = 0$ is the image point of the point $x = p$, $y = 0$, in which the sink is operating, when the axis $x = 0$ is considered as a mirror. This suggests the name 'image method' in association with the technique just described.

As a second example of an application of the method of images, involving a different type of image well, the problem of a sink operating at a distance p from a straight impermeable boundary is considered, see figure 4.4. The aquifer is again confined. The outer boundary of the aquifer is a half-circle

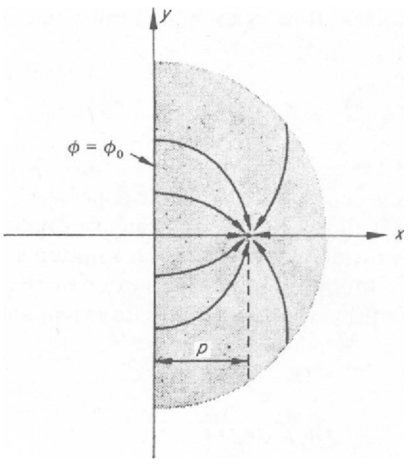

Figure 4.4 *Well near straight impermeable boundary*

of radius R, with $R >> p$. In this case the solution must have the properties that it satisfies Laplace's equation everywhere inside the half-circle to the right of the axis $x = 0$, except at the point $x = p, y = 0$, where a discharge Q_0 is extracted from the soil, the head must be constant ($\phi = \phi_0$) along the circle of radius of R, and the y-axis must be impermeable.

Guided by the observation that the solution found in the preceding section for a system of two sinks in equation 4.5, happened to have the last-mentioned property, one may be tempted to investigate whether that same function perhaps also represents the solution of the problem considered here. It turns out, by verification, that this is indeed the case. It may therefore be concluded that the solution of the present problem is

$$\phi = \phi_0 + \frac{Q_0}{2\pi T} \ln\left(\frac{r_1 r_2}{R^2}\right) \tag{4.14}$$

Here r_2 represents the distance to the imaginary sink at $x = -p, y = 0$. Again, as in the first example, the solution has physical meaning only for $x > 0$.

In this case the image method consists of replacing the original aquifer by a circular aquifer of radius R, and putting an imaginary sink in the image point $x = -p, y = 0$.

In conclusion, it is recalled that a straight line of zero drawdown can be constructed by locating sources in the image points of sinks (and sinks in the image points of sources). A straight impermeable boundary is obtained by locating sinks in the image points of sinks (and sources in the image points of sources).

As a further example of the success of the method of images the case of a sink operating in a semi-confined aquifer bounded by a straight canal and an impermeable boundary perpendicular to it may be mentioned (see figure 4.5). The solution consists of constructing first an impermeable boundary along the x-axis, by locating a sink in the image point $x = p, y = -q$, and then making the y-axis a line of zero drawdown by locating sources in the image points of the sinks (that is, at $x = -p, y = \pm q$). The final solution then is

$$\phi = \phi_0 - \frac{Q_0}{2\pi T} \left[K_0(r_1/\lambda) + K_0(r_2/\lambda) - K_0(r_3/\lambda) - K_0(r_4/\lambda)\right] \tag{4.15}$$

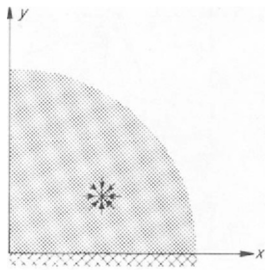

Figure 4.5 *Method of images*

The fact that the problem considered here refers to a semi-confined aquifer does not prohibit the application of the image method, because the basis of the method is the linearity of the differential equation, and the symmetry or antisymmetry of the problem.

As a final example the solution for a well in a circular aquifer will be given, for the case that the eccentricity is not small compared to the radius of the island (see figure 4.6). The solution of this problem is a well-known particular solution from potential theory. In order also to demonstrate the applicability of the method of images to unconfined flow the aquifer is assumed to be unconfined.

The solution consists of three parts. The first part, the most essential one, is the solution for a well in the point $x = p$, $y = 0$, which is actually there. The second part is the solution for an imaginary recharge well at the particular point $x = R^2/p$, $y = 0$, and the third part is a constant. Thus the solution is now supposed to be

$$h^2 = \frac{Q_0}{\pi k} \ln\left(\frac{r_1}{r_2}\right) + C$$

or, in terms of cartesian coordinates

$$h^2 = \frac{Q_0}{2\pi k} \ln \frac{(x-p)^2 + y^2}{(x - R^2/p)^2 + y^2} + C$$

Along the boundary of the aquifer the water level must be $h = H$. By substituting $x = R \cos\theta$ and $y = R \sin\theta$ one now obtains, after some elaboration

$$H^2 = \frac{Q_0}{2\pi k} \ln \frac{p^2}{R^2} + C$$

It appears that by making an appropriate choice for the constant C the boundary condition can indeed be satisfied, for all values of θ, and without any approxi-

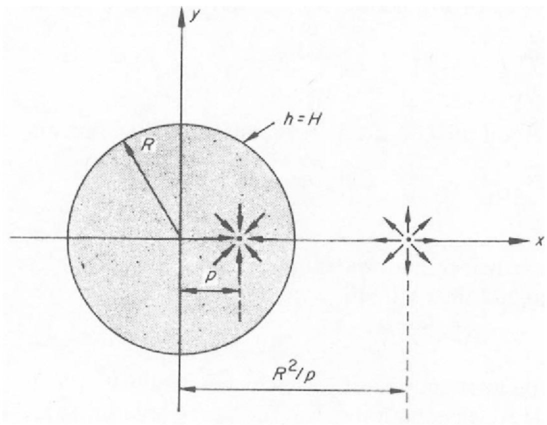

Figure 4.6 *Eccentric well in circular aquifer*

mation. This is due to the particular choice of the location of the image well. The complete solution of the problem is

$$h^2 = H^2 + \frac{Q_0}{\pi k} \ln\left(\frac{r_1 R}{r_2 p}\right) \tag{4.16}$$

In the vicinity of the well one may write $r_2 = R^2/p - p = (R^2/p)(1 - p^2/R^2)$; hence

$$r_1 \ll p, R: \; h^2 = H^2 + \frac{Q_0}{\pi k} \ln\left[\frac{r_1}{R(1 - p^2/R^2)}\right] \tag{4.17}$$

Again it is found that in the vicinity of the well the water level is governed by the standard formula, involving a factor $\ln(r/R_{eq})$, where in this case

$$R_{eq} = R(1 - p^2/R^2) \tag{4.18}$$

When p/R is small, say 0.1 or 0.2, the value of R_{eq} is practically equal to R (the difference being 1 per cent or 4 per cent). This constitutes the formal justification of the practice, already used before in this chapter, of disregarding the influence of the eccentricity p, when it is reasonably small compared to the radius of the aquifer, say $p < 0.2R$.

The method of images can also be extended to aquifers in the shape of an infinite strip, a semi-infinite strip, or a rectangle. In such cases an infinite number of image wells is needed. The convergence of the infinite series is fast enough to allow for breaking them off after a few terms. For these problems the solution can more easily be obtained, however, by using the complex variable method (see chapter 5).

4.3 Potential and Stream Function

In the case of strictly two-dimensional flow Darcy's law states that

$$q_x = -k\frac{\partial \phi}{\partial x}, \qquad q_y = -k\frac{\partial \phi}{\partial y} \tag{4.19}$$

In a homogeneous soil the coefficient k is constant. Then one may write

$$q_x = -\frac{\partial \Phi}{\partial x}, \qquad q_y = -\frac{\partial \Phi}{\partial y} \tag{4.20}$$

where Φ is the so-called (groundwater) potential, defined as the product of the groundwater head and the hydraulic conductivity,

$$\Phi = k\phi \tag{4.21}$$

The purpose of the introduction of this new quantity is that it simplifies the form of Darcy's law (see equation 4.20). It is to be noted that this can only be accomplished if the hydraulic conductivity k is constant. If that is not the case it is not sensible to introduce a quantity defined by equation 4.21, because then the original form of Darcy's law, equation 4.19, is not simplified by the introduction of Φ.

Like the head ϕ, the potential Φ must satisfy Laplace's equation in steady flow

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{4.22}$$

This can of course be derived by substitution of equation 4.20 into the equation of continuity for steady two-dimensional flow

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \tag{4.23}$$

As a consequence of its definition in terms of physically single-valued functions the potential Φ itself is a single-valued function in every point in the x, y-plane occupied by the porous medium. It is possible to represent the function $\Phi(x, y)$ by drawing lines of constant Φ in the x, y-plane (see figure 4.7). It seems natural to choose a constant interval between the values of the potential along successive potential lines

$$\Phi_1 - \Phi_2 = \Phi_2 - \Phi_3 = \ldots = \Delta\Phi \tag{4.24}$$

If in an arbitrary point of the field local coordinates n and s are introduced tangential and perpendicular to the potential lines (see figure 4.7), then in the n, s-coordinates the flow occurs in the s-direction only

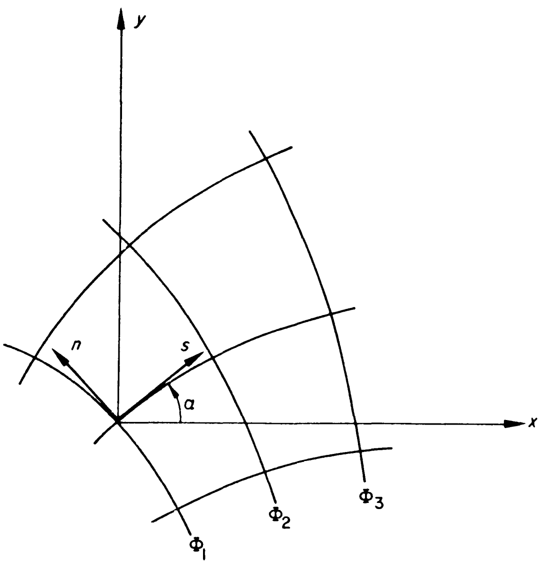

Figure 4.7 *Potential lines and stream lines*

$$\frac{\partial \Phi}{\partial n} = 0, \qquad \frac{\partial \Phi}{\partial s} = -q \tag{4.25}$$

where q is the magnitude of the specific discharge vector, which is perpendicular to the potential lines, because there is no flow in the n-direction, in which direction Φ does not change.

A second function of great importance in the theory of groundwater flow is the so-called stream function Ψ, which can be introduced as follows. Since the specific discharge vector must always satisfy the equation of continuity, equation 4.23, its components q_x and q_y can be derived from a function Ψ by

$$q_x = -\frac{\partial \Psi}{\partial y}, \qquad q_y = +\frac{\partial \Psi}{\partial x} \tag{4.26}$$

Substitution into equation 4.23 shows that this equation is now identically satisfied. It follows from equation 4.20, Darcy's law for a homogeneous field, that

$$\frac{\partial q_x}{\partial y} = \frac{\partial q_y}{\partial x} \tag{4.27}$$

and with equation 4.26 it now follows that

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{4.28}$$

which shows that Ψ is, like the potential Φ, a harmonic function (it should satisfy Laplace's equation).

The physical meaning of Ψ is best understood by noting, from its introduction by equation 4.26, that its variation in one direction determines the flow component in the other direction. Written in terms of n and s-coordinates in the arbitrary point indicated in figure 4.7, this means that

$$\frac{\partial \Psi}{\partial s} = 0, \qquad \frac{\partial \Psi}{\partial n} = -q \tag{4.29}$$

Because there is no flow component in n-direction the variation of Ψ in s-direction vanishes. Equations 4.29 can also be derived formally from equation 4.26 by the usual transformation rules for cartesian coordinate systems.

The first part of equation 4.29 expresses that the function Ψ does not vary in the s-direction, which is the direction of flow. Hence the direction of flow is everywhere tangent to lines of constant Ψ. These lines are therefore called stream lines, and Ψ is called the stream function.

It has now been shown that the lines of constant Φ and Ψ, the potential lines and the stream lines, form a set of orthogonal curved lines, throughout the field. Furthermore, comparison of equations 4.25 and 4.29 shows that

$$\frac{\partial \Phi}{\partial s} = \frac{\partial \Psi}{\partial n} = -q \tag{4.30}$$

This means that if in the x, y-plane lines of constant Φ and Ψ are drawn at intervals $\Delta\Phi$ and $\Delta\Psi$, respectively, then, approximately

$$\frac{\Delta\Phi}{\Delta s} = \frac{\Delta\Psi}{\Delta n} \tag{4.31}$$

where Δs is the distance between two potential lines, and Δn is the distance between two stream lines. By choosing $\Delta\Phi$ and $\Delta\Psi$ equal, which seems a natural thing to do, the distances Δs and Δn are made equal to each other, at least if all distances are taken sufficiently small for equation 4.31 to be an accurate approximation of equation 4.30. Thus, the stream lines and the potential lines are not only orthogonal, but they form elementary, curvilinear, squares. This property is the basis of an approximate graphical method to be presented in section 6.1.

The stream function has a direct relationship with the discharge as can be demonstrated in the following way. Consider two arbitrary points A and B in a two-dimensional field of groundwater flow (see figure 4.8) where C is an auxiliary point, so that $x_C = x_A$ and $y_C = y_B$. The total discharge through a vertical section joining A and C is obtained by integration of the horizontal component of the specific discharge vector. This gives, with equation 4.26

$$Q_{AC} = H \int_A^C q_x \, dy = -H \int_A^C \frac{\partial\Psi}{\partial y} \, dy = H(\Psi_A - \Psi_C)$$

where H is the thickness of the plane in which the groundwater flows. In the same way one obtains for the discharge Q_{CB} through a horizontal section from C to B

$$Q_{CB} = -H \int_C^B q_y \, dx = -H \int_C^B \frac{\partial\Psi}{\partial x} = H(\Psi_C - \Psi_B)$$

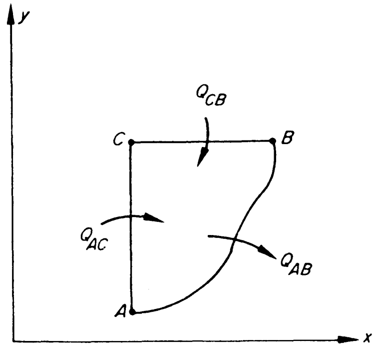

Figure 4.8 *Interpretation of stream function*

Addition of Q_{AB} and Q_{BC} yields the total discharge flowing through the path ABC. Because of continuity the same discharge flows through any continuous path joining the points A and B

$$Q_{AB}/H = \Psi_A - \Psi_B \tag{4.32}$$

This formula indicates that the difference of the values of the stream function in two points represents the amount of water (per unit thickness) flowing through any line joining these two points. The direction of the discharge Q_{AB} in equation 4.32 is from the left side of AB to the right side.

In deriving equation 4.32 it has been stated that because of continuity the discharge through two different curves joining the points A and B is equal. This is true only if in the region enclosed by the two curves no water is injected into or extracted from the soil, for instance by means of sources and sinks. If sinks and sources are acting the derivation fails, indicating that the stream function is not single valued. The use of the stream function then deserves special care.

4.4 Anisotropy

Solutions for two-dimensional flow problems involving a soil having anisotropic (but homogeneous) permeability, can easily be obtained from the solution of corresponding isotropic problems by a simple transformation of the coordinate system. The transformation was first described by Vreedenburgh (1936) and Muskat (1937). The technique for two-dimensional problems will be explained below, but it applies equally well to three-dimensional problems.

If the coordinate axes x and y are taken in the principal directions of permeability Darcy's law states that

$$\begin{aligned} q_x &= -k_{xx} \frac{\partial \phi}{\partial x} \\ q_y &= -k_{yy} \frac{\partial \phi}{\partial y} \end{aligned} \tag{4.33}$$

Substitution of these expressions into the equation of continuity, $\partial q_x/\partial x + \partial q_y/\partial y = 0$, now leads to the differential equation

$$k_{xx} \frac{\partial^2 \phi}{\partial x^2} + k_{yy} \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4.34}$$

where it has been assumed that k_{xx} and k_{yy} are constants. The differential equation 4.34 can be reduced to Laplace's equation by means of the geometrical transformation

$$x = \bar{x}\sqrt{k_{xx}}, \qquad y = \bar{y}\sqrt{k_{yy}} \tag{4.35}$$

Equation 4.34 now becomes

$$\frac{\partial^2 \phi}{\partial \bar{x}^2} + \frac{\partial^2 \phi}{\partial \bar{y}^2} = 0 \tag{4.36}$$

which means that in the transformed system of coordinates the groundwater flow is indeed described by Laplace's equation. The transformation also modifies the location of the boundary of course, but it does not affect the boundary conditions, if these are of the type that the head is given on part of the boundary, whereas the remainder of the boundary is impermeable. If the problem in the transformed plane can be solved it can immediately be retransformed to the original x, y-plane. The orthogonality of potential lines and stream lines in the transformed plane is not preserved in the inverse transformation (see figure 4.9). In an anisotropic soil the stream lines and potential lines are not orthogonal, as was already explained in section 2.1 (see figure 2.2).

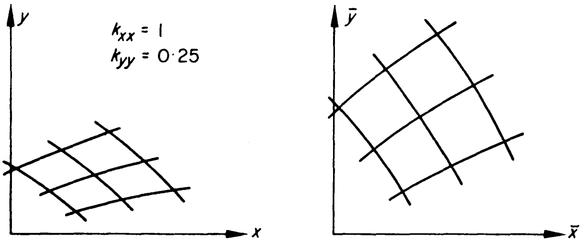

Figure 4.9 *Potential lines and stream lines in anisotropic porous medium*

4.5 Discontinuous Permeability

In this section the conditions along a surface of discontinuity in the permeability will be established. Again it is sufficient to consider the two-dimensional case, the generalisation to the three-dimensional case being self-evident.

Consider a surface separating two regions having different isotropic hydraulic conductivities k_1 and k_2 (see figure 4.10). The directions normal and tangential to the common boundary of the two regions are denoted by n and t, respectively. The specific discharge components in these directions are denoted by q_n^1 and q_t^1 in region 1 and by q_n^2 and q_t^2 in region 2. The problem is to determine the relations between these four quantities.

Physically the transition conditions are that along the boundary the pressure in the fluid, and therefore also the head ϕ, must be continuous, and that all water leaving layer 1 must enter layer 2. This last condition implies that the components of the specific discharge normal to the boundary must be the same on either side

$$q_n^1 = q_n^2 \tag{4.37}$$

The components q_t^1 and q_t^2 are related to the head along the boundary by Darcy's law

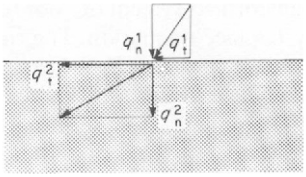

Figure 4.10 *Discontinuity in the permeability*

$$q_t^1 = -k_1 \frac{\partial \phi}{\partial t}, \qquad q_t^2 = -k_2 \frac{\partial \phi}{\partial t} \tag{4.38}$$

Because ϕ is continuous across the boundary, $\partial\phi/\partial t$ is also continuous, hence

$$q_t^1/k_1 = q_t^2/k_2 \tag{4.39}$$

which provides a relation between the tangential components.

A simple rule can be obtained from equations 4.37 and 4.39 by noting that the ratio of q_t and q_n describes the direction of flow, tan α, (see figure 4.10). It now follows that

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{k_1}{k_2} \tag{4.40}$$

This shows that there is an abrupt change in direction of the specific discharge vector across the boundary of two regions of different hydraulic conductivity.

It follows from equation 4.40 that if $k_2 \gg k_1$, say $k_2 = k_1 \times 10^6$, then tan α_2 is very large compared to tan α_1. Thus at the boundary between a clay layer (1) and a sand layer (2) it may well be possible that in the clay layer the flow is almost vertical (say $\alpha_1 = 0.1°$), and yet in the sand layer the flow is practically horizontal ($\alpha_2 = 89.96°$ if $k_2/k_1 = 10^6$). This provides support for the aquifer approach used in section 3.2 for the flow in a semi-confined aquifer. There the flow in the aquifer was considered to be horizontal, even though there is a vertical leakage through the clay layer.

As an example of a problem involving a discontinuity in the hydraulic conductivity, the case of a sink in a confined aquifer, at a distance p from a discontinuity is considered (see figure 4.11). This example may also serve as another illustration of the power of the method of images. The distance p is assumed to be small compared to the radius R of the external boundary of the circular aquifer. Along this boundary the head is kept constant, $\phi = \phi_0$.

The solution of the problem is sought in the form

$$x > 0: \ \phi = \phi_0 + \frac{Q_0}{2\pi k_1 H} \ln\left(\frac{r_1}{R}\right) + \frac{\alpha Q_0}{2\pi k_1 H} \ln\left(\frac{r_2}{R}\right) \tag{4.41}$$

and

$$x < 0: \ \phi = \phi_0 + \frac{\beta Q_0}{2\pi k_1 H} \ln\left(\frac{r_1}{R}\right) \tag{4.42}$$

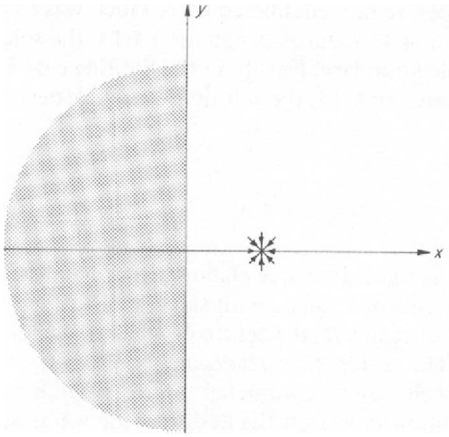

Figure 4.11 *Well in non-homogeneous aquifer*

where α and β are as yet undetermined. Equation 4.41 consists of a superposition of the influences of the actual sink at $x = p, y = 0$ and an imaginary sink at $x = -p, y = 0$. Equation 4.42 does not contain a term of the form $\ln(r_2/R)$ because that would mean a singularity at $x = -p, y = 0$, which is not there in reality.

All that remains is to see whether the values of α and β can be determined so that the continuity conditions along the discontinuity $x = 0$ are satisfied. These conditions are

$$x = 0: \ \phi_1 = \phi_2 \tag{4.43}$$

$$x = 0: \ k_1 \frac{\partial \phi_1}{\partial x} = k_2 \frac{\partial \phi_2}{\partial x} \tag{4.44}$$

With equations 4.41 and 4.42 these conditions give

$$1 + \alpha = \beta$$

$$k_1 (1 - \alpha) = k_2 \beta$$

from which it follows that

$$\alpha = (k_1 - k_2)/(k_1 + k_2) \tag{4.45}$$

$$\beta = 2k_1/(k_1 + k_2) \tag{4.46}$$

This completes the solution of the problem. The guesswork involved in writing down equations 4.41 and 4.42 appears to have been successful (see for example, Kellogg, 1929).

It is interesting to note that the solution presented here includes three particular cases considered earlier in this book. The case $k_1 = k_2$ is that of a single sink, close to the centre of a circular aquifer, which was treated in section 3.1. For $k_2 = 0$ equation 4.41 reduces to equation 4.14, the solution for a sink near a impermeable boundary. Finally in the limiting case $k_2 \to \infty$ equation 4.41 reduces to equation 4.13, the solution for a sink near a boundary of zero drawdown.

4.6 Problems

1. Four wells of the same discharge Q_0 are operating in an unconfined aquifer, in the corner points of a square with sides $2a$. The aquifer is bounded externally by a circle of radius R, the centre of which coincides with the centre of the square. The radius R is so large compared to the dimension a that the individual wells can be considered to be central. Establish, by means of superposition, formulae for the height of the water table in (i) the centre of the square, (ii) the mid-point of a side of the square.

   This system is used to lower the water table to permit the excavation of a building pit in the form of a square with sides 40 m, to a depth of 4 m below the original water table, which was at 10 m above the impermeable base. What should be the discharge of each well in order to keep the bottom of the building pit dry, if $k = 10^{-7}$ m/s and $R = 2000$ m?

2. In a semi-confined aquifer, bounded by a straight impermeable boundary, a well is operating at a distance of 500 m from the impermeable boundary. The radius of the well is $r_w = 0.50$ m, the transmissivity of the aquifer is $T = 2 \times 10^{-3}$ m^2/s, the resistance of the confining layer is $c = 2 \times 10^9$ s, and the discharge of the well is $Q_0 = 10^{-2}$ m^3/s. Calculate the drawdown in the well.

3. A company wishes to extract water at a discharge of $Q_0 = 3.14 \times 10^{-3}$ m^3/s from a completely confined aquifer, which is bounded by a straight canal. The transmissivity of the aquifer is 2×10^{-3} m^2/s. In order to obtain water of optimal quality it is desirable to construct the well as far from the canal as possible. On the other hand, however, local authorities require that at a distance of 400 m from the canal the groundwater head in the aquifer may not be lowered by more than 0.10 m. At what distance from the canal should the well be located?

5

The Complex Variable Method

In this chapter the complex variable method for the solution of two-dimensional groundwater flow problems will be presented. It will be shown that this method provides a technique for the rigorous solution of the general problem for regions bounded by fixed potential lines and stream lines. Many problems involving a free surface or an interface between two immiscible fluids of different density can also be solved by using complex variables. It is in its ability to solve these latter problems that the power of the complex variable method is best illustrated. Some of the major mathematical principles are described briefly in Appendix B. The theory will be presented in this chapter only for strictly two-dimensional flow problems. It can be formulated more generally, however, by introduction of a comprehensive potential (Strack, 1976). It then appears possible to use the theory also for unconfined problems, and even for problems in which the flow is partially confined and partially unconfined.

5.1 Direct Conformal Mapping

In section 4.3 it was shown that for strictly two-dimensional flow in a homogeneous porous medium the components of the specific discharge vector can be expressed in terms of a potential Φ, because of Darcy's law

$$q_x = -\frac{\partial \Phi}{\partial x}, \qquad q_y = -\frac{\partial \Phi}{\partial y} \tag{5.1}$$

They can also be expressed as the derivatives of a stream function Ψ, as a consequence of the continuity equation

$$q_x = -\frac{\partial \Psi}{\partial y}, \qquad q_y = +\frac{\partial \Psi}{\partial x} \tag{5.2}$$

It follows from equations 5.1 and 5.2 that the potential Φ and the stream function Ψ are related by

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \tag{5.3}$$

These are just the Cauchy-Riemann equations (see appendix B, equations B.7). It means that if a complex potential Ω is introduced as

$$\Omega = \Phi + i\Psi \tag{5.4}$$

then Ω is an analytic function of the complex variable $z = x + iy$. This property is the basis of the complex variable methods in the theory of groundwater flow.

The simplest class of problems for which the complex variable method can be used is the class in which the boundary conditions of the problem are such that either the potential Φ or the stream function Ψ is constant along consecutive sections of the boundary. In that case the boundary conditions define a certain region in the Ω-plane (see figure 5.1). The problem then is to find a

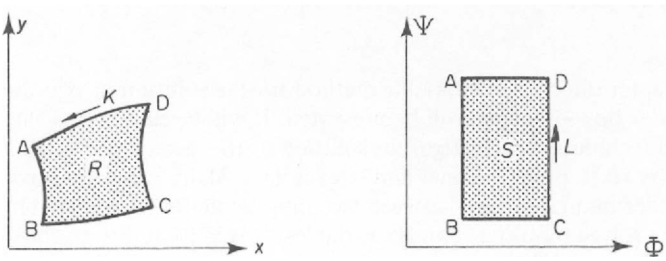

Figure 5.1 *Physical plane and plane of complex potential*

conformal transformation which maps the region in the z-plane on to the region in the Ω-plane, or vice versa. It depends upon the geometry of the two regions whether this conformal mapping is a simple or a complicated matter, perhaps even an impossible task. The procedure may involve the application of standard transformations from a catalogue, or the application of general transformations such as the Schwarz-Christoffel transformation. The technique may best be illustrated by elaborating some examples.

Example 5.1 Flow in an Infinite Layer

The first example concerns the flow of groundwater in an infinite layer from a very far source towards a draining surface (see figure 5.2). On the part BC the layer is covered by a completely impermeable layer, and over the part AB the head is kept constant, say zero. The plane of the complex potential Ω is also represented in figure 5.2. The quantity Q is the total discharge (per unit width) flowing through the aquifer. Very far away, for $x \to \infty$, the potential is infinitely large.

The mathematical problem is to find a conformal transformation mapping the region ABC in the z-plane on to ABC in the Ω-plane. This can most conveniently be done by using auxiliary planes. The infinite strip in the z-plane can be mapped on to the upper half-plane Im $(\zeta) > 0$ by a logarithmic function

$$z = \frac{h}{\pi} \ln \zeta \tag{5.5}$$

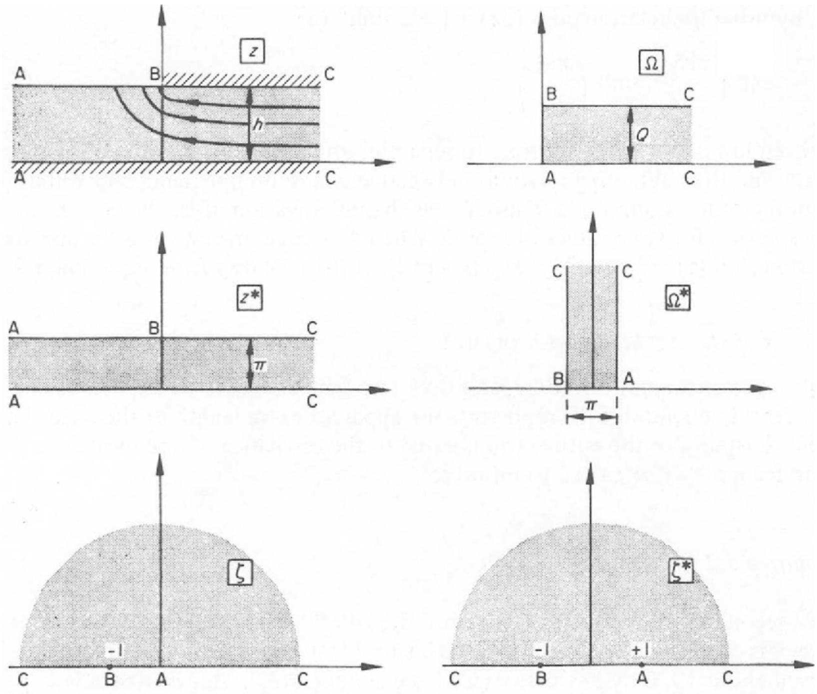

Figure 5.2 *Flow in an infinite layer*

The factor h/π is necessary because the standard transformation $z = \ln \zeta$ maps the half-plane on to a strip of width π. The semi-infinite strip in the Ω-plane can also be mapped on to an upper half-plane. This can most conveniently be done by first rotating and shifting the strip, through the transformation

$$\Omega^* = i\pi\Omega/Q + \pi/2 \tag{5.6}$$

The factor i results in a rotation over $\pi/2$, the factor π/Q transforms the width of the strip from Q into π, and the addition of $\pi/2$ shifts the strip so that it becomes symmetric with respect to the vertical axis (see figure 5.2). The region in the Ω^*-plane can be mapped on to the half-plane Im $(\zeta^*) > 0$ by the standard transformation

$$\zeta^* = \sin(\Omega^*) \tag{5.7}$$

Unfortunately the regions in the ζ-plane and the ζ^*-plane are not identical, because the points A, B, C do not coincide. This can easily be accomplished, however, by the transformation

$$\zeta^* = 1 + 2\zeta \tag{5.8}$$

By elimination of the auxiliary variables ζ^* and Ω^* one now obtains for the relation between z and Ω

$$z = \frac{h}{\pi} \ln \left\{ \tfrac{1}{2} \left[\cosh(\pi\Omega/Q) - 1 \right] \right\}$$

or, by using the relation $\cosh(2z) = 1 + 2\sinh^2(z)$

$$\exp\left(\frac{\pi z}{2h}\right) = \sinh\left(\frac{\pi\Omega}{2Q}\right) \tag{5.9}$$

Written in this form it is possible to separate both sides into real and imaginary parts, and thus obtaining relations between Φ and Ψ on one hand, and x and y on the other. Equation 5.9 also allows the investigation of the behaviour of the solution for large values of x or Φ. When Φ is large one may use the approximation $\sinh(z) \approx \frac{1}{2}\exp(z) = \exp(z - \ln 2)$. It then follows from equation 5.9 that

$$x \gg h: \; z = \Omega h/Q - (2h/\pi)\ln 2 \tag{5.10}$$

This represents a uniform flow, at a flow rate Q/h, as was to be expected. The last term in equation 5.10 represents the apparent extra length of the strip. The total resistance of the entire strip is equal to the resistance of a semi-infinite strip from $z = -(2h/\pi)\ln 2$ to infinity.

Example 5.2 Flow under a Screen

As a second example the flow in an infinite layer underneath a sheet piled screen is considered (see figure 5.3). This problem and many similar ones were solved about 1920 by Pavlovsky (see for example, Polubarinova-Kochina, 1962; Aravin and Numerov, 1965; Harr, 1962). The z-plane and the Ω-plane are represented in figure 5.3, together with an auxiliary ζ-plane. Because of the symmetry of the problem it is sufficient to consider only one half of the flow region, inside ABCD. The value kH, where H is the difference in head of the potential lines BC and AD, is supposed to be given, but the value of the discharge Q is unknown. It can be expected that its value depends upon the ratio ℓ/h, where ℓ is the length of the screen. If $\ell = h$ the discharge will be zero, and if $\ell = 0$ the discharge will be infinite.

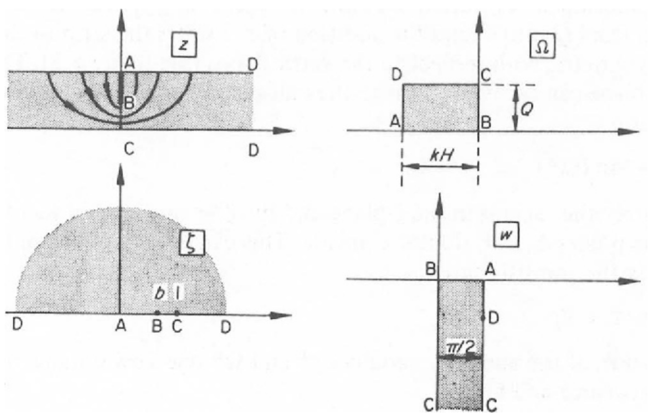

Figure 5.3 *Flow under a screen*

The transformation from the z-plane to the ζ-plane can again be accomplished by a sine function, together with some scaling and shifting. The result is

$$\zeta = \tfrac{1}{2}[1 + \cosh(\pi z/h)] \tag{5.11}$$

It can easily be verified that this is the correct transformation. For $z = 0$ one obtains $\zeta = 1$, which is point C. For $z = ih$ one obtains $\cosh(\pi z/h) = \cosh(\pi i) = \cos(\pi) = -1$, and thus $\zeta = 0$, which is point A. The points at infinity correspond as can be seen by taking $z = x$ or $z = x + ih$, with x very large. One then obtains $\zeta \to \infty$ or $\zeta \to i\infty$. The location of the point B in the z-plane is $z = i \times (h - \ell)$. If the location of that point in the ζ-plane is assumed to be $\zeta = b$ one finds

$$b = \tfrac{1}{2}[1 + \cosh(i\pi(h - \ell)/h)] = \sin^2(\pi\ell/2h) \tag{5.12}$$

The transformation from the Ω-plane to the ζ-plane is more difficult, and requires the use of the Schwarz-Christoffel transformation (see appendix C). In the present case there are four corner points, three of which have been chosen in the ζ-plane to be located at $\zeta = 0$, $\zeta = 1$ and at infinity. Because in each corner the change in direction is $\pi/2$, the powers k_j are all $\frac{1}{2}$ (see figure C.3). The Schwarz-Christoffel formula now gives

$$\frac{d\Omega}{d\zeta} = A[\zeta(\zeta - b)(\zeta - 1)]^{-\frac{1}{2}} \tag{5.13}$$

or

$$\Omega = A\int_b^\zeta [\lambda(\lambda - b)(\lambda - 1)]^{-\frac{1}{2}}\, d\lambda \tag{5.14}$$

where the integration constant has been chosen such that $\zeta = b$ corresponds to $\Omega = 0$. The integral in equation 5.14 is a so-called elliptic integral (see for example, Whittaker and Watson, 1927; Byrd and Friedman, 1971; Abramowitz and Stegun, 1965); it can be expressed into the standard elliptic integral of the first kind (Abramowitz and Stegun, 1965, p. 597)

$$\Omega = -2AF(w|\mathrm{b}) \tag{5.15}$$

where w is defined by

$$\cos^2 w = \frac{\zeta(1 - b)}{b(1 - \zeta)} \tag{5.16}$$

The elliptic integral of the first kind, $F(w|b)$, is defined as

$$F(w|b) = \int_0^w (1 - b\sin^2\theta)^{-\frac{1}{2}}\, d\theta \tag{5.17}$$

The plane of the complex variable w is also represented in figure 5.3. The constant A in equation 5.15 can be determined from the condition that the potential in point A should be $-kH$. Thus, for $\zeta = 0$ one should obtain $\Omega = -kH$. Since for $\zeta = 0$ equation 5.16 gives $w = \pi/2$, one now obtains

$$2A = kH/F(\pi/2\,|\,b) \tag{5.18}$$

The function $F(\pi/2\,|\,b)$ is the complete elliptic integral of the first kind, which is usually denoted by $K(b)$

$$K(b) = \int_0^{\pi/2} (1 - b \sin^2 \theta)^{-\frac{1}{2}}\, d\theta \tag{5.19}$$

Equation 5.15 for the complex potential can now be written as

$$\Omega = -kH \frac{F(w\,|\,b)}{K(b)} \tag{5.20}$$

Equations 5.11, 5.16 and 5.20 together constitute the solution. For any value of the geometrical variable z one can calculate ζ from equation 5.11, then determine w from equation 5.16, and finally calculate the complex potential Ω from equation 5.20, using a table of elliptic integrals. For complex values of w this is a rather cubersome procedure (Abramowitz and Stegun, 1965, pp. 592-3).

A quantity of special interest is the total discharge Q, which can be found by determining $\mathrm{Im}(\Omega)$ for a point on CD, say point C. With $w = -\mathrm{i}\infty$ one obtains from equation 5.20, since $F(-\mathrm{i}\infty\,|\,b) = -\mathrm{i}K(1 - b)$

$$\text{in C:}\quad \Omega = \mathrm{i}kH \frac{K(1 - b)}{K(b)}$$

Because the value of Ω in C is $\mathrm{i}Q$ this means that

$$Q = kH \frac{K(1 - b)}{K(b)} \tag{5.21}$$

Some numerical values of the ratio $K(1 - b)/K(b)$ are given in table 5.1. As expected $Q = 0$ if $\ell = h$ ($b = 1$) and Q tends to infinity if $\ell = 0$ ($b = 0$).

TABLE 5.1

ℓ/H	$b = \sin^2(\pi\ell/2h)$	$K(b)$	$K(1 - b)$	$K(1 - b)/K(b)$
0.0	0.00000	1.5708	∞	∞
0.1	0.02447	1.5805	3.2553	2.0596
0.2	0.09549	1.6105	2.5998	1.6143
0.3	0.20611	1.6627	2.2435	1.3493
0.4	0.34549	1.7415	2.0133	1.1561
0.5	0.50000	1.8541	1.8541	1.0000
0.6	0.65451	2.0133	1.7415	0.8650
0.7	0.79389	2.2435	1.6627	0.7411
0.8	0.90451	2.5998	1.6105	0.6194
0.9	0.97553	3.2553	1.5805	0.4855
1.0	1.00000	∞	1.5708	0.0000

Example 5.3 Flow through a Slit

The third example of an elementary application of conformal mapping concerns the disturbance of a uniform flow by a contraction due to a more or less narrow slit (see figure 5.4). The groundwater flows through a strip of width

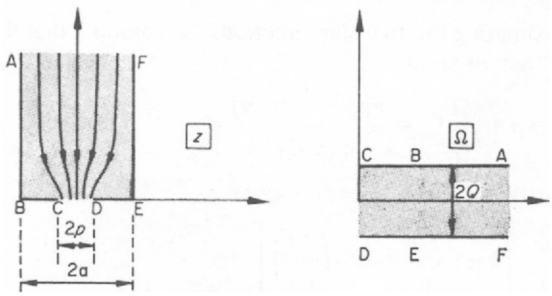

Figure 5.4 *Flow through a slit*

$2h$, at a discharge $2Q$, but the width of the symmetrical opening is only $2p$. The solution of this problem can easily be obtained by mapping both the regions in the z-plane and the Ω-plane on to an auxiliar half-plane Im $\zeta > 0$, by transformations of the type $\zeta = A \sin \alpha z$ and $\zeta = B \sin \beta\Omega$. After elimination of ζ the result is

$$\Omega = -\frac{2iQ}{\pi} \arcsin \frac{\sin(\pi z/2h)}{\sin(\pi p/2h)} \tag{5.22}$$

An interesting result can be obtained from this solution by comparing it with the solution for the flow through a semi-infinite strip of the same width $2h$, having an extra length ℓ instead of the narrowing (see figure 5.5). In this case the solution is easily seen to be

$$\Omega = -\frac{iQ}{h}(z + i\ell) \tag{5.23}$$

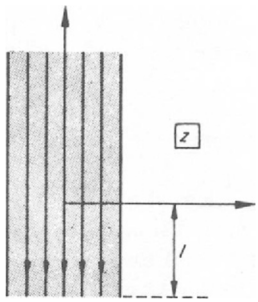

Figure 5.5 *Equivalent length of contraction*

It can be expected that the value of ℓ can be chosen such that very far from the outlet the two solutions are identical. This can be elaborated by writing equation 5.22 as

$$\sin\left(\frac{\pi i\Omega}{2Q}\right) = \frac{\sin(\pi z/2h)}{\sin(\pi p/2h)}$$

and then approximating the two sine functions by assuming that Φ and y are very large. One then obtains

$$y \gg \text{h}: \exp\left(\frac{\pi\Omega}{2Q}\right) = \frac{\exp(-i\pi z/2h)}{\sin(\pi p/2h)}$$

Taking the logarithm of both sides now gives

$$\Omega = -\frac{iQ}{h}\left[1 - i\,\frac{2h}{\pi}\ln\sin\left(\frac{\pi p}{2h}\right)\right] \tag{5.24}$$

Comparison with equation 5.23 shows that both formulae are indeed identical if

$$\ell = -\frac{2h}{\pi}\ln\sin\left(\frac{\pi p}{2h}\right) \tag{5.25}$$

This is a positive quantity, because the sine is smaller than 1, and thus the logarithm is negative. At a large distance from the outlet the effect of the contraction is equivalent to an extra length ℓ of the strip. Therefore ℓ may be denoted as the equivalent length of the contraction.

A great number of other examples of the application of conformal mapping can be found in the literature, especially in the Russian literature (see for example, Polubarinova-Kochina, 1962; Aravin and Numerov, 1965; Harr, 1962).

5.2 Flow Towards Wells

In this section conformal transformations are used to solve some problems for flow towards wells in a region of such shape that it can be mapped onto a half-plane. The basic idea is that the conformal mapping changes the shape of the regions, but boundary conditions of the type Φ = constant or Ψ is constant are not affected (Strack, 1973). Some examples may illustrate the technique.

Example 5.4 Well in an Infinite Strip

The first example refers to the problem of a well of discharge Q (per unit thickness perpendicular to the plane) in an infinite strip of width a (see figure 5.6). The well is located in the point $z = z_0$. If the boundaries $y = 0$ are potential lines, with $\Phi = 0$, the problem is to find an analytic function $\Omega = \Omega(z)$ such that $\Phi = \text{Re}(\Omega) = 0$ for $y = 0$ and $y = a$, and such that near $z = z_0$ the function has a singularity of the type

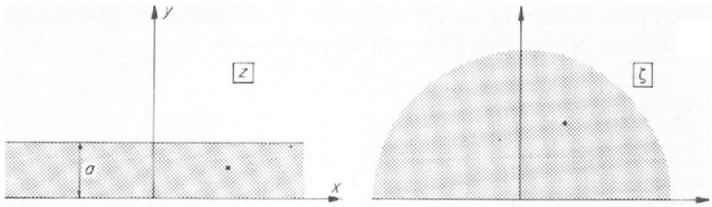

Figure 5.6 *Well in infinite strip*

$$\Omega = \frac{Q}{2\pi} \ln (z - z_0) + \Omega^*(z) \tag{5.26}$$

where $\Omega^*(z)$ is analytic near $z = z_0$. The region in the z-plane is now transformed on to the half-plane $\text{Im}(\zeta) > 0$ by the analytic function

$$\zeta = \exp\left(\frac{\pi z}{a}\right) \qquad z = \frac{a}{\pi} \ln \zeta \tag{5.27}$$

see appendix C. The function Ω can now also be considered as a function of ζ, and it must be analytic in the half-plane $\text{Im}(\zeta) > 0$, except at the image point of z_0, that is, at $\zeta = \exp(\pi z_0/a)$. On the boundary $\text{Im}(\zeta) = 0$ the value of Φ must be zero. Near $z = z_0$ one may write, with equation 5.27 and a Taylor series expansion

$$z = z_0 + (\zeta - \zeta_0) \frac{a}{\pi \zeta_0} + \ldots$$

It now follows from equation 5.26 that near ζ_0, the image point of z_0 in the ζ-plane the function $\Omega(\zeta)$ can be written as

$$\Omega = \frac{Q}{2\pi} \ln (\zeta - \zeta_0) + \Omega^*(\zeta) \tag{5.28}$$

where $\Omega^*(\zeta)$ is analytic near $\zeta = \zeta_0$. Physically this means that in the ζ-plane a well of discharge Q seems to be operating. The conclusion can be, generally: a well in the physical z-plane corresponds to a well of the same discharge in the auxiliary plane.

The solution for $\Omega(\zeta)$ can easily be written down by using the method of images

$$\Omega = \frac{Q}{2\pi} \ln \frac{\zeta - \zeta_0}{\zeta - \overline{\zeta_0}} \tag{5.29}$$

Retransforming this to the z-plane by equation 5.27 gives

$$\Omega = \frac{Q}{2\pi} \ln \left[\frac{\exp(\pi z/a) - \exp(\pi z_0/a)}{\exp(\pi z/a) - \exp(\pi \bar{z}_0/a)}\right] \tag{5.30}$$

This completes the solution.

It is illustrative to investigate the behaviour of the potential near the well somewhat further. Near $z = z_0$ or $\zeta = \zeta_0$ one may write

$$\zeta - \zeta_0 = \frac{\pi\zeta_0}{a}(z - z_0) + \dots$$

For the potential Φ only the real part of Ω is relevant. In equation 5.29 this means that the modulus of the terms under the logarithm is needed. If one writes $|z - z_0| = r$ and $\mathrm{Im}(z_0) = p$, it now follows that near the well

$$\Phi = \frac{Q}{2\pi} \ln \left[\frac{r}{(2a/\pi)\sin(\pi p/a)}\right] \qquad (5.31)$$

Again it appears that the potential contains a factor $\ln(r/R_{eq})$. In this case the equivalent radius is

$$R_{eq} = \frac{2a}{\pi} \sin\left(\frac{\pi p}{a}\right) \qquad (5.32)$$

If the well is located in the middle of the strip $p = \frac{1}{2}a$, and then $R_{eq} = 2a/\pi = 0.64a$, which is only a little more than the distance to each of the boundaries. If the distance to one of the boundaries, say to the lower one, is very small, the sine function can be approximated by its argument. Then $R_{eq} = 2p$, which confirms a result already obtained in equation 4.4.

It goes without saying that for the flow induced by a number of wells one may succesfully apply the principle of superposition.

Eample 5.5 Well in a Square Island

In the case of a well in the central point of a square region (see figure 5.7), it seems more convenient to map the interior of the square on to the interior of the unit circle $|\zeta| = 1$, because then it can be assumed that the symmetry is retained in the transformation. By choosing the image points of the corner points in the z-plane at $\zeta = \exp(i\pi/4)$, $\exp(3i\pi/4)$, $\exp(5i\pi/4)$ and $\exp(7i\pi/4)$ one obtains, using equation C.4 from appendix C, with $k_1 = k_2 = k_3 = k_4 = \frac{1}{2}$

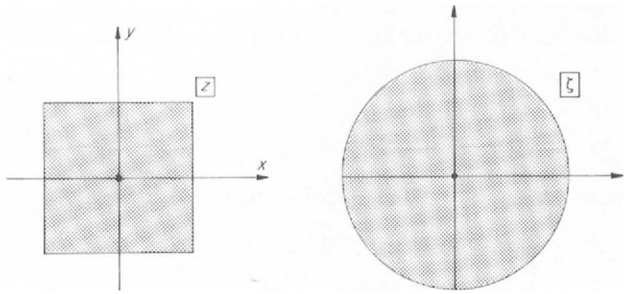

Figure 5.7 *Well in a square island*

$$\frac{dz}{d\zeta} = A\,(1+\zeta^4)^{-\frac{1}{2}} \tag{5.33}$$

Integration of this expression will lead to elliptic integrals. These may be avoided by noting that the expression in equation 5.33 is analytic for all ζ inside the unit circle, and hence it can be expanded in a Taylor series which will converge everywhere inside that circle. Expansion of equation 5.33 gives

$$\frac{dz}{d\zeta} = A\left(1 - \frac{1}{2}\,\zeta^4 + \frac{3}{8}\,\zeta^8 - \frac{5}{16}\,\zeta^{12} + \frac{35}{128}\,\zeta^{16} + \ldots\right)$$

or

$$\frac{dz}{d\zeta} = \mathrm{A}\sum_{j=1}^{\infty} c_j\,\zeta^{4j} \tag{5.34}$$

where $c_0 = 1$ and the other coefficients can be obtained from the recurrent relationship

$$c_j = -\frac{2j-1}{2j}\,c_{j-1}\quad (k = 1, 2, 3\,\ldots\ldots) \tag{5.35}$$

Integrating equation 5.34 term by term gives

$$z = A\sum_{j=0}^{\infty}\frac{c_j}{4j+1}\,\zeta^{4j+1} \tag{5.36}$$

where the integration constant has been omitted in order that $\zeta = 0$ corresponds to $z = 0$. The coefficient A in equation 5.36 determines the magnitude of the square in the z-plane. If the dimensions of this square are $2a \times 2a$, then the point $\zeta = 1$ must correspond to $z = a$. Hence

$$a = A\sum_{j=0}^{\infty}\frac{c_j}{4j+1} = A\sigma \tag{5.37}$$

where σ stands for the sum of the infinite series

$$\sigma = \sum_{j=0}^{\infty}\frac{c_j}{4j+1} = 0.927\,038 \tag{5.38}$$

The value of σ was calculated numerically, by taking 5000 terms into account.

The solution of the flow problem in case of a well in the origin is, in terms of the transformed variable ζ

$$\Omega = \frac{Q}{2\pi}\ln\zeta \tag{5.39}$$

where it has been assumed that the potential Φ is zero along the outer boundary, and where Q represents the discharge of the well, per unit thickness perpendicular to the plane of flow. In the vicinity of the well in the physical plane, the z-plane, one can write

$$\Omega = \frac{Q}{2\pi} \ln \frac{z}{R} + \Omega^*(z) \tag{5.40}$$

where $\Omega^*(z)$ is analytic near $z = 0$ and vanishes in that point. If the conformal mapping is written as

$$z = \omega(\zeta)$$

Then one can write, near $z = 0$, because $\zeta = 0$ is the image of $z = 0$

$$\frac{z}{\zeta} \approx \frac{dz}{d\zeta} = \omega'(0) \tag{5.41}$$

where $\omega'(0)$ is the value of the derivative of the mapping function at $\zeta = 0$, the image of the location of the well. From equations 5.40 and 5.41 one obtains

$$\Omega = \frac{Q}{2\pi} \ln \frac{\zeta\omega'(0)}{R} + \Omega^*(\zeta) \tag{5.42}$$

and comparison with equation 5.39 shows that the value of the equivalent radius R is given by

$$R = \omega'(0)$$

or, with equation 5.34

$$R = A$$

Hence, with equations 5.37 and 5.38

$$R = 1.0787\,a \tag{5.43}$$

which is only 8 per cent more than the value of a itself.

5.3 The Hodograph Method

In the case of flow in a vertical plane with a free surface an essential complication is that the position of the free surface is unknown beforehand. Direct conformal transformation from the z-plane to the Ω-plane is then impossible because the boundary of the region in the z-plane is unknown. It will be shown that in such cases the complex specific discharge $q_x - iq_y$ can be a useful auxiliary variable. In the so-called hodograph plane, which is a representation of the specific discharge components, the boundary will be shown to be well-defined.

5.3.1 *Hodograph*

In every point in the region of flow there exists a certain specific discharge vector, with components q_x and q_y, which define a point in a plane, the so-called hodograph-plane (*hodos* being Greek for way). The set of all points z in a region R will define a region T in the hodograph plane. It will be shown below that the most common types of boundary conditions define certain lines or curves in the hodograph plane.

(1) *Straight Stream Line*

For points on a straight stream line, making an angle α with the x-axis (figure 5.8), the components of the specific discharge are

$$q_x = q \cos \alpha, \qquad q_y = q \sin \alpha \tag{5.44}$$

where q is the magnitude of the vector. The set of all points for which equation 5.44 holds, with variable q, in the q_x, q_y -plane is a straight line through the origin, under an angle α (see figure 5.8).

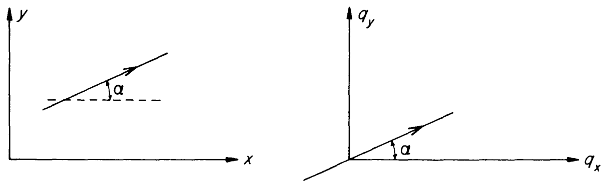

Figure 5.8 *Straight stream line in hodograph*

(2) *Straight Potential Line*

Since the direction of flow is perpendicular to the potential lines, it follows that the specific discharge vectors for points on a straight potential line together constitute a line in the q_x, q_y -plane, through the origin, and perpendicular to the potential line.

(3) *Straight Seepage Surface*

A seepage surface is a boundary of the soil mass along which water flows out of the soil, and then continues its motion downwards in a thin layer over the surface. Such a situation may occur at the tail water side of a dam. The boundary condition along a seepage surface is that the fluid pressure is zero, hence the head equals the height

$$\phi = y \tag{5.45}$$

This holds for all points on the seepage surface, and therefore, in case of a straight seepage surface under an angle γ (see figure 5.9)

$$q_s = -k \frac{\partial \phi}{\partial s} = -k \frac{\partial y}{\partial s} = -k \sin \gamma \tag{5.46}$$

The component of the specific discharge in the direction of the surface appears to be constant. The component perpendicular to it can have any value. Construction of the vectors q such that $q_s = -k \sin \gamma$ and q_n has different values shows that the image in the hodograph plane is a straight line, perpendicular to the seepage surface, and passing through the point $q_x = 0, q_y = -k$ (see figure 5.9). This property can also be deduced formally by expressing q_x and q_y into q_s and q_n, and then using equation 5.46 and eliminating the parameter q_n.

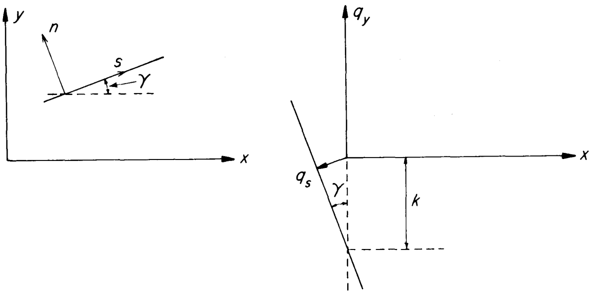

Figure 5.9 *Seepage surface in hodograph*

(4) *Free Surface*

A free surface is a stream line (that is, $q_n = 0$), but its inclination varies. One may now write

$$q_x = q \cos \delta, \qquad q_y = q \sin \delta$$

Along the free surface the pressure is zero, hence $\phi = y$, and therefore

$$q = -k \frac{\partial \phi}{\partial s} = -k \frac{\partial y}{\partial s} = -k \sin \delta$$

The components q_x and q_y can now be expressed as

$$q_x = -q \sin \delta \cos \delta = -\tfrac{1}{2} k \sin (2\delta)$$

$$q_y = -q \sin^2 \delta = -\tfrac{1}{2} k + \tfrac{1}{2} k \cos (2\delta)$$

Elimination of the variable parameter δ now gives

$$q_x^2 + (q_y + \tfrac{1}{2}k)^2 = (\tfrac{1}{2}k)^2 \tag{5.47}$$

This equation represents a circle of radius $\frac{1}{2}k$ around the centre $q_x = 0, q_y = -\frac{1}{2}k$. This circle passes through the origin and through the point $q_x = 0, q_y = -k$ (see figure 5.10).

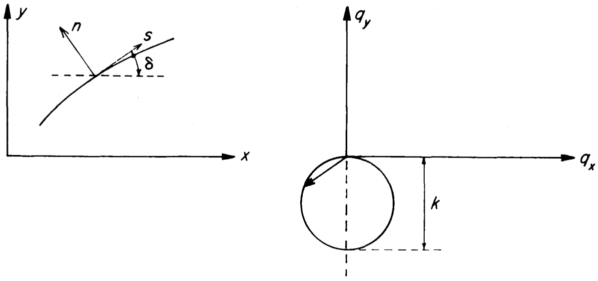

Figure 5.10 *Free surface in hodograph*

(5) *Free Surface with Infiltration*

In the case of a free surface with a constant infiltration at a rate N the free surface is no longer a stream line, $q_n \neq 0$. Because of continuity across the free surface $q_n = -N \cos \delta$, if δ is the inclination of the free surface. Because the relation $q_s = -k \sin \delta$ still applies in this case (it is an immediate consequence of the vanishing of the fluid pressure), one can now write

$$q_x = q_s \cos \delta - q_n \sin \delta = -\tfrac{1}{2}(k - N) \sin (2\delta)$$

$$q_y = q_s \sin \delta + q_n \cos \delta = -\tfrac{1}{2}(k + N) - \tfrac{1}{2}(k - N) \cos (2\delta)$$

Elimination of δ now gives

$$q_x^2 + [q_y + \tfrac{1}{2}(k + N)]^2 = [\tfrac{1}{2}(k - N)]^2 \tag{5.48}$$

This is a circle through the points $q_x = 0, q_y = -k$ and $q_x = 0, q_y = -N$ (see figure 5.11).

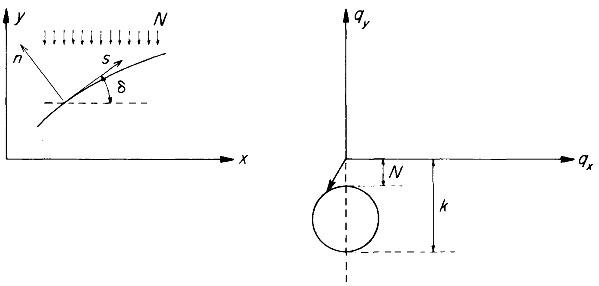

Figure 5.11 *Free surface with infiltration in hodograph*

It has now been shown that for problems whose boundaries consist of straight stream lines, straight potential lines, straight seepage surfaces and free surfaces with or without infiltration, the representation of these boundaries in the hodograph plane is completely defined. This suggest that for such problems the region in the hodograph plane is probably known. That this is indeed the case will be illustrated by the following example.

Example 5.6 Hodograph for Flow through a Dam

The hodograph for the flow of groundwater through a homogeneous dam is shown in figure 5.12. This hodograph was first described by Hamel (1934). The construction of this hodograph can be performed by first tracing the loci of the various boundary segments, and then looking for the intersection points.

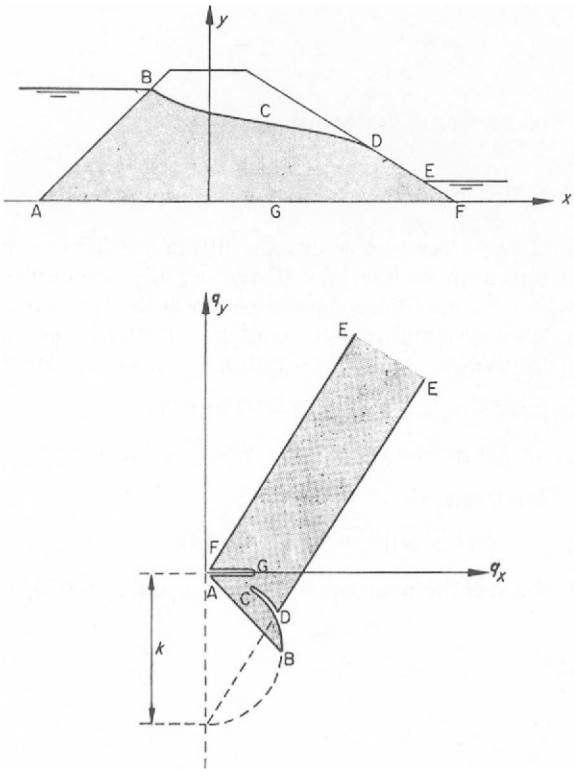

Figure 5.12 *Hodograph for flow through a dam*

In some cases it is not immediately obvious whether the intersection point has to be located at infinity or, for instance, in the origin. This requires some consideration of the flow net in the vicinity of the corner point considered. In this way it is found that points A and F are located in the origin. Point E is found to be located at infinity, because this is the only way that DE and FE intersect. It is interesting to note the hook-shaped parts AGF and BCD. Point G is the point along the lower boundary in which the horizontal velocity is a maximum. That there must exist such a point follows from the vanishing of the velocity at both ends A and F. Points C corresponds to the possible inflection point on the free surface. There need not exist such a point in all cases (probably not if the slope of AB is very steep), but in order to prevent loss of generality this possibility can better be left open.

Some interesting conclusions can be drawn directly from the hodograph. The first is that the existence of the seepage surface DE is a physical necessity. The free surface cannot pass into the potential line EF without giving rise to a nonsensical hodograph. Furthermore it appears that the specific discharge in points B and D is completely determined, in magnitude as well as in direction. The direction of flow in D appears to be perpendicular to the line DE in the hodograph. This means that in D the flow vector is tangent to the slope of the surface in the z-plane. Hence the free surface BCD turns into the seepage surface DE with a common tangent. It also follows from the hodograph that in point E the magnitude of the specific discharge vector is infinitely large, at least theoretically. In reality such an infinitely large velocity is impossible. Either Darcy's law will cease to hold near E, or some of the soil particles will be flushed away, then perhaps the seepage surface becomes somewhat flatter, or the potential line EF becomes somewhat steeper, so that they intersect at some finite distance in the hodograph plane.

5.3.2 *The Hodograph Method*

In section 5.1 it was shown that the complex potential $\Omega = \Phi + i\Psi$ is an analytic function of the complex variable $z = x + iy$. Its derivative is

$$\frac{d\Omega}{dz} = \frac{\partial\Omega}{\partial x} = \frac{\partial\Phi}{\partial x} + i\,\frac{\partial\Psi}{\partial x} = -q_x + i\,q_y$$

where use has been made of the uniqueness of the derivative $d\Omega/dz$ (that is, $d\Omega/dz$ is the same for all possible paths, and the path $dz = dx$ is such a path), and where equations 4.20 and 4.26 have been used to relate the derivatives $\partial\Phi/\partial x$ and $\partial\Psi/\partial x$ to q_x and q_y. If a complex quantity w is now defined as

$$w = q_x - i\,q_y \tag{5.49}$$

then

$$w = -\,d\Omega/dz \tag{5.50}$$

Since the derivative of an analytic function is itself an analytic function, it now follows that $w = q_x - i\,q_y$ is an analytic function of z. This implies that there will exist a conformal transformation from the w-plane to the z-plane, and to the Ω-plane.

The quantity w is the complex conjugate of the quantity $q_x + i\,q_y$, which can be considered to be the variable in the hodograph plane. If the boundary conditions specify a certain region in the hodograph plane, a similar region exists in the w-plane, which is obtained from the hodograph by reflection in the real axis. If the boundary conditions are such that they also define a region in the Ω-plane, it may be possible to determine the relation between w and Ω, by using conformal transformations. By using the inverse of equation 5.50

$$dz/d\Omega = -\,w^{-1} \tag{5.51}$$

the relation between z and Ω can be found from the integral

$$z = -\int w^{-1}(\Omega)\, d\Omega \tag{5.52}$$

which then completes the solution of the problem.

Example 5.7 Flow in a Drained Dam

The technique outlined above can best be illustrated by means of an example. Unfortunately the problem of flow through a dam such as the one sketched in figure 5.12 cannot be solved in this way, because the location of the seepage face in the Ω-plane is unknown. Therefore a somewhat simpler case is considered (see figure 5.13), namely the flow in a drained dam. The purpose

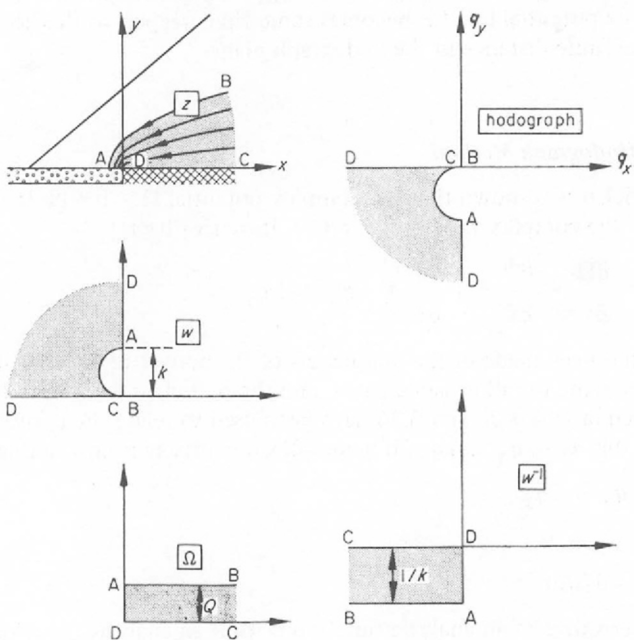

Figure 5.13 *Flow in a drained dam*

of such a drain is to prevent the occurrence of a seepage surface, which might endanger the stability of the slope. The solution of this problem was first obtained by Vreedenburgh (see De Vos, 1929) and Kozeny (1931), who both guessed a solution, without using the hodograph, and then proved that it satisfied all conditions. The straightforward derivation of the solution by means of the hodograph method was given by Pavlovsky in 1937 (see Polubarinova-Kochina, 1962).

The hodograph plane, its complex conjugate the w-plane, and the Ω-plane are all represented graphically in figure 5.13. In this figure the plane of w^{-1}, which is the inverse of w, is also shown. That the half-circle AB in the w-plane is mapped on to a straight line in the w^{-1}-plane can be verified by nothing that for a point on the half-circle

$$w = \tfrac{1}{2}k\ [\mathrm{i} + \exp(\mathrm{i}\theta)], \qquad \pi/2 < \theta < 3\pi/2$$

Hence

$$w^{-1} = \frac{1}{k}\left(\frac{-\mathrm{i} + \exp(-\mathrm{i}\theta)}{1 + \sin\theta}\right) = \frac{1}{k}\left(\frac{\cos\theta}{1 + \sin\theta}\right) - \frac{\mathrm{i}}{k}$$

The represents a straight line at a distance $1/k$ below the real axis. The point A (for which $\theta = \pi/2$) is found to be located at $w^{-1} = -\mathrm{i}/k$ and point B (for which $\theta = 3\pi/2$) is found to be located at $w^{-1} = -\infty - \mathrm{i}/k$.

The transformation from the w^{-1}-plane to the Ω-plane is very simple, since both regions happen to be geometrically similar. The mapping function is

$$w^{-1} = -\Omega/kQ \tag{5.53}$$

With equation 5.52 the relationship between z and Ω is now found to be

$$z = \frac{1}{kQ}\int \Omega\, \mathrm{d}\Omega = \frac{\Omega^2}{2kQ} \tag{5.54}$$

where the integration constant has been taken as zero, in order to obtain that point D (for which $\Omega = 0$) is located in the origin of the z-plane. Equation 5.54 represents the solution of the problem. It can be used to determine the location in the z-plane of any point in the Ω-plane. Of particular interest is the shape of the free surface AB. This can be found by taking $\Omega = \Phi + \mathrm{i}Q$. One then obtains from equation 5.54, after separation into real and imaginary parts

$$x = \frac{\Phi^2 - Q^2}{2kQ}, \qquad y = \frac{\Phi}{k} \tag{5.55}$$

The expression for y indicates that along the free surface $\Phi = ky$, as was required because the pressure is zero on it. Equation 5.55 is a parameter representation of the free surface. An explicit formula can be obtained by elimination of the parameter Φ. This gives

$$\text{AB:}\ \frac{2kx}{Q} = \left(\frac{ky}{Q}\right)^2 - 1 \tag{5.56}$$

This is a parabola. It intersects the horizontal axis at a distance $x = -Q/2k$ (this is point A), and the vertical axis at a distance $y = Q/k$.

Example 5.8 Flow in Triangular Dam

As another example let us consider the case of a triangular dam upon an impermeable base. On one side the water reaches the crest of the dam, and on the other side the water level coincides with the toe of the dam. The base angles are 45° (see figure 5.14). The solution of this problem was obtained by Davison (see Harr, 1962).

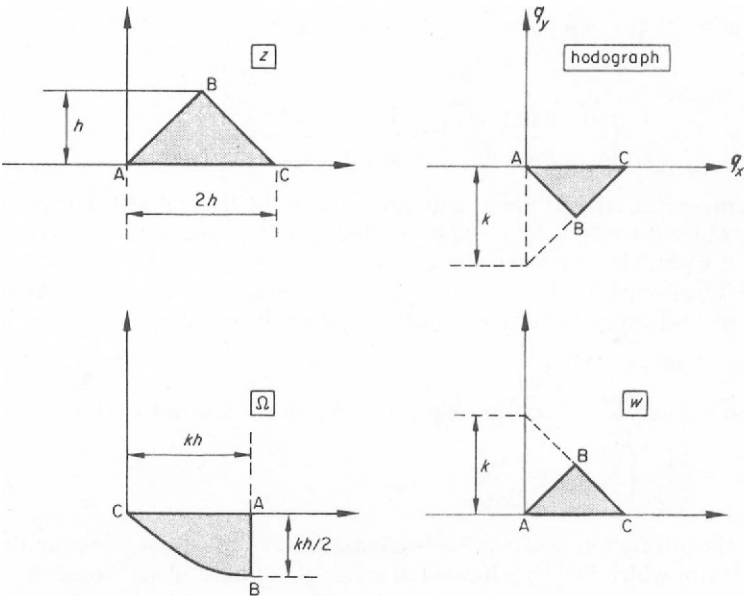

Figure 5.14 *Flow in a triangular dam*

The hodograph plane and its complex conjugate, the w-plane, are also shown in figure 5.14. In this case the entire slope BC is a seepage surface, which means that the region in the Ω-plane is unknown. Fortunately there is no free surface at all, so that the region in the z-plane is known. It also appears that in this special case the mapping from the z-plane to the w-plane is very simple (due to the special choice of the base angles), namely

$$w = kz/2h \tag{5.57}$$

Because $w = \mathrm{d}\Omega/\mathrm{d}z$ one now obtains, by integrating equation 5.57

$$\Omega = -k(z^2 - 4h^2)/4h \tag{5.58}$$

where the integration constant has been determined such that the potential in point C is zero. The value of the total discharge Q through the dam can be found by taking $z = h(1 + i)$. One then obtains for the values of Φ and Ψ in point B

$$\Phi_B = kh, \qquad \Psi_B = -\tfrac{1}{2}kh$$

Hence the total discharge, per unit width of the dam, is

$$Q/B = \tfrac{1}{2}kh \tag{5.59}$$

The two examples given above illustrate that the hodograph method may help to find a closed form solution of problems involving either a free surface or a seepage surface. In many cases, however, the practical utility of the method is limited. This may be due to the fact that the transformation functions, or the integration, are too complicated for analytical evaluation.

5.4 Special Functions

For certain types of problems involving free surfaces, with or without infiltration and/or seepage surfaces special functions facilitating the solution have been presented by Zhukovky (see Polubaronova-Kochina, 1962) and Van Deemter (1951). A more general type of function, incorporating these special functions, was introduced by Strack and Asgian (1978). In this section an elementary example will be given, without going into the more general aspects of the applicability of special mapping functions.

5.4.1 *Zhukovsky's Function*

Problems involving only horizontal potential lines, vertical stream lines, free surfaces and seepage sufaces as boundaries can be solved (if the geometry is simple enough to permit the analytical treatment) by the introduction of a function

$$Z = X + \mathrm{i}Y = \Omega + \mathrm{i}kz \tag{5.60}$$

Separation into real and imaginary parts gives

$$X = \Phi - ky, \qquad Y = \Psi + ky \tag{5.61}$$

Along a horizontal potential line both Φ and y are constant, and therefore so is X. Along a vertical stream line both Ψ and x are constant, and therefore so is Y. Along a free surface or a seepage surface the potential Φ is just ky, and hence $X = 0$. Actually this last property is the reason for the definition of Z, the Zhukovsky function, by equation 5.60.

As an example consider the problem illustrated in figure 5.15, of a drained dam on a homogeneous half-plane, with a free surface generated by infiltration at a rate N. Because of the symmetry of the problem only one half needs to be considered. The plane of the hodograph and the Zhukovsky function are also shown in the figure, together with the auxiliary planes of $w - \mathrm{i}k$ and $(w - \mathrm{i}k)^{-1}$. In this last plane the boundary is composed only of straight lines. Hence it is rather simple to map this plane onto the upper half-plane Im $(\zeta) > 0$. The mapping function is

$$\frac{1}{w - \mathrm{i}k} = \frac{\mathrm{i}}{k - N} - \frac{\mathrm{i}N}{k(k - N)}\left(\frac{\zeta}{\zeta + 1}\right)^{\frac{1}{2}} \tag{5.62}$$

It is easily verified that the point $\zeta = -1$ is mapped on to the point at infinity, $\zeta = 0$ corresponds to $(w - \mathrm{i}k)^{-1} = \mathrm{i}/k$ and $\zeta = \infty$ corresponds to $(w - \mathrm{i}k)^{-1} = \mathrm{i}/(k - N)$. The square root takes care of the correct behaviour of the mapping function near the points A and E.

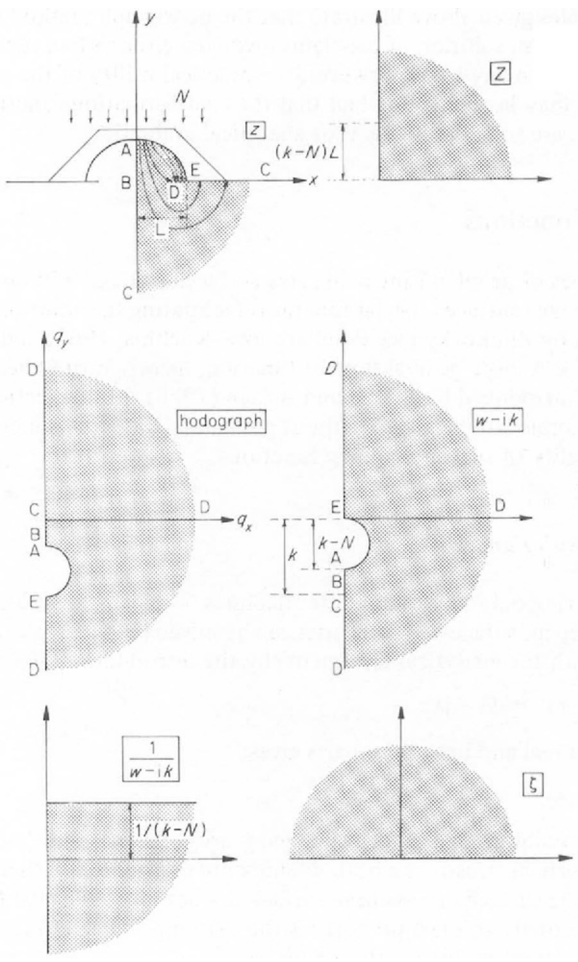

Figure 5.15 *Infiltration on a drained dam*

The mapping of the region in the Z-plane on to the upper half-plane Im $(\zeta) > 0$ is even simpler, because the points at infinity coincide

$$Z = L(k-N)\,\zeta^{\frac{1}{2}} \tag{5.63}$$

Substitution of ζ from equation 5.63 into equation 5.62 gives

$$\frac{1}{w-\mathrm{i}k} = \frac{1}{k-N} - \frac{\mathrm{i}NZ}{kL(k-N)^2}\left\{\frac{1}{1+Z^2/L^2(k-N)^2}\right\}^{\frac{1}{2}} \tag{5.64}$$

This expression gives a relation between the Zhukovsky function and $w = q_x - \mathrm{i}q_y$. In order to determine the relation with the complex variable Ω and the original variable z recourse has to be made to the definition of the Zhukovsky function

$$Z = \Omega + \mathrm{i}k$$

Because in general $w = -\,\mathrm{d}\Omega/\mathrm{d}z$ it now follows that

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = -(w - \mathrm{i}k)$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}Z} = -\frac{1}{w - \mathrm{i}k} \tag{5.65}$$

This means that the relation between z and Z can be found by integration of equation 5.64. That the combination of Z and $(w - \mathrm{i}k)^{-1}$ in both equations 5.64 and 5.65 is no coincidence (as it seems to be here) was shown by Strack and Asgian (1978).

Integration of equation 5.64 gives

$$z = -\frac{\mathrm{i}Z}{k - N} + \mathrm{i}\frac{NL}{k}\,[1 + Z^2/L^2\,(k - N)^2]^{\frac{1}{2}} \tag{5.66}$$

The integration constant has been taken equal to zero in order to obtain that point E is located in $z = L$. Equation 5.66 completes the solution of the problem, because the relation with the complex potential is immediately given by the definition in equation 5.60 (that is, $\Omega = Z - \mathrm{i}kz$). It makes little sense to try to eliminate Z.

Of particular interest is the position of the free surface AE. Then $Z = \mathrm{i} \times \alpha(k - N)\,L$ with $0 < \alpha < 1$ (see figure 5.15). Substitution into equation 5.66 now gives

$$z = \alpha L + \mathrm{i}\frac{NL}{k}\,(1 - \alpha^2)^{\frac{1}{2}} \tag{5.67}$$

or, after separation into real and imaginary parts

$$x = \alpha L, \qquad y = \frac{NL}{k}\,(1 - \alpha^2)^{\frac{1}{2}} \tag{5.68}$$

This is a parameter representation of the free surface. An explicit formula can be obtained by eliminating α. This gives

$$\left(\frac{x}{L}\right)^2 + \left(\frac{y}{NL/k}\right)^2 = 1 \tag{5.69}$$

The free surface turns out to be of elliptic shape. The highest point is point A, where $x = 0$ and $y = NL/k$.

The solution has been described above in terms of a length parameter L, which indicates the position of point E. In reality the location of this point is initially unknown, and it is rather the position of point D that is given, say $z_D = B$. A relation between L and B can be obtained by noting that in point D the value of $(w - ik)^{-1}$ is zero. From equation 5.62 one then obtains $\zeta = -(1 - N^2/k^2)^{-1}$, and then equation 5.63 gives $Z = iL(k - N)/(1 - N^2/k^2)^{1/2}$. Finally substitution into equation 5.66 gives $z = L(1 - N^2/k^2)^{1/2}$. Hence the relation between B, the location of point D in the z-plane, and L, the location of point E in the z-plane is simply

$$B = L(1 - N^2/k^2)^{\frac{1}{2}} \tag{5.70}$$

For given values of B and N/k the value of L can be determined from this equation.

Another special function was introduced by Van Deemter (1951), who realised that for infiltration problems the stream function Ψ varies linearly with x along a free surface, according to the formula $\Psi = A - Nx$, where N is the infiltration rate. This suggests the use of a complex function defined as

$$D = \Omega + iNz \tag{5.71}$$

Separation into real and imaginary parts now gives

$$\text{Re}(D) = \Phi - Ny, \qquad \text{Im}(D) = \Psi + Nx \tag{5.72}$$

The real part of D is constant on horizontal lines, and the imaginary part of D is constant on vertical stream lines and on a free surface with infiltration rate N. This function can be used successfully to solve certain infiltration problems (see also Childs, 1969).

5.5 Interface Problems

A type of boundary closely resembling a free surface is a sharp interface with a fluid of different density, which itself is in equilibrium. For such a sharp interface to be physically possible it is necessary that the two fluids are immiscible. For oil and water this is a reasonable assumption, but in the case of fresh and salt groundwater there usually is a certain transition zone, with brackish water. Yet, as a first approximation, it may be admissible to consider the interface as sharp.

Let the density of the moving fluid be denoted by ρ_f, and let the density of the heavier stationary fluid be denoted by ρ_s. Because in the latter fluid there is no flow, the pressure distribution in it must be hydrostatic. In other words

$$p = p_0 - \rho_s g y \tag{5.73}$$

where the y-axis is directed upwards. Because of equilibrium across the interface the pressure must be the same in both fluids, hence equation 5.73 also holds in the moving fluid, on the interface. For the potential along the interface one now obtains

$$\Phi = k\phi = k\left(y + \frac{p}{\rho_f g}\right) = k\left[\frac{p_0}{\rho_f g} - y\left(\frac{\rho_s - \rho_f}{\rho_f}\right)\right] \tag{5.74}$$

The specific discharge along the interface is (see also figure 5.16)

$$q_s = -\frac{\partial \Phi}{\partial s} = k' \frac{\partial y}{\partial s} = k' \sin \delta \tag{5.75}$$

where k' is a constant, namely

$$k' = k\left(\frac{\rho_s - \rho_f}{\rho_f}\right) \tag{5.76}$$

In equation 5.75 δ represents the local inclination of the interface. Since the interface is a stream line, and thus the flow is in the direction δ, one may now write

$$q_x = k' \sin \delta \cos \delta = \tfrac{1}{2}k' \sin (2\delta)$$

$$q_y = k' \sin^2 \delta = \tfrac{1}{2}k' \, [1 - \cos (2\delta)]$$

Elimination of δ finally gives

$$q_x^2 + (q_y - \tfrac{1}{2}k')^2 = (\tfrac{1}{2}k')^2 \tag{5.77}$$

which represents a circle in the hodograph plane, which passes through the origin and through the point $q_x = 0, q_y = k'$ (see figure 5.16).

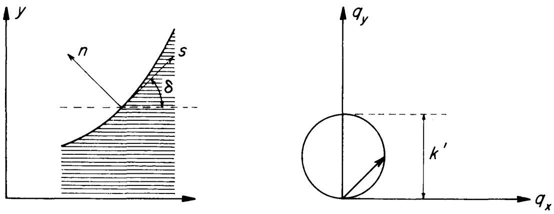

Figure 5.16 *Interface in hodograph*

Example 5.9

As an example, the influence of a well above an initially horizontal interface will be investigated (see figure 5.17). The solution of this problem is due to Strack (1972, 1973). The upper boundary is supposed to be a potential line, and the value of the head along that line is used as a zero level. Hence $\Phi = 0$ along that boundary, which supplies the water extracted by the well. The discharge of the well is denoted by $2Q$. Because of the symmetry of the problem only half of the flow region needs to be considered. Very far from the well the interface is supposed to be located at a depth L below the soil surface; this depth is determined by the level of salt water somewhere very far away, perhaps in a sea to which the aquifer is connected. In figure 5.17 the hodograph is also shown, together with the planes of w^{-1} ($w = q_x - iq_y$) and Ω, and an auxiliary ζ-plane. The functions mapping the regions in the Ω-plane and the w^{-1}-plane on to the upper half plane $\text{Im}(\zeta) > 0$ are

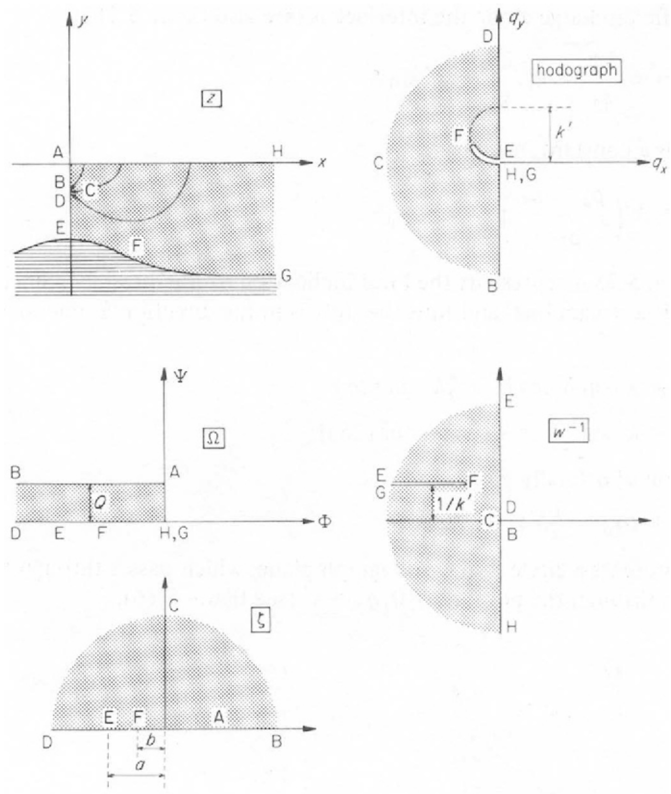

Figure 5.17 *Flow towards a drain above an interface*

$$\Omega = -\frac{2Q}{\pi}\left[\ln\left(\zeta^{\frac{1}{2}} + (\zeta - 1)^{\frac{1}{2}}\right) - \frac{\pi i}{2}\right] \tag{5.78}$$

$$w^{-1} = \frac{i}{k'}\left[1 - \left(\frac{\zeta}{\zeta + a}\right)^{\frac{1}{2}} - \frac{ab}{2b - a}\,\zeta^{-\frac{1}{2}}\,(\zeta + a)^{-\frac{1}{2}}\right] \tag{5.79}$$

Here a and b are parameters, indicating the location of points E and F in the ζ-plane. The occurrence of these two mathematical parameters can be ascribed to the presence of two essential physical parameters in the problem. It seems probable that these are the location of the well and its discharge.

The relation with the physical plane can be obtained in the usual way, by using the relation

$$w = -\,\mathrm{d}\Omega/\mathrm{d}z$$

It now follows that $\mathrm{d}z/\mathrm{d}\Omega = -\,w^{-1}$, and hence

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{\mathrm{d}z}{\mathrm{d}\Omega}\,\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} = -\,w^{-1}\,\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} \tag{5.80}$$

The relation between z and ζ can be found by intergrating $dz/d\zeta$, using the expressions equations 5.78 and 5.79. The integration constant can be determined from the condition that point A ($\zeta = 1$) is located in the origin ($z = 0$). The parameters a and b can be determined by requiring that point B is located in $z = -iH$ and that point G is located in $z = \infty - iL$. The final result is

$$z = \frac{2iQ}{\pi k'} \ln \quad (1+a)^{\frac{1}{2}} \left[\frac{\zeta^{\frac{1}{2}} + (\zeta - 1)^{\frac{1}{2}}}{(\zeta + a)^{\frac{1}{2}} + (\zeta - 1)^{\frac{1}{2}}} \right]$$

$$- \frac{2L}{\pi} \ln \left[\frac{(\zeta + a)^{\frac{1}{2}} + i a^{\frac{1}{2}} (\zeta - 1)^{\frac{1}{2}}}{(1+a)^{\frac{1}{2}} \zeta^{\frac{1}{2}}} \right] \tag{5.81}$$

with the value of a to be determined from the condition

$$\frac{\pi}{2} \frac{H}{L} = \arctan (a^{\frac{1}{2}}) - \frac{Q}{2k'L} \ln (1 + a) \tag{5.82}$$

The details of the derivation of equation 5.81, or of its verification by considering special points, are left to the reader.

The complete solution of the problem is expressed by equations 5.78, 5.79 and 5.81. The shape of the free surface can be determined by taking $\zeta = -\xi$, with $0 < \xi < a$, in equation 5.81. This then leads to an expression that has to be separated into real and imaginary parts to give expressions for x and y. Some possible shapes of the interface are drawn in figure 5.18, for three values of the discharge Q, namely $Q/k'L = 0.824$, 1.320 and 1.490.

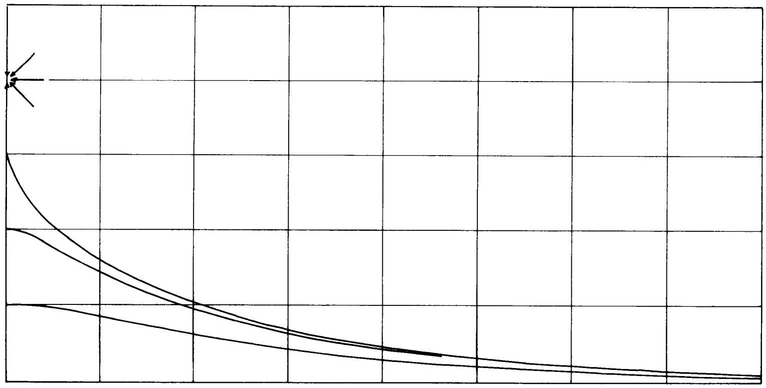

Figure 5.18 *Some steady interfaces*

Some other examples of exact solutions of interface problems, obtained by means of the hodograph method, have been given by Glover (1959), Bear and Dagan (1964), De Josselin de Jong (1965), Verruijt (1969) and Strack (1972, 1973). The last author even succeeded in superimposing solutions, thus enabling the derivation of exact solutions for systems involving several wells.

5.6 Problems

1. Consider the problem illustrated in figure 5.19, which may be considered to represent a schematisation of the flow induced by an infiltration ditch in a shallow aquifer. Show that the complex potential for this case is

$$\Omega = -\frac{2}{\pi} Q \operatorname{arcsinh}\left[\frac{\cosh(\pi z/2a)}{\sinh(\pi b/2a)}\right]$$

where Q is the total discharge. The formula can be derived by some successive conformal transformations, and its validity can be checked by verifying some particular points, for instance $z = \mathrm{i}a$ and $z = -b + \mathrm{i}a$.

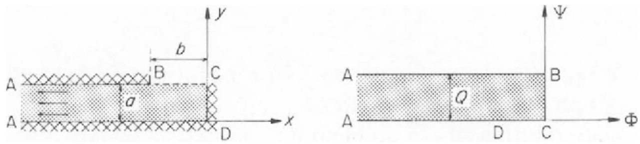

Figure 5.19 *Problem 1*

2. Show that the complex potential

$$\Omega = -q_0 (z + a^2/z)$$

can be used to describe the flow around a circular cylinder of radius a.

3. Let $z = w(\zeta)$ represent the conformal transformation of a region R in the z-plane on to the interior of the unit circle $|\zeta| = 1$, and let a well be operating in the point $z = z_0$, with the water level being constant along the outer boundary. Show that the equivalent radius can be expressed as

$$R_{eq} = (1 - |\zeta_0|^2)\,|\omega'(\zeta_0)|$$

where $\omega'(\zeta) = \mathrm{d}\omega/\mathrm{d}\zeta$, and ζ_0 is the image point of z_0, that is $z_0 = \omega(\zeta_0)$.

4. Show that for a well in the centre of a rectangular island, of dimensions $2a \times 2b$, the equivalent radius is

$$R_{eq} = \frac{2a}{1+\sqrt{m}} \quad \frac{1}{K(m)}$$

where $K(m)$ is the complete elliptic integral, and the value of m must be determined from the equation

$$\frac{K(1-m)}{K(m)} = \frac{2b}{a}$$

A useful table to determine the value of m is given by Abramowitz and Stegun (1965) p. 612.

5. The problem illustrated in figure 5.20 can be solved using the Zhukovsky function. Show that the height of the points E and A are given by

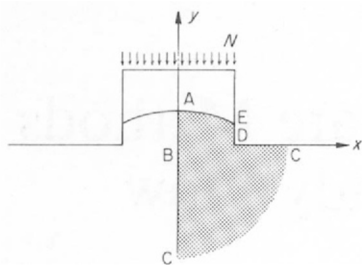

Figure 5.20 *Problem 5*

$$y_E = \frac{2}{\pi} L \ln(1/\cos\alpha)$$

$$y_A = \frac{2}{\pi} L \ln[(1+\sin\alpha)/\cos\alpha]$$

where $\alpha = \pi N/2k$. Note that for small values of N/k the last formula becomes identical to the one for the case illustrated in figure 5.15, (see equation 5.69). Is this a coincidence?

6. Show that for the case of flow in a coastal aquifer illustrated in figure 5.21 (Glover, 1959) the relation between z and Ω is $z = -k'\Omega^2/2Q$, and that $w = Q/k'\Omega$, where Q is the total discharge.

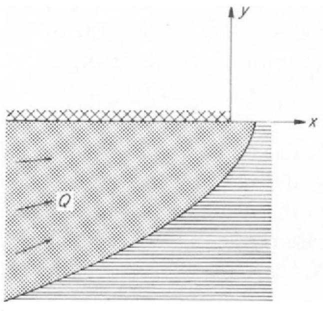

Figure 5.21 *Problem 6*

7. In section 3.3 a very crude derivation was given of Dupuit's formula (see equation 3.36). Apply a similar reasoning to the case considered as example 5.8 (see figure 5.14). Compare the result with the exact expression in equation 5.59.

6

Approximate Methods for Plane Steady Flow

In this chapter some approximate methods for the solution of plane steady groundwater flow problems are presented. These are the graphical method of sketching flow nets, the numerical method of finite differences, and the method of fragments. A powerful and popular approximate method, the finite element method, is treated separately, in chapter 8. Approximate methods not treated explicity in this book are the method of boundary elements (Banerjee and Butterfield, 1977; Brebbia, 1978; Van der Veer, 1978), and the method of singularity distributions (De Josselin de Jong, 1960; Strack, 1980; Haitjema, 1980). These are numerical methods that can be considered as alternatives for the finite element method.

6.1 Flow Net

In section 4.3 it was shown that in two-dimensional flow of groundwater in a homogeneous aquifer the potential lines and the stream lines constitute a network of elementary squares, provided that the intervals between consecutive potential lines and stream lines are chosen equal

$$\Delta\Phi = \Delta\Psi \tag{6.1}$$

This property is the basis for a simple graphical method. In this method a so-called flow net is sketched, and then gradually improved until finally a network of squares is obtained that is also in agreement with the boundary conditions. The most simple case is one of the type illustrated in figure 6.1, in which the boundary consists entirely of given potential lines and stream lines.

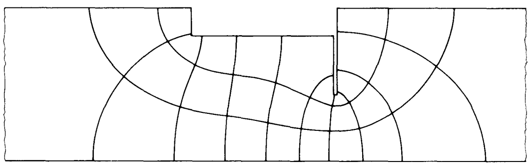

Figure 6.1 *Flow net*

A convenient procedure is to start by sketching a number of stream lines, tracing the paths that one expects the water particles to follow. Next a set of potential lines is drawn perpendicular to the stream lines, trying to keep the distance equal to those of the stream lines. This usually leads to some local difficulties, so that for instance rectangles are obtained rather than squares, which indicates that the original stream lines were not correct. The flow net can then be adjusted, and after several trials and readjustments all requirements may be met. If the flow net forms a network of elementary squares, and complies with the boundary conditions, it can be concluded that the potential lines and stream lines are good approximations, and that their intervals are equal, $\Delta\Phi = \Delta\Psi$. Usually the total potential drop over the entire region is given, say $k(h_1 - h_2)$, where h_1 and h_2 are given water levels. If the flow net contains n intervals of the potential, then apparently

$$\Delta\Phi = k(h_1 - h_2)/n \tag{6.2}$$

In figure 6.1 the value of n is about 9.5 (the value of n need not to be an integer, because the last element near one of the boundaries may turn out to be only part of a square). It was seen in section 4.3 that the difference in stream function between two points represents the total discharge (per unit thickness) flowing through a section between these points (see equation 4.32). Hence the value obtained in equation 6.2 represents the amount of water flowing between two stream lines. If the total number of stream lines is m, then the total discharge is

$$Q = \frac{m}{n} kH(h_1 - h_2) \tag{6.3}$$

where H is the thickness of the plane of flow (perpendicular to the drawing in figure 6.1). In the case illustrated in figure 6.1 the value of m is 3, so that the total discharge is $Q = 0.32kH(h_1 - h_2)$.

It should be noted that in the process of sketching a flow net some difficulties may arise near corner points; in figure 6.1 it seems that there are some five-sided squares. This is of course caused by the corner point in the boundary stream line. The behaviour near such a singularity can be investigated in some detail by the appropriate conformal transformation (see the first two examples in figure C.2 of appendix C), or by drawing a more detailed flow net near these corner points.

A special difficulty arises in the case of groundwater flow with a free surface (see for instance figure 6.2). The boundaries AB and AE present no difficulties. AE is a stream line, and AB is a potential line. The boundaries BCD

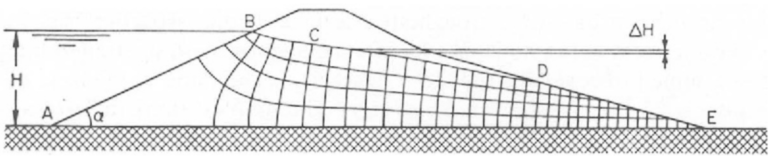

Figure 6.2 *Flow net with a free surface*

and DE are more complicated, however. The boundary BCD is a free surface. This is a stream line, but its location is unknown initially. This indefiniteness is balanced by an extra condition, namely that the pressure along it is atmospheric (that is, $p = 0$). This means that the groundwater head ϕ in any point of the free surface is equal to the height, $\phi = y$, and thus the potential along BCD is $\Phi = ky$. This means that the potential lines, which have a constant difference $\Delta\Phi$, should intersect the free surfaces at constant differences in height $\Delta H = \Delta\Phi/\mathrm{k}$. A possible procedure now is to first assume a position for the free surface, construct a flow net, by trial and error, and then check whether the potential lines intersect the free surface at heights with a constant difference. When this is not the case, as the student will find in his first experience with this method, the free surface must be modified, and the flow net has to be constructed all over again. Along the surface DE water will flow out of the slope. This is a so-called seepage surface. The pressure along DE is atmospheric, hence $\Phi = ky$, which means that the potential lines intersect DE at heights differing from each other by the constant amount $\Delta H = \Delta\Phi/k$. Note that DE is not a stream line, so that the potential lines are not perpendicular to it.

Since the shape of BCD and the location of point D are initially unknown, they have to be adjusted until all conditions are met. This is an essential complication, which may make the process rather time-consuming. The success of the method greatly depends upon the accuracy of the first estimation of the free surface BCD. It may be worth while to note that at point D the free surface is tangential to DE, and that at B it is perpendicular to AB, theoretically. For rather flat slopes experience shows that the free surface rapidly changes its direction near the entry point B (see figure 6.2).

6.2 Finite Differences

In this section a numerical method is presented for the solution of Laplace's equation in two dimensions, based upon the replacement of the differential equations by finite difference expressions. The method was developed originally by Southwell (1940), for calculations by hand. Here the method will be presented for calculations by means of a computer.

The basic differential equation is Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{6.4}$$

where ϕ is the head. Instead of considering ϕ as a continuous variable of variables x and y varying continuously throughout a certain region, restriction is now made to values of ϕ in the nodal points of a rectangular mesh of straight lines, which is assumed to cover the field (see figure 6.3). The value of the head ϕ in a point $x = x_i$, $y = y_j$ is indicated by $\phi(i, j)$. The simplest finite difference approximation of $\partial^2\phi/\partial x^2$ now is (see for example, Carnahan *et al.*, 1969)

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi(i+1,j) - 2\phi(i,j) + \phi(i-1,j)}{(\Delta x)^2} \tag{6.5}$$

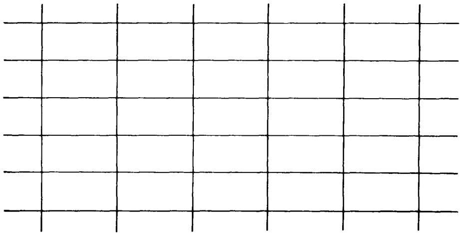

Figure 6.3 *Mesh for finite differences*

where Δx is the constant interval between two consecutive points x_i and x_{i+1}. For a straight line, for which the second derivative vanishes, the approximation in equation 6.5 would indicate that $\phi(i, j)$ is the average of the two neighbouring points, which is then correct. The finite difference approximation of $\partial^2\phi/\partial y^2$ is, similarly

$$\frac{\partial^2 \phi}{\partial y^2} \approx \frac{\phi(i, j+1) - 2\phi(i, j) + \phi(i, j-1)}{(\Delta y)^2} \tag{6.6}$$

where Δy is the constant interval between two consecutive points y_j and y_{j+1}. Adding the two equations 6.5 and 6.6 and setting the result equal to zero, in order to satisfy equation 6.4, now gives

$$\phi(i, j) = \frac{\alpha}{2}\,[\phi(i+1, j) + \phi(i-1, j)] + \frac{1-\alpha}{2}\,[\phi(i, j+1) + \phi(i, j-1)] \tag{6.7}$$

where

$$\alpha = (\Delta y)^2/[(\Delta x)^2 + (\Delta y)^2] \tag{6.8}$$

Equation 6.7 is the basic algorithm for the finite difference method as presented here. For calculations by hand it is convenient to take $\Delta x = \Delta y$. Then $\frac{1}{2}\alpha = \frac{1}{2}(1 - \alpha) = 1/4$, and equation 6.7 expresses that $\phi(i, j)$ is simply the arithmetical average of the values in the four surrounding points. For calculations by means of a computer program the extra flexibility obtained by choosing Δx and Δy unequal is certainly worth the small trouble of having slightly different multiplication factors. In fact, one might as well drop the conditions that Δx and Δy are constant. This would make the computer program even more flexible, at the price of again a somewhat more complicated algorithm.

The algorithm in equation 6.7 holds in every interior point of the field. Points on the boundary need some special attention, because there the appropriate boundary condition has to be satisfied, at least approximately. The simplest type of boundary condition is that the head is prescribed. If this is the case in all points of the boundary the basic algorithm in equation 6.7 can be used to calculate all the interior values, because there are just as many equations as unknown values.

Another important type of boundary condition is the case of an impermeable boundary (see figure 6.4 in which a horizontal impermeable boundary has been sketched). Because no water can flow across this boundary, the tangent to the groundwater head perpendicular to the boundary must be zero. This can most conveniently be expressed by considering, temporarily, an additional

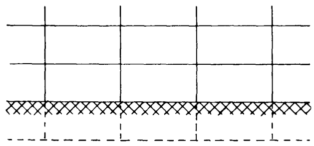

Figure 6.4 *Impermeable boundary*

point on the other side of the boundary, and then requiring that $\phi(i, j - 1) = \phi(i, j + 1)$ if i, j indicates a point on the boundary. This condition makes the head symmetric with respect to the boundary, hence $\partial\phi/\partial y = 0$. Combining this condition with the general formula in equation 6.7 gives for a point on a horizontal impermeable boundary, at the lower side of the region

$$\phi(i, j) = \frac{\alpha}{2} [\phi(i + 1, j) + \phi(i - 1, j)] + (1 - \alpha)\, \phi\, (i, j + 1) \tag{6.9}$$

Similar expressions can be obtained for a horizontal boundary at the upper side of the region ($j + 1$ must than be replaced by $j - 1$ in equation 6.9), and for vertical impermeable boundaries.

For all interior points the appropriate equation is equation 6.7. For points on an impermeable boundary the algorithm is of the form of equation 6.9, and for points on a boundary with given head, this given value acts as a boundary condition. In any case a system of linear equations is obtained, which can be solved by standard numerical methods. A convenient method, which can also be used for hand calculations (denoted as 'relaxation' by Southwell, 1940), is the so-called Gauss-Seidel method. This is an iterative method in which all unknown values are successively updated from some initial estimates, in several cycles. For hand calculations the effort depends very much upon the number of cycles to be performed, and it is worth the trouble to try and use as good an initial estimate as one can get. For calculations by computer, savings in computer time are not so important, so that the initial estimate might well be that $\phi(i, j) = 0$, everywhere.

Program 6.1

As an example a computer program will be presented for the case illustrated in figure 6.5, of flow through a strip with a narrow opening at the left-hand side. The length of the strip is L, the height A, the width of the opening is P, the number of subdivisions in horizontal direction is N, and in vertical direction there are M subdivisions (in figure 6.5, $N = 20$, $M = 8$ and $P/A = 3/16$).

At the right-hand side the head $\phi = 10$ and at the outlet at the left-hand side $\phi = 0$. A program performing the necessary calculations is reproduced below (program 6.1). The program has been written in BASIC, a simple language that can be used even on a very small microcomputer. In this program it has been assumed that several statements can be listed on a single line, when they are separated by a colon (:). The variables are contained in a matrix F. In line 20 the input data are read, where NI is the number of iterations, which is read as 200 by the data-statement 110. The fundamental algorithm is in line 80, and the boundary conditions for the impermeable boundaries are executed in lines 50, 60, 70 and 90. As output the program gives the head along the lower boundary. In figure 6.6 the result of the numerical calculations are compared with the results of an analytical solution for an infinite strip. This analytical solution was given in section 5.1 (see figure 5.4). The value of the discharge Q in this solution was chosen such that over a distance L a head $\Delta\phi = 10$ is obtained. It appears that the results are very close.

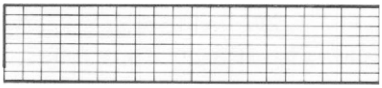

Figure 6.5 *Flow through a strip*

```
10 REM FINITE DIFFERENCES FOR PLANE STEADY FLOW, PROGRAM 6.1
20 READN,M,L,A,P,NI:DX=L/N:DY=A/M:K=INT(P/DY+0.5):DIMF(N,M)
30 FORJ=0TOM:FORI=0TON:A=10*I/N:F(I,J)=A:NEXTI:NEXTJ
40 A=DY*DY:B=2*(A+DX*DX):A=A/B:B=0.5-A:FORIT=1TONI:PRINT"ITERATION";IT
50 FORJ=KTOM-1:F(0,J)=2*A*F(1,J)+B*(F(0,J+1)+F(0,J-1)):NEXTJ
60 F(0,M)=2*A*F(1,M)+2*B*F(0,M-1)
70 FORI=1TON-1:F(I,0)=A*(F(I+1,0)+F(I-1,0))+2*B*F(I,1)
80 FORJ=1TOM-1:F(I,J)=A*(F(I+1,J)+F(I-1,J))+B*(F(I,J+1)+F(I,J-1)):NEXTJ
90 F(I,M)=A*(F(I+1,M)+F(I-1,M))+2*B*F(I,M-1):NEXTI:NEXTIT:PRINT
100 FORI=0TON:PRINT"I=";I;"    PHI=";F(I,0):NEXTI
110 DATA20,8,10,2,0.375,200
```

Program 6.1 *Finite differences for plane steady flow*

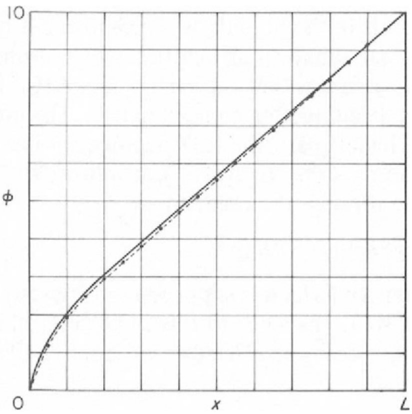

Figure 6.6 *Comparison of numerical and analytical results*

The finite difference method can be extended and generalised in several ways, for instance by introducing variable mesh sizes, or techniques to improve the convergence of the Gauss-Seidel method. Some of these will be used in the finite element method, which is presented in chapter 8.

6.3 Method of Fragments

An effective method for the solution of problems of groundwater flow underneath engineering structures and similar problems is the so-called method of fragments, which was developed by Pavlovsky (see for example, Harr, 1962). In this method the flow region is divided into a number of subregions (fragments) by assuming certain convenient sections to be potential surfaces. The fragments are chosen such that for the flow in each of them an analytic solution is known.

The method will not be treated exhaustively; a single example may illustrate the method, and serve as an introduction to more general problems. Therefore consider the problem of figure 6.7. The boundary consists of straight potential

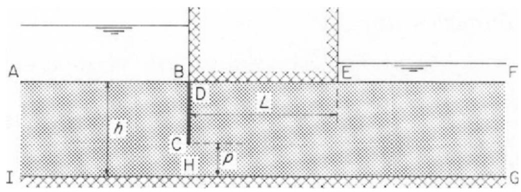

Figure 6.7 *Method of fragments*

lines and stream lines, and hence this problem can in principle be solved by application of the Schwarz-Christoffel transformation. However, since there are so many corner points in the boundary, the Schwarz-Christoffel formula will lead to enormous mathematical difficulties, which prohibits an analytical solution. It is therefore assumed that the vertical line CH is a potential line, which does seem to be an admissible approximation. The problem is now separated into two parts. The left part, the first fragment, is just one half of a problem discussed in section 5.1 (see figure 5.3). It follows from equation 5.21 that the total discharge through the fragment is

$$Q = kB(\phi_1 - \phi_2)K(1 - b)/K(b) \tag{6.10}$$

where ϕ_1 is the head on AB (which is supposed to be given) and ϕ_2 is the head on CH (which is unknown). The value of b is to be determined from the ratio p/h by the expression $b = \cos^2(\pi p/2h)$, (see equation 5.12). Equation 6.10 can also be written as

$$Q = \frac{B\Delta\phi_1}{\alpha_1} \tag{6.11}$$

where α_1 is a parameter representative for the resistance of the fragment to the flow

$$\alpha_1 = \frac{1}{k} \frac{K(b)}{K(1-b)} \tag{6.12}$$

It can be expected that for every type of fragment an expression of the form of equation 6.11 can be established. In fact, for the second fragment one may write

$$Q = \frac{B\Delta\phi_2}{\alpha_2} \tag{6.12a}$$

where $\Delta\phi_2 = \phi_2 - \phi_3$ (with ϕ_3 denoting the given value of the head on EF), and where

$$\alpha_2 = \frac{L + \Delta L_1 + \Delta L_2}{kh} \tag{6.13}$$

The distances ΔL_1 and ΔL_2 in this formula represent the equivalent lengths of the narrowing at the left end and of the extra soil material at the right end. In example 5.1 in section 5.1 it was found (see equation 5.10) that

$$\Delta L_2 = \frac{2h}{\pi} \ln 2 \tag{6.14}$$

On the other hand is was found in example 5.3 in section 5.1 (see equation 5.25) that

$$\Delta L_1 = -\frac{2h}{\pi} \ln \sin\left(\frac{\pi p}{2h}\right) \tag{6.15}$$

Combining equations 6.14 and 6.15 with equation 6.13 gives

$$\alpha_2 = \frac{1}{k}\left[\frac{L}{h} + \frac{2}{\pi} \ln 2 - \frac{2}{\pi} \ln \sin\left(\frac{\pi p}{2h}\right)\right] \tag{6.16}$$

The first term in the expression represents the geometrical resistance factor for a strip of soil of length L and height h. The second term is the extra resistance due to the outlet at the right-hand side, and the third term is the extra resistance due to the narrowing at the left-hand side.

In general one may write for an arbitrary fragment

$$\Delta\phi_j = \frac{Q}{B} \alpha_j \tag{6.17}$$

Because the same discharge flows through all fragments the total head loss $\Delta\phi$ can be expressed as

$$\Delta\phi = \Sigma\, \Delta\phi_j = \frac{Q}{B}\, \Sigma\alpha_j$$

Hence

$$Q = \frac{B\Delta\phi}{\Sigma\alpha_j} \tag{6.18}$$

which expresses that all that has to be done is to add all resistances α_j. In the example of figure 6.7 one obtains from equations 6.12 and 6.16

$$\Sigma\alpha_j = \frac{1}{k}\left[\frac{K(b)}{K(1-b)} + \frac{L}{h} + \frac{2}{\pi}\ln 2 - \frac{2}{\pi}\ln\sin\left(\frac{\pi p}{2h}\right)\right] \tag{6.19}$$

The total discharge can now be found from equation 6.18.

For various types of fragments the values of the resistance parameters have been determined by Pavlovsky (see Harr, 1962). It is to be noted that the method actually amounts to just adding the resistance of a series of fragments. The hydraulic conductivity of each fragment can be different, because this is included in the resistance factor α.

6.4 Problems

1. Consider the flow through a dam with vertical slopes (see figure 3.9). If the length of the base is L, let the water level on the left side be $H_1 = L$, and on the right side $H_2 = \frac{1}{2}L$. Determine the free surface and the total discharge by means of the graphical method of sketching flow nets. Compare the result found for the total discharge Q with Dupuit's formula, equation 3.37.

2. Modify program 6.1 so that it will print the values of the head on the upper boundary. Run the program on a computer, and show that the head in the upper left corner is about $\phi = 2.0$.

3. Modify program 6.1 by including an over-relaxation factor R. This can be done by first calculating for each node the deviation E from the value indicated by the algorithm in equation 6.7, and then adding $R*E$ to the local value of the head, with $1 \leqslant R < 2$. Run the program, and demonstrate that convergence is much more rapid with an over-relaxation factor of say $R = 1.6$.

4. Use the method of fragments to calculate the total discharge underneath the structure sketched in figure 6.1. Compare the result with the value obtained in section 6.1.

7

Non-steady Flow

Non-steady flow of groundwater occurs if water can be stored in the soil. There are two possibilities for storage, the first being when the pores of a soil are not all initially filled with water. In this case, groundwater can be stored by filling up more of the pore space, and this type of storage may especially occur in unconfined aquifers, when the water level varies with time. The phenomenon is usually called phreatic storage. A second possibility of storage is elastic storage, which is the phenomenon that water is stored in a deforming soil. This type of storage, in which the pore space increases by expansion of the soil, may occur in all types of aquifers. In confined and saturated aquifers it is the only possible form of storage. In this chapter the basic equations for non-steady groundwater flow will be established, and some methods of solution will be presented.

7.1 Basic Equations

In this section the basic equations for non-steady flow will be presented, in their simplest form, disregarding non-linear phenomena. This will be done separately for elastic storage and phreatic storage.

7.1.1 *Phreatic storage*

Consider an unconfined aquifer, with a varying water table (see figure 7.1). It is assumed that the flow is mainly horizontal, so that one may write Darcy's law as

$$q_x = -k\frac{\partial h}{\partial x}, \qquad q_y = -k\frac{\partial h}{\partial y} \tag{7.1}$$

where h is the height of the water table above the impermeable base. Equation 7.1 represents Darcy's law, including the Dupuit approximation ($\phi = h$, see section 3.3). In a time interval Δt the net inflow into an element with dimensions Δx, Δy and h is

$$N\Delta x\Delta y\Delta t - \left[\frac{\partial}{\partial x}(hq_x) + \frac{\partial}{\partial y}(hq_y)\right]\Delta t$$

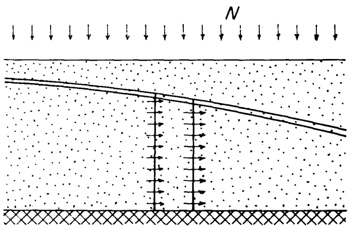

Figure 7.1 *Phreatic storage*

This need not now be zero (as in the case of steady flow) because water can be stored in a thin zone of thickness Δh above the free surface. The amount of water that can be stored in this zone is

$$S_p \Delta h \Delta x \Delta y = S_p \frac{\partial h}{\partial t} \Delta t \Delta x \Delta y$$

where S_p is the part of the pore space that is being filled with water if the water table rises. In a coarse material S_p is equal to the porosity n, but in fine graded materials the value of S_p is usually somewhat smaller than n, say $S_p \approx 0.30$. It is called the phreatic storativity. It now follows that the principle of conservation of mass requires that

$$S_p \frac{\partial h}{\partial t} = N - \frac{\partial}{\partial x}(hq_x) - \frac{\partial}{\partial y}(hq_y) \tag{7.2}$$

Here small effects such as the compression of the water itself have been disregarded. Substitution of equation 7.1 into equation 7.2 gives

$$S_p \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(kh \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(kh \frac{\partial h}{\partial y}\right) + N \tag{7.3}$$

This is the basic differential equation for non-steady groundwater flow with phreatic storage. Unfortunately, it is non-linear, which means that the full analysis of boundary value problems is very complicated mathematically. Dupuit's trick of using h^2 as a variable does not work in this case, because the term $\partial h/\partial t$ cannot be expressed easily in terms of h^2. Therefore a somewhat drastic approximation is necessary to make equation 7.3 amenable to mathematical solutions. This is done by linearisation. First it is assumed that the hydraulic conductivity is constant; and furthermore it is assumed that the derivatives $\partial h/\partial x$ and $\partial h/\partial y$ are relatively small so that one may write

$$\frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) = h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 \approx h \frac{\partial^2 h}{\partial x^2}$$

Under these conditions equation 7.3 reduces to

$$S_p \frac{\partial h}{\partial t} = T\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + N \tag{7.4}$$

where $T = kh$, the transmissivity of the aquifer. Equation 7.4 is the linearised differential equation for non-steady groundwater flow with phreatic storage. This equation will be taken as the starting point for all solutions presented in this chapter. Because it has already been assumed that $\partial h/\partial x$ and $\partial h/\partial y$ are small it seems logical to further simplify the problem by considering the transmissivity T in equation 7.4 as constant. This seems justified if the considerations are restricted to aquifers in which the variations in the water level are small compared to their average value.

7.1.2 *Elastic Storage*

In a completely saturated confined aquifer there can be no phreatic storage. Yet small amounts of water can be stored or released from storage if the pressure in the water fluctuates, resulting in volumetric deformations. It is now necessary to consider the movement of the soil as well as the flow of the groundwater. The basic equation for the phenomenon is the so-called storage equation (see for example, Verruijt, 1969),

$$-\frac{\partial e}{\partial t} = n\beta \frac{\partial p}{\partial t} + \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \tag{7.5}$$

Here e is the volumetric deformation of the soil, $e = \Delta V/V$, n is the porosity, and β is the compressibility of the pore fluid, defined by the relation

$$\Delta V_w = -\beta V_w \Delta p \tag{7.6}$$

which expresses the volume change of a volume of water V_w due to a pressure increment Δp. Equation 7.5 states that the volume of the soil can decrease ($-\partial e/\partial t$ indicates a reduction of the volume because of the minus sign) due to two effects: one is the compression of the pore fluid, which is proportional to n, to β and to the pressure increment; and the other is the expulsion of water due to the flow, which is expressed by the term between brackets in equation 7.5.

Restriction is now made to a horizontal confined aquifer. The term $\partial q_z/\partial z$ can then be omitted from equation 7.5. Together with Darcy's law

$$q_x = -k\frac{\partial \phi}{\partial x}, \qquad q_y = -k\frac{\partial \phi}{\partial y} \tag{7.7}$$

one now obtains, assuming the permeability to be constant

$$-\frac{\partial e}{\partial t} = n\beta\rho g \frac{\partial \phi}{\partial t} - k\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \tag{7.8}$$

Here the pressure term $\partial p/\partial t$ has been transformed into a term $\partial \phi/\partial t$, by using the relation $\phi = z + p/\rho g$.

The left-hand side of equation 7.8 is the rate of volume deformation of the soil. Theoretically speaking one now should go into the analysis of soil deformations due to stresses. This is a separate branch of science, soil mechanics. A complete and correct analysis is available since the pioneering work of Biot (1941) in the so-called theory of consolidation. For the present purpose it is sufficient to follow a more simple (and less accurate) approach, which is due to Jacob (1941) and Terzaghi (1943). This approximation consists of the following steps.

(1) First it is assumed that the vertical total stress does not change with time. The justification of this assumption is the notion that the vertical stress in a soil is primarily due to the weight of the overburden, and in the absence of external loading this weight remains constant. The mathematical formulation of this assumption is

$$\frac{\partial \sigma_V}{\partial t} = 0 \tag{7.9}$$

where σ_V is the vertical normal component of total stress in the soil.

(2) The second step is to use Terzaghi's concept of effective stress, which states that the total stress in a granular material can be decomposed into the pore water pressure p, and the effective stresses, which govern the deformations of the soil. Hence

$$\sigma_V = \sigma_V' + p \tag{7.10}$$

where σ_V' is the effective stress, which is a good measure for the concentrated intergranular forces. The notion that in this decomposition the pore water pressure must be considered to be acting over the entire surface, at least in first approximation, and that in equation 7.10 one should certainly not write np instead of p, is one of the significant contributions of Terzaghi to the development of soil mechanics.

(3) The third assumption, which was introduced by Jacob (1941), is that there are no horizontal deformations in the soil. This means that the volume strain e is equal to the vertical strain ϵ_V,

$$e = \epsilon_V \tag{7.11}$$

(4) Finally it is assumed that there exists a linear relationship between the vertical strain ϵ_V and the vertical effective stress in the soil

$$\epsilon_V = -\alpha \sigma_V' \tag{7.12}$$

The coefficient α is the so-called confined compressibility of the soil (often denoted by m_V in soil mechanics).

From equations 7.9 to 7.12 it now follows that

$$\frac{\partial e}{\partial t} = \alpha \frac{\partial p}{\partial t} = \alpha \rho g \frac{\partial \phi}{\partial t} \tag{7.13}$$

which is a relatively simple relation between the volume strain e and the groundwater head ϕ. Substitution of equation 7.13 into equation 7.8 gives, after multiplication by H, the thickness of the aquifer

$$S_e \frac{\partial \phi}{\partial t} = T\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \tag{7.14}$$

where $T = kH$ and

$$S_e = \rho g H(\alpha + n\beta) \tag{7.15}$$

the so-called elastic storativity.

It is interesting to note that the basic differential equation for non-steady groundwater flow with elastic storage, equation 7.14, is of exactly the same character as the equation describing phreatic storage, equation 7.4, without infiltration of course. The difference is the definition and the physical background of the storativity. In the case of phreatic storage the storativity S_p is of the order of magnitude of the porosity (say 0.3 or 0.4). In the case of elastic storage the value of the storativity S_e depends upon the compressibility of the soil and the fluid, but in general its value is much smaller (say from 0.001 to 0.01). This means that elastic storage is usually insignificant compared to phreatic storage. Only when there is no phreatic storage at all (that is, for confined aquifers) elastic storage is the determining phenomenon for non-steady flow.

The fact that for both elastic and phreatic storage the same basic differential has been obtained, at least as first approximations, means that it is not necessary to distinguish between the two when discussing methods of solution. In the next sections the storativity will be denoted simply by S, and it depends upon the type of aquifer whether this is to be interpreted as phreatic or elastic storativity.

7.2 Some Analytical Solutions

In this section some analytical solutions for non-steady groundwater flow are presented. The most powerful method for determining analytical solutions is by using the so-called Laplace transformation. There exist some interesting particular solutions, however, and one of these will be presented first.

Example 7.1 Propagation of Waves

As a first example the problem of the propagation of a wave (for example, due to tidal fluctuations) in an aquifer is considered, for the one-dimensional case (see figure 7.2). The differential equation for this case can be written as

$$S\frac{\partial\phi}{\partial t} = T\frac{\partial^2\phi}{\partial x^2} \tag{7.16}$$

and the boundary conditions are

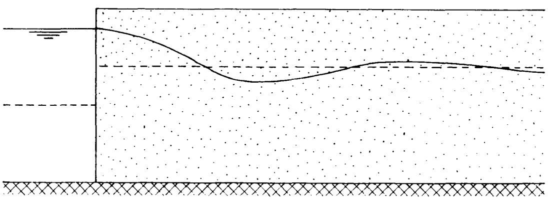

Figure 7.2 *Propagation of a wave*

$$x = 0: \quad \phi = \phi_0 + \Delta\phi \sin(\omega t) \tag{7.17}$$

$$x \rightarrow \infty: \quad \phi = \phi_0 \tag{7.18}$$

where $\Delta\phi$ is the amplitude of the fluctuation and ω is the frequency.

It seems reasonable to assume that the solution can be written as

$$\phi = \phi_0 + f(x)\sin(\omega t) + g(x)\cos(\omega t) \tag{7.19}$$

Substitution of equation 7.19 into equation 7.16 leads to two simultaneous linear differential equations for the unknown functions $f(x)$ and $g(x)$. These equations can easily be solved by the standard method for linear differential equations (see for example, Boyce and DiPrima, 1977). The constants appearing in this solution can be determined from the boundary conditions in equations 7.17 and 7.18. The final result is

$$\phi = \phi_0 + \Delta\phi \exp(-\lambda x)\sin(\omega t - \lambda x) \tag{7.20}$$

where the value of λ is to be determined from the relation

$$\lambda^2 = \omega S/2T \tag{7.21}$$

The solution in equation 7.20 represents a damped sinusoidal wave. It can easily be verified that this solution satisfies equations 7.16 to 7.18. The amplitude has been reduced to 5 per cent of its boundary value when $\exp(-\lambda x) = 0.05$, that is, when $\lambda x = 3$, or $x = 3/\lambda$. For a tidal fluctuation in an unconfined aquifer of transmissivity $T = 10^{-3}$ m^2/s and storativity $S = 0.4$, one obtains $1/\lambda = 5.86$ m. This means that the damping is rather rapid. Along a beach tidal fluctuations influence the groundwater only in a narrow zone. The zone of influence is much larger for slower fluctuations, or for instance in case of elastic storage, when the storativity is much smaller.

Example 7.2 Response to a Stepwise Fluctuation

The second example concerns the response of the groundwater head in an infinite aquifer to a sudden change in water level on the boundary. The differential equation is again equation 7.16, but now the boundary conditions are

$$x = 0: \quad \phi = \phi_0 + \Delta\phi H(t) \tag{7.22}$$

$$x \rightarrow \infty: \quad \phi = \phi_0 \tag{7.23}$$

where $H(t)$ is Heaviside's unit-step function, $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t > 0$. The initial water level in the aquifer is supposed to be given by the initial condition

$$t = 0: \quad \phi = \phi_0 \ (x > 0) \tag{7.24}$$

This problem can most conveniently be solved by application of the Laplace transform (see appendix D). The partial differential equation 7.16 is transformed into the ordinary differential equation

$$s\overline{\phi} - \phi_0 = \frac{T}{S}\frac{d^2\overline{\phi}}{dx^2} \tag{7.25}$$

where $\overline{\phi}$ is the Laplace transform of ϕ. The general solution of equation 7.25 is

$$\overline{\phi} = \frac{\phi_0}{s} + A \exp\,[x(sS/T)^{\frac{1}{2}}] + B \exp\,[-x(sS/T)^{\frac{1}{2}}]$$

The constant A must vanish because of the boundary condition in equation 7.23, and the constant B can be determined from the transformed boundary condition in equation 7.22, that is

$$x = 0: \quad \overline{\phi} = (\phi_0 + \Delta\phi)/s$$

This gives $B = \phi_0/s$, and thus the solution for the Laplace transform is

$$\overline{\phi} = \frac{\phi_0}{s} + \frac{\Delta\phi}{s} \exp\,[-x(sS/T)^{\frac{1}{2}}] \tag{7.26}$$

Inverse transformation is possible by means of a standard transform that can be found in most tables (see appendix D). The result is

$$\phi = \phi_0 + \Delta\phi \,\mathrm{erfc}\,[x/(4Tt/S)^{\frac{1}{2}}] \tag{7.27}$$

where erfc(x) is the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_x^{\infty} \exp(-\lambda^2)\mathrm{d}\lambda \tag{7.28}$$

For a table of values of this function see for instance Abramowitz and Stegun (1965).

An important quantity is the total discharge at the boundary $x = 0$

$$Q_0 = HBq_x = -TB \left.\frac{\partial\phi}{\partial x}\right|_{x=0}$$

With equation 7.27 this gives

$$Q_0 = B\,\Delta\phi \left(\frac{ST}{\pi t}\right)^{\frac{1}{2}} \tag{7.29}$$

This formula, and several similar ones for problems with various types of boundary conditions, were discussed by Edelman (1947).

Example 7.3 Non-steady Flow towards Wells

As a third example the problem of a well in an aquifer of infinite extent is considered (Theis, 1935). Because of the radial symmetry the differential equation can best be expressed in polar coordinates

$$\frac{\partial\phi}{\partial t} = \frac{T}{S}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \tag{7.30}$$

The initial condition is

$$t = 0: \quad \phi = \phi_0 \tag{7.31}$$

The boundary conditions are supposed to be

$$r \to \infty: \quad \phi = \phi_0 \tag{7.32}$$

$$r \to 0: \quad r\frac{\partial \phi}{\partial r} = \frac{Q_0}{2\pi T} \quad (t > 0) \tag{7.33}$$

Here the radius of the well has been assumed to be infinitely small, to simplify the mathematics.

If $\overline{\phi}$ denotes the Laplace transform of the head ϕ equation 7.30 can be transformed into

$$\frac{d^2\overline{\phi}}{dr^2} + \frac{1}{r}\frac{d\overline{\phi}}{dr} - \frac{S}{T}(s\overline{\phi} - \phi_0) = 0 \tag{7.34}$$

The solution of this ordinary differential equation satisfying the transforms of the boundary conditions in equations 7.32 and 7.33 is

$$\overline{\phi} = \frac{\phi_0}{s} - \frac{Q_0}{2\pi Ts} K_0[r(sS/T)^{\frac{1}{2}}] \tag{7.35}$$

The inverse transform of the function $1/s\, K_0(a\sqrt{s})$ is not given in most short tables, but tables usually give the pair of functions $F(s) = K_0(a\sqrt{s})$ and $f(t) = 1/2t \exp(-a^2/4t)$. It is also known (see for example, Churchill, 1958) that multiplication by $1/s$ corresponds to integrating the original function. This means that, after some elementary manipulations, the inverse transform of equation 7.35 can be written as

$$\phi = \phi_0 - \frac{Q_0}{4\pi T} E_1(r^2S/4Tt) \tag{7.36}$$

where $E_1(x)$ is the so-called exponential integral (Abramowitz and Stegun, 1965)

$$E_1(x) = \int_x^\infty \frac{1}{u}\exp(-u)\,du \tag{7.37}$$

It can be verified without much difficulty that the solution in equation 7.36 satisfies all necessary conditions, as expressed by equations 7.30 to 7.33. Therefore it must indeed be the correct solution. It is often called Theis's formula.

For small values of r (that is, close to the well) and large values of t, the exponential integral can be approximated by a logarithm, using the following approximation (see for example, Abramowitz and Stegun, 1965)

$$x \ll 1: \; E_1(x) = -\ln\left(\frac{x}{0.56146}\right) \tag{7.38}$$

This means that for sufficiently small values of r and sufficiently large values of t (such that $r^2S/4Tt$ is small compared to 1) one may write

$$\frac{r^2S}{4Tt} \ll 1: \; \phi = \phi_0 + \frac{Q_0}{2\pi T}\ln\left(\frac{r}{(2.246Tt/S)^{\frac{1}{2}}}\right) \tag{7.39}$$

The importance of this formula is two-fold. Firstly it again shows that in the vicinity of the well the singularity is of logarithmic type, characterised by an equivalent radius

$$R_{eq} = (2.246Tt/S)^{\frac{1}{2}} \tag{7.40}$$

It appears that this equivalent radius (sometimes denoted as radius of influence) increases with time.

Secondly equation 7.39 provides a simple method for the analysis of pumping tests, by rewriting it as follows

$$\frac{\phi}{\phi_1} = \frac{\phi_0}{\phi_1} + \frac{Q_0}{2\pi T\phi_1}\left[\ln\frac{(2.246T_0t/S_0)^{\frac{1}{2}}}{r} + \ln\left(\frac{S_0T}{ST_0}\right)\right] \tag{7.41}$$

where ϕ_1, S_0 and T_0 are arbitrary reference quantities. Equation 7.41 shows that the quantity ϕ/ϕ_1, when plotted against $\ln[(2.246T_0t/S_0)^{1/2}/r]$, should give a straight line. From this graph the value of T can be determined (from the slope of the line) as well as the value of S from the intersection with the vertical axis (see figure 7.3). This method of determining the parameters T and S from a pumping test is due to Cooper and Jacob (1946).

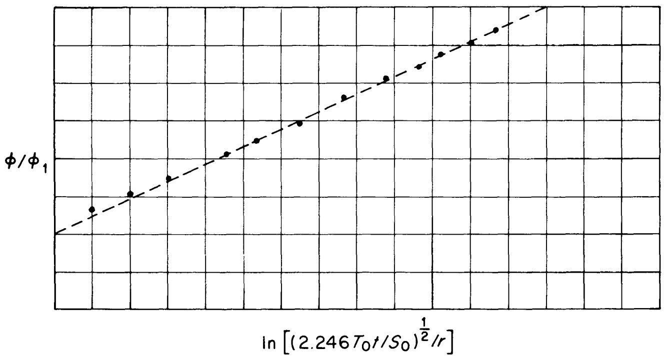

Figure 7.3 *Determination of transmissivity and storativity from pumping test*

For a number of other problems of flow towards wells, for instance involving leakage, the solution can be obtained by using the Laplace transform method, (see for example, Hantush, 1964; Kruseman and De Ridder, 1970). Because of the analogy of the mathematical problem with the problem of conduction of heat in solids it may also be worthwhile to study the literature in that field (see for example, Carslaw and Jaeger, 1959).

7.3 Approximate Solutions

The method of Laplace transforms provides a powerful technique for the solutions of problems of non-steady groundwater flow. The method also allows for approximations for small values of the time and sometimes also for approximations for large values of the time (Carslaw and Jaeger, 1959; Churchill, 1958). A special, very crude, approximation method, due to Schapery (1961) is presented in this section, together with an approximate method for a special class of functions, due to Brakel (1968). General numerical methods are described in section 7.4 and later in section 8.4.

7.3.1 *Schapery's method*

Schapery (1961) has proposed an approximate inversion formula for the Laplace transformation of the following form

$$f(t) = \lim_{s \to 1/2t} [sF(s)] \tag{7.42}$$

The basis for this approximate formula is rather weak. It ensures that for $t = 0$ and $t \to \infty$ the correct result is obtained, and experience has shown that for problems in which the variation is fairly uniform the results are not too bad.

As an example the problem of a strip of land with uniform infiltration, starting at time $t = 0$, is considered (see figure 7.4). In this case the differential equation is, in its linearised form (see equation 7.4)

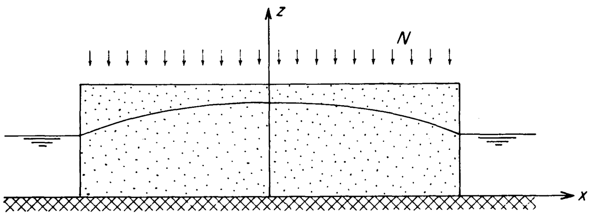

Figure 7.4 *Infiltration problem*

$$S \frac{\partial h}{\partial t} = T \frac{\partial^2 h}{\partial x^2} + N \tag{7.43}$$

with the initial condition

$$t = 0: \; h = H \tag{7.44}$$

and the boundary conditions

$$x = 0: \; \partial h/\partial x = 0 \tag{7.45}$$

$$x = \ell: \; h = H \tag{7.46}$$

If $\bar{h}$ denotes the Laplace transform of the variable h, the transformation of equation 7.43 is, with equation 7.44

$$S(s\bar{h} - H) = T \frac{d^2\bar{h}}{dx^2} + \frac{N}{s} \tag{7.47}$$

The solution satisfying the boundary conditions in equations 7.45 and 7.46 is

$$\bar{h} = \frac{H}{s} + \frac{N}{Ss^2}\left\{1 - \frac{\cosh\,[x\,(Ss/T)^{\frac{1}{2}}]}{\cosh\,[\ell/Ss/T)^{\frac{1}{2}}]}\right\} \tag{7.48}$$

Inversion of this expression is possible by the use of the complex inversion integral (see Churchill, 1958). The detailed calculations will not be presented here. The final result is an expression in the form of an infinite series, namely

$$h = H + \frac{N(\ell^2 - x^2)}{2T} - \frac{16}{\pi^3}\frac{N\ell^2}{T}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$$

$$\cos\left[(2k+1)\frac{\pi x}{2\ell}\right]\exp\left[-(2k+1)^2\frac{\pi^2 Tt}{4S\ell^2}\right] \tag{7.49}$$

Schapery's approximate inversion gives in this case

$$h = H + \frac{N}{S}2t\left\{1 - \frac{\cosh\,[x\,(S/2Tt)^{\frac{1}{2}}]}{\cosh\,[\ell\,(S/2Tt)^{\frac{1}{2}}]}\right\} \tag{7.50}$$

This expression satisfies all boundary conditions and the initial condition. For very large values of t the hyperbolic cosines can be approximated by the first two terms in their Taylor series expansion, and then one obtains

$$t \rightarrow \infty: \quad h = H + \frac{N(\ell^2 - x^2)}{2T} \tag{7.51}$$

which is the correct steady state solution.

In figure 7.5 the approximation in equation 7.50 is compared with the exact solution in equation 7.49, for two values of x/ℓ, namely $x/\ell = 0$ and $x/\ell = 0.8$. Although the correspondence is not too good (in the beginning the approximation is too large, later it is too small), the approximation may well be sufficiently accurate for engineering purposes. The great advantage is, of course, that the inversion of the Laplace transform, which is often very complicated, is avoided. It should be noted, however, that the mathematical basis for the approximation is very weak, and that important details of the time behaviour may get lost.

7.3.2 *Brakel's method*

An approximate method based on some physical assumptions was developed by Brakel (1968). The method bears some resemblance to the method of separation of variables (see for example, Wylie, 1960), with the modification that the spatial behaviour is specified beforehand. A similar method was used by Boussinesq (1904), for a particular problem.

The method can best be illustrated by considering the same example as considered above. The solution of the problem is supposed to be of the form

$$h = H + \frac{N(\ell^2 - x^2)}{2T}f(t) \tag{7.52}$$

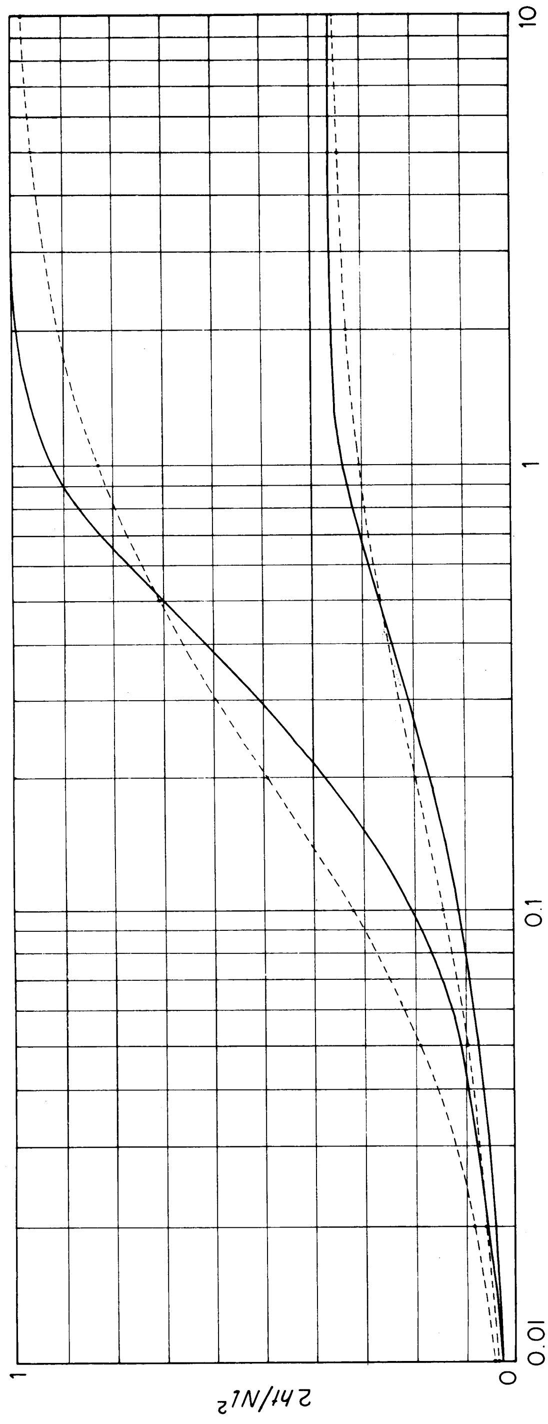

Figure 7.5 *Comparison of approximate and exact solutions*

where $f(t)$ is a function of time only. This formula satisfies the initial condition if $f(0) = 0$, and it also satisfies the final steady state solution if $f(\infty) = 1$. Assuming that the solution is in the form of equation 7.52 means that it is assumed that at any time the solution is geometrically similar to the steady state solution. Substitution into the differential equation 7.43 shows that the assumption is certainly not fully correct, because it is impossible to satisfy that equation for all x and t. It is possible however, to satisfy, at any moment t, the differential equation on the average over the domain. This means that the requirement regarding the differential equation is relaxed into the condition

$$\int_0^{\ell} \left(S \frac{\partial h}{\partial t} - T \frac{\partial^2 h}{\partial x^2} - N \right) dx = 0 \qquad (7.53)$$

Physically this means that only total continuity, over the entire flow domain, is required. Substitution of equation 7.52 into equation 7.53 gives

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{3T}{S\ell^2}(f - 1) \qquad (7.54)$$

This is an ordinary differential equation. The solution vanishing for $t = 0$ is

$$f = 1 - \exp(-3Tt/S\ell^2) \qquad (7.55)$$

Hence the approximate solution of the problem is

$$h = H + \frac{N(\ell^2 - x^2)}{2T}[1 - \exp(-3Tt/S\ell^2)] \qquad (7.56)$$

In figure 7.6 this approximate solution is compared with the exact solution in equation 7.49. It appears that the approximation is somewhat better than the one obtained by Schapery's method. Perhaps this is due to the physical basis of the method, which at least ensures global continuity. The agreement is on the whole not very good, however. Methods of this type are only useful because they give a rapid and simple first insight in the overall behaviour.

7.4 Finite Differences

The differential equation for non-steady groundwater flow

$$S \frac{\partial h}{\partial t} = T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) + N \qquad (7.57)$$

can be solved numerically by using finite elements (see Chapter 8) or by using finite differences. In case of a rectangular mesh with constant intervals Δx and Δy, a simple finite difference approximation to equation 7.57 is

$$S \frac{h'(i,j) - h(i,j)}{\Delta t} = \frac{T}{(\Delta x)^2}[h(i+1,j) + h(i-1,j) - 2h(i,j)]$$

$$+ \frac{T}{(\Delta y)^2}[h(i,j+1) + h(i,j-1) - 2h(i,j)] + N \qquad (7.58)$$

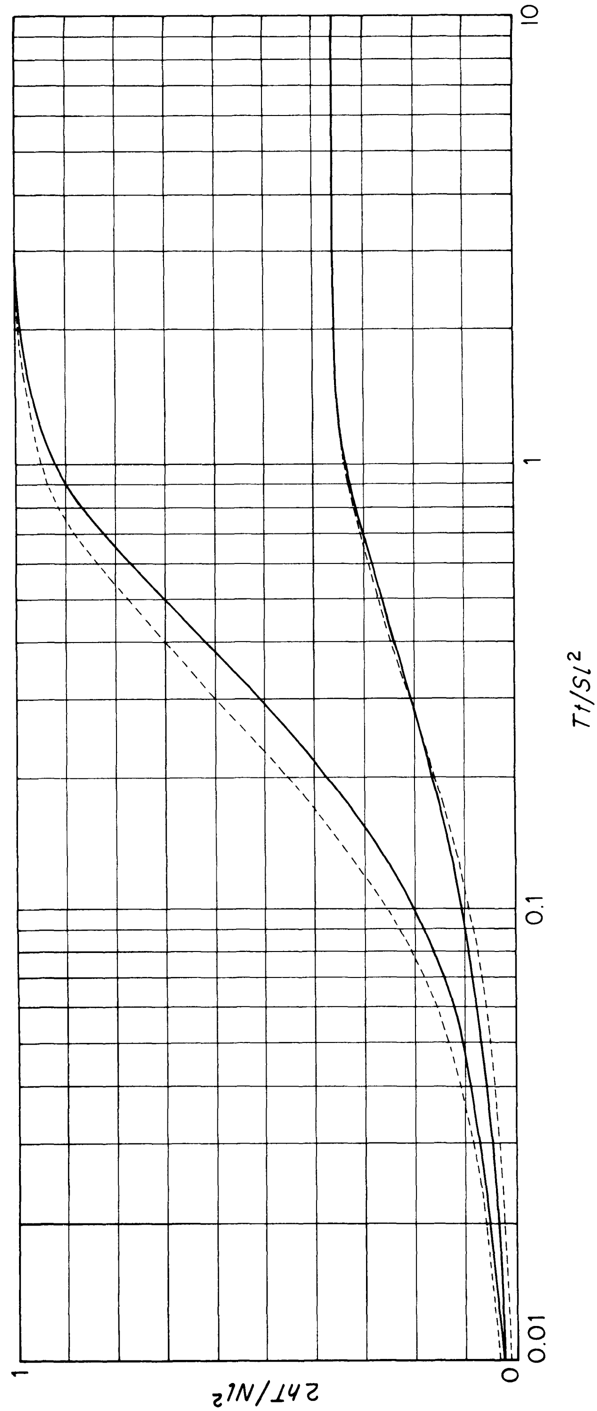

Figure 7.6 *Comparison of approximate and exact solutions*

where h' denotes $h(t + \Delta t)$. From equation 7.58 the value of $h'(i, j)$ can be calculated if all values at the previous value of time are known. In this way an explicit formula is obtained, which is due to the fact that a forward finite difference approximation has been made for the time derivatives. Equation 7.58 can easily be incorporated into a computer program, taking into account the boundary conditions, of course.

Program 7.1

A very simple program, in BASIC, for a rectangular region of dimensions $2L$ and H, is reproduced below (program 7.1). The boundary conditions are supposed to be that $h = 0$ along the left and right side boundaries and $\partial h/\partial y = 0$ along the upper and lower boundaries (see figure 7.7). The values $N = 20$ and $M = 2$ indicate that the length $2L$ is subdivided into 20 equal parts, and the height H into 2 equal parts. The infiltration rate is denoted by P in the program. The values of h are denoted by F(I, J) for the old ones, and FA(I, J) for the new ones, one step

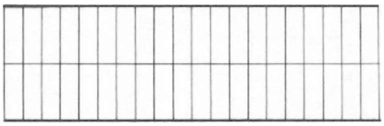

Figure 7.7 *Finite difference mesh*

```
10 REM FINITE DIFFERENCES FOR PLANE NON-STEADY FLOW, PROGRAM 7.1
20 READN,M,L,H,T,S,P,NS:DX=L/N:DY=H/M:DIMF(N,M),FA(N,M):TT=0
30 A=T/(S*DX*DX):B=T/(S*DY*DY):DT=1/(4*A):D=1/(4*B):IFD<DTTHENDT=D
40 A=A*DT:B=B*DT:P=P*DT/S
50 FORK=1TONS:FORI=1TON-1:D=P+A*(F(I+1,0)+F(I-1,0)-2*F(I,0))
60 D=D+B*(2*F(I,1)-2*F(I,0)):FA(I,0)=F(I,0)+D
70 FORJ=1TOM-1:D=P+A*(F(I+1,J)+F(I-1,J)-2*F(I,J))
80 D=D+B*(F(I,J+1)+F(I,J-1)-2*F(I,J)):FA(I,J)=F(I,J)+D:NEXTJ
90 D=P+A*(F(I+1,M)+F(I-1,M)-2*F(I,M))+B*(2*F(I,M-1)-2*F(I,M))
100 FA(I,M)=F(I,M)+D:NEXTI:TT=TT+DT:PRINT:PRINT"TIME=";TT:PRINT
110 FORJ=0TOM:FORI=OTON:PRINT"I=";I,"  J=";J,"  H=";FA(I,J)
120 F(I,J)=FA(I,J):NEXTI:NEXTJ:NEXTK
130 DATA20,2,2,0.6,1,1,2,400
```

Program 7.1 *Finite differences for plane non-steady flow*

in time later. Actually the program applies to the same problem as the one considered in the previous section (see figure 7.4). With this numerical method excellent agreement with the exact analytical solution is obtained (see figure 7.8). The simplicity of the method as illustrated by the short program 7.1, and the accuracy of the results clearly indicate the power of the numerical method.

A detail worthy of some attention is the magnitude of the time step. If the time steps are taken too large, the solution becomes unstable. The criterion for stability of the algorithm in the equation 7.58 is that

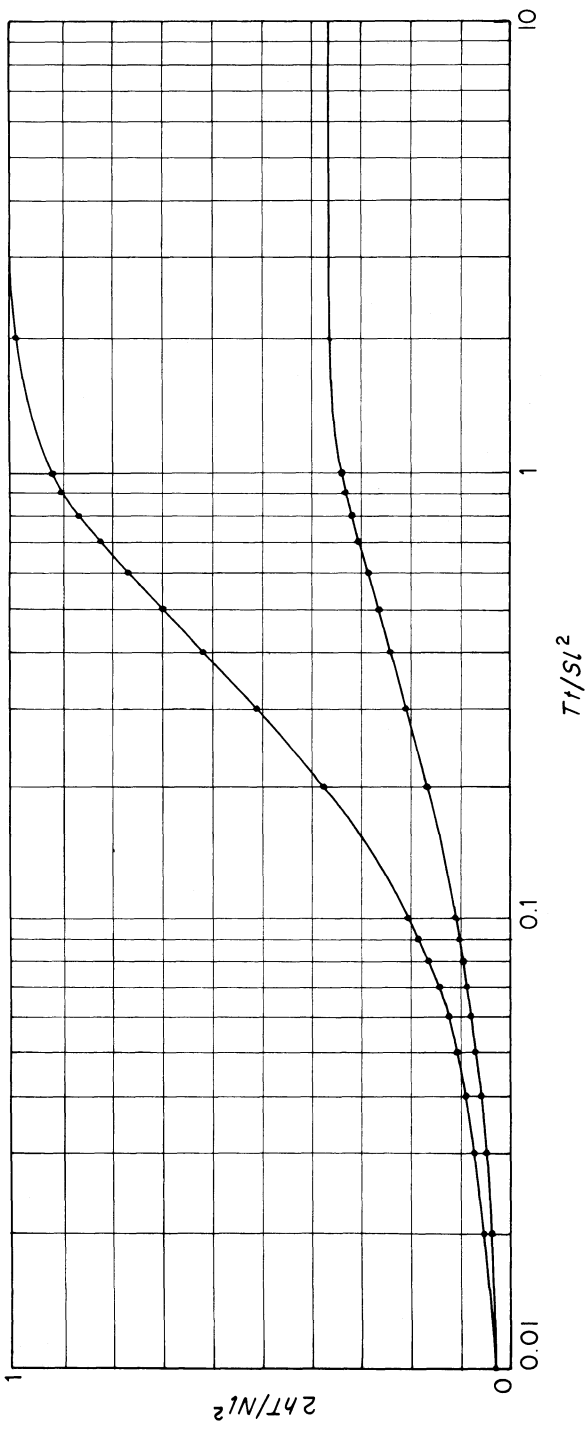

Figure 7.8 *Comparison of numerical and exact solutions*

$$\Delta t < \frac{S}{2T} \frac{1}{[(\Delta x)^{-2} + (\Delta y)^{-2}]} \tag{7.59}$$

A simple derivation of this criterion is by using the algorithm in equation 7.58 with $N = 0$, $h(i,j) = \epsilon$ and $h(i,j+1) = h(i,j-1) = h(i+1,j) = h(i-1,j) = -\epsilon$. This represents a particular type of disturbance, which should disappear with time. Hence the algorithm in equation 7.58 should give $h' < \epsilon$ and $h' > -\epsilon$. The first condition leads to $\Delta t > 0$, and the second to equation 7.59. For a more general treatment of the stability criterion the reader is referred to Forsythe and Wasow (1960), or Carnahan *et al.* (1969).

The stability condition in equation 7.59, which has been incorporated in the computer program, implies that only relatively small time steps are allowed. In order to approach the steady state several thousands of time steps may be necessary. To overcome this difficulty various other numerical schemes have been proposed, generally leading to implicit expressions for the new values of the water level. Some of these, such as the Crank-Nicholson method, which uses a central finite difference for the approximation of the time derivative, are unconditionally stable, and therefore allow for much larger time steps. In section 8.4 such an approach will be followed in the finite element method.

7.5 Interface Problems

For the flow of groundwater in coastal aquifers a relatively simple first-order approach has been developed on the basis of a series of assumptions, which together can be denoted as the Ghyben-Dupuit approximation. This approach is presented in this section.

The first assumptions are that there exists a sharp interface, and that in the salt water the movements are so small compared to the flow in the fresh water (assuming that the main external agents act upon the fresh water) that the pressure distribution in the salt water is hydrostatic, in other words that

$$p = -\rho_s g z \tag{7.60}$$

where the zero level for the pressure has been assumed at $z = 0$ (usually mean sea level). Because of equilibrium this same pressure acts in the fresh water just above the interface. If the location of the interface is supposed to be at $z = -h_s$, and in the fresh water the Dupuit assumption is made, then, because of equilibrium at the interface (see figure 7.9)

$$p = \rho_s g h_s = \rho_f g(h + h_s) \tag{7.61}$$

where h is the groundwater head in the fresh water, for instance manifested by a free surface at a height h above mean sea level. It follows from equation 7.61 that

$$h_s = \frac{\rho_f}{\rho_s - \rho_f} h = \alpha h \tag{7.62}$$

The ratio between the densities of salt and fresh water is of the order of 1.025.

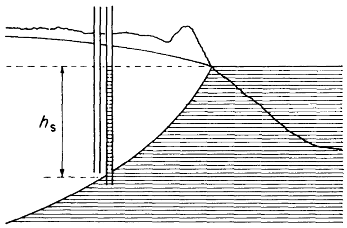

Figure 7.9 *Ghyben-Herzberg relation*

Then equation 7.62 shows that h_s is about 40 times h. In coastal aquifers such as that occurring in the dunes along the coast of the Netherlands this means that for every metre of fresh water above sea level there is a storage of 40 m of fresh water below sea level. Equation 7.62 is usually called the Ghyben-Herzberg relation (Drabbe and Badon Ghyben, 1889; Herzberg, 1901). Darcy's law for the flow in the fresh water now states, using the Dupuit approximation, equation 3.30

$$q_x = -k\frac{\partial h}{\partial x}, \qquad q_y = -k\frac{\partial h}{\partial y} \tag{7.63}$$

The equation of continuity for the flow of the fresh water should show that the storage above the fresh water surface and below the interface is balanced by an inflow of groundwater. Hence

$$S\left(\frac{\partial h}{\partial t} + \frac{\partial h_s}{\partial t}\right) = -\frac{\partial}{\partial x}[(h + h_s)q_x] - \frac{\partial}{\partial y}[(h + h_s)q_y] + N$$

With equations 7.63 and 7.62 this gives

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(kh\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(kh\frac{\partial h}{\partial y}\right) + \frac{N}{1 + \alpha} \tag{7.64}$$

It should be noted that it has been assumed that the Ghyben-Herzberg relation (see equation 7.62) holds throughout the process, also in non-steady conditions. This must be a rather crude approximation, because salt water must be displaced, which violates the underlying assumption that the salt water head is constant.

Equation 7.64 differs from the continuity equation in a normal unconfined aquifer, equation 7.3, only in the last term, by a constant factor $1 + \alpha$. This means that all solutions obtained for unconfined aquifers can immediately be translated into terms of coastal aquifers, which only requires an adaption of the parameters in the solution. Special care has to be taken to make a correct formulation of the entire problem, including the boundary conditions and the initial condition.

All methods presented before in this chapter are also applicable to the flow in coastal aquifers. It is especially interesting to note that Brakel's method (see section 7.3) can now even be used to solve, at least approximately, the original non-linear differential equation. This will be illustrated by considering the example of the generation of a fresh water reservoir by infiltration on an oblong island, see figure 7.10, of width 2ℓ. The differential equation in equation 7.64 now reduces to

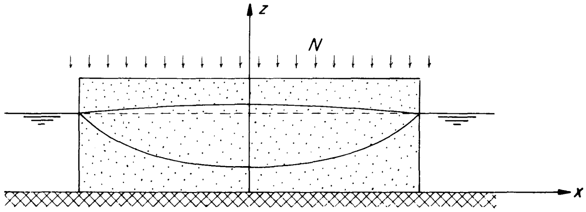

Figure 7.10 *Generation of fresh water reservoir*

$$S\frac{\partial h}{\partial t} - \frac{k}{2}\frac{\partial^2(h^2)}{\partial x^2} - \frac{N}{1+\alpha} = 0 \tag{7.65}$$

The steady state solution in this case is

$$h^2 = h_0^2(1 - x^2/\ell^2) \tag{7.66}$$

where

$$h_0 = \ell[N/k(1+\alpha)]^{\frac{1}{2}} \tag{7.67}$$

It is now assumed that the solution of the non-steady problem can be written as follows (Brakel, 1968)

$$h = f(t)h_0(1 - x^2/\ell^2)^{\frac{1}{2}} \tag{7.68}$$

where $f(0) = 0$ and $f(\infty) = 1$. As in section 7.3 the differential equation can only be satisfied on the average. After substitution of equation 7.68 into equation 7.65, and integrating this over the interval from $x = 0$ to $x = \ell$ one obtains

$$\frac{df}{dt} = \frac{4}{\pi}\frac{kh_0}{S\ell^2}(1 - f^2) \tag{7.69}$$

The solution of this differential equation, satisfying the initial condition $f(0) = 0$, is

$$f = \tanh\left(\frac{4}{\pi}\frac{kh_0t}{S\ell^2}\right) \tag{7.70}$$

The final, approximate, solution of the problem is

$$h = h_0(1 - x^2/\ell^2)^{\frac{1}{2}}\tanh\left(\frac{4}{\pi}\frac{kh_0t}{S\ell^2}\right) \tag{7.71}$$

where h_0, the maximum water level, is given by equation 7.67. The method of solution used here can easily be generalised to islands of more complicated geometrical form. The general character of the approximate solution remains relatively simple.

7.6 Problems

1. Consider the non-steady groundwater flow in a strip of land of width 2ℓ, bounded by two canals of constant water level H. The infiltration rate N fluctuates as $N = N_0 \sin(\omega t)$. Solve the differential equation by assuming that the water level also fluctuates sinusoidally.

2. In a confined aquifer a pumping test has been executed with a constant discharge of 10^{-2} m^3/s. The drawdown of the groundwater head at a distance of 10 m from the well has been recorded. The observations are given below

time (s)	drawdown (cm)
0	0
100	5
400	14
900	22
1 600	28
2 500	32
3 600	36
4 900	39
6 400	42
8 100	44
10 000	46

Determine the values of the transmissivity T and the storativity S of the aquifer.

3. Use Schapery's method of approximate inversion in examples 7.2 and 7.3, and compare the results with the exact solutions given.

4. Investigate the stability of the numerical method used in program 7.1 experimentally, by including a statement DT = 2*DT or DT = 4*DT between lines 30 and 40, and then running the program.

5. Consider the generation of a fresh water reservoir by infiltration, starting at time $t = 0$, on a circular island, surrounded by the sea. Estimate the time needed (in centuries) to reach the steady state (within say 95 per cent) if the radius of the island is $R = 2000$ m, the storativity is $S = 0.4$, the hydraulic conductivity is $k = 10^{-4}$ m/s, and the final water level in the centre of the island (h_0) is 2 m above mean sea level.

8

The Finite Element Method

For the numerical solution of groundwater flow problems by means of a computer the finite element method is particularly suitable. The basic principles of the method and some applications are discussed in this chapter, restricting the considerations to basically two-dimensional problems.

The basic techniques of the finite element method can conveniently be established by using a variational formulation of the problem, rather than a differential equation and the boundary conditions. Although the most general formulation of the finite element method seems to be through the use of the so-called Galerkin method (see for example, Zienkiewicz, 1977) the more classical approach on the basis of a minimum principle will be followed here.

In this chapter several programs will be given for various applications of the finite element method. All programs are in BASIC and have been run successfully on a small microcomputer. They can easily be translated into other languages, for instance FORTRAN, which may be more convenient when a large computer system is available. Actually, all programs were derived from original FORTRAN programs. The programs are given here in BASIC mainly because they are relatively short in this language. An additional reason is, however, to demonstrate that a very simple and cheap computer system is sufficient for relatively complicated calculations. However, for finite element calculations on a microcomputer when very large networks are involved special techniques may be necessary (Verruijt, 1980), to overcome the handicap of limited storage capacity.

8.1 Variational Principle

The fundamental two-dimensional problem of the theory of groundwater flow in isotropic, inhomogeneous soils can be formulated by the following set of equations

$$\frac{\partial}{\partial x}\left(T\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial\phi}{\partial y}\right) + N - \frac{\phi - \phi'}{c} = 0 \qquad (\text{in } R) \qquad (8.1)$$

$$\phi = f \qquad (\text{on } S_1) \qquad (8.2)$$

$$Hq_n = -T\frac{\partial\phi}{\partial n} = g \qquad (\text{on } S_2) \tag{8.3}$$

where $S_1 + S_2$ is the entire boundary of the region R, and $\partial/\partial n$ is the derivative perpendicular to the boundary, the outward direction being considered as positive. In equation 8.1 N represents a given surface supply (for instance by infiltration), and ϕ' is the given head in an adjacent aquifer, separated from the main aquifer by a clay layer of resistance c. In this equation of continuity the last term represents the inflow due to leakage. Equation 8.2 is the boundary condition along the part of the boundary (S_1) where the head is prescribed. Equation 8.3 is the boundary condition along the part of the boundary (S_2) where a given flux g is extracted from the aquifer.

The variational principle equivalent to equations 8.1 to 8.3 is as follows. If a functional U is defined by

$$U = \frac{1}{2}\iint_R \left[T\left(\frac{\partial\phi}{\partial x}\right)^2 + T\left(\frac{\partial\phi}{\partial y}\right)^2 - 2N\phi + \frac{\phi^2 - 2\phi\phi'}{c}\right] \mathrm{d}x\,\mathrm{d}y + \int_{S_2} g\phi\,\mathrm{d}S \tag{8.4}$$

with the constraint

$$\phi = f \qquad (\text{on } S_1) \tag{8.5}$$

then the functional U possesses an absolute minimum for the function ϕ that satisfies the equations 8.1 to 8.3. In order to prove this minimum principle let U_0 denote the value of U if ϕ is indeed the solution of the problem defined by equations 8.1 to 8.3. Now consider a function ψ differing from that solution by a variation ϵF, that is

$$\psi = \phi + \epsilon F \tag{8.6}$$

where ϵ is an arbitrary constant and F represents some arbitrary function of x and y, except that

$$F = 0 \qquad (\text{on } S_1) \tag{8.7}$$

which is a consequence of the fact that both ϕ and ψ satisfy the constraint in equation 8.5. The value of U corresponding to the function ψ can be written as

$$U = U_0 + \epsilon A_1 + \epsilon^2 A_2 \tag{8.8}$$

where

$$A_1 = \iint_R \left[T\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial F}{\partial x}\right) + T\left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial F}{\partial y}\right) - NF + \frac{\phi - \phi'}{c}F\right] \mathrm{d}x\,\mathrm{d}y + \int_{S_2} gF\,\mathrm{d}S \tag{8.9}$$

$$A_2 = \frac{1}{2}\iint_R \left[T\left(\frac{\partial F}{\partial x}\right)^2 + T\left(\frac{\partial F}{\partial y}\right)^2 + \frac{F^2}{c}\right] \mathrm{d}x\,\mathrm{d}y \tag{8.10}$$

For physical reasons the transmissivity T and the resistance c are necessarily positive, hence it is obvious that for any function F

$$A_2 \geqslant 0 \tag{8.11}$$

where the equality sign applies only in the degenerate cases that either $F = 0$ or $T = 0$ and $c = \infty$.

The expression in equation 8.9 can be rewritten as

$$\begin{aligned} A_1 = & \iint_R \left[\frac{\partial}{\partial x}\left(FT\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(FT\frac{\partial \phi}{\partial y}\right)\right] \mathrm{d}x\,\mathrm{d}y \; + \\ & - \iint_R F\left[\frac{\partial}{\partial x}\left(T\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial \phi}{\partial y}\right) + N - \frac{\phi - \phi'}{c}\right] \mathrm{d}x\,\mathrm{d}y \\ & + \int_{S_2} gF\,\mathrm{d}S \end{aligned}$$

The second integral vanishes because of equation 8.1. The first integral can be transformed into a surface integral by means of the divergence theorem (see for instance Wylie, 1960). If this surface integral is separated into two parts, one along S_1 and one along S_2, the result is

$$A_1 = \int_{S_1} FT\frac{\partial \phi}{\partial n}\,\mathrm{d}S + \int_{S_2} F\left(T\frac{\partial \phi}{\partial n} + g\right)\mathrm{d}S$$

The first integral is zero because of equation 8.7, and the second one vanishes because of the boundary condition in equation 8.3. Hence

$$A_1 = 0 \tag{8.12}$$

With equations 8.11 and 8.12 it now follows from equation 8.8 that

$$U \geqslant U_0 \tag{8.13}$$

where the equality sign applies only in irrelevant degenerate cases. This completes the proof that the functional U indeed possesses a minimum for the function ϕ satisfying the equations 8.1 to 8.3.

The alternative formulation of the groundwater flow problem through the variational principle that U must be a minimum, can be used to determine an approximate solution for a certain problem. By investigating a certain class of functions, chosen wide enough to include the most important properties that the solution is expected to have, and evaluating the functional U for all these functions, the 'best' approximation can be considered to be the function that leads to the smallest value of U. In this method it is necessary that all functions investigated satisfy the constraint in equation 8.5. For analytical solutions this often restricts the practical utility of the method. In the finite element method, however, the constraint can easily be taken into account.

8.2 Finite Elements for Steady Flow

In order to present the finite element method in an elementary way restriction is first made to problems without leakage, without infiltration and without surface extraction. This means that $N = 0, c = \infty$ and $g = 0$ in equation 8.4, which then reduces to

$$U = \frac{1}{2} \iint_R \left[T \left(\frac{\partial \phi}{\partial x} \right)^2 + T \left(\frac{\partial \phi}{\partial y} \right)^2 \right] \mathrm{d}x \, \mathrm{d}y \tag{8.14}$$

In the finite element method the region R is subdivided into a great number of elements. Then

$$U = \sum_{j=1}^{m} U_j \tag{8.15}$$

where U_j represents the surface integral over a single element, and where m is the number of elements. A simple and yet flexible form of subdividing the region R is by using triangular elements. The values of the head ϕ in the nodes are taken as the free parameters in the problem, and in the interior of each element a linear interpolation between the values in the three corner points will be used. This means that in a typical element R_j

$$\phi = p_1 x + p_2 y + p_3 \tag{8.16}$$

It seems reasonable to require that the approximation $\phi = \phi(x, y)$ of the groundwater head is continuous. In combination with the linear interpolation rule in equation 8.16 this means that the function $\phi(x, y)$ is approximated by a diamond-shaped surface, with a great many facets (see figure 8.1).

If the node numbers of the typical element R_j are denoted by 1, 2 and 3, at least temporarily, the condition that ϕ should be continuous leads to the equations

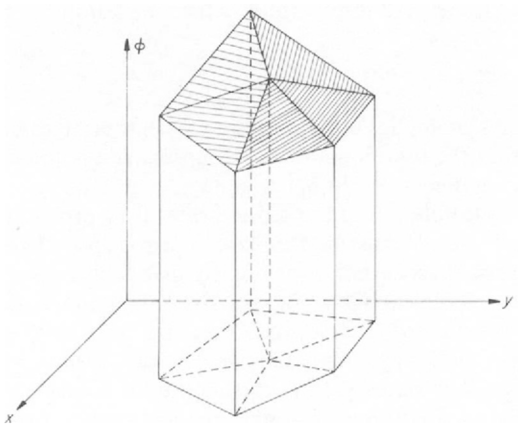

Figure 8.1 *Linear interpolation*

$$\begin{aligned} \phi_1 &= p_1 x_1 + p_2 y_1 + p_3 \\ \phi_2 &= p_1 x_2 + p_2 y_2 + p_3 \\ \phi_3 &= p_1 x_3 + p_2 y_3 + p_3 \end{aligned} \tag{8.17}$$

This means that the three parameters p_1, p_2 and p_3 can be expressed into ϕ_1, ϕ_2 and ϕ_3 by the relations

$$\begin{aligned} p_1 &= (b_1\phi_1 + b_2\phi_2 + b_3\phi_3)/\Delta \\ p_2 &= (c_1\phi_1 + c_2\phi_2 + c_3\phi_3)/\Delta \\ p_3 &= (d_1\phi_1 + d_2\phi_2 + d_3\phi_3)/\Delta \end{aligned} \tag{8.18}$$

where

$$b_1 = y_2 - y_3, \quad b_2 = y_3 - y_1, \quad b_3 = y_1 - y_2 \tag{8.19}$$

$$c_1 = x_3 - x_2, \quad c_2 = x_1 - x_3, \quad c_3 = x_2 - x_1 \tag{8.20}$$

$$d_1 = x_2 y_3 - x_3 y_2, \quad d_2 = x_3 y_1 - x_1 y_3, \quad d_3 = x_1 y_2 - x_2 y_1 \tag{8.21}$$

and where Δ is the determinant of the system of linear equations 8.17, that is

$$\Delta = x_1 b_1 + x_2 b_2 + x_3 b_3 \tag{8.22}$$

It is to be noted that the quantities b_i, c_i, d_i and Δ can easily be calculated if the coordinates of the three nodes of the element R_j are known.

It follows from equation 8.16 that in element R_j the values of $\partial\phi/\partial x$ and $\partial\phi/\partial y$, which are needed in the integral, are constant. Hence

$$U_j = \tfrac{1}{2} \iint_{R_j} \left[T(p_1{}^2 + p_2{}^2) \right] \mathrm{d}x\,\mathrm{d}y$$

In this integral p_1 and p_2 are constants. If it is assumed that in each element the transmissivity T is constant, say $T = T_j$ in R_j, then all that remains is to evaluate the area of the triangle, which can be shown to be just equal to $\frac{1}{2}|\Delta|$. Thus the value of U_j is

$$U_j = \tfrac{1}{4}\, T_j\, |\Delta|\, (p_1{}^2 + p_2{}^2) \tag{8.23}$$

With the first two parts of equation 8.18 this can be elaborated into

$$U_j = \tfrac{1}{2} \sum_{k=1}^{3} \sum_{\ell=1}^{3} P^j_{k\ell}\, \phi_k\, \phi_\ell \tag{8.24}$$

where P^j is a 3 x 3 matrix with coefficients

$$P^j_{kl} = \frac{T_j}{2|\Delta|} (b_k b_\ell + c_k c_\ell) \tag{8.25}$$

The nine coefficients of this matrix can easily be calculated from the geometrical data describing the element (the coordinates of the nodes) and from the transmissivity T_j, which is also supposed to be given.

From each element R_j there will be a contribution to the functional U

of the form of equation 8.25, quadratic in the variables ϕ_k. After summation over all elements a quadratic form for U is obtained

$$U = \frac{1}{2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} P_{k\ell}\, \phi_k\, \phi_\ell \tag{8.26}$$

where n is the total number of nodes. The calculation of the matrix coefficients $P_{k\ell}$ can be performed by a computer, if a program is written that generates its coefficients on the basis of elementary matrices of the form of equation 8.25.

The minimum value of U occurs for those values of ϕ_k for which

$$\frac{\partial U}{\partial \phi_i} = 0, \qquad i = 1, 2, 3, \ldots, n \tag{8.27}$$

With equation 8.26 this leads to the following system of equations

$$\sum_{k=1}^{n} P_{ik}\, \phi_k = 0, \qquad i = 1, 2, 3, \ldots, n \tag{8.28}$$

where use has been made of the symmetry of the matrix ($P_{k\ell} = P_{\ell k}$). The unknown variable ϕ_ℓ can be determined from equation 8.28 by any standard subroutine of solving a system of linear equations. It should be noted that not all values ϕ_ℓ are unknown. In some nodes the value of ϕ is prescribed through the boundary condition, which is in fact the constraint in equation 8.5, which has to be imposed on these variables. Theoretically this could be accomplished by bringing the corresponding terms in equation 8.28 to the right-hand side of the system of equations, but in actual programs this is usually not even necessary, because the boundary conditions can be taken into account more easily.

An effective method of solution of equation 8.28 is by the Gauss-Seidel method (see for example, Carnahan, Luther and Wilkes, 1969). In this iterative method, which was also used in section 6.2, an initial estimation is gradually improved by successive updating by the algorithm

$$\phi_i = \phi_i + R\, (Q_i - \sum_{k=1}^{n} P_{ik}\, \phi_k) \tag{8.29}$$

where Q_i represents the right-hand side of the system of equations, which is zero for all i in equation 8.28. The parameter R is the so-called over-relaxation factor, which must be limited by the condition

$$1 \leqslant R < 2 \tag{8.30}$$

In section 6.2 the value of R was taken equal to 1. It has been found, however, that the convergence of the method is improved by taking $R = 1.5$ or sometimes even somewhat larger. The convergence of the Gauss-Seidel method for the type of problems considered here is ensured because the matrix P is positive definite.

Program 8.1 Steady Flow

An elementary program, in BASIC, is reproduced below. In this program N is the number of nodes, M is the number of elements, NI is the number of interations, and R is the over-relaxation factor. In line 40 the data of the nodes are read, where T% (I) is an indicator for the type of node, which is zero if the head is unknown, and equal to 2 if the head is given, and where F(I) is the head ϕ in node I, either given or estimated. In line 50 the data of the elements are read: the three nodes constituting each element and its transmissivity. In the program the matrix P is generated first, and then the system of equations is solved by Gauss-Seidel interations. The program contains data for a very simple example (see figure 8.2) with six nodes and four elements. The head in nodes 1 and 2 is given to be 10, and the head in nodes 5 and 6 is given to be 0. The transmissivity in elements 1 and 2 is 1, and in elements 3 and 4 the transmissivity is 9. The correct solution of the problem now should be $\phi_3 = \phi_4 = 1$, which is indeed found when the program is run on a computer.

```
10 REM FINITE ELEMENTS FOR PLANE STEADY FLOW, PROGRAM 8.1
20 READN,M,NI,R:DIMXJ(3),YJ(3),B(3),C(3),PJ(3,3)
30 DIMX(N),Y(N),N%(M,3),T%(N),T(M),F(N),Q(N),P(N,N)
40 FORI=1TON:READX(I),Y(I),T%(I),F(I):NEXTI
50 FORJ=1TOM:READN%(J,1),N%(J,2),N%(J,3),T(J):NEXTJ
60 PRINT"GENERATION OF MATRIX":PRINT:FORJ=1TOM:PRINT"    ELEMENT";J
70 FORI=1TO3:K=N%(J,I):XJ(I)=X(K):YJ(I)=Y(K):NEXTI
80 B(1)=YJ(2)-YJ(3):B(2)=YJ(3)-YJ(1):B(3)=YJ(1)-YJ(2)
90 C(1)=XJ(3)-XJ(2):C(2)=XJ(1)-XJ(3):C(3)=XJ(2)-XJ(1)
100 D=XJ(1)*B(1)+XJ(2)*B(2)+XJ(3)*B(3):D=T(J)/(2*ABS(D))
110 FORK=1TO3:FORL=1TO3:PJ(K,L)=D*(B(K)*B(L)+C(K)*C(L)):NEXTL,K
120 FORK=1TO3:FORL=1TO3:U=N%(J,K):V=N%(J,L):P(U,V)=P(U,V)+PJ(K,L):NEXTL,K,J
130 PRINT:PRINT"SOLUTION OF EQUATIONS":PRINT
140 FORIT=1TONI:PRINT"   ITERATION";IT:FORI=1TON:IFT%(I)>0GOTO160
150 A=Q(I):FORJ=1TON:A=A-P(I,J)*F(J):NEXTJ:F(I)=F(I)+R*A/P(I,I)
160 NEXTI:NEXTIT:PRINT
170 FORI=1TON:PRINT"X=";X(I);"   Y=";Y(I);"   PHI=";F(I):NEXTI
180 DATA6,4,50,1.5
190 DATA0,0,2,10,0,1,2,10,1,0,0,0,1,1,0,0,2,0,2,0,2,1,2,0
200 DATA1,2,4,1,1,3,4,1,3,4,6,9,3,5,6,9
```

Program 8.1 *Finite elements for plane steady flow*

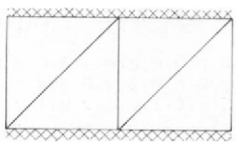

Figure 8.2 *A simple example*

Program 8.2 Steady Flow with Sources

The major disadvantages of program 8.1 are that its memory requirement becomes very large for a large network (because of the N x N matrix P), and that in the Gauss-Seidel iteration process most of the calculations involve multiplications by zero. All this can be avoided by the introduction of a pointer matrix, which keeps track of the non-zero coefficients of the matrix. In most networks each node appears in one element with at the most six or eight other nodes, or some similar number. This means that only six or eight coefficients in each row of the matrix P are non-zero. A program using this property is reproduced below as Program 8.2. The pointer matrix, which is calculated in the program itself, is denoted by K%. The second dimension of this matrix is Z, which is read to be 9. The last column of this matrix is used to indicate the actual number of nodes with which each node appears in any one element. This number should therefore not be greater than 8. The non-zero coefficients

```
10 REM FINITE ELEMENTS FOR PLANE STEADY FLOW, PROGRAM 8.2
20 READN,M,Z,NI,R:DIMXJ(3),YJ(3),B(3),C(3),PJ(3,3)
30 DIMX(N),Y(N),N%(M,3),T%(N),T(M),F(N),P(N,Z),K%(N,Z)
40 FORI=1TON:READX(I),Y(I),T%(I),F(I),P(I,Z):NEXTI
50 FORJ=1TOM:READN%(J,1),N%(J,2),N%(J,3),T(J):NEXTJ
60 PRINT"CALCULATION OF POINTER MATRIX":PRINT:FORI=1TON:PRINT"   NODE";I
70 K%(I,1)=I:K%(I,Z)=1:K=1:FORJ=1TOM:FORH=1TO3:IFN%(J,H)=IGOTO90
80 NEXTH:GOTO120
90 FORH=1TO3:U=N%(J,H):FORL=1TOK:IF(K%(I,L)=U)GOTO110
100 NEXTL:K=K+1:K%(I,K)=U:K%(I,Z)=K
110 NEXTH
120 NEXTJ:NEXTI:PRINT
130 PRINT"GENERATION OF MATRIX":PRINT:FORJ=1TOM:PRINT"   ELEMENT";J
140 FORI=1TO3:K=N%(J,I):XJ(I)=X(K):YJ(I)=Y(K):NEXTI
150 B(1)=YJ(2)-YJ(3):B(2)=YJ(3)-YJ(1):B(3)=YJ(1)-YJ(2)
160 C(1)=XJ(3)-XJ(2):C(2)=XJ(1)-XJ(3):C(3)=XJ(2)-XJ(1)
170 D=XJ(1)*B(1)+XJ(2)*B(2)+XJ(3)*B(3):D=T(J)/(2*ABS(D))
180 FORK=1TO3:FORL=1TO3:PJ(K,L)=D*(B(K)*B(L)+C(K)*C(L)):NEXTL,K
190 FORK=1TO3:U=N%(J,K):H=K%(U,Z):FORL=1TOH:FORV=1TO3
200 IF(N%(J,V)=K%(U,L))THEN220
210 GOTO240
220 P(U,L)=P(U,L)+PJ(K,V)
230 GOTO250
240 NEXTV
250 NEXTL,K,J:PRINT:PRINT"SOLUTION OF EQUATIONS":PRINT:FORIT=1TONI
260 PRINT"   ITERATION";IT:FORI=1TON:IFT%(I)>1GOTO290
270 A=P(I,Z):K=K%(I,Z):FORJ=1TOK:A=A-P(I,J)*F(K%(I,J)):NEXTJ
280 F(I)=F(I)+R*A/P(I,1)
290 NEXTI,IT:PRINT
300 FORI=1TON:PRINT"X=";X(I);"   Y=";Y(I);"   PHI=";F(I):NEXTI
310 DATA13,14,9,50,1.5
320 DATA0,0,0,0.157,1,0,0,0,2,0,0,0,4,0,0,0,8,0,2,0,0,0,1,0,0,0
330 DATA1,1,0,0,0,0,2,0,0,0,2,2,0,0,0,0,4,0,0,0,4,4,0,0,0,0,8,2,0,0,8,8,2,0,0
340 DATA1,2,7,0.1,1,6,7,0.1,2,3,7,0.1,3,7,9,0.1,3,4,9,0.1,4,9,11,0.1
350 DATA4,5,11,0.1,5,11,13,0.1,6,7,8,0.1,7,8,9,0.1,8,9,10,0.1
360 DATA9,10,11,0.1,10,11,12,0.1,11,12,13,0.1
```

Program 8.2 *Finite elements for plane steady flow* of the matrix P are stored in an N x Z matrix, at the locations defined by the pointer matrix. The last column of the matrix P is used to store the right-hand side of the system of equations. This quantity, Q_i in equation 8.29, represents the amount of water supplied to the network at each node, as can be seen as follows. The term representing water being extracted along a part of the boundary in the functional U is, see equation 8.4

$$\Delta U = \int_{S_2} g\phi \, dS \tag{8.31}$$

If in node i a point source of strength Q_i is operating one may write $g = -Q_i/2\pi r$, where r is a very small radius. Furthermore $dS = r\, d\theta$, where θ varies from 0 to 2π so that the contribution to the functional becomes

$$\Delta U = -Q_i \phi_i \tag{8.32}$$

The minimalisation of U requires that this be differentiated with respect to ϕ_i. In the system of equations this leads to a term $-Q_i$ on the left-hand side, or $+Q_i$ on the right-hand side. This means that a source of strength Q_i can be taken into account by simply setting the right-hand side of the appropriate equation equal to Q_i. The computer program determines a value ϕ_i for the head also in points in which a source is operating. It can be shown that this value can be compared to the theoretical value in a continuous field at a distance of about $0.2a$ from the source, where a is the element size near the source (Kono, 1974). The equivalent radius of the well screen in a source is about one-fifth of the local element size.

Program 8.2 contains data for the problem illustrated in figure 8.3. The transmissivity is $T = 0.1$. In nodes 5, 12 and 13 the head is fixed at $\phi = 0$, and in node 1 a source is operating at a strength of $Q_1 = 0.157$. The results may be compared with the analytical solution for a source of strength 4 * 0.157 in a circular aquifer of radius $R = 8$. This solution is $\phi = (4 * 0.157/2\pi * 0.1) \ln (r/R) = \ln (r/8)$, because the coefficient happens to be 1. The numerical and

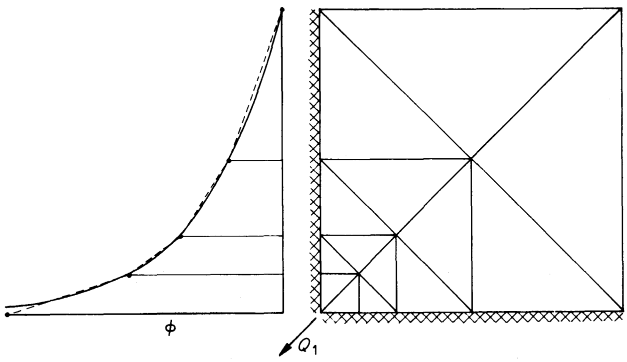

Figure 8.3 *Mesh for radial flow*

analytical results are compared in figure 8.3. The value obtained by the computer program in the source appears to be indeed very close to the exact value at a radius of about one-fifth of the element size.

Program 8.3 Steady Flow in a Dam

The power and flexibility of the finite element manifest themselves in problems such as the flow through a dam, with a free surface (see figure 8.4). The diffi-

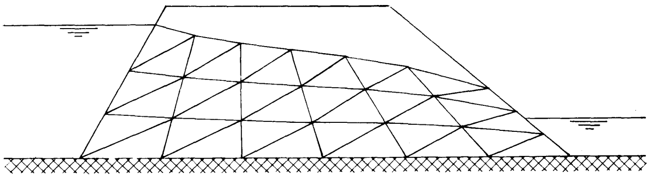

Figure 8.4 *Flow through a dam*

```
10 REM FINITE ELEMENTS FOR PLANE STEADY FLOW IN A DAM, PROGRAM 8.3
20 READW,AL,AR,HL,HR,NL,NH:A=3.1415927/180:AL=A*AL:AR=A*AR:Z=7:E=HL/500
30 N=(NH+1)*(NL+1)-1:M=2*NH*NL-1:DIMX(N),Y(N),T%(N),F(N),K%(N,Z),P(N,Z)
40 DIMN%(M,2),T(M),XJ(2),YJ(2),PJ(2,2),B(2),C(2),XB(NL),XT(NL),YT(NL),CT(NL)
50 CL=COS(AL)/SIN(AL):CR=COS(AR)/SIN(AR):NI=20:R=1.5:IS=1
60 FORI=0TONL:A=I/NL:XB(I)=A*W:CT(I)=CL+A*(CR-CL):YT(I)=HL+0.8*A*(HR-HL)
70 XT(I)=XB(I)+CT(I)*YT(I):NEXTI
80 FORI=0TONL-1:K=I*(NH+1):L=2*I*NH+1:FORJ=0TONH-1:JB=L+2*J:JA=JB-1
90 N%(JA,0)=K+J:N%(JA,1)=N%(JA,0)+NH+1:N%(JA,2)=N%(JA,1)+1:N%(JB,0)=N%(JA,0)
100 N%(JB,1)=N%(JB,0)+1:N%(JB,2)=N%(JA,2):NEXTJ,I
110 PRINT"CALCULATION OF POINTER MATRIX":PRINT:FORI=0TON:PRINT"   NODE";I+1
120 K%(I,0)=I:K%(I,Z)=0:K=0:FORJ=0TOM:FORH=0TO2:IFN%(J,H)=IGOTO140
130 NEXTH:GOTO170
140 FORH=0TO2:U=N%(J,H):FORL=0TOK:IF(K%(I,L)=U)GOTO160
150 NEXTL:K=K+1:K%(I,K)=U:K%(I,Z)=K
160 NEXTH
170 NEXTJ:NEXTI:PRINT
180 FORJ=0TONH:F(J)=HL:T%(J)=2:T%(N-J)=2:F(N-J)=HR:NEXTJ:T%(N)=0
190 FORI=0TONL:K=I*NH+I:FORJ=0TONH:L=K+J:Y(L)=J*YT(I)/NH:IFIS=1THENF(L)=YT(I)
200 X(L)=XB(I)+Y(L)*CT(I):NEXTJ,I:FORI=N-NHTON:F(I)=HR:IFY(I)>HRTHENF(I)=Y(I)
210 NEXTI:PRINT"GENERATION OF MATRIX":PRINT:FORJ=0TOM:PRINT"   ELEMENT";J+1
220 FORI=0TO2:K=N%(J,I):XJ(I)=X(K):YJ(I)=Y(K):NEXTI
230 B(0)=YJ(1)-YJ(2):B(1)=YJ(2)-YJ(0):B(2)=YJ(0)-YJ(1)
240 C(0)=XJ(2)-XJ(1):C(1)=XJ(0)-XJ(2):C(2)=XJ(1)-XJ(0)
250 D=XJ(0)*B(0)+XJ(1)*B(1)+XJ(2)*B(2):D=1/(2*ABS(D))
260 FORK=0TO2:FORL=0TO2:PJ(K,L)=D*(B(K)*B(L)+C(K)*C(L)):NEXTL,K
270 FORK=0TO2:U=N%(J,K):H=K%(U,Z):FORL=0TOH:FORV=0TO2
280 IF(N%(J,V)=K%(U,L))THEN300
290 GOTO310
300 P(U,L)=P(U,L)+PJ(K,V):GOTO320
310 NEXTV
320 NEXTL,K,J:PRINT:PRINT"SOLUTION OF EQUATIONS":PRINT:FORIT=1TONI
330 PRINT"   ITERATION";IT:FORI=0TON:IFT%(I)>1GOTO360
340 A=P(I,Z):K=K%(I,Z):FORJ=0TOK:A=A-P(I,J)*F(K%(I,J)):NEXTJ
350 F(I)=F(I)+R*A/P(I,0)
360 NEXTI,IT:PRINT:PRINT"FREE SURFACE AFTER";IS;"CYCLES":PRINT:L=0
370 FORI=0TONL:K=(I+1)*(NH+1)-1:PRINT"X=";X(K);"  Y=";Y(K)
380 A=F(K)-Y(K):YT(I)=YT(I)+R*A:IFABS(A)>ETHENL=2
390 NEXTI:IS=IS+1:PRINT:IFL>1GOTO190
400 DATA0.162,90,90,0.322,0.084,10,8
```

Program 8.3 *Finite elements for plane steady flow in a dam* culty that the location of the free surface is initially unknown is easily overcome by an iterative process. The location of the free surface is estimated (for instance in the shape of a straight line), and it is considered as a stream line, ignoring the condition that the groundwater head should equal the elevation (because $p = 0$). The distribution of the head is then calculated by means of the finite element method, and at the end the head ϕ along the free surface is compared with the elevation y. If these are not equal (to within a certain acceptable error ϵ) a new estimation for the free surface is made, for instance $y = \phi$. The location of all interior nodes can be defined on the basis of the position of the free surface and the base of the dam, by interpolation. Thus, the entire network is modified in the process, which can easily be incorporated in the finite element method, because the coordinates of the nodes are the basic input variables for the calculation of the matrix. A program performing such calculations for a homogeneous dam on an impermeable base, is reproduced as program 8.3. The input variables are the following. W is the width of the dam. At its toe, AL is the angle of the upstream left-side slope, in degrees, AR is the angle of the downstream right-side slope, HL is the water level at the upstream side, HR is the water level at the downstream side, NL is the number of subdivisions in the horizontal direction, NH is the number of subdivisions in the vertical direction. The program generates its own network, starting with a free surface estimated as a straight line, and a seepage surface initially estimated to have a vertical dimension of 0.2 * (HL - HR). The program performs 20 Gauss-Seidel iterations, with an over-relaxation factor R = 1.5, in each cycle. It stops when the greatest differ-

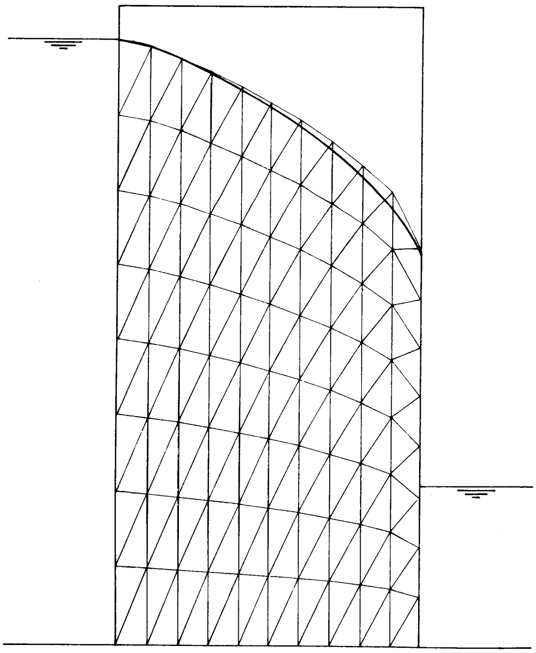

Figure 8.5 *Comparison of numerical and analytical results*

ence between ϕ and y along the free surface is E = HL/500. The program can be run on a small microcomputer with 32 000 bytes memory with NL = 20 and NH = 10. The number of elements then is 400. It is to be noted that in this program full use has been made of the fact that in BASIC the first element of an array X(N) is X(0). When translating the program into FORTRAN, in which the first element usually is X(1), this has to be taken into account.

In figure 8.5 some results for a dam with vertical faces (AL = AR = 90) are compared with known analytical results (Muskat, 1937, p. 314). Even in this extreme case the accuracy seems to be rather good. The program needed 11 cycles of 20 interations each to obtain the required accuracy.

8.3 Non-steady Flow

For non-steady flow the basic differential equation, including infiltration as well as leakage is

$$S \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(T \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T \frac{\partial \phi}{\partial y}\right) + N - \frac{\phi - \phi'}{c} \qquad (\text{in } R) \qquad (8.33)$$

with the boundary conditions

$$\phi = f \qquad (\text{on } S_1) \qquad (8.34)$$

$$-T \frac{\partial \phi}{\partial n} = g \qquad (\text{on } S_2) \qquad (8.35)$$

and with the initial condition

$$\phi = \phi^0 \qquad (\text{for } t = 0) \qquad (8.36)$$

This is the generalisation of the steady state problem given in section 8.1, with S representing either elastic or phreatic storativity. Because of the non-steady character of the problem an initial condition must also be specificed. This is equation 8.36, where ϕ^0 is supposed to be given throughout the region R.

A very simple way to incorporate the term $S\partial\phi/\partial t$ in the finite element method is to integrate equation 8.33 over a short time interval from $t = t_0$ to $t = t_0 + \Delta t$ (Verruijt, 1972). When applied to equation 8.33 this gives

$$\frac{\partial}{\partial x}\left(T \frac{\partial \bar{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(T \frac{\partial \bar{\phi}}{\partial y}\right) + N - \frac{\phi - \phi'}{c} - \frac{S}{\Delta t}(\phi_+ - \phi^0) = 0 \quad (8.37)$$

where $\bar{\phi}$ is the average value of the head over the time interval, ϕ^0 is the value at the beginning, and ϕ_+ is the value at the end of the interval. The average value $\bar{\phi}$ can be approximated by

$$\bar{\phi} = \epsilon\phi_0 - (1 \times \epsilon)\ \phi_+ \qquad (8.38)$$

where ϵ is a parameter representing the type of interpolation, $\epsilon = 0$ corresponds to a backward difference, $\epsilon = 1$ to a forward difference, and $\epsilon = 0.5$ to a central difference (that is, linear interpolation). It can be expected that a value somewhat smaller than 0.5 is most reasonable, in view of the decaying character of the process. With equation 8.38, equation 8.37 can be rewritten as

$$\frac{\partial}{\partial x}\left(T\,\frac{\partial\bar{\phi}}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\,\frac{\partial\bar{\phi}}{\partial y}\right)+N-\frac{\bar{\phi}-\phi'}{c}-\frac{S}{(1-\epsilon)\Delta t}(\bar{\phi}-\phi^0)=0 \tag{8.39}$$

It now appears that the term due to the non-steady flow is of the same nature as the term due to leakage. This implies that the variational formulation can immediately be written down by using the analogy of equation 8.39 with equation 8.1. This means that the functional to be minimised is

$$U=\tfrac{1}{2}\iint_R\left[T\left(\frac{\partial\phi}{\partial x}\right)^2+T\left(\frac{\partial\phi}{\partial y}\right)^2+\frac{\alpha S}{\Delta t}\,\phi^2-\frac{2\alpha S}{\Delta t}\,\phi\,\phi^0-2N\phi+\right.$$
$$\left.+\frac{1}{c}\,\phi^2-\frac{2}{c}\,\phi\,\phi'\right]\mathrm{d}x\,\mathrm{d}y+\int_{S_2}g\phi\,\mathrm{d}S \tag{8.40}$$

where $\bar{\phi}$ has been replaced by ϕ, and where $\alpha = 1/(1-\epsilon)$. The functional in equation 8.40 should be minimised over the class of functions satisfying the constraint

$$\phi = f \qquad (\text{on } S_1) \tag{8.41}$$

Once ϕ has been determined, the value ϕ_+ at the end of the time interval can be determined from equation 8.38, in the following way

$$\phi_+ = \phi^0 + \alpha(\phi - \phi^0) \tag{8.42}$$

A next time step can then be taken, with the values ϕ_+ replacing ϕ^0. In this way the distribution of the head is calculated in successive time steps.

Program 8.4 Non-steady Flow

As in the previous section the domain R is subdivided into triangular elements, with linear interpolation in each element of the functions ϕ, ϕ^0 and ϕ', that is

$$\phi = p_1x + p_2y + p_3 \tag{8.43}$$

$$\phi^0 = q_1x + q_2y + q_3 \tag{8.44}$$

$$\phi' = r_1x + r_2y + r_3 \tag{8.45}$$

To improve the numerical accuracy, and to simplify the analysis in each element a local coordinate system will be used, around the centroid of the element. The parameters $p_1, p_2, \ldots, r_3$ can be expressed in the nodal values in the same way as done in the previous section, using formulae of the type of equation 8.18, involving parameters b_j, c_j, d_j and Δ as defined by equations 8.19 to 8.22. The contribution of a typical element R_j to the value of the functional can now be elaborated into

$$U_j = \sum_{k=1}^{3} \sum_{l=1}^{3} \left\{ \tfrac{1}{2} P^j_{k\ell} \; \phi_k \phi_\ell - \beta R^j_{k\ell} \, \phi_k \, \phi^0_\ell - R^j_{k\ell} \, \phi_k \, \phi'_\ell \right\} -$$

$$- \sum_{k=1}^{3} Q_k \, \phi_k \tag{8.46}$$

where

$$P^j_{k\ell} = \frac{T_j}{2|\Delta|} \left(b_k b_\ell + c_k c_\ell \right) + (1 + \beta) R^j_{k\ell} \tag{8.47}$$

$$R^j_{k\ell} = \frac{1}{2c|\Delta|} \left[d_k d_\ell + Z_{xx} b_k b_\ell + Z_{yy} c_k c_\ell + Z_{xy} (b_k c_\ell + b_\ell c_k) \right] \tag{8.48}$$

$$Q_k = \frac{1}{6} |\Delta| N \tag{8.49}$$

and where

$$\beta = \frac{\alpha c S}{\Delta t} \tag{8.50}$$

In equation 8.48 the constants Z_{xx}, Z_{yy} and Z_{xy} denote the following elementary integrals

$$Z_{xx} = \frac{2}{|\Delta|} \iint x^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{12} (x_1{}^2 + x_2{}^2 + x_3{}^2) \tag{8.51}$$

$$Z_{yy} = \frac{2}{|\Delta|} \iint y^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{12} (y_1{}^2 + y_2{}^2 + y_3{}^2) \tag{8.52}$$

$$Z_{xy} = \frac{2}{|\Delta|} \iint xy \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{12} (x_1 y_1 + x_2 y_2 + x_3 y_3) \tag{8.53}$$

It should be noted that in deriving the formulae presented above use has been made of the fact that the coordinates of the centroid of the element are zero.

After summation over all elements an expression for the functional U is obtained, similar to equation 8.46. The minimisation now leads to the system of equations

$$\sum_{k=1}^{n} P_{ik} \, \phi_k = Q_i + \sum_{k=1}^{n} R_{ik} \, (\phi'_k + \beta \phi^0_k) \tag{8.54}$$

This system of equations can be solved by a standard technique for the solution of linear equations.

The calculations presented above are performed in program 8.4, reproduced below. For simplicity the values of the resistance c and the storativity S are assumed to be constant throughout the region (c = 100 000 and S = 0.5). In that case the value of β is the same in each element (see equation 8.50). This

```
10 REM FINITE ELEMENTS FOR PLANE NON-STEADY FLOW, PROGRAM 8.4
20 READN,M,Z,NI,R,NS,C,S:N=N-1:M=M-1:DIMXJ(2),YJ(2),B(2),C(2),D(2),PJ(2,2)
30 DIMRJ(2,2),N%(M,2),T%(N),T(M),F(N),FA(N),FO(N),K%(N,Z),P(N,Z-1),R(N,Z-1)
40 DIMX(N),Y(N),Q(N),N(M):FORI=0TON:READX(I),Y(I),T%(I),FO(I),FA(I),Q(I):NEXTI
50 FORJ=0TOM:READN%(J,0),N%(J,1),N%(J,2),T(J),N(J)
60 N%(J,0)=N%(J,0)-1:N%(J,1)=N%(J,1)-1:N%(J,2)=N%(J,2)-1:NEXTJ
70 PRINT"CALCULATION OF POINTER MATRIX":PRINT:FORI=0TON:PRINT"   NODE";I+1
80 K%(I,0)=I:K%(I,Z)=0:K=0:FORJ=0TOM:FORH=0TO2:IFN%(J,H)=IGOTO100
90 NEXTH:GOTO130
100 FORH=0TO2:U=N%(J,H):FORL=0TOK:IF(K%(I,L)=U)GOTO120
110 NEXTL:K=K+1:K%(I,K)=U:K%(I,Z)=K
120 NEXTH
130 NEXTJ:NEXTI:PRINT:EP=0.4:G=1/(1-EP)
140 PRINT"GENERATION OF MATRICES":PRINT:FORJ=0TOM:PRINT"   ELEMENT";J+1:ZX=0
150 ZY=0:FORI=0TO2:K=N%(J,I):XJ(I)=X(K):YJ(I)=Y(K):ZX=ZX+X(K):ZY=ZY+Y(K):NEXTI
160 ZX=ZX/3:ZY=ZY/3:FORI=0TO2:XJ(I)=XJ(I)-ZX:YJ(I)=YJ(I)-ZY:NEXTI
170 B(0)=YJ(1)-YJ(2):C(0)=XJ(2)-XJ(1):D(0)=XJ(1)*YJ(2)-XJ(2)*YJ(1)
180 B(1)=YJ(2)-YJ(0):C(1)=XJ(0)-XJ(2):D(1)=XJ(2)*YJ(0)-XJ(0)*YJ(2)
190 B(2)=YJ(0)-YJ(1):C(2)=XJ(1)-XJ(0):D(2)=XJ(0)*YJ(1)-XJ(1)*YJ(0)
200 D=ABS(D(0)+D(1)+D(2)):E=1/(2*C*D):F=T(J)/(2*D)
210 XX=(XJ(0)*XJ(0)+XJ(1)*XJ(1)+XJ(2)*XJ(2))/12
220 YY=(YJ(0)*YJ(0)+YJ(1)*YJ(1)+YJ(2)*YJ(2))/12
230 XY=(XJ(0)*YJ(0)+XJ(1)*YJ(1)+XJ(2)*YJ(2))/12
240 FORK=0TO2:FORL=0TO2:PJ(K,L)=F*(B(K)*B(L)+C(K)*C(L))
250 RJ(K,L)=E*(D(K)*D(L)+XX*B(K)*B(L)+YY*C(K)*C(L)+XY*(B(K)*C(L)+B(L)*C(K)))
260 NEXTL:NEXTK:FORK=0TO2:U=N%(J,K):H=K%(U,Z):FORL=0TOH:FORV=0TO2
270 IF(N%(J,V)=K%(U,L))THEN290
280 GOTO300
290 P(U,L)=P(U,L)+PJ(K,V):R(U,L)=R(U,L)+RJ(K,V):GOTO310
300 NEXTV
310 NEXTL:NEXTK:FORI=0TO2:K=N%(J,I):Q(K)=Q(K)+D*N(J)/6:NEXTI:NEXTJ:PRINT:TN=0
320 FORIS=1TONS:READTT:DT=TT-TN:B=C*S*G/DT:PRINT"SOLUTION OF EQUATIONS":PRINT
330 TN=TT:FORIT=1TONI:PRINT"    ITERATION";IT:FORW=0TON:I=W:H=0
340 IFT%(I)>1GOTO370
350 A=Q(I):K=K%(I,Z):FORJ=0TOK:L=K%(I,J):A=A-P(I,J)*F(L):D=B*FO(L)-(1+B)*F(L)
360 A=A+R(I,J)*(FA(L)+D):NEXTJ:F(I)=F(I)+R*A/(P(I,0)+(1+B)*R(I,0))
370 IFH>1THENNEXTW:NEXTIT:PRINT:GOTO390
380 H=2:I=N-W:GOTO340
390 FORI=0TON:FO(I)=FO(I)+G*(F(I)-FO(I)):F(I)=FO(I):NEXTI:PRINT"  TIME=";TN
400 FORI=0TON:PRINT"X=";X(I);"    Y=";Y(I);"    PHI=";F(I):NEXTI:PRINT:NEXTIS
410 DATA18,20,8,25,1.25,12,100000,0.5
420 DATA0,0,2,0,0,0,0,1,2,0,0,0,0,2,2,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0
430 DATA1,2,0,0,0,0,2,0,0,0,0,0,2,1,0,0,0,0,2,2,0,0,0,0,3,0,0,0,0,0
440 DATA3,1,0,0,0,0,3,2,0,0,0,0,4,0,0,0,0,0,4,1,0,0,0,0,4,2,0,0,0,0
450 DATA5,0,0,0,0,0,5,1,0,0,0,0,5,2,0,0,0,0
460 DATA1,2,4,12.5,1,2,4,5,12.5,1,2,3,6,12.5,1,2,5,6,12.5,1
470 DATA4,5,7,12.5,1,5,7,8,12.5,1,5,6,9,12.5,1,5,8,9,12.5,1
480 DATA7,8,10,12.5,1,8,10,11,12.5,1,8,9,12,12.5,1,8,11,12,12.5,1
490 DATA10,11,13,12.5,1,11,13,14,12.5,1,11,12,15,12.5,1,11,14,15,12.5,1
500 DATA13,14,16,12.5,1,14,16,17,12.5,1,14,15,18,12.5,1,14,17,18,12.5,1
510 DATA0.02,0.04,0.06,0.1,0.2,0.3,0.5,1,2,3,5,10
```

Program 8.4 *Finite elements for plane non-steady flow*

means that the matrices have to be calculated only once. The parameters for the transmissivity T and the infiltration N have been chosen such that the dimensionless parameters $\phi/(N\ell^2/2T)$ and $t/(S\ell^2/T)$ coincide with the head ϕ and the time t. The program calculates the values of the head for 12 values of the time parameter. The first value has been taken such that $T\Delta t/S(\Delta x)^2 =$ 0.5, in accordance with the usual stability and accuracy criteria. Because the process is unconditionally stable, at least for values of ϵ smaller than 0.5, which corresponds to the so-called Crank-Nicholson scheme in finite differences (Carnahan *et al.,* 1969), the time steps can be taken successively larger. The program again uses Gauss-Seidel iteration to solve the equations, using a so-called double sweep technique, in which the nodes are run through both in an ascending and a descending series. In this way the accuracy is less dependent upon the way of numbering the nodes.

In figure 8.6 the numerical results are compared with the analytical solution, for the case of uniform infiltration, without leakage, which is avoided in

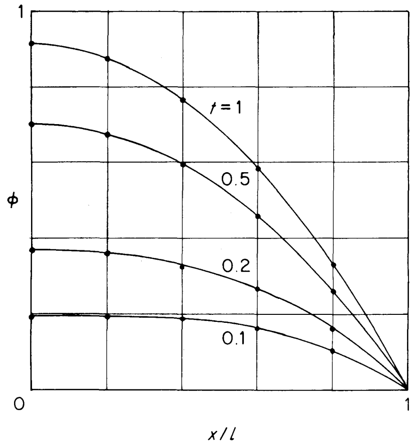

Figure 8.6 *Comparison of numerical and analytical solutions*

the finite element program by taking c very large (see equation 7.49). The agreement appears to be very good, just as when using the finite difference method (see figure 7.8). In fact there is basically little difference between the finite difference method and the finite element method. In engineering practice the finite element method is usually preferred because of its greater flexibility. For further study, and practical examples of applications of numerical methods the reader is referred to the literature (for example, Desai and Christian, 1977; Pinder and Gray, 1977; Remson *et al.*, 1971, Zienkiewicz, 1977).

8.4 Problems

1. Modify program 8.1 or program 8.2 so that the transmissivities in the x- and y-directions are different, assuming that these are the principal directions of permeability.

2. Use program 8.3 to verify the shape of the free surface in the case of the dam illustrated in figure 6.2.

3. Modify program 8.3 so that it can be used for the non-steady flow in a dam, generated by a fluctuating water level on the left side. In this case it is most convenient to put $\phi = y$ along the free surface, and to consider this head to be given. The water to be supplied to maintain the free surface fixed can be calculated using the formula $Q_i = \Sigma P_{ik}\, \phi_k$. Because there is no supply the water level will drop over a distance $Q_i\, \Delta t / SH\Delta x$ in a short

time interval Δt, where Δx is the horizontal distance between the mid-points of the two neighbouring elements, H is the thickness of the plane of flow, and S is the (phreatic) storativity.

4. Change the data in program 8.4 so that it can be used to solve a problem without infiltration and with lėakage. Run the program and check the steady state solution by comparison with an analytical solution.

9

Analogue Methods

The basic equations of groundwater flow are the linear relation between the flux and the gradient of the groundwater head ϕ (Darcy's law), and a conservation equation. Similar equations appear in many places in mathematical physics, for instance in hydrodynamics, thermodynamics (conduction of heat), and electrodynamics. Thus, provided that the boundary conditions are also of a similar nature, the comparison of problems in the various disciplines may be very instructive. Problems solved in one discipline can often be transferred to other disciplines.

There is another aspect to these analogies, namely that some analogue phenomena can so easily be studied in the laboratory that their experimental investigation is an alternative to mathematical calculations. For the translation of the results into the language of the other discipline all that is needed is some insight into analogue quantities and scale factors (Karplus, 1958; Bear, 1972).

From the physical phenomena that are analogous to flow of groundwater the conduction of heat in solids does not lend itself easily to experimental investigation, since thermal isolation (to prevent radiation losses) is difficult to accomplish. Therefore the thermal analogy is of importance only because solutions given in textbooks on conduction of heat in solids (for example, Carslaw and Jaeger, 1959) can often be used as solutions of groundwater flow problems.

Two analogies that are of interest for laboratory investigations are discussed in this chapter. These are the electrical analogue, and the Hele Shaw model. An obvious analogue is the so-called sandbox, which is simply a scale model. This is sometimes used for demonstration purposes, or when the interaction with the soil is of importance (for example, in soil mechanics). Another interesting analogue not considered here is the membrane analogue (see De Josselin de Jong, 1961).

9.1 Electrical Analogue

The electrical analogue is based upon the similarity between Ohm's law for the movement of electric charges through a conducting material, and Darcy's law. The infinitesimal form of Ohm's law is

$$i = -\frac{1}{\rho}\frac{dV}{ds} \tag{9.1}$$

where i is the current density (in A/m^2), ρ is the specific resistance of the material (in Ω m) and V is the voltage (in V). Equation 9.1 is mathematically similar to Darcy's law. Because for the current i a conservation principle also holds (stating in fact that free electrons cannot be lost or produced, just like water), the phenomenon is completely analogous to groundwater flow. The analogue quantities are the head ϕ and the voltage V, the specific discharge q and the current density i, and the hydraulic conductivity k and the inverse $(1/\rho)$ of the specific resistance ρ.

In practical applications the conducting material may be a fluid (for example, water with some added salt to improve the conductivity) or a solid, in particular a specially prepared type of conducting paper (commercially available under the name Teledeltos paper). In figure 9.1 an elementary and simple way of using the electrical analogue is sketched. Impermeable boundaries are simulated by the edges of the paper. The electrodes simulate the boundaries of constant head. These electrodes can be constructed most simply by using copper strips which are pressed on to the paper by clips. Potential lines can be determined by tracing (by means of a probe) the line of points where the electric potential is equal to the one adjusted on the middle terminal of a variable resistor. When the probe is connected to such a point the current meter indicates zero current, which also means that the probe does not disturb the flow. The arrangement of figure 9.1 is very cheap: the battery may be a simple pen-light battery, and the variable resistor can be a set of 10 equal standard resistors, in series. A simple current meter can be obtained in any hobby shop.

With a little effort the electrical analogue can also be used to solve problems involving a free surface, using a trial-and-error technique to satisfy the condition that the groundwater head should be equal to the elevation of the free surface. It is advisable to start by estimating a rather high position for the free surface, because it can easily be lowered, by simply cutting off some material.

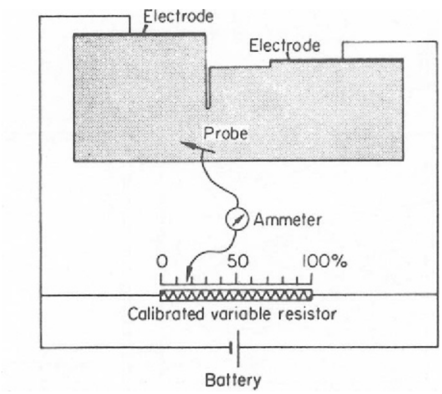

Figure 9.1 *Electrical analogue*

A more sophisticated version of the electrical analogue is by constructing a network of resistors. Then it is a simple matter to vary the permeability, by putting in different resistors. Non-steady problems, involving storage, can be dealt with by adding the possibility of storage of electric charges, through the use of capacitors. Such a network is in fact an analogue computer solving the finite difference equations. Because of the availability of rather cheap digital computers the use of electrical network analogues seems to be a dying art.

9.2 Hele Shaw Model

As already mentioned in section 2.1, Darcy's law in its elementary form

$$q = -k \frac{d\phi}{ds} \tag{9.2}$$

corresponds to the formula of Hagen–Poiseuille for the flow of a viscous fluid through a circular tube, and to the flow of a viscous fluid through the narrow interspace between two closely spaced parallel plates, as investigated by Hele Shaw (Lamb, 1932). In this case the permeability κ is $d^2/12$, where d is the distance between the plates. With equation 2.6 the hydraulic conductivity for such a system is found to be

$$k = \frac{\rho g d^2}{12\mu} \tag{9.3}$$

where ρ is the density of the fluid, μ its dynamic viscosity, and g is the gravity constant.

The direct analogue between two-dimensional groundwater flow and the flow between two parallel plates enables us to study groundwater flow problems by building a model in which a fluid moves between two parallel plates. A sketch of such a model, usually called a Hele Shaw model is given in figure 9.2.

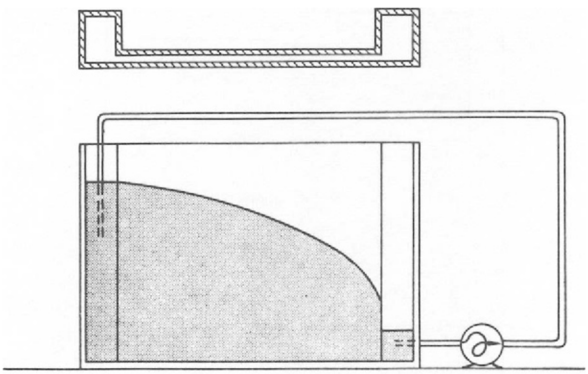

Figure 9.2 *Hele Shaw model*

As the figure suggests, the simultaneous appearance of a free surface and a seepage surface does not lead to difficulties in such a model. On the contrary, the fact that in this model the head is represented by itself results in the flow in the model simulating the flow in the prototype in all its fundamental aspects. Gravity influences the flow in precisely the same way as in the soil. This means that the Hele Shaw model is also particularly suitable for the investigation of problems of multiple fluid flow, such as arise in the cases of simultaneous flow of fresh and salt water, or oil and water. In order that the flow remains laminar it is advisable to use fluids that are somewhat more viscous than water, such as a mixture of glycerine and water or an oil.

Problems involving layers of different permeability can be studied by inserting plates which partially occupy the interspace between the parallel plates. It should be noted that the quantity qH in the prototype (H being the thickness of the plane of flow, for instance 1 m) corresponds to qd in the model. This means that the quantity kH in the prototype corresponds to $\rho g d^3/12\nu$ in the model. Thus by varying the slot width d the permeability in the prototype varies by a factor d^3. This also means that errors in the slot width have a marked influence upon the permeability. It is therefore necessary to take great care that the distance between the parallel plates is maintained at its desired value. This can be achieved with the aid of specially prepared metallic or plastics rings of thickness equal to the desired slot width. These rings are inserted between the plates, at regular intervals, and the plates are then clamped together by means of screws through plates and rings.

Although many problems, including non-steady problems, can be studied in a Hele Shaw model, its practical usefulness is restricted by the circumstances that installation of a model is a time-consuming process, especially when a computer program is available. A Hele Shaw model is an ideal way of visualising certain effects, however, and they are often used successfully for demonstration purposes.

9.3 Problems

1. A problem of steady groundwater flow is being studied by means of an electrical analogue, using conducting paper. The resistance of an arbitrary square of the paper is 2500 Ω. In the prototype the hydraulic conductivity is $k = 10^{-4}$ m/s. The model is a reproduction of the prototype on a scale 1:50. The real difference in head $\Delta\phi = 5$ m, is represented by a voltage difference $\Delta V = 1.5$ V. The total electric current is measured to be $I = 0.1$ mA. Calculate the total discharge (per meter width) through the prototype.

2. In a Hele Shaw model a problem of flow through a dam has to be studied. The dam contains a core which is 20 times less permeable than the rest of the dam. If the distance of the plates in the area representing the main dam is 2 mm, what should be the distance in the area representing the core?

Appendix A

Bessel Functions

The differential equation

$$\frac{d^2 w}{dx^2} + \frac{1}{x}\frac{dw}{dx} - w = 0 \tag{A.1}$$

possesses an elementary solution expressible in a Taylor series expansion around $x = 0$. This solution is denoted by $I_0(x)$, and its definition is

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2} \tag{A.2}$$

That this is indeed a solution can easily be verified by substitution into the differential equation.

A second independent solution is the function $K_0(x)$, defined as

$$K_0(x) = -[\ln(x/2) + \gamma]\, I_0(x) + \sum_{k=1}^{\infty} \left[\sum_{j=1}^{k} (1/j)\right] \frac{(x/2)^{2k}}{(k!)^2} \tag{A.3}$$

where γ is Euler's constant

$$\gamma = \lim_{k \to \infty} \left[\sum_{j=1}^{k} (1/j) - \ln k\right] = 0.577215665\ldots \tag{A.4}$$

The functions $I_0(x)$ and $K_0(x)$ are called modified Bessel functions of the first and second kind, respectively, and of order zero (see for instance Watson, 1944; Abramowitz and Stegun, 1965).

Expressions for the derivatives of $I_0(x)$ and $K_0(x)$ can be obtained by termwise differentiation of their series expansions. These derivatives are denoted by $I_1(x)$ and $-K_1(x)$, respectively

$$\frac{d}{dx} I_0(x) = I_1(x) \tag{A.5}$$

$$\frac{d}{dx} K_0(x) = -K_1(x) \tag{A.6}$$

A short table of the four types of Bessel functions considered here, adapted from the book by Watson (1944) is given in table A.1. Their behaviour is also illustrated in figure A.1.

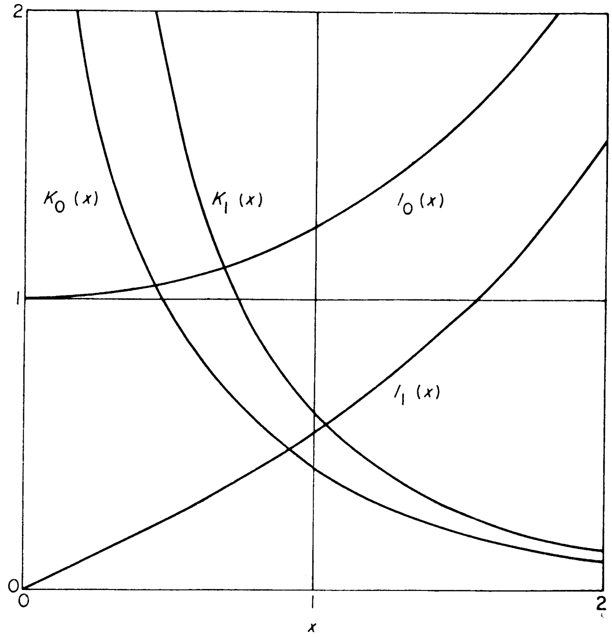

Figure A.1 *Bessel functions*

For small values of the argument x only a few terms of the series expansions suffice to calculate the Bessel functions. Of particular importance are the first approximations for K_0 (x) and K_1 (x), which are both singular for $x \to 0$

$$x \ll 1: \; K_0\,(x) \approx \ln\left(\frac{1.123}{x}\right) \tag{A.7}$$

$$x \ll 1: \; K_1\,(x) \approx \frac{1}{x} \tag{A.8}$$

For large values of x the series expansions converge very slowly. Then it is convenient to use special asymptotic expansions (see Abramowitz and Stegun, 1965).

Table A.1 (Courtesy of Cambridge University Press)

x	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
0.0	1.0000	0.0000	∞	∞
0.1	1.0025	0.0501	2.4271	9.8538
0.2	1.0100	0.1005	1.7527	4.7760
0.3	1.0226	0.1517	1.3725	3.0560
0.4	1.0404	0.2040	1.1145	2.1844
0.5	1.0635	0.2579	0.9244	1.6564
0.6	1.0920	0.3137	0.7775	1.3028
0.7	1.1263	0.3719	0.6605	1.0503
0.8	1.1665	0.4329	0.5653	0.8618
0.9	1.2130	0.4971	0.4867	0.7165
1.0	1.2661	0.5652	0.4210	0.6019
1.1	1.3262	0.6375	0.3656	0.5098
1.2	1.3937	0.7147	0.3185	0.4346
1.3	1.4693	0.7973	0.2782	0.3726
1.4	1.5534	0.8861	0.2436	0.3208
1.5	1.6467	0.9817	0.2138	0.2774
1.6	1.7500	1.0848	0.1880	0.2406
1.7	1.8640	1.1963	0.1655	0.2094
1.8	1.9896	1.3172	0.1459	0.1826
1.9	2.1277	1.4482	0.1288	0.1597
2.0	2.2796	1.5906	0.1139	0.1399
2.1	2.4463	1.7455	0.1008	0.1228
2.2	2.6291	1.8280	0.0893	0.1079
2.3	2.8296	2.0978	0.0791	0.0950
2.4	3.0493	2.2981	0.0702	0.0837
2.5	3.2898	2.5167	0.0624	0.0739
2.6	3.5533	2.7554	0.0554	0.0653
2.7	3.8416	3.0161	0.0493	0.0577
2.8	4.1573	3.3011	0.0438	0.0511
2.9	4.5028	3.6126	0.0390	0.0453
3.0	4.8808	3.9534	0.0347	0.0402
3.1	5.2945	4.3262	0.0310	0.0356
3.2	5.7472	4.7342	0.0276	0.0316
3.3	6.2426	5.1810	0.0246	0.0281
3.4	6.7848	5.6701	0.0220	0.0250
3.5	7.3782	6.2058	0.0196	0.0222
3.6	8.0277	6.7927	0.0175	0.0198
3.7	8.7386	7.4358	0.0156	0.0176
3.8	9.5169	8.1404	0.0140	0.0157
3.9	10.3690	8.9128	0.0125	0.0140
4.0	11.3019	9.7595	0.0112	0.0125

Appendix B

Complex Variables

A complex number is a quantity of the form

$$z = x + \mathrm{i}y \tag{B.1}$$

where x and y are real numbers, and where i is the so-called imaginary unit, which has all the properties of real numbers ($\mathrm{i}a = a\mathrm{i}$, $\mathrm{i} + a = a + \mathrm{i}$, etc.) and in addition to these the special property that $\mathrm{i}^2 = -1$. If $z = x + \mathrm{i}y$ then the complex conjugate of z, denoted by $\bar{z}$, is defined as

$$\bar{z} = x - \mathrm{i}y \tag{B.2}$$

The product of z and its conjugate $\bar{z}$ is always real, $z\bar{z} = x^2 + y^2$.

A complex number $z = x + \mathrm{i}y$ can be represented graphically as a point in a plane (see figure B.1), by considering the real and imaginary parts x and y as cartesian coordinates. The point can also be defined in terms of polar coordinates r and θ, so that

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) \tag{B.3}$$

If to every point in a certain region R in the complex z-plane there are associated one or more values of another variable w (in general also complex, $w = u + \mathrm{i}v$), then w is said to be a function of z

$$w = f(z) \tag{B.4}$$

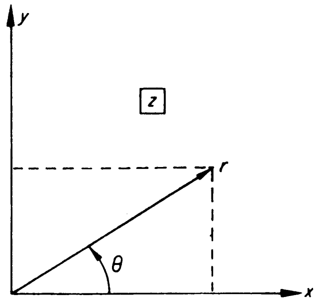

Figure B.1 *Complex plane*

The function $f(z)$ is said to be single-valued when, for every point in R, there is defined one single value of w. Such a function is said to be continuous in the region R if for every point z_0 of R

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{B.5}$$

independent of the path in the z-plane along which z tends to z_0. If the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{B.6}$$

exists, and its value is independent of the path along which $\Delta z \to 0$, then the function $f(z)$ is said to be differentiable (or *analytic*) in z_0. The limit is denoted by $f'(z)$ or $\mathrm{d}f/\mathrm{d}z$, and it is called the derivative of f in z_0. All these definitions are clearly generalisations of similar definitions in real analysis.

It can be shown that for an arbitrary analytic function $w = f(z)$, with $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, the following relations hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \tag{B.7}$$

These are the so-called Cauchy-Riemann relations. They can be derived by requiring that the limit in equation B.6 must give the same result for two perpendicular paths along which $\Delta z \to 0$.

It follows from the Cauchy-Riemann relations (by differentiating one with respect to x, the other with respect to y, and then adding or subtracting the results), that both u and v satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{B.8}$$

Thus, both the real and imaginary parts of any arbitrary analytic function satisfy Laplace's equation. This is the reason why complex analysis is useful for mathematical physics. In fact every solution of Laplace's equation can be considered to be the real (or imaginary) part of an analytic function.

It can easily be shown that functions such as z^2, z^3, z^n, or more in general any polynomial of z, is an analytic function, and that the well-known rules from real analysis also apply to complex variables, for instance

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1} \tag{B.9}$$

Well-known real functions are $\sin(x)$, $\cos(x)$ and $\exp(x)$. It has been found possible to generalise these functions in such a way that they become analytic functions of the complex variable z. The appropriate definitions are

$$\exp(z) = \exp(x)[\cos(y) + \mathrm{i}\sin(y)] \tag{B.10}$$

$$\sin(z) = [\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)]/2\mathrm{i} \tag{B.11}$$

$$\cos(z) = [\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)]/2 \tag{B.12}$$

All familiar rules and formulae holding for the real functions are also valid in complex analysis. For instance

$$\frac{d}{dz} \exp(z) = \exp(z)$$

$$\frac{d}{dz} \sin(z) = \cos(z)$$

$$\frac{d}{dz} \cos(z) = -\sin(z)$$

$$\sin^2(z) + \cos^2(z) = 1$$

It follows from equation B.10, by taking $x = 0$, that

$$\exp(iy) = \cos(y) + i\sin(y)$$

This equality enables to rewrite equation (B.3) for the representation of a complex number in terms of polar coordinates as

$$z = r\exp(i\theta) \tag{B.13}$$

This is a convenient form for multiplication of complex numbers, or for the evaluation of non-integer powers. The angle θ is often called the argument of z

$$\theta = \arg(z) \tag{B.14}$$

and r is the modulus of z

$$r = |z| \tag{B.15}$$

It now follows that an arbitrary power z^α can be written as

$$z^\alpha = r^\alpha \exp(i\alpha\theta) \tag{B.16}$$

The logarithmic function, $w = \ln z$, is defined as the inverse of the exponential function. By using the representation in equation (B.13) one can also write

$$w = \ln z = \ln r + i\theta \tag{B.17}$$

It can also be shown that the properties of the real function $\ln(x)$ have equivalent generalisations in complex analysis, such as

$$\frac{d}{dz} \ln z = \frac{1}{z}$$

Some care is needed in handling logarithmic functions because in general it is not a single-valued function. Two values of θ differing by an integer multiple of 2π, will give the same value for z, but different values for $\ln z$. By resticting the aregument θ to an interval of length 2π (for example $0 \leqslant \theta < 2\pi$) the function can be made single-valued.

Appendix C

Conformal Transformations

An important geometrical application of analytic functions of a complex variable is the conformal transformation from a region in one plane to another. The functional relationship $w = f(z)$ describes, if the function and its inverse are single-valued, a relation between points in the z-plane and points in the w-plane. If in the z-plane a region R is traced, then in the w-plane a region S is traced defined by the function $w = f(z)$. The regions R and S are said to be mappings of each other (see figure C.1).

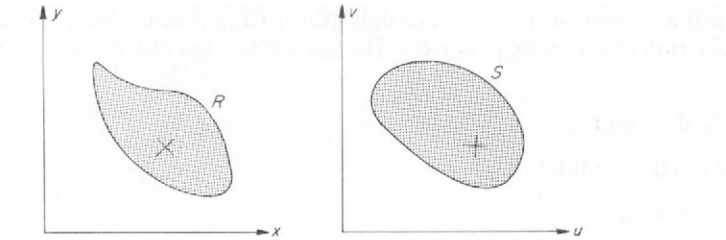

Figure C.1 *Conformal transformation*

If the function $w = f(z)$ is analytic the derivative $\mathrm{d}w/\mathrm{d}z$ is unique, for every direction along which $\Delta z \to 0$ in the z-plane. By writing

$$\frac{\mathrm{d}w}{\mathrm{d}z} \approx \frac{\Delta w}{\Delta z} = \left| \frac{\Delta w}{\Delta z} \right| \exp(\mathrm{i}\theta) \tag{C.1}$$

it now follows that the ratio $|\Delta w|/|\Delta z|$ and the angle θ are unique. This implies that an elementary (infinitely) small figure in the z-plane is transformed into a conformal figure in the w-plane. Mappings by analytic functions are therefore said to be conformal transformations.

The conformal transformation described by a particular function $w = f(z)$ can be found by investigating the mapping of various points or lines in one plane on to the other. Some special care is needed in points where $f'(z) = 0$ or $f'(z) = \infty$. In such points the multiplication factor is zero or infinite, and the argument θ, which represents the rotation from one plane to the other, is undetermined.

The properties of many analytic functions, conformal transformations,

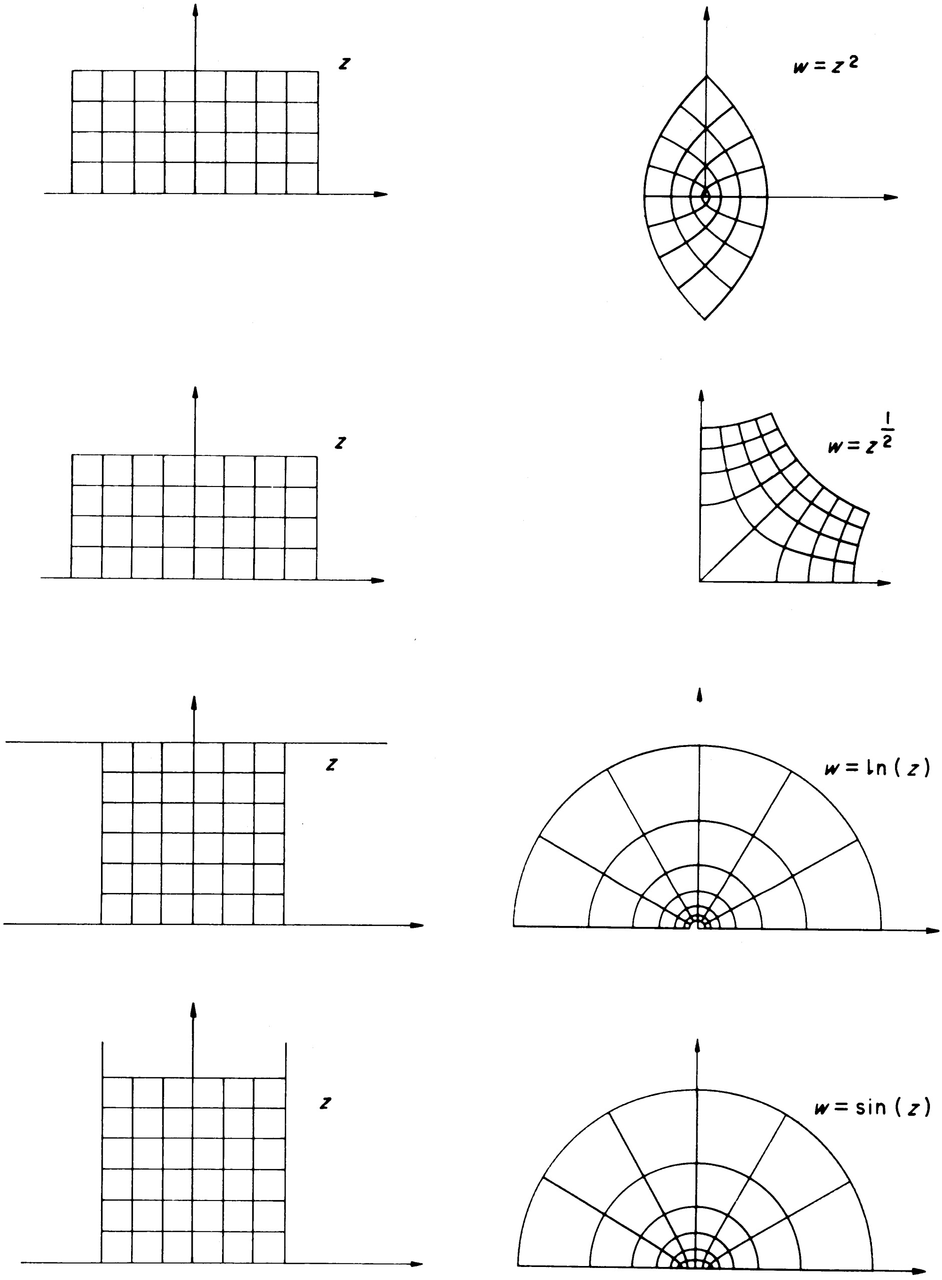

Figure C.2 *Some conformal transformations* have been investigated (see for example, Churchill, 1960; Kober, 1957). A short list of conformal transformations is given in figure C.2. It should be noted that a conformal transformation not only preserves the shape of an elementary figure, but also its orientation. This means that the region to the left of a certain curve in the z-plane is mapped onto a region to the left of the transformed curve in the w-plane.

Schwarz-Christoffel Transformation

A general type of transformation, mapping an arbitrary polygon onto a half-plane is the so-called Schwarz-Christoffel transformation. This mapping function is defined by the formula

$$\frac{dw}{dz} = A(z - x_1)^{-k_1} (z - x_2)^{-k_2} \ldots (z - x_n)^{-k_n} \tag{C.2}$$

where A is a (complex) constant, $x_1, x_2, \ldots, x_n$, are points on the x-axis and $k_1, k_2, \ldots, k_n$ are real constants, to be specified below. The properties of this function can best be examined by considering the argument of dw/dz. It follows from equation C.2 that

$$\arg\left(\frac{dw}{dz}\right) = \arg(A) - k_1 \arg(z - x_1) - k_2 \arg(z - x_2) - \ldots - k_n \arg(z - x_n) \tag{C.3}$$

For real values of z the argument of $z - x_j$ is either 0 or π, and hence $\arg(dw/dz)$ is always constant. If the point z passes one of the points x_j (travelling from the left to the right) the argument of $(z - x_j)$ jumps from the value π to 0, and thus $\arg(dw/dz)$ suddenly increases by $k_j\pi$. This shows that the point w will traverse straight paths, with sudden changes of direction of magnitude πk_j, if z traverses the x-axis. The upper half-plane $\text{Im}\,(z) > 0$ is mapped on to the interior of a polygon, (see figure C.3).

The multiplication factor A in equation C.2 can affect an enlargement of the figure (by its absolute value) and a rotation (by its argument). The integration constant that will appear when integrating dw/dz enables to locate one point of the figure in a prescribed location. Hence the parameters x_j need only to fix the shape of the polygon. This means that three of these parameters

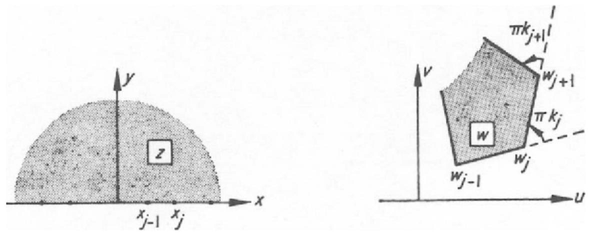

Figure C.3 *Schwarz-Christoffel transformation* can be chosen arbitrarily, because the shape of an n-sided polygon is completely determined by its angles (which are defined by the parameters k_j) and $n - 3$ parameters. In applications it is often convenient to locate one of the points x_j in the origin, one at infinity, and a third one for instance at $x = 1$. It can be shown that the factor $(z - x_j)^{-k_j}$ corresponding to the point $x_j = \infty$ can be omitted from the transformation function (see for example, Churchill, 1960).

The Schwarz-Christoffel transformation has been presented here for the mapping of a polygon onto an upper half-plane. In certain problems it is more convenient to map onto the interior of the unit circle $|\zeta| = 1$. The modification for this transformation can easily be derived from equation C.2 by mapping the upper half-plane Im $(z) > 0$ on to the unit circle $|\zeta| < 1$, by the transformation $\zeta = -(z - \mathrm{i})/(z + \mathrm{i})$. It can be shown (see for example, Churchill, 1960) that this leads to the mapping function

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = A' (\zeta - \mathrm{e}^{\mathrm{i}\theta_1})^{-k_1} (\zeta - \mathrm{e}^{\mathrm{i}\theta_2})^{-k_2} \ldots (\zeta - \mathrm{e}^{\mathrm{i}\theta_n})^{-k_n} \tag{C.4}$$

The meaning of the parameters $k_1, k_2, \ldots, k_n$ is the same as above (πk_j is the change in direction at corner point w_j). The points $\mathrm{e}^{\mathrm{i}\theta_j}$ are the image points of the corner points w_j of the polygon. Again three values of θ_j can be chosen arbitrarily.

Appendix D

Laplace Transforms

The Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$F(s) = \int_0^\infty \exp(-st)\; f(t)\mathrm{d}t \tag{D.1}$$

For many functions $f(t)$ the Laplace transforms have been determined, and collected in tables (Churchill, 1958; Erdelyi *et al.* 1954; Abramowitz and Stegun, 1965). A short table is given below (table D.1). For $f(t) = \exp(-at)$ one obtains for instance

$$F(s) = \int_0^\infty \exp\left[-(s + a)t\right]\,\mathrm{d}t = \frac{1}{s + a} \tag{D.2}$$

The basic property of the Laplace transform is that it transforms the differential operator $\mathrm{d}/\mathrm{d}t$ into multiplication by s. This can be seen as follows, using partial integration

$$\int_0^\infty \exp(-st)\,\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \exp(-st)f(t)\,\Big|_0^\infty$$

$$+\; s\int_0^\infty \exp(-st)f(t)\,\mathrm{d}t = f(0) + sF(s) \tag{D.3}$$

where $f(0)$ is the value of $f(t)$ for $t = 0$, and where it has been assumed that the behaviour of $f(t)$ for $t \to \infty$ is such that, at least for a certain range of values of the parameters s

$$\lim_{t \to \infty} \exp(-st)f(t) = 0$$

The property shown in equation D.3 can be used to solve differential equations. The differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} + 2f = 0 \tag{D.4}$$

with the initial condition $f(0) = 5$ is transformed into

$$(s + 2)\,F - 5 = 0$$

the solution of which is

$$F = \frac{5}{s+2}$$

With equation D.2 one now obtains for the solution of the original differential equation

$$f(t) = 5\exp(-2t) \tag{D.5}$$

The Laplace transform is seen to reduce an ordinary differential equation to an algebraic equation. In the same way a partial differential equation may be reduced to an ordinary differential equation. The success of the method of course depends strongly on the availability of tables of inverse transforms. A straightforward method for the inverse transformation is provided by the complex inversion integral (see for example, Churchill, 1958).

TABLE D.1 *Some Laplace transforms*

$f(t)$	$F(s) = \int_0^\infty \exp(-st)\,f(t)\,dt$
1	$1/s$
t^n	$n!/s^{n+1}$
$\exp(at)$	$1/(s-a)$
$\sin(at)$	$a/(s^2+a^2)$
$\cos(at)$	$s/(s^2+a^2)$
$J_0\,(at)$	$1/(s^2+a^2)^{\frac{1}{2}}$
$J_0\,[2(kt)^{\frac{1}{2}}]$	$\frac{1}{s}\exp(-k/s)$
$\operatorname{erfc}(k/2t^{\frac{1}{2}})$	$\frac{1}{s}\exp(-ks^{\frac{1}{2}})$
$\frac{1}{2t}\exp(-k^2/4t)$	$K_0\,(as^{\frac{1}{2}})$

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