Lecture Notes in Mathematics 1917

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Existence and Regularity Properties of the Integrated Density of States of Random Schrödinger Operators

Springer

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ISBN 978-3-540-72689-0 e-ISBN 978-3-540-72691-3

DOI 10.1007/978-3-540-72691-3

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2007938041

Mathematics Subject Classification (2000): 35J10, 37A30, 35B27, 82D30

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

Random Schrödinger operators are models for the quantum mechanical description of disordered media. The main aim of the analysis of such models is the understanding of the (charge) transport properties of the material. It turns out that many of the properties of interest of random Schrödinger operators are related to a quantity called *integrated density of states*. It measures or 'counts' the number of electron energy levels of the Hamiltonian per unit volume. An alternative name for the integrated density of states is *spectral distribution function* since it is the distribution function of a spectral measure associated to the random family of operators. Many features of this quantity have an intuitive physical interpretation, others play a prominent role in proofs of key theorems. Moreover, the spectral distribution function is an object of study in other fields of mathematics, like differential geometry, group and von Neumann algebras, and homological algebra.

What are the properties of the integrated density of states which have been studied in the literature? It would be hard to give an exhaustive answer, but there are several classes of questions that have drawn the attention of many authors.

The first class is concerned with the definition and construction of the integrated density of states. Can it be expressed as a limit of a sequence of distribution functions associated to the spectra of 'simpler' operators? These operators are usually restrictions of the original Schrödinger operator to some finite volume set. There are various ways how to choose the approximation sequence of operators. Thus another question comes up naturally: Does this choice influence the final outcome, or does one obtain the same distribution function, independently of the approximation procedure? Furthermore, is there a closed formula for the integrated density of states? If there are various such formulas, are some of them better suited for certain applications than others?

Another circle of ideas concerns the continuity properties of the integrated density of states and its set of points of increase: Can one characterise the location and size of the discontinuities? What is the structure of the sets of constancy of the integrated density of states? Since it is a spectral measure

distribution, these questions are intimately related to the spectrum of the random operator. Can one prove quantitative regularity properties of the integrated density of states: Is it log-Hölder, Hölder, or Lipschitz continuous? Does it exhibit even stronger regularity properties like differentiability or analyticity? Is it possible to give upper and lower bounds on the derivative?

Finally, one is interested in the behaviour of the integrated density of states as the energy variable approaches a spectral boundary. The most studied case is the infimum of the spectrum, although the behaviour at very high energies, or at internal spectral edges is of interest, too. Can one give a characteristic law of the asymptotics of the integrated density of states at the boundaries? Is it polynomial, is it exponential? Can one specify or estimate the characteristic exponents?

Of course, the answers to the above questions often depend on the type of random operator one is considering. Thus one my also ask: How do the parameters entering the model influence the abovementioned features? Is there a universal behaviour or some phase transition phenomenon?

We used above the term *random Schrödinger operator* although this terminology refers to just one instance among many types of equivariant Hamiltonians in various geometric settings for which the integrated density of states may be defined. A substantial body of papers is devoted to operators acting on combinatorial graphs, the simplest being the integer lattice $\mathbb{Z}^d$. Others consider operators acting on $L^2(\mathbb{R}^d)$, which includes Schrödinger operators. For models on $\mathbb{R}^d$, one may require a $\mathbb{R}^d$ or a $\mathbb{Z}^d$-equivariance condition. In the latter case, there is still a discrete structure present in the random operator, albeit it acts on continuous space. All settings mentioned so far concern the spaces $\mathbb{Z}^d$ or $\mathbb{R}^d$, and thus Euclidean geometry. Going beyond these, there are interesting related models on covering manifolds, finitely generated groups, as well as combinatorial and metric graphs with a quasi-transitive structure.

In the remainder of the preface we describe briefly the potential audience of the book, the recommended prerequisites, the approach taken to present the material, the selection of topics and the structure of the text.

The aim of the text is to give researchers interested in the subject of random Schrödinger operators an overview of known results and methods. Specialists may find it useful as a guide to further reading. The subject matter of the book draws on various mathematical disciplines. For that reason it was not possible to include all the background material, but the reader can find detailed descriptions of the relevant facts using the references to textbooks and monographs. Thus, the text should be accessible to graduate students who have a working knowledge of selfadjoint operators and quadratic forms, possibly from a course on linear operators in Hilbert space or an advanced functional analysis class. For students without this background any one of the following books is recommended as a reading companion: [18, 19] by Akhiezer and Glazman, [47] by Birman and Solomyak, [110] by Davies, [494] or [495] by Weidmann, or [497] by Werner, the last two references being in German. The reader will find the relevant material also in the treatises [140, 141, 142]

by Dunford and Schwartz, [239] by Kato, or [407, 408, 409, 410] by Reed and Simon. For the reader who wants to know more about the physical background of the models discussed here we recommend the monographs [53, 145, 340, 312, 143] where properties of disordered systems are discussed from the point of view of theoretical physics.

As already mentioned, the integrated density of states can be defined in various geometric settings and for operators with various equivariance types. If one aims at discussing all such models in a text, one could treat them subsequently one by one, or order the text according to various results and properties and discuss each time all models. One could also first develop a general approach which covers all models, and prove theorems on an abstract level. Although all these seem viable options, here we choose a different and more modest way: We consider just one model in detail and refer in remarks to sources in the literature where the proof for other variants may be found. More precisely, we concentrate here on operators on a continuous configuration space with a discrete group structure. The most important example is the alloy-type model, a $\mathbb{Z}^d$-ergodic operator acting on $L^2(\mathbb{R}^d)$, but operators on Riemannian manifolds with non-abelian group actions are also considered.

While most of the relevant aspects of the spectral theory of random Schrödinger operators figure in the text, the presentation is centred around the integrated density of states. A broader view is taken in the monographs [81] by Carmona and Lacroix and [389] by Pastur and Figotin which describe the state of the art at the beginning of the 1990s. There are several other text of a survey nature on the subject from the second half of the 1980s, including the introductory article [247] by Kirsch, a section on random Jacobi matrices in [102] by Cycon, Froese, Kirsch and Simon, and the Lifshitz memorial issue [335]. In recent years there have been two more monographs treating related topics. The theory of Anderson localisation for random Schrödinger operators is exposed in detail in [458] by Stollmann. Many features of the spectral distribution function in the context of geometry, group theory and K-theory are discussed in Lück's book [346].

Let us mention a few recent overview articles which discuss certain aspects of the theory covered only marginally in the present book. A survey of localisation results for one-dimensional random models is provided in [459] by Stolz. A detailed account of $\mathbb{R}^d$-ergodic random Schrödinger operators, which model amorphous media, is given by Leschke, Müller and Warzel in [331]. The present text emphasises the construction of the integrated density of states and its continuity properties, while its asymptotic behaviour at the infimum of the spectrum is discussed only in remarks. An overview of the results devoted to the last mentioned topic can be found in the recent [259] by Kirsch and Metzger. There also spectral properties of random surface models are discussed. We mentioned above that it is possible to introduce the integrated density of states in a general, abstract framework applicable to various types of equivariant operators and geometric settings. Such an approach is taken

for instance in [328], which covers e.g. Hamiltonians on Euclidean space and lattices, on covering manifolds, Delone sets, and on percolation graphs.

Of course, there are many other excellent sources in the literature not mentioned here in the preface. We hope to have them adequately quoted in the main text.

Here is a sketch of the contents and the structure of the book: The introduction explains how one arrives at random Schrödinger operators starting from the quantum mechanical theory of disordered solids. There we also fix frequently used notation, recall basic facts about selfadjoint operators, and define our models. Finally, the introduction explains the relation between spectral and transport properties of Schrödinger operators as well as the notion of spectral fluctuation boundaries.

The second chapter presents two alternative proofs of the approximation of the integrated density of states by its finite volume analoga. One of the approaches is general enough to be applicable to random Schrödinger and Laplace-Beltrami operators on manifolds.

The third chapter explains the relevance of Wegner estimates and regularity properties of the integrated density of states for other aspects of the theory of random Schrödinger operators. A prominent example would be the use of a Wegner bound in the multiscale proof of localisation.

The last two chapters present each a proof of Wegner's estimate for the alloy type or continuum Anderson model on $L^2(\mathbb{R}^d)$. The reason to present two different methods is that each of them has its own advantages when applied to models exhibiting various non-trivial features. We consider alloy type models with long-range or negative correlations, as well as singular and non-monotone dependence on the coupling constants. Several remarks are devoted to similar results for operators on graphs and manifolds mentioned above. Finally, an appendix is devoted to some facts from the theory of the spectral shift function which are used in the main text. More details can be found in the table of contents.

The present text is a revised version of the thesis [486] prepared for the habilitation at the Department of Mathematics of the Technische Universität Chemnitz, which in turn is based on [483]. W. König, P. Stollmann, and S. Teufel have kindly accepted the request of the Department to act as referees of the thesis and I would like to thank them at this occasion. The material presented here draws to a large extent on joint work with colleagues: I have greatly profited from discussions with T. Antunović, D. Borisov, M. Gruber, M. Helm, D. Hundertmark, R. Killip, W. Kirsch, S. Kondej, V. Kostrykin, D. Lenz, P. Müller, S. Nakamura, N. Peyerimhoff, O. Post and P. Stollmann and enjoyed working with them. This work has been made possible through the financial support of the Deutsche Forschungsgemeinschaft. I thank the staff of Springer in charge of the LNM series for their flexibility and efficiency in the course of the preparation of the manuscript.

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September, 2007

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Random Operators

1.1 Physical Background

Random Schrödinger operators are used as models of disordered solids within the framework of quantum mechanics.

A macroscopic solid consists of an order of magnitude of 10^{23} of nuclei and electrons. The resulting Hamiltonian taking into account all interactions is highly complicated. To arrive at a Schrödinger operator which can be studied in some detail one neglects the electron-electron interaction and treats the nuclei in the infinite mass approximation. Thus one arrives at an one-electron Schrödinger operator with an external potential due to the electric forces between the electron and the nuclei, which are assumed to be fixed in space.

In the case that the nuclei are arranged periodically on a lattice, the potential energy of the electron is a periodic function of the space variable.

On the other hand, the electron could be moving in an amorphous medium, in which case there is no large group of symmetries of the Hamiltonian. However, from the physical point of view it is reasonable to assume that the local structure of the medium will be translation invariant *on average*. This means that we consider the potential which the electron experiences as a particular realisation of a random process and assume stationarity with respect to some group of translations. Moreover, physical intuition suggests to assume that the local properties of the medium in two regions far apart (on the microscopic scale) are approximately independent from each other. Therefore the stochastic process describing the potential should have a correlation function which decays to zero, or — more generally — should be ergodic.

There are interesting models which lie between the two extreme cases of lattice-periodic and amorphous media. They still have an underlying lattice structure which is, however, modified by disorder. Probably the best studied Hamiltonian with this properties is the alloy type model. We leave its precise definition for the next section and introduce here a special case on the intuitive level. Consider first the potential

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} u_k(\omega, x)$$

Each k corresponds to a nucleus sitting on a lattice point. The function $u_k(\omega,\cdot)$ describes the atomic or nuclear potential at the site k and depends on the random parameter ω which models the different realisations of the configuration of the nuclei. If there is only one type of atom present, which has a spherically symmetric potential, all the $u_k(\omega,\cdot)$ are the same, and V_ω is periodic. Now assume that there are two kinds a and b of atoms present, which have spherically symmetric atomic potentials of the same shape, but which differ in their nuclear charge numbers q_a and q_b.

In this case the potential looks like

$$V_\omega(x) := \sum_{k \text{ occupied by } a} q_a\, u(x-k) + \sum_{k \text{ occupied by } b} q_b\, u(x-k)$$

If the two sorts of atoms are arranged on the lattice in a regular pattern, this again gives rise to a periodic potential.

However, there are physically interesting situations (e.g. binary alloys) where the type of atom sitting on site k is random, for example obeying the law

$$\mathbb{P}\{k \text{ is occupied by atom } a\} = P, \quad \mathbb{P}\{k \text{ is occupied by atom } b\} = 1 - P$$

with some $P \in]0,1[$. Here $\mathbb{P}\{\dots\}$ denotes the probability of an event. If we furthermore assume that the above probabilities are independent at each site and the parameter P is the same for all k, we arrive at the continuum *Bernoulli-Anderson potential*

$$V_\omega(x) = \sum_k q_k(\omega)\, u(x-k)$$

Here $q_k(\omega) \in \{q_a, q_b\}, k \in \mathbb{Z}^d$ denotes a collection of independent, identically distributed Bernoulli random variables and u is some atomic potential.

This model is a prototype which has motivated much research in the physics and mathematics literature, a part of which we will review in the present work.

1.2 Model and Notation

We introduce here, respectively recall, basic notions on L^p spaces, selfadjoint operators in general and Schrödinger operators in particular, and specify the model Hamiltonian which will be studied throughout the text. We suppose that the reader is acquainted with the theory of selfadjoint operators in Hilbert space. In the Preface one can find a list of monographs which provide the necessary background of this theory.

Let us start with some mathematical notation. The symbols $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ denote the set of reals, the set of integers, the set of natural numbers, and the set of non-negative integers, respectively. For a set $A \subset B$ we denote by $A^c := B \setminus A$ its complement. A measurable subset of $\mathbb{R}^d$ will be often denoted by Λ, and if there is a sequence of such sets its members will be denoted $\Lambda_1, \ldots, \Lambda_l, \ldots$. The symbol $|\Lambda|$ is used for the Lebesgue measure of Λ. We write $|x|$ for the norm of $x \in \mathbb{R}^d$, while the norm of a vector f in a function space is denoted by $\|f\|$.

The Hilbert space of (equivalence classes of) measurable functions on Λ which are square integrable with respect to Lebesgue measure is denoted by $L^2(\Lambda)$. Similarly, $L^p(\Lambda)$ with $p > 0$ stands for the Banach space of measurable functions f such that $|f|^p$ is integrable, while $L^\infty(\Lambda)$ is the set of measurable functions which are essentially bounded with respect to Lebesgue measure. The space of sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $|a_n|^p$ is summable is denoted by $\ell^p(\mathbb{N})$. Note that the case $p \in]0, 1[$ is included in our notation. In our context we will often choose the exponent p dependent on the dimension of the configuration space. In the following we denote by $p(d)$ any number in $[1, \infty[$ which satisfies

$$p(d) \begin{cases} \geq 2 & \text{if } d \leq 3, \\ > d/2 & \text{if } d \geq 4 \end{cases} \tag{1.1}$$

For $\Lambda \subset \mathbb{R}^d$ open, the symbols $C(\Lambda), C^\infty(\Lambda)$ stand for the continuous, respectively smooth, functions on Λ. The subscript $_c$ in $C_c(\Lambda), C_c^\infty(\Lambda), L_c^p(\Lambda)$ means that we consider only those functions which have compact support in Λ. In the sequel we will often consider potentials from the class of functions which are *uniformly locally* in L^p. More precisely, f is in the set of uniformly locally L^p-functions, denoted by $L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$, if and only if there is a constant C such that for each $y \in \mathbb{R}^d$

$$\int_{|x-y|<1} |f(x)|^p \, dx \leq C$$

The infimum over all such constants C equals by definition $\|f\|^p_{p,\,\mathrm{unif,loc}}$. Finally, we introduce Sobolev spaces $W^{k,2}$ of order k. For $\Lambda \subset \mathbb{R}^d$ open, a function $f \in L^2(\Lambda)$ is in $W^{k,2}(\Lambda)$ if all its partial derivatives up to order k exist in the sense of distributions and are elements of $L^2(\Lambda)$. Obviously, $C_c^\infty(\Lambda)$ is a subset of $W^{k,2}(\Lambda)$. Its closure (with respect to the canonical norm of $W^{k,2}(\Lambda)$) is denoted by $W_0^{k,2}(\Lambda)$.

Let Δ denote the Laplacian on $\mathbb{R}^d$. If we choose its *operator domain* $\mathcal{D}(\Delta)$ to be the Sobolev space $W^{2,2}(\mathbb{R}^d)$, it becomes a selfadjoint operator. The restriction of Δ to an open true subset $\Lambda \subset \mathbb{R}^d$ becomes selfadjoint only if we specify appropriate boundary conditions (b.c.). Dirichlet b.c. are defined in Remark 2.2.3. For the definition of Neumann and periodic b.c. see for instance [408].

Let A, B be two densely defined symmetric operators on a Hilbert space $\mathcal{H}$, whose norm we denote by $\|\cdot\|$. The associated scalar product we denoted by $<.,.>$. We say that B is *(relatively) A-bounded* if the domains obey the inclusion $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there are finite constants a and c_a such that for all $f \in \mathcal{D}(A)$

$$\|Bf\| \leq a\|Af\| + c_a\|f\| \tag{1.2}$$

The infimum over all a such that the estimate holds with some c_a is called *relative bound* (of B with respect to A). If B is A-bounded with relative bound zero, we call it *infinitesimally A-bounded*. Let A be selfadjoint, and B symmetric and relatively A-bounded with relative bound smaller than one. Then the operator sum $A + B$ on the domain $\mathcal{D}(A)$ is selfadjoint by the Kato-Rellich Theorem, see e.g. Sect. X.2 in [407]. We will apply this result to the sum of the negative Laplacian and a potential. A multiplication operator by a function $V \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$ is infinitesimally Δ-bounded if $p = p(d)$, cf. Theorem XIII.96 in [408]. Moreover, the constant c_a in (1.2) depends only on $\|V\|_{p,\,\mathrm{unif,loc}}$. Thus the sum $H := -\Delta + V$ is selfadjoint on $W^{2,2}(\mathbb{R}^d)$. In a similar way it is possible to introduce the notion of *relative form-boundedness*. Here we consider A which is selfadjoint and bounded below. Denote by $\mathcal{D}(Q_A)$ its quadratic form domain. A symmetric operator B is said to be (relatively) *A-form bounded* if the quadratic form domains obey the inclusion $\mathcal{D}(Q_A) \subset \mathcal{D}(Q_B)$ and there are finite constants a, C_a such that

$$|\langle\phi, B\phi\rangle| \leq a\langle\phi, A\phi\rangle + C_a\langle\phi, \phi\rangle \tag{1.3}$$

The *relative A-form bound* of B is the infimum of all a which satisfy (1.3). See Sect. VI.1.7 in [239] for more details.

Before we introduce random operators we want to fix notation concerning some terminology in probability theory. The triple $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ stands for a probability space with associated σ-algebra and probability measure, while $\mathbb{E}\{\dots\}$ denotes the expectation value with respect to $\mathbb{P}$. A collection $T_j \colon \Omega \to \Omega, j \in \mathcal{J}$ of measure preserving transformations is called *ergodic* if all measurable sets in Ω which are invariant under the action of all $T_j, j \in \mathcal{J}$ have measure zero or one.

Definition 1.2.1. *Let $p = p(d)$ be as in (1.1), $u \in L^p_c(\mathbb{R}^d)$ and $q_k \colon \Omega \to \mathbb{R}$, $k \in \mathbb{Z}^d$ a sequence of bounded, independent, identically distributed random variables, called* coupling constants*. Then the family of multiplication operators given by the stochastic process*

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} q_k(\omega)\, u(x-k) \tag{1.4}$$

is called alloy type potential*. The function u is called* single site potential*. Let $H_0 := -\Delta + V_{\mathrm{per}}$ be a periodic Schrödinger operator with $V_{\mathrm{per}} \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$. The family of operators*

$$H_\omega := H_0 + V_\omega, \quad \omega \in \Omega \tag{1.5}$$

is called alloy type model.

The distribution measure of the random variable q_0 will be called single site distribution *and denoted by μ. If not stated otherwise, in the sequel we assume that μ is absolutely continuous with respect to the Lebesgue measure and has a bounded density. The density function is denoted by f.*

Due to our assumptions on the boundedness of the coupling constants, for each $a > 0$ there is a constant c_a such that for all ω and all $\psi \in \mathcal{D}(\Delta)$

$$\|V_\omega \psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|, \quad \|V_{\mathrm{per}}\psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|$$

In particular H_0 and all H_ω are selfadjoint on the operator domain of Δ. It will be of importance to us that the constant c_a may be chosen independently of the random parameter ω.

Remark 1.2.2. (a) In several sections we study Hamiltonians as in Definition 1.2.1, but where some of the hypotheses on the single site potential or the coupling constants are relaxed. More precisely, we will consider single site potentials with non-compact support and coupling constants which are unbounded, correlated, or do not have a bounded density.

(b) If the coupling constants are not bounded, one has to impose some moment condition to make sure that the alloy type model still makes sense. The main difference (to the bounded case) is that for ω in a set $\Omega' \subset \Omega$ of full measure the operator H_ω will be (essentially) selfadjoint, however this may fail to hold for ω in the complement $\Omega \setminus \Omega'$. See for example [255, 256, 258] for more details.

(c) There is a group of measure preserving transformations $T_k, k \in \mathbb{Z}^d$ on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that (1.4) obeys

$$V_\omega(x-k) = V_{T_k\omega}(x)$$

In other words, the stochastic process $V\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is *stationary* with respect to translations by vectors in $\mathbb{Z}^d$. Moreover, the group $T_k, k \in \mathbb{Z}^d$ acts ergodically on Ω, therefore we call V an $\mathbb{Z}^d$*-ergodic potential*.

To see that the above statements are true we pass over to the canonical probability space $\Omega = \times_{k\in\mathbb{Z}^d} \mathbb{R}$, equipped with the product measure $\mathbb{P} := \otimes_{k\in\mathbb{Z}^d} \mu$. Now the stochastic process $\{\pi_k\}_{k\in\mathbb{Z}^d}$, defined by $\pi_k(\omega) = \omega_k$ for all $k \in \mathbb{Z}^d$, has the same finite dimensional distributions as $\{q_k\}_k$. It is easily seen that the transformations $(T_k(\omega))_j := \omega_{j-k}$ are measure preserving and that the group $(T_k)_{k\in\mathbb{Z}^d}$ acts ergodically on Ω. See Sect. 3 in [247] or Sect. I.1 in [389] for more details.

Using the stochastic process $\{\pi_k\}_k$ the alloy type potential can be written as

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k) \tag{1.6}$$

We will use notations (1.4) and (1.6) without distinction in the sequel.

Abstracting the properties of stationarity and ergodicity we define general random potentials and operators with a $\mathbb{Z}^d$-ergodic structure.

Definition 1.2.3. *Let $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stochastic process such that for almost all $\omega \in \Omega$ the realisation of the potential obeys $V_\omega \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$, $p = p(d)$ and additionally $\mathbb{E}\{\|V_\omega \chi_\Lambda\|_p^p\} < \infty$, where Λ is a unit cube. Let $T_k, k \in \mathbb{Z}^d$ be a group of measure preserving transformations acting ergodically on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that*

$$V_\omega(x-k) = V_{T_k\omega}(x)$$

*Then we call $\{V_\omega\}_\omega$ a ($\mathbb{Z}^d$-*ergodic*)* random potential *and $\{H_\omega\}_\omega$ with $H_\omega = -\Delta + V_\omega$ a ($\mathbb{Z}^d$-*ergodic*)* random operator.

The restriction of H_ω to an open subset Λ will be denoted by H_ω^Λ if we impose Dirichlet boundary conditions and by $H_\omega^{\Lambda,N}$ in the case of Neumann b.c. While we will be mainly concerned with $\mathbb{Z}^d$-ergodic operators we will give some comments as asides on their counterparts which are ergodic with respect to the group $\mathbb{R}^d$. The overview [331] is devoted to such models that model amorphous media. Insight in the research on almost-periodic operators can be obtained for instance in the papers [431, 432, 32, 40], the literature quoted on page 9, the monographs [102, 389], and the references therein.

Remark 1.2.4. All $\mathbb{Z}^d$-ergodic potentials can be represented in a form which resembles alloy type potentials. In fact, for such $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ there exists a sequence $f_k, k \in \mathbb{Z}^d$ of random variables on Ω taking values in the separable Banach space $L^p(\mathbb{R}^d)$ such that V can be written as

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} f_k(\omega, x-k). \tag{1.7}$$

This representation is of interest because it ensures that after passing to an equivalent probability space and stochastic process one may assume that the sigma algebra on Ω is countably generated. See [245] and Remark 2.8 in [328] for more information.

1.3 Transport Properties and Spectral Types

The main interest in the study of random operators is to understand the transport properties of the materials they model. In the particular case of the quantum mechanical Hamiltonian of an electron in a disordered solid the electric conductance properties are the principal object of interest.

The Hamiltonian governs the equation of motion, i.e. the time dependent Schrödinger equation

$$\frac{\partial \psi(t)}{\partial t} = -iH_\omega \psi(t) \tag{1.8}$$

The time evolution of the vector $\psi(t)$ in Hilbert space describes the movement of the electron. Since we chose the space representation in the Schrödinger picture, we can think of $\psi(t)$ as a wave packet which evolves in time. The square of its absolute value $|\psi(t,\cdot)|^2 \in L^1(\mathbb{R}^d)$ is a probability density. More precisely, $\int_A |\psi(t,x)|^2 dx$ is the probability to find the electron in the set $A \subset \mathbb{R}^d$ at time t. We will be only concerned with Hamiltonians H_ω which are time independent. In this case the solution to the equation (1.8) is given by $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi(0)$ where $\psi(0)$ denotes the initial condition at time 0.

For a given initial state $\psi(0)$ supported in a compact set $A \subset \mathbb{R}^d$ one would like to know whether for large times the function $\psi(t)$ stays (essentially) supported near A, or moves away to infinity. In the first case one speaks of a *bound state*, since it remains localised near its original support for all times. The other extreme case would be that $\psi(t)$ leaves any compact region in $\mathbb{R}^d$ (and never comes back) as time goes to infinity. Such a state is called a *scattering* or *extended state*. By the RAGE theorem it is possible to relate the dynamical properties of states just described to the spectral properties of the Hamiltonian. Roughly speaking, bound states correspond to pure point spectrum and scattering states to (absolutely) continuous spectrum.

For a more precise formulation we assume that the Schrödinger operator H satisfies the following local compactness property: If χ_B is the characteristic function of an arbitrary ball $B = B_R(x) \subset \mathbb{R}^d$ and $P(I)$ the spectral projection of H associated to a bounded interval $I \subset \mathbb{R}$, then the operator $\chi_R P(I)$ is compact. Under this condition a vector ψ is in the subspace associated to the continuous spectrum of H if and only if for arbitrary $B_R(x)$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} dt \int_{B_R(x)} |\psi(t,x)|^2 dx = 0 \tag{1.9}$$

Here we used the notation $\psi(t,x) = (\mathrm{e}^{-\mathrm{i}tH}\psi)(x)$. A vector ψ satisfying (1.9) is called a scattering state in time mean. Under the same local compactness assumption, a vector ψ is in the subspace associated to the pure point spectrum of H if and only if for any $\varepsilon > 0$ these exists a radius $R = R(\psi,\varepsilon)$ such that

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^d \setminus B_R(0)} |\psi(t,x)|^2 \leq \varepsilon \tag{1.10}$$

For a broader discussion consult for instance [409, 102, 458, 496].

The above relation to the dynamical properties of states motivates the systematic study of spectral features of the Schrödinger operators appearing in the time evolution equation (1.8). If a random Schrödinger operator exhibits almost surely only pure point spectrum in an energy region one speaks of *Anderson* or *spectral localisation*. The name goes back to Anderson's seminal work [20]. This property has been established for a variety of random models. In most of those cases one can additionally prove that the corresponding eigenfunctions decay exponentially in configuration space, a phenomenon

called *exponential (spectral) localisation*. The situation is different for random potentials with long range correlations, where sometimes only power-law decay of the eigenfunctions has been established [263, 170, 500].

If an energy interval contains almost surely only pure point spectrum, we call it *localisation interval*. An eigenfunction of H_ω which decays exponentially is called an *exponentially localised eigenstate*. The region or point in space where the localised state has its highest amplitude will be called *localisation centre* (we will not need a mathematically precise definition of this notion).

However, it turns out that the spectrum provides only a rough view on the dynamical properties of the quantum mechanical system. A more detailed understanding can be obtained by studying the time evolution of the moments of the position operators. This led to a formulation of several criteria of *dynamical localisation*. One possible characterisation of this phenomenon, namely *strong dynamical localisation in Hilbert-Schmidt* topology means that for all $q > 0$

$$\mathbb{E}\left\{\sup_{\|f\|_\infty \leq 1} \left\| |X|^{q/2} f(H_\omega) P_\omega(I) \chi_K \right\|_{HS}^2 \right\} < \infty \tag{1.11}$$

Here $P_\omega(I)$ denotes the spectral projection onto the energy interval I associated to the operator H_ω, $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm, $K \subset \mathbb{R}^d$ is any compact set, and $|X|$ denotes the operator of multiplication with the function $|x|$. For the interpretation of (1.11) as non-spreading of wave-packets one chooses $f(y) = \mathrm{e}^{-\mathrm{i}ty}$. Dynamical localisation (1.11) implies in particular that the random Hamiltonian H_ω exhibits spectral localisation in I. In [414] it was first pointed out that, in general, it is important to distinguish between dynamical and spectral localisation. The derivation of various forms of dynamical localisation using the multiscale analysis can be found in [183, 107, 186, 187, 188, 190], while a proof of the same fact based on the fractional moment method is the content of the papers [6, 60]. Actually, dynamical localisation has first been proven for discrete Schrödinger operators using the latter method in [9, 4, 7, 11]. In [414, 111, 235] examples are discussed where spectral localisation occurs, but certain dynamical criteria for localisation are not satisfied.

For the operators discussed in the present volume these distinctions are not crucial. In the case of the alloy type model, to which we devote most attention, spectral and dynamical localisation coincide, cf. [107, 190]. In the sequel we mean by localisation that the considered operator exhibits in a certain energy interval only pure point spectrum, and that the corresponding eigenfunctions decay sufficiently fast.

Since we are dealing not just with a single Hamiltonian, but with a whole family of them, we have to say something on how the spectral properties depend on the parameter ω describing the randomness: many properties of the spectrum of an operator pertaining to the family $\{H_\omega\}_\omega$ hold almost surely, i.e. for ω in a set such that its complement has measure zero in Ω. This is at

least true for the properties we discuss in the present work. There are other quantities which are highly dependent on the randomness, like eigenfunctions of H_ω or the set of its eigenvalues (not its closure!).

We shortly describe what kinds of spectral types one expects from the physical point of view for random Schrödinger operators, say of alloy type. In case there are rigorous results which have confirmed this intuition we quote the reference.

In one space dimension the spectrum is pure point for all energies almost surely. A rigorous proof of this statement was first given in [202] for random potentials generated by Brownian motion on a compact manifold. Later the theorem was extended to Anderson and alloy type models with continuous [306] and Bernoulli randomness [80, 429, 104]. In general one could say that the understanding of the localisation phenomenon is more advanced in the one-dimensional situation than in the higher-dimensional one. This is due to a variety of ideas and methods which are specific — or at least more powerful — for one-dimensional operators, like Kotani theory, subordinacy theory, transfer matrices, Lyapunov exponents, the Prüfer transformation, two parameter spectral averaging, reflection coefficients, inverse spectral theory and scattering theory. These aspects are discussed among others in [303, 252, 304, 444, 306, 305, 198, 197, 365, 73, 271, 296, 111, 445, 459, 74, 105, 106] and surveyed in [460].

In three or more dimensions it is expected that the spectrum is pure point near the boundaries of the spectrum while in the interior it is purely absolutely continuous. In the latter case one speaks also of an energy region with *delocalised* states. However, for alloy type Hamiltonians in Euclidean space the proof of delocalisation is still open. For several different types of random operators results on existence of absolutely continuous spectrum have been obtained: for random operators on trees in [276, 277, 13, 14, 15, 16, 172], for alloy type models with decaying randomness in [315, 253, 219, 316, 250, 317, 171], and for certain one-dimensional quasi-periodic operators in [114, 149, 58, 161, 162, 163, 31]. It is expected that there is a sharp transition between energies with localised and those with delocalised spectrum. The threshold energy is called *mobility edge*. For partial results see [230, 253, 190, 161]. There are also related results in terms of the dynamical properties of the time evolution of states, see [111, 235, 190, 192].

The literature on the existence of pure point spectrum near spectral boundaries is extensive. We discuss it in more detail in Sect. 3.2.

How large the intervals with point or continuous spectrum are, depends on the disorder present in the model. For instance, in (1.5) one could introduce a global coupling constant λ in front of the potential

$$H_\omega = H_0 + \lambda V_\omega$$

Now large λ means large disorder, small λ small disorder. The larger the disorder, the larger is the portion of the spectrum which contains localised states. For a quantitative formulation of this intuition see for instance [491, 287, 286,

187, 188]. Similarly, as the disorder increases the exponentially localised states decay faster, i.e. the localisation length decreases, cf. e.g. [491, 423, 424]. For other types of random Schrödinger operators there are similar ways to introduce a disorder parameter.

1.4 Fluctuation Boundaries of the Spectrum

The phenomenon that localised states emerge at the edges of the spectrum can be understood in terms of the so-called *fluctuation boundaries*. These are the regions of the spectrum which correspond to extremely rare configurations of the potential. Consequently, the *density of states*, which measures the concentration of the spectrum (see the next two chapters for a precise definition) is very thin in this region. Under certain assumptions it can be shown that localisation occurs precisely in those energy regions where the spectrum is sufficiently sparse. See e.g. [190] for a rigorous formulation.

The characteristic behaviour of the integrated density of states near fluctuation boundaries of the spectrum has been first understood on physical grounds by I. M. Lifšic in [336, 337, 338]. For this reason the tails of the integrated density of states at these boundaries bear the name *Lifshitz-asymptotics* or *Lifshitz-tails*. To explain this behaviour, let us denote by E_0 the minimum of the spectrum of a lower-bounded random operator, which is by ergodicity independent of the realisation almost surely. Then the integrated density of states N behaves for energies $E > E_0$ near E_0 like

$$N(E) \sim \exp(-const(E - E_0)^{-d/2}).$$

The most precise bounds for this asymptotics have been obtained for random Schrödinger operators with a potential generated by impurities which are distributed randomly in space according to a Poisson process, see e. g. [128, 387, 379, 465, 467]. Similar results hold for a discrete relative of this operator, namely the *Anderson model* on $\ell^2(\mathbb{Z}^d)$, see e.g. [129, 23, 284, 52, 360, 476]. The reason why these models are amenable to a very precise analysis is the applicability of Brownian motion, respectively random walk techniques and Feynman-Kac functionals. For several other types of random operators a weaker form

$$\lim_{E \searrow E_0} \frac{\log|\log(N(E) - N(E_0))|}{|\log(E - E_0)|} = \frac{d}{2} \tag{1.12}$$

of the exponential law has been established using estimates based on inequalities by Thirring [473] and Temple [472]. In particular, this method can be applied to show that the asymptotics (1.12) holds for the Anderson model on $\ell^2(\mathbb{Z}^d)$ and the alloy type model on $L^2(\mathbb{R}^d)$. This strategy of proof was pursued in the 1980s in several papers [257, 361, 440, 262, 362, 363, 441].

Further works devoted the study of the asymptotics near fluctuation boundaries include [385, 386, 302, 178, 180, 179, 466, 364, 283, 456, 288,

290, 291, 287, 285, 220, 269, 251, 270, 260, 266, 487, 371, 25, 24]. At the end of Sect. 5.7 one can find references to results on Lifshitz tails for random Schrödinger operators with magnetic fields.

The existence of localised states for random Schrödinger operators is in sharp contrast to the features of periodic operators. Indeed, for operators with periodic potential, satisfying some mild regularity assumptions, it is known that the spectrum is purely absolutely continuous, [474, 49, 450, 427, 428, 318]. This difference might seem somewhat surprising, given the similarity of the structure of an alloy type and a periodic operator.

When looking at the integrated density of states one also sees a quite different behaviour for the above discussed random and periodic models. Namely, for periodic Schrödinger operators and their discrete analogues the low energy asymptotics of the integrated density of states is governed by the so-called *van Hove singularities*. This means that the integrated density of states vanishes polynomially when the energy variable approaches the bottom of the spectrum, in the sense that

$$\lim_{E \searrow E_0} \frac{\log N(E)}{\log(E - E_0)} = \frac{d}{2}. \tag{1.13}$$

which is in sharp contrast to the Lifshitz tail behaviour outlined above.

2

Existence of the Integrated Density of States

Intuitively, the integrated density of states (IDS) measures how many electron energy levels can be found below a given energy per unit volume of a solid. An alternative name for this quantity is spectral distribution function. It can be used to calculate the free energy and hence all basic thermodynamic quantities of the corresponding non-interacting many-particle system.

To define the IDS mathematically one uses an exhaustion procedure. More precisely, one takes an increasing sequence Λ_l of open subsets of $\mathbb{R}^d$ such that each Λ_l has finite volume and $\bigcup_l \Lambda_l = \mathbb{R}^d$. Then the operator H_ω^l, which is the restriction of H_ω to Λ_l with Dirichlet boundary conditions, is selfadjoint, bounded below and its spectrum consists of discrete eigenvalues $\lambda_1(H_\omega^l) \leq \lambda_2(H_\omega^l) \leq \cdots \leq \lambda_n(H_\omega^l) \to \infty$. Here $\lambda_n = \lambda_{n+1}$ means that the eigenvalue is degenerate and we take this into account in the enumeration.

The *normalised eigenvalue counting function* or *finite volume integrated density of states* N_ω^l is defined as

$$N_\omega^l(E) := \frac{\#\{n|\, \lambda_n(H_\omega^l) < E\}}{|\Lambda_l|} \tag{2.1}$$

The numerator can equally well be expressed using the trace of the spectral projection $P_\omega^l(I)$ associated to the operator H_ω^l and an energy interval I, namely

$$\#\{n|\, \lambda_n(H_\omega^l) < E\} = \mathrm{Tr}\Big[P_\omega^l\big(]-\infty, E[\big)\Big]$$

Note that $N_\omega^l \colon \mathbb{R} \to [0, \infty[$ is a *distribution function* of a point measure for all $l \in \mathbb{N}$, i.e. $N_\omega^l(E) = \nu_\omega^l(]-\infty, E[)$. Here ν_ω^l is the *finite volume density of states measure* defined by

$$\nu_\omega^l(I) := |\Lambda_l|^{-1}\, \#\{n|\, \lambda_n(H_\omega^l) \in I\}$$

By definition, a distribution function is non-negative, left-continuous and non-decreasing. In particular, it has at most countably many points of discontinuity.

Under specific additional conditions on the random operator and the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ one can prove that

(i) For almost all $\omega \in \Omega$ the sequence N_ω^l converges to a distribution function N_ω as l goes to infinity. This means that we have $N_\omega^l(E) \to N_\omega(E)$ for all continuity points E of the limit distribution N_ω.

(ii) For almost all $\omega \in \Omega$ the distribution functions N_ω coincide, i.e. there is an ω-independent distribution function N such that $N = N_\omega$ for almost all ω. This function N is called the *integrated density of states*. Note that its independence of ω is not due to an explicit integration over the probability space Ω, but only to the exhaustion procedure. This is the reason why the IDS is called *self-averaging*.

(iii) In most cases there is a formula for the IDS as an expectation value of a trace per unit volume of a spectral projection. For $\mathbb{Z}^d$-ergodic operators it reads

$$N(E) := \mathbb{E}\left\{\mathrm{Tr}\big[\chi_\Lambda P_\omega(]-\infty, E[)\big]\right\} \tag{2.2}$$

Here Λ denotes the unit box $]0,1[^d$, which is the periodicity cell of the lattice $\mathbb{Z}^d$. Actually, one could choose certain other functions instead of χ_Λ, yielding all the same result, cf. Formula (2.15). The equality (2.2) holds for $\mathbb{R}^d$-ergodic operators, too. It is sometimes called *Pastur-Shubin trace formula*.

In the following we prove the properties of the IDS just mentioned by two methods. In Sects. 2.2–2.6 a detailed proof is given using the Laplace transforms of the distribution functions N_ω^l, while Sect. 2.7 is devoted to an alternative method of proof. It uses Dirichlet-Neumann bracketing estimates for Schrödinger operators, which carry over to the corresponding eigenvalue counting functions. These are thus super- or subadditive stochastic processes to which an ergodic theorem can be applied.

Actually the proof using Laplace transforms will apply to more general situations than discussed so far, namely to more general geometries than the Euclidean one. To be precise, we will consider random Schrödinger operators on Riemannian covering manifolds, where both the potential and the metric may depend on the randomness. This includes random Laplace-Beltrami operators.

We follow the presentation and proofs in [392, 327]. The general strategy we use was developed by Pastur and Shubin in [384] and [431] for random and almost-periodic operators in Euclidean space. A particular idea of this approach is to prove the convergence of the Laplace transforms $\mathcal{L}_\omega^l$ of the normalised finite volume eigenvalue counting functions N_ω^l instead of proving the convergence of N_ω^l directly. This is actually the main difference to the second approach we outline in Sect. 2.7, which is taken from [254]. The Pastur-Shubin strategy seems to be better suited for geometries with an underlying group structure which is non-abelian.

Indeed, one of the differences between random operators on manifolds and those on $\mathbb{R}^d$ is that the operator is equivariant with respect to a group

which does not need to be commutative. This means that one has to use a non-abelian ergodic theorem to derive the convergence of the distribution functions N_ω^l or, alternatively, of their Laplace transforms $\mathcal{L}_\omega^l$. This imposes some restriction on the strategy of the proof since ergodic theorems which apply to non-abelian groups need more restrictive assumptions than their counterparts for commutative groups, cf. also Remark 2.6.2. For processes which are not additive, but only super- or subadditive, there is a non-abelian maximal ergodic theorem at disposal (cf. 6.4.1 Theorem in [313]) but so far no pointwise theorem. This is also the reason why the Dirichlet-Neumann bracketing approach of Sect. 2.7 does not seem applicable to random operators living on a covering manifold with non-abelian deck-transformation group (covering transformation group).

2.1 Schrödinger Operators on Manifolds: Motivation

In this chapter we study the IDS of random Schrödinger operators on manifolds. Let us first explain the physical motivation for this task.

Consider a particle or a system of particles which are constrained to a sub-manifold of the ambient (configuration) space. The classical and quantum Hamiltonians for such systems have been studied e.g. in [367, 173] (see also the references therein). To arrive at an effective Hamiltonian describing the constrained motion on the sub-manifold, a limiting procedure is used: a (sequence of) confining high-barrier potential(s) is added to the Hamiltonian defined on the ambient space to restrict the particle (system) to the sub-manifold. In [367, 173] one can find a discussion of the similarities and differences between the obtained effective quantum Hamiltonian and its classical analogue.

A important feature of the effective quantum Hamiltonian is the appearance of a so-called *extra-potential* depending on the extrinsic curvature of the sub-manifold and the curvature of the ambient space. This means that even if we disregard external electric forces the relevant quantum mechanical Hamiltonian of the constrained system is not the pure Laplacian but contains (in general) a potential energy term. This fact explains the existence of curvature-induced bound states in quantum waveguides and layers, see [157, 138, 343, 139] and the references therein.

As is mentioned in [367], the study of effective Hamiltonians of constrained systems is motivated by specific physical applications. They include stiff molecular bonds in (clusters of) rigid molecules and molecular systems evolving along reaction paths. From the point of view of the present work quantum wires, wave guides and layers are particularly interesting physical examples. Indeed, for these models (in contrast to quantum dots) at least one dimension of the constraint sub-manifold is of macroscopic size. Moreover, it is natural to assume that the resulting Hamiltonian exhibits some form of translation invariance in the macroscopic direction. E.g. it may be periodic, quasi-periodic or — in the case of a random model — stationary.

For random quantum waveguides and layers the existence of dense point spectrum is expected, cf. the discussion of localisation in Sect. 1.3. Indeed, for a specific type of random waveguide embedded in the Euclidean plane this has been rigorously proven in [274, 275]. The question of spectral localisation due to random geometries has been raised already in [109]. There the behaviour of Laplace-Beltrami operators under non-smooth perturbations of the metric is studied.

While the motivations presented above stem from solid state physics a further stimulus to study the IDS comes from within mathematics itself: For various geometries with a group action it makes sense to define translation invariant or periodic operators. This applies to covering manifolds, Cayley graphs and more generally quasi-transitive graphs, as well as for CW-complexes. These carry naturally defined Laplace operators on functions and more generally on p-forms. Related objects are magnetic Laplacians and Schrödinger operators, for which it is also makes sense to formulate an equivariance condition.

Here, by the term periodic we mean the property that there is a subgroup of the automorphism group of the geometric space such that the operator is invariant under conjugation with unitary transformations which are associated to elements of the subgroup.

Due to the wealth of possibilities of the geometric structure, here even Laplacians without any random perturbation may exhibit intriguing spectral properties, part of which is captured by the IDS, respectively the spectral distribution functions of Laplacians on forms. Instances of such features are L^2-Betti numbers, Novikov-Shubin invariants and other geometric L^2-invariants, the jumps of the IDS, and the gap structure of the spectrum.

Geometric L^2-invariants describe the behaviour of the spectral distribution function at energies near the spectral bottom. For instance, the p^{th} L^2-Betti number is the size of the jump at zero energy of the distribution function of the Laplacian on p-forms, see for instance [30, 118, 126]. Novikov-Shubin invariants correspond to characteristic exponents of the asymptotic behaviour of the IDS near zero, cf. e.g. [381, 380, 144, 208, 344, 160, 433, 347, 382, 25, 24]. L^2-torsion is a generalisation of ordinary torsion and has an analytic as well as a combinatorial variant. These invariants have been introduced in [348, 344, 356, 77] and studied in [349, 75, 76, 103, 71, 321, 272, 62].

Another interesting feature of some periodic Laplace-Beltrami operators is the existence of L^2-eigenfunctions, a phenomenon which cannot happen in Euclidean space. Since the IDS is a spectral measure of the periodic operator, the set of discontinuities of this function is precisely the set of eigenvalues of the operator. These issues have been studied for instance in [126, 461, 292, 462, 325]. For more details see the discussion in Remark 3.1.3.

The analysis of the gap structure of the spectrum of periodic operators of Schrödinger type is a further topic which has attracted attention. More precisely, one is interested whether the spectrum in interrupted by *spectral gaps*, i.e. intervals on the real line which belong to the resolvent set. In case

there are gaps: can one establish upper and lower bounds for the width and number of gaps and the spectral bands separating them? For different types of periodic or gauge-invariant elliptic differential operators on manifolds spectral gaps have been analysed in [463, 70, 69, 238, 393, 394]. See Example 2.2.5 for a particular case. Even for periodic Schrödinger operators in Euclidean space it is not trivial to characterise the gap structure. This is illustrated by works devoted to the Bethe-Sommerfeld conjecture, e.g. [451, 446, 447, 448, 211]. For almost periodic operators the situation is even more difficult and additional questions arise like the gap labelling problem, see [37, 477, 240, 39, 42, 237, 38] and the references therein. Although the gap structure of the spectrum is a mathematically intriguing question for its own sake, it is also important from the physical point of view. The features of gaps in the energy spectrum are relevant for the conductance properties of the physical system cf. e.g.[339].

The periodic operators on manifolds discussed so far are generalised by their random analogues studied in this chapter.

2.2 Random Schrödinger Operators on Manifolds: Definitions

Let us explain the geometric setting in which we are working precisely: let X be a complete d-dimensional Riemannian manifold with metric g_0. We denote the volume form of g_0 by vol_0. Let Γ be a discrete, finitely generated subgroup of the isometries of (X, g_0) which acts freely and properly discontinuously on X such that the quotient $M := X/\Gamma$ is a compact (d-dimensional) Riemannian manifold. Let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which Γ acts by measure preserving transformations. Assume moreover that the action of Γ on Ω is ergodic. Now we are in the position to define what we mean by a random metric and consequently a random Laplace-Beltrami operator.

Definition 2.2.1. *Let $\{g_\omega\}_{\omega\in\Omega}$ be a family of Riemannian metrics on X. Denote the corresponding volume forms by vol_ω. We call the family $\{g_\omega\}_{\omega\in\Omega}$ a* random metric *on (X, g_0) if the following five properties are satisfied:*

(2.3) *The map $\Omega \times TX \to \mathbb{R}$, $(\omega, v) \mapsto g_\omega(v, v)$ is jointly measurable.*

(2.4) *There is a $C_g \in]0, \infty[$ such that*

$$C_g^{-1}\, g_0(v, v) \le g_\omega(v, v) \le C_g\, g_0(v, v) \quad \text{for all} \quad v \in TX.$$

(2.5) *There is a $C_\rho \in]0, \infty[$ such that*

$$|\nabla_0\, \rho_\omega(x)|_0 \le C_\rho \text{ for all } x \in X,$$

where ∇_0 denotes the gradient with respect to g_0, ρ_ω is the unique smooth density of vol_0 with respect to vol_ω, and $|v|_0^2 = g_0(v, v)$.

(2.6) *There is a uniform lower bound $(d-1)K \in \mathbb{R}$ for the Ricci curvatures of all Riemannian manifolds (X, g_ω). Explicitly, $\mathrm{Ric}(g_\omega) \ge (d-1)K g_\omega$ for all $\omega \in \Omega$ and on the whole of X.*

(2.7) *The metrics are compatible in the sense that the deck transformations*

$$\gamma\colon (X, g_\omega) \to (X, g_{\gamma\omega}), \quad \gamma\colon x \mapsto \gamma x$$

are isometries.

Property (2.7) implies in particular that the induced maps

$$U_{(\omega,\gamma)}\colon L^2(X, \mathrm{vol}_{\gamma^{-1}\omega}) \to L^2(X, \mathrm{vol}_\omega), \quad (U_{(\omega,\gamma)}f)(x) = f(\gamma^{-1}x)$$

are unitary operators. The density ρ_ω appearing in (2.5) satisfies by definition

$$\int_X f(x)\, d\mathrm{vol}_0(x) = \int_X f(x)\rho_\omega(x)\, d\mathrm{vol}_\omega(x).$$

It is a smooth function and can be written as

$$\rho_\omega(x) = \big(\det g_0(e^i_\omega, e^j_\omega)\big)^{1/2} = \Big(\det g_\omega(e^i_0, e^j_0)\Big)^{-1/2}$$

Here $e^1_0, \ldots, e^d_0$ denotes any basis of T_xX which is orthonormal with respect to the scalar product $g_0(x)$, and $e^1_\omega, \ldots, e^d_\omega \in T_xX$ is any basis orthonormal with respect to $g_\omega(x)$. It follows from (2.4) that

$$C_g^{-d/2} \le \rho_\omega(x) \le C_g^{d/2} \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.8}$$

which in turn, together with property (2.5) and the chain rule, implies

$$|\nabla_0\, \rho_\omega^{\pm 1/2}(x)|_0 \le C_g^{3d/4}\, |\nabla_0\, \rho_\omega(x)|_0 \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.9}$$

Moreover, for any measurable $\Lambda \subset X$ by (2.8) we have the volume estimate

$$C_g^{-d/2}\, \mathrm{vol}_0(\Lambda) \le \mathrm{vol}_\omega(\Lambda) \le C_g^{d/2}\, \mathrm{vol}_0(\Lambda) \tag{2.10}$$

We denote the Laplace-Beltrami operator with respect to the metric g_ω by Δ_ω.

Associated to the random metric just described we define a random family of operators.

Definition 2.2.2. *Let $\{g_\omega\}$ be a random metric on (X, g_0). Let $V\colon \Omega \times X \to \mathbb{R}$ be a jointly measurable mapping such that for all $\omega \in \Omega$ the* potential *$V_\omega := V(\omega, \cdot) \ge 0$ is in $L^1_{\mathrm{loc}}(X)$. For each $\omega \in \Omega$ let $H_\omega = -\Delta_\omega + V_\omega$ be a Schrödinger operator defined on a dense subspace $\mathcal{D}_\omega$ of the Hilbert space $L^2(X, \mathrm{vol}_\omega)$. The family $\{H_\omega\}_{\omega\in\Omega}$ is called a* random Schrödinger operator *if it satisfies for all $\gamma \in \Gamma$ and $\omega \in \Omega$ the following* equivariance *condition*

$$H_\omega = U_{(\omega,\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)} \tag{2.11}$$

Remark 2.2.3 (Restrictions, quadratic forms and selfadjointness). Some remarks are in order why the sum of the Laplace-Beltrami operator and the potential is selfadjoint. We consider the two cases of an operator on the whole manifold X and on a proper open subset of X simultaneously. The set of all smooth functions with compact support in an open set $\Lambda \subset X$ is denoted by $C_c^\infty(\Lambda)$. For each $\omega \in \Omega$ we define the quadratic form

$$\widetilde{Q}(H_\omega^\Lambda)\colon C_c^\infty(\Lambda) \times C_c^\infty(\Lambda) \to \mathbb{R}, \tag{2.12}$$
$$(f,h) \mapsto \int_\Lambda g_\omega(x)\Big(\nabla f(x), \nabla h(x)\Big)\, d\mathrm{vol}_\omega(x) + \int_\Lambda f(x) V_\omega(x) h(x)\, d\mathrm{vol}_\omega(x)$$

We infer from Theorem 1.8.1 in [108] that this quadratic from is closable and its closure $Q(H_\omega^\Lambda)$ gives rise to a densely defined, non-negative selfadjoint operator H_ω^Λ. Actually, $Q(H_\omega^\Lambda)$ is the form sum of the quadratic forms of the negative Laplacian and the potential. By the very definition, $C_c^\infty(\Lambda)$ is dense in the domain of $Q(H_\omega^\Lambda)$ for all ω. The result in [108] is stated for the Euclidean case $X = \mathbb{R}^d$ but the proof works equally well for general Riemannian manifolds.

The unique selfadjoint operator associated to the above quadratic form is called *Schrödinger operator with Dirichlet boundary conditions*. It is the Friedrichs extension of the restriction $H_\omega^\Lambda|_{C_c^\infty(\Lambda)}$. If the potential term is absent we call it negative *Dirichlet Laplacian*.

There are special subsets of the manifold which will play a prominent role later:

Definition 2.2.4. *For an $x \in X$ the set $O(x) := \{y \in X \,|\, \exists \gamma \in \Gamma : y = \gamma x\}$ is called the Γ-orbit of x. The relation $x \sim y \iff O(x) \cap O(y) \neq \emptyset$ partitions X into equivalence classes. A subset $\mathcal{F} \subset X$ is called* Γ-fundamental domain *if it contains exactly one element of each equivalence class.*

In [2, Sect. 3] it is explained how to obtain a connected, polyhedral Γ-fundamental domain $\mathcal{F} \subset X$ by lifting simplices of a triangularisation of M in a suitable manner. $\mathcal{F}$ consists of finitely many smooth images of simplices which can overlap only at their boundaries. In particular, it has piecewise smooth boundary.

To illustrate the above definitions we will look at some examples. Firstly, we consider covering manifolds with abelian deck-transformation group.

Example 2.2.5 (Abelian covering manifolds). Consider a covering manifold (X, g_0) with a finitely generated, abelian subgroup Γ of the isometries of X. If the number of generators of the group Γ equals r, it is isomorphic to $\mathbb{Z}^{r_0} \times \mathbb{Z}_{p_1}^{r_0} \times \ldots \mathbb{Z}_{p_n}^{r_n}$. Here $\sum r_i = r$ and $\mathbb{Z}_p$ is the cyclic group of order p. Assume as above that the quotient X/Γ is compact. Periodic Laplace-Beltrami and Schrödinger operators on such spaces have been analysed e.g. in [462, 393, 394].

In the following we will discus some examples studied by Post in [393, 394]. The aim of these papers was to construct covering manifolds, such that the corresponding Laplace operator has open spectral gaps. More precisely, for any given natural number N, manifolds are constructed with at least N spectral gaps. For technical reasons the study is restricted to abelian coverings. In this case the Floquet decomposition of the periodic operator can be used effectively. Post studies two classes of examples with spectral gaps. In the first case a conformal perturbation of a given covering manifold is used to open up gaps in the energy spectrum of the Laplacian. The second type of examples in [394] is of more interest to us. There, one starts with infinitely many translated copies of a compact manifold and joins them by cylinders to form a periodic network of 'pipes'. By shrinking the radius of the connecting cylinders, more and more gaps emerge in the spectrum. Such manifolds have a non-trivial fundamental group and are thus topologically not equivalent to $\mathbb{R}^d$. On the other hand their deck-transformation group is rather easy to understand, since it is abelian. In particular, it is amenable (cf. Definition 2.3.4), which is a crucial condition in the study conducted later in this chapter. Some of the examples in [393, 394] are manifolds which can be embedded in $\mathbb{R}^3$ as surfaces. They can be thought of as periodic quantum waveguides and networks. See [393] for some very illustrative figures.

Furthermore, in [394] perturbations techniques for Laplace operators on covering manifolds have been developed, respectively carried over from earlier versions suited for compact manifolds, cf. [85, 22, 177]. They include conformal perturbations and local geometric deformations. Floquet decomposition is used to reduce the problem to an operator on a fundamental domain with quasi-periodic boundary conditions and discrete spectrum. Thereafter the min-max principle is applied to geometric perturbations of the Laplacian.

Related random perturbations of Laplacians are studied in [326, 325] (cf. also Example 2.2.7). In particular a Wegner estimate for such operators is derived.

Now we give an instance of a covering manifold X with non-abelian deck-transformation group Γ.

Example 2.2.6 (Heisenberg group). The Heisenberg group H_3 is the manifold of 3×3-matrices given by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \tag{2.13}$$

equipped with a left-invariant metric. The Lie-group H_3 is diffeomorphic to $\mathbb{R}^3$. Its group structure is not abelian, but nilpotent.

The subset $\Gamma = H_3 \cap M(3, \mathbb{Z})$ forms a discrete subgroup. It acts from the left on H_3 by isometries and the quotient manifold H_3/Γ is compact.

Next we give examples of a random potential and a random metric which give rise to a random Schrödinger operator as in Definition 2.2.2. Both have an underlying structure which resembles alloy-type models (in Euclidean space).

Example 2.2.7. (a) Consider the case where the metric is fixed, i.e. $g_\omega = g_0$ for all $\omega \in \Omega$, and only the potential depends on the randomness in the following way:

$$V_\omega(x) := \sum_{\gamma \in \Gamma} q_\gamma(\omega)\, u(\gamma^{-1}x), \tag{2.14}$$

Here $u : X \to \mathbb{R}$ is a bounded, compactly supported measurable function and $q_\gamma \colon \Omega \to \mathbb{R}$ is a sequence of independent, identically distributed random variables. By considerations as in Remark 1.2.2 the random operator $H_\omega := -\Delta + V_\omega, \omega \in \Omega$ is seen to satisfy the equivariance condition.

(b) Consider the situation where the metric has an alloy like structure. Let (g_0, X) be a Riemannian covering manifold and let a family of metrics $\{g_\omega\}_\omega$ be given by

$$g_\omega(x) = \left(\sum_{\gamma \in \Gamma} r_\gamma(\omega)\, u(\gamma^{-1}x) \right) g_0(x)$$

where $u \in C_c^\infty(X)$ and the $r_\gamma \colon \Omega \to]0, \infty[, \gamma \in \Gamma$ are a collection of independent, identically distributed random variables. Similarly as in the previous example one sees that the operators Δ_ω are equivariant.

Operators of the above type are discussed in [325].

2.3 Non-Randomness of Spectra and Existence of the IDS

Here we state the main theorems on the non-randomness of the spectral components and the existence and the non-randomness of the IDS. They refer to random Schrödinger operators as defined in 2.2.2.

Theorem 2.3.1. *There exists a subset Ω' of full measure in $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ and subsets of the real line Σ and $\Sigma_\bullet$, where $\bullet \in \{disc, ess, ac, sc, pp\}$ such that for all $\omega \in \Omega'$*

$$\sigma(H_\omega) = \Sigma \quad \text{and} \quad \sigma_\bullet(H_\omega) = \Sigma_\bullet$$

for any $\bullet = disc, ess, ac, sc, pp$. If Γ is infinite, $\Sigma_{disc} = \emptyset$.

The theorem is proven in [328], see Theorem 5.1. The arguments go to a large part along the lines of [388, 319, 255]. Compare also the literature on almost-periodic Schrödinger operators, for instance [431, 32].

For the proof of the theorem one has to find random variables which encode the spectrum of $\{H_\omega\}_\omega$ and which are invariant under the action of Γ. By ergodicity they will be constant almost surely. The natural random variables

to use are spectral projections, more precisely, their traces. However, since $\mathbb{R}$ is uncountable and one has to deal also with the different spectral components, some care is needed.

Random operators introduced in Definition 2.2.2 are naturally affiliated to a von Neumann algebra of operators which we specify in

Definition 2.3.2. *A family $\{B_\omega\}_{\omega\in\Omega}$ of bounded operators $B_\omega\colon L^2(X,\mathrm{vol}_\omega)\to L^2(X,\mathrm{vol}_\omega)$ is called a* bounded random operator *if it satisfies:*

(i) $\omega\mapsto\langle g_\omega, B_\omega f_\omega\rangle$ *is measurable for arbitrary $f,g\in L^2(\Omega\times X,\mathbb{P}\circ\mathrm{vol})$.*
(ii) *There exists a ω-uniform bound on the norms $\|B_\omega\|$ for almost all $\omega\in\Omega$.*
(iii) *For all $\omega\in\Omega,\gamma\in\Gamma$ the equivariance condition*

$$B_\omega = U_{(\omega,\gamma)} B_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}$$

holds.

By the results of Sect. 2.4, $\{F(H_\omega)\}_\omega$ is a bounded random operator for any measurable, bounded function F.

It turns out that (equivalence classes of) bounded random operators form a von Neumann algebra. More precisely, consider two bounded random operators $\{A_\omega\}_\omega$ and $\{B_\omega\}_\omega$ as *equivalent* if they differ only on a subset of Ω of measure zero. Each equivalence class gives rise to a bounded operator on $L^2(\Omega\times X,\mathbb{P}\circ\mathrm{vol})$ by $(Bf)(\omega,x):=B_\omega f_\omega(x)$, see Appendix A in [328]. This set of operators is a von Neumann algebra $\mathcal{N}$ by Theorem 3.1 in [328]. On $\mathcal{N}$ a trace τ of type II_∞ is given by

$$\tau(B) := \mathbb{E}\,[\mathrm{Tr}(\chi_{\mathcal{F}}\, B_\bullet)]$$

Here $\mathrm{Tr}:=\mathrm{Tr}_\omega$ denotes the trace on the Hilbert space $L^2(X,\mathrm{vol}_\omega)$. Actually, for any choice of $u\colon\Omega\times X\to\mathbb{R}^+$ with $\sum_{\gamma\in\Gamma} u_{\gamma^{-1}\omega}(\gamma^{-1}x)\equiv 1$ for all $(\omega,x)\in\Omega\times X$ we have

$$\tau(B) = \mathbb{E}\,[\mathrm{Tr}(u_\bullet\, B_\bullet)] \tag{2.15}$$

In analogy with the case of operators which are Γ-invariant [30] we call τ the *Γ-trace*. The spectral projections $\{P_\omega(]-\infty,\lambda[)\}_\omega$ of $\{H_\omega\}_\omega$ onto the interval $]-\infty,\lambda[$ form a bounded random operator. Thus, it corresponds to an element of $\mathcal{N}$ which we denote by $P(]-\infty,\lambda[)$. Consider the normalised Γ-trace of P

$$N_H(\lambda) := \frac{\tau(P(]-\infty,\lambda[)}{\mathbb{E}\,[\mathrm{vol}_\bullet(\mathcal{F})]} \tag{2.16}$$

The following is Theorem 3 in [327], see also [328].

Theorem 2.3.3. *$P(]-\infty,\lambda[)$ is the spectral projection of the direct integral operator*

$$H := \int_\Omega^\oplus H_\omega\, d\mathbb{P}(\omega)$$

and N_H is the distribution function of its spectral measure. In particular, the almost sure spectrum Σ of $\{H_\omega\}_\omega$ coincides with the points of increase

$$\{\lambda \in \mathbb{R} |\, N_H(\lambda + \varepsilon) > N_H(\lambda - \varepsilon) \text{ for all } \varepsilon > 0\}$$

of N_H.

That the IDS can be expressed in terms of a trace on a von Neumann Algebra was known long ago. In [430] and [431] Shubin establishes this relation for almost-periodic elliptic differential operators in Euclidean space.

We want to describe the self-averaging IDS by an exhaustion of the whole manifold X along a sequence $\Lambda_l \nearrow X$, $l \in \mathbb{N}$ of subsets of X. To ensure the existence of a sequence of subsets which is appropriate for the exhaustion procedure, we have to impose additional conditions on the group Γ, which will be discussed next.

Definition 2.3.4. *A group Γ is called* amenable *if it has a left invariant mean m_L.*

Amenability enters as a key notion in Definition 2.3.6 and Theorem 2.3.8. For readers acquainted only with Euclidean geometry, its role is motivated in Remark 2.3.10.

Under some conditions on the group, amenability can be expressed in other ways. A locally compact group Γ is amenable if for any $\varepsilon > 0$ and compact $K \subset \Gamma$ there is a compact $G \subset \Gamma$ such that

$$m_L(G \Delta KG) < \varepsilon\, m_L(G)$$

where m_L denotes the left invariant Haar measure, cf. Theorem 4.13 in [390]. This is a geometric description of amenability of Γ. If Γ is a discrete, finitely generated group we chose m_L to be the counting measure and write instead $|\cdot|$. In this case Γ is amenable if and only if a *Følner sequence* exists:

Definition 2.3.5. *Let Γ be a discrete, finitely generated group.*

(i) *A sequence $\{I_l\}_l$ of finite, non-empty subsets of Γ is called a* Følner sequence *if for any finite $K \subset \Gamma$ and $\varepsilon > 0$*

$$|I_l \triangle K I_l| \leq \varepsilon\, |I_l|$$

for all l large enough.

(ii) *We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the* Tempelman *or* doubling property *if it obeys*

$$\sup_{l \in \mathbb{N}} \frac{|I_l I_l^{-1}|}{|I_l|} < \infty$$

(iii) *We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the* Shulman property *if it obeys*

$$\sup_{l \in \mathbb{N}} \frac{|I_l I_{l-1}^{-1}|}{|I_l|} < \infty$$

(iv) *A Følner sequence* $\{I_l\}_l$ *is called a* tempered Følner sequence *if it has the Shulman property.*

In our setting Γ is discrete and finitely generated. (Actually, $K := \{\gamma \in \Gamma \mid \overline{\gamma\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}$ is a finite generator set for Γ. This follows from the fact that the quotient manifold X/Γ is compact, cf. Sect. 3 in [2].) Under this circumstances a Følner sequence exists if and only if there is a sequence of finite, non-empty sets $J_l \subset \Gamma, l \in \mathbb{N}$ such that $\lim_{l\to\infty} \frac{|J_l \triangle \gamma J_l|}{|J_l|} = 0$ for all $\gamma \in \Gamma$. Moreover, for discrete, finitely generated, amenable groups there exists a Følner sequence which is increasing and exhausts Γ, cf. Theorem 4 in [1].

Both properties (ii) and (iii) control the growth of the group Γ. Lindenstrauss observed in [342] that each Følner sequence has a tempered subsequence. Note that this implies that every amenable group contains a tempered Følner sequence. One of the deep results of Lindenstrauss' paper is, that this condition is actually sufficient for a pointwise ergodic theorem, cf. Theorem 2.6.1. Earlier it was known that such theorems can be established under the more restrictive Tempelman property [470, 313, 471]. Shulman [434] first realised the usefulness of the relaxed condition (iii).

In the class of countably generated, discrete groups there are several properties which ensure amenability. Abelian groups are amenable. More generally, all solvable groups and groups of subexponential growth, in particular nilpotent groups, are amenable. This includes the (discrete) Heisenberg group considered in Example 2.2.6. Subgroups and quotient groups of amenable groups are amenable. On the other hand, the free group with two generators is not amenable.

For the discussion of combinatorial properties of Følner sequences in discrete amenable groups see [1].

Any finite subset $I \subset \Gamma$ defines a corresponding set

$$\phi(I) := \mathrm{int}\Big(\bigcup_{\gamma\in I} \gamma\overline{\mathcal{F}}\Big) \subset X$$

where $\mathrm{int}(\cdot)$ stands for the open interior of a set.

In the following we will need some notation for the thickened boundary. Denote by d_0 the distance function on X associated to the Riemannian metric g_0. For $h > 0$, let $\partial_h\Lambda := \{x \in X \mid d_0(x, \partial\Lambda) \leq h\}$ be the *boundary tube of width* h and Λ_h be the interior of the set $\Lambda \setminus \partial_h\Lambda$.

Definition 2.3.6. *(a) A sequence* $\{\Lambda_l\}_l$ *of subsets of* X *is called* admissible exhaustion *if there exists an increasing, tempered Følner sequence* $\{I_l\}_l$ *with* $\bigcup_l I_l = \Gamma$ *such that* $\Lambda_l = \phi(I_l^{-1})$, $l \in \mathbb{N}$.

(b) A sequence $\Lambda_l, l \in \mathbb{N}$ *of subsets of* (X, g_0) *is said to satisfy the* van Hove property *[478] if*

$$\lim_{l\to\infty} \frac{\mathrm{vol}_0(\partial_h\Lambda_l)}{\mathrm{vol}_0(\Lambda_l)} = 0 \text{ for all } h > 0 \tag{2.17}$$

In our setting amenability of Γ ensures that an admissible exhaustion always exists. It is easy to see (cf. e.g. Lemma 2.4 in [392]) that every admissible exhaustion satisfies the van Hove property. Inequality (2.10) implies that for a sequence with the van Hove property

$$\lim_{l\to\infty} \frac{\mathrm{vol}_\omega(\partial_h \Lambda_l)}{\mathrm{vol}_\omega(\Lambda_l)} = 0 \text{ for all } h > 0$$

holds for all $\omega \in \Omega$. Let us remark that one could require for the sets Λ_l in the exhaustion sequence to have smooth boundary, cf. Definition 2.1 in [392]. Such sequences exist for any X with amenable deck-transformation group Γ, as well. This may be of interest, if one wants to study Laplacians with Neumann boundary conditions.

For groups of polynomial growth it is possible to construct analoga of admissible exhaustions by taking metric open balls $B_{r_l}(o)$ around a fixed point $o \in X$ with increasing radii $r_1, \ldots, r_n, \cdots \to \infty$, cf. Theorem 1.5 in [392].

Remark 2.3.7. In our setting it is always possible to chose the sequences $\{I_l\}_l$ and $\{\Lambda_l\}_l$ in such a way that they exhaust the group, respectively the manifold. However, this is not necessary for our results.

A simple instance where $\cup_l \Lambda_l \neq X$ can be given in one space dimension. Let $X = \mathbb{R}$, $\Gamma = \mathbb{Z}$, $I_l = \{1-l, \ldots, 0\}$, $\mathcal{F} = [0,1[$ and consequently $\Lambda_l =]0, l[$. One can use this sequence of sets to define the IDS of random Schrödinger operators although $\cup_l \Lambda_l = [0, \infty[$. A non-trivial example where the sets Λ_l do not exhaust X can be found in [464, 466]. There Sznitman considers random Schrödinger operators in hyperbolic spaces. In that setting the approach presented here does not work due to lack of amenability. Sznitman constructs the IDS by choosing a sequence of balls Λ_l which converges to a horoball of the hyperbolic space. The resulting IDS corresponds to the restriction of the random operator to the horoball and not to the one on the whole space.

We denote by H_ω^l the Dirichlet restriction of H_ω to Λ_l, cf. Remark 2.2.3, and define the finite volume IDS by the formula

$$N_\omega^l(\lambda) := \mathrm{vol}_\omega(\Lambda_l)^{-1} \#\{n \mid \lambda_n(H_\omega^l) < \lambda\}$$

Now we are able to state the result of [327] on the existence of a self-averaging IDS.

Theorem 2.3.8. *Let $\{H_\omega\}_\omega$ be a random Schrödinger operator and Γ an amenable group. For any admissible exhaustion $\{\Lambda_l\}_l$ there exists a set $\Omega' \subset \Omega$ of full measure such that*

$$\lim_{l\to\infty} N_\omega^l(\lambda) = N_H(\lambda), \tag{2.18}$$

for every $\omega \in \Omega'$ and every continuity point $\lambda \in \mathbb{R}$ of N_H.

Definition 2.3.9. *The distribution function defined by the limit in* (2.18) *is called* integrated density of states.

Thus all properties (i)–(iii) on page 14 can be established for the model under study. In particular, formula (2.18) is a variant of the Pastur-Shubin trace formula in the context of manifolds. Theorem 2.3.8 is proven in Sects. 2.4–2.6. It recovers in particular the result of Adachi and Sunada [2] on the existence of the IDS of periodic Schrödinger operators on manifolds.

Remark 2.3.10. Let us motivate, for readers acquainted only with Euclidean space, why it is natural that the amenability requirement enters the theorem. In the theory of random operators and in statistical mechanics one often considers a sequence of sets $\Lambda_l, l \in \mathbb{N}$ which tends to the whole space. Even in Euclidean geometry it is known that the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ needs to tend to $\mathbb{R}^d$ in an appropriate way, e.g. in the sense of van Hove or Fisher [417]. Convergence in the sense of van Hove [478] means that

$$\lim_{l\to\infty} \frac{|\partial_\varepsilon \Lambda_l|}{|\Lambda_l|} = 0 \tag{2.19}$$

for all positive ε.

If one chooses the sequence $\Lambda_l, l \in \mathbb{N}$ badly, one cannot expect the convergence of the finite volume IDS' N_ω^l to a limit as $l \to \infty$. In a non-amenable geometry, any exhaustion sequence is bad, since (2.19) cannot be satisfied, cf. Proposition 1.1 in [2].

Remark 2.3.11. We have assumed the potentials V_ω to be non-negative and some of our proofs will rely on this fact.

However, the *statements* of Theorem 2.3.1 on the non-randomness of the spectrum and Theorem 2.3.8 on the existence of the IDS carry over to V_ω which are uniformly bounded below by a constant C not depending on $\omega \in \Omega$. Indeed, in this case our results directly apply to the shifted operator family $\{H_\omega - C\}_{\omega\in\Omega}$. This implies immediately the same statements for the original operators, since the spectral properties we are considering transform trivially if a constant is added to the operator.

Remark 2.3.12 (Uniform convergence of the IDS). For many types of random Hamiltonians on discrete geometric structures the convergence (2.18) of the IDS is actually uniform in the spectral parameter almost surely, cf. [323, 330, 147, 148, 324, 329]. Note that this statement is non-trivial, since the IDS may have discontinuities, as discussed in Remark 3.1.3.

Uniform convergence of the IDS has also been established for certain types of random Schrödinger operators on metric graphs, cf. [209].

2.4 Measurability

Since we want to study the operators H_ω^Λ as random variables we need a notion of measurability. To this aim, we extend the definition introduced by

Kirsch and Martinelli [255] for random operators on a fixed Hilbert space to families of operators where the spaces and domains of definition vary with $\omega \in \Omega$.

To distinguish between the scalar products of the different L^2-spaces we denote by $\langle \cdot, \cdot \rangle_0$ the scalar product on $L^2(\Lambda, \mathrm{vol}_0)$ and by $\| \cdot \|_0$ the corresponding norm. Similarly, $\langle \cdot, \cdot \rangle_\omega$ and $\| \cdot \|_\omega$ are the scaler product and the norm, respectively, of $L^2(\Lambda, \mathrm{vol}_\omega)$.

Definition 2.4.1. *Consider a family of selfadjoint operators $\{H_\omega\}_\omega$, where the domain of H_ω is a dense subspace $\mathcal{D}_\omega$ of $L^2(\Lambda, \mathrm{vol}_\omega)$. The family $\{H_\omega\}_\omega$ is called a* measurable family of operators *if*

$$\omega \mapsto \langle f_\omega, F(H_\omega) f_\omega \rangle_\omega \tag{2.20}$$

is measurable for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ and all measurable functions $f\colon \Omega \times \Lambda \to \mathbb{R}$ with $f(\omega, \cdot) = f_\omega \in L^2(\Lambda, \mathrm{vol}_\omega)$ for every $\omega \in \Omega$.

Theorem 2.4.2. *A random Schrödinger operator $\{H_\omega\}_{\omega \in \Omega}$ as in Definition 2.2.2 is a measurable family of operators. The same applies to the Dirichlet restrictions $\{H_\omega^\Lambda\}_{\omega \in \Omega}$ to any open subset Λ of X.*

For the proof of this theorem we need some preliminary considerations.

As the next lemma will show, assumption (2.4) in our setting implies that it is sufficient to establish the weak measurability (2.20) for functions f which are constant in ω. To formulate the precise statement, we first note that the Hilbert spaces $L^2(\Lambda, \mathrm{vol}_0)$ and $L^2(\Lambda, \mathrm{vol}_\omega)$ coincide as sets for all $\omega \in \Omega$, though not in their scalar products. Thus it makes sense to speak about a function $f_\omega \equiv f$ as an element of $L^2(\Lambda, \mathrm{vol}_\omega)$ "=" $L^2(\Lambda, \mathrm{vol}_0)$.

Lemma 2.4.3. *A random Schrödinger operator $\{H_\omega\}_\omega$ is measurable if and only if*

$$\omega \mapsto \langle f, F(H_\omega) f \rangle_\omega \text{ is measurable} \tag{2.21}$$

for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ and all $f \in L^2(\Lambda, \mathrm{vol}_0)$.

Proof. To see this, note that (2.21) implies the same statement if we replace $f(x)$ by $h(\omega, x) = g(\omega) f(x)$ where $g \in L^2(\Omega)$ and $f \in L^2(\Lambda, \mathrm{vol}_0)$. Such functions form a total set in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$.

Now, consider a measurable $h\colon \Omega \times \Lambda \to \mathbb{R}$ such that $h(\omega, \cdot) \in L^2(\Lambda, \mathrm{vol}_\omega)$ for every $\omega \in \Omega$. Then $h_n(\omega, x) := \chi_{h,n}(\omega)\, h(\omega, x)$ is in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$ where $\chi_{h,n}$ denotes the characteristic function of the set $\{\omega \in \Omega \mid \|h(\omega)\|_{L^2(\Lambda, \mathrm{vol}_\omega)} \leq n\}$. Since $\chi_{h,n} \to 1$ pointwise on Ω for $n \to \infty$ we obtain

$$\langle h_n(\omega), F(H_\omega) h_n(\omega) \rangle_\omega \to \langle h(\omega), F(H_\omega) h(\omega) \rangle_\omega$$

which shows that $\{H_\omega\}_\omega$ is a measurable family of operators. □

To prove Theorem 2.4.2 we will pull all operators H_ω^Λ onto the same Hilbert space using the unitary transformation S_ω induced by the density ρ_ω

$$S_\omega \colon L^2(\Lambda, \mathrm{vol}_0) \to L^2(\Lambda, \mathrm{vol}_\omega), \quad (S_\omega f)(x) = \rho_\omega^{1/2}(x) f(x)$$

The transformed operators are

$$\begin{aligned} A_\omega &:= -S_\omega^{-1}\, \Delta_\omega^\Lambda\, S_\omega \\ A_\omega \colon S_\omega^{-1}\, \mathcal{D}(\Delta_\omega^\Lambda) &\subset L^2(\Lambda, \mathrm{vol}_0) \longrightarrow L^2(\Lambda, \mathrm{vol}_0) \end{aligned} \tag{2.22}$$

The domain of definition $S_\omega^{-1}\, \mathcal{D}(\Delta_\omega^\Lambda)$ is dense in $L^2(\Lambda, \mathrm{vol}_0)$ and contains all smooth functions of compact support in Λ.

The first fact we infer for the operators $A_\omega, \omega \in \Omega$ is that they are uniformly bounded with respect to each other, at least in the sense of quadratic forms. This is the content of Proposition 3.4 in [327] which we quote without proof.

Denote the quadratic forms associated to the operators $-\Delta_0^\Lambda$, respectively A_ω, by Q_0 and Q_ω, and the corresponding *quadratic form domains* by $\mathcal{D}(Q_0)$ and $\mathcal{D}(Q_\omega)$.

Proposition 2.4.4. *Let $\mathcal{D} \subset L^2(\Lambda, \mathrm{vol}_0)$ be the closure of $C_c^\infty(\Lambda)$ with respect to the norm $\left(Q_0(f,f) + \|f\|_0^2\right)^{1/2}$. Then*

$$\mathcal{D} = \mathcal{D}(Q_0) = \mathcal{D}(Q_\omega)$$

and there exists a constant C_A such that

$$C_A^{-1} \left(Q_0(f,f) + \|f\|_0^2\right) \leq Q_\omega(f,f) + \|f\|_0^2 \leq C_A \left(Q_0(f,f) + \|f\|_0^2\right) \tag{2.23}$$

for all $f \in \mathcal{D}$ and $\omega \in \Omega$.

In the proof of this proposition the bound (2.5) — more precisely (2.9) — on the gradient of the density ρ_ω is needed. It seems to be a technical assumption and in fact dispensable by using a trick from [109], at least if Λ is precompact or of finite volume.

Since we are now dealing with a family of operators on a fixed Hilbert space, we are in the position to apply the theory developed in [255]. The following result is an extension of Proposition 3 there. It suits our purposes and shows that our notion of measurability is compatible with the one in [255].

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a (fixed) dense subset and $B_\omega \colon \mathcal{D} \to \mathcal{H}$, $\omega \in \Omega$ non-negative operators. Denote by $\tilde{\Sigma} = \overline{\bigcup_\omega \sigma(B_\omega)}$ the closure of the union of all spectra, and by $\tilde{\Sigma}^c$ its complement. To establish the measurability of the family $\{B_\omega\}_\omega$ one can use one of the following classes of test functions:

- $\mathcal{F}_1 = \{\chi_{]-\infty,\lambda[} \,|\, \lambda \geq 0\}$,
- $\mathcal{F}_2 = \{x \mapsto \mathrm{e}^{\mathrm{i}tx} \,|\, t \in \mathbb{R}\}$,
- $\mathcal{F}_3 = \{x \mapsto \mathrm{e}^{-tx} \,|\, t \geq 0\}$,

- $\mathcal{F}_4 = \{x \mapsto (z-x)^{-1} | \, z \in \mathbb{C} \setminus \tilde{\Sigma}\}$,
- $\mathcal{F}_5 = \mathcal{F}_4(z_0) = \{x \mapsto (z_0-x)^{-1}\}$ for a fixed $z_0 \in \mathbb{C} \setminus \tilde{\Sigma}$,
- $\mathcal{F}_6 = C_b = \{f \colon \mathbb{R} \to \mathbb{C} | \, f \text{ bounded, continuous}\}$,
- $\mathcal{F}_7 = \{f \colon \mathbb{R} \to \mathbb{C} | \, f \text{ bounded, measurable}\}$.

The following proposition says, that it does not matter which of the above sets of functions one chooses for testing the measurability of $\{B_\omega\}_\omega$.

Proposition 2.4.5. *For $i = 1, \ldots, 7$ the following statements are equivalent:*

$(\mathbf{F}_i)$ $\qquad \omega \mapsto \langle f, F(B_\omega)h\rangle_{\mathcal{H}}$ *is measurable for all $f, h \in \mathcal{H}$ and $F \in \mathcal{F}_i$*

Proof. It is obvious that $(\mathbf{F}_4) \Rightarrow (\mathbf{F}_5)$, $(\mathbf{F}_7) \Rightarrow (\mathbf{F}_6)$, and $(\mathbf{F}_6) \Rightarrow (\mathbf{F}_3)$. The equivalence of $(\mathbf{F}_1)$, $(\mathbf{F}_2)$ and $(\mathbf{F}_4)$ can be found in [255].

To show $(\mathbf{F}_5) \Rightarrow (\mathbf{F}_4)$, consider the set

$$Z := \{z \in \tilde{\Sigma}^c | \, \omega \mapsto (z - H_\omega)^{-1} \text{ is weakly measurable}\}$$

in the topological space $\tilde{\Sigma}^c$. It is closed, since $z_n \to z$ implies the convergence of the resolvents, see e.g. [410, Theorem VI.5]. A similar argument using the resolvent equation and a Neumann series expansion shows that $z \in Z$ implies $B_\delta(z) \subset Z$ where $\delta := d(z, \tilde{\Sigma})$. Since $\tilde{\Sigma}^c$ is connected, $Z = \tilde{\Sigma}^c$ follows.

$(\mathbf{F}_3) \Rightarrow (\mathbf{F}_1)$: By the Stone-Weierstrass Theorem, see e.g. [410, Thm. IV.9], applied to $C([0,\infty])$ it follows that $\mathcal{F}_3$ is dense in the set of functions $\{f \in C([0,\infty]) \mid f(\infty) = 0\} = C_\infty([0,\infty[\,)$. We may approximate any $\chi_{]-\infty,\lambda[}$ pointwise by a monotone increasing sequence $0 \leq f_n, n \in \mathbb{N}$ in $C_\infty(\mathbb{R})$. Polarisation, the spectral theorem, and the monotone convergence theorem for integrals imply that $\chi_{]-\infty,\lambda[}(H_\omega)$ is weakly measurable. An analogous argument shows $(\mathbf{F}_1) \Rightarrow (\mathbf{F}_7)$, since any non-negative $f \in \mathcal{F}_7$ can be approximated monotonously pointwise by non-negative step functions $f_n, n \in \mathbb{N}$. □

We use the following proposition taken from [458] (Prop. 1.2.6.) to show that $\{A_\omega\}_\omega$ is a measurable family of operators.

Proposition 2.4.6. *Let $B_\omega, \omega \in \Omega$ and B_0 be non-negative operators on a Hilbert space $\mathcal{H}$. Let Q_ω, $\omega \in \Omega$ and Q_0 be the associated closed quadratic forms with the following properties:*

(2.25) *Q_ω, $\omega \in \Omega$ and Q_0 are defined on the same dense subset $\mathcal{D} \subset \mathcal{H}$.*

(2.26) *There is a constant $C > 0$ such that*

$$C^{-1}\left(Q_0(f,f) + \|f\|_0^2\right) \leq Q_\omega(f,f) + \|f\|_0^2 \leq C\left(Q_0(f,f) + \|f\|_0^2\right)$$

for all $\omega \in \Omega$ and $f \in \mathcal{D}$.

(2.27) *For every $f \in \mathcal{D}$ the map $\omega \mapsto Q_\omega(f,f)$ is measurable.*

Then the family $\{B_\omega\}_\omega$ of operators satisfies the equivalent properties of Proposition 2.4.5.

By property ($\mathbf{F}_7$) and Lemma 2.4.3, $\{B_\omega\}_\omega$ as in Proposition 2.4.6 is a measurable family of operators.

We apply the proposition to $B_\omega = A_\omega$, where $\{A_\omega\}_\omega$ is defined in (2.22). To do so, we check that the properties (2.25)–(2.27) are satisfied: Properties (2.25) and (2.26) follow from Proposition 2.4.4. Property (2.27) is obvious for $f \in C_c^\infty(\Lambda)$ and follows by approximation for all $f \in \mathcal{D}$, since $C_c^\infty(\Lambda)$ is dense, again by Proposition 2.4.4.

Proof (of Theorem 2.4.2). We already know that the transformed 'kinetic' part A_ω, $\omega \in \Omega$ of the Hamiltonian is measurable. To deal with the singular potential we introduce the cut off

$$V_{\omega,n}(x) := \min\{n, V_\omega(x)\} \text{ for } n \in \mathbb{N} \text{ and } \omega \in \Omega$$

The auxiliary potential $V_{\omega,n}$ is bounded and in particular its domain of definition is the whole Hilbert space $L^2(\Lambda, \mathrm{vol}_0)$. Thus the operator sum

$$A_{\omega,n} := A_\omega + V_{\omega,n}, \quad \omega \in \Omega$$

is well defined and [255, Prop. 4] implies that it forms a measurable family of operators. To recover the unbounded potential V_ω, we consider the semigroups $\omega \mapsto \exp(-tA_{\omega,n}), t > 0$ which are weakly measurable.

The quadratic forms of $A_{\omega,n}$ converge monotonously to the form of $A_\omega^\infty := A_\omega + V_\omega$. Now Theorems VIII.3.13a and IX.2.16 in [239] imply that the semigroups of $A_{\omega,n}$ converge weakly towards the one of A_ω^∞ for $n \to \infty$. Thus $\exp(-tA_\omega^\infty)$ is weakly measurable, which implies the measurability of the family A_ω^∞.

Finally, since S_ω is multiplication with the measurable function $(x, \omega) \mapsto \rho_\omega(x)$, this implies the measurability of the family $H_\omega = S_\omega A_\omega^\infty S_\omega^{-1}$, $\omega \in \Omega$. □

For later use let us note that the trace of measurable operators is measurable. More precisely we will need the fact that for Λ of finite volume the mappings

$$\omega \mapsto \mathrm{Tr}(\chi_\Lambda \mathrm{e}^{-tH_\omega}) \quad \text{and} \quad \omega \mapsto \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \tag{2.28}$$

are measurable. Note that one can chose an orthonormal basis for $L^2(\Lambda, \mathrm{vol}_\omega)$ which depends in a measurable way on ω, cf. for instance Lemma II.2.1 in [117]. Thus (2.28) follows immediately from the Definition 2.4.1 of measurable operators.

2.5 Bounds on the Heat Kernels Uniform in ω

This paragraph is devoted to heat kernel estimates of the Schrödinger operators H_ω. It consists of four parts. Firstly we discuss existence of L^2-kernels of $\mathrm{e}^{-tH_\omega}, t > 0$ and derive rough upper bounds relying on results in [108].

Secondly, we infer Gaussian off-diagonal decay estimates of the kernels using estimates derived in [332]. We then present an idea of H. Weyl to derive the *principle of not feeling the boundary*, and finally we state a proposition which summarises the information on the heat kernel needed in the next section.

We have to control the dependence on the metric and potential of all these estimates since both the metric and the potential vary with the random parameter $\omega \in \Omega$.

As H_ω is non-negative, the semigroup $\mathrm{e}^{-tH_\omega}, t > 0$ consists of contractions. Moreover, the semigroup satisfies some nice properties formulated in the following definition which enable us to derive estimates on the corresponding heat kernel.

Definition 2.5.1. *Let $\Lambda \subset X$ be open and μ a σ-finite Borel measure on Λ. Let A be a real, non-negative, selfadjoint operator on the Hilbert space $L^2(\Lambda, \mu)$. The semigroup $\mathrm{e}^{-tA}, t > 0$ is called* positivity preserving *if $\mathrm{e}^{-tA} f \geq 0$ for any $0 \leq f \in L^2(\Lambda, \mu)$ and $t > 0$. Furthermore, $\mathrm{e}^{-tA}, t > 0$ is called a* Markov semigroup*, if it is well defined on $L^\infty(\Lambda, \mu)$ and the two following properties hold*

$$\mathrm{e}^{-tA} \colon L^2(\Lambda, \mu) \longrightarrow L^2(\Lambda, \mu) \qquad \text{is positivity preserving for every } t > 0 \tag{2.29}$$

$$\mathrm{e}^{-tA} \colon L^\infty(\Lambda, \mu) \to L^\infty(\Lambda, \mu) \qquad \text{is a contraction for every } t > 0 \tag{2.30}$$

In this case A is called a Dirichlet form.

A Markov semigroup e^{-tA} is called ultracontractive *if*

$$\mathrm{e}^{-tA} \colon L^2(\Lambda, \mu) \to L^\infty(\Lambda, \mu) \text{ is bounded for all } t > 0 \tag{2.31}$$

The above (2.29) and (2.30) are called *Beurling-Deny conditions* [44, 45].

We infer from [108] the following facts: A Markov semigroup is a contraction on $L^p(\Lambda, \mu)$ for all $1 \leq p \leq \infty$ (and all $t > 0$). For all $\omega \in \Omega$ the Schrödinger operator H_ω^Λ on $L^2(\Lambda, \mathrm{vol}_\omega)$ is a Dirichlet form, [108, Thm. 1.3.5]. There the proof is given for $X = \mathbb{R}^d$, but it applies to manifolds, too. By Sobolev embedding estimates and the spectral theorem $\mathrm{e}^{t\Delta_\omega^\Lambda}$ is ultracontractive. Thus by Lemma 2.1.2 in [108] each $\mathrm{e}^{t\Delta_\omega^\Lambda}$ has a kernel, which we denote by $k_\omega^\Lambda(t, \cdot, \cdot)$, such that for almost all $x, y \in \Lambda$

$$0 \leq k_\omega^\Lambda(t, x, y) \leq \|\mathrm{e}^{t\Delta_\omega^\Lambda}\|_{1,\infty} =: C_\omega^\Lambda(t) \tag{2.32}$$

Here $\|B\|_{1,\infty}$ denotes the norm of $B \colon L^1 \to L^\infty$. For $\Lambda = X$ we use the abbreviation $k_\omega = k_\omega^X$.

To derive an analogous estimate to (2.32) for the full Schrödinger operator with potential we make use of the *Feynman-Kac formula*. Using the symbol $\mathbf{E}_x$ for the expectation with respect to the *Brownian motion* b_t starting in $x \in X$ the formula reads

$$(\mathrm{e}^{-tH_\omega} f)(x) = \mathbf{E}_x \left(\mathrm{e}^{-\int_0^t V_\omega(b_s)\, ds} f(b_t) \right)$$

For a stochastically complete manifold X and bounded, continuous V_ω the formula is proven, for instance, in Theorem IX.7A in [150]. It extends to general non-negative potentials which are in L^1_{loc} using semigroup and integral convergence theorems similarly as in the proof of Theorem X.68 in [407]. Since we consider (geodesically) complete manifolds whose Ricci curvature is bounded below, they are all stochastically complete, cf. for instance [206] or Theorem 4.2.4 in [217].

Since the potential is non-negative, the Feynman-Kac formula implies for non-negative $f \in L^1(\Lambda, \mathrm{vol}_\omega)$

$$0 \leq \big(\mathrm{e}^{-tH_\omega^\Lambda} f\big)(x) \leq \big(\mathrm{e}^{t\Delta_\omega^\Lambda} f\big)(x) \leq C_\omega^\Lambda(t)\, \|f\|_{L^1}$$

for almost every $x \in \Lambda$. Thus $\mathrm{e}^{-tH_\omega^\Lambda} \colon L^1(\Lambda, \mathrm{vol}_\omega) \to L^\infty(\Lambda, \mathrm{vol}_\omega)$ has the same bound $C_\omega^\Lambda(t)$ as the semigroup where the potential is absent. This yields the pointwise estimate on the kernel $k_{H_\omega}^\Lambda(t, \cdot, \cdot)$ of $\mathrm{e}^{-tH_\omega^\Lambda}$:

$$0 \leq k_{H_\omega}^\Lambda(t, x, y) \leq C_\omega^\Lambda(t) \quad \text{for almost every } x, y \in X. \tag{2.33}$$

In the following we derive sharper upper bounds on the kernels which imply their decay in the distance between the two space arguments x and y. Such estimates have been proven by Li and Yau [332] for *fundamental solutions* of the heat equation. One would naturally expect that the fundamental solution and the L^2-heat kernel of the semigroup coincide under some regularity assumptions. This is actually the case as has been proven for instance in [119] for vanishing, and in [327] for smooth, non-negative potentials. The proof in the last cited source uses that H_ω is a Dirichlet form.

To formulate the results of Li and Yau [332] which we will be using, we denote by $d_\omega \colon X \times X \to [0, \infty[$ the Riemannian distance function on X with respect to g_ω. Note that the following proposition concerns the heat kernel of the pure Laplacian.

Proposition 2.5.2. *For every $t > 0$ there exist constants $C(t) > 0$, $\alpha_t > 0$ such that*

$$k_\omega(t, x, y) \leq C(t)\, \exp\big(-\alpha_t\, d_0^2(x, y)\big) \tag{2.34}$$

for all $\omega \in \Omega$ and $x, y \in X$.

Proof. For a fixed Schrödinger operator the estimate (with d_0 replaced by d_ω) is contained in Corollary 3.1 in [332]. There the upper bound is given explicitly in terms of the geometric bounds on the manifold. This enables one to show that properties (2.4), (2.8) and

$$C_g^{-1} d_0(x, y) \leq d_\omega(x, y) \leq C_g d_0(x, y)$$

ensure that the constants $C(t)$ and α_t in (2.34) may be chosen uniformly in ω. Moreover, for measuring the distance between the points x and y we may always replace d_ω by d_0 by increasing α_t. □

Let us collect various consequences of Proposition 2.5.2 which will be useful later on.

(i) The pointwise kernel bound on the left hand side of (2.33) can be chosen uniformly in $\omega \in \Omega$.

(ii) We stated Proposition 2.5.2 for the pure Laplacian, although Li and Yau treat the case of a Schrödinger operator with potential. The reason for this is that we want to avoid the regularity assumptions on the potential imposed in [332].
To recover from (2.34) the case where a (non-negative) potential is present we use again the Feynman-Kac formula. We need now a local version of the argument leading to (2.33). More precisely, we consider e^{-tH_ω} as an operator from $L^1(B_\varepsilon(y))$ to $L^\infty(B_\varepsilon(x))$ for small $\varepsilon > 0$. Thus, we obtain

$$0 \le k_{H_\omega}(t,x,y) \le C(t)\, \exp\big(-\alpha_t\, d_0^2(x,y)\big)$$

(iii) The estimates derived so far immediately carry over to the case where the entire manifold is replaced by an open subset $\Lambda \subset X$.

$$0 \le k^\Lambda_{H_\omega}(t,x,y) \le k_{H_\omega}(t,x,y)$$

This is due to *domain monotonicity*, see for example [108, Thm. 2.1.6] where this fact is proven using functional analytic tools. Another way to see that this estimate is true, is to use the probabilistic representation of the heat semigroup, cf. [35, 435].

(iv) The *Bishop volume comparison* theorem controls the growth of the volume of balls with radius r, see for instance [51], [84, Thm. III.6] or [72]. It tells us that the lower bound (2.6) on the Ricci curvature is sufficient to bound the growth of the volume of balls as r increases. The volume of the ball can be estimated by the volume of a ball with the same radius in a space with constant curvature K. The latter volume grows at most exponentially in the radius. For our purposes it is necessary to have an ω-uniform version of the volume growth estimate. Using Properties (2.4), (2.6) and (2.8) we obtain the uniform bound

$$\mathrm{vol}_\omega\big(\{y|\, d_\omega(x,y) < r\}\big) \le C_1\, \mathrm{e}^{C_2 r} \quad \text{for all } x \in X$$

where C_1, C_2 do not depend on x and ω. This implies that for all exponents $p > 0$, there exists a $M_p(t) < \infty$ such that the moment estimate

$$\int_\Lambda [k^\Lambda_{H_\omega}(t,x,y)]^p\, d\mathrm{vol}_\omega(y) \le M_p(t)$$

holds uniformly in $\Lambda \subset X$ open, in $x \in \Lambda$ and $\omega \in \Omega$. We set $M(t) := M_1(t)$.

(v) The heat kernel estimates imply a uniform bound on the traces of the semigroup localised in space. Let $\Lambda \subset X$ be a (fixed) open set of finite volume. There exists a constant $C_{\mathrm{Tr}} = C_{\mathrm{Tr}}(\Lambda, t) > 0$ such that for all $\omega \in \Omega$

$$\mathrm{Tr}\big(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}\big) \leq C_{\mathrm{Tr}}$$

Intuitively this is the same as saying that $\int_\Lambda k_{H_\omega}(t,x,x)\, d\mathrm{vol}_\omega(x)$ is uniformly bounded. However, since the diagonal $\{(x,x) | \, x \in \Lambda\}$ is a set of measure zero, the integral does not make sense as long as we consider k_{H_ω} as an L^2-function. We do not want here to address the question of continuity of the kernel. Instead we use the semigroup property $\mathrm{e}^{-2tH_\omega} = \mathrm{e}^{-tH_\omega}\mathrm{e}^{-tH_\omega}, t > 0$ and selfadjointness to express the trace as

$$\mathrm{Tr}\big(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}\big) = \int_\Lambda \int_\Lambda [k_{H_\omega}(t/2,x,y)]^2 \, d\mathrm{vol}_\omega(x)\, d\mathrm{vol}_\omega(y) \leq M_2(t/2)\, \mathrm{vol}_\omega(\Lambda) \tag{2.35}$$

By (2.10) this is bounded uniformly in $\omega \in \Omega$. Applying domain monotonicity once more, we obtain

$$\mathrm{Tr}\big(\mathrm{e}^{-tH_\omega^\Lambda}\big) \leq M_2(t/2)\, \mathrm{vol}_\omega(\Lambda) \leq M_2(t/2)\, C_g^{d/2}\, \mathrm{vol}_0(\Lambda) \tag{2.36}$$

The following lemma is a maximum principle for Schrödinger operators with non-negative potentials. Combined with the off-diagonal decay estimates in Proposition 2.5.2 it will give us a proof of the principle of not feeling the boundary.

Lemma 2.5.3 (Maximum principle for heat equation with non-negative potential). *Let $\Lambda \subset X$ be open with compact closure, V be a non-negative function, and $u \in C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$ be a solution of the heat equation $\frac{\partial}{\partial t}u + (-\Delta + V)u = 0$ on $]0,T[\times\Lambda$ with non-negative supremum $s = \sup\{u(t,x) \mid (t,x) \in [0,T[\times\overline{\Lambda}\}$. Then,*

$$s = \max\left\{ \max_{x\in\overline{\Lambda}} u(0,x), \sup_{[0,T[\times\partial\Lambda} u(t,x) \right\}$$

Note that regularity of V is not assumed explicitly, but implicitly by the requirements on u. They are e.g. satisfied if V is smooth. Indeed, in that case the heat kernel is smooth, as can be seen following the proof of [108, Thm. 5.2.1].

Now we are in the position to state the second, refined estimate on the heat kernels, the *principle of not feeling the boundary*. It is a formulation of the fact that the heat kernel of the Dirichlet-Laplacian on a (large) open set Λ does not differ much from the heat kernel associated to the Laplacian on the whole manifold, as long as one stays well inside Λ. As before, we derive this estimate first for the pure Laplacian and then show that it carries over to Schrödinger operators with non-negative potential.

Proposition 2.5.4. *For any fixed $t, \varepsilon > 0$, there exists an $h = h(t, \varepsilon) > 0$ such that for every open set $\Lambda \subset X$ and all $\omega \in \Omega$*

$$0 \leq k_\omega(t, x, y) - k_\omega^\Lambda(t, x, y) \leq \varepsilon$$

for all $x \in \Lambda, y \in \Lambda_h$.

Proof. The first inequality is a consequence of domain monotonicity. So we just have to prove the second one.

Fix $\omega \in \Omega$ and $t, \varepsilon > 0$. Choose $h > 0$ such that

$$C(t) \exp\Big(- \alpha_t \big(h/2\big)^2 \Big) \leq \varepsilon$$

Note that the choice is independent of ω. For any $y \in \Lambda_h$ and $0 < \delta < h/2$ denote by $B_\delta(y)$ the open d_0-ball around y with radius δ. Let $f_\delta \in C_0^\infty(B_\delta(y))$ be a non-negative approximation of the δ-distribution at y.

We consider now the time evolution of the initial value f under the two semigroups generated by Δ_ω and Δ_ω^Λ, respectively.

$$u_1(t, x) := \int_X k_\omega(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z) = \int_\Lambda k_\omega(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z).$$
$$u_2(t, x) := \int_\Lambda k_\omega^\Lambda(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z).$$

The difference $u_1(t, x) - u_2(t, x)$ solves the heat equation $\frac{\partial}{\partial t} u = \Delta_\omega u$ and satisfies the initial condition $u_1(0, x) - u_2(0, x) = f_\delta(x) - f_\delta(x) = 0$ for all $x \in \Lambda$. Now, by domain monotonicity we know $k_\omega(t, x, z) - k_\omega^\Lambda(t, x, z) \geq 0$, thus

$$u_1(t, x) - u_2(t, x) = \int_\Lambda \big[k_\omega(t, x, z) - k_\omega^\Lambda(t, x, z)\big] f_\delta(z)\, d\mathrm{vol}_\omega(z) \geq 0$$

for all $t > 0$ and $x \in \Lambda$. The application of the maximum principle yields

$$u_1(t, x) - u_2(t, x) \leq \max_{]0,t] \times \partial\Lambda} \{u_1(s, w) - u_2(s, w)\}\,. \tag{2.37}$$

The right hand side can be further estimated by:

$$\begin{aligned} u_1(s, w) - u_2(s, w) &\leq \int_\Lambda k_\omega(s, w, z) f_\delta(z)\, d\mathrm{vol}_\omega(z) \\ &= \int_{\Lambda_{h/2}} k_\omega(s, w, z) f_\delta(z)\, d\mathrm{vol}_\omega(z). \end{aligned}$$

Since $w \in \partial\Lambda$ and $z \in \Lambda_{h/2}$, we conclude using Proposition 2.5.2:

$$\int_{\Lambda_{h/2}} k_\omega(s, w, z) f_\delta(z) d\mathrm{vol}_\omega(z) \leq C(t) \exp\big(- \alpha_t (h/2)^2 \big) \leq \varepsilon$$

Since the bound is independent of δ we may take the limit $\delta \to 0$ which concludes the proof. □

One can prove the principle of not feeling the boundary by other means too, see for instance [349, 122, 392]. This alternative approach uses information on the behaviour of solutions of the wave equation. Unlike the solutions of the heat equation, they do not have the unphysical property that their support spreads instantaneously to infinity. Actually, the solutions of the wave equation have finite propagation speed [468]. Fourier transforms and the spectral theorem turn this information into estimates on the difference of the solutions of the free and restricted heat equation. Sobolev estimates lead then to the principle of not feeling the boundary. See also Sect. 7 in [406].

Remark 2.5.5. Similarly as in Lemma 2.5.3, one can prove the proposition, if a potential is present. More precisely, Proposition 2.5.4 is valid for Schrödinger operators with potentials V such that for continuous initial and boundary values the solution of the heat equation $\frac{\partial}{\partial t}u = -(-\Delta_\omega + V)u$ is in $C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$. However, Proposition 2.5.4 implies an analogous estimate for the case where a non-negative potential is present, similarly as in (ii) on page 33. This will be explained next.

Consider $\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega^\Lambda}$ as an operator from $L^1(\Lambda_h)$ to $L^\infty(\Lambda)$, and denote by τ_x^Λ the *first exit time* from Λ for a Brownian motion starting in x. By the Feynman-Kac formula, we have for $0 \le f \in L^1(\Lambda_h)$

$$\begin{aligned}[(\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega^\Lambda})f](x) &= \mathbf{E}_x\Big(\mathrm{e}^{-\int_0^t ds V(b_s)} f(b_t)\chi_{\{b|\,\tau_x^\Lambda \le t\}}\Big) \\ &\le \mathbf{E}_x\Big(f(b_s)\chi_{\{b|\,\tau_x^\Lambda \le t\}}\Big) = \int [k_\omega(t,x,y) - k_\omega^\Lambda(t,x,y)] f(y)\, d\mathrm{vol}_\omega \\ &\le \varepsilon \int f(y)\, d\mathrm{vol}_\omega\end{aligned}$$

if we chose h as in Proposition 2.5.4. Thus for almost all $x \in \Lambda, y \in \Lambda_h$

$$k_{H_\omega}(t,x,y) - k_{H_\omega}^\Lambda(t,x,y) \le \|\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega}\|_{L^1(\Lambda_h)\to L^\infty(\Lambda)} \le \varepsilon \qquad (2.38)$$

The upper bounds on the heat kernel and the principle of not feeling the boundary enable us to prove a result on the traces of localised heat-semigroups: In the macroscopic limit, as Λ tends (in a nice way) to the whole of X, the two quantities

$$\mathrm{Tr}(\chi_\Lambda \mathrm{e}^{-tH_\omega}) \quad \text{and} \quad \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda})$$

are approximately the same. The precise statement is contained in the following

Proposition 2.5.6. *Let $\{\Lambda_l\}_{l\in\mathbb{N}}$, be a sequence of subsets of X which satisfies the van Hove property 2.17 and let $\{H_\omega\}_\omega$ be a random Schrödinger operator. Then*

$$\lim_{l\to\infty} \sup_{\omega\in\Omega} \frac{1}{\mathrm{vol}_\omega(\Lambda_l)} \Big|\mathrm{Tr}(\chi_{\Lambda_l}\mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^l})\Big| = 0$$

Proof. We consider first a fixed $l \in \mathbb{N}$ and abbreviate $\Lambda = \Lambda_l$. For the operator $\mathrm{e}^{-tH_\omega^\Lambda}$ we may write the trace in the same way as in (2.35) to obtain

$$\mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) = \int_\Lambda \int_\Lambda [k_{H_\omega}^\Lambda(t/2, x, y)]^2 d\mathrm{vol}_\omega(x) d\mathrm{vol}_\omega(y) \tag{2.39}$$

We express the difference of (2.35) and (2.39) using

$$(k_{H_\omega})^2 - (k_{H_\omega}^\Lambda)^2 = (k_{H_\omega} - k_{H_\omega}^\Lambda)(k_{H_\omega} + k_{H_\omega}^\Lambda)$$

Next we chose $h = h(t/2, \varepsilon) > 0$ as in Proposition 2.5.4 and decompose the integration domain according to

$$\Lambda \times \Lambda = (\Lambda \times \Lambda_h) \cup (\Lambda \times \partial_h \Lambda)$$

The difference of the traces can be now estimated as

$$\begin{aligned} &0 \leq \mathrm{Tr}(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \\ &= \int_\Lambda \int_{\Lambda_h} \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) - k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) + k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] d\mathrm{vol}_\omega(x, y) \\ &+ \int_\Lambda \int_{\partial_h \Lambda} \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) - k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) + k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] d\mathrm{vol}_\omega(x, y) \end{aligned} \tag{2.40}$$

The first term is bounded by $2M(t/2)\,\varepsilon\,\mathrm{vol}_\omega(\Lambda)$ and the second by

$$2M(t/2)\,C(t/2)\,\mathrm{vol}_\omega(\partial_h \Lambda)$$

It follows that

$$\frac{1}{\mathrm{vol}_\omega(\Lambda)} \Big(\mathrm{Tr}(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \Big) \leq 2M(t/2)\varepsilon + 2M(t/2)C(t/2) \frac{\mathrm{vol}_\omega(\partial_h \Lambda)}{\mathrm{vol}_\omega(\Lambda)}$$

Now, we let l go to infinity. Since the sequence Λ_l satisfies the van Hove property (2.17) and since our bounds are uniform in ω, the proposition is proven. □

2.6 Laplace Transform and Ergodic Theorem

This section completes the proof of Theorem 2.3.8. It relies, apart from the results established in Sects. 2.4–2.5, on a general ergodic theorem and a criterion for the convergence of distribution functions.

Lindenstrauss proved in [342, 341] an ergodic theorem which applies to locally compact, second countable amenable groups. It includes as a special case the following statement for discrete groups.

Theorem 2.6.1. *Let Γ be an amenable discrete group and $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space. Assume that Γ acts ergodically on Ω by measure preserving transformations. Let $\{I_l\}_l$ be a tempered Følner sequence in Γ. Then for every $f \in L^1(\Omega)$*

$$\lim_{j\to\infty} \frac{1}{|I_l|} \sum_{\gamma\in I_l} f(\gamma\omega) = \mathbb{E}(f) \tag{2.41}$$

for almost all $\omega \in \Omega$.

In the application we have in mind $f \in L^\infty$, so the convergence holds in the L^1-topology, too.

Remark 2.6.2. Some background on previous results can be found for instance in Sect. 6.6 of Krengel' s book [313], in Tempelman's works [469, 470, 471] or some other sources [152, 204, 26, 151, 383]. The book [471] gives in Sect. 5.6 a survey of Shulman's results [434]. Mean ergodic theorems hold in more general circumstances, see for instance [313, Sect. 6.4] or [471, Ch. 6].

We will apply the ergodic theorem above not to the normalised eigenvalue counting functions N_ω^l, but to their Laplace transforms $\mathcal{L}_\omega^l$. The reason is, that the $\mathcal{L}_\omega^l$ are bounded, while the original N_ω^l are not. The following criterion of Pastur and Shubin [384, 431] says that it is actually sufficient to test the convergence of the Laplace transforms.

Lemma 2.6.3 (Pastur-Shubin convergence criterion). *Let N_n be a sequence of distribution functions such that*

(i) *there exists a $\lambda_0 \in \mathbb{R}$ such that $N_l(\lambda) = 0$ for all $\lambda \le \lambda_0$ and $l \in \mathbb{N}$,*
(ii) *there exists a function $C : \mathbb{R}^+ \to \mathbb{R}$ such that $\mathcal{L}^l(t) := \int \mathrm{e}^{-\lambda t} dN_l(\lambda) \le C(t)$ for all $l \in \mathbb{N}$ and $t > 0$,*
(iii) *$\lim_{l\to\infty} \mathcal{L}^l(t) =: \mathcal{L}(t)$ exists for all $t > 0$.*

Then $\mathcal{L}$ is the Laplace transform of a distribution function N and for all continuity points λ of N we have

$$N(\lambda) := \lim_{l\to\infty} N_l(\lambda)$$

Finally, we present the proof of Theorem 2.3.8 on the existence of a self-averaging IDS:

Proof (of Theorem 2.3.8). We have to check the conditions in the previous lemma for the normalised eigenvalue counting functions N_ω^l. The first one is clearly satisfied for $\lambda_0 = 0$, since all operators we are dealing with are non-negative. To see (ii), express the Laplace transform by the trace of the heat semigroup

$$\mathcal{L}_\omega^l(t) = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \sum_{n,\, \lambda_n\in\sigma} \mathrm{e}^{-t\lambda_n} = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \mathrm{Tr}(\mathrm{e}^{-tH_\omega^l})$$

The sum extends over all eigenvalues λ_n of H_ω^l, counting multiplicities. Now, (2.36) implies condition (ii) of the Pastur-Shubin criterion.

To prove (iii) we will show for all $t > 0$ the convergence

$$\lim_{j\to\infty} \mathcal{L}_\omega^l(t) = \int_{\mathbb{R}} \mathrm{e}^{-t\lambda} dN_H(\lambda)$$

in (L^1 and) $\mathbb{P}$-almost sure-sense. For technical reasons we will deal separately with the convergence of the enumerator and denominator in

$$\mathcal{L}_\omega^l(t) = \mathrm{vol}_\omega(\Lambda_l)^{-1} \operatorname{Tr}(\mathrm{e}^{-tH_\omega^l})$$

However, we need *some* normalisation, to avoid divergences. Consider first the enumerator with an auxiliary normalisation

$$|I_l|^{-1} \operatorname{Tr}(\mathrm{e}^{-tH_\omega^l}) \tag{2.42}$$

Introduce for two sequences of random variables $a_l(\omega), b_l(\omega), l \in \mathbb{N}$ the equivalence relation $a_l \overset{j\to\infty}{\sim} b_l$ if they satisfy $a_l - b_l \to 0$ almost surely for $l \to \infty$. By Proposition 2.5.6, the equivariance, and Lindenstrauss' ergodic theorem 2.6.1

$$\begin{aligned} |I_l|^{-1} \operatorname{Tr}(\mathrm{e}^{-tH_\omega^l}) \overset{j\to\infty}{\sim} |I_l|^{-1} \operatorname{Tr}(\chi_{\Lambda_l} \mathrm{e}^{-tH_\omega}) &= |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \operatorname{Tr}(\chi_{\gamma\mathcal{F}} \mathrm{e}^{-tH_\omega}) \\ &= |I_l|^{-1} \sum_{\gamma\in I_l} \operatorname{Tr}(\chi_{\mathcal{F}} \mathrm{e}^{-tH_{\gamma\omega}}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\operatorname{Tr}(\chi_{\mathcal{F}} \mathrm{e}^{-tH_\bullet})\right\} \end{aligned}$$

Similarly we infer for the normalised denominator

$$|I_l|^{-1}\mathrm{vol}_\omega(\Lambda_l) = |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \mathrm{vol}_\omega(\gamma\mathcal{F}) = |I_l|^{-1} \sum_{\gamma\in I_l} \mathrm{vol}_{\gamma\omega}(\mathcal{F}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\mathrm{vol}_\bullet(\mathcal{F})\right\}$$

Note that by (2.10) all terms in the above line are bounded from above and below uniformly in ω. By taking quotients we obtain

$$\mathcal{L}_\omega^l(t) = \frac{|I_l|^{-1} \operatorname{Tr}(\mathrm{e}^{-tH_\omega^l})}{|I_l|^{-1} \mathrm{vol}_\omega(\Lambda_l)} \overset{j\to\infty}{\sim} \frac{\mathbb{E}\left\{\operatorname{Tr}(\chi_{\mathcal{F}} \mathrm{e}^{-tH_\bullet})\right\}}{\mathbb{E}\left\{\mathrm{vol}_\bullet(\mathcal{F})\right\}}$$

Uniform boundedness implies that the convergence holds also in L^1-sense. The right hand side is the Laplace transform of N_H, see the proof of Theorem 6.1 of [328] for a detailed calculation. □

2.7 Approach Using Dirichlet-Neumann Bracketing

We outline an alternative proof of the existence of the IDS due to Kirsch and Martinelli [254]. It applies to random Schrödinger operators on $\mathbb{R}^d$. It relies on an ergodic theorem for superadditive processes by Akcoglu and

Krengel [17] and estimates on the number of bound states essentially implied by the Weyl asymptotics.

Let us explain the notion of a superadditive process in our context. Denote by Z the set of all multi-dimensional intervals or boxes Λ in $\mathbb{R}^d$ such that $\Lambda = \{x |\, a_j < x_j < b_j, \text{ for } j = 1, \ldots, d\}$ for some $a, b \in \mathbb{Z}^d$ with $a_j < b_j$ for all $j = 1, \ldots, d$. The restriction of H_ω to a $\Lambda \in Z$ with Dirichlet boundary conditions is denoted by H_ω^Λ and with Neumann boundary conditions by $H_\omega^{\Lambda,N}$. Consider a group $\{T_k\}_{k\in\mathbb{Z}^d}$ (or semigroup $\{T_k\}_{k\in\mathbb{N}_0^d}$) of measure preserving transformations on the probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$.

Definition 2.7.1. *A set function $F\colon Z \to L^1(\Omega)$ is called a (discrete)* superadditive process *(with respect to $\{T_k\}_k$) if the following conditions are satisfied*

$$F_\Lambda \circ T_k = F_{\Lambda+k} \text{ for all } k \in \mathbb{Z}^d \text{ (or } \mathbb{N}_0^d), \Lambda \in Z \tag{2.43}$$

$$\text{if } \Lambda_1, \ldots, \Lambda_n \in Z \text{ such that } \Lambda := \operatorname{int}\Big(\bigcup_{k=1}^n \overline{\Lambda}_k\Big) \in Z \text{ then, } \; F_\Lambda \geq \sum_{k=1}^n F_{\Lambda_k} \tag{2.44}$$

$$\gamma := \gamma(F) := \sup_{\Lambda\in Z} |\Lambda|^{-1}\, \mathbb{E}\{F_\Lambda\} < \infty \tag{2.45}$$

F is called subadditive *if $-F$ is superadditive.*

Similarly one can define superadditive processes with respect to an action of $\mathbb{R}^d$ on Ω.

We formulate the main result of [17] in the way it suits our needs (see Theorem 2.4 and the Remark on page 59 in [17] and Sect. 6.2 in [313]).

Theorem 2.7.2. *Let F be a discrete superadditive process. For $l \in \mathbb{N}$ set $\Lambda_l :=]-l/2, l/2[^d$. Then the limit*

$$\lim_{l\to\infty} l^{-d} F_{\Lambda_l} \quad \text{exists for almost all } \omega \in \Omega$$

If $\{T_k\}_k$ acts ergodically on $(\Omega, \mathbb{P})$ we have $\lim_{l\to\infty} l^{-d} F_{\Lambda_l}(\omega) = \gamma(F)$ almost surely.

More generally, one can replace the cubes $\Lambda_l, l \in \mathbb{N}$ by a so-called *regular sequence*, cf. [470, 17, 254] or Sect. 6.2 in [313].

To apply the superadditive ergodic theorem we consider for arbitrary, fixed $\lambda \in \mathbb{R}$ the processes given by the eigenvalue counting functions of the Dirichlet and Neumann Laplacian

$$F_\Lambda^D := F_\Lambda^D(\lambda, \omega) := \#\{n |\, \lambda_n(H_\omega^\Lambda) < \lambda\}, \quad \Lambda \in Z$$
$$F_\Lambda^N := F_\Lambda^N(\lambda, \omega) := \#\{n |\, \lambda_n(H_\omega^{\Lambda,N}) < \lambda\}, \quad \Lambda \in Z$$

where H_ω is a random operator as in Definition 1.2.3. Obviously for $\Lambda = \Lambda_l =]-l/2, l/2[^d$ we have $F_\Lambda^D(\lambda) = l^d N_\omega^l(\lambda)$. We will show that $F_\Lambda^D, \Lambda \in Z$ is a superadditive process, which is also true for $-F_\Lambda^N, \Lambda \in Z$. Property (2.43) follows from the equivariance of $\{H_\omega\}_\omega$, while (2.44) and (2.45) are implied by the following

Lemma 2.7.3. *Let H_ω be a random operator as in Definition 1.2.3 and λ a fixed energy value.*

(i) *For two cubes $\Lambda_{(1)} \subset \Lambda_{(2)}$ we have $F^D_{\Lambda_{(2)}} \geq F^D_{\Lambda_{(1)}}$ and $F^N_{\Lambda_{(1)}} \geq F^D_{\Lambda_{(1)}}$.*

(ii) *If $\Lambda_{(1)}, \Lambda_{(2)} \in Z$ are disjoint such that $\Lambda = \Lambda_{(1)} \cup \Lambda_{(2)} \cup M \in Z$ where $M \subset \mathbb{R}^d$ is a set of measure zero, then*

$$F^D_\Lambda \geq F^D_{\Lambda_{(1)}} + F^D_{\Lambda_{(2)}}$$
$$F^N_\Lambda \leq F^N_{\Lambda_{(1)}} + F^N_{\Lambda_{(2)}}$$

(iii) *There exists a $C_\lambda \in \mathbb{R}$ such that for all $\Lambda \in Z$ and $\omega \in \Omega$ we have $F^D_\Lambda(\omega) \leq C_\lambda |\Lambda|$.*

Proof. The first two statements are known as Dirichlet-Neumann bracketing and are stated e.g. in Proposition XIII.15.4 in [408]. See also Sect. I.5 in [84] for analogous results on manifolds. Lemma A.3.1 in the Appendix and its proof imply property (iii) with $C_\lambda = \left(\frac{\mathrm{e}}{2\pi d}\lambda\right)^{d/2}$. □

More background on bracketing techniques can be found in Sects. XIII.3, 15 and 16 in [408]. The Weyl type bounds are related to the Lieb-Thirring and Cwikel-Lieb-Rozenblum estimates for bound states [416, 334, 333, 101].

Now we can state the main result of [254].

Theorem 2.7.4. *There exists a set $\Omega' \subset \Omega$ of full measure such that*

$$N(\lambda) := \lim_{l\to\infty} N^l_\omega(\lambda) \tag{2.46}$$

exists for every $\omega \in \Omega'$ and every continuity point $\lambda \in \mathbb{R}$ of N.

Proof. For a fixed $\lambda \in \mathbb{R}$ one applies Theorem 2.7.2 to $F^D_\Lambda(\lambda, \omega)$, $\Lambda \in Z$, and denotes the corresponding $\gamma(F)$ by $\gamma(\lambda)$. By definition $F^D_\Lambda(\lambda, \omega) \leq F^D_\Lambda(\tilde\lambda, \omega)$ for all $\lambda \leq \tilde\lambda$ and all $\omega \in \Omega$, $\Lambda \in Z$. Thus $\lambda \mapsto \gamma(\lambda)$ is a non-decreasing function. It has at most a countable set of discontinuity points. We denote its complement by $\mathcal{C}$ and choose a dense countable set $S \subset \mathcal{C}$. Hence γ is continuous at each $\lambda \in S$.

Since our transformation group is ergodic, for each λ there is a set Ω_λ of measure one on which the convergence $\lim_{l\to\infty} l^{-d} F^D_{\Lambda_l}(\omega) = \gamma(\lambda)$ holds. Since S is countable, $\Omega' = \cap_{\lambda\in S}\Omega_\lambda$ has full measure, and the convergence statement of Theorem 2.7.2 holds for all $\lambda \in S$ and $\omega \in \Omega'$. Define the distribution function $N(\lambda) := \lim_{S\ni\tilde\lambda\nearrow\lambda} \gamma(\tilde\lambda)$. Thus, γ and N coincide on $\mathcal{C}$.

The monotonicity of $\lambda \mapsto F^D_{\Lambda_l}(\lambda, \omega)$ and the continuity of N on $\mathcal{C}$ imply the statement of the theorem. To see this, choose a sequence $\lambda_n \in S$, $\lambda_n \geq \lambda \in \mathcal{C}$, $\lim_{n\to\infty} \lambda_n = \lambda$. Then we have

$$l^{-d} F^D_{\Lambda_l}(\lambda, \omega) - N(\lambda) \leq l^{-d} F^D_{\Lambda_l}(\lambda_n, \omega) - N(\lambda_n) + N(\lambda_n) - N(\lambda).$$

For $\omega \in \Omega'$ and $\varepsilon > 0$ we choose first n sufficiently large s.t. $N(\lambda_n) - N(\lambda) \leq \varepsilon/2$ and then l sufficiently large s.t. $l^{-d} F^D_{\Lambda_l}(\lambda_n, \omega) - N(\lambda_n) \leq \varepsilon/2$. Thus one sees that

$$\limsup_{l \to \infty} l^{-d} F^D_{\Lambda_l}(\lambda, \omega) \leq N(\lambda).$$

Similarly one can choose a sequence $\lambda_n \in S$, $\lambda_n \leq \lambda \in \mathcal{C}$, $\lim_{n\to\infty} \lambda_n = \lambda$ and then show that $\liminf_{l\to\infty} l^{-d} F^D_{\Lambda_l}(\lambda, \omega) \geq N(\lambda)$. □

For models which satisfy both the conditions of the previous theorem and of 2.3.8 the two definitions of the IDS coincide.

Under certain regularity assumptions the theorem remains true if Neumann boundary conditions are used to define the IDS, cf. Theorem 3.3.(b) of [254]. In this case one works with the subadditive process $F^N_\Lambda(\lambda, \omega)$, $\Lambda \in Z$. There are versions of the above theorem for $\mathbb{R}^d$-ergodic potentials, cf. for instance [254, 224].

2.8 Independence of the Choice of Boundary Conditions

Consider again the more general setting of Schrödinger operators on a Riemannian covering manifold X. If the open subset $\Lambda \subset X$ of finite volume is sufficiently regular, the Neumann Laplacian $H^{\Lambda,N}_\omega$ on Λ has discrete spectrum. One condition which ensures this is the *extension property* of the domain Λ, see e.g. [108], which is in turn satisfied if the boundary $\partial\Lambda$ is piecewise smooth. Minimal conditions which ensure the extension property are discussed in Sect. VI.3 of [454]. Thus it is possible to define the normalised eigenvalue counting function

$$N^{\Lambda,N}_\omega(\lambda) := \frac{1}{|\Lambda|} \#\{n \in \mathbb{N} |\, \lambda_n(H^{\Lambda,N}_\omega) < \lambda\}$$

Let Λ_l be an admissible exhaustion $\Lambda_l \subset X, l \in \mathbb{N}$ of sets which all have the extension property. Consider the sequence of distribution functions $N^{l,N}_\omega := N^{\Lambda_l,N}_\omega$. It is natural to ask whether it converges almost surely, and, moreover, whether its limit coincides with N as defined in Theorem 2.3.8. If this is true, the IDS is independent of the choice of Dirichlet or Neumann boundary conditions used for its construction. This indicates that boundary effects are negligible in the macroscopic limit.

However, this turns out not to be true for all geometric situations. Sznitman studied in [464, 466] the IDS of a random Schrödinger operator on a horoball in hyperbolic space with potential generated by a Poissonian field. He showed that the IDS does depend on the choice of boundary condition used for its construction. Actually, he computes the Lifshitz asymptotics of the IDS at energies near the bottom of the spectrum and shows that it is different for Dirichlet and Neumann boundary conditions.

In contrast, in the case of Euclidean geometry $X = \mathbb{R}^d$, the question of boundary condition independence has been answered positively already some decades ago [43, 254, 432, 137] for a large class of $\mathbb{Z}^d$ or $\mathbb{R}^d$-ergodic random potentials. More recently, there has been interest in the same question if a magnetic field is included in the Hamiltonian, see also Sect. 5.7. In this case the coincidence of the IDS defined by the use of Dirichlet and Neumann boundary conditions was established for bounded (electric) potentials in [375], for non-negative potentials in [125], and for certain potentials assuming both arbitrarily large positive and negative values in [224] and [222]. The last mentioned approach seems to be extensible to non-Euclidean geometries.

3

Wegner Estimate

In 1981 Wegner [493] proved on a physical level of rigour a lower and upper bound on the *density of states* (DOS) of the (discrete) Anderson model and similar lattice Hamiltonians. The density of states — if it exists — is the density function of the IDS.

Wegner's argument did in particular not rely on any information about the type of the spectrum in the considered energy interval. This was important, since before Wegner's result there where various conjectures in the physics community how the DOS should behave at the *mobility edge*, if it exists. This is the name for the critical energy which is supposed to form the boundary between an interval with pure point spectrum and another one with continuous spectrum. Note however that there is so far no rigorous proof of the existence of continuous spectrum for ergodic random Schrödinger operators. There were arguments suggesting that the DOS should diverge to infinity at the mobility edge, others that it should vanish. Wegner's estimates discarded this misconceptions by establishing upper and lower bounds on the DOS. In a sense it is a negative result: one cannot recognise the borderline of different spectral types by looking at the IDS.

In the sequel we concentrate on alloy type models as defined in 1.2.1 (and Remark 1.2.2). We will be concerned here with upper bounds on the DOS only. It is derived by considering its analoga on finite boxes. So what we are speaking about in this chapter is an estimate on

$$\mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} = |\Lambda_l|\,\mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_1)\right\}$$

where for the moment for notational convenience we only consider half open energy intervals $I = [E_1, E_2[$. By the Čebyšev-inequality one sees

$$\mathbb{P}\left\{\sigma(H^l_\omega) \cap I \neq \emptyset\right\} = \mathbb{P}\left\{\mathrm{Tr}P^l_\omega(I) \neq 0\right\} \leq \mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} \tag{3.1}$$

This means that a bound on the averaged trace of the projection gives immediately a bound on the probability to find an eigenvalue in the considered energy interval. Actually, in the literature on Anderson localisation often the

(weaker) bound on $\mathbb{P}\left\{\mathrm{Tr}P^l_\omega(I) \neq 0\right\}$ is called Wegner estimate, since it is sufficient for the purposes of multiscale analysis, see Sect. 3.2.

In the following we will adopt the following notation: $\Lambda_l :=]-l/2, l/2[^d \subset \mathbb{R}^d$ denotes the cube of side length l centred at zero. Occasionally we suppress the dependence on the size and just write Λ. A cube centred at $x \in \mathbb{R}^d$ is denoted by $\Lambda_l + x = \{y + x | \, y \in \Lambda_l\}$ or $\Lambda_l(x)$. The characteristic function of the unit cube $\Lambda_1 + j$ is abbreviated by χ_j. The symbol $\tilde{\Lambda}_l = \Lambda_l \cap \mathbb{Z}^d$ denotes the lattice points contained in Λ_l. For $l \in 2\mathbb{N} + 1$ we have

$$\mathrm{int}\Big(\bigcup_{j \in \tilde{\Lambda}_l} \overline{\Lambda_1(j)} \Big) = \Lambda_l$$

3.1 Continuity of the IDS

The estimates on the expected number of energy levels in I, which most authors derive or use (for localisation proofs) are 'polynomial', more precisely

$$\mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} \leq C_W \, |I|^a \, |\Lambda_l|^b \tag{3.2}$$

Here $|I|$ and $|\Lambda_l|$ denote the (1-dimensional, respectively d-dimensional) Lebesgue measure of the energy interval I, and the set Λ_l, respectively. The *Wegner constant* C_W depends on the various parameters of the model and for continuum Hamiltonians on the supremum of I. Actually, C_W can be assumed to be a monotone non-decreasing function of $\sup I$. However, once $\sup I$ is fixed, C_W is independent both of the energy interval length and the volume. It is obvious that the energy and volume exponents must satisfy $a \in]0, 1], b \in [1, \infty[$. As far as the exponents are concerned, the Wegner estimate is optimal if the dependence on the volume and energy length is linear, i.e. $a = b = 1$.

For, if $b = 1$, the bound (3.2) carries over to the infinite volume IDS:

$$\lim_{l\to\infty} \mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_2)\right\} = \lim_{l\to\infty} \frac{\mathbb{E}\left\{\mathrm{Tr}P^l_\omega([E_1, E_2[)\right\}}{|\Lambda_l|} \leq C_W \, |E_2 - E_1|^a \tag{3.3}$$

Since we know from Theorem 2.3.8 and dominated convergence that for E_1, E_2 in a dense set of energies

$$\lim_{l\to\infty} \mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_1)\right\} = N(E_2) - N(E_1)$$

it follows

$$N(E_2) - N(E_1) \leq C_W \, |E_2 - E_1|^a$$

where $C_W = C_W(E_2)$. Now the monotonicity of the IDS implies its Hölder continuity. Moreover, if the estimate (3.2) is linear in the energy, i.e. $a = 1$, the IDS is even Lipschitz continuous. Thus its derivative, the DOS

$$n(E) := \frac{dN(E)}{dE} \tag{3.4}$$

exists for almost every energy $E \in \mathbb{R}$ and is locally bounded

$$n(E) \leq C_W(E_2) \quad \text{for all } E \leq E_2$$

So the Wegner constant turns out to be a locally uniform bound on the DOS.

As mentioned above Wegner's paper gives also a lower bounds on the DOS. These have been rigorously established in [234, 215]. In the second reference the extension of this result to the alloy-type model on $L^2(\mathbb{R})$ discussed.

Remark 3.1.1. For certain models the bounds derived on $\mathbb{P}\left\{\sigma(H_\omega^l) \cap I \neq \emptyset\right\}$ are not polynomial in the volume. This is the case for one-dimensional Anderson or alloy type models where the coupling constants $\omega_j, j \in \mathbb{Z}$ are distributed according to the Bernoulli distribution: for some $p \in]0,1[$ the random variable ω_0 takes on the value 1 with probability equal to p and the value 0 with probability $1-p$. Since this disorder regime is highly singular, the 'usual' proofs of the Wegner estimate, see Chapters 4 and 5, fail. The ones that do work, yield somewhat weaker results. Indeed, it is proven in [80] (cf. also [429]) for the discrete Anderson model and in [104] for the continuum alloy type model, that for a fixed compact energy interval I and all $\beta \in]0,1[, \gamma > 0$ there exist $l_0 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\mathbb{P}\{d(\sigma(H_\omega^l), E) \leq \mathrm{e}^{-\gamma l^\beta}\} \leq \mathrm{e}^{-\alpha l^\beta} \tag{3.5}$$

for all $E \in I$ and all $l \geq l_0$. Here in the case of the continuum model it has to be assumed that I is disjoint from a discrete set of exceptional energies.

This estimate obviously does not imply continuity of the IDS. Interestingly, for these models, the Hölder continuity of the IDS is established using other techniques which are specifically tailored for the one dimensional case, see [320], [80, App. to Sect. 5] and [104, Thm. 4.1]. Subsequently, this regularity result is used to derive the finite volume estimate (3.5). In higher dimensions, as we have discussed above, one proceeds in the other direction, carrying over finite volume estimates to the macroscopic limit.

The bound (3.5) is still sufficient as an input for the multiscale analysis which proves localisation, cf. e.g. [136] or our discussion in Sect. 3.2. In the discussion of Spencer's example in Sect. 3.2 we will obtain an insight why subexponentially small eigenvalue splittings are effective enough to prevent resonances.

Remark 3.1.2 (Continuity of the IDS on the lattice and in one dimension). In [116] Delyon and Souillard showed by a very simple argument that the IDS of the Anderson model on the lattice $\mathbb{Z}^d$ is continuous, regardless of the continuity of the distribution of the potential values. The potential may even be correlated over long distances, as long as it is an ergodic stochastic field. Delyon and Souillard use the unique continuation property of the discrete

Schrödinger equation, to prove that no eigenvalue can be sufficiently degenerated to produce a jump of the IDS. At the end of Remark 3.1.3 we contrast their theorem with the situation in graphs other than the lattice. In [99] a quantitative version of the continuity is proven: for random, ergodic lattice Hamiltonians the IDS is actually log-Hölder continuous. See also [345] and [159, 123, 87, 357, 124] for related results for Hamiltonians on combinatorial graphs.

Similarly, the IDS is continuous for one dimensional Hamiltonians, both on $\mathbb{Z}$ and on $\mathbb{R}$, [388, 236, 32]. Again, this result can be strengthened to log-Hölder continuity, cf. [100].

Remark 3.1.3 (Continuity of the IDS and geometry). So far we have only mentioned proven or expected assertions on the continuity of the IDS. One might ask whether there are interesting models which exhibit a discontinuous IDS. It turns out that this phenomenon may occur, if the configuration space has a more complicated geometry than $\mathbb{Z}^d$ or $\mathbb{R}^d$. Another example would be the IDS of the Landau Hamiltonian, cf. e.g. the references in Sect. 5.7, in particular [375].

Maybe the simplest example to illustrate the difference between Euclidean and more general geometry is provided by periodic Schrödinger operators. Under mild assumptions on the $\mathbb{Z}^d$-periodic potential V_{per} the IDS of the Schrödinger operator $H_0 = -\Delta + V_{\mathrm{per}}$ on $\mathbb{R}^d$ is absolutely continuous, see the references at the end of Sect. 1.4. In particular the IDS cannot have jumps. However, precisely this can occur for Laplace-Beltrami operators (even without potential) on a Riemannian covering manifold X, see [461, App. 2] and [326, 325]. This phenomenon can be deduced from the fact that Laplacians on covering manifolds may have eigenvalues, as has been shown in [292]. Furthermore, the size of the jumps of the IDS is related to certain geometric invariants. Examples of such invariants are the order of the torsion subgroup of the deck transformation group Γ of X and the L^2-Betti numbers of X, which can be expressed in terms of the Γ-trace on a certain von Neumann algebra, establishing the connection to the representation of the IDS discussed before Theorem 2.3.3, see [118, 126, 122, 123, 421, 124]. Related material can be found in [30, 462, 345, 433, 420, 205, 422, 357, 121, 346, 351, 350, 120, 329].

Some of the Wegner estimates we present in Chapters 4 and 5 extend to alloy type models on manifolds. A particularly interesting phenomenon occurs if one considers a periodic Laplace-Beltrami operator with discontinuous IDS, and perturbs it randomly such that the IDS of the perturbed operator is continuous. This happens if either an appropriate alloy type potential is added to the Hamiltonian or if the metric is multiplied by an appropriate alloy type perturbation, see [325, 326].

A discontinuous IDS may also occur for models with a random geometry. This is the case for the tight-binding Hamiltonian defined on Delone sets studied in [330, 273]. Ideas related to [273] have been used in [419] and in

Sect. 2 of [357]. The paper [419] is devoted to the proof of the existence of spectral gaps for certain graph Hamiltonians.

We will discuss a different example, the *quantum percolation model*, in some more detail, since it fits readily in the class of models which we have described so far. This model has been studied amongst others in [112, 113, 244, 426, 86, 52, 484, 485, 260, 328, 371, 25, 24, 370, 329]. We sketch the site percolation problem on $\mathbb{Z}^d$ with probability parameter p: let $v_k \colon \Omega \to \{0, \infty\}, k \in \mathbb{Z}^d$ be a sequence of independent, identically distributed random variables which take on the value 0 with probability p and the value ∞ with probability $1-p$. Define X_ω to be the infinite component of the set of *active sites* $\{k \in \mathbb{Z}^d | \, v_k(\omega) = 0\}$. The graph X_ω is called the (active) *infinite cluster*. For p above a certain critical value p_c it is known that almost surely an infinite cluster exists [242, 207] and is unique [8, 181], while for $p < p_c$ there is no infinite cluster almost surely.

One defines the Laplacian h_ω on X_ω as the restriction of the finite difference operator onto $\ell^2(X_\omega)$. For a cube Λ_l one defines X_ω^l to be those active sites in $\Lambda_l \cap \mathbb{Z}^d$ which are connected to the boundary $\partial\Lambda_l$ by a chain of active sites. The finite volume Laplacian h_ω^l is the usual finite difference operator restricted to $\ell^2(X_\omega^l)$.

Although the finite active clusters, which would obviously give rise to bound states, are not taken into consideration, it turns out that h_ω has bound states. This was as first observed in [244]. Eigenstates with finite support in the infinite cluster are called *molecular states*. The existence of such states affects the properties of the IDS of h_ω, which is defined in the following way. For each $l \in \mathbb{N}$ the normalised eigenvalue counting function of the Hamiltonian h_ω^l is defined as

$$N_\omega^l(E) := \frac{1}{\#(\Lambda_l \cap \mathbb{Z}^d)} \, \#\{n | \, \lambda_n(h_\omega^l) < E\}$$

which converges for $l \to \infty$ to a non-random limit almost surely [86, 484, 485]. A construction resembling the Schwarz mirror charge principle in electrostatics shows that there are compactly supported eigenfunctions, cf. [86]. Due to their finite support they depend only on the pattern of X_ω in a bounded region. Consequently the patterns and the associated localised eigenfunctions occur with a non-zero density along the infinite cluster and thus produce jumps of the IDS at the corresponding energy. In [86] it is shown by physical arguments that the discontinuities of N constitute a dense set of energies.

Actually, uniqueness of the infinite cluster is not used in the arguments of [86] and a similar argument for constructing finitely supported eigenfunctions does work on the Bethe lattice as well, although there the infinite cluster is not unique. For the quantum percolation model on the Bethe lattice compactly supported eigenfunctions have been constructed in [80, Sect. 7].

A mathematically rigorous study of the quantum percolation model on amenable graphs is undertaken in [484, 485]. There the discontinuities of the

IDS are explained in terms of the breakdown of the unique continuation property of eigenfunctions of the adjacency operator, see also Remark 3.1.2. Moreover, the set of these energies is characterised in the case $X = \mathbb{Z}^d$. From a wider perspective, the properties of this set are related to the Atiyah conjecture, cf. [121]. The low energy asymptotics of percolation Hamiltonians has been studied in [260, 371, 25, 24, 370].

Remark 3.1.4. While the continuity of the IDS has clearly to do with the distribution of eigenvalues of the random Hamiltonian, it only captures a part of the properties of this distribution. The theory of *level statistics* is concerned with the finer structure of the fluctuations of eigenvalues. It can be studied by an appropriate scaling procedure. This has been carried out for certain one-dimensional and discrete models in [369, 368, 413, 412, 366, 195]. More recently the joint distribution of eigenvalues and eigenfunctions in energy and space has been studied in [378, 243].

Another spectral property related to the continuity of the IDS is the multiplicity of the spectrum. The question is whether the spectrum is simple, whether it has finite multiplicity or is infinitely degenerate. For discrete Anderson type models (with absolutely continuous distribution of coupling constants) it has been proven that exponentially localised spectrum, and indeed singular spectrum in general, is simple almost surely, see [442, 230, 233, 279].

3.2 Application to Anderson Localisation

In the last section the implications of Wegner estimates for the IDS were presented. Now we focus on the second main application of those bounds, namely *Anderson localisation*.

As we discussed earlier in Sect. 1.3, this phenomenon tells us that a random family of Schrödinger operators exhibits in a certain energy interval dense pure point spectrum, almost surely. Moreover, the eigenfunctions of the eigenvalues lying in this interval decay exponentially. Even a stronger property, namely *dynamical localisation*, holds. See Sect. 1.3 for more details and references.

For multi-dimensional configuration space there are two techniques at disposal to prove localisation: the *fractional moment method* and the *multiscale analysis* (MSA). The first one is also called *Aizenman-Molchanov* technique and was introduced in [9, 5, 4, 203, 7]. It was so far applicable only to lattice Hamiltonians, up to the recent extension to continuum configuration space [6, 60]. For discrete models it has in fact been proven [10, 11] that in the energy regime where the MSA applies, the Aizenman-Molchanov method works, too.

However, we will discuss in a little more detail only the MSA, since it has found applications to a variety of models and since the Wegner estimate is a key ingredient in the MSA. We first sketch the basic ideas of the MSA, and then discuss shortly its history.

To carry through the MSA one needs two a priori estimates: the *initial scale estimate* and the *Wegner estimate*. These two conditions essentially determine for which single site potentials u, single site distribution measures μ and which energy intervals localisation can be derived. Note that u and μ are parameters which determine our alloy type potential, see Definition 1.2.1.

In the literature one can find multiscale analyses which are adapted to operators describing the propagation of classical waves or to abstract families of differential operators, see among others [165, 166, 98, 107, 458, 186, 278]. In this context one has also to make sure that certain other conditions are satisfied, like the geometric resolvent inequality, the generalised eigenfunction expansion, a rough upper estimate on the trace of spectral projections of finite box operators (obtained e.g. by the Weyl asymptotics), etc. However, since we discuss here only (random) Schrödinger operators, these conditions are automatically satisfied, cf. [458, Sect. 3.2] or [190, App. A].

The multiscale analysis is an induction argument over a sequence of increasing length scales $l_k, k \in \mathbb{N}$. Each scale l_{k+1} is a power l_k^α of the preceding one, where $\alpha \in]1,2[$. Actually, for technical reasons one truncates the scales so that all l_k lie in $6\mathbb{N}$.

One considers the restriction of the alloy type model H_ω to the open cube $\Lambda^{(k)} := \Lambda_{l_k}(0)$ of side length l_k. The corresponding restricted operator is denoted by $H_\omega^{(k)}$, where Dirichlet, Neumann or periodic boundary conditions ensure its selfadjointness. One wants to study decay properties of the Green's function of $H_\omega^{(k)}$, i.e. the integral kernel of the resolvent operator $R_\omega^{(k)}(z) = (H_\omega^{(k)} - z)^{-1}$, where z is taken from the resolvent set $\mathbb{C} \setminus \sigma(H_\omega^{(k)})$. Since we are not interested in pointwise properties of the kernel of $R_\omega^{(k)}(z)$, and since they tend to be unpleasant near the diagonal, we may consider instead the sandwiched resolvent

$$\chi^{out}(H_\omega^l - E)^{-1}\chi^{in}$$

Here χ^{out} denotes the characteristic function of the boundary belt $\Lambda_{l-1} \setminus \Lambda_{l-3}$, and χ^{in} the characteristic function of the interior box $\Lambda_{l/3}$.

The initial scale estimate is stated in terms of the notion of *regular cubes*. A box $\Lambda_l = \Lambda_l(0)$ is called (E,γ)-*regular* if $l \in 6\mathbb{N}$, $E \notin \sigma(H_\omega^l)$, and

$$\|\chi^{out}(H_\omega^l - E)^{-1}\chi^{in}\| \leq \mathrm{e}^{-\gamma l} \tag{3.6}$$

The exponent has to satisfy $\gamma \geq l^\beta$ for some $\beta > -1$. So regularity describes quantitatively how fast the Green's function on a finite box decays. The exponent β must be greater than -1 otherwise the right hand side of (3.6) would be of order one. Estimate (3.6) can be deduced from so-called *Combes-Thomas estimates* [88, 34, 265] if one knows that $d(\sigma(H_\omega^l), E) > \gamma$.

The *initial scale estimate* is satisfied if there exist a scale $l_1 < \infty$ such that for some $\xi > 0$

$$\mathbb{P}\{\omega | \, \forall E \in I : \, \Lambda_l \text{ is } (E,\gamma)\text{-regular for } \omega\} \geq 1 - l^{-\xi} \tag{3.7}$$

for any $l \geq l_1$.

The *weak form* of the *Wegner estimate* as it is needed for the MSA is:

$$\mathbb{P}\{d(\sigma(H_\omega^l), E) \leq \mathrm{e}^{-l^\theta}\} \leq l^{-q} \text{ for all } l \in 6\mathbb{N} \tag{3.8}$$

where $\theta < 1/2$ and $q > d$. Note that this type of estimate is implied by the bound (3.5) discussed in Remark 3.1.1.

The initials scale estimate (3.7) serves as the induction anchor of the MSA. The induction step uses the Wegner estimate and proves that the exponential decay property holds on the subsequent scale l_2 with even higher probability, and that the decay exponent γ essentially does not change. As one repeats the procedure on the scales $l_1, l_2, \ldots$ one proves that the decay of the Green's functions $\chi^{out}(H_\omega^{(k)} - E)^{-1}\chi^{in}$ holds with probability which tends to one, with error bounded polynomially in l_k^{-1}.

Thus one establishes the exponential decay of the sandwiched resolvent on arbitrary large cubes, where the decay exponent γ is bounded away from zero uniformly in the scales. Now one uses polynomial bounds on the growth of eigenfunctions and subsolution estimates to prove spectral localisation, cf. for instance [458, Sect. 3.3]. To prove dynamical localisation one has to do more work, see e.g. [458, Sect. 3.4] or [183, 107, 186].

The assumptions for the MSA depend on several parameters, and so do the various versions of localisation which may be obtained by it. Germinet and Klein showed in [186] using a bootstrap MSA how to optimise the dependence of the MSA on the various parameters, i.e. how to obtain with the weakest assumptions in the input the strongest conclusions.

The MSA was introduced by Fröhlich and Spencer in [176]. The method applied to the Anderson model on the lattice and experienced various improvements and applications [354, 355, 175, 444, 115].

Based on results from [135] and [453] von Dreifus and Klein presented in [136] a streamlined version of the MSA. Although results on localisation for continuum Hamiltonians existed earlier [353, 306], it was this simplification of the MSA, which made alloy type Schrödinger operators more accessible to systematic research.

There was a series of articles which proved various variants of the MSA for continuum models [280, 89, 282, 265, 183, 170, 107, 458, 186]. Other works concentrated onto identifying energy/disorder regimes where one can prove the assumptions needed to apply the MSA [280, 281, 282, 34, 265, 264, 186, 457, 480, 481, 213, 500, 287, 188, 187]. Using Combes-Thomas estimates, the initial scale estimate (3.7) can be verified in energy intervals where the IDS is very sparse. This is for instance the case in neighbourhoods of spectral fluctuation boundaries, which we discussed in Sect. 1.4.

Remark 3.2.1. One dimensional models play a special role in the game of localisation. Namely, for $d = 1$ there are some specific techniques available which do not exist in higher dimensions, or are not as effective. See the discussion on page 9.

Consequently, in one space dimension localisation has been proven for some models even before the MSA technique was available. See [202, 369, 201, 78, 415, 306] for various models on $\mathbb{Z}$ or $\mathbb{R}$. Furthermore there are certain types of random Schrödinger operators which even now can be treated only in one dimension, like the random displacement model [74, 445] or discrete models with Bernoulli disorder [80, 429]. This restriction to $d = 1$ is partially due to the fact, that there is no Wegner estimate at disposal in these cases. However, for the continuum analog of the Bernoulli-Anderson model, localisation has been proven in arbitrary dimension at low energies in [59], see Sect. 4.8. Based on this work, localisation has been established for Schrödinger operators with Poissonian random potential at low energies in [184, 185].

Let us also mention that one dimensional techniques usually establish localisation on the whole energy axis. For higher dimensional continuum models localisation results are restricted to energies near spectral boundaries.

The following section gives an idea where the Wegner estimate is used in the MSA.

3.3 Resonances of Hamiltonians on Disjoint Regions

Rather than describing precisely how the Wegner estimate enters in the induction step of the MSA we will confine ourselves to present an illuminative example due to Spencer [452]. It was originally formulated for lattice Hamiltonians, but can also be considered in the continuum case, as we have learned from P. Müller.

As we mentioned earlier, (3.2) implies for $0 \leq \delta < 1$

$$\mathbb{P}\{\omega |\, d(\sigma(H_\omega^l), E) \leq \delta\} \leq C_W(E)\, \delta^a\, |\Lambda_l|^b \tag{3.9}$$

This inequality implies that with respect to the parameter ω the eigenvalues of H_ω^l do not cluster on the energy axis. To give a more precise meaning to this statement, consider two Hamiltonians $H^1 = H_\omega^{\Lambda_l(x)}$ and $H^2 = H_\omega^{\Lambda_l(y)}$ which are restrictions of H_ω to the cubes $\Lambda_l(x)$ and $\Lambda_l(y)$ respectively. Assume that the boxes $\Lambda_l(x)$ and $\Lambda_l(x)$ are sufficiently far apart such that H^1 and H^2 are independent. Let I be a bounded interval and consider the event

$$\Omega(\sigma_1, \sigma_2) := \{\omega |\, B_\delta(\sigma_1) \cap B_\delta(\sigma_2) \neq \emptyset\}$$

where σ_i stands for $\sigma(H^i) \cap I$. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of H^1 in I. By Weyl's law we know that $N \leq C_I |\Lambda_l|$, where C_I is independent of ω. Since

$$\Omega(\sigma_1, \sigma_2) \subset \bigcup_{n=1}^{N} \Omega(n, \sigma_2) \quad \text{where } \Omega(n, \sigma_2) := \{\omega |\, B_{2\delta}(\lambda_n(\omega)) \cap \sigma_2 \neq \emptyset\}$$

we may use (3.9) to conclude

$$\mathbb{P}\{\Omega(\sigma_1, \sigma_1)\} \leq C\delta^a\, |\Lambda_l|^{b+1} \tag{3.10}$$

This means that resonances of H^1 and H^2, i.e. the occurrence of approximately the same eigenvalues for both operators, are very unlikely.

The feature which is common to Spencer's example and the MSA is the effect of resonances between Hamiltonians which are localised to disjoint cubes. As we mentioned earlier, in the induction step of the MSA one puts together boxes Λ_l of side length l to form a larger cube Λ_L of side length L. Assume that one knows already that the Green's functions of the operators H^l_ω living on any one of the small cubes Λ_l decays exponentially.

The Schrödinger operator H^L_ω on Λ_L is obtained when we remove the boundary conditions which separate the smaller boxes Λ_l. The question is whether the Green's function on Λ_L will still decay (with approximately the same rate). To answer this question affirmatively it is not enough to know the exponential decay of the individual Green's functions on the small boxes Λ_l, but it has to be ensured that they are not in resonance with each other.

Resonance means in this context that the spectra of two restriction H^1, H^2 of H_ω to disjoint cubes are very close to each other. An appropriate quantitative formulation can be given in terms of

$$d(\sigma_1, \sigma_2) := \inf\{d(\lambda_1, \lambda_2) |\, \lambda_1 \in \sigma_1, \lambda_2 \in \sigma_2\} \tag{3.11}$$

as we will see below.

The model situation we are about to consider is easier than the one occurring in the MSA because we do not introduce boundary conditions but confine ourselves to the analysis of the discrete spectrum below zero.

Example 3.3.1 (Spencer's example [452, p. 903–904]). Consider two smooth potentials $V_1, V_2 \leq 0$ with compact support and set

$$\mathcal{V}_i := \operatorname{supp} V_i \subset B_r(a_i), r > 0, a_i \in \mathbb{R}^d, i = 1, 2 \tag{3.12}$$

It follows $d(\mathcal{V}_1, \mathcal{V}_2) \geq |a_1 - a_2| - 2r =: \varrho$. Consider furthermore the operators

$$H := H_0 + V_1 + V_2, \qquad H_0 := -\Delta \tag{3.13}$$

$$H_i := -\Delta + V_i, \quad i = 1, 2 \tag{3.14}$$

Denote by $\sigma_i := \sigma(H_i) \cap]-\infty, 0[$, $i = 1, 2$ the negative spectra, which are purely discrete. We are interested in the localisation and decay properties of the corresponding eigenfunctions.

We look at two cases, where the first has a special symmetry and the second corresponds to the situation expected to occur in a random medium.

Case (A):
Consider first the *exceptional* case in which V_2 is obtained from V_1 by a reflection along an axis of symmetry. Without loss of generality

$$V_2(x_1, x_2, \ldots, x_d) = V_1(-x_1, x_2, \ldots, x_d) \tag{3.15}$$

Thus H commutes with the reflection operator

$$\Pi\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad (\Pi f)(x_1, x_2, \ldots, x_d) = f(-x_1, x_2, \ldots, x_d) \quad (3.16)$$

In particular, for every eigenfunction ψ

$$H\psi = \lambda\psi, \quad \lambda < 0$$

the reflected function $\Pi\psi$ is an eigenvector of H as well. If ψ is localised around a_1 $\Pi\psi$ will be localised around a_2. Thus a typical vector from $\operatorname{span}\{\psi, \Pi\psi\}$ will have non negligible amplitudes both at a_1 and a_2, even for large distances ϱ. For short, eigenfunctions of H do not need to have just one centre of localisation.

We are dealing with a resonance between the two disjoint regions $\mathcal{V}_1, \mathcal{V}_2$, or more precisely between the spectra of H_1 and H_2. Actually, we encountered the extreme case where the the spectra σ_1 and σ_2 are not only close to each other but identical.

The example we just considered exhibited a special symmetry, namely $[H, \Pi] = 0$. For random potentials we expect generically that such symmetries are absent and that the spectra σ_1 and σ_2 have positive distance. This situation is considered in

Case (B):
We give a condition on $d(\sigma_1, \sigma_2)$ which ensures that the eigenfunctions of H (defined in (3.13)) are localised at *only one* of the potential wells V_1, V_2. Namely, assume that

$$d(\sigma_1, \sigma_2) \geq \mathrm{e}^{-\sqrt{\varrho}} =: \varepsilon \quad (3.17)$$

For an eigenvalue $\lambda \leq -\frac{2}{\varrho}$, with corresponding equation $H\psi = \lambda\psi$, we have either

$$|\lambda_1^j - \lambda| \geq \varepsilon/2, \quad \forall \lambda_1^j \in \sigma_1 \qquad \text{or} \qquad |\lambda_2^j - \lambda| \geq \varepsilon/2, \quad \forall \lambda_2^j \in \sigma_2 \quad (3.18)$$

Assume without loss of generality the first case. The eigenfunction equation implies

$$-\psi = (H_1 - \lambda)^{-1} V_2 \psi$$

Applying twice the resolvent equation we obtain

$$-\psi = [R_0 - R_0 V_1 R_0 + R_0 V_1 R_1 V_1 R_0]\, V_2 \psi, \quad (3.19)$$

where $R_i := (H_i - \lambda)^{-1}$, $i = 0, 1$ denotes the resolvents. We show that the amplitude of ψ on $\mathcal{V}_1$ is exponentially small in the parameter ϱ. Denote with χ^i the characteristic function of $\mathcal{V}_i$ for $i = 1, 2$ and multiply (3.19) with χ^1

$$-\chi^1\psi = \chi^1 R_0 \chi^2 V_2 \psi - \chi^1 R_0 V_1 \chi^1 R_0 \chi^2 V_2 \psi + \chi^1 R_0 V_1 R_1 V_1 \chi^1 R_0 \chi^2 V_2 \psi \quad (3.20)$$

The free resolvent decays exponentially, see e.g. [3] or [407, IX.30],

$$R^{1,2} := \|\chi^1 R_0 \chi^2\| \lesssim \mathrm{e}^{-\varrho\sqrt{-\lambda}},$$

the terms $\|V_2\psi\|, \|\chi^1 R_0 V_1\|$ are bounded uniformly in ϱ, and (3.18) implies

$$\|R_1\| \leq \frac{2}{\varepsilon} = 2\mathrm{e}^{\sqrt{\varrho}}$$

Consequently

$$\begin{aligned}\|\chi^1\psi\| \leq R^{1,2}\ \|V_2\psi\| + \|\chi^1 R_0 V_1\|\ R^{1,2}\ \|V_2\psi\| \\ + \|\chi^1 R_0 V_1\|\ \|R_1 V_1\|\ R^{1,2}\ \|V_2\psi\|\end{aligned}$$

is bounded by a constant times $\exp(-\sqrt{\varrho})\,\|\psi\|$, since $\varrho^{-1} \leq -\lambda$.

Let us finish this section by discussing some aspects and contrasts of the two cases considered in the example.

(i) In general, the spectrum alone describes only general properties of the solution of the eigenvalue equation. In our example it is the additional information contained in (3.15) and (3.17), respectively, which allows us to analyse the eigenfunctions more precisely.

(ii) Obviously, in Case (A), the Green's function decays in space, too. However, this decay is not yet felt at the scale ϱ, since $|\psi(a_1)\psi(a_2)|$ converges to a positive constant for $\varrho \to \infty$. On the contrary, in Case (B), the amplitude is small either at a_1 or a_2, more precisely

$$|\psi(a_1)\psi(a_2)| \lesssim \mathrm{e}^{-\sqrt{\varrho}}$$

(iii) A semiclassical analysis of double well potentials is carried out, for instance, in [216].

4

Wegner's Original Idea. Rigorous Implementation

In this section we present a proof of Wegner's estimate following his original ideas in [493]. His proof was originally formulated for the discrete Anderson model. In the meantime, it has been cast into mathematically rigorous form and adapted for continuum Hamiltonians. We follow mostly the arguments of Kirsch [249]. There are proofs of Wegner's estimate by other authors, which make use of the ideas in [493]. Let us mention [353, 352, 280, 281, 169, 94, 93].

The theorem to be proven is

Theorem 4.0.1. *Let H_ω be as in Definition 1.2.1 and assume additionally that there exists a $\kappa > 0$ such that*

$$u \geq \kappa \chi_{[-1/2,1/2]^d} \tag{4.1}$$

Then for all $E_0 \in \mathbb{R}$ there exists a constant $C_W = C_W(E_0)$ such that for all $l \in \mathbb{N}$, $E \leq E_0$ and all $\varepsilon \in [0,1]$

$$\mathbb{E}\left\{\mathrm{Tr}\left[P^l_\omega([E-\varepsilon, E+\varepsilon])\right]\right\} \leq C_W \ \varepsilon \ l^{2d} \tag{4.2}$$

As before P^l_ω denotes the spectral projection associated to the operator H^l_ω. The theorem is proven in the next section. Its bound with respect to the volume term l^d is quadratic and does not yield a continuity statement for the IDS. Subsequently we show how this estimate was improved in [97]. Denote by ω_+ and ω_- the largest, respectively the smallest value a coupling constant may take.

4.1 Spectral Averaging of the Trace of the Spectral Projection

We show that the expectation over the randomness smears out the eigenvalues of H^l_ω and thus regularises the trace of $P^l_\omega(I)$.

By definition, the spectral projection $P^l_\omega(I) = \chi_I(H^l_\omega)$ is the characteristic function of H^l_ω. For certain purposes it will be necessary to differentiate this

function with respect to the energy parameter, which motivates the introduction of the following smooth 'switch function'.

Let ρ be a smooth, non-decreasing function such that on $]-\infty,-\varepsilon]$ it is identically equal to -1, on $[\varepsilon,\infty[$ it is identically equal to zero and $\|\rho'\|_\infty \leq 1/\varepsilon$. Then

$$\chi_{]E-\varepsilon,E+\varepsilon[}(x) \leq \rho(x-E+2\varepsilon) - \rho(x-E-2\varepsilon) = \int_{-2\varepsilon}^{2\varepsilon} dt\, \rho'(x-E+t)$$

Thus by the spectral theorem

$$P^l_\omega(]E-\varepsilon,E+\varepsilon[) \leq \int_{-2\varepsilon}^{2\varepsilon} dt\, \rho'(H^l_\omega - E + t)$$

in the sense of quadratic forms. Since $B_\varepsilon(E) =]E-\varepsilon,E+\varepsilon[$ is bounded and $\sigma(H^l_\omega)$ discrete, the above operators are trace class and we may estimate:

$$\mathrm{Tr}\Big[P^l_\omega(B_\varepsilon(E))\Big] \leq \mathrm{Tr}\Big[\int_{-2\varepsilon}^{2\varepsilon} dt\, \rho'(H^l_\omega - E + t)\Big] = \sum_{n\in\mathbb{N}} \int_{-2\varepsilon}^{2\varepsilon} dt\, \rho'(\lambda^l_n(\omega) - E + t)$$

where $\lambda^l_n(\omega)$ denotes the eigenvalues of H^l_ω enumerated in non-decreasing order and counting multiplicities. Only a finite number of terms in the sum are non-zero. More generally the above arguments prove the following.

Lemma 4.1.1. *Let H be an operator with purely discrete spectrum. Denote by $\lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of H. Then for $E \in \mathbb{R}$ and $\varepsilon > 0$*

$$\mathrm{Tr}\Big[\chi_{B_\varepsilon(E)}(H)\Big] \leq \sum_{n\in\mathbb{N}} \int_{-2\varepsilon}^{2\varepsilon} dt\, \rho'(\lambda_n - E + t)$$

In the following we analyse the behaviour of the spectrum of the Schrödinger operator under the perturbation $\omega_j\, u(\cdot - j)$. Fix a box-size $l \in \mathbb{N}$, a lattice site $j \in \tilde{\Lambda}_l$ and a configuration of coupling constants $\omega \in \Omega$ and consider the one-parameter family of operators

$$t \mapsto H_t := H + tU, \text{ where } H = H^l_\omega \text{ and } U = u(\cdot - j)$$

By the arguments in Sect. 1.2 the single site potential is infinitesimally bounded with respect to H, thus H_t forms a holomorphic family of type (A) in the sense of Kato [239] for t in a neighbourhood of the real line, cf. e.g. XII. Sect. 2 in [408]. Moreover, H_t has compact resolvent by XII. Sect. 14 in [408]. Hence one may apply a theorem of Rellich [411], see also Theorem VII. Sect. 3.9 in [239]. It says that the eigenvalues and eigenvectors of H_t can be chosen to be real analytic on $\mathbb{R}$. Actually, each eigenvalue is holomorphic on a neighbourhood of $\mathbb{R}$ in the complex plane, but their intersection may contain only $\mathbb{R}$.

If $\lambda_n(t)$ is a non-degenerate eigenvalue of H_t, first order perturbation theory tells us that there exists a normalised eigenfunction $\psi_n(t)$ such that

$$\frac{d\lambda_n}{dt}(t_0) = \langle \psi_n(t_0), U\psi_n(t_0)\rangle \tag{4.3}$$

Remark 4.1.2. This is sometimes called *Hellmann-Feynman formula*, and it holds true also if the eigenvalue λ_n happens to be degenerate at $t = t_0$, cf. for instance [228]. One has however to chose the enumeration of the eigenvalues λ_n and eigenvectors ψ_n in such a way that the pair $\lambda_n(t), \psi_n(t), t < t_0$ continues holomorphically into $\lambda_n(t), \psi_n(t), t > t_0$. Note that this is actually not the case with the enumeration we chose earlier, where $\lambda_n(t)$ denotes the n^{th} eigenvalue of H_t. There are two possibilities to solve the problem: either one chooses a somewhat un-intuitive enumeration of eigenvalues which makes them — together with the eigenvectors — holomorphic functions of t. Or one sums over the eigenvalues. Indeed, formula (4.3) remains true if we sum over all eigenvalues which correspond to a degeneracy. More precisely, for a degenerate eigenvalue $\lambda_n(t_0)$ denote by $l, k \in \mathbb{N}$ the largest numbers such that $\lambda_{n-l}(t_0) = \cdots = \lambda_n(t_0) = \cdots = \lambda_{n+k}(t_0)$ and set $S(t) = \sum_{m=n-l}^{n+k} \lambda_m(t)$. Then we have

$$\frac{dS}{dt}(t_0) = \sum_{m=n-l}^{n+k} \langle \psi_m(t_0), U\psi_m(t_0)\rangle$$

In the application of (4.3) in the next proposition we will be considering all eigenvalues below a certain energy. Thus, if we consider one eigenvalue participating in a degeneracy we will actually take into account all participating eigenvalues.

The results for one parameter families of operators carry over to the multi-parameter family $\omega \mapsto H^l_\omega$. Thus we have

$$\sum_{j\in\tilde{\Lambda}} \frac{\partial \lambda^l_n(\omega)}{\partial \omega_j} = \sum_{j\in\tilde{\Lambda}} \langle \psi_n, u(\cdot - j)\psi_n\rangle$$

where ψ_n are normalised eigenvectors corresponding to $\lambda^l_n(\omega)$ and $\tilde{\Lambda} = \tilde{\Lambda}_l$. By assumption (4.1) we have

$$\sum_{j\in\tilde{\Lambda}} \langle \psi_n, u(\cdot - j)\psi_n\rangle \geq \kappa > 0 \tag{4.4}$$

Now the chain rule

$$\sum_{j\in\tilde{\Lambda}} \frac{\partial \rho(\lambda^l_n(\omega) - E + t)}{\partial \omega_j} = \rho'(\lambda^l_n(\omega) - E + t) \sum_{j\in\tilde{\Lambda}} \frac{\partial \lambda^l_n(\omega)}{\partial \omega_j}$$

implies

$$\rho'(\lambda_n^l(\omega) - E + t) \leq \kappa^{-1} \sum_{j \in \tilde{\Lambda}} \frac{\partial \rho(\lambda_n^l(\omega) - E + t)}{\partial \omega_j} \tag{4.5}$$

Due to monotonicity, when integrating over one coupling constant, we obtain

$$\begin{aligned} \int d\omega_j \, f(\omega_j) \, \frac{\partial \rho(\lambda_n^l(\omega) - E + t)}{\partial \omega_j} &\leq \|f\|_\infty \int d\omega_j \, \frac{\partial \rho(\lambda_n^l(\omega) - E + t)}{\partial \omega_j} \\ &= \|f\|_\infty \big[\rho(\lambda_n^l(\omega, j = \max) - E + t) - \rho(\lambda_n^l(\omega, j = \min) - E + t)\big] \end{aligned}$$

where $\lambda_n^l(\omega, j = \max)$ denotes the n^{th} eigenvalue of the operator

$$H_\omega^l(j = \max) := H_\omega^l + (\omega_+ - \omega_j) \, u(x - j)$$

corresponding to the configuration of the potential where ω_j takes its maximal value. Analogously, we use the notation $\lambda_n^l(\omega, j = \min)$. This proves

Proposition 4.1.3.

$$\begin{aligned} \mathbb{E}\left\{\mathrm{Tr}\big[P_\omega^l([E - \varepsilon, E + \varepsilon])\big]\right\} \leq \frac{\|f\|_\infty}{\kappa} \sum_{n \in \mathbb{N}} \int_{-2\varepsilon}^{2\varepsilon} dt \sum_{j \in \tilde{\Lambda}} \\ \mathbb{E}\left\{\rho[\lambda_n^l(\omega, j = \max) - E + t] - \rho[\lambda_n^l(\omega, j = \min) - E + t]\right\} \end{aligned}$$

The upper bound can be also written as

$$\frac{\|f\|_\infty}{\kappa} \int_{-2\varepsilon}^{2\varepsilon} dt \sum_{j \in \tilde{\Lambda}} \mathbb{E}\left\{\mathrm{Tr}\big[\rho[H_\omega^l(j = \max) - E + t] - \rho[H_\omega^l(j = \min) - E + t]\big]\right\}$$

Since $\rho \leq 0$

$$\begin{aligned} &\sum_{n \in \mathbb{N}} \rho[\lambda_n^l(\omega, j = \max) - E + t] - \rho[\lambda_n^l(\omega, j = \min) - E + t] \\ &\leq - \sum_{n \in \mathbb{N}} \rho[\lambda_n^l(\omega, j = \min) - E + t] \leq C_{E+3\varepsilon} \; l^d \leq C_{E_0+3} \; l^d \end{aligned} \tag{4.6}$$

by bound (iii) in Lemma 2.7.3. This proves Theorem 4.0.1.

Remark 4.1.4. The suboptimality of the volume bound in Theorem 4.0.1 is due to the rough estimate (4.6). The left hand side of the inequality is the net increase of the number of eigenvalues in the energy interval $]E - t - \varepsilon, E - t + \varepsilon[$ due to the increase of the j^{th} coupling constant from its minimal to its maximal value. The average of this quantity is independent of Λ, cf. [92]. However, in (4.6) we estimated it by the total number of eigenvalues below the energy $E + 3\varepsilon$, which is by Weyl's law proportional to the volume of Λ. Thus we get an extra volume factor in the upper bound of the Wegner estimate.

4.2 Improved Volume Estimates

As we mentioned earlier, Theorem 4.0.1 is unsatisfactory with respect to the volume dependence of the upper bound. Since it is quadratic, it does not imply any regularity property of the IDS. There is a number of papers [34, 97, 298, 93, 95, 218] which prove an upper bound which is linear in the volume. Mostly the trade-off is that either the bound is worse with respect to the length of the energy interval $[E-\varepsilon, E+\varepsilon]$, or that it holds only under certain additional hypotheses on the model or energy regime. Only the very recent preprint [91] gives a Wegner estimate with an upper bound which is linear both in the volume and the length of $[E-\varepsilon, E+\varepsilon]$, without adding extra assumptions on the model or energy regime (except the periodicity of the deterministic part of the operator). A different method of proof which yields Wegner estimates with linear volume and energy dependence is discussed in Chapter 5.

We describe results from two of the abovementioned papers in some detail. Let us start with the work [93] of Combes, Hislop and Klopp where they study alloy type models with single site potentials of small support, and establish the Hölder continuity of the IDS at all energies.

They consider the case where the single site potential $u \in L^\infty_c(\mathbb{R}^d)$ is non-negative and not identically equal to zero, and treat three different situations. In all of them the unperturbed background operator $H_0 = (-i\nabla - A)^2 + V_0$ may include a magnetic vector potential A and a (scalar) electric potential V_0. The potentials have to satisfy some regularity conditions such that H_0 is selfadjoint and has $C^\infty_0(\mathbb{R}^d)$ as an operator core. The coupling constants are independent identically distributed with distribution μ, which has a density in $L^\infty_c(\mathbb{R})$.

Here are several hypotheses each of which is sufficient for a continuity result on the IDS.

(i) The background operator H_0 has an IDS N_0, which is Hölder continuous

$$|N_0(E_2) - N_0(E_1)| \leq C_0 |E_2 - E_1|^{\tilde{\alpha}}$$

with Hölder exponent $\tilde{\alpha} \in]0,1]$. The constant $C_0 = C_0(I)$ can be chosen uniformly for E_2, E_1 in a given compact interval I.

(ii) The background operator H_0 is periodic with respect to the lattice $\mathbb{Z}^d$ and has the unique continuation property, cf. Definition 4.2.2 below. The set $\{x \,|\, u(x) > 0\}$ contains an open subset of $\mathbb{R}^d$.

(iii) The space dimension is $d = 2$. The operator $H_0 = (-i\nabla - A)^2 + V_{\mathrm{per}}$ consists of a Landau Hamiltonian with vector potential $A(x_1, x_2) = \frac{B}{2}(-x_2, x_1)$ and a periodic scalar potential V_{per}. Here $B > 0$ is the (constant) magnetic field strength. The magnetic flux trough a unit cell satisfies the rationality condition

$$B \in 2\pi\,\mathbb{Q} \tag{4.7}$$

The scalar potential V_{per} is a $\mathbb{Z}^2$-periodic function in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

In case (i) set $\alpha_c = \frac{\tilde{\alpha}}{\tilde{\alpha}+2}$, otherwise set $\alpha_c = 1$.

Theorem 4.2.1. *Let H_ω be an alloy type model satisfying either one of the above conditions* (i)–(iii)*. Then, for each $a \in]0, \alpha_c[$, and any compact interval $I \subset \mathbb{R}$ there exists a constant $C_{a,I}$ such that the IDS of H_ω satisfies*

$$|N(E_2) - N(E_1)| \leq C_{a,I} \, |E_2 - E_1|^a \quad \text{for all } E_1, E_2 \in I$$

In a follow up work [95] on the Landau Hamiltonian in collaboration with Raikov condition (4.7) in hypothesis (iii) has been removed.

Note that this result is formulated not in terms of the average of the eigenvalue counting function on a finite cube, but rather in terms of IDS itself. The following property appeared in the hypotheses of the theorem.

Definition 4.2.2. *A Schrödinger operator H on $\mathbb{R}^d$ has the* unique continuation property *if the following condition holds: Let $E \in \mathbb{R}$ be arbitrary and ϕ a function in the domain of the operator H satisfying $(H - E)\phi = 0$. If ϕ vanishes on some open set, then it vanishes identically on $\mathbb{R}^d$.*

For a discussion of this property, see for instance [498].

The next two results are taken from [218]. Let μ as before denote the distribution of any of the bounded random coupling constants $\omega_k, k \in \mathbb{Z}^d$.

Theorem 4.2.3. *Let H_ω be an alloy type model as in Definition 1.2.1. Assume that V_{per} has the unique continuation property and is bounded below, μ has a density in $L^\infty_c(\mathbb{R})$ and $0 \leq u \in L^\infty$ is strictly positive on an open set. Then, for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that*

$$N(E + \varepsilon) - N(E - \varepsilon) \leq C_W \; \varepsilon \, |\log \varepsilon|^d \tag{4.8}$$

for all $E \leq E_0$ and $\varepsilon \leq 1/2$.

The next result is formulated in terms of a Wegner estimate, i.e. a bound on the average of a finite volume IDS measure of a small energy interval.

For $\varepsilon > 0$ we define the (global) modulus of continuity of μ by

$$s(\mu, \varepsilon) = \sup \left\{ \mu\Big(\Big[E - \frac{\varepsilon}{2}, E + \frac{\varepsilon}{2} \Big] \Big) \;\Big|\; E \in \mathbb{R} \right\} \tag{4.9}$$

This allows us to formulate the next theorem.

Theorem 4.2.4. *Let H_ω be an alloy type model as in Definition 1.2.1 and $u \geq \kappa \chi_{[-1/2,1/2]^d}$ for some positive κ. Then, for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that, for all $E \leq E_0$ and $\varepsilon \leq 1/2$*

$$\mathbb{E}\left\{ \mathrm{Tr}[\chi_{[E-\varepsilon, E+\varepsilon]}(H^l_\omega)] \right\} \leq C_W \; s(\mu, \varepsilon) \, |\log \varepsilon|^d \; |\Lambda_l| \tag{4.10}$$

Actually, both of these results may be formulated for background operators H_0 which incorporate quite general magnetic vector potentials, see Sect. A.3 and [218]. We will prove Theorem 4.2.4. The proof of Theorem 4.2.3 relies on a combination of results of [93] and [218]. In particular, the unique continuation properties of periodic operators developed in [93] play a crucial role.

Remark 4.2.5 (Analogous results on combinatorial and metric graphs). In [218] one can find also an analogous Wegner estimate for the Anderson model h_ω on $\ell^2(\mathbb{Z}^d)$. Since the model is discrete, single site potentials correspond to finite rank operators. For this reason one has uniform estimates on the spectral shift function at disposal. Consequently, the Anderson model satisfies the following Wegner bound

$$\mathbb{E}\{\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(h_\omega^l)]\} \leq C_W \ s(\mu,\varepsilon) \ |\Lambda_l| \tag{4.11}$$

(See also [314] for similar results.) An adoption of the bound (4.11) to random Schrödinger operators on metric graphs has been derived in [210].

A partial integration estimate concerning probability distributions which do not have densities, which is used in the proof of Theorem 4.2.4, is deferred to Sect. 4.7. A further important tool in the proof are estimates on the spectral shift function (SSF) which are developed in Appendix A, see in particular Corollary A.4.3.

Proof (of Theorem 4.2.4). Let $x \mapsto \rho_{E,\varepsilon}(x) = \rho_{0,\varepsilon}(x-E)$ be a switch function adapted to the interval $[E-\varepsilon, E+\varepsilon]$, see the beginning of Sect. 4.1 or the discussion preceding (A.24). Then

$$\chi_{[E-\varepsilon,E+\varepsilon]}(x) \leq \rho(x+2\varepsilon) - \rho(x-2\varepsilon)$$

We may assume without loss of generality $\kappa = 1$, see Remark 5.0.2, and thus $\sum_k u(\cdot - k) \geq 1$. By the mini-max principle for eigenvalues, we conclude

$$\mathrm{Tr}[\rho(H_\omega^l + \varepsilon)] \leq \mathrm{Tr}\Big[\rho(H_\omega^l + \varepsilon \sum_k u(\cdot - k))\Big] \tag{4.12}$$

Assume without loss of generality that $l \in \mathbb{N}$. Then Λ_l is decomposed in $L := l^d$ unit cubes. We enumerate the lattice sites in Λ_l by $k\colon \{1,\dots,L\} \to \tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$, $n \mapsto k(n)$ and set

$$W_0 \equiv 0, \quad W_n = \sum_{m=1}^{n} u(\cdot - k(m)), \qquad n = 1,2,\dots,L$$

Thus

$$\begin{aligned}
\mathbb{E}\{\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega^l)]\} &\leq \mathbb{E}\{\mathrm{Tr}[\rho(H_\omega^l + 2\varepsilon) - \rho(H_\omega^l - 2\varepsilon)]\} \\
&\leq \mathbb{E}\{\mathrm{Tr}[\rho(H_\omega^l - 2\varepsilon + 4\varepsilon W_L) - \rho(H_\omega^l - 2\varepsilon)]\} \\
&\leq \mathbb{E}\left\{\sum_{n=1}^{L} \mathrm{Tr}[\rho(H_\omega^l - 2\varepsilon + 4\varepsilon W_n) - \rho(H_\omega^l - 2\varepsilon + 4\varepsilon W_{n-1})]\right\}
\end{aligned} \tag{4.13}$$

We fix $n \in \{1,\dots,L\}$, denote

$$\omega^\perp := \{\omega_k^\perp\}_{k\in\tilde{\Lambda}}, \qquad \omega_k^\perp := \begin{cases} 0 & \text{if } k = k(n), \\ \omega_k & \text{if } k \neq k(n), \end{cases}$$

and set

$$\phi_n(t) := \mathrm{Tr}\big[\rho\big(H^l_{\omega^\perp} - 2\varepsilon + 4\varepsilon W_{n-1} + t\cdot u(\cdot - k(n))\big)\big], \quad t \in \mathbb{R}.$$

The function ϕ_n is continuously differentiable, monotone increasing and bounded. By definition of ϕ_n,

$$\mathbb{E}\left\{\mathrm{Tr}[\rho(H^l_\omega - 2\varepsilon + 4\varepsilon W_n) - \rho(H^l_\omega - 2\varepsilon + 4\varepsilon W_{n-1})]\right\} \\ \leq \mathbb{E}\left\{\int [\phi_n(\omega_{k(n)} + 4\varepsilon) - \phi_n(\omega_{k(n)})]\, d\mu(\omega_{k(n)})\right\}$$

Let $a = (\inf \operatorname{supp} \mu) - 1$ and $b = (\sup \operatorname{supp} \mu) + 1$. Using Lemma 4.7.1 together with Corollary A.4.3, we have

$$\int [\phi_n(\omega_{k(n)} + 4\varepsilon) - \phi_n(\omega_{k(n)})]\, d\mu(\omega_{k(n)}) \leq s(\mu, 4\varepsilon)[\phi_n(b + 4\varepsilon) - \phi_n(a)] \\ \leq C_E\, s(\mu, 4\varepsilon)\, (\log(1/\varepsilon))^d$$

which implies that (4.13) is bounded by

$$C_E \sum_{n=1}^{L} s(\mu, 4\varepsilon)\, (\log(1/\varepsilon))^d \leq C_E\, s(\mu, 4\varepsilon)\, (\log(1/\varepsilon))^d\ l^d$$

Note that we apply Corollary A.4.3 successively L times. For this reason it is important to realise that, the constant C_E depends only on the diameter of u and a local norm of the negative part of the background potential. For this local norm exist a uniform estimate independent of Λ_l and the configuration of the coupling constants $\omega_k, k \neq k_{(n)}$. □

The recent preprint [91] contains several Wegner estimates for variants of the alloy type model H_ω. Let us formulate one of them, which concerns the models studied in the present section. Under the assumption that the periodic part of the operator $H_0 = -\Delta + V_{\mathrm{per}}$ has the unique continuation property, that the single site potential $L^\infty_c(\mathbb{R}^d) \ni u \geq 0$ is positive on an open subset of $\mathbb{R}^d$, that the random coupling constants $\omega_k, k \in \mathbb{Z}^d$ are independent identically distributed with distribution μ, and that μ is compactly supported and Hölder continuous, the results of [91] imply that the IDS of H_ω is Hölder continuous with the same Hölder exponent as μ.

4.3 Sparse Potentials

From the physical point of view there are some interesting models which have a potential

$$V_\omega(x) = \sum_{k \in \Gamma} \omega_k u(x - k) \tag{4.14}$$

resembling the alloy type model. However, the set Γ may be much more general than the lattice $\mathbb{Z}^d$. A class of particular interest are surface models

where $\Gamma = \{0\} \times \mathbb{Z}^\nu$ and $\nu < d$ is the dimension of a hyperplane in whose neighbourhood the potential is concentrated. The literature on such models includes [82, 297, 298, 61]. Even earlier [153, 154] so-called interface models, which are closely related to surface potentials, have been studied.

The results in this section are essentially taken from [268]. We consider arbitrary sets Γ, which are *uniformly discrete* in the following sense

$$\sup_{x\in\mathbb{R}^d} \#\{\Gamma \cap B_1(x)\} < \infty$$

For uniformly discrete Γ the number of points of Γ contained in the cube $\Lambda_l(x)$ can be bounded linearly in the volume of the cube and independently of its centre x.

Consider a background Schrödinger operator $H_0 = -\Delta + V_{\rm per}$ with a periodic potential $V_{\rm per} \in L^p_{\rm unif,loc}(\mathbb{R}^d)$ where $p = p(d)$ is as in (1.1). By adding a constant we may assume that $\inf \sigma(H_0) = 0$. Let $H_\omega = H_0 + V_\omega$ be a random operator with a *generalised alloy type potential* V_ω as in (4.14).

As before H^l_ω stands for the restriction of H_ω to the cube Λ_l with Dirichlet boundary conditions (we may as well use Neumann or periodic ones), and P^l_ω denotes the corresponding spectral projection.

Theorem 4.3.1. *Assume that the single site potential $u \in L^\infty_c(\mathbb{R}^d)$ is non-positive, and that the single site distribution μ has a support in $[0, \omega_+]$.*

Then, for any $-E' < 0$ there exists a finite C such that for any $E \in \mathbb{R}$ and $\frac{1}{2} \geq \varepsilon \geq 0$ satisfying $E + 3\varepsilon \leq -E'$:

$$\mathbb{E}\left[\mathrm{Tr} P^l_\omega(B_\varepsilon(E))\right] \leq C\, s\,(\mu, \varepsilon)\; |\log \varepsilon|^d\; |\Lambda_l|$$

The proof of the theorem follows from the arguments of Sect. 4.2, once a replacement for the estimate (4.12) has been established. This is provided by the following lemma. Denote by $\Lambda^+ := \Lambda^+_l := \{k \in \Gamma | \operatorname{supp} u(\cdot - x) \cap \Lambda_l \neq \emptyset\}$ the set of indices whose coupling constants influence the value of the potential in the cube Λ_l. Recall that the supremum of the support of f is denoted by ω_+.

Lemma 4.3.2. *Assume that the n^{th} eigenvalue of the operator H^l_ω satisfies $\lambda^l_n(\omega) \leq -E' < 0$. Then*

$$\rho'(\lambda^l_n(\omega) - E + t) \leq \frac{\omega_+}{E'} \left[-\sum_{k\in\Lambda^+} \frac{\partial \rho(\lambda^l_n(\omega) - E + t)}{\partial \omega_k} \right]$$

For $\varepsilon > 0$ and an isotone function ρ such that $\rho(x) = 0$ for all $x \geq -E'$ we have

$$\mathrm{Tr}\left[\rho(H^l_\omega - \frac{E'}{\omega_+}\varepsilon)\right] \geq \mathrm{Tr}\Big[\rho(H^l_\omega + \varepsilon \sum_{k\in\Lambda^+} u(\cdot - k)\Big]$$

Proof. Let ψ_n be the normalised eigenfunction corresponding to $\lambda_n^l(\omega)$. Then ψ_n satisfies by definition $\langle\psi_n, (H_0^l - \lambda_n^l(\omega))\psi_n\rangle = -\langle\psi_n, V_\omega\psi_n\rangle$. We have

$$\sum_{k\in\Lambda^+} \omega_k \langle\psi_n, -u(\cdot - k)\psi_n\rangle = -\langle\psi_n, V_\omega\psi_n\rangle = \langle\psi_n, (H_0^l - \lambda_n^l(\omega))\psi_n\rangle \geq E'$$

The Hellmann-Feynman theorem implies

$$\begin{aligned} -\sum_{k\in\Lambda^+} \frac{\partial\lambda_n^l(\omega)}{\partial\omega_k} &= \sum_{k\in\Lambda^+} \langle\psi_n, -u(\cdot - k)\psi_n\rangle \\ &\geq \omega_+^{-1} \sum_{k\in\Lambda^+} \omega_k \langle\psi_n, -u(\cdot - k)\psi_n\rangle \geq \frac{E'}{\omega_+} \end{aligned}$$

We conclude the first claim of the Lemma in the following way:

$$\begin{aligned} \rho'(\lambda_n^l(\omega) - E + t) &= -\left[-\sum_{k\in\Lambda^+} \frac{\partial\lambda_n^l(\omega)}{\partial\omega_k}\right]^{-1} \sum_{k\in\Lambda^+} \frac{\partial\rho(\lambda_n^l(\omega) - E + t)}{\partial\omega_k} \\ &\leq \frac{\omega_+}{E'}\left[-\sum_{k\in\Lambda^+} \frac{\partial\rho(\lambda_n^l(\omega) - E + t)}{\partial\omega_k}\right] \end{aligned} \tag{4.15}$$

Now consider the operator $H_\omega^l(t) := H_\omega^l + t \cdot \chi_{\Lambda_l} \sum_{k\in\Lambda^+} u(\cdot - k)$ and its eigenvalues $\lambda_n^l(\omega, t)$. We have for all $\lambda_n^l(\omega) \leq -E'$

$$\lambda_n^l(\omega, \varepsilon) - \lambda_n^l(\omega, 0) = \int_0^\varepsilon \frac{\partial\lambda_n^l(\omega, t)}{\partial t} dt = \int_0^\varepsilon \sum_{k\in\Lambda^+} \frac{\partial\lambda_n^l(\omega)}{\partial\omega_k} dt \leq -\frac{E'}{\omega_+}\,\varepsilon$$

Since ρ is isotone, $\rho(\lambda_n^l(\omega, \varepsilon)) \leq \rho(\lambda_n^l(\omega, 0) - \frac{E'}{\omega_+}\varepsilon)$ and the Lemma is proven. □

Note that since ρ is monotone increasing and u is non-positive, (4.15) is a non-negative real.

Related models and results as presented in this section are discussed in Sect. 3.1 of [94]. In [268] two more classes of generalised alloy type models are analysed. Firstly, the case where the number of points in $\Gamma \cap B_1(x)$ is not uniformly bounded, but grows at a controlled rate as x goes to infinity. Secondly, the case where Γ is itself a random point process, for example of Poissonian type, cf. also [89].

4.4 Locally Continuous Coupling Constants

In this section we present a Wegner estimate which requires the coupling constants ω_k to have a continuous distribution merely in a neighbourhood of their extremal value $\omega_+ = \sup\operatorname{supp} f$. Both the result and its proof are taken from [268], with some improvements from [218].

Consider a background Schrödinger operator $H_0 = -\Delta + V_{\rm per}$ with a periodic potential $V_{\rm per} \in L^p_{\rm unif,loc}(\mathbb{R}^d)$. Let $H_\omega = H_0 + V_\omega$ be a random operator with an alloy type potential V_ω. Assume that the coupling constants $\omega_k, k \in \mathbb{Z}^d$ take values in the bounded interval $[\omega_-, \omega_+]$. By modifying the periodic background potential we may consider only the case that the coupling constants are non-negative. For a value $\omega_c \in [0, \omega_+]$ introduce the auxiliary periodic potential $V_c = \omega_c \sum_{k\in\mathbb{Z}^d} u(x-k)$ and the threshold energy $E_c = \inf \sigma(H_0 + V_c)$.

Theorem 4.4.1. *Assume that the single site potential $u \in L^\infty_c(\mathbb{R}^d)$ is non-positive, and that the restriction of the single site distribution $\mu_c := \mu|_{]\omega_c,\omega_+]}$ has a density $f \in L^\infty$.*

Then, for any $E' < E_c$ there exists a finite C such that for any $E \in \mathbb{R}$ and $\varepsilon \geq 0$ satisfying $E + 3\varepsilon \leq E'$:

$$\mathbb{E}\left[\mathrm{Tr} P^l_\omega(B_\varepsilon(E))\right] \leq C\ \varepsilon |\log \varepsilon|^d\, l^d$$

Proof. The value ω_c is a critical one for the random variable ω_k in the sense that for $\omega_k > \omega_c$ we know that it is continuously distributed, while for smaller values we do not know anything. We introduce a corresponding decomposition of the 'probability' space $\Omega_l := \times_{k\in\Lambda^+}\mathbb{R} \cong \mathbb{R}^L$. This is the part of the randomness on which the restricted Hamiltonian H^l_ω depends. For a given configuration of coupling constants $\{\omega_k\}_{k\in\Lambda^+}$ set

$$\Lambda^{ac}(\omega) = \{k \in \Lambda^+ |\, \omega_k > \omega_c\}$$

This defines an equivalence relation on Ω_l by setting for any $A \subset \Lambda^+$

$$\Omega(A) := \{\omega |\, \Lambda^{ac}(\omega) = A\}$$

Consequently

$$\sum_{A\subset\Lambda^+} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} d\mu(\omega_k)\, \chi_{\Omega(A)}(\omega) = 1. \tag{4.16}$$

Split the potential now into two parts, a singular and an absolutely continuous one. The singular one

$$V^s_\omega(x) := \sum_{k\in\Lambda^+,\omega_k\leq\omega_c} \omega_k u(x-k) + \sum_{k\in\Lambda^+,\omega_k>\omega_c} \omega_c u(x-k) \geq V_c(x)$$

will be considered as part of the background operator, while the absolutely continuous one

$$V^{ac}_\omega(x) := \sum_{k\in\Lambda^+,\omega_k>\omega_c} r_k u(x-k) = \sum_{k\in\Lambda^{ac}} r_k u(x-k), \quad \text{with } r_k = \omega_k - \omega_c > 0$$

will be used for spectral averaging.

Consider an eigenvalue $\lambda_n^l \leq E' < E_c$ and an eigenfunction $H_\omega^l \psi_n = \lambda_n^l \psi_n$ and set $\delta = E_c - E'$. We have

$$-\langle \psi_n, V_\omega^{ac} \psi_n \rangle = \langle \psi_n, (H_0^l + V_\omega^s - \lambda_n^l)\psi_n \rangle \geq \langle \psi_n, (H_0^l + V_c - \lambda_n^l)\psi_n \rangle \geq \delta$$

which implies similarly as in Lemma 4.3.2

$$\begin{aligned} -\sum_{j \in \Lambda^{ac}} \frac{\partial \lambda_n(\omega)}{\partial \omega_j} &\geq \frac{1}{\omega_+ - \omega_c} \sum_{j \in \Lambda^{ac}} r_j \, \langle \psi_n, -u_j(\cdot - j)\psi_n \rangle \\ &= -\frac{\langle \psi_n, V_\omega^{ac} \psi_n \rangle}{\omega_+ - \omega_c} \geq \frac{\delta}{\omega_+ - \omega_c} \end{aligned}$$

Consider first the case $\emptyset \neq A \subset \Lambda^+$ and estimate

$$\begin{aligned} &\int_{\mathbb{R}^L} \prod_{k \in \Lambda^+} d\mu(\omega_k) \chi_{\Omega(A)}(\omega) \sum_{n \in \mathbb{N}} \int_{-2\varepsilon}^{2\varepsilon} dt \, \rho'(\lambda_n(\omega) - E + t) \\ &\leq \frac{\omega_+ - \omega_c}{\delta} \int_{\mathbb{R}^L} \prod_{k \in \Lambda^+} d\mu(\omega_k) \chi_{\Omega(A)} \sum_{n \in \mathbb{N}} \int_{-2\varepsilon}^{2\varepsilon} dt \left[-\sum_{j \in \Lambda^{ac}} \frac{\partial \rho(\lambda_n(\omega) - E + t)}{\partial \omega_j} \right] \end{aligned}$$

As we know that all sites $j \in \Lambda^{ac}$ correspond to coupling constants ω_j with values in the absolutely continuous region of the conditional density f we may estimate as in Sect. 4.1:

$$\begin{aligned} &-\sum_{n \in \mathbb{N}} \int_{\mathbb{R}} d\mu(\omega_j) \, \chi_{\Omega(A)}(\omega) \, \frac{\partial \rho(\lambda_n(\omega) - E + t)}{\partial \omega_j} \\ &= -\sum_{n \in \mathbb{N}} \int_{\omega_c}^{\omega_+} f(\omega_j) d\omega_j \, \frac{\partial \rho(\lambda_n(\omega) - E + t)}{\partial \omega_j} \\ &\leq \|f\|_\infty \sum_{n \in \mathbb{N}} [\rho(\lambda_n(\omega, \omega_j = \omega_c) - E + t) - \rho((\lambda_n(\omega, \omega_j = \omega_+) - E + t)] \end{aligned}$$

which can be estimated as in Sect. 4.2. We have to say something how we deal with the special case $A = \emptyset$. In this situation $V_\omega^{ac} \equiv 0$ and $H_\omega = H_0 + V_\omega^s \geq H_0 + V_c \geq E_c$. Thus there are no eigenvalues in the considered energy interval for this potential configuration.

Finally we use the decomposition (4.16) and Corollary A.4.3 to finish the proof:

$$\begin{aligned} &\mathbb{E}\left(\mathrm{Tr} P_\omega^l([E - \varepsilon, E + \varepsilon]) \right) \\ &\leq \sum_{A \subset \Lambda^+} \int_{\mathbb{R}^L} \prod_{k \in \Lambda^+} d\mu(\omega_k) \, \chi_{\Omega(A)}(\omega) \sum_{j \in A} \frac{(\omega_+ - \omega_c)}{\delta} \, 4\varepsilon \|f\|_\infty C_{E+2\varepsilon} \, |\log \varepsilon|^d \\ &\leq 4 C_{E+2\varepsilon} \frac{(\omega_+ - \omega_c)}{\delta} \|f\|_\infty \, \varepsilon |\log \varepsilon|^d \, |\Lambda^+| \end{aligned}$$

□

4.5 Potentials with Small Support

In Sect. 4.1 we used in a crucial step in the derivation of the Wegner estimate that the single site potentials were lower bounded by a partition of unity

$$\sum_{k\in\mathbb{Z}^d} u(x-k) \geq \kappa \quad \text{on } \mathbb{R}^d$$

It is of natural interest, whether a Wegner estimate holds if this condition is relaxed. In this section we consider the case that u is of fixed sign, but has small support. More precisely, we assume throughout this section merely that there is an open set $\mathcal{O} \subset \mathbb{R}^d$ and a positive κ such that

$$u(x) \geq \kappa \chi_{\mathcal{O}} \tag{4.17}$$

The first Wegner estimates under this relaxed condition on the single site potential were derived for spectral boundaries, i.e. for energies either near the bottom of the spectrum, or near an internal spectral boundary. The case of the infimum of the spectrum was treated e.g. in [282, 249], and internal spectral boundaries in [265]. These works derived a Wegner estimate where the volume dependence of the bound was growing faster than linearly. Thus they were not sufficient to derive a result on the regularity of the IDS, cf. our discussion in Sect. 3.1. A linear bound for the same energy regimes was found in [97].

By now there are Wegner estimates which under the relaxed condition (4.17) derive Wegner estimates valid for any bounded interval on the energy axis. We consider here the one-dimensional case where the result is particularly clear and the proof simple. We follow [267] in the presentation, see [194, 93] for other proofs. The higher-dimensional case is discussed without proofs in Sect. 4.2.

Assume that the single site potential u and the periodic potential V_{per} are bounded.

Theorem 4.5.1. *Let* $\mathrm{d} = 1$. *Assume that* u *is compactly supported and obeys* (4.17). *For any* $E_0 \in \mathbb{R}$ *there exist a constant* C *such that*

$$\mathbb{E}\left[\mathrm{Tr} P^l_\omega(B_\varepsilon(E))\right] \leq C\,\varepsilon\, l, \quad \forall\, \varepsilon \in [0,1], E \leq E_0, l \in \mathbb{N} \tag{4.18}$$

Thus the IDS is Lipschitz-continuous. A similar result can be derived for Schrödinger operators on metric graphs, see [212].

Proof. First we show how to replace (4.4) in the case of small support. By shifting the origin of $\mathbb{R}$ we may assume without loss of generality that there is a $s > 0$ such that $\Lambda_s(0) \subset \mathcal{O}$. Likewise, we may assume $\kappa = 1$ by rescaling the single site potential and the coupling constants.

We set $S = \bigcup_{k\in\tilde{\Lambda}} \Lambda_s(k)$. The Hellmann-Feynman theorem gives us

$$\sum_{k\in\tilde{\Lambda}} \frac{\partial \lambda^l_n(\omega)}{\partial \omega_k} = \sum_{k\in\tilde{\Lambda}} \langle \psi_n, u(\cdot - k)\psi_n\rangle \geq \int_S |\psi_n|^2.$$

where ψ_n is a normalised eigenfunction corresponding to $\lambda^l_n(\omega)$.

If the integral on the right hand side would extend over the whole of Λ_l it would be equal to 1 due to the normalisation of ψ_n. A priori the integral over S could be arbitrary close to zero, but the following Lemma shows that this is not the case.

Lemma 4.5.2. *Let I be a bounded interval and $s > 0$. There exists a constant $c > 0$ such that*

$$\int_{\Lambda_s(k)} |\psi|^2 \geq c \int_{\Lambda_1(k)} |\psi|^2$$

for all $l \in \mathbb{N}$, all $k \in \tilde{\Lambda}_l$ and for any eigenfunction ψ corresponding to an eigenvalue $E \in I$ of H^l_ω.

Proof (of the Lemma). For

$$\phi(y) := \int_{\Lambda_s(k+y)} dx\, |\psi(x)|^2 = \int_{\Lambda_s(k)} dx\, |\psi(x-y)|^2$$

one has

$$\begin{aligned}\left|\frac{\partial}{\partial y}\phi(y)\right| &= \left| \int\limits_{\Lambda_s(k)} dx\, \left[\frac{\partial}{\partial y}\psi(x-y)\right] \overline{\psi(x-y)} + \int\limits_{\Lambda_s(k)} dx\, \psi(x-y)\, \frac{\partial}{\partial y}\overline{\psi(x-y)} \right| \\ &\leq 2\, \|\psi\|_{L^2(\Lambda_s(k+y))}\, \|\psi'\|_{L^2(\Lambda_s(k+y))}\,.\end{aligned}$$

Sobolev norm estimates (e.g. Theorems 7.25 and 7.27 in [196]) imply

$$\|\psi'\|_{L^2(\Lambda_s(k+y))} \leq C_5\, \|\psi\|_{L^2(\Lambda_s(k+y))} + \|\psi''\|_{L^2(\Lambda_s(k+y))}$$

By the eigenvalue equation we have

$$\left|\frac{\partial}{\partial y}\phi(y)\right| \leq C_6\ \|\psi\|^2_{L^2(\Lambda_s(k+y))} = C_6\, \phi(y), \qquad C_6 = C_6(\|V_{\rm per} + V_\omega - E\|_\infty) \tag{4.19}$$

Gronwall's Lemma implies $\phi(y) \leq \exp(C_6|y|)\, \phi(0)$ and thus

$$\int_{\Lambda_1(k)} |\psi|^2 \leq \mathrm{e}^{C_6}\ s^{-1} \int_{\Lambda_s(k)} |\psi|^2$$

□

Thus $\int_S |\psi|^2 \geq c \int_{\Lambda_l} |\psi|^2$ with the same constant as in Lemma 4.5.2.

It remains to estimate the spectral shift

$$\sum_{n\in\mathbb{N}} \left[\rho(\lambda_n^l(\omega, j=\max) - E + t) - \rho(\lambda_n^l(\omega, j=\min) - E + t)\right] \tag{4.20}$$

We may assume without loss of generality that the single site potential u is supported in the interval $[-R, R]$. Introduce now the operator $H_\omega^{l,D}(j = \max)$ which coincides with $H_\omega^l(j = \max)$ up to additional Dirichlet boundary conditions at the points $j - R$ and $j + R$. Likewise, $H_\omega^{l,N}(j = \min)$ coincides with $H_\omega^l(j = \min)$ up to additional Neumann boundary conditions at the same points. Their eigenvalues are $\lambda_n^{l,D}(\omega, j = \max)$ and $\lambda_n^{l,N}(\omega, j = \min)$, respectively. By Dirichlet-Neumann bracketing, the square brackets in (4.20) are bounded by

$$\rho(\lambda_n^{l,D}(\omega, j=\max) - E + t) - \rho(\lambda_n^{l,N}(\omega, j=\min) - E + t) \tag{4.21}$$

Since for both $*=D, N$ the Hamiltonian $H_\omega^{l,*}$ is a direct sum of an operator $H_\omega^{j,*}$ acting on $L^2(j\text{–}R, j\text{+}R)$ and another one $H_\omega^{c,*}$ acting on $L^2(\Lambda_l \setminus [j\text{–}R, j\text{+}R])$ the sum over the terms in (4.21) can be separated:

$$\sum_n \rho(\lambda_n^{c,D}(\omega) - E + t) - \rho(\lambda_n^{c,N}(\omega) - E + t) \tag{4.22}$$

$$+ \sum_n \rho(\lambda_n^{j,D}(\omega, j=\max) - E + t) - \rho(\lambda_n^{j,N}(\omega, j=\min) - E + t) \tag{4.23}$$

Here $\lambda_n^{c,*}(\omega)$ denotes the eigenvalues of $H_\omega^{c,*}$, $\lambda_n^{j,D}(\omega, j = \max)$ the eigenvalues of $H_\omega^{j,D}$ with ω_j maximal, and $\lambda_n^{j,N}(\omega, j = \min)$ the eigenvalues of $H_\omega^{j,N}$ with ω_j minimal. Note that the eigenvalues in (4.22) are independent of $\omega_j u(\cdot - j)$. Since the difference in the boundary conditions is a rank two perturbation in resolvent sense (see e.g. [443]), the interlacing theorem says that

$$\rho(\lambda_n^{c,D}(\omega, j=\max) - E + t) \leq \rho(E_{n+2}^{c,N}(\omega, j=\max) - E + t)$$

A telescoping argument bounds the whole sum in (4.22) by twice the total variation of ρ, which is equal to one. The sum in (4.23) we estimate by

$$\begin{aligned}&\mathrm{Tr}\left[\chi_{[E-3\varepsilon,\infty[}(H_\omega^{j,D}(j=\max)) - \chi_{]E+3\varepsilon,\infty[}(H_\omega^{j,N}(j=\min))\right]\\ &\leq 2+\mathrm{Tr}\left[\chi_{[E-3\varepsilon,\infty[}(H_\omega^{j,D}(j=\min) + \|u_j\|_\infty) - \chi_{]E+3\varepsilon,\infty[}(H_\omega^{j,D}(j=\min))\right]\end{aligned}$$

which is bounded by a constant, that is independent of Λ_l, $j \in \tilde{\Lambda}_l$, $\varepsilon > 0$ and the configuration ω. (Note that the parameter ε determines the support and the derivative of the switch function ρ.) □

In the remainder of this chapter we give an overview of various Wegner estimates which are based or related to techniques presented at the beginning of Chapter 4. However, we refrain from giving the proofs of this results but refer to the original articles.

4.6 Hölder Continuous Coupling Constants

There is special interest to extend Wegner's estimate to coupling constants with singular distribution. The reason is the intuitive interpretation of the coupling consonants as nuclear charge numbers modulating the strength of atomic potentials. In this case their distribution would correspond to a pure point measure.

In Remarks 3.1.1 and 3.2.1 we mentioned already results for one-dimensional models with singular randomness. The first result in this direction for multi-dimensional alloy type models was obtained by Stollmann in [457]. Stollmann's version of Wegner's estimate applies to single site measures μ which have compact support. It is formulated in terms of the modulus of continuity of the single site distribution μ. Stollmann applied this estimate to prove localisation for certain alloy type potentials with Hölder continuous μ. A distribution is called *(uniformly) Hölder continuous with exponent* $\alpha > 0$ if there is a constant C_μ such that the modulus of continuity of μ satisfies

$$s(\mu, \varepsilon) := \sup\{\mu([a,b]) |\, b - a \leq \varepsilon\} \leq C_\mu \; \varepsilon^\alpha \quad \text{for } \varepsilon \geq 0 \tag{4.24}$$

For the following discussion a wider class of distributions will be of interest, which we describe next. We call a measure μ *(uniformly) log-Hölder continuous* with parameter $\alpha > 0$ if there is a constant C_μ such that for all $\frac{1}{2} > \varepsilon > 0$ we have

$$s(\mu, \varepsilon) \leq C_\mu \; |\log \varepsilon|^{-\alpha}$$

We state the main result of [457] without proof:

Theorem 4.6.1. *Let H_ω be an alloy type model as in Definition 1.2.1, but require for the single site measure merely that is has compact support. Assume additionally that the single site potential obeys $u \geq \chi_{[0,1]^d}$. Then for any $E \in \mathbb{R}$ there exists a constant C such that for any open interval $I \subset]-\infty, E[$ and any $l \in 2\mathbb{N}$*

$$\mathbb{P}\{\omega |\, \sigma(H_\omega^l) \cap I \neq \emptyset\} \leq C \, s(\mu, |I|) \cdot l^{2d} \tag{4.25}$$

holds.

By now it has become quite standard to formulate Wegner estimates in terms of the modulus of continuity of the distribution μ. This is for instance the case for the results presented in Sect. 4.2, see also the papers [218, 91].

Since these estimates apply to arbitrary distributions of compact support, one can use them for rather singular μ. This is of interest for applications. As discussed before, Wegner estimates are used to establish regularity properties of the IDS and as an input for the multiscale analysis, cf. Sect. 3.2.

Let us illustrate the usefulness of Wegner estimates expressed in terms of the modulus of continuity by the following lemma. It turns such a bound, under the condition that μ is log-Hölder continuous, into the sort of estimate as it is used for the induction step of the multiscale analysis.

Lemma 4.6.2. *Let $H_\omega, \omega \in \Omega$ be a random Schrödinger operator and $E_0 \in \mathbb{R}$. Assume that there exist constants C_W, l_0 such that for all $\varepsilon \in]0, 1/2]$, all $E \leq E_0$ and all $\mathbb{N} \ni l \geq l_0$*

$$\mathbb{E}\left\{\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega^l)]\right\} \leq C_W \; s(\mu,\varepsilon)\, l^d \tag{4.26}$$

For some $\theta, q > 0$ assume that the measure μ is log-Hölder continuous with parameter $\alpha > (q+d)/\theta > 0$. Set $\delta := \alpha\theta - q - d > 0$ and $l_1 := \max\left(l_0, (C_W C_\mu)^{1/\delta}\right)$. Then for all energies $E \leq E_0$ we have

$$\mathbb{P}\{d(\sigma(H_\omega^l), E) \leq \mathrm{e}^{-l^\theta}\} \leq l^{-q} \quad \text{for all } l \in \mathbb{N}, l \leq l_1 \tag{4.27}$$

Proof. $\mathbb{P}\left\{\omega \mid \mathrm{dist}\left(\sigma(H_\omega^l), E\right) \leq e^{-l^\theta}\right\} \leq \mathbb{E}\left\{\mathrm{Tr}\left[\chi_{]E-e^{-l^\theta},E+e^{-l^\theta}[}(H_\omega^l)\right]\right\}$
$\leq C_W\, l^d\, s(\mu, e^{-l^\theta}) \leq C_W\, l^d\, C_\mu |\log e^{-l^\theta}|^{-\alpha} \leq l^{-q}$ for all $l \geq (C_W C_\mu)^{1/\delta}$. □

Remark 4.6.3. Assumption (4.26) can be verified for various models. For discrete Anderson models on $\ell^2(\mathbb{Z}^d)$ it is established in [218], for continuum alloy type models on $L^2(\mathbb{R}^d)$ it is given in [91], and for alloy type models on metric graphs one can infer it from [210].

If instead of (4.26), a bound as in (4.8) or (4.25) is given, an obvious modification of Lemma 4.6.2 holds.

4.7 A Partial Integration Formula for Singular Distributions

A different way to deal with random coupling constants whose distribution is not absolutely continuous is the following integration by parts formula. It is used in the proof of Theorem 4.2.4. For a similar idea see Lemma 6.1 in [301].

Lemma 4.7.1. *Let μ be a probability measure with support in $]a, b[$, $\phi \in C^1(\mathbb{R})$ be a non-decreasing, bounded function, and $s(\mu,\varepsilon)$ as defined in* (4.9)*. Then for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}} [\phi(\lambda+\varepsilon) - \phi(\lambda)]\, d\mu(\lambda) \leq s(\mu,\varepsilon) \cdot [\phi(b+\varepsilon) - \phi(a)]$$

If $\mathrm{supp}\,\mu \subset]a, \infty[$ is unbounded from above the Lemma still holds if one replaces $\phi(b+\varepsilon)$ by the (well-defined) limit $\lim_{x\to\infty} \phi(x)$.

Proof. We write $d\mu = dM$, where M is the distribution function of μ. In the following, all integrals are defined as Stieltjes integrals. Shifting variables and using that M is constant outside of $[a, b]$ gives

$$\int [\phi(\lambda+\varepsilon) - \phi(\lambda)]\, d\mu(\lambda) = \int_a^{b+\varepsilon} \phi(\lambda)\, d[M(\lambda-\varepsilon) - M(\lambda)]$$

Integrating by parts gives

$$\int [\phi(\lambda+\varepsilon)-\phi(\lambda)]\,d\mu(\lambda) = \Big[\phi(\lambda)[M(\lambda-\varepsilon)-M(\lambda)]\Big]_a^{b+\varepsilon} - \int_a^{b+\varepsilon} \phi'(\lambda)[M(\lambda-\varepsilon)-M(\lambda)]\,d\lambda$$

The first term is zero, since M is constant outside of $[a,b]$ (in case $b=\infty$, one uses boundedness of ϕ and $\lim_{\lambda\to\infty}[M(\lambda-\varepsilon)-M(\lambda)]=0$). The second term is bounded by

$$\begin{aligned}\int_a^{b+\varepsilon} \phi'(\lambda)[M(\lambda)-M(\lambda-\varepsilon)]\,d\lambda &\le \sup_\lambda [M(\lambda)-M(\lambda-\varepsilon)]\cdot\int_a^{b+\varepsilon}\phi'(\lambda)\,d\lambda\\ &\le s(\mu,\varepsilon)\cdot(\phi(b+\varepsilon)-\phi(a))\end{aligned}$$

since $\phi'\ge 0$. □

4.8 Coupling Constants with Bernoulli Disorder

In the recent paper [59] Bourgain and Kenig study alloy type models as in Definition 1.2.1 with Bernoulli disorder. More specifically, they consider the case where the periodic potential V_{per} is absent, the single site potential $u\not\equiv 0$ is smooth, non-negative and has support in the ball of radius $1/10$ around zero, and the coupling constants are independent obeying $\mathbb{P}\{\omega_k=0\}=1-\mathbb{P}\{\omega_k=1\}=P\in]0,1[$ for all $k\in\mathbb{Z}^d$.

They use a quite heavy machinery to prove a form of Wegner estimate similar to the one in (3.8). This bound is much weaker than those discussed in the present work, but still sufficient to prove localisation, see the discussion in Sect. 3.2. To deal with the singular nature of the Bernoulli distribution of coupling constants, the proof in [59] makes use of a strong quantitative version of the unique continuation property, of Sperner's Lemma, and a special multiscale analysis to obtain inductively a Wegner estimate on increasing scales. Recall from the discussion in Sect. 3.2 that usually one uses the multiscale analysis to prove exponential decay of Green's functions, whereas the Wegner estimate is an input for this procedure which is proven beforehand for all length scales simultaneously.

4.9 Single Site Potentials with Changing Sign

First Wegner estimates for indefinite alloy type potentials were derived in [282]. In [213] Hislop and Klopp combine the techniques from [282] and [97] to prove a Wegner estimate valid for general indefinite single site potentials and for energy intervals at edges of $\sigma(H_\omega)$. They assume the single site potential

$u \in C_c(\mathbb{R}^d)$ satisfies $u(0) \neq 0$. The density $f \in L_c^\infty$ of the random variable ω_0 (which may be in fact the conditional density with respect to $\omega^{\perp 0} := (\omega_k)_{k\neq 0}$) is assumed to be piecewise absolutely continuous. For any $\alpha < 1$ and any compact energy interval I strictly below the spectrum of the unperturbed operator H_0 they prove

$$\mathbb{P}\{\sigma(H_\omega^l) \cap I \neq \emptyset\} \leq C\,|I|^\alpha\, l^d$$

where the constant C depends only on α, d and the distance between the interval I and $\sigma(H_0)$. With a sufficiently small global coupling constant λ the same result holds for the operator $H_0 + \lambda V_\omega$ for I in an internal spectral gap of H_0. The results of [213] extend to more general models including certain operators with random magnetic field.

In Sect. 5.5 we discuss in more detail an alternative technique to obtain a Wegner estimate valid for single site potentials which change sign. It applies to a more restricted class of potentials but yields stronger results. In particular, it proves the Lipschitz continuity of the IDS at all energies.

4.10 Uniform Wegner Estimates for Long Range Potentials

Kirsch, Stollmann and Stolz proved in [264] a Wegner estimate for single site potentials which do not need to have compact support, but merely need to decay sufficiently fast. They consider u of polynomial decay

$$|u(x)| \leq C(1 + |x|^2)^{-m/2} \tag{4.28}$$

where $m > 0$ is required to be larger than $3d$. For certain applications they can also deal with the case where m is only larger than $2d$, cf. [263, 500].

For such single site potentials the restrictions of the alloy type potential to two finite cubes may be correlated, even if the cubes are far apart. This makes it necessary to use an enhanced version of the multiscale analysis for the proof of localisation. Among others, this requires a *uniform Wegner estimate*. By this we mean a Wegner estimate for the Hamiltonian H_ω^l restricted to the cube Λ_l which is uniform in the coupling constants ω_k with index $\|k\|_\infty > r$ where r is a function of l.

To formulate the Wegner estimate from [264] let us first introduce some notation. For any cube $\Lambda \subset \mathbb{R}^d$ and $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$ we denote by Π_Λ the projection

$$\Pi_\Lambda \colon \Omega \mapsto \bigtimes_{\tilde{\Lambda}} \operatorname{supp} \mu \qquad \Pi_\Lambda(\omega) := \{\omega_k\}_{k\in\tilde{\Lambda}}$$

For a measurable set $A \subset \Omega$ we denote by A_Λ^* the cylinder set

$$A_\Lambda^* := \Pi_\Lambda^{-1}(\Pi_\Lambda A) = \{\omega \in \Omega |\, \exists \omega' \in A \text{ such that } \Pi_\Lambda(\omega') = \Pi_\Lambda(\omega)\}$$

The following observation plays a crucial role in the enhanced multiscale analysis.

Lemma 4.10.1. *For two disjoint cubes Λ, Λ' and two events $A, B \in \Omega$, the induced events A^*_Λ and $B^*_{\Lambda'}$ are independent.*

The following lemma allows one to turn a 'usual' Wegner estimate, as we have it discussed before, into a uniform Wegner estimate. It relies on the polynomial decay of the single site potential (4.28). Let I be a compact interval, $E \in I$ and $\varepsilon \in]0,1]$. We denote by $A(E,\varepsilon,l)$ the event $\{\omega |\, d(E,\sigma(H^l_\omega)) < \varepsilon\}$ and use the abbreviations $\Pi_l := \Pi_{\Lambda_l}, A^*_l := A^*_{\Lambda_l}$.

Lemma 4.10.2. *Under the above assumptions there exists a finite constant c, independent of $\omega \in \Omega$, $l, r \in \mathbb{N}$ and $\varepsilon \leq 1$ such that*

$$\mathbb{P}\{A(E,\varepsilon,l)^*_{l+r}\} \leq \mathbb{P}\{A(E,\varepsilon + cr^{-(m-d)}, l)\}$$

Proof. By definition, for an $\omega \in A^*_{l+r}$ there exists an $\omega' \in A$ such that

$$\Pi_{l+r}\omega' = \Pi_{l+r}\omega$$

Thus, the coordinates of ω and ω' with index k within the cube of size $l+r$ coincide and we have for $x \in \Lambda_l$

$$\begin{aligned} |V_\omega(x) - V_{\omega'}(x)| &\leq \sum_{|k|_\infty > l+r} |\omega_k - \omega'_k|\, u(x-k) \\ &\leq c' \sum_{|k|_\infty > l+r} |x-k|^{-m} \leq cr^{-(m-d)} \end{aligned}$$

Therefore $d(E,\sigma(H^l_{\omega'})) < \varepsilon$ implies $d(E,\sigma(H^l_\omega)) < \varepsilon + cr^{-(m-d)}$, which proves the lemma. □

Let us have a look on the implications of the preceding lemma for a concrete example. Assume that the single site potential is bounded below on the unit cube around zero by $\kappa > 0$. Then we have by Theorem 5.0.1

$$\mathbb{P}\big\{\omega | \sigma(H^l_\omega) \cap [E-\varepsilon, E+\varepsilon] \neq \emptyset\big\} \leq C_W(I)\ \varepsilon\ l^d$$

for all E, ε such that $[E-\varepsilon, E+\varepsilon]$ is contained in the open interval I. This Wegner estimate implies its uniform analog

$$\mathbb{P}\Big(\big\{\omega | \sigma(H^l_\omega) \cap [E-\varepsilon, E] \neq \emptyset\big\}^*_{l+r}\Big) \leq C_W(I)\ (\varepsilon + cr^{-(m-d)})\ l^d \qquad (4.29)$$

for sufficiently large $r > 0$. In the application in the multiscale analysis, both ε and r are chosen as functions of l. From the estimate in (4.29) it might seem to be sufficient to choose $m > d$. This is also the minimal requirement to make the alloy type model with long range single site potentials well defined as a densely

defined operator. However, for technical reasons, for the multiscale analysis to work one has to assume at least $m > 2d$. Under this assumption one can prove that the spectrum of H_ω is almost surely pure point near its bottom, and that the corresponding eigenfunctions decay faster than any inverse polynomial, cf. [263, 500]. To obtain exponential decay of the eigenfunctions, one has to require $m > 3d$, see [264].

In the paper [500] by Zenk the above results have been extended to a model which incorporates random displacements of the single site potentials.

5

Lipschitz Continuity of the IDS

In [306] Kotani and Simon extended to continuum alloy type models certain arguments previously used for the derivation of Wegner's estimate for the discrete Anderson model. They treated only the case where the single site potential is the characteristic function of the unit cube, but Combes and Hislop showed in [89] that the same argument extends to non-negative single site potentials with uniform lower bound on the unit cube. There also some steps of the proof have been streamlined.

One of the ideas in [306] is that in the same way as rank one perturbations are used for discrete Laplacians, positive perturbations may be used in the continuum case. This is related to the Aronszajn-Donoghue Theory [27, 28, 29, 127]. See [79, 304, 444, 443] for more background and references. This was essential, since a finite rank potential in the continuum may be a Dirac distribution, but not a function.

Theorem 5.0.1. *Let H_ω as in Definition 1.2.1 and assume additionally that there exists a $\kappa > 0$ such that*

$$u \geq \kappa \chi_{[-1/2,1/2]^d}$$

Then for all $E \in \mathbb{R}$ there exists a constant $C_W = C_W(E)$ such that for all $l \in \mathbb{N}$ and all intervals $I \subset]-\infty, E]$

$$\mathbb{E}\left\{ \mathrm{Tr}\left[P^l_\omega(I)\right] \right\} \leq C_W \ |I| \ l^d \tag{5.1}$$

Remark 5.0.2. (a) It is sufficient to prove the theorem for the case $\kappa = 1$. Since $\omega_0 u = \kappa \omega_0 \, \kappa^{-1} u$, the general case follows by rescaling the coupling constants and single site potentials.

(b) The statement of the theorem remains true if one uses Neumann or periodic boundary conditions for H^l_ω.

(c) An explicit formula for the Wegner constant C_W is given in (5.17). Since (5.1) is linear in the volume it follows $|N(E_2) - N(E_1)| \leq C_W \, |E_2 - E_1|$. Thus, as we discussed already in Sect. 3.1, the density of states $n(E) := dN(E)/dE$ exists almost everywhere and is bounded by $n(E) \leq C_W(E_2)$ for all $E \leq E_2$.

The next four sections are devoted to the proof of Theorem 5.0.1. Up to some modifications we follow the line of argument in Sect. 4 of [89].

5.1 Partition of the Trace into Local Contributions

In the present section we derive preparatory estimates on

$$\mathbb{E}\left\{\mathrm{Tr}P_\omega^l(I)\right\} \tag{5.2}$$

where we do not yet use the specific alloy-type structure of the potential. They have two aims. Firstly, to decompose the trace to contributions of unit cubes in Λ_l. This will later facilitate the averaging procedure with respect to random parameters, whose effect on the potential is felt only locally. Secondly, it allows us to reduce the averaging of the trace of the spectral projection to the averaging of the quadratic form of the resolvent. The latter is technically easier to perform.

Denote by Δ^l and Δ_N^l the Laplace operator on Λ_l with Dirichlet, respectively Neumann boundary conditions. In Sect. 1.2 we saw that the potential $V = V_{\mathrm{per}} + V_\omega$ is infinitesimally bounded with respect to $-\Delta$ and that the constants in the bound can be chosen uniformly in $\omega \in \Omega$. This implies that V is infinitesimally form bounded with respect to any of the operators $-\Delta$, $-\Delta^l$ and $-\Delta_N^l$ with bounds uniform in $\omega \in \Omega, l \in \mathbb{N}$ and the choice of Dirichlet or Neumann boundary conditions. Consequently, there is a $C_0 < \infty$ such that for all $\omega \in \Omega$ and $l \in \mathbb{N}$

$$|\langle \phi, V\phi\rangle| \leq \frac{1}{2}\langle \phi, -\Delta_N^l \phi\rangle + C_0\|\phi\|^2$$

which implies

$$\langle \phi, H_\omega^l \phi\rangle \geq \langle \phi, -\frac{1}{2}\Delta_N^l\phi\rangle - C_0\|\phi\|^2 \geq -C_0\|\phi\|^2 \tag{5.3}$$

Thus $H_\omega^l + C_0$ is a non-negative operator.

Definition 5.1.1. *A monotone decreasing, convex function* $r\colon [0,\infty[\to]0,\infty[$ *such that*

$$C_{\mathrm{Tr}} := C_{\mathrm{Tr}}(r) := \sum_{n\in\mathbb{Z}^d, n_j\geq 0} r\left(\frac{\pi^2}{2}\sum_{j=1}^d n_j^2\right) < \infty \tag{5.4}$$

will be called trace regularising.

Throughout the rest of this chapter we denote by Λ the unit cube centred at zero, and by χ_j the characteristic function of its translate $\Lambda + j$, where $j \in \mathbb{Z}^d$.

Remark 5.1.2. The bound (5.4) means that the operator $r(-\frac{1}{2}\Delta_N^\Lambda)$=$r(-\frac{1}{2}\Delta_N^1)$ has finite trace. Namely, the eigenvalues of the Neumann Laplacian on the unit cube are given by

$$\pi^2 \sum_{j=1}^{d} n_j^2 \quad \text{for all } n \in \mathbb{Z}^d \text{ such that } n_j \geq 0, j = 1, \ldots, d$$

cf. for instance [408], page 266. By the spectral mapping theorem the eigenvalues of $r(-\frac{1}{2}\Delta_N^\Lambda)$ are just $r\big(\frac{1}{2}\pi^2 \sum_{j=1}^d n_j^2\big)$.

Examples of functions r which are trace-regularising are the exponential functions $r\colon x \mapsto \mathrm{e}^{-tx}$ for $t > 0$. They have been used in [89] to implement the procedure outlined in this chapter. Another choice for r is a sufficiently high power of the resolvent $x \mapsto (x+1)^{-k}$ for $k > d/2$, which was used in [306]. That the operator $(-\frac{1}{2}\Delta_N^\Lambda + 1)^{-k}$ is actually trace class can be inferred from [439].

The possibility to choose r from a large class of functions is of interest, if one wants to give explicit upper bounds on the density of states. For instance, Sect. 3.2 of [223] is devoted to deriving such explicit upper estimates. However, there, following [89], the function $r(x) = \mathrm{e}^{-tx}$ is used. Due to this choice, the upper bound on the density of states is exponentially growing in the energy. This can be improved to a merely polynomial growing bound. Furthermore, if one studies coupling constants which may take on arbitrarily negative values, the choice of r will determine which moment conditions one has to impose on the negative part of ω_0, see also Sect. 5.7. The role played here by r resembles the one of the function g in Sect. 4.2 and Appendix A.

Proposition 5.1.3. *Let $I = [E_1, E_2]$ be an interval. With C_0 as in* (5.3) *we have*

$$\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} \leq r(E_2 + C_0)^{-1}\, C_{\mathrm{Tr}}(r) \sum_{j \in \tilde{\Lambda}_l} \left\| \mathbb{E}\left\{\chi_j P_\omega^l(I) \chi_j\right\}\right\|$$

Proof. Since $\frac{1}{r}$ is well-defined and bounded on the compact interval I, we have

$$\mathrm{Tr}\left[P_\omega^l(I)\right] = \mathrm{Tr}\left[r(H_\omega^l + C_0)^{-1}\, P_\omega^l(I)\, r(H_\omega^l + C_0)\right]$$

Furthermore, by spectral calculus and since for positive operators A, B we have $\mathrm{Tr}(AB) \leq \|A\|\ \mathrm{Tr}(B)$, the above line is bounded by

$$r(E_2 + C_0)^{-1} \mathrm{Tr}\left[P_\omega^l(I)\, r(H_\omega^l + C_0)\right]$$

According to the direct sum decomposition

$$L^2(\Lambda_l) = \bigoplus_{j \in \tilde{\Lambda}_l} L^2(\Lambda + j)$$

we consider the Laplace operators $-\Delta_{N,j}$ on $L^2(\Lambda+j)$ with Neumann boundary conditions. Dirichlet-Neumann bracketing implies

$$H_\omega^l + C_0 \geq -\frac{1}{2}\Delta_N^l \geq -\frac{1}{2}\bigoplus_{j\in\tilde{\Lambda}_l} \Delta_{N,j} =: \oplus H \tag{5.5}$$

For a normalised eigenfunction ϕ of H_ω^l corresponding to the eigenvalue λ we have by the spectral mapping theorem

$$\langle\phi, r(H_\omega^l + C_0)\phi\rangle = r(\lambda + C_0) = r(\langle\phi, (H_\omega^l + C_0)\phi\rangle) \leq r(\langle\phi, \oplus H\phi\rangle) \tag{5.6}$$

Applying Jensen's inequality to the spectral measure of $\oplus H$ we estimate (5.6) from above by $\langle\phi, r(\oplus H)\phi\rangle$. Let $\phi_n, n \in \mathbb{N}$ be an orthonormal basis of eigenvectors of H_ω^l with corresponding eigenvalues $\lambda_n, n \in \mathbb{N}$. We apply the above estimates to the trace

$$\begin{aligned}\operatorname{Tr}\left[P_\omega^l(I) r(H_\omega^l + C_0)\right] &\leq \sum_{n\in\mathbb{N},\lambda_n\in I} \langle\phi_n, r(H_\omega^l + C_0)\phi_n\rangle \\ &\leq \sum_{n\in\mathbb{N},\lambda_n\in I} \langle\phi_n, r(\oplus H)\phi_n\rangle \leq \operatorname{Tr}\left[P_\omega^l(I) r(\oplus H)\right]\end{aligned}$$

For the next step we write down the trace with respect to different basis. For each $j \in \tilde{\Lambda}_l$ let $\{\psi_n^j | \, n \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\Lambda + j)$, then $\{\psi_n^j | \, n \in \mathbb{N}, j \in \tilde{\Lambda}_l\}$ is an orthonormal basis of $L^2(\Lambda_l)$. Since $r(\oplus H)\psi_n^j = r(-\frac{1}{2}\Delta_{N,j})\psi_n^j$ it follows for the trace

$$\begin{aligned}\operatorname{Tr}\left[P_\omega^l(I) r(\oplus H)\right] &= \sum_{j\in\tilde{\Lambda}_l}\sum_{n\in\mathbb{N}} \langle\psi_{j,n}, P_\omega^l(I) r(\oplus H)\psi_{j,n}\rangle \\ &= \sum_{j\in\tilde{\Lambda}_l}\sum_{n\in\mathbb{N}} \langle\psi_{j,n}, \chi_j P_\omega^l(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\psi_{j,n}\rangle \\ &= \sum_{j\in\tilde{\Lambda}_l} \operatorname{Tr}\left[\chi_j P_\omega^l(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right]\end{aligned}$$

Thus we have decomposed the trace to contributions from each unit cube. We summarise the estimates so far:

$$\operatorname{Tr} P_\omega^l(I) \leq r(E_2 + C_0)^{-1} \sum_{j\in\tilde{\Lambda}_l} \operatorname{Tr}\left[\chi_j P_\omega^l(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right]$$

Since I is a bounded interval and $V_{\mathrm{per}} + V_\omega$ is an infinitesimally small perturbation of $-\Delta^l$ independently of ω, it follows that the dimension of $P_\omega^l(I)L^2(\Lambda_l)$ is bounded by a constant C uniformly in ω. Thus for all $\omega \in \Omega$

$$\operatorname{Tr}\left[\chi_j P^l_\omega(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right] \leq C\, r(0)$$

is an upper bound by an integrable majorant and we are able to interchange the trace and the expectation by Lebesgue's theorem on dominated convergence:

$$\begin{aligned}\mathbb{E}\left\{\operatorname{Tr}\left[\chi_j P^l_\omega(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right]\right\} &= \operatorname{Tr}\left[\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right\}\right] \\ &= \operatorname{Tr}\left[\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\chi_j r(-\tfrac{1}{2}\Delta_{N,j})\chi_j\right] \\ &\leq \left\|\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\right\| \operatorname{Tr}\left[r(-\tfrac{1}{2}\Delta_{N,0})\right]\end{aligned}$$

By assumption, r is trace regularising, so the trace in the last line is finite. □

5.2 Spectral Averaging of Resolvents

Now we consider how resolvents are averaged when integrated over a random parameter. Together with the partition result in Sect. 5.1 this will enable us to complete in Sect. 5.4 the proof of Theorem 5.0.1.

Apart from this, the spectral averaging result bears in itself a meaning. Consider a non-negative operator H with discrete spectrum. Its resolvent $R(E) = (H - E)^{-1}$ has singularities at the eigenvalues of H, which are of the form $(\lambda_n - E)^{-1}, \lambda_n \in \sigma(H)$. Thus for a general vector ψ the function $E \mapsto \langle\psi, R(E)\psi\rangle$ will not be bounded nor even integrable over the energy axis $\mathbb{R}$. Now, if $H = H(\zeta)$ depends on a random parameter ζ, we might hope that the averaged resolvent $\int d\mathbb{P}(\zeta)\langle\psi, R(\zeta, E)\psi\rangle$ will be integrable or even bounded as a function of the variable E. This would mean that the singularities of the resolvent have been smeared out sufficiently by the integral over ζ. The lemma in this section shows that this is actually the case for operators which depend in a specific way on the random parameter.

Consider the following operators on a Hilbert space $\mathcal{H}$. Let H be a selfadjoint operator, W symmetric and infinitesimally bounded with respect to H, and J bounded and non-negative with $J^2 \leq W$. Choose two parameters

$$\begin{aligned} z \in \mathbb{C}_- &:= \{z \in \mathbb{C} |\ \operatorname{Im} z < 0\} \\ \zeta \in \overline{\mathbb{C}_+} &:= \{\zeta \in \mathbb{C} |\ \operatorname{Im} \zeta \geq 0\}\end{aligned}$$

and set

$$H(\zeta) := H + \zeta W,\ K(\zeta, z) := J(H(\zeta) - z)^{-1}J \tag{5.7}$$

The following lemma is a slight generalisation of Lemma 4.1 in [89].

Lemma 5.2.1. *For all $z \in \mathbb{C}_-$, all $t > 0$ and any normalised $\phi \in \mathcal{H}$ we have*

$$\left|\int_{\mathbb{R}} \langle\phi, K(\zeta, z)\phi\rangle \frac{d\zeta}{1 + t\zeta^2}\right| \leq \pi \tag{5.8}$$

Proof. By Pythagoras we have $|\langle\phi,(A+iB)\phi\rangle|^2 = |\langle\phi,A\phi\rangle|^2 + |\langle\phi,B\phi\rangle|^2$ for any two selfadjoint operators A, B. Thus the norm of $K(\zeta,z)$ is bounded by $|\operatorname{Im} z|^{-1}\,\|J\|^2$. On the other hand, the equation

$$-\operatorname{Im} K(\zeta,z) = J[(H(\bar{\zeta})-\bar{z})^{-1}[(\operatorname{Im}\zeta)W - \operatorname{Im} z](H(\zeta)-z)^{-1}]\,J$$

implies

$$\|K(\zeta,z)\| \le |\operatorname{Im}\zeta|^{-1} \tag{5.9}$$

Here we used that $W(H(\zeta)-z)^{-1}$ is a bounded operator. Now observe that for all $z\in\mathbb{C}_-$ the function $\zeta\mapsto K(\zeta,z)$ is holomorphic and bounded on $\overline{\mathbb{C}_+}$. The residue theorem, integration over a closed curve in $\mathbb{C}$ and the bounds on K imply

$$\left|\int_{\mathbb{R}} \langle\phi,K(\zeta,z)\phi\rangle\,\frac{d\zeta}{1+t\zeta^2}\right| = \frac{\pi}{\sqrt{t}}\,|\langle\phi,K\big(i/\sqrt{t},z\big)\rangle| \tag{5.10}$$

Together with (5.9), this completes the proof. □

Remark 5.2.2. The lemma shows that for the particular family of operators $H(\zeta)$ in (5.7), for ζ a random variable with measure $\mathbb{P}(d\zeta) := \frac{d\zeta}{1+t\zeta^2}$, and for vectors ψ in the range of J, the ζ-averaged resolvents $\int d\mathbb{P}(\zeta)\langle\psi,R(\zeta,E)\psi\rangle$ are indeed bounded with respect to the energy variable. Thus the singularities of the resolvent have been smeared out.

5.3 Stone's Formula and Spectral Averaging of Projections

Stone's formula allows one to express the spectral projection in terms of the resolvent. This is handy because the resolvent has some nice analytic properties. In our case we use Stone's formula to derive the analog of (5.8) for spectral projections.

A sequence of bounded operators $A_n, n\in\mathbb{N}$ on the Hilbert space $\mathcal{H}$ converges *strongly* (or in *strong topology*) to A if for every $\phi\in\mathcal{H}$

$$\lim_{n\to\infty}\|A\phi - A_n\phi\| = 0$$

Lemma 5.3.1 (Stone's formula). *Let H be a selfadjoint operator, $I\subset\mathbb{R}$ an interval, and $P(I)$ the corresponding spectral projection. Then the following limit holds in the strong topology*

$$\lim_{\delta\searrow 0}\frac{1}{2\pi i}\int_{E_1}^{E_2}\big[(H-E-\mathrm{i}\delta)^{-1}-(H-E+\mathrm{i}\delta)^{-1}\big]\,dE = \frac{1}{2}\Big[P\big([E_1,E_2]\big)+P\big(]E_1,E_2[\big)\Big]$$

Proof. The function

$$f_\delta(x) := \frac{1}{\pi}\left(\arctan\frac{x-E_1}{\delta} - \arctan\frac{x-E_2}{\delta}\right) \tag{5.11}$$
$$= \frac{1}{2\pi i}\int_{E_1}^{E_2}\left[(x-E-\mathrm{i}\delta)^{-1} - (x-E+\mathrm{i}\delta)^{-1}\right]dE$$
$$= -\frac{1}{\pi}\,\mathrm{Im}\int_{E_1}^{E_2}(x-E+\mathrm{i}\delta)^{-1}dE$$

converges for $\delta \searrow 0$ to

$$\frac{1}{2}\big(\chi_{[E_1,E_2]} + \chi_{]E_1,E_2[}\big)$$

Now one applies the spectral theorem to $f_\delta(H)$. □

More details on Stone's formula can be found in [410], or Sect. 7.3 of [494] where the spectral calculus is actually introduced in this way.

Now let $H(\zeta)$ be as in the last section and $P(\zeta, I)$ the corresponding spectral projection onto an interval I. For a normalised vector ψ in $\mathcal{H}$ denote $\mathcal{P}(\zeta) := \langle\psi, JP(\zeta, I)J\psi\rangle$. The next lemma contains a spectral averaging estimate for $\mathcal{P}$.

Lemma 5.3.2. *Let $\varrho \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \varrho(\zeta)\,\mathcal{P}(\zeta)\,d\zeta \le \|\varrho\|_\infty |I| \tag{5.12}$$

While Combes and Hislop [89] considered compactly supported ϱ, it was first observed in [168] that densities with non-compact support can be treated. There this extension was necessary to derive estimates for Gaussian random potentials.

Proof. We first consider the special density $\frac{1}{1+t\zeta^2}$, for $t > 0$ and an open interval I. By Stone's formula

$$\int_{\mathbb{R}} \frac{d\zeta}{1+t\zeta^2}\,\mathcal{P}(\zeta) \le -\int_{\mathbb{R}} \frac{d\zeta}{1+t\zeta^2}\,\lim_{\delta\searrow 0}\frac{1}{\pi}\,\mathrm{Im}\int_I dE\langle\psi, K(\zeta, E-\mathrm{i}\delta)\psi\rangle \tag{5.13}$$

Note that $\frac{d\zeta}{1+t\zeta^2}$ is a finite, outer regular Borel measure on $\mathbb{R}$ and that (5.11) implies that $|f_\delta(\cdot)|$, and hence $\|f_\delta(H(\zeta))\|$, is bounded by one. Thus we may apply the dominated convergence theorem to interchange the limit and the integration, and bound (5.13) by

$$\frac{1}{\pi}\lim_{\delta\searrow 0}\left|\int_I dE\int_{\mathbb{R}}\frac{d\zeta}{1+t\zeta^2}\,\langle\psi, K(\zeta, E-\mathrm{i}\delta)\psi\rangle\right| \le |I| \tag{5.14}$$

The last inequality follows from Lemma 5.2.1. This implies for all $\varrho \in L^\infty$ with compact support:

$$\begin{aligned}\int_{\mathbb{R}} \varrho(\zeta)\,\mathcal{P}\,(\zeta)\,d\zeta &\leq \sup_{\operatorname{supp}\varrho}\left[\varrho(\zeta)(1+t\zeta^2)\right] \int_{\mathbb{R}} \frac{\mathcal{P}\,(\zeta)}{1+t\zeta^2}\,d\zeta \\ &\leq \sup_{\operatorname{supp}\varrho}\left[\varrho(\zeta)(1+t\zeta^2)\right]\,|I| \\ &\to \|\varrho\|_\infty\,|I| \text{ for } t\to 0\end{aligned}$$

Finally, assume only that $\varrho \in L^\infty \cap L^1$. Set $\varrho^y := \varrho\cdot\chi_{\{x|\,|x|<y\}}$ and decompose $\varrho = \varrho^y + \varrho_y$. For $y\to\infty$, ϱ_y tends to zero pointwise. Since $\mathcal{P}$ is bounded by one, $\varrho \in L^1(\mathbb{R}, d\zeta)$ is a y-uniform majorant for $\varrho_y\mathcal{P}$ and we may apply the dominated convergence theorem to conclude

$$\int_{\mathbb{R}} \varrho(\zeta)\,\mathcal{P}\,(\zeta)\,d\zeta = \lim_{y\to\infty}\int_{\mathbb{R}} \varrho^y(\zeta)\,\mathcal{P}\,(\zeta)\,d\zeta \leq \|\varrho\|_\infty\,|I| \tag{5.15}$$

If I is not open, we write it as an intersection of open, decreasing sets and use monotone convergence to conclude (5.12). □

5.4 Completion of the Proof of Theorem 5.0.1

The results on the localisation of the trace to unit cubes and spectral averaging of projections allow us to assemble the proof of Theorem 5.0.1.

To estimate the operator norm appearing in Proposition 5.1.3 we may as well bound the corresponding quadratic form since

$$\left\|\,\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\right\| = \sup_{\|\phi\|=1}\,\langle\phi,\,\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\phi\rangle \tag{5.16}$$

Now we apply Fubini's Theorem and Lemma 5.3.2 with the choice $\varrho = f$, $\mathcal{H} = L^2(\Lambda_l)$, $J = \chi_j$, $H = H_0 + \sum_{k\in\tilde\Lambda\setminus j}\,\omega_k\,u(\cdot - k)$, $\zeta = \omega_j$, $W = u(x-j)$, and obtain:

$$\langle\phi,\,\mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\phi\rangle \leq \|f\|_\infty\,|I|$$

This bound is j-independent and thus yields

$$\mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} \leq r(E_2+C_0)^{-1}\,C_{\mathrm{Tr}}(r)\,\|f\|_\infty\,|I|\,|\tilde\Lambda_l| \tag{5.17}$$

□

Remark 5.4.1. Now it becomes clear why we introduced the operator $r(H^l_\omega + C_0)^{-1}r(H^l_\omega + C_0) = \mathrm{Id}$ in Proposition 5.1.3: Due to this regularisation we needed merely to derive bounds on the operator norm of $\chi_j P^l_\omega(I)\chi_j$, rather than on its trace. The trace was applied to the single, deterministic operator $r(-\frac{1}{2}\Delta_{N,0})$, where all dependencies on Λ_l, ω, and I drop out. Without the use of a trace regularising function it is still possible to estimate

$$\mathbb{E}\left\{\mathrm{Tr}\,\chi_j P^l_\omega(I)\chi_j\right\} \leq const.\,|I|\,|\tilde\Lambda_l|$$

However, this leads to a Wegner estimate with quadratic volume bound.

5.5 Single Site Potentials with Changing Sign

In Sect. 4.9 we saw an extension of the Wegner-Kirsch approach to single site potentials of changing sign. The Kotani-Simon-Combes-Hislop proof of Wegner's estimate also allows such a generalisation which we present in this section. Its main shortcoming in comparison to the results in Sect. 4.9 is that it is restricted to single site potentials which have a (generalised) step function form. On the other hand, it is valid not only at spectral boundaries, but on the whole energy axis. Furthermore, it yields the Lipschitz continuity of the integrated density of states and thus locally uniform upper bounds on the density of states. The results of this section are taken from [481, 301], see also [479, 300, 482].

First we describe the class of alloy type potentials which can be treated by the methods of this section.

Definition 5.5.1. *Let $L^p_c(\mathbb{R}^d) \ni w \geq \kappa\chi_{\Lambda_1}$ with $\kappa > 0$ and $p = p(d)$ as in (1.1). Let Γ be a finite subset of $\mathbb{Z}^d$. A function of the form*

$$u(x) = \sum_{k\in\Gamma} \alpha_k \, w(x-k) \tag{5.18}$$

will be called a generalised step-function *and the vector $\alpha \in \mathbb{R}^\Gamma$ a* convolution vector. *We set $\alpha_k = 0$ for all $k \in \mathbb{Z}^d \setminus \Gamma$. Thus, we consider α as an element of $c_0(\mathbb{Z}^d)$, the space of all sequences with elements indexed by $j \in \mathbb{Z}^d$ which have finite support. The set Γ will be called the* support *of α,* supp $\alpha = \Gamma$.

A linear operator $A = \{A_{j,k}\}_{j,k\in\mathbb{Z}^d}$ with matrix coefficients $A_{j,k} := \langle \delta_j, A\,\delta_k\rangle$ for $j,k \in \mathbb{Z}^d$ is called (multi-level) Laurent matrix *if it satisfies*

$$A_{j+i,k+i} = A_{j,k} \quad \text{for all } j,k,i \in \mathbb{Z}^d$$

Here $\delta_k(i) = 1$, if $k = i \in \mathbb{Z}^d$, and zero otherwise. So, a Laurent matrix is a doubly-infinite Toeplitz matrix. For a subset $\Theta \subset \mathbb{Z}^d$ denote by A_Θ the truncated matrix with the coefficients $A_{\Theta\,k,j} := \langle \delta_k, A_\Theta\,\delta_j\rangle$ for $k,j \in \Theta$. The matrix A may be considered as a bounded linear operator $A\colon \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$, $p \in [1,\infty[$ acting as

$$(A\phi)(j) = \sum_{j\in\Gamma} A_{j,k}\phi(k) = \sum_{j\in\Gamma} \alpha_{j-k}\phi(k)$$

Similarly, $A_\Theta\colon \ell^p(\Theta) \to \ell^p(\Theta)$ is a bounded operator, which can be related to A by the formula $A_\Theta = \chi_\Theta\, A\, \chi^*_\Theta$. Here

$$\chi_\Theta\colon \ell^p(\mathbb{Z}^d) \to \ell^p(\Theta), \qquad (\chi_\Theta\phi)(j) = \begin{cases} \phi(j), & \text{if } j \in \Theta, \\ 0 & \text{otherwise} \end{cases}$$

is the projection onto the set Θ. If A, respectively A_Θ, are invertible operators, the inverse to A will be denoted by B and the inverse to A_Θ by B_Θ. Each convolution vector α generates a Laurent matrix by the rule $A = \{\alpha_{j-k}\}_{j,k\in\mathbb{Z}^d}$.

Remark 5.5.2 (Notational conventions). Let as before $H_\omega = H_0 + V_\omega$ be an alloy type Schrödinger operator as in Definition 1.2.1. Thus, the single site potential u and the distribution μ are building blocks of the random potential V_ω. We assume throughout this section that $0 \not\equiv u \in L^p_c(\mathbb{R}^d), p = p(d)$ is a generalised step-function as in (5.18). Given such a function u, α will always denote the convolution vector by means of which u is defined, A the Laurent matrix associated to α, and s_α its symbol, cf. Definition 5.5.7. Furthermore we assume that μ has a density $f \in BV_c(\mathbb{R})$ with compact support and finite total variation. We will denote the total variation norm of f by $\|f\|_{\mathrm{Var}}$. Recall that $\Lambda_l^+ = \{k \in \mathbb{Z}^d | \operatorname{supp} u(\cdot - k) \cap \Lambda_l \neq \emptyset\}$ denotes the set of indices whose coupling constants influence the value of the potential in the cube Λ_l.

The following standard approximation result, see e.g. [501], will be useful in the proof of the subsequent theorem.

Lemma 5.5.3. *Let $f\colon \mathbb{R} \to \mathbb{R}^+$ be a function of finite total variation with $\|f\|_1 = 1$. Then there exists a sequence $f_k \in C_0^\infty(\mathbb{R})$ such that $\|f_k\|_1 = 1$ for all $k \in \mathbb{N}$,*

$$\lim_{k\to\infty} \|f_k\|_{\mathrm{Var}} = \|f\|_{\mathrm{Var}}, \qquad \text{and} \qquad \lim_{k\to\infty} \|f_k - f\|_1 = 0.$$

Theorem 5.5.4. *Let H_ω be as in Remark 5.5.2. For a cube Λ_l assume that there exists $\Theta \subset \mathbb{Z}^d$ such that*

(i) $\Lambda_l^+ \subset \Theta$,
(ii) *A_Θ is invertible and $B_\Theta\colon \ell^1(\Theta) \to \ell^1(\Theta)$ is bounded with norm $\|B_\Theta\|_1$.*

Then for all $E \in \mathbb{R}$ and for all intervals $I \subset]-\infty, E]$

$$\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} \leq r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_{\mathrm{Var}}\, \frac{\|B_\Theta\|_1}{\kappa}\, |I|\, l^d \tag{5.19}$$

where we used the same notation as in Proposition 5.1.3.

Proof. We first prove the theorem for densities $f \in W^{1,1}_c(\mathbb{R})$. From Proposition 5.1.3 we infer

$$\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} \leq r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r) \sum_{j \in \check{\Lambda}_l} \left\| \mathbb{E}\left\{\chi_j P_\omega^l(I) \chi_j\right\}\right\|$$

Thus it is sufficient to estimate $\mathbb{E}\left\{\langle \phi, \chi_j P_\omega^l(I)\chi_j \phi\rangle\right\}$ for any normalised $\phi \in L^2(\Lambda_l)$ by a uniform constant. To do this we introduce a transformation of coordinates on the probability space Ω. Since $\Lambda_l^+ \subset \Theta$, the operator H_ω^l depends only on the truncated random vector $(\omega_k)_{k\in\Theta} \in \mathbb{R}^\Theta$. On such vectors acts the truncated Laurent matrix A_Θ, whose inverse $B_\Theta = \{b_{k,j}\}_{k,j\in\Theta}$ is bounded in the column-sum norm $\|B_\Theta\|_1$ by assumption. Set $L := \#\Theta$.

We drop now the subscript Θ and denote with $\eta := A\omega$ the vector of the transformed random coordinates. They have the common density

$$k(\eta) = |\det B|\, F(B\eta) \tag{5.20}$$

where $F(\omega) = \prod_{k\in\Theta} f(\omega_k)$ is the original density of the ω_k. We abbreviate $w_j(x) := w(x-j)$ and calculate the potential V_ω written as a function of η (and $x \in \Lambda_l$):

$$V_\omega(x) = V_{B\eta}(x) = \sum_{j\in\tilde{\Lambda}_l} \eta_j\, w_j(x)$$

In the new representation of the potential the single site potentials are non-negative, so we can make use of the spectral averaging formula in Lemma 5.3.2

$$\int_{\mathbb{R}} d\eta_j\, k(\eta)\, \mathcal{S}(\eta) \le \frac{|I|}{\kappa} \sup_{\eta_j} |k(\eta)|, \qquad \text{where } \mathcal{S}(\eta) := \langle \phi, w_j P^l_{B\eta}(I) w_j \phi\rangle \tag{5.21}$$

Fubini's theorem, (5.21), and the fundamental theorem of calculus give

$$\int_{\mathbb{R}^L} d\eta\, k(\eta)\, \mathcal{S}(\eta) \le \frac{|I|}{\kappa} \int_{\mathbb{R}^{L-1}} d\eta^{\perp j} \sup_{\eta_j} |k(\eta)| \le \frac{|I|}{\kappa} \int_{\mathbb{R}^L} d\eta\, |(\partial_j k)(\eta)| \tag{5.22}$$

Here $\eta^{\perp j}$ is an abbreviation for $\{\eta_k |\, k \in \Theta \setminus j\}$. The last integral equals

$$|\det A| \int_{\mathbb{R}^L} d\omega\, |(\partial_j k)(A\omega)|$$

which is bounded by $\|f'\|_{L^1} \sum_{k\in\Theta} |b_{k,j}|$. Thus, for $f \in W^{1,1}_c(\mathbb{R})$, we have obtained the desired bound

$$\mathbb{E}\left\{ \langle \phi, w_j P^l_\omega(I) w_j \phi\rangle \right\} \le \kappa^{-1} |I|\, \|f'\|_{L^1} \|B\|_1 \tag{5.23}$$

Now let f be a function of finite total variation. Let $\{f_k\}_k$ be an approximation sequence of C^∞_0-functions as in Lemma 5.5.3. We have

$$\begin{aligned}\int_{\mathbb{R}^L} d\omega\; F(\omega)\; \langle \phi, w_j P^l_\omega(I) w_j \phi\rangle &= \int_{\mathbb{R}^L} d\omega\; \prod_{k\in\Theta} f_k(\omega_k)\; \langle \phi, w_j P^l_\omega(I) w_j \phi\rangle \\ &\quad + \int_{\mathbb{R}^L} d\omega\; \left[\prod_{k\in\Theta} f(\omega_k) - \prod_{k\in\Theta} f_k(\omega_k) \right]\; \langle \phi, w_j P^l_\omega(I) w_j \phi\rangle\end{aligned}$$

By our previous considerations, the first integral on the right is bounded by

$$|I|\; \|f_k\|_{\mathrm{Var}}\, \|B\|_1$$

which tends to $|I|\, \|f\|_{\mathrm{Var}} \|B\|_1$ for $k \to \infty$. A telescoping argument shows that the norm of the second integral is bounded by

$$L\, \|w\|^2_\infty\, \|f - f_k\|_1$$

which by Lemma 5.5.3 tends to zero as $k \to \infty$. □

If we are able to find for arbitrarily large cubes $\Lambda_l \subset \mathbb{R}^d$ subsets Θ of $\mathbb{Z}^d$ such that $\Theta \supset \Lambda_l^+$ with bounded B_Θ, Theorem 5.5.4 may be used to obtain a Wegner estimate. A particularly important case is when the B_Θ are uniformly bounded independently of Θ.

Theorem 5.5.5. *Let H_ω be as in Remark 5.5.2. Assume furthermore that there exist a constant $C_B \in \mathbb{R}$ and an exhaustion $\Theta_1 \subset \Theta_2 \subset \dots$ of $\mathbb{Z}^d$ such that $\|B_{\Theta_n}\|_1 \leq C_B$ for all $n \in \mathbb{N}$. Then we have for any $E \in \mathbb{R}$, any interval $I \subset]-\infty, E]$, and all $l \in \mathbb{N}$*

$$\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} \leq C_W \ |I|\, l^d \tag{5.24}$$

where $C_W = r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_{\mathrm{Var}}\, C_B$.

The theorem implies that the density of states, the derivative of the IDS, exists for a.e. E and is locally uniformly bounded:

$$n(E_1) := \frac{dN(E_1)}{dE_1} \leq r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_{\mathrm{Var}}\, C_B \quad \text{for all } E_1 \leq E$$

Proof. For any $l \in \mathbb{N}$ we can find an $n \in \mathbb{N}$ such that $\Lambda_l^+ \subset \Theta_n$. Thus by Theorem 5.5.4 and our assumption we have

$$\begin{aligned}\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} &\leq r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_{\mathrm{Var}}\, \|B_{\Theta_n}\|_1\, |I|\, l^d \\ &\leq r(E + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_{\mathrm{Var}}\, C_B\, |I|\, l^d\end{aligned}$$

□

Now let us turn to the question under which conditions the assumption of Theorem 5.5.5 is satisfied. A simple criterion is the following

Lemma 5.5.6. *Assume that the convolution vector of the single site potential satisfies*

$$\alpha^* := \sum_{k \neq 0} |\alpha_k| < |\alpha_0|$$

Then we have for any $\Theta \subset \mathbb{Z}^d$

$$\|B_\Theta\|_1 \leq \frac{\alpha_0}{1 - \alpha^*}$$

Proof. Since A is the Laurent matrix generated by the vector α, we can write it as

$$A =: \alpha_0\, \mathrm{Id} + S, \quad \|S\|_1 = \sum_{k \neq 0} |\alpha_k| = \alpha^* < |\alpha_0| \tag{5.25}$$

by the assumption on α^*. The decomposition and the bound in (5.25) remain true for any truncation A_Θ. By the Neumann series expansion the inverse B_Θ of A_Θ exists and obeys the bound $\|B_\Theta\|_1 \leq \frac{\alpha_0}{1-\alpha^*}$ for any $\Theta \subset \mathbb{Z}^d$. □

The assumptions on the convolution vector can be further relaxed. To formulate these weaker hypotheses we need the notion of the symbol of a Laurent matrix.

Definition 5.5.7. *The* symbol *of a convolution vector α, respectively the associated Laurent matrix A, is defined as the function*

$$s_\alpha \colon [-\pi,\pi[^d \to \mathbb{C}, \qquad s_\alpha(\theta) = \sum_{k\in\mathbb{Z}^d} \alpha_k \, \mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}, \qquad \theta = (\theta_1,\dots,\theta_d)$$

The symbol s_α is called sectorial *if there is a $\phi \in [-\pi,\pi[$ such that*

$$\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\phi} s_\alpha(\theta)) \geq 0 \text{ for all } \theta \in [-\pi,\pi[^d$$

Since we assume that only finitely many components of α are different from zero, s_α is actually a trigonometric polynomial. Thus it is a uniformly continuous and bounded function with a finite number of zeros, unless it vanishes identically.

If the symbol function s_α has no zeros, we define for every $i = 1,\dots,d$ the i^{th} winding number

$$\mathrm{wn}_i(s_\alpha) = \frac{1}{2\pi\mathrm{i}} \int_{[-\pi,\pi[} \frac{d}{dt} \log s_\alpha(\theta_1,\dots,\theta_i = t,\dots,\theta_d)\; dt$$

This number is an integer independent of θ. The vector of winding numbers

$$\mathrm{wn}(s_\alpha) := (\mathrm{wn}_1(s_\alpha),\dots,\mathrm{wn}_d(s_\alpha)) \in \mathbb{Z}^d$$

is called also the topological index *of the symbol s_α.*

Remark 5.5.8. Let H_ω be as in Remark 5.5.2. If the symbol function s_α associated to u does not vanish on $[-\pi,\pi[^d$ we can assume without loss of generality that $\mathrm{wn}(s_\alpha) = 0$.

Proof. First observe that a translation of the convolution vector α by an arbitrary $j_0 \in \mathbb{Z}^d$ leaves the operator H_ω unchanged up to unitary equivalence. To see this, let us set $\tilde{\alpha}_k := \alpha_{k-j_0}$ and denote by $\tilde{u}(x) = \sum_{k\in\Gamma+j_0} \tilde{\alpha}_k \, w(x-k)$ the associated single site potential of generalised step-function form. It gives rise to an alloy type potential

$$\tilde{V}_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j \sum_{k\in\Gamma+j_0} \tilde{\alpha}_k \, w(x-k-j)$$

which equals

$$\sum_{j\in\mathbb{Z}^d} \omega_j \sum_{k\in\Gamma+j_0} \alpha_{k-j_0} \, w(x-k-j) = \sum_{j\in\mathbb{Z}^d} \omega_j \sum_{l\in\Gamma} \alpha_l \, w(x-l-j_0-j)$$
$$= \sum_{j\in\mathbb{Z}^d} \omega_j \sum_{k\in\Gamma} \alpha_k \, w((x-j_0)-k-j) = V_\omega(x-j_0) = (U_{j_0} V_\omega U_{j_0}^*)(x)$$

where U_{j_0} denotes the unitary translation operator by j_0. Note that the periodic part H_0 commutes with U_{j_0}.

On the other hand, supp $\tilde{\alpha} =$ supp $\alpha + j_0$ and the symbol of $\tilde{a}$ is given by

$$s_{\tilde{\alpha}}(\theta) = \mathrm{e}^{-\mathrm{i}\langle j_0, \theta\rangle} s_\alpha(\theta)$$

By the product rule for winding numbers we obtain

$$\mathrm{wn}(s_{\tilde{\alpha}}) = \mathrm{wn}(s_\alpha) - j_0. \tag{5.26}$$

Now choose $j_0 = \mathrm{wn}(s_\alpha)$. Then the winding number of $s_{\tilde{\alpha}}$ vanishes and the corresponding alloy type model $H_0 + \tilde{V}_\omega$ is unitarily equivalent to the original alloy type model H_ω. □

Recall that in our situation the total potential energy $V_{\mathrm{per}} + V_\omega$ is relatively bounded with respect to the Laplace operator. This is expressed in a quantitative way in (5.3) using the constant C_0. Now we are ready to state the main result of [301].

Theorem 5.5.9. *Let H_ω be as in Remark 5.5.2. If $d \leq 2$ and $s_\alpha(\theta) \neq 0$ for all $\theta \in [-\pi, \pi[^d$, then there is a constant C independent of E, l, and f such that for all $\varepsilon \geq 0$ we have*

$$\mathbb{E}\{\mathrm{Tr}\ P_\omega^l([E-\varepsilon, E])\} \leq C\, \mathrm{e}^{E+C_0} \|f\|_{\mathrm{Var}}\, \varepsilon\, l^d \tag{5.27}$$

Proof. By Theorem 5.5.5 it is sufficient to show that there exists an exhaustion $\Theta_1 \subset \Theta_2 \subset \ldots$ of $\mathbb{Z}^d$ such that $\|B_{\Theta_n}\|_1 \leq C_B$ for all $n \in \mathbb{N}$. In Sect. 5.6 it is shown that such an exhaustion always exist, if $d \in \{1, 2\}$ and the symbol s_α has no zeros on $[-\pi, \pi[^d$ and has vanishing winding number. Without loss of generality we can drop the last condition by Remark 5.5.8. □

Remark 5.5.10. It seems that Theorem 5.5.9 holds for arbitrary dimension. This result requires a modification of the proof: instead of finite truncations A_Θ of A one uses (multi-dimensional) circulant matrices. They have better invertibility properties than approximands obtained by the finite section method.

Now we relax the hypotheses on the convolution vector α, respectively its symbol s_α, further. There is a set of conditions under which the methods of [301] are not strong enough to establish the Lipschitz continuity of the integrated density of states, but only a weaker form of Wegner estimate. Still, this may be useful to prove localisation for certain alloy-type random potentials with single site functions u of changing sign, cf. the discussion in Sect. 3.2 and Theorem 4 in [301].

This regime concerns convolution vectors whose symbol s_α does vanish, but is sectorial and has the property that its real part has only a finite number of nodal points. We infer from [301] without proof the following result. Let us just mention that the arguments of the proof heavily rely on Sect. 8 in [54].

Theorem 5.5.11. *Let H_ω be as in Remark 5.5.2. Assume that the symbol s_α associated to u is sectorial and* Re s_α *has at most finitely many zeros. Then there exist $C, b \in \mathbb{R}$ such that for all $E \in \mathbb{R}$, $l \in \mathbb{N}$ and $\varepsilon \geq 0$*

$$\mathbb{E}\{\mathrm{Tr}\ P_\omega^l([E-\varepsilon, E])\} \leq C\, \mathrm{e}^{E+C_0} \|f\|_{\mathrm{Var}}\, \varepsilon\, l^{bd} \tag{5.28}$$

holds, where C is a constant independent of f.

Remark 5.5.12 (Anderson model). For the discrete Anderson model $h_\omega = h_0 + V_\omega$ there is a result analogous to Theorem 5.5.5. Here h_0 is the finite difference Laplacian on $\ell^2(\mathbb{Z}^d)$ and $(V_\omega\psi)(n) = V_\omega(n)\psi(n)$, $\forall\, n \in \mathbb{Z}^d$, a multiplication operator as in the continuum case. This is not surprising, since the arguments in Sects. 5.2 and 5.3 rely only on abstract functional analysis. If fact, as we mentioned earlier, Kotani and Simon were motivated in their treatment [306] of the alloy type model by its discrete counterpart. Moreover, since on $\ell^2(\mathbb{Z}^d)$ the trace can be expressed using the canonical basis as

$$\mathrm{Tr}[P_\omega^l(I)] = \sum_{j \in \tilde{\Lambda}_l} \langle \delta_j, P_\omega^l(I)\delta_j \rangle$$

the use of a trace regularising function is not necessary. Here P_ω^l denotes the spectral projection of the truncation h_ω^l of the Anderson model h_ω. More precisely, h_ω^l is the finite matrix $\{\langle \delta_j, h_\omega \delta_k \rangle\}_{j,k \in \tilde{\Lambda}_l}$.

Note that in the discrete case χ_j is just δ_j. Under the assumptions of Theorem 5.5.5 on the coupling constants $\{\omega_j\}_j$ and the single site potential u we have the following Wegner estimate for the Anderson model:

$$\mathbb{E}\left\{\mathrm{Tr}[P_\omega^l(I)\right\} \leq C_B \|f\|_{\mathrm{Var}}\, |I|\, |\tilde{\Lambda}_l| \tag{5.29}$$

Remark 5.5.13. Theorems 5.5.4 and 5.5.5 can also be understood as a Wegner estimate for the alloy type potential

$$V_\eta(x) = \sum_{k \in \mathbb{Z}^d} \eta_k\, \chi_k$$

where the coupling constants $\{\eta_j\}_j$ are not any more independent, but correlated, satisfying certain conditions. See Sect. 4.2 in [482] for a precise formulation. Wegner estimates for correlated coupling constants can also be found in [96] (cf. [223], too).

The use of the common density F, respectively k, in the proof of Theorem 5.5.4 is conceptually new. One could try to use conditional densities instead by considering the indefinite potential V_ω in its representation $V_{B\eta}$ as an alloy type potential with dependent coupling constants. However, this would require to have uniform upper bounds on the conditional densities, cf. [96, 223]. They do not seem to be easy to establish for the model considered in this section, and in fact sometimes fail to hold as can be seen in the following example.

Example 5.5.14. It is sufficient to consider only one space dimension $d = 1$. Let the density function be $f = \chi_{[0,1]}$ and the single site potential $u = \chi_{[0,1]} - \alpha\chi_{[1,2]}$ with $-\alpha \in]-1, 0[$. To this model the results of Theorem 5.5.5 and Lemma 5.5.6 apply.

We analyse the restriction of H_ω to an interval of length l. Instead of $\Lambda_l =]-1/2, l-1/2[$ we consider the translation thereof $\Lambda_l + \frac{l}{2} =]0, l[$. The restricted Schrödinger operator $H_\omega^{]0,l[}$ depends only on the coupling constants ω_j with indices $j \in \{-1, \ldots, l-1\} =: \Lambda_l^+$. They are transformed by the Toeplitz matrix A into new random variables $\eta_j, j \in \{-1, \ldots, l-1\}$, as in the proof of Theorem 5.5.4. Here the convolution vector is given by $\alpha_0 = 1, \alpha_1 = -\alpha$.

The conditional density $\rho_j(\eta) = \rho_j^l(\eta)$ of the variable η_j with respect to the remaining coupling constants $\eta^{\perp j} = (\eta_k)_{k \in \Lambda_l^+ \setminus j}$ in Λ_l^+ is given by $\rho_j(\eta) = \frac{k(\eta)}{g_j(\eta)}$. Here $g_j(\eta) = \int k(\eta) d\eta_j$ denotes the marginal density. The question is whether $\sup_j \rho(\eta)$ is finite.

One calculates the common density to be

$$k(\eta) = \prod_{k=-1}^{l-1} \chi_{[0,1]}\Big(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\Big)$$

For $\eta_{j+1} \in [0,1], \eta_k = 0, \forall k \neq j+1$ we have

$$k(\eta) = \prod_{k=-1}^{l-1} \chi_{[0,1]}(\alpha^{k-j-1}\eta_{j+1}) = 1.$$

The marginal density

$$\begin{aligned} g_j(\eta) &= \prod_{k=-1}^{j-1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) \int \prod_{k=j}^{l-1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) d\eta_j \\ &\leq \int d\eta_j \prod_{k=j}^{j+1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) \end{aligned}$$

has for $\eta_{j+1} \in [0,1], \eta_k = 0, \forall k \notin \{j, j+1\}$ the upper bound

$$\int_0^1 \chi_{[0,1]}(\alpha\eta_j + \eta_{j+1}) d\eta_j \leq \alpha^{-1}(1 - \eta_{j+1})$$

In particular, $g_j(\eta) \searrow 0$ for $\eta_{j+1} \nearrow 1$ and thus

$$\sup_\eta \rho_j(\eta) = \infty$$

Therefore, proofs of a Wegner estimate which require the conditional density to be bounded cannot be applied to this alloy type potential. See also Sect. 4.3 of [482] for another example.

5.6 The Finite Section Method for Multi-Level Laurent Matrices

In this section we collect certain results which are used in Sect. 5.5. They concern the theory of finite sections for multi-level Laurent matrices. In fact, the finite section operators of a Laurent matrix are the same as the ones of the Toeplitz matrix generated by the same convolution vector. We are concerned with criteria for (uniform) bounded invertibility of the finite section operators. These questions have a long history of research in the one dimensional case. For an accessible introduction to this subject see for instance [199], [56], and for a detailed account [55]. In the higher-dimensional case, i.e. for *multi-level* Laurent or Toeplitz matrices much less is known. Results of the type which are relevant for us can be found in [309, 308, 307, 310, 311, 54], see also [134].

Throughout this section we use the notation of Definition 5.5.1. We need some further notions to formulate the results.

Definition 5.6.1. *A subset $M \subset \mathbb{Z}^d$ is called a* canonical discrete half-space *if M and $\mathbb{Z}^d \setminus M$ are sub-semigroups of $\mathbb{Z}^d$. A set M is* called discrete half-space *if there exists $j \in \mathbb{Z}^d$ such that $M + j$ is a canonical discrete half-space.*

Let $F \subset \mathbb{Z}^d$ be finite and $\tilde{F}$ its convex hull in $\mathbb{R}^d$. Then the set $\tilde{F} \cap \mathbb{Z}^d$ is called a convex lattice polygon.

Denote by $B_r(x) \subset \mathbb{R}^d$ an open ball of radius r centred at the point x. Let $\mathfrak{M}(r, R)$ be the set of all convex lattice polygons Θ in $\mathbb{Z}^d$ satisfying the following conditions

(i) for any $x \in \mathbb{R}^d$ there is a discrete half-space M such that

$$\Theta \cap B_r(x) = M \cap B_r(x)$$

(ii) $\Theta \supset B_R(0) \cap \mathbb{Z}^d$.

Here is a special case of the main result of the paper [311].

Theorem 5.6.2. *Let A be a multi-level Laurent matrix on $\ell^p(\mathbb{Z}^d)$, $p \in [1, \infty[$ with non-vanishing symbol $s(\theta)$. If $s(\theta)$ is a trigonometric polynomial and its topological index is zero, then there exist positive numbers r, R, and $C_{r,R}$ such that for any $\Theta \in \mathfrak{M}(r, R)$ the associated truncated Toeplitz matrix A_Θ is invertible and its inverse satisfies $\|A_\Theta^{-1}\| \leq C_{r,R}$.*

If we apply the theorem to two different values of allowed radii $R = R_1$ and $R = R_2$, such that $R_1 \leq R_2$, we may chose the constants appearing in Theorem 5.6.2 to obey $C_{r,R_2} \leq C_{r,R_1}$, since

$$\mathfrak{M}(r, R_1) \supset \mathfrak{M}(r, R_2) \tag{5.30}$$

To be able to use Theorem 5.6.2 in the context of Sect. 5.5, one has to settle the question whether for a pair $r, R > 0$ as in Theorem 5.6.2 the set $\mathfrak{M}(r, R)$ contains an exhaustion of $\mathbb{Z}^d$. By (5.30), such an exhaustion exists

if for arbitrary $r, R > 0$ the set $\mathfrak{M}(r, R)$ is non-empty. For $d \geq 3$ this seems to be an open question. In one space dimension, $\mathfrak{M}(r, R)$ contains all discrete intervals $\Theta \supset B_R(0)$ of diameter greater or equal to $2r$. By a discrete interval we mean the intersection of an ordinary interval with $\mathbb{Z}$. For the case $d = 2$ the following fact has been stated in [311] and proven in [301].

Proposition 5.6.3. *If $d = 2$, then for any $r > 0$ and $R > 0$ the set $\mathfrak{M}(r, R)$ is non-empty.*

Thus in dimensions one or two an exhaustion sequence $\Theta_1 \subset \Theta_2 \subset \dots$ of discrete lattice polygons in $\mathfrak{M}(r, 1), r > 0$ exists and the corresponding truncated Toeplitz matrices A_{Θ_n} have inverses B_{Θ_n} which are uniformly bounded in $\ell^p(\mathbb{Z}^d)$. Thus we have proven the assumptions required for Theorem 5.5.5.

Let us give one historical comment. For one space dimension a criterion on uniformly bounded invertibility of finite section Toeplitz operators in $\ell^1(\mathbb{Z})$ was proven in [36], see also Theorem III.2.1 in [199]: For a convolution vector $\alpha \in c_0(\mathbb{Z})$ whose symbol s_α does not vanish anywhere on $[-\pi, \pi[$ and has zero winding number there exist a constant $C_B \in \mathbb{R}$ such that

$$\|B\| \leq C_B, \qquad \sup_{n \in \mathbb{N}} \|B_{\Theta_n}\| \leq C_B \tag{5.31}$$

where $\Theta_n = \{-n, -n+1, \dots, n\}$. This result may be seen as a special case of the above Theorem 5.6.2.

5.7 Unbounded Coupling Constants and Magnetic Fields

Motivated by certain physical applications, e.g. the study of the quantum hall effect (see for instance [41, 377, 174, 425, 241, 146]), it is desirable to extend the results on the (Lipschitz) continuity of the IDS to include Hamiltonians with magnetic fields. This is, for instance, done in the papers [97, 213, 223, 91].

We discuss here the results on alloy type potentials obtained in [223] by Hupfer, Leschke, Müller, and Warzel, since they are build on the method presented in Sects. 5.1–5.4. Moreover, their result allows the coupling constants to be unbounded, as long as very negative fluctuations are exponentially rare. Actually, the primary interest of their research are Hamiltonians with Gaussian random potentials, so they need to cope with unbounded fluctuations of the potential. The proof is based on earlier techniques from [168] — which in turn use [89] — and Dirichlet-Neumann bracketing for magnetic Schrödinger operators, as discussed in Appendix A of [223]. The results concerning alloy type potentials are summarised in Sect. 4.1 of their paper, which we review shortly.

Let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be a measurable vector potential with the property $|A|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Denote by H_A the selfadjoint closure of $\sum_{i=j}^d (i\partial_j + A_j)^2$ defined on smooth functions with compact support. The alloy type Schrödinger operator

$H_\omega = H_A + V_\omega$ now incorporates a magnetic field. In [223] it is proven that Theorem 5.0.1 essentially remains true if the magnetic field is included. (Their conditions on the single site potential are slightly different.)

Moreover, two cases are discussed, where the coupling constants are unbounded random variables, and the theorem still remains true. In the first one, it is assumed that ω_0 is non-negative and a certain moment condition is satisfied, roughly $\mathbb{E}\{\omega_0^{2d+2}\} < \infty$.

The second one concerns the case where ω_0 is distributed according to the Laplace distribution: $\mathbb{P}\{\omega_0 \in I\} = \frac{1}{a}\int_I dx\, \mathrm{e}^{|x|/a}$. Note that the probability that ω_0 assumes very negative values is exponentially small. In fact, in [223] it is noted, that this is a necessary requirement for their techniques to work. The reason for this is that they use $r(x) = \mathrm{e}^{-tx}$ as the trace regularising function, cf. Definition 5.1.1. A different choice of r would allow for more general distributions with support unbounded from below.

For the study of spectral properties of random Schrödinger operators with or without magnetic fields the spectral shift function (SSF) is a convenient tool. In Appendix A.3 and A.4 we derive bounds on the SSF for Schrödinger operators which may include rather general magnetic vector potentials. See also the survey article [403].

We conclude this section by listing further literature on random Hamiltonians with magnetic fields. Works treating the regularity of the IDS of random Schrödinger operators with magnetic field include [488, 33, 490, 227, 213, 218, 91], while the question of the (in)dependence of the IDS on boundary conditions for these models has been treated in [375, 125, 223, 222]. A related problem is the analysis of the semigroup kernels of magnetic operators [65, 66]. In [261, 404, 67, 63] the behaviour of the IDS in a strong magnetic field is identified.

The asymptotic behaviour of the IDS near the boundaries of the spectrum in the presence of random magnetic fields was the object of study of the articles [373, 374, 376] which prove high energy and Lifshitz asymptotics for certain models. The high energy asymptotics has been analysed already in [358, 475].

For Schrödinger operators with constant magnetic field and random potential generated by a Poissonian process the different possible behaviours of the IDS at the bottom of the spectrum are analysed in [64, 155, 225, 226, 156, 492, 220]. An analogous study for Landau Hamiltonians plus alloy type random potentials is carried out in [289].

The analysis of Landau Hamiltonians in the single band approximation is done in [130, 131, 395, 396, 227]. Examples of localisation proofs which allow for magnetic fields can be found in [130, 90, 131, 132, 489, 33, 183, 133, 170, 396, 187, 6]. For a result on delocalisation in this context, see [192].

A

Properties of the Spectral Shift Function

In this appendix certain properties of the spectral shift function which are relevant for the study of the integrated density of states are collected. Let us note that this function is sometimes called Lifshitz-Krein SSF since Lifshitz and Krein were the first to devote attention to it, see e.g. the discussion in the introduction of [50].

For an exposition of the theory of the spectral shift function (SSF) see [50, 293].

An survey of the role played by the SSF in scattering theory can be found in the last chapter of [499]. The SSF has proven useful in the study of random operators, particularly in problems related to surface models, e.g. the definition of the density of surface states [82, 83, 297, 298]. In [443] it is used in the analysis of rank one perturbations of operators, in particular in the random context. More recently, it has found applications in the study of quantum graphs, cf. [299, 209, 295, 210].

Various of the properties of the SSF are discussed in the literature: monotonicity and concavity [182, 193, 294], the asymptotic behaviour in the large coupling constant limit [400, 418, 401] and semiclassical limit [372], and some other bounds [397, 398, 399]. For magnetic Schrödinger operators the SSF is analysed in [68, 164, 67, 402, 403].

A.1 The SSF for Trace Class Perturbations

For two selfadjoint operators A, B such that the difference $A-B$ is trace class the SSF $\xi(\cdot, A, B)$ may be defined (up to an additive constant) by the formula

$$\operatorname{Tr}(\rho(A) - \rho(B)) = \int \rho'(\lambda)\, \xi(\lambda, A, B)\, d\lambda \tag{A.1}$$

for all functions $\rho \in C^1(\mathbb{R})$ such that ρ' is the Fourier transform of a complex measure on $\mathbb{R}$ with finite total variation, cf. Theorem 3.2 in [50] or Theorem

8.3.3 in [499]. Actually, the assumption that ρ is in the Besov space $B^1_{\infty,1}(\mathbb{R})$ is sufficient to guarantee that $\rho(A) - \rho(B)$ is trace class and that the relation (A.1) holds, see [391] for details. Equality (A.1) is called *Krein trace formula*. If the operators A, B are semi-bounded, the mentioned additive constant can be normalised to be equal to zero, and thus the SSF becomes unique.

The SSF is related to the perturbation determinant from scattering theory by the formula

$$\xi(\lambda, A, B) = \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \arg \det[1 + (A - B)(B - \lambda - i\varepsilon)^{-1})] \tag{A.2}$$

for almost all values of $\lambda \in \mathbb{R}$. For the right side to be well defined it is actually sufficient to assume only that $(A - B)(B + i)^{-1}$ is trace class. So one can interpret (A.2) as an extension of the definition of the SSF to the class of relative trace class perturbations.

The SSF can be bounded in terms of Schatten-von Neumann ideal properties of $A - B$, namely

$$\|\xi(\cdot, A, B)\|_1 \leq \|A - B\|_{J_1} \tag{A.3}$$

Here J_1 denotes the ideal of trace class operators and $\|\cdot\|_{J_1}$ the *trace norm*. In particular, for trace class perturbations $A - B$, the SSF ξ is in $L^1(\mathbb{R})$. On the other hand, if $A - B$ is finite rank

$$\|\xi(\cdot, A, B)\|_\infty \leq \operatorname{rank}(A - B) \tag{A.4}$$

Since we have an estimate on ξ in the L^1 and L^∞-norms, it is natural to ask whether an estimate for the L^p-norm, $p \in]1, \infty[$, may be derived. This indeed turns out to be true and can be understood as an interpolation result, cf. the proof of Theorem 2.1 in [97].

To formulate this bound we have to introduce ideals of 'better than trace class' operators, defined in terms of summability properties of singular values. Recall that the singular values of a compact operator C are the square-roots of the eigenvalues of C^*C. We enumerate them in non-increasing order $\mu_1(C) \geq \mu_2(C) \geq \cdots \geq \mu_n(C) \cdots \geq 0, n \in \mathbb{N}$ counting multiplicities. If C is trace class, the sum of the singular values is finite and equals $\|C\|_{J_1}$. We denote by J_β the class of compact operators such that

$$\|C\|_{J_\beta} := \Big(\sum_{n \in \mathbb{N}} \mu_n(C)^\beta \Big)^{1/\beta} < \infty \tag{A.5}$$

The theory of such operators is classical for $\beta \geq 1$, see for instance [437]. However, since we want to interpolate between (A.3) and (A.4), we need to consider operators whose singular values converge not slower, but *faster* than an ℓ^1-sequence to zero. This leads us to consider operators such that $\|C\|_{J_\beta}$ is finite, for β *smaller than one*. In particular, all such operators are trace class, which explains why they are sometimes called *super-trace class* operators.

It follows that the SSF may be defined for perturbations of this type. The classes $J_\beta, \beta < 1$ have been studied in [200, 46, 48], while their relevance in the context of random operators was recognised in [97].

From these sources we infer the following properties of J_β. For any compact operator A and bounded B the singular values of the products obey the relation

$$\mu_n(AB) \leq \|B\|\,\mu_n(A) \quad \text{and} \quad \mu_n(BA) \leq \|B\|\,\mu_n(A) \tag{A.6}$$

If we assume that B is also compact, the Ky-Fan inequalities [158, 437] establish for the singular values of the sum $A + B$ the bounds

$$\mu_{n+m+1}(A+B) \leq \mu_{n+1}(A) + \mu_{m+1}(B) \quad \text{for all } n, m = 0, 1, 2, \ldots \tag{A.7}$$

In particular the set J_β is a two-sided ideal in the algebra of bounded operators for all $\beta > 0$. For $\beta \geq 1$ the functional $A \mapsto \|A\|_{J_\beta}$ is a norm, which is not true for $\beta < 1$. More precisely, in the latter case we have only

$$\|A+B\|_{J_\beta}^\beta \leq \|A\|_{J_\beta}^\beta + \|B\|_{J_\beta}^\beta$$

This property implies that $\|\cdot\|_{J_\beta}$ is a *quasi-norm* and that

$$\mathrm{dist}_\beta(A,B) = \|A-B\|_{J_\beta}^\beta$$

is a well defined metric on J_β. The pair $(J_\beta, \mathrm{dist}_\beta)$ forms a complete, separable, linear metric space, in which the finite rank operators form a dense subset.

In [97] the following L^p-bound on the SSF was proven.

Theorem A.1.1. *Let $p \geq 1$ and A, B be selfadjoint operators whose difference is in J_β, where $\beta = 1/p$. Then the spectral shift function $\xi(\cdot, A, B)$ is in $L^p(\mathbb{R})$ and*

$$\|\xi(\cdot, A, B)\|_{L^p} \leq \big\|A - B\big\|_{J_\beta}^\beta \tag{A.8}$$

A sharp bound on the SSF was proven by Hundertmark and Simon in [221], which in fact includes Theorem A.1.1 as a special case.

Theorem A.1.2. *Let $F\colon [0,\infty[\to [0,\infty[$ be a convex function such that $F(0) = 0$. Let A, B be bounded and C a non-negative compact operator such that for all $N \in \mathbb{N}$*

$$\sum_{n=N}^{\infty} \mu_n(|A-B|) \leq \sum_{n=N}^{\infty} \mu_n(C) \tag{A.9}$$

Then

$$\int F\big(|\xi(\lambda, A, B)|\big)d\lambda \leq \int F\big(|\xi(\lambda, C, 0)|\big)d\lambda = \sum_{n\in\mathbb{N}} \big[F(n) - F(n-1)\big]\,\mu_n(C)$$

Condition (A.9) is in particular satisfied if $|A - B| \leq C$. Of course, to be able to apply Theorem A.1.2 one needs to have appropriate estimates on the singular values of the operator C. In the context of Schrödinger operators such estimates are derived in Sect. A.2.

In certain situations one can show that an operator C belongs to a Schatten-von Neumann class J_β by writing it as a product of operators $C_1, \ldots, C_N$ for which one already knows that they belong to a larger class J_α with $\alpha > \beta$. Then the product $C = C_1 \cdots C_N$ will enjoy much better summability properties than the individual factors.

For this purpose it is useful to note that the Hölder inequality extends also to the case of exponents smaller than one: let $a_i \colon \mathbb{N} \to \mathbb{C}$, $i = 1, \ldots, N$ be such that $|a_i(n)|^{p_i}$ is summable, where $p_i > 0$ for all $i = 1, \ldots, N$, and set $\frac{1}{r} := \sum_{i=1}^N \frac{1}{p_i}$. Then the pointwise product $\prod_{i=1}^N a_i$ is in $\ell^r(\mathbb{N})$ and

$$\Big\| \prod_{i=1}^N a_i \Big\|_r \leq \prod_{i=1}^N \|a_i\|_{p_i}$$

By applying this to the sequence of singular values of compact operators, we obtain the following

Lemma A.1.3. *Let $C_i \in J_{p_i}$ for $i = 1, \ldots, N$, then $\prod_{i=1}^N C_i$ is in J_r where $\frac{1}{r} := \sum_{i=1}^N \frac{1}{p_i}$ and*

$$\Big\| \prod_{i=1}^N C_i \Big\|_{J_r} \leq \prod_{i=1}^N \|C_i\|_{J_{p_i}} \tag{A.10}$$

See also [48], Corollary 11.11.

A.2 The SSF for Schrödinger Operators and the Invariance Principle

We want to apply the formalism of the SSF to Schrödinger operators. In this setting we cannot use the results of the previous section on trace class perturbations, since any potential given by a proper function is not in this class of operators.

It turns out that the SSF can be defined easily for a pair of selfadjoint, lower bounded operator with purely discrete spectrum. This would cover Schrödinger operators restricted to finite cubes Λ. In the application in Sect. 4.2 we use only the SSF for this type of operators. However, since we are not only interested in the *existence* of the SSF as a function, but also on *upper bounds*, we have to resort to more powerful techniques, which then allow us along the way to define the SSF for a pair of Schrödinger operators on the whole of $\mathbb{R}^d$, under the assumption that they differ by a compactly supported potential.

For a selfadjoint, lower bounded operator H with purely discrete spectrum the eigenvalue counting function $\mathcal{N}(H,E) := \#\{n \mid \lambda_n(H) \leq \lambda\}$ is well defined. Here $\lambda_n(H), n \in \mathbb{N}$ enumerates the spectrum of H, counting multiplicities, in increasing order. Thus for two such operators H_1, H_2 the SSF can be defined as the difference of the eigenvalue counting functions,

$$\xi(\lambda, H_2, H_1) := \mathcal{N}(H_2, E) - \mathcal{N}(H_1, E)$$

If the difference $H_2 - H_1$ happens to be trace class, this definition produces (almost everywhere) the same function ξ as (A.1). To see this, one chooses in (A.1) a sequence ρ_ε of switch functions which converges to the step function $\chi_{]-\infty,E]}$ as $\varepsilon \to 0$.

Now we consider the case that the operators H_1 and H_2 are selfadjoint and lower bounded, but we neither assume that the difference $H_2 - H_1$ is trace class nor that the operators H_1 and H_1 have purely discrete spectrum. Let $g\colon \mathbb{R} \to [0, \infty[$ be a C^2-function such that g' is everywhere negative. In particular, the function g is bounded on the spectra of H_1 and H_2 and thus $g(H_2) - g(H_1)$ is a bounded operator. Assume that $g(H_2) - g(H_1)$ is trace class. Then the SSF for the operator pair $g(H_1), g(H_2)$ is well defined by (A.1) and we may set

$$\xi(\lambda, H_2, H_1) := -\xi\big(g(\lambda), g(H_2), g(H_1)\big) \tag{A.11}$$

This definition is independent of the choice of g, as long as it has the abovementioned properties, see e.g. [50, Sect. 1.4]. Formula (A.11) is called the *invariance principle* in analogy to the relation in scattering theory. This last definition will be sufficiently general to cover the type of Schrödinger operators we are considering. Natural candidates for the function g are the following families of functions

$$g(x) = (x + C)^{-k},\ \mathbb{N} \ni k > \frac{d}{2} + 2,\ C > -\inf \sigma(H)$$

corresponding to powers of resolvents, respectively

$$g(x) = \exp(-tx),\ t > 0 \tag{A.12}$$

corresponding to semigroups of Schrödinger operators. Of course we will still have to impose certain regularity assumptions on the Schrödinger operators H_1, H_2 such that $g(H_2) - g(H_1)$ really turns out to be trace class. In the remainder of the appendix we choose the function g as in (A.12).

A.3 Singular Value Estimates

There is a short and transparent way to prove the super-trace class estimates needed for the bound of the SSF. It uses the decay of singular values of certain auxiliary operators. The basic observation is that the singular values of the difference of two Schrödinger semigroups decay almost exponentially and the semigroup difference is therefore in any super-trace class ideal. There are

essentially two ingredients in the proof of this statement: a Weyl-type bound on eigenvalues and exit time estimates for Brownian motion. We follow here the presentation from the paper [218] by Hundertmark, Killip, Nakamura, Stollmann and the author.

Weyl's law gives the asymptotic behaviour of the n^{th} eigenvalue of the Laplacian on an open ball B for large n. For our purposes it is necessary to have a lower bound of this type valid for *all* eigenvalues. It is provided in the following lemma which applies for rather general Schrödinger operators with electromagnetic field.

We consider magnetic Schrödinger operators

$$H = H_A + V, \quad H_A = (-i\nabla - A)^2 \tag{A.13}$$

acting on $\mathbb{R}^d$ with magnetic potential A and electric potential $V = V_+ - V_-$, where $V_+ := \max(0, V)$ and $V_- := \max(0, -V)$.

Lemma A.3.1. *Let H be as in* (A.13)*. Assume that each component of A is in $L^2_{\text{loc}}(\mathbb{R}^d)$, that $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$, and that V_- is $-\Delta$ bounded with relative bound $\delta < 1$. Furthermore, let $H^{\mathcal{U}}$ be the Dirichlet restriction of H to an arbitrary open set $\mathcal{U} \subset \mathbb{R}^d$ with finite volume $|\mathcal{U}|$. Then there exists a constant C, such that the n^{th} eigenvalue of $H^{\mathcal{U}}$ satisfies*

$$\lambda_n \geq \frac{2\pi(1-\delta)d}{e} \Big(\frac{n}{|\mathcal{U}|}\Big)^{2/d} - C \qquad \textit{for all } n \in \mathbb{N} \tag{A.14}$$

Under these assumptions on the vector and scalar potential, H can be defined by the use of quadratic forms, see for instance Sect. 2 of [222]. The Dirichlet restriction $H^{\mathcal{U}}$ to the set $\mathcal{U}$ is defined in the same way. If A and V_- vanish, we have already used this way to define Schrödinger operators in Remark 2.2.3.

Proof. Since the Sobolev space with Dirichlet b.c. $W^{1,2}_0(\mathcal{U})$ is a natural subset of $W^{1,2}(\mathbb{R}^d)$, V_- is also relatively form bounded w.r.t. $-\Delta^{\mathcal{U}}$, the Dirichlet Laplacian on $\mathcal{U}$, with relative bound δ. The diamagnetic inequality, cf. [436], then implies that V_- is also relatively form bounded with respect to the Dirichlet restriction $H_A^{\mathcal{U}}$ of H_A to $\mathcal{U}$. In other words, there exists a constant $C \in \mathbb{R}$ such that

$$V_- \leq \delta H_A^{\mathcal{U}} + C$$

in the sense of quadratic forms. In particular, since V_+ is non-negative,

$$H^{\mathcal{U}} \geq H_A^{\mathcal{U}} - V_- \geq (1-\delta) H_A^{\mathcal{U}} - C$$

which implies the bound

$$\begin{aligned} \operatorname{Tr}(\mathrm{e}^{-2tH^{\mathcal{U}}}) &\leq \mathrm{e}^{2tC} \operatorname{Tr}(\mathrm{e}^{-2t(1-\delta)H_A}) = \mathrm{e}^{2tC} \|\mathrm{e}^{-t(1-\delta)H_A}\|^2_{\text{HS}} \\ &= \mathrm{e}^{2tC} \iint_{\mathcal{U}\times\mathcal{U}} |\mathrm{e}^{-t(1-\delta)H_A}(x,y)|^2 dx\, dy \end{aligned}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes as before the Hilbert-Schmidt norm. Using once more the diamagnetic inequality for Schrödinger semigroups, e.g., [436, 222], one obtains the pointwise bound $|\mathrm{e}^{-t(1-\delta)H_A}(x,y)| \leq \mathrm{e}^{t(1-\delta)\Delta^{\mathcal{U}}}(x,y)$. In particular,

$$\|\mathrm{e}^{-t(1-\delta)H_A}\|_{\mathrm{HS}}^2 \leq \|\mathrm{e}^{t(1-\delta)\Delta^{\mathcal{U}}}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\mathrm{e}^{2t(1-\delta)\Delta^{\mathcal{U}}}) \leq |\mathcal{U}| \left(8\pi t(1-\delta)\right)^{-d/2}$$

In the last line we used that by domain monotonicity the kernel of the Dirichlet semigroup $\mathrm{e}^{\beta\Delta^{\mathcal{U}}}$ on the diagonal is bounded by the heat kernel on the whole of $\mathbb{R}^d$, i.e.,

$$\mathrm{e}^{\beta\Delta^{\mathcal{U}}}(x,x) \leq \mathrm{e}^{\beta\Delta}(x,x) = (4\pi\beta)^{-d/2} \quad \text{for all } \beta > 0 \text{ and } x \in \mathcal{U}$$

Thus

$$\mathrm{Tr}(\mathrm{e}^{-2tH^{\mathcal{U}}}) \leq |\mathcal{U}| \left(8\pi t(1-\delta)\right)^{-d/2} \tag{A.15}$$

Let $\mathcal{N}^{\mathcal{U}}(\lambda) := \#\{n \mid \lambda_n(H^{\mathcal{U}}) \leq \lambda\}$ be the number of eigenvalues of $H^{\mathcal{U}}$ smaller or equal to λ. By Čebyšev's inequality and (A.15),

$$\begin{aligned}\mathcal{N}^{\mathcal{U}}(\lambda) &\leq \mathrm{e}^{2t\lambda} \int_{-\infty}^{\lambda} \mathrm{e}^{2ts} d\mathcal{N}^{\mathcal{U}}(s) \leq \mathrm{e}^{2t\lambda}\mathrm{Tr}(\mathrm{e}^{-2tH^{\mathcal{U}}}) \\ &\leq |\mathcal{U}| \left(8\pi(1-\delta)\right)^{-d/2} t^{-d/2} \mathrm{e}^{2t(\lambda+C)}\end{aligned}$$

for arbitrary $t > 0$. The last expression is for $t := \frac{d}{4(\lambda+C)}$ equal to

$$|\mathcal{U}| \left(\frac{e(\lambda+C)}{2\pi(1-\delta)d}\right)^{d/2}$$

This estimate implies together with $n \leq \mathcal{N}^{\mathcal{U}}(\lambda_n)$ the lower bound

$$\lambda_n \geq \frac{2\pi(1-\delta)d}{e} \left(\frac{n}{|\mathcal{U}|}\right)^{2/d} - C$$

on the eigenvalues. □

In the next theorem we consider a pair Schrödinger operators which obey slightly stronger assumptions regarding the negative part V_- of their scalar potential. More precisely, we assume that V_- is in the Kato class. A general discussion of the Kato-class can be found in [102]. For its relevance to the Feynman-Kac formula see, e.g., [12, 65, 438]. In particular, V_- is in the Kato-class, if it is in $L^p_{\mathrm{loc,unif}}(\mathbb{R}^d)$ for $p = 1$, if $d = 1$ and $p > d/2$, if $d \geq 2$, cf. Sect. 1.2. Under these hypotheses, one may define H via the corresponding quadratic form with core $C_c^\infty(\mathbb{R}^d)$, similarly as in Remark 2.2.3. By the same method, one can define the Dirichlet restriction H^l of H to the cube $\Lambda_l =]-l/2, l/2[^d, l \geq 1$.

Let H_1 be a Schrödinger operator of the form just described and let $H_2 = H_1 + u$. Assume that $u = u_+ - u_-$ has compact support, that $u_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$

and that u_- is in the Kato class. Our aim is to obtain an estimate on the singular values of $V_{\text{eff}} := \mathrm{e}^{-H_1} - \mathrm{e}^{-H_2}$ and on the corresponding object in the finite volume case, namely $V_{\text{eff}}^l := \mathrm{e}^{-H_1^l} - \mathrm{e}^{-H_2^l}$.

Theorem A.3.2. *There exists a constant c depending only on the dimension d, and a constant C depending on the Kato-class norms of u_-, V_- and on the diameter of the support of u_+ such that the singular values of the operators V_{eff} and V_{eff}^l obey for all $n \in \mathbb{N}$ and $l \geq 1$ the relations*

$$\mu_n(V_{\text{eff}}) \leq C\,\mathrm{e}^{-cn^{1/d}} \quad \text{and} \quad \mu_n(V_{\text{eff}}^l) \leq C\,\mathrm{e}^{-cn^{1/d}} \tag{A.16}$$

Remark A.3.3. (i) Note that the estimate (A.16) depends on the positive part of u only through $\operatorname{supp} u$. Thus, for $u = \kappa\tilde{u}$ where $\tilde{u} \geq 0$ and κ is a non-negative coupling constant, the estimate is independent of the choice of κ.

(ii) Actually, u may be taken to $+\infty$ on its support. In this case H_2 equals the restriction of H_1 to $\mathbb{R}^d \setminus \operatorname{supp} u$ with Dirichlet boundary conditions, provided the boundary of $\operatorname{supp} u$ obeys some mild regularity conditions, see for instance [455].

(iii) Similarly, H_1, H_2 may be defined on a set strictly smaller than $\mathbb{R}^d$: Let $\mathcal{U} \subset \mathbb{R}^d$ be open, $H_A^{\mathcal{U}}$ the Dirichlet restriction of H_A on $\mathcal{U}$, and $H_1 = H_A^{\mathcal{U}} + V$, $H_2 = H_A^{\mathcal{U}} + V + u$ where V and u satisfy the same conditions as before. In this case H_j^l is the Dirichlet restriction of $H_j, j = 1, 2$ to the set $\Lambda_l \cap \mathcal{U}$.

Remark A.3.4. The decay estimate in (A.16) is almost optimal with respect to the exponent. In [405] Raikov and Warzel analyse the perturbation of the free Schrödinger operator with a constant magnetic field in two dimensions, i.e. the Landau Hamiltonian, by a compactly supported potential. The results in [405] (and similarly those in [359]) show that for general magnetic Schrödinger operators one cannot obtain a faster decay than

$$\mu_n(V_{\text{eff}}) \leq C\,\mathrm{e}^{-cn^{2/d}}$$

This is explained in some detail in the first Section of [218]. See also Remark A.4.2.

Proof (of Theorem A.3.2). We give the proof for V_{eff}, the adaption to V_{eff}^l requires only minor changes. We will use the symbols c and C for constants that vary from line to line; however, their dependence on H_1 and H_2 will always be as stated in the Theorem.

Without loss of generality, we can assume that the origin is contained in the support of u. We will estimate the n^{th} singular value by Dirichlet decoupling at an n-dependent radius R. To this end, let R be sufficiently large so that $\operatorname{supp} u$ is contained strictly inside the ball of radius R centred at the origin, which we will denote by B_R.

Let H_j^R ($j = 1$ or 2) be the Dirichlet restriction of H_j to the ball B_R, and let

$$A_R := \mathrm{e}^{-H_2^R} - \mathrm{e}^{-H_1^R} \quad \text{and} \quad D_R := V_{\text{eff}} - A_R. \tag{A.17}$$

As any Kato-class potential is relatively form bounded with respect to the Laplacian with relative bound zero, we may apply Lemma A.3.1 to deduce that $\mu_n(\mathrm{e}^{-H_j^n}) \leq C\exp(-cn^{2/d}R^{-2})$ for both $j = 1$ and $j = 2$. Since A_R is the difference of two *non-negative* operators by the min-max theorem its singular values obey the same type of bound:

$$\mu_n(A_R) \leq C\exp(-cn^{2/d}R^{-2}) \tag{A.18}$$

If D_n is bounded, then $\mu_n(V_{\text{eff}}) \leq \mu_n(A_R) + \|D_n\|$. We now proceed to estimate the norm of D_n by using the Feynman-Kac-Itô formula for magnetic Schrödinger semigroups with Dirichlet boundary conditions, see [65, 436].

Let $\mathbf{E}_x$ and $\mathbf{P}_x$ denote the expectation and probability for a Brownian motion, b_t starting at x. Let $\tau_R = \inf\{t > 0 | b_t \notin B_R\}$ denote the exit time from the ball B_R and set $\tau_n := \tau_{R_n}$. Then

$$(D_n f)(x) = \mathbf{E}_x\left[\mathrm{e}^{-\mathrm{i}S_A(b)}\big(\mathrm{e}^{-\int_0^1 (V+u)(b_s)ds} - \mathrm{e}^{-\int_0^1 V(b_s)ds}\big)\chi_{\{\tau_n \leq 1\}}(b) f(b_1)\right]$$

where S_A^t is real valued stochastic process corresponding to the purely magnetic part of the Schrödinger operator. Actually, for this representation one first chooses a suitable gauge, for instance $\mathrm{div} A = 0$, and then uses gauge invariance for the general case, see [322].

By taking the modulus and using the triangle inequality, one sees that the magnetic vector potential can be eliminated:

$$|D_n f|(x) \leq \mathbf{E}_x\big[\mathrm{e}^{-\int_0^1 V(b_s)ds}\big|\mathrm{e}^{-\int_0^1 u(b_s)ds} - 1\big|\chi_{\{\tau_n \leq 1\}}(b)|f(b_1)|\big]$$

Moreover, only Brownian paths which both visit $\operatorname{supp} u$ and leave B_{R_n} within one unit of time contribute to the expectation. Thus if τ_u is the hitting time for $\operatorname{supp} u$ and $\mathcal{B} = \{\tau_n \leq 1, \tau_u \leq 1\}$, then

$$|D_n f|(x) \leq \mathbf{E}_x\big[\mathrm{e}^{-\int_0^1 V(b_s)ds}\big|\mathrm{e}^{-\int_0^1 u(b_s)ds} - 1\big|\chi_{\mathcal{B}}(b)|f(b_1)|\big]$$

so, applying Hölder's inequality,

$$\begin{aligned}|D_n f|(x) \leq &\Big(\mathbf{E}_x\big[\mathrm{e}^{-8\int_0^1 V(b_s)ds}\big]\Big)^{1/8}\Big(\mathbf{E}_x\big[\big|\mathrm{e}^{-\int_0^1 u(b_s)ds} - 1\big|^8\Big)^{1/8} \\ &\cdot \Big(\mathbf{E}_x\big[\chi_{\mathcal{B}}(b)\big]\Big)^{1/4}\Big(\mathbf{E}_x\big[|f(b_1)|^2\big]\Big)^{1/2}\end{aligned}$$

Since V_- and u_- are in the Kato class, Kashminskii's lemma implies that the first two terms are bounded uniformly in x, see for instance [12, 438].

Levy's inequality combined with elementary estimates imply $\mathbf{P}_{x=0}\{\tau_R \leq 1\} \leq 2\mathbf{P}_{x=0}\{|b_1| \geq R\} \leq C\mathrm{e}^{-R^2/4}$. As any path in $\mathcal{B}$ must cover the distance

r between supp u and the complement of the ball B_R, we can deduce that $\mathbf{P}_x(\mathcal{B}) \leq C\mathrm{e}^{-r^2/4} \leq C\mathrm{e}^{-R^2/8}$ where we chose without loss of generality $r \geq R/\sqrt{2}$. Thus

$$|D_n f|(x) \leq C\mathrm{e}^{-R^2/32}\big\{\mathbf{E}_x|f(b_1)|^2\big\}^{1/2} = C\mathrm{e}^{-R^2/32}\big\{(\mathrm{e}^{\Delta}|f|^2)(x)\big\}^{1/2}$$

in particular, using the fact that e^{Δ} is an L^1 contraction,

$$\|D_n f\|_2 \leq C\mathrm{e}^{-R^2/32}\big\|(\mathrm{e}^{\Delta}|f|^2)\big\|_1^{1/2} \leq C\mathrm{e}^{-R^2/32}\|f^2\|_1^{1/2} = C\mathrm{e}^{-R^2/32}\|f\|_2$$

To balance the two bounds obtained for $\mu_n(A_R)$ and $\|D_n\|$ one chooses $R_n := n^{1/2d}$, which leads to (A.16). □

A.4 Bounds on the SSF for Schrödinger Operators

Let H_1, H_2 be as in the last Section. Theorem A.3.2 implies in particular that V_{eff} and V_{eff}^l are trace class. Thus the SSF for the operator pair H_1, H_2 is well defined by formula (A.11) with the choice $g(x) = \mathrm{e}^{-x}$.

The explicit estimates obtained in Theorem A.3.2 for the singular values of Schrödinger semigroups differences together with the abstract results from Sect. A.1 allow us to infer the desired bounds on the SSF for a pair of Schrödinger operators. The results and the presentation in this section are taken from the paper [218] by Hundertmark, Killip, Nakamura, Stollmann and the author.

Theorem A.4.1. *Let ξ be the spectral shift function for the pair H_1, H_2 or H_1^l, H_2^l. There exists constants K_1, K_2 depending only on d,* diam supp u_+ *and the Kato class norms of V_-, u_-, such that for any bounded compactly supported function f,*

$$\int f(\lambda)\,\xi(\lambda)\,d\lambda \leq K_1\mathrm{e}^b + K_2\,\{\log(1+\|f\|_\infty)\}^d\|f\|_1 \tag{A.19}$$

with $b = \sup\operatorname{supp} f$.

Remark A.4.2. (i) Theorem A.4.1 implies that the spectral shift function can have at most logarithmic local singularities. One might think that, at least for smooth compactly supported perturbations, the SSF should always be locally bounded. However, this is not the case. In the paper [405] already mentioned in Remark A.3.4, Raikov and Warzel consider the free Schrödinger operator with a constant magnetic field in dimension two. For a perturbation of this operator by a compactly supported potential they showed that the SSF diverges at each Landau level λ_q like

$$|\xi(\lambda_q + \lambda)| \sim \Big(\frac{|\ln(\lambda)|}{\ln|\ln\lambda|}\Big)^{d/2} \qquad \text{as } \lambda \downarrow 0 \tag{A.20}$$

See [359] for the generalisation to even dimensions.

(ii) An example without magnetic fields, where the SSF shows unexpected divergencies, was given by Kirsch in [246, 248]. Denote by Δ^l the Laplace operator on the cube Λ_l with Dirichlet b.c. For $\lambda > 0$, $u\colon \mathbb{R}^d \to \mathbb{R}$ a non-negative, bounded function with compact support, which is not identically equal to zero, and a function $a\colon [0,\infty[\to]0,\infty[$, set $\xi_l(\cdot) := \xi(\cdot, -\Delta^l, (-\Delta + a(l)u)^l)$. Then $\limsup_{l\to\infty} \xi_l(\lambda) = \infty$, for any λ, a and u as above. This result relies on the high degeneracy of eigenvalues of the pure Dirichlet Laplacian on a cube. In this respect it is related to the example in Remark (i) with the Landau Hamiltonian which has infinitely degenerate eigenvalues. Kirsch shows that there is, however, a set of full measure $\mathcal{E} \subset \mathbb{R}$ with dense complement such that

$$\lim_{\mathbb{N}\ni l\to\infty} \xi_l(\lambda) = 0, \text{ for all } \lambda \in \mathcal{E}, \text{ if } a(l) \le l^{-k}, k > 3$$

(iii) In contrast to the above unboundedness results, Sobolev, [449] showed that for the pair $H_1 = -\Delta$ and $H_2 = -\Delta + u$ with $|u(x)| \le const.\,(1 + |x|)^{-\alpha}$ and $\alpha > d$, the spectral shift function ξ is, indeed, locally bounded. However, this type of result seems to require very strong hypotheses on H_1, for example, a trace-class limiting absorption principle and in particular, that H_1 has absolutely continuous spectrum on the positive real axis.

(iv) For certain alloy type Schrödinger operators Combes, Hislop and Klopp obtain in Theorem 2.1 of [92] a local boundedness result for an associated *averaged* SSF.

Proof (of Theorem A.4.1). Let the two Schrödinger operators $H_2 = H_1 + u$ be as in the statement of the Theorem.

For $t > 0$ define $F_t : [0,\infty[\to [0,\infty[$ by

$$F_t(x) = \int_0^x (\exp(ty^{1/d}) - 1)\, dy \tag{A.21}$$

As the integrand is increasing, F_t is a convex function. We show first that there exists a constant K_1, depending on t, such that for small enough $t > 0$,

$$\int_{-\infty}^{T} F_t(|\xi(\lambda)|)\, d\lambda \le K_1 \mathrm{e}^T < \infty \tag{A.22}$$

for all $T < \infty$. To see this, we use the invariance principle and a change of variables, to obtain

$$\begin{aligned}\int_{-\infty}^{T} F(|\xi(\lambda, H_2, H_1)|)\, d\lambda &= \int_{-\infty}^{T} F(|\xi(\mathrm{e}^{-\lambda}, \mathrm{e}^{-H_2}, \mathrm{e}^{-H_1})|)\, d\lambda \\ &\le \mathrm{e}^T \int_{\mathrm{e}^{-T}}^{\infty} F(|\xi(s, \mathrm{e}^{-H_2}, \mathrm{e}^{-H_1})|)\, ds\end{aligned}$$

By Theorem A.1.2 the integral on the right hand side is bounded by

$$\int_{-\infty}^{\infty} F(|\xi(s, \mathrm{e}^{-H_2}, \mathrm{e}^{-H_1})|)\, ds \leq \sum_{n=1}^{\infty} \mu_n(V_{\mathrm{eff}})(F(n) - F(n-1))$$
$$\leq \sum_{n=1}^{\infty} \mu_n(V_{\mathrm{eff}}) \int_{n-1}^{n} (\mathrm{e}^{ts^{1/d}} - 1) ds \leq C \sum_{n=1}^{\infty} \mathrm{e}^{(t-c)n^{1/d}}$$

which is finite, if we chose t smaller than the constant c from Theorem A.3.2. Thus we have proven (A.22).

Now we deduce the bound (A.19) from (A.22) with the help of Young's inequality for an appropriate pair of functions. Note that F_t is non-negative, convex with $F_t'(0) = 0$ and hence its Legendre transform G is well defined and satisfies

$$G(y) := \sup_{x \geq 0} \{xy - F(x)\} \leq y \Big(\frac{\log(1+y)}{t} \Big)^d \text{ for all } y \geq 0$$

Thus, by the very definition of G, Young's inequality holds: $yx \leq F(x) + G(y)$. So, with $b = \sup \operatorname{supp} f$,

$$\int f(\lambda)\xi(\lambda)\, d\lambda \leq \int_{-\infty}^{b} F(|\xi(\lambda)|)\, d\lambda + \int G(|f(\lambda)|)\, d\lambda \tag{A.23}$$

Using the estimate (A.22), the first integral is bounded by $K_1 \mathrm{e}^b$. For the second integral in (A.23), we estimate

$$\int G(|f(\lambda)|)\, d\lambda \leq \int |f(\lambda)| \Big(\frac{\log(1 + |f(\lambda)|)}{t} \Big)^d d\lambda \leq t^{-d} |\log(1 + \|f\|_\infty)|^d \, \|f\|_1$$

This finishes the proof of Theorem A.4.1. □

To apply Theorem A.4.1 in the situation of Sect. 4.2, we take f to be the the derivative of a smooth, monotone switch function $\rho_{E,\varepsilon} \colon \mathbb{R} \to [-1, 0]$. By a switch function we mean that for a positive $\varepsilon \leq 1/2$ it has the following properties: $\rho_{E,\varepsilon} \equiv -1$ on $]-\infty, E-\varepsilon]$, $\rho_{E,\varepsilon} \equiv 0$ on $[E+\varepsilon, \infty[$ and $\|\rho'_{E,\varepsilon}\|_\infty \leq 1/\varepsilon$, similarly as in Sect. 4.1. Theorem A.4.1 and the Krein trace formula (A.1) imply

Corollary A.4.3. *Let H_1, H_2 and $\rho_{E,\varepsilon}$ be as above. There is a constant C_E depending only on E, d,* diam supp u_+ *and the Kato class norms of V_-, u_-, such that*

$$\operatorname{Tr}\left[\rho_{E,\varepsilon}(H_2) - \rho_{E,\varepsilon}(H_1)\right] \leq C_E \, |\log \varepsilon|^d \tag{A.24}$$

The function $E \mapsto C_E$ is monotone and continuous.

The estimate (A.24) improves upon a bound derived by Combes, Hislop and Nakamura in [97]. They prove that for any exponent $\alpha < 1$, there is a constant $\tilde{C}_E(\alpha)$ depending only on d, C_0, diam supp u, $E + \varepsilon$ and α such that

$$\operatorname{Tr}\left[\rho_{E,\varepsilon}(H_2) - \rho_{E,\varepsilon}(H_1)\right] \leq \tilde{C}_E(\alpha)\, \varepsilon^{-\alpha} \tag{A.25}$$

Remark A.4.4. In the context of random Schrödinger operators H_1 and H_2 appear as particular members of a random family $\{H_\omega\}_\omega$. If two configurations $\omega, \omega' \in \Omega$ differ only in one coordinate $\omega_j \neq \omega'_j$ and coincide in all the others, i.e. $\omega_k = \omega'_k$ for all $k \in \mathbb{Z}^d \setminus \{j\}$, then the pair $H_1 = H_\omega$, $H_2 = H_{\omega'}$ differs only by a single site potential. If this is compactly supported, the pair H_1, H_2 fits in the framework considered in this and the preceding sections.

Specialising further, one is often interested in the case that on the lattice site $j \in \mathbb{Z}^d$ one of the operators, say H_1, has a coupling constant taking the lowest possible value, i.e.

$$\omega_j = \omega_- = \inf \operatorname{supp} \mu$$

where μ denotes as before the distribution of ω_j, and the other operator H_2 has at this site a coupling constant with the largest possible value, i.e.

$$\omega'_j = \omega_+ = \sup \operatorname{supp} \mu$$

Let $\Lambda_l \subset \mathbb{R}^d$ be a cube such that $j \in \Lambda_l^+$. Then we can write the two operators H_1 and H_2 as

$$\begin{aligned} H_1 = H_\omega^l(\omega_j = \min) &:= H_\omega^l + (\omega_- - \omega_j)\, u(\cdot - j) \\ H_2 = H_\omega^l(\omega_j = \max) &:= H_\omega^l + (\omega_+ - \omega_j)\, u(\cdot - j) \end{aligned} \tag{A.26}$$

and consequently the difference $H_2 - H_1 = (\omega_+ - \omega_-)\, u(\cdot - j)$ is a compactly supported potential. Note that in the formulae (A.26) it is possible to express the two operators H_1 and H_2 using only one of the two configurations ω and ω'.

The upshot of this considerations is that the trace

$$\operatorname{Tr}\big[\rho[H_\omega^l(\omega, j = \max) - E] - \rho[H_\omega^l(\omega, j = \min) - E]\big] \tag{A.27}$$

can be estimated by Corollary A.4.3. It is precisely for such a pair of operators that this corollary is used in the proof of Theorem 4.2.4.

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