Contributions to Management Science

David Mueller
Ralf Trost *Editors*

Game Theory in Management Accounting

Implementing Incentives and Fairness

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Implementing Incentives and Fairness

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Preface

In the last decades, game theory has experienced growing interest and numerous applications in a wide variety of areas, e.g. economics, political science, law and psychology. This strong response results not least from the fact that games are models of social organisations and the solutions are possible stable standards of behaviour. In this way, game theory applies social standards to design universal rules and solutions which yield important and novel insights.

The relevance and rigour of game theoretic approaches for economic modelling have been highlighted by awarding the 'The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel'—better known as the 'Nobel Prize in Economics'—to researchers in this field (e.g. 1994, John F. Nash Jr., John C. Harsanyi and Reinhard Selten; 1996, William S. Vickrey; 2005, Robert J. Aumann and Thomas C. Schelling; 2007, Leonid Hurwicz, Eric S. Maskin and Robert B. Myerson; 2012, Alvin E. Roth and Lloyd S. Shapley).

Distinguishing between cooperative and non-cooperative game theory, the first assumes players who have different goals and are unwilling or unable to make binding agreements. This leads to the question of strategic behaviour and strategic decisions of the players. In contrast, the second assumes the players have identical targets and are willing and able to commit themselves: this reveals a totally different range of problems. One of these is the question of sharing fairly the jointly generated result. The demand to share fairly is well accepted, but the crucial question of how to define 'fairness' is a very complicated issue.

Management accounting provides an important background for corporate decision making. Game theory models are suitable instruments in order to assist the management's operations. These also include models of game theory which are widespread in the field of management accounting too. The main focus traditionally has been on the field of non-cooperative behaviour, but the area of cooperative game theory has developed rapidly in the last years and has received increasing attention as companies are forced to cooperate and to deal with more complex products and increasing customer needs. Generating joint economies of scope and joint economies of scale challenges management accounting. This indicates the two-sided relationship between game theory and management accounting: new game theoretic models offer new fields of applications, and these applications raise new questions for the theory.

Intensive research, in combination with the changing culture of publishing, has produced a nearly unmanageable number of publications in the areas concerned. But we know of no available volume that provides an intensive analysis of the intersection of these areas. Therefore, one main purpose of this volume is filling this research gap. In addition, we want to strengthen the relationship between the theory and the practical applications. We would like to demonstrate what kind of problems, originating in a management accounting setting, may be solved with game theoretic models. In addition, we want to indicate the restrictions of the models and the demands which are placed by management accountants on the game theoretic instruments.

Needless to say, we cannot present a universal survey over all models and research directions or even a good portion of them. Therefore, we would like to present a subjectively chosen overview of important and interesting results and developments from management accounting point of view.

We would like to provide a short explanation of the organisation of this volume. The main structure follows the traditional classification into **non-cooperative** and **cooperative** game theory. Within the non-cooperative section, we have separated two subclasses of topics: incentives and preferences. Several models and mechanisms to incentivise agents have been developed in the last decades. These models and the underlying assumptions and interpretations are introduced and discussed in the **first six chapters of Part I**. In this context, the questions arise on how to use and how to reveal the individual preferences of the agents. These questions are answered in the **following three chapters**.

The first chapter by Trost and Heim links the aspect of setting incentives for managers, an important objective in managerial accounting, with mathematical game theory via the construct of 'incentive compatibility', a construct stemming from economic mechanism theory. It shows that theoretical incentive setting models in managerial accounting may be consistent with this definition although a different wording is used. On the other hand, issues not consistent with this definition are also called 'incentive compatible'.

In the second chapter, Kunz provides a broad structured literature review of current findings in game theory regarding incentive mechanisms. The author critically evaluates the practical applicability of these findings to the design of managerial incentive systems.

The third chapter by Lukas summarises and elaborates some of the topical findings on the design of incentive contracts, concerning the measurement of the manager's performance in a multiperiod setting and the consideration of aspects known from behavioural economics. As this chapter shows, the popularity of bonus contracts or the efficacy of flat wage contracts can be explained through behavioural aspects: loss aversion and identification with the firm offer explanations though other factors, e.g. the costs of writing and administering more complex contracts, certainly contribute to their popularity.

The forth chapter by Löffler presents a model assuming asymmetric information for the case when an intermediate product is sold internally by transfer pricing but concurrently on an external market also. Three frequently applied transfer pricing systems that are applied to achieve coordination and to offer incentives within a decentralised firm are examined with respect to their properties in this specific setting.

In the fifth chapter, Bamberg and Krapp provide a formal multiperiod analysis of investment incentives allowing managers to be impatient, i.e. because of earlier ending contracts, they won't wait until the last period for their remuneration. The authors pursue an approach which uses extended incentive contracts to induce risk-neutral managers to truthfully report the net present value of investment projects ex ante. If the managers are risk-averse, this expectation-eliciting contract is biased. This bias is analysed and quantified.

Aust, Dominko, and Buscher focus in the sixth chapter on the management accounting of R&D alliances. The authors analyse the research efforts, the distribution of costs and the division of the resulting profits. The results show that some of the studied scenarios lead to non-viable equilibria and that the research effort will always be higher under an equal distribution of power than for a Stackelberg game.

In the centre of the next three chapters stands the roles and the impacts of the agents' preferences. Moreover, mechanisms which may be employed by a principal to reveal the true preferences of the agents are analysed.

In the seventh chapter, Küpper and Sandner analyse the impact of agents' heterogeneous social preferences in rivalry, pure self-interest and altruism, on the weighting and combination of incentive performance measures and on a firm's profitability. It is shown that firms maximise their profits when they maximise the difference between two agents' individual social preferences.

The eighth chapter by Patzenhauer provides a broad overview of the applications of auction theory, especially in a management accounting context. The main purpose of that article is to present the state of the art and to indicate future fields of research.

Woskowski analyses in the ninth chapter the problem of revealing the true preferences in rostering arrangements. Knowing the true preferences is crucial for the appreciation and the robustness of the roster. Thereby, the problem occurs in which way the preferences can be modelled and particularly on how to reveal individuals' valuations. Therefore, the paper presents an approach for how to design an appropriate auction.

We have decided to structure **Part II** dealing with **cooperative models** into two subparts: the **first five chapters** are dedicated to the different approaches to mirroring the notion of 'fairness', whereas the **following five chapters** present detailed applications and specific models.

In the tenth chapter, Zelewski criticises the axiomatic grounding of cooperative game theory for corporate purposes. To overcome these difficulties, the notion of 'fairness' is characterised by six requirements, which should mirror a corporate understanding of fairness.

Fairness stands in the centre of attention of the eleventh chapter by Meinhardt too, but the author uses a different way of addressing it. Using a set of principles, the pre-kernel is described and classified as an attractive fair division rule which is easy to compute. The author reviews the generalised conjugation theory from convex analysis to offer a better understanding and broader interpretation of the pre-kernel solution.

In the twelfth chapter, Moreno-Ternero uses the Babylonian Talmud as a background for discussing fairness. The author analyses classical and recent papers that constitute this Talmudic approach to bankruptcy problems. He presents some families of rules which emerge from the classical Talmudic rule but may serve as solutions for managerial accounting problems.

Hougaard discusses in the thirteenth chapter the problem of sharing the costs of access to public goods. The author surveys some recent axiomatic characterisations of relevant allocation rules and provides an overview of how the problem of fair division can be approached and structured. The author presents a model which captures the central aspects of several classes of practical problems and therefore has many potential applications.

The fourteenth chapter by Arin and Katsev introduces and analyses the solution concepts that are based on the fair distribution of the surplus. The surplus distributor-prenucleolus and the surplus distributor-pre-kernel for TU games are discussed. It is shown that these solutions have some nice properties.

All of these five chapters scrutinise the notion of fairness. Their authors choose different tools with which to prospect for an answer. **The subsequent papers** are dedicated to concrete applications or specialised developments of cooperative game theory models in a management accounting environment.

Gallardo, Jiménez, and Jiménez-Losada take in the fifteenth chapter the corporate hierarchy as well as hierarchical structures on the set of players into account. Such games are referred to as games with permission structure. Several models have been developed in the literature. The authors describe one model, which allows dealing with non-hierarchical or nontransitive dependency relationships. In addition, it can be adapted to consider fuzzy dependency relationships.

In the sixteenth chapter Saavedra-Nieves, García-Jurado, and Fiestras-Janeiro analyse a multi-agent inventory system where each agent has a deterministic demand and a warehouse with constant holding costs. Shortages are not allowed, the lead time is constant and the cost of placing an order has two components: a fixed cost and a variable cost. For this model, the authors derive the optimal policy and propose an allocation rule for the joint ordering costs.

The seventeenth chapter by Meca and Varela-Peña is inspired by the Spanish tax system and presents an application of linear cost games. The authors define the class of tax games, and they prove that these games are balanced. They prove that investors have strong incentives to cooperate instead of being tax evaders.

In the centre of the eighteenth chapter by Zelewski and Heeb stands the analysis of two compromise values: the τ-value and the χ-value. It is shown that the τ-value and the χ-value mostly, but not completely, fulfil the requirements of being a fair solution. Moreover, the χ-value proves to be superior to the τ-value.

In the last chapter, Mueller provides a detailed literature review of the properties of the Shapley value, nucleolus, τ-value and the Dutta–Ray solution. This serves as a basis for the evaluation of these solutions with respect to management accounting purposes.

We would like to thank and express our gratitude to all the authors for their contributions, their understanding and their patience! We hope that the volume meets their expectations and that they enjoy and benefit from reading it.

We have created this volume with a view towards colleagues both in the field of game theory and in the field of management accounting. Moreover, we would like to recommend the volume to students of business administration at the graduate or postgraduate level.

We wish a pleasant and enlightening reading!

Cottbus, Germany — David Mueller
Ilmenau, Germany — Ralf Trost
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David Mueller is Professor of Management Accounting and Control at Brandenburg University of Technology (BTU) Cottbus-Senftenberg. After completing a 2-year-banking apprenticeship, he studied business administration at Osnabrück, San Sebastián (Spain) and Zittau. Before joining the BTU, he was a junior professor at the Ilmenau University of Technology. He received his Ph.D. (Dr. rer. pol.) in 2004 and his habilitation in 2011. His research interests are cost accounting, life-cycle costing, environmental management, small and medium-sized enterprises, product design, and real options.

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Part I
Non-cooperative Models: The Design of Incentives and the Analysis of Preferences

Setting Incentives for Managers: Incentive Compatibility, Similarity Rule, and Goal Congruence

Ralf Trost and Sebastian Heim

Abstract The terms 'incentives' and 'incentive compatibility' are widely used in economics. Incentives are an important object in managerial accounting. Nevertheless—or just because?—there is a notable ambiguity in their exact meaning, especially of 'incentive compatibility'. Likewise, in many cases it is not clear whether 'setting the right incentives' means 'ensuring incentive compatibility' or something else. This chapter depicts some important relationships between different definitions in differing economic models dealing with incentives, starting with the game theoretic definitions of incentive compatibility.

Keywords Agency • Allocation mechanism • Core • Goal congruence • Incentives • Incentive compatibility • Mechanism theory • Principal agent model • Similarity rule

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1 Management Accounting and the Postulate of Incentive Compatibility

Following Horngren et al. (2014, p. 21), management accounting 'is the process of identifying, measuring, accumulating, analysing, preparing, interpreting, and communicating information that helps managers fulfill organizational objectives'. In contrast to financial accounting, management accounting does not generate information for external parties, such as banks or stockholders (Drury 2015, pp. 6–9; Horngren et al. 2014, pp. 21–23). It is rather for the purpose of helping managers within an organization to 'make better decisions and improve the efficiency and effectiveness of existing operations' (Drury 2015, p. 6). Thereby, an important subject in management accounting is the informational asymmetry between managers and stockholders or between managers on different levels of a hierarchy, as known from the standard agency model.

As in other areas of economics, agency theory[1] plays an important role in management accounting—cf. for example, Baiman (2006), Lambert (2007)—too, as a well-established instrument for modelling situations in which a group of persons has to make decentralized decisions in an environment of incomplete information, whereby some sort of higher-ranking objectives instead of individual self-interests should (but probably will not) guide the decisions. Therefore the question arises whether certain rules and arrangements can mitigate this problem, setting 'the right incentives' for the individuals (the so-called 'agents') to act as they should (in the sense of an existing or hypothetical 'principal'). Incentives which reward managerial effort are necessary to guide the actions of the managers and to motivate their behaviour (Horngren et al. 2014, p. 411 f.; Atkinson 2001, p. 577). Hence, 'incentives' is one of the most common keywords in management accounting. In the meantime, the notion of 'contract theory' instead of 'agency theory' is quite common in the field of managerial accounting, cf. Baiman (2006) and Lambert (2001).

A lot of survey articles accompanied the developing theory. To name some of them: Nearly three decades ago by now, Stanley Baiman gave an well-known overview of agency research in managerial accounting, already 'a second look', following his first survey 8 years before, cf. Baiman (1982, 1990). Again about one decade later came an overview of the field by Lambert (2001). Currently, for example, we have the contribution by Kunz (2017).

It is common use in management accounting to speak about 'incentives', but strictly speaking, this is an incomplete formulation. Naturally, the incentives should cause a 'compatibility' of the agents' actions with the goals of the principal, which leads us to the notion of 'incentive compatibility'. Incentive compatibility originally was a construct in social choice theory and mechanism design theory, but

[1]The use of this term goes back to Stephen Ross, cf. Ross (1973, 1974).

the underlying thoughts are the same in the context of management accounting.[2] In a quite vague manner, incentive compatibility may be described as in the *Encyclopædia Britannica*: 'State in game theory and economics that occurs when the incentives that motivate the actions of individual participants are consistent with following the rules established by the group ... Problems may arise when the participant with more information has an incentive to use information for personal benefit at the expense of others. However, when the interaction is structured so that the participant with more information is motivated to act in the interest of the other party (or has less incentive to exploit an informational advantage), the result is incentive compatibility.'[3]

Undoubtedly this statement describes the basic idea of incentive compatibility appropriately. However, a lot of details remain open, for instance: How many participants? Have they symmetric or differing roles in the game? What are the objectives standing behind the determination of the rules, whose aims are these and who sets the rules? What kind of interactions between the participants will be considered? What are the determining factors underlying the participants' preferences and therefore influencing their actions? And, lastly, the crucial point in this chapter: How shall the above formulation 'the participant ... is motivated' be translated into the terms of a formal model?

Conditional on the answer to these and other questions, a lot of differing models have arisen[4] and some differing definitions of incentive compatibility, too. Furthermore, while following the basic idea, some definitions nevertheless lead to concepts usually not captured by the notion 'incentive compatibility' (at least in the English language). This chapter deals with the meaning of incentive compatibility. It presents and compares some definitions which can be interpreted as incentive compatibility in the broader and in the narrower sense. On the other hand, it will only secondarily and sporadically discuss the optimality of certain rules suggested as the solution for the one or the other special model in this context.

Section 2 will deal with the case of a group of managers faced with a decentralized investment decision in which, despite the individual information and interests of the managers, the overall interest of the firm should be considered as well as possible. The conceptual framework for modelling this agency situation with one principal and multiple agents is established by mechanism design theory, which on its part relies on noncooperative game theory and will be outlined in the first part of the section. In Sect. 3 we deal with the case of one agent and a non-hypothetical principal. This covers the situation where an employed manager has to make investment decisions on behalf of the firm's owner. After a short description of the standard principal agent model and its relation to the notion of incentive

[2]There are many facts confirming this statement. Both fields deal with the actions of so-called agents, for instance, and the revelation principle of mechanism theory (cf. Sect. 2.1) is an important element of budgeting mechanisms in management accounting (cf. Sect. 2.2).

[3]www.britannica.com/topic/incentive-compatibility [16.02.2017].

[4]For an earlier discussion with a narrower scope, cf. Trost (1999).

compatibility, we will explicate the simplified concepts of goal congruence and preference similarity. Unfortunately, the label ‘incentive compatible’ may be used for solutions to either of the different conditions. Especially this statement holds for Germany and the corresponding term ‘*Anreizkompatibilität*’, which is used by a quite large group of lecturers and researchers in the sense of one of the simplified approaches. The short Sect. 4 points to the fact that occasionally the term incentive compatibility is used in the context of cooperative game theory as well.

2 The Game Theoretical Approach to Incentive Compatibility

2.1 Defining Incentive Compatibility in Mechanism Design Theory

2.1.1 Basic Definitions

We consider a group $N = \{1, 2, \ldots, n\}$ of individuals, called *participants*, faced with the task of coming to a joint *decision*. D denotes the set of potential decisions. Every participant $i \in N$ has some *private information* ϑ_i taken from a set Θ_i. $\vartheta \in \Theta = \Theta_1 \times \ldots \times \Theta_n$ describes the vector of all participants' information. A *decision rule* depending on the participants' information is a mapping $d\colon \Theta \to D$. Note that here the information ϑ may be true just as well as false. For the sake of convenience, in the sequel, the information that participant i possesses will be called the *type* of i.

Depending on their respective types, the participants have individual preferences concerning the outcomes[5] and therefore concerning the decision rules, too. In order to find decision rules which are acceptable to all participants, a *transfer function* $t\colon \Theta \to \mathbb{R}^n$ may also be considered. It describes how the participants may be taxed or subsidized by certain amounts of money. If there is neither an outside source nor an outside sink, the transfer function has to be *balanced*, i.e. $\sum_{i=1}^{n} t_i(\vartheta) = 0$ for all $\vartheta \in \Theta$.[6]

The pair $f = (d, t)$, or rather $f(\vartheta) = (d(\vartheta), t(\vartheta))$, is called a *social choice function*. It stylizes the decision making in a social institution, for instance voting systems. Originally the objectives of this theory were outside the scope of management accounting, but obviously various subjects of management accounting concerning decentralized decisions under the existence of private informations fit into this setting as well. The following parts of this chapter will deal with this aspect.

[5] Note that at present it is left open whether there is a stochastic influence on the outcome or not.

[6] A weaker condition is that the transfer function t be *feasible*, i.e. all of these sums are less than or equal to zero. In this case there could be a surplus which would be wasted from the participants' point of view.

The information about the types of the participants is incomplete: Every participant knows his own type (the so-called private information) but has no or at most incomplete information about the types of the other participants. Consequently, since the participants are assumed to be selfish utility maximizers, strategic behaviour has to be taken into account. Every participant influences the outcome since the joint decision necessarily relies on the messages she submits, which may be truthful or not. A *mechanism* consists of n message spaces M_i $(i \in N)$, $M = M_1 \times \ldots \times M_n$, and an outcome function $g : M \to D \times \mathbb{R}^n$ which assigns to every combination of received messages $m \in M$ a decision $d \in D$ and individual transfer payments t_i. Every participant will choose her message in order to maximize her individual utility from the outcome. A mechanism has to be interpreted as an instrument for implementing a social choice function.

The link to game theory is now established by the obvious fact that every mechanism (M, g) induces a *strategic game*, when the outcomes are replaced by their utilities, derived from the preferences, for the participants now to be named *players*. Game theory here acts as a tool for analysing decision rules. Conversely to the standard game theoretic reasoning, where the reactions of players to given rules are considered, mechanism design theory searches for rules which evoke a certain desired behaviour from the players. The usual noncooperative game theoretic solution concepts are *Nash equilibria* and *equilibria in dominant strategies*. In the first case, for every player it is optimal to stay with the equilibrium strategy supposing the other players do so as well, whereas in the latter case, a certain strategy is optimal regardless of the actions taken by the other players. If for a given social choice function $f = (d, t)$ there exist functions $m_i : \Theta_i \to M_i$ with the property that all $m_i(\vartheta_i)$ are dominant strategies, then the mechanism (M, g) consisting of these message functions m_i and the outcome function $g(\cdot)$ given by $g(m(\vartheta)) = f(\vartheta)$ implements this social choice function in dominant strategies.

2.1.2 The Case of Ignorance About the Other Players' True Types

This setting originally was suggested by Hurwicz (1972). The agents have no knowledge about the other agents' true types, so that they can't take into account these objectively given factors. The definition of incentive compatibility may rely on one of the two above mentioned game theoretic solution concepts. Of course, equilibria in dominant strategies are much more stable, and therefore more convincing solutions, than mere Nash equilibria. Unfortunately, in many games there are no dominant strategies. So it is in mechanism theory. But before we return to this point, a well known property should be mentioned, which makes the analysis of mechanisms much more comfortable and establishes the definition of incentive compatibility. Assume that $M_i = \Theta_i$ for all $i \in N$, i.e. every player i truly or untruly announces his type. This is a so-called *direct mechanism*, and for the sake of convenience we can identify the direct mechanism $(M, g) = (\Theta, f)$ with the social choice function $f = (d, t)$ itself. The *revelation principle* asserts that we can restrict to direct mechanisms while searching for equilibria in dominant strategies, cf. Myerson (1979): Whenever a mechanism (M, g) implements a social choice function $f = (d, t)$ in dominant strategies, in the direct mechanism truth-telling $m_i = \vartheta_i$ is a dominant strategy: If we denote by $u_i(m, \vartheta, d(m), t(m))$ the utility for player $i \in N$ under the direct mechanism $f(\cdot) = (d(\cdot), t(\cdot))$ with the players' true types $\vartheta \in M$ and messages $m \in \Theta$, resp., then the condition reads

$$u_i((\vartheta_i, m_{-i}), \vartheta, d(m), t(m)) \geq u_i((m_i, m_{-i}), \vartheta, d(m), t(m)) \quad \forall m_i \quad \forall m_{-i} \quad \forall \vartheta \quad \forall i,$$

where (m_i, m_{-i}) is a vector of messages with the specific message m_i by player i. Usually the utility function is assumed to be quasi-linear in the following sense: $u_i(m, \vartheta, d(m), t(m)) = v_i(m, \vartheta, d(m)) + t_i(m)$ for suitably defined utility functions v_i for all $i \in N$.

This is called *dominant strategy incentive compatibility*, or, *strong incentive compatibility of the direct mechanism* $f = (d, t)$. Hence, if we search for a mechanism fulfilling certain additional properties in equilibrium, it suffices to look for a direct mechanism that does so. Thereby, incentive compatibility describes no original aim. In fact, it is a tool to characterize stable solutions. Further restrictions then impose certain favourable properties, such as, for example, the Pareto efficiency of the allocations, or, in the case of decentralized investment decisions, maximizing the firm's overall profit.

It may be desirable that a mechanism make no use of transfer payments, i.e. $t_i(\cdot) \equiv 0$ for all $i \in N$.[7] If such a direct mechanism $f = (d, 0)$ is strongly incentive compatible, we refer to d as a *strongly incentive compatible decision rule*. Unfortunately, the famous Gibbard–Satterthwaite theorem [cf. Gibbard (1973) and Satterthwaite (1975)] tells us that in general the search for such decision rules will yield no reasonable results: under some slight regularity conditions, a decision rule can be strongly incentive compatible only if it is dictatorial.[8] This means that the decision rule generates results that are always in favour of one particular participant.

Transfer functions may establish strongly incentive compatible mechanisms in certain settings. If such transfers are regarded as unacceptable, solutions may arise by adding more structure to the model, i.e. by applying the general model in special frameworks with more restrictions on the feasible solutions. This will be the case in the decentralized investment budgeting process discussed in the next part of this chapter.

Intuitively, it might be appealing to weaken the condition of strongly incentive compatibility to that of Nash equilibrium, rather than equilibrium of dominant strategies:

$$u_i((\vartheta_i, \vartheta_{-i}), \vartheta, d(\vartheta), t(\vartheta)) \geq u_i((m_i, \vartheta_{-i}), \vartheta, d((m_i, \vartheta_{-i})), t((m_i, \vartheta_{-i}))) \quad \forall m_i \quad \forall \vartheta \quad \forall i.$$

[7] In terms of game theory, this is the case of no side-payments.

[8] The proof is based on the even more famous impossibility theorem of Arrow (1950).

This we may call *weak incentive compatibility*. However, in the general model, these two conditions are equivalent,[9] so that the Gibbard–Satterthwaite theorem holds as well. Strong and weak incentive compatibility only differ in more specialized settings.

2.1.3 Bayesian Incentive Compatibility

Harsanyi proposed the following model of games, called *Bayesian games* (Harsanyi 1967, 1968a,b; Fudenberg and Tirole 1993, pp. 243–318). Incentive compatibility is to be defined in this framework if all or at least some players possess incomplete information about the other players' true type, in contrast to the previous section, where complete ignorance was assumed. From a formal point of view, an additional player, called 'nature', is introduced. This player chooses the types of the participants. It is assumed that all participants share some a priori knowledge about the possible types of all players and their joint distribution, the so-called *common prior*. When 'nature' has drawn the vector ϑ of true types, player i can observe only her own type ϑ_i.[10] According to the Bayes theorem, this information, in conjunction with the a priori distribution of types, generates an a posteriori distribution of types, describing the individual *beliefs* of player i about the other players' types. Let $F(\vartheta_{-i}|\vartheta_i)$ be this distribution function on the space of the other players' types, given the known type of player i. Since in this game the players get (incomplete) information when they learn about their own type, messages have to be modelled as functions $m_i : \Theta_i \to \Theta_i$, describing the strategy of player i as a reaction to the information about her type. To keep the following formulas as short as possible, we set, for a given direct mechanism $f = (d, t)$,

$$U_i^f(m_i, m_{-i}(\vartheta_{-i}), \vartheta) := u_i((m_i, m_{-i}(\vartheta_{-i})), \vartheta, d(m_i, m_{-i}(\vartheta_{-i})), t((m_i, m_{-i}(\vartheta_{-i})))).$$

This denotes the utility of player i under the vector ϑ of true types, the reaction functions $m_{-i}(\cdot)$ of the other players, and the own strategy (i.e. message) m_i. Now player i evaluates the utility of a strategy m_i of her own under the direct mechanism $f = (d, t)$ by calculating the corresponding expected utility:

$$\begin{aligned} E_i^f(m_i, m_{-i}, \vartheta_i) &:= \mathrm{E}\, U_i^f(m_i, m_{-i}(\vartheta_{-i}), \vartheta) \\ &= \int U_i^f(m_i, m_{-i}(\vartheta_{-i}), (\vartheta_i, \vartheta_{-i}))\, \mathrm{d}F(\vartheta_{-i}|\vartheta_i). \end{aligned}$$

[9]Cf., for instance, d'Aspremont and Gérard-Varet (1979a, p. 31). It has to be assumed that the plausibility condition is satisfied, i.e. the space of potential messages M is equal to the space Θ_i of types for every player $i \in N$. In our above description we have presumed this implicitly.

[10]Maybe she may gain some additional but in any case incomplete information. For the sake of simplicity we will omit this aspect and restrict the information to one's own type.

Using this notation, incentive compatibility again can be defined in two variants—cf., for instance, d'Aspremont and Gérard-Varet (1979a,b). *Strong Bayesian incentive compatibility* requires $m_i = \vartheta_i$ to be a dominant strategy,

$$E_i^f(\vartheta_i, m_{-i}, \vartheta_i) \geq E_i^f(m_i, m_{-i}, \vartheta_i) \quad \forall m_i \quad \forall m_{-i} \quad \forall \vartheta_i,$$

whereas *weak Bayesian incentive compatibility* relies on the concept of Nash equilibrium, here called *Bayesian Nash equilibrium*:

$$E_i^f(\vartheta_i, \vartheta_{-i}, \vartheta_i) \geq E_i^f(m_i, \vartheta_{-i}, \vartheta_i) \quad \forall m_i \quad \forall (\vartheta_i, \vartheta_{-i}).$$

While the condition for strong Bayesian incentive compatibility is, as always, much more rigorous than the condition for weak Bayesian incentive compatibility, the verification of the latter here may be much more difficult because it would require knowledge about the other players' reaction functions m_{-i}.[11] Furthermore, d'Aspremont/Gérard-Varet show that under some measurability conditions strong Bayesian incentive compatibility is equivalent to strong incentive compatibility, cf. d'Aspremont and Gérard-Varet (1979a).

2.2 *Incentive Compatibility in Allocation Mechanisms*

One prominent application of mechanism design theory is the allocation of a resource, because this requires knowledge about the welfare the resource creates as a consequence of certain allocations and because this information has to be elicited at least partially from the (potentially) selfish participants. In this context, messages consist of revealing the utility function as a function of the allocation: $m_i = v_i(\cdot)$, where $v_i(\cdot)$ is that part of the quasi-linear utility function not dealing with transfer payments but with the assigned resource (see above). Following Groves and Ledyard (1987), such a model has four characteristic elements:

(1) the 'environment', i.e. the parameters describing the specific situation,
(2) the mechanism, a mapping from the space of potential messages of the participants to the space of potential allocations times $\mathbb{R}^n$, the space of potential transfer payments,
(3) the game theoretic description of the selfishness of the participants' actions, i.e. a definition of incentive compatibility, and
(4) a criterion or a set of criteria for the measurement of an allocation's quality.

[11] With respect to an allocation mechanism (cf. the next section) Groves and Ledyard (1987, p. 58) state: 'There is wide acceptance of the presumption that if there exist dominant strategies, then agents will adopt them [...] No sophisticated prediction of others' behaviour is necessary.'

The task is now to find a mechanism, as in (2), that maximizes the quality or satisfies a required quality level according to (4), while (3) has also to be considered and (1) defines the interactions between the different variables, functions, and so on. As a consequence, this may lead either to an optimization program for the mechanism (2) with a goal function according to (4), or to the search for a feasible solution respecting the constraints according to (4). In this way, incentive compatibility together with the 'environment' provides the so-called *incentive constraints*. The adopted concept of incentive compatibility may follow the Hurwicz or the Harsanyi (Bayes) concept, and may rely either on dominance or merely on Nash equilibrium. Additional constraints according to (4) may force the transfer function to be balanced (especially when there is no principal ready to play the role of a sink or a source) or may require the Pareto efficiency of the allocation. In addition, in most cases there will be required *individual rationality*, i.e. the participants shouldn't be better off if they refuse cooperation. This constitutes the so-called *participation constraints*.

Unfortunately, one shouldn't expect too much from an allocation mechanism: In general, there is no Pareto efficient mechanism with balanced transfer function which at the same time induces truth-telling, i.e. which is strongly incentive compatible, cf. Groves and Ledyard (1987, pp. 65–66). Again, this is why sometimes one has to be content with weak incentive compatibility.

Quite famous is a family of allocation mechanisms named *Groves mechanisms*, suggested by Groves (1973).[12] They require quasi-linear utility functions as defined above. The 'type' of participant i here is the utility generated by the allocation to her. After the participants have sent their messages $m_i(\cdot)$, the mechanisms work as follows:

- Allocations $x_1^*, x_2^*, \ldots, x_n^*$ to the agents are determined in order to maximize $\sum_{i=1}^{n} m_i(x_i)$, the overall utility as reported by the agents.
- The outcome for agent i is the sum of his utility $v_i(x_i^*)$ plus a transfer payment

$$t_i(m) = \sum_{j \neq i} v_j(x_j^*) + h_i(m_{-i}).$$

  $h_i(\cdot)$ is an arbitrary real-valued function not depending on the message of i but possibly on the messages of the other agents.

Groves mechanisms induce truth-telling but they are not balanced. At least they are feasible because the sum of transfer payments will be negative in general.

[12]For an overview of allocation mechanisms with special attention to Groves mechanisms, cf. Green and Laffont (1979). Together with the Vickrey auction and the Clarke mechanism, the class of mechanisms which induce truth-telling is called the class of Vickrey-Clarke-Groves (VCG) mechanisms; cf. Vickrey (1961) and Clarke (1971).

2.3 Incentive Compatibility in Decentralized Investment Decision Making

In order to stress the relation between the setting of incentives in management accounting and game theoretical reasoning and in order to create a common framework, in Sects. 2.1 and 2.2 we have introduced the notion of incentive compatibility as it is known from mechanism design theory and social choice theory. For the rest of this chapter, we will stick to the scope of management accounting more closely. The process of decentralized investment decisions[13] in a divisionalized firm can be seen as a special case of the above described allocation of a resource, namely the resource 'investment capital':

1. The specific decision situation ('environment'): The firm has to allocate the scarce good 'money' to its divisions (which are identified with their leading managers) for the purpose of investing it into profitable assets. The division managers know about the potential surplus $\vartheta_i(x_i)$ they can gain in their respective division with alternative allocations x_i. This information is private to each manager.[14] Hence, the centre asks the divisions for information about their potential profits and subsequently calculates the allocation presumed to be optimal on the basis of this messages. As before, the messages of the managers are denoted by functions $m_i(\cdot)$.
2. The principle of the considered mechanisms: Managers get compensation linked to the outcome. In principle, the manager's remuneration has a fixed component and a variable share of an assessment base dependent on the concrete mechanism chosen, i.e. it is $F_i + s_i \cdot AB_i$ with the fixed part F_i of salary, the assessment base AB_i, and the sharing parameter s_i for manager i.
3. The constraints:
   (a) Participation constraints: The fixed component of the managers' remuneration is a vehicle to ensure their participation. So it is obsolete to model this aspect explicitly.
   (b) The transfer payments don't have to be balanced, since the principal gets the net surplus.
   (c) The incentive constraints reflect the managers' strategic behaviour, i.e. the fact that they will send the message maximizing their utility under a given mechanism and with respect to their assumptions about the others' strategies.
4. The measure for an allocation's quality: The centre wants to maximize the net profit, i.e. the overall revenue less the compensation of the managers and the cost of capital.

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[13]For an overview of capital budgeting under asymmetric information, cf. Gordon et al. (2006).

[14]$\vartheta(\cdot)$, the type of agent i, is here the production function of her division. It plays the role of the utility function $v(\cdot)$ in the foregoing section.

As is well known, the Groves mechanism as described in the recent section can be applied to this situation, too, with the assessment base of manager *i*'s remuneration according to Groves (1976) and Groves and Loeb (1979):

$$AB_i = AB_i(x, \vartheta_i, m_{-i}) = \vartheta_i(x_i) + \sum_{j \neq i} m_j(x_j) \quad \forall x = (x_1, x_2, \ldots, x_n) \quad \forall i.$$

The assessment base consists of the real surplus gained by manager *i*'s division and of the surpluses that the other managers have announced (!) they would gain with the capital assigned to their division. Under this mechanism, truth-telling is a dominant strategy for all managers, i.e. this mechanism is strongly incentive compatible. Despite the fact that the model doesn't capture aspects like stochastic influences or the dynamics of repetition over time, it seems remarkable that there is a mechanism with the compelling property that truth-telling is a dominant strategy. Moreover, in principle the Groves mechanism should be easy to implement. The main, and presumably crucial, drawback in practice may be due to the lack of understanding of how the mechanism works. It could be really difficult to explain to a manager that his remuneration will depend on the unverified ex ante assertions of his colleagues about their potential gains, and it is just as difficult for the managers to recognize that truth-telling is the optimal reaction to this rule. Obviously the concept of incentive compatibility requires that the parties involved are able to notice the incentives at all.

Relaxation to weak incentive compatibility brings a very common and broadly used payment scheme into play, namely the so-called *profit sharing*, where the assessment base for every division manager is the firm's overall surplus. Clearly, if all the managers tell the truth and the centre calculates the appropriate allocation, then a single manager has no incentive to deviate, because this would reduce her own remuneration. So truth-telling is a Nash equilibrium, but unfortunately in many cases there will also be a lot of other Nash equilibria in this game. So the truth-telling equilibrium may be not very stable. It can be assumed that the popularity of this mechanism in practice is caused mostly by its intuitive plausibility in spite of its secondary quality from the theoretical point of view of incentive compatibility.[15]

We close this section with two remarks: First, the decentralized budgeting problem shows that the equivalence of strong and weak incentive compatibility doesn't hold in special applications that have a richer structure than the general model. Second, the Groves mechanism possesses one disadvantage: It is prone to collusive behaviour, i.e. cooperating division managers can increase their respective remunerations simultaneously at the expense of the firm's overall net profit (Bamberg and Trost 1995, p. 224). This assertion holds even without considering side payments between the managers: The trick is that all managers send too high

[15]Profit sharing becomes even more problematic if effort and effort aversion of the agents are taken into account and the free rider problem arises. This aspect is an essential element of principal agent theory; cf. the next section.

potentials to the centre (within a plausible range, of course). Thereby every manager causes a too high remuneration for all other managers, while the net surplus for the centre sinks simultaneously. Clearly this works only in a cooperative setting. So the question arises, for which situations would cooperative game theory rather than noncooperative game theory be the adequate theoretical framework?

2.4 From Incentive Compatible Allocation Mechanisms to Principal Agent Theory

Principal agent theory incorporates one party, the principal, which follows her own selfish interests as well and has less detailed information than the agents—but on the other hand the power to set the payment schedule for the agents. Agents can either agree or leave. This applies to capital budgeting with multiple divisions where the asymmetry of information exists ex ante. In agency theory, constellations of this kind are called *hidden information*. Agency models with hidden information incorporate specific communication and signaling structures between the agents and the principal, influencing the outcomes for all involved parties. The agents and the principal maximize their respective utilities. Because of the right of the principal to set the rules of the agents' remuneration, her utility maximization gives the goal function (in decentralized investment decision making, it is the net gain after capital costs), whereas the utility maximization of the agents is reflected in the incentive constraints. The communication between the parties defines the functions: The remuneration for the agents (the division managers) as well as the net surplus for the principal (the centre) depend on the allocation of capital calculated by the centre (the principal), and the allocation in turn depends on the messages (the potential gains) by the agents. The participation constraints set lower bounds on the agents' respective utilities.

Mechanism design as a starting point for approaching principal agent theory should stress the game theoretic background of such models. Of course there are differences between these theories concerning intention, the details of the models, and the gained insights, but the correspondences are obvious. The common element is the concept of incentive compatibility, even though it is not usual to use the term 'compatibility' in agency theory.[16] In the next section we will, on the one hand, have a look at the standard principal agent model with only one agent.[17] In the context of this chapter, this appears to be a degenerate case of the models depicted so far. On

[16] Rees (1985a, p. 5) uses this to distinguish principal agent theory from 'the literature on incentive compatibility'.

[17] In their overview of the agency literature dealing with capital budgeting, Gordon et al. (2006) discuss the multi-agent case as well as the case of one agent, which latter has been, presumably because of its lesser mathematical complexity, much more exhaustively analysed.

the other hand we will briefly discuss two other concepts, sometimes referred to as incentive compatibility, too.

3 The Case of One Principal and One Agent

3.1 The Principal Agent Model

In the standard principal agent model[18] there is just one agent. The principal delegates a certain task to that agent. As before, a problem arises, because the agent has the opportunity to act in favour of his own interests and in contradiction to the interests of the principal. So far, this is like the situations described above. What is different is the shift towards a game between the principal and solely one opponent. The multi-agent system of incentive constraints reflecting the interactions between the participants besides their selfishness reduces to one simple incentive constraint. Consequently, the incentive constraint in the standard principal agent model can be interpreted as the remaining 'rest' of incentive compatibility in this 'degenerate' model; it seems to be justified to speak of incentive compatibility in this model, too. As in the decentralized investment decision, 'compatibility' refers to the principal's aims, merely the complications in interaction with the other players have vanished.

The parallelism can be pointed out by a look at the basic principal agent model. It consists just of elements known from allocation mechanisms. Unlike in the foregoing sections, we will refer to the model version called *hidden action* or *moral hazard*, because this model can be depicted in more simplicity than the hidden information case and suffices for our purposes. In this model the information asymmetry concerns the action of the agent. He chooses an *effort* level $e \in \mathfrak{E}$, unobservable to the principal, which positively affects the random variable $\tilde{r}(e)$, representing the result of the agent's activity. Due to the stochastic influence, the principal is not able to infer the agent's action ex post from the realized outcome. Again, the mechanism is a compensation scheme, denoted by $c(\cdot)$, where $\tilde{c}(e) := c(\tilde{r}(e))$ is a random variable itself. The residual result is the (random) surplus for the principal: $\tilde{s}(e) := \tilde{r}(e) - \tilde{c}(e)$. The utility function u_P of the principal is a function of the monetary surplus, while the agent's utility function is two-dimensional and a function of his compensation and the effort he has spent. Due to *effort aversion*, his utility function decreases with increasing effort, provided the first argument remains unchanged. Let u_{min} be the minimum expected utility the agent demands for participation. To evaluate the strategies, both players have to calculate the respective

[18]For an early but thorough overview see Rees (1985a,b). For seminal papers on agency theory, cf., for example, Jensen and Meckling (1976), Harris and Raviv (1979), Holmström (1979), Grossman and Hart (1983). Gordon et al. (2006, p. 153) stress that Jensen and Meckling (1976), Harris and Raviv (1979), Holmström (1979) and Baiman (1982) 'have had a major impact in promoting the use of agency theory to examine managerial accounting issues.'

expected utilities. By these definitions, the three components of the optimization problem are[19]:

- the goal function of the principal: $\mathrm{E}\, u_P(\tilde{s}(e)) \underset{c}{\longrightarrow} \max!$

subject to

- the incentive constraint: $\mathrm{E}\, u_A(\tilde{c}(e), e) \geq \mathrm{E}\, u_A(\tilde{c}(\hat{e}), \hat{e}) \;\; \forall \hat{e} \in \mathfrak{E}$
- the participation constraint: $\mathrm{E}\, u_A(\tilde{c}(e), e) \geq u_{min}$

We have argued that incentive compatibility should be the denomination for the principal agent principle of designing optimal incentive contracts. However, in the context of managerial accounting, other, simplified concepts of incentive compatibility can be found. Two related approaches will be briefly presented in the next section.

3.2 Simplified Approaches to Incentive Compatibility: Similarity Rule and Goal Congruence

As noted, Ross (1973) coined the term ‘agency’ for the very common economic relationship discussed in this chapter and introduced a property he called the *principle of similarity* or the *similarity rule* [cf. Ross (1974)]. The idea had appeared in a slightly differing context already some years before in Wilson (1968). Seemingly this property shall enforce behaviour similar to that we here call incentive compatibility: ‘Since the agent chooses an act by maximizing the expected utility of his utility function [...], we need only choose a fee schedule that makes the agent’s and the principal’s evaluations of payoffs equivalent to ensure that, given the fee schedule, the agent takes the action the principal would wish him to’ (Ross 1974, p. 220). In his model the utility function of the agent depends only on the compensation payment. If we denote this function by $\hat{u}_A(\cdot)$, Ross’s similarity rule reads

$$\exists\, a > 0, b: \quad u_P(r - c) = a \cdot \hat{u}_A(c) + b$$

with the result r and the compensation c. It is a requirement on the compensation function $c(\cdot)$. If the compensation scheme satisfies the similarity rule, ‘the agent will always choose the act that maximizes the principal’s expected utility’ (Ross 1974, p. 220).[20] A priori it is not guaranteed that such a compensation function exists at all, but, surprisingly, linear compensation schemes $s \cdot r + F$ are good candidates:

[19]For existence, uniqueness, and manageability, suitable specifications of the model are necessary.

[20]As Ross remarks, the ‘constants a and b can be chosen to satisfy outside constraints’. Cf. Ross (1974, p. 220). This could be, for example, a participation constraint.

Ross shows that every two of the three conditions

- similarity rule,
- Pareto efficiency, and
- linearity of compensation

imply the third one (Ross 1974, p. 221). Bad news is that only under certain utility functions—namely the members of the HARA (hyperbolic absolute risk aversion) class[21]—can these three conditions be met simultaneously. Since we aren't particularly interested in the solutions of the models in this chapter, we restrict ourselves to one remark about further considerations in this direction: If we allow non-linear compensation schemes, we have to take their curvature into account. It has to be convex, resp., concave, depending on whether the agent is more risk averse than the principal or vice versa (Ross 1974, p. 227).

If we defined incentive compatibility by the similarity rule, how would this fit into the framework we have delineated before? The answer is: not at all. Characteristic for this early model is the fact that the agent feels no effort aversion, which is an essential part of the standard hidden action model. Without effort aversion, the optimization problem formulated in the section above would be trivial. It is not clear in what the selfishness of the agent lies, because this is no part of the model. Ross' model is a starting point which has been developed further. But misunderstanding becomes likely when in Germany the similarity rule or similar conditions[22] are called '*anreizkompatibel*', which is the direct translation of 'incentive compatible'. This is done in many of the publications and textbooks published by Laux and his school.[23] Furthermore, Laux calls the similarity rule (translated) 'strongly incentive compatible'—cf., for example, Laux (2006a, p. 73) and Laux (2006b, p. 231)—and the condition cited in the footnote 22 is called by Laux 'weakly incentive compatible'. This vocabulary was originally used for the different game theoretic solution concepts in the incentive compatibility literature. Quite confusing, but admittedly logical in his framework, Laux distinguishes between '*Anreizkompatibilität*' (which is the similarity rule) and 'optimal compensation schemes' (which is incentive compatibility in agency theory).

The criterion of *goal congruence* is formulated in the context where a manager has to make investment decisions on behalf of the principal and where the quality of the decisions is measured by net present value.[24] Financial theory tells us that different persons, because of their different preferences (regardless of possible different information), in general will have different ideas about the adequate

[21] At least the HARA class contains such important categories of utility functions as the exponential, the logarithmic, and the linear utility functions.

[22] For example: The expected utility of the agent should be a positive monotone transformation of the expected gross return before (!) the agents' remuneration; cf., for example, Laux (2006a, p. 376).

[23] Because of the different language, we have cited only two of these books, cf. Laux (2006a,b).

[24] The fundamental papers are Itami (1975) and Reichelstein (1997).

discount rate to use. If moral hazard should be included, effort aversion had to be considered in addition. Goal congruence claims that in this situation, the manager should be compensated in such a way that he realizes exactly those investment projects the principal would realize too, i.e. he uses the same rate of return for discounting. Note, since the discount rate used by the principal reflects none of the difficulties possibly associated with the engagement of the agent, that under this criterion the compensation for the agent is not considered in the assessment base of the remuneration, the gross returns. In other words, the criterion is the one cited above (‘*schwache Anreizkompatibilität*’).[25]

In the end, the similarity rule and goal congruence may be regarded as possible and meaningful formal translations of incentive compatibility as has been formulated verbally in, e.g. the *Encyclopædia Britannica* (cf. the first section). But definitely they are foreign objects in the context of the formal models of incentive compatible mechanisms and standard principal agency theory. They are alternative and simplified approaches.

4 Incentive Compatibility and Cooperative Game Theory

The concept of incentive compatibility originally was a child of noncooperative game theory. But a closer look shows that in the meantime it has also been used in cooperative game theory. This shows, for example, in a glance at Forges et al. (2002). They give an overview of papers analysing exchange economies in the framework of cooperative game theory, with the classic solution concept of the *core*, flanked by a property called incentive compatibility.

In the basic model a *cooperative game* there is a pair $\Gamma = (N, w)$, where the *characteristic function* w assigns to every *coalition* $S \subseteq N$ of players the collective outcome $w(S)$. A *solution concept* φ on the other hand assigns to every game Γ a subset $\varphi(\Gamma)$ of $\mathbb{R}^n$, the solutions.[26] Every $x \in \varphi(\Gamma)$ is an allocation in the cooperative setting and apportions the collectively generated result. A ‘good’ solution concept should lead to ‘fair’ allocations which in addition are stable, so that no player wants to leave the game. The *core* is a simple and common version of such a solution concept. It is defined by three requirements:

(a) *efficiency*, i.e. the value of the grand coalition is fully exhausted by the sum of payments to the players[27]

$$\sum_{i=1}^{N} x_i = w(N),$$

[25] As stated by Laux (2006b, p. 324).

[26] $\mathbb{R}^n$ may contain one or more solutions or may be empty.

[27] This corresponds with the balancedness of mechanisms in the noncooperative case.

(b) *individual rationality*, i.e. every player gets at least as much as he could achieve without taking part in any coalition

$$x_i \geq w(\{i\}) \ \ \forall i,$$

(c) *group rationality*, i.e. every group of players gets together at least as much as by building a coalition of their own

$$\sum_{i \in S} x_i \geq w(S) \ \ \forall S \subseteq N.$$

(b) and (c) together are sometimes called incentive compatibility [cf. Forges et al. (2002, p. 12)].[28] Maybe the monotonicity conditions discussed in this volume by Mueller (2017, pp. 439–441) and by Arin and Katsev (2017, p. 331), reflect similar thoughts. In fact, they refer to the question of how a solution concept reacts to changes in the game, i.e. changes in input data.

In a more sophisticated modelling, the basic model is then enriched by asymmetric information about the players' types, which leads to a signaling process before building coalitions.[29] In this situation, incentive compatibility is defined just as seen above in mechanism theory[30] with the denomination *incentive compatible core* (in cooperative games with incomplete information) (McLean and Postlewaite 2003, p. 222). In the same spirit, Balog et al. (2016) analyse different methods of risk capital allocation in financial institutions, such as a bank, in terms of several desirable properties. One group of these is related to cooperative game theory and one of these again is incentive compatibility.

5 Concluding Remarks

Two short remarks shall close this chapter. First, the term incentive compatibility is ambiguous. Its provenance, at least in a scientific context, is undoubtedly (noncooperative) game theory with the solution concepts of dominance equilibrium and Nash equilibrium. The main applications and variations can be traced back to that origin. But nowadays the term incentive compatibility is used for a variety of meanings in practice, but in science as well. Therefore, whenever this term is used, the recipient should scrutinize the exact meaning the talker insinuates. Maybe

[28] Karagök (2006, p. 9) shows an analogous usage of the German language term '*Anreizkompatibilität*' in the definition of the core.

[29] In fact it could be discussed whether this is still a cooperative game. In our opinion, we have here a hybrid of cooperative and noncooperative modelling.

[30] Cf. Forges et al. (2002, p. 9) for the Hurwicz-like setting and Forges et al. (2002, p. 22) for Bayesian incentive compatibility.

the lowest common denominator indeed is a verbal description like the one by *Encyclopædia Britannica* cited at the beginning of this chapter.

Second, a question about the practical implication of incentive compatible solutions arises: Above, we discussed whether the Groves mechanisms, despite their desirable advantages, may be unimplementable in practice because its mode of action is not easy to understand and may contradict intuition—unlike the (at least in theory) less advantageous profit sharing mechanism. It seems possible that a similar argumentation holds in other circumstances: Maybe simplified approaches like, e.g. goal congruence, work better in practice than theoretically founded concepts, here the solution of the moral hazard optimization problem. This becomes even more striking if we consider the enormous simplifications (for example, the assumption of normal distributions and exponential utility functions) necessary for guaranteeing the solvability of the optimization problem. Whether this solution then can be implemented in practice remains an open question, too.

References

Arin J, Katsev I (2017) The SD-prenucleolus for TU-Games: coalitional Monotonicity and Core Stability. In: Mueller D, Trost R (eds) Game theory in management accounting – implementing incentives and fairness. Springer, pp 27–57

Arrow K (1950) A difficulty in the concept of social welfare. JPolE 58(4):328–346

Atkinson AA (2001) Management accounting, 3rd edn. Prentice Hall, Upper Saddle River, NJ

Baiman S (1982) Agency research in managerial accounting: a survey. JAL 1(Spring):154–213

Baiman S (1990) Agency research in managerial accounting: a second look. AOS 15(4):341–371

Baiman S (2006) Contract theory analysis of managerial accounting issues. In: Bhimani A (ed) Contemporary issues in management accounting. Oxford University Press, pp. 20–41

Balog D, Batyi TL, Csoka P, Pinter M (2016) Properties and comparison of risk capital allocation methods. EJOR 259(2):614–625

Bamberg G, Trost R (1995) Wahrheitsinduzierende Mechanismen, Fehlallokationen und kollusives Verhalten bei der Investitionsbudgetierung. In: Rinne H, Rüger B, Strecker H (eds) Grundlagen der Statistik und ihre Anwendungen. Physica-Verlag, Heidelberg, pp. 219–230

Clarke E (1971) Multipart pricing of public goods. Public Choice 8(1):19–33

d'Aspremont C, Gérard-Varet L-A (1979a) Incentives and incomplete information. JPubE 11(1):25–45

d'Aspremont C, Gérard-Varet L-A (1979b) On Bayesian incentive compatible mechanisms. In: Laffont J-J (ed) Aggregation and revelation of preferences. North- Holland, Amsterdam/London, pp 269–288

Drury CM (2015) Management and cost accounting, 9th edn. Cengage Learning Emea

Forges F, Minelli E, Vohra R (2002) Incentives and the core of an exchange economy: a survey. JME 38(1–2):1–41

Fudenberg D, Tirole J (1993) Game Theory. MIT Press, Cambridge, MA

Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41(4):587–601

Gordon LA, Loeb MP, Tseng C-Y (2006) Capital budgeting and informational impediments: a management accounting perspective. In: Bhimani A (ed) Contemporary issues in management accounting. Oxford University Press, Oxford/New York, pp 146–165

Green JR, Laffont J-J (1979) Incentives in public decision making. North-Holland, Amsterdam/London

Grossman SJ, Hart OD (1983) An analysis of the principal-agent problem. Econometrica 51(1): 7–45

Groves T (1973) Incentives in teams. Econometrica 41(4):617–631

Groves T (1976) Incentive compatible control of decentralized organizations. In: Ho YC, Mitters SK (eds) Directions in large-scale systems: many person optimization and decentralized control. Plenum Press, New York/London

Groves T, Ledyard J (1987) Incentive compatibility since 1972. In: Groves T, Radner R, Reiter S (eds) Information, incentives, and economic mechanisms. University of Minnesota Press, Minneapolis, pp 48–111

Groves T, Loeb MP (1979) Incentives in a divisionalized firm. MS 25(3):221–230

Harris M, Raviv A (1979) Optimal incentive contracts with imperfect information. JET 20(2):231–259

Harsanyi JC (1967) Games with incomplete information played by "'Bayesian" players, I–III. Part I: the basic model. MS 14(3):158–182

Harsanyi JC (1968a) Games with incomplete information played by "'Bayesian" players, I–III. Part II: Bayesian equilibrium points. MS 14(5):320–334

Harsanyi JC (1968b) Games with incomplete information played by "'Bayesian" players, I–III. Part III: the basic probability distribution of the game. MS **14**(7):486–502

Holmström B (1979) Moral hazard and observability. TBJE 10(1):74–91

Horngren C, Sundem G, Burgstahler D, Schatzberg J (2014) Introduction to management accounting, 16th edn. Pearson Education

Hurwicz L (1972) On informationally decentralized systems. In: McGuire CB, Radner R (eds) Decisions and organizations. North-Holland, pp 297–336

Itami H (1975) Evaluation measures and goal congruence under uncertainty. JAR 13(Spring): 73–96

Jensen MC, Meckling WH (1976) Theory of the firm: managerial behavior, agency costs, and ownership structure. J Financ Econ 3:305–360

Karagök Y (2006) Methoden zur Berechnung des Nekleolus kooperatiover Spiele mit einer Anwendung für die Schweiz, Réseau des Bibliothèques de Suisse occidentale (Library Network of Western Switzerland); available under https://doc.rero.ch/record/5627/files/1_KaragoekY.pdf [2017-01-26]

Kunz J (2017) Reflections on the practical applicability of strategic game theory to managerial incentivation. In: Mueller D, Trost R (eds) Game theory in management accounting – implementing incentives and fairness. Springer, pp. 27–57

Lambert RA (2001) Contracting theory and accounting. JAE 32(1–3):3–87

Lambert RA (2007) Agency theory and management accounting. In: Chapman CS, Hopwood AG, Shields MD (eds) Handbook of accounting research. Elsevier, Oxford/Amsterdam, pp 247–268

Laux H (2006a) Wertorientierte Unternehmenssteuerung und Kapitalmarkt, 2nd edn. Springer

Laux H (2006b) Unternehmensrechnung, Anreiz und Kontrolle, 3rd edn. Springer

McLean RP, Postlewaite A (2003) Informational size, incentive compatibility, and the core of a game with incomplete information. GEB 45(1):222–241

Mueller D (2017) The usability and validity of allocation schemes for corporate cost accounting. In: Mueller D, Trost R (eds) Game theory in management accounting – implementing incentives and fairness. Berlin, pp. 431–462

Myerson RB (1979) Incentive-compatibility and the bargaining problem. Econometrica 47(1): 61–73

Rees R (1985a) The theory of principal and agent - Part 1. Bull Econ Res 37(1):3–26

Rees R (1985b) The theory of principal and agent - Part 2. Bull Econ Res 37(2):75–95

Reichelstein S (1997) Investment decisions and managerial performance evaluation. RAS 2(2):157–180

Ross SA (1973) The economic theory of agency: the principal's problem. Am Econ Rev 63(2):134–139

Ross SA (1974) On the economic theory of agency and the principle of similarity. In: Balch MS, McFadden DL, Wu SY (eds) Essays in economic behavior under uncertainty. North-Holland Publishing Company, Amsterdam, pp 215–237

Satterthwaite M (1975) Strategy-proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. JET 10(2):187–217
Trost R (1999) What is "incentive compatibility"? In: Gaul W, Locarek-Junge H (eds) Classification in the information age. Berlin, pp 482–489
Vickrey WS (1961) Counterspeculation, auctions, and competitive sealed tenders. TJF 16(1):8–37
Wilson RB (1968) On the theory of syndicates. Econometrica 36(1):119–132

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Reflections on the Practical Applicability of Strategic Game Theory to Managerial Incentivation

Jennifer Kunz

Abstract On the one hand, game theory has proven to effectively address a wide range of economic problems. In general, it analyses the impact of incentives of different kinds on human decision making and behaviour. Thereby, it has found mechanisms to effectively induce specific behaviour, like the Groves mechanism. On the other hand, in management accounting the design of effective incentive systems plays a major role. The aim of these incentives is to induce decision makers to act in the interest of their firms. Consequently, the question rises whether and how game theory can inform the design of these incentive systems. The present paper provides an overview over the current findings in game theory regarding such incentive mechanisms and critically evaluates the practical applicability of these findings to the design of incentive systems in the area of management accounting. The paper concludes with an overview of aspects that should be addressed in future research.

Keywords Behavioural agency model • Empirical evidence • Monitoring • Performance measurement • Personal characteristics • Principal agent model

Contents

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1 Introduction

In the last decades the body of research on the optimal incentive structure to motivate managers has grown considerably. In this context, many scholars have applied models based on strategic game theory. They mainly focused on principal-agent models containing a variety of game theoretical mechanisms. The core assumption of these models is that the separation of ownership and decision making competencies results in agency costs, as the decision makers' actions have to be aligned with the owners' interests (Cuevas-Rodríguez et al. 2012, p. 526). This alignment typically is achieved by linking managerial compensation to firm performance (Fogarty et al. 2009, p. 169). Building on seminal papers like Harris and Raviv (1979), Holmström (1979), Grossman and Hart (1983), Lambert (1983), Murphy (1986), and Gibbons and Murphy (1992) this research stream yielded a broad range of findings. With respect to executive pay it is also said to be the dominant research stream (Pepper et al. 2013, p. 37). However, due to the application of mathematical models with rather narrow assumptions, this research also has been criticized in general for its simplicity and lack of practical applicability (e.g. Eisenhardt 1989, p. 57; Cuevas-Rodríguez et al. 2012, p. 527) and it has been critically remarked that "[a]gency theory, by legitimating the pursuit of self-interest by agents, offers a deeply cynical perspective on human behaviour" (Fogarty et al. 2009, p. 179).

Furthermore, some specific assumptions of agency theory have been debated: Several scholars have criticized that in reality executive pay is not determined by a principal but rather by the executives at the top firm level themselves (Bertrand and Mullainathan 2000, p. 203; Bertrand and Mullainathan 2001, p. 902). Other scholars have pointed to the fact that the focus on material incentives in principal-agent research falls short of leading to optimal managerial incentive plans (e.g. Ellingsen and Johannesson 2007, p. 135). The debate about managerial compensation and the applicability of agency theoretical findings to its design has recently been further fueled by the last financial crisis, which was partially caused by inappropriate incentive systems for high level decision makers (Chen et al. 2011, p. 1779). Thus, the practical usefulness of models based on strategic game theory for the design of managerial incentive systems is still unclear. The present book chapter picks up on this discussion and addresses the question whether strategic game theory provides insights that can be fruitfully applied to the practical design of managerial incentive plans. Existing literature comprises several further articles, which also have critically evaluated game theoretical and agency theoretical research, e.g. Levinthal (1988), Bol (2008), and O'Reilly and Main (2010). However, the present chapter provides further insights, as it focuses explicitly on the *practical applicability* of these models.

In order to investigate this practical applicability, a broad literature overview is necessary that provides comprehensive insights into the main topics of game theoretical and principal-agent theoretical research regarding managerial incentive systems and into the empirical relevance of its findings. The chapter proceeds thus as follows: In Sect. 2 the methodological approach to select the literature is discussed. In Sect. 3 the identified literature will be categorized and the key findings of the selected articles will be worked out. Moreover, these findings are analysed regarding two dimensions: the *practical relevance* of the discussed topics and the *(un)ambiguity of the findings* within each discussed subcategory. The latter is of importance to the present analysis, because ambiguous results and findings that contradict each other impede the practical application of research outcomes, as they make it harder for practitioners to decide which results are actually relevant and correct. Drawing on the conclusions from Sect. 3, Sect. 4 is dedicated to the analysis of the general applicability of the research findings in practical contexts and the identification of research gaps. The book chapter closes with some concluding remarks in Sect. 5.

2 Method

As the considered research field—the application of strategic game theory to management incentive systems—is rather broad, the following literature overview can only cover parts of the existing literature. However, instead of grounding the selection of the discussed literature on subjective criteria and in order to get a quite broad scope and a representative sample, a more objective strategy was chosen and a structured literature review was performed.

The search concentrated on the data base ABI/Inform Collection, as this data base covers about 4000 journals. The search was undertaken from 9th until 18th of May, 2016, by the author.

During this search the following keywords were applied: *game theory, agency theory* or *principal agent* combined with *compensation, incentive, gratification, management control system, pay, payment, remuneration, rewards, salary* or *wages*.

The search was limited to peer-reviewed journal articles in English language and it focused on incentives for *managers*. Thus, it did not consider articles that dealt with incentives for workers providing insights, which could not be transferred to managers. Additionally, articles covering team incentives only were considered, if they could be applied to a managerial context. The search further concentrated on articles that provided generalizable results, i.e. for example research dealing with specific questions in the international context was not included. Furthermore, both, *mathematical models* and *experimental research* were considered. The latter mainly does not contain analyses intended to bring forward mathematical modelling. Thus, they do not yield core findings of mathematical strategic game theory and principal-agent theory. However, they provide insights into the closeness to reality of these models and into their practical applicability.

In total 141 articles were identified that met the mentioned criteria. The fact, that e.g. Harris and Raviv (1979), Holmström (1979), and Grossman and Hart (1983) are not part of this sample, shows that this overview is far from being complete, which results from the limited number of key words and the search in only one data base. Moreover, as the search and selection were only performed by the author herself, their outcome is subjective. However, as this literature review is not intended to cover the complete set of existing literature in this research field, but only aims at generating a reasonable sample to ground the discussion on, the mentioned limitations should only have a minor impact on the quality of the following discussion.

3 Discussion of Selected Literature

3.1 Framework

The selected articles were analysed regarding their main research focus. This analysis yielded the following classification: The main categories comprise the constituting elements of incentive systems, i.e. *incentive types, performance measures*, and the incorporation of *risk*. They are complemented by two additional categories called *monitoring* and *psychological and social aspects*. The categories are further structured in subcategories, like *specific incentive types, specific performance measures*, and *psychological versus social aspects*. The lowest level of categorization is the differentiation in *mathematical* and *empirical* research.

The **first category** concentrates on research regarding the optimal incentive type in compensation plans. Studies that fall into this category examine the selection of optimal monetary incentive types in general, the application of options and shares or the efficiency of non-monetary incentives. The **second category** contains articles that discuss the selection of optimal performance measures and their characteristics. They analyse non-contractible and unobservable information, precision, distortion, and subjectivity. Moreover, in this category articles are presented that focus on specific measures, as well as on aggregate, team-based, and relative measures. Finally, it comprises articles that cover further aspects. The **third category** summarizes research that concentrates on the effects of risk on optimal incentives. The **fourth category** presents articles that analyse the impact of monitoring as a replacement of or a complement to rewards in the context of managerial incentives. The **fifth category** comprises research that focuses on personal characteristics, social and interactional aspects, as well as on the behavioural agency model.

To keep the following discussion traceable, each article is classified in one category based on its main findings, which in some cases admittedly entails some degree of subjectivity, as several articles cover a broad range of topics.

3.2 Incentive Type

3.2.1 Performance-Based Pay and Monetary Incentives

Mathematical Models

This subcategory concentrates on *the effective and efficient design of performance-based pay and monetary incentives*. It can be further divided. The present section discusses research based on mathematical models, while the following sections look at the empirical research in this area. In the sample several articles focus on the design of performance-based pay in a *multi-task setting*. The following discussion contains only some of these articles, as several of them were classified in other categories, because they cover further topics.

In this research stream, one focus lies on the substitution between activities that are beneficial to the firm and activities that are beneficial to the agent: Ellingsen (1997) explores optimal incentive schemes in this context under the assumption of agents' limited liability. Dewatripont et al. (2000) provide an overview over research on incentives in this setting with respect to effort substitution, conflicts between tasks, implicit incentives, and missions. Alles and Gupta (2009) analyse the effectiveness of incentive systems in a situation with production-related and non-production-related effort.

Other scholars concentrate on non-contractible or unobservable input and output in the multi-task context: Chambers and Quiggin (2005) examine the optimal weight put on variable versus fixed payment depending on non-contractible outputs. Chen (2012) derives an optimal incentive structure called all-or-nothing payments that is feasible in contexts with unobservable inputs, verifiable inputs, and observable but unverifiable inputs.

One further body of research explores the effects of *dynamic, repeated and history dependent principal-agent settings* on incentives: Chassang (2013) provides optimal incentive contracts in a dynamic agency model with limited liability, moral hazard, and adverse selection. Shin and Strausz (2014) show that delegation can improve dynamic incentives under certain conditions. Opp and Zhu (2015) examine optimal contracts under the assumption of an impatient agent and find a contract that oscillates between the agent and the principal. Williams (2015) studies the optimal incentives in a continuous time dynamic moral hazard model.

Several scholars analyse *further aspects*: Prendergast (2000) discusses reasons for the absence of a clear trade-off between risks and incentives in practice. Dutta (2003) studies a principal-agent model in which an agent can carry out a project, for whose implementation he has specific skills, either within a firm our outside that firm. Dependent on the severity of the retention problem the principal either should implement a residual income-based bonus (less severe retention problem) or an option-based compensation plan (severe retention problem). Among other things, Kocabiyikoğlu and Popescu (2007) find that increasing salary does not stimulate managerial effort, while under certain circumstances a higher share of variable pay does not have a positive impact on effort either. Jewitt et al. (2008) examine solutions for optimal contracts with upper and lower limits and prove their uniqueness and existence. Verdier and Woo (2011) find that sanctions should be avoided and promises to reward should be preferred to induce the wished for behaviour. Bakó and Kálecz-Simon (2013) study factors that influence the effective application of quota bonuses. Chen (2013) also examines the reasons for the mixed evidence in empirical research regarding the risk-incentive trade-off. To explain this evidence the author incorporates the assumption of agents who are heterogeneous in risk aversion and of agents who can search for outside options. Meng and Tian (2013) analyse why in practice low powered incentive schemes are more often found than high powered incentive schemes in a situation of moral hazard and adverse selection. Kräkel and Müller (2015) show that CEOs rather propose merger targets with low potential for synergies to get high powered incentives. Veldman and Gaalman (2015) find in the context of cost reducing process improvements that bonuses are only optimal under certain circumstances. Carroll and Meng (2016) analyse how to design a contract that is robust against a small amount of environmental uncertainty.

The mentioned models cover a broad range of questions, where multi-task settings, dynamic agency relations, and the risk-incentive trade-off are three core themes. These topics are highly relevant to the practical implementation of managerial incentives, as they reflect main aspects of the managerial work context. Moreover, several papers explicitly try to explain phenomena found in practice, e.g. Meng and Tian (2013) and Kräkel and Müller (2015). Other articles can at least be linked to phenomena that can be found in practice, e.g. Verdier and Woo (2011). Thus, the presented research covers themes that are of *practical relevance*. With respect to the *(un)ambiguity of results* the following aspect can be observed: Several authors pick up on ambiguous empirical results with respect to agency theoretical predictions and try to adjust the models to these results, e.g. Prendergast (2000) and Chen (2013).

Application of Performance-Based Pay in Practice

Articles in this category mainly focus on the *risk-incentive trade-off*: Umanath et al. (1993) test several hypothesis in an experimental setting regarding the relation between perceived environmental uncertainty and perceived agent effectiveness on the one hand and compensation on the other hand. One finding contradicts the predictions of agency theory, as the authors do not find evidence that higher perceived environmental uncertainty leads to a lower level of performance-based pay. Umanath et al. (1996) pick up on this result and find experimentally that information (a)symmetry could be one possible reason for the mentioned result. Stroh et al. (1996) show in their survey, that variable pay is more extensively used, when the managerial tasks are less programmable and the organisation faces a high level of risk. In contrast, long-term relationships reduce the level of variable pay. Moreover, the results do not indicate a risk premium for managers in turbulent environments, i.e. the firms do not compensate the managers for the higher risk as recommended by agency theory, but just increase the variable pay. Kraft and Niederprüm (1999) analyse the weight that is put on a profit-based variable pay component as compared to a fixed component in managerial compensation in Germany. As predicted by agency theory, they find a negative relation between the weight put on the profit-based component and the variance in profits. Foss and Laursen (2005) find, among other things, only a weak relation between environmental uncertainty and the application of performance-based pay. Gao (2010) analyses mathematically and empirically the optimal incentive structure depending on the possibility of CEOs to hedge their personal portfolio. Cao and Wang (2013) investigate mathematically and empirically pay-performance sensitivity and show, among other things, that it depends on a firm's idiosyncratic and systematic risk.

Three articles discuss *some further aspects*: Gayle and Miller (2009) explore the development of managerial compensation over several decades, i.e. its increase in value and variability under the premise of the importance of moral hazard in managerial positions. Goktan (2014) investigates the relation between green management practices and CEO compensation and finds a negative relation between green management and the CEOs' base pay and no significant relation between green management and CEOs' bonuses. Daljord et al. (2016) analyse the reasons why in practice firms typically use uniform incentive contracts despite a heterogeneous work force.

In sum, again, this research stream covers a broad range of topics, which are *relevant to the practical implementation* of managerial incentive systems. However, it exhibits a rather *low level of unambiguous* results, as it provides mixed evidence with respect to the applicability of agency theory in practice. While the findings by Kraft and Niederprüm (1999) conform to agency theory, Stroh et al. (1996) receive results that contradict agency theory at least partly.

Empirical Relation Between Performance-Based Pay, Managerial Behaviour, and Firm Performance

Various scholars explore the *empirical relation between performance-based pay* on the one hand and *managerial behaviour and firm performance* on the other hand.

Several studies rather *confirm*, at least partly, the predictions of agency theory: Banker et al. (1996) find a positive long-term impact of a performance-based incentive plan on sales in retailing. In a sample of Korean firms, Byun et al. (2009) show a positive relation between deferred compensation and performance-based pay on the one hand and several measures of firm value on the other hand, where deferred compensation has a positive relation to ROA and labour productivity, while performance-based pay has a positive relation to sales growth. Nyberg et al. (2010) find, among other things, a positive alignment between managerial incentives and owner interests. Young et al. (2012) observe in a field study in the context of a network of physician practices a positive effect of financial incentives on performance. This effect is moderated by the physicians' attitudes, like their attitude towards the importance of the performance goals. Kishore et al. (2013) apply a field experiment to analyse the different effects of quota-based bonuses and commissions. In their study the switch from a bonus to a commission-based incentive system improved productivity depending on the employees' ability, while commissions also increase the neglect of tasks that are not incentivized. Gayle and Miller (2015) study the empirical content of pure moral hazard and hybrid moral hazard principal-agent models.

Other studies provide evidence that is rather *contrary* to agency theoretical predictions or show that *they only hold under specific conditions*: Kosnik and Bettenhausen (1992) analyse an interaction effect between managerial compensation, the board of directors' supervision and the situation in the managerial labour market on managerial opportunistic behaviour. Among other things, they observe the lowest level of opportunistic behaviour in the case of high board control and low equity ownership, which according to the authors contradicts the predictions of agency theory (Kosnik and Bettenhausen 1992, pp. 325–326). Keser and Willinger (2000) explore experimentally the behaviour of principals and agents in a hidden action setting. They find that while agents act to maximize their expected profits, principals offer incentive schemes that deviate from the theoretical optimal schemes. Bertrand and Mullainathan (2000) provide evidence that a firm's governance has an impact on that firm's behaviour: Better-governed firms rather behave according to agency theory than worse governed ones, while the latter rather behave according to a skimming model, i.e. here executives determine their pay themselves by manipulating the compensation committee. Bertrand and Mullainathan (2001) pick up on this discussion and further elaborate on the skimming theory in contrast to the agency theory. Chen et al. (2011) analyse the reasons for the recent financial crisis and conclude "that the executive remuneration design derived from a single agency perspective is insufficient to provide convincing explanation to the real business world during the financial crisis" (Chen et al. 2011, p. 1779). They argue, that prospect theory, real option theory and the managerial power approach should be applied to complement agency theory. Amzaleg et al. (2014) conclude from their findings of a model and an empirical investigation of Israeli firms a link between pay-performance sensitivity in CEO compensation and firm performance that is reversal to the predictions of agency theory: They find that powerful CEOs, who expect a good firm performance, try to get a contract with high pay-performance sensitivity.

Finally, Bremzen et al. (2015) put agency theory into a new perspective. They test in an experimental setting whether agents interpret high rewards in the sense that the task to perform is difficult. The authors actually can find such an effect and decompose the overall effect of incentives in a negative informational effect and a positive incentive effect.

In sum, the presented articles cover a broad range of topics that are of *high practical relevance*. However, the findings are also rather *ambiguous*.

3.2.2 Options and Stocks

Mathematical Models

The second subcategory deals with the application of specific incentive types, which are prototypical in the business context: *options and stocks*. Again, this research stream can be further differentiated. The present section deals with mathematical models in this context, while the following section focuses on empirical research.

Several models provide evidence that stock options are at least under specific conditions an *effective component* of compensation plans: Hemmer (1993) finds that under certain circumstances the introduction of stock options as one component can result in an optimal incentive plan. The findings by Hemmer et al. (2000) indicate that depending on agents' risk aversion the introduction of stock options into incentive plans can lead to an optimal contract. Oyer (2004) explains the application of stock options in incentive contracts in practice by the participation constraint postulated by agency theory. Choe and Yin (2006) show that under certain conditions option-based incentive plans weakly dominate stock-based incentive plans. Wu (2011) demonstrates the optimality of stock options in comparison to restricted stock in incentive plans under certain conditions. Flor et al. (2014) exhibit circumstances under which the combination between stocks, options, and capping pay mechanisms is an effective way of compensation. Chaigneau (2015a) studies the optimality of incentive contracts for risk averse and for prudent managers. He shows that in case of risk averse managers a concave contract is superior, while in case of a prudent manager a convex contract is better. The author concludes that the second result can explain the fact that in practice stock options are still part of compensation plans (Chaigneau 2015a, p. 1357).

In contrast, Chaigneau (2015b) calls into question the effectiveness of convex incentive contracts and thus also stock options as component in compensation plans.

Finally, Benmelech et al. (2010) analyse the detrimental effects of stock-based incentives.

In sum, this subsection deals with a topic of *high practical relevance* and provides mainly *positive evidence* regarding the optimality of options, while several authors are rather critical regarding the application of stocks as incentive, e.g. Choe and Yin (2006) and Benmelech et al. (2010).

Empirical Research

A number of studies explores the *application of options and stock empirically*.

Jensen and Murphy (1990) study the effects of different incentive components, like stock options, bonuses or performance-based dismissals. They conclude that their results are inconsistent with agency theoretical predictions (Jensen and Murphy 1990, p. 227). Welbourne and Cyr (1999) explore the effect of ownership of managers and employees on firm performance. They find a negative relation between both CEO ownership and executive team ownership on the one hand and firm performance on the other hand, while employee ownership has a positive relation with firm performance. Lam and Chng (2006) find that in practice the motivation to offer stock options in a managerial incentive plan are in line with agency theoretical predictions. Pendleton (2006) investigates whether incentive plans containing shares substitute for direct monitoring and individual incentives in the UK. Among other things, he finds that individual incentives are complementary to share plans and not substitutional. Dittmann and Maug (2007) show mathematically that option-based incentives should be abandoned and stock-based incentives should be combined with low base salaries. However, this result is in contrast to the contracts that the authors observe in practice. Narayanan and Seyhun (2008) provide evidence for the purposeful picking of option grant dates by managers. Among other things, Fogarty et al. (2009) critically evaluate the application of options in executive incentive plans with respect to the case of Nortel.

Also empirical research regarding the optimality of options and stocks covers a broad range of *practically relevant* research questions. Moreover, among other things, Lam and Chng (2006) provide evidence in favor of agency theory, while other studies find that either practice acts contrary to agency theoretical suggestions, e.g. Dittmann and Maug (2007), or that the agency theoretical predictions do not completely hold, because firm ownership as incentive component does not always result in higher firm performance, e.g. Welbourne and Cyr (1999). Thus, in sum, also this research stream provides rather ambiguous evidence.

3.2.3 Non-Monetary Incentives

Mathematical Models

Beside financial incentives, like options and stocks, *non-monetary incentives* play an important role in the discussion of optimal managerial incentive plans. In the sample of the literature review several research directions can be identified with respect to these incentives. In the present section articles are presented that analyse the effect of non-monetary incentives mathematically. The following section deals with experimental research in this area.

One important type of non-monetary incentives are *career concerns*. In the sample several mathematical articles focus on this aspect. Kwon (2006) analyses the optimality of dismissal options versus wage contracts and he finds that dismissal is an efficient part of incentive contracts only under certain conditions. Englmaier et al. (2010) study the joint effect of compensation-based and reputational incentives in a setting, where the principal has to decide how much of authority he wants to cede to the agent whose ability is uncertain. Schöttner and Thiele (2010) examine the joint effects of promotion and individual performance-based pay for selection and incentive purposes. Koch and Peyrache (2011) explore the trade-off between short-term monetary and long-term career concerns and show that optimal contracts should generate ambiguity about the agent's abilities and that the application of relative performance measures in incentive plans can be optimal. Miklós-Thal and

Ullrich (2015) investigate the effect of the precision of beliefs about abilities in a promotion context.

Furthermore, several scholars cover *various other non-monetary incentives*: Zou (1997) analyses the role of fringe benefits as one component in incentive plans and finds that they can reduce agency costs. Radhakrishnan and Ronen (1999) explore the effect of both monetary incentives and job challenge. Under information asymmetry and further certain conditions they show that job challenge can substitute for monetary incentives, that through job challenge risk can be imposed on the agent to cope with the hidden action problem in a cheaper way than when it is introduced via monetary incentives and that the additional application of job challenge might result in a more informative final outcome. Gürtler and Kräkel (2010) find that under certain contextual factors the threat of litigation can result in the first-best solution.

Overall, the mentioned models try to explain empirically observable phenomena and thereby provide evidence that agency theory can lead to practical relevant insights. Moreover, they demonstrate the value of non-monetary incentives. In sum, this subsection contains research that is of *practical relevance* and provides rather *unambiguous* results.

Experimental Research

There are several studies in the sample that examine non-monetary incentives in an *experimental setting*.

Brüggen and Moers (2007) investigate the effects of financial and social incentives in a multi-task setting. Their results highlight the importance of social incentives in this context. Dugar (2010) finds in a minimum effort game that agents' statements of disapproval lead to better results than agents' expressions of approval. Kelly and Tan (2010) examine the joint effects of profit-sharing and feedback regarding the individual cooperativeness on cooperation sustainability. Altmann et al. (2012) show in an experimental setting the motivating effects of promotion. Dugar (2013) analyses the effect of different communication strategies to mitigate free riding in a public good game. He finds that the possibility to distribute either approval or disapproval points results in the highest contribution rate.

In total, the presented papers exhibit the importance of non-monetary incentives for practice and try to improve practical incentive provision, e.g. Kelly and Tan (2010). Thus, they are of *practical relevance*. However, they also partly *call the predictions of standard strategic game theory into question*, e.g. Dugar (2010).

3.3 *Performance Measures*

3.3.1 Effect of Non-Contractible and Unobservable Information, Precision, Distortion, and Subjectivity

Mathematical Models

With respect to performance measures several subcategories can be identified. The first subcategory focuses on the effect of *non-contractible and unobservable information, precision, distortion, and subjectivity*. Articles dealing with these different topics were subsumed under one subcategory, as these articles often do not concentrate on one of these aspects but cover several of them. The present section focuses on the mathematical models, while the following section presents empirical research.

Hermalin and Katz (1991) concentrate on the effects of renegotiating on the actions that a principal can induce and on the costs to induce these actions in the case that an agent's actions are observable but not verifiable. They find a welfare increasing effect of renegotiation in case that the principal observes an unverifiable signal regarding the agent's action. Baker (1992) analyses a situation in which a principal's objective is not contractible and, thus, performance measures are used that do not fully reflect a principal's objective. He also provides a statistical measure to assess the efficiency of a performance measure. Baiman and Rajan (1995) explore the use of non-contractible information through the application of a bonus pool. The authors show that such a bonus plan leads to a strict Pareto improvement, if the non-contractible information is informative. Kaarbøe and Olsen (2008) investigate the joint effect of the distortion of performance measures and implicit dynamic incentives, like career concerns and the ratchet effect, and show that an increase in distortion or a stronger ratchet effect result in higher optimal monetary incentives. Schnedler (2008) examines the application of an aligned versus an unaligned, but otherwise identical measure in a multi-task setting. He finds that "the optimal measure is not aligned but tilted toward tasks that the agent finds easy" (Schnedler 2008, p. 595). Moav and Neeman (2010) find a non-monotonic relation between the degree of precision of a performance measure and the incentive effect of this measure. According to this result, very imprecise and very precise measures both are detrimental. Maestri (2014) analyses the optimal contract in a situation, where the principal evaluates the agent after several periods of time using a subjective assessment and either pays a bonus and continues the employment or does not pay a bonus and terminates the employment. The efficiency of this kind of payment depends on the parties' patience.

In sum, this literature stream again covers *highly relevant topics from a practical perspective*. For example, managerial tasks comprise activities that are difficult to measure objectively and, thus, subjective evaluation is a valuable alternative. Furthermore, the results do not provide *ambiguous evidence*, but they call for a more differentiated perspective on performance measures, e.g. Moav and Neeman (2010).

Empirical Research

The sample also contains several *empirical studies* and one *literature overview* with respect to the mentioned aspects: Bushman et al. (1996) explore the application of individual performance evaluation, which according to the authors is characterized by subjectivity and discretion. They find that the application of this performance assessment type is increasingly applied with growth opportunities and product time horizon. Hayes and Schaefer (2000) examine the relation between performance measures that contain information, which is unobservable to outsiders of the firm, and a firm's future performance. The authors conclude from their results that boards of directors apply performance measures that are not observable by third parties in top executives' incentive plans to reward executives for actions that benefit the firm, but that are not reflected in current performance measures (Hayes and Schaefer 2000, p. 274). Moers (2006) studies the relation between financial performance measures and delegation depending on that performance measures' sensitivity, precision, and verifiability. He finds that the application of financial performance measures which are characterized by a high degree of these three aspects in incentive plans results in an increase in delegation.

Bol (2008) provides a literature review over the research on subjectivity in performance measurement and discusses that subjectivity can have both, positive and negative incentive effects.

The mentioned articles provide a better understanding of the effects of performance measures' characteristics on the effectiveness of managerial incentives and, thus, are of *practical relevance*. Moreover, the presented findings *do not provide ambiguous evidence*. Yet, Bol (2008) points to the need for a more differentiated perspective on the application of subjective performance assessment.

3.3.2 Application of Specific Measures

Beside the investigation of characteristics that apply to any kind of measures, several scholars focus on *specific measures*, like accounting information. Again, the sample contains both, mathematical and empirical research. Because of the small number of empirical articles the two subcategories are discussed in one section.

In the articles applying *mathematical models* the following topics are covered: Zou (1991) shows that under certain conditions a target-incentive system can more effectively attenuate the ratchet effect than a price-incentive system. Bushman and Indjejikian (1993) explore the effects of earnings and price on incentive plans dependent on the fact whether earnings and price either reflect the same underlying information or earnings are only partly linked to price. Brown et al. (1994) explore the ratchet effect and show that under certain conditions this effect provides information to offset the costs incurred to implement it. Wagenhofer (1996) examines the application of distorted overhead costs to calculate an accounting profit as performance measure. Bontems and Bourgeon (2000) study the efficiency of output- versus input-based performance measures in incentive systems. Kwon (2005) finds that the application of conservative accounting measures as opposed to aggressive measures can reduce agency costs that can occur due to the implementation of suboptimal decisions. Şabac (2007) analyses the influence of renegotiation on incentives in case that the performance information is serially correlated. Moreover, he examines the optimal managerial tenure depending on this serial correlation. Raith (2008) investigates the use of input- and output-based performance measures and finds that optimally a combination of both measures is applied to minimize income risk and to maximize the application of an agent's personal knowledge.

With respect to the *empirical research*, the following aspects are discussed in the sample: Core et al. (2003) explore the weighting of price and non-price measures in CEO cash and total compensation. They find some evidence that contradicts standard agency theory, as in case of total CEO compensation "the relative incentives from price and non-price measures are *positively* related to the relative variances of the performance measures" (Core et al. 2003, p. 959). The authors discuss this result critically (Core et al. 2003, p. 959). O'Connell and O'Sullivan (2014) investigate the application of customer satisfaction as performance measure in managerial incentive plans. Dutcher et al. (2015) show in an experimental setting that incentives which induce agents to be the first and to avoid to be the last lead to agents' largest effort.

The mentioned articles concentrate on performance measures that are typically part of managerial incentive plans. Thus, again, the presented research exhibits a *high degree of practical relevance*. Furthermore, the empirical findings partly contradict agency theory, e.g. Core et al. (2003), which yields *further ambiguity* to the body of evidence within this research field.

3.3.3 Aggregate, Team-Based and Relative Performance Measures

Several articles provide insights into the efficiency of measures that go beyond individual performance like *aggregate, team-based* and *relative performance measures*. Again, as the number of empirical articles in the sample is small, mathematical and empirical research is discussed in one section.

The following articles analyse the effect of these measures on a *mathematical basis*. Meyer and Vickers (1997) show that comparative performance information has an ambiguous effect in dynamic settings, as it can either reinforce or mitigate the insurance effect that is observed in static settings. Dur and Tichem (2010) analyse the effect of incentives on work climate and provide evidence that both team-based and relative incentives positively affect work climate. Kvaløy and Olsen (2012) study why in human-capital-intense industries individual performance-based pay prevails over group-based pay. They find in the context of complementary tasks that the degree of indispensability has a negative effect on the applicability of group-based incentives. Kim (2015) studies the optimal incentive contract between a principal and several agents with differing beliefs about success. The article shows that the optimal contract contains relative performance evaluation in case that the principal is more optimistic than the agents, and joint evaluation in case that the principal is less optimistic.

Furthermore, the sample contains the following *empirical articles*: Janakiraman et al. (1992) examine whether the implications of relative performance evaluation that can be derived from standard agency theory are actually applied in the practical design of CEOs' cash compensation. The authors do not find any strong evidence that these implications are followed in practice. Bushman et al. (1995) explore the application of aggregate measures in comparison to more disaggregate measures in the case of intrafirm interdependencies. They find that the relative application of aggregate measures increases with these interdependencies.

In sum, these articles cover research questions of *practical relevance*, e.g. Kvaløy and Olsen (2012) pick up on an empirical observable phenomenon. But again, the findings *are not unambiguous*, e.g. one article provides evidence that contradicts predictions of standard agency theory (Janakiraman et al. 1992).

3.4 Effects of Risk

Risk is one of the constituting elements of a principal-agent relation. Therefore, articles that explicitly focus on this aspect are of high importance to the understanding of principal-agent models as such. Within the sample the following articles could be identified that deal mainly with this topic.

Androkovich (1990) analyses mathematically an incentive mechanism that induces the agent to bear more risk than predicted by the agent's risk aversion. Chouikhi and Ramani (2004) study mathematically the relation between shirking and information asymmetry under the premise of the firm's and the agent's risk aversion. They find that "under incomplete information about the degree of risk aversion of the worker, shirking can emerge as an equilibrium phenomenon" (Chouikhi and Ramani 2004, p. 53).

Gray and Cannella (1997) show empirically that, among other things, CEOs' total compensation and compensation risk vary with firms' financial and strategic context. By applying a mathematical model and a certain empirical methodology Chaigneau (2013) explores whether the structure of CEOs' incentive plans in practice fits to preferences with hyperbolic absolute risk aversion. Ekins et al. (2014) show in an experimental setting applying functional magnetic resonance imaging that humans perceive risk from a lottery differently than risk in an incentive contract.

In total, the presented articles indicate the importance of a deeper understanding regarding the effects of risk in the context of incentives to design optimal compensation plans in *practice*. Especially Ekins et al. (2014) provide valuable insights for practice, as they indicate the need for a *more differentiated perspective* on an agent's perception of different risk types. Furthermore, Gray and Cannella (1997) show the context dependence of managerial compensation.

3.5 Monitoring

Monitoring is not an incentive in the narrower sense. However, because in literature it is often discussed in combination with incentive systems, research dealing with it has also been incorporated into the present discussion. Within the sample the following mathematical and empirical studies could be identified:

Cowen and Glazer (1996) demonstrate *mathematically* that under certain context factors a better information state through monitoring of the principal might result in more shirking by an agent.

The following scholars apply an *empirical approach*: Based on empirical data Evans and Weir (1995) analyse incentives and monitoring separately. They argue that the best monitoring frequency is monthly and that there is no linear relation between the provision of performance-based pay for divisional managers and firm performance.

Among other things, Tosi et al. (1997) show experimentally that incentives are a better mechanism to align the agent's activities with the principal's objectives than monitoring. Hoskisson et al. (2009) argue based on empirical evidence that increased monitoring conveys increased compensation. Applying a laboratory experiment and a field study Boly (2011) shows that monitoring increases subjects' effort. Moreover, the author finds a significant, disciplining effect of monitoring for selfish subjects, but no effect for intrinsically motivated subjects.

In total, the previously discussed articles shed more light on the joint effects of incentives and monitoring and, thus, cover a topic of *practical relevance*. However, the results are rather *mixed and ambiguous*. Some scholars indicate that incentives might be a better mechanism than monitoring to align agents' activities to principals' interests, e.g. Tosi et al. (1997), while other scholars find that monitoring conveys certain compensation schemes, e.g. Hoskisson et al. (2009), and that it has either a motivating impact, e.g. Boly (2011), or can lead to shirking, e.g. Cowen and Glazer (1996).

3.6 Psychological and Social Aspects

3.6.1 Personal Characteristics

To structure the following discussion, the articles are clustered in three subcategories. The present section deals with *personal characteristics of agents and principals* in terms of motivation, knowledge, beliefs, and other aspects, while the following section covers *social and interactional aspects*. A third category concentrates on the *behavioural agency theory*.

Several articles in the sample discuss the impact of agents' *loss aversion*. Dittmann et al. (2010) study the optimal compensation plan for CEOs with loss aversion. They adjust a principal-agent model to an empirical set of incentive plans and show that it explains the practical application of incentive schemes with options and high base salaries quite well. They also derive an optimal contract under the assumption of loss aversion. Herweg et al. (2010) derive mathematically a binary payment scheme that is optimal under the assumption of an agent's loss aversion.

Beside loss aversion further *agent's characteristics* are considered *mathematically*: Goering (1996) shows that different managerial styles in terms of aggressiveness of beliefs regarding competitor behaviour should be considered when designing compensation contracts. Santos-Pinto (2008) studies the effect of an agent's mistaken beliefs regarding her abilities and finds that in case of observable effort, these beliefs are favourable to the principal, while in case of unobservable effort these beliefs can be favourable or unfavourable. Makris (2009) investigates the optimal incentive plan for an intrinsically motivated agent in a situation that is characterized by limited liability. Von Thadden and Zhao (2012) examine optimal incentives for an agent who is unaware of the whole action space. Immordino et al. (2015) explore the optimal incentive contract in case of a wishful thinking agent. Müller and Weinschenk (2015) analyse the effect of a supervisor's rater bias and show that a rater bias can lead to stronger incentive effects in early periods through implicit incentives, while in later periods it weakens the incentive effect. In case of a moderate bias, the mentioned positive effect can offset the negative effect.

Beside these mathematical models, the sample contains further articles that apply *empirical research methods*. Fong and Tosi (2007) test the interaction effect between conscientiousness on the one hand and incentive alignment and monitoring on the other hand on agent's behaviour. Among other things, they show that the effect of incentive alignment on effort and task performance depends on conscientiousness in the sense that agents with a low level of conscientiousness react positively to incentive alignment while more conscious people are less affected. Dubois and Vukina (2009) study mathematically and empirically the optimal incentive contract in case of heterogeneous agents.

In sum, the presented articles contain a broad range of findings regarding the effects of psychological aspects on effective incentive provision and thereby provide valuable insights for the *practical application of incentives*. The evidence in this subsection is rather *unambiguous*.

3.6.2 Social and Interactional Aspects

A row of articles covers *social and interactional aspects*, i.e. situations, in which the agents' social context or their interaction with other agents, the principal or third parties becomes important. In this research stream social comparison, fairness and justice, as well as gaming are the most widely discussed phenomena in the sample.

With respect to *social comparison* the following topics are covered in the sample: Alewell and Nicklisch (2009) explore in an ultimatum game experiment the effect of available information regarding the ultimatum offer made by others on the agent's own acceptance behaviour. They find under certain conditions a detrimental effect on the acceptance of offers that comprise an unfavourable discrimination against the agent or others. Gächter et al. (2012) analyse in an experimental setting the effect of social comparison with respect to pay and effort on reciprocity towards an employer. They find that "employees are more willing to reciprocate by choosing high effort in response to a high wage if they observe others doing so" (Gächter et al. 2012, p. 1346). Nosenzo (2013) examines the impact of pay comparison between agents in an experimental setting and finds a detrimental effect of transparent pay on effort, if the agents are not treated equally. Nafziger and Schumacher (2013) study mathematically the effect of transparency regarding colleagues' performance on an agent's provision of effort and find that under certain circumstances transparency can result in unfavourable incentive effects.

One further aspect that is discussed in several articles is *fairness and justice*. Welbourne et al. (1995) show in a field study an effect of perceived distributional and procedural justice of a gainsharing plan on agents' monitoring of others. Scarpello and Jones (1996) examine the importance of justice in compensation decision making by applying a questionnaire-based survey. Englmaier and Wambach (2010) demonstrate mathematically, among other things, that agents' inequity aversion modifies the optimal contract in situations of moral hazard, i.e. it converges to a linear sharing rule. Bruttel and Eisenkopf (2012) find experimentally in a sequential prisoner's dilemma game, among other things, that any contract—even if it is unfair—improves welfare.

Several articles in the sample cover *gaming*. These articles are also subsumed under the present subcategory, because gaming can be interpreted as an interactive process between the agent and the principal, where the agent tries to deceive the principal. Courty and Marschke (2004) investigate gaming in the sense that agents strategically report their performance outcomes. The authors find that in the analysed empirical context the incentive system prompts gaming behaviour and that this gaming is detrimental to organisational effectiveness. Foster and Young (2010) show mathematically that in the context of portfolio managers there exists no incentive contract that is able to separate the skilled managers from the unskilled ones based on their past returns and that, thus, there exists no contract that could impede in this situation gaming completely. Benson (2015) studies empirically how sales managers game the behaviour of their subordinates and thereby act in contrast to the firm's interest.

Moreover, scholars provide evidence regarding several *further topics*: Güth et al. (1998) study experimentally a dynamic principal-agent relationship and thereby focus on aspects like trust and reciprocity. Grabke-Rundell and Gomez-Mejia (2002) try to enrich agency theory by combining it with the resources dependency theory to analyse conceptually the effect of power on executive compensation. Ellingsen and Johannesson (2008) introduce the effect of social esteem and the impact of the audience that provides this esteem in a principal-agent model and discuss a situation in which through incentive provision a principal reveals information about himself, which in turn influences an agent's esteem about the principal and, thus, the agent's behaviour. In turn, in this situation the agent's behaviour reveals information about his character and thereby affects the principal's esteem for the agent. Rigdon

(2009) explores in an experimental setting the effects of specific ex post incentives on informal contracts.

In sum, the studies exhibit that research tries to enrich the strategic game theoretical reasoning with a broad range of social aspects to make its assumptions more realistic and to find better suggestions for the practical design of incentive systems. Moreover, the literature on social comparison indicates the complex effects of transparency. Finally, the articles on gaming shed a rather critical light on incentive systems. In sum, this subcategory contains articles that cover topics of *practical relevance* and provide rather *unambiguous* but differentiated evidence.

3.6.3 The Behavioural Agency Model

Various authors work explicitly with a *behavioural agency model*, i.e. they try to enrich the traditional agency concept with behavioural aspects.

Ellingsen and Johannesson (2007) discuss conceptually the behavioural agency model. Larraza-Kintana et al. (2007) focus empirically on the analysis of risk taking. Pepper et al. (2013) show in an empirical study that due to certain characteristics of executives' evaluation processes long-term incentive plans are inefficient and ineffective in motivating executives. Pepper and Gore (2014) analyse the effect of long-term incentives in executive pay schemes and conclude that they are not an efficient instrument. Pepper and Gore (2015) develop the behavioural agency model further by incorporating time discounting, inequity aversion, and the trade-off between intrinsic and extrinsic motivation. Pepper et al. (2015) investigate empirically executives' fairness considerations and discuss the implications of their results for the behavioural agency theory.

This research stream underlines the willingness of scholars to not only partly enrich the agency framework with psychological and social aspects, as observed in the previous sections, but to find a coherent theoretical basis to renew the concepts of agency theoretical reasoning.

4 Evaluation of Practical Applicability and Identification of Further Research

The present literature review indicates that strategic game theory and its application in agency theoretical settings cover a wide range of research questions. Moreover, the previous brief discussion of the presented articles also reveals that overall the application of strategic game theory and agency theory is intended to provide valuable insights for the practical implementation of incentive plans.

This section picks up on this observation and elaborates further on it by exploring the presented sample along three dimensions.

The first dimension focuses on the *mathematical modelling approach*. The previous discussion demonstrated the broad range of topics that is covered by the mathematical articles in the sample. Moreover, these articles mainly focus on the explication of phenomena that can be observed in practice or try to solve practical problems. Hence, they actually should be of high practical value. However, in many articles a *translation of the mathematical results to the practical context* is missing. Research that applies mathematical models borrowed from strategic game theory typically show that under certain conditions a specific solution exists. But these conditions are rather seldom exactly met in practice. Thus, although this research actually covers topics of highly practical relevance, the transmission of the findings to practical application is difficult. Consequently, further research is needed, that fosters this transmission by further translating the mathematical results into implementable incentive systems.

The second dimension concentrates on the *content*. The previous literature review shows that research on incentive systems which applies strategic game theory is broad and that it has begun to enrich its perspective by incorporating psychological and social aspects. Thereby it improves the practical relevance of its findings. Yet, while from a scientific perspective a broad range of analysed topics is positive, as it fosters scientific progress, it also may hinder the practical application of these results, as it is difficult for a practitioner to get an overview of the core findings. Consequently, a second research gap arises, which points to the need of further structuring the existing research findings and of the derivation of core findings to foster practical application.

The third dimension comprises the *empirical foundation*. The previous discussion exhibits that a broad literature exists that tries to test the empirical validity of assumptions and predictions of strategic game theoretical models. Here, different empirical research strategies (like experimental and survey research) are applied. Within the sample through many categories mixed evidence regarding the empirical validity of strategic game theoretical and agency theoretical reasoning was observed. This further impedes the practical application of the findings of strategic game theory, which leads to a third need for research, i.e. the empirical evidence of many findings of the mathematical models has to be further structured and the results of these analyses should be translated into recommendations for incentive systems in practice.

5 Conclusion

The present book chapter deals with the question whether the results of strategic game theory and agency theory have practical value and can inform the design of managerial incentive plans in practice. In order to answer this question a structured literature review was undertaken. Due to the limited number of key words and the application of only one data base, this review is not comprehensive. However, it covers a wide range of both mathematical and empirical research and thereby provides a broad overview of the existing literature allowing for a critical evaluation of the general applicability of the findings of this research stream to practice.

The analysis uncovered that scholars applying strategic game theory and agency theory in the context of managerial incentives actually try to provide practical valid insights. However, as this research is very specialized and fragmented and as the empirical validation provides rather mixed evidence, further research is needed that structures the existing results and translates them into core findings, which then can be picked up by practice to improve managerial incentive systems.

References

Alewell D, Nicklisch A (2009) Wage differentials and social comparison: an experimental study of interrelated ultimatum bargaining. Int Rev Law Econ 29(3):210–220
Alles M, Gupta M (2009) Non-production-related effort and the limits of contracting. Asia-Pac J Account Econ 16(2):145–170
Altmann S, Falk A, Wibral M (2012) Promotions and incentives: the case of multistage elimination tournaments. J Labor Econ 30(1):149–174
Amzaleg Y, Azar OH, Ben-Zion U, Rosenfeld A (2014) CEO control, corporate performance and pay-performance sensitivity. J Econ Behav Organ 106:166–174
Androkovich RA (1990) Relative risk aversion, incentive effects, and risk sharing. Atl Econ J 18(4):38–41
Baiman S, Rajan MV (1995) The informational advantages of discretionary bonus schemes. Account Rev 70(4):557–579
Baker GP (1992) Incentive contracts and performance measurement. J Polit Econ 100(3):598–614
Bakó B, Kálecz-Simon A (2013) Quota bonuses with heterogeneous agents. Econ Lett 119(3): 316–320
Banker RD, Lee S-Y, Potter G (1996) A field study of the impact of a performance-based incentive plan. J Account Econ 21(2):195–226
Benmelech E, Kandel E, Veronesi P (2010) Stock-based compensation and CEO (dis)incentives. Q J Econ 125(4):1769–1820
Benson A (2015) Do agents game their agents' behavior? Evidence from sales managers. J Labor Econ 33(4):863–890
Bertrand M, Mullainathan S (2000) Agents with and without principals. Am Econ Rev 90(2): 203–208
Bertrand M, Mullainathan S (2001) Are CEOs rewarded for luck? The ones without principals are. Q J Econ 116(3):901–932
Bol JC (2008) Subjectivity in compensation contracting. J Account Lit 27:1–24
Boly A (2011) On the incentive effects of monitoring: evidence from the lab and the field. Exp Econ 14(2):241–253
Bontems P, Bourgeon J-M (2000) Creating countervailing incentives through the choice of instruments. J. Public Econ. 76(2):181–202
Bremzen A, Khokhlova E, Suvorov A, van de Ven J (2015) Bad news: an experimental study on the informational effects of rewards. Rev Econ Stat 97(1):55–70
Brown PC, Miller JB, Thornton JR (1994) The ratchet effect and the coordination of production in the absence of rent extraction. Economica 61(241):93–114
Brüggen A, Moers F (2007) The role of financial incentives and social incentives in multi-task settings. J Manag Account Res 19:25–50
Bruttel L, Eisenkopf G (2012) No contract or unfair contract: what's better? J Socio-Econ 41(4):384–390

Bushman RM, Indjejikian RJ (1993) Accounting income, stock price, and managerial compensation. J Account Econ 16(1–3):3–23
Bushman, RM, Indjejikian RJ, Smith A (1995) Aggregate performance measures in business unit manager compensation: the role of intrafirm interdependencies. J Account Res 33(Supplement):101–128
Bushman RM, Indjejikian RJ, Smith A (1996) CEO compensation: the role of individual performance evaluation. J Account Econ 21(2):161–193
Byun DH, Kim J, Shin JY (2009) The effects of deferred compensation and performance-based compensation on firm value. Asia-Pac J Account Econ 16(1):49–67
Cao M, Wang R (2013) Optimal CEO compensation with search: theory and empirical evidence. J Financ 68(5):2001–2058
Carroll G, Meng D (2016) Locally robust contracts for moral hazard. J Math Econ 62(C):36–51
Chaigneau P (2013) Explaining the structure of CEO incentive pay with decreasing relative risk aversion. J Econ Bus 67:4–23
Chaigneau P (2015a) Risk aversion, prudence, and compensation. Eur J. Financ 21(15):1357–1373
Chaigneau P (2015b) Aversion to the variability of pay and the structure of executive compensation contracts. J Bus Econ Manag 16(4):712–732
Chambers RG, Quiggin J (2005) Incentives and standards in agency contracts. J Public Econ Theory 7(2):201–228
Chassang S (2013) Calibrated incentive contracts. Econometrica 81(5):1935–1971
Chen B (2012) All-or-nothing payments. J Math Econ 48(3):133–142
Chen Y-J (2013) Risk-incentives trade-off and outside options. OR Spectr 35(4):937–956
Chen JJ, Zhang H, Xiao X, Li W (2011) Financial crisis and executive remuneration in banking industry – an analysis of five british banks. Appl Financ Econ 21(23):1779–1791
Choe C, Yin X (2006) Should executive stock options be abandoned? Aust J Manag 31(2):163–179
Chouikhi O, Ramani SV (2004) Risk aversion and the efficiency wage contract. Labour 18(1): 53–73
Core JE, Guay WR, Verrecchia RE (2003) Price versus non-price performance measures in optimal CEO compensation contracts. Account Rev 78(4):957–981
Courty P, Marschke G (2004) An empirical investigation of gaming responses to explicit performance incentives. J Labor Econ 22(1):23–56
Cowen T, Glazer A (1996) More monitoring can induce less effort. J Econ Behav Organ 30(1):113–123
Cuevas-Rodríguez G, Gomez-Mejia LR, Wiseman RM (2012) Has agency theory run its course?: Making the theory more flexible to inform the management of reward systems. Corp Gov 20(6):526–546
Daljord Ø, Misra S, Nair HS (2016) Homogeneous contracts for heterogeneous agents: aligning sales force composition and compensation. J Mark Res 53(2):161–182
Dewatripont M, Jewitt I, Tirole J (2000) Multitask agency problems: focus and task clustering. Eur Econ Rev 44(4–6):869–877
Dittmann I, Maug E (2007) Lower salaries and no options? On the optimal structure of executive pay. J Financ 62(1):303–343
Dittmann I, Maug E, Spalt O (2010) Sticks or carrots? Optimal CEO compensation when managers are loss averse. J Financ 65(6):2015–2050
Dubois P, Vukina T (2009) Optimal incentives under moral hazard and heterogeneous agents: evidence from production contracts data. Int J Ind Organ 27(4):489–500.
Dugar S (2010) Nonmonetary sanctions and rewards in an experimental coordination game. J Econ Behav Organ 73(3):377–386
Dugar S (2013) Non-monetary incentives and opportunistic behavior: evidence from a laboratory public good game. Econ Inq 51(2):1374–1388
Dur R, Tichem J (2010) Social interaction, co-worker altruism, and incentives. Games Econ Behav 69(2):293–301
Dutcher EG, Balafoutas L, Lindner F, Ryvkin D, Sutter M (2015) Strive to be first or avoid being last: an experiment on relative performance incentives. Games Econ Behav 94:39–56

Dutta S (2003) Capital budgeting and managerial compensation: incentive and retention effects. Account Rev 78(1):71–93
Eisenhardt KM (1989) Agency theory: an assessment and review. Acad Manag Rev 14(1):57–74
Ekins WG, Brooks AM, Berns GS (2014) The neural correlates of contractual risk and penalty framing. J Risk Uncertain 49:125–140
Ellingsen T (1997) Efficiency wages and x-inefficiencies. Scand J Econ 99(4):581–596
Ellingsen T, Johannesson M (2007) Paying respect. J Econ Perspect 21(4):135–149
Ellingsen T, Johannesson M (2008) Pride and prejudice: the human side of incentive theory. Am Econ Rev 98(3):990–1008
Englmaier F, Wambach A (2010) Optimal incentive contracts under inequity aversion. Games Econ Behav 69(2):312–328
Englmaier F, Filipi A, Singh R (2010) Incentives, reputation and the allocation of authority. J Econ Behav Organ 76(2):413–427
Evans J, Weir C (1995) Decision processes, monitoring, incentives and large firm performance in the UK. Manag Decis 33(6):32–38
Flor CR, Frimor H, Munk C (2014) Options in compensation: promises and pitfalls. J Account Res 52(3):703–732
Fogarty T, Magnan ML, Markarian G, Bohdjalian S (2009) Inside agency: the rise and fall of nortel. J Bus Ethics 84(2):165–187
Fong EA, Tosi HL (2007) Effort, performance, and conscientiousness: an agency theory perspective. J Manag 33(2):161–179
Foss NJ, Laursen K (2005) Performance pay, delegation and multitasking under uncertainty and innovativeness: an empirical investigation. J Econ Behav Organ 58(2):246–276
Foster DP, Young HP (2010) Gaming performance fees by portfolio managers. Q J Econ 125(4):1435–1458
Gächter S, Nosenzo D, Sefton M (2012) The impact of social comparisons on reciprocity. Scand J Econ 114(4):1346–1367
Gao H (2010) Optimal compensation contracts when managers can hedge. J Financ Econ 97(2):218–238
Gayle G-L, Miller RA (2009) Has moral hazard become a more important factor in managerial compensation? Am Econ Rev 99(5):1740–1769
Gayle G-L, Miller RA (2015) Identifying and testing models of managerial compensation. Rev Econ. Stud 82(3):1074–1118
Gibbons R, Murphy KJ (1992) Optimal incentive contracts in the presence of career concerns: theory and evidence. J Polit Econ 100(3):468–505
Goering GE (1996): Managerial style and the strategic choice of executive incentives. Manag Decis Econ 17(1):71–82
Goktan AB (2014): Impact of green management on CEO compensation: interplay of the agency theory and institutional theory perspectives. J Bus Econ Manag 15(1):96–110
Grabke-Rundell A, Gomez-Mejia LR (2002) Power as a determinant of executive compensation. Hum Resour Manag Rev 12(1):3–23
Gray SR, Cannella AA (1997) The role of risk in executive compensation. J Manag 23(4):517–540
Grossman SJ, Hart OD (1983) An analysis of the principal-agent problem. Econometrica 51(1): 7–45
Gürtler O, Kräkel M (2010) Double-sided moral hazard, efficiency wages, and litigation. J Law Econ Org 26(2):337–364
Güth W, Klose W, Königstein M, Schwalbach J (1998): An experimental study of a dynamic principal-agent relationship. Manag Decis Econ 19(4/5):327–341
Harris M, Raviv A (1979) Optimal incentive contracts with imperfect information. J Econ Theory, 20(2):231–259
Hayes RM, Schaefer S (2000) Implicit contracts and the explanatory power of top executive compensation for future performance. RAND J Econ 31(2):273–293
Hemmer, T. (1993): Risk-free incentive contracts: eliminating agency cost using option-based compensation schemes. J Account Econ 16(4):447–473

Hemmer T, Kim O, Verrecchia RE (2000) Introducing convexity into optimal compensation contracts. J Account Econ 28(3):307–327
Hermalin BE, Katz ML (1991) Moral hazard and verifiability: the effects of renegotiation in agency. Econometrica 59(6):1735–1753
Herweg F, Müller D, Weinschenk P (2010) Binary payment schemes: moral hazard and loss aversion. Am Econ Rev 100(5):2451–2477
Holmström B (1979) Moral hazard and observability. Bell J Econ 10(1):74–91
Hoskisson RE, Castleton MW, Withers MC (2009) Complementarity in monitoring and bonding: more intense monitoring leads to higher executive compensation. Acad Manag Perspect 23(2):57–74
Immordino G, Menichini AMC, Romano MG (2015) Contracts with Wishful Thinkers. J Econ Manag Strateg 24(4):863–886
Janakiraman SN, Lambert RA, Larcker DF (1992) An empirical investigation of the relative performance evaluation hypothesis. J Account Res 30(1):53–69
Jensen MC, Murphy KJ (1990) Performance pay and top-management incentives. J Polit Econ 98(2):225–264
Jewitt I, Kadan O, Swinkels JM (2008) Moral hazard with bounded payments. J Econ Theory 143(1):59–82
Kaarbøe OM, Olsen TE (2008) Distorted performance measures and dynamic incentives. J Econ Manag Strateg 17(1):149–183
Kelly K, Tan PM-S (2010) The effects of profit-sharing contract and feedback on the sustainability of cooperation. J Manag Account Res 22:251–269
Keser C, Willinger M (2000) Principals' principles when agents' actions are hidden. Int J Ind Organ 18(1):163–185
Kim J (2015) Managerial Beliefs and Incentive Policies. J Econ Behav Organ 119:84–95
Kishore S, Rao RS, Narasimhan O, John G (2013) Bonuses versus commissions: a field study. J Mark Res 50(3):317–333
Kocabiyikoğlu A, Popescu I (2007) Managerial motivation dynamics and incentives. Manag Sci 53(5):834–848
Koch AK, Peyrache E (2011) Aligning ambition and incentives. J Law Econ Org 27(3):655–688
Kosnik RD, Bettenhausen KL (1992) Agency theory and the motivational effect of management compensation: an experimental contingency study. Group Org Manag 17(3):309–330
Kraft K, Niederprüm A (1999) Determinants of management compensation with risk-averse agents and dispersed ownership of the firm. J Econ Behav Organ 40(1):17–27
Kräkel M, Müller D (2015) Merger efficiency and managerial incentives. Int J Ind Organ 41:51–63
Kvaløy O, Olsen TE (2012) The rise of individual performance pay. J Econ Manag Strateg 21(2):493–518
Kwon YK (2005) Accounting conservatism and managerial incentives. Manag Sci 51(11): 1626–1632
Kwon I (2006) Optimal contracts for long-term decisions and the threat of dismissal. J Econ Manag Strateg 15(3):735–761
Lam S-S, Chng B-F (2006) Do executive stock option grants have value implications for firm performance? Rev Quant Finan Acc 26(3):249–274
Lambert RA (1983) Long-term contracts and moral hazard. Bell J Econ 14(2):441–452
Larraza-Kintana M, Wiseman RM, Gomez-Mejia LR, Welbourne TM (2007) Disentangling compensation and employment risks using the behavioral agency model. Strateg Manag J 28(10):1001–1019
Levinthal D (1988): A survey of agency models of organizations. J Econ Behav Organ 9(2):153–185
Maestri L (2014) The efficiency of bonus-or-terminate incentive schemes under subjective evaluations. Games Econ Behav 87:412–418
Makris M (2009) Incentives for motivated agents under an administrative constraint. J Econ Behav Organ 71(2):428–440

Meng D, Tian G (2013) Multi-task incentive contract and performance measurement with multidimensional types. Games Econ Behav 77(1):377–404
Meyer MA, Vickers J. (1997) Performance comparisons and dynamic incentives. J Polit Econ 105(3) 547–581
Miklós-Thal J, Ullrich H (2015) Belief precision and effort incentives in promotion contests. Econ J 125(589):1952–1963
Moav O, Neeman Z (2010) The quality of information and incentives for effort. J Ind Econ 58(3):642–660
Moers F (2006) Performance measure properties and delegation. Account Rev 81(4):897–924
Müller D, Weinschenk P (2015) Rater bias and incentive provision. J Econ Manag Strateg 24(4):833–862
Murphy KJ (1986) Incentives, learning, and compensation: a theoretical and empirical investigation of managerial labor contracts. RAND J Econ 17(1):59–76
Nafziger J, Schumacher H (2013) Information management and incentives. J Econ Manag Strateg 22(1):140–163
Narayanan MP, Seyhun HN (2008): The dating game: do managers designate option grant dates to increase their compensation? Rev Financ Stud 21(5):1907–1945
Nosenzo D (2013) Pay secrecy and effort provision. Econ Inq 51(3):1779–1794
Nyberg AJ, Fulmer IS, Gerhart B, Carpenter MA (2010) Agency theory revisited: CEO return and shareholder interest alignment. Acad Manag J 53(5):1029–1049
O'Connell V, O'Sullivan D (2014) The influence of lead indicator strength on the use of nonfinancial measures in performance management: evidence from CEO compensation schemes. Strateg Manag J 35(6):826–844
Opp MM, Zhu JY (2015) Impatience versus incentives. Econometrica 83(4):1601–1617
O'Reilly III CA, Main BGM (2010): Economic and psychological perspectives on CEO compensation: a review and synthesis. Ind Corp Chang 19(3):675–712
Oyer P (2004): Why do firms use incentives that have no incentive effects? J Financ 59(4):1619–1649
Pendleton A (2006) Incentives, monitoring, and employee stock ownership plans: new evidence and interpretations. Ind Relat 45(4):753–777
Pepper A, Gore J (2014) The economic psychology of incentives: an international study of top managers. J World Bus 49(3):350–361
Pepper A, Gore J (2015) Behavioral agency theory: new foundations for theorizing about executive compensation. J Manag 41(4):1045–1068
Pepper A, Gore J, Crossman A (2013) Are long-term incentive plans an effective and efficient way of motivating senior executives? Hum Resour Manag J 23(1):36–51
Pepper A, Gosling T, Gore J (2015) Fairness, envy, guilt and greed: building equity considerations into agency theory. Hum Relat 68(8):1291–1314
Prendergast C (2000) What trade-off of risk and incentives? Am Econ Rev 90(2):421–425
Radhakrishnan S, Ronen J (1999) Job challenge as a motivator in a principal-agent setting. Eur J Oper Res 115(1):138–157
Raith M (2008) Specific knowledge and performance measurement. RAND J Econ 39(4):1059–1079
Rigdon M (2009) Trust and reciprocity in incentive contracting. J Econ Behav Organ 70(1/2):93–105
Şabac F (2007) Dynamic agency with renegotiation and managerial tenure. Manag Sci 53(5):849–864
Santos-Pinto L (2008) Positive Self-image and incentives in organisations. Econ J 118(531):1315–1332
Scarpello V, Jones FF (1996) Why justice matters in compensation decision making. J Organ Behav 17(3):285–299
Schnedler W (2008) When is it foolish to reward for A while benefiting from B? J Labor Econ 26(4):595–619

Schöttner A, Thiele V (2010) Promotion tournaments and individual performance pay. J Econ Manag Strateg 19(3):699–731
Shin D, Strausz R (2014) Delegation and dynamic incentives. RAND J Econ 45(3):495–520
Stroh LK, Brett JM, Baumann JP, Reilly AH (1996) Agency theory and variable pay compensation strategies. Acad Manag J 39(3):751–767
Tosi HL, Katz JP, Gomez-Mejia LR (1997) Disaggregating the agency contract: the effects of monitoring, incentive alignment, and term in office on agent decision making. Acad Manag J 40(3):584–602
Umanath NS, Ray MR, Campbell TL (1993) The impact of perceived environmental uncertainty and perceived agent effectiveness on the composition of compensation contracts. Manag Sci 39(1):32–45
Umanath NS, Ray MR, Campbell TL (1996) The effect of uncertainty and information asymmetry on the structure of compensation contracts: a test of competing models. Manag Sci 42(6):868–874
Veldman J, Gaalman GJC (2015) Competitive investments in cost reducing process improvement: the role of managerial incentives and spillover learning. Int J Prod Econ 170:701–709
Verdier D, Woo B (2011) Why rewards are better than sanctions. Econ Polit 23(2):220–238
Von Thadden E-L, Zhao X (2012) Incentives for unaware agents. Rev Econ Stud 79(3):1151–1174
Wagenhofer A (1996) The value of distorting overhead cost allocations in an agency setting. Manag Account Res 7(4):367–385
Welbourne TM, Cyr LA (1999) Using ownership as an incentive. Group Org Manag 24(4):438–460
Welbourne TM, Balkin DB, Gomez-Mejia LR (1995) Gainsharing and mutual monitoring: a combined agency-organizational justice interpretation. Acad Manag J 38(3):881–899
Williams N (2015) A solvable continuous time dynamic principal-agent model. J Econ Theory 159:989–1015
Wu YW (2011) Optimal executive compensation: stock options or restricted stocks. Int Rev Econ Financ 20(4):633–644
Young GJ, Beckman H, Baker E (2012) Financial incentives, professional values and performance: a study of pay-for-performance in a professional organization. J Organ Behav 33(7):964–983
Zou L (1991): The target-incentive system vs. the price-incentive system under adverse selection and the ratchet effect. J Public Econ 46(1):51–89
Zou L (1997) Incentive roles of fringe benefits in compensation contracts. J Econ 65(2):181–199

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Optimal Design of Incentive Contracts: Behavioural and Multi-Period Performance Measurement Aspects

Christian Lukas

Abstract In this chapter I summarize and elaborate on some of the findings from the analytical literature on incentive contract design. After a short introduction to the standard agency model, behavioural extensions of the model are discussed. Both the analysis of loss aversion and identity utility can offer explanations for the popularity of bonus contracts or low-powered incentive schemes. Incentive contracts usually span multiple periods. For this reason the multi-period extension of the standard model is presented with focus on the frequency of performance evaluations. Depending on the specific long-term effect of effort, a high informativeness or low informativeness of performance signals leads to optimality of infrequent performance evaluations. The solution concept to incentive contract design problems is subgame-perfection. I present thoughts about possible problems associated with that concept and about available alternatives with an eye on incentive contract design.

Keywords Agency • Aggregation • Behavioural accounting • Frequency • Multi period • Single period

Contents

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1 Introduction

Research on the optimal design of incentive contracts has attracted much attention over the past three decades. Many empirical studies provided insights about the types of contracts used in business and their effectiveness in directing managerial actions; theoretical work helped to understand practice and to gain insight about how contracts should be optimally designed. For extensive overviews see, e.g., Prendergast (1999), Lambert (2001), Jensen et al. (2004). The process of designing an incentive contract can be thought of as a strategic game played between two contracting parties: the employer (principal) who offers the contract and the employee (agent) who decides about signing the contract and subsequently how to act under the contract. The strategic nature of the game manifests itself in the principal's notional foreclosure of the agent's behaviour when designing the contract. Hence the underlying idea is to choose *ex ante* a proper contract that induces a desirable behaviour. *Ex post* information then serves to determine the claims from the contract.

Employment contracts usually span multiple years. An obvious conclusion would be that game-theoretic models of contract design should span several periods as well. However, not every recurring incentive problem poses a new problem. For example, it is known from the literature that the optimal long-term contract for a twice repeated so-called static incentive problem is equivalent to two short-term contracts (Fellingham et al. 1985). This implies that insights gained from single-period analysis are useful for designing multi-period contracts.

The standard agency model of moral hazard has proven useful for analysing a variety of incentive problems. In recent years, behavioural aspects have been included in the analysis. This has helped, for example, to identify reasons for the popularity of bonus contracts, or why and when low-powered incentives can be effective in controlling employee behaviour. A bonus contract is a simple contract that consists of a fixed payment and a bonus which is contingent on achieving a predetermined performance level. Performance improvements above the threshold level do not lead to higher compensation (just as a further reduction below that level does not lead to lower compensation). Stated differently, available performance information is to some extent *not* used for contracting purposes. Loss aversion, which has been shown to be a characteristic of many individuals' preferences can account for this, at least in part. Even if a bonus contract is optimal the strength of the incentive, i.e., the bonus amount, needs to be determined. The higher the amount the higher is the incentive to exert more effort and work with more dedication. Yet if employees identify with the firm low-powered incentives may suffice.

As argued before the insights obtained from the analysis of single-period incentive problems are of course relevant for designing multi-period contracts. However, two new, distinct questions emerge: (1) how often should an employee's performance be measured, and (2) should, and if so, how should performance-contingent compensation in a given period depend on performance in earlier periods? Concerning (1), the frequency of performance evaluations determines the nature of the game. If performance is measured only once at the end of a multi-period contract, the incentive problem collapses to a single-period problem with simultaneous effort choices. However, if performance is measured during the term of contract the incentive problem becomes truly dynamic and (2) gains relevance. The agent takes into account interim performance information when making effort decisions. In addition, both principal and agent may wish to renegotiate a contract during its duration. Then both agent and principal repeatedly decide and possibly alternate in their decisions. Such a game structure is of course different from the one of the single-period problem—and so is the effect of additional information. We know from the seminal work of Holmström (1979) that more (costless) information, i.e., more informative signals, always increase the value of the single-period agency. In multi-period incentive problems, more information has two effects: First, it allows the principal to fine-tune incentives; second, it allows the agent to make effort decisions contingent on prior outcomes and this implies more room for opportunistic behaviour. Therefore, whether more or less performance evaluations are optimal within the term of contract is subject to a trade-off.

The solution concept usually applied in contract design problems—both in single-period problems and in multi-period problems—is the subgame-perfect equilibrium for sequential games. Determining the equilibrium can become very challenging. Possible reasons include limited information-processing capabilities or unavailability of information. For practical purposes the subgame-perfect equilibrium may not be relevant despite its theoretical appeal as a benchmark. It raises the question about alternatives; and more importantly, how would the equilibrium predictions based on alternative concepts differ from the subgame-perfect equilibrium? Can they help us to better understand what we observe in business or will recommendations for contract design change?

In this contribution I pursue two objectives. First, I want to summarize and elaborate on some of the findings from the literature on incentive contract design. Second, on the basis of that analysis I direct attention to an issue—the equilibrium concept—which I believe should receive or maintain a top position on the research agenda in management accounting. Given the vast literature on incentive contract design I had to select rigorously what to include. The selection is subjective and therefore excludes subdomains that are both relevant and topical. However, inclusion of more subdomains would have come at the cost of less depth. I opted for depth. Therefore, in what follows I will discuss the above issues in detail. Section 2 presents the standard single-period agency model and two behavioural extensions of it. The following Sect. 3 deals with contract design for multi-period incentive problems. This includes the analysis of performance evaluation frequency in a static environment with independent periods and in a dynamic environment with interdependent periods. In addition, consequences of contract renegotiations are sketched.[1] Since many models of incentive contract design apply the subgame-perfect equilibrium concept I feel that it can be helpful to present thoughts about

[1]Formal proofs of results are not presented. I refer to the original papers instead.

possible problems associated with that concept and about available alternatives in Sect. 4; this of course with an eye on incentive contract design.

2 Single-Period Incentive Contracts

2.1 The Standard Agency Model

In the standard agency model the principal (she) is the owner of a production technology. She hires a risk-averse agent (he) to perform a task on her behalf. The agent incurs private effort costs to perform the task. If effort is not observable the principal cannot directly compensate the agent for his effort. Instead she offers outcome-contingent compensation specified in a contract. The contract is designed to achieve two objectives: (i) the contract should be sufficiently advantageous for the agent so that he actually accepts it; and (ii) the contract should induce the agent to select the effort that maximizes the principal's expected surplus net of compensation costs. The contract design problem can be studied as a sequential game (see Fig. 1).

In the game, the principal moves first by offering a contract $s(x)$ that defines a certain compensation s dependent on the outcome $x, x \in \{x_1, x_2, \ldots, x_n\}$ with $x_1 < x_2 < \ldots < x_n$. The agent then decides to either accept the contract or to reject the contract. In the latter case, the game ends. If the agent accepts the contract, he selects effort $e, e \in \{e_1, e_2, \ldots, e_m\}$ with $e_1 < e_2 < \ldots < e_m$ at cost $c \cdot e, c > 0$. The dotted line in Fig. 1 indicates the multitude of effort choices. The probability that effort e_j leads to outcome x_i is $p_i(e_j)$ such that $\sum_{i=1}^{n} p_i(e_j) = 1 \forall j$. Contingent on the actual outcome x_i payoffs for agent $(s(x_i))$ and principal $(x_i - s(x_i))$ are realised. The optimal contract represents a subgame-perfect Nash equilibrium of the contract design game. It is standard to apply backward induction to determine the equilibrium. Following Grossman and Hart (1983) it is convenient to divide the contract design problem into two parts. In the first part, the contracts that induce every possible effort $\hat{e}$ at minimum costs are identified. Then, in the second part, the optimal effort e^*—the effort that maximizes the principal's expected net payoff—is selected from the set of effort choices. The corresponding contract is optimal.

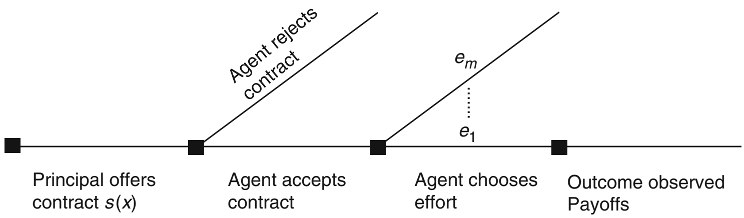

Fig. 1 Contract design problem as a sequential game. Own representation

Let $u_i = u(s(x_i))$ denote the agent's utility from receiving a payment $s(x_i)$ with $u' > 0, u'' \leq 0$; and define $h(u) = u^{-1}$. The principal's decision problem in the first part is

$$min_{u_i} \sum_{i=1}^{n} p_i(\hat{e}) \cdot h(u_i) \tag{1}$$

subject to

$$\sum_{i=1}^{n} p_i(\hat{e})u_i - c\hat{e} \geq \underline{U} \tag{2}$$

$$\sum_{i=1}^{n} p_i(\hat{e})u_i - c\hat{e} \geq \sum_{i=1}^{n} p_i(e_j)u_i - ce_j \ \forall e_j \tag{3}$$

where $\underline{U}$ represents the agent's reservation utility. Constraints (2) and (3) formalize the requirements for an optimal contract expressed in (i) and (ii) above. The cost minimizing contract to induce $\hat{e}$ is characterised by the following Kuhn-Tucker conditions:

$$h'(u_i) = \lambda + \sum_{e_j \neq \hat{e}} \mu_j \left(1 - \frac{p_i(e_j)}{p_i(\hat{e})}\right) \ \forall i \tag{4}$$

where λ and μ are Lagrange multipliers for the participation constraint (2) and the relevant incentive compatibility constraint (3) for effort e_j. Let $C(e_j) = \sum_{i=1}^{n} p_i(e^j) \cdot h(u_i)$ denote minimum expected compensation cost to induce effort e_j. Then the principal's problem in the second part is to select $e^* = argmax_{e_j} \sum_{i=1}^{n} p_i(e^j) \cdot x_i - C(e_j)$. The optimal contract is characterised by $C(e^*)$.

As (4) shows the level of pay for a particular outcome depends on the likelihood ratios $\frac{p_i(e_j)}{p_i(\hat{e})}$. The more likely it is that a particular outcome results from the effort $\hat{e}$ to be induced, the higher is the payment for that outcome. Under which condition(s) will the optimal contract be non-decreasing in the outcome? Grossman and Hart (1983) prove that two conditions—the monotone likelihood ratio property (MLRP) and the convexity of the distribution function property (CDFC)—are necessary and sufficient for a contract that is non-decreasing in outcome. MLRP means that higher outcomes become more likely with higher effort; CDFC requires that higher effort increases the likelihood for high outcomes at a decreasing rate (Rogerson 1985).

A non-decreasing optimal contract in general uses much of the available information: different outcomes lead to different pay levels. In contrast, a fixed wage contract is not incentive compatible. However, empirical studies show the popularity of bonus contracts (that make use of available performance information to a limited extent) and fixed wage contracts. Research in behavioural agency theory can offer possible explanations. For a review of behavioural contract theory see Köszegi (2014). The two extensions to the standard model presented next introduce changes in the utility from compensation (Sect. 2.2) and in the effort cost function (Sect. 2.3).

2.2 Contract Design and Loss Aversion

Loss aversion is the term often used to describe the phenomenon of a reference point dependent utility (Kahneman and Tversky 1979). Let r be a reference point, e.g., the compensation an employee expects to receive (or to be worth). If actual compensation s exceeds r by Δ, this positive deviation from the reference point is valued less than a negative deviation by the same amount. Simply said, the utility function is steeper for $s < r$ than for $s > r$. A possible utility function with that property is:

$$u(s \mid r) = \begin{cases} u(s) + \eta(u(s) - u(r)) & \text{if } s \geq r \\ u(s) + \eta\lambda(u(s) - u(r)) & \text{if } s < r \end{cases} \tag{5}$$

with $\eta > 0$ and $\lambda > 1$ (cf. e.g. Köszegi 2014, p. 1078; Köszegi and Rabin 2006, 2007). Utility consists of the conventional utility derived from compensation, $u(s)$, and the sensation that a gain (or loss) relative to the reference point has. The parameter λ captures the degree of loss aversion, and η is the weight that the gain-loss utility possesses relative to the conventional utility.

In contract design problems, different actions generate different outcome distributions which in turn influence compensation. This raises the question what a proper reference point can be. Köszegi and Rabin (2006, 2007) allow for a reference point to be stochastic. For a particular outcome, all other possible outcomes are weighted against this outcome taking into account how likely the other outcomes are. Then using (5) the agent's utility for a particular outcome x_i is:[2]

$$u_i + \sum_{\{\kappa \mid \kappa < i\}} p_\kappa(e)\eta(u_i - u_\kappa) + \sum_{\{\kappa \mid i \geq \kappa\}} \eta\lambda(u_i - u_\kappa) - ce \tag{6}$$

The agent's expected utility from effort e is then given by

$$EU(e) = \sum_{i=1}^{n} p_i(e) \cdot u_i - \eta(\lambda - 1) \sum_{i=1}^{n} \sum_{\{\kappa \mid u_\kappa > u_i\}} p_\kappa(e) p_i(e)(u_\kappa - u_i) - ce \tag{7}$$

The outcome distribution generated by effort e represents the reference point as a distribution. To simplify the analysis of the impact of loss aversion on the optimal contract the set of effort levels consists of two levels, $e \in \{0, 1\}$. It is assumed that

[2]The exposition emanates from a non-decreasing compensation function.

parameters are such that inducing $e = 1$ is optimal for the principal. This reduces the two-part problem from Sect. 2.1 to just one part—finding the compensation contract that implements $e = 1$ at minimum expected costs. The principal's program is

$$min_{u_i} \sum_{i=1}^{n} p_i(e = 1) \cdot h(u_i) \tag{8}$$

subject to

$$EU(e = 1) \geq \underline{U} \tag{9}$$

$$EU(e = 1) \geq EU(e = 0) \tag{10}$$

Inspection of (7) shows that reducing the variation in compensation—induced by utility differences $(u_i - u_\kappa)$—relaxes the participation constraint (9): The second term in (7) becomes zero if $u_\kappa = u_i \forall \kappa, i$. Writing (10) explicitly and arranging terms properly gives

$$\sum_{i=2}^{n} [p_i(e = 1) - p_i(e = 0)] \cdot (u_i - u_1)$$
$$- \eta(\lambda - 1) \sum_{i=1}^{n} \sum_{\{\kappa | u_\kappa > u_i\}} [p_\kappa(e = 1) p_i(e = 1) - p_\kappa(e = 0) p_i(e = 0)](u_\kappa - u_i)$$
$$\geq c. \tag{11}$$

Lowering variation in compensation reduces the second term on the left-hand side of (11) which relaxes the incentive constraint. However, some variation is necessary to establish incentive compatibility. The optimal contract minimizes variation in payments while maintaining incentive compatibility; this can be achieved by a bonus contract, i.e. $u_1 = \ldots = u_\tau = 0 < u_{\tau+1} = \ldots = u_n$ for some τ.[3]

Introducing loss aversion into the standard agency model does not change the basic structure of the contract design game but the change in preferences has a pronounced impact on the subgame-perfect equilibrium prediction. Though the information space is possibly rich a threshold level of outcome is defined so that outcomes below (above) that level are rewarded equally leading to only two different payments.

[3] See Herweg et al. (2010) for detailed analysis and proof of the result, and Köszegi (2014, pp. 1078–1079) for an example based on Herweg et al. (2010).

2.3 Contract Design and Identity

From time to time survey results about how much employees identify with their companies can be found in the news.[4] Identification with the company means that employees share—at least to some extent—values and goals of the company and basically make company goals their own goals. Consequently, identification may mitigate or even erase the conflict of interest (as it is represented in the standard agency model). This section demonstrates how (non-)identification with the company affects the optimal incentive contract. The analysis uses the same model setup as the previous section.

Following Akerlof and Kranton (2005), let there be two categories of employees in a company: (1) employees may engage with or identify with the company, these employees are called *insider*; (2) employees who do not identify with the company are called *outsider*. Belonging to either group generates an identity utility I_k, $k \in \{N, O\}$, where N indicates insider and O outsider. In addition, there is an identity effect on the cost of effort function: if an employee deviates from the "standard effort" for his category it increases personal costs of effort. The term $t_k|e_k^* - e|$ with $t_k > 0 \forall k$ captures these additional costs. Standards are $e_N^* = 1$ and $e_O^* = 0$; the standard for the insider is high effort, and it is low effort for the outsider. The agent's expected utility from choosing effort e is $EU(e;k) = \sum_{i=1}^{n} p_i(e) \cdot u_{i,k} + I_k - t_k|e_k^* - e| - ce$, where $u_{i,k}$ denotes utility from compensation paid for outcome x_i to an agent with identity k. Then the principal's program is

$$min_{u_i} \sum_{i=1}^{n} p_i(e=1) \cdot h(u_i) \tag{12}$$

subject to

$$EU(e=1;k) \geq \underline{U} \; \forall k \tag{13}$$

$$EU(e=1;k) \geq EU(e=0,k) \; \forall k \tag{14}$$

Solving program (12) leads to optimal contracts for the insider and the outsider, respectively. It is useful to write the incentive compatibility constraints (14) explicitly in rearranged form:

$$\sum_{i=2}^{n} (p_i(e=1) - p_i(e=0)) \cdot (u_{i,N} - u_{1,N}) \geq c - t_N \tag{15}$$

$$\sum_{i=2}^{n} (p_i(e=1) - p_i(e=0)) \cdot (u_{i,O} - u_{1,O}) \geq c + t_O \tag{16}$$

[4]So called engagement reports are available, for example, from consulting firms and they try to measure emotional ties between employees and their company.

It is evident that the incentive compatibility constraint for the insider, (15), is less stringent than the one for the outsider, (16). This means that the utility spreads $(u_{i,N} - u_{1,N})$ necessary for the insider to induce high effort are lower than those for the outsider, $(u_{i,O} - u_{1,O})$. For any $t_N \geq c$, a flat wage for the insider would be optimal. And as long as $t_N \geq t_O$, expected compensation costs to induce the insider to choose high effort are lower than those for the outsider to induce high effort. To see this note that the outsider contract creates slack in the incentive constraint of the insider; and with $t_N \geq t_O$ the participation constraint of the insider is also slack. Hence, a better (lower cost) contract for the insider is feasible.

The analysis assumes a binary effort choice for the agent. If the set of effort choices is richer (as in Sect. 2.1), it is not clear whether the insider contract shows less variation in payments than the outsider contract. From the above analysis one can only conclude that any *given* effort can be induced at lower costs for the insider compared to the outsider. This leaves open the question which effort level should be optimally induced for each category. Here the division of the principal's program into two parts in Sect. 2.1 reappears. If identity utility of the insider allows for lower incentives to implement a given effort, then implementing a higher effort may be optimal. This can lead to *higher* incentives, i.e. more variation in compensation, for the insider compared to the outsider (Akerlof and Kranton 2005).[5]

3 Multi-Period Incentive Contracts

3.1 Stationary Production Technology

In business life many labor contracts span multiple years or have unlimited term. It raises the question whether and which insights from the analysis of single-period problems carry over to multi-period problems. The simplest case of a multi-period incentive problem is the repetition of the standard agency. Assume a two-period agency where each period features an incentive problem as in Sect. 2.1 without any interdependencies between periods. Then the production technology is stationary. If, in addition, the agent's utility function does not show time preferences and wealth effects it is a well-known result that the optimal multi-period (or long-term) contract is the repetition of the optimal single-period contract (Amershi et al. 1985; Fellingham et al. 1985). The contracting game as depicted in Fig. 1 repeats itself.

The contracting game changes if the principal has the option to withhold performance information from the agent after the first period or to delay performance evaluation until the end of the second period. The principal may offer a contract $s(x_1 + x_2)$ that rewards aggregate performance over two periods; there is no

[5] This may also be optimal in multi-task problems. See Heinle et al. (2012) which also includes a discussion of possible implications of identity-based incentives for performance measurement in firms.

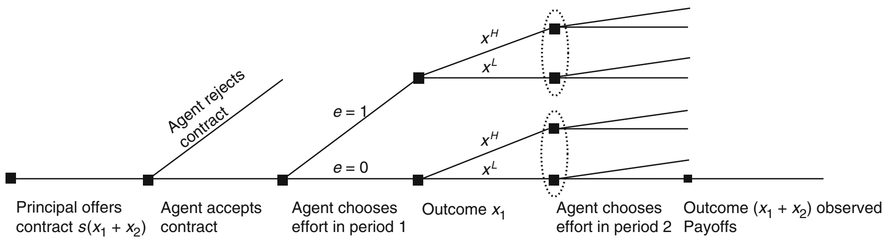

Fig. 2 Timeline of two-period contracting game. Own representation

performance information available after the first period. An alternative would be to offer a contract $s(x_1, x_2)$ and measure performance after each period so that compensation can be tied to the sequence of outcomes. Stated differently, the principal can choose between frequent performance evaluation (FPE) and offer $s(x_1, x_2)$, or infrequent performance evaluation (IPE) and offer $s(x_1 + x_2)$.[6] Figure 2 exemplifies the change in the game. Given $s(x_1 + x_2)$ the agent is unsure about the outcome in the first period when he decides about second-period effort. His information set (represented by the dotted ellipse) is no longer a singleton as it is with observation of first-period outcome under $s(x_1, x_2)$.

The principal's objective is to determine the contract that induces the agent to choose high effort in both periods at minimum expected costs. For $i, j \in \{L, H\}$ let

$$EU(e_1, e_2)) = \sum_{ij} P(x_1 = x^i \mid e_1) P(x_2 = x^j \mid e_1, e_2(x_1)) \cdot u(s^{ij}) - c(e_1) - c(e_2)$$

denote the agent's expected utility under FPE, and let

$$EU(e_1, e_2) = \sum_{ij} P(x_1 = x^i \mid e_1) P(x_2 = x^j \mid e_1, e_2) \cdot u(s^{i+j}) - c(e_1) - c(e_2)$$

represent expected utility under IPE for the agent. To determine optimal payments under FPE the principal solves

$$min_{u^{ij}} \sum_{ij} P(x_1 = x^i \mid e_1 = 1) P(x_2 = x^j \mid e_2 = 1) \cdot h(u^{ij}) \quad (17)$$

subject

$$EU(e_1 = e_2 = 1) \geq \underline{U} \quad (18)$$

$$EU(e_1 = e_2 = 1) \geq EU(e_1 = m, e_2(x_1) = n), \forall m, n \in \{0, 1\}, x_1 \in \{x^L, x^H\} \quad (19)$$

[6] Labels are adopted from Arya et al. (2004).

where (18) represents the participation constraint and (19) represents the incentive compatibility constraint. The program under IPE is

$$min_{u^{i+j}} \sum_{ij}^{n} P(x_1 = x^i \mid e_1 = 1)P(x_2 = x^j \mid e_2 = 1) \cdot h(u^{i+j}) \tag{20}$$

subject to

$$EU(e_1 = e_2 = 1) \geq \underline{U} \tag{21}$$

$$EU(e_1 = e_2 = 1) \geq EU(e_1 = m, e_2 = n), \forall m, n \in 0, 1 \tag{22}$$

where (21) and (22) are participation constraint and incentive compatibility constraint, respectively.

The important difference between program (17) and program (20) is—as illustrated by Fig. 2—that incentive compatibility constraints given FPE condition second-period effort on first-period outcome whereas given IPE they do not. The principal can capitalize on the agent's uncertainty about first-period outcome. This will be shown by means of a numerical example.

Example 1 Assume a model setup as in Sect. 2.1 with the simplification of a binary outcome distribution in each period, $x_t \in \{x^L, x^H\}, x^L < x^H, t = 1, 2$, and a binary effort choice $e_t \in \{0, 1\}, t = 1, 2$. The agent's utility is represented by an exponential function $u(s, e) = -e^{-r(s-c(e_1)-c(e_2))}$. Payments for outcome sequences $\{x_1, x_2\}$ are denoted s^{ij}, payments for aggregate outcomes $(x_1 + x_2)$ are denoted $s^{i+j}, i, j \in \{L, H\}$ and $E[s]$ indicates expected compensation costs.

Parameters are as follows:

$P(x_t = x^H \mid e_t = 1) = 0.8, P(x_t = x^H \mid e_t = 0) = 0.35, t = 1, 2;, r = 0.0001; c(e_t = 1) = 10.000; c(e_t = 0) = 5.000, \underline{U} = -0.13535.$[7]

Outcome contingent payments and expected compensation costs both for FPE and IPE are listed in Table 1.[8]

The contract under frequent performance evaluation (FPE) shows only three distinct payment although there are four outcome sequences. It reflects the property of the optimal contract $s(x_1, x_2)$ being the repetition (or sum) of two optimal single-period contracts. In a single-period incentive problem a difference $s^H - s^L = 11.071$ between payments for the high outcome and for the low outcome would be required to ensure incentive compatibility of the contract. Note that $s^{HL} - s^{LL} = s^{HH} - s^{HL} = s^{LH} - s^{LL} = 11.071$; differences in outcome-contingent payments in the two-period contract und FPE are the same in each period. Accordingly, the reward for the

[7]For the examples in this section it is convenient to directly set effort costs for each effort level; results can be carved out clearer. Maintaining the multiplicative effort cost function from Sect. 2.1 would require resizing of effort levels.

[8]It is straightforward to implement programs (17) and (20) and solve the example with the help of a spreadsheet.

Table 1 Optimal payments and expected compensation costs $E[s]$ by contract

Frequent evaluation $s(x_1, x_2)$	Infrequent evaluation $s(x_1 + x_2)$
$s^{LL} = 24.660$	$s^{L+L} = 21.576$
$s^{HL} = 35.731$	$s^{H+L} = 38.922$
$s^{LH} = 35.731$	
$s^{HH} = 46.802$	$s^{H+H} = 44.923$
$E[s]$=42.374	$E[s]$=42.070

high outcome in the second period does not depend on the previous outcome—the contract is memoryless (Amershi et al. 1985; Fellingham et al. 1985).[9] Given $s^{HL} = s^{LH}$ in contract $s(x_1, x_2)$ under FPE, the contract is also feasible under IPE. However, it is not optimal as the comparison of expected compensation costs in Table 1 shows; expected compensation costs given IPE are lower than those given FPE. The result holds in general.

Proposition 3.1 *(Arya et al. 2004) Given a stationary production technology and binary outcome distribution. The principal prefers infrequent performance evaluation over frequent performance evaluation.*

Proof Arya et al. (2004, p. 649–650).

Since the agent does not know first-period outcome when selecting effort for the second period the principal can decrease the premium for another high outcome in the second period, $s^{H+H} - s^{L+H} = 6.001 < 11.071$ compared with the one for achieving the high outcome only once, $s^{H+L} - s^{L+L} = 17.346$; Demski (1998) calls it decreasing returns to good news. The feasibility of the contract structure critically depends on the information rationing effect of intertemporal aggregation of performance measures.

3.2 Non-Stationary Production Technology

In the model from the previous section (intertemporal) aggregation is costless (Nikias et al. 2005) because the aggregate outcome $(x_1 + x_2)$ is a sufficient statistic for the sequence of outcomes, $\{x_1, x_2\}$. As soon as interdependencies between periods arise aggregation in general comes at the cost of less information. Interdependencies may be caused by correlation in random effects on outcomes or by tasks that have long-term effects. The latter interdependency can be interpreted as a link between production (or value creation) and performance measurement.[10] Tasks

[9] The memoryless-property disappears if the agent has a time preference. Cf. Lambert (1983).

[10] A link that is often missing in agency models because they focus on pure information content of signals. Cf. van der Stede (2015, p. 174).

entail lasting effects even if incentives are removed later on.[11] Task characteristics and their distinct effects on performance measures influence the optimal frequency of performance evaluations, or, in terms used in the previous section, whether FPE or IPE is efficient.

To analyse effects of interactive effort (or tasks) the model setup from the previous Sect. 3.1 is modified in the following way (Lukas 2010):

$$P(x_1 = x^H) = p_{e_1} \tag{23}$$

$$P(x_2 = x^H) = p_{e_1}^{\kappa} \cdot p_{e_2}, \tag{24}$$

with $e_t \in \{0, 1\}$, $p_{e_t=1} > p_{e_t=0}$ and $p_{e_t} > 0$ for $t = 1, 2$, and $0 \leq \kappa \leq 1$. According to (23) and (24) higher effort leads to higher chances to succeed in each period. The long-term effect of first-period effort is captured by κ. For $\kappa = 0$, there is no long-term effect and the model is identical to the one in Arya et al. (2004) introduced in the previous section. For any $\kappa > 0$, an interdependency between periods exists, and it becomes stronger with κ increasing. A possible interpretation of the probability structure determined by (23) and (24) is skill acquisition by the agent. By providing high effort in the first period the agent prepares himself for the task in the second period.

The principal's programs (17) and (20) to determine optimal payments under FPE and IPE change accordingly. Optimality of FPE or IPE, respectively, depends on both the strength of the interdependency κ and the informativeness of the first-period outcome measured by $\iota = (p_{e_1=1} - p_{e_1=0})$.

Proposition 3.2 *(Lukas 2010) Given a non-stationary production technology with complementary effort and binary outcome distribution. The principal prefers infrequent performance evaluation over frequent performance evaluation only if the informativeness of first-period outcome $\iota = (p_{e_1=1} - p_{e_1=0})$ is sufficiently low.*

Proof Lukas (2010, p. 171ff)

Intuitively, if both κ and ι are high, there is distinct long-term benefit of higher effort in the first-period. Due to that long-term benefit, incentives offered for the second-period are very effective in motivating the agent in the first period. This allows the principal to reduce first-period incentives in turn. However, this incentive structure requires a differentiation between outcome sequences $\{x^L, x^H\}$ and $\{x^H, x^L\}$, a differentiation that is not possible under IPE. In contrast, if ι is low the (long-term) effect of first-period effort on outcomes is almost the same so that it requires fairly high incentives in the first period under FPE. The resulting contract under FPE entails high variation in payments. IPE generates lower variation in payments as it allows for a spillover of first-period incentives into the second period. Hence, for low levels of informativeness of first-period outcome, IPE is optimal.

[11] See, for example, Lukas (2007) or Campbell (2008). Interactive effects between tasks influence performance measurement also in single-period problems. Cf. Dikolli et al. (2009).

Besides complementary tasks there may be cases where a substitution effect occurs, for example, if more effort in a given period reduces the probability to succeed in the next period due to exhaustion or disenchantment. To model this effect the following probability structure is assumed (Lukas 2015).

$$P(x_1 = x^H) = p_{e_1} \tag{25}$$

$$P(x_2 = x^H) = (1 - p_{e_1}) \cdot p_{e_2}, \tag{26}$$

with $e_t \in \{0, 1\}$, $p_{e_t=1} > p_{e_t=0}$ and $p_t > 0$ for $t = 1, 2$. Note that $P(x_2 = x^H)$ is decreasing in first-period effort e_1. Again, the principal's programs (17) and (20) to determine optimal payments under FPE and IPE change accordingly. Whether FPE or IPE is optimal depends on both (i) the informativeness of outcomes and (ii) the strength of the substitution effect. Inspection of (25) and (26) shows that (i) and (ii) to some extent go hand in hand. A high informativeness of the first-period outcome implies a strong substitution effect. In this case, (i) could tip the scale towards FPE, because under FPE the informative outcome in the first period is observed and becomes part of the contract; however, (ii) would call for IPE, because a strong substitution effect would require fairly high incentives in the second period under FPE causing costly variation in payments, a variation that can be reduced via IPE. The tradeoff between observing informative outcomes, variation in payments due to the strong substitution effect and risk sharing cannot be easily solved. Example 2 illustrates the point.

Example 2 Assume again an exponential utility function for the agent with $r = 0.0001; c(e_t = 1) = 5.500; c(e_t = 0) = 5.000$, and $\underline{U} = -0.13535$. Table 2 presents parameters for three different scenarios and the expected compensation costs $E[s(\cdot)]$ for each performance evaluation regime.

Parameters for cases 1–3 in Table 2 are chosen such that infrequent evaluation is efficient in the absence of a substitution effect. To analyse the impact of the substitution effect, consider case 1 in Table 2. The outcome in the first period is informative but given $p_{e_1=1} = 0.45$ the substitution effect is not too strong. Yet infrequent performance evaluation is efficient. In case 2, the marginal effect of first-period effort on first-period outcome is kept constant but the substitution effect is stronger than in case 1; infrequent performance evaluation remains efficient. Moving from case 2 to case 3 both the informativeness of first-period outcome and the substitution effect become weaker. Here frequent performance evaluation

Table 2 Expected compensation costs $E[s]$ by contract

Case	Parameters				Expected compensation costs	
	$p_{e_1=1}$	$p_{e_1=0}$	$p_{e_2=1}$	$p_{e_2=0}$	$E[s(x_1, x_2)]$	$E[s(x_1 + x_2)]$
1	0.45	0.10	0.45	0.10	31.172	31.168
2	0.65	0.30	0.65	0.30	31.414	31.399
3	0.50	0.30	0.50	0.30	31.688	31.740

is efficient. Apparently, the interaction of informativeness and substitution effect in this case is dominated by informativeness. Cases 1–3 do not suggest a clear-cut pattern of effects from a change in informativeness and/or the substitution effect on the optimality of (in)frequent evaluation.

The indeterminacy of a solution in the setting with substitution effect and risk aversion is somewhat unsatisfactory. Absent any risk considerations in an otherwise identical setting with risk neutrality and the agent being protected by limited liability a clear result obtains: a high informativeness of first-period outcome leads to optimality of IPE (Lukas 2015). High informativeness implies a strong substitution effect, requiring high incentives in the second period unter FPE. Again, IPE lowers variation in payments as second-period incentives spill back into the first period. The result contrasts the finding for complementary effort where *low* informativeness of first-period outcomes entails optimality of IPE. For both settings, however, results are sensitive to the specific probability structure. Nikias et al. (2005) employ a similar probability structure that differs from the one in this chapter only by the effect of low effort in the first period. While in this chapter both high and low first-period effort affect the probability for the high outcome in the second period, in Nikias et al. (2005) only high effort has an effect on the second period. Nikias et al. (2005) find that IPE is weakly preferred for complementary effort but Proposition 3.2 in this section shows that IPE is preferred only if the first-period outcome is sufficiently uninformative. Similarly, Nikias et al. (2005) find a weak substitution effect leads to optimality of IPE, it can remain optimal even for a strong substitution effect in Lukas (2015).

The purpose of explicating the differences in probability structures and results is it to point out that robustness of results cannot be taken for granted. While analysing effects of effort interdependency can generate interesting and important insights into optimal long-term contract design care should be taken to fit the probability structure to the setting the researcher has in mind. This very likely applies to other contracting problems as well but the analysis in this chapter suggests that it is of special importance in settings with effort interdependency.

3.3 Renegotiation of Multi-Period Contracts

Multi-period contracts determine outcome-contingent compensation for a possibly large number of periods in advance. When these periods approach, there may be the desire for a contract change to the benefit of both contracting parties. A possible reason could be that compensation payments initially agreed upon impose too much risk on the agent which is costly to the principal. To achieve a change in the contract a renegotiation of the initial contract takes place.

In the two-period model with effort complementarity from Sect. 3.2 first-period effort entails a long-term benefit. For this reason the optimal multi-period contract shifts incentives into the second period. The agent does not receive an immediate reward for a high outcome in the first-period but the prospects of high payments in the second period motivate the agent. Technically, the shift of incentives is accomplished by creating slack in second-period incentive constraints (or sequential rationality constraints) as given by (19), which read as follows after rearranging terms:

$$s^{iH} - s^{iL} \geq \frac{c(e_2 = 1) - c(e_2 = 0)}{p^{\kappa}_{e_1}(p_{e_2=1} - p_{e_2=0})}, \ i = \{L, H\} \tag{27}$$

Whenever contracting parties can commit to the two-period contract, the relation in (27) can be strict. If, in contrast, the parties cannot commit they renegotiate the contract at the end of the first period so that (27) holds as equality. The latter is the (correct) prediction for the contract valid in the second period whenever contracting parties foresee that a renegotiation will take place. However, the argument is not as straightforward as it appears. At the point of renegotiation the principal does not know the agent's first-period effort choice. It implies the principal's information set is not a singleton as the start of a subgame would require. Therefore the principal's *beliefs* about the agent's first-period effort are used to determine the optimal contract by backward induction. The Bayesian equilibrium requires that beliefs are correct. From the renegotiation-proof principle (Fudenberg and Tirole 1990), if actual renegotiations are expected, the contract design can be restricted to those contracts that are sequentially rational. The so-called renegotiation-proof contract does not leave room for mutually beneficial contract changes along the time path.

From a technical perspective, renegotiation enters the principal's program as an additional constraint. It raises the question when and to what extent are renegotiation-proof contracts inferior to optimal long-term contracts (where parties can fully commit to an initially signed contract)? By now, there is a rich literature on contract renegotiation that addresses the question.[12] It is beyond the scope of this chapter to summarize this literature. Instead I want to highlight a single issue that basically unites the subsections in this section. Does the decision about the frequency of performance evaluations relate to lack of commitment and renegotiations? The short answer is: yes. Infrequent performance evaluation can be substituted for lack of commitment, or, stated differently, IPE can be used as a commitment device (Lukas 2010). If performance is *not* measured at interim dates no information is received by contracting parties that could substantiate renegotiation. Infrequent evaluation can be advantageous because it permits a spillover (or spillback) of incentives to improve the intertemporal allocation of incentives; a shift of incentives that cannot be achieved by FPE. As soon as the intertemporal allocation of incentives varies less in comparison to the renegotiation-proof contract IPE is optimal.

[12]See, for example, Demski and Frimor (1999), Indjejikian and Nanda (1999), Christensen et al. (2003), Schöndube (2008), Lukas (2010), Ohlendorf and Schmitz (2012). Şabac (2015) analyses performance measure choice under renegotiation and derives sufficient conditions such that there are no losses from renegotiation.

The effectiveness of infrequent evaluation as a commitment device rests on the information rationing effect. It may have benefits beyond that. Infrequent evaluation implies that equal aggregate performance is rewarded equally. This may speak to employees' fairness perceptions and could have a motivating effect that is so far not modeled in multi-period contract design problems.

4 Subgame-Perfect Equilibria—and Alternatives?

The vast majority of papers analysing incentive contract design uses subgame-perfection as the equilibrium concept. Reliance on subgame-perfection has been criticised in the past because informational demands are high, the concept does not consider the *development* of the equilibrium (March 1987) and its value is likely to depend on the context that is analysed (Binmore and Samuelson 1994). In addition there is experimental evidence that people may not be able to apply backward induction (Binmore et al. 2002; Johnson et al. 2002) or they do not plan ahead (Hey and Knoll (2007)). The question arises if, when, and how the standard analysis that tries to identify a subgame-perfect equilibrium should be modified or supplemented. It is not the intent in this section to present ready algorithms or analytical options to deal with the problems just mentioned. Rather some thoughts are developed to encourage research on these issues.

If individuals in the laboratory have difficulties with backward induction or with planning ahead it is very likely that individuals outside the laboratory face similar difficulties (although they may receive help from consultants). Technically, one can summarize observed behaviour as strategy selection with error and/or selecting myopic best strategies.[13] It poses the problem how short-term best responses can be identified and which equilibrium eventually occurs. Kandori et al. (1993) analyse the development (evolution) of long-run equilibria in a simultaneous game under three assumptions that characterize bounded rationality of agents: (1) inertia, (2) myopia, and (3) mutation, meaning that (1) not all agents quickly adjust to changes in the environment, (2) agents are short-term optimizer, and (3) agents change their strategies at random. The analysis shows that in games with multiple strict Nash equilibria not all equilibria are equally likely to occur. Related to the result is the finding by Samuelson (1993) that in evolutionary models the exclusion of weakly dominated strategies from equilibrium considerations (or predictions) may not be as innocuous as is often assumed; in other words, weakly dominated strategies may survive in repeated games.

[13]In a fast changing environment, short-term optimization can prove superior to attempts of long-term optimization if only a small fraction of people involved quickly adjusts its behaviour to those changes (Kandori et al. 1993, p. 31). Or, as March (1987, p. 159) puts it: *"When competition takes place over long periods of time ... exclusive reliance on cleverness is by no means guaranteed to evolve as a dominant style of behaviour."*

Parallels to contract design problems are evident. Theoretically optimal multi-period contracts are often difficult to identify and may not possess a simple structure. Limited foresight or suboptimal specification of contracts could be practical consequences. For these reasons research on contract design would likely benefit from modifying modelling assumptions along the lines of Kandori et al. (1993).[14] Perhaps more insight can be gained into reasons for differing compensation structures between firms, or when and why more focus on short-term or long-term incentives is optimal. Related to the latter is the search for a rationale of infrequent performance evaluation (in this chapter's terminology) being superior to frequent performance evaluation given *observability* of interim outcome. In business, employees and managers typically have access to performance information that is relevant to their long-term incentives. The argument that rests on information rationing as in Sects. 3.1 and 3.2 cannot explain superiority of infrequent evaluation.[15] While project selection represents one possible explanation (Bhojraj and Libby 2005; Gigler et al. 2014), not all employees have responsibilities for project choice and hence other explanations are necessary.

Informational demands to determine subgame-perfect equilibria are potentially very high. The previous two paragraphs basically broach the issue of limited information processing. However, unavailability of information is problematic, too. How can the problem be solved? A natural requirement for contract design would be that a contract is robust, i.e., the contract remains optimal for different utility functions of the agent or the contract guarantees a minimum level of surplus for the principal. Carroll (2015) addresses the issue in the standard agency model with risk neutrality and limited liability. The model has a strikingly simple solution: a linear contract is optimal in the setting where the principal does not know all actions that are available to the agent. In other words, the linear contract is robust to changes in the action set of the agent. The intuition is as follows (Carroll 2015, p. 537): Any given contract in combination with a given action from the action set known to the principal determines a minimum expected net payoff for the agent (expected compensation less effort costs). If the resulting expected payoff for the principal (expected outcome less expected compensation) is to be turned into a minimum expected payoff for the principal, an increase in the outcome—due to actions not

[14]Evolutionary games are two-player games played between equal players. The equality assumption at first sight seems implausible for contract design problems. However, one can think of principal and agent as being equally sovereign in contract closure decisions. Both may have a contract structure in mind that they prefer. This structure defines the type of player in the evolutionary game. If a principal player encounters an agent player of equal type they contract, otherwise not.

[15]The Holmström and Milgrom (1987) result that linear contracts are optimal is based on the observability of interim outcomes, a stationary production technology, and a binary outcome distribution (success or failure). In this case the total number of successes is a sufficient statistic for the sequence of successes/failures. It is a restatement of the argument on page 59 that the optimal contract under frequent evaluation is the repetition of the optimal single-period contract.

known to the principal—must leave the proportion of expected payoffs between principal and agent unaffected. This can only be achieved by a linear contract.

The approach might also prove helpful in the analysis of multi-period incentive problems. It seems plausible to assume that actions sets or other parameter sets (e.g., demand, cost, technology) are not fully known to the principal, may change over time, or the principal learns about them over time; that would create a dynamic environment. Now if a linear contract is optimal for single periods, a contract linear in total outcome over periods could prove optimal (at least intuitively). But even if the proof of the result cannot be produced in this particular case the endeavour of modelling unquantifiable uncertainty in multi-period contracting problems promises to lead to interesting insights.

5 Conclusion

The single period agency model has proven a workhorse for analysing contract design problems and it has been applied in many different settings. With more attention being given to behavioural aspects in recent years the basic agency model has been extended to include behavioural phenomena. As this chapter shows, the popularity of bonus contracts or the efficacy of flat wage contracts can be explained with behavioural aspects; loss aversion and identification with the firm offer explanations though other factors, e.g., the costs of writing and administering more complex contracts, certainly contribute to their popularity.

Since many labor contracts are multi-period contracts, research in contract design addresses issues of performance evaluation frequency or use of past performance information for current performance evaluations. Designing a multi-period contract sets up a sequential game played between employer (principal) and employee (agent) and the subgame-perfect equilibrium determines the optimal contract. It is a well-known result that, in general, long-term contracts feature memory and they condition current period compensation on past outcomes. Many research papers focus on the pure information content of performance signals. Less attention is given to links between productive activities and performance measurement. This chapter includes the analysis of some possible features of effort interdependency and its impact on optimal performance measurement. The multi-period setting creates a truly dynamic game that could become quite complex. It raises the question how the researcher can deal with the complexity and model coping strategies of individuals when making decisions. Strategy selection with error, short-term optimization or robust contracts are presented as some of the certainly many ways to do so. Inclusion of these aspects in multi-period incentive contract design will definitely bring interesting and relevant insights for the curious researcher and practitioner.

References

Akerlof G, Kranton R (2005) Identity and the economics of organizations. J Econ Perspect 19(1): 9–32
Amershi AH, Demski JS, Fellingham JC (1985) Sequential bayesian analysis in accounting settings. Contemp Account Res 1(2):176–192
Arya A, Glover J, Liang PJ (2004) Intertemporal aggregation and incentives. Eur Account Rev 13(4):643–657
Bhojraj S, Libby R (2005) Capital market pressure, disclosure frequency-induced earnings/cash flow conflict, and managerial myopia. Account Rev 80(1):1–20
Binmore K, Samuelson L (1994) An economist's perspective on the evolution of norms. J Inst Theor Econ 150(1):45–63
Binmore K, McCarthy J, Ponti G, Samuelson L (2002) A backward induction experiment. J Econ Theory 104(1):48–88
Campbell D (2008): Nonfinancial performance measures and promotion-based incentives. J Account Res 46(2):297–332
Carroll G (2015) Robustness and linear contracts. Am Econ Rev 105(2):536–563
Christensen PO, Feltham G, Şabac F (2003) Dynamic incentives and responsibility accounting: a comment. J Account Econ 35(3):423–436
Demski JS (1998) Performance measure manipulation. Contemp Account Res 15(3):261–285
Demski J, Frimor H (1999) Performance measure garbling under renegotiation in multi-period agencies. J Account Res 37(Supplement):187–214
Dikolli S, Hofmann C, Kulp S (2009) Interrelated performance measures, interactive effort, and incentive weight. J Manag Account Res 21(1):125–149
Fellingham JC, Newman DP, Suh YS (1985) Contracts without memory in multi-period agency models. J Econ Theory 37(2):340–355
Fudenberg D, Tirole J (1990) Moral hazard and renegotiation in agency contracts. Econometrica 58(6):1279–1319
Gigler F, Kanodia C, Sapra H, Venugopalan R (2014) How frequent financial reporting causes managerial short-termism: an analysis of the costs and benefits of reporting frequency. J Account Res 52(2):357–387
Grossman SJ, Hart OD (1983) An analysis of the principal-agent problem. Econometrica 51(1): 7–45
Heinle M, Hofmann C, Kunz A (2012) Identity, incentives, and the value of information. Account Rev 87(4)1309–1334
Herweg F, Müller D, Weinschenk P (2010) Binary payment schemes: moral hazard and loss aversion. Am Econ Rev 100(5):2451–2477
Hey J, Knoll J (2007) How far ahead do people plan? Econ Lett 96(1):8–13
Holmström B (1979) Moral hazard and observability. Bell J Econ 10(1):74–91
Holmström B, Milgrom PR (1987) Aggregation and linearity in the provision of intertemporal incentives. Econometrica 55(2):303–328
Indjejikian RJ, Nanda D (1999) Dynamic incentives and responsibility accounting. J Account Econ 27(2):177–201
Jensen MC, Murphy KJ, Wruck EG (2004) Remuneration: where we've been, how we got there, what are the problems, and how to fix them. ECGI working paper series in finance No. 44
Johnson E, Camerer CF, Sen S, Rymon T (2002) Detecting failures of backward induction: monitoring information search in sequential bargaining. J Econ Theory 104(1):16–47
Kahneman D, Tversky A (1979) Prospect theory: an analysis of decision under risk. Econometrica 47(2):263–291
Kandori M, Mailath G, Rob R (1993) Learning, mutation, and long run equilibria in games. Econometrica 61(1):29–56
Köszegi B (2014) Behavioral contract theory. J Econ Lit 52(4):1075–1118

Köszegi B, Rabin M (2006) A Model of reference-dependent preferences. Q J Econ 121(4):1133–1165

Köszegi B, Rabin M (2007) Reference-dependent risk attitudes. Am Econ Rev 97(4):1047–1073

Lambert RA (1983): Long-term contracts and moral hazard. Bell J Econ 14(2):441–452

Lambert RA (2001) Contracting theory and accounting. J Account Econ 32(1–3):3–81

Lukas C (2007) Managerial expertise, learning potential and dynamic incentives: get more for less. Manag Decis Econ 28(3):195–211

Lukas C (2010) Optimality of intertemporal aggregation in dynamic agency. J Manag Account Res 22(1):157–174

Lukas C (2015) On reporting frequency and intertemporal substitution effects. Working paper

March JG (1987) Ambiguity and accounting: the elusive link between information and decision making. Acc Organ Soc 12(2):153–168

Nikias A, Schwartz S, Young R (2005) Optimal performance measures with task complementarity. J Manag Account Res 17(1):53–73

Ohlendorf S, Schmitz P (2012) Repeated moral hazard and contracts with memory: the case of risk-neutrality. Int Econ Rev 53(2):433–452

Prendergast C (1999) The provision of incentives in firms. J Econ Lit 37(1):7–63

Rogerson WP (1985) The first-order approach to principal-agent problems. Econometrica 53(6):1357–1367

Şabac F (2015) Ranking performance measures when contracts are renegotiated. Working paper

Samuelson L (1993) Does evolution eliminate dominated strategies? In: Binmore K, Kirman A, Tani P (eds) Frontiers of game theory. MIT Press, Cambridge, pp 213–235

Schöndube JR (2008) Early versus late effort in dynamic agencies with unverifiable information. Bus Res 1(2):165–186

van der Stede WA (2015) Management accounting: where from, where now, where to? J Manag Account Res 27(1):171–176

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Transfer Prices for Coordination Under Decentralized Decision Making

Clemens Löffler

Abstract This paper provides an overview of selected transfer pricing systems that are applied to achieve coordination within a decentralized firm. Specifically, we highlight the specific properties of transfer pricing systems when an intermediate product is sold internally via the transfer price but concurrently also sold on an external market. We adopt an incomplete contracting framework with asymmetric information at the trading stage. In this stylized model, transfer pricing guides intra-company trade and provides incentives for value-enhancing specific investments. Dependent on the distribution of information within the firm, we illustrate how these transfer pricing schemes are able to achieve coordination.

Keywords Cost-based transfer pricing • Intra-company coordination • Market-based transfer pricing • Specific investments

Contents

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1 Introduction

Based on neoclassical analysis (Cook 1955; Hirshleifer 1956), the managerial accounting literature frequently proposes to coordinate decentralized intra-company decisions using market-based transfer prices when a perfectly competitive intermediate market exists. Although the scenario of a perfectly competitive intermediate market does not seem to be descriptive for most intermediate products, firms nevertheless use market-based transfer prices, as indicated by several empirical studies. For a comprehensive overview see for example Oyelere and Turner (2000), Tang (2002), Abu-Serdaneh (2004) and Emmanuel et al. (1996). According to these studies 25–57% of the interviewed firms adopt a market-based approach to coordinate decentralized firms.

In this paper, we analyse market-based transfer pricing when the transfer price is based on the upstream division's market price for its intermediate product. Providing a comprehensive overview of empirical studies, Emmanuel et al. (1996) find that 6.8–39.9% of the firms that use market-based transfer prices use the market price of the upstream division, while 23.4–71% use a rival's market price to coordinate a decentralized firm. Although the upstream division is monopolist in its market and therefore has potential to manipulate its price and consequently the transfer price, we illustrate possible benefits of applying such a transfer pricing system even in this situation.

For the purpose of our analysis, we investigate a decentralized firm consisting of two divisions. The upstream division produces an intermediate product that is sold externally and that is also used by the downstream division to produce a final product. Both divisions act as monopolists on their respective markets. Synthesizing transfer pricing literature on coordination, asymmetric information, and incomplete contracting (e.g. Baldenius et al. 1999; Baldenius and Reichelstein 2006; Pfeiffer et al. 2011, among others) in our model the transfer price headquarters provides up-front investment incentives and guides intra-company trade. The upstream division can reduce its costs by undertaking specific investments up-front. Using the upstream division's market price as the basis for the internal transfer price, this creates incentives to behave opportunistically as the upstream division's pricing decision influences the transfer price and thus the internal trade decision. The upstream division raises the price on the intermediate market to increase its internal profit, inducing inefficient internal trade and consequently underinvestments.[1] To restore trade and investment efficiency, the optimal transfer price entails a markup over expected marginal costs which reduces the upstream division's inefficiently high external price. The downside of such a transfer pricing system is that, because of the upstream division's opportunistic behaviour, expected firm profits will be inefficiently low. On the positive side, however, the market price to a certain extend reflects actual costs of the intermediate product, facilitating to transfer this

[1]In this vein, Feinschreiber (2004) notes that market-based transfer prices can induce opportunistic pricing behaviour.

information implicitly via the transfer price to the downstream division. This fact may give market-based transfer pricing an advantage compared cost-based transfer pricing systems if they are not able to efficiently pass on information regarding actual costs.

Several empirical studies indicate that a significant amount of trade occurs within firms. The OECD estimates that 60% of worldwide trade is intra-firm trade (OECD 2012). Using US data, Bernard et al. (2010) and Lanz and Miroudot (2011) find that between 30% and 50% of US trade accounts to intra-firm trade. Principles developed in managerial accounting have structured how practitioners perceive the usefulness of different transfer price methods to govern intra-company trade. Typically the literature in managerial accounting suggests to refrain from market-based transfer pricing and to use cost-based transfer prices if the intermediate market becomes "too imperfect", e.g. Al-Eryani et al. (1990), Tomkins (1990), Brickley et al. (2009), Horngren et al. (2009), Schuster and Clarke (2010). Providing a performance comparison of the market-based transfer price with standard cost-based and actual cost-plus transfer prices allows a refinement of this suggestion. In particular, we find that even for a monopolistic intermediate market reasons exist for applying a market-based approach. In fact, we find that cost-plus transfer prices are effective if the cost uncertainty is rather high, i.e. when cost information becomes important, and the cost information is accessible to downstream timely and in sufficient quality. Standard cost-based transfer prices are effective if the cost uncertainty is low. Market-based transfer pricing remains effective for intermediate cost uncertainty and if cost information transfer within the firm is inefficient.

Beside the neoclassical analysis (Cook 1955; Hirshleifer 1956) this paper is related to the extant literature on market-based transfer pricing. Analysing properties of market-based transfer pricing, Baldenius and Reichelstein (2006) already find that market-based transfer pricing distorts intra-company trade. Complementing this key finding, the present paper shows how specific investments and different information structures influence such behaviour. This paper illustrates that the intra-company trade problem is alleviated with increasing amount of investments. Investigating competition on the intermediate market, Johnson et al. (2016) find justification for using the market price of a competitor instead of the upstream division as the basis for a market-based transfer price. Focusing on the competitive effects on the intermediate market, Arya and Mittendorf (2008) find that market-based transfer pricing always dominates cost-based transfer pricing. In the absence of an external intermediate market, previous literature has studied the effectiveness of various cost-based transfer pricing methods to provide trade and specific investment incentives (cf. Baldenius et al. (1999), Baldenius (2000), Sahay (2003), Johnson (2006), Pfeiffer et al. (2011) and Baldenius (2008) for an excellent overview). The present model shows that investments that influence costs for different products of the firm create interlinkages between markets that are significantly different under the individual transfer pricing schemes.

The reminder of the paper is organized as follows. Section 2 presents the basic setup. Section 3 studies as a benchmark the first-best solution if headquarters would have all informations and could make every decision itself. Section 4 presents and analyses the different transfer pricing regimes. Section 5 provides a performance comparison. Section 6 concludes.

2 Model

We consider a decentralized firm that consists of two divisions. Division 1 produces an intermediate product that is sold in both an external market and to Division 2, which uses it to produce a final product that is also sold externally. To distinguish between the two external markets, we refer to the external market for Division 1's intermediate product as the 'intermediate market' and to Division 2's external market for the final product as the 'final market'. The intermediate and final products are assumed to be different products that do not compete in the external markets. The firm is vertically integrated in order to provide Division 2 with a reliable source for the intermediate product, which is a specific input used in its manufacturing process. For simplicity, Division 2 converts each unit of the intermediate product into one unit of the final product, with the conversion cost normalized to zero. Division 1 can reduce the production costs by undertaking specific investments—for instance, installing more efficient equipment. This investment decision must be made up-front, when costs are still uncertain. Both divisions enjoy monopoly power on their respective markets with the inverse demand functions

$$p_i(q_i) = a_i - q_i, \text{ with } i = 1, 2 \tag{1}$$

where p_i denotes the price that can be charged for selling q_i units and a_i denotes the consumers' maximum willingness to pay on the respective market. For simplicity, we assume $a_1 = a_2 = a$. Figure 1 depicts the sequence of events.

At date 0, the headquarters specifies the particular transfer pricing method, which we will discuss subsequently. At date 1, Division 1 undertakes specific investments, $I \in [0, \bar{I}]$, that reduce Division 1's cost per unit, $C(I) = (c - x \cdot I)$, where c denotes the basic costs and x denotes the productivity of the investment, i.e. $a > C(\bar{I}) \geq 0$.

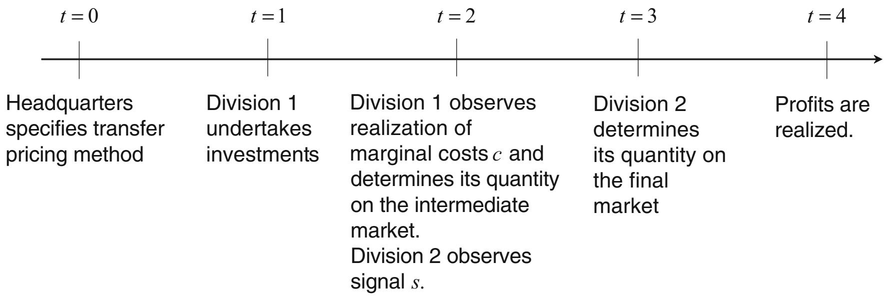

Fig. 1 Time line of the decision. Own representation

The basic costs are uncertain, $c \in [\underline{c}, \overline{c}]$, with $\mu_c = E[c]$ and $\sigma^2 = Var[c]$. The investments generate non-contractible fixed costs of $w(I) = I^2/2$. The investment level is observable for Division 2 but not for headquarters. At date 2, Division 1 observes the realization of the basic costs c and determines the quantity q_1 on the intermediate market. Division 2 makes only an imperfect observation of c in that it observes a signal s. We assume that c equals $s + \eta$ where η is an additive noise term with $E[\eta] = 0$ and $Cov[s, \eta] = 0$. There is symmetric cost information if $Var[\eta] = 0$ or, equivalently, $Var[c] = Var[s]$. Division 2 has no cost information if $Var[s] = 0$. A greater variance $Var[s]$ indicates, ceteris paribus, that Division 2 has better cost information. At date 3, Division 2 orders a quantity q_2 that must be delivered by Division 1.[2] Finally, at date 4, revenues and production costs are realized. Profits are calculated according to the transfer pricing rule. Since the state variable, the signal and the investment level cannot be verified by headquarters, the transfer pricing rules cannot be contingent on I, c, and s. To rule out trivial solutions, we assume $x^2 < 1$.

Each division is run by a risk-neutral manager who seeks to maximize the divisional profit. Given the transfer price t, the profits of Division 1 and 2, π_1 and π_2, are given by:

$$\begin{aligned} \pi_1 &= [p_1 - C(I)] \cdot q_1(p_1) + [t - C(I)] \cdot q_2(p_2) - w(I) \\ \pi_2 &= [p_2 - t] \cdot q_2(p_2). \end{aligned} \tag{2}$$

The firm's corporate profit is given by: $\Pi = \pi_1 + \pi_2$.

3 Benchmark: The Integrated First-Best Solution

As a benchmark, we first derive the first-best solution assuming centralized decision-making by headquarters under full information about the divisions' costs and revenues. The firm-wide profit function is:

$$\Pi = [p_1 - (c - xI)] \cdot q_1 + [p_2 - (c - xI)] \cdot q_2 - \frac{I^2}{2}. \tag{3}$$

The profit-maximizing quantities in the two markets are the monopoly quantities, i.e.

$$q_1^{FB}(I) = \frac{a-(c - xI)}{2} \text{ and } q_2^{FB}(I) = \frac{a-(c - xI)}{2}, \tag{4}$$

[2] We do not formally restrict Division 2's ability to source the intermediate good from Division 1 via the external market. However, as it turns out, Division 2 never has an incentive to buy the intermediate product via the market.

with the associated first-best prices $p_i^{FB} = (c - xI) + [a - (c - xI)]/2$. Clearly, optimal quantities increase with increasing investment I and productivity of investment x. The efficient investment level is then given by

$$I^{FB} = x \cdot E[q_1^{FB} + q_2^{FB}] = x \cdot \frac{a - E[c]}{1 - x^2}. \tag{5}$$

Interpreting the difference $a - E[c]$ as the expected net market size of each product without the investment, one can see that investments increase in this difference. Substituting (5) into (4) yields the optimal quantities

$$q_1^{FB} = q_2^{FB} = \frac{(a - c) - (E[c] - c)\, x^2}{2\,(1 - x^2)} = q^{FB}. \tag{6}$$

Since higher expected initial costs reduce the optimal investment level, also actual quantities decrease in $E[c]$.

The properties of the linear setting allow to express the contribution margins, $p^{FB} - (c - xI^{FB})$, for each product as $p - C = q^{FB}$. Additionally, the investment level can be expressed as $I^{FB} = 2xE[q^{FB}]$. Then the expected firm-wide profit $E[\Pi^{FB}]$ has the following simple form:

$$\begin{aligned} E[\Pi^{FB}] &= E\left[\left(q^{FB}\right)^2 + \left(q^{FB}\right)^2 - \frac{\left(I^{FB}\right)^2}{2} \right] \\ &= 2E\left[q^{FB}\right]^2 \left(1 - x^2\right) + \frac{Var\,[c]}{2} \\ &= B^{FB} + \frac{Var\,[c]}{2}. \end{aligned} \tag{7}$$

The expression $B^{FB} = 2E\left[q^{FB}\right]^2 \left(1 - x^2\right)$ depicts the firm's expected basic profit that could be achieved if headquarters ignores the informations regarding the basic costs c when determining the quantities. Since the expected first-best quantity $E\left[q^{FB}\right]$ increases in x, also the basic profit B^{FB} increases in x. The expression $Var\,[c]\,/2$ quantifies the additional flexibility value that the firm can achieve when adapting optimal quantities to the actual state of c.

4 Transfer Pricing Methods

Assuming that headquarters is neither able to determine the specific investments nor has the ability to make production decisions, it has to decentralize decision making. In order to maximize expected profits within the decentralized firm, headquarters then tries to implement appropriate transfer pricing schemes that shall achieve coordination between divisions. Subsequently, we highlight the advantages and disadvantages of three frequently applied transfer pricing schemes, that are standard cost-based transfer pricing, actual cost-based transfer pricing, and market-based transfer pricing.

4.1 Standard Cost-Based Transfer Pricing

We first study standard cost-based transfer pricing under which headquarters determines the transfer price ex-ante.

Backward induction provides the subgame perfect equilibrium. Maximizing its divisional profit at date 3, $\max_{q_2}\{[a-q_2-t]\cdot q_2\}$, Division 2 determines the monopoly quantity based on the pre-specified transfer price t,

$$q_2^{SC}(t) = \frac{a-t}{2}. \tag{8}$$

An inefficiency arises when the standard cost-based transfer price does not equal the realized cost because the first-best quantity is based on the realized cost, i.e. $q_2^{FB} = [a-C(I)]/2$.

At date 2, Division 1 determines the quantity on the intermediate market, yielding:

$$q_1^{SC}(I) = \frac{a-C(I)}{2}. \tag{9}$$

Division 1's quantity decision is efficient, i.e. $q_1^{SC}(I) = q_1^{FB}(I)$, because Division 1's objective is perfectly aligned with headquarters at this stage.

Anticipating the decisions of the subsequent stages, i.e. (8) and (9), Division 1 maximizes its expected divisional profit at date 1, $\max_I\{E[(p_1^{SC}(I)-C(I))\cdot q_1^{SC}(I) + [t-C(I)]\cdot q_2^{SC}(t)] - w(I)\}$, and determines the optimal investment decision, I^{SC}, as follows:

$$w'(I^{SC}) = E\left[\frac{\partial C(\cdot)}{\partial I}\cdot\left(q_1^{SC}\left(I^{SC}\right) + q_2^{SC}(t)\right)\right] = x\cdot E\left[q_1^{SC}(\cdot) + q_2^{SC}(\cdot)\right]. \tag{10}$$

Recognizing that Division 1 is the residual claimant of the return from its investment, $x\cdot E[q_1^{SC}+q_2^{SC}]$, depicts Division 1's direct marginal benefit of reducing its cost. One might suspect that Division 1 as the residual claimant determines the investment efficiently. This conjecture is only correct if the decisions of the subsequent stages are efficient in expectation. That is, $I^{SC} = I^{FB}$ if $E[q_2^{SC}] = E[q_2^{FB}]$.

At date 0, anticipating the decisions of the subsequent stages, i.e. (8), (9) and (10), headquarters determines the standard cost-based transfer price that maximizes the expected corporate profit:

$$\max_t\left\{E[[p_1^{SC}(I^{SC}(t))-C(I^{SC}(t))]\cdot q_1^{SC}(I^{SC}(t))+[p_2^{SC}(t)-C(I^{SC}(t))]\cdot q_2^{SC}(t)]-w(I^{SC}(t))\right\}.$$

Applying the Envelope Theorem shows that the optimal standard cost-based transfer price equals the expected marginal costs at the optimal investment level,

$$t^{SC}{=}E[C(I^{FB})] = E[c] - x^2 \frac{a - E[c]}{1 - x^2}. \tag{11}$$

The optimal transfer price avoids the creation of a double marginalization problem in expectation and thus the investment decision is efficient, i.e. $I^{SC} = I^{FB}$.[3]

Equilibrium quantities are given by:

$$\begin{aligned} q_1^{SC} &= \frac{(a-c) - (E[c]-c)\,x^2}{2\,(1-x^2)} \\ q_2^{SC} &= \frac{a - E[c]}{2\,(1-x^2)} \\ I^{SC} &= x \cdot \frac{a - E[c]}{1 - x^2}. \end{aligned} \tag{12}$$

The equilibrium quantity on the intermediate market equals the first-best quantity, i.e. $q_1^{SC} = q_1^{FB}$. However, the equilibrium quantity on the final market differs once actual costs differ from expected costs, i.e. $q_2^{SC} \neq q_2^{FB}$ if $c{\neq}E[c]$. The reason is that Division 2 bases its quantity on the transfer price and the standard cost-based transfer price does not reflect actual cost realization.

Under standard cost-based transfer pricing, the maximum expected corporate profit, $E[\Pi^{SC}(\theta)]$, is given by:

$$\begin{aligned} E\left[\Pi^{SC}(\theta)\right] &= E\left[\left(q_1^{SC}\right)^2 + \left(q_2^{SC}\right)^2 - \frac{\left(I^{SC}\right)^2}{2}\right] \\ &= 2E\left[q^{FB}\right]^2\left(1-x^2\right) + \frac{Var\,[c]}{4} \\ &= B^{FB}(\theta) + \frac{Var\,[c]}{4}. \end{aligned} \tag{13}$$

The basic value equals the first-best basic value, $B^{FB}(\theta)$. Because the decisions on the final market are based on the expected marginal costs rather than on actual marginal costs, standard cost-based transfer pricing induces a flexibility loss compared to the first-best situation of $Var\,[c]\,/2 - Var\,[c]\,/4 = Var\,[c]\,/4$.

[3]Studying standard cost-based transfer pricing in the absence of an intermediate market, Pfeiffer et al. (2011) find also that the optimal standard cost-based transfer price equals the expected marginal costs at the optimal investment level. In a seminal work, Baldenius (2000) finds for a binary trade setting that the optimal standard cost-based transfer price includes a positive markup.

4.2 Cost-Plus Transfer Pricing

In this section, we outline how our results change if headquarters uses actual costs as transfer price. Under actual cost-based transfer pricing, the intra-company price is determined when costs have been recognized by headquarters after trade has taken place (see Pfeiffer et al. 2011). Thus, Division 2 relies on its own imperfect observation s when determining its output choice.[4] The transfer price is set equal to actual marginal costs plus a markup, m, that headquarters determines ex-ante at date 0, i.e. $t^+ = C(I) + m$.

To elaborate, Division 2 determines the monopoly price based on the transfer price, t^+, and its imperfect signal s, yielding a monopoly quantity of $q_2^+(I, m) = [a - (s - xI) - m]/2$. Since the cost-plus transfer price cannot be influenced by the pricing decision on the intermediate market, the equilibrium quantity and price at date 2 are the same as the one in the benchmark case, i.e. $q_1^+(I) = q_1^{FB}(I)$, and $p_1^+(I) = p_1^{FB}(I)$ from (4).

At first glance, cost-plus transfer pricing seems to separate the pricing decisions on the external intermediate and final market. However, the imposed markup induces investment incentives and thus links the two markets. At date 1, Division 1's optimal investment decision is given by:

$$w'(I^+) = xE\left[q_1^+(I^+) + \frac{m}{2}\right]. \tag{14}$$

The term, $xE[q_1^+ + m/2]$, depicts Division 1's marginal benefit of reducing its costs. Since Division 1 receives a profit from internal trade of $[t - C] \cdot q_2^+ = m \cdot q_2^+$, it benefits only indirectly from its investments in that lower costs increase the internally traded quantity, $\partial q_2^+ / \partial I = x/2$. Since the investment level influences also Division 1's quantity on the intermediate market, the two markets become interlinked.

Recognizing that a higher mark-up increases the level of investment, but induces a double marginalization problem, headquarters determines the optimal markup as follows:

$$m^+ = x^2 \cdot \frac{(a - E[c])\left(2 - x^2 + x^4\right)}{4 - 2x^2 - x^4} = x^2 \cdot \frac{2}{2 - x^2} \cdot E\left[q_2^+(I^+, m^+)\right]. \tag{15}$$

In the absence of specific investments, $x = 0$, headquarters determines $m^+ = 0$ to avoid a double marginalization problem. A key insight of cost-plus transfer pricing is that the optimal markup increases with increasing trade at the final market, q_2^+, and the productivity of the investments x (Sahay 2003). Equilibrium quantities and

[4] Division 2's observation s could be interpreted as an imperfect attempt to deduce Division 1's marginal costs from the market price on the intermediate market. In this respect, cost-plus transfer pricing is sometimes interpreted as an attempt to approximate a market price. For example, Eccles (1985, p. 5) states that "one kind of market-based transfer price is cost-plus-profit markup."

investments are then given by:

$$q_1^+ = \frac{4(a-c) - (E[c]-c)x^2(2+x^2)}{2(4-2x^2-x^4)}$$
$$q_2^+ = \frac{2(2-x^2)(a-s) + (E[c]-s)x^4}{2(4-2x^2-x^4)} \tag{16}$$
$$I^+ = x \cdot \frac{(2+x^2)(a-E[c])}{4-2x^2-x^4}.$$

The maximum expected corporate profit is given by:

$$\begin{aligned} E[\Pi^+] &= E\left[(q_1^+)^2 + (q_2^+)^2 + m^+ q_2^+ - \frac{(I^+)^2}{2}\right] \\ &= 2E\left[q^{FB}\right]^2 \cdot \frac{(1-x^2)^2(4+x^2)}{4-2x^2-x^4} + \frac{Var[c] + Var[s]}{4} \\ &= \underbrace{B^{FB} \cdot \frac{(1-x^2)(4+x^2)}{4-2x^2-x^4}}_{B^+} + \frac{Var[c]}{4} + \frac{Var[s]}{4}. \end{aligned} \tag{17}$$

If Division 2 has perfect information, i.e. $Var[s] = Var[c]$, the flexibility value equals the first-best flexibility value. If Division 2 has no cost information, i.e. $Var[s] = 0$, actual cost-based transfer pricing adds no information advantage compared to standard cost-based transfer pricing and yields the same flexibility value. In the presence of a specific investment problem, cost-plus pricing exhibits an inefficiency due to the double marginalization problem. Consequently, the basic value is below the one of the benchmark case, $B^+ \leq B^{FB}$.

4.3 Market-Based Transfer Pricing

In this section, we investigate whether market-based transfer pricing can mitigate the intra-company coordination problem. In essence, the question is whether headquarters can use the cost information incorporated in the market price for the intermediate product to timely transfer the information to Division 2. Then Division 2 does not rely on its own imperfect observation of c, but on a signal provided by the market price. Our analysis will show that market-based transfer pricing causes quite different properties in terms of investment, pricing and trade incentives than the cost-based systems.

Applying the setup of Baldenius and Reichelstein (2006), under market-based transfer pricing the transfer price equals Division 1's external market price less an ex-ante determined discount δ, i.e. $t = p_1 - \delta$. We solve the problem with backward induction. At date 3, Division 2 determines the monopoly quantity,

$$q_2^M(p_1,\delta) = \frac{a-t}{2} = \frac{a-p_1+\delta}{2}. \tag{18}$$

The main difference to cost-based transfer pricing is that Division 1 can influence the transfer price by distorting the price p_1. Market-based transfer pricing thus connects the pricing and trading decisions of the two divisions.

Anticipating Division 2's response from (18) Division 1's determines its quantity according to the following reaction function at date 2,

$$0 = \frac{d\pi_1}{dq_1} = a-(c-xI)-2q_1+\left(t(p_1)-\frac{a+(c-xI)}{2}\right). \tag{19}$$

The first term, $a-(c-xI)-2q_1$, represents the standard maximization problem of a monopolist. The second term, $t(p_1)-[a+(c-xI)]/2$, represents Division 1's incentive to reduce its quantity, q_1, in order to increase the market price and therefore its profit from internal trade (note that in equilibrium, the transfer price is below the monopoly price, i.e. $t(p_1) < [a+(c-xI)]/2$). Thus, market-based transfer pricing provides incentives for Division 1 to behave opportunistically in the sense that it reduces the profitability of selling its product on the intermediate market in order to increase its benefits from internal trade. The associated equilibrium quantity and price are given by:

$$\begin{aligned} q_1^M(I,\delta) &= \frac{3(a-(c-xI))-2\delta}{6} \\ p_1^M(I,\delta) &= (c-xI)+\frac{3(a-(c-xI))+2\delta}{6}. \end{aligned} \tag{20}$$

The discount δ affects the quantity and the price. To offset a high discount, Division 1 reduces its quantity to increase the market price and thereby benefits from internal trade.

At date 1, anticipating the optimal responses at the subsequent stages, Division 1 maximizes its expected divisional profit, $\max_I\{E[[p_1^M(I,\delta)-(c-xI)]\cdot q_1^M(I,\delta)+[p_1^M(I,\delta)-\delta-(c-xI)]\cdot q_2^M(p_1^M(I,\delta)]-w(I)\}$, and determines the optimal level of investments as follows:

$$w'(I^M) = E\left[\frac{\partial C(\cdot)}{\partial I}\cdot\left(q_1^M(\cdot)+q_2^M(\cdot)\right)\right] = x\cdot E\left[q_1^M(\cdot)+q_2^M(\cdot)\right]. \tag{21}$$

As in the first-best case, the term $x\cdot E[q_1^M+q_2^M]$ reflects Division 1's direct marginal benefit of reducing its costs. Thus, given the output choices the investment decision is efficient.

At date 0, anticipating the decisions of the subsequent stages, headquarters determines the optimal discount that maximizes the expected corporate profit,

$\max_\delta\{E[[p_1^M(I^M(\delta),\delta) - C(I^M(\delta))] \cdot q_1^M(I^M(\delta),\delta) + [p_2^M(I^M(\delta),\delta) - C(I^M(\delta))] \cdot q_2^M(p_1^M(I^M(\delta),\delta),\delta)] - w(I^M p_1^M(I^M(\delta),\delta))\}$. Applying the Envelope-Theorem reveals that the expected transfer price exhibits a cost-plus structure, $E[p_1^M] - \delta^M = E\left[C(I^M)\right] + M^M$ and $\delta^M \geq 3(a - E[c])/2(4 - 3x^2)$, with a positive markup,[5]

$$M^M = \frac{3\,(a - E[c])}{2\,(4 - 3x^2)} = \frac{2E\left[q_2^M\right]}{3}. \tag{22}$$

The optimal transfer price entails a markup of $2E[q_2^M]/3$ in order to mitigate Division 1's incentives to reduce its quantity to gain from internal trade. Hence, trade is distorted in expectation. This transfer pricing system yields equilibrium quantities and the investment level

$$\begin{aligned} q_1^M &= \frac{3\,(a-c) + (E[c]-c)\left(1-3x^2\right)}{2\,(4-3x^2)} \\ q_2^M &= \frac{6\,(a-c) - (E[c]-c)\left(2+3x^2\right)}{4\,(4-3x^2)} \\ I^M &= x \cdot \frac{3\,(a-E[c])}{4-3x^2}. \end{aligned} \tag{23}$$

Finally, the associated maximum expected corporate profit, $E\left[\Pi^M\right]$, can be stated in terms of an associated basic value plus a flexibility value,

$$\begin{aligned} E\left[\Pi^M\right] &= E\left[\left(q_1^M\right)^2 + \frac{2}{3}q_2^M\left(q_1^M + q_2^M\right) + \left(q_2^M\right)^2 - \frac{\left(I^M\right)^2}{2}\right] \\ &= 2E\left[q^{FB}\right]^2 \cdot \frac{\left(1-x^2\right)^2 3\left(5-3x^2\right)}{\left(4-3x^2\right)^2} + \frac{7Var[c]}{16} \\ &= \underbrace{B^{FB} \cdot \frac{\left(1-x^2\right) 3\left(5-3x^2\right)}{\left(4-3x^2\right)^2}}_{B^M} + \frac{7Var[c]}{16}. \end{aligned} \tag{24}$$

Because of Division 1's opportunistic behaviour, expected quantities on both markets are generally below the first-best level, i.e. $E\left[q_1^M\right] < E\left[q_1^{FB}\right]$ and $E\left[q_2^M\right] < E\left[q_2^{FB}\right]$. Accordingly, the implemented behaviour causes that the basic value of the market-based transfer price is below the first-best level, i.e. $B^M < B^{FB}$. Also the flexibility value is below the first-best level, since the market price for the intermediate product is less sensitive to cost changes than the direct costs i.e. $7Var[c]/16 < Var[c]/2$.

[5] Division 2 never purchases the intermediate good externally at the market price, p_1^M, because $t^M = p_1^M - \delta^M \leq p_1^M$.

5 Performance Comparison

This section compares the effectiveness of the three considered transfer pricing methods. Market-based and actual cost-plus transfer pricing are more elaborated transfer pricing methods in that these methods condition the transfer price on actual circumstances. In the following, we investigate under which circumstances headquarters can improve the effectiveness of actual cost-based transfer pricing by using Division 1's market price as a benchmark. Additionally, we ask if there are circumstances under which centralized standard-cost pricing dominates the other two methods, although it is the least elaborate of the three.

Table 1 provides a summary of the properties of the presented alternative transfer pricing methods.

In the following we represent the precision of the signal s in terms of an α-factor, where $Var[s] = \alpha \cdot Var[c]$, with $\alpha \in [0,1]$. If $\alpha = 0$, Division 2 has no cost information, and if $\alpha = 1$, Division 2 has perfect cost information. Comparing expected profits of the three different transfer pricing methods of (13), (17), and (24), and solving for $Var[c]$ yields the following cut-off values:

$$\begin{aligned} Var[c]_{S+} &= \frac{4x^2}{(4-2x^2-x^4)\,\alpha} \cdot B^{FB} \\ Var[c]_{SM} &= \frac{16x^2}{3\,(4-3x^2)^2} \cdot B^{FB} \\ Var[c]_{M+} &= \frac{16\left(1-x^2\right)\left(4-14x^2+9x^4\right)}{(4-3x^2)^2\,(4-2x^2-x^4)\,(3-4\alpha)} \cdot B^{FB}. \end{aligned} \tag{25}$$

5.1 Performance Comparison in the Absence of a Specific Investment Problem

As shown above, in the absence of the specific investment problem, the markup under cost-plus transfer pricing converges to marginal costing, i.e. $m^+ = 0$, and the basic value is identical to first-best. Under these circumstances cost-plus transfer pricing dominates market-based transfer pricing as long as Division 2's observation is sufficiently precise because cost-plus transfer pricing provides better internal trade incentives. Therefore, the basic value under cost-plus transfer pricing exceeds the basic value under market-based transfer pricing, i.e. $B^+(x = 0) = B^{FB}(x = 0) > B^M(x = 0)$. Thus, market-based transfer pricing can outperform cost-plus transfer pricing only if the flexibility value becomes sufficiently larger than under cost-plus transfer pricing. This happens only when the signal s becomes uninformative. Since standard cost-based transfer pricing imports no actual cost information at all, standard cost-based transfer pricing is unambiguously dominated by cost-plus transfer pricing.

Table 1 Properties of transfer pricing methods

Property / Pricing method	Standard cost-based transfer pricing	Cost-plus transfer pricing	Market-based transfer pricing
Transfer price	t^{SC}	$t^{+} = C(I) + m$	$t^{M} = p_1 - \delta$
Equilibrium structure of the expected transfer price	Expected marginal cost, i.e. $t^{SC} = E[C(I^{SC})]$	Expected cost-plus, i.e. $t^{+} \geq E[C(I^{+})]$	Expected cost-plus, i.e. $t^{M} \geq E[C(I^{M})]$
Expected quantity on final market (given investments)	Expected first-best quantity, i. e. $E[q_2^{SC}(I)] = E[q_2^{FB}(I)]$	Underproduction, i. e. $E[q_2^{+}(I)] \leq E[q_2^{FB}(I)]$	Underproduction, i. e. $E[q_2^{M}(I)] \leq E[q_2^{FB}(I)]$
Expected quantity on intermediate market (given investments)	Expected first-best quantity, i.e. $E[q_1^{SC}(I)] = E[q_1^{FB}(I)]$	Expected first-best quantity, i.e. $E[q_1^{+}(I)] = E[q_1^{FB}(I)]$	Underproduction, i.e. $E[q_1^{M}(I)] \leq E[q_1^{FB}(I)]$
Investment level	First-best investment, i.e. $I^{SC} = I^{FB}$	Underinvestment, i.e. $I^{+} \leq I^{FB}$	Underinvestment, i.e. $I^{M} \leq I^{FB}$
Flexibility value	$\frac{Var[c]}{4}$	$\frac{Var[c]}{4} + \frac{Var[s]}{4}$	$\frac{7\,Var[c]}{16}$

Corollary 4.1 *In the absence of a specific investment problem,*

(i) cost-plus transfer pricing dominates market-based transfer pricing for a sufficiently informative signal s. i.e. $\alpha \geq \dfrac{3Var[c] - B^{FB}}{4Var[c]}$ *(and vice versa).*

(ii) cost-plus transfer pricing unambiguously dominates standard cost-based transfer pricing.

5.2 Performance Comparison in the Presence of a Specific Investment Problem

Returning to the general case, recall that with a specific investment problem inherent cost-plus pricing exhibits inefficient trade incentives so that $B^{+} \leq B^{FB}$. The difference between B^{+} and B^{FB} increases in the productivity of the investment x. As a consequence, cost-plus transfer pricing does not unambiguously dominate standard cost-based transfer pricing. In fact, cost-plus transfer pricing dominates standard-cost transfer pricing only if cost uncertainty is sufficiently high. Intuitively, the cut-off value $Var[c]_{S+}$ is decreasing in α, since higher precision of the signal s increases the information advantage of cost-plus transfer pricing.

Proposition 4.1 *In the presence of a specific investment problem, cost-plus transfer pricing dominates the standard cost-based transfer pricing for sufficiently high cost uncertainty, i.e.* $Var[c] \geq \dfrac{4x^2 B^{FB}}{(4 - 2x^2 - x^4)\,\alpha}$.

The basic value of cost-plus transfer pricing can decline even below the basic value of market-based transfer pricing. Comparison of B^{+} and B^{M} yields that $B^{+} < B^{M}$ if $x > (7 - \sqrt{13})^{1/2}/3$. In this situation, cost-plus transfer pricing dominates the market-based transfer pricing only for sufficiently high cost uncertainty, i.e. when cost information becomes important, and when the signal s is sufficiently informative. On the other hand, as long as $B^{+} > B^{M}$, cost-plus transfer pricing unambiguously dominates market-based transfer pricing for a sufficiently informative signal s. In fact, the flexibility value generated by cost-plus transfer pricing is larger than the flexibility value generated by market-based transfer pricing once $\alpha > 3/4$. This yields the following finding.

Proposition 4.2 *In the presence of a specific investment problem, cost-plus transfer pricing dominates market-based transfer pricing*

(i) if $\alpha < 3/4$ *and* $Var[c] < \dfrac{16\left(1 - x^2\right)\left(4 - 14x^2 + 9x^4\right) B^{FB}}{\left(4 - 3x^2\right)^2 (4 - 2x^2 - x^4)(3 - 4\alpha)}$.

(ii) if $\alpha > 3/4$ *and* $Var[c] > \dfrac{16\left(1 - x^2\right)\left(4 - 14x^2 + 9x^4\right) B^{FB}}{\left(4 - 3x^2\right)^2 (4 - 2x^2 - x^4)(3 - 4\alpha)}$.

Finally, market-based transfer pricing only can dominate standard cost-based transfer pricing for sufficiently high cost uncertainty. Since market-based transfer

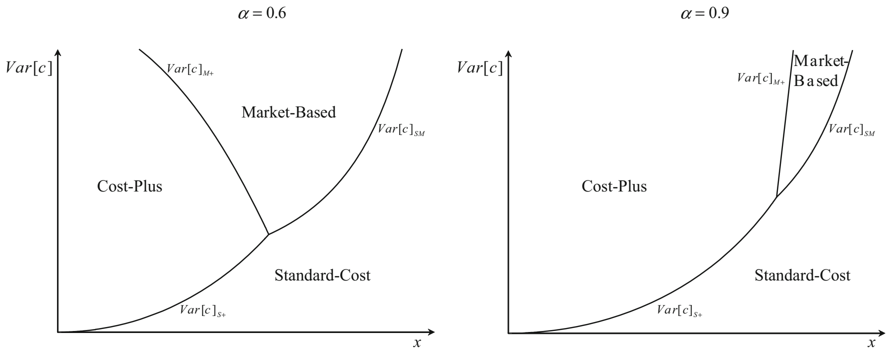

Fig. 2 Performance comparison. Own representation

pricing induces a coordination problem due to Division 1's opportunistic behaviour, this transfer pricing method can only legitimate its existence if the market based transfer price provides sufficiently important cost information.

Proposition 4.3 *In the presence of a specific investment problem, market-based transfer pricing dominates standard cost-based transfer pricing for sufficiently high cost uncertainty, i.e.* $Var[c] \geq 16B^{FB}/3\left(4 - 3x^2\right)$.

Figure 2 illustrates the outcome of the performance comparison in terms of the productivity of investments x and the cost uncertainty $Var[c]$.

Figure 2 highlights that standard cost-based transfer pricing becomes more useful (i) if cost uncertainty is rather low and for increasing productivity of investments x. This is quite intuitive, since standard cost-based transfer pricing provides no cost information for the final market but prevents the creation of coordination problems in expectations that increase under the other two transfer pricing methods with increasing productivity of investments x. The underinvestment problem of cost-plus transfer pricing increases because Division 1 is not equipped with sufficient investment incentives. Headquarters has to trade-off the creation of investment incentives with concurrently creating a double marginalization problem. Under market-based transfer pricing, the opportunistic behaviour problem increases because a more profitable final market increases Division 1's incentives to shift rents via an appropriate transfer price.

Finally, Fig. 2 illustrates that the benefit of market-based transfer pricing generally diminishes if Division 2's signal becomes more informative.

6 Conclusion

In the presence of an intermediate market for their products, firms frequently use as a transfer price the market price that the upstream division charges to its external costumers (e.g. Emmanuel et al. 1996). From a coordination perspective, our analysis shows that market-based transfer pricing can be optimal although it provides the upstream division with incentives to distort the market price for the intermediate product.

In a seminal case study on implementation issues of various transfer pricing methods, Eccles (1983, p. 2) notes that using a "market price isn't the best approach in imperfectly competitive markets." In this context, cost-based transfer prices are frequently proposed as an alternative to market-based transfer pricing. Empirical studies indeed show that cost-based transfer prices are widely used in practice, e.g. Borkowski (1990), Oyelere and Turner (2000), Tang (2002). According to these studies 36–52% of the firms use cost-based transfer prices.

Conducting a performance comparison, our analysis provides the straightforward proposal of using cost-based transfer pricing in quite distinctive situations. While actual cost-plus transfer pricing is the correct cost-based transfer pricing method in the absence of a specific investment problem and perfect information transmission within the firm, standard cost-based transfer pricing is the correct cost-based transfer pricing method when the upstream division's costs are deterministic and a specific investment problem arises. In particular, standard cost-based transfer pricing can only be optimal if a specific investment problem arises. In this case, we find briefly stated that (i) market-based transfer pricing is optimal for sufficiently high cost uncertainty and if the precision of information transmission is not extremely high, (ii) standard cost-based transfer pricing is optimal if the productivity of investments is high and cost uncertainty is rather low, and (iii) actual cost-plus transfer pricing is optimal if cost uncertainty is sufficiently high and the precision of information transmission is rather high.

Our analysis provides insights into the determination of adjustments that are frequently applied in practice for transfer prices (e.g. Drury 2009; Sahay 2013) generating interesting empirical predictions. As Merchant and van der Stede (2012) state: *"…many firms use quasi market-based transfer prices by allowing deviations from observed market prices"* (Merchant and van der Stede 2012, p. 271). Indeed, empirical studies identify that 38.2–48.4% of the firms use adjusted market-based transfer prices rather than directly prevailing market prices, see for example Borkowski (1990), Tang (2002), Abu-Serdaneh (2004). These adjustments are usually explained to reflect internal cost savings, potential internal synergies, imperfect comparables, and price distortions arising from imperfectly competitive markets, e.g. Zimmerman (2004), Baldenius and Reichelstein (2006), Drury (2009), Merchant and van der Stede (2012), Sahay (2013).

Market-based transfer pricing entails a markup over expected marginal costs in order to decrease the division's price on the intermediate market. The markup increases when the productivity of investments increases, since Division 1 has an increasing incentive to extract rents from Division 2. Cost-plus transfer pricing exhibits a markup over actual cost that also increases when the productivity of investments increases.

References

Abu-Serdaneh JAR (2004) Transfer pricing in UK manufacturing companies. University of Huddersfield, Huddersfield
Al-Eryani MF, Alam P, Akhter SH (1990) Transfer pricing determinants of US multinationals. J Int Bus Stud 21(3):409–425
Arya A, Mittendorf B (2008) Pricing internal trade to get a leg up on external rivals. JEMS 17(3):709–731
Baldenius T (2000) Intrafirm trade, bargaining power, and specific investments. RAS 5(1):27–56
Baldenius T (2008) Internal pricing. Found Trends Account 3(4):223–313
Baldenius T, Reichelstein S (2006) External and internal pricing in multidivisional firms. JAR 44(1):1–28
Baldenius T, Reichelstein S, Sahay SA (1999) Negotiated versus cost-based transfer pricing. RAS 4(2):67–91
Bernard AB, Jensen JB, Redding SJ, Schott PK (2010) Intra-firm trade and product contractibility. Working Paper
Borkowski SC (1990) Environmental and organizational factors affecting transfer pricing: a survey. JMAR 2:78–99
Brickley JA, Smith CW, Zimmerman JL (2009) Managerial economics and organizational architecture, 5th edn. McGraw-Hill, New York
Cook PW (1955) Decentralization and the transfer price problem. JB 28(2):87–94
Drury CM (2009) Management and cost accounting for business, 4th edn. Cengage, Hampshire
Eccles RG (1983) Control with fairness in transfer pricing. HBR 61(6):149–161
Eccles RG (1985) The transfer pricing problem: a theory for practice. Lexington Books, New York
Emmanuel C, Otley D, Merchant KA (1996) Accounting for management control, 2nd edn. Chapman and Hall, London
Feinschreiber R (2004) Transfer pricing methods: an applications guide. Wiley, New York
Hirshleifer J (1956) On the economics of transfer pricing. JB 29:172–184
Horngren C, Foster G, Datar S, Rajan M, Ittner C (2009) Cost accounting: a managerial emphasis, 13th edn. Upper Saddle River, New Jersey
Johnson N (2006) Divisional performance measurement and transfer pricing for intangible assets. RAS 11(2/3):339–365
Johnson N, Löffler C, Pfeiffer T (2016) Market-based transfer pricing: an evaluation of alternative market prices. Working Paper.
Lanz R, Miroudot S (2011) Intra-firm trade: patterns, determinants and policy implications. OECD Trade Policy Papers 114, OECD Publishing
Merchant KA, van der Stede WA (2012) Management control systems. 3rd edn. Pearson, Harlow
OECD (2012): OECD insights; debate the issue; price fixing. Available at: http://oecdinsights.org/2012/03/26/price-fixing/
Oyelere PB, Turner JD (2000) A survey of transfer pricing practices in UK banks and building societies. Eur Bus Rev 12(2):93–99
Pfeiffer T, Schiller U, Wagner J (2011) Cost-based transfer pricing. RAS 16(2):219–246
Sahay SA (2003) Transfer pricing based on actual cost. JMAR 15(1):177–192
Sahay SA (2013) Financial control and transfer pricing. In: Lee C-F, Lee AC (eds) Encyclopedia of finance, 2nd edn. Springer, New York, pp 783–794
Schuster P, Clarke P (2010) Transfer prices: functions, types, and behavioral implications. Manag Account Q 11(2):22–32
Tang RY (2002) Current trends and corporate cases in transfer pricing. Quorum Books, Westport
Tomkins C (1990) Making sense of cost-plus transfer prices where there are imperfect intermediate good markets by a 'pragmatic-analytical' perspective. MAR 1(3):199–216
Zimmerman JL (2004) Accounting for decision making and control, 4th edn. McGraw-Hill, New York

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Managerial Compensation, Investment Decisions, and Truthfully Reporting

Günter Bamberg and Michael Krapp

Abstract This paper provides a formal analysis of investment decisions with special emphasis to mechanisms which induce managers to reveal their knowledge truthfully. In a one-period context 'knowledge' usually means the profit ratio. In a multi-period setting 'knowledge' is referred to the (multivariate) cash flow stream or the (univariate) net present value. Both situations are analysed in the paper. We start with the basic case 'one firm, one manager' and continue with the case 'divisional firm, division managers'. With respect to the first case, we criticise two approaches (Rogerson, JPolE 105(4):770–795, 1997; Reichelstein, RAS 2(2):157–180, 1997) and develop a solution based on extended incentive contracts. To tackle the second case, we analyse pros and cons of Groves schemes.

Keywords Extended incentive contracts • Groves mechanism • Goal congruence • Impatient manager • Investment decisions • Managerial compensation • Preinreich/Lücke-theorem

Contents

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1 Introduction

Investment planning is often characterized by asymmetric information. Managers are frequently better informed about the technology and market opportunities than the corporate headquarters. Therefore, incentive mechanisms are needed to limit the scope of opportunistic managers. When selecting and implementing investment projects, managers shall act according to the corporate objectives. In particular, they shall report the profitability of investment opportunities truthfully ahead of investment decision making.

Incentive mechanisms discussed in the literature are—with only few exceptions—based on one-period models. On the other hand, typical investment projects span a multi-period planning horizon T (for example, 10 years). What is more, a real dynamic model should also consider e.g. changes in the economic environment, the development of other (later starting) projects, whether interactions between projects exist etc. As soon as stochastics and different risk attitudes are taken into account, the risk of misspecification increases and practicality decreases.

This paper strives to study a compromise between the overly restrictive one-period models and the complex multi-period models. This compromise is based on

- the examination of investment projects by the (deterministic or stochastic) net present value (NPV) and
- the remuneration of managers by payments in the periods $t = 1$ to $t = T$ proportional to residual income (RI_t).

The last point is of particular interest from a practical point of view. Many incentive mechanisms determine managers' compensation depending on the realized NPV or the deviation between the actual NPV and $\widehat{NPV}$, i.e. the NPV reported to central management at date $t = 0$. Both, the NPV as well as its deviation from $\widehat{NPV}$, cannot be evaluated without major dissent until date $t = T$ (for example, in 10 years). A remuneration only at the planning horizon $t = T$ without interim payments at dates $t = 1, 2, \ldots, T$ is problematic in practise. It seems reasonable (cf. Sect. 2.2) to make these interim payments proportional to residual income RI_t. However, the ongoing determination of project-specific RI_t involves high requirements to be met by the accounting system.

Section 2 sums up the foundations of NPV from the perspectives of money market and utility theory as well as the interrelations of net present value and residual income. In Sect. 3, the case 'one firm, one manager' is treated. Special attention is paid to the problem of the impatient manager, i.e. when the duration of the manager's contract is shorter than that of her proposed projects. Section 4 analyses incentives within a divisionalized company in which the various divisional managers compete for the scarce resource investment capital. Section 5 concludes. All proofs are in the Appendix.

2 Net Present Value, Utility Theory, and Residual Income

2.1 Money Market Invariance

The task under consideration is to examine an investment opportunity that generates the cash flow stream $\mathbf{c} = (c_0, c_1, \ldots, c_T)$. Here, $c_1, \ldots, c_T$ are cash flows at dates $1, \ldots, T$ and $c_0 < 0$ is the initial net investment due at date $t = 0$, cf. Fig. 1.

When $\mathbf{c}$ is risky, one may strive to evaluate it by means of a (scalar) certainty equivalent $CE(\mathbf{c})$. The latter will, in general, depend on the date $0, 1, \ldots, T$ of evaluation. We, however, restrict our considerations to date 0 as this is usually the date of evaluation in the context of investment accounting. Furthermore, if a perfect money market exists, the decision maker has the opportunity to transform the risky cash flow stream $\mathbf{c}$, for instance by borrowing or lending certain amounts z_t (with $t = 0, 1, \ldots, T-1$) at rate r for one period each. When doing so, she can transform $\mathbf{c}$ into

$$\mathbf{c} + \mathbf{z} = (c_0 - z_0, c_1 + qz_0 - z_1, c_2 + qz_1 - z_2, \ldots, c_T + qz_{T-1}), \tag{1}$$

where $q = 1 + r$ and $\mathbf{z} = (-z_0, qz_0 - z_1, \ldots, qz_{T-1})$. Since $\mathbf{z}$ is non-stochastic and already projectable at date 0, it seems natural to demand that both cash flow streams, $\mathbf{c}$ as well as $\mathbf{c} + \mathbf{z}$, should be assigned the same value, i.e.

$$CE(\mathbf{c}) = CE(\mathbf{c} + \mathbf{z}) \tag{2}$$

for all $\mathbf{c}$ and $\mathbf{z}$. Multiattributive utility functions $u(\mathbf{x})$ (where $\mathbf{x} = (x_0, x_1, \ldots, x_T)$ is the vector of attributes) that suffice this condition are termed money market invariant. The respective certainty equivalent $CE(\mathbf{c})$ is then characterized by the indifference $\mathbf{c} \sim (CE(\mathbf{c}), 0, \ldots, 0)$. Theorem 5.1 clarifies the structure of multiattributive utility functions that are money market invariant in this sense.

Theorem 5.1 *A multiattributive utility function $u(\mathbf{x})$ is money market invariant in the sense of condition (2) if and only if it evaluates $\mathbf{x}$ on the basis of the net present value $NPV(\mathbf{x}) = \sum_{t=0}^{T} x_t \cdot \gamma^t$ (with $\gamma = q^{-1}$) only, i.e. iff*

$$u(\mathbf{x}) = u(NPV(\mathbf{x}), 0, \ldots, 0) \tag{3}$$

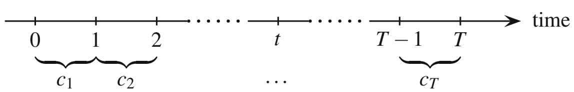

Fig. 1 Cash flow stream. Own representation

holds true. Then, $u(\mathbf{x})$ *de facto simplifies to an uniattributive utility function* $u_0(x)$ *that evaluates payments* x *due at date 0, i.e.* $u_0(x) = u(x, 0, \dots, 0)$.
The proof can be found in the Appendix (section "Proof of Theorem 5.1").

2.2 Net Present Value and Residual Income

The well-known Theorem by Preinreich (1937) and Lücke (1955) captures the interrelation of cash flows and residual incomes. As this interrelation is fundamentally important for our considerations, we sketch the Preinreich-Lücke Theorem in the following. Given the cash flow stream $c_0, c_1, \dots, c_T$ with $c_0 < 0$, the residual income in period t is defined by

$$RI_t = NI_t - r \cdot EC_{t-1} \,. \tag{4}$$

Here, $NI_t = c_t - d_t \cdot |c_0|$ denotes the net income in period t, where d_t is the depreciation factor relevant in period t. Further, r is the cost of equity, and EC_{t-1} is the equity capital in the preceding period $t-1$. The latter resembles the difference of net incomes and cash flows cumulated up to period $t-1$, i.e. $EC_{t-1} = (NI_1 + \cdots + NI_{t-1}) - (c_0 + c_1 + \cdots + c_{t-1})$ for $t > 1$ (otherwise, $EC_{-1} = 0$ and $EC_0 = -c_0$, respectively). In period 0, $RI_0 = 0$ holds true. We are now able to formulate the Preinreich/Lücke Theorem 5.2.

Theorem 5.2 *Given the total sum of net incomes* $NI_1 + \cdots + NI_T$ *equals the total sum of cash flows* $c_0 + c_1 + \cdots + c_T$ *(and, hence,* $EC_T = 0$*), discounting the stream of residual incomes and discounting the stream of cash flows lead to the same result, i.e.*

$$\sum_{t=1}^{T} RI_t \cdot \gamma^t = \sum_{t=0}^{T} c_t \cdot \gamma^t \,. \tag{5}$$

As the right-hand side of (5) is the net present value, the latter can also be computed on the basis of residual incomes.
The proof can be found in the Appendix (section "Proof of Theorem 5.2").

3 One Firm, One Manager

Regarding the manager's planning horizon τ, we distinguish the cases $\tau \geqq T$ and $\tau < T$. In the first case, the ('patient') manager's contract does not expire within the duration T of the investment project under consideration, whereas in the second case, the ('impatient') manager plans to leave or retire before all the benefits of the investment are realized. In addition, various scenarios regarding the level of information on the cash flow stream are conceivable. For example, the cash flows or their expected values may be common knowledge. Contrariwise, corporate headquarters may only know the expected net present value or cash flows (or the associated net present value) reported by the manager. Finally, one can distinguish whether manager and/or company are risk neutral or risk averse.

3.1 The Case of the Patient Manager

In the following, we assume $\tau \geqq T$, meaning that the manager's planning horizon exceeds the duration of the investment project under consideration. If one wishes to achieve goal congruence in the sense that the manager maximizes her discounted remuneration by selecting a project that maximizes the company's *NPV*, then the scheme described in Sect. 3.1.1 is the contract of choice. If, however, corporate headquarters are—for the sake of planning certainty—primarily eager to learn the *NPV*'s value, extended incentive contracts as described in Sect. 3.1.2 should be preferred.

3.1.1 Remuneration Based on Residual Income

As outlined in Sect. 2.2, the Preinreich/Lücke Theorem 5.2 implies

$$NPV = \sum_{t=0}^{T} c_t \cdot \gamma^{\,t} = \sum_{t=1}^{T} RI_t \cdot \gamma^{\,t}. \tag{6}$$

If in period t (with $t = 1, \ldots, T$) a remuneration proportional to residual income RI_t is provided to the manager, i.e. $\beta \cdot RI_t$ with $\beta > 0$, goal congruence can be achieved: Provided manager and company apply the same discount factor γ, the right-hand side of (6) is proportional to the manager's present value of remuneration. Hence, maximizing this present value leads to the maximal *NPV*, implying goal congruence.

This reasoning tacitly assumes the cash flow stream $c_0, c_1, \ldots, c_T$ to be deterministic, regardless of who has what information about it. Only then 'maximize the *NPV*' or 'maximize the present value of remuneration' are sensible directives. If, on the other hand, $c_0, c_1, \ldots, c_T$ are (in part) stochastic, (6) turns out to be a correct relationship between random variables. However, maximization then need to be applied to expected utilities or certainty equivalents, cf. Sect. 2.1. Then the certainty equivalent of *NPV* has to be evaluated by means of the company's utility function, whereas the certainty equivalent of the remuneration's present value needs to be evaluated by means of the manager's utility function. That's why remuneration based on residual income cannot generally assure goal congruence in the presence of risk aversion. An attempt to tackle this problem by implementing a rather sophisticated risk allocation schedule was recently suggested by Grottke and Schosser (2014).

In the special case of risk neutrality goal congruence can be obtained, as can be seen from (6) by the formation of expectations, i.e.

$$\mathrm{E}(NPV) = \sum_{t=0}^{T} \mathrm{E}(c_t) \cdot \gamma^{\,t} = \sum_{t=1}^{T} \mathrm{E}(RI_t) \cdot \gamma^{\,t} = \mathrm{E}\Big(\sum_{t=1}^{T} RI_t \cdot \gamma^{\,t} \Big) . \tag{7}$$

Obviously, the expected present value of remuneration is maximal if and only if the expected NPV of the investment project is maximal.

3.1.2 Remuneration Based on Extended Incentive Contracts

A remuneration based on $\beta \cdot RI_t$ (or, more general, $\alpha + \beta \cdot RI_t$) ensures goal congruence provided cash flows are deterministic and company as well as manager are risk neutral. However, such incentive schemes are not capable of extracting the attainable NPV or its expected value ex ante. If such information is desired, e.g. for the sake of planning certainty, extended incentive contracts (Reichelstein and Reichelstein 1987; Bamberg 1991) are a promising alternative. Then, remuneration is proportional to

$$g(\widehat{NPV}) + s(\widehat{NPV}) \cdot [NPV - \widehat{NPV}] , \tag{8}$$

where $\widehat{NPV}$ denotes the project's net present value as reported by the manager at date $t = 0$, whereas NPV is the actual net present value. The latter value is random from an ex ante perspective (i.e., at date $t = 0$), but certain from an ex post perspective (i.e., at date $t = T$). Finally, $g(\cdot)$ and $s(\cdot)$ are design functions determined by the company. Regarding $g(\cdot)$ and $s(\cdot)$ the requirements

$$s(\cdot) = g'(\cdot), \quad s'(\cdot) > 0, \quad s(\cdot) > 0 \tag{9}$$

ensure that the manager maximizes her expected remuneration if and only if she reports $\widehat{NPV}_{\mathrm{opt}} = \mathrm{E}(NPV)$ (Bamberg et al. 2013). Hence, information on the investment project's expected net present value can be extracted truthfully if the manager is risk-neutral. Furthermore, $s(\cdot) > 0$ also induces the manager to strive for the highest possible realization of NPV when implementing the project, even if $\widehat{NPV}$ was biased.

In the following, we apply the presumably most simple specification of $g(\cdot)$ and $s(\cdot)$ satisfying (9), namely $g(x) = x^2$ and hence $s(x) = 2x$. Then, remuneration is given by

$$\beta \cdot \{(\widehat{NPV})^2 + 2\widehat{NPV} \cdot [NPV - \widehat{NPV}]\} \tag{10}$$

with expectation

$$\beta \cdot \{(\widehat{NPV})^2 + 2\widehat{NPV} \cdot [\mathrm{E}(NPV) - \widehat{NPV}]\} , \tag{11}$$

where $\beta > 0$ is a constant of proportionality.

Some remarks regarding formulae (10) and (11) are in order. First, it can easily be verified that reporting the expected net present value is indeed optimal for a risk-neutral manager. As the first derivative of (11) with respect to $\widehat{NPV}$ is $\beta \cdot [-2\widehat{NPV} + 2 \cdot \mathrm{E}(NPV)]$, evaluating the first-order condition immediately yields $\widehat{NPV}_{\mathrm{opt}} = \mathrm{E}(NPV)$. The second-order condition sufficient for a maximum is fulfilled since the second derivative of (11) with respect to $\widehat{NPV}$, -2β, is clearly negative.

Second, if the manager manages to forecast NPV without any deviation, i.e. $\widehat{NPV} = NPV$, she will earn a remuneration of exactly $\beta \cdot (\widehat{NPV})^2$. This scenario is, of course, unrealistic as NPV is ex ante random. If, however, target-actual comparison reveals a rather low discrepancy, i.e. $\widehat{NPV} \approx NPV$, the manager's remuneration will approximately be $\beta \cdot (\widehat{NPV})^2$, providing a hint for the determination of β. If a salary level of for example 200,000 € per year is considered to be realistic in the manager market, and the project duration is $T = 5$ years, $\beta \cdot (\widehat{NPV})^2 = 10^6$ € seems a reasonable specification. Please note that such a specification does not constitute a fixed remuneration, even if progress payments amounting to for example 200,000 € at dates $t = 1, 2, 3, 4$ might be agreed. The reason is the final (positive or negative) payment at date $t = 5$, taking into account possibly accrued earlier payments.

Given the assumption $\beta \cdot (\widehat{NPV})^2 = 10^6$, one can rewrite (10) as follows

$$10^6 \cdot \left[2 \cdot \frac{NPV}{\widehat{NPV}} - 1\right], \tag{12}$$

yielding the final payment

$$10^6 \cdot \left[2 \cdot \frac{NPV}{\widehat{NPV}} - 1\right] - 800,000 \tag{13}$$

or

$$10^6 \cdot \left[2 \cdot \frac{NPV}{\widehat{NPV}} - 1\right] - 200,000 \cdot (q + q^2 + q^3 + q^4)\,. \tag{14}$$

If the actual net present value observed at date $T = 5$ exceeds the ex ante forecast by 10%, i.e. $NPV = 1{,}1 \cdot \widehat{NPV}$, the manager earns a final payment of 400,000 € according to (13). If, on the other hand, $NPV = 0{,}9 \cdot \widehat{NPV}$ turns out to be true, the final payment will be zero.

Third, risk neutrality serves as a decision-theoretical prerequisite for the implementation of extend incentive contracts. In this regard it is worthwhile to note that the company's risk attitude refers to the net present value evaluated at date $t = 0$ (whose realization cannot be observed until date $t = T$). On the other hand, the manager's risk attitude refers to payments due at date $t = T$. What is more, the design of the interim payments is, although important for the acceptance in practice, irrelevant from a decision-theoretical perspective. In order to incorporate these payments into the formal analysis, one need to apply multiattributive utility functions (as in Sect. 2.1). This, however, seems hardly be practicable in real-world settings.

Let us finally consider the case of a risk-averse (rather than a risk-neutral) manager. For the sake of convenience, we adopt the well-known LEN setting (Holmström and Milgrom 1987; Spremann 1987), i.e. we assume NPV to be normally distributed and the manager to be risk-averse with positive constant absolute risk aversion ϱ. The manager then maximizes the certainty equivalent of her remuneration (10) with respect to $\widehat{NPV}$. As can easily be verified, her optimal report is

$$\widehat{NPV}_{\text{opt}} = \frac{\mathrm{E}(NPV)}{1 + 2\varrho\beta \cdot \mathrm{Var}(NPV)} . \tag{15}$$

Obviously, the optimal report systematically falls below the true expected net present value. This bias is more pronounced, the greater the manager's risk aversion ϱ is and the greater the project's risk $\mathrm{Var}(NPV)$ is. It can be shown that such biases generally occur when the manager is risk-averse (Bamberg 1991).

3.2 *The Case of the Impatient Manager*

We now turn to the case of the impatient manager who plans to leave or retire before all the benefits of the investment are realized. Formally, $\tau < T$, where τ is the manager's planning horizon and T is the duration of the investment project under consideration. The central question is then how to provide incentives for a manager whose contract runs until period τ to only select investment projects with positive net present values, even if $\tau < T$.

Reichelstein (1997) and Rogerson (1997) can be considered the main contributions to this stream of literature. Similar to Sect. 3.1.1, Reichelstein (1997) and Rogerson (1997) suggest the remuneration in period t to be proportional to residual income RI_t, i.e. $\beta \cdot RI_t$. However, the solution outlined in Sect. 3.1.1 does not rely on any specific assumption regarding RI_t aside from $NI_1 + \cdots + NI_T = c_0 + c_1 + \cdots + c_T$. Since net incomes can be computed according to $NI_t = c_t - d_t \cdot |c_0|$, the latter premise only imposes the restriction $d_1 + \cdots + d_T = 1$. In contrast to that, implementing the Reichelstein/Rogerson solution requires quite specific depreciation factors, namely

$$d_t = \frac{c_t}{\sum_{j=1}^{T} c_j \cdot \gamma^j} - r \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right) . \tag{16}$$

Although not obvious, these depreciation factors indeed add to one, as stated in the following property.

Property 5.1 The depreciation factors (16) needed for implementing the Reichelstein/Rogerson solution add to one, i.e.

$$\sum_{t=1}^{T} d_t = \sum_{t=1}^{T} \left[\frac{c_t}{\sum\limits_{j=1}^{T} c_j \cdot \gamma^j} - r \cdot \left(1 - \sum_{j=1}^{t-1} d_j \right) \right] = 1 \,. \tag{17}$$

Hence, the premises of the Preinreich/Lücke Theorem 5.2 are fulfilled.
The proof can be found in the Appendix (section "Proof of Property 5.1").

In order to implement the Reichelstein/Rogerson solution, special informational requirements regarding the cash flow stream have to be met. Sections 3.2.1 and 3.2.2 discuss these requirements in detail.

3.2.1 Cash Flows Known

The most restrictive conceivable premise regarding the cash flow stream is that it is (up to a common factor, which cancels in (16)) known for each investment project. This should be the case in this section. Furthermore, we must limit ourselves to investment projects that suffice the conditions

$$c_0 < 0, \quad c_1 > 0, \quad c_2 > 0, \quad \ldots, \quad c_T > 0 \,. \tag{18}$$

The manager's present value of remuneration is then proportional to

$$\sum_{t=1}^{\tau} RI_t \cdot \gamma^t \,, \tag{19}$$

where residual incomes RI_t are computed according to the depreciation scheme (16). Theorem 5.3 states that this solution ensures goal congruence.

Theorem 5.3 *Given an investment project with known cash flows meeting conditions (18), a remuneration proportional to residual income based on the depreciation scheme (16) provides goal congruence in the following sense: The manager's present value of remuneration is positive if and only if the investment project's NPV is positive.*
The proof can be found in the Appendix (section "Proof of Theorem 5.3").

3.2.2 Risky Cash Flows with Known Expectation

In this section, we assume the cash flows $c_1, \ldots, c_T$ to be stochastic with known expected values $\mathrm{E}(c_1), \ldots, \mathrm{E}(c_T)$. The same applies to the (deterministic) initial net investment c_0. Furthermore, we assume manager and company to be risk neutral.

Then, the manager strives to maximize the expectation of her remuneration's present value, $\mathrm{E}(RI_1 \cdot \gamma^{1} + \cdots + RI_T \cdot \gamma^{T})$, and the company's goal is to maximize $\mathrm{E}(NPV)$. Similar to (18), the conditions

$$\mathrm{E}(c_1) > 0, \quad \ldots, \quad \mathrm{E}(c_T) > 0 \tag{20}$$

shall be met. If this is the case, the analysis of the previous section can easily be translated into the setting considered here. The depreciation factors (16) must of course not be interpreted as random variables, but as known figures

$$d_t = \frac{\mathrm{E}(c_t)}{\sum_{j=1}^{T} \mathrm{E}(c_j) \cdot \gamma^{j}} - r \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right). \tag{21}$$

Then, goal congruence can also be achieved, cf. Theorem 5.4.

Theorem 5.4 *Given an investment project with risky cash flows meeting conditions (20) and known* $c_0, \mathrm{E}(c_1), \ldots, \mathrm{E}(c_T)$*, a remuneration proportional to residual income based on the depreciation scheme (21) provides goal congruence in the following sense: The risk neutral manager's expected present value of remuneration is positive if and only if the investment project's* $\mathrm{E}(NPV)$ *is positive.*
The proof can be found in the Appendix (section "Proof of Theorem 5.4").

3.2.3 Cash Flows Reported by the Manager

When the company only knows the initial net investment c_0, it relies on reports or forecasts $\hat{c}_1, \ldots, \hat{c}_T$ regarding the cash flows $c_1, \ldots, c_T$ provided by the manager. Then, several residual income based remuneration schemes of the Reichelstein/Rogerson type are conceivable. Amongst others, depreciation factors may solely depend on the manager's reports, i.e.

$$d_t = \frac{\hat{c}_t}{\sum_{j=1}^{T} \hat{c}_j \cdot \gamma^{j}} - r \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right). \tag{22}$$

Alternatively, cash flows realized over time may be exploited. At date t, the company knows $c_1, \ldots, c_t$ and insofar only relies on reports regarding $c_{t+1}, \ldots, c_T$. Then depreciation factors

$$d_t = \frac{c_t}{\sum_{j=1}^{t} c_j \cdot \gamma^{j} + \sum_{j=t+1}^{T} \hat{c}_j \cdot \gamma^{j}} - r \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right) \tag{23}$$

may be applied. However, such approaches cannot guarantee goal congruence, as the following example illustrates.

Example 5.1 The investment project under consideration involves an initial net investment of $c_0 = -1000$ € and spans $T = 2$ periods. The impatient manager, whose planning horizon is $\tau = 1$, reports the cash flows $\hat{c}_1 = \hat{c}_2 = 600$, whereas the true values are $c_1 = 700$ and $c_2 = 300$. Using the discount factor $\gamma = 0.95$, one predicts

$$\widehat{NPV} = -1000 + 600 \cdot (0.95 + 0.95^2) = 111.5 \tag{24}$$

based on the manager's reports, whereas the true value

$$NPV = -1000 + 700 \cdot 0.95 + 300 \cdot 0.95^2 = -64.25 \tag{25}$$

is negative. Consequently, the company would prefer to forgo this investment. However, when applying depreciation scheme (23), the manager's present value of remuneration would be proportional to

$$RI_1 \cdot \gamma = \left(700 - \frac{700}{700 \cdot 0.95 + 600 \cdot 0.95^2} \cdot 1000\right) \cdot 0.95 = 113.82 \tag{26}$$

and, hence, positive. Accordingly, there is a considerable disincentive for the manager to propose a project with great c_1. □

To the best of our knowledge, no incentive-compatible mechanism to cope with the situation of an impatient manager ($\tau < T$) who reports potentially biased cash flows $\hat{c}_t$ has been proposed in the literature. This suggests that such a mechanism may indeed not exist. However, literature does also not provide evidence of such an impossibility.

4 Divisionalized Firm, Division Managers

In the preceding sections we assumed investment capital to be available in unlimited quantities. In practice, however, interest rates rise with investment volumes. But even when an uniform interest rate r exists (as assumed in this paper), a hard credit limit may have to be taken into account. We want to focus on this case in the following. Then, divisional managers will compete for the scarce resource investment capital. As less profitable, yet acceptable projects may crowd out perhaps better projects, central management crucially relies on truthful reports regarding potential returns provided by the divisional managers. As has been seen in the last section, truthful reports $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_T$ are (too) difficult to achieve. Therefore, we again limit the analysis to reports regarding the net present value.

Let $NPV_i(b_i)$ denote the net present value attainable by division i (with $i = 1, \ldots, n$) when equipped with an investment budget of b_i. This function is assumed to be monotonic increasing and concave. Further, let B denote the credit limit, i.e.

$$\sum_{i=1}^{n} b_i \leqq B \tag{27}$$

must be met. If $\widehat{NPV}_i(b_i)$ is divisional manager i's report regarding $NPV_i(b_i)$ (again, not a number, but a function of b_i), central management will maximize

$$\sum_{i=1}^{n} \widehat{NPV}_i(b_i) \quad \text{subject to} \quad \sum_{i=1}^{n} b_i \leqq B\,. \tag{28}$$

As $\widehat{NPV}_i(b_i)$ is also monotonic increasing and concave, the budget constraint will hold in equality in the optimum. We can therefore rewrite (28) as follows: Maximize

$$\sum_{i=1}^{n} \widehat{NPV}_i(b_i) \quad \text{subject to} \quad \sum_{i=1}^{n} b_i = B\,. \tag{29}$$

There is an unique solution to this problem, given by $(b_1^*; \ldots; b_n^*)$. This capital allocation is, however, only the real optimum when divisional managers have supplied truthful reports. The latter can be achieved when the remuneration of divisional manager i is a monotonic increasing function of the evaluation measure

$$EM_i = NPV_i(b_i^*) + \sum_{j \neq i} \widehat{NPV}_j(b_j^*)\,. \tag{30}$$

Accordingly, the evaluation measure of divisional manager i depends on the actual net present value of her own division as well as on the reports of all other divisions. Then, it is optimal for each divisional manager to provide truthful reports, regardless whether the other managers also convey the truth or distorted messages (Groves 1976; Bamberg and Trost 1998). Truthful reporting is hence not 'only' a Nash equilibrium strategy, but a dominant strategy. The following example inspired by Locarek and Bamberg (1994) serves to illustrate this so-called Groves mechanism.

Example 5.2 A company consisting of $n = 2$ divisions faces the credit limit $B = 10^8$ €. Furthermore, we assume

$$NPV_i(b_i) = 10^6 \cdot \ln b_i \quad \text{and} \quad \widehat{NPV}_i(b_i) = e_i \cdot \ln b_i\,. \tag{31}$$

Accordingly, $\widehat{NPV}_i(b_i)$ is distorted if $e_i \neq 10^6$. Central management maximizes $\widehat{NPV}_1(b_1) + \widehat{NPV}_2(b_2)$ subject to $b_1 + b_2 = B$. Maximizing the Lagrangian

$$\mathscr{L}(b_i, b_j, \lambda) = e_i \cdot \ln b_i + e_j \cdot \ln b_j - \lambda \cdot (b_i + b_j - B) \tag{32}$$

with respect to b_i immediately yields

$$\frac{e_i}{b_i} = \lambda\,. \tag{33}$$

Taking into account $b_i + b_j = B$, (33) implies

$$\frac{e_i + e_j}{\lambda} = B \iff \lambda = \frac{e_i + e_j}{B} \Rightarrow b_i^* = \frac{B}{e_i + e_j} \cdot e_i\,. \tag{34}$$

Divisional manager i maximizes

$$\begin{aligned} 10^6 \cdot \ln b_i^* + e_j \cdot \ln b_j^* &= (10^6 + e_j) \cdot \ln \tfrac{B}{e_i + e_j} + 10^6 \cdot \ln e_i + e_j \cdot \ln e_j \\ &= (10^6 + e_j) \cdot [\ln B - \ln(e_i + e_j)] + 10^6 \cdot \ln e_i + e_j \cdot \ln e_j \end{aligned} \tag{35}$$

with respect to e_i. Evaluating the first-order condition

$$-\frac{10^6 + e_j}{e_i + e_j} + \frac{10^6}{e_i} = 0 \tag{36}$$

immediately yields the prospect solution $e_i = 10^6$. A more detailed analysis of the function (35) reveals that it is indeed maximized by $e_i = 10^6$ and that this solution is unique. Hence, it is optimal to report the truth regardless of e_j, i.e. no matter what the other manager reports. Truthful reporting is thus a dominant strategy of divisional manager i. Since this applies to both divisional managers, pursuing these strategies establishes a dominant strategy equilibrium in the game played by the divisional managers, providing a substantially more stable solution than an 'ordinary' Nash equilibrium. Then, both divisions will be equipped with an investment budget amounting to $b_i^* = 0.5B$ and the company's net present value will be $2 \cdot 10^6 \cdot \ln(0.5 \cdot 10^8) = 35{,}455{,}067 \approx 35.46$ million €. □

Granted all divisional managers possess the mental capacities to grasp this rather sophisticated mechanism, the evaluation measures for all managers' remunerations will be the same, namely $NPV_1(b_1^*) + \cdots + NPV_n(b_n^*)$. Its precise value is, however, unknown until date $t = T$, the planning horizon. As interim payments are needed in practice, implementing such schemes induces great challenges for accounting: The evaluation measures are relevant at the time of reporting. On the other hand, according to the Preinreich/Lücke Theorem 5.2, net present value NPV and the stream of residual incomes $RI_1, \ldots, RI_T$ are equivalent. Insofar, evaluation measure and residual income RI_t (with respect to all projects) are identical in period t. In the case of rolling application, accounting has to determine RI_t for each new project. This is, of course, a rather involved task. On the other hand, no applicable dynamic version of the Groves mechanism is known until now.

5 Conclusion

We conclude gathering some critical remarks on the approaches presented above. First, as already indicated, mechanisms like the Groves scheme require rather pronounced mental capacities on the side of the managers in order to work properly. If managers fail to comprehend the mechanism, 'good solutions' (i.e. goal congruence, truthful reporting etc.) cannot be guaranteed. This is, of course, a problem inherent to all incentive mechanisms.

Another potential obstacle when implementing incentive mechanisms like the ones studied here are their premises. Almost all rely on restrictive assumptions like risk neutrality or even absence of risk, which are hardly fulfilled in practise. As soon as risk-aversion is taken into account, mechanisms based on risk neutrality fail to work properly. In addition to that, the hard to tackle task of measuring the risk attitudes (especially of the managers) arises.

Regarding the discount factor, we have tacitly assumed the same constant and uniform value for company and managers. If the manager uses a different discount factor, the problem of determination arises again. Furthermore, the Preinreich/Lücke Theorem cannot be adopted anymore.

Finally, let us note that the Reichelstein/Rogerson solution relies on rather specific depreciation factors d_t (cf. also the explicit formulae in the Appendix, section "Proof of Property 5.1"). These factors do not resemble common depreciation methods like straight-line or annuity depreciation and, hence, may disturb in practice. From a financial accounting and tax perspective, such modes of depreciation will almost surely violate national commercial or tax law.

Appendix

Proof of Theorem 5.1

Proof '$\Rightarrow$': Assume (2) holds true. Since this condition covers all possible cash flow streams, it also applies to non-stochastic streams $\mathbf{x}$. Therefore, $CE(\mathbf{x}+\mathbf{z})$ needs to be independent of the specific value of $\mathbf{z}$. In particular, it is allowed to determine $\mathbf{z}$ so that all components of $\mathbf{x}+\mathbf{z}$ except the first one, $x_0 - z_0$, are set to zero, i.e. $z_t = (-x_{t+1} + z_{t+1}) \cdot \gamma$ for all $t = 0, 1, \ldots, T-1$.

Following the recursion

$$\begin{aligned}
z_{T-1} &= -x_T \cdot \gamma \\
z_{T-2} &= -(-x_{T-1} + z_{T-1}) \cdot \gamma = -x_{T-1} \cdot \gamma - x_T \cdot \gamma^2 \\
z_{T-3} &= -(-x_{T-2} + z_{T-2}) \cdot \gamma = -x_{T-2} \cdot \gamma - x_{T-1} \cdot \gamma^2 - x_T \cdot \gamma^3 \\
&\cdots \\
z_0 &= -(-x_1 + z_1) \cdot \gamma \qquad = -x_1 \cdot \gamma - x_2 \cdot \gamma^2 \cdots - x_T \cdot \gamma^T
\end{aligned} \tag{37}$$

one arrives at $x_0 - z_0 = NPV(\mathbf{x})$. Therefore, the certainty equivalents $CE(\mathbf{x}+\mathbf{z})$ and $CE(NPV(\mathbf{x}), 0, \ldots, 0)$ coincide. Assuming (2) holds true, this implies $CE(\mathbf{x}) = CE(NPV(\mathbf{x}), 0, \ldots, 0)$, stating the indifference $\mathbf{x} \sim (NPV(\mathbf{x}), 0, \ldots, 0)$. Since $\mathbf{x}$ is non-stochastic, the structure asserted in (3) follows immediately.

'$\Leftarrow$': Assume the structure asserted in (3) applies to the multiattributive utility function. Then, the date 0 certainty equivalent $CE(\mathbf{c}+\mathbf{z})$ is implicitly given by

$$\begin{aligned} u(CE(\mathbf{c}+\mathbf{z}), 0, \ldots, 0) &= \mathrm{E}u(\mathbf{c}+\mathbf{z}) = \mathrm{E}u(NPV(\mathbf{c}+\mathbf{z}), 0, \ldots, 0) \\ &= \mathrm{E}u(NPV(\mathbf{c}) + NPV(\mathbf{z}), 0, \ldots, 0)\,. \end{aligned} \tag{38}$$

Since

$$\begin{aligned} NPV(\mathbf{z}) &= -z_0 + \frac{qz_0 - z_1}{q} + \frac{qz_1 - z_2}{q^2} + \cdots + \frac{qz_{T-2} - z_{T-1}}{q^{T-1}} + \frac{qz_{T-1}}{q^T} \\ &= -z_0 + z_0 - \frac{z_1}{q} + \frac{z_1}{q} - \frac{z_2}{q^2} + \cdots + \frac{z_{T-2}}{q^{T-2}} - \frac{z_{T-1}}{q^{T-1}} + \frac{z_{T-1}}{q^{T-1}} = 0\,, \end{aligned} \tag{39}$$

$CE(\mathbf{c}+\mathbf{z})$ does not depend on $\mathbf{z}$ and, hence, condition (2) is fulfilled. □

Proof of Theorem 5.2

Proof Evaluating the shift in equity capital from period $t-1$ to period t, $EC_t - EC_{t-1}$, one immediately arrives at

$$EC_t - EC_{t-1} = NI_t - c_t \iff NI_t = c_t + EC_t - EC_{t-1}\,. \tag{40}$$

Hence, the sum of discounted residual incomes can be written as

$$\begin{aligned} \sum_{t=0}^{T} RI_t \cdot \gamma^t &= \sum_{t=0}^{T} (c_t + EC_t - EC_{t-1} - r \cdot EC_{t-1}) \cdot \gamma^t \\ &= \sum_{t=0}^{T} c_t \cdot \gamma^t + \sum_{t=0}^{T} EC_t \cdot \gamma^t - \sum_{t=0}^{T} qEC_{t-1} \cdot \gamma^t \\ &= \sum_{t=0}^{T} c_t \cdot \gamma^t + \sum_{t=0}^{T} EC_t \cdot \gamma^t - \sum_{t=0}^{T} EC_{t-1} \cdot \gamma^{t-1} \\ &= \sum_{t=0}^{T} c_t \cdot \gamma^t + EC_T \cdot \gamma^T. \end{aligned} \tag{41}$$

Taking $RI_0 = EC_T = 0$ into account, (5) follows immediately. □

Proof of Property 5.1

Proof Resolving the recursion, one arrives at

$$d_t = \zeta^{-1} \cdot \left[c_t + \sum_{j=1}^{t-1} c_j \cdot \sum_{i=1}^{t-j} \binom{t-j-1}{i-1} \cdot r^i \right] - \sum_{i=1}^{t} \binom{t-1}{i-1} \cdot r^i , \tag{42}$$

where

$$\zeta = \sum_{j=1}^{T} c_j \cdot \gamma^{\,j} . \tag{43}$$

Please note

$$\sum_{i=1}^{t-j} \binom{t-j-1}{i-1} \cdot r^i = r \cdot (1+r)^{t-j-1} = (1-\gamma) \cdot \gamma^{\,j-t} \tag{44}$$

and

$$\sum_{i=1}^{t} \binom{t-1}{i-1} \cdot r^i = r \cdot (1+r)^{t-1} = (1-\gamma) \cdot \gamma^{\,-t} . \tag{45}$$

Hence, (42) can be rewritten as follows

$$\begin{aligned} d_t &= \zeta^{-1} \left[c_t - (1-\gamma) \cdot \gamma^{\,-t} \cdot \zeta + (1-\gamma) \cdot \sum_{j=1}^{t-1} c_j \cdot \gamma^{\,j-t} \right] \\ &= \zeta^{-1} \left[\gamma \cdot c_t - (1-\gamma) \cdot \gamma^{\,-t} \cdot \zeta + (1-\gamma) \cdot \sum_{j=1}^{t} c_j \cdot \gamma^{\,j-t} \right] . \end{aligned} \tag{46}$$

We are now able to evaluate the sum

$$\begin{aligned} \sum_{t=1}^{T} d_t &= \zeta^{-1} \cdot \left[\gamma \cdot \sum_{t=1}^{T} c_t - (1-\gamma) \cdot \zeta \cdot \sum_{t=1}^{T} \gamma^{-t} + (1-\gamma) \cdot \sum_{t=1}^{T} \sum_{j=1}^{t} c_j \cdot \gamma^{\,j-t} \right] \\ &= \zeta^{-1} \cdot \left[\gamma \cdot \sum_{t=1}^{T} c_t + (1-\gamma^{-T}) \cdot \zeta + (1-\gamma) \cdot \sum_{t=1}^{T} \sum_{j=1}^{t} c_j \cdot \gamma^{\,j-t} \right] \\ &= 1 + \zeta^{-1} \cdot \left[\sum_{t=1}^{T} c_t \cdot (\gamma - \gamma^{\,t-T}) + (1-\gamma) \cdot \sum_{t=1}^{T} \sum_{j=1}^{t} c_j \cdot \gamma^{\,j-t} \right] . \end{aligned} \tag{47}$$

In order to prove $d_1 + \cdots + d_T = 1$ it is sufficient to verify that the term in square brackets is equal to zero. Since

$$\sum_{t=1}^{T}\sum_{j=1}^{t} c_j \cdot \gamma^{j-t} = \sum_{t=1}^{T} c_t \cdot \sum_{j=0}^{T-t} \gamma^{-j} = \sum_{t=1}^{T} c_t \cdot \frac{\gamma^{t-T} - \gamma}{1-\gamma} \tag{48}$$

this is in fact the case. □

Proof of Theorem 5.3

Proof Since

$$\begin{aligned}
\sum_{t=1}^{\tau} RI_t \cdot \gamma^t &= \sum_{t=1}^{\tau}\left[c_t + d_t \cdot c_0 + r \cdot c_0 \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right)\right] \cdot \gamma^t \\
&= \sum_{t=1}^{\tau}\Bigg[c_t + \underbrace{c_0 \cdot \frac{c_t}{\sum_{j=1}^{T} c_j \cdot \gamma^j} - r \cdot c_0 \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right)}_{=d_t \cdot c_0} + r \cdot c_0 \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right)\Bigg] \cdot \gamma^t \\
&= \sum_{t=1}^{\tau}\left[c_t + c_0 \cdot \frac{c_t}{\sum_{j=1}^{T} c_j \cdot \gamma^j}\right] \cdot \gamma^t = \sum_{t=1}^{\tau}\left[1 + \frac{c_0}{\sum_{j=1}^{T} c_j \cdot \gamma^j}\right] \cdot c_t \cdot \gamma^t \\
&= \frac{1}{\sum_{j=1}^{T} c_j \cdot \gamma^j} \cdot \sum_{t=1}^{\tau}\left[c_0 + \sum_{j=1}^{T} c_j \cdot \gamma^j\right] \cdot c_t \cdot \gamma^t = \frac{\sum_{t=1}^{\tau} c_t \cdot \gamma^t}{\sum_{j=1}^{T} c_j \cdot \gamma^j} \cdot NPV,
\end{aligned} \tag{49}$$

manager's present value of remuneration and the investment project's *NPV* have the same sign as long as $c_1 > 0, \ldots, c_T > 0$. The latter is ensured when condition (18) is met. □

Proof of Theorem 5.4

Proof Since

$$\mathrm{E}\left(\sum_{t=1}^{\tau} RI_t \cdot \gamma^t\right) = \sum_{t=1}^{\tau} \mathrm{E}(RI_t) \cdot \gamma^t = \sum_{t=1}^{\tau} \mathrm{E}\left[c_t + d_t \cdot c_0 + r \cdot c_0 \cdot \left(1 - \sum_{j=1}^{t-1} d_j\right)\right] \cdot \gamma^t$$

$$= \sum_{t=1}^{\tau} \mathrm{E}\Bigg[c_t + \underbrace{c_0 \cdot \frac{\mathrm{E}(c_t)}{\sum_{j=1}^{T} \mathrm{E}(c_j)\cdot\gamma^j} - r\cdot c_0 \cdot \Big(1-\sum_{j=1}^{t-1} d_j\Big)}_{=d_t\cdot c_0} + r\cdot c_0\cdot\Big(1-\sum_{j=1}^{t-1} d_j\Big)\Bigg]\cdot\gamma^t$$

$$= \sum_{t=1}^{\tau} \mathrm{E}\Bigg[c_t + c_0 \cdot \frac{\mathrm{E}(c_t)}{\sum_{j=1}^{T} \mathrm{E}(c_j)\cdot\gamma^j}\Bigg]\cdot\gamma^t = \sum_{t=1}^{\tau}\Bigg[1+\frac{c_0}{\sum_{j=1}^{T} \mathrm{E}(c_j)\cdot\gamma^j}\Bigg]\cdot \mathrm{E}(c_t)\cdot\gamma^t$$

$$= \frac{1}{\sum_{j=1}^{T} \mathrm{E}(c_j)\cdot\gamma^j}\cdot\sum_{t=1}^{\tau}\Bigg[c_0+\sum_{j=1}^{T}\mathrm{E}(c_j)\cdot\gamma^j\Bigg]\cdot\mathrm{E}(c_t)\cdot\gamma^t$$

$$= \frac{\sum_{t=1}^{\tau}\mathrm{E}(c_t)\cdot\gamma^t}{\sum_{j=1}^{T}\mathrm{E}(c_j)\cdot\gamma^j}\cdot\mathrm{E}(NPV)\,, \tag{50}$$

manager's expected present value of remuneration and the investment project's expected NPV have the same sign as long as $\mathrm{E}(c_1) > 0, \ldots, \mathrm{E}(c_T) > 0$. The latter is ensured when condition (20) is met. □

References

Bamberg G (1991) Extended contractual incentives to reduce project cost. OR Spectr 13(2):95–98

Bamberg G, Trost R (1998) Anreizsysteme und kapitalmarktorientierte Unternehmenssteuerung. In: Möller HP, Schmidt F (eds) Rechnungswesen als Instrument für Führungsentscheidungen. Schäffer-Poeschel, Stuttgart, pp 91–109

Bamberg G, Coenenberg AG, Krapp M (2013) Betriebswirtschaftliche Entscheidungslehre, 15th edn. Vahlen, München

Groves T (1976) Incentive compatible control of decentralized organizations. In: Ho YC, Mitters SK (eds) Directions in large-scale systems: many person optimization and dezentralized control. Plenum Press, New York, pp 149–185

Grottke M, Schosser J (2014) Goal congruence and preference similarity between principal and agent with differing time horizons – setting incentives under risk. GEABA Discussion Paper DP 14–29

Holmström B, Milgrom PR (1987) Aggregation and linearity in the provision of intertemporal incentives. Econometrica 55(2):303–328

Locarek H, Bamberg G (1994) Anreizkompatible Allokationsmechanismen für divisionalisierte Unternehmungen. Wirtschaftswissenschaftliches Studium 23(1):10–14

Lücke W (1955) Investitionsrechnung auf der Basis von Ausgaben oder Kosten? Zeitschrift für handelswissenschaftliche Forschung/N. F. 7(3):310–324

Preinreich GAD (1937) Valuation and amortization. TAR 12(3):209–226

Reichelstein S (1997) Investment decisions and managerial performance evaluation. RAS 2(2):157–180

Reichelstein H-E, Reichelstein S (1987) Leistungsanreize bei öffentlichen Aufträgen. Zeitschrift für Wehrtechnik 19(3):44–49

Rogerson WP (1997) Intertemporal cost allocation and managerial investment incentives: a theory explaining the use of economic value added as a performance measure. JPolE 105(4):770–795
Spremann K (1987) Agent and principal. In: Bamberg G, Spremann K (eds) Agency theory, information, and incentives. Springer, Berlin, pp 3–37

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Interorganizational Resource Sharing in Research and Development Alliances

Gerhard Aust, Dora Dominko, and Udo Buscher

Abstract Shorter life cycles as well as increasing complexity of new product developments can induce companies to form research and development alliances. This study especially focuses on the management accounting perspective of such alliances and analyses the companies' research efforts, the distribution of costs, and the division of resulting profits. The analytically obtained results as well as the numerical studies show that: (a) not each scenario leads to a viable equilibrium in which both companies would like to participate; (b) research effort will be higher under an equal distribution of power; (c) total profit under equal distribution of power and individual profit maximization yields the same profit as a joint profit maximization, which is always higher than with a dominant company.

Keywords Distribution of power • Nash • Profit sharing • Research and development alliance • Resource sharing • Stackelberg

Contents

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1 Introduction

Cooperations between companies play an important role in many industries and occur in various business areas like collective procurement, advertising, or distribution of products. Depending on the subject, members of cooperations can either belong to different echelons of a supply chain or the same echelon, which means that competitors decide to work together for a common purpose. Especially in the automotive industry, alliances between competitors can also be found in the area of research and development. Reasons for this behaviour are manifold: Shorter development periods following from shorter life cycles of products and technologies, growing complexity of new vehicle concepts, and cost pressure through the globalization of markets (Wallentowitz et al. 2009, p. 14; Hensel 2007, pp. 1 et seq.). This becomes particularly apparent when considering the currently ongoing transition to electric mobility, which necessitates (to some extent) the development of completely new technologies.

The relevance of research and development alliances in practice shifts this topic also in the focus of research. Thereby, mathematical approaches in this area often apply game theory, because—in contrast to the individual optimization of classical operations research methods—it allows to analyse the strategic behaviour of the members of an alliance, the interdependencies between their decisions, as well as different forms of distribution of power within the alliance. For a recent and comprehensive review of game-theoretic studies on knowledge creation and sharing in product development, we refer the reader to Arsenyan et al. (2015, pp. 2074 et seg.), which lists various subjects like knowledge spillover or knowledge sharing incentives.

Our aim is to study the research and development alliances from a management accounting perspective, i.e., the resource sharing, the distribution of costs, and the division of the resulting profits in an already existing partnership. Closely related to this is also the decision on the research intensity or the efforts that should be made within the collaboration, which is also a common subject in literature. For example, Katz (1986) proposes a very comprehensive four-stage approach consisting of the formation of an alliance, the agreeing on the cost sharing between the members, the decision on the research intensity, and, lastly, the production quantity. The effects of knowledge spillover on the determination of research investment and production quantity in a research and development joint venture are studied in d'Aspremont and Jacquemin (1988), Nakao (1989), and Suzumura (1992). These studies show that a cooperation is especially beneficial compared to a competitive behaviour when the risk of a knowledge spillover to other companies is high. Samaddar and Kadiyala (2006) omit the formation process as well as the determination of production quantities and concentrate on knowledge creation effort and cost sharing in an alliance consisting of two companies that act in a leader-follower structure. Based on that, two scenarios without and with knowledge spillover are analysed. The latter is extended by Ding and Huang (2010) by the possibility of positive and negative spillovers. Recently, Arsenyan et al. (2015) design a negotiation framework based upon a bargaining model that shall help to determine collaboration level, cost sharing, and revenue sharing in collaborative product development.

In the following, we would like to draw the attention to the impact of the relationship between the companies that form a research and development alliance on the performance and outcome of this alliance, with special focus on the cost and profit sharing. Therefore, we take the model of Samaddar and Kadiyala (2006) as basis, because it concentrates on the financial aspects of a research alliance without taking into account other topics like production quantities. We extend this analysis, which is limited to a leader-follower scenario, by other forms of relationship like equal distribution of power and joint profit maximization in order to get insights into the changes of strategies that result from those configurations of an alliance. Thus, the remainder is organized as follows.

Section 2 will describe the basic assumptions of the mathematical model like decision variables, parameters, and profit functions of the participating companies. On this basis, four scenarios of relationship within the research and development alliance are considered and closed-form expressions of the respective equilibria are derived: first, two leader-follower structures with different sequence of decisions (Sects. 3.1 and 3.2); then, an symmetric distribution of power between the companies with individual (Sect. 3.3) and joint profit maximization (Sect. 3.4). The results will be compared in Sect. 4 by means of numerical examples and sensitivity analyses. Section 5 will finally summarize the main findings of our research and will give some directives for future research.

2 Model Formulation

Following Samaddar and Kadiyala (2006, pp. 197 et seq.), we consider a research and development alliance consisting of two companies $i = \{1, 2\}$ (see Table 1 for a complete list of symbols). Within this cooperation, the partners jointly invest into a certain research and development activity. This total investment is called knowledge creation effort and is represented by a (with $a \geq 0$). Please note that this does not only comprehend monetary investments, but also the involvement of human capital, technical equipment, etc., which can be converted in monetary units, though. The total effort a is shared by both companies, with participation rate t_1 (with $0 \leq t_1 \leq 1$) denoting the percentage that is attributable to company $i = 1$ and $t_2 = 1 - t_1$ being the share of company $i = 2$, respectively.

Table 1 List of symbols

Variables		Parameters	
a	Knowledge creation effort	α	Saturation asymptote
t_i	Participation rate of company i	γ	Research performance elasticity
$P(a)$	Research performance function	ρ_i	Marginal profit of company i
π_i	Profit of company i		
π	Total profit		

Depending on the amount of the total investment a, the alliance achieves a research outcome $P = P(a)$, which can be interpreted as performance of the alliance in the considered research field that results from the joint activities. To account for the saturation effect that commonly occurs with increasing investments, we assume an increasing and concave function

$$P(a) = \alpha - a^{-\gamma}. \tag{1}$$

The positive parameter α denotes the saturation asymptote of the performance function $P(a)$, while γ can be seen as elasticity parameter. Hence, it describes the relative change of research performance, $\mathrm{d}P/P$, compared to the relative change of knowledge creation effort, $\mathrm{d}a/a$, and controls the gradient of $P(a)$: For $\gamma = 0$, the effort a has no effect on the resulting performance P, whereas $0 < \gamma < 1$ ($\gamma > 1$) means that the performance increases to a lesser (greater) extent than the effort; for $\gamma = 1$, a variation of a leads to an equal variation of P.

To measure the utility of the achieved research performance, we introduce the parameter ρ_i that specifies the marginal profit of company i for each additional unit of P. Subtracting the costs for this performance, i.e., the share in the total knowledge creation effort, we are able to formulate the companies' profit functions π_i:

$$\pi_1 = \rho_1 P(a) - t_1 a \tag{2}$$

$$\pi_2 = \rho_2 P(a) - (1 - t_1)a. \tag{3}$$

By means of the performance function in Eq. (1), one can formulate the complete profit functions of both companies

$$\pi_1 = \rho_1 \left(\alpha - a^{-\gamma}\right) - t_1 a \tag{4}$$

$$\pi_2 = \rho_2 \left(\alpha - a^{-\gamma}\right) - (1 - t_1)a, \tag{5}$$

as well as the profit function of the entire research alliance:

$$\pi = \pi_1 + \pi_2 = (\rho_1 + \rho_2)\left(\alpha - a^{-\gamma}\right) - a. \tag{6}$$

Obviously, the participation rate t is not part of the alliance's profit function π, because it is only relevant for the division of research and development effort between the two companies and, consequently, for the division of profits between them. Therefore, it has no effects on the total profit.

3 Four Scenarios of Relationship Within the Alliance

In this section, we analyse four different scenarios of relationship between the two companies forming a research and development alliance, which differ in the underlying distribution of power and in the intensity of collaboration. This extends previous research of Samaddar and Kadiyala (2006), who only consider an asymmetric distribution of power where company $i = 1$ is assumed to obtain the lead and first sets the participation rate t_1, while company $i = 2$ subsequently decides on the total knowledge creation effort a (Scenario I).

Similar to that, we analyse the reverse situation in Scenario II, with company $i = 2$ first deciding on the effort variable a and company $i = 1$ choosing its participation rate t_1 afterwards in order to investigate the effects of the sequence of decisions (total effort, participation rate) on the resulting profits.

In addition to that, two scenarios without power imbalances are studied, one with individual profit maximization (Scenario III) and one with joint profit maximization (Scenario IV). We mainly focus on the changes within the resulting strategies that emerge from the aforementioned different ways of decision-making to find out if there are better forms of managing a research and development alliance compared to the approach proposed by Samaddar and Kadiyala (2006).

3.1 *Scenario I: Asymmetric Distribution of Power*

We start our analysis with the asymmetric distribution of power studied, where the participation rate is set before the total research effort (Samaddar and Kadiyala 2006). This situation can be modeled by means of a Stackelberg game, in which company $i = 1$ obtains the Stackelberg leadership and company $i = 2$ is called Stackelberg follower. Mathematically, the solution of this game can be calculated by backward induction: We start with the follower's decision problem to determine its response function, which then serves as a constraint for the leader's optimization problem. Accordingly, company $i = 2$'s decision problem is

$$\begin{aligned} \max \quad & \pi_2 = \rho_2\,(\alpha - a^{-\gamma}) - (1 - t_1)a \\ \text{s.t.} \quad & a \geq 0. \end{aligned} \tag{7}$$

Setting the first order derivative $\mathrm{d}\pi_2/\,\mathrm{d}a$ to zero leads to

$$\rho_2 \gamma a^{-\gamma-1} - (1 - t_1) = 0, \tag{8}$$

which can be reformulated to the response function of company $i = 2$ that indicates the follower's best decision on a depending on the leader's decision variable t:

$$a(t) = \left(\frac{1 - t_1}{\rho_2 \gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)}. \tag{9}$$

According to perfect information, which is an inherent assumption of the Stackelberg game, company $i = 1$ as the leader is able to anticipate the follower's response in its own decision problem:

$$\begin{aligned} \max \quad & \pi_1 = \rho_1 (\alpha - a^{-\gamma}) - t_1 a \\ \text{s.t.} \quad & a(t_1) = \left(\frac{1-t_1}{\rho_2\gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)} \\ & 0 \leq t_1 \leq 0. \end{aligned} \tag{10}$$

Therefore, we first have to insert Eq. (9) into company $i = 1$'s profit function before we can calculate the first order condition. Thereby we obtain a profit function which solely depends on the participation rate t_1:

$$\pi_1(t_1) = \rho_1 \left\{\alpha - \left[\left(\frac{1-t_1}{\rho_2\gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)}\right]^{-\gamma}\right\} - t_1 \left(\frac{1-t_1}{\rho_2\gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)}. \tag{11}$$

Solving the corresponding first order condition $d\pi_1/\,dt_1 = 0$ for the participation rate t_1 and inserting the resulting expression into Eq. (9), we derive the Stackelberg equilibrium given in the following proposition:

Proposition 6.1 *Given an asymmetric distribution of power between the members of an alliance, where the leader first sets its participation rate and the follower sets the total research effort, a Stackelberg equilibrium will be reached with:*

(i) $a = \begin{cases} [\gamma(\rho_1 - \rho_2\gamma)]^{\frac{1}{\gamma+1}} & \textit{for} \quad \frac{\rho_1}{\rho_2} \geq \gamma + 1 \\ (\rho_2\gamma)^{\frac{1}{\gamma+1}} & \textit{otherwise.} \end{cases}$

(ii) $t_1 = \begin{cases} \frac{\rho_1 - \rho_2(\gamma+1)}{\rho_1 - \rho_2\gamma} & \textit{for} \quad \frac{\rho_1}{\rho_2} \geq \gamma + 1 \\ 0 & \textit{otherwise.} \end{cases}$

(iii) $t_2 = \begin{cases} \frac{\rho_2}{\rho_1 - \rho_2\gamma} & \textit{for} \quad \frac{\rho_1}{\rho_2} \geq \gamma + 1 \\ 1 & \textit{otherwise.} \end{cases}$

(iv) $\pi_1 = \begin{cases} \rho_1 \left[\alpha - [\gamma(\rho_1 - \rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - \frac{\rho_1 - \rho_2(\gamma+1)}{\rho_1 - \rho_2\gamma}[\gamma(\rho_1 - \rho_2\gamma)]^{\frac{1}{\gamma+1}} & \textit{for} \quad \frac{\rho_1}{\rho_2} \geq \gamma + 1 \\ \rho_1 \left[\alpha - (\rho_2\gamma)^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] & \textit{otherwise.} \end{cases}$

(v) $\pi_2 = \begin{cases} \rho_2 \left[\alpha - [\gamma(\rho_1 - \rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - \frac{\rho_2}{\rho_1 - \rho_2\gamma}[\gamma(\rho_1 - \rho_2\gamma)]^{\frac{1}{\gamma+1}} & \textit{for} \quad \frac{\rho_1}{\rho_2} \geq \gamma + 1 \\ \rho_2 \left[\alpha - (\rho_2\gamma)^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - (\rho_2\gamma)^{\frac{1}{\gamma+1}} & \textit{otherwise.} \end{cases}$

(vi) $\pi = \begin{cases} (\rho_1 + \rho_2) \left[\alpha - [\gamma(\rho_1 - \rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - [\gamma(\rho_1 - \rho_2\gamma)]^{\frac{1}{\gamma+1}} & \textit{for} \quad \frac{\rho_1}{\rho_2} > \gamma + 1 \\ (\rho_1 + \rho_2) \left[\alpha - (\rho_2\gamma)^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - (\rho_2\gamma)^{\frac{1}{\gamma+1}} & \textit{otherwise.} \end{cases}$

As visible from Proposition 6.1, a case-by-case analysis is necessary, because the resulting participation rate t_1 given in Part (i) may take negative values when the ratio ρ_1/ρ_2 goes below $(\gamma + 1)$. A negative participation rate would contradict the domain of definition $0 \leq t_1 \leq 1$ stated in Sect. 2, for which reason we set the participation of company $i = 1$ to the lowest possible value $t_1 = 0$ (Samaddar and Kadiyala 2006, p. 198). In this case, company $i = 2$ will have to bear the entire research effort.

3.2 Scenario II: Asymmetric Distribution of Power with Reversed Sequence of Decisions

Now we reverse the distribution of power within the alliance so that company $i = 2$, which sets the knowledge creation effort a, takes the lead and company $i = 1$, which decides on its participation rate t_1, follows. By this means, we intend to analyse the effects of the sequence of decisions and whether the company obtaining the dominant position should set the research effort or the participation rate. As aforementioned, this asymmetric distribution of power corresponds to a Stackelberg game, which can be solved mathematically via backward induction.

We start with the decision problem of the follower, company $i = 1$:

$$\begin{aligned} \max \quad & \pi_1 = \rho_1 (\alpha - a^{-\gamma}) - t_1 a \\ \text{s.t.} \quad & 0 \leq t_1 \leq 1. \end{aligned} \tag{12}$$

As visible, participation rate t_1 occurs only once within the profit function of company $i = 1$. Due to its negative sign, the profit π_1 will be maximum for the smallest possible value of t_1, which is, according to the domain of definition,

$$t_1 = 0. \tag{13}$$

This behaviour can be explained by the fact that the follower has no information about the effects of his decision on the leader's strategy (i.e., the setting of a). Therefore, from a theoretical point of view, a participation rate $t_1 > 0$ would only cause costs without increasing the research and development output of the alliance and, consequently, the profit.

In contrast, the leading company $i = 2$ is able to include the follower's supposed strategy into its own decision problem, which is:

$$\begin{aligned} \max \quad & \pi_2 = \rho_2 (\alpha - a^{-\gamma}) - (1 - t_1) a \\ \text{s.t.} \quad & t_1 = 0 \\ & a \geq 0. \end{aligned} \tag{14}$$

Inserting Eq. 13 into the profit function leads to the leader's objective function

$$\pi_2 = \rho_2 \left(\alpha - a^{-\gamma}\right) - a, \tag{15}$$

which has to be differentiated with respect to a and set to zero. By solving this equation for the research effort a, we derive the Stackelberg equilibrium stated in the following proposition:

Proposition 6.2 *Given an asymmetric distribution of power between the members of an alliance, where the leader first sets the total research effort and the follower sets its participation rate, a Stackelberg equilibrium will be reached with:*

(i) $a = (\rho_2\gamma)^{\frac{1}{\gamma+1}}$.
(ii) $t_1 = 0$.
(iii) $t_2 = 1$.

We can see that the reversed sequence of decisions, in which the leader sets the total research effort and the follower specifies its participation rate afterwards, also implicates that the follower has no incentive to share the research costs. Consequently, this game setting cannot be seen as a viable approach for determining the optimal strategy of a research and development alliance, because the entire effort would have to be borne by one single company. Therefore, resulting profits are omitted in Proposition 6.2, though they equal the profits given in Proposition 6.1 for $\rho_1/\rho_2 < (\gamma + 1)$.

3.3 Scenario III: Symmetric Distribution of Power with Individual Profit Maximization

In this section we turn away from dominance within the research and development alliance and consider a scenario with two coequal partners. Assuming an individual profit maximization of the members of the alliance, it is common to apply a Nash game, which bases on a simultaneous decision process. Hence, both individual optimization problems have to be solved simultaneously and without including information about the partner's strategy.

First, we keep the assignment of decision variables as before, that means company $i = 1$ sets its participation rate t_1 in the total research and development effort a, which is (simultaneously) determined by company $i = 2$. The resulting decision problems of both partners are as follows:

$$\begin{aligned} \max \quad & \pi_1 = \rho_1 \left(\alpha - a^{-\gamma}\right) - t_1 a \\ \text{s.t.} \quad & 0 \le t_1 \le 1. \end{aligned} \tag{16}$$

$$\begin{aligned} \max \quad & \pi_2 = \rho_2 \left(\alpha - a^{-\gamma}\right) - (1 - t_1) a \\ \text{s.t.} \quad & a \ge 0. \end{aligned} \tag{17}$$

Similar to Scenario II described in the previous section, company $i = 1$ has no incentive to participate in the costs for the knowledge creation effort, because it has no information about the effects of its participation on the level of efforts. Therefore, company $i = 1$ will set

$$t_1 = 0, \tag{18}$$

independently of the actual research effort a. To find the response function of company $i = 2$, we set the first order derivative $\mathrm{d}\pi_2/\,\mathrm{d}a$ to zero:

$$\frac{\mathrm{d}\pi_2}{\mathrm{d}a} = \rho_2\gamma a^{-\gamma-1} - (1 - t_1) = 0. \tag{19}$$

The corresponding response function of company $i = 2$, which describes the best strategy (i.e., a) for any t_1 set by company $i = 1$, is

$$a(t_1) = \left(\frac{1 - t_1}{\rho_2\gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)}. \tag{20}$$

Both response functions given in Eqs. (18) and (20) constitute a system of equations which describes the Nash equilibrium of this game, i.e., the combination of strategies in which no player is able to increase its profit without reducing the partner's profit. It is easy to see that inserting Eq. (18) into Eq. (20) leads to the same equilibrium as in the previous Scenario III, which is stated in the following proposition:

Proposition 6.3 *Given a symmetric distribution of power between the members of an alliance, where the companies individually maximize their profits, a Nash equilibrium will be reached with:*

(i) $a = (\rho_2\gamma)^{\frac{1}{\gamma+1}}$.
(ii) $t_1 = 0$.
(iii) $t_2 = 1$.

We can see that this scenario leads to the same results that we already observed in Proposition 6.1 for $\rho_1/\rho_2 < (\gamma + 1)$ and in Proposition 6.2. Although this would be the mathematically correct solution for the described Nash game, its practical realization might be as questionable as the solution of Scenario II (see Sect. 3.2). Therefore, we present an alternative game setting in the following.

In contrast to the previous approaches, we now cancel the stringent assignment of decision variables within the alliance. Instead of that, the idea is to first investigate which level of total knowledge creation effort each company would prefer itself, independently of the individual participation in that total effort. That means, the

Nash game consists of the two individual decision problems:

$$\begin{aligned} \max \quad & \pi_i = \rho_i \left(\alpha - a^{-\gamma}\right) - t_i a \\ \text{s.t.} \quad & a \geq 0 \end{aligned} \qquad \text{for } i = 1, 2. \tag{21}$$

The solution can be calculated by setting the first order derivative $\mathrm{d}\pi_i / \mathrm{d}a$ to zero and solving the resulting equation with respect to a:

$$a^{(i)} = \left(\frac{t_i}{\rho_i \gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)} \qquad \text{for } i = 1, 2. \tag{22}$$

Thereby, $a^{(i)}$ denotes the total knowledge creation effort preferred by company i. As visible, it still depends on the own participation rate t_i and can of course differ between the two companies. However, in order to reach an equilibrium within the alliance, it is necessary that both companies agree on the same effort level. That means, under symmetric distribution of power, where no company is able to dominate its partner, an alliance is only possible when both companies have the same opinion on how much effort should be invested into the common project. Hence, the preferred knowledge creation effort derived in Eq. (22) has to be equal:

$$\begin{aligned} a^{(1)} &= a^{(2)} \\ \left(\frac{t_1}{\rho_1 \gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)} &= \left(\frac{t_2}{\rho_2 \gamma}\right)^{-\left(\frac{1}{\gamma+1}\right)}. \end{aligned} \tag{23}$$

Substituting the participation rate of company $i = 2$ by $t_2 = 1 - t_1$, we obtain an equation with t_1 being the only remaining variable. Solving this equation with respect to t_1 leads to:

$$t_1 = \frac{\rho_1}{\rho_1 + \rho_2} \tag{24}$$

Finally, we can obtain the Nash equilibrium given in the following proposition by inserting Eq. (24) into Eq. (22):

Proposition 6.4 *Given a symmetric distribution of power between the members of an alliance, where the companies individually maximize their profits, a Nash equilibrium will be reached with:*

(i) $a = \left[\gamma\left(\rho_1 + \rho_2\right)\right]^{\frac{1}{\gamma+1}}$.
(ii) $t_1 = \frac{\rho_1}{\rho_1+\rho_2}$.
(iii) $t_2 = \frac{\rho_2}{\rho_1+\rho_2}$.
(iv) $\pi_1 = \rho_1 \left[\alpha - \left[\gamma(\rho_1 + \rho_2)\right]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right] - \frac{\rho_1}{\rho_1+\rho_2}\left[\gamma(\rho_1 + \rho_2)\right]^{\frac{1}{\gamma+1}}$.

(v) $\pi_2 = \rho_2 \left[\alpha - [\gamma(\rho_1 + \rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)} \right] - \frac{\rho_2}{\rho_1+\rho_2} [\gamma(\rho_1 + \rho_2)]^{\frac{1}{\gamma+1}}$.

(vi) $\pi = (\rho_1 + \rho_2) \left[\alpha - [\gamma(\rho_1 + \rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)} \right] - [\gamma(\rho_1 + \rho_2)]^{\frac{1}{\gamma+1}}$.

Obviously, the participation rate of each company depends strongly on the marginal profit parameters ρ_i and, especially, on the share of marginal profit obtained by each company compared to the total marginal profit $(\rho_1 + \rho_2)$. Hence, this approach is able to provide a reasonable alternative to the Nash equilibrium presented in the beginning of this section, because both companies participate in the total research effort.

3.4 Scenario IV: Symmetric Distribution of Power with Joint Profit Maximization

The last scenario to be analysed is a joint profit maximization which solely concentrates on the total profit of the research and development alliance π instead of the individual profits. Commonly, this form of cooperation is used as a benchmark to rate the performance of the previously described game scenarios and equilibria. The underlying optimization problem is:

$$\begin{aligned} \max \quad & \pi = (\rho_1 + \rho_2)\,(\alpha - a^{-\gamma}) - a \\ \text{s.t.} \quad & a \geq 0. \end{aligned} \tag{25}$$

Please note that the only remaining decision variable is the knowledge creation effort a, because the participation rate solely effects the division of profits between the two members of the alliance and not the total profit π. Hence, to calculate the equilibrium, it is only necessary to set the first order derivative $\mathrm{d}\pi/\,\mathrm{d}a$ to zero and to solve the equation with respect to a. This leads to the following result:

Proposition 6.5 *Given a symmetric distribution of power between the members of an alliance, where the companies jointly maximize their profits, an equilibrium will be reached with:*

(i) $a = [\gamma\,(\rho_1 + \rho_2)]^{\frac{1}{\gamma+1}}$.

(ii) $\pi = (\rho_1 + \rho_2) \left[\alpha - [\gamma(\rho_1 + \rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)} \right] - [\gamma(\rho_1 + \rho_2)]^{\frac{1}{\gamma+1}}$.

At this point, only the total effort a as well as the total profit π is identified, but neither the companies' participation rates t_1 and t_2 nor the companies' individual profits π_1 and π_2, respectively, because the alliance was seen as a whole. Obviously, such an intense cooperation will only be established when both companies can obtain higher profits through a cooperation $\left(\pi_i^{\mathrm{C}}\right)$ compared to the situations with

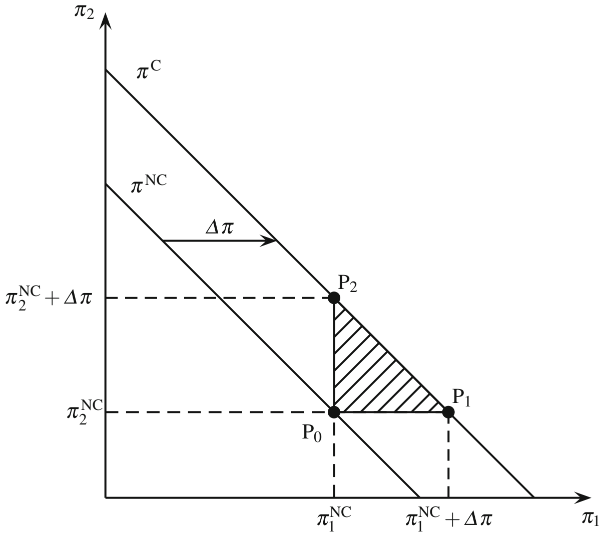

Fig. 1 Distribution of extra profit in joint profit maximization. Own representation

individual profit maximization $\left(\pi_i^{NC}\right)$:

$$\Delta\pi_i = \pi_i^{C} - \pi_i^{NC} \geq 0 \qquad \text{for } i = 1, 2. \tag{26}$$

Summarized for the entire alliance, this corresponds to:

$$\Delta\pi = \Delta\pi_1 + \Delta\pi_2 = \pi^{C} - \pi_1^{NC} - \pi_2^{NC} \geq 0. \tag{27}$$

If these two conditions given in Eqs. (26) and (27) are fulfilled, a cooperation with joint profit maximization is viable. In this case, the next step is to agree on the division of $\Delta\pi$ between the two members of the alliance. Figure 1 illustrates this graphically: It shows iso-profit lines for π^{NC} and π^{C}, which indicate the total profit obtainable through individual and joint profit maximization, respectively, and the possible allocation between the two companies. P_0 can be seen as starting point for the distribution of total extra profit $\Delta\pi$, because it represents the profits both companies could receive without joint profit maximization. The two points P_1 and P_2 mark the maximum for company $i = 1$ and $i = 2$, respectively, because there the entire additional profit $\Delta\pi$ is attributed to one single company. Hence, the hatched triangle can be seen as area of feasible results of the profit distribution.

One possibility to determine also the individual variables is the application of a bargaining game, which is a game-theoretic concept that allows to divide a certain pay-off between the involved parties. Depending on bargaining game applied, the division can be carried out according to the risk behaviour and/or the bargaining power (Aust and Buscher 2014, pp. 9–11). Besides the bargaining model, it is

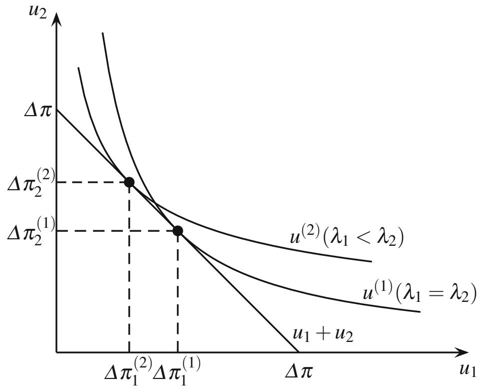

Fig. 2 Asymmetric Nash bargaining model. Own representation based on (Kunter 2009, p. 25)

furthermore necessary to define the utility function $u_i\,(\Delta\pi_i)$, which convert the individual extra profit $\Delta\pi_i$ into an individual utility u_i. For the sake of simplicity, we assume:

$$u_i\,(\Delta\pi_i) = \Delta\pi_i \qquad \text{for } i = 1, 2. \tag{28}$$

Here, we apply the asymmetric Nash bargaining model (Nash 1950; Harsanyi and Selten 1972; Kalai 1977), which allows to incorporate the bargaining power λ_i of the companies (with $\lambda_1 + \lambda_2 = 1$). It is visualized in Fig. 2 and can virtually be seen as a detail of Fig. 1, with P_0 being the origin. According to Eq. (28), the maximum total utility $u = u_1 + u_2$ is given by $\Delta\pi$, while the companies' utilities depend on the individual shares $\Delta\pi_1$ and $\Delta\pi_2$ as well as on the shape of the utility function. As expected, for equal bargaining power $\lambda_1 = \lambda_2$, which is shown by the iso-utility line $u^{(1)}$, the maximum utility arises from an equal distribution of extra profit, while in the second example, company $i = 2$ with the higher bargaining power receiver a higher share compared to company $i = 1$.

The mathematical formulation of the asymmetric bargaining model is as follows:

$$\begin{aligned} \max \quad & u = (u_1)^{\lambda_1} \cdot (u_2)^{\lambda_2} = (\Delta\pi_1)^{\lambda_1} \cdot (\Delta\pi_2)^{\lambda_2} \\ \text{s.t.} \quad & \Delta\pi = \Delta\pi_1 + \Delta\pi_2 \\ & \Delta\pi_1, \Delta\pi_2 \geq 0. \end{aligned} \tag{29}$$

By means of the first constraint, we can eliminate $\Delta\pi_2$ within the objective function:

$$u = (\Delta\pi_1)^{\lambda_1} \cdot (\Delta\pi - \Delta\pi_1)^{\lambda_2}. \tag{30}$$

Thereby, we can determine the utility-maximizing extra profit of company $i = 1$, $\Delta\pi_1$, by calculating the first order derivative $\mathrm{d}u/\,\mathrm{d}\Delta\pi_1$ and setting it to zero:

$$\frac{\mathrm{d}u}{\mathrm{d}\Delta\pi_1} = \lambda_1 \Delta\pi_1^{\lambda_1 - 1} \cdot (\Delta\pi - \Delta\pi_1)^{\lambda_2} + \Delta\pi_1^{\lambda_1} \cdot \lambda_2 (\Delta\pi - \Delta\pi_1)^{\lambda_2 - 1} \cdot (-1) = 0. \tag{31}$$

Solving this equation with respect to $\Delta\pi_1$ and transferring the result to $\Delta\pi_2$, we derive the following fair division of extra profits within the research and development alliance under joint profit maximization:

$$\Delta\pi_1 = \lambda_1 \Delta\pi \tag{32}$$

$$\Delta\pi_2 = \lambda_2 \Delta\pi. \tag{33}$$

Hence, the complete companies' profits after bargaining are:

$$\pi_1^{\mathrm{C}} = \pi_1^{\mathrm{NC}} + \Delta\pi_1 = \pi_1^{\mathrm{NC}} + \lambda_1 \Delta\pi \tag{34}$$

$$\pi_2^{\mathrm{C}} = \pi_2^{\mathrm{NC}} + \Delta\pi_2 = \pi_2^{\mathrm{NC}} + \lambda_2 \Delta\pi. \tag{35}$$

4 Discussion and Numerical Studies

The analysis of four scenarios in the previous section showed that not every considered approach leads to viable results. In Scenario II, where the leader sets the knowledge creation effort a and the follower sets the participation rate t_1, the model reveals that company $i = 1$ has no incentive to participate in the research cost, which is why company $i = 2$ would have to bear the entire cost of research. This would clearly be an infeasible behaviour in a research and development alliance in practice. The same result and, consequently, the same criticism holds for the first approach in Scenario III, which assumes an equal distribution of power between the two members of the alliance.

Therefore, those approaches are omitted below and we concentrate on the comparison of strategies that seem to be realistic in practice. This applies for

- the Stackelberg equilibrium in Scenario I for $\rho_1/\rho_2 \geq \gamma + 1$ (Proposition 6.1),
- the second Nash equilibrium in Scenario III (Proposition 6.4), and
- the equilibrium under joint profit maximization in Scenario IV (Proposition 6.5).

The identified strategies for knowledge creation effort and participation rates as well as the resulting profits of those scenarios are summarized in Table 3.

With regard to Scenario I, the example in Table 2 shows that the follower (company $i = 2$) has to bear two third of the research effort in this case, whereas the leader is able to realize a much higher profit, which follows from both the less research cost and the higher marginal profit. Interestingly, Scenario III, which bases

Table 2 Numerical example with $\alpha = 10$, $\rho_1 = 0.8$, $\rho_2 = 0.2$, and $\gamma = 2.5$

	a	t_1	t_2	π_1	π_2	π
Scenario I	0.921	0.333	0.667	6.710	1.140	7.851
Scenario III	1.299	0.800	0.200	6.545	1.636	8.181
Scenario IV	1.299	–	–	–	–	8.181

Table 3 Summary of (feasible) equilibria

	Scenario I	Scenario III	Scenario IV
a	$[\gamma(\rho_1-\rho_2\gamma)]^{\frac{1}{\gamma+1}}$	$[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$	$[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$
t_1	$\frac{\rho_1-\rho_2(\gamma+1)}{\rho_1-\rho_2\gamma}$	$\frac{\rho_1}{\rho_1+\rho_2}$	–
t_2	$\frac{\rho_2}{\rho_1-\rho_2\gamma}$	$\frac{\rho_2}{\rho_1+\rho_2}$	–
π_1	$\rho_1\left[\alpha-[\gamma(\rho_1-\rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-\frac{\rho_1-\rho_2(\gamma+1)}{\rho_1-\rho_2\gamma}[\gamma(\rho_1-\rho_2\gamma)]^{\frac{1}{\gamma+1}}$	$\rho_1\left[\alpha-[\gamma(\rho_1+\rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-\frac{\rho_1}{\rho_1+\rho_2}[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$	–
π_2	$\rho_2\left[\alpha-[\gamma(\rho_1-\rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-\frac{\rho_2}{\rho_1-\rho_2\gamma}[\gamma(\rho_1-\rho_2\gamma)]^{\frac{1}{\gamma+1}}$	$\rho_2\left[\alpha-[\gamma(\rho_1+\rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-\frac{\rho_2}{\rho_1+\rho_2}[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$	–
π	$(\rho_1+\rho_2)\left[\alpha-[\gamma(\rho_1-\rho_2\gamma)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-[\gamma(\rho_1-\rho_2\gamma)]^{\frac{1}{\gamma+1}}$	$(\rho_1+\rho_2)\left[\alpha-[\gamma(\rho_1+\rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$	$(\rho_1+\rho_2)\left[\alpha-[\gamma(\rho_1+\rho_2)]^{-\left(\frac{\gamma}{\gamma+1}\right)}\right]-[\gamma(\rho_1+\rho_2)]^{\frac{1}{\gamma+1}}$

on an equal distribution of power, proposes that the research effort should be divided between the companies according to the ratio of marginal profit parameter, which seems to be a reasonable solution if no company is able to dominate its counterpart. However, the advantage in marginal profit again leads to higher profit for company $i = 1$. Lastly, we can see that Scenario IV leads to the same knowledge creation effort and total profit as Scenario III, while the division of the costs for research effort and, therefore, the companies' individual profits are not determined by the model.

A comparison of the three scenarios indicates that, at least for the underlying set of parameters, the research and development alliance makes more knowledge creation effort under both an equal distribution of power and a joint profit maximization. In the Stackelberg game (Scenario I), company $i = 2$ sets a considerably minor effort a, which is presumably caused by the leader's low participation rate t_1. Regarding the individual profits, it is visible that company $i = 1$ is better off when obtaining the leadership, while company $i = 2$ would receive a higher profit under an equal distribution of power. The same holds for the total profit of the research alliance, which is the highest in Scenario III and IV. Furthermore, we can see that the proposed approach for an equal distribution of power (see Sect. 3.3) yields the same research effort and total profit as the joint profit maximization.

In Sect. 3.4, it was stated in Eqs. (26) and (27) that such an intense cooperation as a joint profit maximization would only be possible if both companies are able to realize higher individual profits than non-cooperative scenarios. In this example, we have

$$\Delta\pi^{\mathrm{I}} = \pi^{\mathrm{C}} - \pi_1^{\mathrm{I}} - \pi_2^{\mathrm{I}} = 8.181 - 6.710 - 1.140 = 0.331 \geq 0$$

for Scenario I, which means that a total extra profit $\Delta\pi^{I} = 0.331$ could be divided between both companies via bargaining. Hence, both members of the alliance could benefit from a cooperation instead of retaining the leader-follower-structure.

Obviously, there is no incentive to cooperate by a joint profit maximization in Scenario III as it already yields the same profit and a redistribution would always reduce the profit of one company. Consequently, it is not possible that the companies claim higher profits than in any other game, because the sum of maximum individual profits $\pi_1^{max} + \pi_2^{max} = 6.710 + 1.636 = 8.346$ would exceed the total profit in case of a cooperation $\pi^{C} = 8.181$.

Of course, the aforementioned findings are only based on this first numerical example. Therefore, it is necessary to prove whether the insights hold only for the assumed set of parameters or if they are valid in general. Reviewing the closed-form expressions in Table 3, one can see that knowledge creation effort a will always be higher in Scenario III and IV than Scenario I. Hence, the research and development alliance will invest more in the common research project when both members are equal and no company is able to obtain a dominant position. A similar conclusion can be affirmed for the total profit of the alliance π: Due to the lower level of research effort, the Stackelberg equilibrium (Scenario I) will always result in less total profit compared to the scenarios with equal distribution of power.

By reason of complexity, further interpretation is carried out with numerical studies and sensitivity analysis of the four model parameters α, which is the saturation asymptote of the performance function $P(a)$, γ, which denotes the elasticity of the research performance to changes of knowledge creation effort a, and the two marginal profit parameters ρ_1 and ρ_2. According to the formulae given in Table 3, the decision variables a, t_1 and t_2 are only affected by the parameters γ, ρ_1, and ρ_2, whereas the saturation asymptote only effects the absolute height of companies' profits. Consequently, we do not expect any insights thereof and limit our sensitivity analysis to γ, ρ_1, and ρ_2.

We start with the discussion of parameter γ, which determines the elasticity of the research performance with respect to the knowledge creation effort a and, hence, controls the shape of the function $P(a)$. As explained in Sect. 2, for $\gamma < 1$ ($\gamma > 1$), the research performance $P(a)$ increases to a lesser (greater) extent than the research effort a. Keeping the parameters $\alpha = 10$, $\rho_1 = 0.8$, and $\rho_2 = 0.2$ constant, a sensitivity analysis of a, t_1, π, π_1, and π_2 with respect to $0 < \gamma \leq 3$ is given in Figs. 3, 4, 5 (higher values of γ would violate the condition $\rho_1/\rho_2 \geq \gamma + 1$, which is required for a participation of company $i = 1$ in Scenario I). Please note that Scenario III and IV are summarized within Figs. 3 and 5, as both lead to the same values for a and π.

The left diagram in Fig. 3 proves the aforementioned observation that the research effort is lower under unequal distribution of power within the alliance (Scenario I). Furthermore, it is visible that the effort is an increasing function of γ under equal distribution of power (Scenario III), whereas a^{I} first increases and then slightly decreases when approaching $\gamma = 3$ (which corresponds to ρ_1/ρ_2). This can be explained by the participation rate t_1 of company $i = 1$, which strictly decreases

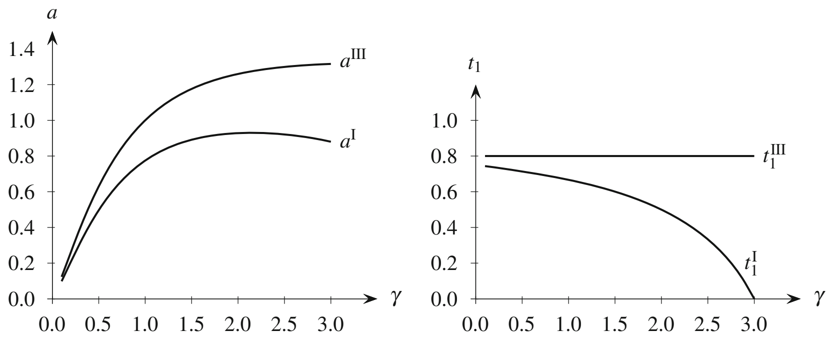

Fig. 3 Sensitivity analysis of research effort a and participation rate t_1 with respect to γ. Own representation

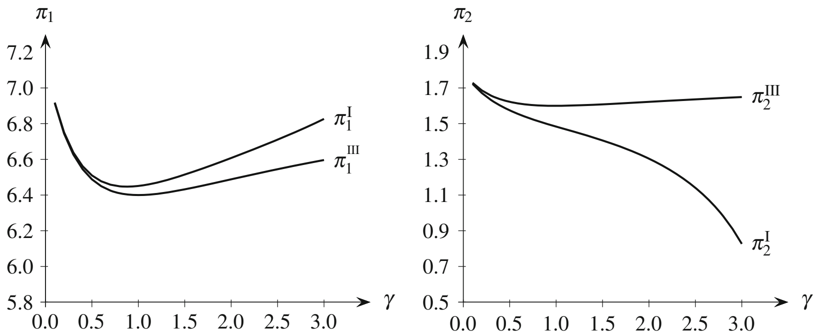

Fig. 4 Sensitivity analysis of companies' profits π_1 and π_2 with respect to γ. Own representation

until $t_1 = 0$ for $\gamma = 3$: When company $i = 2$ has to bear the major part of research cost, which is for $\gamma > 2$ in this example, it decides to make less effort. Similarly, the profit π_2^{I} of company $i = 2$ in Fig. 4 shows the negative effect of the decreasing participation rate t_1, which leads to higher costs for company $i = 2$ and, consequently, to less profit.

Interestingly, each profit function in Figs. 4 and 5 has its maximum for low values of γ, which imply a low level of knowledge creation effort as aforementioned. That means, when the effort a has only little effect on the research performance $P(a)$, it is better to reduce the effort in order to reduce the cost, which is a comprehensible result to some extent. However, it is questionable that the profits do not assume higher values for greater γ, for which the investment within the alliance is considerably higher. This might indicate a weakness of the proposed model and could be due to an overemphasising of the costs that arise from the research effort within the profit functions of the companies, which could be corrected by an additional cost parameter that specifies the cost associated with one unit of research effort. Nevertheless, the profit functions confirm the finding of the numerical

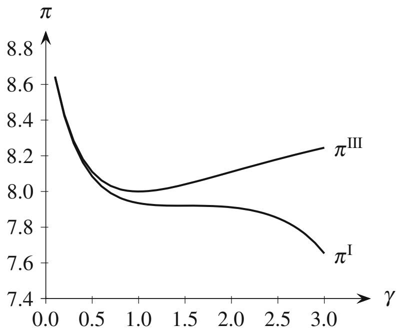

Fig. 5 Sensitivity analysis of total profit π with respect to γ. Own representation

Table 4 Sensitivity analysis with respect to ρ_2

ρ_2	Sc.	a	t_1	t_2	π_1	π_2	π
0.1	I	0.837	0.857	0.143	6.327	0.761	7.088
	III	0.949	0.889	0.111	6.313	0.789	7.103
0.2	I	0.775	0.667	0.333	6.451	1.484	7.934
	III	1.000	0.800	0.200	6.400	1.600	8.000
0.4	I	0.632	0.000	1.000	6.735	2.735	9.470
	III	1.095	0.667	0.333	6.539	3.270	9.809

example that company $i = 1$ will always prefer to be the leader of the alliance (Scenario I), while company $i = 2$'s as well as the total profit are higher under equal distribution of power. Thereby, it should be born in mind that the joint profit maximization with a subsequent bargaining on the total extra profit could further increase the individual profit of company $i = 1$, which would, however, implicate less profit for company $i = 1$ compared to an equal distribution of power with individual profit maximization.

Lastly, we consider the effects of a variation of the ratio of marginal profit parameters, wherefore we set the parameters $\alpha = 10$, $\rho_1 = 0.8$, and $\gamma = 1$ to constant values and change parameter ρ_2 to three different values (thereby, we ensure that condition $\rho_1/\rho_2 \geq \gamma + 1$ is observed). By doing so, we can see similar results as aforementioned, e.g., that company $i = 1$ receives higher profits when obtaining the leadership compared to Scenario III or that the total profit is always higher under equal distribution of power (see Table 4). Furthermore, an increase of marginal profit ρ_2 is followed by less participation of company $i = 1$ and, therefore, a higher participation of company $i = 2$. This increase also leads to a considerably higher profit π_2, while the profit of company $i = 1$ is only less affected, but still increases. Hence, also the total profit of the research and development alliance is higher, independently of the underlying scenario.

5 Conclusion

This analysis addresses resource sharing in a research and development alliance. Following a model proposed by Samaddar and Kadiyala (2006), we consider an alliance consisting of two companies, which make a common knowledge creation effort leading to a certain research performance. The costs of this effort are shared by both companies. Our aim was to study different forms of relationship within the alliance and their consequences on both the determination of the research effort and the individual participation rates of the companies in the resulting costs. We analysed four different scenarios: (I) an asymmetric distribution of power between the companies where the leader sets its participation rate and the follower sets the research effort; (II) an asymmetric distribution of power with a reserved sequence of decision, i.e., the leader sets the research effort and the follower its participation rate; (III) a symmetric distribution of power where the companies individually maximize their profits; and (IV) a symmetric distribution of power with a joint profit maximization. For each scenario, we derived closed-form expressions for the decision variables, research effort and participation rates, as well as for the resulting companies' profits.

During our analysis, we found that some of the aforementioned scenarios lead to non-viable equilibria, where one of the companies would have to bear the entire costs of the research, because the other company is not willing to participate in the effort. Though being mathematically correct, this would not be a possible strategy for a research and development alliance in practice. By means of numerical examples and sensitivity analyses, we showed that the research effort will always be higher under an equal distribution of power compared to a Stackelberg game. Furthermore, the company setting the participation rate prefers to obtain the leadership or a joint profit maximization with subsequent bargaining, because it receives a higher individual profit in this cases, while the company deciding on the common research effort is better off under a symmetric distribution of power. Interestingly, the total profit under equal distribution of power and individual profit maximization already yields the same profit as a joint profit maximization.

Future research could address the problems of non-viable equilibria under certain scenarios by considering a model which is more in step with actual practice. For example, it could be discussed whether it is really reasonable to assume that one company sets the common research effort, while the other company sets its participation. Instead of that, it could be interesting to see the changes of the strategies when both companies decide on their individual research efforts. Similarly, a more elaborated research performance and profit function would be necessary to further enable the model to serve as decision support for research and development alliances in practice, e.g., by introducing additional parameters for the cost impact of research effort. Lastly, the current model only considers a partnership of two companies, wherefore it could be interesting to study alliances consisting of more members.

References

Arsenyan J, Büyüközkan G, Feyzioğlu O (2015) Modeling collaboration formation with a game theory approach. Expert Syst Appl 42(4):2073–2085
Aust G, Buscher U (2014) Cooperative advertising models in supply chain management: a review. EJOR 234(1):1–14
d'Aspremont C, Jacquemin A (1988) Cooperative and noncooperative R&D in duopoly with spillovers. TAER 78(5):1133–1137
Ding X-H, Huang R-H (2010) Effects of knowledge spillover on interorganizational resource sharing decision in collaborative knowledge creation. EJOR 201(3):949–959
Harsanyi JC, Selten R (1972) A generalized Nash solution for two-person bargaining games with incomplete information. MS 18(5):80–106
Hensel J (2007) Netzwerkmanagement in der Automobilindustrie: Erfolgsfaktoren und Gestaltungsfelder. Deutscher Universitäts-Verlag, Wiesbaden
Kalai E (1977) Nonsymmetric Nash solutions and replications of 2-person bargaining. IJGT 6(3):129–133
Katz ML (1986) An analysis of cooperative research and development. TRJE 17(4):527–543
Kunter M (2009) Absatzkanalkoordination durch Hersteller-Handels-Konditionen. Gabler, Wiesbaden
Nakao T (1989) Cost-reducing R&D in oligopoly. JEBO 12(1):131–148
Nash JF (1950) The bargaining problem. Econometrica 18(2):155–162
Samaddar S, Kadiyala SS (2006) An analysis of interorganizational resource sharing decisions in collaborative knowledge creation. EJOR 170(1):192–210
Suzumura K (1992) Cooperative and noncooperative R&D in an oligopoly with spillovers. TAER 82(5):1307–1320
Wallentowitz H, Freialdenhoven A, Olschewski I (2009) Strategien in der Automobilindustrie: Technologietrends und Marktentwicklungen. Springer Vieweg, Wiesbaden

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Differences in Social Preferences: Are They Profitable for the Firm?

Hans-Ulrich Küpper and Kai Sandner

Abstract We analyse the impact of agents' heterogeneous social preferences in rivalry, pure self-interest, and altruism on the weighting and combination of incentive performance measures and on a firm's profitability. We show that firms maximize their profits when they maximize the difference between two agents' individual social preferences. However, for such an increase to happen, two prerequisites must apply: first, there must be no disadvantaged dominant agent; and second, the principal must reallocate participation in performance measures in such a way that competitive agents are privileged over altruistic agents.

Keywords Altruism • LEN model • Moral hazard • Reservation utilities • Rivalry

Contents

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1 Introduction

In studies of organizational behaviour, economics experts usually concentrate on analysing the structure of management problems in a situation of moral hazard, under the assumption that employees behave in a purely selfish manner (Holmström 1979, 1982). However, studies from different fields within the behavioural sciences, including psychology (Fehr and Falk 2002), neuroscience (Camerer et al. 2004; Fehr et al. 2005), and experimental decision theory (Fehr and Fischbacher 2002; Fehr and Schmidt 2003), suggest that in many cases, the economic decisions that individuals make are also determined by social preferences such as altruism, rivalry, and reciprocity (Fehr and Fischbacher 2002; Fehr and Schmidt 2003).

Hence, in this paper we analyse the influence of social preferences in a decentralized organization. Our focus is on two questions: which factors determine how differences in preferences affect profitability, and if we want to guarantee the best possible performance from a superordinate's point of view, how can we best select a group of managers and assign them to different divisions? To answer these questions, we use a principal/agent framework, with one principal, i.e., the owner or the top manager of the firm, and two agents, each of whom manages a division of that firm. We first show how a principal reacts to his or her agents' homogeneous and heterogeneous preferences of varying strength when the principal provides incentives. We then present analytically derived statements about the effects of social preferences on firm profitability. In our formal analysis, our focus is on three extreme types of preferences: altruism, pure self-interest, and rivalry. Using this focus makes it possible for us to draw conclusions on the principal's responses to each agent's individual social preferences.

We show that a principal can use variety in the agents' motivational structures to increase his or her own profits, as long as the agents are similar in aspects other than their RAS preferences.[1] This constraint in similarity is important, since other characteristics such as degree of risk aversion, exposure to risk, and marginal

[1] We note that in this paper, when we use the term *"RAS preferences"*, we refer to preferences that comprise rivalry, altruism, and pure self-interest. When we use the term *"RA preferences"*, we refer to preferences that include only rivalry and altruism. We still use the term "social preferences" whenever we speak of these types of preferences in a more general sense that is less directly related to the specific problem that we examine in our study.

costs of effort but also reservation utilities, have an influence on the impact of RAS preferences on firm profitability. Therefore, to analyse the primary effect of differences in social preferences, in a first step we will not take into account the differences in other factors that impact the individual contributions of agents to firm profits. In a second step we examine how the results of the initial analysis are altered as soon as differences in these other factors are introduced.

To exploit differences in the agents' RA preferences, the principal must shift the weighting of performance measures between agents so that the competitive agent gains at the expense of the other agent, or, put differently, so that the more altruistic agent suffers a loss in favor of the competitive agent. This scheme serves as an optimal way of reacting to the agents' RA preferences. In the two-agent model described above, if the "disfavored"[2] (in terms of variable wage compensation components) agent views such measures as positive, this translates into higher profits for the firm as long as that agent is not dominant. In other words, the degree to which "disadvantaged" agents perceive the consequences of the principal's measures as negative in such a situation is inversely proportional to the firm's profits, which implies that principals can potentially make use of differences in motivational structures, but cannot profit from strongly homogeneous RA preferences. By contrast, if the disfavored agent is dominant—that is, if his or her contribution to firm profits is significantly higher than that of the other agent—a reduction in effort may have negative consequences on firm profitability. In this scenario, homogeneous RA preferences are more appealing.

Our analysis shows that the increase in a firm's profits is maximized when the agents are as diverse as possible in their preferences, as long as there is no disfavored dominant agent. This finding contradicts the common intuition that altruistic agents, who do not begrudge their hierarchically equal colleagues their prosperity, make the largest contributions, and are therefore the most important members of a group. Instead, as will become evident, competitive agents may be equally significant as part of a group with mixed social preferences.

In an earlier version of this contribution (Küpper and Sandner 2008), we have shown that stochastic interdependencies leave the results unaltered. The findings obtained by Sandner (2009), who only considers rivalry, show that technological dependencies do not affect the impact of social preferences on firm profitability as long as separate performance measures for each of the agents are available (pp. 459–460). Building on these insights, it is safe to conclude that our results also hold in cases of stochastic or technological interdependences. This, however, may not be true when considering pure team production in the sense of Alchian and Demsetz (1972).

[2]Throughout our analysis, whenever we use the term "disfavored" or "disadvantaged" we refer to variable wage-compensation components. By construction, fixed wage-compensation components are set in such a way that participation constraints in our model are binding. This setting implies that in terms of utility, both agents receive their exact reservation utilities and neither one is advantaged compared to the other. However, because of social preferences, there are reallocations between the agents in terms of their variable wage-compensation components.

Our analysis contributes to the literature on "Behavioral Contract Theory" (Itoh 2004; Englmaier and Wambach 2010). There are now many studies that include various social preferences concerning the horizontal comparisons of agents on the same horizontal layer in models with moral hazard (Demougin and Fluet 2006; Demougin et al. 2006; Neilson and Stowe 2010; Bartling and von Siemens 2010a,b). Most of these papers like Itoh (2004), Rey Biel (2008), and Bartling and von Siemens (2010a,b) assume identical and/or risk-neutral agents. Results on the profitability of social preferences are mixed, depending on whether limited liability constraints are imposed or not. It turns out that under the assumptions of identical agents and/or risk neutrality social preferences can primarily be exploited in cases of limited liability (Itoh 2004; Rey Biel 2008; Bartling and von Siemens 2010b).

The effects of different social preferences on firm profitability are best analysed in a LEN model, which also allows for risk-averse agents but does not include the limited liability constraints. To the best of our knowledge, Mayer and Pfeiffer (2004) were the first authors to incorporate a linear formulation of social preferences similar to that of Fehr and Schmidt (1999) in a LEN model. Mayer/Pfeiffer study a single agent who is envious of his or her boss. Building on the Mayer/Pfeiffer work, several other articles have used similar social preference specifications in multi-agent models (Mayer 2006; Sandner 2008, 2009; Küpper and Sandner 2011).

Sandner (2009) and Dierkes and Harreiter (2010) focus on competing agents to show that a principal may benefit from one agent's rivalry when the other agent behaves entirely selfishly and is not dominant in terms of contributions to firm profits. Sandner (2009) focuses on the detailed analysis of the underlying variable wage compensation components in a situation with different types of technological interdependencies; Dierkes and Harreiter (2010) analyse a more basic setting without technological or stochastic interdependencies and also scrutinize the first-best situation, which Sandner (2009) does not examine. Their model is similar to the one presented here. However, altruism is not considered by either Sandner (2009) or by Dierkes and Harreiter (2010).

In this study, we simplify the social preference specifications in Mayer and Pfeiffer (2004), Sandner (2008, 2009) and Dierkes and Harreiter (2010). Doing so makes it possible for us to include altruism, self-interest, and rivalry, using the same formal specification and differentiating between the considered types of social preferences only through the relevant ranges of one social preference parameter. This version is the clearest way to show our results in a setting of moral hazard. However, we also include a section in which we discuss the robustness of these results with respect to different social preferences specifications, particularly in cases of altruism.

Another closely related article is Brunner and Sandner (2012). Building on results in an earlier version of this paper (Küpper and Sandner 2008), Brunner and Sandner (2012) generalize the setting to an arbitrary number of agents, comparing their own remuneration either to the average remuneration of all other members of their group or to a focal agent. Although the underlying mechanisms of action turn out to be the same, these authors formulate a model in vector/matrix notation and shift attention to special and more complicated cases of more than two agents. In our present article, we have a stronger focus on the careful demonstration and evaluation of the subtleties of the basic mechanisms of action. Therefore, we limit our attention to the two-agent case, which mathematically is a special case of the n-agent case discussed in Brunner and Sandner (2012). However, we do present more detail, background, and variation here than the authors do in Brunner and Sandner (2012).

Our study shows the underlying effects in depth and in the clearest possible way. Furthermore, our scope is also broader than that of Brunner and Sandner (2012), since we discuss to some extent how differences in the agents' degrees of risk aversion, exposures to risk, and their marginal costs of effort, and, as a consequence, their unequal effectiveness in manipulating firm profits, affect the principal's opportunities to take advantage of the agents' diversity in relation to their altruism, pure self-interest, and rivalry. We also examine the impact of heterogeneous reservation utilities and the stability of our results when we model altruism in a different and more general way.

The paper is organized as follows: In Sect. 2 we describe conceptually the reasoning behind our selection of RAS preferences. We outline the theoretical model and present its formal solution. We also briefly examine the consequences of different RA preferences on the structure of the optimal incentive system. In Sect. 3 we present our main results. We also examine the impact of homogeneous or heterogeneous RAS preferences on firm profitability. In Sect. 4 we discuss assumptions and further aspects of interest concerning modelling choices. In Sect. 5 we explore the implications of our theoretical analysis and highlight opportunities for further research.

2 Prerequisites and Structure of the Analytical Model

2.1 Systematization and Selection of Social Preferences

Generally, the term "social preferences" describes the interest that individual agents take in both the "material resources" allocated to them and to "relevant reference agents" (Fehr and Fischbacher 2002). On the basis of this definition, Fehr and Fischbacher (2002) distinguish between various types of motivational structures in the economics literature. Their main criterion is the effect that changes in one person's payoff have on another person's own utility. *Joy*, *envy*, *schadenfreude*, and *guilt* are positive or negative feelings about the fortune or success of others; Macaulay and Berkowitz (1970, p. 3) define *altruism* as "behavior that helps others and does not occur in expectation of an external reward." Another type of behaviour, *reciprocity*, includes the principle of mutuality (Kolm 2000; Fehr and Fischbacher 2002). In turn, reciprocity is related to *inequity aversion*, according to which agents strive to achieve an equal distribution of rewards (Fehr and Schmidt 1999, 2003).

In a theoretical analysis of contracts it is reasonable to focus on behaviour rather than the underlying emotions. Decision-theory models that represent behaviour use utility functions that include preference parameters as variables in the problem

Table 1 Characteristics of RAS preferences

		Effect on agent *A*		
Change in agent *B*'s result		Emotion	Preference	Utility
	Increase	Joy	Altruism	Increase
		No effect	Pure self-interest	No effect
		Envy	Rivalry	Decrease
	Decrease	Guilt	Altruism	Decrease
		No effect	Pure self-interest	No effect
		Schadenfreude	Rivalry	Increase

formulation. Emotions such as *envy* and *schadenfreude* are often related to, and can result in, a preference for *rivalry* (van Dijk et al. 2006); the emotions of *joy* and *guilt* may generate a willingness to share benefits with others and might result in a form of *altruistic behaviour*. It is possible to use these tendencies to systematize altruism, self-interest, and rivalry, according to how a change in the result of another person B affects the preference of person A. We show this effect in Table 1 and note that the effect of changing the average result of a reference group on individual utility is equivalent to the effect of changing the result of another person.

Within the spectrum of possible preferences, these three types of preferences are the cornerstones of an interval between two polar evaluations of another agent's payoff. At the one extreme, in the case of rivalry, the other agent's payoff is always evaluated negatively, and at the other extreme, in the case of altruism, it is always evaluated positively. All three types of preferences play an important role in many business areas, in particular in providing incentives in situations of moral hazard. These preferences can be interpreted as both extremes and the central point of a continuum of social preferences. An important reason for our choices is that we are interested in the possible effects of differences between the preferences of two agents on the principal's provision of incentives and on firm profitability. Therefore, social preferences such as reciprocity and inequity aversion, which aim at minimizing differences (e.g., in wages), are not a suitable starting point for such an investigation.

Although there is overwhelming evidence that self-interest is a ubiquitous motive that affects to a greater or lesser degree human behaviour (Fehr and Schmidt 2003), the ongoing discussion on wage justice in Germany, as well as in North America, indicates that rivalry also significantly affects how people evaluate the appropriateness of their compensation (Krapp and Sandner 2016). The impact of altruism on incentive provision, as well as on firm profitability, has been intensively discussed in the context of family firms over the last few years (Chrisman et al. 2005; Karra et al. 2006). However, the evidence from the results of most studies, such as those by van den Berghe and Carchon (2003) and Schulze et al. (2003), is ambiguous.

At first glance, altruism does not seem to be a preference that is important in economics, because competitive markets tend to promote rivalry and self-interest. However, in real life we often observe that altruistic behaviour also plays a role in business situations. There is a long tradition of social entrepreneurs, such as Robert Bosch, and companies engaging in social activities, which are marketed and researched under the label of "Corporate Social Responsibility (CSR)." Especially in small and medium-sized firms, and also in family firms, personal relationships and altruism often play a vital role (Fröhlich and Pichler 1988). Further, religious beliefs and moral attitudes influence the behaviour of some people in a way that is, or can at least appear to be, altruistic. These observations imply that our results may be more relevant for small, medium-sized, and family firms. From our point of view, it is desirable to collect empirical information on firm members' RAS preferences, differentiated according to the different types of enterprises in which they are employed as well as by their hierarchical positions. As the results of psychological, neurobiological, and experimental research indicate, we believe that it is reasonable to assume that there is a spectrum of different RAS preferences, which range from strong rivalry over self-interest to at least some degree of altruism.

In our model of altruism, pure self-interest, and rivalry we apply a distributional approach similar to that of Fehr and Schmidt (1999), which specifies different types of social preferences according to how differences in the two agents' remuneration are evaluated by these agents. By specifying these differences we can use the same formal specification to capture altruism, pure self-interest, and rivalry, thus making it possible for us to study differences in intensity among both homogeneous and heterogeneous types of RAS preferences within a unified framework.

2.2 *Profit Functions, Effort Costs, and Incentive Contracts*

To analyse the impact of diverse RAS preferences on both the structure of the optimal incentive system and firm profitability, the underlying theoretical framework ideally allows the explicit computation of utilities and, therefore, a precise solution. Hence, the best framework for our purposes is to follow Holmström and Milgrom's (1987) continuously formulated principal/multi-agent linear exponential normal model (LEN). The LEN model also fulfils the requirement of explicit solvability and allows for heterogeneity between agents.

We consider a situation of moral hazard with one principal P and two agents i $(i = A, B)$, each of whom leads his or her own decentralized division. The profits x_i of the two decentralized divisions, which depend on the agents' non-observable efforts a and b and on the error terms ε_i, which represent stochastic environmental influences, add up to the firm profits, denoted by x. Therefore, the profit functions of the divisions (i.e., agents) A and B, based on (isolated) production functions, are:

$$x_A = a + \varepsilon_A \quad ; \quad x_B = b + \varepsilon_B. \tag{1}$$

The error terms are normally distributed with a mean of zero, variance σ_i^2, and correlation coefficient ρ, which we assume to be equal to zero. The agents' efforts cause nonmonetary quadratic disutility V_i, given by

$$V_A = \frac{1}{2} c_A a^2 \quad ; \quad V_B = \frac{1}{2} c_B b^2. \tag{2}$$

In (2) c_A, c_B are measures for the marginal costs. Both agents are offered linear contracts, where the total amount of wage compensation S_i comprises a fixed payment α_0 (β_0), and a proportional fraction in the profits of each of the two decentralized divisions, which are determined by share rates α_A, α_B and β_A, β_B:

$$S_A = \alpha_0 + \alpha_A x_A + \alpha_B x_B \quad ; \quad S_B = \beta_0 + \beta_A x_A + \beta_B x_B. \tag{3}$$

The principal optimizes his or her objective function for the wage-compensation coefficients in (3). Depending on the endogenously determined values of α_B and β_A, the principal might implement one of three different compensation schemes: individual compensation for $\alpha_B = 0$ $(\beta_A = 0)$, relative performance evaluation for $\alpha_B < 0$ $(\beta_A < 0)$, or team-based compensation for $\alpha_B > 0$ $(\beta_A > 0)$. Since we assume that the principal is risk-neutral, he or she maximizes the expected value of his or her residuum after wage compensations ($E(\cdot)$ denotes the expectation operator):

$$U_P = E[(1 - \alpha_A - \beta_A)x_A + (1 - \alpha_B - \beta_B)x_B - \alpha_0 - \beta_0]. \tag{4}$$

We assume that both agents are strictly risk averse with exponential utility functions. We measure the strength of their risk aversion by using the constant coefficients $r_A > 0$ and $r_B > 0$ where higher values of r_i imply a higher degree of risk aversion. Risk-averse agents make it necessary for the risk-neutral principal to strike a trade-off between optimal provision of incentives and optimal risk-sharing in a situation of moral hazard. By contrast, the principal holds the risk-neutral selfish agents fully responsible for their division results. In reality, risk aversion is more plausible, since agents are usually less well diversified compared to principals, who tend to hold shares in several firms. Consequently, opting for the more realistic assumption of risk-averse agents broadens the applicability of our main results and adds to their informational value.

2.3 Modeling Altruism, Self-interest, and Rivalry Within a Unified Framework

We assume that the term social preferences refers only to the agents' remuneration and that it does not include parameters that are harder to observe, such as costs of effort. By means of $S_i, S_j; i, j = A, B$ and $i \neq j$, we define the function

$$G_i(S_i(\cdot), S_j(\cdot)) = S_i(\cdot) - k_i(S_j(\cdot) - S_i(\cdot)). \tag{5}$$

Mayer and Pfeiffer (2004) introduced a similar specification, as did Mayer (2006). However, we leave out their aspiration level parameter, thus making it possible for us to include altruism in the same framework. The formulation in (5) consists of an agent's own remuneration $S_i(\cdot)$, and a social preference term $k_i(S_j(\cdot) - S_i(\cdot))$, which has a negative sign. We differentiate the three types of RAS preferences considered according to parameter k_i, which denotes the range of RAS preferences and which we restrict to the interval $]-0.5;\infty[$. For $k_i \in]-0.5;0[$ the equation in (5) represents altruism, for $k_i = 0$ we get pure self-interest and $k_i \in]0;\infty[$ yields rivalry. Therefore, the specification in (5) represents a continuum of different behavioural types in which k_i measures the strength of RA preferences, i.e., either altruism or rivalry. Higher absolute values for either case indicate stronger RA preferences.

For the case of rivalry, i.e., $k_i \in]0;\infty[$, if the resulting value of the social preference term $k_i(S_j(\cdot) - S_i(\cdot))$ in Eq. (5) is positive, i.e. $S_j > S_i$, then agent i suffers a disutility. Therefore, the principal must increase agent i's wage compensation if he or she wants that agent to cooperate. But if the result has a negative value, i.e. $S_i > S_j$, then agent i perceives that agent j is at a disadvantage and draws additional utility from such a situation. Consequently, agent i accepts lower remuneration, which suggests that he or she works harder, even though there is no change in the incentives provided by the principal. Recently, Charness et al. (2010) conducted a real-effort experiment, which points to the importance of such competitive preferences, which we specify in our model in the context of incentive provision.

Unlike rivalry, altruism, which is described by the formula given in Eq. (5) for $k_i \in]-0.5;0[$, suggests that an agent is content when the other agent receives a greater monetary reward. This specification is identical to that used by Bester and Güth (1998), which was introduced by Edgeworth (1881). It turns out to be the clearest way to show our results for altruism, and is without loss of generality. If $S_i > S_j$, then the social preference term $k_i(S_j(\cdot) - S_i(\cdot))$ in Eq. (5) is positive and agent i, as a result of being ahead of agent j, suffers additional disutility. As a consequence, the principal has to either decrease agent i's or increase agent j's remuneration so that agent i cooperates. For $S_j > S_i$, by contrast, the social preference term is negative and the utility of agent i increases because agent j does well in terms of remuneration. This change in utility increases agent i's willingness to exert greater effort, although there is no change in that agent's remuneration. The utility function of an altruist strictly increases in both agents' wage compensations S_i and S_j. Therefore, both agents strive to maximize the weighted sum of monetary payoffs. For each agent, their respective wage payments S_i and S_j are perfect substitutes. The weighting factor $(1 + k_i) < 1$ in front of S_i indicates that when one agent attaches greater weight to the other agent's remuneration, the subjective importance of the first agent's own reward decreases. However, restricting k_i to values > -0.5 guarantees that agent i's own remuneration always matters more to him or her than does agent j's remuneration. Thus, we exclude selfless behaviour from our analysis. Andreoni and Miller (2002) use a similar model of altruism in which they explain the altruistic behaviour of 20% of the participants in a dictator game.

2.4 Solving the Analytical Model in a Situation of Moral Hazard

Given the above assumptions, we can write the agents' utility functions by using certainty-equivalent notation. We find that:

$$CE_i = E[(1+k_i)\cdot S_i(\cdot) - k_i \cdot S_j(\cdot)] - V_i(\cdot) - \frac{r_i}{2}Var[(1+k_i)\cdot S_i(\cdot) - k_i S_j(\cdot)], \tag{6}$$

where $Var(\cdot)$ denotes the variance operator. Both agents choose effort levels a and b to maximize their certainty equivalents (6), which, using Eqs. (1), (2), and (3), we can also write as (the example refers to agent A):

$$CE_A = (1+k_A)\alpha_0 + (1+k_A)\alpha_A a + (1+k_A)\alpha_B b - k_A\beta_0 - k_A\beta_A a -$$
$$k_A\beta_B b - \frac{1}{2}c_A a^2 - \frac{r_A}{2}\cdot\left\{[\alpha_A(1+k_A) - k_A\beta_A]^2\sigma_A^2 + [\alpha_B(1+k_A) - k_A\beta_B]^2\sigma_B^2\right\}. \tag{7}$$

The principal anticipates this behaviour and restricts the optimization problem by using incentive compatibility constraints, thus considering that effort levels cannot be negative. Furthermore, the principal must consider the participation constraints, which imply that the fixed wage-compensation parameters are set such that the agents receive at least their reservation utility, which is $\overline{CE_A} \geq 0$ for agent A and $\overline{CE_B} \geq 0$ for agent B in our model. Given these limitations, the principal's optimization problem is:

$$\max_{\alpha_0,\alpha_A,\alpha_B,\beta_0,\beta_A,\beta_B} U_P = E[(1-\alpha_A-\beta_A)x_A + (1-\alpha_B-\beta_B)x_B - \alpha_0 - \beta_0] \tag{8}$$

s.t.

Incentive Compatibility Constraints:

$$a = max\left(\frac{[(1+k_A)\alpha_A - k_A\beta_A]}{c_A};\ 0\right)$$
$$b = max\left(\frac{[(1+k_B)\beta_B - k_B\alpha_B]}{c_B};\ 0\right) \tag{9}$$

Participation Constraints:

$$CE_A \geq \overline{CE_A};\ CE_B \geq \overline{CE_B} \tag{10}$$

Performing the typical optimization steps yields:

$$\alpha_A = \frac{1+k_B}{1+2k_B}\cdot\frac{1}{1+r_A\sigma_A^2 c_A} \quad ; \quad \alpha_B = \frac{k_A}{1+2k_A}\cdot\frac{1}{1+r_B\sigma_B^2 c_B} \tag{11}$$

$$\beta_A = \frac{k_B}{1+2k_B} \cdot \frac{1}{1+r_A\sigma_A^2 c_A} \quad ; \quad \beta_B = \frac{1+k_A}{1+2k_A} \cdot \frac{1}{1+r_B\sigma_B^2 c_B} \tag{12}$$

Equations (11) and (12) show that the principal reacts to the agents' comparisons of remuneration by reallocating their variable wage-compensation components. The principal achieves this reallocation by shifting the shares in performance measures that each agent receives, where α_A and β_A, and α_B and β_B are interdependent. The principal divides the weightings of performance measures differently between the two agents according to the type of RA preferences each exhibits. Doing so gives rise to significant changes in the optimal design of the incentive system. As a result of one agent's RA preferences, the shares of both agents in the second agent's performance measure are affected, i.e., agent A's RA preference affects α_B and β_B, while agent B's RA preference affects α_A and β_A.

In cases of rivalry in which there is one competing agent and one selfish agent, the competing agent has the advantage over the selfish agent, while the selfish agent's share in his or her own performance measure is reduced. Thus, extrinsic incentives remain unchanged for the competing agent but are reduced for the agent with whom the competing agent compares him- or herself. But at the same time, the share of the competing agent in his or her counterpart's performance measure is increased, resulting in team-based compensation. The same reasoning applies when both agents compete, which leads to overlapping effects (Dierkes and Harreiter 2010, pp. 546–547).

For cases of altruism in which one agent is altruistic and the other is selfish, because of the altruism of one agent the selfish agent's share in his or her own performance measure increases. As a result, the selfish agent's direct incentive intensity also increases. At the same time, the altruist's share in the selfish agent's performance measure is reduced, resulting in relative performance evaluation. Again, there is an overlap of effects when both agents simultaneously embrace a preference for altruism. In general, the stronger the RA preferences of both agents, the more pronounced these effects become.

3 The Influence of RA Preferences on the Profitability of the Firm

3.1 The Profitability of RA Preferences With Equally Effective Agents

Here, we examine the maximum attainable firm profits in a situation of moral hazard. We do so for all possible combinations of the different types of RAS preferences considered in our model. The results can also be derived when analysing the model in Brunner and Sandner (2012) for special cases of two agents. We assume that the principal designs the wage-compensation system optimally according to the principles outlined in Sect. 2.4, and that agents are identical apart from their RAS preferences, i.e., $r_A = r_B$, $\sigma_A = \sigma_B$, $c_A = c_B$, $\overline{CE_A} = \overline{CE_B}$. This assumption implies that when both agents are completely selfish, they are equally effective with respect to manipulating firm profits, so we can only attribute any differences to their RAS preferences. Our analysis leads us to:

Proposition 1 (Firm Profitability in Cases of Equally Effective Agents)

(1a) If agents exhibit identical preferences, i.e., if both exhibit altruism or pure self-interest or rivalry, then in all three cases firm profits are the same, regardless of whether both agents are altruists, self-interested, or competitive. From a firm's point of view, if the preferences are identical, then none of the three types of preferences confers an advantage to the firm compared to the other two.

(1b) For the principal, in a two-agent setting the optimal combination of RAS preferences consists of an agent who exhibits altruism and an agent who exhibits rivalry, and whose respective RA preferences are as strong as possible.

Proof of Proposition 1: We calculate the maximum value of the principal's objective function by assigning to the share rates in Eq. (8) their optimal values, which are given in Eqs. (11) and (12). The explicit solutions for the fixed wage compensation components are:

$$\alpha_0 = -\frac{1}{2c_A} \cdot \frac{(1+k_B)[(1+k_A)(1+k_B) - k_A k_B]}{(1+2k_B)^2} \left[\frac{1-r_A\sigma_A^2 c_A}{(1+r_A\sigma_A^2 c_A)^2}\right] - \frac{1}{2c_B} \cdot \frac{k_A[(1+k_A)(1+k_B) - k_A k_B]}{(1+2k_A)^2} \cdot \left[\frac{1-r_B\sigma_B^2 c_B}{(1+r_B\sigma_B^2 c_B)^2}\right] + \frac{k_A\overline{CE_B} + (1+k_B)\overline{CE_A}}{(1+k_A)(1+k_B) - k_A k_B} \tag{13}$$

$$\beta_0 = -\frac{1}{2c_B} \cdot \frac{(1+k_A)[(1+k_A)(1+k_B) - k_A k_B]}{(1+2k_A)^2} \cdot \left[\frac{1-r_B\sigma_B^2 c_B}{(1+r_B\sigma_B^2 c_B)^2}\right] - \frac{1}{2c_A} \cdot \frac{k_B[(1+k_A)(1+k_B) - k_A k_B]}{(1+2k_B)^2} \cdot \left[\frac{1-r_A\sigma_A^2 c_A}{(1+r_A\sigma_A^2 c_A)^2}\right] + \frac{k_B\overline{CE_A} + (1+k_A)\overline{CE_B}}{(1+k_A)(1+k_B) - k_A k_B}. \tag{14}$$

Plugging the expressions given in Eqs. (11)–(14) into (8) and re-arranging yields:

$$U_P^*(k_A, k_B) = \frac{1}{2c_A} \cdot \frac{(1+k_A)(1+k_B) - k_A k_B}{1+2k_B} \cdot \frac{1}{1+r_A\sigma_A^2 c_A} + \frac{1}{2c_B} \cdot \frac{(1+k_A)(1+k_B) - k_A k_B}{1+2k_A} \cdot \frac{1}{1+r_B\sigma_B^2 c_B} - \frac{1+2k_B}{(1+k_A)(1+k_B) - k_A k_B} \cdot \overline{CE_A} - \frac{1+2k_A}{(1+k_A)(1+k_B) - k_A k_B} \cdot \overline{CE_B}. \tag{15}$$

Since we are only interested in the effect of differences in RA preferences, we can rewrite (15) by establishing $k_B = k_A + \varsigma, \varsigma \geq 0$ and suppress the indexes to get:

$$U_P^*(k,\varsigma) = \frac{1}{2c_A} \cdot \frac{(1+k)(1+k+\varsigma) - k(k+\varsigma)}{1+2(k+\varsigma)} \cdot \frac{1}{1+r_A\sigma_A^2 c_A} + \frac{1}{2c_B} \cdot \frac{(1+k)(1+k+\varsigma) - k(k+\varsigma)}{1+2k} \cdot \frac{1}{1+r_B\sigma_B^2 c_B} - \frac{1+2(k+\varsigma)}{(1+k)(1+k+\varsigma) - k(k+\varsigma)} \cdot \overline{CE_A} - \frac{1+2k}{(1+k)(1+k+\varsigma) - k(k+\varsigma)} \cdot \overline{CE_B}. \quad (16)$$

We can prove part (a) of the proposition by deriving the limiting value for $\varsigma \to 0$:

$$\lim_{\varsigma \to 0} U_P^*(k,\varsigma) = \frac{1}{2c_A} \cdot \frac{1}{1+r_A\sigma_A^2 c_A} + \frac{1}{2c_B} \cdot \frac{1}{1+r_B\sigma_B^2 c_B} - \overline{CE_A} - \overline{CE_B}, \quad (17)$$

which is independent of k.

To establish part (b) of the proposition, we calculate the first derivative of (16) for ς. The result is:

$$\frac{\partial U_P^*(k,\varsigma)}{\partial \varsigma} = -\frac{1}{2c_A} \cdot \frac{1+2k}{(1+2k+2\varsigma)^2} \cdot \frac{1}{1+r_A\sigma_A^2 c_A} + \frac{1}{2c_B} \cdot \frac{1}{1+2k} \cdot \frac{1}{1+r_B\sigma_B^2 c_B} - \frac{1+2k}{(1+2k+\varsigma)^2}(\overline{CE_A} - \overline{CE_B}). \quad (18)$$

We assume that $\frac{1}{c_A(1+r_a\sigma_A^2 c_A)} = \frac{1}{c_B(1+r_B\sigma_B^2 c_B)}$, $\overline{CE_A} = \overline{CE_B}$ and since $\varsigma > 0$, it follows that $\frac{1+2k}{(1+2k+2\varsigma)^2} < \frac{1}{1+2k}$, which implies $\frac{\partial U_P^*(k,\varsigma)}{\partial \varsigma} > 0$. Thus, firm profits unambiguously increase with the difference in the strength of RA preferences. These considerations establish part (b) of the proposition and therefore complete the proof. ■

The explanation for part (a) of this result is that the principal can only make use of the agents' rivalry or altruism if he or she reacts to their social behaviour by appropriately reallocating the variable wage-compensation components between the two agents. However, the measures that a principal must take in response to each agent's rivalry or altruism are interdependent. For example, agent A's rivalry means that the principal increases that agent's share α_B in agent B's performance measure x_B at the expense of a reduced incentive intensity β_B for B. The stronger agent A's competitiveness, the more intense the principal's response. In this situation, if B also behaves competitively, then the principal's response to agent A's rivalry has negative effects on the motivation of agent B. Thus, when both agents simultaneously behave competitively, the principal cannot achieve a trade-off that is advantageous to the firm by using a properly designed wage-compensation system. Every reallocation of the variable wage-compensation components leads to an improvement in the performance of one agent but, at the same time, has a negative effect on the performance of the other agent. This observation has also been made by Sandner (2009, pp. 455–460) and Dierkes and Harreiter (2010, pp. 547–550).

The same argument holds true for a situation of two-sided altruism. For this case, to profit from each agent's altruism, the principal has to reduce each agent's share in his or her own performance measure and simultaneously increase the other agent's incentive intensity. But because the second agent is also an altruist, this reduction decreases the other agent's intrinsic motivation. Therefore, the positive effects of the increased incentive intensity are at least partly wasted. Since both agents are altruists, possibly to the same degree, neither agent wants to be privileged over the other. Consequently, the reallocation of the variable wage-compensation components, which is prompted by either agent's altruism, has negative effects on the other agent. In both cases, if the principal acts as described above in response to each agent's rivalry or altruism, then a conflict will ensue. As a consequence, if the agents exhibit rivalry or altruism of equal strength, then the behaviour that results from those conflicting actions will not benefit the principal. Figure 1 depicts an example of firm profits as a function of the strength of both agents' homogeneous RAS preferences in cases of rivalry. *A* similar graph could be drawn for cases of altruism.

The graph shows that from the firm's perspective, equally strong homogeneous RA preferences cannot be beneficial. By contrast, firm profits are increased when the agents exhibit either rivalry or altruism of unequal strength, which means that the principal can exploit any such differences in strength in cases of two-sided altruism or rivalry.

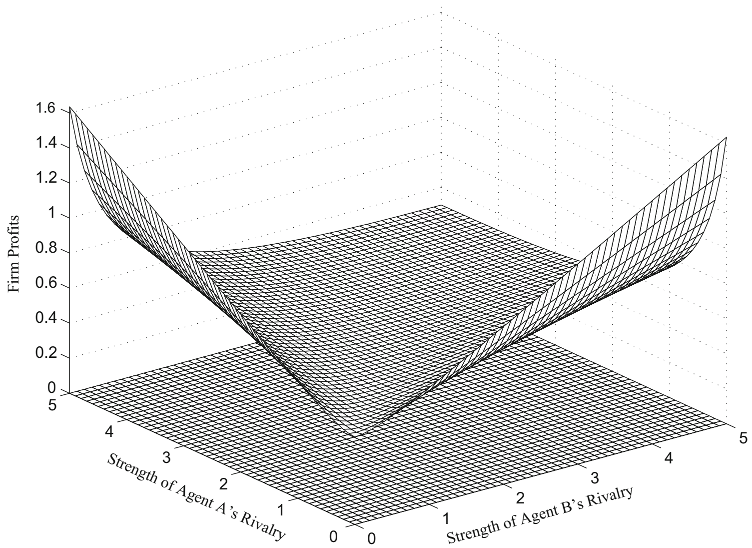

Fig. 1 Impact of two-sided rivalry on firm profits ($r_A = r_B = \sigma_A^2 = \sigma_B^2 = c_A = c_B = 1; \overline{CE_A} = \overline{CE_B} = 0$). Own representation

This observation serves as a basis for the thinking behind part (b) of Proposition 1. Agent *B*'s rivalry encourages the reduction of agent *A*'s variable wage compensation and therefore his or her incentives, while enhancing agent *B*'s own compensation. In such a situation, if agent *A* is altruistic, then he or she will partly internalize the relevant external effect by drawing satisfaction from the fact that agent *B* receives higher remuneration. Thus, the principal's measure of favoring agent *B* to make use of that agent's rivalry simultaneously enhances the utility of altruistic agent *A*, who attributes to *B* the improvement of his or her own situation. At the same time, as a result of agent *A*'s altruism, the principal reallocates variable wage-compensation components between the decentralized divisions in favor of *B* and at the expense of *A*. This action is in competing agent *B*'s interest, since agent *B* aims at outdoing agent *A* to as great an extent as possible in terms of wage compensation. Therefore, the principal's responses to agent *A*'s altruism and agent *B*'s rivalry, which aim at balancing the two agents' RA preferences, are complementary. In general, the complementarity of a principal's reactions to the agents' RAS preferences increases with the heterogeneity of those preferences. Thus, the larger the difference in RAS preferences, the greater the firm's profits. This effect, which we illustrate in Fig. 1, becomes more pronounced in two-agent settings where one agent is altruistic and the other is competitive.

Figure 2 depicts firm profits in relation to the strength of agent *A*'s altruism and agent *B*'s rivalry. The graph shows that firm profits are maximized when agent *A* behaves as altruistically as possible and agent *B* exhibits the strongest degree of rivalry.

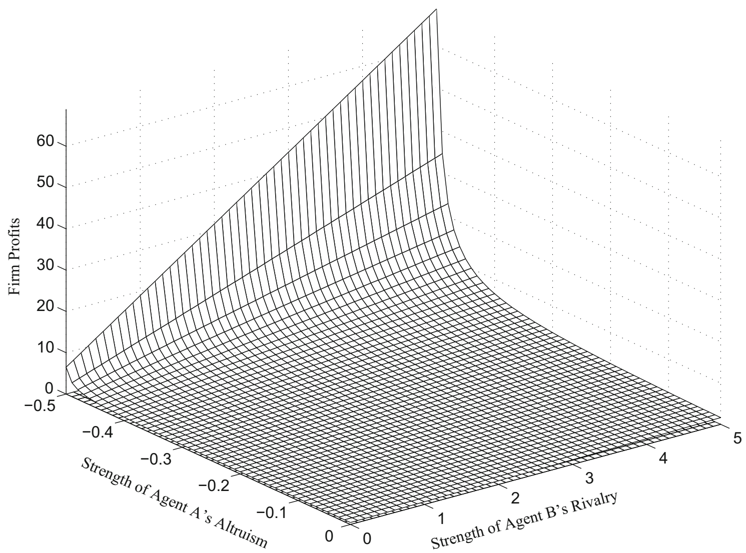

Fig. 2 The dependence of firm profits on Agent A's strength of altruism and agent B's strength of rivalry ($r_A = r_B = \sigma_A^2 = \sigma_B^2 = c_A = c_B = 1; \overline{CE_A} = \overline{CE_B} = 0$). Own representation

3.2 The Profitability of RAS Preferences with Unequally Effective Agents

In this section we look specifically at $\overline{CE_A} = \overline{CE_B} = 0$ and allow for $r_A \neq r_B$, $\sigma_A \neq \sigma_B$, $c_A \neq c_B$ to examine cases in which, due to lower risk aversion, exposure to risk, and marginal costs of effort, one agent is more effective than the other in generating firm profits. Our analysis leads us to:

Proposition 2 (Firm Profitability When There are Unequally Effective Agents) *From a firm's point of view, homogeneous RAS preferences may be preferable when one of the two agents is simultaneously disfavored in terms of variable wage compensation components and more effective than the other in terms of contributing to firm profits.*

Proof of Proposition 2: Setting Eq. (18) equal to zero, assuming $\overline{CE_A} = \overline{CE_B}$, and solving for the difference in RAS preference parameter ς yields:

$$\varsigma = \frac{(1+2k)\cdot\left(-\frac{1}{c_B(1+r_B\sigma_B^2 c_B)} \pm \sqrt{\frac{1}{c_A(1+r_A\sigma_A^2 c_A)}\cdot\frac{1}{c_B(1+r_B\sigma_B^2+c_B)}}\right)}{2\cdot\frac{1}{c_B(1+r_B\sigma_B^2 c_B)}} \tag{19}$$

Since $\varsigma > 0$, there exists no extreme point when $\frac{1}{c_A(1+r_A\sigma_A^2 c_A)} = \frac{1}{c_B(1+r_B\sigma_B^2 c_B)}$.

The first derivative of (18) for ς is greater than zero. Therefore, any extreme point for $\frac{1}{c_A(1+r_A\sigma_A^2 c_A)} \neq \frac{1}{c_B(1+r_B\sigma_B^2 c_B)}$ marks a minimum of firm profits. It is clear from (19) that such a minimum always exists for $k_B = k_A + \varsigma$ and $\frac{1}{c_A(1+r_A\sigma_A^2 c_A)} > \frac{1}{c_B(1+r_B\sigma_B^2 c_B)}$, i.e., when the more effective agent has weaker RAS preferences. In this case, differences in RAS preferences are detrimental to firm profits unless ς is sufficiently large. These considerations complete the proof. ■

From this proposition, we derive the following insight: the total impact of differences in RAS preferences depends not only on the (relative) strength of the two agents' RAS preferences, but also on both agents' degree of risk aversion, exposure to risk, and marginal costs of effort. As we point out above, the principal's opportunity to benefit from the agents' RA preferences depends on how he or she designs the wage compensation system, the aim being to disfavor the less competitive agent or the more altruistic agent in terms of variable wage compensation parameters compared to the other agent. In such a situation, if the disadvantaged agent is more effective in generating firm profits, then differences in RAS preferences may prove detrimental.

In the example illustrated in Fig. 3, one agent is rivalrous (agent A) while the other one is selfish and therefore less competitive (agent B). The dotted line shows that if agent B contributes more effectively to firm profits (which is the case in the example, because $r_A >> r_B$, $\sigma_A >> \sigma_B$, and $c_A = c_B$), then A's ambition can prove detrimental. The reason is that if the principal favors A over B by increasing A's share in performance measure x_B, doing so will reduce the effort of the more capable

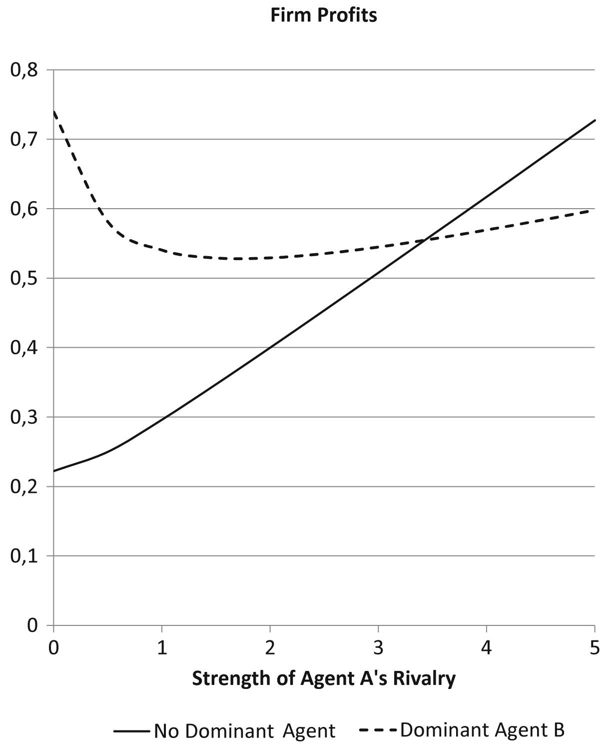

Fig. 3 Impact of agent A's rivalry on firm profits (No dominant agent: $r_A = r_B = \sigma_A^2 = \sigma_B^2 = 2$; Dominant agent B: $r_A = \sigma_A = 3$; $r_B = \sigma_B = 0.75$; Both cases: $c_A = c_B = 1$; $\overline{CE_A} = \overline{CE_B} = 0$). Own representation

agent B. Therefore, competing behaviour on the part of agents who make only small contributions to firm profits can have a negative effect, since this behaviour requires the principal to pay those agents greater attention in terms of variable wage compensation and to discriminate against the high performers. By contrast, when rivalrous agent A contributes as much as or more than selfish agent B, then agent A's ambition amplifies his or her effectiveness. Hence, the corresponding reduction in incentive intensity for B is less important. Thus, as is indicated by the continuous line in Fig. 3, when the two agents are very alike in terms of attitudes, exposure to risk, and marginal costs of effort, then regardless of any differences in self-interest or rivalry, the principal benefits from agent A's rivalry.

The same reasoning applies to any setting in which there are differences in the agents' RAS preferences. In particular, if one agent is altruistic and dominant and the other agent is competitive, then if the difference in RAS preferences is not large, the reduction in the altruist's variable wage compensation (and therefore effort) can have negative consequences on firm profits. Consequently, if the principal could choose which of the two agents should exhibit a stronger RA preference, the greater contribution (expressed as lower risk aversion, smaller exposure to risk, and smaller marginal costs of effort) would come from the more competitive (or less altruistic) agent. More generally, since the agent who behaves competitively has a greater positive impact on firm profits, it follows that from the firm's perspective, the best scenario involves both a rivalrous agent and an altruistic (or at least less competitive) agent who makes a smaller contribution to firm profits, i.e., is more risk averse, has a greater exposure to risk, and exhibits higher marginal costs of effort compared to the competitive agent.

The question of whether it is better to exploit the diversity of preferences by hiring a (more) altruistic and less effective agent or to hire a high performer with preferences that are more like those of the other agent depends on the strength(s) of both agents' RA preferences and on their effectiveness in manipulating firm profits. From an applied point of view, the important and relevant insight is that a firm's profit can be influenced not only by the effectiveness of employees, but also by their differences in RAS preferences.

3.3 Analyzing the Impact of Heterogeneous Reservation Utilities

So far, we assume that the reservation utilities of both agents are equal. However, this is not always the case in practice. It is easy to imagine that because of better qualifications, a better education, better contacts, a job profile which is more attractive for other employers, or simply a better overall curriculum vitae, one agent may have better outside alternatives than the other. Formally, we capture these factors by allowing for differences in the reservation utilities of both agents, which leads to $\overline{CE_A} \neq \overline{CE_B}$. At first, this setting seems to be similar to our analysis of the agents' effectiveness in manipulating firm profits. However, from an applied point of view, unequal outside options are completely different from agents being unequally effective in manipulating firm profits, and both phenomena can occur simultaneously in practice. Although possibly correlated, we can think of situations in which one agent is more valuable in terms of contributions to firm profits but at the same time has weaker outside options. For example, the agent might have private reasons for not being so flexible. Therefore, we examine the situation in which agents are heterogeneous with respect to their outside options. Since formally, this requires solving an equation of the fourth grade, we restrict our analysis to simulation analysis and logically derived remarks.

First, our asymptotical result—that for a sufficiently large difference in RAS preferences this difference will benefit firm profitability—continues to hold when agents are unequal in their reservation utilities. Second, if we assume equally effective agents, i.e., $r_A = r_B, \sigma_A = \sigma_B, c_A = c_B$, then, as is illustrated by the straight line in Fig. 4 (which simulates equation 16), as long as the agent with the stronger rivalry (or weaker altruism) also has the larger reservation utility, a greater difference in RAS preferences is still unambiguously beneficial from a firm point of view.

However, the dotted line in Fig. 4 shows that as soon as this is not the case, differences in RAS preferences are detrimental at first and only become beneficial when a critical value of the difference in the RAS preference parameter is exceeded.

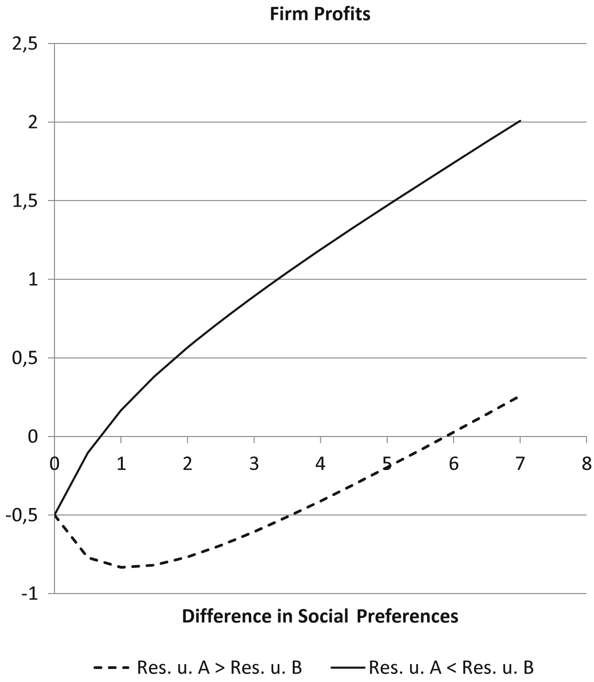

Fig. 4 Impact of the difference in social preferences on firm profits in cases of unequal reservation utilities (Res. u. A > Res. u. B: $\overline{CE_A} = 1$; $\overline{CE_B} = 0$; Res. u. A < Res. u. B: $\overline{CE_A} = 0$; $\overline{CE_B} = 1$; Both Cases: $r_A = r_B = \sigma_A^2 = \sigma_B^2 = c_A = c_B = 1$; $k = 0$). Own representation

The reasoning behind this result is straightforward. Equations (13) and (14) show that higher reservation utilities imply higher fixed wage payments to each of the two agents. When the other agent has stronger rivalry (or weaker altruism), this effect is amplified by the differences in RAS preferences, making them less profitable from the firm's point of view. This negative impact of unequal reservation utilities on firm profits will only be counterbalanced if the motivating effects stemming from differences in RAS preferences described in the previous sections are strong enough to dominate the negative effects caused by asymmetric reservation utilities. By contrast, when the agent who is more competitive (or less altruistic) has a larger reservation utility, then both effects point in the same direction and greater differences in RAS preferences are unambiguously beneficial with respect to the profitability of the firm.

These insights continue to hold for unequally effective agents. However, in these cases, there may be additional trade-offs that the principal must consider. First, when the more competitive (or less altruistic) agent is more effective and has a larger reservation utility, then from the firm's point of view a larger difference in RAS preferences is always beneficial. Second, in cases in which the less competitive (or more altruistic) agent is more effective and has the larger reservation utility, differences in RAS preferences are detrimental to firm profitability until a certain cutoff value is exceeded. Finally, whenever the effectiveness of both agents and their reservation utilities point in different directions on the impact of differences in RAS preferences on the firm's profitability, the overall impact of differences in RAS preferences on firm profitability depends on the relative strength of these effects. For example, we look at a situation in which one agent is much more effective in manipulating firm profits than the other, who is more altruistic (or less competitive) and possibly also has a larger reservation utility. As a reference point, from which we start varying the more altruistic agent's reservation utility, consider a situation with equal reservation utilities. In such a situation differences in RAS preferences are at first detrimental to firm profits. The differences only become beneficial if the difference in RAS preferences is large enough. However, as soon as the greater reservation utility of the agent with stronger altruism (or weaker competitiveness) exceeds a certain cutoff value, differences in RAS preferences become unambiguously profitable for the firm.

4 Discussion

4.1 *Robustness of the Results Against a Different Modeling of Altruism and Further Aspects Concerning the Modeling of Social Preferences*

Our choice of the way in which we model social preferences departs slightly from earlier models. Therefore, we show that our simplification makes it possible to analyse earlier formulations of altruism, self-interest, and rivalry within a unified framework. We do so by differentiating behavioural types solely through the relevant ranges of one social preference parameter without loss of generality. The most general linear specification of altruism proposed in previous studies yields:

$$CE_i = E[m_i S_i(\cdot) + n_i S_j(\cdot)] - V_i(\cdot) - \frac{r_i}{2} Var[m_i S_i(\cdot) + n_i S_j(\cdot)]. \tag{20}$$

The utility function is strictly increasing for both agents' wage compensations S_i and S_j, where the weighting factor $0 \leq m_i \leq 1$ reflects the fact that an agent's higher concentration on the other agent's remuneration through $n_i \geq 0$ can lead to a lessened subjective importance of his own reward. Using the general specification (20), we can differentiate among the special cases of altruism, which are used in the related literature:

$$m_i = m_j = 1 \tag{21a}$$

as well as

$$m_i + n_i = 1. \tag{21b}$$

For (21b), the specification in (20) corresponds to that in (5), assuming negative values for k_i. Since we want to rule out cases in which the other agent's remuneration is more important for that agent's well-being than his or her own remuneration, we restrict n_i to lie in the interval [0; 1] for cases of (21a) and to lie in the interval [0; 0, 5] for cases of (21b). It is clear that the only difference between the case in (21a) and that in (21b) is a shift in the reference point, i.e., the value of the social preference parameter n_i for which selfless behaviour begins. The same effect arises for varying m_i between zero and one. Since absolute values of parameters are without meaning in a model like ours, all of our insights, which are derived from comparative statics type analyses, continue to hold when we model altruism according to the general specification in Eq. (20).

4.2 Empirical Aspects of Designing Contracts and Our Assumptions with Respect to the Production Technology

One aspect that is inherent to our model construction is that for cases of very strong altruism, when the altruist puts almost the same weight on his or her own wage compensation as on the other agent's wage compensation ($k_i \to -0.5$), the altruist's share in the other agent's performance measure can become strongly negative, resulting in a forceful form of relative performance evaluation. We can observe this type of compensation in practice, e.g., in sales departments. Depending on both agents' RA preferences and their effectiveness in manipulating firm profits, the fixed wage-compensation components can also become negative.

By construction, the LEN model usually does not impose limited liability constraints, which means that agents' expected total remuneration is in principle allowed to become negative. This property is also known as the "deep pockets assumption," which is standard in models like ours. But by construction, the LEN model controls only for utility. In this respect, the fixed wage-compensation parameters serve as a means to constrain the agents to their reservation utilities, which implies that all surplus in terms of utility is shifted towards the principal. Therefore, we can interpret the goal function value in Eq. (15) as a measure of efficiency. Thus, interpreting the fixed wage-compensation parameters is not usually the focus of the analysis in a LEN model, so reservation utilities for simplification purposes are often normalized to zero. The result is that α_0 and β_0 can also embrace negative values in models that do not include social preferences in general and altruism in particular. However, expressions (13) and (14) show that for the fixed wage-compensation components, when reservation utilities $\overline{CE_A}$ and $\overline{CE_B}$ are large enough, the fixed wage-compensation components will usually be greater than zero. This is what we normally observe in practice. (However, we note that this argument does not hold for both agents when one agent has a much higher reservation utility than does the more altruistic agent and when the second agent's altruism is sufficiently large.)

Since the absolute values of $\overline{CE_A}$ and $\overline{CE_B}$ have no meaningful interpretation, for most cases, we can think of the absolute values as being sufficiently high without imposing any restrictions to the empirical relevance of our model. What may also be of relevance in this context is the expected amount of total compensation, which we calculate as:

$$S_i = \frac{k_i\overline{CE_j} + (1+k_j)\overline{CE_i}}{(1+k_i)(1+k_j) - k_ik_j} + \frac{1}{2c_i}\cdot\frac{(1+k_j)[(1+k_i)(1+k_j) - k_ik_j]}{(1+2k_j)^2}\cdot\frac{1}{1+r_i\sigma_i^2c_i} + \frac{1}{2c_j}\cdot\frac{k_i[(1+k_i)(1+k_j) - k_ik_j]}{(1+2k_i)^2}\cdot\frac{1}{1+r_j\sigma_j^2c_j} \quad ; \quad i,j = [A,B]. \tag{22}$$

This expression is usually positive, at least when $\overline{CE_A}$ and $\overline{CE_B}$ are sufficiently large. However, as noted, our argument that greater reservation utilities technically avoid hard-to-interpret negative wage-compensation components and negative expected wages is not valid under all circumstances. Due to the linear specification, our model incorporates both limits for the strength of altruism on the one side and for the strength of rivalry on the other, i.e., $k_i \to -0.5$ and $k_i \to \infty$. Depending on the values of all other parameters, these limits might lead to infinitely large or small share parameters, fixed wage-compensation components, expected amounts of compensation, and utilities. However, although these limits are a technical property of our continuous formulation of the model, this takes nothing away from any of our qualitative insights. Again, it is not the interpretation of absolute values or the extreme cases, but the comparative statics analysis that delivers the analytical background for our results. However, for many cases, these results accompany contracts that comprise positive fixed wage-compensation components and expected amounts of total compensation—at least when $\overline{CE_A}$ and $\overline{CE_B}$ are sufficiently large.

There is another aspect that concerns our setting. For reasons of clarity, we assume stochastic and technological independence. The only interrelation between the agents stems from their social preferences. It is an accepted fact that social preferences matter most in frequent and personal interactions, but they are less prevalent in anonymous competitive markets. People tend to choose which persons to use as a reference. However, social comparison as a matter of individual choice does not necessarily imply that these people work closely together, which would be the case when modelling technological interdependencies. However, what is relevant is that these people know each other and that there is some sort of interaction, which is usually the case when both agents are part of the same group. However, interaction does not necessarily mean that there must be technological interdependencies. Additionally, the results of both the Sandner (2009) and the Brunner and Sandner (2012) studies suggest that all of our insights continue to hold in settings where there are stochastic and/or technological interdependencies.

4.3 Observability of Remuneration Parameters and Preferences

To best exploit the agents' diverse RAS preferences, the principal must rely on both agents knowing each other's contracts and acting accordingly. As we noted in the previous section, people tend to compare themselves to others, people whom they know personally and with whom they frequently interact. The literature in "Behavioral Economics" points out that people also tend to compare themselves to equals (Fehr and Fischbacher 2002; Fehr and Schmidt 2003). Therefore, we can assume that they also talk about their remuneration and their contracts. Although the differences in contracts that individual employees in a firm receive are usually kept secret from outsiders, because of interest, curiosity, or because of their social preferences employees may talk to each other about their contract design and disclose this information accurately. Furthermore, due to their frequent interaction and their interest in the other person, an individual may be able to infer the other's contract compared to his or her own contract. Bonuses are paid at similar times, and when a person observes spending behaviour, the mood, or what people talk about in the halls, the individual might have quite an accurate impression of what his or her contract looks like compared to the other's contract.

Additionally, as a prerequisite for our results, the principal needs to know his or her agents' RAS preferences. Since these are psychological factors, they cannot be observed directly, but instead must be inferred and approximated. Hence, this assumption will just barely be entirely fulfilled in practice. There are two ways that principals can acquire information on agents' preferences in practice. The first way is through the explicit use of instruments such as assessment centers, intelligence, or other written tests, through interviews with trained employees or external service providers, or any combination of these factors. Such evaluations are often part of the hiring process, but they can also take place during internal restructuring processes. These practices seem to be quite widespread, especially in bigger enterprises. This evidence points to the relevance of psychological factors such as social preferences for the success of a company. The second way that principals can gain information is through the frequent and personal interaction that happens during the process of daily work. Employers normally get to know their employees quite well over the years. This acquaintanceship includes information about employees' behaviour towards others during work or in their spare time, and therefore reveals character traits such as social preferences.

From a psychological point of view it is not clear that people know exactly what their own preferences are, since many of these preferences are subconscious. Depending on the instruments a principal uses, the knowledge of human nature and experience that he or she has, and on how well the agents know themselves, there may be informational asymmetries on employer preferences such that either the principal or the agents are the better informed party. Against such a background we can reasonably assume symmetric information over agent preferences as an approximation of reality.

Finally, normative analytical models like ours cannot reproduce reality exactly. Models such as ours are a means of analysing problems and conditions that are empirically relevant. To capture the relations that researchers suspect are important, most models reduce the analysis to the parameters and relations that are necessary for narrowly showing certain causal relation. Thus, solutions of these analytical models are quite extreme. However, although their results cannot be adopted directly, they still yield important insights for practice. The important assumptions of our model include that the principal has a sufficiently well-developed perception of his or her agents' RAS preferences, their relative effectiveness (risk aversion, exposure to risk, and marginal costs of effort), and the other components of their utility functions.

5 Implications and Conclusion

5.1 Main Findings

Whether insights gained from fields such as experimental decision theory, behavioural psychology, and neuroscience can have an impact on economic theories is an important question in modern research. In this paper, we study how RAS preferences influence the incentive system in a decentralized organization. In our analysis we take into account that in real life not all people behave completely selfishly.

Unlike other research in the field, we examine altruism as one of three types of preferences (rivalry, pure self-interest, and altruism) and the differences between them. In our analysis, in which we consider a two-agent setting, altruism does not turn out to be inherently advantageous. This finding seems to be counterintuitive, but it becomes plausible in the context of the principal/agent situation. In such a situation we can assume that the principal must use incentives to prompt an agent to contribute maximally to firm profits. We express firm profits in terms of the principal's utility. If there are altruistic agents, then their altruism complicates the target-oriented compensation of the firm. This result is significant for practical applications, since it implies that in competitive environments in which profitability is key to firm success, it may not always be beneficial to compose a group comprising only altruistic, and therefore unambitious, members.

The central result of our paper is the insight that a firm may increase its profits by taking advantage of differences in the preferences of its agents and, more specifically, in the (relative) strength of those preferences. The reason for this is that RA preferences imply intrinsic motivation. Agents compare their compensation with that of other agents and are intrinsically motivated to change the difference between these compensations, but the kind of change agents will aim at depends on their RA preferences. A competitive agent strives to maximize that difference, while an altruistic agent may draw utility, and therefore motivation, from other agents who receive greater wage compensation. In the latter case, the principal does not need to use high-powered incentives to motivate the altruist. Instead, the principal can exploit his or her intrinsic motivation by varying the composition of the group of agents. When there is rivalry, this type of motivation drives competition in many areas of social interaction, including sports and other games, education, and the arts, as well as economics. As soon as a group of agents includes altruists, it is not necessary for the principal to use incentives to the same extent that would be necessary to motivate the agents if all agents exhibited pure self-interest. Therefore, in such settings incentives can be shifted partly towards the less altruistic, selfish, or competitive agent, who can thus be expected to invest greater effort. This important result indicates that since firms are hierarchically structured organizations, they have the opportunity to exploit intrinsic motivation by combining agents with different RAS preferences and optimizing the remuneration system. However, in a two-agent setting, these findings do not hold when the firm, in response to RA preferences, disfavors, in terms of variable wage-compensation components, the agent who contributes much more to profits than does the other agent. In this special case, the firm should combine agents with homogeneous preferences.

The results in Sect. 3.2 and proposition 2 show that in reality, if a firm aspires to maximize its profitability, then it must consider not only the effectiveness of managers, but also their RAS preferences. Therefore, the firm must scrutinize the impact of differences in RAS preferences, depending on the effectiveness of agents. Our model results show that combining both criteria can lead to a different decision than if both are considered separately and without taking the other influence factor into account. Although not an easy task, acquiring the necessary information about manager characteristics may be satisfactorily achieved in specially designed assessment centers. However, we can conclude that it is just as important to acquire information on manager RAS preferences as it is to gain information about their effectiveness, which, e.g., is achieved by looking at grades or attitudes towards taking risks.

A significant, and perhaps unexpected, result of our analysis is that usually people who share equally competitive or altruistic preferences are not more efficient as managers of different divisions in a firm than is a group composed exclusively of managers with selfish preferences, and in fact may be less efficient than a group of managers with different preferences. The insight that differences in preferences can be profitable and hence can be “managed” by a firm is certainly relevant to the issue of remuneration. However, at the same time, it raises the issue of ethical aspects, which should be considered in problems of economics as well as in business administration (Küpper 2011). The results of our analysis are important for determining organizational structure, the distribution of tasks and decision authority within a firm, and for selecting the right personnel for managerial positions and the members of business teams, including the company’s board. We believe that since preferences determine human behaviour to a high degree, different types of preferences are likely to affect many aspects of a firm.

5.2 Prerequisites and Ethical Aspects of Our Results

Our findings raise two questions. First, we ask if firms should combine persons with extremely different RAS preferences and employ them as managers of decentralized divisions. Second, we ask to what degree such a policy can be realized. Answers to these questions should take the prerequisites of our theoretical model into account, i.e., that the number of performance measures should equal the number of available agents, and that the principal is sufficiently insightful as to observe to some extent his or her agents' RAS preferences.

As soon as these prerequisites are fulfilled to at least some degree, the next question is whether and under what circumstances such a team is desirable. Generally speaking, anecdotical evidence and casual observations in practice support our results, since diversity in preferences is often recommended and propagated. However, it is important to differentiate between situations in which a group of managers control different divisions with several organizational tasks, and situations in which the members of a group must fulfil a common task. In the first situation, a firm can use the intrinsic motivation, which is caused by rivalry and differences in the preferences, to reduce extrinsic incentives. In most cases, it will not be possible to utilize the extreme form of our theoretical result by combining an entirely competitive agent with one who is altruistic, since the altruist will most probably feel exploited in the long run. Nevertheless, our result points to the possible benefits of diversity in preferences as well as in other personal characteristics of a team. The basic mechanisms of action behind this result continue to hold in cases of more than two agents, as is shown in Brunner and Sandner (2012). By contrast, in the second situation, when all members work on a common task, some empirical research indicates that teams whose members have similar preferences and shared norms seem to be most successful (Hobman et al. 2003)

Another question that arises relates to the legal and moral aspects of our results. Most societies and some laws and collective labour agreements accept the principles of equality. This compliance becomes particularly apparent in the form of general principles of remuneration (Krapp and Sandner 2016). In ethics, the principle of equality does not imply identical treatment, but rather a treatment that is in accordance with individual preferences (Küpper 2011, p. 100). The principles of equal treatment are more widespread in remuneration agreements for workers and lower-tier employees. Additionally, firms may use other sources of compensation or extrinsic motivation in general, such as decision authority, career opportunities, office space, parking space, and much more, to distinguish between the incentives for managers with different RAS preferences, particularly in higher management. Therefore, our result is more relevant for the selection of managers at higher levels. It implies that the assumption of pure self-interest for all managers is not particularly well suited when one is striving to maximize firm profits. Instead, a firm must consider individual preferences and can influence profits by selecting its managers according to their differences in both their RAS preferences and for other relevant characteristics such as risk aversion and costs of effort or productivity.

At first it may seem counterintuitive to make use of the differences in individual preferences to increase the profit of a firm. However, this result implies that it is necessary to pay even more attention to personal characteristics. The emphasis on the need to consider individual preferences provides many links to questions that are discussed in the field of business ethics. If the researcher does interpret the results of our model in a more general sense, there even is an unexpected similarity to the so-called "business case" of CSR, since it may be profitable to account for individual preferences when hiring and designing contracts. The researcher must then interpret the principle of equality as 'suum cuique'.

5.3 Further Research

All of the aspects we discuss indicate that further research is needed to analyse the relevance and practicability of the assumptions presented in our paper. For example, we believe it is important that we use a theoretical model without imposing limited liability constraints, as doing so allows compensation to become negative (debt position), in conjunction with the assumption that a firm can use different compensation parameters for each of the two types of agents. Since previous research suggests that results hinge on whether or not the limited liability constraints are imposed, this aspect calls for further analytical research that tests our results in different settings and against different modelling assumptions on the one hand and for empirical research on whether limited liability is a reasonable assumption or not on the other hand. Future research should also examine the degree to which moral ideas and principles of equality, justice, etc., influence behaviour in firms, and therefore limit the design of wage-compensation systems. Such principles relate to individual preferences in reciprocity and inequity aversion. Therefore, we believe that it is necessary to try to include those principles in future studies, and to analyse in greater detail how they influence the incentive system and the profitability of the firm. Krapp and Sandner (2016) provide a first such study. They do, however, only consider the case of homogeneous agents, which has its limits given the background of the contribution we make in this study. Our results also need to be scrutinized through empirical investigations. Such researches would identify in which areas and to what degree firms can use differences in personal preferences to increase their profits.

To conclude, we add that in reality, people's behaviour may be guided by a mixture of competitive, selfish, and altruistic motives. Assuming that selfish preferences always prevail is certainly a simplification of real-life behaviour. Nevertheless, if we define self-interest as the mean between rivalry and altruism, which represent extreme types of behaviour, on average this assumption might yield satisfactory results when modelling the behaviour of big groups. But at the same time, this assumption may not be correct in cases of individual behaviour, which is far more pertinent to decision-making in firms. As our study shows, principal/agent models that do not take into account differences in social preferences can overvalue the importance of providing firm managers with explicit incentives. Therefore, since behavioural science research constantly illuminates differences in people's choices and because it provides ongoing insights into the circumstances under which different preferences are reflected in personal behaviour, such insights should also be taken into greater consideration in future theoretical and empirical research in the fields of economics and business administration. The issues raised in our paper demand empirical studies on the relevant questions on topics such as the distribution, determinants, and stability of selfish and social preferences in companies.

References

Alchian AA, Demsetz H (1972) Production, information costs, and economic organization. TAER 62(5):777–795

Andreoni J, Miller J (2002) Giving according to GARP. An experimental test of the consistency of preferences for altruism. Econometrica 70(2):737–753

Bartling B, von Siemens F (2010a) Equal sharing rules in partnerships. J Theor Inst Econ 166(2):299–320

Bartling B, von Siemens F (2010b) The intensity of incentives in firms and markets: moral hazard with envious agents. Labour Econ 17(3):598–607

Bester H, Güth W (1998) Is altruism evolutionarily stable? JEBO 34(2):193–209

Brunner M, Sandner KJ (2012) Social comparison, group composition, and incentive provision. IJGT 41(3):565–602

Camerer CF, Loewenstein G, Prelec D (2004) Neuroeconomics. Why economics needs brains. ScanJE 106(3):555–579

Charness G, Masclet D, Villeval MC (2010) Competitive preferences and status as an incentive: experimental evidence. IZA Working Paper No. 5034

Chrisman JJ, Chua JH, Sharma P (2005) Trends and directions in the development of a strategic management theory of the family firm. Entrep Theory Pract 29(5):555–575

Demougin D, Fluet C (2006) Group vs. individual performance pay when workers are envious. In: Demougin D, Schade C (eds) Contributions to entrepreneurship and economics. First Haniel-Kreis meeting on entrepreneurial research. Duncker and Humblot, Berlin, pp 39–47

Demougin D, Fluet C, Helm C (2006) Output and wages with inequity averse agents. Can J Econ 39(2):399–413

Dierkes S, Harreiter B (2010) Profit-Center-Steuerung unter Berücksichtigung von sozialen Präferenzen. BFuP 62(5):534–557

Edgeworth FY (1881) Mathematical physics. An essay on the application of mathematics to the moral sciences. C. Kegan Paul, London

Englmaier F, Wambach A (2010) Optimal incentive contracts under inequity aversion. GEB 69(2):312–328

Fehr E, Falk A (2002) Psychological foundations of incentives. EER 46(4–5):687–724

Fehr E, Fischbacher U (2002) Why social preferences matter. The impact of non-selfish motives on competition, cooperation and incentives. TEJ 112(478):C1–C33

Fehr E, Fischbacher U, Kosfeld M (2005) Neuroeconomic foundations of trust and social preferences. Initial evidence. TAER 95(2):346–351

Fehr E, Schmidt KM (1999) A theory of fairness, competition and cooperation. TQJE 114(3): 817–868

Fehr E, Schmidt KM (2003) Theories of fairness and reciprocity. Evidence and economic applications. In: Dewatripont M, Hansen LP, Turnovsky SJ (eds) Advances in economics and econometrics, 8th World Congress. Cambridge, pp 208–257
Fröhlich E, Pichler JH (1988) Werte und Typen mittelständischer Unternehmer. Duncker and Humblot, Berlin
Hobman EV, Bordia P, Gallois C (2003) Consequences of feeling dissimilar from others in a work team. J Bus Psychol 17(3):301–325
Holmström B (1979) Moral hazard and observability. TBJE 10(1):74–91
Holmström B (1982) Moral hazard in teams. TBJE 13(2):324–340
Holmström B, Milgrom PR (1987) Aggregation and linearity in the provision of intertemporal incentives. Econometrica 55(2):303–328
Itoh H (2004) Moral hazard and other-regarding preferences. JapER 55(1):18–45
Karra N, Tracey P, Phillips N (2006) Altruism and agency in the family firm. Exploring the role of family, kinship, and ethnicity. Entrep Theory Pract 30(6):861–877
Kolm S-C (2000) The theory of reciprocity. In: Gérard-Varet L-A, Kolm S-C, Mercier-Ythier J (eds) The economics of reciprocity, giving and altruism. Macmillan, London, pp 115–141
Krapp M, Sandner KJ (2016) Impact of an equal pay norm on the optimal design of incentive contracts. J Bus Econ 86(3):301–338
Küpper H-U (2011) Unternehmensethik. Hintergründe, Konzepte, Anwendungsbereiche. 2nd edn. Schäffer Poeschel, Stuttgart
Küpper H-U, Sandner KJ (2008) Differences in social preferences: are they profitable for the firm? Working Paper 2008-3, LMU Munich
Küpper H-U, Sandner KJ (2011) Anreizsysteme und Unternehmensethik. In: Küpper H-U, Schreck P (eds): Unternehmensethik in Forschung und Lehre, ZfB - Sonderheft 1, pp 119–148
Macaulay JR, Berkowitz L (1970) Altruism and helping behavior. Social psychological studies of some antecedents and consequences. Academic Press, New York
Mayer B (2006) Prinzipien der Anreizgestaltung bei sozialen Interaktionen. Eine theoriegeleitete Analyse. Unpublished dissertation
Mayer B, Pfeiffer T (2004) Prinzipien der Anreizgestaltung bei Risikoaversion und sozialen Präferenzen. ZfB 74(10):1047–1075
Neilson WS, Stowe J (2010) Piece rate contracts for other-regarding workers. EI 48(3):575–586
Rey Biel P (2008) Inequity aversion and team incentives. ScanJE 110(2):297–320
Sandner KJ (2008) Behavioral contract theory. Einfluss sozialer Präferenzen auf die Steuerung dezentraler Organisationseinheiten. Berlin
Sandner KJ (2009) Impacts of rivalry on types of compensation. Competition vs. co-operation between multiple agents under technological interdependencies. ZfB 79(4):427–471
Schulze WS, Lubatkin MH, Dino RN (2003) Toward a theory of agency and altruism in family firms. J Bus Ventur 18(4):473–490
van den Berghe LAA, Carchon S (2003) Agency relations within the family business system: an exploratory approach. CG 11(3):171–180
van Dijk WW, Ouwerkerk JW, Goslinga S, Nieweg M, Gallucci M (2006) When people fall from grace: reconsidering the role of envy in Schadenfreude. Emotion 6(1):156–160

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Applications and Potentials of Auction Theory in Management Accounting

Max Patzenhauer

Abstract In economics, the applications of auction theory are very broad. The present paper examines the applications of auction theory, especially in a management accounting context. Auctions are widely used in different applications. Often auctions have been used as procurement auctions with a reverse auction design, for instance to choose the best supplier. In addition, auctions have been used in supply chain management. In particular, the present paper will take a look at internal auctions, which are used to allocate scarce resources within a firm. The main purpose of the present paper is to exposit the state of the art of the applications of auctions and auction theory. But also there is already a huge potential for auction theory as an instrument of management accounting. So the present paper will also take a look at auctions that are used to generate information or to coordinate firms, for example, auctions could be used to determine transfer prices or budgets.

Keywords Auction • Budgeting • Common value • Dutch auction • English auction • Interdependent value • Private value • Reverse auction • Supply chain management • Transfer prices • Vickrey auction

Contents

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1 Introduction

Everyone has an idea of auctions in general. For example, auctions for antiques and art or online auctions like eBay. But there are also applications of auctions in further areas. There is also a very extensive theory of auctions, with its game theoretic concepts and analyses to describe how auctions really work.

The present paper will provide a literature overview of the previous and current applications of auction theoretic approaches as an instrument in the management accounting. Furthermore, the potential of auction theory in management accounting will be pointed out. One of the main problems in economics and business is to allocate and handle scarce resources. Auction theory can provide a solution for the efficient and incentive-compatible allocation of scarce resources.

In Sect. 2 there will be a brief summary of the current approaches in management accounting. It will include an overview of the functions of management accounting. Then there will be an overview of the essentials of auction theory in Sect. 3. In that section, the most popular auction designs will be considered. Section 4 will be the main part of this paper. There, the applications of auctions will be pointed out. In Sect. 4.1 there will be a description of the applications of auctions in general economics, for example to the arts or to allocate spectrum licences. Section 4.2 is more important: there, a literature overview of the previous applications of auction theory in the management accounting context will be provided. For instance, auctions have been applied in a supply chain management context, staff planning as transfer auctions, and to allocate internal scarce resources, risk capital, and truckload capacities. Furthermore, auctions have been used as procurement auctions with a reverse auction design, to determine the best supplier. Moreover, there are applications of auctions in corporate finance or asset pricing. In addition, auctions can be used as emission trading instruments or to optimize airport slot time allocation. Whenever there are scarce resources to allocate, one can use an auction theoretic design that is incentive-compatible and efficient, so that the bidders will have a truthful bidding behaviour. Thus, the resource will be awarded to the bidder who values the item or the resource the most. After this overview of the previous approaches of using auctions, there will be mentioned, in Sect. 5, the potential of auction theory. Thus, in that section, an outlook for the future will be presented.

2 Fundamentals of Management Accounting

Before we can consider, in the next section, the applications of auctions or auction theory, especially in a management accounting context, we should represent the current development of management accounting theory. There are very detailed overviews of the literature on management control systems: Hared et al. (2013), Berry et al. (2009), Malmi and Brown (2008), Rom and Rohde (2007), and Langfield-Smith (1997). In addition, Malmi (2016) reviews the studies in management accounting of the last 25 years; for the last 35 years, see Otley (2016). Furthermore, Nielsen et al. (2015) consider outsourcing decisions in a management accounting context. This could be important when we take a look at auction theoretic approaches to the selection of suppliers.

Management accounting is, next to financial accounting, one of the most important information systems in a company (Nilsson and Stockenstrand 2015, pp. 2, 17). So the application of auctions to management accounting should generate information for the company with a focus on the internal users (Schuster 2015, p. 1). Such information about the management accounting system could be used by decision makers for the planning, coordination and internal control of the company (Schuster 2015, p. 1). Management accounting, as an internal accounting, is free of legal and other restrictive rules (Schuster 2015, p. 1). Schuster (2015) considers two main functions of management accounting: decision making and behavioural control.

One of the first characterizations of management accounting was provided by Anthony (1988) or, earlier, by Anthony (1965). Anthony (1965) differentiates the internally oriented processes in strategic planning, management control, and operational control, as opposed to the externally oriented process: financial accounting (Anthony 1965, p. 22). In 1988 he described the management functions as planning, control, directing, and coordinating (Anthony 1988, pp. 26–27). Furthermore, he defines management control as a process of the manager to influence the members of an organization and to implement the strategies of this organization (Anthony 1988, p. 34). Also he considered information systems in a management control context (Anthony 1988, pp. 121–144). A more current work in management accounting and management control is by Anthony et al. (2014). Emmanuel et al. (1997, p. 1) see management accounting as a part of the process of organizational control. Also they consider accounting information as an important aspect of management control (Emmanuel et al. 1997, pp. 34–36). Kaplan and Atkinson (1998) see the functions of management accounting information in companies as a way to improve decision making, guide the development of strategy and evaluate existing strategies, and focus efforts related to better organizational performance and evaluating the contribution and performance of individual divisions and members (Kaplan and Atkinson 1998, p. 12).

In summary, the main functions of a management accounting system are the generation of information and the coordination of the firm. Thus, in Sect. 4.2, when we will investigate the use of auctions and auction theory in a management accounting context, we will consider operations research as a huge part of management accounting, and we will consider the application of auctions as an instrument to provide information and to coordinate internal processes within firms. But we can also use auctions to generate information about the firm, its environment, and suppliers, for instance. Furthermore, it is possible to coordinate the processes and exchanges between firms, in a supply chain for instance, with the application of auction theoretic approaches. Whenever it is necessary to provide information about the true value of scarce resources, auctions could be useful to solve this problem and to coordinate the system with the right or efficient allocation of the resources. So auctions could be perceived as an instrument for incentive-compatible management, control, and coordination.

3 Basics of Auction Theory

3.1 Fundamentals of Auction Theory

After the presentation of the current development and problems of management accounting, we want to consider solutions of management accounting problems with the tools of auction theory. So our first question is what characterizes an auction. The market with the strongest tradition of auctions is for antiques and art. Shubik (1983) and Cassady (1967) provided good reviews of the history of auctions. From a linguistic point of view, 'auction' is derived from the Latin *augeo* (increase). That is a hint of the pricing rule used in traditional ascending auctions (Mochón and Sáez 2015, p. 2).

We understand an auction as a market mechanism. We determine, with an auction, to whom one or more items will be awarded and at what price. Especially when it is difficult to set a market price we can use auctions, because auctions can tell us how much a specific picture, for example, is worth. In history the first known auctions were described by the Greek historian Herodotus of Halicarnassus. He pointed out that the Babylonians auctioned women of marriageable age (Mochón and Sáez 2015, p. 1; Klemperer 2004, p. 1; Cassady 1967, p. 26). Very detailed surveys of auction theory are provided by Klemperer (1999), Wolfstetter (1996), Samuelson (2014), and Milgrom (1989); or for a brief overview, see Salvatore (2015, pp. 560–562). Vickrey (1961) has also provided an early investigation of auction theory.

An auction can be defined as a market institution in which an explicit set of rules determine the allocation and prices of resources by the bids of the market participants (McAfee and McMillan 1987, p. 701). In other words, an auction is a way to sell an item of unknown value to interested buyers. So an auction is a market mechanism to determine a price for a good or service and allocate it in most cases to the highest bidder, but there are also other allocation rules, which will not be considered in the present paper. The question is how to design an auction so that the items will be allocated optimally (Krishna 2010, p. 62)

3.2 Types of Auctions

For auctions we can classify the values of the bidders as either private, interdependent, or common values. We have **private values** or privately known values when each bidder has a well-known personal value of the item and the values to the other bidders are unknown. Also, the private value would not change if knowledge about the values to the other bidders were obtained. Private values are strongly based on personal and emotional preferences. The bidders purchase the items for personal use. We only have private values when there is no opportunity to resell the items and obtain a profit (Mochón and Sáez 2015, p. 3; Krishna 2010, p. 3; Klemperer 2004, p. 13) . Art or antiques are good examples of this kind of valuation.

The next possible case is that of **interdependent values** in which each bidder has his own personal estimate of the item's value. But now the information about the valuation of the other bidders influences his own valuation. That means that if the bidder gets some signals about the other bidders' estimates this information affects the own-value (Mochón and Sáez 2015, p. 3; Krishna 2010, p. 3).

Another possibility of valuation is that of **common values**. This is a special case of interdependent values. Here, each bidder has his own estimated value of the item before the auction. But after the auction and with complete information about the item, all of the bidders have the same value (Mochón and Sáez 2015, p. 3; Klemperer 2004, p. 13). For example, with licences for oil wells, the bidders can only ex post determine the true value.

It is also possible to subdivide auctions into different forms of auction. There are open-bid and sealed-bid auctions. As an **open-bid auction**, we have the open ascending-bid auction called the English auction and the open descending-bid auction called the Dutch auction. The **sealed-bid auctions** can be differentiated in first-price and second-price auctions, also called a Vickrey auction, because Vickrey (1961) investigated this auction mechanism and pointed out that if we assume private values, there is an incentive for truth telling and so it is the dominant strategy of all bidders to bid their true valuation for the good (Vickrey 1961, p. 21). In addition to this issue, Rothkopf et al. (1990) consider the Vickrey auction in more detail. These auction types are the four traditional forms of an auction (Krishna 2010, p. 2; Klemperer 2004, p. 11; Matthews 1995, p. 4).

An **ascending-bid auction** or an **English auction** is one in which the seller or the auctioneer sets a starting price (which is low and can be zero) and then the price increases as long as there is more than one bidder. When only one bidder is left, he wins the auction. There are different ways to increase the price. The seller or the bidders could increase the price, or the price could increase continuously with time. The English auction is a dynamic auction. The winner pays the second highest bid plus the bid increment. This auction design is easy to understand and implement (Mochón and Sáez 2015, pp. 12–14; Klemperer 2004, p. 11; Matthews 1995, pp. 15–16; Cassady 1967, pp. 56–60). The English auction is one of the most popular auction formats (Krishna 2010, p. 129). Figure 1 illustrate the auction format of the English auction. There are three bidders, who make their bids until only one bidder is left.

In the **descending-bid auction** or **Dutch auction**, we start with a high price and decrease the price until one bidder submits a bid. The Dutch auction is also a dynamic auction, but with a first-price rule (Mochón and Sáez 2015, pp. 14–15; Klemperer 2004, p. 12; Matthews 1995, pp. 29–31; Cassady 1967, pp. 60–63). Figure 2 provides an illustration of the Dutch auction format.

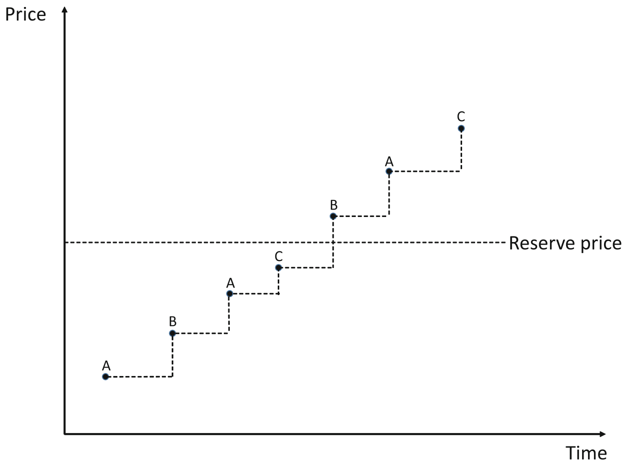

Fig. 1 English (ascending-bid) auction (Cassady 1967, p. 58)

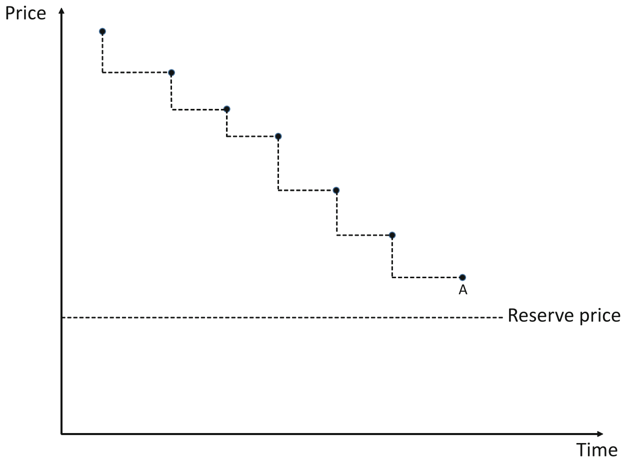

Fig. 2 Dutch (descending-bid) auction (Cassady 1967, p. 61)

In the **first-price sealed-bid auction**, all bidders offer their bids simultaneously and the highest bidder wins the auction and pays his bid (Mochón and Sáez 2015, p. 15; Klemperer 2004, p. 12; Matthews 1995, pp. 16–18).

In the **second-price auction**, also called a **Vickrey auction**, the bidders also bid simultaneously and the winner has the highest bid, but only has to pay the second highest bid (Mochón and Sáez 2015, pp. 15–16; Klemperer 2004, p. 12; Matthews 1995, pp. 8–14) . For a comprehensive review of Vickrey auctions, see Ausubel and Milgrom (2006).

It is very important to think about the incentives and behaviour in auctions. Coppinger et al. (1980) provide a good overview. We can show that there are

Table 1 Standard auctions

	Open-bid	Sealed-bid
Ascending auction	English auction	Vickrey auction
Descending auction	Dutch auction	First-price auction

strategic equivalences between the four auction formats. The first-price sealed-bid auction is strategically equivalent to the Dutch auction. In both auctions, we have the same bidding strategy with the same surplus. For private values, the second-price sealed-bid auction or Vickrey auction is strategically equivalent to the English auction. We can show, for any bidder, that it is best to bid their personal value (Mochón and Sáez 2015, pp. 16–17; Klemperer 2004, pp. 13–14). We can sum up by saying that in Vickrey auctions, it is the weakly dominant strategy for a bidder to bid his value. This holds even if the bidders are asymmetric. It follows that the bidder who will win the auction is the one who values the item most. So, the Vickrey auction always leads to an ex post efficiency under the premise of private values (Krishna 2010, p. 53). In a Vickrey auction the dominant strategy for all bidders is to bid their true evaluation of the item. If one bidder were to bid more than his reserve price, he would face the risk of getting the item for a higher price than his willingness to pay, if the second highest price is higher than his reserve price. If one bidder were to bid less than his reserve price, he would face the risk of not getting the item if he bids less than the bidder with the second highest reserve price. So there is an incentive for every bidder to bid truthfully. Because of the strategical equivalence of the English and the Vickrey auction, it follows that both are efficient mechanisms. So in the English and the Vickrey auction, the winning bidder is that one, who values the item the most; thus we have an efficient allocation (Krishna 2010, p. 129; Krishna 2010, pp. 188–189).

In contrast, a first-price sealed-bid auction yields, in the presence of asymmetries, an inefficient allocation. Thus, the winner of the auction is not the bidder with the highest value (Krishna 2010, pp. 53–54). Table 1 summarizes the four traditional auction formats.

Besides the four traditional auction formats, there are more exotic auction formats. These formats are studied from a theoretical point of view, but are not very practicable because of their complexity and incentives. An example is the all-pay auction, which is an auction in which every bidder pays their bid and the highest bid is be awarded with the item. We can model lobbying activities with an *all-pay auction*: different interest groups spend money (their 'bids') because they want to affect the government in a certain direction. The group that spends the most (the highest 'bidder') can push the government in the preferred direction (Krishna 2010, p. 29; Mochón and Sáez 2015, pp. 28–29; Klemperer 2004, p. 17). For an overview of all-pay auctions, see Amann and Leininger (1995), Baye et al. (1996), Che and Gale (1996), and Krishna and Morgan (1997).

Another exotic auction format is the *loser-pay auction*. Here the highest bidder will be awarded with the item, but pay nothing: only the losing bidders pay their own bids (Krishna 2010, p. 34).

Also possible are *hybrid auctions*, i.e., combinations of the auction formats. For example, an Anglo-Dutch auction, which combines a dynamic phase with a sealed-bid phase. In the first step we have an ascending-bid auction. So, the price is increasing until only two bidders are left. In the second step, these two bidders bid in a Dutch auction (Mochón and Sáez 2015, pp. 25–26; Klemperer 2004, pp. 116–117).

We can also do this the other way round, with the Dutch-English auction. In the first step we have a Dutch auction until one bidder bids. That bid will be used in the second step as the opening bid for the English auction (Mochón and Sáez 2015, pp. 26–27).

Other possibilities are the third-price auction, in which the winner pays the third highest bid or the average of the bids placed, so the winner with the highest bid pays the average over all bidders (Mochón and Sáez 2015, p. 27). We also can design auctions with different closing rules, for example, deadline auctions in which the auction ends at a certain point in time. In such an auction there is a last minute bidding or 'sniping' strategy (Mochón and Sáez 2015, p. 28).

An important auction design includes the **combinatorial auctions**. This means that there is not only one item to allocate, but several items in the auction and the bidders can bid on packages or bundles of the items. Lehmann et al. (2002) provide an article about truth telling and efficiency in combinatorial auctions. Therefore they consider a generalized Vickrey auction. To get a further impression of the generalized Vickrey auction see MacKie-Mason and Varian (1994), Ausubel (1999), and Ausubel and Milgrom (2006). The bidders could bid for several items in one auction, and they can bid for the items or for combinations of the items (Mochón and Sáez 2015, p. 87). For an overview of combinatorial auctions, see Mochón and Sáez (2015, pp. 87–120). Hunsberger and Grosz (2000) provide an approach to collaborative planning using a combinatorial auction. They presented a combinatorial auction mechanism which agents could use to find a solution for the initial-commitment decision problem (Hunsberger and Grosz 2000). The agents bid on roles in the group activity, each role comprising constituent subtasks that must be done by the same agent.

3.3 Desiderata for Auctions

It is very important to talk about what an optimal and an efficient auction is. An **optimal auction** is an auction in which we have the maximum expected revenue for the seller, or the auctioneer. Because of the revenue equivalence theorem, we can assert that under the premise of private values, the four traditional single-unit auctions lead to the same expected revenue to the seller (Mochón and Sáez 2015, p. 30; Klemperer 2004, p. 2; Klemperer 2004, p. 17). Myerson (1981), Riley and Samuelson (1981), Bulow and Roberts (1989) and Armstrong (2000) consider the problem of optimal auction design. Furthermore, Maskin and Riley (1984) investigate optimal auctions with risk averse buyers.

When the item is awarded to the bidder with the highest value, we have an **efficient auction** (Mochón and Sáez 2015, p. 31). For an overview of efficient auctions, see Ausubel (1997), Krishna and Perry (1998), Dasgupta and Maskin (2000), Pesendorfer and Swinkels (2000), Ausubel (2004), Ausubel et al. (2014) and Jackson (2003). Furthermore, Matthews (1980) considers efficient auctions under risk aversion for first- and second-price auctions. Close to this, we can define *incentive compatible auctions* as those auctions in which for all bidders it is the dominant strategy to bid their own true values. So the bidders will have a truthful bidding behaviour. Such an auction must be efficient, because the bidder who values the item the most will be the highest bidder and so win the auction. To consider the bidding behaviour, the dominant strategy and the incentives for truthful bidding, we have to assume that they are rational players (or bidders). We remark that the Vickrey auction is an efficient auction mechanism because it fulfills these conditions.

A first-price sealed-bid auction is not efficient, because it is not the dominant strategy for all bidders to bid their value: if they bid their value they always have a surplus of zero while winning the auction. So all bidders bid less than their value. Then it could probably happen that the bidder with the highest valuation of an item does not submit the highest bid. For the analysis of the dominant strategies and incentive compatibility, see Mookherjee and Reichelstein (1992), and for the dominant strategies in double auctions, see McAfee (1992). For the efficiency of double auctions, see Wilson (1985). There is a conflict between the goals of revenue maximization and efficient allocation (Mochón and Sáez 2015, p. 31). But from the overall company perspective, the revenue in internal auctions does not matter, so efficient allocation should be the focus. Ausubel and Cramton (1999) provide an investigation of the conflicting goals of optimal and efficient auctions. Thus this section has considered the fundamentals for the theory of auctions. Next, in Sect. 4, the applications of auction theory will be considered.

4 Applications of Auction Theory

4.1 A Framework

This section is divided into two parts. First we consider the applications of auction theory in general economic theory; second, in management accounting. In the general economics section, especially auctions for spectrum licenses will be the focus. For such auctions, we can assume almost common values for the bidders, in contrast to the other part, the applications of auctions in management accounting, where almost private values can be assumed. The section on auctions in management accounting will be subdivided into internal and external auctions. For an overall overview, see Table 2.

Table 2 Different types of auctions and their use in different branches

Use of different auction types	
Types of auctions	References
– Procurement auctions	– Huang et al. (2013), Bichler et al. (2006), Sheffi (2004), Lorentziadis (2014), Walsh and Wellman (1998), Emiliani and Stec (2002), Huang and Xu (2013), Xu and Huang (2015), Grimm (2004, 2007), Grimm et al. (2006a), Arozamena and Cantillon (2004), Jofre-Bonet and Pesendorfer (2000), Samuelson (1986), Engel (2009), Engel and Wambach (2005), and Campo (2012)
– Reverse auctions	– Emiliani (2000), Jap (2007), Leong (2008), Emiliani and Stec (2002), Emiliani (2004, 2005), Smart and Harrison (2002, 2003), Lösch and Lambert (2007), Stein et al. (2003), Caniëls and van Raaij (2009), Smeltzer and Carr (2003), Wagner and Schwab (2004), Kros et al. (2011), Yu et al. (2007), Chang (2007), Razuk et al. (2009), and Ağralıet al. (2008)
Fields of applications	
Fields of application	References
– Allocation of internal resources	– Hodak (1997), Baiman et al. (2007), and Harris et al. (1982)
– Corporate finance and asset pricing	– Baule (2012), Dasgupta and Hansen (2007), and Breuer et al. (2015)
– Airport slot time allocation	– Rassenti et al. (1982) and Balinski and Sand (1985)
– Allocation of truckload capacities	– Caplice and Sheffi (2006) and Chen et al. (2009)
– Wine & arts	– Ashenfelter (1989) and Ashenfelter and Graddy (2003)
– Distributed resource scheduling	– Kutanoglu and Wu (1999) and Adhau et al. (2012)
– Emission trading	– Grimm and Ilieva (2013), Sturm (2008), Cramton and Kerr (2002), and Muller et al. (2002)
– Staff planning	– Chiaramonte and Chiaramonte (2008), De Grano et al. (2009), Koeppel (2004), De Grano and Medeiros (2007), and Sasaki and Nishi (2015)
– Coordination of supply chains	– Lorentziadis (2014), Ertogral and Wu (2000), Morales and Steinberg (2014), Walsh et al. (2000), Emiliani (2004), Wu (2001), and Hobbs (1996)
– Spectrum licenses	– Grimm et al. (2001, 2003), Wolfstetter (2001), Börgers and Dustmann (2005), Binmore and Klemperer (2002), Klemperer (2002); van Damme (2002), Cramton and Ockenfels (2015), Doyle and McShane (2003), Cramton (2013), Sutter et al. (2007), Ewerhart and Moldovanu (2002), Milgrom (1989), McMillan (1995), Gebhardt and Wambach (2008), and Ma et al. (2014)

4.2 *Applications to Economics in General*

In this section the applications of auction theory will be presented. Therefore, we consider general economic issues before subsequently, in the following section, investigating auction theory more particularly with regard to its applications in management accounting. A very general overview is provided by Ashenfelter (1989) and Ashenfelter and Graddy (2003). They especially consider the auctions of wine and art.

Auctions have often been used to allocate spectrum licences and so on. This is an important issue, because these licence auctions could yield high revenues for the seller and it is important to think about the design of such an auction to get high gains. Grimm et al. (2001) consider the auction of the third generation spectrum of the Universal Mobile Telecommunications System (UMTS) in Germany. They explain the design of this auction. They point out that a spectrum license auction could lead to high revenue, for example, €37.5 billion in the United Kingdom and €50.8 billion in Germany (Grimm et al. 2001, p. 2). But there could be disappointing license auctions, for example, in Italy, the Netherlands, and Switzerland, where the revenues were significantly lower than expected (Grimm et al. 2001, p. 2).

In addition, Grimm et al. (2003) investigate the second generation (GSM) spectrum auction in Germany in 1999. They point out that there was a very low price equilibrium; they consider the reasons why such a low price was realized. Furthermore, Wolfstetter (2001) take a look at the flopped UMTS spectrum auction in Switzerland. He points out the reasons for the low realized revenue, and identifies weaknesses and problems in the auction design. Thus he provides advice on how to improve spectrum auctions. Besides these issues, Börgers and Dustmann (2005) investigated the bidding behaviour in the third generation auction that took place in the United Kingdom. Closely related to this, Binmore and Klemperer (2002) also investigated the United Kingdom's third generation auction. Klemperer (2002) also analyses the European UMTS auctions. He especially considered the spectrum auctions in the United Kingdom, the Netherlands, Italy, and Switzerland, with simple ascending auctions as well as the variable price ascending auctions in Germany and Austria in the year 2000. Moreover he took a look at the spectrum auctions in the year 2001 in Belgium, Greece, and Denmark. He points out the weaknesses and strengths of these auctions.

Closely related to this investigation is van Damme (2002), who examines the European UMTS auctions, too. Furthermore, Cramton and Ockenfels (2015) analysed the 4G spectrum auction in Germany. But it is not only in Europe that spectrum licences have been allocated with the use of auctions. Doyle and McShane (2003), for instance, provide an investigation of the GSM spectrum auction in Nigeria. Other papers on this issue include Cramton (2013), Sutter et al. (2007), Ewerhart and Moldovanu (2002), Milgrom (1998), and McMillan (1995). McMillan (1995), for instance, considered the reasons for auctioning the spectrum. Cramton (2013) took a look at the designs of spectrum auctions. Sutter et al. (2007) provide an experimental investigation of UMTS auctions, examining the bidding behaviour, and especially the opportunity of collusive bidding via team building instead of individual bidding.

Gebhardt and Wambach (2008) provide an approach to implement an efficient market structure with auctions. They used jumping English auctions instead of standard English auctions, to create an efficient market structure to allocate licences, for instance in UMTS licence auctions in Europe.

Besides these approaches for spectrum licences, Ma et al. (2014) considered the incentives for demand side management in a smart grid. They used auctions as an incentive mechanism.

Moreover, Harris and Raviv (1981) provide an investigation of the allocation mechanism and the design of auctions. They investigate the advantages of auctions and why to use them to allocate resources in some environments. Furthermore, they want to find an auction that is both Pareto optimal and optimal from the seller's perspective. So they thought about how to design an auction. This issue is also important for the next section, in which we want to consider applications of auction theory in a management accounting context.

4.3 Applications in Management Accounting

4.3.1 Applications to Internal Markets

We now want to present previous applications of auction theory in management accounting to internal markets. This means that we want to consider auctions within the firm. Thus, the allocation of scarce resource between the divisions, for instance, is the issue.

Hodak (1997) investigates transfers in organizations with transfer auctions. He gives the example of AT&T's using transfer auctions for a cost acceptance process to determine whether network upgrades are profitable, with a sealed bid auction design (Hodak 1997, p. 121). Transfer auctions could be used for investments in any shared service, for example, legal work, human resources, and IT services—or even core operations (Hodak 1997, p. 123) An interesting approach is to use an auction mechanism instead of to negotiate the transfer prices.

With the Vickrey auction, we can create an efficient allocation of the transferred goods and services, because we know that a Vickrey auction is an efficient auction mechanism (Krishna 2010, p. 53). Furthermore, Bichler et al. (2002) investigate auctions as an allocation mechanism and provide a framework for IBM. They provided an allocation algorithm for multidimensional auctions and a winner determination algorithm for electronic auctions. Huang et al. (2013) provide an analysis of the auction and bargaining mechanism in a procurement auction. Other research which compares auctions versus negotiations are, for example, Gretschko and Wambach (2014), Gretschko and Wambach (2013), Bulow and Klemperer (1996), and Bajari et al. (2009). Already in 1982, Harris et al. (1982) mentioned the idea of designing an auction mechanism to allocate internal resources.

Baiman et al. (2007) analyse an internal market by using an auction mechanism for resource allocation. They used a second-price sealed-bid auction, the Vickrey auction. They chose this auction design because, as we have already mentioned, it is the dominant strategy to bid the true value for the resource (Baiman et al.

2007, p. 917). This approach is related to transfer pricing, but the difference is that the resource is not transferred between divisions, rather the divisions compete for a centrally held resource (Baiman et al. 2007, p. 918). They create a model that considers two division managers, who compete by bidding for the resource. The division managers' dominant strategy is to bid their true values, so the resource allocation will be efficient: the payment to the division managers depends on the success of the divisions.

Baule (2012) presents an approach in which economic risk capital is allocated by a sequential auction in which the investment allowances are based on the marginal risk contributions. He used a Vickrey auction, because it allocates the units efficiently, because of its truthful bidding behaviour (Baule 2012, p. 16). He used the fact that in a Vickrey auction the bidders bid rationally their willingness to pay, because the winner only has to pay the second highest price (Baule 2012, p. 18). With this approach, he wants to solve the problem of asymmetric information (Baule 2012, p. 24).

It is important to mention the work of Jose and Ungar. They provide an auction-driven coordination for plantwide optimization (Jose and Ungar 1998). They compare a resource slack auction with Lagrangian-based coordination, and come to the result that the slack resource auction is the better approach. With slack prices, they can identify bottlenecks (Jose and Ungar 1998, p. 5). In an extension of this approach, they used slack auctions for the pricing of interprocess streams (Jose and Ungar 2000). They used auctions to optimize the profitability of the divisions in a chemical plant and also to compare the solutions of a Lagrangian approach with those of the slack auction method. Furthermore, Jose et al. (1997) consider the coordination of locally constrained agents by using augmented pricing. They provide an auction-theoretic approach and they point out that augmented prices are optimal to coordinate agents if they have concave objectives and compact feasible sets. They investigate the existence conditions for augmented prices and consider why augmented pricing works.

Kutanoglu and Wu (1999) also investigate combinatorial auctions, focussing on distributed resource scheduling. They consider a classical job scheduling problem and show a possible solution based on multi-item combinatorial auctions (Kutanoglu and Wu 1999, p. 815). In this setting, the auctioneer is the coordinating agent. They investigate a regular and an augmented *tâtonnement* (Kutanoglu and Wu 1999, p. 816). Furthermore, Adhau et al. (2012) investigate an auction-theoretic negotiation approach for distributed multi-project scheduling in a multi-agent system. Another approach is provided by Wellman et al. (2001), who consider decentralized scheduling with auctions. They investigate an ascending single-unit auction and a generalized Vickrey auction for a combinatorial auction design. There is a note on this issue provided by Hall and Liu (2011).

Auctions also could be applied to the staff planning problem, as a form of staff accounting or personnel control, for instance, to solve the nurse rostering problem, also known as the nurse scheduling problem. See, for example Chiaramonte and Chiaramonte (2008), De Grano et al. (2009), Koeppel (2004), and De Grano and Medeiros (2007).

Furthermore, in the edited volume published by Cramton et al. (2006), there is much about the applications of combinatorial auctions. To be mentioned are Ball et al. (2006), Bichler et al. (2002), Cantillon and Pesendorfer (2006), and Caplice and Sheffi (2006). Ball et al. (2006) used combinatorial auctions to allocate airspace system resources with a focus on safety and efficiency. Sasaki and Nishi (2015) presented an auction theoretic approach for an airline crew scheduling problem while using combinatorial auctions with a price adjustment mechanism. Another approach is provided by Caplice and Sheffi (2006), who consider combinatorial auctions for a truckload transportation problem and allocate truckload capacities. Cantillon and Pesendorfer (2006) investigate the auctioning of bus routes; they consider the case of London. Lastly, Bichler et al. (2006) investigate procurement auctions as a possible application for combinatorial auctions.

A further application of combinatorial auctions is provided by Rassenti et al. (1982). Their analyses are one of the first applications of auction theory in operations research. They used a combinatorial auction mechanism to optimize airport time slot allocation. Furthermore, Balinski and Sand (1985) used auctions to allocate the landing rights in congested airports.

Chen et al. (2009) also consider a truckload transportation problem with auctions. Sheffi (2004) considered the procurement of transportation services with combinatorial auctions. He points out that 'Dozens of leading companies, such as Colgate-Palmolive Company, Compaq Computers Inc., Ford Motor Company, The Home Depot Inc., International Paper Company, Lucent Technologies Inc., Nestlé S. A., The Procter and Gamble Company, Quaker Oats (a unit of Pepsico Beverages and Foods Inc.), Sears Roebuck and Co., and Wal-Mart Stores Inc., have used combinatorial auctions to obtain low transportation rates and high levels of service.' (Sheffi 2004, p. 251). Besides these issues, Ertem and Buyurgan (2011) provide an approach to solve a resource allocation problem for disaster relief, with auction-based methods.

Pekeç and Rothkopf (2003) provide a good review of the design of combinatorial auctions. There is an approach provided by Song and Regan (2005) to approximate a solution for the NP-hard problems which are faced in combinatorial auctions. Closely related to this is Sandholm et al. (2005), who provide an issue with CABOB, which solves the winner determination problem of a combinatorial auction with a fast optimal algorithm. They consider an optimization problem and they compare two algorithms, CPLEX 8.0 and the CABOB. Especially they point out the advantages of CABOB. Earlier, there were considered approximations for the winner determination problems in combinatorial auctions, see for example Anandalingam et al. (2002).

Dasgupta and Hansen (2007) consider auctions in the context of corporate finance. They point out that there is an intersection of auction theory with corporate finance theory (Dasgupta and Hansen 2007, p. 69). Furthermore, Breuer et al. (2015) investigate double auctions for endogenous leverage and asset pricing.

Table 3 summarizes the applications mentioned for the use of auction theoretic mechanisms in internal markets.

Table 3 Applications of auctions to internal markets

Fields of application	References
– Allocation of internal resources	– Hodak (1997), Baiman et al. (2007), and Harris et al. (1982)
– Corporate finance and asset pricing	– Baule (2012), Dasgupta and Hansen (2007), and Breuer et al. (2015)
– Airport slot time allocation	– Rassenti et al. (1982) and Balinski and Sand (1985)
– Allocation of truckload capacities	– Caplice and Sheffi (2006) and Chen et al. (2009)
– Distributed resource scheduling	– Kutanoglu and Wu (1999) and Adhau et al. (2012)
– Staff planning	– Chiaramonte and Chiaramonte (2008), De Grano et al. (2009), Koeppel (2004), De Grano and Medeiros (2007), and Sasaki and Nishi (2015)

4.3.2 Applications to External Markets

In this section, the applications of auctions in management accounting will be considered for external markets. For instance, there are applications of auctions to supply chain management, procurement auctions, reverse auctions, and transportation auctions.

It is also possible to use auction theory in the context of supply chain management. Supply chain management can be seen in the context of the control of production with respect to management accounting. There are detailed reviews of the literature on supply chain management, for instance Tan (2001), Croom et al. (2000), Lambert and Cooper (2000), and Thomas and Griffin (1996). For example, Moyaux et al. (2010) consider a supply chain as a network of auctions. They used a JASA (Java Auction Simulator API) to simulate the auctions and their results.

Another investigation, written by Du et al. (2006), investigates a two-firm joint venture. In the created scenario, they have one firm with the technology and know-how for a new product and the other firm has the necessary capital to finance the venture. They present an approach based on auction theory to value the technology. They determine the optimal bidding strategies for both firms.

Moreover Lorentziadis (2014) also investigates supply chains with an auction mechanism. He considers an independent private value sealed-bid lowest-price procurement auction. He assumes information asymmetry, so each bidder knows his own costs, but not the costs of the opponents. The costs of the other bidder are assumed to be independent and identically distributed random variables (Lorentziadis 2014, p. 873). Another use of an auction mechanism in supply chains comes from Chen et al. (2005). They used the Vickrey auction as an efficient allocation mechanism and pointed out the advantages of this auction mechanism, especially the truth telling incentive, which means that for every bidder it is the dominant strategy to bid their true value. They used multi-unit sealed-bid procurement auctions in supply chains and they took into account, in addition to the production costs, also the transportation costs. Ertogral and Wu (2000) used an auction-theoretic coordination mechanism for production planning in supply chains.

To avoid the inherent problems of a Lagrangian decomposition, they developed an adaptive auction *tâtonnement* with an augmented payment function with respect to Kutanoglu and Wu (1999). Indeed, Ertogral and Wu (2000) use the auction mechanism to coordinate the production planning in a supply chain.

Another approach is provided by Walsh et al. (2000). They consider combinatorial auctions to coordinate and control a supply chain. Walsh and Wellman (1998) have also presented a market protocol for decentralized task allocation using an auction mechanism and analysing the bidding policies. Morales and Steinberg (2014) investigate the revenue deficiency in a supply chain with second-price auctions. They compare two auction formats: the second-price procurement auction, where the least costly supplier wins the contract and gets the second lowest bid, and, in contrast, the multiple winner auction, in which the least costly set of suppliers wins the auction. Another article with a similar approach has been written by Emiliani (2000). He considers business to business online auctions. Carter et al. (2004) define an electronic reverse auction as 'an online, real-time auction between a buying organization and two or more invited suppliers, where suppliers can submit multiple bids during the time period of the auction, and where some degree of visibility exists among suppliers regarding the actions of their competitors.' They point out that the bid prices are lower when there is more competition among the suppliers and there is more competition with a higher number of suppliers. This is called a reverse auction, because it is the other way round, so the sellers bid and not the buyers, and the goal of this auction is to obtain a low price instead of a high price (Jap 2007, p. 146; Leong 2008, p. 18). Hence, reverse auctions are also known as downward price auctions (Emiliani and Stec 2002, p. 12).

Thus, reverse auctions could be used as procurement auctions for the selection of suppliers. Emiliani and Stec (2002) consider why managers use online reverse auctions (Emiliani and Stec 2002, p. 20) and therefore they focus on the savings with this method of purchasing. They point out that online reverse auctions are used for the local optimization of the business system along functional, managerial, or financial dimensions (Emiliani and Stec 2002, p. 22). Moreover, Emiliani (2004) investigates the use of online reverse auctions in supply chains for sourcing in the field of global aerospace. In addition, Emiliani (2005) studies how voluntary codes of conduct could regulate business-to-business (B2B) online reverse auctions.

Smart and Harrison (2002, 2003) investigate online reverse auctions with respect to supply chain management and the buyer–supplier relationship. They provide a case study to consider the questions of how reverse auctions influence the price levels for suppliers and how could reverse auctions impact on the buyer–supplier relationship (Smart and Harrison 2003, p. 258).

Also, Jap (2007) considers the influence of online reverse auction design on the buyer–supplier relationship. Therefore, they run a regression. One of the main results was that an increasing number of bidders generates pressure on the suppliers to lower their margins and so there is an increase in the participation, bidding, and savings in the auction. This issue could influence the buyer–supplier relationship, because there is an increasing suspicion of opportunism, because the suppliers could receive the impression that the buyer is increasing the price competition to enforce lower supply margins (Jap 2007, p. 156). Closely related to these, (Lösch and Lambert 2007) also provide an investigation of the buyer–supplier relationship and the behaviour in electronic reverse auctions.

Besides this issue, Stein et al. (2003) carried out a case study to investigate the efficacy of reverse auctions. They point out that online reverse auctions yield significant cost savings and can replace an existing in-house procurement, but they criticized it in that there is an induced increase in the mistrust by the suppliers (Stein et al. 2003, p. 17). Closely related to this, Caniëls and van Raaij (2009) investigated the advantages and disadvantages of online reverse auctions. They ran a multivariate analysis to investigate several effects of electronic reverse auctions. Furthermore, Smeltzer and Carr (2003) investigate electronic reverse auctions and they point out the risks of reverse auctions and the conditions for a successful reverse auction. They used interviews as their research method. They point out that reverse auctions could raise the efficiency of strategic sourcing, but they also mentioned the risks for the buyer–supplier relationship (Smeltzer and Carr 2003, p. 487). Closely related to this, Wagner and Schwab (2004) provided a study to point out the conditions which have an impact on the success of electronic reverse auctions. This investigation can support purchasing managers in their decisions as to whether and how to use and design an online reverse auction. Moreover, Wagner and Schwab (2004, pp. 12–15) provided a very detailed literature review of electronic reverse auctions.

Kros et al. (2011) show in their study how supply chain managers use online auctions to manage costs and profitability. Furthermore, they consider the way the use of an online auction impacts the collaborative relationships with suppliers. Yu et al. (2007) consider B2B online reverse auctions in Taiwan and they investigate the factors which influence the performance of e-procurement. Closely related to this, Chang (2007) provided a case study about Taiwan's high-tech industry and their use of online reverse auctions. Also, Razuk et al. (2009) provided a field study of the use of online reverse auctions in Brazilian companies with a focus on supply processes. Beside these issues of procurement auctions in supply chains, Wu (2001) has provided an approach that considers knowledge management to coordinate supply chains and auctions.

Another approach is provided by Xu et al. (2015) for efficient intermodal transportation auctions. They therefore take the transaction costs into account and they use auctions to optimize the transportation costs. Also, Hobbs (1996) considers transaction costs in a supply chain management context, among others, with auctions. Furthermore, Xu and Huang (2013) investigate transportation procurement with sealed double auctions under stochastic demand. In addition, Xu (2014) considered an auction-based transportation procurement problem. He provided, among other things, optimization-based efficient auctions for the transportation procurement problem. Furthermore, Xu and Huang (2014) presented allocatively efficient auction mechanism for the distributed transportation procurement problem. Closely related to this, and in addition, Huang and Xu (2013) provided an approach using multi-unit procurement auctions with truthful bidding behaviour and incentive compatibility for logistics e-marketplaces. Beyond these issues, Xu and Huang (2015) investigated an auction-based procurement process. Furthermore, Ağralıet al.

(2008) consider and model an auction-based logistic market. Their investigation is motivated by a logistics auction market in Turkey (Ağralıet al. 2008, p. 290). There, a reverse auction is applied to match carriers with shippers.

In addition, Remli and Rekik (2013) provided an approach that considers combinatorial auctions for transportation procurement, and they take into account that the shipment volumes are uncertain. Liu et al. (2012) point out that a procurement auction is multi-attribute, which means the bid contains both price and non-price attributes, for instance, quality, time of delivery, and service levels (Liu et al. 2012, p. 408). Also they consider risk averse suppliers. Furthermore, Grimm et al. (2006b) investigate competition in procurement with auctions and they consider firms that divide procurement contracts into lots. Another issue to examine in procurement auctions is provided by Grimm (2004). Also, Arozamena and Cantillon (2004) consider procurement auctions and they take the firm's incentives into account. Closely related to this, Grimm et al. (2006a) provided an experiment with auctions, especially procurement auctions, to investigate investment incentives. Furthermore, Grimm (2007) provided an approach to examine recurring procurement with sequential and bundle auctions. Moreover, Jofre-Bonet and Pesendorfer (2000) also investigate repeated procurement auctions for highway paving contracts. In addition to this, an early approach with procurement auctions was written by Samuelson (1986).

Engel (2009) investigates the risk in procurement auctions that the winning supplier could be bankrupt. So he provides different approaches why suppliers bid too aggressively, and he considers different auction mechanisms and designs so as to investigate when the risk of bankrupt bidders is higher or lower. He pointed out that it need not be optimal that the lowest bidder wins the auction and gets the contract, because there is the risk of the winner's curse, so it can be the case that the lowest bidder underestimates his true costs, thus the supplier could go bankrupt and then the project can not be completed. For an overview of the issue of the winner's curse, see Kagel and Levin (2002), Bulow and Klemperer (2002), Hong and Shum (2002), Hansen and Lott (1991), Spulber (1990), Decarolis (1990), Kagel and Levin (1986), and Bazerman and Samuelson (1983). Harstad and Rothkopf (1995) presented an approach to avoid the winner's curse. They used an auction with withdrawable bids as an insurance against winner's curse. The trade-off between the low prices generated by the procurement auction mechanism and the risk that the awarded supplier can go bankrupt was already mentioned by Engel and Wambach (2005). They consider different mechanisms and methods to find how best to handle the trade-off between low prices and low bankruptcy probability (Engel and Wambach 2005, p. 7). In contrast, Campo (2012) investigates an approach with asymmetric and risk averse bidding behaviour in procurement auctions.

Auctions have also been used for experimental investigations. For instance, the strategic disclosure of risky prospects was considered by Hobson and Kachelmeier (2005) with experimental methods and with the use of auctions. Other experimental investigations with auctions have been written by Engelmann and Grimm (2009), Kagel (2008), Avery and Kagel (1997), and Cox et al. (1985). An overview of experiments with auctions is provided by Kagel and Levin (1986). Furthermore, Grimm and Ilieva (2013) provide an experimental investigation of the use of auctions and other allocation mechanisms in an emission trading system. Closely related to this, Sturm (2008) investigates emission trading with experimental double auctions.

Then, emissions trading is another possible application of auctions. Betz et al. (2010) investigates the greenhouse gas emission permits in Australia. They take a look at the allocation of permits as a form of emissions trading, therefore they consider auction mechanisms to allocate the permits. Other investigations of trading emissions with auctions are, for example, provided by Cramton and Kerr (2002) and Muller et al. (2002).

Besides the aforementioned literature, Klemperer (1998) considers auctions with almost common values. He investigates applications for these auctions, for instance, airwaves auctions and auctioning a firm or a company in a takeover process. Furthermore, Argoneto and Renna (2011) consider, among other issues, the application of auctions in a business-to-business environment.

Table 4 summarizes the mentioned applications of auctions, and approaches with the use of auction theoretic mechanisms, to external markets.

Table 4 Applications of auctions to external markets

Use of different auction types	
Types of auctions	References
– Procurement auctions	– Huang et al. (2013), Bichler et al. (2006), Sheffi (2004), Lorentziadis (2014), Walsh and Wellman (1998), Emiliani and Stec (2002), Huang and Xu (2013), Xu and Huang (2015), Grimm (2004, 2007), Grimm et al. (2006b), Arozamena and Cantillon (2004), Jofre-Bonet and Pesendorfer (2000), Samuelson (1986), Engel (2009), Engel and Wambach (2005), and Campo (2012)
– Reverse auctions	– Emiliani (2000), Jap (2007), Leong (2008), Emiliani and Stec (2002), Emiliani (2004, 2005), Smart and Harrison (2002, 2003), Lösch and Lambert (2007), Stein et al. (2003), Caniëls and van Raaij (2009), Smeltzer and Carr (2003), Wagner and Schwab (2004), Kros et al. (2011), Yu et al. (2007), Chang (2007), Razuk et al. (2009), and Ağralıet al. (2008)
Fields of applications	
Fields of application	References
– Emission trading	– Grimm and Ilieva (2013), Sturm (2008), Cramton and Kerr (2002), and Muller et al. (2002)
– Coordination of supply chains	– Lorentziadis (2014), Ertogral and Wu (2000), Morales and Steinberg (2014), Walsh et al. (2000), Emiliani (2004), Wu (2001), and Hobbs (1996)

5 Present and Future Applications in Management Accounting

In the previous chapter, we have seen many applications of auction theory to management accounting. Now we want to consider the potential of auction theory in a management accounting context. In general, auctions could be used when there are unknown evaluations of the items or resources and the aim is to find out the willingness to pay of the bidders. From an internal perspective, efficiency should be the focus, so that the scarce resources will be allocated in such a way that the firm generates the highest utility. Thus, in internal auctions, the price does not matter from the perspective of the firm as a whole. If auctions with external bidders are used, for example to select the best supplier, the realized price does matter, because the aim of the firm is to get a low price. So there are different aims to consider in the design of an auction, and the auctioneer has to decide which aim should be the more important, because efficiency and optimality in auctions are conflicting goals.

There are some areas of possible applications for auction theory in management accounting that have not been examined so far. Summarizing, we may conclude that auctions may be employed to:

- reveal the true preferences and/or
- allocate scarce resources.

Auctions are a possible mechanism to quantify the true value or the incentive compatible value of an item. This is necessary, e.g. in the area of

(a) transfer pricing
(b) budgeting
(c) scheduling
(d) variable remuneration
(e) supplier selection.

Designing a **transfer price system** with auction theoretic mechanisms enables the evaluation of the transferred item with an incentive compatible value. In this way, information is generated about the resource within the firm and the preferences of the divisions. An auction mechanism:

- may provide relevant information to each division in the firm to point to the optimum trade-off between company costs and revenues,
- may lead to goal congruent or incentive compatible decisions,
- may measure the economic performance of the individual divisions, and
- should be simple to implement so that it is easy to understand.

All these points are conditions which have to be fulfilled by an effective transfer price system. For a general overview of transfer prices in management accounting cf. Schuster (2015), Anthony et al. (2014, pp. 291–306), Kaplan and Atkinson (1998, pp. 453–457), Weygandt et al. (2015, pp. 247–253), Drury 2015, pp. 525–557, and Bhimani et al. (2015, pp. 565–575).

Very important for revealing the true evaluation is the design of an auction which generates truthful reports. In a single unit auction with private values (cf. p. 166) we could use the Vickrey auction (cf. p. 167) as an efficient mechanism to allocate the transferable goods or services. We can assume private values in the context of transfer prices, because every division has its own private evaluation of the resources, because the resources will be used for different projects in each division and so each division has a different utility created by the allocated items. But it also might be possible to assume interdependent or common values. Then the Vickrey auction is not efficient and so we should use another auction design, for example, a generalized Vickrey auction (cf. p. 170). Also, it is conceivable that there is not only one resource to allocate. Then a multi-unit or combinatorial auction could be used. On the issue of transfer prices, there is a huge potential for the use of auctions as allocation mechanisms and to design these auctions as efficient auctions with the incentive for the division to bid truthfully. We have seen in Sect. 4.3.1 that a transfer auction could work and this should be an approach with the potential for further research.

Another possible area of application could be **budgeting** (for an overview cf. Nilsson and Stockenstrand 2015, pp. 87–93; Anthony et al. 2014, pp. 331–355; Drury 2015, pp. 368–393; Weygandt et al. 2015, pp. 276–314; Bhimani et al. 2015, pp. 424–447). Here we are faced with a principal agent problem, because we have asymmetric information and we could use auctions to reduce these information asymmetries. With the use of auction mechanisms we could determine the true value and benefit for the divisions for the budget. So the auctions help us to generate information about an incentive compatible evaluation. This information could be used to allocate the budget efficiently, with the highest benefit for the whole organisation. Very important is the incentive mechanism for the divisions to implement a truthful bidding behaviour. It is possible to design auction theoretic approaches, such as efficient auctions, so that there would be an incentive compatible mechanism which leads to truthful bidding behaviour. Then we could control and direct the budget in the right way to increase the overall efficiency of the firm. Moreover, in the case of budgeting, there is an issue in that it is not useful to have monetary bids. The auction should be designed in such a way that the divisions bid with their performance, for example, with their expected profitability. Moreover, truthful bidding should be ensured and therefore there should be a mechanism to penalize, in the next budget negotiation, a division that has not attained the offered profitability.

A further application field of auctions is **scheduling** (cf. p. 175). We have already seen, for example, personnel scheduling and distributed resource scheduling. Now we want to consider personnel scheduling in more detail. We are faced with the issue that personnel resources are scarce, and the staff has different preferences for their shifts. The idea is to implement an efficient auction to figure out the true preferences of the staff for the different shifts so as to efficiently allocate the shifts. It is important to mention that in this auction the staff should not bid with their money. It should be possible to use a fictitious contingent of points to bid on the preferable shifts or to bid with minus points on the disliked shifts. The auction mechanism could be used to allocate the shifts with respect to the preferences of the staff. So the staff has the possibility of influencing the roster with their bids. This could increase the motivation and satisfaction of the personnel and in consequence the performance of the personnel could improve. Moreover, it is also possible to allocate resources with auctions in other scheduling issues.

Closely related to the previous consideration of personnel scheduling, **variable remuneration** could be an issue to be solved by using auctions. In this scenario we have several agents, who are able to execute one job, and we want to determine the agent with the best performance. For example, the agents could bid the time in which they could finish the job and so the quickest agent will get the job, assuming the same quality. And the variable remuneration could depend on the jobs which were done by the agents. It is important to design an auction with an efficient design, so that there is no incentive for untruthful bidding behaviour. Furthermore, the agents should be penalized if they can not maintain the offered time for finishing the job.

Another important problem that can be solved with auctions is the already mentioned problem of **supplier selection** (see p. 178 for reverse auctions and procurement auctions). In the previous section, reverse and procurement auctions were already considered. These approaches could be used to select the best supplier. For example, in a reverse auction, the suppliers bid, taking into consideration their costs, the price that they would accept. And so the company could run an auction for a project and let the suppliers bid and choose the cheapest one, while assuming the same quality of the suppliers. As an extension of such a procurement auction, it might be possible to bid with another dimension than the monetary one. For example, it is conceivable to let the suppliers bid with their time of delivery and determine the price and quality of the goods or services, or let the suppliers bid with other dimensions of their performance, for example, quality. It is important that an ex post penalization of the suppliers be possible: if they do not fulfill their bids, for example, if they not maintain the time of delivery that they bid.

Thus, as we have seen in this section, there is a huge potential for auctions in management accounting for further research.

6 Conclusion

In the present paper, the recent literature on management accounting and auction theory was surveyed. First, the basics of management accounting and auction theory were presented. Then in the main part of the present paper the existing applications of auctions were reviewed with the relevant literature. It was mentioned that auctions have been used for different applications, for instance, for art and for spectrum licences, and for transfer auctions and procurement auctions as well. Furthermore, auctions have been used as an allocation mechanism for internal resources, economic risk capital, and staff planning. Besides this, auctions can be applied for the optimization of the allocation of airport time slots and of the allocation of airspace resource. A huge field for auctions were mentioned, with supply chain management. In addition, auctions could be used for asset pricing and corporate finance. After this review of the current literature in the last part of the present paper, the potential for auction theory in management accounting was mentioned. So there has been a view into the future of the applications of auctions in this context. Especially the concepts of transfer prices, budgeting, scheduling, variable remuneration, and supplier selection were in the focus of this contemplation.

References

Adhau S, Mittal ML, Mittal A (2012) A multi-agent system for distributed multi-project scheduling: an auction-based negotiation approach. Eng Appl Artif Intell 25(8):1738–1751
AğralıS, Tan B, Karaesmen F (2008) Modeling and analysis of an auction-based logistics market. EJOR 191(1):272–294
Amann E, Leininger W (1995) Expected revenue of all-pay and first-price sealed-bid auctions with affiliated signals. J Econ 61(3):273–279
Anandalingam G, Kwon R, Ungar LH (2002) An efficient approximation algorithm for combinatorial auctions. Working paper
Anthony RN (1965) Planning and control systems: a framework for analysis. Harvard University Press, Boston
Anthony RN (1988) The management control function. Harvard Business School Press, Boston
Anthony RN, Govindarajan V, Hartmann FGH, Kraus K, Nilsson G (2014) Management control systems. McGraw-Hill, London
Argoneto P, Renna P (2011) Innovative tools for business coalitions in B2B applications: how negotiation, auction and game theory can support small-and medium-sized business in E-business. Springer, London
Armstrong M (2000) Optimal multi-object auctions. RES 67(3):455–481
Arozamena L, Cantillon E (2004) Investment incentives in procurement auctions. RES 71(1):1–18
Ashenfelter O (1989) How auctions work for wine and art. JEP 3(3):23–36
Ashenfelter O, Graddy K (2003) Auctions and the price of art. JEL 41(3):763–787
Ausubel LM (1997) An efficient ascending-bid auction for dissimilar objects. Working paper
Ausubel LM (1999) A generalized Vickrey auction. Working paper, University of Maryland
Ausubel LM (2004) An efficient ascending-bid auction for multiple objects. TAER 94(5):1452–1475
Ausubel LM, Cramton PC (1999) The optimality of being efficient. Working paper, University of Maryland.
Ausubel LM, Milgrom PR (2006) The lovely but lonely Vickrey auction. In: Cramton PC, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 57–95
Ausubel LM, Cramton PC, Pycia M, Rostek M, Weretka M (2014) Demand reduction and inefficiency in multi-unit auctions. RES 81(4):1366–1400
Avery C, Kagel JH (1997) Second price auctions with asymmetric payoffs: an experimental investigation. JEMS 6(3):573–603
Baiman S, Fischer P, Rajan MV, Saouma R (2007) Resource allocation auctions within firms. JAR 45(5):915–946
Bajari P, McMillan R, Tadelis S (2009) Auctions versus negotiations in procurement: an empirical analysis. JLEO 25(2):372–399
Balinski ML, Sand FM (1985) Auctioning landing rights at congested airports. In: Young HP (ed) Cost allocation: methods, principles, applications. North-Holland, Amsterdam, pp 179–192

Ball M, Donohue G, Hoffman K (2006) Auctions for the safe, efficient, and equitable allocation of airspace system resources. In: Cramton PC, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 949–1015
Baule R (2012) Allocation of risk capital on an internal market. EJOR 234(1):186–196
Baye MR, Kovenock D, De Vries CG (1996) The all-pay auction with complete information. ET 8(2):291–305
Bazerman MH, Samuelson WF (1983) I won the auction but don't want the prize. J Confl Resol 27(4):618–634
Berry AJ, Coad AF, Harris EP, Otley DT, Stringer C (2009) Emerging themes in management control: a review of recent literature. Br Account Rev 41(1):2–20
Betz R, Seifert S, Cramton PC, Kerr S (2010) Auctioning greenhouse gas emissions permits in Australia. Aust J Agric Res Econ 54(2):219–238
Bhimani A, Horngren CT, Datar SM, Rajan M (2015) Management and cost accounting, 6th edn. Pearson, Harlow
Bichler M, Kalagnanam J, Lee HS, Lee J (2002) Winner determination algorithms for electronic auctions: a framework design. In: Bauknecht K, Tjoa AM, Quirchmayr G (eds) E-commerce and web technologies. Springer, Berlin, pp 37–46
Bichler M, Davenport A, Hohner G, Kalagnanam J (2006) Industrial procurement auctions. In: Cramton PC, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 1116–1147
Binmore K, Klemperer P (2002) The biggest auction ever: the sale of the British 3G telecom licences. TEJ 112(478):C74–C96
Börgers T, Dustmann C (2005) Strange bids: bidding behaviour in the United Kingdom's third generation spectrum auction. TEJ 115(505):551–578
Breuer T, Jandačka M, Summer M, Vollbrecht HJ (2015) Endogenous leverage and asset pricing in double auctions. J Econ Dyn Control 53:144–160
Bulow J, Klemperer P (2002) Prices and the Winner's curse. TRJE 33(1):1–21
Bulow J, Klemperer P (1996) Auctions versus negotiations. Am Econ Rev 86(1):180–194
Bulow J, Roberts J (1989) The simple economics of optimal auctions. JPolE 97(5):1060–1090
Campo S (2012) Risk aversion and asymmetry in procurement auctions: identification, estimation and application to construction procurements. J Econometrics 168(1):96–107
Caniëls MCJ, van Raaij EM (2009) Do all suppliers dislike electronic reverse auctions? JPSM 15(1):12–23
Cantillon E, Pesendorfer M (2006) Auctioning bus routes: the London experience. In: Cramton PC, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 1076–1115
Caplice C, Sheffi Y (2006) Combinatorial auctions for truckload transportation. In: Cramton PC, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 1016–1075
Carter CR, Kaufmann L, Beall S, Carter PL, Hedrick TE, Petersen K (2004) Reverse auctions - grounded theory from the buyer and supplier perspective. TRPE 40(3):229–254
Cassady R (1967) Auctions and auctioneering. University of California Press, Berkeley and Los Angeles
Chang S (2007) Reverse electronic auctions: cases in Taiwan's high-tech industry. Technol Soc 29(4):490–496
Che Y, Gale I (1996) Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. EL 50(3):373–379
Chen RR, Roundy RO, Zhang RQ, Janakiraman G (2005) Efficient auction mechanisms for supply chain procurement. MS 51(3):467–482
Chen RLY, AhmadBeygi S, Cohn A, Beil DR, Sinha A (2009) Solving truckload procurement auctions over an exponential number of bundles. Transp Sci 43(4):493–510
Chiaramonte MV, Chiaramonte LM (2008) An agent-based nurse rostering system under minimal staffing conditions. IJPE 114(2):697–713
Coppinger VM, Smith, VL, Titus JA (1980) Incentives and behavior in English, Dutch and sealed-bid auctions. EI 18(1):1–22

Cox JC, Smith VL, Walker JM (1985) Experimental development of sealed-bid auction theory; calibrating controls for risk aversion. TAER 75(2):160–165
Cramton PC (2013) Spectrum auction design. Rev Ind Organ 42(2):161–190
Cramton PC, Kerr S (2002) Tradeable carbon permit auctions: how and why to auction not grandfather. Energy Policy 30(4):333–345
Cramton PC, Ockenfels A (2015) The German 4G spectrum auction: design and behavior. Working paper, Mimeo
Cramton PC, Shoham Y, Steinberg R (eds) (2006) Combinatorial auctions. MIT Press, Boston
Croom S, Romano P, Giannakis M (2000) Supply chain management: an analytical framework for critical literature review. EJPSM 6(1):67–83
Dasgupta P, Maskin ES (2000) Efficient auctions. TQJE 115(2):341–388
Dasgupta S, Hansen RG (2007) Auctions in corporate finance. In: Eckbo BE (ed) Handbook of corporate finance: empirical corporate finance - vol 1. Elsevier North-Holland, Amsterdam, pp 87–143
De Grano ML, Medeiros DJ (2007) An auction system for nurse scheduling with "On" and "Off" bids. In: IIE annual conference. Proceedings. Institute of Industrial Engineers-Publisher, Norcross, pp 151–156
De Grano ML, Medeiros DJ, Eitel D (2009) Accommodating individual preferences in nurse scheduling via auctions and optimization. HCMS 12(3):228–242
Decarolis F (1990) When the highest bidder loses the auction: theory and evidence from public procurement. Working Paper, Bank of Italy, No. 717
Doyle C, McShane P (2003) On the design and implementation of the GSM auction in Nigeria—the World's first ascending clock spectrum auction. Telecommun Policy 27(5):383–405
Drury CM (2015) Management and cost accounting, 9th edn. Cengage, London
Du L, Hu Q, Liu L (2006) A profit sharing scheme for a two-firm joint venture. EJOR 170(1):277–292
Emmanuel C, Otley D, Merchant KA (1997) Accounting for management control. Pitman, Boston
Emiliani ML (2000) Business-to-business online auctions: key issues for purchasing process improvement. Supply Chain Manag 5(4):176–186
Emiliani ML (2004) Sourcing in the global aerospace supply chain using online reverse auctions. IMM 33(1):65–72
Emiliani ML (2005) Regulating B2B online reverse auctions through voluntary codes of conduct. IMM 34(5):526–534
Emiliani ML, Stec DJ (2002) Realizing savings from online reverse auctions. Supply Chain Manag 7(1):12–23
Engel AR (2009) Procurement auctions and the risk of bankrupt bidders. In: Bogaschewsky R, Eßig M, Lasch R, Stölzle W (eds) Supply management research. Gabler, Wiesbaden, pp 31–46
Engel AR, Wambach A (2005) Risk management in procurement auctions. In: Quaderni Consip, vol 10. Rome, Italy
Engelmann D, Grimm V (2009) Bidding behaviour in multi-unit auctions - an experimental investigation. TEJ 119(537):855–882
Ertem MA, Buyurgan N (2011) An auction-based framework for resource allocation in disaster relief. J Humanitarian Logist Supply Chain Manag 1(2):170–188
Ertogral K, Wu SD (2000) Auction-theoretic coordination of production planning in the supply chain. IIE Trans 32(10):931–940
Ewerhart C, Moldovanu B (2002) The German UMTS design: insights from multi-object auction theory. Working paper, CESifo, No. 680
Gebhardt G, Wambach A (2008) Auctions to implement the efficient market structure. Int J Ind Organ 26(3):846–859
Gretschko V, Wambach A (2013) Auctions vs. negotiations: the case of favoritism. Beiträge zur Jahrestagung des Vereins für Socialpolitik 2013: Wettbewerbspolitik und Regulierung in einer globalen Wirtschaftsordnung - Session: Issues in Auctions, No. A16-V1
Gretschko V, Wambach A (2014) Procurement under public scrutiny: auctions vs. negotiations. Working paper, University of Cologne

Grimm V (2004) On procurement auctions of complementary goods. Working papers, Documentos de trabajo: Serie AD, vol 2, pp 1–36
Grimm V (2007) Sequential versus bundle auctions for recurring procurement. J Econ 90(1):1–27
Grimm V, Ilieva L (2013) An experiment on emissions trading: the effect of different allocation mechanisms. J Regul Econ 44(3):308–338
Grimm V, Riedel F, Wolfstetter E (2001) The third generation (UMTS) spectrum auction in Germany. Discussion paper
Grimm V, Riedel F, Wolfstetter E (2003) Low price equilibrium in multi-unit auctions: the GSM spectrum auction in Germany. IJIO 21(10):1557–1569
Grimm V, Mengel F, Ponti G, Viianto LA (2006) Investment incentives in auctions: an experiment. Working paper series in economics, vol 25. University of Cologne
Grimm V, Pacini R, Spagnolo G, Zanza M (2006) Division into lots and competition in procurement. In: Dimitri N, Piga G, Spagnolo G (eds) Handbook of procurement. Cambridge University Press, Cambridge, pp 168–192
Hall NG, Liu Z (2011) On auction protocols for decentralized scheduling. GEB 72(2):583–585
Hansen RG, Lott JR (1991) The Winner's curse and public information in common value auctions: comment. TAER 81(1):347–361
Hared BA, Abdullah Z, Huque SMR (2013) Management control systems: a review of literature and a theoretical framework for future researches. Eur J Bus Manag 5(26):1–13
Harris M, Raviv A (1981) Allocation mechanisms and the design of auctions. Econometrica 49(6):1477–1499
Harris M, Kriebel CH, Raviv A (1982) Asymmetric information, incentives and intrafirm resource allocation. MS 28(6):604–620
Harstad RM, Rothkopf MH (1995) Withdrawable bids as Winner's curse insurance. Oper Res 43(6):983–994
Hobbs JE (1996) A transaction cost approach to supply chain management. Supply Chain Manag 1(2):15–27
Hobson JL, Kachelmeier SJ (2005) Strategic disclosure of risky prospects: a laboratory experiment. TAR 80(3):825–846
Hodak M (1997) The end of cost allocations as we know them. J Appl Corp Financ 10(3):117–124
Hong H, Shum M (2002) Increasing competition and the winner's curse: evidence from procurement. RES 69(4):871–898
Huang GQ, Xu SX (2013) Truthful multi-unit transportation procurement auctions for logistics E-marketplaces. TRPB 47:127–148
Huang H, Xu H, Kauffman RJ, Sun N (2013) Analyzing auction and bargaining mechanisms in E-procurement with supply quality risk. ORL 41(4):403–409
Hunsberger L, Grosz BJ (2000) A combinatorial auction for collaborative planning. In: Proceedings fourth international conference on multiagent systems, 2000, IEEE, pp 151–158
Jackson MO (2003) Efficiency and information aggregation in auctions with costly information. RED 8(2):121–141
Jap SD (2007) The impact of online reverse auction design on buyer-supplier relationships. J Market 71(1):146–159
Jofre-Bonet M, Pesendorfer M (2000) Bidding behavior in a repeated procurement auction: a summary. EER 44(4):1006–1020
Jose RA, Ungar LH (1998) Auction-driven coordination for plantwide optimization. In: Proceedings of foundations of computer-aided process operation FOCAPO
Jose RA, Ungar LH (2000) Pricing interprocess streams using slack auctions. AIChE J 46(3):575–587
Jose RA, Harker PT, Ungar LH (1997) Coordinating locally constrained agents using augmented pricing. In: Wharton Financial Institutions Center Working Paper
Kagel JH (2008) Auctions: a survey of experimental research. In: Kagel JH, Roth AE (eds) Handbook of experimental economics, 2nd edn. Princeton University Press, Princeton
Kagel JH, Levin D (1986) The winner's curse and public information in common value auctions. TAER 76(5):894–920

Kagel JH, Levin D (eds) (2002) Common value auctions and the winner's curse. Princeton University Press, Princeton
Kaplan RS, Atkinson AA (1998) Advanced management accounting, 3rd edn. Prentice Hall, Upper Saddle River, NJ
Klemperer P (1998) Auctions with almost common values: the "Wallet Game" and its applications. EER 42(3):757–769
Klemperer P (1999) Auction theory: a guide to the literature. J Econ Surv 13(3):227–286
Klemperer P (2002) How (not) to run auctions: the European 3G telecom auctions. EER 46(4):829–845
Klemperer P (2004) Auctions: theory and practice. The Toulouse lectures in economics. Princeton University Press, Princeton
Koeppel D (2004) Nurses bid with their pay in auctions for extra work. In: The New York Times, June 6
Krishna V (2010) Auction theory, 2nd edn. Academic Press, Amsterdam
Krishna V, Morgan J (1997) An analysis of the war of attrition and the all-pay auction. JET 72(2):343–362
Krishna V, Perry M (1998) Efficient mechanism design. Working paper
Kros JF, Nadler SS, Chen H (2011) The adoption and utilization of online auctions by supply chain managers. TRPE 47(2):105–114
Kutanoglu E, Wu SD (1999) On combinatorial auction and Lagrangean relaxation for distributed resource scheduling. IIE Trans 31(9):813–826
Lambert DM, Cooper MC (2000) Issues in supply chain management. IMM 29(1):65–83
Langfield-Smith K (1997) Management control systems and strategy: a critical review. AOS 22(2):207–232
Lehmann D, O'Callaghan LI, Shoham Y (2002) Truth revelation in approximately efficient combinatorial auctions. J ACM 49(5):577–602
Leong L (2008) Value chain management in online reverse auction: towards strategic and operational excellence. Acad Inform Manag Sci J 11(1):13–28
Liu S, Li J, Liu D (2012) Multi-attribute procurement auctions with risk averse suppliers. EL 115(3):408–411
Lorentziadis PL (2014) Pricing in a supply chain for auction bidding under information asymmetry. EJOR 237(3):871–886
Lösch A, Lambert JS (2007) E-reverse auctions revisited: an analysis of context, buyer-supplier relations and information behavior. J Supply Chain Manag 43(4):47–63
MacKie-Mason JK, Varian HR (1994) Generalized Vickrey auctions. Working paper, University of Michigan
Ma J, Deng J, Song L, Han Z (2014) Incentive mechanism for demand side management in smart grid using auction. IEEE Trans Smart Grid 5(3):1379–1388
Malmi T (2016) Managerialist studies in management accounting: 1990–2014. MAR 31:31–44
Malmi T, Brown DA (2008) Management control systems as a package - opportunities, challenges and research directions. MAR 19(4):287–300
Maskin ES, Riley JG (1984) Optimal auctions with risk averse buyers. Econometrica 52(6):1473–1518
Matthews SA (1980) Risk aversion and the efficiency of first and second price auctions. Working paper, Mimeo, Northwestern University
Matthews SA (1995) A technical primer on auction theory I: independent private values. Discussion paper No. 1096, Northwestern University
McAfee RP (1992) A dominant strategy double auction. JET 56(2):434–450
McAfee RP, McMillan J (1987) Auctions and bidding. JEL 25(2):699–738
McMillan J (1995) Why auction the spectrum? Telecommun Policy 19(3):191–199
Milgrom PR (1989) Auctions and bidding: a primer. JEP 3(3):3–22
Milgrom PR (1998) Game theory and the spectrum auctions. EER 42(3):771–778
Mochón A, Sáez Y (2015) Understanding auctions. Springer, Cham

Mookherjee D, Reichelstein S (1992) Dominant strategy implementation of Bayesian incentive compatible allocation rules. JET 56(2):378–399
Morales DR, Steinberg R (2014) Revenue deficiency under second-price auctions in a supply-chain setting. EJOR 233(1):131–144
Moyaux T, McBurney P, Wooldridge M (2010) A supply chain as a network of auctions. Decis Support Syst 50(1):176–190
Muller RA, Mestelman S, Spraggon J, Godby R (2002) Can double auctions control monopoly and monopsony power in emissions trading markets? J Environ Econ Manag 44(1):70–92
Myerson RB (1981) Optimal auction design. MOR 6(1):58–73
Nielsen LB, Mitchell F, Nørreklit H (2015) Management accounting and decision making: two case studies of outsourcing. Account Forum 39(1):64–82
Nilsson F, Stockenstrand A (2015) Financial accounting and management control. In: The tensions and conflicts between uniformity and uniqueness. Springer, Cham
Otley D (2016) The contingency theory of management accounting and control: 1980–2014. MAR 31:45–62
Pekeç, Rothkopf MH (2003) Combinatorial auction design. MS 49(11):1485–1503
Pesendorfer W, Swinkels JM (2000) Efficiency and information aggregation in auctions. In: TAER 90(3):499–525
Rassenti SJ, Smith VL, Bulfin RL (1982) A combinatorial auction mechanism for airport time slot allocation. TBJE 13(2):402–417
Razuk de RO, Arkader R, Braga AR (2009) Reverse E-auction usage in supply processes of Brazilian companies: an exploratory survey. Lat Am Bus Rev 10(1):41–60
Remli N, Rekik M (2013) A robust winner determination problem for combinatorial transportation auctions under uncertain shipment volumes. Transp Res Part C: Emerg Technol 35:204–217
Riley JG, Samuelson WF (1981) Optimal auctions. TAER 71(3):381–392
Rom A, Rohde C (2007) Management accounting and integrated information systems: a literature review. Int J Account Inform Syst 8(1):40–68
Rothkopf MH, Teisberg TJ, Kahn EP (1990) Why are Vickrey auctions rare? JPolE 98(1):94–109
Salvatore D (2015) Managerial economics in a global economy - principles and worldwide applications, 8th edn. Oxford University Press, New York
Samuelson WF (1986) Bidding for contracts. MS 32(12):1533–1550
Samuelson WF (2014) Auctions: advances in theory and practice. In: Chatterjee K, Samuelson WF (eds) Game theory and business applications, 2nd edn. Springer, Berlin, pp 323–366
Sandholm T, Suri S, Gilpin A, Levine D (2005) CABOB: a fast optimal algorithm for winner determination in combinatorial auctions. MS 51(3):374–390
Sasaki D, Nishi T (2015) Combinatorial auction algorithm with price-adjustment mechanism for airline crew scheduling problems. In: Proceeding of the 2015 IEEE IEEM, pp 1183–1187
Schuster P (2015) Transfer prices and management accounting. Springer, Cham
Sheffi Y (2004) Combinatorial auctions in the procurement of transportation services. Interfaces 34(4):245–252
Shubik M (1983) Auctions, bidding, and markets: an historical sketch. In: Shubik M, Engelbrecht R, Stark R (eds) Auctions, bidding and contracting: uses and theory. New York University Press, New York, pp 33–52
Smart A, Harrison A (2002) Reverse auctions as a support mechanism in flexible supply chains. Int J Logist 5(3):275–284
Smart A, Harrison A (2003) Online reverse auctions and their role in buyer–supplier relationships. JPSM 9(5):257–268
Smeltzer LR, Carr AS (2003) Electronic reverse auctions: promises, risks and conditions for success. IMM 32(6):481–488
Song J, Regan A (2005) Approximation algorithms for the bid construction problem in combinatorial auctions for the procurement of freight transportation contracts. TRPB 39(10):914–933
Spulber DF (1990) Auctions and contract enforcement. JLEO 6(2):325–344
Stein A, Hawking P, Wyld DC (2003) The 20% solution?: a case study on the efficacy of reverse auctions. Manage Res News 26(5):1–20

Sturm B (2008) Double auction experiments and their relevance for emissions trading. In: Antes R, Hansjürgens B, Letmathe B (eds) Emissions trading: institutional design, decision making, and corporate strategies. Springer, New York, pp 49–68
Sutter M, Kocher M, Strauß S (2007) Individuals and teams in UMTS-license auctions. Working papers in economics and statistics 2007–23, University of Innsbruck
Tan KC (2001) A framework of supply chain management literature. EJPSM 7(1):39–48
Thomas DJ, Griffin PM (1996) Coordinated supply chain management. EJOR 94(1):1–15
van Damme E (2002) The European UMTS-auctions. EER 46(4):846–858
Vickrey WS (1961) Counterspeculation, auctions, and competitive sealed tenders. TJF 16(1):8–37
Wagner SM, Schwab, AP (2004) Setting the stage for successful electronic reverse auctions. JPSM 10(1):11–26
Walsh WE, Wellman MP (1998) A market protocol for decentralized task allocation. In: Multi agent systems, 1998, Proceedings, International conference on IEEE, pp 325–332
Walsh WE, Wellman MP, Ygge F (2000) Combinatorial auctions for supply chain formation. In: Proceedings of the 2nd ACM conference on Electronic commerce, ACM
Wellman MP, Walsh WE, Wurman PR, MacKie-Mason JK (2001) Auction protocols for decentralized scheduling. GEB 35(1):271–303
Weygandt JJ, Kimmel PD, Kieso DE (2015) Managerial accounting: tools for business decision making, 7th edn. Wiley, Singapore
Wilson, RB. (1985): Incentive Efficiency of Double Auctions. In: Econometrica, Vol. 53, No. 5, pp. 1101–1115.
Wolfstetter E (1996) Auctions: an introduction. J Econ Surv 10(4):367–420
Wolfstetter E (2001) The Swiss UMTS spectrum auction flop: bad luck or bad design? In: Interdisciplinary research project 373. Humboldt University of Berlin
Wu DJ (2001) Software agents for knowledge management: coordination in multi-agent supply chains and auctions. Expert Syst Appl 20(1):51–64
Xu SX (2014) Truthful, efficient auctions for transportation procurement. PhD Thesis, The University of Hong Kong
Xu SX, Huang GQ (2013) Transportation service procurement in periodic sealed double auctions with stochastic demand and supply. TRPB 56:136–160
Xu SX, Huang GQ (2014) Efficient auctions for distributed transportation procurement. TRPB 65:47–64
Xu SX, Huang GQ (2015) Auction-based transportation procurement in make-to-order systems. IIE Trans 47(11):1236–1251
Xu SX, Cheng M, Huang GQ (2015) Efficient intermodal transportation auctions for B2B E-commerce logistics with transaction costs. TRPB 80:322–337
Yu Y, Lee C, Yu H (2007) B2B online reverse auction: factors that affect the E-procurement performance in Taiwan. J Inform Optim Sci 28(4):505–521

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The Use of Auction in Nurse Rostering

Benno Woskowski

Abstract Hospitals are confronted with many challenges. On the one hand, demographic changes demand additional services. On the other hand, hospitals have to profitably manage the supply of services. That requires the efficient use of resources. These resources include the nurses who perform the services. Nurses, as human beings, cannot be managed in the same way as any other resource. They have, in terms of balancing between their work and life, certain ideas about their working schedule. The creation of a solid roster, known as the problem of nurse rostering, is a challenging task in operations research. In particular, considering nurses' preferences is crucial for the appreciation and the robustness of the roster. Thereby the problem occurs in what way the preferences can be modeled and particularly how to discover the private valuations of the nurses. Auctions make it possible to reveal true preferences. This paper presents an approach for designing an auction to include nurses' preferences in the nurse rostering problem.

Keywords Collusion • Preference scheduling • Resource allocation • Set of auction rules • Uniform-price auction

Contents

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1 Introduction

Once an enterprise has to abandon single-shift operation, it is confronted with the challenging question of staff allocation. When and for how long does every single person have to be on duty in the planning period to satisfy the demand for a certain service? In addition, staffing and personnel constraints have to be met.

For solving this personal scheduling problem, two basic solution approaches can be distinguished: cyclic scheduling and self scheduling. (cf. Bard and Purnomo 2007; Hung 2002).

Following the cyclic approach, a schedule is created fulfilling all required constraints of the planning period. The schedule created describes certain tours of duty (cf. Levner et al. 2010). Each staff member is then assigned to one tour. For the next planning period, the same schedule will be used but the staff members will be assigned to another tour within the schedule in such a way that they roll (typically forward) from one individual tour plan of one schedule to another individual tour plan of the next schedule. The changes of individual tour plan in the sequence of schedules is planned so that after a certain number of schedules, each staff member will repeat the same individual tour. Once the schedule with all the tours is planned, it is easy to use in the daily routine. Staff members will be simply assigned to the "next" tour of the following schedule. Therefore each staff members can work out at what time they will be on duty even months ahead. This predictability goes along with a rigidity, which is the major downside of cyclic scheduling. Individual preferences, such as getting a certain duty or day off, will be ignored in this approach. The created schedules are thus not in favor of all the staff members per se, and job dissatisfaction is abetted.

The self scheduling approach in contrast is focused on supporting all staff members. The schedule will be created in a dialog with all stuff members (cf. Baily et al. 2007). Every staff member has a veto right. If a staff member does not support a possible schedule, the staff member will not vote for this solution. As a consequence, this specific solution will be prohibited. Hence the individual preferences of every single staff member have to be considered in the solution identification stage. The whole process is characterized by finding a consensus. But finding a consensus is getting harder, if not impossible, with the increasing number of veto participants. Moreover, the schedule finding process is particularly time consuming, and every staff member has to participate. To fulfill the requirements for a dialog within the daily routine is a challenging task and only possible with a small number of staff.

An alternative approach, preference scheduling, tries to combine the advantages of cyclic scheduling and self scheduling (cf. Bard and Purnomo 2005). Here the methods of operations research are applied. By including the preferences of the staff members, a feasible solution is found through heuristics or optimization techniques. The assignment of staff members to a given shift schedule in a 24/7 context is a challenging task of operation research, and is known as the nurse rostering problem. Note that the shift schedule is given, meaning that the shift schedule denotes the demand for a certain number of nurses (staff members) with certain qualification for every point of time within the planning period. This requested number of nurses is person non-specific. The aim of this resource allocation problem is to find a roster that satisfies the demand of nurses. The roster itself is a plan that gives the times at which every single nurse is on duty, as well as when not on duty. The crux of the nurse rostering problem is that the allocation is confronted with the mentioned staffing and coverage constraints (cf. Glass and Knight 2010). Now if all constraints are considered, the solution space is empty. In consequence, some of the constraints are softened in such a way that hard and soft constraints are distinguished. For finding a feasible roster, a sensible trade-off between the constraints is wanted.

The fact that the allocated resources are persons with individual preferences gives a special character to the nurse rostering problem. If a roster is created, the nurses may either approve of it or not. If they do not approve of the roster, they will perform their duties reluctantly, with a consequent low job satisfaction (cf. Larrabee et al. 2007). However, low job satisfaction is somehow related with sick leave (cf. Chan et al. 2013; Schreuder et al. 2011). The health care sector is confronted with a high level of sick leave (cf. Carlsson et al. 2013; Steenstra et al. 2005).

The nurse rostering problem has been known since the 1970s. And even though the problem could not be solved at that time because of the lack of computer power, different solution approaches were formulated (cf. Burke and Curtois 2014).

The much noted publications Cheang et al. (2003), Burke et al. (2004), and Ernst et al. (2004), give a broad overview of the state of the art at that time. They show that only a minority of treatments related to the nurse rostering problem were practically relevant for that problem.

The literature review of De Causmaecker and Vanden Berghe (2011) draws a different picture. By that date, a variety of publications had offered solutions for practically oriented nurse rostering problems. But even if the presented solutions were practically relevant, the created rosters did not contribute to a high level of job satisfaction.

In the last couple of years, the discussion in the relevant literature has revolved around the fairness of rosters, with the argument that the quality of a roster has to be high to cause a high level of job satisfaction (cf. Ouelhadj et al. 2012; Smet et al. 2012). It is at least questionable whether fairness is the criterion on which to focus for creating a high quality roster, because an answer to the question what exactly is fair in not finally given. For instance, Rönnberg and Larsson (2010) pursue a uniform distribution of undesired shift/duty constellations among the nurses, to handle the fairness question, to induce a high quality roster.

But is it a necessity to find an answer to the fairness question to create a high quality roster? After all, each nurse is the judge of a roster's quality. If the preferences of a nurse are considered as far as possible in the rostering process, then the nurse should certify the roster as being of high quality. That does not mean that the nurse should work on weekdays from nine to five. A 24/7 shift system is still a given, and the willingness to work on certain undesirable shifts is presupposed. Therefore, it should be possible get around the fairness question to create a high quality roster by considering, as much as possible, the preferences of the nurses. However, this raises another difficulty. How can one reveal the true preferences of the nurses? Interviewing the nurses will fail, because the preferences are private informations of the nurses and they have an incentive to conceal their information. This situation is identical to price negotiations, whereby each party tries to maximize its own surplus.

Well designed auctions can reveal the willingness to pay of the consumers. This context can be transferred to the nurse rostering context for identifying the true preferences of the nurses. A well designed auction should reveal the needed private informations of the nurses, which can be thereafter be included in the rostering process to create a hight quality roster. But which kind of auction can be implemented in the nurse rostering process to consider the preferences of nurses in the mentioned way? Giving an answer to that question is the purpose of this paper.

2 Fundamental Auction Classifications

Designing auctions requires an awareness of the advantages and disadvantages of the different kinds of auctions and their combinations.

Every auction design is described by its policies. The set of rules contains the allocation rule and the pricing rule. The allocation rule specifies which bidder will win the auction. In most cases, this would be the bidder submitting the highest bid. However, it is conceivable that the bidder with the lowest or any bid in between the highest and lowest bid will be declared the winner. If the allocation rule determines the highest bid as the winning bid, then this type of auction is called a standard auction.

The pricing rule specifies in the narrow sense which price the winner has to pay for the good won. If the winner has to pay their own bid, the highest price, then the auction is called a first price auction. If the price corresponds with the second highest bid, the auction is referred to as a second price auction. Note that any other rules fixing the final price are also possible.

In the broader sense, the pricing rule determines how the final price is found. This concerns, in the first place, whether the bidder can only bid once or several times. Besides, another difference is whether bidders submit their bids in public in a way that their bid can be observed by other bidders (open auction) or if the bids are submitted in a sealed way (sealed bid auctions).

In addition, the price can either rise or fall. Rising prices are common and obvious against the background that a seller is usually interested in high prices. However, if the price falls, bids as low as possible are wanted. Thereby, minimal increments or decrements are set. Moreover, the pricing rule specifies whether the bidders hand in their own bids or whether they can only signal to be willing to pay a certain raised or dropped price for the good. Of course further regulations are possible. If the pricing rule is understood only in the narrow sense, then an additional set of rules to specify the auction design is necessary.

Now it is not always the case that only one good is intended to be auctioned off: there can be more than one. Hence another distinction can be made: single unit auctions versus multiple unit auctions. In the multi-unit case, the different objects are in relation to each other. That implies the objects are either substitutes or complements for each other (cf. Krishna 2009, p. 165). Which auction form will be applied depends on the following case dependent reflections.

3 Requirements for Designing Auctions for the Nurse Rostering Problem

3.1 Basic Reflections for the Design of an Auction for the Nurse Rostering Problem

In designing an auction, it is important to be aware of the answers to the following questions: What exactly is auctioned off? What are the existing approaches to the nurse rostering context? What lessons can be learned from these approaches? How can payment methods be designed for the nurse rostering problem?

- *What exactly is auctioned off?*
  First of all it needs to be clarified what are the goods allocated through the auction, within the context of nurse rostering. In general, nurses have two kinds of interests: to be on or to be off duty at a certain time.

  The motivation to be off duty is some kind of opportunity from the social background of each nurse. For example, a nurse might have a private meeting or family commitments requiring being off duty.

  On the one hand, the motivation to be on duty can be the consequence of having an interest in working at that specific time. Some nurses prefer to work at night time, because the workflow is not as busy as in the early or late shift. On the other hand, a nurse might be interested in being on duty at a certain time to ensure being off duty at a related time. A nurse might want to work on an early shift to be off duty for the following late shift.

  Generally, nurses are interested in having the right to do something (to be on or off duty) at a certain time. If nurses have a demand for the right to do something, then this right should be a good allocated through the needed auction. Hence, two kinds of goods can be provided by a possible auction for the nurse rostering problem: permit rights, being allowed to be on duty, and omission rights, ensuring obtaining private opportunities.
- *What are existing approaches to the nurse rostering context?*
  De Grano et al. (2009) present an auction based approach to allocate permit rights and omission rights through a two stage optimization. As far as it is know to the author know this is the only approach to include permit rights and omission right in the nurse rostering context in the sense of an allocated good by an two stage auction based approached. The first optimization stage enables a selection of the auction winners for the demanded shifts. The second optimization stage ensures that all staffing constrains are satisfied and the remaining unfilled shift

will be operated. This approach shows that the including of auction into the nurse rostering enables the consideration of nurses preferences and a solution is found in a reasonable amount of time.

However De Grano et al. (2009) use for there approach self scheduled timetables of the nurses. In there case study it is routine to inform the nurses about the schedule a few weeks in advance. Each nurse then is submitting a preferred plan of duty. It can be assumed that this handed in information corresponds to the nurses preferences, because a nurse should have a weak dominate strategy to tell the truth. By varying its submitted preferences the nurse risks to be considered for an undesired shift or not to be considered for a desired shift. Now De Grano et al. (2009) convert the self-scheduling requests of the nurses into bids. This converting is done by weighting the requested shift according the shift length.

- *What lessons can be learned out of this contributions?*
This conversion of bids resembles the idea of using penalty costs in conventional nurse rostering approaches. The motivation for the use of penalty points is to find the above mentioned trade-off between hard and soft constraints. The non-consideration of soft constraints requires a price, which is enabled by the penalty points. Against the backdrop of an auction, for revealing the nurses' preferences, the existence of prices is desirable. By assigning penalty costs to the different soft constraints, it is possible to distinguish between the constraints and therefore the problem can be handled.

  But neither penalty costs nor converting submitted preferences into bids can represent the felt harm of a nurse. The planner can only guess how the consequences will appear to a nurse. While the conversion into bids enables some kind of communication between the planner and the effected nurses, the use of penalty costs forbears any kind of communication between the planner and the nurses. To reveal the felt harm of a nurse, an auction design is needed, which is known to the nurses in advance so that each nurse will anticipate the behaviour of the other nurses and therefore strategic no bidding will occur. In that way the true preferences of nurses can be divulged.

  In consequence, one requirement needs to be kept in mind. The auction mechanism, the bidding itself, has to be designed in such a way that the nurses still reveal their true preferences. Only if the bidding is made according to the true preferences can an efficient allocation be ensured by the auction. Therefore, an efficient auction is needed. An auction is efficient if the auctioned good is allocated to the bidder with the highest willingness to pay.
- *How can payment methods be designed for the nurse rostering?*
Next to the requirement of an efficient auction, a little thought should be given to the payment funds. To create the needed high quality roster, it is unavoidable to install a bidding mechanism to receive bids, bids that can be treated as prices, and that are anticipated as prices of the bidder while bidding. However, it is not clear what is the dimension of the bids in an auction for the nurse rostering problem. Letting the nurses offer monetary units to receive permit or omission rights will not work in practice, because the opportunity cost will be priced in by the nurses. These opportunity costs vary among the nurses. The idea of De

Grano et al. (2009) to assign each nurse a certain amount of points seems to be charming. In that way, nurses have a limited budget. In addition, the different policies of hospitals can be taken into account. For instance, nurses with more years of service will get a higher budget of points.

Summarizing the design of an auction has to consider the following basic requirements: (1) The auctioned goods should be certain rights. (2) The auction design itself has to be efficient to achieve as high a quality of the roster as possible. (3) Certain rights could be paid out of the individual budget of points of each nurse, to enable the allocation of the nurse rostering problem.

3.2 Special Features of Auctions for Nurse Rostering

The daily routine on stations in hospitals with its regulations and demands causes special reflections in that way that the basic requirements have to be specified. By the answers to the following question the catalog of the requirements for a suitable auction form for the nurse rostering problem will be extended. Does it matter if permit rights or omission rights are actioned off? If multi-unit auctions are applied should combinatorial auctions considered? What is the object of the auction? Thereby what is the comprehension of auctions? What are the limiting facts of daily routine?

- *Does it matter whether it is permit rights or omission rights that are auctioned off?*
  The motivation for designing an auction for the nurse rostering context is to get specific information about the nurses' preferences, which will be used to create a roster. Again, not all nurses can be assigned to duties in a way that they would work Monday till Friday from nine to five. Assigning nurses to unpopular shifts causes harm. Omission rights can help to indicate the magnitude of the felt harm of a nurse, because the corresponding price can be understood as a measurement of the felt harm, and so expresses the nurse's preferences in that way. If a nurse is willing to pay a high (the highest) price to get a specific omission right, and therefore the right to enjoy time off, then it is sensible not to assign that specific nurse to the contemplated shift. In that way, and because a roster has to be created, the nurses' preferences should be considered. As a consequence, the minimization of the suffering of all the nurses can be intended and thereby a high quality of the roster can be produced.

  However, permit rights are of little help to reveal the nurses' preferences. What does the planner, intending to solve the nurse rostering, know about the nurses' preferences from a submitted bid for a permit right? It is divulged that the winning nurse did bid the maximum price (assuming a standard auction). In this way it is possible to achieve a high level of satisfaction by assigning the permit right efficiently. But even if the assigning is efficient, it is pretty much safe to say that some nurses will still be assigned to unpopular shifts, because it is unlikely that the demand for permit rights among the nurses is distributed according to the supplied permit rights. Therefore, and because the reduction of job dissatisfaction and sick leave is intended, the offered good by the auction designed should only be omission rights.

- *If multi-unit auctions are employed, should combinatorial auctions be considered?*
  The intended auctioned goods need to be described precisely to stimulate bidding. An omission right has to be specified by its beginning and ending points in advance. As mentioned above, the demand for services is known in advance, and thereby the different shifts with their starting and ending points are also known in advance. To keep the service quality high, the duties are not congruent with the shifts. It is intended that not all duties start or end with the shifts, as in the automobile industry, for example. Besides, finding the duties' location in time is an aim of the nurse rostering problem. In consequence, omission rights can only be referred to shifts, not to duties. This implies that an omission right for one shift is not the same as an omission right for the immediately following shift, because of the different starting and ending points. An omission right can only be a substitute for another omission right when both of these rights refer to the same shift. Obviously there exists the above wanted relation between different omission rights among themselves in the form of substitutes. But only if they refer to the same shift. One can preliminarily conclude that the intended relation between the goods exists, because omission rights are substitutes for each other. Hence some kind of multi-unit auction should be designed.

  Moreover it is worthwhile so consider omission rights as complements. Goods can be understood as complements if the principle of superadditivity can be applied. This means that a bidder values the sum of two goods higher than the sum of the values of each single good. Thus a bidder should be able to consider getting a bundle of goods instead of just a single units. Transferring this to the context of nurse rostering, it can not be ruled out that a nurse might be interested in a combination of omission rights instead of single omission rights only. If a nurse could win a couple of successive omission rights, she could achieve a few combined days off. Consequently the auction should offer the possibility of biding for conjoined omission rights.

  A combinatorial auction gives the bidders the possibility of bidding for certain sets of goods. Hereby a distinctions can be made. (1) The auctioneer combines some of the goods into a bundle. (2) The bidders are bidding for some packages or goods (cf. Krishna 2009, pp. 223–230). Still, given the lack of knowledge about the nurses' preferences, the first case is not suitable, because a bundle of the goods on behalf of the auctioneer can only be based on more or less good guessing. And as mentioned the motivation for implementing auctions in the nurse rostering problem is to get around guessing. Hence the means of choice seems to be to allow the bidders to bid for any packages they are interested in, if it is guaranteed that the auction is efficient.

  Unfortunately, allowing packaging causes an unwanted problem. The motivation to inaugurate an auction is to unveil the nurses' preferences. If nurses submit their bids for packages of omission rights, it is hard to identify the nurses' valuation for the omission right for a specific shift, simply because of the design of the combinatorial auction. Given an efficient auction design, a bid for a package can be understood as the nurse's valuation for the combination of omission rights, but drawing conclusions about the nurse's valuation of any shift among the package is false. This leads to the situation of first place. The nurses valuations of specific omission rights are needed to assign duties to nurses in a preferable and efficient way. Therefore, a combinatorial auction should be out of the question.

- *What is the object of the auction?*
  Up to this point, the interdependence of winning the auction(s)—whether or not a combinatorial auction is installed—in the nurse rostering context has not been discussed. The daily routine might prevent winning the auction!

  Usually the planning period amounts to a couple of weeks (cf. Burke et al. 2004). Within that time, hard constraints like minimum/maximum hours of work time and rest times for each nurse have to be ensured (cf. Wu et al. 2015). The different auction designs can not by themselves guarantee the consideration of hard constraints of the nurse rostering problem. For example, it cannot be ruled out that a nurse might win a combination of omission rights in such a way that she is unable to fulfill her contract while retaining all statutory provisions of working time.

  The aim, then, of applying auctions to the nurse rostering problem cannot be to allocate every single omission right to the highest-bidding nurse, assuming an efficient auction design, but to find, as much as possible, auction winners while all hard constraints are considered. As a consequence, the efficiency of the allocation will pursued according to the Pareto criterion.
- *What is the concept of the auction?*
  Considering the planning period with its hard constraints and the fact that bidding the most for an omission right does not result in winning the auction per se raises the question of the understanding of auctions. It is obvious that such a design contradicts the intention of a standard auction. Is it correct to talk about an auction in general in the described nurse rostering context? Note that by an auction, an allocation of goods is enabled. The allocation takes place by a set of rules. This applies to the described situation of nurse rostering also. Hence the use of the concept of "auction" can not be denied. But it is necessary to prove that, for the specific auction design, the intended incentives are still valid.
- *What are the limiting facts of daily routine?*
  In addition to securing the incentives of truthful bidding, the daily routine of nurses in hospital requires a preferably easy handling in the whole decision process. Remember that it is one downside of self scheduling that it is hard if not impossible to bring all the nurses together at one time. Therefore it is hard to see how an open auction form, where the actions of others can be observed, can be employed. This extends to sequential auction forms where the bidding is done in successive rounds and more and more information can be gained. Especially if a one round only auction design could result in an efficient allocation as well.

Again, the intention is to find as many auction winners as possible over every auction.

The *requirements on a specific auction design for the nurse rostering context* are that some efficient multiple-unit standard auction in a closed single round form is wanted. The allocated goods are omission rights, which will be paid for with points.

4 Supply and Demand in Nurse Rostering

To illustrate the fundamental understanding of market clearing in multi-unit standard auctions for the regarded nurse rostering context, it has to be clarified how can the supply and demand be ascertained. Thereby the interdependency of different omission rights will be ignored at first, so that only an omission right for a shift will be considered.

In an auction with multiple goods to sell, the auctioneer can decide if the goods are sold one by one in multiple auctions or if (somehow related) goods are sold in one go (cf. Krishna 2009, p. 165). In the following, the term multi-unit is used for the latter case. Thereby it will be assumed that multiple bidders have a demand for the goods. Due to the mentioned unsuitability of open auction forms, the discussion will concentrate on sealed-bid single-round auction forms.

The goods that will be auctioned off are omission rights for one specific shift. Nurses can bid for these goods to achieve the right to not be considered for duties on that shift. The amount of the omission rights depends on the total number of nurses needed. Typically, a nurse is employed at one hospital station only. So that the budget of nurses conforms to the total number of nurses employed at that specific hospital station. Nurses on vacation are obviously not available and the demand for services, a certain minimum for the number of nurses, has to be ensured. Hence the supply of omission rights is the difference between the budget of nurses and the nurses on vacation and a minimum service demand.

Independently, there is a special feature that needs to be regarded as to the specific amount of omission rights for one shift in nurse rostering. Nurses might be interested in achieving a omission right to pursue private affairs, but nurses do not attach a positive valuation to a second omission right. (A nurse's valuations of the omission rights is of a private nature, and independently and identically distributed.) If the nurse is getting one omission right she has no use for another one. This is connected to the assumption that no side payments are possible. The medium of exchange consists of the virtual points which are registered to every nurse's fictive account and cannot be transferred. Therefore the singularity of single-unit demand is relevant for the nurse rostering context. Whether the (single) bids of the nurses correspond with their private valuation of the good or not, or, in other words, whether the auction is efficient, depends on the auction design.

The equilibrium of supply and demand is reached by ordering (the submitted) bids. Therefore the individual demand functions (cf. Fig. 1) of each bidder

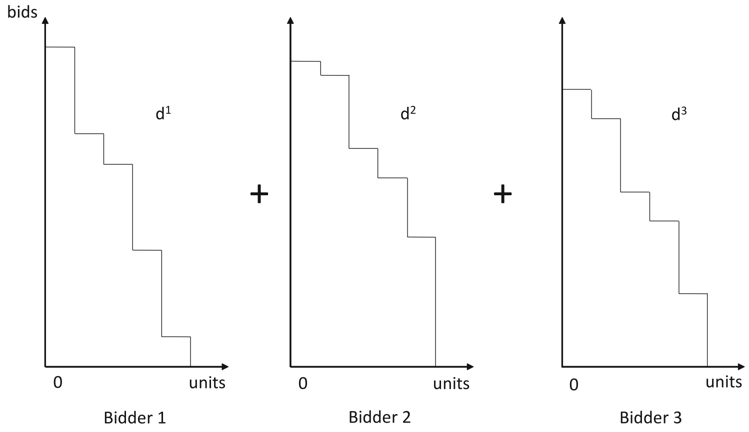

Fig. 1 Individual demand functions (Krishna 2009, p. 166)

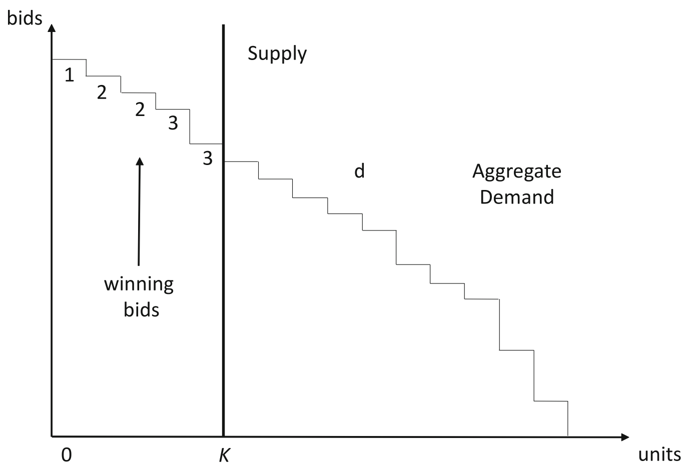

Fig. 2 Aggregated demand and supply (Krishna 2009, p. 167)

b^i $i \in (1, 2, 3)$ are assembled into one aggregated demand function (cf. Fig. 2). Given the case of a standard auction, the supply of K omission rights will be matched with the K highest bids. With single unit-demand, the supply is matched with the K highest bidders. Now two cases can be distinguished. (1) The bidders among the bidders with the highest bids according to the amount of offered omission rights will be declared as the winners of one omission right. (2) The number of offered omission rights exceeds the number of interested nurses. If it is clear that the amount of submitted bids (the number of bidders) is less than the number of offered omission rights, then the number of omission rights will be reduced to the number of submitted bids and every bidder will win one good. This can be justified by the overall intention of allocating the duties to the nurses. Considering un-demanded goods leads to an unnecessary reduction of the solution space. In either case, the market clears when the number of offered goods will be assigned to the bidders with the highest bids.

5 Designing the Nurse Rostering Problem Based on Uniform-Price Auctions

When it comes to efficient sealed bid standard auctions, the Vickrey auction is surely one often regarded auction design, due to its theoretically efficient outcome (cf. Vickrey 1961). However the Vickrey auction has its downsides in terms of practical application (cf. Rothkopf 2007). Ausubel and Milgrom (2006) describe that it is hard to explain to bidders that they will pay very different prices for the same goods. In view of the fact that the auction design for the nurse rostering should not be unnecessarily difficult, the Vickrey auction does not come into question. Instead, the criticism referring to the different prices is adopted, and the uniform-price auction is employed.

One advantage of the uniform price auction design is the relatively simple set of rules. The uniform price auction is a standard auction in the described setting. The payment rule determines that every winner pays the same (uniform) price for the good. The relevant uniform price conforms to the market clearing price, which could be any price in between the lowest winning bid and the highest losing bid. In the following, the market clearing price will be kept down to the lowest winning bid. This is explained as the lowest possible demand of one bidder. If only one bidder is willing to pay for the good, she should pay her bid. Otherwise it is hard to explain to other nurses why the winner did not pay anything. Besides, the easy understanding of the set of rules would be ruined.

With the uniform-price auction, the worked out requirements are met. The uniform-price auction is a standard sealed-bid auction for multiple units, it can be performed in one single round, and the installation of points as the payment method is possible.

A disadvantage of the uniform price auction is that it does not guarantee an efficient allocation per se. But this applies to common multi-unit demand. Bidders tend to obtain a benefit which can be achieved by bid shading (cf. Fig. 3). By bid shading, the bidders do not bid according their (private) valuation for the good x. Instead, bidders bid below their valuation b. This occurs if the own bid influences the final price (Krishna 2009, pp. 185–190). In a uniform price auction, bidders tend to use bid shading for every unit demanded but the first one. The explanation is the same as for the Vickrey auction. On the one hand, bidding above the own valuation for the good would not guarantee winning the bid, but gives rise to the possibility that the own bid is the price determining uniform price. In consequence, the bidder could pay a price above their valuation, which causes a negative benefit. On the other hand, bidding below the valuation could result in not bidding high enough to be declared a winner. Thus the bidders have a (weakly) dominant strategy

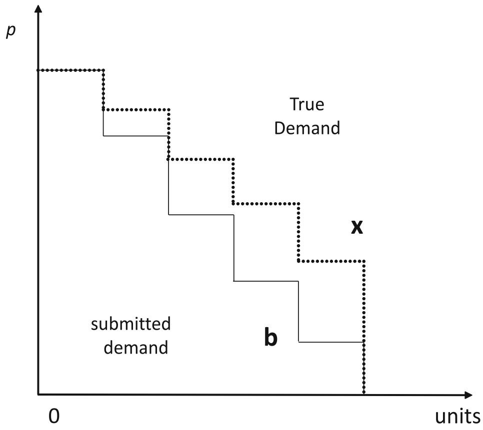

Fig. 3 Demand reduction in uniform-price auctions (Krishna 2009, p. 189)

to bid for the first unit (single-unit demand) according to the own valuation. Bidding according to the own valuation is exactly what is intended by implementing auctions in the nurse rostering context. Hence the uniform-price auction seems to be suitable for this kind of problem. It reveals the nurses' preferences. The nurses have only to submit their bids once in a sealed way. For submitting the bids a time period could be determined in such a way that bidding besides work and life is enabled. Finally every winner has to pay the same price for the same good. As currency, virtual point are defined. As mentioned above, the nurses will get a certain number of points for bidding in a period.

6 Discussion

6.1 Multiple Multi-Unit Auction

It is conceivable that the uniform-price auction with single-unit demand leads to an efficient allocation. But as mentioned, the nurse rostering problem is more complex. There are several different types of shifts in the planning period. The proposed uniform-price auction refers only to one shift with its referred omission rights. Now the question is how to extend the uniform-price auction so that the nurses still bid according to their own valuations and the design stays as simple as it is.

At first, it seems to be possible to conducting several of the uniform-price auctions, one by one. Now the problem occurs that this management increases the number of transactions and the communication in general, not to mention possible strategic bidding behaviour based on the successively revealed information.

Conducting several uniform-price auctions all at once seems to be suitable, so that only one multiple multi-unit uniform-price auction has to be performed. Again the nurse rostering problem is complex and difficult to solve. For finding a solution, the methods of operations research should be applied with the described intention of finding as many winners of the single uniform-price auction. Note that the way of solving the problem does not change the nurses' preferences. The nurses would still submit bids for which shift they are interested in getting an omission right, and at what price. This is what is more or less done in practice already, as in the case study of De Grano et al. (2009). Except that the nurses assign their preferences a price.

6.2 Truthful Bidding

But the aforementioned interdependence of omission rights according to different shifts and the considering of hard constraints could result in not assigning a omission right to a nurse who had bid high enough to be actually among the winners. Does this effect the nurses' bidding? Would a nurse have an incentive to differ from the strategy of truthful bidding if there are many parallel uniform price auctions?

For diverging from truthful bidding, a nurse has to have an incentive. In the very first place, a nurse is confronted with two decisions concerning every shift. The first decision is binary. Either the nurse is interested in the corresponding omission right or not. If she is interested, the nurse has to decide (in the second step) how much to bid. By differing from truthful bidding, the nurse has to change at least one decision. If the nurse decides against her preferences not to bid for a omission right, she will definitely not be awarded that omission right. Note that this does not mean the nurse has to work on that shift. The final roster can assign the nurse to that shift or not. The outcome would be rather based on coincidence. Bidding in the case that the nurse might not be interested in an omission right and would prefer to work on that shift could result in winning the shift with the consequence of not being allowed to work on that desired shift and paying unnecessarily the uniform price. Differing from the first decision to bid or not to bid only causes possibly worse outcomes for the nurse, but no positive. The second decision is associated with the first. If the nurse decides not to bid, the second decision is obsolete. In the case that the nurse decides to bid, the incentive does not change from the single uniform price. Therefore the nurse should not have an incentive to differ from bidding truthfully. The point is that a nurse has no possibilities for inter-shift action, even if a nurse is considering the whole planning period.

6.3 Collusion

When it comes to designing auctions, collusion needs to be considered. Arrangements among the bidders can destroy possible intended incentives. In the nurse rostering context, arrangements among nurses are quite common in daily routine. A conventional roster might not consider all the preferences of the nurses. Hence nurses swap shifts to improve the situation of at least one participating nurse. This common practice is in the interests of the hospital, because in this way the services at that relevant hospital station can be maintained. Note that statutory provisions still need to be satisfied. The installation of auctions for the nurse rostering context does not intend to make a profit, but to find a solution for the nurse rostering problem based on the nurses' preferences. If the nurses are making arrangements, this will reflect their preferences, which is the intention of the auction.

Besides auctions, revenue reducing price agreements are insignificant, simply because the auction's revenue does not matter. The awarded winners will pay the price, the uniform price, in terms of points. These points will be deducted from their virtual accounts. But for the overall intention of solving the rostering problem, the number of points that will be paid is of no interest. Moreover, if nurses find a way to reduce the final prices by arrangements, then this improves their situation, which should be in the sense of job satisfaction.

7 Conclusion

That designing an auction has to be done carefully is already known from many cases of application. The nurse rostering context has its own characteristics that need to be considered while designing auctions. Thereby it is necessary to make sure that nurses submit their bids according to their preferences. The presented multiple uniform-price auction enables a procedure that can be implemented in the daily routine of nurses, because submitting the bid is only done once in a sealed way. The auction form allows a (weak) dominant strategy of truthful bidding. In that way, the nurses' anticipation of their submitting is ensured so that the bids reflect the nurses' preferences.

Note that the presented multiple uniform-price auction is only one possible auction design for the nurse rostering problem. This paper does not claim to present the only applicable auction form, but to present one possible auction design.

Of course further research needs to be done. The multiple uniform-price auction needs to be implemented in a model of operations research. Hence the anyway NP-hard nurse rostering problem will not be easier. The nurse rostering problem with its hard constraints results in a situation where the concept of the standard auction can not be maintained. Nevertheless, the term "auction" with its set of rules is justified. Possible concerns about divergences from the truthful bidding strategy can be ruled out because a nurse can not improve their situation by doing so.

References

Ausubel LM, Milgrom PR (2006) The lovely but lonely Vickrey auction. In: Cramton P, Shoham Y, Steinberg R (eds) Combinatorial auctions. MIT Press, Boston, pp 57–95

Baily L, Collins R, Song Y (2007) Self-scheduling for hospital nurses: an attempt and its difficulties. JNM 15(1):72–77

Bard JF, Purnomo HW (2005) Preference scheduling for nurses using column generation. EJOR 164(2):510–534
Bard JF, Purnomo HW (2007) Cyclic preference scheduling of nurses using a Lagrangian-Bases heuristic. JS 10(1):5–23
Burke EK, Curtois T (2014) New approaches to nurse rostering benchmark instances. EJOR 237(1):71–81
Burke EK, De Causmaecker P, Vanden Berghe G, Van Landeghem H (2004) The state of the art of nurse rostering. JS 7(6):441–499
Carlsson L, Englund L, Hallqvist J, Wallman T (2013) Early multidisciplinary assessment was associated with longer periods of sick leave: a randomized controlled trial in a primary health care centre. ScanJE 31(3):141–146
Chan ZC, Tam WS, Lung MK, Wong WY, Chau CW (2013) A systematic literature review of nurse shortage and the intention to leave. JNM 21(4):605–613
Cheang B, Li H, Lim A, Rodrigues B (2003) Nurse rostering problems - a bibliographic survey. EJOR 151(3):447–460
De Causmaecker P, Vanden Berghe G (2011) A categorisation of nurse rostering problems. JS 14(1):3–16
De Grano ML, Medeiros DJ, Eitel D (2009) Accommodating individual preferences in nurse scheduling via auctions and optimization. HCMS 12(3):228–242
Ernst AT, Jiang H, Krishnamoorthy M, Sier D (2004) Staff scheduling and rostering: a review of applications, methods and models. EJOR 153(1):3–27
Glass CA, Knight RA (2010) The nurse rostering problem: a critical appraisal of the problem structure. EJOR 202(2):379–389
Hung R (2002) A note on nurse-self-scheduling. Nursing Economics 20(1):37–39
Krishna V (2009) Auction theory, 2nd edn. Academic Press, Amsterdam
Larrabee JH, Sions J, Fanning M, Withrow ML, Ferretti A (2007) Evaluation of a program to increase evidence-based practice change. J Nur Adm 37(6):302–310
Levner E, Kats V, de Pablo DAL, Cheng TE (2010) Complexity of cyclic scheduling problems: a state-of-the-art survey. Comput Ind Eng 59(2):352–361
Ouelhadj D, Martin S, Smet P, Ozcan E, Vanden Berghe G (2012) Fairness in nurse rostering. Working paper, University of Portsmouth
Rönnberg E, Larsson T (2010) Automating the self-scheduling process of nurses in Swedish healthcare: a pilot study. Health Care Manag Sci 13(1):35–53
Rothkopf MH (2007) Thirteen reasons why the Vickrey-Clarke-Groves process is not practical. Oper Res 55(2):191–197
Schreuder JA, Roelen CA, Van Zweeden NF, Jongsma D, Van der Klink JJ, Groothoff JW (2011) Leadership effectiveness and recorded sickness absence among nursing staff: a cross-sectional pilot study. JNM 19(5):585–595
Smet P, Martin S, Ouelhadj D, Ozcan E, Vanden Berghe G (2012) Investigation of fairness measures for nurse rostering. In: International conference on the practice and theory of automated timetabling (PATAT 2012)
Steenstra IA, Koopman FS, Knol DL, Kat E, Bongers PM, De Vet HC, Van Mechelen W (2005) Prognostic factors for duration of sick leave due to low-back pain in Dutch health care professionals. J Occup Rehabil 15(4):591–605
Vickrey WS (1961) Counterspeculation, auctions, and competitive sealed tenders. TJF 16(1):8–37
Wu TH, Yeh JY, Lee YM (2015) A particle swarm optimization approach with refinement procedure for nurse rostering problem. Comput Oper Res 54:52–63

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Part II
Cooperative Models: Models of Fairness and Its Applications

Fair Distribution of Cooperation Gains in Supply Chains: A Justification Program from an Economic Point of View

Stephan Zelewski

Abstract The fair distribution of the gains from cooperation presents a challenge to economic research as well as to business practice. This is based, above all, on two reasons. First, fairness is a very vague term that can be interpreted in very different ways so that this term needs to be operationalized. Second, the term "fairness" cannot be derived from "objective" or "empirical" data, but needs a substantive justification based, ultimately, on subjective judgements about fairness or justice. This twofold challenge is elaborated on in this paper. On the one hand, the term "fairness" is operationalized from the perspective of cooperative game theory. On the other hand, a program for its justification is presented that aims at evaluating game theoretic solution concepts. As a result, we present a program for its justification which consists of six requirements.

Keywords Acceptability • Communicability • Existence • Rationality • Threat point • Uniqueness • Usability

Contents

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1 The Fair Distribution of Cooperative Benefits in Supply Chains as a Real Problem: Explication and Limitation

The economic rationality of the development and the maintenance of cooperation resides in the attainment of a cooperation gain. Such a cooperation gain can be attained if two requirements are met. First, several partner that are economically independent of each other have to be willing to fulfill a "challenging" task together. Second, because of their joint fulfillment of this task, they realize a profit which is altogether bigger than the sum of the partial profits that could have been made by the partners in question in isolated actions. The cooperation gain is the difference between the collectively earned profit and the sum of the partial profits made in isolation instead. Other common descriptions, like, e.g. gains in efficiency and cooperation-induced added value, are being treated as synonyms in this paper. For convenience, the cooperation gain will be frequently referred to briefly as the "profit" that is attained through the cooperation of the partners due to their joint fulfillment of a task.

The real problem (versus the later discussed scientific problem) of this paper consists of the challenge of distributing the collectively generated cooperation gain to the participants in the cooperation (which are referred to as partners) in a way that the cooperation partners perceive as fair, and consequently accept it. For simplicity, this will be hereinafter be referred to as the problem of a cooperation gain distribution perceived as fair, the problem of a fair cooperation gain distribution, or, for short, the distribution problem. Every distribution of the gain to the partners constitutes a *solution*. The partners will be frequently referred to as the participants or players with respect to the later regarded game theoretic modeling of the distribution problem.

To limit—but also illustrate—the previously specified real problem, 5 basic premises shall be granted. **First**, only "reasonable", i.e. economically rational, cooperations are considered. They are characterized by the expectation on the part of the affected partners of attaining a positive gain (the premise of positive cooperation gains). This only involves an expectation "ex ante". The considerations thus apply independently of the fact that "ex post" it turns out that the initially intended profits have not been generated by means of the cooperation. In contrast, cooperations that provide no prospective of cooperation induced added value from the onset, seem to be "futile" from an economic point of view and thus will not be considered more closely.

Second, supply chains—or, here taken synonymously, supply networks and added value networks—are only being given as a "paradigmatic example". They are well suited for emphasizing the economic relevance of the real problem of a fair distribution of collectively attained cooperation gains to the partners. Because it is often a controversial problem in economic practice how the "cooperation advantages" – concretized in this paper as cooperation gains—are distributed to the partners of a supply chain in a "just" or "fair" manner (the terms "justice" and "fairness" are being used synonymously in this paper). For example, this presents a subtle challenge for a "focal" company, which dominates a supply chain because of its market power, not to pocket the cooperation gains totally for itself, but to give "fair" parts of it to its cooperation partners so that these partners do not leave the cooperation or even join a competing supply chain. The treatment of the suppliers in supply chains of the automotive industry with one automobile producer as a focal company provides a vivid application example for the real problem examined here.

Transfers of this "paradigmatic example" of supply chains to other fields of business cooperation are obvious (the terms "business" and "economic" are treated equivalently in this paper). In this regard, for example, airline companies that cooperate by means of "code sharing" as part of an "alliance" come into consideration. Likewise, virtual companies can be looked at, in which several legally autonomous companies cooperate, but which virtually act as one company at the interface with their customers. Such examples of extending the field of the "intended applications" of the here presented concept of a fair distribution of collectively attained cooperation gains to the partners could be increased almost at will. They emphasize the economic relevance of the distribution problem presented here.

Third, this paper focuses on the "institutional" sphere of legally and economically independent companies that cooperate in a supply chain. Each of these companies constitutes an partner (or synonymously, a player) of the cooperation that can threaten to stop cooperating at any time, as a "defective" behaviour, and can also carry out this behaviour, because of the partner's legal and economic independence.

Because of this focus, no pseudo-cooperations are considered, on the one hand. In such pseudo-cooperations, the individual partners are condemned to collaborate due to their legal or economic dependence, because they cannot threaten in a credible manner to operate economically successfully outside of the cooperation.

On the other hand, this focus has once again only a "paradigmatic" character. For the reflections made here can be also transferred to a real context where economically independent partners cooperate with each other for the purpose of jointly accomplishing a challenging task. In this wording, the legal independence of the partners is being purposely omitted, because the real problem being analysed here, of a gain distribution perceived as fair, can also arise in the case of legally dependent partners, insofar as they—at least within the given (e.g. legal) bounds—are able to adjust their behaviour semi-autonomously according to their individual economic interests. In the light of principal agent theory, what is especially being kept in mind are those members of a company that, although they find themselves in a dependent employment relationship from the legal perspective, nevertheless have some scope for activities that they can use for their own economic interests, due to real information and control defects on the part of the principal (management). In this case, too, the cooperation of members within a company essentially depends on whether the distribution of the cooperation induced added value is being perceived as fair; otherwise they can threaten, e.g. an "inner emigration" or even leaving the company.

Beyond that, also "macro-economic" and "transnational" contexts can be imagined for the real problem of a gain distribution perceived as fair, where economically independent partners cooperate in order to accomplish a challenging task together.

Hereto belong, e.g. cooperations between employees' associations and employers' associations that negotiate a "fair" wage-setting in the face of anticipated productivity growth (as a cooperation gain). Also, negotiations between national governments that consult about how the charges—which would be treated as "negative" cooperation gains—from transnational programs can be fairly distributed to the partners in the "community of states" can be looked at. For this, one has, for instance, combating famines, medical epidemics such as, e.g. the Ebola or the current Zika virus, or of climate-damaging greenhouse gas emissions.

Fourth, in this paper only the real problem is being considered to what extent a gain distribution perceived as fair can contribute to maintain an already existing cooperation. Thus, all that is being analysed is how an "economically rational" incentive can be provided to every cooperation partner via a gain distribution so that the partner does not leave the cooperation, i.e., does not "defect". The focus of the analyses is on the stability of already established cooperations (premise of stability). However, all considerations concerning the "economically rational" origin of a cooperation are left aside. The economic relevance of this aspect of cooperation formation is anything but denied. But this aspect cannot be even rudimentarily covered here, due to the brevity required of this paper. Besides, the concepts of cooperative game theory, which are the focus of this paper, do not suffice to explain the development of a cooperation between originally "autonomously" acting partners. Instead, the concepts of non-cooperative game theory are necessary for this, which are not further considered in this paper. For this subject, one may refer to the groundbreaking papers of Axelrod that have significantly contributed to explain rationally the formation of alliances via experiments, like, e.g. the strategy "tit for tat" (cf. Axelrod 1984, pp. 13–14, 20, 31–54 and 170–190; Dluhosch and Horgos 2013, pp. 155–166 and 176–177; Duersch et al. 2014, pp. 25–28 and 30–34; Matsushima 2014, pp. 116–121). Furthermore, there are recent papers, that have proved to be very interesting, that show within multilevel, at least two-level games, how out of the behaviour of initially competing partners—on the first level of the game modelled by means of the concepts of non-cooperative game theory—cooperative behaviour of the partners can develop in a later, e.g. second game level as an "emergent phenomenon". For this, cf., for example, Bergantiños et al. (2000, pp. 278–280), Bergantiños and Méndez-Naya (2000, pp. 32–41) (with a non-cooperative foundation of the χ-value of cooperative game theory that is looked at closely).

Fifth, the special perspective of the author on the real problem of a gain distribution perceived as fair or just has to be emphasized. This author assumes that the "fairness" or "justice" of the distribution of a gain to the participants does not have any "objectively" ascertainable distributional characteristics. For the terms fairness and justice (here used synonymously, and in the following, will, for convenience, often be referred to simply as "fairness") cannot be empirically observed or "measured". Instead, "fairness" is an *imputation term* that can be attributed by several subjects, e.g. the partners in a cooperative activity, to a proposed distribution of a gain—but does not have to be so attributed. Thus, it depends on the subjective value judgments of the several judging subjects (partners in the cooperation), whether the distribution of a gain is perceived as fair, and thus accepted, or is rejected because of a subjectively perceived unfairness. Because of this genuinely subjective component of the real problem of a gain distribution perceived as fair, every concept needs a justification that cannot be "empirically" derived from "objective" data. The currently prevalent "main stream paradigm" of data—or evidence-based—analyses "pepped up" with econometric methods, therefore proves to be insufficient for the real problem studied here; strictly speaking, this paradigm is even completely unsuitable, epistemologically. Instead, "good reasons", with which a proposed profit distribution can be rationally justified, are necessary.

The present paper and a related publication (Zelewski and Heeb 2017) apply themselves especially to the task of reconstructing two closely related game theoretic concepts for a fair distribution of cooperation gains—the τ-value and the χ-value—in such a way that the "good reasons" referred to just now become evident. These reasons should at least prove to be sufficient for the gain distribution, that is being proposed by means of these concepts, will be perceived by all cooperation partners as fair, and thus be accepted. These good reasons, however, cannot suffice to show that both of these game theoretic concepts for a fair distribution of cooperation gains are necessary. Hence, one may concede a priori that other game theoretic concepts could also offer similar—or even better—solutions for the real problem of the distribution of cooperation gains that is perceived as fair. A comparative evaluation of alternative game theoretic concepts regarding this real problem cannot be treated in the present paper due to the brevity required here.

2 The Exposition of the Scientific Problem

2.1 Conceptual Requirements for a Fair Distribution from an Economic Point of View

As already explained in the preceding chapter, good reasons, from an economic point of view, should be adduced in order to justify rationally a game theoretic concept for the real problem of the fair distribution of cooperation gains. These good reasons can be compiled in various—and surely also controversial—ways.

This paper refrains from a meta-discourse about which good reasons, and in what kind of combination, should be used for an "appropriate" or "adequate" solution of this real problem. For this, superordinated appropriateness or adequacy criteria are necessary that have hardly been discussed in relevant economic and game theoretic literature up to now. Instead, a group of requirements will be presented, in the sense of a suggestion for a discussion, that qualify, from the point of view of the author, as good reasons for recommending a game theoretic concept as the solution of the real problem of a gain distribution perceived as fair in supply chains—or in short to recommend a solution concept.

These requirements prove to be "unorthodox" insofar as they predominantly—two exceptions present the commonly used existence and uniqueness requirements—do not correspond with those requirements that are being discussed in the prevailing game theoretic literature. However, this tension towards the established literature is being purposely created. For the author does not aim at orienting himself towards formal-mathematical requirements of game theory. Instead, he wants to give priority to the needs of business practice regarding the real problem of a profit distribution perceived as fair. These requirements from an economical perspective have already been hinted at in previous publications (cf. e.g. Zelewski 2009, pp. 22–23, 67–71; Jene 2015, pp. 122–127), but will be specified and unfolded in the following. Moreover, in Fromen (2004, pp. 153–180), a very profound notion and explanation of criteria for the assessment of the solution concepts of cooperative game theory can be found. Those assessment criteria partly overlap with the following catalogue of requirements, but partly place their focus on other, chiefly game theoretically motivated, points, such as several variants of monotonicity. Furthermore, the requirements on a game theoretic concept for the solution of the real problem of the distribution of cooperation gains perceived as fair are formulated by Jene (2015, pp. 287–297), and Müller (2015, pp. 460–462) (from a specific management accounting perspective), which are partly, but not completely, consistent with the following catalogue of requirements.

Especially from a firmly economical point of view, the following six "transconceptual", i.e. conceptually non-specific requirements, are to be met, in the estimation of the author, by every concept meant to solve the distribution problem:

1. *The requirement of rationality*: A solution concept shall be structured in such a way that the scope of action for seemingly reasonable distribution outcomes is constructed in a systematically comprehensible fashion. The starting point is, trivially, the scope of all combinatorially permissible solutions for the distribution problem. This initial scope of action is to be successively restricted by adding seemingly reasonable *postulates of rational action*. The reasonableness of each rationality postulate shall be either directly intuitively comprehensible, or be justified by good reasons. The good reasons shall refer directly to the respective concrete behavioural repertoire of the concerned partners. By means of this last requirement, those reasons are excluded as insignificant that lack this direct, concrete behavioural reference but merely justify an abstract postulate by deriving a specific game theoretic solution concept by means of this postulate. Such abstract and "teleological" postulates are often adduced as grounds of justification in formal game theory. We will come back to this later.
2. *The requirement of uniqueness*: On the one hand, no guarantee is given (such as in some sense of "pre-established harmony") that a unique distribution outcome can be achieved by means of the successive restriction of the scope of action. Past experience with such reconstructions of game theoretic solution concepts for distribution problems argues against such expectations. On the other hand, no acceptance of a concept for solving the distribution problem can be expected in business practice that does not guarantee a unique distribution outcome. For a

solution concept that reveals several distribution outcomes as "equivalent" solutions would not be perceived by the practitioners as "solutions" of the original distribution problem but merely as shifting the problem to the derivative problem of now having to choose from the multiple-element solution set exactly one distribution outcome. This attitude could be put crudely like this: "practice craves uniqueness". Consequently, it is demanded that a solution concept yield a unique distribution outcome in every situation where the concept is to be applicable.

3. *The requirement of existence*: A solution concept helps business practice only if at least one problem solution exists that can be suggested as a distribution outcome for each specific instance. This seemingly trivial requirement is, however, by no means always fulfilled. Instead, it can be shown that for each game theoretic solution concept that basically comes into question for the distribution problem, there are many specific numerical constellations to which it cannot be applied, because the formulae for the calculation of the solution (in the sense of this concept) fail for that constellation, i.e. are not correctly defined. Those numeric constellations to which a game theoretic solution concept can be correctly applied, so that at least one problem solution exists, are clearly grouped by game theory through the so-called "game classes". A solution concept is thus all the more effective the more comprehensive is the largest game class for which the solution concept can guarantee the existence of at least one problem solution. Since the extension (the number of elements) of this largest game class can be interpreted as the application range of a solution concept, this requirement of existence can be defined more precisely by the *requirement that the application range be as great as possible*.
4. *The requirement of acceptability*: This requirement is the focus of the real problem considered here: to distribute a collectively realized profit to the partners in a way that the distribution outcome is perceived as *fair* and consequently accepted by the partners. In order to fulfill this requirement, it is necessary to explicate the *good reasons* that seem to be suitable for justifying a distribution outcome as fair.
5. *The requirement of communicability*: A solution concept might basically fulfill the requirement of acceptability because good reasons for justifying the concept-induced distribution outcomes can be pointed out. But these good reasons and their argumentative context might be so complex that they can barely be conveyed ("communicated") to the partners in business practice. In this case, the fulfillment of the requirement of acceptability does not suffice for the successful application of a solution concept to business practice. Instead, the good reasons that argue for the rationality and fairness of a solution concept must be communicated as convincingly as possible. This extremely "soft" requirement can be specified only with difficulty. Nevertheless, it proves to be very important from an economic point of view. Therefore, two indicators that are far from sufficient, but are necessary, for the fulfillment of the requirement of communicability shall be stated. First, the solution concept shall be accounted for by mathematical calculation formulae that are as simple as possible (*the requirement of simplicity*), because complicated "formula conglomerates" are

only rarely accepted. Second, the overall context of the formulaic apparatus of the solution concept shall be capable of being understood as easily as possible (*the requirement of intelligibility*), because the author assumes that an issue can be communicated in an "inter-personally" convincing way if it can be "intra-personally" easily understood. The possibility of visualizing the overall context of a formulaic apparatus via a clear chart (*the requirement of visualizability*) serves as a sub-indicator for the easy intelligibility of a solution concept.

6. *The requirement of usability*: A solution concept can fail because diverse information, that is necessary for the concept's application, is neither directly available to the partners nor can be ascertained with a "practically acceptable" effort. This aspect has not yet been acknowledged in the game theoretic literature, because it is insinuated there that all necessary information is "given". This attitude is, however, often far from economic reality. Especially, it is presumed, from a game theoretic perspective, that for each imaginable coalition of the partners that participate in a cooperation, the values of the so-called characteristic function, which assigns to each coalition the success that it could realize on its own (of which the particulars later), are known. Since there are numerous imaginable coalitions for which these coalition-related values of the characteristic function are not known in business practice, and neither can be ascertained with a "practically acceptable" effort, the usability of a solution concept depends especially on whether the values of the characteristic function only need to be known for a number of coalitions that is small enough to be practicable (*the requirement of minimal coalition knowledge*).

The requirements listed above constitute a "pragmatic demand" on the way a game theoretic concept for the solution of the real problem of a gain distribution perceived as fair—according to the estimation of the author—can be justified from an economic point of view. Of course, such a *justification program* can be criticized with arguments from multiple angles. The author concedes this from the very start. His concern is not to submit a research concept that is "immune to criticism". Rather, it is to suggest a research concept that launches a, from an economic point of view, "fruitful" critical-constructive discourse about which requirements should be imposed on a game theoretic solution concept.

A few comments shall be allowed in order to, in a self-critical way, point to some weaknesses of the previously outlined justification program and to encourage the corresponding advance. First, the six requirements mentioned above—including their subordinate subsidiary requirements—prove to be mostly very vague, because they are not yet specified through precise formal linguistic phrases. Exempted from this are only the requirements of uniqueness and existence. The remaining requirements evade—at least for now—a precise specification. The author, however, does not regard this aspect as a basic disadvantage, but rather as an "invitation" to the research community to further specify those remaining requirements, in fact, to specify them formally linguistically if possible.

Second, the six requirements mentioned above are not, as is often demanded in the "classic" theories of decision and valuation, independent of each other ("orthogonal"). Rather, the requirements depend in many ways on each other. For example, the requirement of acceptability has recourse to good reasons that are at least partly subject to the requirement of rationality. Therefore, it lends itself to distinguishing between good reasons that are, on the one hand, indicative only of the rationality of a game theoretic solution concept, and, on the other hand, also to the acceptability of a distribution outcome (acceptability reasons). In addition, the aspects of simplicity, intelligibility and visualization ability that have been mentioned regarding the requirement of communicability, can be understood as good reasons for the requirement of acceptability. With regard to such dependencies between these requirements for game theoretic concepts for the solution, what should be considered in further studies is the application of multi-criterion assessment concepts that are tailored to make allowances for such dependencies between the requirements and criteria in a systematic way. Hereto belongs above all the concept of the "Analytic Network Process" (ANP).

Third, the six requirements mentioned above are not to be understood as a closed list. Rather, they shall encourage the contribution of more requirements from other perspectives that should be fulfilled from the perspective of third parties in order to deal with the real problem of a profit distribution that is perceived as fair in an economically convincing manner. Therefore, the present paper is to be understood as a kind of "invitation *ad offerendum*" that wishes to encourage an economic discourse.

2.2 An Overview of the State of the Art Concerning Fair Distributions

On the part of cooperative game theory, a wide range of concepts has been offered, that basically come into consideration for this real problem. It would go beyond the scope of this survey paper to present the abundance of these solution concepts of cooperative game theory in detail. Instead the reader is referred to, for instance, the survey papers (Binmore 2007, pp. 521–539; Peleg and Sudhölter 2007, pp. 1–199; Brânzei et al. 2008, pp. 13–42; Peters 2008, pp. 121–150 and 229–306; Gilles 2010, pp. 12–106; Narahari 2014, pp. 363–458). There, numerous solution concepts are, from an economic point of view, presented and evaluated regarding their applicability.

Due to the brevity required here, only a short qualitative survey can be given of the extent to which the solution concepts of cooperative game theory contribute to the fulfillment of the six requirements for concepts for a fair distribution adduced above from an economic point of view. This overview will confine itself to three "established" solution concepts of cooperative game theory, which are frequently mentioned in economic contexts: the Shapley value, the nucleolus, and the so-called core of a cooperative game. This list of game theoretic solution concepts could be "expanded at will", but that would go beyond the prescribed scope of the present paper.

First of all, it is to be noticed that the fulfillment of the requirements of uniqueness and existence can be precisely answered for these three solution concepts. The core proves to be "hopelessly" ambiguous, because for the gain distribution problems of cooperative game theory, there are an infinite number of solutions located, as distribution outcomes, in the core. The Shapley value and the nucleolus, in contrast, are uniquely defined and their existence is guaranteed for the class of essential games. From an economic point of view, this game class secures a "sufficient" application range for solving the distribution problem, because it encompasses all games in which cooperation is "profitable". A game is *essential* for a distribution problem if a greater profit can be made through the cooperation of all the partners (according to the characteristic function; to be explained in detail later on) than the resulting sum of profits of all partners operating individually; cf. Gilles (2010), p. 18. The Shapley value and the nucleolus exhibit, however, another grave disadvantage. Neither fulfills the requirements of rationality, acceptability, communicability and usability to a desirable extent.

The requirement of rationality is only rudimentarily, but not nearly completely, fulfilled. On the one hand, this is due to the fact that the idea, outlined above, of a successive restriction of the scope of action, is not at all further pursued. On the other hand—and this weighs far more heavily—these two solution concepts are only rudimentarily mirrored by rationality postulates that are either directly intuitively comprehensible or justified by means of good reasons that refer directly to the specific behavioural repertoire of the partners concerned. These rare explicit rationality postulates restrict themselves above all to the aspect of individual rationality. It will be subsequently investigated how such postulates of collective rationality can be precisely formulated, in the context of the τ-value. In addition, a similar point of view is used for discussing the χ-value in Zelewski and Heeb (2017).

However, it could be objected that the rationality of solution concepts like the Shapley value and the nucleolus can be justified even if they are not explicitly related to "rationality postulates", but only to important concept attributes. This justification approach is frequently to be found in the relevant game theoretic literature, especially in the form of the axiomatic foundation of a solution concept. In the case of such a foundation of the concept, a small number of axioms is predetermined and subsequently it is proven that the corresponding solution concept results as a "logical" consequence from these axioms. Regarding this axiomatic justification approach, cf., for example, Tijs (1987, pp. 179–181) (for the τ-value), Driessen (1988, pp. 70–73) (for the τ-value), Casas-Méndez et al. (2003, pp. 499–502) (for the τ-value), Gilles (2010, pp. 76–89) (for the Shapley value), Narahari (2014, pp. 385–388) (for the Shapley value), Heilmann and Wintein (2015, pp. 5–9 and 12–13).

It can be held that such axioms express the "rationality" of a game theoretic solution concept, therefore each axiom can be interpreted as a rationality postulate. The author wants to contradict this opinion. A closer look at the respective concept-specific axioms shows that these axioms—apart from a few exceptions (like, e.g. the already addressed individual rationality, provided that it is being explicitly presumed as an axiom at all)—are usually neither directly intuitively comprehensible nor can be justified via good reasons that refer to the specific behavioural scope of the partners concerned.

Instead this is a case of postulates, very abstract in appearance, that merely fulfill the purpose of deducing a game theoretic solution concept from their presupposition. The additivity axiom which has been employed to derive the Shapley value may serve as an example. These axioms are teleological presuppositions without having any directly comprehensible connection to reality.

For these reasons, the game theoretic axioms that are often quoted for the derivation of a solution concept of cooperative game theory, do not make a convincing contribution to the fulfillment of the rationality requirement in most cases.

The requirements of acceptability and communicability are not explicitly addressed in game theoretic contributions involving the Shapley value or the nucleolus. This is not surprising, because neither requirement belongs to the established game theoretic discourse, but rather are being contributed to the discussion of solution concepts of cooperative game theory by the author from a specifically economic point of view. However, it has to be taken into account that contributions concerning the fulfillment of these two requirements may exist without explicitly referring to the acceptability or communicability requirement. A corresponding examination on the part of the author of this paper has, however, not yielded any indication of this. Good reasons, that could support the acceptability of a distribution outcome beyond the rationality postulates already mentioned, cannot be found in the relevant literature. Communicability aspects, such as, e.g. the simplicity of the calculation formulae as well as the intelligibility and visualizability of the overall context of the apparatus of formulae of a solution concept, are not taken into account.

By way of example, it shall be mentioned that the Shapley value as well as the nucleolus violate the requirement of simplicity of the calculation formulae, so that the intelligibility of these solution concepts is also indirectly impaired. For the calculation of the Shapley value, arithmetic operations with the factorial operator "!" are necessary, which is hardly going to be met with comprehension in economic practice. For the nucleolus, algorithmically demanding calculations have to be performed in order to maximize the distribution outcome for the worst situated partner(s). Admittedly, this "minimax" problem has proven to be very interesting from a conceptual perspective, because it reminds one of the ethical considerations of Rawls concerning distributive justice. According to Rawls, it is, in distribution problems, an ethical imperative to rank as highly as possible those partners that are the most disadvantaged (cf. Rawls 1999). But the mathematical formulation and solution of this "minimax" problem poses considerable formal, and especially algorithmic, difficulties. This leads to the fact that the nucleolus can not establish itself in business practice as a solution concept for distribution problems.

Regarding the usability requirement, it has to be observed that it is not being discussed in papers concerning the Shapley value and the nucleolus, especially not regarding their specification with respect to minimal coalition knowledge. Instead, it is assumed in the game theoretic literature that the values of the characteristic function are known for all imaginable coalitions. Practical problems of information acquisition regarding these values are ignored.

Based on the preceding deliberations, merely roughly outlined, concerning the state of the art, one can now record the scientific problem to be treated in this paper: for the real problem of distributing a collectively realized gain to the partners in a way that the distribution outcome is perceived as fair by those partners and is consequently accepted, a solution concept is needed that fulfills more than just the requirements of uniqueness and existence, which have already been comprehensively taken into account in the state of the art of game theory.

Rather, it is also necessary for the solution concept to meet two challenges. First, the requirement of rationality, of the successive restriction of the scope of the action, for seemingly reasonable distribution outcomes, has to be fulfilled more comprehensively than is currently the case in the relevant game theoretic literature. Second, this solution concept should render substantial contributions to fulfilling the requirements of acceptability, communicability, and usability, whose discussion has, so far, been completely insufficient. Moreover, it is to be mentioned as desirable regarding the requirement of existence that a game theoretic solution concept should have an application range as great as possible. This application range is to be operationalized through the idea of the largest game class for which the solution concept can guarantee at least one solution for each instance of the distribution problem.

3 The Compromise Values of Cooperative Game Theory

3.1 Overview

For this real problem, of a benefit distribution perceived as fair in supply chains, and the scientific problem derived from it, principally a wide range of solution concepts of cooperative game theory come into the picture. This conceptual diversity cannot be exhausted here, for reasons of space. That's why the treatment will focus on two aspects

First, there will be a concentration on those solution concepts of cooperative game theory that belong to the so-called class of compromise values (*compromise desideratum*). They are characterized by the fact that a distribution outcome is being proposed that represents a compromise between two extremes. These two *extreme values* constitute either a lower or an upper bound for a seemingly reasonable distribution of the profit to every partner. If these extreme values can be justified by good reasons, they provide an interesting foundation for the scientific problem, especially with regard to the requirement of rationality, since such extreme values are suited for limiting the scope of action for seemingly reasonable distribution outcomes through rationality postulates to the top and to the bottom in an inter-subjectively comprehensible way. However, the solution concepts of cooperative game theory that belong to the class of compromise values vary greatly. Therefore, another selection criterion it needed regarding this class of concepts.

Second, those solution concepts of cooperative game theory are preferred that satisfy—in a way to be more fully explained regarding their calculation basis—the intuitive notion that fairness and justice are closely connected with the concept of proportionality (the *proportionality desideratum*). Regarding this close association between fairness and justice on the one side as well as the proportionality desideratum on the other, cf., e.g. Heilmann and Wintein (2015, pp. 2–9, 12–13 and 17–19); besides Díaz et al. (2005, p. 485).

Two solution concepts of cooperative game theory that fulfill the compromise as well as the proportionality desideratum, are the τ-value and the χ-value. Both values admittedly lead a niche existence in the relevant literature that deals with the application of the solution concepts of cooperative game theory to economic real problems—like, e.g. the distribution problem treated in the present paper. Apart from a few publications that originated in the surroundings of the author of this paper concerning the τ- and the χ-values (cf., for example, Zelewski and Peters 2010, pp. 5–18 and 21–24; 2012, pp. 151–165; Zelewski 2009, pp. 91–271; Jene and Zelewski 2011a, pp. 321–335; 2011b, pp. 303–313; 2011c, pp. 118–130; 2012, pp. 171–184; 2013, pp. 31–36 and 41–46; Zelewski and Jene 2011, pp. 3–18; Jene 2015, pp. 128–317), the τ-value is only rarely mentioned in *economic* publications. Regarding the few exceptions to this, cf. Fromen (2004, pp. 129–132 and 187–188), Müller (2015, pp. 469–470 and 477–478), Mueller (2016, pp. 205–206). The χ-value, which is based conceptually on the τ-value, has not yet been elaborated outside the purely game theoretic literature, as far as the knowledge of the author extends.

Nevertheless, to the author it seems to be a promising endeavour to deal more intensively with the solution concepts of the τ-value and the χ-value for the scientific problem treated in the present paper, for they provide interesting approaches to the better fulfillment of these requirements, rationality, acceptability, communicability and usability, than has been the case up to now on the part of the "established" game theoretic solution concepts, like, e.g. the Shapley value or the nucleolus. In addition to this, in a recently published scientific paper, the τ-value has been especially acknowledged from a philosophical perspective regarding its suitability for the fair solution of distribution problems; cf. Heilmann and Wintein (2015, pp. 16–19, especially pp. 17 and 19). Having this flank philosophically protected encourages the author to focus on the two solution concepts of the τ- and the χ-value (Zelewski and Heeb 2017).

3.2 The τ-Value as a Basis for Argumentation

3.2.1 Basics

The formal linguistic elaborations concerning the specification of the τ-value will be very brief in what follows, because they are mostly the recapitulation of known facts. For game theoretic presentations of the τ-value, cf., for example, Tijs (1981, pp. 127–131, 1987, pp. 177–181), Driessen and Tijs (1982, pp. 397–406, 1984, pp. 252–260, 1985, pp. 230–246, 1990, pp. 7–14), Tijs and Driessen (1983, pp. 10–17, 1986, pp. 1020–1026, 1987, pp. 150–155), Driessen (1985, pp. 27–67, 1986, pp. 226–228, 1987, pp. 209–213, 1988, pp. 57–110 and 200–202), Borm et al. (1992, pp. 179–180 and 186–189), Tijs and Otten (1993, pp. 3–9 and 18–20), Bergantiños and Méndez-Naya (2000, pp. 31–33), Sánchez-Soriano (2000, pp. 471–473), Brânzei and Tijs (2001, pp. 4–16), Bilbao et al. (2002, pp. 71–77), Núñez and Rafels (2002, pp. 411–417), Otten and Peters (2002, pp. 199–202), Brânzei et al. (2003, pp. 8–12 and 14–15, 2008, pp. 31–33), Casas-Méndez et al. (2003, pp. 495–512), Díaz et al. (2005, pp. 483–485), Timmer (2006, pp. 96 and 100); besides Zelewski (2009, pp. 91–271) (with additional references on pp. 91–92); Jene (2015, pp. 205–286).

Instead, greater value is being placed here on the reconstruction of the solution concept of the τ-value in such a way that the justification program is satisfied in terms of the six characteristic requirements which should be fulfilled by a solution concept. For reasons of space, this reconstruction will not be elaborated in all its details. For a more detailed elaboration, cf. Zelewski (2009, pp. 91–213).

From the perspective of cooperative game theory, the *cooperation gain distribution problem* consists of the task of distributing a collectively realized gain $G \in \mathbb{R}_{>0}$ (here, $\mathbb{R}_{>0}$ denotes the set of positive real numbers) to the N partners A_n of a supply chain with $n \in \{1, \ldots, N\}$, $N \in \mathbb{N}_+$ and $N \geq 2$, in a way that the distribution outcome is perceived as fair and consequently accepted by every member of the set of partners $A = \{A_1, \ldots, A_N\}$.

For the solution of this problem, a *distribution function* v is to be established that assigns to every partner A_n that partner's share v_n in the gain G to be distributed. If it is additionally assumed that the distribution function v is not allowed to assign an partner a negative amount as the share in the gain, the distribution function v can be specified as follows: $v : A \to \mathbb{R}_{\geq 0}$ with $A_n \mapsto v(A_n) = v_n$ for every partner A_n (here, $\mathbb{R}_{\geq 0}$ is the set of all non-negative real numbers).

A *solution* is any N-tuple (row vector) $\underline{v}$ with $\underline{v} = (v_1, \ldots, v_N)$ that assigns to every partner A_n from the set of partners A that partner's share v_n in the gain G. The *solution point* S_v corresponds to this N-tuple in the N-dimensional solution space $\mathbb{R}^N_{\geq 0}$. It can be represented as an N-dimensional column vector $S_v = \overrightarrow{v} = (v_1, \ldots, v_N)^T$. Here, the symbol "^{T}" denotes the transposed manner of representation of a vector. For cooperative game theory, a two-step procedure is characteristic for determining such a solution.

In the *first step*, a *characteristic function c* is considered. It represents the "complete" knowledge that is known about the action possibilities of the partners A_n for the real problem considered (or that is being supposed as known). The characteristic function c is thus not fitted to one particular game theoretic solution concept. Instead, merely the knowledge about the partners and their action possibilities is being described that is needed for a problem solution in the sense of cooperative game theory independently of the specific solution concept being applied. The characteristic function c always refers to the entirety of all *coalitions* which the partners of the regarded supply chain can form. As a borderline case, also degenerate coalitions are allowed, that contain exactly one partner. A *coalition* C_m is thus any non-empty subset of the set of all partners A, i.e. $\varnothing \subset C_m \subseteq A$.

For a characteristic function c, one has $c : \varphi(A) \to \mathbb{R}$ with $C_m \mapsto c(C_m)$ for each coalition C_m (here, φ denotes the operation of taking the power set). For the borderline case of $\varnothing$, i.e. the empty set, $\varnothing \mapsto c(\varnothing) = 0$. The characteristic function c assigns to every coalition C_m of the partners that amount $c(C_m)$ which this coalition can claim for itself reasonably, i.e. "with good reasons". In case of the grand coalition $C_0 = A = \{A_1, \ldots, A_N\}$, this amount is simply the collectively realized gain G that is to be distributed: $c(C_0) = G$. For any other coalition C_m with $\varnothing \subset C_m \subset A$, it is the amount $c(C_m)$ that this coalition could realize on its own outside the grand coalition—and thus in competition with the residual coalition $C_0 \setminus C_m$.

In the *second step* the specific shape of the distribution function v is to be determined by calculating for each partner A_n from the set of partners A its value $v(A_n) = v_n$. The calculation v_n is to draw on "only" two sources of information. One is the amount which each coalition can claim for itself based on the characteristic function c from the first step. The other is the corresponding applied solution concept, which decides how the values of the distribution function v are to be calculated from the values $c(C_m)$ of the characteristic function c for all coalitions C_m, where $m = 0, 1, \ldots, 2^N-2$. As soon as all values v_n of the distribution function have been calculated, a solution $\underline{v}$ for this instance is available, as the N-tuple $\underline{v} = (v_1, \ldots, v_N)$.

3.2.2 A Requirement-Based Reconstruction of the τ-Value

The basic idea of the reconstruction of the solution concept of the τ-value consists in—as already mentioned—gradually restricting the solution space for the distribution problem by successively adding five requirements that are guided by the real problem of the fair distribution of a cooperation gain. These 5 *concept-specific requirements* are to be strictly distinguished from the previously presented 6 trans-conceptual or concept unspecific requirements, which have been established in order to evaluate the quality of a game theoretic solution concept from an economic point of view. These 5 other requirements, which will prove to be characteristic for the solution concept of the τ-value, are primarily about the postulates of rational action. However, they do not suffice to guarantee the existence of the τ-value. Therefore, another, sixth, concept-specific requirement has to be added (Zelewski and Heeb 2017, p. 379).

In order to have consistent terms for the different concept-specific requirements of the τ-value and at the same time to have set them apart from the previously presented concept-unspecific requirements for assessing the quality of a solution concept, the concept-specific requirements of the τ-value will from now on be referred to as *conditions* that have to be fulfilled in accordance with the τ-value.

At the end of the following line of reasoning, the τ-value will emerge as a "reasonable" solution. For this problem solution, good reasons can be adduced for perceiving the distribution outcome of the problem solution as fair and consequently for accepting it.

The *first condition* for a solution concept is the *condition of individual rationality*. This condition assumes that every partner in a supply chain acts rationally in the sense of the conventional concept of perfect rationality (*homo oeconomicus*). This means in particular that each partner maximizes his individual utility—measured by the share in profits that he receives. Furthermore, each partner acts without envy effects, i.e. his own sense of utility does not depend on the profit shares received by the other partners. Finally, it is being assumed that no partner is hindered from calculating his individual maximum utility by restrictions of his information processing capacity.

From the condition of individual rationality follows the first restriction of the solution space $\mathbb{R}^N_{\geq 0}$ for the problem. For a single partner it would not be rational to cooperate with the rest of the partners of a supply chain in a grand coalition C_0 if he would realize a smaller individual utility within this coalition than if he were to leave ("defect") the cooperation. If leaving the coalition, the partner A_n could realize the amount $c(\{A_n\})$ outside of the supply chain by his own efforts. Therefore, the condition of individual rationality can be expressed by means of the characteristic function c for every solution point S_v as follows:

$$\forall S_v \in \mathbb{R}^N_{\geq 0}: \quad S_v = \begin{pmatrix} v_1 \\ \dots \\ v_N \end{pmatrix} \geq \begin{pmatrix} c(\{A_1\}) \\ \dots \\ c(\{A_N\}) \end{pmatrix} \tag{1}$$

As the *second condition*, the *efficiency condition* will now be formulated. This condition postulates that the collectively realized cooperation gain G is being entirely distributed to all partners of the grand coalition $C_0 = \{A_1, \dots, A_N\}$. Therefore, for each solution point S_v,

$$\forall S_v \in \mathbb{R}^N_{\geq 0}: \quad S_v = \begin{pmatrix} v_1 \\ \dots \\ v_N \end{pmatrix} \rightarrow \sum_{n=1}^{N} v_n = G \tag{2}$$

The efficiency condition can be also characterized as the requirement of *Pareto efficiency* with respect to the grand coalition C_0. This can be easily shown: If, on the one hand, the condition $\sum_{n=1}^{N} v_n < G$ is fulfilled, this would be a Pareto inferior or "suboptimal" solution, since an alternative cooperation gain distribution could be proposed where each partner would receive at least the amount v_n—i.e. would be just as well situated as in the considered solution, but at least one partner would be better situated by additionally receiving the positive differential amount $G - \sum_{n=1}^{N} v_n > 0$. Consequently, the partners of the grand coalition have no reason to hold onto such a Pareto inferior solution. On the other hand, a solution of the cooperation gain distribution problem would be invalid if $\sum_{n=1}^{N} v_n > G$. For there cannot be more distributed to the partners of the grand coalition than is available from the collectively realized cooperation gain G. Consequently, the solution that is defined by the above mentioned efficiency condition remains as the only valid and at once Pareto optimal solution for the problem.

The formulation of the efficiency condition means a further restriction of the solution space for the problem, since all solutions that fulfill the efficiency condition are solution points S_v on a hyperplane H in the N-dimensional solution space $\mathbb{R}^N_{\geq 0}$. So every point on that hyperplane is a solution point and represents a solution to the problem that fulfills the efficiency condition. This hyperplane H is the set of all solutions $\underline{v} = (v_1, \ldots, v_N)$ that fulfill the restriction $\sum_{n=1}^{N} v_n = G$.

Solutions that fulfill the condition of individual rationality as well as the efficiency condition are also called *imputations*. These two conditions are regarded as basic requirements of a "reasonable" solution concept for a distribution problem throughout the cooperative game theory literature, because they seem to be directly comprehensible; cf., e.g. Gilles (2010), pp. 18–19. Every solution of this problem should belong to the set of imputations, which restricts considerably the solution space $\mathbb{R}^N_{\geq 0}$.

Up to this point, the solution concept of the τ-value turns out to be conventional, i.e. it does not (yet) deviate from the other solution concepts of cooperative game theory. This does not, however, apply to the subsequently adduced requirements through which the solution concept of the τ-value increasingly differentiates itself from other solution concepts. The third condition, which follows, is, in itself, shared by the solution concept of the τ-value with a great number of alternative solution concepts.

As the *third condition* for a solution concept, a *rationality condition* for *maximally allocable profit shares* will be posited. This condition has the character of a *condition of collective rationality*. It mirrors the rational consideration of all $N - 1$ partners of the so-called *marginal coalition* MC_n with $MC_n = C_0 \setminus \{A_n\} = \{A_1, \ldots, A_{n-1}, A_{n+1}, \ldots, A_N\}$, to grant partner A_n at most the share $v_{n.max}$ in the profit G that is to be distributed, by which this cooperation gain G would decrease if partner A_n left the grand coalition $C_0 = \{A_1, \ldots, A_N\}$. For this rationality condition,

$$\begin{aligned} &\forall n = 1, \ldots, N \;\; \forall v_n \in \mathbb{R}_{\geq 0} : v_n \leq v_{n.max} \\ \wedge \quad & v_{n.max} = c(C_0) - c(MC_n) = G - c(MC_n) \end{aligned} \tag{3}$$

In the solution space $\mathbb{R}^N_{\geq 0}$, the solution point at which each partner A_n receives his maximally allocable share in the profit $v_{n.max}$ is called the *upper bound* (*UB*) or the *utopia point* (or *ideal point*) for the distribution of the cooperation gain G.

The *fourth condition* for a solution concept involves a *rationality condition* for *minimally allocable profit shares*. This condition also has the character of a *collective rationality condition*. It reflects the rational consideration of all $N - 1$ partners of the marginal coalition MC_n to grant partner A_n at least the share in the profit G with which he could credibly threaten to found at least one so-called *outsider coalition*, $OC_{n.q}$. An outsider coalition is a coalition $OC_{n.q}$ of former partners of a supply chain which "defect", i.e. terminate their cooperation in the grand coalition $C_0 = \{A_1, \ldots, A_N\}$—at least hypothetically—and to which at least the partner A_n belongs, as the "leader" of the outsider coalition. Since the partner A_n can lead several outsider coalitions $OC_{n.q}$, the second index q with $q = 1, \ldots, Q_n$ is used in order to distinguish these outsider coalitions. It is always true that $Q_n \geq 1$, because an partner A_n can always threaten with at least the "degenerate" outsider coalition $OC_{n.1} = \{A_n\}$, which complies with the above mentioned condition of individual rationality. Here, $IN_{n.q}$ denotes the set of all indices of partners that belong to an outsider coalition $OC_{n.q}$. Lastly, it is to be taken into account that an outsider coalition can never contain all the partners of the grand coalition C_0. Otherwise, no non-empty *residual coalition* $RC_{n.q} = C_0 \setminus OC_{n.q}$ would exist, whose partners could realize a cooperation gain G that is to be distributed in the reduced supply chain. Consequently, no distribution problem would exist.

Central to the solution concept of the τ-value is the question, with what kind of outsider coalitions $OC_{n.q}$ can a partner A_n *credibly* threaten. First, it is assumed that it is known what amount $c(OC_{n.q})$ an outsider coalition $OC_{n.q}$ can attain on its own outside of the supply chain (i.e. outside the grand coalition C_0). Then it is assumed that the partner A_n who leads an outsider coalition $OC_{n.q}$ possessing several partners offers all the other partners an optimal incentive to defect, which the other partners cannot reject in the case of rational behaviour. The partner A_n offers the other partners of an outsider coalition $OC_{n.q}$ side payments as good as they would receive in the *best case* if they stayed in the grand coalition C_0. These side payments to the other partners A_m are already, from the third requirement, known to be the components $v_{m.max}$ of the utopia point. In this case, the other partners of an outsider coalition $OC_{n.q}$ have no reason to remain in the grand coalition C_0. For through their defection to the outsider coalition $OC_{n.q}$, they can never fare worse than if staying in the grand coalition C_0. However, in numerous situations, they are better situated than in the grand coalition C_0. The betterment occurs always when the partners would not receive the profit share in the grand coalition that is the best possible for them, but a smaller amount instead.

The amount $c(\{A_n\}|OC_{n.q})$ represents the *threat potential* of the partner A_n. It is that amount which the partner A_n could believably achieve through the foundation of an outsider coalition $OC_{n.q}$ that is led by him. With the help of the above mentioned side payments to all the other partners within the outsider coalition, this threat potential can be operationalized in a first approach, that ignores the complications that will be discussed later, as follows:

$$\forall \varnothing \subset OC_{n.q} \subset A : \{A_n\} \subset OC_{n.q} \rightarrow \ldots$$
$$c(\{A_n\}|OC_{n.q}) = c(OC_{n.q}) - \sum_{m \in (IN_{n.q} \setminus \{n\})} v_{m.max} \tag{4}$$

Through the condition $\{A_n\} \subset OC_{n.q}$ it is secured that the partner A_n belongs to the outsider coalition $OC_{n.q}$ as their leader, so that $A_n \in OC_{n.q}$ also. However, the borderline case that the degenerate outsider coalition $OC_{n.1} = \{A_n\}$ only consists of the partner A_n has not yet been taken into account. In this case, no side payments have to be made to the "other" partners. Instead, this borderline case is consistent with the already presented condition of individual rationality, so that this condition can be understood as a borderline case of the rationality condition regarded here, for minimally allocable profit shares.

This overlap in the content of two different requirements for a solution concept can be handled in three ways. First, it is possible to additionally demand for each outsider coalition $OC_{n.q} : |OC_{n.q}| \geq 2$. Second, the condition of individual rationality according to formula (1) can be abstained from, because it is included in the rationality condition for minimally allocable profit shares. Third, one can unalteredly continue to work with the conditions of individual rationality and the rationality condition for minimally allocable profit shares if an overlap in content, i.e. a partial redundancy of the conditions, is acceptable. The author has decided in favour of the last alternative, because it allows a complete specification of the two concerned conditions.

Another complication occurs because of the fact that the amounts $c(\{A_n\}|OC_{n.q})$ with which a partner A_n can threaten to form an outsider coalition $OC_{n.q}$ can also be *negative*. This can be, on the one hand, because the sum $\sum_{m \in (IN_{n.q} \setminus \{n\})} v_{m.max}$ of the side payments to the other defecting partners is greater than the amount $c(OC_{n.q})$ that the outsider coalition $OC_{n.q}$ can attain. In that case the leading partner A_n would have to add the partial amount $\sum_{m \in (IN_{n.q} \setminus \{n\})} v_{m.max} - c(OC_{n.q})$ of the side payments to the other partners from "savings", borrowing money, or other sources. On the other hand, if the outsider coalition $OC_{n.q}$ contains only the one partner A_n and the above mentioned side payments are not necessary, the amount $c(\{A_n\})$ that the partner can realize by his own efforts might be negative. This is possible, e.g. if it would not be competitively viable for the partner A_n without being embedded in a supply chain, and consequently would incur a loss in the market. In both previously mentioned cases with $c(\{A_n\}|OC_{n.q}) < 0$, a threat of partner A_n to form an outsider coalition $OC_{n.q}$ would not be believable. Therefore, these cases must be excluded from the rationality condition for minimally allocable profit shares.

In consideration of the exclusion of not believable threats and the borderline case of degenerate outsider coalitions that consist of only one partner, the complete rationality condition for minimally allocable profit shares from the fourth requirement for a solution concept is

$$\forall\, n = 1, .., N \;\; \forall v_n \in \mathbb{R}_{\geq 0} : v_n \geq v_{n.min} \wedge v_{n.min} = \max\{d_{n.1}, d_{n.2}, 0\}$$
$$\textit{with}$$
$$d_{n.1} = \max \left\{ \begin{matrix} c(\{A_n\}|OC_{n.q}) = c(OC_{n.q}) - \sum_{m \in (IN_{n.q} \setminus \{n\})} v_{m.max} | \ldots \\ q = 1, \ldots, Q_n \wedge \varnothing \subset OC_{n.q} \subset A \wedge \{A_n\} \subset OC_{n.q} \end{matrix} \right\} \tag{5}$$
$$d_{n.2} = c(\{A_n\}|OC_{n.q}) = c(\{A_n\}) \text{ for } OC_{n.q} = \{A_n\}$$

The *lower bound LB* for the distribution of the cooperation gain G is that point in the solution space $\mathbb{R}^N_{\geq 0}$ at which each partner A_n is assigned his minimally allocable share in the profit $v_{n.min}$. This lower bound is also called the *threat point* (or *disagreement point*), because it combines in its components $v_{n.min}$ the *believable threat potentials* $c(\{A_n\}|OC_{n.q})$ of all partners A_n to leave the grand coalition C_0 and to form at least one outsider coalition $OC_{n.q}$ each.

Up to this point in the argument, the solution space for the problem has only been restricted by means of rationality postulates that have each been justified with good reasons. This concerns the conditions of individual and collective rationality as well as the efficiency condition of Pareto optimality that will be met by common consent—at least prima facie—because of their “reasonableness”. Later on, one exception will be elaborated on Zelewski and Heeb (2017, pp. 407–429).

The *fifth condition* for a solution concept consists in bringing the not yet considered *fairness aspect* into effect. This happens by generally demanding, for every problem solution, that it can be perceived as fair and thus is accepted by the cooperation partners concerned (in the following, referred to as a “fair problem solution”, for short); that it fulfills the previously discussed compromise and proportionality desiderata. To this end, a twofold operationalization of these two desiderata is being added, from the specific perspective of the τ-value. First, it is demanded by the compromise desideratum that every fair solution of the problem is a compromise between the maximum profit shares $v_{n.max}$ of the partners A_n at the utopia point UB and the minimum profit shares $v_{n.min}$ of the partners A_n at the threat point LB. Second, it is demanded by the intelligibility of the solution concept, mentioned at the beginning, that this compromise is expressed in a way—depending on the perspective—that is as simple or elegant as possible.

The simplest or most elegant combination of two extreme values is, from the perspective of the author, a convex linear combination. It ensures that the profit shares v_n suggested for the partners A_n behave *proportionally* to the partner-specific components $v_{n.max}$ of the utopia point UB and $v_{n.min}$ of the threat point LB. The coefficient γ will serve as the weight factor; it is uniquely defined by the system of formulas of the τ-value. Both previously mentioned operationalizations of the fairness aspect can be summarized with the following fairness condition from a game theoretic point of view:

$$\exists \gamma \in \mathbb{R} \;\; \forall n = 1, \ldots, N : \quad v_n = \gamma \cdot v_{n.max} + (1 - \gamma) \cdot v_{n.min} \wedge 0 \leq \gamma \leq 1 \tag{6}$$

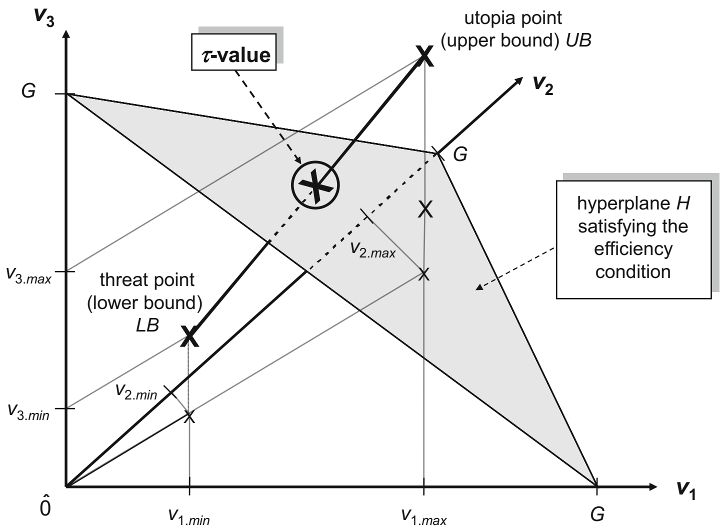

Fig. 1 Visualization of the position of the τ-value in the solution space. Own representation

The characteristic compromise and proportionality desiderata, that the τ-value fulfills because of the preceding formula (6), can be visualized in an directly intuitive and catchy manner with Fig. 1.

4 Conclusion and Outlook

In this paper, a "programmatic approach" has been outlined, which shows how a game theoretic concept for the solution of the real problem of the distribution of the cooperation gain that will be perceived as fair can be justified from an economic point of view. This justification program has been specified by means of 6 trans-conceptual requirements that should be met by every concept of cooperative game theory for solving the distribution problem.

In addition, it has been illustrated, by means of a reconstruction of the τ-value, how this justification program can be "filled with life". In this first, still rough and incomplete reconstruction of the τ-value, the fulfillment of the "condition of rationality" has been stressed by successively restricting the scope of action for seemingly reasonable distribution outcomes by adding seemingly reasonable postulates of rational action.

This first step to the reconstruction of the τ-value, however, does not yet suffice, because only one of the above mentioned 6 trans-conceptual conditions was explicitly treated. The necessary analysis is realized by Zelewski and Heeb (2017). It will become apparent that the τ-value is faced with a great challenge regarding the fulfillment of the condition of existence. Since the τ-value does not satisfactorily fulfill the corresponding requirement that a game theoretic solution concept should have an application range as great as possible, it will be elaborated to the closely related, but still barely known, χ-value as a promising alternative.

References

Axelrod R (1984) The evolution of cooperation. Basic Books, New York

Bergantiños G, Casas-Méndez B, Vázquez-Brage M (2000) A non-cooperative bargaining procedure generalising the Kalai-Smorodinsky bargaining solution to NTU games. IGTR 2(4):273–286

Bergantiños G, Méndez-Naya L (2000) Implementation of the τ-value. Top 8(1):31–41

Bilbao JM, Jiménez-Losada A, Lebrón E, Tijs SH (2002) The τ-value for games on Matroids. Top 10(1):67–81

Binmore K (2007) Playing for real – a text on game theory. Oxford University Press, Oxford

Borm P, Keiding H, McLean RP, Oortwijn S, Tijs SH (1992) The compromise value for NTU-games. IJGT 21(2):175–189

Brânzei R, Tijs SH (2001) Additivity regions for solutions in cooperative game theory. Discussion Paper No. 2001-81, Tilburg University

Brânzei R, Solymosi T, Tijs SH (2003) Type monotonic allocation schemes for multi-glove games. Discussion Paper No. 2003-34, Tilburg University

Brânzei R, Dimitrov, D, Tijs SH (2008) Models in cooperative game theory, 2nd edn. Springer, Berlin

Casas-Méndez B, García-Jurado I, van den Nouweland A, Vázquez-Brage M (2003) An extension of the τ-value to games with coalition structures. EJOR 148(3):494–513

Díaz JG, Borm P, Hendrickx R, Quant M (2005) A geometric characterisation of the compromise value. MMOR 61(3):483–500

Dluhosch B, Horgos D (2013) (When) Does tit-for-tat diplomacy in trade policy pay off? World Econ 36(2):155–179

Duersch P, Oechssler J, Schipper BC (2014) When is tit-for-tat unbeatable? IJGT 43(1):25–36

Driessen TSH (1985) Contributions to the theory of cooperative games: the τ-value and k-convex games. Ph.D. Dissertation, University of Nijmegen

Driessen TSH (1986) Solution concepts of k-Convex n-Person games. IJGT 15(4):201–229

Driessen TSH (1987) The τ-value: a survey. In: Peters HJM, Vrieze OJ (eds) Surveys in game theory and related topics. CWI Tract 39. Centrum voor Wiskunde an Informatica, Amsterdam, pp 209–213

Driessen TSH (1988) Cooperative games, solutions and applications. Kluwer Academic Publishers, Dordrecht

Driessen TSH, Tijs SH (1982) The τ-Value, the nucleolus and the core for a subclass of games. In: Loeffel H, Stähly P (eds) Methods of operations research, vol XLVI. Athenäum, Königstein, pp 395–406

Driessen TSH, Tijs SH (1984) Extensions and modifications of the τ-value for cooperative games. In: Hammer G, Pallaschke D (eds) Selected topics in operations research and mathematical economics. Springer, Berlin, pp 252–261

Driessen TSH, Tijs SH (1985) The τ-value, the core and semiconvex games. IJGT 14(4):229–247

Driessen TSH, Tijs SH (1990) The core and the τ-value for cooperative games with coalition structures. Memorandum No. 848. Faculty of Applied Mathematics, University of Twente

Fromen B (2004) Faire Aufteilung in Unternehmensnetzwerken – Lösungsvorschläge auf der Basis der kooperativen Spieltheorie. Deutscher Universitäts-Verlag, Wiesbaden

Gilles, RP (2010) The cooperative game theory of networks and hierarchies. Springer, Berlin

Heilmann C, Wintein S (2015) How to be fairer. In: Synthese. doi 10.1007/s11229-015-0967-y
Jene S (2015) Die faire Verteilung von Effizienzgewinnen in Kooperationen – Eine kritische Analyse der Eignung des τ-Werts und des χ-Werts. Springer Gabler, Wiesbaden
Jene S, Zelewski S (2011a) Fair distribution of added values in networks of autonomous actors. In: Jodlbauer H, Olhager J, Schonberger RJ (eds) Modelling value: pre-proceedings, vol II. Shaker, Aachen, pp 317–337
Jene S, Zelewski S (2011b) Fair distribution of profit in supply chains. In: Kersten W, Blecker T, Jahn C (eds) International supply chain management and collaboration practices. Eul, Lohmar, pp 299–313
Jene S, Zelewski S (2011c) Distributive justice in supply chains – fair distribution of collectively earned profits in supply chains. In: Sucky E, Asdecker B, Dobhan A, Haas S, Wiese J (eds) Logistikmanagement – Herausforderungen, Chancen und Lösungen, vol III. University of Bamberg Press, Bamberg, pp 115–132
Jene S, Zelewski S (2012) Fair distribution of added values in networks of autonomous actors. In: Jodlbauer H, Olhager J, Schonberger RJ (eds) Modelling value: selected papers of the 1st international conference on value change management. Springer, Berlin, pp 167–186
Jene S, Zelewski S (2013) Practical application of cooperative solution concepts for distribution problems: an analysis of selected game theoretic solution concepts from an economic point of view. In: Meunier F, Carre F (eds) Game theory – economics, theoretical concepts and finance applications. Nova Science, New York, pp 25–49
Matsushima H (2014) Interlinkage and generous tit-for-tat strategy. JapER 65(1):116–121
Müller D (2015) Erweiterung des Instrumentariums des Kooperationscontrollings durch Lösungskonzepte der kooperativen Spieltheorie – eine Analyse. BFuP 67(4):457–481
Mueller D (2016) What's in it for me? – An Analysis of environmental investments through a game theoretic lens. Int J Innov Sustain Dev 10(2):198–218
Narahari, Y (2014) Game theory and mechanism design. IISc Press World Scientific, New Jersey
Núñez, M, Rafels C (2002) The assignment game: the τ-value. IJGT 31(3), 411–422
Otten G-J, Peters H (2002) The Shapley transfer procedure for NTU-games. In: Borm P, Peters, HJ (eds) Chapters in game theory – in honor of Stef Tijs. Kluwer Academic Publishers, Boston, pp 183–203
Peleg B, Sudhölter P (2007) Introduction to the theory of cooperative games, 2nd edn. Springer, Berlin
Peters H (2008) Game theory – a multi-leveled approach. Springer, Berlin
Rawls J (1999) A theory of justice, 2nd edn. The Belknap Press, Cambridge
Sánchez-Soriano J (2000) A note on compromise values. MMOR 51(3):471–478
Tijs SH (1981) Bounds for the core and the τ-value. In: Moeschlin O, Pallaschke D (eds) Game theory and mathematical economics. North-Holland Publisher, Amsterdam, pp 123–132
Tijs SH (1987) An axiomatization of the tau-value. MSS 13(2):177–181
Tijs SH, Driessen TSH (1983) The τ-value as a feasible compromise between utopia and disagreement. Report 8312. University of Nijmegen: Department of Mathematics
Tijs SH, Driessen TSH (1986) Game theory and cost allocation problems. MS 32(8):1015–1028
Tijs SH, Driessen TSH (1987) The tau-value as a feasible compromise between utopia and disagreement. In: Paelinck JHP, Vossen PH (eds) Axiomatics and pragmatics of conflict analysis. Gower, Aldershof, pp 142–156
Tijs SH, Otten G-J (1993) Compromise values in cooperative game theory. Top 1(1):1–36
Timmer J (2006) The compromise value for cooperative games with random payoffs. MMOR 64(1):95–106
Zelewski S (2009) Faire Verteilung von Effizienzgewinnen in Supply Webs – ein spieltheoretischer Ansatz auf der Basis des τ-Werts. Logos, Berlin
Zelewski S, Heeb T (2017) Characteristics of the τ-value and the χ-value. In: Mueller D, Trost R (eds) Game theory in management accounting – implementing incentives and fairness. Springer, Berlin, pp 407–429
Zelewski S, Jene S (2011) Fairness: a challenge for the distribution of cooperation gains in value chains. Int J Manag Value Supply Chains 2(1):1–20

Zelewski S, Peters ML (2010) Fair distribution of efficiency gains in supply networks from a cooperative game theory point of view. Int J Inf Syst Supply Chain Manag 3(2):1–24
Zelewski S, Peters ML (2012) Fair distribution of efficiency gains in supply networks from a cooperative game theory point of view. In: Wang J (ed) Information technologies, methods, and techniques of supply chain management. IGI Global, Hershey, pp 149–171

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The Pre-Kernel as a Fair Division Rule for Some Cooperative Game Models

Holger I. Meinhardt

Abstract Rather than considering fairness as some private property or a subjective feeling of an individual, we study fairness on a set of principles (axioms) which describes the pre-kernel. Apart from its appealing axiomatic foundation, the pre-kernel also qualifies in accordance with the recent findings of Meinhardt (The pre-kernel as a tractable solution for cooperative games: an exercise in algorithmic game theory. Springer, Berlin, 2013b) as an attractive fair division rule due to its ease of computation by solving iteratively systems of linear equations. To advance our understanding of compliance on non-binding agreements, we start our analysis with a Cournot situation to derive four cooperative game models well introduced in the literature, where each of it represents different aspiration levels of partners involved in a negotiation process of splitting the monopoly proceeds. In this respect, we demonstrate the bargaining difficulties that might arise when agents are not acting self-constraint, and what consequences this impose on the stability of a fair agreement.

Keywords Convex analysis • Convex games • Fenchel-Moreau conjugation • Indirect function • Pre-kernel • Stable agreement

Contents

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1 Introduction

Involved in negotiation people often have strong concern on the execution and stability of an agreed-upon contract. Indeed, there are also agreements where implementing is not problematic, because technical conditions or adequate institutions guarantee the fulfilment of the contract. Though we focus on non-binding agreements, simply referred to as agreements, so that the issue of compliance becomes crucial. This question immediately leads the focus on how arguments of power and fairness can be based on the structure of the game, and how fairness and stability properties of a negotiated (non-binding) agreement can be judged.

In particular, in areas where multilateral agreements are non-binding, compliance becomes a main issue in order to avoid its obstruction. The execution of an agreement can be achieved when agents exercise self-constraint and refrain from using their powers to exploit one another. Then a solution can be obtained that is acceptable for all participants: a fair compromise. Such a compromise will be considered a fair outcome when it produces a common virtual world where compliance is reality and obstruction is held to account. An often expressed hypothesis on compliance is that people reduce fulfilment if they feel that they have been treated unfair. It is worthwhile to examine the role fairness can play in explaining the evolution of bargaining.

In accordance with fairness standards, a compromise must be reached by means, so that fraud is excluded from scratch. If partners agree on honest and respectful behaviour of each other, they may agree to disagree—but they will not accept a proposal they regard as unjust. Apart from fair play, not only selfish motivations, but also the feeling of being treated unfair, may cause obstruction. Fairness refers to non-discriminatory treatment of partners. Imagine a situation in which there is a wealthy and powerful individual as well as a poor one. Equality rules would discriminate the rich, and the extensive use of power would impair the poor—it hurts their feelings, harms their individual rights, and does them material damage. Rules, which establish mutual respectful behaviour and offer an outcome that can be freely accepted by both sides, permit a fair compromise.

In the forthcoming analysis, we investigate how communication may induce a fair compromise and stable outcome between actors in Cournot situations. Observed cooperative behaviour in the field often allows for communication; often it provides meetings for their members. It is more than natural that these meetings are also used for reflection on the state and process of the negotiation and on the discussion of joint strategies, as well as to resolve conflicts. Analysing solely the incentive structure of non-cooperative games and non-cooperative solution concepts like the Nash- and sub-game perfect equilibria can neither satisfactory explain that agreements are more efficient than expected nor the extent to which actors obey fairness standards. That non-cooperative solution concepts fail to explain observed cooperative behaviour is mainly due by taking not into account the actor's incentives to cooperate. Moreover, non-cooperative analysis provides no instrument to quantify dissatisfaction of agents with respect to a proposed payoff distribution in order to judge if a proposal may be significantly considered as unfair. Having now the opportunity to bargain offers an environment where the prospects of cooperation can be discussed. But then, the arguments of power and fairness must be related to solution concepts, which are well known from cooperative game theory, to fill the gap left open by non-cooperative game theory.[1]

The more formalized the settings we study, the more the bargaining for agreements on future action becomes central. In the hypothesis of cheap talk, it was erroneously assumed that, in the absence of binding contracts, only non-cooperative concepts should be considered. Moreover, many scholars did not recognize that the introduction of communication in a model will change the game theoretical framework in which subjects are involved. Communication allows subjects to make proposals of how they want self-organize, and enables them to discuss joint strategies. But this means that, at this stage, concepts from cooperative game theory have to be used to analyse the underlying situation. In communication, not only proposals and claims can be exchanged, but also supporting and demanding arguments that may motivate the opponent to move. The exchange of proposals and arguments creates a common virtual world beside the basic relations found in reality. If the partners agree that a specific compromise is justified and binds them, then they change the situation into one in which compliance is not problematic.

Rather than considering fairness as some private nebulous feeling of an individual, we study fairness on a set of principles (axioms) which describes the pre-kernel. Apart from its appealing axiomatic foundation, the pre-kernel also qualifies in accordance with the recent findings of Meinhardt (2013b) as an attractive fair division rule due to its ease of computation by solving iteratively systems of linear equations. To advance our understanding of fulfilment on non-binding agreements, we start our analysis with a Cournot situation to derive four cooperative game models well introduced in the literature, where each of it represents different aspiration levels of partners involved in a negotiation process of splitting the monopoly proceeds. In this respect, we demonstrate the bargaining difficulties that might arise when agents are claiming excessive demands, and what consequences this impose on the stability of a fair agreement. Contrasting these results with the conditions of stable bargaining scenarios which make an unproblematic settlement of agreements possible.

This treatise is organized as follows: Sect. 2 gives a short refresher of some important cooperative game theoretical solution concepts and properties. Though Sect. 3 reviews the axiomatization of the pre-kernel on which our discussion of fairness is based on. In Sect. 4 we are going to introduce the concept of the indirect function and present a dual pre-kernel characterization in terms of solution sets. This allows us to give some simple and efficient computation methods for the pre-kernel to qualify apart from its appealing axiomatic foundation as a suitable solution scheme of cooperative game theory. Resulting in Sect. 5 in the discussion of a method—which is based on the indirect function approach—to determine a

[1] See also the study of fairness by Ostmann and Meinhardt (2007, 2008).

pre-kernel point for a minimum cost spanning tree game. These are the building blocks to start in Sect. 6 an investigation of a Cournot situation from which several cooperative game models are derived to advance our understanding of compliance and stable outcomes referring to a pre-kernel agreement. We close this treatise by some final remarks in Sect. 7.

2 Preliminaries

A n-person cooperative game with side-payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \ldots, n\}$ represents the player set and v is the characteristic function with $v : 2^N \rightarrow \mathbb{R}$, and the convention that $v(\emptyset) := 0$. Elements of N are denoted as players. A subset S of the player set N is called a coalition. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. However, the cardinality of the player set N is given by $n := |N|$, and that for a coalition S by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. Formally, we identify a cooperative game by the vector $v := (v(S))_{S\subseteq N} \in \mathscr{G}^n = \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$.

A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a feasible vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) := \sum_{k\in S} x_k$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathscr{I}^0(v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}, \tag{1}$$

where an element $\mathbf{x} \in \mathscr{I}^0(v)$ is called a pre-imputation. The set of pre-imputations which satisfies in addition the **individual rationality property** $x_k \geq v(\{k\})$ for all $k \in N$ is called the **imputation set** $\mathscr{I}(v)$.

Given a vector $\mathbf{x} \in \mathscr{I}^0(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \mathbf{x}) := v(S) - x(S). \tag{2}$$

A non-negative (non-positive) excess of S at $\mathbf{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S unless the members of S do not accept the payoff distribution $\mathbf{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game $v \in \mathscr{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathscr{I}^0(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S\in\mathscr{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathscr{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{3}$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathscr{I}^0(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathscr{P}r\mathscr{K}(v) := \left\{\mathbf{x} \in \mathscr{I}^0(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\right\}. \tag{4}$$

The **core** of a game $\mathscr{C}(v)$ is the set of imputations satisfying besides the individual rationality property as well as the coalitional rationality property, i.e. the core of a game $v \in \mathscr{G}^n$ is given by

$$\mathscr{C}(v) := \{\mathbf{x} \in \mathscr{I}(v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) \;\forall\; S \subset N\}. \tag{5}$$

The core of a n-person game may be empty. Whenever it is non-empty we have some incentive for mutual cooperation in the grand coalition. A core agreement is preferable over imputations outside the core, since the grand coalition can distribute to its members a value that exceeds the value that the intermediate coalitions can produce to their members. Hence, the formation of a smaller coalition is unattractive. In this sense, a payoff distribution located in the core cannot be blocked by any coalition. Moreover, the nucleolus, denoted as $\nu(v)$, is contained in the core of the game whenever the core is non-empty, i,e., $\nu(v) \in \mathscr{C}(v)$ if $\mathscr{C}(v) \neq \emptyset$.

Imposing on the worth of any proper coalition—namely the set of coalitions excluding the grand coalition N and the empty set—the same cost $\epsilon \in \mathbb{R}$, then we can define the **strong ϵ-core** $\mathscr{C}_\epsilon(v)$ through

$$\mathscr{C}_\epsilon(v) := \{\mathbf{x} \in \mathscr{I}(v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) - \epsilon \;\forall\; \emptyset \neq S \subset N\}. \tag{6}$$

with $\mathscr{C}_0(v) = \mathscr{C}(v)$. For $n \geq 2$ we note that $\mathscr{C}_\epsilon(v) \neq \emptyset$ if ϵ is large enough and $\mathscr{C}_\epsilon(v) = \emptyset$ for small enough ϵ. Furthermore, if $\epsilon_0 < \epsilon_1$ then $\mathscr{C}_{\epsilon_0}(v) \subseteq \mathscr{C}_{\epsilon_1}(v)$ and $\mathscr{C}_{\epsilon_0}(v) \subset \mathscr{C}_{\epsilon_1}(v)$ whenever $\mathscr{C}_{\epsilon_1}(v) \neq \emptyset$. Similar to the core of the game, we have $\nu(v) \in \mathscr{C}_\epsilon(v)$ whenever $\mathscr{C}_\epsilon(v) \neq \emptyset$ and $\epsilon \leq 0$.

The least core formalized by Maschler et al. (1979) is the $\epsilon_0(v)$-core of the game v, where $\epsilon_0(v)$ is the critical number at which the ϵ-core still exists, that is, for $\epsilon < \epsilon_0(v)$ the ϵ-core is empty. This critical number is specified by

$$\epsilon_0(v) := \min_{\mathbf{x} \in \mathscr{I}^0(v)} \max_{S \neq \emptyset, N} e(\mathbf{x}, S). \tag{7}$$

An **objection** of player i against a player j w. r. t. a payoff vector $\mathbf{x} \in \mathbb{R}^n$ in game $v \in \mathscr{G}^n$ is a pair $(\mathbf{y}_S, S)$ with $S \in \mathscr{G}_{ij}$ and $\mathbf{y}_S := \{y_k\}_{k \in S}$ satisfying the following properties:

$$v(S) = \sum_{k \in S} y_k \qquad \text{and} \qquad y_k > x_k \quad \text{for } k \in S. \tag{8}$$

A **counter-objection** to the objection $(\mathbf{y}_S, S)$ is a pair $(\mathbf{z}_T, T)$ with $T \in \mathscr{G}_{ji}$ and $\mathbf{z}_T := \{z_k\}_{k \in T}$ satisfying

$$v(T) = \sum_{k \in T} z_k \qquad \text{and} \qquad \begin{aligned} z_k &\geq x_k \quad \text{for } k \in T \backslash S \\ z_k &\geq y_k \quad \text{for } k \in T \cap S. \end{aligned} \tag{9}$$

Thus, if the pair $(\mathbf{y}_S, S)$ is an objection against vector $\mathbf{x}$, then any member of coalition $S \in \mathscr{G}_{ij}$ can improve upon rather than accepting proposal $\mathbf{x}$. Acceptance would mean that players in $S \in \mathscr{G}_{ij}$ would accept a loss due to $e^v(S, \mathbf{x}) > 0$. Hence, a player i can formulate an objection against player j using coalition $S \in \mathscr{G}_{ij}$ w. r. t. the proposal $\mathbf{x}$ iff the excess $e^v(S, \mathbf{x})$ is positive.

In contrast, a counter-objection $(\mathbf{z}_T, T)$ of player j against player i w. r. t. objection $(\mathbf{y}_S, S)$ uses a coalition T without player i, i.e. $T \in \mathscr{G}_{ji}$, to formulate a proposal that cannot strictly be improved upon to the precedent proposal for players belonging to the set $S \cap T$ and which can also not strictly be improved upon w. r. t. $\mathbf{x}$ for all $k \in T \backslash S$. This means, that player j can only use a coalition $T \in \mathscr{G}_{ji}$ with non-negative excess $e^v(T, \mathbf{x})$ to formulate a counter-objection against player i.

An imputation $\mathbf{x} \in \mathscr{I}(v)$ is an element of the **bargaining set** $\mathscr{M}(v)$ of game $v \in \mathscr{G}^n$ whenever for any objection of a player against another player w. r. t. $\mathbf{x}$ in $v \in \mathscr{G}^n$ exists a counter-objection.

Be reminded that the following property $\mathscr{C}(v) \subseteq \mathscr{M}(v)$ holds for all $v \in \mathscr{G}^n$. This means, that for core allocations the excesses described by formula (2) are non-positive, implying that for core allocations there are no objections w. r. t. other core allocations, and for allocations outside the core it is always possible to formulate against an objection a counter-objection. Hence, core allocations can be stabilized by an abstract bargaining procedure while formulating objections and counter-objections. Moreover, the bargaining set $\mathscr{M}(v)$ is non-empty, since $\emptyset \neq \mathscr{K}(v) \subseteq \mathscr{M}(v)$ for all $v \in \mathscr{G}^n$ (Davis and Maschler 1965). This implies that whenever $\mathscr{C}(v) = \emptyset$ is valid, we may fail to achieve cooperation into the grand coalition, however, the bargaining set $\mathscr{M}(v)$ is non-empty there exist allocations which can be stabilized on the basis of the bargaining set. As a consequence, cooperation in a subgroup of players in N is always possible.

In addition, we want to discuss some important game properties. A game $v \in \mathscr{G}^n$ is said to be **monotonic** if

$$v(S) \leq v(T) \qquad \forall \emptyset \neq S \subseteq T. \tag{10}$$

Thus, whenever a game is monotonic, a coalition T can guarantee to its member a value at least as high as any sub-coalition S can do. This subclass of games is referred to as $\mathscr{M}\mathscr{N}^n$. A game $v \in \mathscr{G}^n$ satisfying the condition

$$v(S) + v(T) \leq v(S \cup T) \qquad \forall S, T \subseteq N, \text{ with } S \cap T = \emptyset, \tag{11}$$

is called **superadditive**. This means, that two disjoint coalitions have some incentive to join into a mutual coalition. This can be regarded as an incentive of merging economic activities into larger units. We denote this subclass of games by $\mathscr{SA}^n$. However, if a game $v \in \mathscr{G}^n$ satisfies

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \qquad \forall S, T \subseteq N, \tag{12}$$

then it is called **convex**.

In this case, we will observe a strong incentive for a mutual cooperation in the grand coalition, due to its achievable over proportionate surpluses while increasing the scale of cooperation. This subclass of games has been introduced by Shapley (1971), and we denote it by $\mathscr{CV}^n$. Convex games having a non-empty core and the Shapley value is the centre of gravity of the extreme point of the core (cf. Shapley 1971), that is, a convex combination of the vectors of marginal contributions, which are core imputations for convex games. It should be evident that $\mathscr{CV}^n \subset \mathscr{SA}^n$ is satisfied. Finally, note that whenever $v \in \mathscr{CV}^n$, then $\mathscr{C}(v) = \mathscr{M}(v)$.

3 Axiomatization of the Pre-Kernel

Fairness is for many people an opaque concept. As a consequence, people do not have an uniform understanding of what they regard to be fair or unfair. To avoid that fairness is solely understood as a rule without any objective foundation, we need to base fairness standards on impartial norms on which people can agree or disagree. Then one can classify an agreement as a fair outcome by referring to a rule of distributive justice. The pre-kernel establishes a solid foundation of upright principles that even apply in situations with unequal partners. It provides a division of benefit—generated by mutual cooperation—that can be freely accepted by each party, and can, therefore, be accepted by partners as a fair compromise. It hurts neither the most powerful nor the weakest negotiating party. Though it allocates the benefit in accordance with the productivity of actors.

The axiomatic foundation of the pre-kernel and its related solutions imposes some requirements on their consistency. A cooperative solution is considered consistent when it distributes the same payoff to any appropriately defined reduced game as in the original game. An outcome is viewed as consistent under this specific rule. No subgroup has an incentive to deviate from the original proposal and to play with its members its proper game to improve their situation. Consistency is widely regarded as a desirable property and can be interpreted as indicating subgame perfection of a cooperative solution concept.

Let be a game $\langle N, v \rangle \in \mathcal{G}$, $\emptyset \neq S \subseteq N$ and let $\mathbf{x} \in \mathscr{I}^0(N, v)$. The **reduced game** w. r. t. S and $\mathbf{x}$ is the game $\langle S, v_{S,\mathbf{x}} \rangle$ as given by

$$v_{S,\mathbf{x}}(T) := \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \backslash S) & \text{if } T = S \\ \max_{Q \subseteq N \backslash S} \, (v(T \cup Q) - x(Q)) & \text{otherwise.} \end{cases}$$

This game type has been introduced by Davis and Maschler (1965) to study the kernel.

Denote by $\mathscr{U}$ the infinite universe of players, clearly $N \subset \mathscr{U}$. Fix any n and identify with $\mathcal{G}$ a subset of all n-person games with player set N. Note that $\mathcal{G} \subset \mathcal{G}^n \subset \mathcal{G}_{\mathscr{U}}$ holds. A solution on $\mathcal{G}$ is a correspondence σ on $\mathcal{G}$ such that $\sigma(N, v) \subset \mathscr{I}^0(N, v)$. A value on $\mathcal{G}$ is a function σ on $\mathcal{G}$ such that $\sigma(v) \in \mathbb{R}^n$, whenever $\langle N, v \rangle \in \mathcal{G}$.

The pre-kernel is well understood and justified by various axiomatic foundations. To understand that fairness can be related to an axiomatization of the pre-kernel, we give now a summary of some important axioms.

Single-Valuedness (SIVA) A solution σ on $\mathcal{G}$ is single valued (SIVA), whenever $|\,\sigma(N, v)\,| = 1$ for every $\langle N, v \rangle \in \mathcal{G}$.

Anonymity (AN) A solution σ on $\mathcal{G}$ satisfies the AN property, if for $\langle N, v \rangle \in \mathcal{G}$, for an injection $\vartheta : N \to \mathscr{U}$ and for $\langle \vartheta(N), \vartheta v \rangle \in \mathcal{G}$ implying $\sigma(\vartheta(N), \vartheta v) = \vartheta(\sigma(N, v))$.

Nonemptiness (NE) A solution σ on $\mathcal{G}$ satisfying nonemptiness, if $\sigma(N, v) \neq \emptyset$ for every $\langle N, v \rangle \in \mathcal{G}$.

Individual Rationality (IR) A solution σ on $\mathcal{G}$ fulfills the IR property, if $\langle N, v \rangle \in \mathcal{G}$ and $\mathbf{x} \in \sigma(N, v)$, then $x_k \geq v(\{k\})$ for all $k \in N$.

Equal Treatment Property (ETP) If $\langle N, v \rangle \in \mathcal{G}$, $\mathbf{x} \in \sigma(N, v)$ and if k and l are substitutes, i.e., $v(S \cup \{k\}) = v(S \cup \{l\})$ for all $S \subseteq N \backslash \{k, l\}$, then $x_k = x_l$.

Covariance with Strategic Equivalence (COV) A solution σ on $\mathcal{G}$ fulfills the COV property, if for $\langle N, v_1 \rangle, \langle N, v_2 \rangle \in \mathcal{G}$, with $v_2 = t \cdot v_1 + \mathfrak{m}$ for some $t \in \mathbb{R}_{++}, \mathfrak{m} \in \mathbb{R}^{2^n}$, then $\sigma(N, v_2) = t \cdot \sigma(N, v_1) + \mathbf{m}$, whereas $\mathbf{m} \in \mathbb{R}^n$ and $\mathfrak{m}$ is the vector of measures obtained from $\mathbf{m}$.

Reduced Game Property (RGP) A solution σ on $\mathcal{G}$ satisfies the RG property, if for $\langle N, v \rangle \in \mathcal{G}, \emptyset \neq S \subseteq N$ and $\mathbf{x} \in \sigma(N, v)$, then $\langle S, v_{S,\mathbf{x}} \rangle \in \mathcal{G}$ and $\mathbf{x}_S \in \sigma(S, v_{S,\mathbf{x}})$.

Converse Reduced Game Property (CRGP) A solution σ on $\mathcal{G}$ possesses the CRG property, if for $\langle N, v \rangle \in \mathcal{G}$ with $|N| \geq 2$, $\mathbf{x} \in \mathscr{I}^0(N, v)$, $\langle S, v_{S,\mathbf{x}} \rangle \in \mathcal{G}$ and $\mathbf{x}_S \in \sigma(S, v_{S,\mathbf{x}})$ for every $S \in \{S \subseteq N \,|\, |S| = 2\}$, then $\mathbf{x} \in \sigma(N, v)$.

The related solution, the kernel, satisfies **COV, AN** and **IR**. There is a unique solution σ on $\mathcal{G}_{\mathscr{U}}$ satisfying **NE, EFF, COV, ETP, RGP** and **CRGP**, which is the pre-kernel. Moreover, there exists a unique solution σ on $\mathcal{G}_{\mathscr{U}}$ which fulfills **SIVA, COV, AN** and **RGP**. This particular point alluded from the pre-kernel is the pre-nucleolus.

The axioms of the pre-kernel can be interpreted as follows: **NE** assures its existence in every TU game. **EFF** distributes the proceeds of mutual cooperation in accordance with efficiency—none of the total proceeds should be wasted. **COV** claims that the pre-kernel solution of the default game is strategically equivalent to a solution of an affine transformed game. **ETP** requires that equal partners should be treated equally. **RGP** and **CRGP** impose the required consistency properties. For a rigorous axiomatic treatment we refer the reader to Peleg and Sudhölter (2007).

Furthermore, the pre-kernel satisfies the property that all pairs of bilateral claims of partners are bisected (see Maschler et al. 1979). Thus, their bilateral demands are in equilibrium, and there is no room to renegotiate the proposal by claiming an additional share from opponents (cf. also with Meinhardt 2013b, pp. 21–24). Again, equal partners are equally treated. From this perspective, negotiating partners can accept those norms of distributive justice as a fair outcome.

In addition, the pre-kernel does not depend on "interpersonal comparisons of utility" that consider many scholars as a nebulous concept (see Maschler et al. 1979). Following this view, Serrano (1997) as well as Chang and Hu (2016) established a pairwise bargaining process in a non-cooperative game theoretical model resulting in a unique subgame perfect equilibrium, which is the pre-kernel under the condition of zero-monotonicity. There, the reduced game property is used in a non-cooperative bargaining game to give a reinterpretation of the kernel that does not make any use of interpersonal utility comparisons. A non-cooperative foundation of the pre-kernel solution can be given while constructing from a zero-monotonic game an extensive game, which formulates a bargaining process that will lead actors to the proposed solution whenever they follow the described rules. Thus, the pre-kernel results as a solution of a Nash program.

4 A Dual Pre-Kernel Representation

In this section, we review a procedure of how one can characterize the pre-kernel of TU games by means of convex analysis. Our approach is based on a generalized conjugation theory from convex analysis that provides a dual representation of TU games, called an indirect function. This approach was invented by Martínez-Legaz (1996). The representation of a TU game by the indirect function provides the same information as the characteristic function (cf. Martínez-Legaz 1996). An indirect function has the appealing property of being a non-increasing polyhedral convex function. In this respect, it was Meseguer-Artola (1997) who has first recognized that from the indirect function a simplified pre-kernel representation can be obtained from which a pre-kernel element can be computed as a solution of an over-determined system of non-linear equations. From this over-determined system an equivalent minimization problem can be constructed, whose set of global minima coalesces with the pre-kernel set.

However, imposing some additional conditions was enough to present a new characterization of the pre-kernel in terms of a finite collection of restricted solution sets obtained by solving convex programming problems. This characterization makes it possible to describe a practical method of the pre-kernel computation to iteratively solve some systems of linear equations converging in a finite number of iteration steps to a pre-kernel element (cf. Meinhardt 2013b).

Theorem 11.1 (Martínez-Legaz 1996) *The indirect function $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial\pi(\mathbf{x}) \cap \{-1, 0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,
(ii) $\{-1, 0\}^n \subset \bigcup_{\mathbf{x}\in\mathbb{R}^n} \partial\pi(\mathbf{x})$, *and*
(iii) $\min_{\mathbf{x}\in\mathbb{R}^n} \ \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (i)-(iii) then there exists a unique n-person TU game $\langle N, v\rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\mathbf{x}\in\mathbb{R}^n} \left\{\pi(\mathbf{x}) + \sum_{k\in S} x_k\right\} \qquad \forall\, S \subset N. \tag{13}$$

According to the above result, the associated **indirect function** $\pi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is given by:

$$\pi(\mathbf{x}) = \max_{S\subseteq N} \left\{v(S) - \sum_{k\in S} x_k\right\} \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{14}$$

whereas $\partial\pi$ is the subdifferential of the function π. Hence, $\partial\pi(\mathbf{x})$ is the set of all subgradients of π at $\mathbf{x}$, which is a closed polyhedral convex set. A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). Here, we present this representation in its most general form, although we restrict ourselves to the trivial coalition structure $\mathscr{B} = \{N\}$.

The pre-imputation that comprises the possibility of compensation between a pair of players $i, j \in N, i \neq j$, is denoted as $\mathbf{x}^{\,i,j,\delta} = (x_k^{i,j,\delta})_{k\in N} \in \mathscr{I}^0(v)$, with $\delta \geq 0$, which is given by

$$\mathbf{x}^{i,j,\delta}_{N\setminus\{i,j\}} = \mathbf{x}_{N\setminus\{\,i,j\}},\; x_i^{i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{\,i,j,\delta} = x_j + \delta$$

Proposition 11.1 (Meseguer-Artola 1997) *For a TU game with indirect function π, a pre-imputation $\mathbf{x} \in \mathscr{I}^{\,0}(v)$ is in the pre-kernel of $\langle N, v\rangle$ for the coalition structure $\mathscr{B} = \{B_1, \ldots, B_l\}$, $\mathbf{x} \in \mathscr{P}r\mathscr{K}(v, \mathscr{B})$, if, and only if, for every $k \in \{1, 2, \ldots, l\}$, every $i, j \in B_k,\ i < j$, and some $\delta \geq \delta_1(v, \mathbf{x})$, one gets*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

whereas $\delta_1(\mathbf{x}, v) := \max_{k\in N, S\subset N\setminus\{k\}} \ |v(S \cup \{k\}) - v(S) - x_k|$.

Proof For a proof see Meinhardt (2013b, pp. 53–55).

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 11.1 a pre-kernel element can be derived as a solution of an over-determined system of non-linear equations. Every over-determined system can be equivalently expressed as a minimization problem. The set of global minima coalesces with the pre-kernel set. For the trivial coalition structure $\mathscr{B} = \{N\}$ the over-determined system of non-linear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i,j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{15}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{i,j,\delta}) - \pi(\mathbf{x}^{j,i,\delta}) \qquad \forall i,j \in N, i < j, \tag{15a}$$

and

$$f_0(\mathbf{x}) := \sum_{k \in N} x_k - v(N). \tag{15b}$$

$$h(\mathbf{x}) := \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{16}$$

For further details see Meinhardt (2013b, Chap. 5 and 6). Then one can establish the subsequent result:

Corollary 11.1 (Meinhardt 2013b) *For a TU game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y} \in \mathscr{I}^0(v)} h(\mathbf{y}) = 0, \tag{17}$$

if, and only if, $\mathbf{x} \in \mathscr{P}r\mathscr{K}(v)$.

To identify a partition of the domain of function h into payoff equivalence classes we first define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathscr{C}_{ij}(\mathbf{x}) := \left\{ S \in \mathscr{G}_{ij} \;\middle|\; s_{ij}(\mathbf{x}, v) = e^v(S, \mathbf{x}) \right\}. \tag{18}$$

When we gather for all pair of players $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w. r. t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathscr{C}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathscr{C}_{ij}(\mathbf{x}). \tag{19}$$

Notice that the set $\mathscr{C}_{ij}(\mathbf{x})$ for all $i,j \in N, i \neq j$ does not have cardinality one, which is required to identify a partition on the domain of function h. Now let us choose for each pair $i,j \in N, i \neq j$ a descending ordering on the set of most effective coalitions in accordance with their size, and within such a collection of most effective coalitions having smallest size the lexicographical minimum is singled out, then we obtain the required uniqueness to partition the domain of h. This set is denoted by $\mathscr{S}_{ij}(\mathbf{x})$ for all pairs $i,j \in N, i \neq j$, and gathering all these collections we are able to specify the set of lexicographically smallest most effective coalitions w. r. t. $\mathbf{x}$ through

$$\mathscr{S}(\mathbf{x}) := \left\{ \mathscr{S}_{ij}(\mathbf{x}) \;\middle|\; i,j \in N, i \neq j \right\}. \tag{20}$$

This set will be denoted in short as the set of **lexicographically smallest coalitions**. Given the correspondence $\mathscr{S}$ on the payoff space we say that two payoff vectors $\mathbf{x}$ and $\mathbf{y}$ are equivalent w. r. t. the binary relation $\sim$ iff $\mathscr{S}(\mathbf{x}) = \mathscr{S}(\mathbf{y})$. In case that the binary relation $\sim$ is reflexive, symmetric and transitive, then it is an **equivalence relation** and it induces **equivalence classes** $[\boldsymbol{\gamma}]$ on $dom\,h$ which we define through $[\boldsymbol{\gamma}] := \{\mathbf{x} \in dom\ h \mid \mathbf{x} \sim \boldsymbol{\gamma}\}$. Thus, if $\mathbf{x} \sim \boldsymbol{\gamma}$, then $[\mathbf{x}] = [\boldsymbol{\gamma}]$, and if $\mathbf{x} \nsim \boldsymbol{\gamma}$, then $[\mathbf{x}] \cap [\boldsymbol{\gamma}] = \emptyset$. This implies that whenever the binary relation $\sim$ induces equivalence classes $[\boldsymbol{\gamma}]$ on $dom\,h$, then it partitions the domain $dom\,h$ of the function h. The resulting collection of equivalence classes $[\boldsymbol{\gamma}]$ on $dom\,h$ is called the quotient of $dom\,h$ modulo $\sim$, and we denote this collection by $dom\,h/\sim$. We indicate this set as an equivalence class whenever the context is clear, otherwise we apply the term payoff set or payoff equivalence class.

Proposition 11.2 (Meinhardt 2013b) *The binary relation $\sim$ on the set $dom\,h$ defined by $\mathbf{x} \sim \boldsymbol{\gamma} \iff \mathscr{S}(\mathbf{x}) = \mathscr{S}(\boldsymbol{\gamma})$ is an equivalence relation, which forms a partition of the set $dom\,h$ by the collection of equivalence classes $\{[\boldsymbol{\gamma}_k]\}_{k \in J}$, where J is an arbitrary index set. Furthermore, for all $k \in J$, the induced equivalence class $[\boldsymbol{\gamma}_k]$ is a convex set.*

Proof For a proof see Meinhardt (2013b, Chap. 5 and 6).
This binary relation induces a partition on the payoff space. Having identified payoff equivalence classes, we can select an arbitrary payoff vector to get a unique quadratic and convex function. To see this, select payoff vector $\mathbf{x}$ from payoff equivalence class $[\boldsymbol{\gamma}]$, then we get the set $\mathscr{S}(\mathbf{x})$ from which a rectangular matrix $\mathbf{E}$ can be constructed through $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n$, $\forall i,j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Notice that in this respect the characteristic vector for $\mathbf{x} \in \mathbb{R}^n$ is given by $x_k = 1$ if $k \in S$ and $x_k = 0$ whenever $k \notin S$. Let $q = \binom{n}{2} + 1$; combining

these q-column vectors, we can construct matrix $\mathbf{E}$ as an $(n \times q)$-matrix in $\mathbb{R}^{n \times q}$, which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n \times q}. \tag{21}$$

A matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ can now be expressed as $\mathbf{Q} = 2 \cdot \mathbf{E}\ \mathbf{E}^\top$, a column vector $\mathbf{a}$ as $2 \cdot \mathbf{E}\, \boldsymbol{\alpha} \in \mathbb{R}^n$. Notice, that the transpose of a vector $\mathbf{x}$ or a matrix $\mathbf{Q}$ is denoted by the symbols $\mathbf{x}^\top$, and $\mathbf{Q}^\top$. Moreover, defining $\alpha_{ij} := v(S_{ij}) - v(S_{ji}) \in \mathbb{R}\ \forall i,j \in N, i < j$ and $\alpha_0 := v(N)$. Finally, the scalar α is given by $\|\boldsymbol{\alpha}\|^2$, whereas $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ and $\boldsymbol{\alpha} \in \mathbb{R}^q$. For the details to construct the above set, matrix and vector we refer the reader to Meinhardt (2013b, Chap. 5 and 6).

From vector $\boldsymbol{\gamma}$ the set (20) is constructed and then matrix $\mathbf{Q}$, column vector $\mathbf{a}$, and scalar α are induced from which a quadratic and convex function can be specified through

$$h_{\boldsymbol{\gamma}}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\,\mathbf{x} \,\rangle + \langle\, \mathbf{x}, \mathbf{a} \,\rangle + \alpha \qquad \mathbf{x} \in \mathbb{R}^n. \tag{22}$$

In view of Proposition 6.2.2 Meinhardt (2013b) function h as defined by (16) is composed of a finite family of quadratic and convex functions of type (22). For the details, we again refer the interested reader to Meinhardt (2013b, Chap. 5 and 6). In accordance with Theorem 7.3.1 by Meinhardt (2013b, p. 137) a dual representation of the pre-kernel is obtained as a finite union of convex and restricted solution sets $M(h_{\boldsymbol{\gamma}_k}, \overline{[\boldsymbol{\gamma}_k]})$ of a quadratic and convex function of type $h_{\boldsymbol{\gamma}_k}$, that is,

$$\mathcal{P}r\mathcal{K}(v) = \bigcup_{k \in \mathcal{J}'} M(h_{\boldsymbol{\gamma}_k}, \overline{[\boldsymbol{\gamma}_k]}), \tag{23}$$

where $\mathcal{J}'$ is a finite index set such that $\mathcal{J}' := \{k \in \mathcal{J} \mid g(\boldsymbol{\gamma}_k) = 0\}$. In addition, $g(\boldsymbol{\gamma}_k)$ is the minimum value of a minimization problem under constraints of function $h_{\boldsymbol{\gamma}_k}$ over the closed convex payoff set $\overline{[\boldsymbol{\gamma}_k]}$. For the index set it is claimed that this minimum value is equal to zero on the closed payoff set $\overline{[\boldsymbol{\gamma}_k]}$. The solution sets $M(h_{\boldsymbol{\gamma}_k}, \overline{[\boldsymbol{\gamma}_k]})$ are convex. Taking the finite union of convex sets may give us a non-convex set. Hence, the pre-kernel set is generically a non-convex set for games with more than 4 players. By the characterization of (23) we observe that it can be even disconnected. An example of a disconnected pre-kernel was discussed by Kopelowitz (1967) and Stearns (1968). This example was recently reconsidered in Meinhardt (2013b, Sect. 8.5).

For the class of convex games and three person games we have $|\ \mathcal{J}'\ | = 1$, which implies that the pre-kernel must be a singleton. Meinhardt (2015b) has established that whenever a default game has a singleton pre-kernel satisfying the non-empty interior condition for a payoff set, then on a restricted subset of the game space constituted by the default game and a set of related games, this point is the sole pre-kernel element. The pre-kernel correspondence is single-valued and constant on this subset.[2]

5 Determining a Pre-Kernel Element

Having reconsidered the results which allow us to characterize the pre-kernel of TU games in terms of a finite union of convex solutions sets, we pass now to a method of computing an element of the pre-kernel by solving iteratively systems of linear equations by means of an example. We base our discussion on a minimum cost spanning tree example borrowed from the literature (cf. Curiel 1997, pp. 132–134).

To this end we consider a mapping that sends a point $\boldsymbol{\gamma}$ to a point $\boldsymbol{\gamma}_{\circ} \in M(h_{\gamma})$ through

$$\Gamma(\boldsymbol{\gamma}) := -\left(\mathbf{Q}^{\dagger}\,\mathbf{a}\right)(\boldsymbol{\gamma}) = -\left(\mathbf{Q}_{\gamma}^{\dagger}\,\mathbf{a}_{\gamma}\right) = \boldsymbol{\gamma}_{\circ} \in M(h_{\gamma}) \qquad \forall \boldsymbol{\gamma} \in \mathbb{R}^{n}, \tag{24}$$

where $\mathbf{Q}_{\gamma}$ and $\mathbf{a}_{\gamma}$ are the matrix and the column vector induced by vector $\boldsymbol{\gamma}$, respectively. Notice that matrix $\mathbf{Q}_{\gamma}^{\dagger}$ is the pseudo-inverse of matrix $\mathbf{Q}_{\gamma}$. In addition, the set $M(h_{\gamma})$ is the solution set of function h_{γ}. Under a regime of orthogonal projection this mapping induces a cycle free method to evaluate a pre-kernel point for any class of TU games. We restate here Algorithm 8.1.1 of Meinhardt (2013b) in a more succinctly written form through Method 5.1.

Meinhardt (2013b, Theorem 8.1.2) establishes that this iterative procedure converges towards a pre-kernel point. In view of Meinhardt (2013b, Theorem 9.1.2) we even know that at most $\binom{n}{2} - 1$-iteration steps are sufficient to successfully terminate the search process. However, we have some empirical evidence that generically at most $n + 1$-iteration steps are needed to determine an element from the pre-kernel set (cf. Meinhardt 2013b, Appendix A).[3]

Now, we are in the position to introduce the minimum cost spanning tree game of Curiel (1997). There, the player set given by $N = \{1, 2, 3\}$ represents the users of a common good provided by a common supplier 0. Then the distribution system consists of links among members $N_0 = \{0\} \cup N$. The costs associated to buildup the links is given by the following cost matrix

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 2 & 6 \\ 2 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 6 & 2 & 2 & 0 \end{bmatrix}, \tag{25}$$

[2]This topic will be reviewed in the forthcoming Sect. 6.5.

[3]Algorithm 5.1 is implemented in our MATLAB toolbox MatTuGames 2015a. The documentation of the toolbox is given by Meinhardt (2013a) and ships with the toolbox.

Table 1 Minimum cost spanning tree and savings game

Game	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	N
c	2	2	6	3	4	4	5
$v^{a,b,c}$	0	0	0	1	4	4	5

[a]Kernel: (2/3, 2/3, 11/3)
[b]Nucleolus: (2/3, 2/3, 11/3)
[c]Shapley Value: (7/6, 7/6, 8/3)

Algorithm 5.1: Procedure to seek for a Pre-Kernel element

Data: Arbitrary TU-Game $\langle N, v \rangle$, and a payoff vector $\boldsymbol{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\boldsymbol{\gamma}_{k+1} \in \mathscr{PrK}(v)$.
begin
0 $k \longleftarrow 0, \quad \mathscr{S}(\boldsymbol{\gamma}_{-1}) \longleftarrow \emptyset$
1 Select an arbitrary starting point $\boldsymbol{\gamma}_0$
if $\boldsymbol{\gamma}_0 \notin \mathscr{PrK}(v)$ **then** Continue
else Stop
2 Determine $\mathscr{S}(\boldsymbol{\gamma}_0)$
if $\mathscr{S}(\boldsymbol{\gamma}_0) \neq \mathscr{S}(\boldsymbol{\gamma}_{-1})$ **then** Continue
else Stop
repeat
3 **if** $\mathscr{S}(\boldsymbol{\gamma}_k) \neq \emptyset$ **then** Continue
else Stop
4 Compute $\mathbf{E}_k$ and $\boldsymbol{\alpha}_k$ from $\mathscr{S}(\boldsymbol{\gamma}_k)$ and v
5 Determine $\mathbf{Q}_k$ and $\mathbf{a}_k$ from $\mathbf{E}_k$ and $\boldsymbol{\alpha}_k$
6 Calculate by Formula (24) $\mathbf{x}$
7 $k \longleftarrow k + 1$
8 $\boldsymbol{\gamma}_{k+1} \longleftarrow \mathbf{x}$
9 Determine $\mathscr{S}(\boldsymbol{\gamma}_{k+1})$
until $\mathscr{S}(\boldsymbol{\gamma}_{k+1}) = \mathscr{S}(\boldsymbol{\gamma}_k)$
end

where each entry denotes the cost of constructing the link $\{i, j\}$. In the next step, let us define a savings game by

$$v(S) := \sum_{k \in S} c(\{k\}) - c(S) \qquad \forall\ S \subseteq N. \tag{26}$$

From the cost matrix (25), we derive a minimum cost spanning tree game from which a savings game is obtained through formula (26). The derived minimum cost spanning tree and savings game are given by Table 1.

To illustrate how Algorithm 5.1 works, let us focus on the pre-selected imputation vector $\mathbf{y}_0 = ((5, 5, 5)/3)^\top$ to see how we can apply this method for our specific example. From the vector $\mathbf{y}_0$, we get the following excess vector $exc(\mathbf{y}_0) = (0, -5/3, -5/3, -5/3, -7/3, 2/3, 2/3, 0)$.

In the next step, we look on the maximum surpluses for all pair of players. For any pair of players $i, j \in N, i \neq j$, the maximum surplus of player i over player j with respect to any pre-imputation $\mathbf{x}$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S \in \mathscr{G}_{ij}} e^{v}(S, \mathbf{x}) \qquad \text{where } \mathscr{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}.$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j.

From this excess vector $exc(\mathbf{y}_0)$ we get now the following set of lexicographically smallest coalitions for each pair of players:

$$\mathscr{S}(\mathbf{y}_0) = \{\{1,3\}, \{1\}, \{2,3\}, \{2\}, \{2,3\}, \{1,3\}\}$$

whereas the order of the pairs of players in $\mathscr{S}(\mathbf{y}_0)$ is given by

$$\{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}.$$

For instance, for the pair of players $(1,2)$, we find out these coalitions that support the claim of player 1 without counting on the cooperation of player 2, these are the coalitions $\{\{1\}, \{1,3\}\}$ having excess $(-5/3, 2/3)$. We see here that coalition $\{1,3\}$ has maximum surplus. If this set is not unique, we determine the coalitions that have smallest cardinality, and from this set the coalition that has lexicographical minimum. To observe this, let us assume that $n = 4$, then the set of coalitions supporting player 1 without counting on the cooperation of player 2 is $\{\{1\}, \{1,3\}, \{1,4\}, \{1,3,4\}\}$. Moreover, let us assume that the coalitions $\{\{1,3\}, \{1,4\}, \{1,3,4\}\}$ have maximum surpluses, then the smallest cardinality is 2 and we single out the coalitions $\{\{1,3\}, \{1,4\}\}$ and taking finally the lexicographical minimum, which is $\{1,3\}$.

For the reverse pair $(2,1)$ we find out that coalition $\{2,3\}$ supports best the claim of player 2 without taking into account the cooperation of player 1. Proceeding in the same way for the remaining pairs, then we derive a matrix $\mathbf{E}$ by $\mathbf{E}_{ij} = \mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}$ for each $i, j \in N, i < j$, and $\mathbf{E}_0 = 1_N$. Notice that $\mathbf{1}_S : N \mapsto \{0,1\}$ is the characteristic vector given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$. Then matrix $\mathbf{E}$ is defined by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{2,3}, \mathbf{E}_0] \in \mathbb{R}^{3\times 4}.$$

We realize that vector $\mathbf{E}_{1,2}$ is given by $(0,1,1)^{\top} - (1,0,1)^{\top} = (-1,1,0)^{\top}$ and $\mathbf{E}_0 = (1,1,1)^{\top}$. Proceeding in an analogous way for the remaining pair of players $(1,3)$ and $(2,3)$, matrix $\mathbf{E}$ is quantified by

$$\mathbf{E} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

A column vector $\mathbf{a}$ can be obtained by $2 \cdot \mathbf{E}\,\boldsymbol{\alpha} \in \mathbb{R}^n$ whereas the vector $\boldsymbol{\alpha}$ is given by $\alpha_{ij} := (v(S_{ji}) - v(S_{ij})) \in \mathbb{R}$ for all $i, j \in N, i < j$, and $\alpha_0 := v(N)$. Therefore, vector $\boldsymbol{\alpha}$ is given by $(0, 4, 4, 5)^\top$.

From this matrix, we construct matrix $\mathbf{Q}$ by $2 \cdot \mathbf{E}\ \mathbf{E}^\top$, inserting its numbers, matrix $\mathbf{Q}$ is specified by

$$\mathbf{Q} = 2 \cdot \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

The column vector $\mathbf{a}$ is given by $\mathbf{a} = 2 \cdot (5, 5, 13)^\top$.

Solving this system of linear equations $\mathbf{Q}\ \mathbf{x} - \mathbf{a} = \mathbf{0}$, or alternatively $\mathbf{E}^\top\ \mathbf{x} - \boldsymbol{\alpha} = \mathbf{0}$, we get as a solution $\mathbf{y}_1 = (1/2, 1/2, 4)^\top$. The corresponding excess vector is given through

$$exc(\mathbf{y}_1) = (0, -1/2, -1/2, -4, 0, -1/2, -1/2, 0).$$

We observe that the maximum surpluses are not balanced. Hence, we need at least an additional iteration step to complete.

For the second iteration step we use the vector $\mathbf{y}_1 = (1/2, 1/2, 4)^\top$ while applying the procedure from above to get matrix $\mathbf{E}$ by

$$\mathbf{E} = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

with $\boldsymbol{\alpha}$ is given by $(0, 3, 3, 5)^\top$. Constructing again matrix $\mathbf{Q}$ through

$$\mathbf{Q} = 2 \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The column vector $\mathbf{a}$ is given by $\mathbf{a} = 2 \cdot (2, 2, 11)^\top$. Solving this system of linear equations $\mathbf{Q}\ \mathbf{x} - \mathbf{a} = \mathbf{0}$, we get as a solution $\mathbf{y}_2 = (2/3, 2/3, 11/3)^\top$. The corresponding excess vector is given through

$$exc(\mathbf{y}_2) = (0, -2/3, -2/3, -11/3, -1/3, -1/3, -1/3, 0).$$

We can check out that the maximum surpluses are balanced, hence the vector $\mathbf{y}_2 = (2/3, 2/3, 11/3)^\top$ is a pre-kernel element of the game. Notice that in this specific case, we needed only two iteration steps to complete. This is the theoretical expected upper bound of iteration steps, since by Theorem 9.2.1 of Meinhardt (2013b, p. 222), we have $\binom{3}{2} - 1 = 3 - 1 = 2$. Notice, this method is applicable for any n-person TU game (cf. Meinhardt 2013b, Appendix A), and has also been proven to be useful in finding a N-shaped pre-kernel (cf. Meinhardt 2014).

6 Some Cooperative Game Models

In this section we want to discuss in more details the pre-kernel for certain cooperative oligopoly games with transferable technologies. Oligopoly games are very often studied in a non-cooperative game context while deriving from an oligopoly situation its corresponding normal form game. In this section, we go a step further by obtaining the associated cooperative games with a homogeneous good from the same Cournot oligopoly normal form game. The literature has discussed a couple of ways of converting a non-cooperative game into a game of characteristic function form. In our analysis we confine ourselves to the so-called $\alpha, \beta, \gamma, \delta$-value and s-type games.

6.1 Oligopoly Situation and Games

Consider an oligopoly situation $\langle N, (\omega_k)_{k\in N}, (c_k)_{k\in N}, p\rangle$ where $\omega_k > 0, \; k \in N$ denote the capacity of the firm k. Furthermore, let $c_k : \mathbb{R}_+ \to \mathbb{R}_+, \; k \in N$ denote the arbitrary once differentiable cost function of firm k with $c_k(0) = 0$. Notice that the assumption $c_k(0) = 0$ for all $k \in N$ does not impose any loss of generality, since Driessen and Meinhardt (2005) have established that fixed cost arguments aren't crucial for mutual cooperation and the formation of larger cartels (coalitions). However, the variable cost structure is the decisive argument when it comes to joining or from being deterred of entering a cartel. Fixed costs are only crucial for the market entry decision. Fixed costs must already be incurred due to the market entry decisions made by the firms, and therefore the costs are sunk. In addition, let $p : \mathbb{R}_+ \to \mathbb{R}$ be an arbitrary inverse demand function satisfying the canonical assumption in oligopoly situations of being weakly decreasing i.e. $\frac{\partial p}{\partial q}(q) \leq 0, \; \forall q \geq 0$. The corresponding normal form game $\Gamma := \langle N, (\Upsilon_k, x_k)_{k\in N}\rangle$ of the oligopoly situation $\langle N, (\omega_k)_{k\in N}, (c_k)_{k\in N}, p\rangle$ is defined by the payoff (profit) functions $x_k, k \in N$, such that

$$x_k((\upsilon_l)_{l\in N}) := p(q) \cdot \upsilon_k - c_k(\upsilon_k) \qquad \text{with} \quad q := \sum_{l\in N} \upsilon_l. \tag{27}$$

The strategy $\upsilon_k \in \Upsilon_k$ for any firm $k \in N$ represents the quantity sold by firm k. The vector of quantities sold by all firms is given by $\boldsymbol{\upsilon} \in \prod_{k\in N} \Upsilon_k$, though the strategy vector of opponents of firm k is denoted by $\upsilon_{-k} := (\upsilon_i)_{i\in N\backslash\{k\}} \in \Upsilon_{N\backslash\{k\}}$. Note that in a Cournot market the market price is determined by the total quantity sold in the market, i.e. $q := \sum_{l\in N} \upsilon_l$, and therefore under oligopolistic rivalry (cf. Vives 1999). Formula (27) captures the fact that the payoff to the firm k depends on its individual output decision υ_k and on the total production of its opponents $q_{-k} := \sum_{l \in N\setminus\{k\}} \upsilon_l$, whereas the expression $p(q) \cdot \upsilon_k$ represents the revenue of firm k.

When encountered with tacitly or non-tacitly colluding firms in a Cournot market, the usual prediction is that firms which reached an explicit or implicit contract have an incentive to cheat and to deviate from the agreement. However, firms have an incentive to reach agreements whenever a cartel or merger can distribute to its member firms at least as much as each firm can obtain by operating independently. Firms can be better off through cooperation than by acting alone. In contrast to the predicted instability of cartels in Cournot situations by non-cooperative game theory, there is, nevertheless, empirical evidence that firms stick to a cooperative arrangement on output decisions despite of the incentive scheme. This view has been theoretically supported by Zhao (1999b), Norde et al. (2002), and Driessen and Meinhardt (2001, 2005, 2010), who have established that the associated α-value games are convex games under regular economical conditions.

Recall that convexity offers strong incentives for mutual cooperation due to over proportional surpluses which are attainable by large scale operation. Moreover, the two preconditions of a monopoly merger (i.e., profitability and non-empty core, Zhao (2009) are both satisfied, we can expect that firms want to form a monopoly, and we can deal with the question of the conditions of a stable cartel agreement. This requires the introduction of communication among member firms to permit them to discuss their joint strategies to produce larger profits. Through communication, proposals, claims, and arguments can be exchanged to distribute the monopoly proceeds to everyone's satisfaction. Then partners can agree that a specific compromise is justified and binds them, i.e., they attain a situation of compliance (cf. Ostmann and Meinhardt 2007, 2008).

6.2 *Characteristic Function Forms*

To obtain from the normal form game Γ, as given by (27), a game of characteristic function form (with transferable utility) we first focus on the α- and β-value games. Observe that for every coalition (cartel,trust) $S \subseteq N$ we denote its strategy set by $\Upsilon_S := \prod_{k \in S} \Upsilon_k = \prod_{k \in S} [0, \omega_k]$. A possible payoff distribution of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) = \sum_{k \in S} x_k$.

For any $S \subseteq N$, $S \neq \emptyset$, write $\boldsymbol{\upsilon}_S := (\upsilon_k)_{k \in S} \in \Upsilon_S$ and $\mathbf{y}_{N\setminus S} := (y_k)_{k \in N\setminus S} \in \Upsilon_{N\setminus S}$ and let the objective function $f_S : \Upsilon_S \times \Upsilon_{N\setminus S} \to \mathbb{R}$ be given by

$$f_S(\boldsymbol{\upsilon}_S, \mathbf{y}_{N\setminus S}) := [p(\upsilon(S) + y(N\setminus S))] \cdot \upsilon(S) - c_S \cdot \upsilon(S). \qquad (28)$$

Note that the cost term in the preceding equation captures the fact that the technology is transferable, that is, synergy effects are possible among the firms. The smallest marginal cost $c_S := \min_{k \in S} c_k$ is accessible to all member firms of trust S. The trust-wide production technology is determined by the most efficient firm in the cartel.

This class of games has been studied by Zhao (1999a,b,c), Norde et al. (2002), and Driessen and Meinhardt (2010). Oligopoly games without synergy effects have been studied by Norde et al. (2002) and Driessen and Meinhardt (2005). However, the subclass of oligopoly games with transferable technologies, the so-called common pool games, have been studied by Driessen and Meinhardt (2001), and Meinhardt (1999a,b, 2002).

The α**-characteristic function** $v_\alpha : 2^N \to \mathbb{R}$ derived from the normal form game Γ is defined by

$$v_\alpha(S) := \max_{\boldsymbol{v}_S \in \Upsilon_S} \min_{\mathbf{y}_{N\setminus S} \in \Upsilon_{N\setminus S}} f_S(\boldsymbol{v}_S, \mathbf{y}_{N\setminus S}), \tag{29}$$

for all $S \subseteq N, S \neq \emptyset$.

However, the β**-characteristic function** $v_\beta : 2^N \to \mathbb{R}$ derived from the normal form game Γ is defined by

$$v_\beta(S) := \min_{\mathbf{y}_{N\setminus S} \in \Upsilon_{N\setminus S}} \max_{\boldsymbol{v}_S \in \Upsilon_S} f_S(\boldsymbol{v}_S, \mathbf{y}_{N\setminus S}), \tag{30}$$

for all $S \subseteq N, S \neq \emptyset$. These game types have been studied for the first time in a non-cooperative framework by von Neumann and Morgenstern (1944) and in a cooperative setting by Aumann (1959, 1961).

In general, the β-value is equal to or greater than the α-value, this implies a weak incentive to passively react by awaiting the action of the opponents. This means, we have a second mover advantage. Waiting or reaction pays extra while negotiating. Implying that the α- and β-cores may be different whenever they are non-empty. This has some negative side effects for stabilizing an agreement within $\mathscr{C}(v_\alpha)$ that doesn't belong to $\mathscr{C}(v_\beta)$. In this case, we can expect some bargaining difficulties (cf. Meinhardt 2002, pp. 81–83). For cooperative Cournot oligopoly games these values are equal (cf. Zhao (1999b)), and no bargaining difficulties will arise through a passive behaviour. This kind of games are called clear games, since no determinant gap occur that would give an advantage to the second mover to wait for the proposal of the opponents (cf. Jentzsch 1964).

Moreover, the cooperative α- or β-value Cournot oligopoly games with synergy effects are also convex or super-modular, this implies that their respective cores are non-empty and large (cf. Zhao 1999b; Driessen and Meinhardt 2001, 2010). Recall that this gives strong incentives for mutual cooperation, and to find an agreement point inside of the core. An allocation that cannot be blocked by any sub-coalition.

Though another type of arguing can be formulated to justify larger claims, the s-types introduced by Moulin (1981, 1988) for two person games, and generalized for $n \geq 2$ by Ostmann (1986, 1994). For these games, the opposition does not rely on the complete strategy set to stabilize proposals as it is in the case of the α- and β-games; it relies rather on the best response set. Strategies that hurt a coalition are not selected by its members. This kind of argumentation is based on the Stackelberg concept of leader and follower. To introduce this cooperative game model, we need to define the set of best replies of a cartel S w. r. t. the joint actions $\mathbf{y}_{N\backslash S}$ of the opponents $N\backslash S$ given by

$$B_S(\mathbf{y}_{N\backslash S}) := \left\{(\upsilon_l)^*_{l\in S} \in \Upsilon_S \mid f_S(\boldsymbol{\upsilon}^*_S, \mathbf{y}_{N\backslash S}) = \max_{\boldsymbol{\upsilon}_S\in\Upsilon_S} f_S(\boldsymbol{\upsilon}_S, \mathbf{y}_{N\backslash S}\right\}. \tag{31}$$

The **s-type-characteristic function** $v_s : 2^N \to \mathbb{R}$ derived from the normal form game Γ is defined by

$$v_{st}(S) := \max_{\boldsymbol{\upsilon}_S\in\Upsilon_S} \min_{\mathbf{y}_{N\backslash S}\in B_{N\backslash S}(\boldsymbol{\upsilon}_S)} f_S(\boldsymbol{\upsilon}_S, \mathbf{y}_{N\backslash S}), \tag{32}$$

for all $S \subseteq N, S \neq \emptyset$. Notice that we have in general the following relation among these cooperative games $v_\alpha \leq v_\beta, v_{st}$. Though for cooperative Cournot oligopoly games we have $v_\alpha = v_\beta \leq v_{st}$, hence, $\mathscr{C}(v_{st}) \subseteq \mathscr{C}(v_\alpha) = \mathscr{C}(v_\beta)$, which gives a first mover advantage (cf. Moulin 1981, 1988). For convenience sake's, we set $v := v_\alpha = v_\beta$.

It was established by Meinhardt (2002) for the subclass of symmetric common pool games that the resultant coalitional s-type values are too large to be satisfied simultaneously. Subjects are referring on too excessive demands within this bargaining scenario. This implies that the s-type-core is empty, if some overuse is possible. The implementation of a stable contract within a monopoly cannot be achieved, and obstruction is not held to account.

A cooperative game with transferable utility represents a virtual bargaining situation where arguments as claims or proposals can be exchanged through communication to reach an agreement. Although the communicational aspect is not visible by the characteristic function, it is, nevertheless, of fundamental importance of how we have to read and understand a TU game. A value of the characteristic function reflects the bargaining power of a coalition, and by looking on all coalitions to which a particular player can belong, we implicitly specify the bargaining power of the players. Since the value of a coalition reflects its power, it is therefore the value from which the coalition can not be prevented from under a certain kind of arguing by the joint actions of the opponents. Only the three game types introduced above fall within the realm of the reasoning of exchanging arguments through communication.

It is, of course, questionable of how realistic it is that the opponents are relying on arguments that would hurt them if they are carried out as in the case of α and β-value. But they capture the aspect of exchanging arguments by proposal and counter-proposal. In the Cournot case, the opponents could argue as follows to get a cartel agreement in the grand coalition: *"We propose a sharing of the market which gives you x, this is more than what you could get by $v(S)$ when we flood the market if the negotiation comes to halt. So it is better for you to accept the proposal."* We observe that this threat is only used to encourage the cartel S to accept the proposal, nothing is carried out here. No flooding of the market occurs at this stage. It should be clear that this is only an argument to motivate the partner to act. It says nothing about what will happen when the negotiation comes to halt. What happened afterwards is of any importance. It doesn't matter, if the opponents carry out their threat or if a non-cooperative Nash equilibrium materializes. The crucial part is that the establishing of a cartel in the grand coalition has failed.

6.3 The Breakup and Loyal Beliefs

In the above scenario, the exchange of the extreme argument to flood the market encourages partners to move and allows them to refrain from larger claims, which makes an agreement within this bargaining framework more likely. This is due that people are able to evaluate opponents, and can therefore better estimate the aspiration level that can still be enforced against the opponents. Thus, the exchange of arguments reveal to actors important information on the underlying bargaining situation to establish mutual trust. This allows them to behave self-constraint.

However, some will ask how fair such a bargaining situation can be when it is based on the exchange of threats, so that people were constraint to behave modest. In contrast, it is not better to rely on a scenario that is based on best responses given the joint action of opponents rather than on incredible threats? Of course, such a scenario can be studied. Even though there cannot exist a specific one, since one has to model of how it is in the interest of outsiders to act. In particular, their possibility to organize themselves is crucial. Nevertheless, the complexity of this issue requires a concentration on certain beliefs of outsiders' behaviour. We characterize the coalitional beliefs on outsiders' behaviour by the γ- and δ-beliefs invented by Hart and Kurz (1983). In the former case, insiders S believe that the outsiders $N\backslash S$ will breakup into singletons, whereas in the latter, insiders S believe that the outsiders $N\backslash S$ stay in or are loyal to coalition $N\backslash S$. This implies that if a certain belief about the formation of outsiders have been materialized, actors within their proper coalition will coordinate their strategies in accordance with their belief. Though subjects belonging to different coalitions have no possibility to negotiate a joint action at this stage.[4]

We employ the concept of Nash equilibrium to a cooperative game theoretical model. This means that an exchange of proposal and counter-proposal cannot anymore be incorporated, since all actors would act simultaneously. No communication among actors of different coalitions is realizable, and a post-merger (Nash)

[4]For other concepts of beliefs of how outsiders are self-organizing, see for an overview (Zhao 2016) or, in particular (Lekeas 2013), for the notion of j-belief.

equilibrium will occur (cf. Zhao 2016).[5] Therefore, no exchange of arguments take place and no negotiation can occur that would motivate partners to move or to grasp additional information about the bargaining situation. As a consequence, actors will not act self-constraint and demand larger claims for their coalition. As we will see, this shrinks the feasible set of agreement points on the second stage of negotiation to obtain acceptable merger condition to form a monopoly.

Recall that a Nash equilibrium fulfills $(\upsilon_k, \mathbf{y}_{-k}) \in (B_k(\mathbf{y}_{-k}), \mathbf{y}_{-k})$ for all $k \in N$. In an oligopoly market with an homogeneous good this is called the Cournot equilibrium or by Zhao (2016) a pre-merger equilibrium.

An arbitrary collection of sets $\Delta := \{S_1, \ldots, S_r\}$ is called a partition, if it satisfies (i) $N = \cup_{k=1}^{r} S_k$, (ii) $S_k \neq \emptyset$ for all k, and (iii) $S_i \cap S_j = \emptyset$ for all $i \neq j$. Representing in a Cournot model the market structure. In addition, the collection of all partitions Δ of N is denoted as Ξ, hence, $\Delta \in \Xi$. Furthermore, define the set of embedded coalitions as $\Lambda := \{(S, \Delta) \,|\, S \in \Delta \in \Xi\}$, and consider the mapping $w : \Lambda \to \mathbb{R}$, that assigns a real number $w(S, \Delta)$ to each embedded coalition $(S, \Delta) \in \Lambda$. This mapping is called a partition function game invented by Thrall and Lucas (1963), and formally represented through the ordered pair $\langle N, w \rangle$. The real number $w(S, \Delta)$ represents the coalition value of S given that the coalition structure Δ has formed.

Under the γ-belief, we have a partition $\Delta_\gamma := \{S, \{k_{s+1}\}, \ldots, \{k_{n-s}\}\} \in \Xi$. However, under the δ-belief, we have a partition $\Delta_\delta := \{S, N\backslash S\} \in \Xi$. Defining $v_\gamma(S) := w(S, \Delta_\gamma)$ for each $S \in 2^N$, we get the γ-coalitional game denoted as $\langle N, v_\gamma \rangle$. Similar defining $v_\delta(S) := w(S, \Delta_\delta)$ for each $S \in 2^N$, we get the δ-coalitional game specified by the ordered pair $\langle N, v_\delta \rangle$.

Assuming that firms in S coordinates their actions from their best response set given the joint strategies $\mathbf{y}^*_{\Delta_\gamma}$ of outsiders $N\backslash S$ under the belief that they are breakup into singletons Δ_γ, the payoff of coalition S under coalition structure Δ_γ is given by $w(S, \Delta_\gamma) = f_S(\boldsymbol{\upsilon}^*_S, \mathbf{y}^*_{\Delta_\gamma})$, whereas $\mathbf{y}^*_{\Delta_\gamma} \in B_{\Delta_\gamma}(\boldsymbol{\upsilon}^*_S)$ is the best response vector of outsiders $N\backslash S$ under partition Δ_γ and the joint action $\boldsymbol{\upsilon}^*_S$ of S. In addition, we claim for the joint strategies $\boldsymbol{\upsilon}^*_S$ of trust S that $\boldsymbol{\upsilon}^*_S \in B_S(\mathbf{y}^*_{\Delta_\gamma})$ holds. Thus, the ordered pair of $(\boldsymbol{\upsilon}^*_S, \mathbf{y}^*_{\Delta_\gamma})$ is a so-called post-merger equilibrium under the γ-belief.

Analogously, if firms in S coordinates their actions from their best response set given the joint strategies $\mathbf{y}^*_{\Delta_\delta}$ of outsiders $N\backslash S$ under the belief that they are loyal to each other Δ_δ, the payoff of coalition S under coalition structure Δ_δ is given by $w(S, \Delta_\delta) = f_S(\boldsymbol{\upsilon}^*_S, \mathbf{y}^*_{\Delta_\delta})$, whereas $\mathbf{y}^*_{\Delta_\delta} \in B_{\Delta_\delta}(\boldsymbol{\upsilon}^*_S)$ is the best response vector of outsiders $N\backslash S$ under partition Δ_δ and the joint action $\boldsymbol{\upsilon}^*_S$ of S. Similar to the γ-belief from above, we claim for the joint strategies $\boldsymbol{\upsilon}^*_S$ of trust S that $\boldsymbol{\upsilon}^*_S \in B_S(\mathbf{y}^*_{\Delta_\gamma})$ holds. Thus, the ordered pair of $(\boldsymbol{\upsilon}^*_S, \mathbf{y}^*_{\Delta_\delta})$ is a so-called post-merger equilibrium under the δ-belief.

[5]Notice that at this stage is nothing produced or that a Nash equilibrium materializes. This is simply a virtual world that allows actors to receive information about the bargaining power from one another to enter in a negotiation of splitting the proceeds of mutual cooperation.

6.4 Pre-Kernel Outcomes Under Different Cooperative Game Models

To advance our understanding of compliance on agreements referring to the pre-kernel division rule, we start with an oligopoly situation $\langle N, (\omega_k)_{k\in N}, (c_k)_{k\in N}, p\rangle$ with four firms $N = \{1, 2, 3, 4\}$ producing a homogeneous good and having production capacities given by the capacity vector $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \{5, 8, 15, 20\}$. The individual marginal costs are given by $\{c_1, c_2, c_3, c_4\} = \{1, 2, 3/2, 4\}$. Thus, the cost functions are increasing in its arguments. Furthermore, we assume that the parameters of the linear inverse demand function $a - b \cdot q$ are given by $\{a, b\} = \{20, 1/2\}$. Then the profit function is specified for each firm k by $x_k(q) := p(q) \cdot v_k - c_k \cdot v_k$. From which we derive the corresponding normal form game, and then its Cournot or pre-merger equilibrium is quantified through $(55/2, 36, 95/2, 45/4)$.[6]

From the underlying normal form game, we compute four different cooperative oligopoly games reflecting a different exchange of proposals and beliefs of insiders of how outsiders will self-organize. Table 2 shows the coalitional values of the α-,γ-,δ-value, and s-type game as well as the corresponding pre-kernel or pre-nucleolus solutions.

The expected relation among the α-,γ-, and δ-value game is $v := v_\alpha \leq v_\gamma \leq v_\delta$ implying that $\mathscr{C}(v_\delta) \subseteq \mathscr{C}(v_\gamma) \subseteq \mathscr{C}(v_\alpha)$ must hold. This relation can only

Table 2 List of different game types with pre-kernel solutions

Game	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4\}$	
v_α^{a}	0	0	9/8	2	25/2	18	81/2	2	
v_γ^{b}	55/2	36	2025/32	95/2	72	1369/18	968/9	45/4	
v_δ^{c}	85/2	52	169/2	72	800/9	72	968/9	338/9	
v_{st}^{d}	85/2	52	169/2	315/4	100	81	121	169/4	
Game	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N	CV	ZM
v_α^{a}	225/8	32	529/8	72	225/2	128	361/2	Y	Y
v_γ^{b}	961/18	50	169/2	72	225/2	128	361/2	N	N
v_δ^{c}	169/2	578/9	169/2	72	225/2	128	361/2	N	N
v_{st}^{d}	1521/16	289/4	1521/16	81	100	128	361/2	N	N

CV Convex Game ; *ZM* Zero-Monotonic Game
Note: Computation performed with TuOligopoly, and MatTuGames
[a]Pre-Kernel/Pre-Nucleolus: $(105/4, 34, 931/16, 993/16)$
[b]Pre-Kernel/Pre-Nucleolus: $(40, 2729/54, 1675/27, 754/27)$
[c]Pre-Kernel/Pre-Nucleolus: $(721/18, 128/3, 188/3, 316/9)$
[d]Pre-Kernel/Pre-Nucleolus: $(1365/32, 1771/48, 3055/48, 3581/96)$

[6]The forthcoming results can be replicated with our Mathematica package *TuOligopoly* (2016b) conceived for modelling Industrial Cooperation, and *TuGames* (2016a). The former can be made available upon request.

be expected for unlimited capacities within a Cournot situation (cf. Zhao 2016). Though, we observe from Table 2 that for limited capacities, like in ours, the relation $v_{\gamma} \leq v_{\delta}$ does not hold. We notice that for trust $S = \{2, 3\}$, we have $v_{\gamma}(S) = 76.05 > 72 = v_{\delta}(S)$. This is due that the opponent firm $\{1\}$ is producing at its capacity level with $\upsilon_1 = \omega_1 = 5$, whereas it is not enough that firm $\{4\}$ produces within its constraint set at $\upsilon_4 = 22/3 < \omega_4 = 20$. This allows cartel S under the γ-belief to improve its capacity utilization to achieve higher returns. In contrast, under the δ-belief the firms 1 and 4 form the anti-trust $\{1, 4\}$ that can produce at its interior solution $\upsilon(\{1, 4\}) = 13$.

Similar, we obtain $v_{\delta} \not\leq v_{st}$ where we notice that for coalition $S = \{1, 3, 4\}$ the relation $v_{\delta}(S) = 225/2 > 100 = v_{st}(S)$ is fulfilled, which is caused by coalition $\{2\}$ producing at its capacity level at the post-merger equilibrium, i.e., $\upsilon_2 = \omega_2 = 8$. This allows cartel $\{1, 3, 4\}$ under the δ-value game to exploit better its capacity level producing a higher profit than in the s-type game. Under the latter belief firm $\{2\}$ produces at a first-order condition, which coincides with its capacity constraint, hence $\upsilon_2 = 8$. However, if we only allow interior solutions, then it is very likely that we obtain $v_{\delta} \leq v_{st}$. Finally, we have to assert that $\mathscr{C}(v_{\delta}) = \emptyset$, whereas $\mathscr{C}(v_{\gamma})$ and $\mathscr{C}(v_{\alpha})$ are non-empty. More precisely, the volume of $\mathscr{C}(v_{\gamma})$ is 5.78% of $\mathscr{C}(v_{\alpha})$. Thus, the γ-core is relatively small in comparison to the α-core.

Moreover, it should be evident by comparing the coalitional values of the δ-game and s-type game that the core of the latter is also empty, hence $\mathscr{C}(v_{st}) = \emptyset$. Recall that the pre-kernel solutions of the first two games must belong to their respective core. Due to the largeness of the core $\mathscr{C}(v_{\alpha})$, it contains also the pre-kernel solutions derived from the other games. The inclined reader may want to check that this not the case for $\mathscr{C}(v_{\gamma})$.

Notice that the claims for δ-value and s-type game are too large implying that the associated cores are empty. This does not account for mutual cooperation into the grand coalition (monopoly). Ostmann (1986) argued that subjects may judge an argumentation as invalid if they notice that persisting on this reasoning makes it impossible to reach an agreement. Whenever subjects want to realize the proceeds of a mutual cooperation, they have to refrain from claims that can be considered too excessive by the opponent, which can subsequently lead to an immediate end of the negotiation. As we realize, the δ-value or s-type oligopoly game is an inappropriate concept to justify a proposal that could be acceptable for each party under the monopoly situation.

In Fig. 1 we have depicted the $\mathscr{C}(v_{\alpha})$ (coloured in yellow) in connection with the imputation set given by the triangular skeleton, the Shapley value (enlarged blue point), the pre-kernel (enlarged red point), which is also the pre-nucleolus of the α-value game. We notice, that the Shapley value is the centre of gravity of the extreme point of $\mathscr{C}(v_{\alpha})$.

Similar in Fig. 2, there the $\mathscr{C}(v_{\gamma})$ is drawn together with the mentioned solution schemes. Noticeable is that the γ-core is small in proportion to the α-core, even though the γ-core is relatively large in comparison to the imputation set, i.e., it infills 60% of the latter. Again the Shapley value is in the core, and can therefore qualify as an alternative fair division rule under a γ-bargaining scenario.

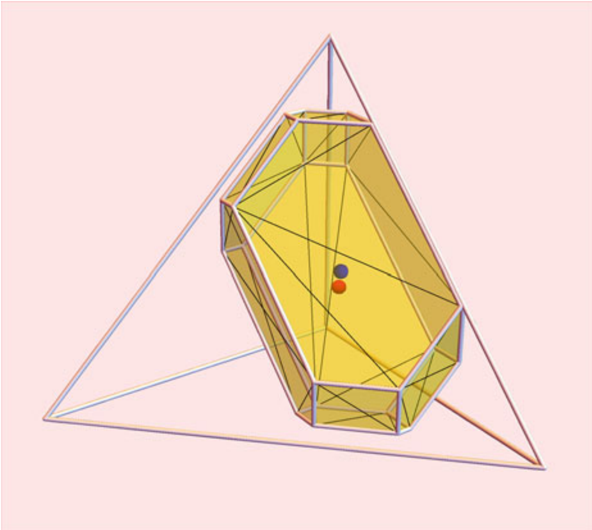

Fig. 1 Pre-kernel, Shapley value, imputation set, and core of game v_α. Own representation

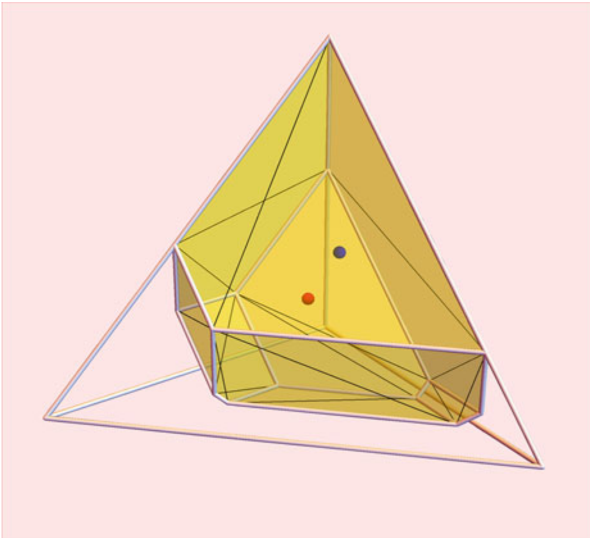

Fig. 2 Pre-kernel, Shapley value, imputation set, and core of game v_γ. Own representation

Moreover, by Fig. 3 we notice that the volumes of both imputation sets are different even though they seem to be equal in the Figs. 1 and 2. This figure also confirms our theoretical expectation that the γ-core must be included in the α-core. In addition, we observe that the pre-kernel (small red dot) of the γ-value game is part of the larger α-core, but not vice versa.

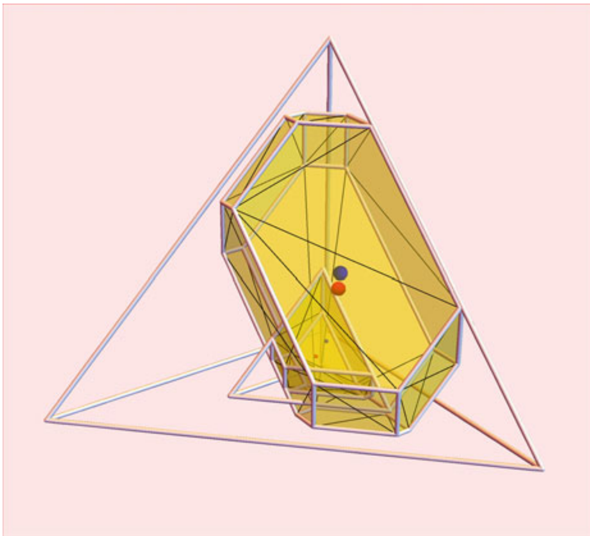

Fig. 3 Interlocking α- and γ-core with imputation sets, pre-kernel/Shapley value solutions. Own representation

Even though we have in each case a distribution of the monopoly proceeds in accordance with the norms of distributive justice (set of axioms) referring to the pre-kernel, we have identified, nevertheless, some bargaining difficulties to establish such a fair division. If subjects change the bargaining agenda from the α-value setting to a γ-,δ-value or s-type context, then a formerly agreement on the pre-kernel division rule is vulnerable by its own norms of distributive arbitration. Thus, in order to stabilize an agreement point, it is not only important to find a consensus on the principles of justice, but also on the bargaining agenda to avoid any form of obstruction. Referring, for instance, to **RG**- or **CRG**-properties is not sufficient to enforce a pre-kernel agreement. Here, consistency must be defined much broader.

6.5 A Stable Pre-Kernel Agreement

By the foregoing discussion, we have established that a pre-kernel agreement is not robust against a change in the bargaining agenda. Therefore, we shall study the stability of an agreement within a fixed game setting. In this respect, the α-value game offers a scenario in which actors' bargaining power is too weak to exploit one another. Consequently, we can expect stable outcomes.

From Table 2 we realize that game v is convex, and has therefore a large core. It is a common held belief that a large core makes it impossible to find an allocation inside the core on which subjects can agree upon, since there are infinite many conceivable agreement points. However, the largeness of the core is not a drawback. It allows to stabilize a proposal by objection and counter-objection. For instance, if subjects made terms—while exchanging proposals on the α-argumentation—that they want to distribute the proceeds of mutual cooperation in accordance with the principles of distributive justice related to the pre-kernel, then this solution cannot only be stabilized on these grounds, but also on the norms of distributive arbitration based on the core. This is due that the pre-kernel of game v is in the core. In addition, the largeness of the core implies that a small perturbation of the underlying market structure does not destroy the core.[7] Therefore, an agreement point selected from it cannot be obstructed by principles of distributive justice related to the core. The incentive to cooperate remains valid.

From these considerations we notice that an agreement on a rule of distributive justice needs not to be binding. Having made their choice, for instance, through a Rawlsian type of decision device about their preferred division scheme reflecting their objective norms of fairness, subjects must obey the underlying system of axioms by their self-interest. There is no possibility to renegotiate the outcome while formulating an objection. Such an objection must be based on a different set of unaccepted principles, and can therefore not be considered as valid in accordance with the underlying rule of distributive justice. In contrast, if the agreement is binding, the proposed solution can be enforced by an unbiased arbitrator or a legal system. This implies that an obstruction is impossible, even under the existence of strong incentives to deviate from it.

In addition, a pre-kernel point of a TU game cannot only be supported by means of objection and counter-objection, or on the basis of distributive arbitration, such a point is also robust against a certain redistribution of claims among actors.

From this perspective, it has been established by Meinhardt (2013b, Sect. 7.6) that a pre-kernel element of a TU game is replicable as a pre-kernel solution of a related game, whenever the pre-kernel element of the default game belongs to a payoff equivalence class, which satisfies the non-empty interior property. Then a full dimensional ellipsoid can be inscribed from which some parameter bounds can be specified within coalitional values can be varied without destroying the pre-kernel properties of the solution from the default game. These bounds specify a redistribution of the bargaining power among coalitions while supporting the selected pre-imputation still as a pre-kernel point. By the work of Meinhardt (2015b), we can even say that this pre-kernel solution is also the sole pre-kernel point for a set of linear independent games whenever the default pre-kernel is a singleton. We quote this important result through:

Theorem 11.2 (Meinhardt 2015b) *Assume that the payoff equivalence class $[\boldsymbol{\gamma}]$ induced from TU game $\langle N, v \rangle$ has non-empty interior. In addition, assume that game $\langle N, v \rangle$ has a singleton pre-kernel such that $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v) \subset [\boldsymbol{\gamma}]$ is satisfied, then the pre-kernel $\mathcal{P}r\mathcal{K}(v^{\mu})$ of a related TU game $\langle N, v^{\mu} \rangle$ consists of a single point, which is given by $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v^{\mu})$.*

Proof The proof is given by Meinhardt (2015b).

[7] In this sense, the strong ϵ-core can be understood.

Let us now illustrate on the α-value game from Table 2 the stability of a pre-kernel agreement against a redistribution of the coalitional claims. For the convex game v the pre-kernel coalesces with the pre-nucleolus. It is a singleton, which is given by: $\nu(v) = \mathcal{P}r\mathcal{K}(v) = \{105/4, 34, 931/16, 993/16\}$. Moreover, this imputation is an interior point, thus the non-empty interior property of the payoff equivalence class is fulfilled. Hence, by Meinhardt (2013b, Thm. 7.6.1) a redistribution of power among coalitions can be attained while supporting the imputation $\{105/4, 34, 931/16, 993/16\}$ still as a pre-kernel element for a set of related games (see also Meinhardt 2015b, Thm. 4.1). Getting a null space $\mathcal{N}_{\mathcal{W}}$ with maximum dimension, we set the scaling parameter μ to 6. Then the rank of the null space $\mathcal{N}_{\mathcal{W}}$ is 4, and we can derive at most 11-linear independent games which replicate the point $\{105/4, 34, 931/16, 993/16\}$ as a pre-kernel element. Theorem 11.2 even states that this point is also the sole pre-kernel point, hence the pre-kernel coincides with the pre-nucleolus for these games. This can be figured out through Table 3. Moreover, from the set of games $\{v, \ldots, v_{11}\}$, a convex set in the game space can be constructed, where each TU game in this set has exactly the above element as its sole pre-kernel point (cf. Meinhardt 2015b, Thm. 5.1). In addition, non of the TU games v_{γ}, v_{δ} or v_{st} from Table 2 belongs to $conv\{v, \ldots, v_{11}\}$.

Notice that from these 12-linear independent related games 9 are convex, and that game v_{12} is the Chebyshev centre derived from set of linear independent games. Only three games, namely $\{v_3, v_5, v_7\}$ are not zero-monotonic. However, all games have a non-empty core and are semi-convex. The cores of the games have between 22 and 24-vertices, and have volumes that range from approximately 95–125% of the default core. TU game v_2 has the largest and v_3 the smallest core.[8]

By the foregoing discussion, we have identified a stable bargaining scenario where a settlement of an agreement is not problematic while referring to the principles of distributive justice from the pre-kernel.

7 Concluding Remarks

We illustrated of how an agreement related to the pre-kernel solution can be understood as a fair division rule. In this respect, we have reviewed the generalized conjugation theory from convex analysis to offer a better understanding and broader interpretation of the pre-kernel solution. From this building blocks we passed to a study of several cooperative game models obtained from a Cournot situation to give some understanding of fulfilment and on the robustness of non-binding agreements based on the norms of distributive justice referring to the pre-kernel scheme. Even though we have confined ourselves to cooperative Cournot models, we invite the

[8]The example can be reproduced while using our MATLAB toolbox *MatTuGames* (2015a). The results can also be verified with our Mathematica package *TuGames* (2016a).

Table 3 List of games which possess the same unique pre-kernel[a] as v_α ($\mu = 6$)

Game	{1}	{2}	{1, 2}	{3}	{1, 3}	{2, 3}	{1, 2, 3}	{4}	
v_α	0	0	9/8	2	25/2	18	81/2	2	
v_1	−31/11	27/16	15/7	305/76	88/7	137/7	2603/62	25/13	
v_2	−24/7	−14/5	−89/18	−7/9	65/9	194/15	103/3	30/13	
v_3	−3/16	3	9/10	−2/7	201/17	239/13	317/8	47/23	
v_4	0	0	25/21	75/38	199/18	439/28	1111/28	40/7	
v_5	0	0	−5/9	43/16	51/5	143/8	297/8	6/7	
v_6	0	0	1/2	9/4	111/8	15	426/11	13/19	
v_6	0	0	16/3	0	4	−5/47	143/19	0	
v_7	0	0	−5/4	104/35	13	86/5	217/6	11/6	
v_7	0	0	16/3	0	4	−5/47	143/19	0	
v_8	0	0	33/10	10/9	71/7	193/12	912/23	10/17	
v_9	0	0	14/9	11/6	65/7	128/7	408/11	19/11	
v_{10}	0	0	13/5	7/5	350/27	77/5	387/10	14/9	
v_{11}	0	0	6/7	19/9	109/9	88/5	253/7	27/10	
v_{12}	−3/11	1/6	4/5	45/23	117/10	103/6	267/7	31/15	
Game	{1, 4}	{2, 4}	{1, 2, 4}	{3, 4}	{1, 3, 4}	{2, 3, 4}	*N*	CV	ZM
v_α	225/8	32	529/8	72	225/2	128	361/2	Y	Y
v_1	1997/74	356/11	477/7	73	1827/16	1377/11	361/2	Y	Y
v_2	703/27	241/8	1837/29	1325/19	1097/10	872/7	361/2	Y	Y
v_3	428/15	569/17	383/6	1563/22	231/2	2045/16	361/2	N	N
v_4	301/10	757/22	661/10	526/7	225/2	128	361/2	Y	Y
v_5	249/8	309/10	1069/16	644/9	225/2	128	361/2	N	N
v_6	441/17	35	531/8	569/8	225/2	128	361/2	Y	Y
v_7	728/27	1543/49	671/10	1103/15	225/2	128	361/2	N	N
v_8	469/18	208/7	1109/17	1118/15	225/2	128	361/2	Y	Y
v_9	352/13	161/5	1319/20	1135/16	225/2	128	361/2	Y	Y
v_{10}	279/10	303/10	1638/25	3314/47	225/2	128	361/2	Y	Y
v_{11}	347/12	164/5	1126/17	875/12	225/2	128	361/2	Y	Y
v_{12}	2715/97	257/8	793/12	940/13	338/3	1405/11	361/2	Y	Y

CV Convex Game ; *ZM* Zero-Monotonic Game
Note: Computation performed with MatTuGames
[a]Pre-Kernel/Pre-Nucleolus: $(105/4, 34, 931/16, 993/16)$

reader to extend this kind of analysis to issues which arise from Management Accounting like transfer pricing to allocate the resultant proceeds by a profit- and fairness-related method to the associated business units.

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References

Aumann RJ (1959) Acceptable points in general cooperative n-person games. In: Tucker AW, Luce RD (eds) Contributions to the theory of games, vol IV. Princeton University Press, Princeton, pp 287–324

Aumann RJ (1961) A survey on cooperative games without side payments. In: Shubik M (ed) Essays in mathematical economics in honor of Oskar Morgenstern. Princeton University Press, Princeton, pp 3–27

Chang C, Hu C-C (2016) A non-cooperative interpretation of the kernel. IJGT, doi:10.1007/s00182-016-0529-7

Curiel I (1997) Cooperative game theory and applications. Kluwer Academic Publishers, Boston

Davis M, Maschler M (1965) The kernel of a cooperative game. NRLQ 12(3/4):223–259

Driessen TSH, Meinhardt HI (2001) (Average-) convexity of common pool and oligopoly TU-games. IGTR 3(2):141–158

Driessen TSH, Meinhardt HI (2005) Convexity of oligopoly games without transferable technologies. MSS 50(1):102–126

Driessen TSH, Meinhardt HI (2010) On the supermodularity of homogeneous oligopoly games. IGTR 12(4):309–337

Hart S, Kurz M (1983) Endogenous formation of coalitions. Econometrica 51(4):1047–1064

Jentzsch G (1964) Some thoughts on the theory of cooperative games. In: Dresher M, Shapley LS, Tucker AW (eds) Advances in game theory. Princeton University Press, Princeton, pp 407–442

Kopelowitz A (1967) Computation of the kernels of simple games and the nucleolus of N-person games. Technical report, RM 31, Research Program in Game Theory and Mathematical Economics, The Hebrew University of Jerusalem, 1967. Mimeo

Lekeas PV (2013) Coalitional beliefs in cournot oligopoly TU games. IGTR 15(1):1–21

Martínez-Legaz J-E (1996) Dual representation of cooperative games based on Fenchel-Moreau conjugation. Optimization 36(4):291–319

Maschler M, Peleg B, Shapley LS (1979) Geometric properties of the kernel, nucleolus, and related solution concepts. MOR 4(4):303–338

Meinhardt HI (1999a) Common pool games are convex games. JPubET 1(2):247–270

Meinhardt HI (1999b) Convexity and k-convexity in cooperative common pool games. Discussion Paper 11, Institute for Statistics and Economic Theory, University Karlsruhe

Meinhardt HI (2002) Cooperative decision making in common pool situations. Springer, Heidelberg

Meinhardt HI (2013a) The Matlab game theory toolbox MatTuGames version 0.4: an introduction, basics, and examples. Technical report, Karlsruhe Institute of Technology (KIT)

Meinhardt HI (2013b) The pre-kernel as a tractable solution for cooperative games: an exercise in algorithmic game theory. Springer, Berlin

Meinhardt HI (2014) A note on the computation of the pre-kernel for permutation games. Technical Report MPRA-59365, Karlsruhe Institute of Technology (KIT). http://mpra.ub.uni-muenchen.de/59365/

Meinhardt HI (2015a) MatTuGames: A Matlab toolbox for cooperative game theory. Karlsruhe Institute of Technology (KIT), Karlsruhe. http://www.math\discretionary-works.com/matlabcentral/fileexchange/35933-mattugames. Version 0.8

Meinhardt HI (2015b) On the single-valuedness of the pre-kernel. ArXiv e-prints, http://arxiv.org/abs/1510.03705

Meinhardt HI (2016a) TuGames: A Mathematica package for cooperative game theory. Karlsruhe Institute of Technology (KIT). http://library.wol\discretionary-fram.com/in\discretionary-focenter/MathSource/5709. Version 2.4

Meinhardt HI (2016b) TuOligopoly: A Mathematica package for industrial cooperation. Karlsruhe Institute of Technology (KIT). Version 0.1, Mimeo

Meseguer-Artola A (1997) Using the indirect function to characterize the kernel of a TU-game. Technical report, Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona. Mimeo
Moulin H (1981) Deterrence and cooperation: a classification of two person games. EER 15(2):179–193
Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge
Norde H, Do KHP, Tijs SH (2002) Oligopoly games with and without transferable technologies. MSS 43(2):187–207
Ostmann A (1986) Limits of rational behavior in cooperatively played normalform games. In: Tietz R, Albers W, Selten R (eds) Bounded rational behavior in experimental games and markets. Springer, Berlin, pp 317–332
Ostmann A (1994) A note on cooperation in symmetric common dilemmas. Mimeo
Ostmann A, Meinhardt HI (2007) Non-binding agreements and fairness in commons dilemma games. CEJOR 15(1):63–96
Ostmann A, Meinhardt HI (2008) Toward an analysis of cooperation and fairness that includes concepts of cooperative game theory. In: Biel A, Eek D, Garling T, Gustafson M (eds) New issues and paradigms in research on social dilemmas. Springer, Berlin, pp 230–251
Peleg B, Sudhölter P (2007) Introduction to the theory of cooperative games, 2nd edn. Springer, Berlin
Serrano R (1997) Reinterpretating the kernel. JET 77(1):58–80
Shapley LS (1971) Cores of convex games. IJGT 1(1):11–26
Stearns RE (1968) Convergent transfer schemes for N-person games. Trans Am Math Soc 134(3):449–459
Thrall RM, Lucas WF (1963) N-person games in partition function form. NRLQ 10(1):281–298
Vives X (1999) Oligopoly pricing: old ideas and new tools. MIT Press, Cambridge
von Neumann JM, Morgenstern O (1944) Theory of games and economic behavior. Princeton University Press, Princeton
Zhao J (1999a) A β-core existence result and its application to oligopoly markets. GEB 27(1):153–168
Zhao J (1999b) A necessary and sufficient condition for the convexity in oligopoly games. MSS 37(2):189–204
Zhao J (1999c) The existence of TU α-core in normal form games. IJGT 28(1):25–34
Zhao J (2009) Estimating merging costs by merger preconditions. Theor Decis 66(4):373–399
Zhao J (2016) TU oligopoly games and industrial cooperation. In: Corchon L, Marini M (eds) Handbook of game theory and industrial organization. Edward Elgar, Cheltenham, pp 1–41

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A Talmudic Approach to Bankruptcy Problems

Juan D. Moreno-Ternero

Abstract Bankruptcy problems arise when agents hold claims against a certain (perfectly divisible) good, and the available amount is not enough to satisfy them all. A great source of inspiration to solve these problems emanates from the Talmud. We survey classical and recent contributions to the literature that constitute this Talmudic approach to bankruptcy problems.

Keywords Bankruptcy problems • Equal awards • Equal losses • TAL-family • Talmud rule

Contents

1 Introduction

The Babylonian Talmud is an ancient collection of writings that constitutes a central text of Rabbinic Judaism. Therein, several instances of what we call bankruptcy problems, and specific recommendations to solve them, are presented (see, for instance, Aumann (2002) for a leisurely discussion).

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A canonical case is the so-called contested garment problem, in which two men disagree on the ownership of a garment. The first man claims half of it, and the other claims it all. Assuming both claims are made in good faith, the Talmud recommends that the first agent gets one fourth of the garment, whereas the second agent gets three fourths of the garment.

Another well-known case is the following. There are three creditors; the debts are 100, 200 and 300. When the estate is 100, it should be divided equally. If the estate is 300 it should be divided proportionally. Finally, if the estate is 200, the recommendation is to allocate the first creditor 50 and the other two creditors 75.

It was not until 30 years ago that a rationale for these, apparently unrelated, recommendations was provided. Aumann and Maschler (1985) presented what is now dubbed as the *Talmud* rule, which explains all those recommendations.

This survey is about the Talmud rule, and the ramifications that originated in the sizable literature on bankruptcy problems. For more general reviews and surveys of that literature, whose seminal work is O'Neill (1982), the reader is referred to Thomson (2003, 2014, 2015).

A bankruptcy problem refers to a situation in which one has to distribute a good whose available amount is not enough to cover all agents' demands (claims) on it. A variety of situations, like the bankruptcy of a firm (our running interpretation throughout this survey), the collection of a given amount of taxes, or the division of an insufficient estate fit this definition. Obvious ways to solve these problems amount to allocate awards proportionally to claims, or to impose equal awards or losses (subject to the condition that agents neither receive a negative amount not a higher amount than their claims). The Talmud rule proposes an alternative (and ingenious) procedure to solve these problems. More precisely, it applies equal division until the claimant with the smallest claim has obtained one half of her claim. Then, that agent stops receiving additional units and the remaining amount is divided equally among the other agents until the claimant with the second smallest claim gets one half of her claim. The process continues until every agent has received one half of her claim, or the available amount is distributed. If there is still something left after this process, agents are invited back to receive additional shares. Now agents receive additional amounts sequentially starting with those with larger claims and applying equal division of their losses.

There exist axiomatic as well as game-theoretical foundations for this rule and we shall survey the main ones here. We shall also be concerned with several alternatives and generalizations of this rule that have been considered in the literature.

2 The Model

We study bankruptcy problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of

N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i\in N}$ the claims profile.[1] A *(bankruptcy) problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. Let $C \equiv \sum_{i\in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the domain of bankruptcy problems with population N and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i\in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *rule* on $\mathcal{D}$, $R: \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Each rule R has a *dual* rule R^d defined as $R^d(N, c, E) = c - R(N, c, C - E)$, for each $(N, c, E) \in \mathcal{D}$. A rule is *self-dual* if it coincides with its dual.

We now introduce some axioms that formalize standard properties of rules within the literature.

Equal Treatment of Equals is arguably the most basic axiom one could consider in this model. It states that agents with equal claims should receive equal amounts. Formally, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, we have $R_i(N, c, E) = R_j(N, c, E)$, whenever $c_i = c_j$.

We now consider two independence properties, known as *Claims Truncation Invariance* and *Minimal Rights First*.[2] The former postulates that the part of a claim that is above the endowment should be ignored. That is,

$$R(N, c, E) = R(N, t(N, c, E), E),$$

where $t_i(N, c, E) = \min\{E, c_i\}$ for each $i \in N$. The latter ensures each agent the portion of the endowment that is left to her when the claims of all other agents are fully honored (provided this amount is nonnegative) and divides the remainder according to revised claims. Formally,

$$R(N, c, E) = m(N, c, E) + R(N, c - m(N, c, E), E - M(N, c, E)) ,$$

where $m_i(N, c, E) = \max\{0, E - \sum_{j\in N\setminus\{i\}} c_j\}$, for each $i \in N$, and $M(N, c, E) = \sum_{i\in N} m_i(N, c, E)$.

We now move to axioms modelling the concept of lower and upper bounds, which have a long tradition of use within the theory of fair allocation. A focal lower bound is the so-called *Average Truncated Lower Bound on Awards*, which is somewhat related to the *Claims Truncation Invariance* axiom considered above. It ensures each agent a minimal share of her individual claim, no matter what the other claims are. In particular, for a problem involving n agents, it establishes that any agent holding a feasible claim (a claim not larger than the endowment) will get at least one nth of her claim. And also that those agents whose individual claims

[1]For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i\in M}$.

[2]These two axioms were studied first by Curiel et al. (1987).

are unfeasible will get at least one nth of the endowment.[3] Formally, a rule R satisfies *Average Truncated Lower Bound on Awards* if, for each $(N, c, E) \in \mathscr{D}$, $R_i(N, c, E) \geq \frac{1}{n} \min\{c_i, E\}$. Its dual property is also an interesting one. This property provides an upper bound to each claimant involved in the problem. Formally, a rule R satisfies *Average Truncated Lower Bound on Losses* if, for each $(N, c, E) \in \mathscr{D}$, $R_i(N, c, E) \leq c_i - \frac{1}{n} \min\{c_i, C - E\}$.

We conclude our inventory of axioms with a principle that has played a fundamental role in axiomatic analysis (e.g., Thomson 2012). *Consistency* states that if some claimants leave with their awards and the problem of dividing among the remaining claimants what is left is considered, these claimants should receive the same awards as initially. Formally, a rule R is *consistent* if for each $(N, c, E) \in \mathscr{D}$, each $M \subset N$, and each $i \in M$, we have $R_i(N, c, E) = R_i(M, c_M, E_M)$, where $E_M = \sum_{i \in M} R_i(N, c, E)$.

3 The Talmud Rule

The Talmud rule, introduced by Aumann and Maschler (1985), focusses on equal awards or equal losses depending on whether the endowment falls short or exceeds one half of the aggregate claim, using half-claims instead of claims. Formally,

Talmud rule, T: For each $(N, c, E) \in \mathscr{D}$, and each $i \in N$,

$$T_i(N, c, E) = \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \leq \frac{1}{2}C \\ \max\left\{\frac{c_i}{2}, c_i - \mu\right\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} T_i(N, c, E) = E$.

The Talmud rule can also be given the following representation, which will be useful for the ensuing discussion. For each $(N, c, E) \in \mathscr{D}$,

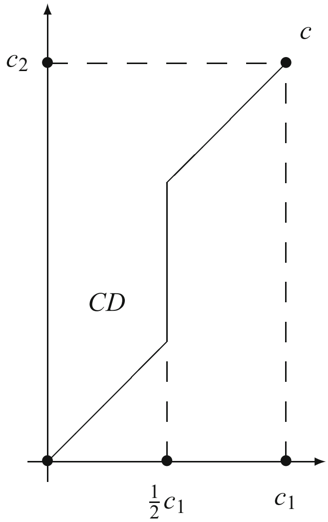
$$T(N, c, E) = \begin{cases} A(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + L(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

That is, for "small" values of E the Talmud rule behaves as the constrained equal awards rule (A) and for "large" values of E as the constrained equal losses rule (L).[4]

[3] The property was introduced by Moreno-Ternero and Villar (2004) under the name of *Securement*.

[4] The constrained equal-awards rule, A, selects, for each $(N, c, E) \in \mathscr{D}$, the vector $(\min\{c_i, \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$. The constrained equal-losses rule, L, selects, for each $(N, c, E) \in \mathscr{D}$, the vector $(\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

Fig. 1 Concede-and-divide



Its two-agent version, also known as *concede-and-divide*, has a particularly appealing form, which we describe next.[5]

Concede-and-divide, *CD*: For each $E \in \mathbb{R}_+$, and each $c = (c_1, c_2) \in \mathbb{R}^2_+$,

$$\begin{cases} CD_1(c,E) = \max\{0, E - c_2\} + \frac{1}{2}(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \\ CD_2(c,E) = \max\{0, E - c_1\} + \frac{1}{2}(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \end{cases}$$

That is, *CD* first concedes to each agent her *minimal rights* and then divides the remainder equally. Figure 1 illustrates the behaviour of *CD* when the vector of claims is fixed and the endowment grows from zero to the aggregate claim (i.e., its "path of awards").

Figure 1 illustrates the "path of awards" of concede-and-divide. A point in the drawing corresponds to the awards that agents receive for a given endowment. The schedule relative to a typical claim c follows the 45° line until it gives both agents half of the smallest claim, then it continues vertically until the endowment equals the highest claim, then again, it follows a line of slope 1 until it reaches the vector of claims.

The following characterization results of concede-and-divide were proved by Dagan (1996) and Moreno-Ternero and Villar (2004, 2006c).[6]

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[5]The name was coined by Thomson (2003). To ease its presentation, we assume $N = \{1, 2\}$, but dismiss it from the definition.

[6]See also Moreno-Ternero (2006).

Theorem 1 *Concede-and-divide is characterized by*

1. *Self-duality and Minimal Rights First.*
2. *Self-duality and Claims Truncation Invariance.*
3. *Equal treatment of equals, Minimal Rights First and Claims Truncation Invariance.*
4. *Self-duality and Average Truncated Lower Bound on Awards.*
5. *Self-duality and Average Truncated Lower Bound on Losses.*
6. *Average Truncated Lower Bound on Awards and Average Truncated Lower Bound on Losses.*
7. *Average Truncated Lower Bound on Awards and Minimal Rights First.*
8. *Average Truncated Lower Bound on Losses and Claims Truncation Invariance.*

Several rules coincide with concede-and-divide in the two-agent case. Among them, only the Talmud rule is consistent. Thus, by means of the so-called Elevator Lemma (e.g., Thomson 2014), we can extend the previous characterizations to the Talmud rule, just appending each statement of Theorem 1 with the axiom of consistency.

4 The TAL-Family

One natural way of generalizing the Talmud rule is obtained by moving the threshold in its definition from one half to any other possible fraction (of the aggregate and individual claims). In doing so, we would obtain a non-countable set of piece-wise linear rules ranging from the constrained equal-awards rule to the constrained equal-losses rule, and having the Talmud rule in the middle. Such a family, known as the TAL-family, was introduced by Moreno-Ternero and Villar (2006a). Formally:

TAL-family, R^{θ}: For each $\theta \in [0, 1]$, each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$$R_i^{\theta}(N, c, E) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \leq \theta C \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} R_i^{\theta}(N, c, E) = E$.

A systematic analysis of the TAL-family was provided by Moreno-Ternero and Villar (2006a,b). Regarding the properties introduced in Sect. 2, all rules within this family satisfy equal treatment of equals and consistency. It is interesting to remark that the family behaves in a perfectly symmetric way regarding the remaining properties stated above. More precisely, $\theta = \frac{1}{2}$ is the precise value of the parameter that separates the rules in the family that satisfy Claims Truncation Invariance from those that satisfy Minimal Rights First. An analogous behaviour occurs for the Average Truncated Lower Bounds.

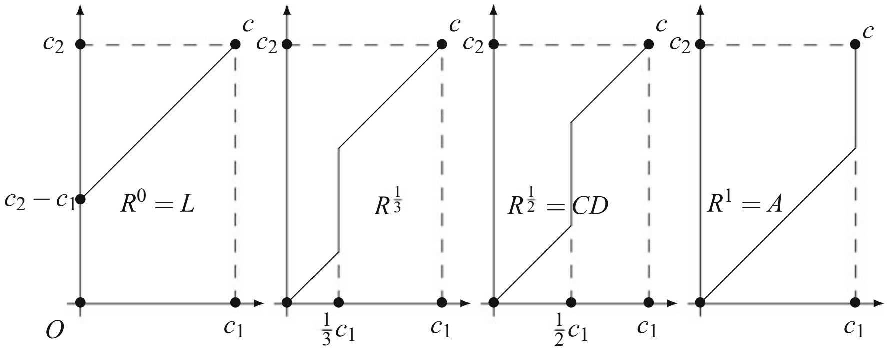

Fig. 2 The TAL-family of rules

Theorem 2 *The following statements hold:*

(i) For each $\theta \in [0, \frac{1}{2}]$, R^{θ} *satisfies Minimal Rights First and Average Truncated Lower Bound on Losses.*

(ii) For each $\theta \in [\frac{1}{2}, 1]$, R^{θ} *satisfies Claims Truncation Invariance and Average Truncated Lower Bound on Awards.*

Figure 2 illustrates the "path of awards" of some rules within the TAL-family for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of awards for c of $R^0 = L$ follows the vertical axis until the average loss coincides with the lowest claim, i.e., until $E = c_2 - c_1$. After that, it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/3}$ follows the 45° line until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = c_2 - \frac{1}{3}c_1$, from where it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/2}$ is that of CD. Finally, the path of awards of $R^1 = A$ follows the 45° line until claimant 1 obtains her whole claim. Then, it is a vertical line until it reaches the vector of claims.

The parameter θ that generates the TAL-family can actually be interpreted as an index of progressivity of the rules within the family. More precisely, given $x, y \in \mathbb{R}^n$ satisfying $x_1 \leq x_2 \leq \ldots \leq x_n$, $y_1 \leq y_2 \leq \ldots \leq y_n$, and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, we say that x *is greater than* y *in the Lorenz ordering* if $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$, for each $k = 1, \ldots, n-1$, with at least one strict inequality. This criterion induces a partial ordering on allocations which reflects their relative spread. When x is greater than y in the Lorenz ordering, the distribution x is unambiguously " more egalitarian" than the distribution y.

We say that a rule R *Lorenz dominates* a rule R', which we write as $R \succsim_L R'$, when for each $(N, c, E) \in \mathscr{D}$, $R(N, c, E)$ is greater than $R'(N, c, E)$ in the Lorenz ordering. The following result, which is due to Moreno-Ternero and Villar (2006b), says that all rules within the TAL-family are fully ranked in terms of the Lorenz dominance criterion.

Theorem 3 *For each pair* $\theta_1, \theta_2 \in [0, 1]$ *with* $\theta_1 \geq \theta_2$, $R^{\theta_1} \succsim_L R^{\theta_2}$.

5 The Generalized TAL-Family

Bankruptcy rules can also be interpreted as taxation rules. In the usual parlance of taxation, the Talmud rule yields two possible types of tax schedules. If the aim is to collect a tax revenue below one half of the aggregate income, the tax rate is one half up to some income level (which is endogenously determined), and zero afterwards. If, on the contrary, the tax revenue is above one half of the aggregate income, the tax rate is one half first and then one. The rules within the TAL-family, interpreted as tax rules, would also yield two possible types of tax schedules that could be described similarly to those originating from the Talmud method. More precisely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and zero afterwards. For tax revenues above such a fraction, the tax rate would be θ first and then one.

In order to accommodate less restrictive methods too, while preserving the principle behind the Talmud method, Moreno-Ternero (2011a) allowed for other minimum and maximum tax rates, instead of always imposing zero and one for those values. More precisely, tax methods yielding two possible types of tax schedules; namely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and $\theta_{\min}$ afterwards. For tax revenues above such a fraction, the tax rate would be θ first and then $\theta_{\max}$. Formally,

Generalized TAL-family, GR^{θ}: For each $\theta_{\min}, \theta_{\max} \in [0, 1]$ with $\theta_{\min} < \theta_{\max}$, each $\theta \in [\theta_{\min}, \theta_{\max}]$, each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$$GR_i^{\theta}(N, c, E) = \begin{cases} \min\{\theta c_i, \max\{\theta_{\min} c_i + \lambda, 0\}\} & \text{if } E \leq \theta C \\ \max\{\theta c_i, \min\{c_i, \theta_{\max} c_i - \mu\}\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} GR_i^{\theta}(N, c, E) = E$.

In Fig. 3 the path of awards for c of $R_{0,\frac{3}{4}}^{\frac{1}{3}}$ follows the 45° line until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = \frac{3}{4}(c_1 + c_2) - 2(\frac{3}{4} - \frac{1}{3})c_1$. After that, it follows the line of slope 1 until claimant 1 obtains the whole of her claim. Then, it follows a vertical line until it reaches the vector of claims. The path of awards of $R_{\frac{1}{4},\frac{3}{4}}^{\frac{1}{3}}$ follows the vertical axis until claimant 2 obtains one fourth of the difference between claims. After that, it follows a line of slope 1, until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = \frac{3}{4}(c_1 + c_2) - 2(\frac{3}{4} - \frac{1}{3})c_1$, from where it follows the line of slope 1 until claimant 1 obtains the whole of her claim. Then, it follows a vertical line until it reaches the vector of claims. The path of awards of $R_{\frac{1}{4},1}^{\frac{1}{2}}$ follows the vertical axis until claimant 2 obtains one fourth of the difference between claims. After that, it follows a line of slope 1, until claimant 1 obtains one half of her claim. Then, it is a vertical line until $E = (c_1 + c_2) - 2(1 - \frac{1}{2})c_1$, from where it follows the line of slope 1 until it reaches the vector of claims.

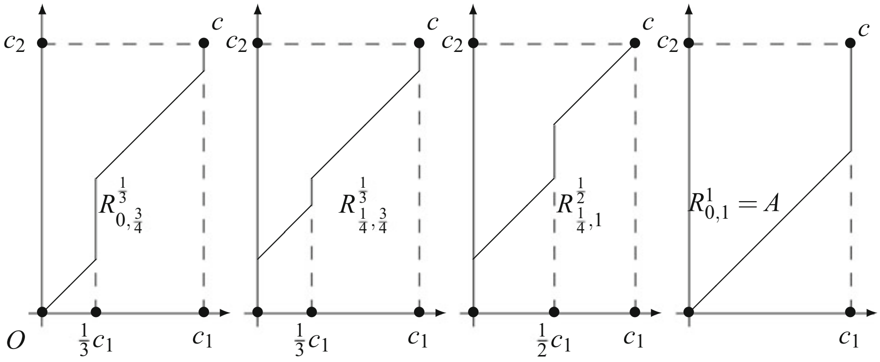

Fig. 3 The generalized TAL-family of rules

As shown by Moreno-Ternero (2011a), rules within this family satisfy the so-called *single-crossing* property, which allows one to separate those agents who benefit from the application of one rule or the other, depending on the rank of their claims. More precisely, let $0 \leq \theta_{\min} \leq \theta_1 \leq \theta_2 \leq \theta_{\max} \leq 1$, with $\theta_{\min} < \theta_{\max}$, and $(N, c, E) \in \mathscr{D}$ be given. For ease of exposition, assume that $N = \{1, \ldots, n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Then, there exists $i^* \in N$ such that:

(i) $R_i^{\theta_1}(N, c, E) \leq R_i^{\theta_2}(N, c, E)$ for each $i = 1, \ldots, i^*$ and
(ii) $R_i^{\theta_1}(N, c, E) \geq R_i^{\theta_2}(N, c, E)$ for each $i = i^* + 1, \ldots, n$.

This property has strong implications for the decentralization of the choice of rules. More precisely, suppose agents vote for rules according to majority rule. Suppose too that voters are self-interested: given a pair of alternatives, an agent votes for the alternative that gives her the greatest award. We say that a rule R is a majority voting equilibrium for a domain of rules $\mathscr{R}$ if, for each $(N, c, E) \in \mathscr{D}$, there is no other rule $R' \in \mathscr{R}$ such that $R'_i(N, c, E) > R_i(N, c, E)$ for the majority of voters.

Theorem 4 *There is a majority voting equilibrium for the Generalized TAL-family.*

Another important implication of the single-crossing property is to guarantee that rules within the Generalized TAL-family are completely ranked according to the Lorenz dominance criterion, as stated in Theorem 3 for the case in which $\theta_{min} = 0$ and $\theta_{max} = 1$.

6 The Reverse Talmud

The Talmud rule has a natural counterpart rule in which the equal awards and equal losses principles are applied in the reverse order. More precisely, the so-called *reverse Talmud* rule (e.g., Chun et al. 2001) originates when, for each claims vector, we apply the equal losses principle in the lower half of the range of the endowment, and the equal awards principle to the upper half. As with the Talmud rule, half-claims are used instead of the claims themselves.

Reverse Talmud, RT: For each $(N, c, E) \in \mathscr{D}$, and each $i \in N$,

$$RT_i\,(N, c, E) = \begin{cases} \max\left\{\frac{c_i}{2} - \lambda, 0\right\} & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c_i + \min\left\{\frac{c_i}{2}, \mu\right\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} RT_i\,(N, c, E) = E$.

Alternatively, the Reverse Talmud rule can also be given the following representation. For each $(N, c, E) \in \mathscr{D}$,

$$RT\,(N, c, E) = \begin{cases} L(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + A(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

The same natural idea considered above to generalize the Talmud rule could be considered to generalize the reverse Talmud rule, as was suggested by van den Brink et al. (2013).[7] That process gives rise to a new family of rules, the *reverse TAL-family*. Such a family also comprises a non-countable set of piece-wise linear rules, ranging from the constrained equal-awards rule to the constrained equal-losses rule, but this time having the reverse Talmud rule in the middle. Formally,

Reverse TAL-family, RT^{θ}: For each $\theta \in [0, 1]$, each $(N, c, E) \in \mathscr{D}$, and each $i \in N$,

$$RT_i^{\theta}\,(N, c, E) = \begin{cases} \max\{\theta c_i - \lambda, 0\} & \text{if } E \leq \theta C \\ \theta c_i + \min\{(1-\theta)c_i, \mu\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} RT_i^{\theta}\,(N, c, E) = E$.

7 The Talmudic Operator

Thomson and Yeh (2008) introduced the concept of operators on the space of bankruptcy rules. They focussed on three operators in order to uncover the structure of such a space. The above discussion partly inspires the following definition of a different operator from the cartesian product of the space of rules onto itself. More precisely, for a given $\theta \in [0, 1]$, the *talmudic operator* T^{θ} is the operator assigning to each pair of rules (R, S), the rule $T^{\theta}(R, S)$ defined as follows. For each $(N, c, E) \in \mathscr{D}$,

$$T^{\theta}(R, S)\,(N, c, E) = \begin{cases} R(N, \theta c, E) & \text{if } E \leq \theta C \\ \theta c + S(N, (1-\theta)c, E - \theta C) & \text{if } E \geq \theta C \end{cases} \qquad (1)$$

[7] See also van den Brink and Moreno-Ternero (2016).

A consequence of (1) is that $T^{\theta}(R, S)$ yields allocations satisfying $x_i \leq \theta c_i$ for each $i \in N$ if and only if $E \leq \theta C$, and $x_i \geq \theta c_i$ for each $i \in N$ if and only if $E \geq \theta C$. In words, the operator T^{θ} imposes a rationing of the same sort for each individual and the whole society, which somewhat reflects the Talmudic dictum for these problems.

It is straightforward to see that, for each $\theta \in [0, 1]$, $T^{\theta}(A, L)$ yields the corresponding member of the TAL-family of rules, whereas $T^{\theta}(L, A)$ yields the corresponding member of the reverse TAL-family of rules. Similarly, $T^{\frac{1}{2}}(A, A)$ is the so-called Piniless' rule (e.g., Thomson 2015), whereas $T^{\theta}(A, A)$ gives rise to a sort of generalized Piniless' rules (in the same form as the TAL-family does with respect to the Talmud rule). Likewise, $T^{\frac{1}{2}}(L, L)$ is the dual of the Piniless' rule, whereas $T^{\theta}(L, L)$ gives rise to the corresponding generalized rules. Finally, $T^{\theta}(P, P) = P$, for each θ, i.e., the proportional rule is a fixed point for such an operator. Note that if the operator T^{θ} applies to the same rule (or to a rule and its dual), then it simply becomes a member of the family of composition operators studied by Hougaard et al. (2012, 2013a,b).

This operator has interesting properties when combined with the so-called duality operator in specific ways, as stated in the next result.

Proposition 1 *The following statements hold:*
$(T^{\theta}(R, R^d))^d = T^{1-\theta}(R, R^d)$*, for each rule R and* $\theta \in [0, 1]$.
$(T^{\theta}(R, R))^d = T^{1-\theta}(R^d, R^d)$*, for each rule R and* $\theta \in [0, 1]$.

In words, the first statement of the proposition says that the dual of the rule the operator associated to θ yields for a pair of dual rules is the rule the operator associated to $1 - \theta$ yields for the same pair of rules. Likewise, the second statement of the proposition says that the dual of the rule the operator associated to θ yields for a pair made of a replicated rule is the rule the operator associated to $1 - \theta$ yields for the pair made of the replicated dual rule.

8 Game-Theoretical Foundations

In the seminal paper on bankruptcy problems, O'Neill (1982) not only explored the axiomatic approach to these problems, but also a game-theoretical approach. He suggested to use solutions to (transferable utility) coalitional games to generate rules for bankruptcy problems, by means of a natural procedure. More precisely, for each $(N, c, E) \in \mathcal{D}$, its associated (transferable utility) coalitional game is the one defined by the characteristic function $v(S) = \max\{0, E - \sum_{j \notin S} c_j\}$, for each $S \subset N$, with $v(N) = E$ and $v(\emptyset) = 0$. In words, the worth of each coalition $S \subset N$ is the difference between the endowment and the sum of the claims of the members of the complementary coalition, if this difference is non-negative, and 0 otherwise.[8]

[8]This is a rather pessimistic assessment of what a coalition can achieve. Other more optimistic proposals have been considered in the literature (e.g., Driessen 1998).

In this context, individual rationality is given by $x_i \geq v(\{i\}) = m_i(N,c,E)$. Moreover, for each pair $i, j \in N$ such that $c_i, c_j \geq E$ it follows that $v(\{i\}) = v(\{j\}) = 0$ (that is, players i and j are "permuted players" whenever they claim more than there is). Therefore, the limits given by the properties of *Minimal Rights First* and *Claims Truncation Invariance* turn out to be the natural bounds to the characteristic function of the associated game. Note that the core of this game is given by all allocations $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = E$ and $m_i(N,c,E) \leq x_i \leq t_i(N,c,E)$. We say that a bankruptcy rule is associated to a coalitional game solution if the recommendation made by the rule coincides with the recommendation made by the solution when applied to the coalitional game associated with the problem.[9]

The following result was proved by Aumann and Maschler (1985).

Theorem 5 *The Talmud rule is associated to the pre-nucleolus solution.*[10]

Dagan and Volij (1993) showed how to associate each bankruptcy problem with a bargaining problem and, actually, they derived some rules proceeding accordingly, as others did later. Thomson (2003) summarized some of the existing results along those lines. No result connecting the Talmud rule with a known bargaining solution exists. Except for the domain of two-agent problems, for which the Talmud rule (i.e., concede-and-divide) is associated to the so-called Perles-Maschler bargaining solution, which, in the two-player case, when the undominated boundary of the problem is a segment, selects the middle of the segment (e.g., Perles and Maschler 1981).

Aumann and Maschler (1985) also considered a coalitional procedure to explain the awards obtained by the Talmud rule. We present here a generalization of their procedure (originally introduced by Moreno-Ternero 2011b) which explains the awards obtained by each of the rules within the TAL-family.[11]

Fix some $\theta \in [0,1]$, and consider the following procedure. First, in the case of a two-agent problem, we apply the two-agent version of the corresponding rule within the family. Formally,

$$R^{\theta}(N,c,E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E - \theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 - c_1 + 2\theta c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 - c_1 + 2\theta c_1 \leq E \end{cases} \tag{2}$$

Suppose now that we have a problem with three creditors. Then, we proceed in the following way. First, creditors 2 and 3 pool their claims an act as a single agent vis-a-vis 1. The solution (2) of the resulting problem yields awards to agent

[9]Curiel et al. (1987) showed that the necessary and sufficient condition for a bankruptcy rule to be associated with a coalitional game is precisely *Claims Truncation Invariance*.

[10]The pre-nucleolus (e.g., Schmeidler, 1969) is the set of payoff vectors at which the vector of dissatisfactions is minimized in the lexicographic (maximin) order among all efficient payoff vectors.

[11]Quant and Borm (2011) proposed a different generalization of Aumann and Maschler's procedure.

1, and to the coalition of agents 2 and 3; to divide its award among its members, the coalition again applies solution (2). The result is order preserving if and only if $3\theta c_1 \leq E \leq C - 3(1-\theta)c_1$. To see this, note that if $3\theta c_1 > E$, then the award of creditor 1, θc_1, would be strictly greater than the one of creditor 2, which is $\frac{E-\theta c_1}{2}$, as a result of the awards sharing in the coalition of creditors 2 and 3. Analogously, if $E > C - 3(1-\theta)c_1$, then the loss of creditor 1, $(1-\theta)c_1$, would be greater than $\frac{c_2+c_3-E+\theta c_1}{2}$, the resulting loss associated to creditor 2, after dividing the awards in the coalition. If one divides the awards equally when $E \leq 3\theta c_1$, and the losses equally when $E \geq C - 3(1-\theta)c_1$, it is obtained, precisely, the solution provided by the rule R^θ, over the entire range $0 \leq E \leq C$.

By using induction, one may generalize this in a natural way to an arbitrary n. Suppose we already know the solution for $(n-1)$-agent problems. Depending on the values of the endowment and the vector of claims, we treat a given $n-$person problem in one of the following three ways:

(i) Divide E between $\{1\}$ and $M = \{2, \ldots, n\}$, in accordance with the solution (2) to the two-agent problem $(\{1, M\}, (c_1, c_2 + \ldots + c_n), E)$, and then use the $(n-1)$-agent rule, which we know by induction, to divide the amount assigned to the coalition M between its members.
(ii) Assign equal awards to all creditors.
(iii) Assign equal losses to all creditors.

Specifically, (i) is applied whenever it yields an order-preserving result, which is precisely when $n\theta c_1 \leq E \leq C - n(1-\theta)c_1$. We apply (ii) when $E \leq n\theta c_1$. Finally, we apply (iii) when $E \geq C - n(1-\theta)c_1$. We call this generalization the $\theta-$*coalitional procedure*. In the particular case of $\theta = \frac{1}{2}$, the θ-*coalitional procedure* corresponds to the coalitional procedure stated by Aumann and Maschler (1985). To summarize the previous discussion, we can state the following result:

Theorem 6 *For each* $\theta \in [0, 1]$, *and for each bankruptcy problem, the* $\theta-$*coalitional procedure and the rule* R^θ *in the TAL-family yield the same solution to the problem.*

Theorem 6 describes an orderly step-by-step process, which by its very definition must lead to a unique result, therefore characterizing the TAL-family of rules.

9 Coalitional Manipulation

An important strategic aspect of the bankruptcy model is the manipulation by merging and splitting agents' claims. We say that a rule is merging-proof, if there is no incentive for a coalition $Q \subset N$ to consolidate their claims $(c_i)_{i\in Q}$ into a single one $c_Q = \sum_{i\in Q} c_i$. Similarly, we say that a rule is splitting-proof if no single agent $i \in N$ has incentives to represent her claim c_i as a collection of several claims, $c_{i1}, c_{i2}, \ldots, c_{iK}$. Formally, *A rule R is* ***merging-proof*** *on* $\widehat{\mathscr{D}} \subseteq \mathscr{D}$*, if for all* (M, c', E) *and* $(N, c, E) \in \widehat{\mathscr{D}}$*, with* $M \subset N$*, and such that there is some* $i \in M$ *such that* $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ *and for each* $j \in M \setminus \{i\}$*,* $c'_j = c_j$*, then* $R_i(M, c', E) \leq R_i(N, c, E) + \sum_{j \in N \setminus M} R_j(N, c, E)$*.*

A rule R is ***splitting-proof*** *on* $\widehat{\mathscr{D}} \subseteq \mathscr{D}$*, if for each* (M, c', E) *and* $(N, c, E) \in \widehat{\mathscr{D}}$*, with* $M \subset N$*, and such that there is some* $i \in M$ *such that* $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ *and for each* $j \in M \setminus \{i\}$*,* $c'_j = c_j$ *then* $R_i(M, c', E) \geq R_i(N, c, E) + \sum_{j \in N \setminus M} R_j(N, c, E)$*.*

A rule R is ***non manipulable*** *on* $\widehat{\mathscr{D}} \subseteq \mathscr{D}$*, if it is simultaneously merging-proof on* $\widehat{\mathscr{D}}$ *and splitting-proof on* $\widehat{\mathscr{D}}$*.*

Let $\tau(N, c, E) = \frac{E}{C}$ stand for the share of the endowment in the aggregate claim of a given problem, and define

$$\mathscr{D}^{\delta} = \{(N, c, E) \in \mathscr{D} : \tau(N, c, E) = \delta\},$$

for each $\delta \in (0, 1)$. In other words, $\mathscr{D}^{\delta}$ is the domain of problems whose ratio between the endowment and the aggregate claim is δ.

The following result was proved by Moreno-Ternero (2007):

Theorem 7 *Let* $\{R^{\theta}\}_{\theta \in [0,1]}$ *denote the TAL-family, and let* $\delta \in (0, 1)$ *be given. The following statements hold:*

(i) If $\theta < \delta$ *then* R^{θ} *is splitting-proof on* $\mathscr{D}^{\delta}$*.*
(ii) If $\theta > \delta$ *then* R^{θ} *is merging-proof on* $\mathscr{D}^{\delta}$*.*
(iii) If $\theta = \delta$ *then* R^{θ} *is non manipulable on* $\mathscr{D}^{\delta}$*.*

Besides determining whether a rule is non-manipulable or not, one could also be interested in comparing the relative non-manipulability of different rules in terms of their outcomes. This can be done by introducing an index of non-manipulability that measures the difference between the resulting and primitive outcomes of the claimants who incurred in the manipulation. Such a difference can be contemplated as the magnitude of the incentive against the manipulation.

Formally, we say that a rule F is **more merging-proof than** G on $\widehat{\mathscr{D}}$ (which we write $\mathscr{M}^{\widehat{\mathscr{D}}}(F) \geq \mathscr{M}^{\widehat{\mathscr{D}}}(G)$) if, for each $(N, c, E) \in \widehat{\mathscr{D}}$ and (M, c', E), with $M \subset N$, and such that there is some $i \in M$ such that $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ and for each $j \in M \setminus \{i\}$, $c'_j = c_j$, then

$$\left(\sum_{j \in (N \setminus M) \cup \{i\}} F_j(N, c, E) \right) - F_i(M, c', E) \geq \left(\sum_{j \in (N \setminus M) \cup \{i\}} G_j(N, c, E) \right) - G_i(M, c', E).$$

Similarly, we say that a rule F is **more splitting-proof than** G on $\widehat{\mathscr{D}}$ (which we write $\mathscr{S}^{\widehat{\mathscr{D}}}(F) \geq \mathscr{S}^{\widehat{\mathscr{D}}}(G)$) if, for each $(N, c, E) \in \widehat{\mathscr{D}}$ and (M, c', E) in the above conditions,

$$\left(\sum_{j \in (N \setminus M) \cup \{i\}} F_j(N, c, E)\right) - F_i(M, c', E) \leq \left(\sum_{j \in (N \setminus M) \cup \{i\}} G_j(N, c, E)\right) - G_i(M, c', E).$$

The following result, also proved by Moreno-Ternero (2007), is obtained:

Theorem 8 *Let* $\{R^{\theta}\}_{\theta \in [0,1]}$ *denote the TAL-family, and let* θ_1 *and* $\theta_2 \in [0, 1]$ *such that* $\theta_1 \geq \theta_2$ *be given. Let* $\delta \in (0, 1)$ *be fixed. The following statements hold:*

(i) If $\theta_1 \geq \theta_2 \geq \delta$ *then* $\mathscr{M}^{\mathscr{D}^{\delta}}(R^{\theta_1}) \geq \mathscr{M}^{\mathscr{D}^{\delta}}(R^{\theta_2}) \geq \mathscr{M}^{\mathscr{D}^{\delta}}(R^{\delta})$.
(ii) If $\theta_2 \leq \theta_1 \leq \delta$ *then* $\mathscr{S}^{\mathscr{D}^{\delta}}(R^{\theta_2}) \geq \mathscr{S}^{\mathscr{D}^{\delta}}(R^{\theta_1}) \geq \mathscr{S}^{\mathscr{D}^{\delta}}(R^{\delta})$.
(iii) If $\theta_2 \leq \delta \leq \theta_1$ *then* $\mathscr{M}^{\mathscr{D}^{\delta}}(R^{\theta_1}) \geq \mathscr{M}^{\mathscr{D}^{\delta}}(R^{\delta})$ *and* $\mathscr{S}^{\mathscr{D}^{\delta}}(R^{\theta_2}) \geq \mathscr{S}^{\mathscr{D}^{\delta}}(R^{\delta})$.

Without loss of generality, we may assume that if R is a non-manipulable rule on a domain $\widehat{\mathscr{D}}$, then $\mathscr{M}^{\widehat{\mathscr{D}}}(R) = \mathscr{S}^{\widehat{\mathscr{D}}}(R) = 0$. Consequently, we may also assume that $\mathscr{M}^{\widehat{\mathscr{D}}}(R) < 0$ for each manipulable-by-merging rule R on $\widehat{\mathscr{D}}$ and $\mathscr{S}^{\widehat{\mathscr{D}}}(R) < 0$ for each manipulable-by-splitting rule R on $\widehat{\mathscr{D}}$. This convention and Theorem 8 provide us with a precise interpretation of the parameter θ that generates the TAL-family as an index of relative non-manipulability. More precisely, fix some $\delta \in (0, 1)$ and consider its corresponding domain of problems $\mathscr{D}^{\delta}$. Up to affine transformations, the indexes can be expressed as follows:

- $\mathscr{M}^{\mathscr{D}^{\delta}}(R^{\theta}) = \theta - \delta$ for all $\theta \in [0, 1]$.
- $\mathscr{S}^{\mathscr{D}^{\delta}}(R^{\theta}) = \delta - \theta$ for all $\theta \in [0, 1]$.

Thus, $\mathscr{M}^{\mathscr{D}^{\delta}}(R^{\theta}) < 0$ for each $\theta \in [0, \delta)$, which means that they are all manipulable (by merging) rules on $\mathscr{D}^{\delta}$. Furthermore, the rules corresponding to the remaining values of the parameter θ increase the degree of merging-proofness from R^{δ}, which coincides with the proportional rule on $\mathscr{D}^{\delta}$, to $R^1 = A$. Similarly, $\mathscr{S}^{\mathscr{D}^{\delta}}(R^{\theta}) < 0$ for each $\theta \in (\delta, 1]$, which means that they are all manipulable (by splitting) rules on $\mathscr{D}^{\delta}$. Furthermore, the rules corresponding to the remaining values of the parameter θ increase the degree of splitting-proofness from R^{δ}, which coincides with the proportional rule on $\mathscr{D}^{\delta}$, to $R^0 = L$.

10 Final Remarks

To conclude, we report on some other families of rules that have emerged in the literature, while extending the Talmud rule in other directions.

Hokari and Thomson (2003) introduced a family of consistent rules meeting the two characteristic properties of the Talmud rule described above (*Minimal*

Rights First and *Claims Truncation Invariance*), while dismissing *Equal Treatment of Equals*. The resulting rules are *weighted* versions of the Talmud rule, defined by partitioning the set of potential claimants into priority classes, and selecting reference weights for all potential claimants. The rules, which could also be seen as hybrid rules between weighted versions of the constrained equal awards and constrained equal losses rules, consistently extend a one-parameter family in the two-agent case, dubbed as *weighted concede-and-divide* rules. These rules also endorse the Talmudic dictum of conceding minimal rights to each claimant, but then divide the remainder unequally (and according to the weights). Formally, let $\alpha \in (0, 1)$.[12]

Weighted concede-and-divide, CD^{α}: For each $E \in \mathbb{R}_+$, and each $c = (c_1, c_2) \in \mathbb{R}^2_+$,

$$\begin{cases} CD_1^{\alpha}(c, E) = \max\{0, E - c_2\} + \alpha(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \\ CD_2^{\alpha}(c, E) = \max\{0, E - c_1\} + (1 - \alpha)(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \end{cases}$$

Thomson (2008) introduced the so-called ICI family, which constitutes a further generalization of the TAL-family. Rules within the ICI-family impose that the evolution of each claimant's award, as a function of the endowment, is increasing first, constant next and finally increasing again.[13]

Formally, let $\mathscr{G}^N$ be the family of lists $G \equiv \{E_k, F_k\}_{k=1}^{n-1}$, where $n = |N|$, of real-valued functions of the claims vector, satisfying for each $c \in \mathbb{R}^N_+$, the following relations:

$$\begin{aligned} \frac{E_1(c)}{n} + \frac{C - F_1(c)}{n} &= c_1 \\ c_1 + \frac{E_2(c) - E_1(c)}{n-1} + \frac{F_1(c) - F_2(c)}{n-1} &= c_2 \\ &\vdots \\ c_{k-1} + \frac{E_k(c) - E_{k-1}(c)}{n-k+1} + \frac{F_{k-1}(c) - F_k(c)}{n-k+1} &= c_k \\ &\vdots \\ c_{n-1} + \frac{-E_{n-1}(c)}{1} + \frac{F_{n-1}(c)}{1} &= c_n \end{aligned}$$

[12]To ease its presentation, we assume $N = \{1, 2\}$, but dismiss it from the definition.

[13]More recently, Huijink et al. (2015) have identified the rules in such a family as *claim-and-right* rules, which give a specific interpretation to the concept of *baselines* formalized earlier by Hougaard et al. (2012, 2013a,b). See also Timoner and Izquierdo (2016) for a related notion.

The ICI rule relative to $G \equiv \{E_k, F_k\}_{k=1}^{n-1} \in \mathscr{G}^N$, is defined as follows. For each $c \in \mathbb{R}_+^N$, the awards vector is given as the following function of the amount available E, as it varies from 0 to C. As E increases from 0 to $E_1(c)$, equal division prevails; as it increases from $E_1(c)$ to $E_2(c)$, claimant 1's award remains constant, and equal division of each new unit prevails among the other claimants. As E increases from $E_2(c)$ to $E_3(c)$, claimants 1 and 2's awards remain constant, and equal division of each new unit prevails among the other claimants, and so on. This process goes on until E reaches $E_{n-1}(c)$. The next units go to claimant n until E reaches $F_{n-1}(c)$, at which point equal division of each new unit prevails among claimants n and $n-1$. This goes on until E reaches $F_{n-2}(c)$, at which point equal division of each new unit prevails among claimants n through $n-2$. The process continues until E reaches $F_1(c)$, at which point claimant 1 re-enters the scene and equal division of each new unit prevails among all claimants.

This is a large family encompassing many rules. Nevertheless, if we impose the property of *consistency* introduced above, as well as the innocuous condition of *scale invariance*, then the family shrinks precisely to the TAL-family, which we have thoroughly described here.

Thomson (2008) also introduced the so-called CIC-family, which imposes that the evolution of each claimant's award, as a function of the endowment, is constant first, increasing next and finally constant again. Its formal definition is a *reverse* parallel of that of the ICI-family just described. Imposing *consistency* and *scale invariance* to the rules within the family, it shrinks to the reverse TAL-family (also described above).

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References

Aumann RJ (2002) Game theory in the talmud. In: Research bulletin series on jewish law and economics. Bar-Ilan University, Ramat Gan

Aumann RJ, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the talmud. JET 36(2):195–213

Chun Y, Schummer J, Thomson W (2001) Constrained egalitarianism: a new solution for claims problems. Seoul J Econ 14(3):269–297

Curiel I, Maschler M, Tijs SH, (1987) Bankruptcy Games. Z Oper Res 31(5):A143–A159

Dagan N (1996) New characterizations of old bankruptcy rules. SCW 13(1):51–59

Dagan N, Volij O (1993) The bankruptcy problem: a cooperative bargaining approach. Math Soc Sci 26(3):287–29

Driessen TSH (1998) The greedy bankruptcy game: an alternative game theoretic analysis of a bankruptcy problem. In: Petrosjan LA, Mazalov VV (eds) Game theory and applications, vol IV. Nova Science Publishers, New York, pp 45–61

Hokari T, Thomson W (2003) Claims problems and weighted generalizations of the Talmud rule. ET 21(2):241–261

Hougaard JL, Moreno-Ternero JD, Østerdal LP (2012) A unifying framework for the problem of adjudicating conflicting claims. JME 48(2):107–114

Hougaard JL, Moreno-Ternero JD, Østerdal LP (2013a) Rationing in the presence of baselines. SCW 40(4):1047–1066
Hougaard JL, Moreno-Ternero JD, Østerdal LP (2013b) Rationing with baselines: the composition extension operator. Ann Oper Res 211(1):179–191
Huijink S, Borm P, Kleppe J, Reijnierse J (2015) Bankruptcy and the per capita nucleolus: the claim-and-right rules family. Math Soc Sci 77:15–31
Moreno-Ternero JD (2006) Composition, securement, and concede-and-divide. Span Econ Rev 8(3):227–237
Moreno-Ternero JD (2007) Bankruptcy rules and coalitional manipulation. IGTR 9(1):105–118
Moreno-Ternero JD (2011a) Voting over piece-wise linear tax methods. JME 47(1):29–36
Moreno-Ternero JD (2011b) A coalitional procedure leading to a family of bankruptcy rules. ORL 39(1):1–3
Moreno-Ternero JD, Villar A (2004) The Talmud rule and the securement of agents' awards. Math Soc Sci 47(2):245–257
Moreno-Ternero JD, Villar A (2006a) The TAL-family of rules for bankruptcy problems. SCW 27(2):231–249
Moreno-Ternero JD, Villar A (2006b) On the relative equitability of a family of taxation rules. JPubET 8(2):283–291
Moreno-Ternero JD, Villar A (2006c) New characterizations of a classical bankruptcy rule. RED 10(2):73–84
O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2(4):345–371
Perles M, Maschler M (1981) The super-additive solution for the Nash bargaining game. IJGT 10(3):163–193
Quant M, Borm P (2011) Random conjugates of bankruptcy rules. SCW 36(2):249–266
Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appli Math 17(6):1163–1170. doi:10.1137/0117107
Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45(3):249–297
Thomson W (2008) Two families of rules for the adjudication of conflicting claims. SCW 31(4):667–692
Thomson W (2012) On the Axiomatics of resource allocation: interpreting the consistency principle. Econ Philos 28(3):385–421
Thomson W (2014) How to divide when there isn't enough: from the Talmud to game theory. Book Manuscript, University of Rochester
Thomson W (2015) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Math Soc Sci 74(3):41–59
Thomson W, Yeh C-H (2008) Operators for the adjudication of conflicting claims. JET 143(1):177–198
Timoner P, Izquierdo JM (2016) Rationing problems with ex-ante conditions. Math Soc Sci 79(1):46–52
van den Brink R, Funaki Y, van der Laan G (2013) Characterization of the reverse Talmud bankruptcy rule by exemption and exclusion properties. EJOR 228(2):413–417
van den Brink R, Moreno-Ternero J (2016) The reverse TAL-family or rules for bankruptcy problems. Ann Oper Res. Forthcoming. doi:10.1007/s10479-017-2455-x

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Sharing the Costs of Access to a Set of Public Goods

Jens Leth Hougaard

Abstract A group of agents share assess to a set of public goods. Each good has a cost and the total cost of all goods must be shared among the agents. Agents preferences are described by subsets of goods that provides the agent with service. As such, demands are binary, and it is further assumed that agents prefer a low cost share, but other differences in their individual preferences are irrelevant, making demand fully inelastic. The model captures central aspects of several classes of practical problems and therefore has many potential applications.

The paper surveys some recent axiomatic characterizations of relevant allocation rules and provides a overview of how the problem of fair division can be approached and structured subject to the richness inherent in the description of agents service constraints.

Keywords Allocation rules • Core • Cost sharing • Public goods • Service constraint • Shapley value

Contents

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1 Introduction

1.1 The Problem

We consider situations where a group of agents share assess to a set of public goods in the sense that all agents can obtain "service" from these goods without rivalry. Each good has a cost and the total cost of all goods must be shared among the agents. The benefits from having access to the set of public goods are described by a set of *service sets* for each agent, i.e., subsets of the set of goods that provides the agent with service. As such, demands are binary: either the agent is served or not. Agents prefer a low cost share, but other differences in their individual preferences are irrelevant, making demand fully inelastic.

The set of public goods is not necessarily designed to match the agents service demands. Rather the situation can be construed as the agents "taking over" an existing set of goods. That is, there may be some goods which are redundant and even useless for everyone. In this sense our model deviates from the vast majority of models in the cost sharing literature where the problem can be construed as a two step process: first an optimal (cost minimizing) set of goods is determined given agents service demands; and second, the total cost of the optimal set of goods is shared. Examples of conventional problems include well-known models from combinatorial optimization such as the minimum cost spanning tree model, the traveling salesman model, and other routing models, see e.g., Moulin (2013).

To picture the kind of situation we have in mind think of a group of friends going out for dinner. The food they order is shared among them and there is plenty so consumption can be seen as non-rival. The chef decides the menu which is not necessarily designed to optimize their demands (like a buffet) so often there will be dishes that are redundant but nevertheless available and enjoyed. Each dish has a cost and the total bill is shared among the friends. Worst case scenario is that one guy pays for everyone which is implicitly accepted when going out, but obviously we may find more fair ways to share the bill. In particular ways that differs from equal split because we know about individual service demands: why should Adam pay for the lobster when he is allergic to seafood? why should Ben pay the same as Carl when Ben is completely flexible and can be satisfied by any one dish while Carl is the gourmet needing the entire range of dishes available to feel happy?

As such, we are not concerned with the welfare of the agents, they are all happy getting access independent of their cost share. Admittedly, this a somewhat extreme assumption from the outset, but it may nevertheless seem reasonable in cases where the benefits of having access far exceeds the costs (as in the potential applications we have in mind). Moreover, it allows us to highlight the issue of fairness in allocation given the individual service constraints of the agents.

1.2 Potential Applications

The model captures central aspects of several classes of practical problems and therefore has many potential applications. One such important class of problems relates to cost allocation in existing network structures: here agents share public links (edges) in the network and each link is connected with a cost, for instance, due to maintenance. This situation may occur in some power grids where consumers access an existing network structure that often involves redundancies due to the historical evolution of the grid structure itself. Yet, redundant links provide consumers with increased connectivity and the associated maintenance costs needs to be shared.

Commodity trading networks constitutes another example. Historically, associations of merchants have shared the cost of obtaining trading privileges and of keeping transportation networks safe for commodity transport. Here the existing markets, roads and sea ways are the public goods that are accessed by the merchants and the costs of ensuring smooth trade (which may be substantial, but nevertheless outranked by the potential gains from trade) are shared among members of the association according to their trading patterns. A classical historical example is that of the German Hansa and its nearly 500 years of often direct political involvement in the common interest of sustaining beneficial trade monopolies, see e.g., Dollinger (1970). A more recent example is that of the multinational coalition "CTF-150" and their anti-piracy operations off the coast of Somalia involving more than 25 nations and their direct commercial interests in safe sea transport.

The model further has relevance at company level. Streaming services like Netflix and Spotify all sell access to a collection of digital goods (in the form of movies, e-books or music). Since consumption is based on streaming it is non-rival (unlike traditional libraries) and the costs of covering Intellectual Property rights may differ between the products while it is the total cost that has to be covered by the consumers. As it is now, consumers pay a flat (monthly) access fee, but observed consumption history could potentially justify different types of allocations along the lines we suggest here. Moreover, companies (for example in the pharmaceutical industry) may find it beneficial to enter into joint ventures such as common R and D projects where each company in the group has access to the obtained results, but may have heterogeneous needs. Again, our results shed some light on potentials ways of sharing the associated costs also when success of the projects is uncertain.

1.3 The Literature

This paper will mainly survey results from three recent papers: Moulin and Laigret (2011), Hougaard and Moulin (2014) and Hougaard and Moulin (2016). These papers fall within the strand of modern cost sharing literature that focusses on axiomatic characterization of model-specific allocation rules, where the conventional approach of using the tools from cooperative game theory is inferior for one reason or the other. For general texts on axiomatic cost allocation, see e.g., Moulin (2002), Young (1994) and Hougaard (2009).

As mentioned above, a particularly relevant strand of literature concerns fair allocation in network structures where the only externalities present are the positive cost externalities (i.e., that the per capita usage cost decreases in the number of agents in the network). Surveys include Sharkey (1995) or more recently, Moulin (2013). Classic models include the fixed tree model, e.g., Koster et al. (2001), the minimum cost spanning tree model, e.g. Bergantiños and Vidal-Puga (2007), the minimum cost Steiner tree model, e.g. Meggido (1978), the traveling salesman model, e.g. Potters et al. (1992), and, the Chinese postman model, e.g. Hamers et al. (1999). These models have often inspired related work on pricing, e.g. Hougaard et al. (2010) and more general connection demands and network structures, e.g. Bogomolnaia et al. (2010), Moulin (2013), and Hougaard and Tvede (2015).

1.4 Content

The remaining text is organized as follows: Sect. 2 describes the formal model. Section 3 discusses the conventional game theoretic approach to cost sharing and explains why we deviate. Section 4 presents the new approach of Moulin and Laigret (2011) and Hougaard and Moulin (2014) and review their main results. Section 5 modifies the model to encompass limited reliability of the public goods and discuss how this affects the central notion of fairness. It further reviews the analysis and main results of Hougaard and Moulin (2016).

2 The Model

Formally, a description of a problem entails a finite set of agents N and a finite set of public goods A. Each good $a \in A$ has a cost c_a and total costs $c_A = \sum_{a \in A} c_a$ must be shared by the agents in N. Let $c \in \mathbf{R}^A_+$ denote vector of costs. The needs of every agent $i \in N$ are described by a (non-empty) set of (non-empty) subsets of goods $\mathscr{D}^i \subseteq 2^A \setminus \emptyset$. That is, sets in $\mathscr{D}^i$ all provide agent i with service and are hence dubbed the *service constraint* of agent i. The sets in $\mathscr{D}^i$ are inclusion monotonic since if i is served by $D \in \mathscr{D}^i$ any superset of D must also serve i. We call $D \in \mathscr{D}^i$ *minimal* if removing any good $a \in D$ leads to service failure for agent i. Denote by $\bar{\mathscr{D}}^i$ the minimal service constraints of agent i. For the sake of notational simplicity we will often represent the service constraints of agent i by a list $[D, D', D'' \ldots]$ of minimal service sets. Thus, a *cost allocation problem* is a list $(N, A, \{\mathscr{D}^i\}_{i \in N}, c)$ and a *cost allocation rule*, Y, assigns to each such problem a vector of payments $Y(N, A, \{\mathscr{D}^i\}_{i \in N}, c) \in \mathbf{R}^N_+$ satisfying budget-balance $\sum_{i \in N} Y_i(N, A, \{\mathscr{D}^i\}_{i \in N}, c) = c_A$.

We say that a problem is *non-redundant* if deleting one good from A will lead to service failure of at least one agent in N. Otherwise, we say the problem contains *redundancies*. The analysis in Moulin and Laigret (2011) focusses on non-redundant problems while the analysis in Hougaard and Moulin (2014, 2016) include problems with redundancies.

When considering a given agent i and his service constraint $\mathscr{D}^i$, we say that i has *substitutable* resp. *complementary* needs in the two extreme cases consisting of singleton minimal sets $[a, b, c, \dots]$ and the full minimal set $[A]$ respectively: if i has substitutable needs she can be served by any single good in A while an agent with complementary needs, has to have access to all the goods in A to be served. A good which is not part of any of agent i's minimal service sets is called *useless* to i: otherwise the good is called *useful*.

Moreover, we say that agent i is more *flexible* than agent j if $\mathscr{D}^i \supseteq \mathscr{D}^j$. Clearly, for instance, an agent with substitutable needs is more flexible than an agent with complementary needs.

3 The Conventional Approach and Why It is Questionable in the Present Context

The typical approach to cost sharing problems is to model the situation as a cooperative game describing the cost associated with each subset (coalition) of the agents involved and then allocate the total costs according to some game theoretic solution concept like the Shapley value or the Nucleolus, see e.g. Moulin (2002) and Hougaard (2009). The compelling concept of the *core* with its inherent rationality for all coalitions (the *stand-alone core conditions*) seems to offer exactly the kind of stable allocations that sustain cooperation because no group of agents can do better on their own.

This approach has proved very fruitful for a broad range of allocation problems including cost allocation in large public investment projects and price regulation in case of natural monopolies with joint production, see e.g. Young (1994). In particular, it has inspired a large strand of literature based on classical problems from combinational optimization, see for a survey e.g. Curiel (1997), stressing its usefulness in situations where the cost structure is found by minimizing the cost of every subgroup of agents.

In the present context, however, it is much less natural for several reasons. First, the coalitional values (which are defined as the cost of the cost minimizing subset of goods servicing all members of the coalition) are counterfactual since when accessing the pool of public goods (or taking it over) it is "all or nothing". Therefore it is irrelevant whether groups of agents could have been satisfied with a (less costly) subset of the goods. Recall that the set of public goods is not designed to optimize the satisfaction of the grand coalition either. So, because of eventual redundancies there can be huge differences between the cost of the grand coalition and the coalitional costs leaving the core of the associated game empty in many cases. Second, even if the cooperative game is found relevant then applying conventional solution concepts like the Shapley value often leads to counter intuitive results.

Consider the following simple example. **Example 1:** Consider a situation with two goods $\{a, b\}$ and two agents, Ann and Bob. Ann can only be served by a, i.e., $\mathscr{D}^{Ann} = [a]$, while Bob needs both goods available in order to be served, i.e., $\mathscr{D}^{Bob} = [ab]$. Assume that $c_a \leq c_b$, then the Shapley value of the corresponding game is

$$Y^{Sh} = (\frac{c_a}{2}, \frac{c_a}{2} + c_b).$$

So according to the Shapley value they share the cost of the cheapest good a equally while Bob alone pays for b. This seems quite compelling since Ann has no need for b and hence should not pay for it, while they both need a and hence split its cost equally. However, if Ann could in fact also be served by b (that is, if $\mathscr{D}^{Ann} = [a, b]$) the Shapley value remains the same. This time, however, much less compelling since Ann has access to b for free despite her potential benefit from using it.

To emphasize this point further assume that $\mathscr{D}^{Ann} = [a]$ while Bob now has substitutable needs, i.e., $\mathscr{D}^{Bob} = [a, b]$. So, as originally, Ann can only be served by a while Bob this time can be served by any of the two goods. In this case the Shapley value shares the total cost equally between Ann and Bob,

$$Y^{Sh} = (\frac{c_a + c_b}{2}, \frac{c_a + c_b}{2}),$$

but why should Ann pay for b when she has no use of it while Bob can potentially benefit from it. □

Using the Shapley value is not only problematic in certain specific situations. It is also problematic in a broader perspective since it fails the compelling, and indeed widely accepted—see e.g. Moulin (2002)—property of cost additivity: in our context stating that $Y(N, A, \{\mathscr{D}^i\}_{i \in N}, c + c') = Y(N, A, \{\mathscr{D}^i\}_{i \in N}, c) + Y(N, A, \{\mathscr{D}^i\}_{i \in N}, c')$. By focusing on cost additive allocation rules we emphasize that a fair cost share of a given good does not depend on its cost, or the cost of other goods, but rather on the structure of the service sets of all agents: that is, the way the agents benefit from using the goods.

The relevance of cost additivity is crucially linked to the fact that the set of public goods is determined exogenously and not designed to fit the needs of the agents. To illustrate, consider the example below.

Example 1 (con't) Recall the situation where Bob has substitutable needs, i.e., $\mathscr{D}^{Bob} = [a, b]$ while Ann, as initially, can only be served by a. If we allow a central planner to find the cost minimizing set of goods that satisfies the demand of all agents, the set $A = \{a\}$ would then be chosen. If $c_a \leq c_b$ we can forget about b and share the cost of a equally between Ann and Bob. If $c_a > c_b$, however, the fact that Bob could have been served by the cheaper good b ought to play a role when they share the cost of a, so that fairness conflicts with cost additivity. □

4 A New Approach Based on the Structure of Service Constraints

In case of non-redundant problems Moulin and Laigret (2011) demonstrate that the only reasonable cost additive allocation rule satisfying the stand alone core conditions is a rule dubbed the *Equal Need* rule: here the cost of a good a is shared equally among all the agents for whom a is needed for service, i.e., among $N(a) = \{i \in N | A \setminus \{a\} \notin \mathscr{D}^i\}$. The Equal Need rule often results in extreme solutions: For instance in benchmark cases involving an agent with substitutable needs versus an agent with complementary needs, the agent with substitutable needs is allowed to free ride (indeed, if $\mathscr{D}^{Ann} = [a, b]$ and $\mathscr{D}^{Bob} = [ab]$, Ann needs neither a nor b and hence pays nothing according to the Equal Need rule). So even in non-redundant problems (where it is likely that the core is non-empty) there are good arguments in favor of abandoning the stand alone core conditions when focussing on cost additive rules.

Hougaard and Moulin (2014) also focus on cost additive rules, but allow for redundancies in the problem. The point of departure is a search for allocation rules for which the individual cost shares are proportional to a so-called *liability index*. This index is intended to measure the extend to which a given agent is liable for a given good based solely on the agent's own service constraints: that is, how "important" the good is for the agent compared to the agent's general ability for being served by the different goods in A.

Hougaard and Moulin (2014) give several examples of relevant liability indices, but end up focussing on a particular class of *counting indices*: agent i's liability for good a is given by the ratio between the number of minimal service sets containing a over the total number of minimal service sets in i's service constraint. Clearly, the counting index rewards flexibility as the ratio decreases when the denominator increases.

Example 1 (cont') Returning to the situation in Example 1 and the case where $\mathscr{D}^{Ann} = [a, b]$ and $\mathscr{D}^{Bob} = [ab]$, we have already noted that Ann free rides if we use the Equal Need rule. Even though Ann is more flexible than Bob (and hence ought to pay less) this solution is arguably somewhat extreme. Rather, we can use the counting liability index and allocate costs in proportion to that: Ann has two minimal service sets and each good enters into one of them giving her liability $(\frac{1}{2}, \frac{1}{2})$ while Bob has one minimal service set where both goods enters once giving him liability $(1, 1)$. The proportional cost shares of the counting rule hence make Ann pay $\frac{1}{3}$ (and Bob pay $\frac{2}{3}$) of the cost for both goods a and b. □

Formally, for any number $\pi \geq 0$, any item a and serving constraints $\mathscr{D}^i$, we can define i's generalized counting liability index as

$$\theta^{\pi}(\mathscr{D}^i, a) = \left(\frac{|\overline{\mathscr{D}}^i(a)|}{|\overline{\mathscr{D}}^i|} \right)^{\pi}$$

with the convention $(0^0) = 0$ and where $\overline{\mathscr{D}}^i(a)$ are the minimal service sets containing a. Clearly, the counting index used in the example above sets $\pi = 1$.

The main result in Hougaard and Moulin (2014) characterizes the class of generalized counting rules,

$$Y_i^\pi(N, R, \{\mathscr{D}^j\}_{j\in N}, c) = \sum_{a\in A} \frac{\theta^\pi(\mathscr{D}^i, a)}{\sum_{j\in N} \theta^\pi(\mathscr{D}^j, a)} c_a, \text{ for all } i. \tag{1}$$

by standard axioms of anonymity, neutrality, consistency, cost additivity and replication together with an axiom dubbed irrelevance of supplementary goods. The latter property states that if we add a good, say b, to the set A which is *supplementary to i's needs* in the sense that if D is a minimal service set in i's service constraint originally then adding b makes either D or $D \cup \{b\}$ a minimal service set in the new problem, then i's cost share of all original goods in A should remain the same. In other words, adding goods which do not create new service opportunities for agent i, should not affect i's liability for the original goods.

It is further shown that if payments should obey the familiar *unanimity lower bound* (here stating that $Y_i \geq \frac{1}{n} \min_{D\in\mathscr{D}^i} c_D$) the choice of π is limited to the interval $[0, 1]$: If $\pi = 0$ the rule (1) divides the cost of good a equally among all agents for whom a is useful; If $\pi = 1$ we have the canonical counting rule as illustrated in Example 1 above.

Example 1 (cont') Recall the situation where $\mathscr{D}^{Ann} = [a]$ and $\mathscr{D}^{Bob} = [a, b]$ and using the Shapley value (when $c_a \leq c_b$) would make Ann and Bob split costs equally. Now, using the counting rule (with $\pi = 1$) in this case gives the following: Ann has liability 1 for a and 0 for b while Bob has liability $\frac{1}{2}$ for both goods yielding payments

$$(\frac{2}{3}c_a, \frac{1}{3}c_a + c_b)$$

So compared to the conventional game theoretic approach the counting rule acknowledges that only Bob has potential use of b (and hence has to cover its cost alone) while a is more important to Ann and consequently she has to cover the major part of its cost. Note that in this case Bob is actually more flexible than Ann, but because b is useless to Ann (and we argue that this ought to result in 0 liability) there may be case like this one where the more flexible agent ends up paying more than the less flexible agent in total. □

5 Limited Reliability and Its Impact on the Notion of Fairness

If the public goods are links in a network, digital goods that can be streamed from a database, or any other kind of project which may turn out successful or not, it becomes relevant to consider issues of *reliability*. Indeed, the reason that redundant connections in network may be useful to the agents (and therefore worth paying for) is because they represent alternative connectivity options in case some connections fail; likewise, if streaming my favorite movie fails I may try another (seemingly redundant) one with more success, and so forth.

Including the possibility that goods can fail has a fundamental impact on the notion of fairness. There are at least two reasons for that:

- It seems to matter whether we consider the situation before (ex-ante), or after (ex-post) the resolution of uncertainties. In the context of axiomatic bargaining this was pointed out by Myerson (1981). Indeed, ex-ante it may seem fair that there is a connection between payment and the probability of obtaining service. Yet, ex-post (once the uncertainty is resolved) we may have a clear idea of fairness in payment and this fairness notion can be applied ex-ante by computing the expected payments. As demonstrated in Example 2 below, these two approaches may result in quite different payments.
- When goods may fail the relation between flexibility in need and liability in payment becomes ambiguous. Indeed, the more flexible agent now has a bigger chance of being served and this may be a good argument in favor of higher payment despite that higher flexibility in need generally ought to result in lower liability in payment. So how are these effects balanced in the choice of allocation rule? Example 2 below illustrates.

It is straightforward to modify the model in order to encompass limited reliability: Say, nature chooses the set X, $\emptyset \subseteq X \subseteq A$, of successful goods with probability distribution p. Irrespective of the realization of X, agents must still cover the total cost c_A. As such we deal with sunk costs: access fee to a streaming service has to be paid whether each individual streaming works or not; R and D costs has to be covered whether the projects are successful or not, and so forth.

Thus, a *problem with limited reliability* is a list $(N, A, \{\mathscr{D}^i\}_{i \in N}, p, c)$.

Example 2 Recall the situation where $\mathscr{D}^{Ann} = [a, b]$ and $\mathscr{D}^{Bob} = [ab]$, but now each good has a probability of success q. Assuming p is IID we thus get the information in Table 1.

So, ex-ante, Ann has a probability $1-(1-q)^2 = q(2-q)$, of being served, while Bob's probability is q^2. Fairness from an ex-ante perspective could, for instance, entail that the agents should pay in proportion to these probabilities—dubbed the Ex-ante rule in Hougaard and Moulin (2016)—making Ann pay the share $y^{Ann} = 1 - \frac{q}{2}$ of the total cost (and hence Bob pay the share $1 - y^{Ann}$). So with very small probability of success (q close to 0) Ann basically pays for everything. Then, as $q \to 1$ Ann's share decreases to 0.5 in the deterministic case $q = 1$.

Table 1 Ex-ante success probabilities

X	$\emptyset$	a	b	ab
$p(X)$	$(1-q)^2$	$q(1-q)$	$q(1-q)$	q^2

Now, if we view fairness as a matter of splitting costs equally among those getting service at X (that is, once the uncertainty is resolved) then they should split equally in the cases $X = \emptyset$ and $X = \{a, b\}$, while Ann alone should pay in the cases where $X = \{a\}$ and $X = \{b\}$. Taking the expectation of this ex-ante—dubbed the Ex-post rule in Hougaard and Moulin (2016)—gives Ann the cost share $y^{Ann} = \frac{1}{2}(1-q)^2 + 2q(1-q) + \frac{1}{2}q^2 = 0.5 + q - q^2$. Clearly, this differs from the approach above: for $q < 0.5$ Ann pays a larger share using the Ex-ante rule than the Ex-post rule, while it is the other way around for $q > 0.5$ (for $q = 0.5$ both approaches yields $y^{Ann} = 0.75$).

Finally, notice that Ann is more flexible than Bob, yet using both the ex-ante and the ex-post rule above she will always pay weakly more than Bob. As argued above this is not necessarily unfair since she always has at least as big a chance of being served (and often much bigger) than Bob. Yet, as $q \to 1$ it can be argued that Ann should indeed pay less than Bob due to her bigger flexibility.

Note, that given X we can also use, for instance, the counting rule (1) to compute the shares, as if this was a deterministic case. Clearly, we need to project the service constraints of each agent onto X, but given that is done we can again compute the expected shares ex-ante—dubbed the Expected Counting Liability rule in Hougaard and Moulin (2016). In the present case they should still split equally when $X = \emptyset$ and Ann pay it all when $X = \{a\}$ (sharing the cost of a) and $X = \{b\}$ (sharing the cost of b), but recall that when $X = \{a, b\}$ the counting rule resulted in shares $(\frac{1}{3}, \frac{2}{3})$ for both goods a and b. Thus, Ann's share using the Expected Counting Liability rule becomes $y^{Ann} = \frac{1}{2}(1-q)^2 + \frac{1}{2}q(1-q) + q(1-q) + \frac{1}{3}q^2 = \frac{1}{2} + \frac{1}{2}q - \frac{2}{3}q^2$. Now, for $q > \frac{3}{4}$ this makes Ann's share smaller than Bob's, just as we would like it to be when approaching the deterministic case due to her flexibility. □

As argued in Hougaard and Moulin (2016) a suitable allocation ought to make the agents indifferent between dividing costs before or after the uncertainties are resolved. This will be the case if the cost shares computed ex-ante coincide with the expectation of those computed ex-post: an axiom dubbed *independence of timing*. Cost allocation rules that are cost additive and satisfies Independence of Timing will have shares, y, of the general form,

$$y(a; N, A, \{\mathscr{D}^j\}_{j\in N}, p, c) = \sum_{\emptyset\subseteq X\subseteq A} p(X) \times y^*(a; N, X, \{\mathscr{D}^j\}_{j\in N}, c) \tag{2}$$

where $y^*(a; N, X, \{\mathscr{D}^j\}_{j\in N}, c)$ is dividing the cost of good a in the deterministic case where only the goods in X are available.

Note that the Ex-ante rule, where cost shares are found in proportion to the probability of being served, does *not* satisfy Independence of Timing. The Ex-post rule, on the other hand, does. Formally, it is defined as

$$Y^{xp}(N, X, \{\mathscr{D}^j\}_{j\in N}, p, c) = \left\{ \sum_{X\in 2^A} p(X) \times e[S(X)] \right\} \times c_A \tag{3}$$

where $S(X) = \{i \in N | X \in \mathscr{D}^i\}$ is the set of agents served by X, and $e[S]$ is the uniform lottery on S, with the important convention $e[\varnothing] = e[N]$.

Also, it is clear from the general form (2) that any cost additive rule from the deterministic model can be used given service constraints are projected onto X as, for instance, in the Expected Counting Liability rule mentioned in Example 2 above.

Hougaard and Moulin (2016) furthermore argue in favor of a third independence property called *Separability Across Items* (stating that costless deterministic goods should only influence cost shares through their impact on the service constraints) and determine the class of allocation rules satisfying cost additivity, independence of timing and separability across items. The ex-post rule is among those (but not the expected counting liability rule which fails separability across items).

Within this class an additional rule is singled out: the needs priority rule. This rule assigns specific cost shares for each good based on the idea that, ex-post, agents for whom the good in question is pivotal (that is, agents served if and only if this good is available) should share equally, and if no such agents exist then all agents served should share equally. Formally,

$$Y^{np}(N, X, \{\mathscr{D}^j\}_{j \in N}, p, c) = \sum_{a \in A} \{ \sum_{\varnothing \subseteq X \subseteq \mathscr{A}} p(X) \times e[S(X) \diagdown S(X \diagdown a); S(X)] \} c_a \quad (4)$$

with the notation $e[S^1; S^2] = e[S^1]$ if $S^1 \neq \varnothing$, and $e[\varnothing; S^2] = e[S^2]$.

Hougaard and Moulin (2016) provide axiomatic characterizations of both the Ex-post and the Needs Priority rule. In particular, a property stating that the more flexible agent should pay the most separates the two rules: it is satisfied by Y^{xp}, but not by Y^{np}. As discussed above the role of flexibility is ambiguous under limited reliability, which seems to speak in favor of the Needs Priority rule. Yet, both rules are known for their extreme outcomes in deterministic situations so some kind of weighted average of the two rules may prove even more desirable. In this connection it is worth mentioning that the class of rules satisfying Cost Additivity, Independence of Timing and Separability Across Items, is closed by convex combinations.

Example 3 Consider the following network consisting of a loop with source 0 (as supplier). Three agents Ann, Bob and Carl all want connection to the source and are located at nodes 1, 2 and 3 respectively as illustrated in Fig. 1.

Edges a, b, c, d, are the public goods, all with limited reliability q and p is IID as in Example 2 above. Given connection demands we get the following (minimal) service constraints for each agent;

$\mathscr{D}^{Ann} = [a, bcd]$, $\mathscr{D}^{Bob} = [ab, cd]$ and $\mathscr{D}^{Carl} = [d, abc]$.

In the deterministic cases ($q \in \{0, 1\}$) the three agents should clearly split costs equally since all have 2-connectivity in the loop. Yet, with limited reliability it gets more complicated.

Using the Ex-Post Service rule results in cost shares;

$$y^{xp} = \frac{1}{3}(1 + q - 2q^2 + q^3, 1 - 2q + 4q^2 - 2q^3, 1 + q - 2q^2 + q^3)$$

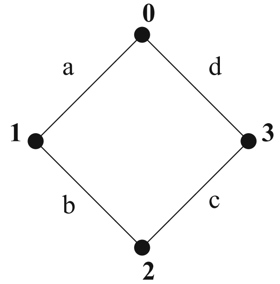

Fig. 1 Public good network. Own representation

Clearly, Ann and Carl are located symmetrically so they pay the same. In the deterministic cases $q \in \{0, 1\}$ all share equally as they should. For all other values of q, Bob pays less than Ann and Carl since he has a lower probability of successful connection to the source.

Using the Needs Priority rule gives the following cost shares for edges a and b;

$$y^{np}(a) = \frac{1}{6}(2 + 2q - q^2 + q^3 - 2q^4, 2 - q - q^2 + q^3 + q^4, 2 - q + 2q^2 - 2q^3 + q^4)$$

$$y^{np}(b) = \frac{1}{6}(2 + 2q - 7q^2 + 12q^3 - 7q^4, 2 - 4q + 11q^2 - 3q^3 - 4q^4, 2 + 2q - 4q^2 - 9q^3 + 11q^4)$$

and similarly (due to symmetry) for edges c and d.

Focussing on payment for the edges a and b: For edge a, Ann pays more than agent Carl, who pays more than agent Bob for all values of $q \in (0, 1)$. This makes sense since Ann ought to be more liable than Carl for her direct connection to the source. The picture is more complicated looking at the edge b where payments vary substantially between agents according to the value of q: For low reliabilities ($q < 0.4$) Ann pays more than Carl, who pays more than Bob (Ann and Carl both have higher probability of getting service than Bob and should accordingly pay more). For high reliabilities ($q > 0.6$) Bob pays more than Ann, who pays more than Carl (Bob is now the agent who needs b the most and should accordingly pay more than both Ann and Carl). □

6 Final Remarks

We close with a few remarks on replacing agents service constraints, $\mathscr{D}^i$, with utility functions $u_i : 2^A \to \mathbf{R}$.

For each agent $i \in N$, this induces a cooperative game (A, u_i) with the public goods as "players". Thus, using a value (e.g., such as the Shapley value) of the game to allocate the total utility $u_i(A)$ onto the individual goods provides a set of straightforward candidates for liability indices of the goods for a given agent. Again, agents can share the cost of each good in proportion to these agent specific liability indices in line with approach above.

Note that even though we could use a similar idea based directly on the service constraints (where agents get utility 1, if served, and 0 if not, resulting in a simple game for each agent with the goods as "players") it is much less natural due to the implicit normalization of the induced simple games. The problems which arise are briefly discussed in Hougaard and Moulin (2014).

References

Bergantiños G, Vidal-Puga J (2007) A fair rule in minimum cost spanning tree problems. JET 137(1):326–352

Bogomolnaia A, Holzman R, Moulin H (2010) Sharing the cost of a capacity network. MOR 35(1):173–192

Curiel I (1997) Cooperative game theory and applications. Kluwer Academic Publishers, Boston

Dollinger P (1970) The german hansa. Macmillan, London

Hamers H, Borm P, van de Leensel R, Tijs SH (1999) Cost allocation in the chinese postman problem. EJOR 118(1):153–163

Hougaard JL (2009) An introduction to allocation rules. Springer, Berlin

Hougaard JL, Moulin H (2014) Sharing the cost of redundant items. GEB 87:339–352

Hougaard JL, Moulin H (2017) Sharing the cost of risky projects. Economic Theory (in press)

Hougaard JL, Tvede M (2015) Minimum cost connection networks: truth-telling and implementation. JET 157:76–99

Hougaard JL, Moulin H, Østerdal LP (2010) Decentralized pricing in minimum cost spanning trees. ET 44(2):293–306

Koster M, Molina E, Sprumont Y, Tijs SH (2001) Sharing the cost of a network: core and core allocations. IJGT 30(4):567–599

Meggido N (1978) Cost allocation for Steiner trees. Networks 8(1):1–6

Moulin H (2002) Axiomatic cost and surplus sharing. In: Arrow KJ, Sen AK, Suzumura K (eds) Handbook of social choice and welfare. Elsevier, Amsterdam, pp 289–357

Moulin H (2013) Cost sharing in networks: some open questions. IGTR 15(2):134–144

Moulin H, Laigret F (2011) Equal-need sharing of a network under connectivity constraints. GEB 72(1):314–320

Myerson RB (1981) Utilitarianism, egalitarianism, and the timing effect in social choice problems. Econometrica 49(4):883–897

Potters J, Curiel I, Tijs SH (1992) Traveling salesman games. MP 53(1):199–211

Sharkey WW (1995) Network models in economics. In: Handbook of operations research and management science. North-Holland, Amsterdam, pp 713–765

Young HP (1994) Cost allocation. In: Aumann RJ, Hart S (eds) Handbook of game theory with economic applications, vol II. Reprint, Amsterdam, 2007, pp 1193–1235

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The SD-Prenucleolus for TU-Games: Coalitional Monotonicity and Core Stability

Javier Arin and Ilya Katsev

Abstract The chapter introduces and analyses the Surplus Distributor-prenucleolus for TU games, a lexicographic value that satisfies core stability, strong aggregate monotonicity and null player out property in the class of balanced games. The solution is characterized in terms of balanced collection of sets and can be easily computed in the class of monotonic games with veto players and in the class of bankruptcy games. The SD-prenucleolus stands out as the only known core solution that satisfies coalitional monotonicity in the class of convex games and in the class of veto balanced games. Further, the SD-prekernel for TU games is introduced and analysed.

Keywords Bankruptcy games • Coalitional monotonicity • Convex games • Games with veto players • Surplus distribution

Contents

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1 Introduction

The Shapley value (Shapley 1953) and the prenucleolus (Schmeidler 1969) stand out as the best-known, most widely analysed single-valued solutions for coalitional games with transferable utility (TU games). One of the main reasons for the attractiveness of the Shapley value lies in the fact that it respects the principle of monotonicity, i.e. if a new TU game w is obtained from a given TU game v by increasing the worth of a coalition S then the members of S receive a payoff in game w that is no lower than in game v.

On the other hand, the prenucleolus respects the core stability principle, i.e. the prenucleolus selects a core allocation whenever the game is balanced. A core allocation provides each coalition with at least the worth of the coalition, i.e. the amount that the members of the coalition can obtain by themselves. It seems very attractive to ask for a solution that fulfils both principles, since they share a kind of incentive compatibility principle that can be summarized in the following idea: the higher the worth of a coalition the better for its members. However, in the class of balanced games they are not compatible (Young 1985) which means that the Shapley value does not respect core stability and the prenucleolus fails to satisfy coalitional monotonicity.

The prenucleolus is a lexicographic value that selects the vector of satisfactions of coalitions that lexicographically dominates any other vector of satisfactions of coalitions. When this vector is selected its associated allocation is automatically selected and this proves to be the prenucleolus of the game. When the satisfactions of coalitions are weighted by using a system of weights for the size of the coalitions this procedure will generate the various weighted prenucleoli. In the per capita prenucleolus (Grotte 1970) the total surplus of the coalition (the difference between the total payoff received by the coalition and its worth) is divided by the size (cardinality) of the coalition. In this way, each member of the coalition receives an equal part of the surplus.

Following this approach, Arin and Katsev (2014) introduce a different way of dividing the surplus of coalitions among their members. This alternative way of computing the satisfaction vector is used to define a new lexicographic value: the Surplus Distributor Prenucleolus or just SD-prenucleolus for short. Once the satisfaction vector is computed for any allocation, the SD-prenucleolus arises as the lexicographic optimal value in the set of vectors of satisfactions of coalitions.

Apart from the interpretation of the vector of satisfactions, the attractiveness of the SD-prenucleolus lies in the properties that it satisfies. Like the prenucleolus and the per capita prenucleolus, the SD-prenucleolus satisfies continuity, desirability, covariance and core stability. It also satisfies strong aggregate monotonicity, the Null Player Out property in the class of balanced games. However, the main difference with respect to any other known solution concept is that the SD-prenucleolus is a core solution that satisfies coalitional monotonicity in the class of convex games and in the class of veto monotonic games. Therefore, the solution arises as a natural candidate whenever there is compatibility between core stability and coalitional monotonicity.

2 Preliminaries: TU Games

A *cooperative n-player game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued function defined on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v is called the characteristic function of the game. Any subset S of N is called a *coalition*. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ Denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an *allocation*, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum\limits_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(N, v)$. Denote $\sum\limits_{i \in S} x_i$ by $x(S)$. The *core of a game* is the set of efficient allocations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

A game is *balanced* when it has a nonempty core (Bondareva 1963; Shapley 1967). A game (N, v) is said to be *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subseteq N.$$

Given a game (N, v), player $i \in N$ is said to be a *veto player* if $i \notin S$ implies that $v(S) = 0$. A balanced game with at least one veto player is called a veto balanced game. Let Γ_0 be a nonempty set of games. A *solution* φ on Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if the previous inequality holds with equality. A solution satisfies *core stability in* Γ_0 if it selects core allocations whenever the game is balanced and belongs to Γ_0. The solution is *single-valued* if the set contains a *single* element for each game in the set of games. A *core solution* is a single-valued solution concept that selects a core allocation whenever the game is balanced.

The vector x *weakly lexicographically dominates* the vector y (denoted by $x \succeq_L y$) if either $\widetilde{x} = \widetilde{y}$ or there is k such that $\widetilde{x}_i = \widetilde{y}_i$ for all $i \in \{1, 2, \ldots, k-1\}$ and $\widetilde{x}_k > \widetilde{y}_k$ where $\widetilde{x}$ and $\widetilde{y}$ are the vectors with the same components as the vectors x, y, but rearranged in a non decreasing order ($i > j \Rightarrow \widetilde{x}_i \geq \widetilde{x}_j$).

Given a game (N, v), a coalition $S \subset N$ and $x \in X(N, v)$, the *satisfaction of S with respect to x* is defined as $f(S, x) = x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in non decreasing order. Schmeidler (1969) introduced the *prenucleolus*[1] of a game v, denoted by $PN(N, v)$, as the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of allocations. In formula:

$$PN(N, v) = \{x \in X(N, v) \,|\, \theta(x) \succeq_L \theta(y) \text{ for all } y \in X(N, v)\}.$$

The *per capita prenucleolus* (Grotte 1970) is defined analogously by using the concept of per capita satisfaction instead of satisfaction. Given S, $S \neq \emptyset$, and x the *per capita satisfaction* of S at x is

$$f^{pc}(S, x) = \frac{x(S) - v(S)}{|S|}$$

The prenucleolus and the per capita prenucleolus are core solutions. Other *weighted prenucleoli* can be defined in a similar way whenever a weighted satisfaction function is defined. For two-*player* games the prenucleolus and the per capita prenucleolus coincide with the *standard solution* that allocates to each player the sum of the worth of his/her individual coalition and half of the surplus of the game (the difference between the worth of the grand coalition and the sum of the worth of individual coalitions).

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following.

- φ satisfies **anonymity** if for each (N, v) in Γ_0 and each bijective mapping $\tau : N \longrightarrow N$ such that $(N, \tau v)$ in Γ_0 it holds that $\varphi(N, \tau v) = \tau(\varphi(N, v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ ($x \in \mathbb{R}^N, j \in N, T \subseteq N$)). In this case (N, v) and $(N, \tau v)$ are equivalent games.
- φ satisfies the **equal treatment property (ETP)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ interchangeable players i, j are treated equally, i.e. $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.
- φ satisfies **desirability** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$, $x_i \geq x_j$ if i is more desirable than j in v. In a game (N, v) a player i is said to be more desirable than a player j if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subset N \backslash \{i, j\}$.

[1]The solution concept was defined using the notion of excess instead of satisfaction. Given a game (N, v) and an allocation x, the *excess of a coalition S with respect to x* in game (N, v) is defined as follows: $e(S, x) = v(S) - x(S)$.

Note that desirability implies ETP. Given a game (N, v), a real number $\alpha > 0$ and a vector $\beta \in \mathbb{R}^N$ the game $(N, w^{\alpha,B,v})$ can be defined as follows:

$$w^{\alpha,B,v}(S) = \alpha v(S) + \sum_{i \in S} \beta_i \text{ for all } S \subseteq N.$$

- φ satisfies **covariance** if the following condition is satisfied. If $\alpha > 0$, $\beta \in \mathbb{R}^N$ and $(N, v), (N, w^{\alpha,B,v}) \in \Gamma_0$ then $\varphi(N, w^{\alpha,B,v}) = \alpha\varphi(N, v) + \beta$.
- φ satisfies the **null player property** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ null players receive 0. Here, a player is a null player if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$.
- φ satisfies the **null player out property (NPO)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ it holds that $(x_i)_{i \in N \setminus T} \in \varphi(N \setminus T, v)$. Here T is the set of null players in game (N, v).
- φ satisfies **core stability** if for each (N, v) in Γ_B, $\varphi(N, v) \subset C(N, v)$.

The following properties are defined for single-valued solutions:

- φ satisfies *coalitional monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(T) < w(T)$ for some $T \subseteq N$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{T\}$ then $\varphi_i(N, v) \leq \varphi_i(N, w)$ for all $i \in T$.
- φ satisfies *aggregate monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(N) < w(N)$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{N\}$ then $\varphi_i(N, w) - \varphi_i(N, v) \geq 0$ for all $i \in N$.
- φ satisfies *strong aggregate monotonicity in* Γ_0 if the following condition holds: If $v, w \in \Gamma_0$, $v(N) < w(N)$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{N\}$ then $\varphi_i(N, w) - \varphi_i(N, v) = \varphi_j(N, w) - \varphi_j(N, v) \geqslant 0$ for all $i, j \in N$.

Young (1985) proves that no core solution satisfies coalitional monotonicity in the class of balanced games. However there are core solutions, including the per capita prenucleolus and the SD-prenucleolus (defined in Sect. 3.1), that satisfy strong aggregate monotonicity. The prenucleolus does not satisfy aggregate monotonicity in the class of convex games (Hokari 2000). The per capita prenucleolus does not satisfy coalitional monotonicity in the class of convex games (Arin 2013).

The following notation is widely used here. (N, b_S) denotes the game whose characteristic function is

$$b_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

Given two TU games (N, v) and (N, w) with the same set of players $(N, v + w)$ denotes a new game where $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. Given a game (N, v) and a real number λ $(N, \lambda v)$ denotes a new game where $(\lambda v)(S) = \lambda v(S)$ for all $S \subseteq N$.

3 The SD-Prenucleolus

3.1 Definition

The definition of the SD-prenucleolus is based on the concept of satisfaction of a coalition with respect to an allocation. Given a game (N, v) and an allocation $x \in X(N, v)$ a new satisfaction vector is calculated $(F(S, x))_{S \subset N}$. The components of this vector are defined recursively by defining an algorithm. The algorithm has several steps (at most $2^{|N|} - 2$) and at each step the collection of coalitions $\mathscr{H}$ that has obtained the new satisfaction is identified. In the first step this collection $\mathscr{H}$ is empty. The algorithm ends when $\mathscr{H} = 2^N \setminus \{N\}$.

For a collection $\mathscr{H} \subset 2^N \setminus \{N\}$ and a function $F : \mathscr{H} \to \mathbb{R}$ the function $F_{\mathscr{H}} : 2^N \setminus \{\mathscr{H} \cup \{N\}\} \to \mathbb{R}$ is now defined.

To this end, some notation is introduced. For $\mathscr{H} \subset 2^N \setminus \{N\}$ and $S \subset N$, denote

$$\sigma_{\mathscr{H}}(S) = \bigcup_{T \in \mathscr{H}, T \subset S} T.$$

For $S \subset N$ denote by $f_{\mathscr{H},F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathscr{H}$ $(i \in \sigma_{\mathscr{H}}(S))$:

$$f_{\mathscr{H},F}(i, S) = \min_{T: T \in \mathscr{H}, i \in T \subset S} F(T).$$

Now a value $F_{\mathscr{H}}(S)$ is defined for all $S \in 2^N \setminus \{\mathscr{H} \cup \{N\}\}$. Two cases are considered (since it is evident that $\sigma_{\mathscr{H}}(S) \subseteq S$):

1. Relevant coalitions. $\sigma_{\mathscr{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathscr{H}}(S) = \frac{x(S) - v(S) - \sum_{i \in \sigma_{\mathscr{H}}(S)} f_{\mathscr{H},F}(i, S)}{|S| - |\sigma_{\mathscr{H}}(S)|}. \tag{1}$$

   Note that if collection $\mathscr{H}$ is empty then the current satisfaction of coalition S coincides with its per capita satisfaction:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}.$$

2. Completed coalitions. $\sigma_{\mathscr{H}}(S) = S$. In this case the satisfaction of S is

$$F_{\mathscr{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathscr{H},F}(i, S) + \max_{i \in S} f_{\mathscr{H},F}(i, S). \tag{2}$$

The algorithm that computes the new satisfaction vector, whose components are denoted by $F(S) = F(S,x)$, is the following:

Let (N, v) be a TU game and $x \in X(N, v)$.
Step 1: Set $k = 0$, $\mathscr{H}_0 = \emptyset$ and $F_\emptyset(S) = \frac{x(S)-v(S)}{|S|}$.
Step 2: Set

$$\mathscr{H}_{k+1} = \mathscr{H}_k \cup \{S \notin \mathscr{H}_k : F_{\mathscr{H}_k}(S) = \min_{T \notin \mathscr{H}_k} F_{\mathscr{H}_k}(T)\}.$$

Define for each $S \in \mathscr{H}_{k+1} \setminus \mathscr{H}_k$:

$$F(S,x) = F_{\mathscr{H}_k}(S).$$

Step 3: If $\mathscr{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k+1$ and go to Step 2. Otherwise go to Step 4.
Step 4: Stop. Return the vector $(F(S,x))_{S \subset N}$.

Given a TU game (N, v) and $x \in X(N, v)$, a coalition $S \subset N$ is said to be *relevant* (*completed*) with respect to x if S was a relevant (completed) coalition at the step where its satisfaction $F(S,x)$ was determined. Given a TU game (N, v), a player $i \in S \subset N$ and $x \in X(N, v)$ denote by $f_i(S,x) = \min_{T: i \in T \subseteq S} F(T,x)$ and $z_i(S,x) = x_i - f_i(S,x)$ the *satisfaction* and the *coalitional payoff of player i in coalition S at x*.

The following 3-player game is used to illustrate how this algorithm works.

Example 14.1 Let (N, v) be a game where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1,3\},\{1,2\}\} \\ -10 & \text{if } S = \{2,3\} \\ 6 & \text{if } S = N \end{cases}$$

and consider the allocation $x = (5, 1, 0)$.

In the first step min $\frac{x(S).v(S)}{|S|}$ is computed and it is concluded that coalition $\{3\}$ has the lowest per capita satisfaction and therefore $F(\{3\}, x) = 0$ and $\mathscr{H}_1 = \{\{3\}\}$. That is, in the first step coalition $\{3\}$ is the only coalition that has received its satisfaction with respect to x.

In the second step when computing the satisfaction of coalitions containing coalition $\{3\}$ it needs to be taken into account that $F(\{3\}, x) = 0$ and $f_{\mathscr{H}_1,F}(3, S) = 0$ for any S containing player 3. In this step, the satisfaction of coalition $\{1, 3\}$ results

$$F_{\mathscr{H}_1}(\{1,3\},x) = \frac{x(\{1,3\}) - v(\{1,3\}) - \sum_{i \in \sigma_{\mathscr{H}}(\{1,3\})} f_{\mathscr{H},F}(i,\{1,3\})}{|\{1,3\}| - |\sigma_{\mathscr{H}}(\{1,3\})|} = \frac{5-4-0}{2-1}.$$

since $\sigma_{\mathscr{H}}(\{1,3\}) = \{3\}$. Similarly, $F_{\mathscr{H}_2}(\{2,3\},x) = 11$ and for $S \in \{\{2\},\{1,2\},\{1\}\}$ it holds that $F_{\mathscr{H}_2}(S,x) = F_{\mathscr{H}_1}(S,x)$.

In this way, $F(\{2\},x) = F(\{1,3\},x) = 1$ and $\mathscr{H}_2 = \{\{2\},\{1,3\},\{3\}\}$:

The procedure is summarized in the following table:

k	Coalitions in $\mathscr{H}_k \backslash \mathscr{H}_{k-1}$	$F(S,x)$
1	$\{3\}$	0
2	$\{2\},\{1,2\},\{1,3\}$	1
3	$\{1\}$	5
4	$\{2,3\}$	11.

Note that coalition $\{2,3\}$ is a completed coalition. The rest of the coalitions are relevant coalitions.

Therefore, given a TU game (N,v) for each $x \in X(N,v)$ a satisfaction vector $(F(S,x))_{S\subset N}$ is computed.

Definition 14.1 The satisfaction vector $F^x = \{F(S,x)\}_{S\subset N}$ is said to *dominate* the satisfaction vector $F^y = \{F(S,y)\}_{S\subset N}$ if there is $k \geq 1$ such that

1. $\tilde{F}_i^x = \tilde{F}_i^y$ for all $i < k$
2. $\tilde{F}_k^x > \tilde{F}_k^y$,

where $\tilde{F}^x$ and $\tilde{F}^y$ are the vectors with the same components as the vectors F^x, F^y, but rearranged in a nondecreasing order ($i > j \Rightarrow \tilde{F}_i^x \leq \tilde{F}_j^x$).

The allocation x is said to belong to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector. Denote the SD-prenucleolus of game (N,v) by $SD(N,v)$.

Definition 14.2 Let (N,v) be a TU game. Then $x \in SD(N,v)$ if and only if for any $y \in X(N,v)$ it holds that $F^x \succeq_L F^y$.

Similarly to the prenucleolus, the SD-prenucleolus satisfies single-valuedness in the class of all TU games.

Theorem 14.1 *Let (N,v) be a TU game. Then $|SD(N,v)| = 1$.*

The SD-prenucleolus is a core solution. Clearly, if a game is balanced then any core allocation has a non negative satisfaction vector.

3.2 Properties

Like the prenucleolus, the SD-prenucleolus satisfies desirability (and therefore the equal treatment property), anonymity, covariance and core stability. Unlike the prenucleolus and similarly to the per capita prenucleolus, the SD-prenucleolus satisfies strong aggregate monotonicity.

The SD-prenucleolus and the per capita prenucleolus do not satisfy the null player property since there is incompatibility between strong aggregate monotonicity, the null player property and core stability.

Proposition 14.1 *If a solution φ defined in the class of all TU games satisfies core stability and the null player property then φ does not satisfy the strong aggregate monotonicity property.*

Clearly, in the class of balanced games a solution that satisfies core stability must satisfy the NP property. But this is not necessarily true for the NPO property.

Proposition 14.2 *The SD-prenucleolus satisfies the NPO property in the class of balanced games.*

The per capita prenucleolus does not satisfy the NPO property in the class of balanced games as the following example shows. Let $N = \{1, 2, 3, 4\}$ and consider the games (N, v_1) and $(N \backslash \{4\}, v_2)$ where

$$v_1(S) = \begin{cases} 7 \text{ if } S \in \{\{1,2,3\}, N\} \\ 4 \text{ if } S \in \{\{1,2\}, \{1,2,4\}\} \\ 0 \text{ otherwise,} \end{cases}$$

$$v_2(S) = \begin{cases} 7 \text{ if } S = N \backslash \{4\} = \{1,2,3\} \\ 4 \text{ if } S = \{1,2\} \\ 0 \text{ otherwise.} \end{cases}$$

In game (N, v_1) player 4 is a null player and game $(N \backslash \{4\}, v_2)$ results after eliminating player 4 from game (N, v_1). The per capita prenucleolus of game (N, v_1) is $(2.9, 2.9, 1.2, 0)$ and the per capita prenucleolus of game $(N \backslash \{4\}, v_2)$ is $(3, 3, 1)$.

3.3 Characterization

The SD-prenucleolus has a characterization in terms of balanced collections of coalitions which is the equivalent of Kohlberg's characterization for the prenucleolus (Kohlberg 1971). The theorem is useful for checking whether an allocation is the prenucleolus of a game or not.

Given an allocation x and a real number α the following set of coalitions is defined:

$$\mathscr{B}_\alpha = \{S \subset N : F(S, x) \leq \alpha\}.$$

Let N be a finite set of players and let $\mathscr{C}$ be a collection of distinct nonempty subsets. Define $\mathscr{C}_i = \{S \in \mathscr{C} : i \in S\}$. $\mathscr{C}$ is said to be a balanced collection of sets if there are positive numbers $(\lambda_S)_{S \in \mathscr{C}}$ such that $\sum_{S \in \mathscr{C}_i} \lambda_S = 1$ for all $i \in N$.

Let (N, v) be a TU game and $x \in X(N, v)$. Denote by $\mathscr{B}(x)$ the set of coalitions with minimal satisfaction with respect to x.

Theorem 14.2 *Let (N, v) be a TU game and x be an allocation. Then $x = SD(N, v)$ if and only if the collection of sets $\mathscr{B}_\alpha(x)$ is empty or balanced for every α.*

This theorem allows it to be checked whether an allocation is the SD-prenucleolus of the SD-prenucleolus of the game or not. In general, the computation of the SD-prenucleolus of a game is not immediately apparent.

4 Applications

Although in general the computation of the SD-prenucleolus of a game may involve some complexity there are classes of games where this aspect has been satisfactorily solved, such as games with veto players and bankruptcy games.

4.1 Games with Veto Players

The class of games with veto players has been widely used to model economic situations where the presence of special players is needed in order to achieve some positive outcome. An easy way to compute the SD-prenucleolus of games with veto players is provided here.

Arin and Feltkamp (2012) introduce and characterize the Serial Rule for the class of veto balanced games. Let (N, v) be a game with veto players and let player 1 be a veto player. Define for each player i a value d_i as follows:

$$d_i = \max_{S \subseteq N \setminus \{i\}} v(S).$$

Clearly, $d_1 = 0$. Let $d_{n+1} = v(N)$ and rename players according to the nondecreasing order of those values. That is, player 2 is the player with the lowest value apart from player 1 and so on. The solution SR associates the following payoff vector with each game with veto players:

$$SR_l(N, v) = \sum_{i=l}^{n} \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1, \ldots, n\}.$$

Since $d_1 = 0$, the solution is efficient. If there is no veto player the solution is not efficient.

Example 14.2 Let $N = \{1, 2, 3, 4\}$ and consider a 4-*player* game (N, v) defined as follows:

$$v(S) = \begin{cases} 1 \text{ if } S \in \{\{1,3,4\}, \{1,2,4\}\} \\ 4 \qquad \text{if } S = \{1,2,3\} \\ 8 \qquad \text{if } S = N \\ 0 \qquad \text{otherwise.} \end{cases}$$

Then $(d_1, d_2, d_3, d_4, d_5) = (0, 1, 1, 4, 8)$ and applying the formula

$$\begin{aligned} SR_1 &= \tfrac{d_2-d_1}{1} + \tfrac{d_3-d_2}{2} + \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 3 \\ SR_2 &= \tfrac{d_3-d_2}{2} + \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 2 \\ SR_3 &= \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 2 \\ SR_4 &= \tfrac{d_5-d_4}{4} = 1. \end{aligned}$$

Theorem 14.3 *Let (N, v) be a monotonic veto game. Then $SR(N, v) = SD(N, v)$.*

Corollary 14.1 *The SD-prenucleolus satisfies coalitional monotonicity in the class of monotonic veto games.*

4.2 Bankruptcy Games

Bankruptcy problems (O'Neill 1982) model situations where a finite set of claimants need to share an endowment that is not enough to satisfy fully all their claims. Formally, consider an infinite set of potential claimants, indexed by the natural numbers $\mathbb{N}$. Each given bankruptcy problem involves a finite number of claimants. Let $\mathcal{N}$ denote the class of non-empty finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $i \in N$, let c_i be claimant i's claim and $c = (c_i)_{i \in N}$ the claims vector and let E be the endowment to be divided among the claimants in N. A *bankruptcy problem* (or just *problem*) is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$, where N $\in \mathcal{N}$, such that $\sum_{i \in N} c_i \geq E$. Let $\mathscr{B}^N$ denote the class of all problems with claimants set N. An *allocation* for $(c, E) \in \mathscr{B}^N$ is a vector $x \in \mathbb{R}^N$ such that it satisfies the non-negativity and claim boundedness conditions, *i.e.* $0 \leqq x \leqq c$ and the efficiency condition $\sum_{i \in N} x_i = E$. Let $X(c, E)$ be the set of allocations of (c, E). A *bankruptcy rule* (or just *rule*) is a function defined on $\cup_{N \in \mathcal{N}} \mathscr{B}^N$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathscr{B}^N$ an allocation in $X(c, E)$. Given a problem $(c, E) \in \mathscr{B}^N$, denote the total claim by $C = \sum_{i \in N} c_i$ and the total loss by $L = C - E$.

- The constrained equal awards rule divides the endowment equally among the claimants under the constraint that no claimant receives more than his/her claim.

Constrained equal awards rule, CEA: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$CEA_i(c, E) = \min\{\beta, c_i\} \text{ where } \beta \in \mathbb{R} \text{ solves} \sum_{i \in N} \min\{\beta, c_i\} = E.$$

- The constrained equal losses rule divides the total loss equally among the claimants under the constraint that no claimant receives a negative amount.

  Constrained equal losses rule, CEL: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$CEL_i(c, E) = \max\{0, c_i - \beta\} \text{ where } \beta \in \mathbb{R} \text{ solves} \sum_{i \in N} \max\{0, c_i - \beta\} = E.$$

- The **Minimal overlapping rule, MO:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$MO_i(c, E) = \begin{cases} CEA_i(SR(c), E) & \text{if } E \leq c_n \\ SR_i(c) + CEL_i(c - SR(c), E - c_n) & \text{otherwise.} \end{cases}$$

where the vector of claims has been ordered such that $c_1 \leq \ldots \leq c_n$ and $SR_i(c) = \sum_{j=0}^{l-1} \frac{c_{j+1} - c_j}{n-j}$ for all $l \in N$ and $c_0 = 0$.

According to this definition, $SR(c)$ is a reference vector. No claimant receives less (more) than $SR_i(c)$ if the endowment is greater (smaller) than the highest claim.

Finally, the TU game associated with a bankruptcy problem $(c, E) \in \mathcal{B}^N$ is a pair (N, v) where $v(S) = \max\left\{E - \sum_{l \notin S} c_l, 0\right\}$ for all $S \subset N$.

The following example considers several bankruptcy problems with the same vector of claims.

Example 14.3 Let $N = \{1, 2, 3, 4\}$ and $c = (4, 7, 9, 10)$. Therefore $SR(c) = (1, 2, 3, 4)$ and,

$$\begin{aligned} MO(c, 4) &= (1, 1, 1, 1) = CEA(c, 4) \\ MO(c, 7) &= (1, 2, 2, 2) \\ MO(c, 10) &= (1, 2, 3, 4) = SR(c, 10) \\ MO(c, 12) &= (1, 2, 4, 5) \\ MO(c, 22) &= (2, 5, 7, 8) = CEL(c, 22) \end{aligned}$$

Aumann and Maschler (1985) prove that the prenucleolus[2] of bankruptcy games and the Talmud rule coincide. Huijink et al. (2015) introduce the claim-and-right rules family. They provide a formula for computing the per capita prenucleolus and show that it is a member of the said family which also includes the Talmud rule and the Minimal Overlapping rule.[3]

Theorem 14.4 *Let* $(c, E) \in \mathscr{B}^N$ *and* (N, v) *be its associated TU game. Then* $MO(c, E) = SD(N, v)$.

5 Coalitional Monotonicity and Core Stability

Young's theorem implies that every core solution, including the SD-prenucleolus, violates monotonicity in the class of all TU games. Arin (2013) investigates when such a violation of monotonicity by a given core solution is justified. It is argued that in the class of convex games and in the class of monotonic veto games there is compatibility between core stability and coalitional monotonicity. The fact that the SD-prenucleolus satisfies coalitional monotonicity in both classes of games reinforces the arguments.

5.1 SD-Relevant Games

In the class of convex games (Shapley 1971) core stability and coalitional monotonicity are compatible since in this class the Shapley value satisfies both properties. The Shapley value is not a core solution in the class of monotonic veto games. Therefore, the issue of whether there is a core solution that satisfies coalitional monotonicity in the class of convex games has remained an open question. The answer is yes: Convex games are SD-relevant games (Arin and Katsev 2016a) and the SD-prenucleolus satisfies coalitional monotonicity in this class of games.

Definition 14.3 A TU game (N, v) is SD-relevant if any $S,\ S \subset N$, is relevant with respect to $SD(N, v)$.

Equivalently, a game (N, v) is SD-relevant if

$$F(S \cup T, x) \leq \max\{F(S, x), F(T, x)\} \text{ for all } S, T \subset N,\ S \cup T \neq N$$

where $x = SD(N, v)$.

The following example shows that there are TU games that are not SD-relevant.

[2]They prove the coincidence of the nucleolus and the Talmud rule. In the class of balanced games the (per capita) prenucleolus and the (per capita) nucleolus coincide.

[3]In the class of two-claimant problems the Talmud rule and the Minimal Overlapping rule coincide.

Example 14.4 Let (N, v) be a TU game where $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 0 \text{ if } |S| = 1 \text{ or } S \in \{\{1,2\}, \{3,4\}\} \\ 4 \qquad\qquad \text{if } S = N \\ 2 \qquad\qquad \text{otherwise.} \end{cases}$$

Then $SD(N, v) = (1, 1, 1, 1)$, $F(\{1, 2, 3\}) = 1$ and $F(\{1, 3\}) = F(\{2, 3\}) = 0$. Therefore, coalition $\{1, 2, 3\}$ is completed with respect to the allocation $(1, 1, 1, 1)$. Consequently, (N, v) is not SD-relevant.

Arin and Katsev (2016b) also introduce the notion of totally relevant games.

Let (N, v) be a TU game and $x \in X(N, v)$. Then the game is x-relevant if any S, $S \subset N$, is relevant with respect to x. A TU game (N, v) is totally relevant if it is x-relevant for any $x \in X(N, v)$.

Convex games are totally relevant games.

The proof that the SD-prenucleolus is coalitionally monotonic in the class of SD-relevant games is based on other results that are of interests in themselves and are set out in the following two subsections.

5.2 Reduced Game Property

Like the prenucleolus and other lexicographic values, the SD-prenucleolus satisfies a consistency property called the SD-reduced game property. This property is based on the notion of the SD-reduced game, which is introduced here.

Definition 14.4 Let (N, v) be a TU game, $S \subset N$ and $x \in X(N, v)$. A game (S, v^x) is the SD-reduced game with respect to S and x if

1. $v^x(S) = v(N) - x(N \setminus S)$
2. for every $T \subset S$

$$F^{(S,v^x)}(T, (x_i)_{i \in S}) = \min_{U \subseteq N \setminus S} F^{(N,v)}(U \cup T, x)$$

where $F^{(S,v^x)}(T, (x_i)_{i \in S})$ is the satisfaction in game (S, v^x) of coalition T at $(x_i)_{i \in S}$ and $F^{(N,v)}(U \cup T, x)$ is the satisfaction in game (N, v) of coalition $U \cup T$ at x.

The following lemma guarantees that the SD-reduced game (S, v^x) exists and is unique for any game (N, v), any coalition $S \subset N$ and any allocation $x \in X(N, v)$.

Lemma 14.1 *Let N be a finite set of players, $x \in \mathbb{R}^N$, let V be a real number and $f \in \mathbb{R}^{2^N \setminus \{N\}}$. Then there is a unique TU game (N, v) such that*

1. $v(N) = V$
2. $(F(S, x))_{S \subset N} = f$.

Definition 14.5 Solution ϕ satisfies the *SD-!reduced game property* on Γ, *SD-RGP*, if for every game $(N, v) \in \Gamma$ then $(x_i)_{i \in S} \in \phi(S, v^x)$ for any $S \subset N$ and any $x \in \phi(N, v)$.

The SD-prenucleolus satisfies the SD-reduced game property in the class of all TU games. This type of property[4] plays a determinant role in the characterization of lexicographic values such as the prenucleolus (Sobolev 1975) and the per capita prenucleolus (Kleppe 2010). The reduced games associated with the prenucleolus and the per capita prenucleolus can be reformulated explicitly.

If a game is SD-relevant then any SD-reduced game with respect to the SD-prenucleolus of the game is also SD-relevant. Therefore, in the class of SD-relevant games the SD-reduced games with respect to the SD-prenucleolus belong to this class.

Lemma 14.2 *Let (N, v) be an SD-relevant game, $S \subset N$ and $x = SD(N, v)$. Then (S, v^x) is SD-relevant.*

Lemma 14.2 allows for a different interpretation of the SD-reduced game of an SD-relevant game. SD-reduced games with respect to the SD-prenucleolus of the game can be easily computed according to the result established by the following lemma.

Lemma 14.3 *Let (N, v) be an SD-relevant game, $S \subset N$ and $x = SD(N, v)$. Consider the SD-reduced game (S, v^x) and $T \subset S$. Then*

$$v^x(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S), x) = \sum_{i \in T} z_i(T \cup (N \setminus S), x).$$

Corollary 14.2 *Let (N, v) be an SD-relevant game, $x = SD(N, v)$ and $S \in \mathscr{B}(x)$. Consider the SD-reduced game $(N \backslash S, v^x)$ and coalition $T \subset N \backslash S$. Then $v^x(T) = v(T \cup S) - v(S)$.*

5.3 Antipartitions and SD-Equivalent Games

The notion of the *antipartition* (Arin and Iñarra 1998) also plays a central role in the proof of Theorem 14.5. A collection of sets $\mathscr{C} = \{S : S \subset N\}$ is called an *antipartition* if the collection of sets $\{N \backslash S : S \in \mathscr{C}\}$ is a partition of N. An antipartition is a balanced collection of sets. In order to balance an antipartition $\mathscr{Q}$ each coalition receives the same weight, i.e. $\frac{1}{|\mathscr{C}|-1}$. Given a TU game (N, v) the *satisfaction of an antipartition* $\mathscr{C}$ is defined by

$$\mathscr{F}(\mathscr{C}, v) = \frac{v(N) - \sum_{S \in \mathscr{C}} \frac{1}{|\mathscr{C}|-1} v(S)}{|N|}. \tag{3}$$

[4]Note that the definition of this reduced game depends on the definition of the satisfaction vector. If the vector considered is $(x(S) - v(S))_{S \subset N}$ then the associated reduced game property is satisfied by the prenucleolus.

Lemma 14.4 *Let* (N, v) *be a TU game and* $x \in X(N, v)$*. Let* $\mathscr{C}$ *be an antipartition contained in* $\mathscr{B}(x)$*. Then* $F(S) = \mathscr{F}(\mathscr{C},)$ *for all* $S \in \mathscr{B}(x)$*.*

Note that if the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game contains an antipartition then the satisfaction of those coalitions only depends on the characteristic function of the game.

Arin and Iñarra (1998) prove that, given a convex game, the collection of coalitions with minimal satisfaction with respect to the prenucleolus of the game contains either a partition or an antipartition. In the case of the SD-prenucleolus of SD-relevant games only antipartitions should be considered, as the following lemma shows.

Lemma 14.5 *Let* (N, v) *be an SD-relevant game. Then* $\mathscr{B}(SD(N, v))$ *contains an antipartition.*

Note that Lemma 14.5 also holds for the per capita prenucleolus since the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game and the set of coalitions with minimal satisfaction with respect to the per capita prenucleolus of the game coincide. The notion of SD-equivalent games which is also used in the proof of Theorem 14.5 is now introduced.

Definition 14.6 TU games (N, v) and (N, w) are said to be SD-equivalent if $\mathscr{B}(SD(N, v)) \cap \mathscr{B}(SD(N, w))$ contains an antipartition.

The following lemma enables only SD-equivalent games to be considered when analysing the coalitional monotonicity of the SD-prenucleolus in the class of SD-relevant games.

Lemma 14.6 *Let* (N, v) *be a TU game such that for some* $S \subset N$ *and all* $\gamma \in [0, \alpha]$ *the game* $(N, v + \gamma b_S)$ *is SD-relevant. Then there exists* $\beta \in (0, \alpha]$ *such that:*

1 (N, v) *and* $(N, v + \beta b_S)$ *are SD-equivalent games.*
2 $(N, v + \beta b_S)$ *and* $(N, v + \alpha b_S)$ *are SD-equivalent games.*

5.4 Coalitional Monotonicity

In the class of SD-relevant games the SD-prenucleolus satisfies coalitional monotonicity. This result is based on the following facts:

1. Only SD-equivalent games need to be considered.
2. Given an SD-relevant game, the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game contains an antipartition. The satisfaction of the coalitions included in the antipartition only depends on the characteristic function of the game.
3. Any SD-reduced game with respect to the SD-prenucleolus of an SD-relevant game is SD-relevant and can be easily computed.

Theorem 14.5 *Let (N, v) be a TU game such that for some $S \subset N$ and all $\gamma \in [0, \alpha]$ the game $(N, v + \gamma b_S)$ is SD-relevant. Then $SD_i(N, v + \alpha b_S) \geq SD_i(N, v)$ for all $i \in S$.*

The theorem is not necessarily true if the games are not SD-relevant.

Example 14.5 Consider two 4-player games (N, v) and (N, w) defined as follows:

$$v(S) = \begin{cases} 4 & \text{if } S = N \\ 0 \text{ if } & |S| = 1 \text{ or } S \in \{\{1,2\},\{3,4\}\} \\ 2 & \text{otherwise} \end{cases}$$

and

$$w(S) = \begin{cases} 4 & \text{if } S = \{1,2,3\} \\ v(S) & \text{otherwise.} \end{cases}$$

It can be checked that $SD(N, v) = x = (1, 1, 1, 1)$ and $SD(N, w) = y = (2, 2, 0, 0)$. Since $\{\{1,3\},\{2,4\}\} = \mathscr{B}(x) \cap \mathscr{B}(y)$, (N, v) and (N, w) are SD-equivalent games. However the two games are not SD-relevant. Note that

$$F(\{1,2,4\}, x, v) = 1 > \max(F(\{1,4\}, x, v), F(\{2,4\}, x, v)) = 0$$

and similarly,

$$F(\{1,2,4\}, y, w) = 2 > \max(F(\{1,4\}, y, w), F(\{2,4\}, y, w)) = 0.$$

Coalition $\{1,3\}$ belongs to the antipartition included in $\mathscr{B}(x) \cap \mathscr{B}(y)$. Consider the SD-reduced games $(\{1,3\}, v^x)$ and $(\{1,3\}, w^y)$. Then

$$v^x(\{1\}) = 1 \neq v(\{1,2,4\}) - v(\{2,4\}) = 0$$

and

$$w^y(\{1\}) = 2 \neq w(\{1,2,4\}) - w(\{2,4\}) = v(\{1,2,4\}) - v(\{2,4\}) = 0.$$

The worth of coalitions in games $(\{1,3\}, v^x)$ and $(\{1,3\}, w^y)$ depends on x and y and not only on the characteristic functions v and w.

6 The SD-Prekernel

The prenucleolus of a game is contained in its prekernel and this makes the study of this solution concept attractive. Given a TU game (N, v), the prekernel selects allocations where for each pair of players, $\{i,j \subset N,\}$, a complaint by player i

against player j equals a complaint by j against i. A complaint by i against j is the minimal satisfaction obtained by a coalition containing i and excluding j. In some cases the prekernel is single-valued and coincides with the prenucleolus. The coincidence holds for convex games (Maschler et al. 1971).

Similarly to the classic prekernel, the SD-prekernel arises whenever the new satisfaction vector is considered. Let (N, v) be a game and $x \in X(N, v)$. Define a complaint by a player i against a player j as the minimal satisfaction obtained using coalitions that contain player i but not player j. Formally,

$$s_{ij}(x) = \min_{S:i\in S,\, j\notin S} F(S, x).$$

Unlike the prekernel, the following remark identifies coalitions that may be used as complaints of the players.

Remark 14.1 Let (N, v) be a game and $x \in X(N, v)$. Then:

$$\min_{S:i\in S,\, j\notin S} F(S, x) = f_i(N\setminus\{j\}, x).$$

The SD-prekernel of a game $(N, vI$, denoted by $SD\text{-}PK(N, v)$ is now introduced.

Definition 14.7 Let (N, v) be a TU game. Then

$$SD\text{-}PK(N, v) = \left\{x \in X(N, v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\right\}$$

The SD-prenucleolus of a game is contained in its SD-prekernel. In some cases this inclusion is strict.

Example 14.6 Consider a 4-player game (N, v) defined as follows:

$$v(S) = \begin{cases} 4 & \text{if } S = N \\ 0 \text{ if } |S| = 1 & \text{or } S \in \{\{1, 2\}, \{3, 4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

Then $SD\text{-}PK(N, v) = \{(x, x, 2-x, 2-x) : 0 \leq x \leq 2\}$ and the $SD(N, v) = (1, 1, 1, 1)$.

The following example shows that in some cases the SD-prekernel is multivalued while the prekernel is single-valued.

Example 14.7 Consider a 4-*player* game (N, v) defined as follows:

$$v(S) = \begin{cases} 24 & \text{if } S = N \\ 0 \text{ if } |S| = 1 & \text{or } S \in \{\{1, 2\}, \{3, 4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

Note that $SD\text{-}PK(N, v) = \{(5+x, 5+x, 7-x, 7-x) : 0 \leq x \leq 2\}$ and that $SD\text{-}PN(N, v) = (6, 6, 6, 6)$ which is the only allocation contained in the prekernel.

Remark 14.2 Let $(N, v), (N, w)$ be two games such that $w(N) > v(N)$ and $w(S) = v(S)$ if $S \neq N$. If $x \in SD\text{-}PK(N, v)$ then $x + (\frac{w(N)-v(N)}{|N|}, \ldots, \frac{w(N)-v(N)}{|N|}) \in SD\text{-}PK(N, w)$.

Finally, it is shown that, in general, the SD-prekernel does not need to be a subset of the core.

Example 14.8 Let $N = M_1 \cup M_2 = \{1, 2, 3\} \cup \{4, 5\}$ and consider a 5-*player* glove game (N, v) defined as follows:

$$v(S) = \begin{cases} 0 & \text{if } S = N \\ 0 & \text{if } |S| = 3 \text{ and } |S \cap M_2| = 1 \\ -10 & \text{otherwise.} \end{cases}$$

Since $(0, 0, 0, 0, 0) \in C(N, v)$, the game is balanced. Let $x = (-2, -2, -2, 3, 3)$. Since $F(S, x) = -\frac{1}{6}$ if $v(S) = 0$ and the rest of coalitions have a positive satisfaction, $x \in SD\text{-}PK(N, v)$.

For convex games, the prekernel and the prenucleolus coincide (Maschler et al. 1971). The SD-prekernel is single-valued in this class.

Theorem 14.6 *In the class of convex games the SD-prekernel and the SD-prenucleolus coincide.*

In the class of two-player games the SD-prekernel, the SD-prenucleolus and the prenucleolus coincide.

Peleg (1986) characterizes the prekernel using the Davis-Maschler reduced game property, among other properties. The SD-prekernel is analogously characterized by replacing the Davis-Maschler reduced game property by the SD-reduced game property.

7 Conclusion

It seems desirable for a single-valued solution to satisfy core stability and monotonicity whenever they are compatible. A core solution may be considered monotonic whenever any violation of monotonicity is justified. To justify such a violation is to show that there was an incompatibility between core stability and coalitional monotonicity. Since no such incompatibility exists in the class of convex games or in the class of veto monotonic games, a monotonic core solution should satisfy coalitional monotonicity in both classes.

The SD-prenucleolus may be considered as a first positive contribution in the search and analysis of single-valued continuous[5] solutions that accommodate the principles of monotonicity and core stability as far as possible. In this sense, the notion that there may be a continuous single-valued solution that satisfies coalitional monotonicity in the class of all TU games and core stability in the aforementioned classes of games seems an interesting issue.

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References

Arin J (2013) Monotonic core solutions: beyond Young's theorem. IJGT 42(2):325–337
Arin J, Feltkamp V (2012) Coalitional games: monotonicity and core. EJOR 216(1):208–213
Arin J, Iñarra E (1998) A characterization of the nucleolus for convex games. GEB 23(1):12–24
Arin J, Katsev I (2014) The SD-prenucleolus for TU games. MMOR 80(3):307–327
Arin J, Katsev I (2016a) A monotonic core solution for convex TU games. Int J Game Theory 45(4):1013–1029
Arin J, Katsev I (2016b) The SD-prekernel for TU games. WP IL. 96/16
Aumann RJ, Maschler M (1985) Game theoretical analysis of a bankruptcy problem from the talmud. JET 36(2):195–213
Bondareva O (1963) Nekotorjye primeneniya metodov linynogo programmorovaniya k teorii kooperativnikh igr. Problemi Kibernetiki 10:119–139. Translated with: Some applications of linear programming methods to the theory of cooperative games. In: Morgenstern O (ed) Selected Russian papers on game theory 1959–1965. Princeton, pp 79–114
Grotte J (1970) Computation of and observations on the nucleolus, the normalised nucleolus and the central games. Ph.D. thesis, Cornell University, Ithaca
Hokari T (2000) The nucleolus is not aggregate-monotonic on the domain of convex games. IJGT 29(1):133–137
Huijink S, Borm P, Kleppe J, Reijnierse JH (2015) Bankruptcy and the per capita nucleolus: the claim-and-right rules family. MSS 77:15–31
Kleppe J (2010) Modeling interactive behavior, and solution concepts. Ph.D. thesis, Tilburg University
Kohlberg E (1971) On the nucleolus of a characteristic function game. SIAP 20(1):62–66
Maschler M, Peleg B, Shapley LS (1971) The kernel and the bargaining set for convex games. IJGT 15(1):73–93
O'Neill B (1982) A problem of rights arbitration from the talmud. MSS 2(4):345–371
Peleg B (1986) On the reduced game property and its converse. IJGT 15(3):187–200
Schmeidler D (1969) The nucleolus of a characteristic function game. SIAP 17(6):1163–1170
Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games - vol. II. Princeton University Press, Princeton, pp 307–317

[5]Requiring continuity avoids core solutions defined as follows. Let ϕ be a solution that coincides with the Shapley value if the game is convex and with the prenucleolus otherwise. Then ϕ satisfies coalitional monotonicity in the class of convex games but is not continuous. To our knowledge there is not other continuous core solution that satisfies this requirement of monotonicity.

Shapley LS (1967) On balanced sets and cores. NRLQ 14(4):453–460
Shapley LS (1971) Cores of convex games. RAND, Santa Monica
Sobolev A (1975) The characterization of optimality principles in cooperative games by functional equations. In: Vorobiev N (ed) Mathematical methods in the social sciences. Academy of Science of the Lithuanian SSR, Vilnius, pp 95–151
Young HP (1985) Monotonic solutions of cooperative games. IJGT 14(2):65–72

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A Shapley Value for Games with Authorization Structure

José M. Gallardo, Nieves Jiménez, and Andrés Jiménez-Losada

Abstract A cooperative TU-game consists of a set of players and a characteristic function which determines the maximal gain or minimal cost that every subset of players can achieve when they decide to cooperate, regardless of the actions that the other players take. It is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to cooperate within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. To do this, several models have been introduced in literature. In this chapter we describe one of those models for games with restricted cooperation. This model is more general than others in several ways. For instance, it allows us to deal with non-hierarchical or non-transitive dependency relationships. In addition, it can be adapted to consider fuzzy dependency relationships, which arise in situations in which each player has a degree of freedom to cooperate within a coalition.

Keywords Authorization structure • Choquet integral • Fuzzy coalition • Restricted cooperation • Shapley value

Contents

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1 Introduction

In a general way, game theory studies cooperation and conflict models, using mathematical methods. This paper is about cooperative game theory. A cooperative game over a finite set of players is defined as a function establishing the worth of each coalition. Given a cooperative game, the main problem that arises is how to assign to each player a payoff in a reasonable way. In this setting, it is often assumed that all players are socially identical.

In real life, however, political or economic circumstances may impose certain restraints on coalition formation. This idea has led several authors to develop models of games in which relationships among players must be taken into account. Depending on the nature of such relationships, different structures in the set of players have been considered. Myerson studied games in which communication between players is restricted (Myerson 1977, pp. 225–229). He considered graphs to model those restraints. Subsequently, different kinds of limitations on cooperation among players have been studied, and various structures have been used for that, like convex geometries (Bilbao 1998, pp. 368–376), matroids (Bilbao et al. 2001, pp. 333–348), antimatroids (Algaba et al. 2004, pp. 1–15) or augmenting systems (Bilbao and Ordóñez 2009, pp. 1008–1014).

A particularly interesting case of limited cooperation arises when we consider a hierarchical structure on the set of players. This is the origin of the so called games with permission structure. A permission structure consists of a set of players, a cooperative game and a mapping that assigns to every player a subset of direct subordinates. In this case, the power of a player over a subordinate can be of different kinds. In the conjunctive approach (Gilles et al. 1992, pp. 277–293) it is assumed that every player needs the permission of all his superiors, whereas in the disjunctive approach (van den Brink 1997, pp. 27–43) the permission of any of those superiors will suffice. In each case they consider a new characteristic function, which collects the information given by both the original characteristic function and the permission structure. This allows them to define a value for games on conjunctive (or respectively disjunctive) permission structures. They provide an intuitive characterization for each value. Subsequently, those approaches where generalized by considering the so called restrictions (Derks and Peters 1993, pp. 351–366).

The first part of this chapter is devoted to the description of a model for games on structures that restrict the participation of players within coalitions. In Sect. 2 we recall some basic definitions, as well as certain properties of the Shapley value. We also describe some structures that model situations in which cooperation among players is restricted. Moreover, we recall some concepts regarding fuzzy sets, which will be needed in the second part of the chapter. In Sect. 3 we introduce authorization structures, which model situations in which some players need permission from other players in order to cooperate within a coalition. We also introduce games with authorization structure. For each game with authorization structure we will define a new characteristic function, the restricted game, that will allow us to define, in Sect. 4, a Shapley value for this kind of games. In this section we also provide a characterization of the value obtained. We give an example as well.

In all of the models mentioned before the dependency relationships are considered to be complete, in the sense that either a player is allowed to fully cooperate within a coalition or they cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within a coalition. In order to model these situations, in Sect. 5 we introduce games with fuzzy authorization structure (Gallardo et al. 2015, pp. 115–125). In Sect. 6 we define a Shapley value for these games. A characterization of this value is provided. An example is described as well.

2 Preliminaries

2.1 Cooperative TU-Games

A *transferable utility cooperative game* or TU-game is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. The elements of $N = \{1, \ldots, n\}$ are called players, and the subsets of N coalitions. Given a coalition E, $v(E)$ is the worth of E, and is interpreted as the maximal gain or minimal cost that the players in this coalition can achieve by themselves against the best offensive threat by the complementary coalition. Frequently, a TU-game (N, v) is identified with the function v. A game v is monotone if for every $F \subseteq E \subseteq N$, it holds that $v(F) \leq v(E)$. The family of games with set of players N is denoted by $\mathcal{G}^N$. This set is a $(2^n - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N,\ F \neq \emptyset\}$ where for a coalition $F \subseteq N,\ F \neq \emptyset$, the unanimity game u_F is a monotone game given by

$$u_F(E) = \begin{cases} 1 & \text{if } F \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

Every game $v \in \mathcal{G}^N$ can be written as a linear combination of them,

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E)\, u_E$$

where $\triangle_v(E)$ is the dividend of the coalition E in the game v.

A solution or value on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns to each game a vector $(\psi_1(v), \ldots, \psi_n(v))$ where the real number $\psi_i(v)$ is the payoff of the player i in the game (N, v).

Many values have been defined for different families of games in literature. The best known of them is the Shapley value (Shapley 1953, pp. 307–317). The *Shapley value* $\phi(v) \in \mathbb{R}^N$ of a game $v \in \mathcal{G}^N$ is a weighted average of the marginal contributions of each player to the coalitions and formally it is defined by

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[v(E) - v(E \setminus \{i\})\right], \quad \text{for all } i \in N,$$

where

$$p_E = \frac{(n - |E|)!\,(|E| - 1)!}{n!}$$

and $|E|$ denotes the cardinality of E.

Some desirable properties for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ are the following:

Efficiency: $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.
Additivity: $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \mathcal{G}^N$.
Null player property: A player $i \in N$ is a null player in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. If $i \in N$ is null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$.
Necessary player property: A player i is a necessary player in $v \in \mathcal{G}^N$ if $v(E) = 0$ for $E \subseteq N \setminus \{i\}$. If i is a necessary player in a monotone game $v \in \mathcal{G}^N$, then $\psi_i(v) \geq \psi_j(v)$ for all $j \in N$.

The Shapley value satisfies all these properties (van den Brink 1994).

2.2 Permission Structures

A *permission structure* on N is represented by a mapping $S : N \to 2^N$ where the players in $S(i)$ are the successors of player $i \in N$, that is, $S(i)$ contains all agents that are dominated directly by agent i. Let $\hat{S}$ denote the transitive closure of S, i.e. $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^{q}$ such that $i_0 = i,\ i_q = j$ and $i_p \in S(i_{p-1})$ for $1 \leq p \leq q$. Thus, the players in $\hat{S}(i) \setminus S(i)$ are those dominated indirectly by i. The set of predecessors of player i is $P_S(i) = \{j \in N : i \in S(j)\}$ and the set of superiors of i in S is denoted by $\hat{P}_S(i) = \left\{j \in N : i \in \hat{S}(j)\right\}$. The collection of all permission structures on N is denoted by $\mathcal{S}^N$. The family of permission structures on N can be identified with the set of directed graphs (digraphs) on N. The vertex set is N and the pair (i,j) is a link if $j \in S(i)$.

If $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$, the pair (v, S) is said to be a *game with permission structure on* N. Several assumptions can be made about how a permission structure restricts the formation of coalitions in a TU-game. In the conjunctive approach (Gilles et al. 1992, pp. 277–293) it is assumed that every player needs permission from all his superiors in order to be allowed to cooperate. So, the conjunctive sovereign part of a coalition E contains those players in E whose superiors are in E, that is

$$A_c^S(E) = \left\{ i \in E : \hat{P}_S(i) \subseteq E \right\}.$$

The conjunctive approach has been generalized (Gallardo et al. 2014, pp. 510–519) to deal with a wider range of dependency relationships.

In the disjunctive approach (van den Brink 1997, pp. 27–43) it is assumed that each player only needs permission from one of his predecessors (if he has any). In this case, a coalition is autonomous if for any player in the coalition either he does not have any predecessors or at least one of his predecessors is in the coalition. The disjunctive sovereign part of a coalition E, denoted by $A_d^S(E)$, is the largest autonomous subset of E.

In order to find reasonable payoff vectors for games with permission structure, Gilles, Owen and van den Brink proposed to modify the characteristic function $v \in \mathcal{G}^N$ taking account of the limited possibilities of cooperation determined by the permission structure $S \in \mathcal{S}^N$. The conjunctive and disjunctive restricted games are defined, respectively, as $v_c^S(E) = v\left(A_c^S(E)\right)$ and $v_d^S(E) = v\left(A_d^S(E)\right)$ for every coalition $E \subseteq N$. A *value for games with permission structure on* N is a map $\psi : \mathcal{G}^N \times \mathcal{S}^N \to \mathbb{R}^N$. The *conjunctive permission value* and the *disjunctive permission value* are defined as $\phi^c(v, S) = \phi\left(v_c^S\right)$ and $\phi^d(v, S) = \phi\left(v_d^S\right)$ respectively.

Permission structures were generalized by introducing the so called restrictions (Derks and Peters 1993, pp. 351–366). A *restriction* on N is a mapping $\rho : 2^N \to 2^N$ satisfying:

1. $\rho(E) \subseteq E$ for any $E \subseteq N$,
2. If $E \subset F$ then $\rho(E) \subseteq \rho(F)$,
3. $\rho(\rho(E)) = \rho(E)$.

They interpreted the coalitions in $Im(\rho)$ as the only formable coalitions. We could also interpret $\rho(E)$ as the set of players that are allowed to play within coalition E. For any restriction ρ, they introduced and characterized the so called restricted Shapley value $\psi(v) = \phi(v \circ \rho)$.

2.3 Fuzzy Sets and the Choquet Integral

Fuzzy sets (Zadeh 1965, pp. 338–353) were introduced as a tool to mathematically deal with uncertainties. A *fuzzy subset* of N is a mapping $e : N \longrightarrow [0, 1]$ where e assigns to $i \in N$ a degree of membership. A fuzzy subset of N is identified with a vector in $[0, 1]^N$. Given $e \in [0, 1]^N$ the *support* of e is the set $supp(e) = \{i \in N : e_i > 0\}$ and the *image* of e is the set $im(e) = \{e_i : i \in N\}$. If $t \in [0, 1]$ the *t-level set* of e is $[e]_t = \{i \in N : e_i \geqslant t\}$. Given $e,f \in [0, 1]^N$ standard union and intersection are defined, respectively, by $(e \cap f)_i = \min\{e_i,f_i\}$,

$(e \cup f)_i = \max\{e_i, f_i\}$ for all $i = 1, \ldots, n$. The fuzzy sets $e, f \in [0,1]^N$ are called *comonotone* if $(e_i - e_j)(f_i - f_j) \geq 0$ for all $i, j \in N$.

Regarding cooperative game theory, the concept of fuzzy set led to consider fuzzy coalitions (Aubin 1981, pp. 1–13). A *fuzzy coalition* in N is a fuzzy subset e of N where, for all $i \in N$, the number $e_i \in [0,1]$ is regarded as the degree of participation of player i in e. Every coalition $E \subseteq N$ can be identified with the fuzzy coalition $\mathbf{1}^E \in [0,1]^N$ defined by $\mathbf{1}^E_i = 1$ if $i \in E$ and $\mathbf{1}^E_i = 0$ otherwise.

Different Shapley values for games with fuzzy coalitions have been studied (Butnariu 1980, pp. 63–72; Tsurumie et al. 2001, pp. 596–618).

The Choquet integral (Choquet 1953, pp. 131–295) was originally defined for capacities. Later on it was studied for all set functions (Schmeidler 1986, pp. 255–261). Given $v : 2^N \to \mathbb{R}$ and $e \in [0,1]^N$, the Choquet *integral* of e with respect to v is defined as

$$\int e\,dv = \sum_{p=1}^{q} \left(s_p - s_{p-1}\right) v\left([e]_{s_p}\right), \tag{1}$$

where $im\,(e) \cup \{0\} = \{s_p\}_{p=0}^{q}$ and $0 = s_0 < s_1 < \ldots < s_q$.

It will be useful, when dealing with several fuzzy coalitions, to rewrite the expression above using a superset of $im(e)$, that is,

$$\int e\,dv = \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left([e]_{t_l}\right), \tag{2}$$

where $im(e) \subseteq \{t_l\}_{l=0}^{m}$ and $0 = t_0 < t_1 < \ldots < t_m$.

The following properties of the Choquet integral are known:

(C1) $\int \mathbf{1}^E\,dv = v\,(E)$, for all $E \subseteq N$.
(C2) $\int te\,dv = t\int e\,dv$, for all $t \in [0,1]$.
(C3) $\int e\,dv \leq \int f\,dv$, whenever $e \leq f$ and v is monotone.
(C4) $\int e\,d(cv) = c\int e\,dv$, for $c \in \mathbb{R}$.
(C5) $\int e\,d\,(v_1 + v_2) = \int e\,dv_1 + \int e\,dv_2$.
(C6) $\int (e + f)\,dv = \int e\,dv + \int f\,dv$, when $e + f \leq \mathbf{1}^N$ and e, f are comonotone.

3 Games with Authorization Structures

In order to model situations in which players may need permission from other players to actively participate within a coalition, we introduce the concept of authorization operator.

Definition 15.1 A function $A : 2^N \to 2^N$ is an authorization operator on N if it satisfies the following requirements:

1. $A(E) \subseteq E$ for any $E \subseteq N$,
2. If $E \subset F$ then $A(E) \subseteq A(F)$.

The pair (N, A) will be called authorization structure. The set of all authorization operators on N will be denoted by $\mathscr{A}^N$.

If $A \in \mathscr{A}^N$ and $E \subseteq N$ we will interpret $A(E)$ as the set of elements in E to which the coalition E can grant authorization to play.

The concept of authorization structure extends those of conjunctive or disjunctive structure. The following example can be illustrative.

Example On the Table 1 we have defined three different authorization operators A_1, A_2 and A_3 on $\{1, 2, 3, 4\}$.

Let us consider the digraph in Fig. 1.

It is plain to see that, for any $E \subseteq \{1, 2, 3, 4\}$, $A_1(E)$ and $A_2(E)$, are respectively, the conjunctive and disjunctive sovereign part of E in the permission structure represented by the digraph.

Finally, the structure determined by A_3 is also hierarchical (and induced by the digraph above), but neither conjunctive or disjunctive. In this structure, a coalition is autonomous (note that A_3 is, as well as A_1 and A_2, an interior operator) if for any element in the coalition, the majority of his predecessors are in the coalition too.

Table 1 Different authorization operators

E	$A_1(E)$	$A_2(E)$	$A_3(E)$
E such that $4 \notin E$	E	E	E
$\{4\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1, 4\}$	$\{1\}$	$\{1, 4\}$	$\{1\}$
$\{2, 4\}$	$\{2\}$	$\{2, 4\}$	$\{2\}$
$\{3, 4\}$	$\{3\}$	$\{3, 4\}$	$\{3\}$
$\{1, 2, 4\}$	$\{1, 2\}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$
$\{1, 3, 4\}$	$\{1, 3\}$	$\{1, 3, 4\}$	$\{1, 3, 4\}$
$\{2, 3, 4\}$	$\{2, 3\}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$
$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4\}$

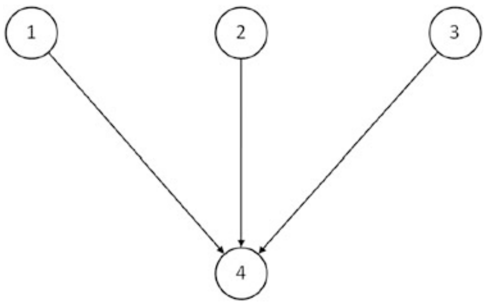

Fig. 1 Simple permission tree. Own representation

In the following we study cooperative games with an authorization structure on the set of players.

Definition 15.2 A game with authorization structure is a triple (N, v, A) where (N, v) is a game and (N, A) is an authorization structure.

If the set of players is fixed, the triple (N, v, A) is identified with the pair (v, A).

Definition 15.3 A value for games with authorization structure is a function that assigns to every game with authorization structure a payoff vector.

We introduce for each game with authorization structure a new game involving the relationships among players.

Definition 15.4 Let $v \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$. The restriction of v on A is the game $v^A \in \mathscr{G}^N$ given by

$$v^A(E) = v(A(E)) \quad \text{for all } E \subseteq N.$$

Note that $v^A(E)$ is the profit that coalition E can actually make if we consider both the original characteristic function and the authorization structure on the set of players.

4 The Shapley Authorization Value

In this section we use the Shapley value and the restricted game defined above to introduce a value for games with authorization structure.

Definition 15.5 The Shapley authorization value is the value Φ that assigns to each game with authorization structure the payoff vector given by

$$\Phi(v, A) = \phi\left(v^A\right) \quad \text{for all } (v, A) \in \mathscr{G}^N \times \mathscr{A}^N.$$

Let Ψ be a value for games with authorization structure. In order to characterize the Shapley authorization value we consider the following properties.

Efficiency *For every $v \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$ it holds that*

$$\sum_{i \in N} \Psi_i(v, A) = v(N).$$

Additivity *For every $v, w \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$ it holds that*

$$\Psi(v + w, A) = \Psi(v, A) + \Psi(w, A).$$

As we aforementioned in Sect. 2, the Shapley value satisfies the Null-player property. If we want to determine an "adequate" value for games with authorization structure, it would not be a good idea to look for a value satisfying that property. That is due to the fact that we have to take the structure into consideration, since a player could make profit not only by playing, but also by giving permission to play. Suppose that we have $v \in \mathcal{G}^N$, an authorization structure on N and $i \in N$ a null player in v. There might be other players depending on the authorization from i, in which case player i would reasonably expect to be rewarded. But if all players depending on i are also null players in v, i should not expect anything but a zero-payoff. That is the Irrelevant-player property, that we state next.

Given $A \in \mathscr{A}^N$ and $i, j \in N$, player j *depends partially* on i according to A if there exists $E \subseteq N$ such that $j \in A(E) \setminus A(E \setminus \{i\})$. Given $v \in \mathcal{G}^N$, $A \in \mathscr{A}^N$ and $i \in N$, player i is an *irrelevant player in* (v, A) if for every $j \in N$ such that j depends partially on i according to A it holds that j is a null player in v.

Irrelevant-Player Property *For every $v \in \mathcal{G}^N$, $A \in \mathscr{A}^N$ and $i \in N$ such that i is an irrelevant player in (v, A) it holds that*

$$\Psi_i(v, A) = 0.$$

Note that if A is the trivial authorization structure (that is, $A(E) = E$ for every $E \subseteq N$) then the irrelevant players in (v, A) are just the null players in v. From this point of view, the Irrelevant-player property is a generalization of the Null-player property.

Similarly to the necessary player property of the Shapley value, our value should satisfy this "natural" property.

Necessary-Player Property *For every monotone game $v \in \mathcal{G}^N$, $A \in \mathscr{A}^N$ and $i \in N$ such that i is a necessary player in v it holds that, for all $j \in N$,*

$$\Psi_i(v, A) \geqslant \Psi_j(v, A).$$

Fairness *For every $v \in \mathcal{G}^N$, $A \in \mathscr{A}^N$, $T \in 2^N \setminus \{\emptyset\}$ and $i \in T$ it holds that*

$$\Psi_j\left(v, A^{T,i}\right) - \Psi_j(v, A) = \Psi_i\left(v, A^{T,i}\right) - \Psi_i(v, A) \quad \text{for all } j \in T,$$

where $A^{T,i} \in \mathscr{A}^N$ is given, for all $E \subseteq N$, by

$$A^{T,i}(E) = \begin{cases} A(E) & \text{if } T \not\subseteq E, \\ A(E) \cup \{i\} & \text{if } T \subseteq E. \end{cases}$$

Notice that if $i \in A(T)$ then $A^{T,i} = A$. Therefore, the expression above is non trivial only if $i \notin A(T)$.

Although at first sight the fairness property might seem a little contrived, it is actually quite natural and intuitive. Suppose that we have a game and an authorization structure on N, $T \subseteq N$ and $i \in T$ such that in case coalition T were formed i could not play. Imagine now that somehow coalition T acquires the power to authorize i to play. It seems reasonable to think that all the players in T will benefit equally from that fact.

In the following theorem we will show that the five properties seen before uniquely determine the Shapley authorization value.

Theorem 15.1 *A value for games with authorization structure is equal to the Shapley authorization value if and only if it satisfies the properties of additivity, efficiency, irrelevant player, necessary player and fairness.*

Proof Firstly, we are going to show that the Shapley authorization value satisfies the five properties.

Efficiency. Let $v \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$. It holds that

$$\sum_{i \in N} \Phi_i(v, A) = \sum_{i \in N} \phi_i\left(v^A\right) = v^A(N) = v(A(N)) = v(N).$$

Additivity. Let $v, w \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$. We can derive that

$$\begin{aligned}\Phi(v + w, A) &= \phi\left((v + w)^A\right) = \phi(v^A + w^A) \\ &= \phi(v^A) + \phi(w^A) = \Phi(v, A) + \Phi(w, A).\end{aligned}$$

Irrelevant-player property. Let $v \in \mathscr{G}^N$, $A \in \mathscr{A}^N$ and $i \in N$ an irrelevant player in (v, A). We must prove that $\Phi_i(v, A) = 0$. Taking into consideration that $\Phi_i(v, A) = \phi_i\left(v^A\right)$ and that ϕ satisfies the Null-player property, it will be enough to show that i is a null player in v^A. For that, take $E \subseteq N$. We have to show that $v^A(E) = v^A\left(E \setminus \{i\}\right)$. First, notice that

$$A(E) \setminus A(E \setminus \{i\}) \subseteq \{j \in N : j \text{ depends partially on } i \text{ according to } A\}.$$

Since i is an irrelevant player in (v, A), every player that depends partially on i according to A is a null player in v. So we can derive that

$$A(E) \setminus A(E \setminus \{i\}) \subseteq \{j \in N : j \text{ is a null player in } v\}$$

and, hence $v(A(E)) = v\left(A\left(E \setminus \{i\}\right)\right)$ or, equivalently $v^A(E) = v^A\left(E \setminus \{i\}\right)$.

Necessary-player property. Let $v \in \mathscr{G}^N$ be a monotone game, $A \in \mathscr{A}^N$ and $i \in N$ such that i is a necessary player in v. We must prove that $\Phi_i(v, A) \geqslant \Phi_j(v, A)$ for all $j \in N$. Keeping in mind that the Shapley value satisfies the Necessary-player property and the fact that $\Phi(v, A) = \phi\left(v^A\right)$, it will be enough to prove that $v^A \in \mathscr{G}^N$ is monotone and i is a necessary player in v^A. Firstly, the monotonicity of v^A derives directly from the monotonicity of v and the definition of authorization operator. It only remains to prove that i is a necessary player in v^A. Since v^A is monotone it is enough to prove that $v^A\left(N \setminus \{i\}\right) = 0$. Since $i \notin A\left(N \setminus \{i\}\right)$ and i is a necessary player in v we obtain that $v\left(A\left(N \setminus \{i\}\right)\right) = 0$, or, equivalently, $v^A\left(N \setminus \{i\}\right) = 0$.

Fairness. Let $v \in \mathscr{G}^N$, $A \in \mathscr{A}^N$, $T \in 2^N \setminus \{\emptyset\}$ and $i, j \in T$. We must prove that

$$\Phi_j\left(v, A^{T,i}\right) - \Phi_j(v, A) = \Phi_i\left(v, A^{T,i}\right) - \Phi_i(v, A). \tag{3}$$

Let us focus on the left-hand side.

$$\Phi_j\left(v, A^{T,i}\right) - \Phi_j(v, A) = \phi_j\left(v^{A^{T,i}}\right) - \phi_j\left(v^A\right) =$$

$$= \sum_{\{E \subseteq N: j \in E\}} p_E\left[v\left(A^{T,i}(E)\right) - v\left(A^{T,i}(E \setminus \{j\})\right) - v(A(E)) + v(A(E \setminus \{j\}))\right].$$

As $j \in T$, it is clear, from the definition of $A^{T,i}$ that $A^{T,i}(E \setminus \{j\}) = A(E \setminus \{j\})$ for all $E \subseteq N$. Notice also that if we take $E \subseteq N$ such that $T \nsubseteq E$ it holds that $A^{T,i}(E) = A(E)$. So the sum above is equal to

$$\sum_{\{E \subseteq N: j \in E\}} p_E\left[v\left(A^{T,i}(E)\right) - v(A(E))\right] = \sum_{\{E \subseteq N: T \subseteq E\}} p_E\left[v\left(A^{T,i}(E)\right) - v(A(E))\right],$$

that does not depend on the player $j \in T$ chosen, thus obtaining (3).

We have already seen that the Shapley authorization value satisfies the five properties. Now we will see that such properties uniquely determine this value.

Let Ψ be a value for games with authorization structure that satisfies the properties of additivity, efficiency, irrelevant player, necessary player and fairness. We must prove that $\Psi = \Phi$.

Let $n \in \mathbb{N}$. Our first goal will be to show that $\Psi(cu_E, A) = \Phi(cu_E, A)$ for all $c > 0$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathscr{A}^N$. For this, for every $A \in \mathscr{A}^N$ we denote

$$m(A) = \sum_{F \subseteq N} |A(F)|$$

and we prove $\Psi(cu_E, A) = \Phi(cu_E, A)$ by induction on $m(A)$. Let E be a nonempty coalition and $c > 0$.

BASE CASE. Let $A \in \mathscr{A}^N$ be such that $m(A) = 0$. It is clear that all the players in N are irrelevant players in (cu_E, A). Since Φ and Ψ satisfy the Irrelevant-player property, it holds that $\Phi(cu_E, A) = 0 = \Psi(cu_E, A)$.

INDUCTIVE STEP. Let $A \in \mathscr{A}^N$. We consider the following set:

$$H = \{i \in N : i \text{ is an irrelevant player in } (cu_E, A)\}.$$

Since Φ and Ψ satisfy the Irrelevant-player property it holds that

$$\Phi_i(cu_E, A) = \Psi_i(cu_E, A) = 0 \quad \text{for all } i \in H. \tag{4}$$

From the Necessary-player property we can derive that there exist $b, b' \in \mathbb{R}$ such that

$$\Phi_j(cu_E, A) = b \quad \text{and} \quad \Psi_j(cu_E, A) = b' \quad \text{for all } j \in E. \tag{5}$$

Now suppose that $i \in N \setminus H$. There must exist $j \in E$ such that j depends partially on i according to A. This means that there exists $F \subseteq N$ such that $j \in A(F) \setminus A(F \setminus \{i\})$. Take T minimal such that $T \subseteq F$ and $j \in A(T)$. It is clear that $i \in T$. We define the function $\tilde{A} : 2^N \to 2^N$ given by

$$\tilde{A}(S) = \begin{cases} A(S) & \text{if } S \neq T, \\ A(T) \setminus \{j\} & \text{if } S = T. \end{cases}$$

It is straightforward to check that $\tilde{A} \in \mathscr{A}^N$ and $\tilde{A}^{T,j} = A$. By using the fairness property we obtain

$$\Phi_i(cu_E, A) - \Phi_i(cu_E, \tilde{A}) = \Phi_j(cu_E, A) - \Phi_j(cu_E, \tilde{A}),$$
$$\Psi_i(cu_E, A) - \Psi_i(cu_E, \tilde{A}) = \Psi_j(cu_E, A) - \Psi_j(cu_E, \tilde{A}).$$

Since $j \in E$ we know from (5) that $\Phi_j(cu_E, A) = b$ and $\Psi_j(cu_E, A) = b'$. If we substitute those values into the equalities above we have

$$\Phi_i(cu_E, A) = b + \Phi_i(cu_E, \tilde{A}) - \Phi_j(cu_E, \tilde{A}),$$
$$\Psi_i(cu_E, A) = b' + \Psi_i(cu_E, \tilde{A}) - \Psi_j(cu_E, \tilde{A}).$$

As $m(\tilde{A}) = m(A) - 1$ it follows by induction hypothesis that $\Psi(cu_E, \tilde{A}) = \Phi(cu_E, \tilde{A})$ and hence

$$\Phi_i(cu_E, A) - \Psi_i(cu_E, A) = b - b'.$$

So we have proved that

$$\Phi_i(cu_E, A) - \Psi_i(cu_E, A) = b - b' \quad \text{for all } i \in N \setminus H. \tag{6}$$

Now, on the one hand, from (4) and (6), we can obtain

$$\sum_{i \in N} \Phi_i(cu_E, A) - \sum_{i \in N} \Psi_i(cu_E, A) = (b - b')|N \setminus H|,$$

and, on the other hand, as Φ and Ψ are efficient we know that

$$\sum_{i \in N} \Phi_i(cu_E, A) = \sum_{i \in N} \Psi_i(cu_E, A).$$

Therefore, it follows that $(b - b')|N \setminus H| = 0$. So either $b = b'$ or $N = H$. In either case, taking into consideration (4) and (6), we conclude that $\Psi(cu_E, A) = \Phi(cu_E, A)$.

So, fixed $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathscr{A}^N$, $\Psi(cu_E, A) = \Phi(cu_E, A)$ is satisfied for all $c > 0$. Moreover, by additivity and the Irrelevant-player property it holds that

$$\Psi(-cu_E, A) = -\Psi(cu_E, A) \quad \text{and} \quad \Phi(-cu_E, A) = -\Phi(cu_E, A).$$

We conclude that $\Psi(-cu_E, A) = \Phi(-cu_E, A)$.

So we already know that $\Psi(cu_E, A) = \Phi(cu_E, A)$ for all $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathscr{A}^N$.

Finally, take $v \in \mathscr{G}^N$ and $A \in \mathscr{A}^N$. It holds that

$$\Psi(v, A) = \Psi\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E) u_E, A\right) = \sum_{\{E \subseteq N : E \neq \emptyset\}} \Psi\left(\Delta_v(E) u_E, A\right)$$

$$= \sum_{\{E \subseteq N : E \neq \emptyset\}} \Phi\left(\Delta_v(E) u_E, A\right) = \Phi\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E) u_E, A\right) = \Phi(v, A).$$

Example Imagine the following situation. A consumer electronics company wants to make a new product. To do this, the company needs several components from various suppliers. We will focus on three of those suppliers. For $i = 1, 2, 3$ supplier i produces component i. The company has signed an agreement with the three suppliers that establishes the following:

- The company will pay i dollars to supplier i for every unit of component i delivered before the deadline.
- The company will pay a total of $2(i + j)$ dollars to suppliers i and j for every pair made up of a unit of component i and a unit of component j delivered before the deadline.
- The company will pay a total of 20 dollars to the three suppliers for every set made up of a unit of each component delivered before the deadline.

Each supplier has calculated that it would be able to produce one million units of the corresponding component before the deadline. This situation can be modeled with a cooperative game $(\{1, 2, 3\}, v)$, where, for every $E \subseteq \{1, 2, 3\}$, $v(E)$ is the revenue (in millions) obtained by coalition E.

$$v(\{1\}) = 1, \qquad v(\{2\}) = 2, \qquad v(\{3\}) = 3,$$
$$v(\{1,2\}) = 6, \qquad v(\{1,3\}) = 8, \qquad v(\{2,3\}) = 10, \quad v(\{1,2,3\}) = 20.$$

Imagine now the following. Factory 2 sues factory 1 for violation of patent rights and factory 3 sues factory 2 for the same reason. So, after a few months, the scenario is this: factory 1 cannot use or sell the component they make without

Table 2 Representation of operator A

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$A(E)$	$\emptyset$	$\emptyset$	$\{3\}$	$\{1\}$	$\{3\}$	$\{2,3\}$	$\{1,2,3\}$

the authorization from factory 2, and the latter cannot use or sell their component without the authorization from factory 3. Let us consider an operator A that assigns to each coalition $E \subseteq \{1,2,3\}$ the set of players in E that do not need the authorization from a player in $\{1,2,3\} \setminus E$. We can represent A with Table 2.

Let us calculate the restricted game v^A.

$$\begin{aligned}
v^A(\{1\}) &= v(\emptyset) = 0,\\
v^A(\{2\}) &= v(\emptyset) = 0,\\
v^A(\{3\}) &= v(\{3\}) = 3,\\
v^A(\{1,2\}) &= v(\{1\}) = 1,\\
v^A(\{1,3\}) &= v(\{3\}) = 3,\\
v^A(\{2,3\}) &= v(\{2,3\}) = 10,\\
v^A(\{1,2,3\}) &= v(\{1,2,3\}) = 20.
\end{aligned}$$

Now we use the definition of the Shapley authorization value

$$\Phi(v,A) = \phi(v^A) = (3.5, 7, 9.5).$$

We can obtain an expression of the Shapley authorization value that does not involve the restricted game, but the game and the authorization operator separately.

Lemma 15.1 *Let $v \in \mathcal{G}^N$ and $A \in \mathcal{A}^N$. Then, if $\Phi(v,A)$ is considered as a column matrix, it holds that*

$$\Phi(v,A) = Z_A \cdot \Delta_v$$

where Z_A is the matrix in $\mathcal{M}_{n,2^n-1}(\mathbb{R})$ defined by $(Z_A)_{i,E} = \Phi_i(u_E, A)$ for every $i \in N$ and $E \in 2^N \setminus \{\emptyset\}$ and Δ_v is the column matrix given by the dividends of v.

Proof Making use of the linearity of the Shapley authorization value we can write

$$\begin{aligned}
\Phi_i(v,A) = \Phi_i\left(\sum_{E\in 2^N\setminus\{\emptyset\}} \Delta_v(E)u_E, A\right) &= \sum_{E\in 2^N\setminus\{\emptyset\}} \Delta_v(E)\Phi_i(u_E, A)\\
&= \sum_{E\in 2^N\setminus\{\emptyset\}} (Z_A)_{i,E}\,\Delta_v(E).
\end{aligned}$$

Example Let us use the expression given in the preceding result to calculate $\Phi(v,A)$ in the previous example. Firstly we calculate u_E^A for each nonempty $E \subseteq \{1,2,3\}$ in Table 3.

Table 3 Calculation of u_E^A for each nonempty $E \subseteq \{1,2,3\}$

F	$u^A_{\{1\}}(F)$	$u^A_{\{2\}}(F)$	$u^A_{\{3\}}(F)$	$u^A_{\{1,2\}}(F)$	$u^A_{\{1,3\}}(F)$	$u^A_{\{2,3\}}(F)$	$u^A_{\{1,2,3\}}(F)$
{1}	0	0	0	0	0	0	0
{2}	0	0	0	0	0	0	0
{3}	0	0	1	0	0	0	0
{1,2}	1	0	0	0	0	0	0
{1,3}	0	0	1	0	0	0	0
{2,3}	0	1	1	0	0	1	0
{1,2,3}	1	1	1	1	1	1	1

We calculate $\Phi\,(u_E, A)$ for each nonempty $E \subseteq N$

$$
\begin{aligned}
\Phi\left(u_{\{1\}}, A\right) &= \phi\left(u^A_{\{1\}}\right) &&= \left(\frac{1}{2}, \frac{1}{2}, 0\right)\\
\Phi\left(u_{\{2\}}, A\right) &= \phi\left(u^A_{\{2\}}\right) &&= \left(0, \frac{1}{2}, \frac{1}{2}\right)\\
\Phi\left(u_{\{3\}}, A\right) &= \phi\left(u^A_{\{3\}}\right) &&= (0, 0, 1)\\
\Phi\left(u_{\{1,2\}}, A\right) &= \phi\left(u^A_{\{1,2\}}\right) &&= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\\
\Phi\left(u_{\{1,3\}}, A\right) &= \phi\left(u^A_{\{1,3\}}\right) &&= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\\
\Phi\left(u_{\{2,3\}}, A\right) &= \phi\left(u^A_{\{2,3\}}\right) &&= \left(0, \frac{1}{2}, \frac{1}{2}\right)\\
\Phi\left(u_{\{1,2,3\}}, A\right) &= \phi\left(u^A_{\{1,2,3\}}\right) &&= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}
$$

and we write Z_A

$$
Z_A = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3}\\
0 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3}
\end{pmatrix}.
$$

Now we calculate Δ_v

$$\Delta_v = \begin{pmatrix} \Delta_v(\{1\}) \\ \Delta_v(\{2\}) \\ \Delta_v(\{3\}) \\ \Delta_v(\{1,2\}) \\ \Delta_v(\{1,3\}) \\ \Delta_v(\{2,3\}) \\ \Delta_v(\{1,2,3\}) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix}.$$

Finally,

$$\Phi(v,A) = Z_A \cdot \Delta_v = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 7 \\ \frac{19}{2} \end{pmatrix}.$$

5 Games with Fuzzy Authorization Structure

In the previous sections we studied games in which there are dependency relationships among the players. These dependency relationships were considered to be complete, in the sense that, when a coalition is formed, a player in the coalition either can fully cooperate within the coalition or cannot cooperate at all. Nevertheless, in some situations it is possible to consider another option: that a player has a degree of freedom to cooperate within the coalition. In the rest of the chapter a model for these situations will be described.

Definition 15.6 A fuzzy authorization operator on N is a function $a : 2^N \to [0,1]^N$ that satisfies the following requirements:

1. $a(E) \leqslant \mathbf{1}^E$ for any $E \subseteq N$,
2. If $E \subseteq F$ then $a(E) \leqslant a(F)$.

The pair (N, a) will be called a fuzzy authorization structure. The set of fuzzy authorization operators on N will be denoted by $\mathscr{F}\mathscr{A}^N$.

Given $a \in \mathscr{F}\mathscr{A}^N$, we will denote

$$im(a) = \bigcup_{E \subseteq N} im(a(E)).$$

Suppose that a is a fuzzy authorization operator and v is a game on N. Then, given $E \subseteq N$ and $i \in N$, we will interpret $a_i(E)$ as the proportion of the whole operating capacity of player i that he is allowed to use within coalition E. Or, equivalently, $1 - a_i(E)$ is the fraction of the operating capacity of player i that is under control of coalition $N \setminus E$.

Definition 15.7 A game with fuzzy authorization structure is a triple (N, v, a) where (N, v) is a game and (N, a) is a fuzzy authorization structure.

If the set of players is fixed, the triple (N, v, a) is identified with the pair (v, a).

Definition 15.8 A value for games with fuzzy authorization structure is a function that assigns to every game with fuzzy authorization structure a payoff vector.

The restricted game will be the tool used to amalgamate the information from the game and the information from the fuzzy authorization structure.

Definition 15.9 Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. The restriction of v on a is the game $v^a \in \mathscr{G}^N$ defined as

$$v^a(E) = \int a(E)\, dv \qquad \text{for all } E \subseteq N.$$

Suppose we want to determine the worth of a coalition E in a game with fuzzy authorization structure (v, a). The calculation of the corresponding Choquet integral implies summation over a set of indexed numbers. A priori, this set depends on the coalition E. Nevertheless, using (2) consider the same set for all the coalitions, as we see in the following remark.

Remark Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. Let $im(a) = \{t_l : l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$. It holds that

$$v^a(E) = \sum_{l=1}^{m} (t_l - t_{l-1})\, v\left([a(E)]_{t_l}\right) \qquad \text{for all } E \subseteq N.$$

Let $a \in \mathscr{F}\mathscr{A}^N$ and $t \in [0, 1]$. We define $a^t \in \mathscr{A}^N$ as follows

$$a^t(E) = [a(E)]_t = \{k \in E : a_k(E) \geqslant t\} \qquad \text{for all } E \subseteq N.$$

The following expression of the restricted game will turn out to be very useful in order to prove the results in this section.

Remark Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. Let $im(a) = \{t_l : l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$. Then it holds that

$$v^a = \sum_{l=1}^{m} (t_l - t_{l-1})\, v^{a^{t_l}}. \tag{7}$$

6 A Shapley Value for Games with Fuzzy Authorization Structure

We apply the Shapley value to the restricted game in order to define a value for games with fuzzy authorization structure.

Definition 15.10 The Shapley fuzzy authorization value on the set of players N is the allocation rule $\varphi^N : \mathscr{G}^N \times \mathscr{F}\mathscr{A}^N \to \mathbb{R}^N$ given by

$$\varphi^N(v, a) = \phi\,(v^a) \qquad \text{for all } v \in \mathscr{G}^N \text{ and } a \in \mathscr{F}\mathscr{A}^N.$$

We will write φ (rather than φ^N) and say just *Shapley fuzzy authorization value* as long as there is no possibility of confusion.

Lemma 15.2 *Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. Let $im(a) = \{t_l : l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$. It holds that*

$$\varphi(v, a) = \sum_{l=1}^{m} (t_l - t_{l-1})\, \Phi(v, a^{t_l}),$$

where Φ is the Shapley authorization value.

Proof Taking into account (7) and the linearity of the Shapley value, we have that

$$\varphi(v, a) = \phi\,(v^a) = \phi\left(\sum_{l=1}^{m}(t_l - t_{l-1})\, v^{a^{t_l}}\right) = \sum_{l=1}^{m}(t_l - t_{l-1})\ \phi\left(v^{a^{t_l}}\right)$$
$$= \sum_{l=1}^{m}(t_l - t_{l-1})\, \Phi(v, a^{t_l}).$$

We aim to prove that the Shapley fuzzy authorization value has good properties with respect to both the game and the fuzzy authorization structure. To do this, we will consider the properties described below. The five first properties considered will be the fuzzy versions of the analogous crisp properties seen for the Shapley authorization value.

If $a \in \mathscr{F}\mathscr{A}^N$ with $im(a(N)) \subseteq \{0, 1\}$, which means that when the grand coalition is formed each player can use either his full capacity or no capacity at all, the set $supp(a(N))$ can be seen as a carrier. In that case, we can consider the following efficiency property:

Efficiency *For every $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$ with $im(a(N)) \subseteq \{0, 1\}$ it holds that*

$$\sum_{i \in N} \psi_i(v, a) = v(supp(a(N))).$$

Additivity *For every* $v, w \in \mathscr{G}^N$ *and* $a \in \mathscr{F}\mathscr{A}^N$ *it holds that*

$$\psi(v + w, a) = \psi(v, a) + \psi(w, a).$$

Given $a \in \mathscr{F}\mathscr{A}^N$ and $i, j \in N$, player j depends partially on i according to a if there exists $E \subseteq N$ such that $a_j(E) > a_j(E \setminus \{i\})$. Given $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$, a player $i \in N$ is an irrelevant player in (v, a) if for every $j \in N$ such that j depends partially on i according to a it holds that j is a null player in v. Notice that a null player in v is not necessarily an irrelevant player in (v, a). The Null-player property is generalized now in the following way:

Irrelevant Player *For every* $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$ *and* $i \in N$ *such that* i *is an irrelevant player in* (v, a) *it holds that*

$$\psi_i(v, a) = 0.$$

Necessary-Player Property *For every monotone game* $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$ *and* $i \in N$ *such that* i *is a necessary player in* v *it holds that, for all* $j \in N$,

$$\psi_i(v, a) \geqslant \psi_j(v, a).$$

Let $a \in \mathscr{F}\mathscr{A}^N$, $\emptyset \neq T \subseteq N$ and $i \in T$. The fuzzy authorization operator a describes a situation in which some players may need the permission from other players in order to use a fraction of their operating capacity. In such situation, if coalition T is formed, player i will be allowed to use a proportion of his capacity equal to $a_i(T)$. Now suppose that somehow the players in T acquire the power to authorize player i to use a bigger proportion of his capacity, say $s \in (a_i(T), 1]$. The new situation would be described by the fuzzy authorization operator $a^{T,i,s}$ defined as

$$a^{T,i,s}(E) = \begin{cases} a(E) \cup (s \cdot \mathbf{1}^{\{i\}}) & \text{if } T \subseteq E, \\ a(E) & \text{otherwise.} \end{cases}$$

In this case, it would be reasonable to expect that all the players in T will benefit equally from the change. This is what the following property states:

Fairness *For every* $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ *and* $s \in [0, 1]$ *it holds that*

$$\psi_j\left(v, a^{T,i,s}\right) - \psi_j(v, a) = \psi_i\left(v, a^{T,i,s}\right) - \psi_i(v, a) \quad \text{for all } j \in T.$$

Notice that if $s \leqslant a_i(T)$ then $a^{T,i,s} = a$. Therefore, the expression above is non trivial only if $s \in (a_i(T), 1]$. This property is the fuzzy version of the fairness property of the Shapley authorization value. If a coalition acquires the power to increase the capacity of cooperation of one of the players within the coalition then all the players in the coalition will benefit equally.

Reduction For every $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$ and $t \in (0, 1)$ it holds that

$$\psi(v, a) = t\,\psi(v, a^{[0,t]}) + (1 - t)\,\psi(v, a^{[t,1]}),$$

where, for all $i \in N$ and $E \subseteq N$,

$$a_i^{[0,t]}(E) = \min\left(1, \frac{a_i(E)}{t}\right),$$
$$a_i^{[t,1]}(E) = \max\left(0, \frac{a_i(E) - t}{1 - t}\right).$$

Theorem 15.2 *An allocation rule $\psi : \mathscr{G}^N \times \mathscr{F}\mathscr{A}^N \to \mathbb{R}^N$ is equal to the Shapley fuzzy authorization value if and only if it satisfies the properties of efficiency, additivity, irrelevant player, necessary player, fairness and reduction.*

Proof Firstly it will be proved that φ satisfies the properties in the theorem.

Efficiency. Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$ with $im(a(N)) \subseteq \{0, 1\}$. It holds that

$$\sum_{i \in N} \varphi_i(v, a) = \sum_{i \in N} \phi_i(v^a) = v^a(N) = \int a(N)\, dv$$
$$= \int \mathbf{1}^{supp(a(N))}\, dv = v(supp(a(N))),$$

where we have used the efficiency of the Shapley value and property (C1).

Additivity. Let $v, w \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. From (C5) it follows that for every coalition E it holds that

$$(v + w)^a(E) = \int a(E)\, d(v + w) = \int a(E)\, dv + \int a(E)\, dw = v^a(E) + w^a(E).$$

Therefore, $(v + w)^a = v^a + w^a$. From this fact and the additivity of the Shapley value we get

$$\varphi(v + w, a) = \phi\left((v + w)^a\right) = \phi(v^a + w^a)$$
$$= \phi(v^a) + \phi(w^a) = \varphi(v, a) + \varphi(w, a).$$

Irrelevant player. Let $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$ and $i \in N$ an irrelevant player in (v, a). We must prove that $\phi_i(v, a) = 0$. Let $im(a) = \{t_l : l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$. Taking into consideration that Φ satisfies the Irrelevant-player property, it is clear from Lemma 15.2 that it is enough to prove that i is an irrelevant player in (v, a_{t_l}) for every $l = 1, \ldots, m$. So take $l \in \mathbb{N}$ with $l \leqslant m$.

Suppose that $j \in N$ depends partially on i in (N, a^{t_l}). This means that there exists $E \subseteq N$ such that $j \in a^{t_l}(E) \setminus a^{t_l}(E \setminus \{i\})$. Therefore, $a_j(E) \geqslant t_l > a_j(E \setminus \{i\})$. It follows that j depends partially on i in (N, a). Since i is an irrelevant player in (v, a), we conclude that j is a null player in v. So we have proved that i is an irrelevant player in (v, a^{t_l}).

Necessary player. Let $v \in \mathscr{G}$ be a monotone game, $a \in \mathscr{F}\mathscr{A}^N$ and $i \in N$ such that i is a necessary player in v. Let $im(a) = \{t_l : l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$. Since Φ satisfies the Necessary-player property, it holds that $\Phi_i(v, a^{t_l}) \geqslant \Phi_j(v, a^{t_l})$ for every $j \in N$ and $l = 0, \ldots, m$. Using Lemma 15.2 we obtain that $\varphi_i(v, a) \geqslant \varphi_j(v, a)$ for every $j \in N$.

Fairness. Let $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$, $T \in 2^N \setminus \{\emptyset\}$, $i \in T$ and $s \in [0, 1]$. Take $j \in T$. We must prove that

$$\varphi_j\left(v, a^{T,i,s}\right) - \varphi_j(v, a) = \varphi_i\left(v, a^{T,i,s}\right) - \varphi_i(v, a). \tag{8}$$

Using the definition of the Shapley value, we can write

$$\begin{aligned}\varphi_j\left(v, a^{T,i,s}\right) = \phi_j\left(v^{a^{T,i,s}}\right) &= \sum_{\{E \subseteq N : j \in E\}} p_E\left[v^{a^{T,i,s}}(E) - v^{a^{T,i,s}}(E \setminus \{j\})\right] \\ &= \sum_{\{E \subseteq N : j \in E\}} p_E\left[\int a^{T,i,s}(E)\, dv - \int a^{T,i,s}(E \setminus \{j\})\, dv\right]. \end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned}\varphi_j(v, a) = \phi_j(v^a) &= \sum_{\{E \subseteq N : j \in E\}} p_E\left[v^a(E) - v^a(E \setminus \{j\})\right] \\ &= \sum_{\{E \subseteq N : j \in E\}} p_E\left[\int a(E)\, dv - \int a(E \setminus \{j\})\, dv\right]. \end{aligned} \tag{10}$$

Taking into account that $a^{T,i,s}(F) = a(F)$ if $T \not\subseteq F$, we obtain, subtracting (10) from (9), that

$$\varphi_j\left(v, a^{T,i,s}\right) - \varphi_j(v, a) = \sum_{\{E \subseteq N : T \subseteq E\}} p_E\left[\int a^{T,i,s}(E)\, dv - \int a(E)\, dv\right].$$

Finally, (8) follows from the fact that the last expression does not depend on the player $j \in T$ chosen.

Reduction. Let $v \in \mathscr{G}^N$, $a \in \mathscr{F}\mathscr{A}^N$ and $t \in (0, 1)$. Notice that

$$a(E) = t a^{[0,t]}(E) + (1 - t) a^{[t,1]}(E) \quad \text{for every } E \subseteq N. \tag{11}$$

Now take $E \subseteq N$ and $i, j \in N$. Notice that if $a_i(E) \geqslant a_j(E)$ then $a_i^{[0,t]}(E) \geqslant a_j^{[0,t]}(E)$ and $a_i^{[t,1]}(E) \geqslant a_j^{[t,1]}(E)$. This leads to the fact that $a^{[0,t]}(E)$ and $a^{[t,1]}(E)$ are co-monotone fuzzy set for every $E \subseteq N$. Therefore $ta^{[0,t]}(E)$ and $(1-t)a^{[t,1]}(E)$ are co-monotone too. Using (C6) we have that

$$\begin{aligned}\int \left(ta^{[0,t]}(E) + (1-t)a^{[t,1]}(E)\right) dv &= \int ta^{[0,t]}(E)dv + \int (1-t)a^{[t,1]}(E)dv \\ &= t\int a^{[0,t]}(E)dv + (1-t)\int a^{[t,1]}(E)dv \\ &= tv^{a^{[0,t]}}(E) + (1-t)v^{a^{[t,1]}}(E) \qquad (12)\end{aligned}$$

for every $E \subseteq N$. From (11) and (12) we obtain that

$$v^a = tv^{a^{[0,t]}} + (1-t)v^{a^{[t,1]}}.$$

We have that

$$\begin{aligned}\varphi(v,a) = \phi(v^a) &= \phi\left(tv^{a^{[0,t]}} + (1-t)v^{a^{[t,1]}}\right) \\ &= t\phi\left(v^{a^{[0,t]}}\right) + (1-t)\phi\left(v^{a^{[t,1]}}\right) \\ &= t\varphi\left(v, a^{[0,t]}\right) + (1-t)\varphi\left(v, a^{[t,1]}\right).\end{aligned}$$

We have proved that the Shapley fuzzy authorization value satisfies all of the properties mentioned in the theorem. Now we will show that such properties uniquely determine the Shapley fuzzy authorization value.

Let $\psi : \mathscr{G}^N \times \mathscr{F}\mathscr{A}^N \to \mathbb{R}^N$ be such that it satisfies the properties of efficiency, additivity, irrelevant player, necessary player, fairness and reduction. We must prove that $\psi = \varphi$.

We proceed by strong induction on $\lceil(a)$ where

$$\lceil(a) = |im(a) \setminus \{0,1\}| \quad \text{for all } a \in \mathscr{F}\mathscr{A}^N.$$

BASE CASE. $\lceil(a) = 0$.
Notice that we can identify $\mathscr{A}^N$ with the set $\{a \in \mathscr{F}\mathscr{A}^N : im(a) \subseteq \{0,1\}^N\}$. From this point of view, we can say that the restriction of ψ to the set of games with fuzzy authorization structure (v, a) with $\lceil(a) = 0$ is an allocation rule for games with authorization structure. It is easy to check that such restriction satisfies the properties of efficiency, additivity, irrelevant player, necessary player and fairness. Therefore, using Theorem 15.1 we conclude that

$$\psi(v,a) = \varphi(v,a) \quad \text{for every } n \in \mathbb{N},\ v \in \mathscr{G}^N \text{ and } a \in \mathscr{F}\mathscr{A}^N \text{ with } \lceil(a) = 0.$$

INDUCTIVE STEP. Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$ with $\lceil(a) > 0$. We want to prove that $\psi(v, a) = \varphi(v, a)$. Take $t \in im(a) \setminus \{0, 1\}$. Since ψ satisfies the reduction property it holds that

$$\psi(v, a) = t\,\psi(v, a^{[0,t]}) + (1-t)\,\psi(v, a^{[t,1]}).$$

Since $\lceil(a^{[0,t]}) < \lceil(a)$ and $\lceil(a^{[t,1]}) < \lceil(a)$ it follows by induction hypothesis that $\psi(v, a^{[0,t]}) = \varphi(v, a^{[0,t]})$ and $\psi(v, a^{[t,1]}) = \varphi(v, a^{[t,1]})$. Hence

$$\psi(v, a) = t\,\varphi(v, a^{[0,t]}) + (1-t)\,\varphi(v, a^{[t,1]}) = \varphi(v, a).$$

Example Let us go back to the example in Sect. 4. We will keep the game v, but we will introduce some changes in the dependency relationships. Imagine the following situation. Supplier 1 admits that they use a technology patented by 2. However, 1 demonstrates that they are capable of producing component 1 without using that technology. But in that case they would only be able to produce six hundred thousand units before the deadline. Something similar happens with the other dispute. Supplier 2 can produce component 2 without the technology patented by 3, but if they do so then they only will be able to produce nine hundred thousand units within the stipulated time.

For every $E \subseteq \{1, 2, 3\}$ and $i \in \{1, 2, 3\}$, $a_i(E)$ indicates the fraction of its maximal productive capacity that player i can reach if it does not have the authorization of the players in $\{1, 2, 3\} \setminus E$ (cf. Table 4).

In a similar way as we did in the crisp case, we can obtain an expression of the Shapley fuzzy authorization value that does not involve the restricted game, but the game and the fuzzy authorization operator separately.

Table 4 Situation in the productivity game

E	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$a(E)$	$(0.6, 0, 0)$	$(0, 0.9, 0)$	$(0, 0, 1)$	$(1, 0.9, 0)$	$(0.6, 0, 1)$	$(0, 1, 1)$	$(1, 1, 1)$

We calculate the restricted game:

$v^a(\{1\}) = 0.6\,v(\{1\}) = 0.6,$

$v^a(\{2\}) = 0.9\,v(\{2\}) = 1.8,$

$v^a(\{3\}) = v(\{3\}) = 3,$

$v^a(\{1, 2\}) = 0.9\,v(\{1, 2\}) + 0.1\,v(\{1\}) = 5.5,$

$v^a(\{1, 3\}) = 0.6\,v(\{1, 3\}) + 0.4\,v(\{3\}) = 6,$

$v^a(\{2, 3\}) = v(\{2, 3\}) = 10,$

$v^a(\{1, 2, 3\}) = v(\{1, 2, 3\}) = 20.$

Finally, a payoff vector for the suppliers is

$\varphi(v, a) = \phi(v^a) = (4.65, 7.25, 8.1).$

Lemma 15.3 *Let $v \in \mathscr{G}^N$ and $a \in \mathscr{F}\mathscr{A}^N$. Then, if $\varphi(v,a)$ is considered as a column matrix, it holds that*

$$\varphi(v,a) = \zeta_a \cdot \Delta_v$$

where ζ_a is the matrix in $\mathscr{M}_{n,2^n-1}(\mathbb{R})$ defined by $(\zeta_a)_{i,E} = \varphi_i(u_E, a)$ for every $i \in N$ and $E \in 2^N \setminus \{\emptyset\}$ and Δ_v is the column matrix given by the Harsanyi dividends of v. Moreover, if $im(a) = \{t_l : \ l = 0, \ldots, m\}$ with $0 = t_0 < \ldots < t_m$ then

$$\zeta_a = \sum_{l=1}^{m} (t_l - t_{l-1})\, Z_{a^{t_l}}.$$

Example Let us use the expression given in the preceding result to calculate $\varphi(v,a)$ in the previous example. Firstly we calculate the characteristic function of u_E^a for each non-empty $E \subseteq \{1,2,3\}$ (cf. Table 5).
We calculate $\varphi\,(u_E, a)$ for each non-empty $E \subseteq \{1,2,3\}$

$$\begin{aligned}
\varphi\left(u_{\{1\}}, a\right) &= \phi\left(u^a_{\{1\}}\right) &&= \left(\frac{4}{5}, \frac{1}{5}, 0\right)\\
\varphi\left(u_{\{2\}}, a\right) &= \phi\left(u^a_{\{2\}}\right) &&= \left(0, \frac{19}{20}, \frac{1}{20}\right)\\
\varphi\left(u_{\{3\}}, a\right) &= \phi\left(u^a_{\{3\}}\right) &&= (0,0,1)\\
\varphi\left(u_{\{1,2\}}, a\right) &= \phi\left(u^a_{\{1,2\}}\right) &&= \left(\frac{29}{60}, \frac{29}{60}, \frac{1}{30}\right)\\
\varphi\left(u_{\{1,3\}}, a\right) &= \phi\left(u^a_{\{1,3\}}\right) &&= \left(\frac{13}{30}, \frac{2}{15}, \frac{13}{30}\right)\\
\varphi\left(u_{\{2,3\}}, a\right) &= \phi\left(u^a_{\{2,3\}}\right) &&= \left(0, \frac{1}{2}, \frac{1}{2}\right)\\
\varphi\left(u_{\{1,2,3\}}, a\right) &= \phi\left(u^a_{\{1,2,3\}}\right) &&= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

Table 5 Characteristic function of the productivity game

F	$u^a_{\{1\}}(F)$	$u^a_{\{2\}}(F)$	$u^a_{\{3\}}(F)$	$u^a_{\{1,2\}}(F)$	$u^a_{\{1,3\}}(F)$	$u^a_{\{2,3\}}(F)$	$u^a_{\{1,2,3\}}(F)$
{1}	0.6	0	0	0	0	0	0
{2}	0	0.9	0	0	0	0	0
{3}	0	0	1	0	0	0	0
{1,2}	1	0.9	0	0.9	0	0	0
{1,3}	0.6	0	1	0	0.6	0	0
{2,3}	0	1	1	0	0	1	0
{1,2,3}	1	1	1	1	1	1	1

and we write ζ_a

$$\zeta_a = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{29}{60} & \frac{13}{30} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{19}{20} & 0 & \frac{29}{60} & \frac{2}{15} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{20} & 1 & \frac{1}{30} & \frac{13}{30} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Finally,

$$\varphi(v,a) = \zeta_a \cdot \Delta_v = \begin{pmatrix} \frac{4}{5} & 0 & 0 & \frac{29}{60} & \frac{13}{30} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{19}{20} & 0 & \frac{29}{60} & \frac{2}{15} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{20} & 1 & \frac{1}{30} & \frac{13}{30} & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{93}{20} \\ \frac{29}{4} \\ \frac{81}{10} \end{pmatrix}.$$

References

Algaba E, Bilbao JM, van den Brink R, Jiménez-Losada A (2004) Cooperative games on antimatroids. Discrete Math 282(1–3):1–15

Aubin JP (1981) Cooperative fuzzy games. MOR 6(1):1–13

Bilbao JM (1998) Axioms for the Shapley value on convex geometries. EJOR 110(2):368–376

Bilbao JM, Driessen TSH, Jiménez-Losada A, Lebrón EA (2001) The Shapley value for games on matroids: the static model. MMOR 53(2):333–348

Bilbao JM, Ordóñez M (2009) Axiomatizations of the Shapley value for games on augmenting systems. EJOR 196(1):1008–1014

Butnariu D (1980) Stability and Shapley value for an n-persons fuzzy game. Fuzzy Sets Syst 4(1):63–72

Choquet G (1953) Theory of capacities. Ann de l'Institut Fourier 5:131–295

Derks J, Peters H (1993) A Shapley value for games with restricted coalitions. IJGT 21(4):351–366

Gallardo JM, Jiménez N, Jiménez-Losada A, Lebrón EA (2014) Games with fuzzy permission structure: the conjunctive approach. Inform. Sci. 278(10):510–519

Gallardo JM, Jiménez N, Jiménez-Losada A, Lebrón EA (2015) Games with fuzzy authorization structure: a Shapley value. Fuzzy Sets Syst 272(1):115–125

Gilles RP, Owen G, van den Brink R (1992) Games with permission structures: the conjunctive approach. IJGT 20(3):277–293

Myerson RB (1977) Graphs and cooperation in games. MOR 2(3):225–229

Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games - vol. II. Princeton University Press, Princeton, pp 307–317

Schmeidler D (1986) Integral representation without additivity. Proc Am Math Soc 97(2):255–261

Tsurumi M, Tanino T, Inuiguchi M (2001) A Shapley Function on a class of cooperative fuzzy games. EJOR 129(3):596–618

van den Brink R (1994) Relational power in hierarchical organizations. Ph.D. Thesis
van den Brink R (1997) An axiomatization of the disjunctive permission value for games with a permission structure. IJGT 26(1):27–43
Zadeh LA (1965) Fuzzy sets. Inform Control 8(3):338–353

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Placing Joint Orders When Holding Costs Are Negligible and Shortages Are Not Allowed

Alejandro Saavedra-Nieves, Ignacio García-Jurado, and M. Gloria Fiestras-Janeiro

Abstract In this paper we analyse multi-agent inventory systems where each agent has a deterministic demand and a capacitated warehouse with constant holding costs. Additionally, shortages are not allowed, the leadtime is constant and the cost of placing an order has two components: a fixed cost and a variable cost. We consider that agents cooperate by placing joint orders and that the variable cost is not necessarily additive. For this model we obtain the optimal policy and propose an allocation rule for the joint ordering costs.

Keywords Airport game • Core allocation • Cost game • Inventory problem • Order coalition

Contents

1 Introduction

Multi-agent inventory systems have received a lot of attention over recent years. In one of these systems, a group of agents facing similar inventory problems agree to cooperate by placing joint orders to reduce ordering costs. Dror and Hartman (2011) and Fiestras-Janeiro et al. (2011a) are two recent surveys in this field. When analysing a multi-agent inventory system one must deal with two main issues: (a)

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identify an optimal inventory policy, and (b) choose an allocation rule to distribute the joint ordering costs. The issue (b) is usually approached using cooperative game theory. Fiestras-Janeiro et al. (2011b), review some applications of cooperative game theory to cost allocation problems. Owen (2013), is an introduction to game theory stressing the cooperative models.

Fiestras-Janeiro et al. (2014), study a particular multi-agent inventory system arising in a farming community in Northwestern Spain. In their system:

1. There is a finite set of agents N. The agents in N consider the possibility of forming one or several *order coalitions*. An order coalition is a set of agents $S \subset N$ who place joint orders for a product they buy from a common supplier.
2. Orders placed by an agent or by a group of agents have a fixed cost $a > 0$. They also have variable (transportation) costs which depend on the agents involved in the orders. $A(S) \geq 0$ denotes the variable costs of joint orders placed by the agents in $S \subset N$. When an order coalition S forms, the supplier charges the cost $a + A(S)$ each time an order is placed. By convention $A(\emptyset) = 0$.
3. All the agents are located in the same line route, which means that $A(S) = \max_{i \in S} A(i)$, for all $S \subset N$.
4. The lead time from the moment an order is placed until it is received is deterministic and thus can be assumed to be zero.
5. Each agent $i \in N$ has a deterministic demand of $d_i > 0$ units per time unit. Each agent has a warehouse to store the product; agent i's warehouse can store at most $K_i > 0$ units. Holding costs are either non-existent or fixed (i.e. independent of the units stored) and thus irrelevant for the optimization and the allocation problems. No shortages are allowed.

Notice that Assumption 3 is adequate in some circumstances, but it is certainly restrictive. For instance, it does not hold in a situation like the one depicted in Fig. 1, for which $A(S) = \sum_{i \in S} A(i)$, for all $S \subset N$. Furthermore, when the supplier and the agents are located in a circular route or when they are located in most types of graphs, Assumption 3 cannot be formulated.

In this paper we analyse the above model when Assumption 3 is dropped. So, we deal with *basic EOQ systems without holding costs and with general transportation costs*, i.e. with systems $g = (N, a, A, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$ satisfying

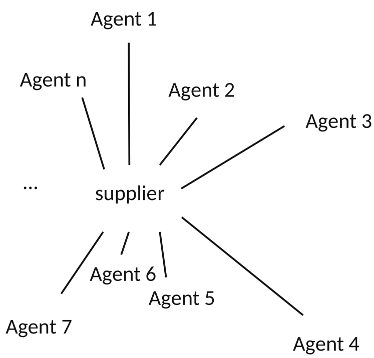

Fig. 1 Agents located in a star route

Assumptions 1, 2, 4, 5. For this model we identify an optimal inventory policy when the agents form an order coalition (see Sect. 2), and we propose an allocation rule for the joint ordering costs (see Sect. 3).

2 The Optimal Policy

Let $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with general transportation costs.

When there is only one agent i (i.e. when $N = \{i\}$), the optimal policy is very simple. Since holding costs are irrelevant, agent i will place the largest possible order (i.e. of size K_i) when the stock level reaches zero. Thus, the length of a cycle is K_i/d_i and the minimum cost per time unit is:

$$\frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{a + A(\{i\})}{K_i/d_i} = (a + A(\{i\}))\frac{d_i}{K_i}.$$

Since shortages are not allowed, when the agents in $S \subset N$ place joint orders the length of a cycle is $\min_{j\in S}\{K_j/d_j\}$ and then the minimum joint cost per time unit is:

$$\frac{\text{cost of a cycle}}{\text{length of a cycle}} = \frac{a + A(S)}{\min_{j\in S}\{K_j/d_j\}} = \big(a + A(S)\big)\max_{j\in S}\{\frac{d_j}{K_j}\}.$$

Thus, we can associate with every basic EOQ system without holding costs and with general transportation costs g the cost game c^g which is given by:

$$c^g(S) = \big(a + A(S)\big)\max_{j\in S}\{\frac{d_j}{K_j}\},$$

for all $S \subset N$.

A relevant question for a system g is the following. Is it reasonable for the agents of N to be willing to form an order coalition? The answer to this question is positive when the cost game c^g is subadditive, i.e. when it satisfies that

$$c^g(S \cup T) \leq c^g(S) + c^g(T)$$

for all $S, T \subset N$ with $S \cap T = \emptyset$. In words, the subadditivity condition means that if any two disjoint order coalitions merge, the costs associated with the new order coalition are not greater than the sum of the costs associated with the two initial order coalitions. The following result studies the subadditivity of c^g.

Theorem 16.1 *Let $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with general transportation costs and take the corresponding cost game c^g. Then c^g is subadditive if and only if, for every pair of non-empty coalitions $S, T \subset N$ with $S \cap T = \emptyset$ and satisfying that $\max_{i\in S}\{\frac{d_i}{K_i}\} = \max_{i\in S\cup T}\{\frac{d_i}{K_i}\}$, it holds* *that*

$$\frac{A(S \cup T) - A(S)}{a + A(T)} \leq \frac{\max_{j \in T}\{\frac{d_j}{K_j}\}}{\max_{j \in S \cup T}\{\frac{d_j}{K_j}\}}.$$

Proof Take $S, T \subset N$ in the conditions above. Then, $c^g(S \cup T) \leq c^g(S) + c^g(T)$ if and only if

$$(a + A(S \cup T)) \max_{j \in S \cup T}\{\frac{d_j}{K_j}\} \leq (a + A(S)) \max_{j \in S}\{\frac{d_j}{K_j}\} + (a + A(T)) \max_{j \in T}\{\frac{d_j}{K_j}\}$$

or, equivalently, if and only if

$$A(S \cup T) \max_{j \in S \cup T}\{\frac{d_j}{K_j}\} \leq A(S) \max_{j \in S}\{\frac{d_j}{K_j}\} + (a + A(T)) \max_{j \in T}\{\frac{d_j}{K_j}\}. \quad (1)$$

Now, (1) is equivalent to

$$A(S \cup T) \leq A(S) + (a + A(T)) \frac{\max_{j \in T}\{\frac{d_j}{K_j}\}}{\max_{j \in S \cup T}\{\frac{d_j}{K_j}\}},$$

and to

$$\frac{A(S \cup T) - A(S)}{a + A(T)} \leq \frac{\max_{j \in T}\{\frac{d_j}{K_j}\}}{\max_{j \in S \cup T}\{\frac{d_j}{K_j}\}}.$$

□

Theorem 16.1 gives a condition under which agents in N may be willing to form an order coalition, i.e. to make an agreement to place joint orders regularly. Now, a crucial issue is to identify an allocation rule to distribute the joint ordering costs adequately within this context. We do this in the next section.

3 How to Distribute the Joint Ordering Costs

If we are to distribute the joint ordering costs when the agents involved in a basic EOQ system without holding costs and with general transportation costs form an order coalition, it would be convenient to have an allocation rule that provides core allocations for a large class of systems. Formally, an allocation rule in this context is an application Ψ which assigns each system $g = (N, a, A, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$ an allocation $\Psi(g) \in \mathbb{R}^N$. The core of c^g is the set

$$Core(c^g) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c^g(N), \sum_{i \in S} x_i \leq c^g(S) \text{ for all } S \subset N\}.$$

The core allocations are *stable*, in the sense that abandoning the grand coalition and acting independently represents no improvement to coalition S. So, it would be convenient that $\Psi(g) \in Core(c^g)$ for a large class of systems g. Notice that $Core(c^g)$ may be an empty set. The following example provides a subadditive c^g with $Core(c^g) = \emptyset$.

Example 16.1 Take $g = (N, a, A, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$, a basic EOQ system without holding costs and with general transportation costs so that:

- $N = \{1, 2, 3\}$, $a = 0.5$, A is given in Table 1.
- $\frac{d_i}{K_i} = 1$ for all $i \in N$.

It is clear that the associated cost game c^g is the one in Table 2.
Notice that c^g is a subadditive game; however, it is easy to check that $Core(c^g) = \emptyset$.

An interesting issue is to find conditions under which a system has stable allocations. In this section we provide a sufficient condition for a system g so that $Core(c^g) \neq \emptyset$. First, we define the concept of submodularity for a map. In words, a map $f : 2^N \to \mathbb{R}$ is submodular when the marginal contribution of an agent to a coalition does not increase when the coalition enlarges.

Definition 16.1 Let N be a finite set and denote by 2^N the collection of all the subsets of N. A map $f : 2^N \to \mathbb{R}$ is said to be *submodular* if, for all $S, T \subset N$ with $S \subset T$, and for all $i \in N \setminus T$, it holds that

$$f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T).$$

Next, we introduce some notations that will later be useful. $\Pi(N)$ denotes the set of permutations[1] of N. If $\sigma \in \Pi(N)$, $\sigma(i) = j$ indicates that agent i occupies the j-th position in the ordering given by σ. For every $i \in N$, the set of predecessors of i

Table 1 The function A

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$A(S)$	0	4	4	4	4.5	4.5	4.5	7.5

Table 2 The cost game c^g

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^g(S)$	0	4.5	4.5	4.5	5	5	5	8

[1] A permutation of N is a bijective map from N to $\{1, \ldots, n\}$, n being the cardinality of N.

with respect to $\sigma \in \Pi(N)$ is

$$P_i^{\sigma} = \{j \in N : \sigma(j) < \sigma(i)\}.$$

Now take $\sigma \in \Pi(N)$ and a cost game c involving the agents in N. The marginal vector in c corresponding to σ is the vector $m^{\sigma}(c) = (m_i^{\sigma}(c))_{i \in N}$ such that $m_i^{\sigma}(c) = c(P_i^{\sigma} \cup i) - c(P_i^{\sigma})$ for all $i \in N$. For every $g = (N, a, A, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$ denote

$$M^g = \{i \in N \mid \frac{d_i}{K_i} \geq \frac{d_j}{K_j} \text{ for all } j \in N\}.$$

To finish this collection of notations, notice that, for all $S \subset N$,

$$c^g(S) = a \max_{j \in S}\{\frac{d_j}{K_j}\} + A(S) \max_{j \in S}\{\frac{d_j}{K_j}\}.$$

Denote by $c_1^g(S)$ and $c_2^g(S)$ the first and the second terms of the sum. This way we obtain two costs games c_1^g and c_2^g such that $c^g = c_1^g + c_2^g$.

Below we formally state and prove the main result in this section. In words it says that every system g with a submodular A and a subadditive c_2^g has a non-empty $Core(c^g)$. However, let us make a couple of previous remarks.

Remark 16.1 Observe that c_1^g is an airport game and thus $Core(c_1^g) \neq \emptyset$. See, for instance, González-Díaz et al. (2010, p. 256), for a definition of airport games and for the main properties of this type of games.

Remark 16.2 Notice that the submodularity of A is not a weak condition; however, there are many real problems in which it is fulfilled. For instance, in situations like the one depicted in Fig. 1, the corresponding A is submodular; also, it is submodular when Assumption 3 holds. Moreover, the submodularity condition arises naturally in cooperative game theory and in inventory models (see, for instance, Cheung et al. 2016). One may wonder whether the submodularity of A is also a necessary condition for guaranteeing the non-emptiness of $Core(c^g)$. The answer to this question is negative. For instance, take a system $\hat{g}$ that is identical to g in Example 16.1 with the only exception being that the fixed cost is $\hat{a} = 4$; then the corresponding cost game $c^{\hat{g}}$ is the one in Table 3. Clearly, $c^{\hat{g}}$ is subadditive, A is not submodular and $Core(c^{\hat{g}}) \neq \emptyset$ (for instance $(3, 3, 5.5) \in Core(c^{\hat{g}})$).

Theorem 16.2 *Take a basic EOQ system without holding costs and with general transportation costs* $g = (N, a, A, \{d_i\}_{i \in N}, \{K_i\}_{i \in N})$. *If* A *is submodular and* c_2^g *is subadditive, then* $Core(c^g) \neq \emptyset$.

Table 3 The cost game $c^{\hat{g}}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^{\hat{g}}(S)$	0	8	8	8	8.5	8.5	8.5	11.5

Proof The proof is constructive. Under the conditions of the statement we find a collection of elements of $Core(c_2^g)$ and, then, a collection of elements of $Core(c^g)$. Take $\sigma \in \Pi(N)$ satisfying that $\sigma^{-1}(1) \in M^g$. Let us now check that $m^\sigma(c_2^g) \in Core(c_2^g)$. It is enough to prove that for all $S \subset N$ it holds that $\sum_{i\in S} m_i^\sigma(c_2^g) \leq c_2^g(S)$. We distinguish two cases.

- $\sigma^{-1}(1) \in S$. In this case

$$\sum_{i\in S} m_i^\sigma(c_2^g) = c_2^g(\sigma^{-1}(1)) + \sum_{j\in S\setminus\{\sigma^{-1}(1)\}} (c_2^g(P_j^\sigma \cup j) - c_2^g(P_j^\sigma))$$
$$= A(\sigma^{-1}(1))\frac{d_{\sigma^{-1}(1)}}{K_{\sigma^{-1}(1)}}$$
$$+ \sum_{j\in S\setminus\{\sigma^{-1}(1)\}} \left(A(P_j^\sigma \cup j) \max_{i\in P_j^\sigma\cup j}\{\frac{d_i}{K_i}\} - A(P_j^\sigma) \max_{i\in P_j^\sigma}\{\frac{d_i}{K_i}\}\right).$$

Now, since $\sigma^{-1}(1) \in M^g$,

$$\sum_{i\in S} m_i^\sigma(c_2^g) = \sum_{j\in S} \big(A(P_j^\sigma \cup j) - A(P_j^\sigma)\big) \max_{i\in S}\{\frac{d_i}{K_i}\}.$$

Finally, using the submodularity of A,

$$\sum_{i\in S} m_i^\sigma(c_2^g) \leq \sum_{j\in S} \Big(A\big((P_j^\sigma \cup j) \cap S\big) - A\big(P_j^\sigma \cap S\big)\Big) \max_{i\in S}\{\frac{d_i}{K_i}\}$$
$$= A(S) \max_{i\in S}\{\frac{d_i}{K_i}\} = c_2^g(S).$$

- $\sigma^{-1}(1) \notin S$. Denote $\bar{S} = S \cup \{\sigma^{-1}(1)\}$; in view of the first case, considered above, it is clear that

$$\sum_{i\in \bar{S}} m_i^\sigma(c_2^g) \leq c_2^g(\bar{S}).$$

Notice that $m_{\sigma^{-1}(1)}^\sigma(c_2^g) = c_2^g(\sigma^{-1}(1))$. Then, the subadditivity of c_2^g implies[2]

$$c_2^g(\sigma^{-1}(1)) + \sum_{i\in S} m_i^\sigma(c_2^g) = \sum_{i\in \bar{S}} m_i^\sigma(c_2^g) \leq c_2^g(\bar{S}) \leq c_2^g(\sigma^{-1}(1)) + c_2^g(S),$$

and thus $\sum_{i\in S} m_i^\sigma(c_2^g) \leq c_2^g(S)$.

[2]Notice that the full subadditivity of c_2^g is not necessary. It is enough that the subadditivity is fulfilled when one of the coalitions is $\{\sigma^{-1}(1)\}$.

Now, in view of Remark 16.1, $Core(c_1^g) \neq \emptyset$. Then, it is clear that $x + m^\sigma(c_2^g) \in Core(c^g)$ for all $x \in Core(c_1^g)$. □

Now we state an immediate corollary of the proof of Theorem 16.2. For a system g, denote

- $\Pi_1^g(N) = \{\sigma \in \Pi(N) : \sigma^{-1}(1) \in M^g\}$.
- $\Pi_n^g(N) = \{\sigma \in \Pi(N) :$ for all $i \in \{1, \ldots, n-1\}$, *that* $\frac{d_{\sigma^{-1}(i)}}{K_{\sigma^{-1}(i)}} \geq \frac{d_{\sigma^{-1}(i+1)}}{K_{\sigma^{-1}(i+1)}}\}$.

Corollary 16.1 *Take a basic EOQ system without holding costs and with general transportation costs* $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$. *If* A *is submodular and* c_2^g *is subadditive, then*

$$\{x + m^\sigma \mid x \in Core(c_1^g) \text{ and } \sigma \in \Pi_1^g(N)\} \subset Core(c^g).$$

In view of Corollary 16.1, we define the allocation rule R that has the three following attractive properties: (a) it is a variation of the Shapley value, (b) it provides core allocations for all systems g with a submodular A and a subadditive c_2^g, and (c) it can be easily computed.

Definition 16.2 For every basic EOQ system without holding costs and with general transportation costs $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ and for every $i \in N$,

$$R_i(g) := \frac{1}{|\Pi(N)|} \sum_{\sigma\in\Pi(N)} m_i^\sigma(c_1^g) + \frac{1}{|\Pi_n^g(N)|} \sum_{\sigma\in\Pi_n^g(N)} m_i^\sigma(c_2^g). \tag{2}$$

Notice that R is in fact a variation of the Shapley value, in the sense that the first term is precisely the Shapley value of c_1^g and the second term is a linear convex combination of some marginal vectors of c_2^g. Statement (b) is a consequence of Corollary 16.1 and the fact that the Shapley value of an airport game always belongs to its core (see, for instance, González-Díaz et al. 2010, p. 256). Finally, R can be easily computed taking into account that there is a simple formula of the Shapley value of an airport game (see, for instance, González-Díaz et al. 2010, pp. 257–258) and that the second term in the definition of R is the average of $|\Pi_n^g(N)|$ marginal vectors, and $|\Pi_n^g(N)|$ is often a small number (sometimes equal to one). The following example illustrates the behaviour of R for a particular system.

Example 16.2 Take $\tilde{g} = (N, \tilde{a}, \tilde{A}, \{\tilde{d}_i\}_{i\in N}, \{\tilde{K}_i\}_{i\in N})$, a basic EOQ system without holding costs and with general transportation costs so that:

- $N = \{1, 2, 3\}$, $\tilde{a} = 10$, $\tilde{A}$ is given in Table 4.
- $\frac{\tilde{d}_1}{\tilde{K}_1} = 1$, $\frac{\tilde{d}_2}{\tilde{K}_2} = 0.95$, $\frac{\tilde{d}_3}{\tilde{K}_3} = 0.9$.

It is easy to check that the cost games $c_1^{\tilde{g}}$, $c_2^{\tilde{g}}$ and $c^{\tilde{g}}$ are the ones in Table 5.

Table 4 The function $\tilde{A}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$\tilde{A}(S)$	0	10	10	10	20	20	20	30

Table 5 The cost games $c_1^{\tilde{g}}$, $c_2^{\tilde{g}}$ and $c^{\tilde{g}}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c_1^{\tilde{g}}(S)$	0	10	9.5	9	10	10	9.5	10
$c_2^{\tilde{g}}(S)$	0	10	9.5	9	20	20	19	30
$c^{\tilde{g}}(S)$	0	20	19	18	30	30	28.5	40

Notice that $\tilde{A}$ is additive and thus submodular. It is easy to check that $R(c^{\tilde{g}}) = (3.75, 3.25, 3) + (10, 10, 10) = (13.75, 13.25, 13)$. This seems to be a reasonable allocation. Observe that $c_2^{\tilde{g}}$ is not subadditive but that $R(c^{\tilde{g}}) \in Core(c^{\tilde{g}})$ anyway.

In Example 16.2 the three agents are indistinguishable in $\tilde{A}$. In fact, they only differ in their ratios: $\frac{\tilde{d}_1}{\tilde{K}_1} > \frac{\tilde{d}_2}{\tilde{K}_2} > \frac{\tilde{d}_3}{\tilde{K}_3}$. R takes this into account, in the sense that $R_1(c^{\tilde{g}}) > R_2(c^{\tilde{g}}) > R_3(c^{\tilde{g}})$. This is a good property satisfied by R for all systems with a submodular A.

Proposition 16.1 *Take a basic EOQ system without holding costs and with general transportation costs $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$. Assume that A is submodular and that $i, j \in N$ are indistinguishable in A (i.e. for $S \subset N\backslash\{i, j\}$, $A(S\cup i) = A(S\cup j)$). Then,*

$$R_i(c^g) \geq R_j(c^g) \text{ if and only if } \frac{d_i}{K_i} \geq \frac{d_j}{K_j}.$$

Proof "$\Rightarrow$" Assume that $\frac{d_i}{K_i} < \frac{d_j}{K_j}$. Then the i-th component of the Shapley value of c_1^g is smaller than the j-th component of the Shapley value of c_1^g (this is an obvious consequence of the formula of the Shapley value of an airport game). Now take $\sigma \in \Pi_n^g(N)$. Then,

$$\begin{aligned} m_i^\sigma(c_2^g) = (A(P_i^\sigma \cup i) - A(P_i^\sigma)) \max_{l\in N}\{\frac{d_l}{K_l}\} &\leq (A(P_j^\sigma \cup i) - A(P_j^\sigma)) \max_{l\in N}\{\frac{d_l}{K_l}\} \\ &= (A(P_j^\sigma \cup j) - A(P_j^\sigma)) \max_{l\in N}\{\frac{d_l}{K_l}\} \\ &= m_j^\sigma(c_2^g), \end{aligned}$$

since A is submodular, $P_j^\sigma \subset P_i^\sigma$, and i, j are indistinguishable in A. Thus $R_i(c^g) < R_j(c^g)$, which is impossible.

"⇐" If $\frac{d_i}{K_i} = \frac{d_j}{K_j}$, then it is clear that $m_i^{\sigma}(c_1^g) = m_j^{\sigma}(c_1^g)$ for every $\sigma \in \Pi(N)$ and, besides, $m_i^{\sigma}(c_2^g) = m_j^{\sigma}(c_2^g)$ for every $\sigma \in \Pi_n^g(N)$ because i and j are indistinguishable in A. If $\frac{d_i}{K_i} > \frac{d_j}{K_j}$ then, as proved above, $R_i(c^g) > R_j(c^g)$. □

Proposition 16.1 provides a good property of R when agents are indistinguishable from the point of view of the transportation costs given by A. Proposition 16.2 below provides a good property of R when agents are indistinguishable from the point of view of the ratios $\frac{d_i}{K_i}$.

Proposition 16.2 *Take a basic EOQ system without holding costs and with general transportation costs* $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$. *Assume that* $\frac{d_i}{K_i} = \frac{d_j}{K_j}$ *for all* $i, j \in N$. *Then* $R(c^g)$ *is the Shapley value of* c^g.

Proof The result is obvious if we take into account (2), the additivity of the Shapley value, and the fact that $\Pi_n^g(N) = \Pi(N)$ when $\frac{d_i}{K_i} = \frac{d_j}{K_j}$ for all $i, j \in N$. □

4 Conclusions

In this paper we have introduced and studied a basic EOQ system without holding costs and with general transportation costs, a generalization of a model considered in Fiestras-Janeiro et al. (2014). We have identified an optimal inventory policy when the agents form an order coalition, and we have proposed and analysed an allocation rule for the joint ordering costs.

We now make two final comments on possible lines for future research concerning the proposed allocation rule.

- We have already indicated that R has a collection of attractive properties. A further matter of interest would be to find an axiomatic characterization of R. To this respect, the characterization of the so-called *line rule* in Fiestras-Janeiro et al. (2012), may provide some ideas.
- An extensive study comparing R with other rules in this context may be of interest. In particular, one could consider the possibility of using the proportional rule (proportional to the individual costs) instead of R. Notice however that the conditions in Theorem 16.2 do not guarantee that the proportional rule may provide core allocations. See the following example. Take $g = (N, a, A, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$, a basic EOQ system without holding costs and with general transportation costs such that:

  - $N = \{1, 2, 3, 4\}$, $a = 2$, A is an additive map satisfying that $A(1) = 2$, $A(2) = 3, A(3) = 5, A(4) = 10$.
  - $\frac{d_i}{K_i} = 1$ for all $i \in N$.

  It is easy to check that the associated cost games c_1^g, c_2^g and c^g are the ones in Table 6. Notice that A and c_2^g are additive. So, according to Theorem 16.2, $R(c^g) \in Core(c^g)$. It is clear that $R(c^g) = (0.5, 0.5, 0.5, 0.5) + (2, 3, 5, 10) =$

Table 6 The cost games c_1^g, c_2^g, c^g

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$
$c_1^g(S)$	0	2	2	2	2	2	2	2	2
$c_2^g(S)$	0	2	3	5	10	5	7	12	8
$c^g(S)$	0	4	5	7	12	7	9	14	10
S	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N		
$c_1^g(S)$	2	2	2	2	2	2	2		
$c_2^g(S)$	13	15	10	15	17	18	20		
$c^g(S)$	15	17	12	17	19	20	22		

$(2.5, 3.5, 5.5, 10.5)$, which seems to be a natural allocation in this example. However, the proportional rule proposes $(3.143, 3.929, 5.5, 9.429)$, which is not a core allocation (for instance, $3.143 + 3.929 > c^g(12) = 7$).

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References

Cheung M, Elmachtoub AN, Levi R, Shmoys DB (2016) The submodular joint replenishment problem. MP 158(1):207–233
Dror M, Hartman BC (2011) Survey of cooperative inventory games and extensions. J Oper Res Soc 62(4):565–580
Fiestras-Janeiro MG, García-Jurado I, Meca A, Mosquera MA (2011a) Cooperative game theory and inventory management. EJOR 210(3):459–466
Fiestras-Janeiro MG, García-Jurado I, Mosquera MA (2011b) Cooperative Games and cost allocation problems. Top 19(1):1–22
Fiestras-Janeiro MG, García-Jurado I, Meca A, Mosquera MA (2012) Cost allocation in inventory transportation systems. TOP 20(2):397–410
Fiestras-Janeiro MG, García-Jurado I, Meca A, Mosquera MA (2014) Centralized inventory in a farming community. J Bus Econ 84(7):983–997
González-Díaz J, García-Jurado I, Fiestras-Janeiro MG (2010) An introductory course on mathematical game theory. American Mathematical Society, Providence
Owen G (2013) Game theory, 4th edn. Emerald Group Publishing, Bingley

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Corporation Tax Games: An Application of Linear Cost Games to Managerial Cost Accounting

Ana Meca and J. Carlos Varela-Peña

Abstract Everybody knows that any Government, as a social institution, is concerned to provide and ensure a stable legal framework within which the investors can perform their capital investment in an environment of legal certainty that allows them to reduce their costs. Its primary function is to secure some form of cooperative benefit. Motivated by the Spanish Tax system, in this paper, we present an application of linear cost games to a corporate tax reduction system: corporation tax games. We prove that the grand coalition is always stable in the sense of the core.

Keywords Benefactor • Beneficiaries • Core stability • Corporation tax games • Cost distribution • Economic core • Grand coalition • Linear cost games

Contents

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1 Introduction

Linear cost games were introduced as a particular case of k-BBCM games, when $k = 1$ (cf. Meca and Sošić 2014). The authors, motivated by supply chain collaborations in practice, introduce a class of cost-coalitional problems, which are based on a priori information about the cost faced by each agent in each set that it could belong to. Then, they focus on problems with decreasingly monotonic coalitional costs. Their paper study the effects of giving and receiving, on cost-coalitional problems, when there exist players whose participation in an alliance always contributes to the savings of all alliance members (benefactors), and there also exist players whose cost decreases in such an alliance (beneficiaries). They use linear and quadratic norm cost games to analyse the role played by benefactors and beneficiaries in achieving stability of different cooperating alliances. Different notions of stability, the core and the bargaining set, are considered there and provided conditions for stability of an all-inclusive alliance of agents which leads to minimum value of total cost incurred by all agents.

Subsequently, Meca and Sošić (2016) present a new class of cooperative cost games, supremum-norm cost games, which emerges as a natural extension of k-norm games introduced by Meca and Sošić (2014). Supremum-norm cost games are k-BBCM games with $k = \infty$. They show that it is reasonable to expect formation of the grand coalition in such setting, and describe allocations that lead to stability of the grand coalition and reduce the individual cost of each agent.

Two applications of linear cost games are (1) purchasing games, presented by Schaarsberg et al. (2013), and (2) knowledge-sharing games, introduced by Bernstein et al. (2015). Both models illustrate benefits from cooperation in productions and operations management. The first analyses cooperation in purchasing interactive situations, while the second study cooperation in assembly systems and focuses on the role of knowledge sharing network.

Schaarsberg et al. (2013) consider a group buying model in which each buyer requires a certain amount of goods, and the purchasing price depends on quantity ordered. If buyers form a group, each member of the group pays a unit price that corresponds to that faced by the group member with the largest order quantity. Thus, members of the group who need fewer items (beneficiaries) benefit by joining a buyer with a larger order quantity (benefactor). The cost incurred by the group is obtained by simply adding all individual costs faced by its members.

Bernstein et al. (2015) examine the role of process improvement in a decentralized assembly system in which a buyer purchases components from several first-tier suppliers. These components are assembled into a finished product, which is sold to the downstream market. The sequence of decisions is modeled as a two-stage game—the first stage is a non-cooperative game, in which suppliers determine the amount of investment in effort to reduce their fixed costs, while the second stage is a cooperative game among suppliers (knowledge-sharing game), in which each supplier faces an EOQ inventory problem and the cost of a cooperative alliance is the sum of all individual costs of its members. As a result of cooperation, all suppliers achieve a level of cost reduction in their fixed costs equal to that of the most efficient supplier(s) (i.e., the one(s) with the lowest fixed cost). Thus, through knowledge sharing all suppliers indirectly benefit from the process improvement initiatives (beneficiaries) instituted by the more efficient suppliers (benefactors) in the network.

The idea of giving and receiving is part of our lives and is continuously present in our behaviour. Depending on the situation we face, sometimes we are benefactors and others are beneficiaries, but always seems to be benefits of cooperation. These benefits are generally tangible, but could also be intangible (for instance, under an altruistic behaviour). A cooperative situation, which often generates benefits, is that of social institutions. It is commonly accepted that the primary function of a social institution is to secure some form cooperative benefit. A social institution may be defined as an organizational system that operates to satisfy basic social needs by establishing a set of norms that codify certain constraints over the behaviour of everyone involved. We could say that it provides an ordered framework linking the individual to the larger culture. The basic institutions are Family, Religion, Government, Education and Economics.

The specific functions of a Government are (1) the Institutionalization of norms (Laws), (2) the enforcement of laws, (3) the adjudication of conflict (Court), (4) provide for the welfare of members of society, (5) protection of Society from external threat. A way to create welfare is to develop a competitive corporate tax system. If the Government wants the country to be a place for businesses to invest, it has to provide certainty and good incentives to invest in it. Thus, investors, through their account manager, will use all the available prior information to make optimal financial decisions related to that corporate tax system.

Motivated by the above ideas, we realized that cost-coalitional problems can also be viewed from the perspective of managerial cost accounting. Indeed, any corporate tax system contains a big amount of apriori information on tax reductions than can be collected as a cost-coalitional vector. The system generally provides different group investment options (e.g. corporation, cooperative, SICAV, etc.) that generate certain tax payments on benefits obtained by the group. Then, the cost generated by a group of investors is usually obtained by adding all individual costs faced by its members. Hence, an analysis of the impact of corporation tax reductions can be done by using linear cost games.

In this paper, we present an application of linear cost games to the corporate tax reduction system. It is inspired by the Spanish Tax system where the Government is concerned to provide and ensure a stable legal framework within which the investors can perform their capital investment in an environment of legal certainty that allows them to reduce their costs. The Government is considered the benefactor, as it keeps costs at the same level, zero cost, while reduce the costs of those investors who act legally (beneficiaries). Investors may decide to cooperate or not cooperate with the Government.

If they decide to cooperate, the Government will provide a framework of legal certainty, which is in their benefit. The Spanish tax system offers several figures of corporation tax reductions, being the most common corporation, cooperative and

SICAV. The corporation may be individual or consists of two investors. To create a cooperative need to be at least 3 investors,[1] while in a SICAV the Government would require at least 100 investors together[2] and 2.4 million € as total capital invested.[3] On the contrary, if investors decide not cooperate with the Government and try to defraud the system by tax evasion, they can be detected and charged with unlawful behaviour. Once this irregular behaviour is demonstrated, they will be punished and required to return all amount defrauded plus a penalty. This means that the costs of not cooperating with the Government would be higher than cooperate, and so all investors are willing to pay the lowest taxes under legal protection of the Government.

The plan of the article is as follows. We collect and discuss the main well known results for the core of k-BBCM games (games defined through cost-coalitional vectors with decreasingly monotonic coalitional costs) in Sect. 2. Section 3 analyses the linear case and describe under which conditions the core is small or large, as they have different implication on stability of the all-included alliance. In Sect. 4 we study the impact of corporate tax reductions by means of corporation tax games, a particular class of linear cost games. Finally, we conclude in Sect. 5. All details for the calculation of the penalty parameter in the Spanish tax system are in the Appendix.

2 Giving and Receiving on Cost-Coalitional Problems

We begin describing the effects of giving and receiving on cost-coalitional problems. For these problems there exist always a player or a group of players, benefactors, whose participation in an alliance contributes to the saving of all alliance members. There also exist beneficiaries, a player or a group of players whose cost decreases in such an alliance. Following Meca and Sošić (2014), we use k-BBCM games to analyse the role played by benefactors and beneficiaries in achieving stability of an all-inclusive alliance. We consider just one notions of stability, the core, and provide conditions for stability of an all-inclusive alliance of agents which leads to minimum value of total cost incurred by all agents.

[1] Article 8 of the Law 27/1999 concerning cooperations; Boletin Oficial de Estado (BOE) No. 170, 17. July 1999, pp. 27027–27062.

[2] Article 5 of the Law 35/2003, concerning cooperative investment institutions; BOE No. 265, 5. November 2003. Article 6 of the Royal Decree 1082/2012, 13. July 2012, approving the regulation of the development of the Law 35/2003, 4. November, concerning cooperative investment institutions; BOE No. 173, 20. July 2012.

[3] Article 80 of the Royal Decree 1082/2012, 13. July, approving the regulation of the law 35/2003, from 4. November, concerning cooperative investment institutions, BOE No. 173, 20. July 2012.

2.1 Cost-Coalitional Problems

Let $N = \{1, 2, \ldots, n\}$ denote the set of all agents and $S \subseteq N$ be an arbitrary set of agents in N. We assume that each member of S incurs certain non-negative cost, which depends on the subset S to which he belongs; we denote this cost by c_i^S for $i \in S, S \subseteq N$. We assume that $c_i^S \geq 0$ for all $i \in S$ and, for all $S \subseteq N$, there exists $i \in S$ such that $c_i^S > 0$.

To simplify the notation, we use c_i to denote an agent's stand-alone cost, $c_i = c_i^{\{i\}}$. In addition, we denote the vector of individual agents' costs in all possible subsets by $c^N = \left(c_i^S\right)_{i \in S, \emptyset \neq S \subseteq N}$. A cost-coalitional problem is a pair $\left(N, c^N\right)$, where N is the set of all agents involved in the problem and c^N is the cost-coalitional vector. For this kind of problems, cooperation is beneficial only if agents' costs in larger subsets do not exceed their costs in smaller ones. This is a desirable property, well known as Cost Monotonicity, that we formalize as follows: a cost-coalitional vector c^N satisfies cost monotonicity if each agent's cost in a given set does not exceed his cost in its subset, $c_i^S \leq c_i^T$, for all $i \in T,\ T \subset S$.

In the set of all agents, we identify first the subset of benefactors. For a given cost-coalitional problem $\left(N, c^N\right)$, let us denote by $\mathscr{B}\left(N, c^N\right)$ the set of agents who, by joining a subset, reduce the cost for its members without changing their own cost,

$$\begin{aligned}\mathscr{B}\left(N, c^N\right) = \big\{ & i \in N \mid c_i^S = c_i^T \forall T, S \subseteq N; T, S \ni i; \\ & c_j^S \leq c_j^T \forall j \in N \setminus \{i\}, S \ni i, j, T \subseteq N \setminus \{i\}, T \ni j \big\}\end{aligned} \tag{1}$$

We use β to denote cardinality of $\mathscr{B}\left(N, c^N\right)$, $\beta = \left|\mathscr{B}\left(N, c^N\right)\right|$. Note that agents from $\mathscr{B}\left(N, c^N\right)$ have an important role in bringing down the cost of a set; for that reason, we refer to member(s) of $\mathscr{B}\left(N, c^N\right)$ as benefactor(s). In many instances, the case $\beta = 1$ is likely to have very different implications from the case $\beta \geq 2$.

Meca and Sošić (2014) show that when we have multiple benefactors, an agent sees the same individual costs in any set that contains at least one benefactor and is not all-inclusive. Thus, with a single benefactor all the members of a set may see their cost increase if he leaves the group; we can say that he is irreplaceable. On the other hand, when there are several benefactors, the cost of a member of the set remains the same as long as there is another benefactor in the set; they say then that each benefactor in this case is replaceable. For this reason, they usually consider the two cases separately in their analysis.

We denote by $b\left(N, c^N\right)$ the set of beneficiaries from cooperation—the set of agents whose cost decreases in the all-inclusive set of players:

$$b\left(N, c^N\right) = \left\{i \in N : c_i > c_i^N\right\}. \tag{2}$$

Note that definitions (1) and (2) imply that an agent cannot be a benefactor and a beneficiary at the same time—$\mathscr{B}\left(N, c^N\right) \cap b\left(N, c^N\right) = \emptyset$. We now define cost games related to those cost-coalitional problems.

2.2 k-BBCM Games are Balanced

2.2.1 Preliminaries

To study the effects of giving (by benefactors) and receiving (as beneficiaries) on a cost-coalitional problem, Meca and Sošić (2014) introduce a new class of cooperative games, k-norm cost games, and study their main properties. They show that if firms consider cooperation in their cost-reducing efforts, they can always find a way to distribute cost of the grand coalition in a stable way that discourage defections, and describe how beneficiaries may use *giving* a share of their savings (which induces *receiving* of savings on the benefactors' side) to motivate benefactors to cooperate in cost-reducing efforts. In other words, Meca and Sošić (2014) show that k-norm games are balanced whenever $k \geq 1$ and describe the set of all core allocations.

For a given cost-coalitional problem, $\left(N, c^N\right)$, we can assemble a family of k-norm cost games for $k \in \mathbb{R}$. Each set $S \subseteq N$ is referred to as a coalition, and $S = N$ is referred to as the grand coalition. Let (N, c) denote a cost game, where $c(\cdot)$ denotes the cost function defined on the set of all subsets of N. We can reduce the game (N, c) to any coalition $S \subseteq N$ by means of the subgame $(S, c|_S)$, where $c|_S(T) := c(T)$ for all $T \subseteq S$.

A cost game (N, c) is k-norm if the cost of each coalition is obtained by taking a kth-root of the sum of kth powers of individual costs that coalition members incur in that coalition,

$$c(S)^k = \sum_{i \in S} \left(c_i^S\right)^k, \qquad k \in \mathbb{R} \tag{3}$$

While the definition of a cooperative cost game assigns a cost to each specific coalition, in this model it is also known the cost of each member of a specific coalition. This additional information is useful when we study the role played by benefactors and beneficiaries in achieving stability of the grand coalition.

A special case of k-norm cost games arises when $k = 1$, and they are called linear cost games. (N, c) is a linear cost game when the cost of a coalition is obtained by simply adding individual costs that coalition members incur when they belong to that particular coalition such that $c(S) = \sum_{i \in S} c_i^S$.

Notice that each cost-coalitional vector c^N can lead to infinitely many possible cost games (a different k-norm game for each choice of k). However, for a given value of k (say, linear cost, $k = 1$), each cost-coalitional vector is associated with a unique cost game. Hereafter, we focus on games that satisfy the following assumptions:

1. $k \geq 1$,
2. there is at least one benefactor ($\beta \geq 1$),
3. there is at least one beneficiary $\left(\left|b\left(N, c^N\right)\right| \geq 1\right)$,
4. cost-coalitional vectors satisfy cost monotonicity,

and refer to all of them as k-BBCM games. Note than Meca and Sošić (2014) focus on $k = 1$ and $k = 2$, meanwhile Meca and Sošić (2014) extend their study to supremum-norm cost games, i.e., $k = \infty$.

We know that, for the class of k-BBCM games, the grand coalition always minimize the total cost incurred by all players; i.e., we can reasonably expect the agents to form the grand coalition since k-BBCM games are subadditive (see Proposition 3 in Meca and Sošić (2014)). We also know that the grand coalition is always stable, in the sense of the core (k-BBCM games are balanced).

Next, we focus on describing in detail what we mean by stability of the grand coalition, in the sense of the core of k-BBCM games.

2.2.2 The Core Coincides with the Set of k-Reallocations

Consider a k-BBCM game (N, c) and assume that the agents form the grand coalition. A vector $\varphi \in \mathbb{R}^n$ is called an allocation for the game (N, c), with $|N| = n$. We say that an allocation φ is a member of the core (Gillies 1959) of (N, c) if it discourages the agents from seceding and forming smaller alliances, $\sum_{i \in S} \varphi_i \leq c(S)$ for all $S \subseteq N$, and if it is efficient, $\sum_{i=1}^{n} \varphi_i = c(N)$. We denote the core of the game (N, c) by $C(N, c)$. It is well known that the core is a polyhedron; hence it can be small (singleton) or large (contains infinitely many allocations). In addition, concave cost games have a non-empty core; however, although k-BBCM games are subadditive, they are not, generally, concave, as illustrated by a counterexample in Bernstein et al. (2015). Thus, by knowing that the games are subadditive, it cannot be concluded that they have a non-empty core. Meca and Sošić (2014) prove that the core of k-BBCM games is not empty by first introducing an allocation and then showing that it belongs to the core whenever $k \geq 1$. They define k-proportional allocation for k-BBCM games as $\Phi^k(c) = \left(\Phi_i^k(c)\right)_{i \in N}$, with

$$\Phi_i^k(c) = \frac{\left(c_i^N\right)^k}{c(N)^{k-1}} \tag{4}$$

Under this allocation each player is allocated the ratio of *kth* power of the cost that he incurs in the grand coalition and the $(k-1)$ *th* power of the total cost of the grand coalition. Meca and Sošić (2014)) show that $\Phi^k(c)$ belongs to the core and use the fact that the core of a game is non-empty if and only if the game is balanced (Bondareva 1963; Shapley 1967). Thus, they identify one core element of k-BBCM games and show that the games are balanced (see their Theorem 17.1). They next investigate some properties of the k-proportional allocation. With $k = 1$, a k-proportional allocation corresponds to the altruistic allocation, defined in Bernstein et al. (2015); it is the allocation in which each player is assigned exactly his own cost in the grand coalition. As a result, because players who are not beneficiaries do not see a reduction in their costs when they join the grand coalition, $c_i = c_i^N$ for

$i \notin b\left(N, c^N\right)$, each nonbeneficiary pays his stand-alone cost, while the remaining players pay strictly less:

$$\Phi_i^1(c) = \begin{cases} c_i^N = c_i, & i \in N \setminus b\left(N, c^N\right) \\ c_i^N < c_i & i \in b\left(N, c^N\right) \end{cases} \tag{5}$$

Recall that benefactors are not beneficiaries, $\mathscr{B}\left(N, c^N\right) \subset N \setminus b\left(N, c^N\right)$; hence they do not see any benefits from belonging to the grand coalition under the altruistic allocation. If the core contains more than one allocation, are there other core members that do not reward benefactors? Meca and Sošić (2014) show that this is not the case. In particular, they show that there is at most one core allocation for a k-BBCM game (N, c) in which all benefactors are allocated exactly their stand-alone costs, the k-proportional allocation (see their Proposition 4).

When $k = 2$, the k-proportional allocation corresponds to a minimum square proportional allocation for generalized holding cost games used in Meca (2007). In this case, the players do benefit not only from the presence of a benefactor but also from the economies of scale. As a result, when $k = 2$, benefactors who are replaceable ($\beta \geq 2$) have more weight than in the scenario with $k = 1$; this is illustrated in the example 1 given by Meca and Sošić (2014).

Next, we go further in order to better understand the set of allocations that could support the stability of the grand coalition. We know that the k-proportional allocations belong to the core of a k-BBCM game (N, c). To facilitate the analysis of the other core members for k-BBCM games, Meca and Sošić (2014) introduce the set of allowed reallocations. It tells us how "far" from the k-proportional allocations we may go and still remain in the core. First, they define a distance function $d_i^k(S, T)$ for any two coalitions $S, T \subseteq N$ and $i \in T$:

$$d_i^k(T, S) = \begin{cases} \frac{\left(c_i^T\right)^k}{c(T)^{k-1}} - \frac{\left(c_i^S\right)^k}{c(S)^{k-1}}, & T \subset S, i \in T \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

In other words, distance function $d_i^k(S, T)$ measures the improvement in allocation to player $i \in T$ under k-proportional allocations when coalition T adds members $S \setminus T$. As the cost is monotone decreasing, it is easy to verify that $d_i^k(T, S) \geq 0$ for any $T \subset S \subseteq N$. Then, they define the distance between T and S as the total improvement in allocations to members in T when they join $S \setminus T$:

$$d^k(T, S) = \begin{cases} \sum_{i \in T} \left[\frac{\left(c_i^T\right)^k}{c(T)^{k-1}} - \frac{\left(c_i^S\right)^k}{c(S)^{k-1}} \right], & T \subset S \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Next, for a set $S \subseteq N$, they define the set of allowed k-reallocations in S:

$$\begin{aligned} D^k(S, c) := \Big\{ \varphi \in \mathbb{R}^S \mid \varphi_i = \Phi_i^k(c) + d_i, i \in S; \textstyle\sum_{i \in S} d_j = 0; \\ \textstyle\sum_{i \in T} d_i \leq d^k(T, S), \text{ for all } T \subseteq S, T \neq \emptyset \Big\} \end{aligned} \tag{8}$$

Starting from a k-proportional allocation among members of S, we can increase or decrease allocations to specific players as long as we remain efficient (first condition (8)) and do not increase the distance between S and any of its subsets (second condition (8)). Note that the latter conditions assure that by increasing cost allocations to some players we do not make their defection to a subcoalition desirable, because no subcoalitions could generate lower cost.

The following theorem states that the core of k-BBCM games corresponds to the set of allowed k-reallocations.

Theorem 17.1 (Meca and Sošić 2014) *For a k-BBCM game* (N, c)*, the core coincides with the set of allowed k-reallocations,* $C(N, c) = D^k(N, c)$.

Meca and Sošić (2014) point out that the size of the core of k-BBCM games can differ significantly—in the case of $k = 2$, the core has infinitely many members, but when $k = 1$ the core could shrink to a single member, the altruistic allocation (see their Example 2). In both cases, the core corresponds to the set of allowed k-reallocations. The following proposition shows that the core is in general large when $k = 2$ (i.e., contains an infinite number of allocations).

Proposition 17.1 (Meca and Sošić 2014) *For a 2-BBCM game* (N, c)*, the core contains infinitely many allocations,* $|C(N, c)| > 1$.

However, for $k = 1$, the core does not necessarily shrink to a single member. Characterization of the size of the core is more complex for cost games with a linear norm; we focus on these games in the next section, and Theorem 17.2 provides conditions under which the core of 1-BBCM games is small (singleton) or large.

3 Linear Cost Games

In this section we study the linear case—cost game in which the cost of a coalition is the sum of the individual cost of its members (i.e. 1-BBCM games). Following Meca and Sošić (2014), we provide conditions under which the core of linear cost games is small (a singleton) or large (infinite), as they have different implications on stability.

Before we state the main result in this paper, we need some additional definitions. We use Δ_i to measure what is the best that beneficiaries can do if no benefactors were involved:

$$\Delta_i = c_i^{N \setminus \mathscr{B}(N, c^N)} - c_i^N \text{ for all } i \in N \setminus \mathscr{B}\left(N, c^N\right) \tag{9}$$

We further define strict beneficiaries as agents who need the inclusion of benefactors in order to achieve the lowest possible cost (all other beneficiaries have $\Delta_i = 0$), $Sb\left(N, c^N\right) = \{i \in N : \Delta_i > 0\}$. We can now state the theorem which addresses the size of the core of 1-BBCM games with respect to the number of benefactors and beneficiaries.

Theorem 17.2 (Meca and Sošić 2014) *Let (N, c) be a 1-BBCM game, and let $\Phi^1(c)$ be the altruistic cost allocation. Then, the following statements hold:*

1. *if the core is a singleton, $C(N, c) = \{\Phi^1(c)\}$, then all players receive the same allocations in all coalitions that contain a benefactor,*

$$c_j^N = c_j^T, \ \forall T \subseteq N, j \in T, T \cap \mathscr{B}\left(N, c^N\right) \neq \emptyset \tag{10}$$

2. *if the players receive equal allocations in all coalitions that contain a benefactor,*

$$c_j^N = c_j^T, \ \forall T \subseteq N, j \in T, T \cap \mathscr{B}\left(N, c^N\right) \neq \emptyset \tag{11}$$

   then the core is a singleton, $C(N, c) = \{\Phi^1(c)\}$, whenever:

   (a) benefactors are replaceable ($\beta \geq 2$), or
   (b) benefactors are irreplaceable ($\beta = 1$) and there are no strict beneficiaries $\left(Sb\left(N, c^N\right) = \emptyset\right)$;

3. *if benefactors are irreplaceable ($\beta = 1$) and the game possesses strict beneficiaries $\left(Sb\left(N, c^N\right) \neq \emptyset\right)$, then the core has infinitely many elements, $|C(N, c)| > 1$.*

Theorem 17.2(1) states that whenever $k = 1$ and the core contains a single element, the altruistic allocation, agents receive identical allocations in all coalitions that contain at least one benefactor, and they correspond to their allocations in the grand coalition. Thus, beneficiaries are indifferent between participating in the grand coalition and any subcoalition containing at least one benefactor, while the benefactors do not see any differences between their stand-alone costs and costs they incur when they join any set of players. This is a source of disincentive for benefactors to participate in the grand coalition, in which they improve payoffs for other players but not for themselves. This fact prompted the interest of Meca and Sošić (2014) in studying alternative allocations and stability concepts but here we are not dealing with them.

Theorem 17.2(2) identifies instances in which benefactors are disincentivized—when all players receive identical allocations in all coalitions with benefactors, the altruistic allocation is the only core member whenever the benefactors are replaceable (i.e., beneficiaries can achieve their lowest costs whenever the coalition contains at least one benefactor, so the grand coalition is not the only cost-minimizing outcome for them—Theorem 17.2(2a)), or there are no strict beneficiaries (i.e., beneficiaries can achieve their lowest cost without benefactors—Theorem 17.2(2b)). Note that in the second case it cannot be found a way to incentivize the sole benefactor's participation in the grand coalition, as the beneficiaries' costs can be minimized without him. However, in the first case there are additional savings that are realized whenever a benefactor joins a coalition of beneficiaries, and this can be used to generate another allocation which is not a core member, but could satisfy some alternative stability criteria.

Finally, Theorem 17.2(3) provides conditions for a "large" core (with infinitely many elements) in which we can use the set of allowed reallocations to find an allocation that makes the grand coalition stable (i.e., the allocation belongs to the core) and allocates positive savings to the benefactor—it happens whenever we have an irreplaceable benefactor, and beneficiaries cannot achieve their lowest costs without his participation. Thus, we are able to identify instances in which the core has infinitely many allocations, and benefactors can be rewarded for their contribution to reduction of overall costs.

Next, we present one of these instances in the framework of managerial cost accounting. It is an application of linear cost games to a corporate tax reduction system.

4 The Impact of Corporate Tax Reductions

We analyse a corporation tax reduction model that is inspired in the Spanish tax system. Our starting point is the fact that the Spanish Government is concerned to provide and ensure a stable legal framework within which the investors can perform their capital investment in a an environment of legal certainty that allows them to reduce their costs. Therefore, the Government can be considered a benefactor, as it keeps costs at the same level, zero cost, while reduce the costs of those investors who act legally (beneficiaries).

Of course, investors could not cooperate with the Government and try to defraud the system by tax evasion, but this is an illegal behaviour which involve several risks including that of be detected. If their illicit behaviour is demonstrated, then not only they will be required to return all the defrauded amount, but also be punished. This means that the costs of not cooperating with the Government would be higher than cooperate.

As a hypothesis we might consider that fraudsters always end up being detected. Today it may be an extreme assumption, but if we compare the existing fraud controls today that there were a few decades ago, we realize that the current global system is tending to a high level of fraud control. In fact, in recent years we are witnessing a string of revelations that show as though the number of offenders is very high, impunity and anonymity is not at all guaranteed, but quite the opposite. Defraud in a fiscal year means taking a risk, which is not limited to that fiscal year, but remains dormant until your prescription. Nevertheless, and despite the prescription, social condemnation of such behaviour remains in time, and have caused the fall of a number of political and business leaders, in some cases for evidence of illegal behaviour carried out over 20 years ago.

Summarizing, we assume that all those offenders will be detected, so that the costs of not cooperating with the Government will be higher than those of cooperating.

Next, we formalize the model of corporate tax reductions. It is a corporate tax model in which costs depend on the size of corporations.

Let $N = \{1, 2, \ldots, n\}$ be a set of investors and 0 the Government. We denote by $N_0 = N \cup \{0\}$ the set of all agents in the system. Each agent $i \in N$ has an amount, $I_i > 0$, that want to invest in a certain time and which will have some benefits $B_i > 0$ at that time. We assume that $B_0 = I_0 = 0$. Our model is static and therefore independent of any evolution over time. Each agent must pay some taxes to the Government by the benefits obtained from the investment; that is, each agent $i \in N$ must pay $\alpha \cdot 100\%$ of his benefits, that is his cost of investment will be $\alpha \cdot B_i$, with $0 < \alpha < 1$. To simplify, we assume that the Government offers three options of investment: (1) Individual investment: α^I, (2) Investment in a cooperative: α^C, (3) Investment in a grand coalition: α^N, where $\alpha^I > \alpha^C > \alpha^N$.

The first decision to be taken by the investors is to cooperate or not with the Government. If they do not cooperate and take an individual behaviour that violates the rules of the tax system, they will be detected and must pay a percentage α^P of their benefits. This percentage includes the payment of the amount owed, plus a penalty or interest charge α^*, i.e., $\alpha^P = \alpha^I \cdot (1 + \alpha^*)$, where $\alpha^P \gg \alpha^I$.

On the contrary, if they decide to cooperate with the Government, it will provide a framework of legal certainty, which is in their benefit. We consider that each investor can put their money into three separate entities: (1) a corporation, (2) a cooperative or (3) a grand coalition. We further assume that the percentages of benefits to be paid are not always equal, but depend on the specific entity created (size of the coalition). That is, must be met certain conditions, such as the corporation may be individual but to create a cooperative needs to be at least k investors and the government, while in a grand coalition all the investors would be required besides the Government.

This corporation tax model can be described as a cost-coalitional problem (N_0, c^{N_0}), where the cost coalitional vector c^{N_0} is given by $c_0^{S\cup\{0\}} = 0$, and for all $S \subseteq N$, and all $i \in S$

$$c_i^S = \alpha^P \cdot B_i, \tag{12}$$

$$c_i^{S\cup\{0\}} = \begin{cases} \alpha^I \cdot B_i, & |S| < k \\ \alpha^C \cdot B_i, & |S| \geq k \\ \alpha^N \cdot B_i, & S = N \end{cases} \tag{13}$$

Notice that, in the Spanish tax system, the corporation may be individual, but to create a cooperative need to be at least $k = 3$ investors, while in a SICAV (or grand coalition) the Government would require at least 100 investors together and 2.4 million € as total capital invested. If the investors take individual behaviours that violates the rules of the tax system, contrary to what the article 31.1 of the Spanish Constitution,[4] then it will be detected and must pay a percentage α^P of

[4]Article 31.1. of the Spain Constitution; BOE No. 311, from 29. December 1978: *"Todos contribuirán al sostenimiento de los gastos públicos de acuerdo con su capacidad económica mediante un sistema tributario justo inspirado en los principios de igualdad y progresividad que, en ningún caso, tendrá alcance confiscatorio."* [Everybody has to contribute to finance the public expenditures according to his/her economic capacity based on a tax system which respects the principles of equality and progressive taxation, but which will under no circumstances implies confiscatory effects.]

its benefits. As we already announced, this percentage includes the payment of the amount plus a penalty or interest charge α^*. The calculation of α^* is a relatively laborious process, which depends on the specific situation of each fraudster, and we have modeled[5] as $\alpha^* = \xi_G \cdot \gamma$, where ξ_G reflects the degree of infringement, while γ is an incentive for the offender to accept the penalty (see Appendix for details).

If investors decide to cooperate with the government, then they will enjoy legal protection and receive some economic stability and legal certainty. According they decide to invest in a corporation, or a cooperative, or a SICAV, they will pay a different percentage of their profits: $\alpha^I = 0.25, \alpha^C = 0.20, \alpha^N = 0.01$, respectively.

It is easy to prove that the cost coalitional vector c^{N_0} satisfies cost monotonicity property, that there is a single benefactor, the Government, and that all investors are strict beneficiaries. That is,

$$\begin{aligned} &\mathscr{B}\left(N_0, c^{N_0}\right) = \{0\}, \\ &b\left(N_0, c^{N_0}\right) = N = Sb\left(N_0, c^{N_0}\right). \end{aligned} \tag{14}$$

Notice that the Government is an irrepaceable benefactor ($\beta = 1$) and no investors can achieve the lowest cost without Government involvement $\left(\left|Sb\left(N_0, c^{N_0}\right)\right| = n\right)$. We are ready now to define the corresponding linear cost game that we will call *corporation tax game*. It is a cost game (N_0, c) such that $c(\{0\}) = 0$, and for all $S \subseteq N$,

$$c(S) = \alpha^P \cdot B_S, \tag{15}$$

$$c(S \cup \{0\}) = \begin{cases} \alpha^I \cdot B_S, & |S| < k \\ \alpha^C \cdot B_S, & |S| \geq k \\ \alpha^N \cdot B_S, & S = N \end{cases} \tag{16}$$

where B_S is the aggregate benefit for coalition S, i.e. $B_S = \sum_{i \in S} B_i$.

Corporation tax games are balanced, i.e., the grand coalition is always stable in the sense of the core. We know by Theorem 17.2(3) that the core has infinitely many elements, $|C(N_0, c)| > 1$. More precisely, we know, by Theorem 17.1, that the core coincides with the set of allowed 1-reallocations, $C(N_0, c) = D^1(N_0, c)$.

The 1-proportional allocation is given by

$$\begin{aligned} &\Phi_0^1(c) = c_0^{N_0} = 0 \\ &\Phi_i^1(c) = c_i^{N_0} = \alpha^N \cdot B_i, \forall i \in N. \end{aligned} \tag{17}$$

This is a natural and suitable allocation that support the stability (in the sense of the core) of the grand coalition. With this allocation, $\Phi^1 = (0, \alpha^N \cdot B_1, \ldots, \alpha^N \cdot B_n)$,

[5]For more information see the Spanish legislation, especially: Law 58/2003, General Taxation Law; BOE No. 302, from 18. December 2003. Royal Decree 2063/2004, from 15. October, approving the Regulation BOE No. 260, from 28. October 2004, pp. 35598–35612.

all investors must pay the lowest taxes under legal protection of the Government (which pays nothing). Recall that $\alpha^N < \alpha^C < \alpha^I \ll \alpha^P$.

Hence, the core is given by

$$C(N_0, c) = \left\{\varphi \in \mathbb{R}^{n+1} \mid \varphi_i = \Phi_i^1(c) + d_i, i \in N; \sum_{i \in N_0} d_i = 0; \right. \\ \left. \sum_{i \in T} d_i \le d^1(T, N_0), \text{ for all } T \subseteq N_0, T \neq \emptyset \right\} \tag{18}$$

where

$$d^1(T, N_0) = \sum_{i \in T} \left[c_i^T - c_i^{N_0}\right] = \sum_{i \in T} c_i^T - \alpha^N \cdot B_T. \tag{19}$$

We could then find other reallocations that allowed the Government to further reduce costs to the point that can be negative, i.e. the Government could get some positive savings. These reallocations would be

$$\begin{aligned} \varphi_0(c) &= d_0 \\ \varphi_i(c) &= \alpha^N \cdot B_i + d_i, \forall i \in N, \end{aligned} \tag{20}$$

such that

$$\begin{aligned} d_0 &= -\sum_{i \in N} d_i, \\ 0 &< d_i \le \left(\alpha^P - \alpha^N\right) \cdot B_i, \forall i \in N, \\ \sum_{i \in S \cup \{0\}} d_i &\le \sum_{i \in S \cup \{0\}} c_i^{S \cup \{0\}} - \alpha^N \cdot B_S, \forall S \subset N. \end{aligned} \tag{21}$$

From an economic point of view, the only reallocations that make sense are those for which $0 < d_i \le \left(\alpha^C - \alpha^N\right) \cdot B_i, \forall i \in N$, and so $\left(\alpha^N - \alpha^C\right) \cdot B_N \le d_0 < 0$. We collect all of them with the 1-proportional allocation, and define the *Economic core* as follows:

$$C^E(N_0, c) = \left\{\varphi \in \mathbb{R}^{n+1} \mid \varphi_i(c) = \alpha^N \cdot B_i + d_i, \forall i \in N, \varphi_0(c) = -\sum_{i \in N} d_i; \right. \\ \left. 0 \le d_i \le \left(\alpha^C - \alpha^N\right) \cdot B_i \forall i \in N \right\} \tag{22}$$

Notice that $C^E(N_0, c) \subset C(N_0, c)$. When $d_i = 0, \forall i \in N$, we get the 1-proportional allocation Φ^1. However, if we take $d_i = \left(\alpha^C - \alpha^N\right) \cdot B_i, \forall i \in N$, then $d_0 = \left(\alpha^N - \alpha^C\right) \cdot B_N$, and the corresponding reallocation is

$$\begin{aligned} \varphi_0(c) &= \left(\alpha^N - \alpha^C\right) \cdot B_N \\ \varphi i(c) &= \alpha^C \cdot B_i, \forall i \in N, \end{aligned} \tag{23}$$

With this allocation $\varphi(c) = \left(\left(\alpha^N - \alpha^C\right) \cdot B_N, \alpha^C \cdot B_1, \ldots, \alpha^C \cdot B_n\right)$, all investors must pay the second lowest taxes under legal protection of the Government which gets $\left(\alpha^C - \alpha^N\right) \cdot B_N$ as positive savings.

5 Conclusion

Everybody knows that any Government, as a social institution, is concerned to provide and ensure a stable legal framework within which the investors can perform their capital investment in an environment of legal certainty that allows them to reduce their costs. Its primary function is to secure some form of cooperative benefit. Many philosophers have written about the social benefits that can be generated by cooperation. Rawls (1999), for example, states that *"social cooperation makes possible a better life for all than any would have if each were to live solely by his own efforts."* Heath (2006) proposes five different mechanisms (ways in which individuals can help each other to achieve each other's objectives, whatever those objectives may be) of cooperative benefit.

Motivated by the Spanish Tax system, in this paper, we have presented an application of linear cost games to a corporate tax reduction system: *corporation tax games*. Here, the Government, as benefactor, provides different group investment options which reduce the costs of those investors who act legally (beneficiaries). Investors may decide to cooperate or not cooperate with the Government. If they decide to cooperate, the Government will provide a framework of legal certainty, which is in their benefit. On the contrary, if investors decide not cooperate with the Government and try to defraud the system by tax evasion, they are always detected and charged with unlawful behaviour. Thus, they are punished and required to return all amount defrauded plus a penalty. This implies that the costs of not cooperating with the Government are always higher than cooperate, and so all investors are willing to pay the lowest taxes under legal protection of the Government.

We have proved that *corporation tax games* are balanced, i.e., the grand coalition is always stable in the sense of the core. This means that investors have strong incentives to cooperate with the Government instead of being fraudsters. A natural and suitable cost distribution, that support the stability of the grand coalition, is one in which all investors pay the lowest taxes under legal protection of the Government which pays nothing. However, as the core is large, there are other cost distributions allowing the Government to generate positive savings, those belonging to the Economic core. One of them is the cost distribution in which all investors must pay the second lowest taxes under legal protection of the Government which gets its maximum positive savings.

Notice that the model we have presented is mere simplification of reality, always more complex, but allows us to begin to understand corporate tax systems and realize the important role that cooperative game theory can play in the study of the impact of corporate tax reductions. Some extension of this simple model we are planning to address in the near future are: (1) a model with the Government as exogenous benefactor; i.e. the only players are investors, (2) a dynamic model that takes into account the evolution of the benefits over time, (3) include the risk factor in the model, instead of assuming than all offenders are detected, (4) expand investment options proposed by the Government.

Appendix

As we pointed out in Section 4, if an investor decides not to cooperate with the Government, he takes a chance to receive a financial penalty,[6] whose amount depends on several factors and calculation can be relatively complicated. To show how the amount of the penalty would be obtained, without incurring excessive artificiality, we'll stick to the simplified situation most suited to our situation. To do this we consider the concepts that we define next.

Let $\mathbb{B}$ be *"penalty basis"*, consisting of the amount not entered for committing infringement and P_{HP} the damage to Public Finances, a variable that is obtained by dividing the basis of the penalty among the defrauded amount.[7]

The base rates to consider are ξ_G, ξ_{CR} y ξ_{per}. The first, ξ_G, reflects the degree of infringement, and so it starts with a minimum value and increases with the severity of the offense.

$$\xi_G = \begin{cases} 0.50, & \text{minor infringement} \\ 0.01 \cdot \left(50 + \xi_{CR} + \xi_{per}\right), & \text{mayor infringement} \\ 0.01 \cdot \left(100 + \xi_{CR} + \xi_{per}\right), & \text{very serious infringement} \end{cases} \tag{24}$$

The second, ξ_{CR} has to be added only in case of recidivism.[8]

$$\xi_{CR} = \begin{cases} 5, & \text{minor infringement} \\ 15, & \text{mayor infringement} \\ 25, & \text{very serious infringement} \end{cases} \tag{25}$$

[6]The Government sanctioning powers is under Title IV of Ley 58/2003, de 17 de diciembre, General Tributaria. BOE núm. 302, de 18/12/2003.

[7]In our case, we could simplify the concept assuming that the penalty basis $\mathbb{B}$ always be the amount no entered by commission of the offense $\mathbb{B} = \alpha^I \cdot B_i$, thus the injury will always be the 100%.

[8]It is understood that there has been backsliding, when the offender had been penalized for an infringement of the same nature, by a final administrative decision within four years prior to the commission of the offense.

The latter, ξ_{per}, reflects the damage caused to Public Finances.

$$\xi_{per} = \begin{cases} 10, & 10 < P_{HP} \leq 25 \\ 15, & 25 < P_{HP} \leq 50 \\ 20, & 50 < P_{HP} \leq 75 \\ 25, & 75 < P_{HP} \leq 100 \end{cases} \qquad with\ P_{HP} = \frac{\mathbb{B}}{B_i} \cdot 100 \tag{26}$$

To simplify ξ_{per}, we define the following unit step function, $a \in \mathbb{R}$,

$$u(t-a) = \begin{cases} 0, & 0 < t \leq a \\ 1, & t > a \end{cases} \tag{27}$$

and then,

$$\xi_{per} = 10 \cdot u(P_{HP} - 10) + 5 \cdot [u(P_{HP} - 25) + u(P_{HP} - 50) + u(P_{HP} - 75)] \tag{28}$$

We analyse next how to encourage the offender to accept the sanction.

Let t_a be the time allowed to pay the penalty fee, and t the time actually paid, both measured in days. The variable γ reflects some reductions by acceptance of the sanction. The amount to be paid will be reduced by a percentage γ_1 if the fraudster agree with the penalty.

$$\gamma_1 = \begin{cases} 30 & \text{agreement} \\ 50 & \text{records with agreement} \\ 0 & \text{another case} \end{cases} \tag{29}$$

Besides this amount it is still reduced by 25% if the total income of the remaining amount of the penalty in time, and also waiving the application for review or appeal against the sanction is made.

$$\gamma_2 = 25 \cdot u(t - t_a) \cdot \delta \qquad with\ \delta = \begin{cases} 0, & \text{sanction is claimed} \\ 1, & \text{sanction not claimed} \end{cases} \tag{30}$$

Hence,

$$\gamma = \left(1 - \frac{\gamma_1}{100}\right) \cdot \left(1 - \frac{\gamma_2}{100}\right) \tag{31}$$

and finally, we obtain the penalty (or interest charge) coefficient

$$\alpha^* = \xi_G \cdot \gamma. \tag{32}$$

References

Bernstein F, Kok G, Meca A (2015) Cooperation in assembly systems: the role of knowledge sharing networks. EJOR 240(1):160–171

Bondareva O (1963) Nekotorjye primeneniya metodov linynogo programmorovaniya k teorii kooperativnikh igr. In: Problemi Kibernetiki 10, pp 119–139 (Some applications of linear programming methods to the theory of cooperative games. In: Morgenstern O (ed) Selected Russian papers on game theory 1959–1965). Princeton University Press, Princeton, pp 79–114

Gillies DB (1959) Solutions to general non-zero-sum-games. In: Tucker AW, Luce RD (eds) Contributions to the theory of games, vol IV. Princeton University Press, Princeton, pp 47–87

Heath J (2006) The benefits of cooperation. Philos Public Aff 34(4):359–374

Meca A (2007) A core-allocation family for generalized holding cost games. MMOR 65(3): 499–517

Meca A, Sošić G (2014) Benefactors and beneficiaries: the effects of giving and receiving on cost-coalitional problems. POM 23(9):1549–1560

Meca A, Sošić G (2016) On supremum-norm cost games. ORL 44(1):54–58

Rawls J (1999) A theory of justice. 2nd edn. Harvard University Press, Cambridge

Schaarsberg MG, Borm P, Hamers H, Reijnierse JH (2013) Game theoretic analysis of maximum cooperative purchasing situations. NRL 60(8):607–624

Shapley LS (1967) On balanced sets and cores. NRLQ 14(4):453–460

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Characteristics of the τ-Value and the χ-Value

Stephan Zelewski and Tatjana Heeb

Abstract In the centre of this paper stands the analysis of two compromise values: the τ-value and the χ-value. This analysis is based on a broad justification program which aims at evaluating game theoretic solution concepts for the problem of the fair distribution of cooperation gains. This justification program consists of six requirements that should be fulfilled by game theoretic solution concepts in order to be able to be convincing from an economic perspective. We show that the τ-value as well as the χ-value mostly, but not completely, fulfill the requirements of this justification program. The χ-value proves to be superior to the τ-value in terms of the existence requirement, i.e. regarding its application range.

Keywords Bargaining power • Compromise values • Fairness conditions • Outsider coalition • τ-value • χ-value

Contents

1 Introduction

In this volume, a programmatic approach has been outlined that shows how a game theoretic concept for the solution of the real problem of a cooperation gain distribution perceived as fair can be justified from an economic point of view (Zelewski 2017, pp. 233–257).

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In this paper, it is at first deepened and critically analysed to what extent the game theoretic solution concept of the τ-value is able to fulfill the six trans-conceptual requirements of the aforementioned justification program. For the τ-value solution concept, what especially proves to be crucial is the requirement of existence. It is only insufficiently fulfilled, because the τ-value violates the relevant requirement that a game theoretic solution concept should have an application range as great as possible. This will be illustrated by the necessity to postulate the special integrity condition of quasi-balancedness in order to “save” the τ-value. Subsequently, it will be shown that the game theoretic solution concept of the τ-value has, despite this weakness, the potential for yielding a highly interesting insight, from an economic point of view: the τ-value can be reformulated in an equivalent manner by means of a specific *fairness condition* so that—from an economic perspective especially transparent—*good reasons* result that seem to be suitable for justifying a distribution outcome as fair.

Lastly, the χ-value. will be elaborated on. It represents a still barely known compromise value of cooperative game theory that is closely related to the τ-value. The χ-value is, mostly, structurally consistent with the τ-value. Therefore, the χ-value also fulfills the five concept-specific requirements (except for the interpretation of the two conditions of collective rationality) (Zelewski 2017, p. 225). Especially the possibility of an equivalent reformulation by means of a special fairness condition, which is going to be demonstrated in an example for the τ-value from an economic perspective, can be transferred to the χ-value, without any requirement of amendment, because of its broad structural consistence. However, the χ-value represents a promising alternative to the τ-value, for the χ-value overcomes the handicap of the τ-value, that it is only applicable to those instances of the distribution problem that fulfill the special integrity condition of *quasi-balancedness*. Nevertheless, the χ-value does not represent an “ideal” solution concept: the χ-value proves to be inferior to the τ-value with respect to the requirement of usability. This relation between the τ-value and χ-value will be elaborated on in the following.

2 The τ-Value Revisited

2.1 *Limitation of the τ-Value with Regard to the Existence Requirement*

Profound analyses show that the thus far developed solution of the τ-value can lead to complications. These complications are based on the fact that instances of the distribution problem exist that admittedly belong to the very broad class of essential games (here interpreted as cooperation gain distribution games), but for which the τ-value is not defined in a mathematically “clean” manner, i.e. it does not exist. These complications are discussed in, for example, Zelewski (2009, pp. 163–167).

In order to exclude these complications, as the *sixth requirement* and (for the time being) the last one for the solution concept of the τ-value, the following *integrity condition* for the relation between the lower and upper bounds LB and UB, respectively, regarding the shares in the cooperation gain G as well as for the hyperplane H for the purpose of complying with the efficiency condition, one can establish that

$$\forall S_v \in \mathbb{R}^N_{\geq 0} \forall LB \in \mathbb{R}^N_{\geq 0} \forall UB \in \mathbb{R}^N_{\geq 0} \forall G \in \mathbb{R}_{>0} :$$
$$\left(S_v = \begin{pmatrix} v_1 \\ \dots \\ v_N \end{pmatrix} \wedge LB = \begin{pmatrix} v_{1.min} \\ \dots \\ v_{N.min} \end{pmatrix} \wedge UB = \begin{pmatrix} v_{1.max} \\ \dots \\ v_{N.max} \end{pmatrix} \wedge c(C_0) = G \right) \tag{1}$$
$$\rightarrow \left(\sum_{n=1}^{N} v_{n.min} \leq G \leq \sum_{n=1}^{N} v_{n.max} \wedge LB \leq UB \right)$$

All games from the domain of cooperative game theory that fulfill the integrity condition for a (cooperation gain) distribution problem will be referred to as *quasi-balanced* games. Therefore, in the following, the phrase *integrity condition of quasi-balancedness* will be used for this purpose.

The integrity condition of quasi-balancedness can be motivated as follows: The cooperation gain G that is to be distributed has to be, on the one hand, big enough in order to cover the minimally allocable profit shares of all partners A_n from the grand coalition of a supply chain, but it is not allowed, on the other hand, to be bigger than would be necessary for covering the maximally allocable profit shares of all partners A_n from the grand coalition of a supply chain. This first partial condition only refers to *scalar* amounts of the cooperation gain that is to be distributed, as well as to the sums of minimally or maximally allocable profit shares of all affected partners. In contrast, the second partial condition expresses, in a vectorial manner, that for every single partner A_n, his maximally allocable profit share $v_{n.max}$ at the utopia point UB has to be at least as big as his minimally allocable profit share $v_{n.min}$ at the threat point LB. These two partial conditions appear—at least prima facie—to be directly comprehensible. A possible critique of this estimation will be treated later on.

The integrity condition of formula (1) can be also graphically illustrated. This condition mirrors the hyperplane H with $\sum_{n=1}^{N} v_n = G$ on which all efficient solutions are located in the normal case *above* the threat point LB and *below* the utopia point UB. In this normal case, the τ-value is always defined as a convex linear combination of the threat point LB and the utopia point UB. But also in the two special cases where the threat point LB or the utopia point UB lie by chance on the hyperplane H, i.e. $\sum_{n=1}^{N} v_{n.min} = G$ or respectively $\sum_{n=1}^{N} v_{n.max} = G$, the τ-value still exists provided that the second partial condition $LB \leq UB$ is complied with. Even in the extreme case $\sum_{n=1}^{N} v_{n.min} = G = \sum_{n=1}^{N} v_{n.max}$, where LB and UB coincide, the existence of the τ-value is not endangered. In this extreme case, the utopia as well as the threat point lie on the hyperplane of all efficient solution points.

2.2 Condensed Definition of the τ-Value

The essence of the solution concept of the τ-value consists in being able to prove the following theorem: There exists exactly one solution to the distribution problem (and also for all other analogously constructed distribution problems, such as, e.g. cost distribution problems), that fulfills the aforementioned 6 concept-specific requirements of individual and collective rationality, Pareto efficiency, fairness from a game theoretic perspective, and quasi-balancedness—i.e. the formulae (1)–(3) on pp. 226–227 as well as formulae (5)–(6) on pp. 230–230. This solution is specified by the solution concept of the τ-value. For the theorem's proof, which cannot be described here for reasons of space, cf. Zelewski 2009, pp. 153–163.

This essence of the solution concept of the τ-value can be compactly as well as precisely explicated by means of the following formula (2). The terms $v^{\tau}_{n.max}$, respectively $v^{\tau}_{n.min}$, represent those shares in the cooperation gain G which the partners A_n can expect at most, respectively, at least, according to the solution concept of the τ-value, because of the formula (3) on p. 227, respectively formula (5) on p. 230, i.e. $v^{\tau}_{n.min} = v_{n.min}$ and $v^{\tau}_{n.max} = v_{n.max}$. For the unique existence of the τ-value,

$$\forall G \in \mathbb{R}_{>0} \forall LB \in \mathbb{R}^N_{\geq 0} \forall UB \in \mathbb{R}^N_{\geq 0} :$$

$$\left(\begin{array}{c} LB = \begin{pmatrix} v^{\tau}_{1.min} \\ \dots \\ v^{\tau}_{N.min} \end{pmatrix} \wedge UB = \begin{pmatrix} v^{\tau}_{1.max} \\ \dots \\ v^{\tau}_{N.max} \end{pmatrix} \\ \wedge \sum_{n=1}^{N} v^{\tau}_{n.min} \leq G \leq \sum_{n=1}^{N} v^{\tau}_{n.max} \wedge LB \leq UB \end{array} \right)$$

$$\rightarrow \dot{\exists} S_{\tau} \in \mathbb{R}^N_{\geq 0} \exists \gamma \in \mathbb{R}_{\geq 0} : \tag{2}$$

$$\left(\begin{array}{c} S_{\tau} = \begin{pmatrix} v^{\tau}_{1} \\ \dots \\ v^{\tau}_{N} \end{pmatrix} \wedge \sum_{n=1}^{N} v^{\tau}_{n} = G \wedge \dots \\ S_{\tau} = \gamma \cdot LB + (1-\gamma) \cdot UB \wedge 0 \leq \gamma \leq 1 \end{array} \right)$$

In addition, the following definition of the convex linear factor γ that results from the proof of this theorem, is to be taken into account for the calculation of the

τ-value:

$$\forall n = 1,\ldots,N: \quad v_n^\tau = \gamma \cdot v_{n.max}^\tau + (1-\gamma) \cdot v_{n.min}^\tau \qquad \text{with}$$

$$\begin{cases} \gamma = \dfrac{G - \sum_{n=1}^{N} v_{n.min}^\tau}{\sum_{n=1}^{N} v_{n.max}^\tau - \sum_{n=1}^{N} v_{n.min}^\tau} & ;\mathit{if} \sum_{n=1}^{N} v_{n.max}^\tau \neq \sum_{n=1}^{N} v_{n.min}^\tau \\ \gamma \in [0; 1] & ;\mathit{if} \sum_{n=1}^{N} v_{n.max}^\tau = \sum_{n=1}^{N} v_{n.min}^\tau \end{cases} \tag{3}$$

It shall be pointed out that the special, marginal, case $\sum_{n=1}^{N} v_{n.max}^\tau = \sum_{n=1}^{N} v_{n.min}^\tau$, that is in no way excepted from the requirements in the above mentioned formulae, is often not considered in the relevant literature concerning the τ-value.

Besides, it is to be noted that the uniqueness of the τ-value is also guaranteed for the aforementioned special case with $\sum_{n=1}^{N} v_{n.max}^\tau = \sum_{n=1}^{N} v_{n.min}^\tau$ despite the indeterminacy of the convex linear factor γ with $\gamma \in [0; 1]$, for it can be shown that for this special case, exactly one solution point S_τ exists that fulfills the formula (2), cf. Zelewski (2009, pp. 154–156).

2.3 *Accentuation of the Fairness Condition of the τ-Value from an Economic Perspective*

The solution concept of the τ-value does not yet prove to be convincing from an economic perspective in the form that prevails so far in the relevant literature of cooperative game theory (albeit mainly in a considerably shorter seemingly "apodictic" form). For the fairness condition (cf. formula (6) on p. 230) may seem to be plausible as an abstract requirement from a game theoretic point of view. But this condition is lacking a "behavioural scientific" foundation that considers economic, especially business-oriented behavioural motives. Therefore, a seventh requirement for the solution concept of the τ-value will be added.

This *seventh requirement* represents a *fairness condition from an economic perspective*. This fairness condition has two essential features.

On the one hand, it does not exceed the six requirements already presented for a solution concept. This seventh requirement is not genuinely independent from the other six requirements. Instead, it can be shown that it is fulfilled whenever the first six requirements are fulfilled; this point will be dealt with shortly. In this respect, this fairness condition does not represent anything new.

On the other hand, this fairness condition proves to be interesting, because it permits interpreting the abstract game theoretic solution concept of the τ-value in a concrete manner from an economic perspective with regard to the *bargaining power* of a partner: The greater the bargaining power of the partner A_n, the greater the share v_n in the cooperation gain G that is to be distributed.

The bargaining power of a partner is measured by two opposed effects:

- On the one hand, this bargaining power is measured by his *cooperation contribution* that would be caused if he were to join in the already existing marginal coalition MC_n with $MC_n = C_0 \setminus \{A_n\}$ and thus would complete this marginal coalition to the grand coalition C_0. This *positive network effect* has already been established within the third requirement on the τ-value as the maximally allocable profit share $v_{n.max}$.
- On the other hand, the bargaining power of a partner A_n is measured by his greatest *threat potential* that he could build by founding at least one outsider coalition $OC_{n.q}$. This *negative network effect* has also already been established, namely within the fourth requirement on the τ-value, as the minimally allocable profit share $v_{n.min}$.

The good reasons to accept a distribution of the cooperation gain G to the partners A_n of a supply chain—and thus a solution $\underline{v} = (v_1, \ldots, v_N)$ for the underlying distribution problem—as fair, can be thus specified as follows: It is being deemed as *fair* to concede a partner A_n an all the greater share v_n in the cooperation gain G, the greater his cooperation contribution to the realization of the grand coalition (positive network effect) and the greater his threat potential to prevent the realization of the grand coalition through the founding of at least one outsider coalition (negative network effect).

The preceding characterizations of a fair distribution happened predominantly in a qualitative manner, i.e. with primarily natural linguistic means of expression. Thus, it still leaves room for interpretation regarding the concrete, numerical determination of the profit shares v_n for all partners A_n of a supply chain. Therefore, it needs a far-reaching quantification of the fairness criterion in the form of a calculation rule for the τ-value. This calculation rule should be as simple as possible and thus easily understandable as well as allow for a clear interpretation of the calculated distribution outcomes in the interest of the acceptability of the τ-value solution concept.

As a calculation rule that takes into account the preceding expectations, the following innovative τ-value formula will be introduced for the τ-value solution concept with regard to a cooperation gain G that is to be distributed. It conveys the economic content of this solution concept in a considerably more distinct manner than was the case in formula (6) on p. 230 for the fairness condition from a game theoretic point of view. This τ-value formula represents the quantitatively operationalized *fairness condition from an economic perspective* $\forall\ n = 1, \ldots, N$:

$$v_n^{\tau} = \begin{cases} \alpha \cdot \frac{v_{n.max}^{\tau}}{\sum_{n=1}^{N} v_{n.max}^{\tau}} \cdot G + \beta \cdot \frac{v_{n.min}^{\tau}}{\sum_{n=1}^{N} v_{n.max}^{\tau}} \cdot G & ;\text{if } \sum_{n=1}^{N} v_{n.max} \neq \sum_{n=1}^{N} v_{n.min} \\ v_{n.max}^{\tau} = v_{n.min}^{\tau} & ;\text{if } \sum_{n=1}^{N} v_{n.max}^{\tau} = \sum_{n=1}^{N} v_{n.min}^{\tau} \end{cases} \tag{4a}$$

with

$$\alpha = \frac{G - \sum_{n=1}^{N} v^{\tau}_{n.min}}{\sum_{n=1}^{N} v^{\tau}_{n.max} - \sum_{n=1}^{N} v^{\tau}_{n.min}} \cdot \frac{\sum_{n=1}^{N} v^{\tau}_{n.max}}{G}$$
$$\beta = \frac{\sum_{n=1}^{N} v^{\tau}_{n.max} - G}{\sum_{n=1}^{N} v^{\tau}_{n.max} - \sum_{n=1}^{N} v^{\tau}_{n.min}} \cdot \frac{\sum_{n=1}^{N} v^{\tau}_{n.min}}{G} \tag{4b}$$

This calculation rule for the τ-value is equivalent to the fulfillment of the first 6 requirements for a solution concept: A solution point S_v in the solution space $\mathbb{R}^N_{\geq 0}$ fulfills the fairness condition from an economic perspective that is expressed with the help of formulae (4a) and (4b), exactly when this solution point fulfills the requirements according to the formulae (1)–(3) on pp. 226–227 as well as formulae (5)–(6) on pp. 230–230 and formula (1) on p. 381 by means of which the existence and uniqueness of the τ-value for quasi-balanced (cooperation gain distribution) games are guaranteed in a conventional manner from a game theoretic perspective. The corresponding proof of equivalence cannot be presented here for reasons of space; cf. instead Zelewski (2009, pp. 168–178); Zelewski and Peters (2010, pp. 21–24, 2012, pp. 168–171); Zelewski and Jene (2011, pp. 10–12); Jene and Zelewski (2012, pp. 178–180).

In summary, the solution concept of the τ-value can be characterized by four special features. First, when determining the share v_n of every partner A_n in the overall cooperation gain G that is to be distributed, the *bargaining power* of the partner is captured through two summands in formula (4a). The first summand describes the bargaining power of partner A_n based on his cooperation contribution $v^{\tau}_{n.max}$ to the grand coalition (positive network effect) by means of the profit share $v^{\tau}_{n.max} \cdot G$. The second summand represents the bargaining power of partner A_n based on the threat potential $v^{\tau}_{n.min}$ to break up the grand coalition (negative network effect) by means of the profit share $v^{\tau}_{n.min} \cdot G$. Such a direct reference of the calculation of a distribution outcome to the bargaining power of the partners, that is captured through positive and negative network effects, is neither known from the other solution concepts of cooperative game theory nor from the conventional game theoretic definitions of the τ-value solution concept. By means of the construct "bargaining power", these desiderata for a "behavioural scientific" foundation of the τ-value solution concept is being honoured.

Second, the cooperation contribution and the threat potential of partner A_n are not absolutely captured, but relativized with reference to the sums of the cooperation contributions and threat potentials of all the partners of a supply chain. This corresponds to a *normalization* of the cooperation contribution and threat potential of each partner A_n with regard to the upper bound *UB* and the lower bound *LB* of the solution space for the problem.

Third, the cooperation contribution and the threat potential of a partner A_n are weighted by means of the factors α and β when determining his profit share v_n. This weighting allows expressing the τ-value in a more compact, but equivalent, manner of representation than the convex linear combination of the upper bound *UB* (utopia point) and lower bound *LB* (threat point) that lies exactly on the hyperplane *H* for the fulfillment of the efficiency condition in the solution space $\mathbb{R}^N_{\geq 0}$ for the problem. This is expressed through the formula (6) on p. 230 and formula (3) on p. 383 for the conventionally defined τ-value because of this equivalence.

Fourth, the τ-value represents a *compromise solution* for the problem. This compromise solution is characterized by two features. On the one hand, it proves to be Pareto optimal in the sense of the efficiency condition mentioned above. On the other hand, it represents intuitively the simplest compromise between the utopia and the threat point, for no easier connection can be constructed between these two points in the solution space $\mathbb{R}^N_{\geq 0}$ than the direct, linear connection that is defined through the convex linear combination as per the formula (6) on p. 230. Furthermore, the positive and the negative network effects of a partner are each being considered in a proportional manner. This characterization of the τ-value as a compromise solution for the problem with a proportional consideration of the positive and the negative aspects of the bargaining power of all the involved partners yields another good reason to accept, as fair, the distribution of the cooperation gains by means of the τ-value solution concept.

2.4 A Critical Analysis of the τ-Value

The τ-value has been presented as a solution concept for the assigning problem (cf. Zelewski 2017). This problem consists in distributing the gain that results from the cooperation of several partners in a supply chain to the partners concerned, in a manner for which good reasons can be given for accepting this distribution as fair. In order to assess the suitability of the τ-value for this problem, six trans-conceptual requirements have been established as assessment criteria. Their fulfillment will be examined in the following.

The *rationality requirement* is satisfied by the solution concept of the τ-value. At first, three rationality requirements are to be adduced that are fulfilled by the τ-value: the condition of individual rationality, the collective rationality condition for maximally allocable profit shares, and the collective rationality condition for minimally allocable profit shares. With the help of each of these three rationality conditions, the scope of the action for seemingly reasonable distribution outcomes has been successively restricted. Thus, the scope of the action has been restricted, and this can also be interpreted as an expression of rational action in the sense of Pareto optimality.

The *uniqueness requirement* is admittedly fulfilled by the solution concept of the τ-value. But this only applies under the restrive precondition that the existence of the τ-value has been secured. Therefore, the analysis of the uniqueness requirement cannot be substantially separated from the examination of the existence requirement.

The *existence requirement* is the Achilles' heel of the τ-value solution concept, for the τ-value is not defined for all conceivable instances of the distribution problem that can be defined in the class of essential games. Instead, its existence is only guaranteed for a proper subset of the class of essential games, because of the integrity condition of quasi-balancedness. In this respect, the τ-value solution concept is inferior to competing solution concepts. For example, the nucleolus as well as the Shapley value exist for any essential game. Therefore, the *requirement of an application range as great as possible* is much less fulfilled by the τ-value than, for example, by solution concepts like the nucleolus or the Shapley value.

It has been shown that the violation of the existence requirement within the class of essential (cooperation gain distribution) games merely extends to three cases with respect to the τ-value solution concept in which the requirement of quasi-balancedness is not being fulfilled; cf. Zelewski (2009, pp. 163–166). Up to now, it has been thought that these three "pathological" cases normally prove to be of little relevance for business practice; cf. Zelewski (2009, p. 164); Jene and Zelewski (2012, pp. 181–182). This opinion has been also adopted, for example, by Heilmann and Wintein (2015, p. 17, footnote 9).

This holds at least when the cooperation gain G that is to be distributed does not remain under the smallness threshold, i.e. as long as $G \geq \sum_{n=1}^{N} v_{n.min}^{\tau}$. Every time the cooperation gain G that is to be distributed stays below this smallness threshold, these "pathological" cases can occur when the requirement of the quasi-balancedness is not fulfilled and thus the τ-value does not exist. This smallness threshold, however, is fulfilled by all instances of the distribution problem that belong to the class of essential games (strictly speaking even for the class of all slightly essential games). This game class has been marked out at an earlier point as a sufficient application range for game theoretic solution concepts that should be applied to the problem. Consequently, this smallness threshold is not a substantial restriction of the τ-value solution concept.

However, it is to be conceded self-critically by one of the authors (Zelewski) of the present paper that his earlier estimation, that the violation of the existence requirement by the τ-value solution concept with respect to non-quasi-balanced games, is (from the perspective of applications to business practice) at best of peripheral importance, needs a revision. For by Jene (2015, pp. 226–229), an equally simple and instructive example has been presented for a distribution problem that, at first glance, does not seem to be lacking in practical relevance, but does not fulfill the requirement of quasi-balancedness. Based on such elementary evidence, it has to be conceded that the τ-value solution concept does not fulfill the requirement of having an application range as great as possible to the desired extent (covering the complete class of essential games). In this respect, the τ-value proves to be considerably inferior to alternative game theoretic solution concepts like the nucleolus and the Shapley value.

From the perspective of the *acceptability requirement*, it is to be expected of a solution concept that it explicates good reasons a distribution outcome to be perceived as fair and consequently accepted. Regarding this requirement, the τ-value solution concept shows specific strengths as well as considerable weaknesses.

The strength of the τ-value solution concept lies in the fact that it reveals—in contrast to most other game theoretic solution concepts, such as the nucleolus and the Shapley value—the good reasons mandated for making it seem plausible to accept the τ-value as a solution for an instance of the distribution problem as a fair distribution outcome. First, these good reasons consist in justifying the share of a partner in the cooperation gain by the fact that this share turns out to be greater, the greater the bargaining power of this partner is, for it is plausible to attribute partners with "strong" bargaining positions a greater profit share than other partners with less bargaining power.

Second, there are good reasons for arguing for measuring the bargaining power of a partner based, on the one hand, on his contribution that he would make if he were to complete an already existing marginal coalition to the grand coalition C_0, and, on the other hand, by means of his greatest threat potential that he could build by credibly threatening to form at least one outsider coalition $OC_{n.q}$. For the cooperation contribution and the threat potential of a partner can be understood as a positive and a negative network effects. This partner contributes this twofold network effect to the realization of the cooperation gain that is to be distributed among the members of a supply chain in a "comprehensible"—but not source-specific—manner.

Third, the fairness condition of the τ-value solution concept, that has been presented from an economic perspective by means of a special calculation rule for the τ-value, is based on a proportional and efficient solution that lies between the utopia and the threat point in the solution space of all conceivable solutions for every instance of the distribution problem (provided that the existence condition is fulfilled). Since notions of fairness are normally linked to the intuition of a compromise between conflicting interests, this compromise feature of the τ-value is also a good reason for the acceptability of the τ-value solution concept.

However, these arguments in support of the τ-value solution concept open gateways for questioning the fulfillment of the acceptability requirement. For every solution concept that contributes to the explication and specification of the good reasons that (can) prompt the acceptance of a distribution outcome as fair, exposes itself to the risk of being criticized because of this explication and specification. Thus, one can argue about whether it is "reasonable" to define the size of the profit shares according to the bargaining power of a partner and to measure this bargaining power by means of the cooperation contribution and the threat potential of a partner, as was established in formulae (4a) and (4b) above for the calculation of the τ-value. For example, it can be discussed whether to assess the bargaining power of a partner in some other way than by formulae (4a) and (4b), or even take some aspect other than the bargaining power of a partner as a basis for the calculation of his share in the cooperation gain that is to be distributed.

In this possibility of criticizing the “reasonableness” lies, on the one hand, the basic weakness of the τ-value solution concept. However, this also pertains to every other solution concept that explicates a fairness criterion in a similarly precise manner. On the other hand, the authors would appreciate it if other solution concepts, that do not fulfill these requirements of explicitness and precision, or at least not in a similar manner, would first reach the explication and specification level of the τ-value solution concept. Only after this would it seem to be advisable to compare the explication and specification of the good reasons suitable for accepting a distribution outcome as fair, with each other in a critical manner regarding the τ-value and other solution concepts of cooperative game theory.

Furthermore, it is appropriate to discuss under what conditions the reasons that are being quoted for or against the fairness of a distribution outcome, are to be accepted as “good” reasons. In this paper, the authors have indeed adduced reasons that, in their estimation, speak for the fairness of the distribution outcomes of the τ-value solution concept. However, it remains open whether these reasons will also be accepted as “good” reasons by the recipients of this paper. The authors hope that they have at least launched an open, controversial as well as productive discussion of this. Such a discussion could contribute to determine which reasons for the fairness of a distribution outcome find recognition in the scientific community as “good” reasons.

For example, the rationality condition for minimally allocable profit shares is a critical aspect of the τ-value solution concept according to formula (5) on p. 230. This rationality condition is based on an operationalization of the intuitive notion about what could characterize a believable threat to defect from the grand coalition. The credibility of such a threat always involves a judgement call, that depends on subjective estimations and can thus always—from the perspective of other, but equally subjective estimations—be doubted. Therefore, the solution concept of the τ-value exhibits an open flank at this point: it just takes the questioning of its operationalization of the meaning of a “believable” threat by an outsider coalition for undermining the solution concept as a whole. In order to close this open flank as much as possible, the credibility of threats through outsider coalitions is being conceptualized in an especially rigid manner by the solution concept of the τ-value.

This has been elaborately explained in connection with formulae (4) and (5) on pp. 229–230. Nevertheless, it basically remains correct that the operationalization of the meaning of “believable” threats through outsider coalitions could take place differently than has happened by means of these formulae. In the case of another operationalization, a solution concept for the gain distribution problem other than the τ-value would probably result.

With regard to the requirement of communicability, a solution concept shall be helpful to not only generate solutions for problem instances, but also to communicate them to third parties. This requirement is also met by the τ-value solution concept. The communicability of the τ-value as a solution for an instance of the distribution problem is especially supported by four aspects.

First, it can be illustrated in a manner easy to explain and comprehend through the reconstruction of the τ-value that the τ-value lies within the solution space that is being gradually restricted based on a series of rationality, fairness and integrity conditions to a narrower—even single-unit—field of “reasonable” solutions (cf. Zelewski 2017).

Second, the specification of the τ-value can be convincingly presented as a compromise solution that, on the one hand, is located between a threat and an utopia point in the solution space and, on the other hand, lies on the hyperplane of Pareto efficiency. The real meanings of the three “anchors” for the specification of the τ-value—the threat point, the utopia point, and the hyperplane of Pareto optimality—can be explained, from an economic perspective, with arguments that refer directly to the distribution problem that is to be solved. Both previously mentioned aspects contribute considerably to the fulfillment of the requirement of intelligibility.

Third, the formal calculation rule for the τ-value presents a seemingly simple convex linear combination of the threat and utopia point that can be explained to third parties without “intellectual pull-ups” due to proportional considerations with respect to the bargaining power of all involved partners. This once again fulfills the requirement of intelligibility.

Fourth, the requirement of visualizability can be fulfilled (cf. Fig. 1 on p. 231).

Due to the preceding arguments, the solution concept of the τ-value fulfills the requirement of communicability far better than all other solution concepts of cooperative game theory. For example, neither the nucleolus nor the Shapley value have a similar support in terms of a substantial explanation with which help the solutions for instances of the distribution problem can be believable communicated. Therefore, a particular strength of the τ-value solution concept lies in the fulfillment of the requirement of communicability.

The usability requirement, specified by the requirement of minimal coalition knowledge, stems from the information-economic interest to have to presume as little knowledge as possible about those plausible claims that can be made by partners of coalitions of a supply chain regarding their shares in the cooperation gain that is to be distributed. The τ-value solution concept remarkably satisfies this requirement *prima facie*, because it requires knowledge about plausibly claimable profit shares of only three coalition types: the grand coalition C_0, all marginal coalitions MC_n, and all outsider coalitions $OC_{n.q}$. Upon closer examination, however, it becomes clear that the outsider coalitions can extend to all conceivable coalitions except for the grand coalition. Therefore, the solution concept of the τ-value does not fulfill the requirement of minimal coalition knowledge, at least not better than alternative solution concepts from cooperative game theory.

However, an effective reduction of knowledge, which is necessary for plausibly claimable profit shares of coalitions, can be realized when one succeeds in considerably reducing the number of outsider coalitions that are to be actually considered compared to the set of all combinatorially possible outsider coalitions. For this reduction, an interesting approach exists, which is based on the selection of unimportant partners and insignificant outsider coalitions. For reasons of space, this substantial advancement of the τ-value solution concept cannot be elaborated on. Cf. instead Zelewski (2009, pp. 244–255); Jene (2015, pp. 235–242).

To sum up, the six transconceptual requirements for a concept for solving the distribution problem are fulfilled in a decent, but incomplete manner, by the τ-value. Four of these six requirements are directly fulfilled.

With respect to the acceptability and communicability requirements, the τ-value solution concept has its particular strengths, because these two requirements are, in the authors' estimation, better fulfilled by it than by all other solution concepts of cooperative game theory.

Considerable weaknesses of the τ-value solution concept become apparent, however, with regard to the usability requirement, specified by the requirement of minimal coalition knowledge, and to the existence requirement, specified here by the requirement of an application range as great as possible. While justified prospects exist to overcome the weak fulfillment of the requirement of minimal coalition knowledge through an advance on the τ-value solution concept, the restriction of its secure existence and thus its applicability, the requiring of the integrity condition of quasi-balancedness, remains as the Achilles' heel of this solution concept. Because of this restriction of the τ-value solution concept to the class of quasi-balanced (distribution) games, the requirement of an application range as great as possible is not convincingly fulfilled compared to the competing solution concepts of cooperative game theory.

Beyond these criticisms, two additional aspects are to be noted. They do not present specific weaknesses of the τ-value solution concept, but apply to all the solution concepts of cooperative game theory for the distribution problem considered here.

On the one hand, it is always implicitly presumed that the cooperation gain G that the partners of a supply chain can collectively realize, is known. This common presupposition of game theoretic research ("a cooperation gain $G \ldots$ is given") is, however, only rarely and above all only partly fulfilled in business practice. This genuine economic problem of the determination of a cooperation gain is only rarely explicitly and concretely addressed. One of the few reflections that deals with the economic determination of a cooperation gain in a broad and deeper manner can be found in Jene (2015, pp. 25–87, and 261–286); cf. tangentially also Otto and Obermaier (2009, pp. 135–145); Zelewski (2009, pp. 62–66); Hofmann and Wessely (2010, pp. 98–118).

On the other hand, an "error of the third kind" (cf. Mitroff and Featheringham 1974, p. 383) usually happens in practical examples of the application of game theoretic solution concepts to distribution problems. This also applies with the examples of applications of the τ-value that have been created with the involvement of one of the authors (Zelewski) (cf., for example, Zelewski 2009, pp. 214–222; Zelewski and Jene 2011, pp. 13–16; Jene and Zelewski 2013, pp. 38–44). This error of the third kind is based on the fact that a "wrong" problem is being "correctly" solved. The crucial point of this error is the interpretation and numerical specification of the characteristic function c that not only underlies the τ-value, but every solution concept of cooperative game theory. The values of the characteristic function c usually do not represent cooperation gains, but merely the success that a coalition C_m can collectively achieve. In order to determine the cooperation gain from this, the sum of all individual profits of all partners of a supply chain that are cooperating with each other would have to be subtracted from the collectively achieved profit of a coalition. The value that is to be subtracted can vary from coalition to coalition. Therefore, the cooperation gain that a coalition C_m can collectively realize is fixed only after subtracting the individual profits $c(\{A_n\})$ of all partners of the coalition C_m from the collectively achieved profit $c(C_m)$. Consequently, an adjusted characteristic function c^*, with index set IM_m the set of all partners that belong to the coalition C_m, would have to be defined as follows:

$$\forall C_m \subseteq A: \quad c^*(C_m) = c(C_m) - \sum_{n \in IM_m} c(\{A_n\}) \tag{5}$$

Strictly speaking, the τ-value as well as the other solution concepts of cooperative game theory should employ this adjusted characteristic function c^*.

The non-linear transformation of the characteristic function c by formula (5) to the adjusted characteristic function c^* does not cause fundamental difficulties, provided that the individual profits $c(\{A_n\})$ of all partners of a coalition C_m and their collectively achieved profit $c(C_m)$ are known for each coalition C_m (the information premise). The fulfillment of this information premise is admittedly not trivial; see the previous explanations concerning the requirement of minimal coalition knowledge. But every time this information premise is fulfilled, this conversion can be implemented according to formula (5). Therefore, the previously outlined error of the third kind does not represent a fundamental weakness of the τ-value solution concept, but can be corrected in the case of the τ-value—as in the case of every other solution concept of cooperative game theory—if the information premise is fulfilled.

For example, it results from formula (5) that the adjusted characteristic function c^* has to take on the value $c^*(C_m) = 0$ for every degenerate coalition C_m, which consists of exactly one partner A_n:

$$\forall C_m \subseteq A \forall A_n \in A : C_m = \{A_n\} \rightarrow c^*(C_m) = c(\{A_n\}) - c(\{A_n\}) = 0 \tag{6}$$

Even if the preceding transformation of the characteristic function c is not included in examples of the application of cooperative game theory, this issue is not a fundamental defect for the application of game theoretic solution concepts to distribution problems. Rather, the two formulae (5) and (6) merely need to be understood as "memorandum items" for the numerically correct formulation of the examples of applications for the solution concepts of cooperative game theory, such as, e.g. the τ-value.

3 The χ-Value as an Advance on the τ-Value

The χ-value goes back to papers of Bergantiños/Massó, cf. Bergantiños and Massó (1994, pp. 5–24, 1996, pp. 5–24, 2002, pp. 272–285). Up to now, the χ-value solution concept has only occasionally been picked up on (e.g. Bergantiños et al. 2000, pp. 276–277; Sánchez-Soriano 2000, pp. 473–474) and is, at least in the field of economic analyses, widely unknown. Among the rare exceptions are Zelewski (2009, p. 151) ; Jene and Zelewski (2011a, pp. 120–129, 2011b, pp. 120–129, 2013, pp. 31–36 and 41–46); Jene (2015, pp. 128–204 and 258–317). Nevertheless, it is a remarkable game theoretic solution concept treated here. Two reasons are especially indicative for this judgement.

First, the χ-value can be understood as a generalization of the τ-value, because the χ-value exhibits the same structure as the τ-value and replaces only one central requirement in the field of collective rationality, with a generalized rationality condition. In this context, by the same structure it is meant that in the case of the χ-value as well as in the case of the τ-value it is about so-called compromise solution concepts. They can be characterized through the fact that the solution for an instance of the distribution problem is being determined as a compromise value that mediates between an upper bound for the maximally allocable shares in the cooperation gain and a lower bound for the minimally allocable shares in the cooperation gain. The mediation between the upper and the lower bound is operationalized by the calculation of a convex linear combination of both bounds.

Second, the χ-value addresses the criticism that the application of the τ-value is restricted to instances of the distribution problem that fulfill the integrity condition of quasi-balancedness, for it can be shown that the existence of the χ-value can be ensured without presuming this integrity condition; cf. Bergantiños and Massó (1996, pp. 280–281). Instead, it is enough to restrict the application range of the χ-value regarding the existence requirement to the class of the slightly essential (cooperation gain distribution) games. Such slightly essential games exist precisely when, with G denoting the cooperation gain that is to be distributed and with $c(C_0) = G$ for the grand coalition C_0,

$$\begin{aligned} & c(C_0) \geq \sum_{n=1}^{N} c(\{A_n\}) \quad // \quad c(C_0) = G \\ \Rightarrow & G \geq \sum_{n=1}^{N} c(\{A_n\}) \end{aligned} \tag{7}$$

This restriction of the application range of the χ-value solution concept to the class of slightly essential games guarantees a very great application range of the χ-value, because it is even less restrictive than the "reasonable" restriction of the distribution problem to the class of all essential games where cooperation is worth it, for now, all that is being demanded, to be a slightly essential game, is that cooperation "is not detrimental".

The restriction of the application range of the χ-value solution concept to the class of slightly essential games merely demands that the cooperation gain G is enough to concede every partner A_n of the grand coalition $C_0 = A$ that amount $c(\{A_n\})$ that this partner A_n could achieve outside the supply chain, on his own, by leaving ("defecting") the grand coalition. This means that the requirement to be a slightly essential game is already fulfilled when the condition of individual rationality according to formula (1) on p. 226 is fulfilled.

Otherwise, if the requirement of slightly essential games were not to apply, the cooperation gain G that is to be distributed would be so small that it would be impossible to concede every partner A_n that amount $c(\{A_n\})$ that he could realize—on his own—outside the supply chain. Consequently, in this case, it is certain that at least one partner would defect, i.e. leave the grand coalition. As a result, the grand coalition would be unstable from the outset if the requirement of being a slightly essential game was not fulfilled. In the context of the distribution problem, however, only those distributions of cooperation gains are being analysed that have been collectively realized by all the partners of a supply chain in the grand coalition or that shall be collectively attained in the future. Therefore, the stability of the grand coalition has to be taken for granted for the analysis of the distribution problem from the beginning. Consequently, self-contradictions while analysing the distribution problem can only be avoided if it is assumed that the requirement of slightly essential games is fulfilled. This is why this requirement does not present a "substantial" restriction of the application range for the χ-value solution concept as long as the χ-value is being applied as a solution concept to the problem.

It may be surprising that the χ-value does not link directly to the rationality condition for minimally allocable profit shares according to formula (5) on p. 230, which—as already explained—represents a particularly critical aspect of the τ-value. Instead, the basic approach for the operationalization of the meaning of a "believable" threat involving an outsider coalition is being adopted by the solution concept of the χ-value from that of the τ-value. Only the specific mode of calculation of the believable threat of an outsider coalition has been modified, and that only slightly. This will be returned to shortly.

The central modification of the χ-value compared to the τ-value happens with regard to the rationality condition for maximally allocable profit shares according to formula (3) on p. 227. In the case of the τ-value, the maximally allocable profit share $v^{\tau}_{n.max}$ for a partner A_n is only measured by the amount $c(C_0) - c(C_0 \setminus \{A_n\})$ by which the cooperation gain G of the grand coalition $C_0 = \{A_1, \ldots, A_N\}$ would decline if the partner A_n were to leave. In contrast, in the case of the χ-value, the maximally allocable profit share $v^{\chi}_{n.max}$ is determined for a partner A_n with respect to every coalition C_m which includes the partner A_n as a member. Regarding such a coalition C_m, what is being considered is by which amount $c(C_m) - c(C_m \setminus \{A_n\})$ the cooperation gain of this coalition C_m would decline if the partner A_n were to leave this coalition.

Since the partner A_n always belongs to the grand coalition C_0, the restriction of the calculation of $v^{\tau}_{n.max}$ in the case of the τ-value to the grand coalition C_0 can be understood as a special case of the calculation of $v^{\chi}_{n.max}$ in the case of the χ-value with $C_m = C_0$. Put another way, since in the case of the χ-value other coalitions C_m with $C_m \subset C_0$ besides the grand coalition C_0 are being included in the calculation of $v^{\chi}_{n.max}$, the χ-value represents a generalization of the τ-value because of these additionally included coalitions.

The third requirement for a solution concept is being replaced, with respect to the rationality condition for maximally allocable shares $v^{\chi}_{n.max}$ in the cooperation gain G analogously to formula (3) on p. 227, by the following generalized rationality condition (because of the preceding elaborations concerning the χ-value compared to the τ-value):

$$\begin{aligned} &\forall n = 1, \ldots, N \forall v_n \in \mathbb{R}_{\geq 0} : v_n \leq v^{\chi}_{n.max} \wedge \ldots \\ &\qquad v^{\chi}_{n.max} = \max\{c(C_m) - c(C_m \setminus \{A_n\}) | \varnothing \subset C_m \subseteq A \wedge \{A_n\} \subset C_m\} \end{aligned} \tag{8}$$

Since the basic approach of the operationalization of the meaning of a “believable” threat of an outsider coalition is being transferred from the τ-value, the following formula results for the χ-value analogously to formula (5) on p. 230 as a rationality condition for minimally allocable shares $v^{\chi}_{n.min}$ in the cooperation gain G:

$$\begin{aligned} &\forall n = 1, \ldots, N \forall v_n \in \mathbb{R}_{\geq 0} : v_n \geq v^{\chi}_{n.min} \wedge v^{\chi}_{n.min} = \max\left\{d_{n.1}, d_{n.2}, 0\right\} \\ &\quad \textit{with} \\ &d_{n.1} = \max\left\{\begin{matrix} \{c(\{A_n\}|OC_{n.q}) = c(OC_{n.q}) - \sum_{m \in (IN_{n.q} \setminus \{n\})} v^{\chi}_{m.max} | \ldots \\ q = 1, \ldots, Q_n \wedge \varnothing \subset OC_{n.q} \subset A \wedge \{A_n\} \subset OC_{n.q} \end{matrix}\right\} \\ &d_{n.2} = c(\{A_n\}|OC_{n.q}) = c(\{A_n\}) \text{ für } OC_{n.q} = \{A_n\} \end{aligned} \tag{9}$$

In this connection, it is to be noted that formula (9), for the calculation of the minimally allocable shares $v^{\chi}_{n.min}$ in the cooperation gain G for the χ-value, is not identical with the corresponding formula (5) on p. 230 for the calculation of the minimally allocable shares $v_{n.min}$ in the gain G for the τ-value despite the obvious structural equality. The difference consists in the fact that the maximally allocable profit shares $v_{m.max}$ of formula (5) on p. 230 are replaced here in formula (9) by the maximally allocable profit shares $v^{\chi}_{m.max}$ that, according to formula (8), are being calculated differently in the case of the χ-value—as explained above—than the they were for the τ-value.

Other than this, the χ-value does not differ from the τ-value. Especially the χ-value is also being determined as a convex linear combination of the utopia point (upper bound) UB and the threat point (lower bound) LB, which are defined in the same structural manner as in the case of the τ-value: by the maximally allocable and the minimally allocable share in the cooperation gain G. It only needs to be kept in mind that these maximally or minimally allocable profit shares are to be calculated in the new, and characteristic for the χ-value, manner by means of the terms $v^{\chi}_{n.max}$ according to formula (8) and $v^{\chi}_{n.min}$ according to formula (9).

As a rule for the calculation of the χ-value with special regard to the frequently ignored degenerated case $\sum_{n=1}^{N} v^{\chi}_{n.max} = \sum_{n=1}^{N} v^{\chi}_{n.min}$, this yields, analogously to formula (3) for the τ-value,

$$\forall n = 1, \ldots, N : v^{\chi}_{n} = \gamma \cdot v^{\chi}_{n.max} + (1 - \gamma) \cdot v^{\chi}_{n.min}$$

with

$$\begin{cases} \gamma = \dfrac{G - \sum\limits_{n=1}^{N} v^{\chi}_{n.min}}{\sum\limits_{n=1}^{N} v^{\chi}_{n.max} - \sum\limits_{n=1}^{N} v^{\chi}_{n.min}} & ; \text{if } \sum\limits_{n=1}^{N} v^{\chi}_{n.max} \neq \sum\limits_{n=1}^{N} v^{\chi}_{n.min} \\ \gamma \in [0; 1] & ; \text{if } \sum\limits_{n=1}^{N} v^{\chi}_{n.max} = \sum\limits_{n=1}^{N} v^{\chi}_{n.min} \end{cases} \tag{10}$$

Analogously to the τ-value, it can be shown that exactly one solution point S_{χ} exists in the solution space $\mathbb{R}^{N}_{\geq 0}$ that fulfills the 5 concept-specific requirements in the form of the conditions of individual and collective rationality, the condition of Pareto optimality, and the condition of fairness in the game theoretic sense, i.e. formulae (1) and (2) on p. 226, Eq. (8) on p. 395 [instead of formula (3) on p. 227], formula (9) [instead of formula (5) on p. 230] and Eq. (10) [instead of formula (6) on p. 230]. The integrity condition of quasi-balancedness, however, does not need to be imposed anymore, because it is fulfilled from the outset by the χ-value for all slightly essential games—and thus all the more for all essential games.

The proof of the fulfillment of the aforementioned five requirements by the χ-value was originally provided by Bergantiños and Massó (1994, pp. 6–8, 1996, pp. 281–282, 2002, p. 272), but is only very briefly outlined there. Especially the a priori fulfillment of the integrity condition of the quasi-balancedness by the χ-value is merely hinted at. A detailed derivation of this outstanding feature of the χ-value solution concept, that, unlike the solution concept of the τ-value, expands its application range to the class of all slightly essential games, is to be found in Jene (2015, pp. 175–181).

Due to the preceding explanations, the χ-value can be characterized as the unique solution that—as in case of the τ-value—presents a convex linear combination of the utopia point (upper bound) *UB* of maximally allocable shares and the threat point (lower bound) *LB* of minimally allocable shares in the cooperation gain *G*, but that is determined in a specific ("χ-value-specific") manner regarding those profit shares:

$$\forall G \in \mathbb{R}_{>0} \forall LB \in \mathbb{R}^{N}_{\geq 0} \forall UB \in \mathbb{R}^{N}_{\geq 0} :$$

$$\left(LB = \begin{pmatrix} v^{\chi}_{1.min} \\ \ldots \\ v^{\chi}_{N.min} \end{pmatrix} \wedge UB = \begin{pmatrix} v^{\chi}_{1.max} \\ \ldots \\ v^{\chi}_{N.max} \end{pmatrix} \wedge G \geq \sum_{n=1}^{N} c(\{A_n\}) \right) \tag{11}$$

$$\rightarrow \exists S_\chi \in \mathbb{R}^N_{\geq 0} \exists \gamma \in \mathbb{R}_{\geq 0}:$$

$$\left(\begin{array}{c} S_\chi = \begin{pmatrix} v_1^\chi \\ \dots \\ v_N^\chi \end{pmatrix} \wedge \sum_{n=1}^N v_n^\chi = G \wedge \dots \\ S_\chi = \gamma \cdot LB + (1-\gamma) \cdot UB \wedge 0 \leq \gamma \leq 1 \end{array} \right)$$

From formula (11) it becomes immediately evident that, unlike the τ-value, the χ-value does not require the quasi-balancedness of a game anymore. Instead, the χ-value can be still applied even to an instance of the distribution problem that does not belong to the class of quasi-balanced games in its game theoretic reconstruction, but does belong to the class of slightly essential games. Therefore, the χ-value has a greater application range than the τ-value. Consequently, the χ-value is to be clearly favoured from the perspective of its application range.

However, this preference for the χ-value with regard to the usability requirement is affected by the fact that the effort of calculating the χ-value for an instance of the distribution problem is considerably greater than the effort that would arise for calculating the τ-value for the same problem. (This, however, only applies under the condition that the τ-value exists with certainty, i.e. that the problem instance belongs to the class of quasi-balanced games.) The considerably greater calculational effort of the χ-value is caused by formula (8) for the determination of the maximally allocable profit shares $v^\chi_{n.max}$. When applying this formula, all the coalitions C_m that include the partner A_n have, in general, to be taken into account. In contrast, in the case of the corresponding formula (3) on p. 227 for the maximally allocable profit shares $v_{n.max}$ of the τ-value, only the grand coalition C_0 needs to be considered.

Merely for the special case in which the instance of the cooperation gain distribution problem belongs to the class of the so-called *convex* games, the following facts can be shown: For determining the maximally allocable profit shares $v^\chi_{n.max}$ in the case of the χ-value, too, only the grand coalition C_0 has to be accounted for; cf. Bergantiños and Massó (1996, pp. 280 and 282). Consequently, for this special case $v^\chi_{n.max} = v^\tau_{n.max}$. From this it follows, through the insertion of this identity into formulae (8) and (9), that the χ-value is identical to the τ-value for all instances of the distribution problem that belong to the class of convex games.

Based on these explanations, the χ-value and the τ-value coincide on the class of convex games. In the class of non-convex, but slightly essential games, however, a "trade-off" exists between the χ-value and the τ-value. On the one hand, the χ-value has a greater application range than the τ-value, because the existence of the χ-value is guaranteed for slightly essential and non-convex games, thus also for non-quasi-balanced games, while this existence guarantee does not hold for the τ-value with regard to non-convex and non-quasi-balanced games. In this respect, the existence requirement is better fulfilled by the χ-value than by the τ-value. On the other hand, the τ-value—provided that it exists at all, i.e. the integrity condition of quasi-balancedness is fulfilled—can be calculated with less effort than the χ-value. Therefore, for the class of quasi-balanced games, the τ-value fulfills the usability requirement better than the χ-value. For these reasons, neither the χ-value nor the τ-value prove to be clearly superior to the other.

4 Conclusion and Outlook

We have presented a "programmatic demand" for how a game theoretic concept for the solution of the real problem in supply chains, of establishing a distribution of the cooperation gains that will be perceived as fair, should be justified from an economic perspective. This demand has been honored in an illustrative manner by a *justification program* in which the requirements for a game theoretic solution concept have been formulated from a firmly economic point of view. The fulfillment of these requirements yields "good reasons" for perceiving the distribution outcome that is being proposed due to a game theoretic solution concept, as fair and thus for accepting it.

This economically inspired justification program differs clearly from the usual contributions to cooperative game theory. There, solution concepts for distribution problems are either presented by means of operational formulae for calculation or are justified by means of abstract axioms. These axioms can hardly be prescribed for the real problem of interest, i.e. the problem of a fair distribution of the cooperation gains in supply chains.

It has been shown that the τ-value as well as the χ-value, as typical compromise values, mostly fulfill, but not completely so, the requirements of this justification program. The χ-value proves to be superior to the τ-value regarding the existence requirement, i.e. regarding its application range. However, the τ-value exhibits advantages compared to the χ-value regarding the usability requirement, although only if the integrity condition of quasi-balancedness is fulfilled.

Regarding future research concerning the question of how widely "convincing" solutions for the here regarded problem can be achieved by means of the solution concepts of cooperative game theory, at least three desiderata are to be emphasized.

First, it is necessary to improve the usability of the χ-value and the τ-value with respect to the condition of minimal coalition knowledge by sorting out unimportant partners and insignificant outsider coalitions. An approach concerning this matter has already been mentioned in the present paper, but would have to be further developed in order to be applicable to business practice.

Second, it would be very interesting to examine other solution concepts of cooperative game theory with regard to the extent to which they are also able to fulfill the requirements of the justification program presented here. This was not possible here due to space limitations. It shall be merely hinted that the fairness condition of a compromise value according to formula (6) on p. 230 and to formula (10) on p. 396 is not fulfilled by any other of the established solution concepts, such as the nucleolus or the Shapley value. Against this exemplary background, the authors of this paper regard it as a "challenge" to the scientific community to examine the fulfillment of the requirements of the justification program presented here in detail by as large as possible a range of solution concepts of cooperative game theory.

Third, the requirements of the presented justification program for perceiving the distribution as fair should be critically scrutinized. These requirements are neither “objectively given” nor “generally obligatory”. Surely, other requirements can also be compiled in order to operationalize the fairness of a distribution of cooperation gains. From the perspective of the authors, it is merely desirable to explicate and specify such fairness requirements in order to be able to rationally discuss alternative fairness concepts. This explication and specification postulate, however, has barely been considered up to now in most game theoretic contributions to the solution of (cooperation gain) distribution problems. Therefore, this paper should be understood as an incentive for contributing to the explicit and precise specification of the requirements to be placed on a distribution of cooperation gains (or other collectively achieved results, like, e.g. general expenses) to be perceived as fair. Only if this desideratum for this specification is fulfilled will it be possible to challenge different—especially competing—requirements for the distribution outcome to be perceived as fair in an open and unbiased discourse.

References

Bergantiños G, Massó J (1994) Notes on a new compromise value: the χ-value. Working Paper 266.94. Universitat Autonòma de Barcelona

Bergantiños G, Massó J (1996) Notes on a new compromise value: the χ-value. Optimization 38(3):277–286

Bergantiños G, Massó J (2002) The Chi-compromise value for non-transferable utility games. MMOR 56(2):269–286

Bergantiños G, Casas-Méndez B, Vázquez-Brage M (2000) A non-cooperative bargaining procedure generalising the Kalai-Smorodinsky bargaining solution to NTU games. IGTR 2(4): 273–286

Heilmann C, Wintein S (2015) How to be fairer. Synthese. doi:10.1007/s11229-015-0967-y

Hofmann E, Wessely P (2010) Quantifizierung des Wertbeitrags von supply chain-initiativen in Zulieferer-Abnehmer-Beziehungen. In: Bogaschewsky R, Eßig M, Lasch R, Stölzle W (eds) Supply management research—Aktuelle Forschungsergebnisse 2009. Gabler, Wiesbaden, pp 97–123

Jene S (2015) Die faire Verteilung von Effizienzgewinnen in Kooperationen—Eine kritische Analyse der Eignung des τ-Werts und des χ-Werts. Springer Gabler, Wiesbaden

Jene S, Zelewski S (2011a) Distributive justice in supply chains—fair distribution of collectively earned profits in supply chains. In: Sucky E, Asdecker B, Dobhan A, Haas S, Wiese J (eds) Logistikmanagement—Herausforderungen, Chancen & Lösungen, Band III. University of Bamberg Press, Bamberg, pp 115–132

Jene S, Zelewski S (2011b) Fair distribution of profit in supply chains. In: Kersten W, Blecker T, Jahn C (eds) International supply chain management and collaboration practices. Eul, Lohmar, pp 299–313

Jene S, Zelewski S (2012) Fair distribution of added values in networks of autonomous actors. In: Jodlbauer H, Olhager J, Schonberger RJ (eds) Modelling value. Selected papers of the 1st international conference on value change management. Springer, Berlin, pp 167–186

Jene S, Zelewski S (2013) Practical application of cooperative solution concepts for distribution problems: an analysis of selected game theoretic solution concepts from an economic point of view. In: Meunier F, Carre F (eds) Game theory—economics, theoretical concepts and finance applications. Nova Science Publishers, New York, pp 25–49

Mitroff II, Featheringham TR (1974) On systemic problem solving and the error of the third kind. Behav Sci 19(6):383–393

Otto A, Obermaier R (2009) How can supply networks increase firm value? A causal framework to structure the answer. Logist Res 1(3–4):131–148

Sánchez-Soriano J (2000) A note on compromise values. MMOR 51(3):471–478

Zelewski S (2009) Faire Verteilung von Effizienzgewinnen in Supply Webs—ein spieltheoretischer Ansatz auf der Basis des τ-Werts. Logos, Berlin

Zelewski S (2017) Fair distribution of cooperation gains in supply chains—a justification program from an economic point of view. In: Mueller D, Trost R (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp 233–257

Zelewski S, Jene S (2011) Fairness: a challenge for the distribution of cooperation gains in value chains. Int J Manag Value Supply Chains 2(1):1–20

Zelewski S, Peters ML (2010) Fair distribution of efficiency gains in supply networks from a cooperative game theory point of view. Int J Inf Syst Supply Chain Manag 3(2):1–24

Zelewski S, Peters ML (2012) Fair distribution of efficiency gains in supply networks from a cooperative game theory point of view. In: Wang J (ed) Information technologies, methods, and techniques of supply chain management. Business Science Reference, Hershey, pp 149–171

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The Usability and Suitability of Allocation Schemes for Corporate Cost Accounting

David Mueller

Abstract Several instruments may be employed by accountants to assist management's operations. These include models of game theory. Game theoretic solution concepts are based on different argumentations, from which follow different properties and—different results. From the viewpoint of management accounting, the question arises whether these allocation schemes are suitable for solving corporate cost allocation problems. Therefore, this paper provides a detailed literature review of the crucial properties of some well-known solution concepts. This serves as the basis for the evaluation of these solutions with respect to management accounting purposes.

Keywords Aggregate monotonicity • Coalitional monotonicity • Core selection • Dutta-Ray solution • Egalitarian solution • Nucleolus • Populational monotonicity • Shapley value • Strong monotonicity • τ-value

Contents

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1 Fundamentals of Cost Accounting

The relevance of jointly generated costs has grown in the last decades for several reasons. On the one hand, the share of overhead costs within firms has increased significantly (Gunasekaran et al. 2005, p. 524). On the other hand, vertical and horizontal cooperative activities between firms have been multiplied, e.g. logistics, administrative routines, production facilities, or the development of product platforms (Alenius et al. 2015; Lozano et al. 2013; Meca and Sošić 2014; Roma and Perrone 2016; Timmer et al. 2013). These developments present a challenge to corporate cost accounting.

Cost accounting, as a substantial part of management accounting, provides detailed information for cost-based decisions and analyses. A costing system has to meet the following requirements:

- suitability and
- usability.

Costing systems may serve different purposes, e.g. price calculation, profitability analysis, behavioural control (Drury 2015, pp. 48–52). Therefore, the suitability of a system has to be interpreted against the background of the intended purpose. Suitability can be characterized by (Uyar and Kuzey 2016, p. 2):

- relevance to decision-making,
- level of detail and accuracy.

The allocation system has to provide that data which support and guide the decision to be made. The necessary level of accuracy and detail is determined by the intended purpose. The usability of a costing system is determined by:

- the complexity, transparency and communicability of the method and of its results,
- the compatibility of the procedure with existing structures, norms and conventions,
- its comparability and flexibility: the results should be comparable to historical data as well as to the results of other departments,
- theoretical efficiency: the existence, stability and computability of a solution procedure,
- practical efficiency: respecting the cost–benefit ratio (e.g. the costs for the set-up and the use of the model or the availability of the necessary data),
- closeness to reality (e.g. the necessary assumptions and simplifications).

All these requirements aim at the acceptance of the procedure by the participants (e.g. employees, cost centre managers, external partners), which should enable a stable cooperation.

Besides collecting, recording, classifying and analysing costs, the *assignment of costs* is of crucial importance. One of the basics of cost assignment is the costs-by-cause principle. This principle could not serve as a basis for cost allocation in the case of internal or external cooperation, as the costs of these operations are generated jointly. Therefore the costs-by-cause principle has to be replaced by the *principle of fairness*.

In the context of management accounting, the notion of fairness has been discussed for a long time (Choudhury 1990; Pazner 1977). Cooperative game theory offers several solutions which translate the term "fairness" in different ways. To give an overview and to guide the discussion we study the characteristics of selected solution concepts.

2 Characteristics of Cooperative Games and Solutions

2.1 Characteristics and Families of Cooperative Games

To model a cooperative game in a mathematical way we have to make the following assumptions concerning the behaviour of the players and the nature of their relationship (Maschler 1992, p. 594; Maschler et al. 2013, pp. 659–662):

- The players make binding and enforceable agreements to reach a target by cooperation.
- The result of the cooperation is the result of the players' actions.
- Every player tries to maximise his/her utility, there is no emotional behaviour, such as mercy or gloating.
- The result generated by a coalition is *completely transferable*, so that it can be entirely divided between the members of a coalition. Such games are referred to as transferable utility games, *TU-games*.
- Transferability of the result presupposes a *linear utility function* of each player (Aumann 1960). Without a linear utility function, it would be possible to increase or to decrease the jointly generated result by transferring the utility. Assuming linearity makes the detailed analysis of individual utility functions redundant.
- The result of every coalition is known ex ante, which implies *truthful reporting* on the part of the players as well as *certainty* concerning the data.
- There is a kind of *central authority* which collects the information as well as the cooperation's result and which applies/determines the allocation mechanism to realize the transfer.

Definition 19.1 A TU game Γ in characteristic function form is an ordered pair (N, v), where N denotes the set of players $N = \{1, 2, \ldots, n\}$ and the characteristic function v assigns a value to every coalition $v : 2^N \to \mathbb{R}$.

Not only the set of the players, N, is important here, but also all subsets of N. Such a subset $S \subseteq N$ is referred to as a coalition, whereas N itself is described as the grand coalition. Each coalition is marked by a value function $v\ (S)$. The function v assigns a value to each subset S, which represents the economic performance of this coalition. The result of an empty coalition is zero, therefore: $v\{\emptyset\} = 0$. The set of all cooperative TU-games with the player set N will be denoted by Γ^N.

Cooperation may be successful or not. To concretise the term “success”, some *desirable properties* of games may be defined. One goal at which every cooperation aims is the generation of a result which is not worse than the results of isolated actions. This is referred to as *superadditivity*.

Definition 19.2 A game (N, v) is superadditive if $v(R \cup S) \geq v(R) + v(S)\ \forall\ R, S \subseteq N$ *with* $R \cap S = \emptyset$.

Looking at the above definition we want to analyse these situations in which for all coalitions yields $v(R \cup S) = v(R) + v(S)$. Obviously the result of the grand coalition will be $v(N) = \sum_{i \in N} v(\{i\})$. From this relation neither results a motivation for forming the grand coalition nor arises any distribution problem. That’s why we concentrate on situations in which for at least one coalition yields $v(R \cup S) > v(R) + v(S)$. In consequence, the grand coalition generates a better result than the sum of all stand-alone coalitions.

Definition 19.3 A game (N, v) is essential if $v(N) > \sum_{i \in N} v(\{i\})$.

By E^N we denote the set of all essential TU-games with the set of players N. In the following essential games only are analysed. Superadditivity describes the relationship of coalitions of disjoint elements. A similar effect may be claimed for coalitions of conjoint elements. This is called convexity.

Definition 19.4 A game (N, v) is convex if $v(S \cup \{i\}) - v(S) \leq v(R \cup \{i\}) - v(R)$ for all $S \subseteq R \subseteq N \setminus \{i\}$.

This definition claims that a greater coalition generates a better result. It becomes apparent, that a convex game is superadditive. This class of games was introduced by Shapley (1971). We will denote the set of all convex TU-games by C^N. This class is an interesting class as it “…expresses a sort of increasing marginal utility for coalition membership…” (Shapley 1971, p. 6) and mirrors a desirable characteristic.

Beside these properties, the property of balancedness has to be introduced. To do so, we use the (0,1)-vector z_S with $z_S = (z_S(1), z_S(2), \ldots . z_S(n))$ to describe whether a player $\{i\}$ belongs to a coalition or not (Peleg and Sudhölter 2007, p. 28) as follows:

$$z_S(i) = \begin{cases} 1, \textit{if}\ i \in S \\ 0, \textit{if}\ i \in N \setminus S \end{cases}$$

In the following we refer to a set of non-empty coalitions as a collection of coalitions $\mathbb{B}$ with $\mathbb{B} = \{S_1, S_2, \ldots, S_m\}$. Moreover we use a weighting factor α_j with $j = 1, \ldots, m$, which mirrors the share of the coalition S_j in the overall activities of player $\{i\}$.

Definition 19.5 A collection $\mathbb{B}$ of non-empty subsets of N is called balanced if there is an α_j with $0 \leq \alpha_j \leq 1$ for all $S \in \mathbb{B}$ such that $\sum_{j=1}^{m} \alpha_j z_{S_j}(i) = 1 \; \forall \; S_j \in \mathbb{B}$.

This concept now allows defining a balanced TU game (Shapley 1967, pp. 457–458).

Definition 19.6 A game (N, v) is called a balanced game if for each balanced collection $\mathbb{B}$ with $\alpha_j \ldots .\alpha_m > 0$ one has $\sum_{j=1}^{m} \alpha_j v(S_j) \leq v(N)$.

Every balanced game has a non-empty core (Bondareva 1963; Shapley 1967). By B^N we denote the set of all balanced TU-games with the set of players N. The balancedness of a game may be extended to all of its subgames.

Definition 19.7 A *subgame* of (N, v) is a game (T, v^T), where $\emptyset \neq T \subseteq N$ and $v^T = v(S)$ for all $S \subseteq T$. This subgame will be denoted by (T, v).

If all subgames of a game (N, v) are balanced, then the game (N, v) is called *totally balanced* (Peleg and Sudhölter 2007, p. 33).

Beside these properties of games, some special classes of games have been established. Of special interest for the following are *cost games*, also called *cost saving games*. In these games the function v is replaced by the function c which assigns a cost value to each subset S. In cost games, the superadditivity property of characteristic function is replaced by the *subadditivity* property (Young 1994, pp. 1197–1198; Fiestras-Janeiro et al. 2011, p. 4).[1] Assuming fixed revenues, the subadditivity of costs leads to the superadditivity of profits.

A special type of cost game is the *airport game*, introduced some decades ago (Littlechild and Owen 1973; Littlechild and Thompson 1977). The problem which has to be solved is a fair sharing of the cost of a landing strip between different airlines. We assume (González-Díaz et al. 2010, p. 256) m different types of aircraft and the costs of constructing and maintain a runway for the aircraft of type k is denoted by c_k with $1 \leq k \leq m$. N_k denotes the set of type k aircraft that are landing, and $N = \bigcup_{k=1}^{m} N_k$ denotes the set of aircraft landings in a period of time.

The airlines need runways of different lengths, which are determined by the type of the aircraft. The cost of a runway is essentially determined by the largest aircraft (Peleg and Sudhölter 2007, p. 15; González-Díaz et al. 2016, p. 106).

Definition 19.8 The cost function of an airport game with player set N is defined by $c(\emptyset) = 0$ and $c(S) = max \{c_k | S \cap N_k \neq \emptyset\}$.

By A^N we denote the set of all airport games with player set N. We can interpret airport games as special types of cost games for constructing or maintaining a facility, where the preferences of the coalitions are linearly ordered. Airport games are concave and have, therefore, a non-empty core (Potters and Sudhölter 1999, p. 88; González-Díaz et al. 2016, p. 107).

[1]Using this procedure for designing cost savings games for public goods is criticised by Hougaard 2017, p. 291.

2.2 Properties of a Fair Solution

Looking at a game, the question arises of how to share the jointly generated result between the partners, which is equivalent to an allocation of the result.

Definition 19.9 A function f which assigns to a game (N, v) a, possible empty, subset $f(v)$ of $\mathbb{R}^n$ is called a solution concept.

The function f distributes $v(N)$ and generates a payoff vector $x = (x_1, x_2, x_3 \dots x_n)$ with $x \in \mathbb{R}^n$. Such a function is referred to in an corporate setting as allocation scheme.

The first and very crucial property of a solution concept is that this solution exists. This may seem to be a triviality, but the following discussion will show that the existence of a solution is not a matter of course. With respect to applications in an economic environment it is desirable that the allocation scheme would generate one solution.

Definition 19.10 A solution f is a single-valued solution if $|f(v)| = 1$ for every v.

With the allocation of the jointly generated result, the problem of fairness arises. "In any collaboration, rules for how much each firm should contribute and for how the profits are to be shared are difficult to establish. Firms may each feel that they contributed more than any other…" (Tripsas et al. 1995, p. 371). Only a fair distribution ensures a stable and successful cooperation.

The main aim of cooperative game theory is the identification of a solution mechanism which leads to a fair allocation of the jointly realized outcomes. The crucial question of *how to define* and how to quantify *fairness* is answered in cooperative game theory by demanding *desirable properties* of a solution. These properties mirror a common understanding of fairness. "In this connection we emphasize again that any game is a model of a possible social or economic organization and any solution is a possible stable standard of behavior in it." (von Neumann and Morgenstern 1944, p. 436). Zelewski identifies six requirements for a fair allocation scheme: rationality, uniqueness, existence, acceptability, communicability and practicability (Zelewski 2017, p. 216). The most important and most relevant properties which have been established in this field are presented in the following.

A basic requirement is that the solution mechanism distributes the whole profit of the game. This is called *efficiency* (EFF).

Definition 19.11 A single-valued solution f is efficient if $\sum_{i \in N} f_i(v) = v(N)$.

Another fundamental basic requirement for a fair solution is *individual rationality* (IR). This criterion means that no single player can improve, without cooperation, the payoff proposed by the solution mechanism.

Definition 19.12 A single-valued solution f fulfils IR if $f_i(v) \geq v(\{i\}) \;\; \forall \; i \in N$.

The sum of these requirements leads to payoffs which are called *imputations* (von Neumann and Morgenstern 1944, pp. 34–37).

Definition 19.13 The set of imputations $I(v)$ of a game (N, v) is defined by

$$I(v) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \textit{ and } x_i \geq v(\{i\}) \ \forall \, i \in N \right\}.$$

In the class of essential games the set of imputations is never empty, i.e. $I(v) \neq \emptyset$. For solving a distribution problem, only those imputations that are not dominated by another imputation are of interest. The set of non-dominated imputations forms the core (Gillies 1959; González-Díaz et al. 2010, p. 218).

Definition 19.14 The core $Core(v)$ of a game (N, v) is defined by

$$Core(v) = \left\{ x \in I(v) | \sum_{i \in S} x_i \geq v(S) \forall \, S \subseteq N \right\}.$$

The core of a game contains all solutions which are justified as fair and, therefore, are stable. It may be small, very large, or empty. The core of a convex game is never empty (Shapley 1971, p. 24). With respect to an allocation scheme, it would be a desirable property if the scheme would identify only these solutions which belong to the non-empty core. In other words: we want to indicate if the mechanism is able to identify the core-solution whenever the core is not empty. We call this property core selection (CS). A solution satisfies core selection if it selects a core element for any game with a non-empty core (Housman and Clark 1998, p. 611; Calleja et al. 2012, p. 901).

Definition 19.15 A single-valued solution f satisfies CS if $f(v) \in Core(v)$ for all $v \in B^N$.

The core as a solution concept does not deliver a detailed suggestion, but indicates whether there exists one stable solution or even a set of stable solutions. With respect to cost allocation schemes, it has been proved that a distribution of overhead costs which is based on *activity-based costing (ABC)* always lies in the core, whereas the *functional-based costing (FBC)* does not. FBC assignments can fail to be imputations and even when they are imputations it is not assured that these are core elements. As a result, FBC assigns too high a cost to some products and too low a cost to other products (Charles and Hansen 2008, p. 292).

Another requirement for a fair solution results from the fact that only players who contribute to the result deserve and should receive a profit share. Those players who do not contribute, do not deserve any share of it, and are called *dummy players*.

Definition 19.16 A player i is called a dummy player if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N$ with $i \notin S$.

A dummy player deserves only that result which he would generate in a stand-alone coalition. That is why a fair solution should not reward these players, which is referred to as the *dummy player property* (DPP).

Definition 19.17 A single-valued solution f satisfies the DPP if $f_i(v) = v(\{i\})$.

Cooperation with such players does not offer a coalition any synergetic gain. Moreover it has to be stated that only the economic performance determines the allocation of the gain. That means that neither the name nor the origin of the players have an influence on their profit shares. This property is referred to as anonymity, but is referred to as symmetry as well. We use the term *equal treatment property* (ETP) in the following (Shapley 1953, p. 309; Peleg and Sudhölter 2007, p. 91; Curiel 1997, p. 9; Fiestras-Janeiro et al. 2011, p. 4).

Definition 19.18 A single-valued solution f satisfies ETP if for the players i and j, for which holds: $v\ (S \cup \{i\}) = v\ (S \cup \{j\})\ \forall\ S \subseteq N$ with $i, j \notin S$ yields: $f_i(v) = f_j(v)$.

Looking at several possibilities of forming games and subgames, the allocation mechanism must prevent the increase or decrease of the allocation of the players by splitting one game into two games. This is expressed by the additivity axiom (ADD), because adding the solutions of two games has to generate the same result as the sum of these games ("law of aggregation") (Shapley 1953, p. 309).

Definition 19.19 A single-valued solution f satisfies ADD if for any two games v, w follows $f(v + w) = f(v) + f(w)$.

This property was criticised early on (Luce and Raiffa 1957, pp. 250–252). It is important to assume that the same set of players is playing the two separate games, so that these games may be regarded as a single game. To discuss this property, we analyse a production process for which N managers are responsible. All these managers deserve, as an incentive,[2] part of the annual profit of this division, which is $u(N)$. Consider manager i, so we can determine his incentive amount, denoted by $f_i(u)$.

Assume that this division is split into two centres, e.g. to increase the accuracy of the calculation. We denote the relationship of these games as $(N, u) = (N, v) + (N, w)$. The production process remains the same, as well as the responsibilities. Therefore we can say the managers are playing a split game $(N, v) + (N, w)$. The incentive of manager i should be $f_i(v + w)$, which is equal to $f_i(u)$. The incentive of manager i does not depend on the formal structure of the production process but only on the real result. The very crucial question remains, if these games are indeed separate from each other (Spinetto 1975, p. 486).

Looking at the costs of the firm and the allocation of these costs to its managers, it has to be stated that an allocation mechanism is fair if changing the costs in one direction—rising or falling—leads to changing the managers' allocation in the same direction. This is called *monotonicity*. The following main types of monotonicity have been established in the last decades:

- aggregate monotonicity,
- coalitional monotonicity,
- strong monotonicity, and
- population monotonicity.

[2]For a deep and broad discussion of incentives in this volume cf. Trost and Heim (2017).

Aggregate monotonicity demands that an increasing value of the game must lead to a non-decreasing payoff for the players. This frequently encountered type of monotonicity was introduced very early (Megiddo 1974). Aggregate monotonicity refers to the value of the grand coalition and seems to be a very crucial and intuitive demand (Young 1985, p. 66; Megiddo 1974, p. 355; Hokari 2000, p. 135).

Definition 19.20 A single-valued solution f is referred to as an aggregate monotonic allocation scheme (AMAS) if for all games v, w with $v(N) > w(N)$ and $v(S) = w(S)$ for all $S \subsetneq N$ there follows $f_i(v) \geq f_i(w)$ for all $i \in N$.

In an economic context, it follows from this property that an increasing profit of the whole company has to generate at least a non-decreasing payoff for all managers. With respect to the core selection property, it has to be stated that *core selection* and *aggregate monotonicity* are compatible for *all games* (Calleja et al. 2009, p. 742).

Besides an increasing result of the grand coalition, i.e. the whole company, it is possible that the values of different coalitions, e.g. cost centres or profit centres, increase. In these cases it has to be demanded that every player which belongs to such a coalition receives not less than before. As the reference points are the values of the coalitions, this demand is called *coalitional monotonicity*. This idea was introduced by Shubik with respect to the allocation of joint costs and profits (Shubik 1962). "There should always be an incentive for a manager to implement an efficiency or report a new idea if it benefits the firm as a whole no matter what changes it may cause to take place in his own operation" (Shubik 1962, p. 328). With respect to this manager and his contribution to the overall profit, a fair incentive mechanism should not damage the manager's individual profit.

An allocation mechanism satisfies coalitional monotonicity if an increase in the worth of a particular coalition implies no decrease in the allocation to any member of that coalition. This demand is stronger than aggregate monotonicity, and is the most intuitive expression of the idea that a higher result leads to higher payoffs for the players (Young 1985, p. 68; Housman and Clark 1998, p. 611).

Definition 19.21 A single-valued solution f is referred to as a coalitional monotonic allocation scheme (CMAS) if for all games v, w with satisfying $v(T) > w(T)$ for some coalition T and $v(S) = w(S)$ for all S, $S \neq T$ it follows that $f_i(v) \geq f_i(w)$ for all $i \in T$.

We have to point out that the core-selection property and the coalitional monotonicity property are incompatible to some extent (Hougaard et al. 2005, p. 432). Young proved that there does not exist any core allocation which satisfies coalitional monotonicity for games with $|N| \geq 5$ (Young 1985, p. 69). Some years later it was shown that for games with $|N| \geq 4$, there is no core allocation scheme that fulfils coalitional monotonicity (Young 1994, p. 1211; Housman and Clark 1998, p. 612; Hougaard 2009, p. 85). Maschler (1992, p. 614) stated: "There is no escape from this fact: if you want a unique outcome in the core you must face some undesirable nonmonotonicity consequences." But in the class of convex games two solutions - the Shapley value (cf. section 3.1) and the Dutta-Ray-solution (cf. section 3.4) - fulfil as well core selection as coalitional monotonicity (Hokari 2000, p. 332; Hougaard et al. 2005, p. 432; Arin and Katsev 2016, p. 1026). For a deeper discussion of the relationship between coalitional monotonicity and core stability, cf. Arin 2013; Arin and Feltkamp 2012; Arin and Katsev 2016 as well as Arin and Katsev 2017, p. 313.

Coalitional monotonicity refers to absolute changes in the values of coalitions which contain player *i*. With respect to the relative changes of the coalitional values, it has to be required that the same relationship holds. This means that if the values of the coalitions containing player *i* increase relative to the values of the coalitions not containing *i*, then this player should not be penalized. This is covered by *strong monotonicity*, which is referred to as contributional monotonicity as well (Young 1985, p. 69; Hokari and van Gellekom 2002, p. 597).

Definition 19.22 A single-valued solution f is referred to as a strongly monotonic allocation scheme (SMAS) if for all v, w it holds that $m_i v(S) \geq m_i w(S)$ for all $S \subseteq N$ leads to $f_i(v) \geq f_i(w)$. The marginal contribution m_i is defined by

$$m_i v(S) = \begin{cases} v(S) - v\,(S\backslash\{i\}) \ \ \textit{if } i \ \in S \\ v\,(S \cup \{i\}) - v(S) \ \textit{if } i \ \notin S. \end{cases}$$

This kind of monotonicity claims that higher individual marginal contributions imply a higher pay-off. With respect to the axiomatisation of the Shapley value, Young proved that strong monotonicity replaces both the *dummy player and additivity properties* (Young 1985, p. 71; Gilles 2010, p. 81; Carreras and Owen 2013, p. 701).

Now we consider a game with a given number of players, which is extended by some additional players, so that we are enlarging the original game. This kind of monotonicity is called *populational monotonicity* and was introduced by Thomson as "monotonicity with respect to changes in the number of agents" (Thomson 1983a, p. 321; Thomson 1983b, p. 214). In such a constellation, it should be claimed that the payoffs of the original players should be affected in the same direction (Sprumont 1990, p. 379; Grafe et al. 1998, p. 72). The criterion for evaluating the populational monotonicity of a solution concept is "... if they yield allocations for the augmented games that are not inferior, for any player, to the allocation generated for the original game" (Rosenthal 1990, p. 46).

Definition 19.23 A single-valued solution f is called a populational monotonic allocation scheme (PMAS) if $\sum_{i \in S} f_i(S) = v(S)\ \forall\ S \in 2^N$ and $f_i(S) \leq f_i(T)\ \forall\ S, T \in 2^N$ with $S \subseteq T$ and $i \in S$.

This condition claims that each player of T does not receive more in any subcoalition S in which he would take part (Getán and Montes 2010, p. 495). Looking at the process of coalition formation, it is imaginable that some players may not necessarily achieve full efficiency. If the game is superadditive, it is efficient to form the grand coalition. Populational monotonicity specifies how to allocate the value $v(S)$ of every coalition $S \in 2^N$. It can be noticed that the total balancedness of a game is a *necessary condition* for the existence of a PMAS, i.e., the class of games with an PMAS is included in that of totally balanced games (Sprumont 1990,

p. 380; Brânzei et al. 2008, p. 45). Total balancedness is a *sufficient condition* for an existing PMAS for games with at most three players. The total balancedness does not guarantee the existence of an PMAS if the number of players is at least four (Sprumont 1990, p. 381)

Moreover, we can state that the convexity of a game is a sufficient condition for the existence of a PMAS (Sprumont 1990, p. 382; Brânzei et al. 2008, p. 49), as the populational monotonicity of marginal contribution values follows from the characteristics of convex games. Population monotonicity in combination with efficiency imply the dummy-player property (Hokari and van Gellekom 2002, p. 598).

We can summarize that the properties "existence and single-valuedness" and "core selection" mirror the theoretical efficiency of a solution concept whereas the remaining properties are measurements of fairness.

We want to analyse some solution concepts of cooperative game theory against the background of these properties. Several solution concepts have been elaborated in the last decades (for an overview, cf. Kruś and Bronisz 2000; Fiestras-Janeiro et al. 2011). From among these concepts, the Shapley value, the nucleolus, the τ-value, and the egalitarian solution will be discussed in the following.

3 Selected Solutions and Their Properties

3.1 The Shapley Value

In order to determine a fair share for player i, the following thought is worth noting: each player receives a part depending on that player's contributions to the theoretically possible, thus imaginable, coalitions. The contribution of the player consists in the increase in value caused by that player's participation in the coalition. The question that has to be answered is which value the coalition has with player i and which it would have without player i. This difference is called the marginal contribution (cf. p. 410). Assuming all orders of forming a coalition to have the same probability results in the weighted average of the marginal contributions of a player, which is commonly described as the Shapley value (Shapley 1953, p. 311; Peleg and Sudhölter 2007, p. 154).

Definition 19.24 The Shapley value of a player i in a game (N, v) is defined by
$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-1-|S|)!}{n!}[v(S \cup \{i\}) - v(S)].$$

It has been shown that the Shapley value is the unique value which satisfies (Shapley 1953, pp. 311–312; Peleg and Sudhölter 2007, pp. 152–154):

- efficiency,
- equal treatment property,
- additivity, and
- dummy player property.

Using the property of strong monotonicity, it can be concluded that the Shapley value is the unique value which satisfies (Young 1985, p. 70; Peleg and Sudhölter 2007, pp. 156–158; Gilles 2010, pp. 80–82):

- efficiency,
- equal treatment property, and
- strong monotonicity.

The main problem with the Shapley value is the fact that it does belong only in convex games always to the non-empty core (Shapley 1971, p. 29; Sudhölter and Peleg 1998, p. 386). If the game is not convex, it is possible that the Shapley value does not belong to the core. In such a case, it is not a stable and fair solution.

With respect to airport games (cf. p. 405) the proposed solution procedure for this class of games is as follows (Littlechild and Owen 1973, p. 370; González-Díaz et al. 2010, p. 257). The first step consists in dividing the cost of catering to the smallest type of aircraft equally among the number of landings of all aircraft. In the next step the incremental costs of catering for the second smallest type of aircraft (i.e., above the cost of the smallest type) are divided equally among the number of landings of all but the smallest type of aircraft. Continue thus until finally the incremental cost of the largest type of aircraft is divided equally among the number of landings made by the largest type of aircraft. This sequential allocation procedure leads to the Shapley value of an airport game. We summarize that the Shapley value:

- exists as a singleton for each $v \in \Gamma^N$(Shapley 1953, pp. 311–312),
- satisfies CS for each $v \in C^N$(Shapley 1971, p. 29),
- fulfils IR for each $v \in E^N$ (Moulin 1991, p. 122),
- satisfies for each EFF, DPP, ETP and ADD for each $v \in \Gamma^N$ by definition (Shapley 1953, p. 309),
- is an AMAS for each $v \in \Gamma^N$ (Megiddo 1974, p. 355),
- is a CMAS for each $v \in \Gamma^N$ (Shubik 1962, p. 336; Young 1985, p. 69; Maschler 1992, p. 614),
- is a SMAS (Young 1985, p. 70; Gilles 2010, p. 82),
- is a PMAS for each $v \in C^N$ (Sprumont 1990, p. 382; Rosenthal 1990, pp. 49–50; Hokari 2000, p. 328),
- ensures compatibility of coalitional and populational monotonicity on the one hand and core selection on the other hand for each $v \in C^N$ (Rosenthal 1990, p. 50; Hougaard et al. 2005, p. 432; Arin and Katsev 2016, p. 1026).

The number of publications on the Shapley value is nearly unmanageable. It is the most important single-valued concept of cooperative game theory and therefore has been employed several times for fair sharing (for early treatments, cf. Littlechild and Owen 1973; Loehman and Whinston 1976; Suzuki and Nakayama 1976; Hamlen et al. 1980; Tijs and Driessen 1986; Rideout and Wagner 1988; Moulin 1992. For an early discussion cf. the volume edited by Roth 1988a and for overviews of business applications cf. Winter 2002; Zelewski 2009, p. 43; Serrano 2013; Guajardo and Rönnqvist 2016, p. 380).

Based on the Shapley value, the Aumann–Shapley mechanism and some related pricing schemes have been developed to solve the problem of cost allocation (Aumann and Shapley 1974, pp. 175–251; Moulin 2002; Friedman 2012). These approaches presuppose known cost functions for production that are continuously differentiable, which excludes fixed costs. That's why we do not consider these approaches in the following.

3.2 Nucleolus

Another well-known and important solution concept of cooperative game theory is the nucleolus, which was introduced by Schmeidler in 1969. The nucleolus involves searching for a fair distribution by minimizing the maximal dissatisfaction of every player. For this, the dissatisfaction of a coalition with a concrete payoff vector is referred to in this connection as the excess (Moulin 1991, pp. 121–123; Driessen 1988, pp. 37–38). What has to be calculated is how unhappy a coalition would be with a payoff vector (Schmeidler 1969, p. 1163).

Definition 19.25 In a game (N, v), the excess of a coalition $S \subseteq N$ with respect to a payoff vector $x \in I(v)$ is defined by $e(S, x) = v(S) - \sum_{i \in S} x_i$.

To derive the nucleolus, the payoff vectors with the highest unhappiness for every player are sought in the next step. To do so, these excess values are sorted in nonincreasingly order (González-Díaz et al. 2010, p. 231). The excess of a coalition S_i with respect to a payoff vector x is denoted by $e(S_i, x) = \theta_i(x)$.

Definition 19.26 The vector of nonincreasingly ordered excess values $\Theta(x)$ is defined by $\Theta(x) = (\theta_1(x); \theta_2(x); \theta_3(x); \ldots .; \theta_{2^n}(x))$ with $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq 2^n$.

To compare two payoffs, their vectors of nonincreasingly ordered excess values are compared based on the lexicographic order. The vector which is lexicographically smaller than the other one is chosen as this vector offers the minimum of the maximal dissatisfaction for all players resulting from the two payoffs.

Definition 19.27 Vector $\Theta(x)$ is lexicographically smaller than vector $\Theta(y)$ if there is an index number m such that *a)* $\theta_i(x) = \theta_i(y)\ \forall\ 1 \leq i < m$ and *b)* $\theta_m(x) < \theta_m(y)$.

In this case, it is denoted by $\Theta(x) <_{Lex} \Theta(y)$. By $\Theta(x) \leq_{Lex} \Theta(y)$ we express that either $\Theta(x) = \Theta(y)$ or $\Theta(x) <_{Lex} \Theta(y)$. With these explanations, the nucleolus of a game can be defined (Curiel 1997, p. 12; Peleg and Sudhölter 2007, p. 83; Maschler et al. 2013, pp. 802–805).

Definition 19.28 In a game (N, v) with $v \in E^N$ the nucleolus $nuc(v)$ is defined by

$$nuc(v) = \{x \in I(v) | \Theta(x) \leq_{Lex} \Theta(y)\ \forall y \in I(v)\}.$$

Another interpretation is the following: an arbitrator has to bargain and to decide how the result of a coalition will be divided. To do so, he generates a proposal, which will be discussed by the players. Every player calculates its excess regarding this first proposal. The player for which results the largest excess has the greatest incentive to leave the coalition. To avoid this, in the second step, the arbitrator tries to minimise the highest value of unhappiness by suggesting another allocation. This second allocation proposal increases the portion of the player with the highest excess at the expense of the other members of the coalition. Based on this, the values of unhappiness are calculated again and the player with the highest unhappiness will be identified. This procedure is carried out until there is no possibility of reducing the value of unhappiness for all coalitions.

The main advantage of the nucleolus, in comparison with the Shapley value, is that it always belongs to the core if the core is not empty (Schmeidler 1969, p. 1165).

We can summarize, that the nucleolus is a rule for identifying a specific imputation. From this follows, that the nucleolus exists, if the set of imputations is not empty (González-Díaz et al. 2010, p. 233). We have seen, this is the case for superadditive games and for essential games (cf. p. 407). For any superadditive game the nucleolus exists and is a singleton (Chalkiadakis et al. 2012, p. 32).

It has been shown very early for a non-balanced game that the nucleolus does not fulfil aggregate monotonicity (Megiddo 1974). The open question if the nucleolus fulfils this property in the class on convex games was studied by Hokari. He proved that for convex games with $|N| \geq 4$ the nucleolus does not fulfil aggregate monotonicity (Hokari 2000).

It can be stated that the nucleolus does not fulfil populational monotonicity for the class of convex games but it does for the class of airport games (Sönmez 1994, pp. 4–5). This is due to the fact that airport games are a subclass of convex games. We conclude that the nucleolus:

- exists as a singleton for each $v \in E^N$ (Schmeidler 1969, pp. 1165–1166; Driessen 1988, pp. 38–47; Maschler et al. 1979, pp. 331–335),
- satisfies CS for each $v \in E^N$ (Schmeidler 1969, pp. 1165–1166),
- fulfils EFF, IR, DPP and ETP for each $v \in E^N$ (Schmeidler 1969, pp. 1163–1164; Maschler et al. 1979, p. 335),
- does not fulfil ADD for each $v \in E^N$ (Maschler et al. 2013, p. 812),
- is neither for each $v \in E^N$ nor for each $v \in C^{|N| \geq 4}$ an AMAS (Megiddo 1974, p. 358; Hokari 2000, p. 135),
- is not an CMAS for each $v \in C^N$ (Hougaard et al. 2005, p. 432),
- is not an SMAS for each $v \in C^N$ (Young 1985, p. 70; Gilles 2010, p. 82),
- is neither for each $v \in C^N$ a PMAS nor for each $v \in E^N$, but is a PMAS for each $v \in A^N$ (Sönmez 1994, pp. 4–5; Potters and Sudhölter 1999, p. 95; Hokari 2000, p. 328; Grafe et al. 1998, p. 76).

The nucleolus is well established in the field of game theory, so that a wide range of applications have been developed (cf. Bhakar et al. 2010; Massol and Tchung-Ming 2010; Songhuai et al. 2006; Stamtsis and Erlich 2004), also in the area of cost and profit management (cf. Frisk et al. 2010; Guajardo and Rönnqvist

2016; Homburg and Scherpereel 2008; Kimms and Çetiner 2012; Legros 1986; Littlechild and Thompson 1977; Mueller 2016; Rideout and Wagner 1988; Suzuki and Nakayama 1976, p. 380).

3.3 The τ-Value

The concept of the τ-value was developed some time ago (Driessen and Tijs 1982; Tijs 1981) and is alternatively described as the Tijs value (Bilbao, 2000, p. 6). This concept has been known in the research field of game theory for many years (Driessen and Tijs 1984; Driessen 1988; Tijs and Driessen 1986) but has only recently been put to use for solving management issues.

The τ-value is characterized by the fact that it has been developed on the basis of a bargaining situation. In this concept, the first step specifies an upper and lower limit. The upper limit is set as the vector of the marginal contributions of the players to the grand coalition. No higher payoff is granted to the player than the value which it contributes by its participation in the grand coalition (Tijs 1981, pp. 123–124; Driessen 1988, p. 57).

Definition 19.29 In a game (N, v) the upper vector b is defined by $b_i = v(N) - v(N \setminus \{i\})\ \forall\ i \in N$.

The i-th coordinate b_i of this vector—which is referred to as utopia vector too—is the marginal contribution of player i with regard to the grand coalition (Tijs 1981, p. 124). To determine the lower limit, the following consideration is used: in the case that the player i does not participate in the grand coalition, there is the opportunity for i to participate in another coalition or form the so-called outsider coalition. In this constellation, i receives not less than that amount with which it is able to credibly threaten to receive by founding at least one outsider coalition. However, in order to attract other players to the outsider coalition, the player i has to offer each of those members at least the amount they would be able to achieve in the best case within the grand coalition. The amount resulting after these side payments remains for i and represents the lower limit, also referred to as the threat point or concession limit. Player i would strive for that outsider coalition in which the residual income is maximum (Driessen and Tijs 1982, p. 3; Mueller 2016, p. 206).

Definition 19.30 In a game (N, v), the lower vector a is defined by

$$a_i = \max_{S \subseteq N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} b_j \right\} \forall i \in N.$$

In an outsider coalition S, into which i might enter, the remaining companies could each maximally claim their marginal contributions. In the worst case, the payment would be due to i. Player i rationally strives for that coalition in which this difference is maximal.

The vectors a and b do not necessarily represent imputations. If there is to be an imputation between these two vectors, the class of quasi-balanced games has to be introduced.

Definition 19.31 A game (N, v) is quasi-balanced if $\sum_{i \in N} a_i \leq v(N) \leq \sum_{i \in N} b_i$ and $a_i \leq b_i \; \forall \; i \in N$.

By QB^N we denote the set of all quasi-balanced games with player set N. For quasi-balanced games, one clearly determined imputation exists, which is between the upper and the lower vector, and which is referred to as the τ-value (Tijs 1981, p. 127) .

Definition 19.32 The τ-value of a quasi-balanced game (N, v) is defined by

$$\tau = a + \lambda(b - a), \text{ with } \lambda = 0, \; \textit{if} \;\; a = b, \;\; \text{otherwise: } \lambda = \frac{v(N) - \sum_{i \in N} a_i}{\sum_{i \in N} b_i - \sum_{i \in N} a_i}.$$

Balancedness implies quasi-balancedness (Driessen 1988, p. 62). With respect to the *core-selection property*, we point out that the τ-value lies in the core of every quasi-balanced game with two or three players (Tijs 1981, p. 128; Driessen and Tijs 1985, p. 234; Driessen 1988, p. 65). For necessary and sufficient conditions that the τ-value satisfies core-selection property for games with more players, we refer to the appropriate literature (Driessen 1988; Driessen and Tijs 1985, pp. 62–70).

The field of applying the τ-value is remarkable (Zelewski 2017, pp. 223–224; Zelewski 2009; Mueller 2016) but compared with the Shapley value and with the nucleolus, there has been less attention and fewer applications. We can conclude that the τ-value:

- exists as a singleton for each $v \in QB^N$ (Tijs 1981, p. 127),
- ensures CS for each $v \in QB^2$ and each $v \in QB^3$, but does not satisfy CS for each $v \in C^N$ (Tijs 1981, p. 128; Driessen and Tijs 1985, p. 234; Driessen 1988, p. 65),
- fulfils IR, EFF, DPP and ETP for each $v \in QB^N$ (Tijs 1981, p. 127; Driessen and Tijs 1982, p. 4; Driessen and Tijs 1985, p. 232),
- does not fulfil ADD for each $v \in QB^N$ (Tijs 1981, p. 128),
- is not an AMAS for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600; Tijs and Driessen 1986, p. 1022),
- is not a CMAS for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600),
- is not a SMAS for either each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600) or each $v \in A^N$ (Potters and Sudhölter 1999, p. 100),
- is neither for each $v \in C^N$ nor for each $v \in A^N$ a PMAS (Sönmez 1994, p. 7; Potters and Sudhölter 1999, p. 100; Hokari and van Gellekom 2002, p. 600).

Based on a similar argumentation, another solution concept, the χ-value, was developed later (cf. Bergantiños and Massó 1996, 2002). This concept is not discussed here, but is introduced elsewhere in this book (cf. Zelewski and Heeb 2017, pp. 407–429).

3.4 The Egalitarian Solution, by Dutta and Ray

This solution is grounded on the theory of egalitarianism established by Rawls (1999) and employs the concept of Lorenz dominance, which was established very early to mirror the concentration of wealth in a quantitative way (Lorenz 1905). Starting point of the argumentation is a society of n individuals in which the total income of I is distributed by the allocation x. The vector resulting from rearranging the allocation according to $(\hat{x}_1 \leq \hat{x}_2 \leq, \ldots, \leq \hat{x}_n)$ is denoted with $\hat{x}$. The vector x **Lorenz dominates** the vector y for any $x, y \in \mathbb{R}^n_+$ with $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = I$ if $\sum_{i=1}^{p} \hat{x}_i \geq \sum_{i=1}^{p} \hat{y}_i$ for all $p \in \{1, \ldots, n-1\}$ with at least one strict inequality (Moulin 1991, p. 48; Brânzei et al. 2008, p. 37). Lorenz dominance implies an allocation with less inequality.

Some concepts of egalitarian solution have been developed based on Lorenz dominance (Arin et al. 2008, pp. 569–571). We are referring here to the solution which has been developed by Dutta and Ray (Dutta 1990; Dutta and Ray 1989). They indicated "…that we do not employ the concept to build up a theory of social norms (e.g., justice) from individual values alone, as Rawls does. Rather, we use it to apply an already existing social ethic (egalitarianism) to the design of rules." (Dutta and Ray 1989, p. 615).

The Dutta–Ray solution on the class of all games has no necessary relation to the core. On the one hand it is possible that the core is not empty, but the Dutta–Ray solution does not exist. On the other hand it is possible that the core is empty, but the egalitarian solution exists. Even if the core is not empty and the Dutta–Ray solution exists, it is not assured that this solution is a part of the core (Dutta and Ray 1989, p. 618).

Therefore we restrict the following discussion to convex games (for defining the Dutta-Ray solution on the class of all games cf. Dutta and Ray 1989; Dutta 1990). For convex games, the Dutta–Ray solution can be defined by a simple expression (Hokari and van Gellekom 2002, p. 596).

Definition 19.33 The Dutta–Ray solution for all $v \in C^N$ is defined by:

$$DR(v) \equiv arg \min_{x \in Core(v)} \sum_{i \in N} \left(x_i - \frac{v(N)}{|N|} \right)^2.$$

For the class of convex games the Dutta–Ray belongs always to the non-empty core. Moreover, it Lorenz dominates every other allocation in the core (Dutta 1990, p. 154). For convex games, there is only one egalitarian solution, which is unique (Arin et al. 2003, p. 330). Moreover, the single-valuedness for the egalitarian solution is only assured for convex games (Dutta and Ray 1989, pp. 627–628; Hougaard et al. 2001, p. 159). Summarizing, we can state that the Dutta–Ray solution:

- exists as a singleton for each $v \in C^N$ (Dutta and Ray 1989, pp. 627–628; Arin et al. 2003, p. 330),

- does not satisfy CS for each $v \in B^N$ but does for each $v \in C^N$ (Dutta and Ray 1989, pp. 624–625; Dutta 1990, p. 158),
- fulfils IR and EFF for each $v \in C^N$ (Dutta and Ray 1989, pp. 620–621; Dutta 1990, p. 158),
- satisfies ETP and DPP for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600),
- does not fulfil ADD for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 597),
- is an AMAS for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600),
- is a CMAS for each $v \in C^N$ (Hokari 2000, p. 332),
- is not an SMAS for each $v \in C^N$ (Hokari and van Gellekom 2002, p. 600),
- is a PMAS for each $v \in C^N$ (Hokari 2000a, p. 328; Hougaard et al. 2005, p. 432),
- ensures compatibility of coalitional and populational monotonicity on the one hand and core selection on the other hand for each $v \in C^N$ (Hougaard et al. 2005, p. 432).

Despite its nice properties on the class of convex games, the Dutta–Ray solution has received only scant attention in the literature (for an exception cf. Hougaard and Smilgins 2016). We have to emphasize that there are no real applications with a management background or even a cost accounting focus (i.e. compared with the other solution concepts).

4 Summarizing Discussion

4.1 General Applicability

As pointed out in the first section (cf. p. 402) the usability of an allocation method can be judged based on several aspects. Aspects of the overall usability, such as the complexity and transparency of the method, the compatibility of the procedure with existing structures, and the cost–benefit ratio, depends on the specific company and on the situation (see Table 1).

The closeness to reality of employing cooperative game theory for allocating the costs of joint operations depends on the relation between the assumptions/solution schemes and reality.[3] Therefore, the necessary criticism may be separated into two parts: One part deals with game theoretic concepts in general, and the other part concentrates on the economic argumentation for the allocation schemes. The general assumptions of cooperative game theory have already been highlighted (cf. p. 403) and will now be discussed briefly.

The assumption of binding and enforceable agreements between players is fulfilled if we look at the cost allocation within one firm. In the case of the cost

[3]Cf. Kunz (2017) for analysing the applicability of strategic game theory to management accounting purposes.

Table 1 Usability of cooperative game theory for corporate cost accounting. Own representation

Overall usability criteria	Fulfilment in corporate cost accounting setting
– Complexity, transparency, timeliness and communicability of the method and of its results	Depending on specific company
– Comparability and flexibility	
– Compatibility of the procedure with existing structures, norms and conventions	
– Practical efficiency (cost–benefit ratio)	
Assumptions of cooperative TU-games	Fulfilment in corporate cost accounting setting
– Binding and enforceable agreements between players, implying a suitable authority	– Fulfilled for in-house cooperation. In the case of external cooperation the agreements have to be contracted but enforceability is not assured in any case
– Every player tries to maximise his/her utility	– Standard assumption in economic modelling, but questionable from theoretical and practical points of view
– Results are completely transferable	– Fulfilled
– Linear utility function of each player	– Standard assumption in economic modelling, but questionable from theoretical and practical points of view
– Truthful reporting of the players as well as certainty concerning the data	– Not fulfilled
– Existence of a central authority	– Fulfilled for in-house cooperations, for external cooperation fulfilled rarely
Underlying argumentation	Fulfilment in corporate cost accounting setting
Shapley value: Expected marginal contribution of a player, if all orderings of coalitions are equally likely (Shapley 1953) or expected utility of bargaining for a player with neutrality to ordinary risk over games and neutrality to strategic risk (Roth and Verrecchia 1979)	Original argumentation is not convincing, as cost centres do not permute over all orders, neither in-house centres nor external centres. Extended argumentation holds only for risk neutrality and implies a bargaining process
Nucleolus: Minimizing the maximal dissatisfaction of every player with an imputation by applying an adjudicator-analogy	Arbitrator or central authority required, which may be identified for internal, but rarely for external cooperation
τ*-value:* Point on the efficiency plane between the utopia point (marginal contribution of a player to the grand coalition) and the threat point (remaining result of establishing an outsider-coalition and incentivising other outsider-players)	Questionable if a cost centre is able to form an outsider coalition (internal or external) with which it can credibly threaten. In the case of external cooperation, the question arises which firms can form the outsider coalition
Dutta–Ray solution: Lorenz dominance as concept of egalitarianism. This leads, for the class of convex games, to a targeted equal distribution of the core imputation(s)	Questionable if egalitarianism as an "social target state" may serve as a tool for mirroring economic performance

allocation problem between two or more companies, the agreements have to be established by contracts, which ensures the binding nature of the cooperation.

Assuming utility maximizing participants is a standard assumption in economic modelling. Research has shown that this presupposition is questionable from the theoretical and practical points of view.

Establishing the characteristic function on the basis of costs seems to fulfil the transferability assumption. We have indicated that the assumption of a transferable utility game implies that utilities are interpersonally comparable (Shapley 1988), which leads to the presupposition of linear utility functions.

Speaking of establishing the characteristic function, we have presupposed that these values are given or can be truthfully revealed/disclosed by the players. Besides, it can be doubted whether a characteristic function ever represents a game adequately (Luce and Raiffa 1957, pp. 190–192). This is the most questionable assumption, as it offers the possibility of strategic behaviour for the players[4] and for untruthful reporting.[5] Therefore game theoretic concepts ar rarely suitable for supporting open book accounting (OBA). The nature of OBA is more a strategic situation between the partners in a supply chain or in a supply network (Agndal and Nilsson 2010; Alenius et al. 2015). The partners are forced to cooperate, but the question is how to encourage the players to reveal their costs truthfully. This has to be assured by an incentive system. But if this system is based on the characteristic function, a circular reference results (Jarimo et al. 2005).

To reveal the characteristic function of the game and to "collect the utility", some kind of central authority is necessary. Moreover, we have seen that there is a broad range of solution concepts that may be used. So this authority has to choose and to employ one of these solutions. If we look at in-house cost allocation problems, we can state that such an authority exists, due to the organization of the firm. But analysing external cooperation, it seems to be difficult to find such an authority, except in some special cases (Mueller 2017a, b).

Beside these general assumptions, the underlying argumentation of every solution concept has to be criticised. The Shapley value is based on the expected marginal contribution of each player to all possibilities of forming coalitions. This is merely theoretically realizable and therefore seems to be questionable in a cost allocation setting. Shapley's idea of checking and calculating every possible order of coalition formation is not that convincing. By assuming an individual who is both neutral to ordinary risk over games and neutral to strategic risk it is possible to model corporate cost allocation as a bargaining process and to interpret the Shapley value as the expected utility of bargaining (Roth 1988b; Roth and Verrecchia 1979).

The nucleolus determines the lexicographic order of unhappiness of all coalitions and introduces, at least implicitly, the concept of an arbitrator. This concept seems to be from an economical point of view more realistic than the argumentation of the Shapley value. It has to be pointed out that the figure of the arbitrator incorporates

[4]Cf. the contributions in the part "Non-cooperative Models" in this volume.

[5]Cf. Zelewski and Heeb (2017), p. 391 for a similar critique.

the above mentioned central authority explicitly. Concerning the calculational effort of the nucleolus, the very involved procedure of establishing lexicographic orders of excess vectors for games with many players must be mentioned. There are some mistakes in computing the nucleolus caused by overlooking the possibility that a linear program can have multiple solutions (Guajardo and Jørnsten 2015). Nevertheless the nucleolus has been correctly computed in several publications (e.g. Fromen 1997; Hallefjord et al. 1995; Kimms and Çetiner 2012).

From an economic point of view, the τ-value is very interesting, as it posseses the advantage over the Shapley value that its notion is closer to economic reality. With the definition of the upper and lower limits, the opportunities of a player are mirrored in a way which is closer to the firm's decision-making process. Either the player is a part of the grand coalition or forms an outsider coalition. Of crucial importance for the τ-value is the question, with what kind of outsider coalitions can a player make a credible threat (Zelewski 2017, p. 228). But with respect to a corporate cost allocation problem, it is very questionable how to interpret the outsider coalition. Cost centres are designed against the background of their contribution to the overall production process. If a cost centre would be able to form some kind of outsider coalition, then this would mean that this centre would create some kind of outsider production process. This is not a very convincing scenario.

The constrained egalitarian solution by Dutta and Ray is based on egalitarianism. The origin of this concept is based on the assumption, or rather on the aim, of the equality of all players. So it mirrors fairness from the point of the distribution of a result in an equitable society, but not against the background of generating a result in a firm. Looking at its usability, another disadvantage is the very involved calculation procedure for the class of non-convex games and the fact that this solution has received scant attention in the literature.

4.2 Stability and Fairness Criteria

After discussing the overall usability of game theoretic concepts, we have to summarize the results with respect to the theoretical efficiency and the fairness of the solution concepts. The results are summarized in Table 2.

With regard to the Shapley value, we can state that its main disadvantage is the deficient core-selection property for all essential games. In comparison with the other solution concepts, this disadvantage is compensated for by the fulfilment of the other properties.

The nucleolus is the only solution concept which possesses the very crucial core-selection property for all essential games. This advantage is bought dearly, by its not fulfilling important monotonicity properties.

The τ-value suffers from deficiencies with respect to the existence and stability of a solution. With respect to the four types of monotonicity, it can be detected that this solution performs the worst.

Table 2 Properties of allocation schemes. Own representation based on Hokari and van Gellekom 2002, p. 600; Calleja et al. 2012, pp. 903–904

Property / Solution concept	Shapley value	Nucleolus	τ-value	Dutta–Ray solution
Properties which mirror the theoretical efficiency				
Existence and single-valuedness	Satisfied for each $v \in \Gamma^N$	Satisfied for each $v \in E^N$	Satisfied for each $v \in QB^N$	Satisfied for every $v \in C^N$
Core selection	Satisfied for each $v \in C^N$	Satisfied for each $v \in E^N$	Not satisfied for each $v \in C^N$ but satisfied for all $v \in QB^2$ and for all $v \in QB^3$	Satisfied for each $v \in C^N$
Properties which mirror the fairness				
Efficiency	Satisfied for each $v \in \Gamma^N$	Satisfied for each $v \in E^N$	Satisfied for each $v \in QB^N$	Satisfied for each $v \in C^N$
Individual rationality	Satisfied for each $v \in E^N$	Satisfied for each $v \in E^N$	Satisfied for each $v \in QB^N$	Satisfied for each $v \in C^N$
Dummy player	Satisfied for each $v \in \Gamma^N$	Satisfied for each $v \in E^N$	Satisfied for each $v \in QB^N$	Satisfied for each $v \in C^N$
Equal treatment	Satisfied for each $v \in \Gamma^N$	Satisfied for each $v \in E^N$	Satisfied for each $v \in QB^N$	Satisfied for each $v \in C^N$
Additivity	Satisfied for each $v \in \Gamma^N$	Not satisfied for each $v \in E^N$	Not satisfied, even for $v \in QB^N$	Not satisfied for each $v \in C^N$
Aggregate monotonicity	Satisfied for each $v \in \Gamma^N$	Satisfied neither for each $v \in E^N$ nor for each $v \in C^{\|N\| \geq 4}$	Not satisfied for each $v \in C^N$	Satisfied for each $v \in C^N$
Coalitional monotonicity	Satisfied for each $v \in \Gamma^N$	Not satisfied for each $v \in C^N$	Not satisfied for each $v \in C^N$	Satisfied for each $v \in C^N$
Strong monotonicity	Satisfied for each $v \in \Gamma^N$	Not satisfied, neither for each $v \in C^N$, nor for each $v \in A^N$	Not satisfied, neither for each $v \in C^N$, nor for each $v \in A^N$	Not satisfied, even for each $v \in C^N$
Populational monotonicity	Satisfied for each $v \in C^N$ and for $v \in A^N$	Not satisfied for each $v \in C^N$ but is satisfied for each $v \in A^N$	Not satisfied, neither for each $v \in C^N$, nor for each $v \in A^N$	Satisfied for each $v \in C^N$

Γ^N—set of all cooperative TU-games with player set N; E^N—set of all essential TU-games with player set N; C^N—set of all convex TU-games with player set N; QB^N—set of all quasi-balanced TU-games with player set N; A^N—set of all airport TU-games with player set N

The egalitarian solution by Dutta and Ray offers some nice properties on the class of convex games. For this class of games, it seems to be a very competitive solution concept.

The fact that different allocation schemes aim at a fair cost allocation but deliver different results for identical input data may be astonishing. This has to be interpreted against the background of the different underlying argumentations. It becomes apparent that rationality and fairness are not absolute, but rather relative measures. Each presented solution is fair according to its underlying argumentation. They cannot replace the concluding decision concerning a fair cost allocation, but they may guide debates about the identification of the most suitable solution.

References

Agndal H, Nilsson U (2010) Different open book accounting practices for different purchasing strategies. MAR 21(3):147–166

Alenius E, Lind J, Strömsten T (2015) The role of open book accounting in a supplier network: creating and managing interdependencies across company boundaries. IMM 45(2):195–206

Arin J (2013) Monotonic core solutions: beyond young's theorem. IJGT 42(2):325–337

Arin J, Feltkamp V (2012) Coalitional games: monotonicity and core. Eur J Oper Res 216(1): 208–214

Arin J, Katsev I (2016) A monotonic core solution for convex TU games. IJGT 45(4):1013–1029

Arin J, Katsev I (2017) The SD-prenucleolus for TU-games: coalitional monotonicity and core stability. In: Mueller D, Trost R (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp. 327–347

Arin J, Kuipers J, Vermeulen D (2003) Some characterizations of egalitarian solutions on classes of TU-games. MSS 46(3):327–345

Arin J, Kuipers J, Vermeulen D (2008) An axiomatic approach to egalitarianism in TU-games. IJGT 37(4):565–580

Aumann RJ (1960) Linearity of unrestrictedly transferable utilities. NRLQ 7(3):281–284

Aumann RJ, Shapley LS (1974) Values of non-atomic games. Princeton University Press, Princeton

Bergantiños G, Massó J (1996) Notes on a new compromise value: the χ-value. Optimization 38(3):277–286

Bergantiños G, Massó J (2002) The Chi-compromise value for non-transferable utility games. MMOR 56(2):269–286

Bhakar R, Sriram VS, Padhy NP, Gupta HO (2010) Probabilistic game approaches for network cost allocation. IEEE Trans Power Syst 25(1):51–58

Bilbao JM (2000) Cooperative games on combinational structures. Kluwer, Dordrecht

Bondareva O (1963) Nekotorjye primeneniya metodov linynogo programmorovaniya k teorii kooperativnikh igr. In: Problemi Kibernetiki 10, pp. 119–139 (Some applications of linear programming methods to the theory of cooperative games. In: Morgenstern O (ed) Selected russian papers on game theory 1959–1965). Princeton University Press, Princeton, pp 79–114

Brânzei R, Dimitrov D, Tijs SH (2008) Models in cooperative game theory, 2nd edn. Springer, Berlin

Calleja P, Rafels C, Tijs SH (2012) Aggregate monotonic stable single-valued solutions for cooperative games. IJGT 41(4):899–913

Calleja P, Rafels C, Tijs SH (2009) The aggregate-monotonic core. GEB 66(2):742–748

Carreras F, Owen G (2013) Pure bargaining problems and the shapley rule. In: Holler MJ, Nurmi HN (eds) Power, voting, and voting power: 30 years after. Springer, Berlin, pp 681–702

Chalkiadakis G, Elkind E, Wooldridge M (2012) Computational aspects of cooperative game theory. Morgan & Claypool Publishers, San Rafael

Charles SL, Hansen DR (2008) An evaluation of activity-based costing and functional-based costing: a game-theoretic approach. IJPE 113(1):282–296

Choudhury N (1990) Is cost allocation just? MAR 1(3):217–232

Curiel I (1997) Cooperative game theory and applications. Kluwer, Boston

Driessen TSH, Tijs SH (1984) Game-theoretic solutions for some economic situations. In: Steckhan H, Bühler W, Jäger KE, Schneeweiß C, Schwarze J (eds) DGOR. Operations Research Proceedings, vol 1983. Springer, Berlin/Heidelberg, p 557

Driessen TSH (1988) Cooperative games, solutions and applications. Kluwer, Dordrecht

Driessen TSH, Tijs SH (1982) The [tau]-value, the nucleolus and the core for a subclass of games. Report 8215, University of Nijmegen

Driessen TSH, Tijs SH (1985) The τ-value, the core and semiconvex games. IJGT 4(4): 229–248

Drury CM (2015) Management and cost accounting. 9th edn. Cengage Learning, London

Dutta B (1990) The egalitarian solution and reduced game properties in convex games. IJGT 19(2):153–169

Dutta B, Ray D (1989) A concept of egalitarianism under participation constraints. Econometrica 57(3):615–635

Fiestras-Janeiro MG, García-Jurado I, Mosquera MA (2011) Cooperative games and cost allocation problems. TOP 19(1):1–22

Friedman EJ (2012) Asymmetric cost sharing mechanisms. GEB 75(1):139–151

Frisk M, Göthe-Lundgren M, Jørnsten K, Rönnqvist M (2010) Cost allocation in collaborative forest transportation. EJOR 205(2):448–458

Fromen B (1997) Reducing the number of linear programs needed for solving the nucleolus problem of n-person game theory. EJOR 98(3):626–636

Getán J, Montes J (2010) On cooperative games with large monotonic core. TOP 18(2) :493–508

Gilles RP (2010) The cooperative game theory of networks and hierarchies. Springer, Berlin.

Gillies DB (1959) Solutions to general non-zero-sum-games. In: Tucker AW, Luce RD (eds) Contributions to the theory of games, vol IV. Princeton University Press, Princeton, pp 47–87

González-Díaz J, Mirás Calvo MÁ, Quinteiro Sandomingo C, Sánchez Rodríguez E (2016) Airport games: the core and its center. MSS 82:105–115

González-Díaz J, García-Jurado I, Fiestras-Janeiro MG (2010) An introductory course on mathematical game theory. American Mathematical Society, Providence

Grafe F, Iñarra E, Zarzuelo J (1998) Population monotonic allocation schemes on externality games. MMOR 48(1):71–80

Guajardo M, Jørnsten K (2015) Common mistakes in computing the nucleolus. EJOR 241(3):931–935

Guajardo M, Rönnqvist M (2016) A review on cost allocation methods in collaborative transportation. Int Trans Oper Res 23(3):371–392

Gunasekaran A, Williams HJ, McGaughey RE (2005) Performance measurement and costing system in new enterprise. Technovation 25(5):523–533

Hallefjord Å, Helming R, Jørnsten K (1995) Computing the nucleolus when the characteristic function is given implicitly: a constraint generation approach. IJGT 24(4):357–372

Hamlen SS, Hamlen WA, Tschirhart J (1980) The use of the generalized shapley allocation in joint cost allocation. TAR 55(2):269–287

Hokari T (2000) The nucleolus is not aggregate-monotonic on the domain of convex games. IJGT 29(1):133–137

Hokari T (2000) Population monotonic solutions on convex games. IJGT 29(3):327–338

Hokari T, van Gellekom A (2002) Population monotonicity and consistency in convex games: some logical relations. IJGT 31(4):593–607

Homburg C, Scherpereel P (2008) How should the cost of joint risk capital be allocated for performance measurement? EJOR 187(1):208–227

Hougaard JL (2009) An introduction to allocation rules. Springer, Berlin

Hougaard JL (2017) Sharing the costs of access to a set of public goods. In: Mueller D, Trost R (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp 287–299
Hougaard JL, Peleg B, Østerdal LP (2005) The Dutta–Ray solution on the class of convex games: a generalization and monotonicity properties. IGTR 7(4):431–442
Hougaard JL, Peleg B, Thorlund-Petersen L (2001) On the set of Lorenz-maximal imputations in the core of a balanced game. IJGT 30(2):147–165
Hougaard JL, Smilgins A (2016) Risk capital allocation with autonomous subunits: the Lorenz set. Insur Math Econ 67:151–157
Housman D, Clark L (1998) Core and monotonic allocation methods. IJGT 27(4): 611–616
Jarimo T, Pulkkinen U, Salo A (2005) Encouraging suppliers to process innovations: a game theory approach. Int J Technol Intell Plan 1(4):403–423
Kimms A, Çetiner D (2012) Approximate nucleolus-based revenue sharing in airline alliances. EJOR 220(2):510–521
Kruś L, Bronisz P (2000) Cooperative game solution concepts to a cost allocation problem. EJOR 122(2):258–271
Kunz J (2017) Reflections on the practical applicability of strategic game theory to managerial incentivation. In: Mueller D, Trost R (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp 23–48
Legros P (1986) Allocating joint costs by means of the nucleolus. IJGT 15(2):109–119
Littlechild SC, Owen G (1973) A simple expression for the shapley value in a special case. MS 20(3):370–372
Littlechild SC, Thompson GF (1977) Aircraft landing fees: a game theory approach. TBJE 8(1):186–204
Loehman ET, Whinston AB (1976) A generalized cost allocation scheme. In: Lin SAY (ed) Theory and measurement of economic externalities. Academic Press, London, pp 87–101, 103–106
Lorenz MO (1905) Methods of measuring the concentration of wealth. Publ Am Stat Assoc 9(70):209–219
Lozano S, Moreno P, Adenso-Díaz B, Algaba E (2013) Cooperative game theory approach to allocating benefits of horizontal cooperation. EJOR 229(2):444–452
Luce RD, Raiffa H (1957) Games and decisions—introduction and critical survey. Courier Corporation, New York
Maschler M (1992) The bargaining set, kernel and nucleolus. In: Aumann RJ, Hart S (ed) Handbook of game theory with economic applications, vol I. Reprint 2006, North-Holland Publishing Company, Amsterdam, pp 591–667
Maschler M, Peleg B, Shapley LS (1979) Geometric properties of the kernel, nucleolus, and related solution concepts. MOR 4(4):303–338
Maschler M, Solan E, Zamir S (2013) Game theory. Cambridge University Press, Cambridge
Massol O, Tchung-Ming S (2010) Cooperation among liquefied natural gas suppliers: is rationalization the sole objective? Energy Econ 32(4):933–948
Meca A, Sošić G (2014) Benefactors and beneficiaries: the effects of giving and receiving on cost-coalitional problems. POM 23(9):1549–1560
Megiddo N (1974) On the nonmonotonicity of the bargaining set, the kernel, and the nucleolus of a game. SIAP 27(2):355–358
Moulin H (1991) Axioms of cooperative decision making. Cambridge University Press, Cambridge
Moulin H (1992) An application of the shapley-value to fair division with money. Econometrica 60(6):1331–1349
Moulin H (2002) Axiomatic cost and surplus sharing. In: Arrow KJ, Sen AK, Suzumura K (eds) Handbook of social choice and welfare. Elsevier, New York, pp 289–357
Mueller D (2016) What's in it for me? an analysis of environmental investments through a game theoretic lens. Int J Innov Sustain Dev 10(2):198–218
Mueller D (2017a) Grid extension in german backyards: a game-theory rationale. J Environ Plan Manag 60(3):437–461

Mueller D (2017b) ... and justice for all – cost fairness in Volkswagen's product family. J App Acc Res 18(2):261–271
Pazner EA (1977) Pitfalls in the theory of fairness. JET 14(2):458–466
Peleg B, Sudhölter P (2007) Introduction to the theory of cooperative games, 2nd edn. Springer, Berlin
Potters J, Sudhölter P (1999) Airport problems and consistent allocation rules. MSS 38(1):83–102
Rawls J (1999) A theory of justice, 2nd edn. Harvard University Press, Cambridge
Rideout D, Wagner JE (1988) Testing cost-sharing techniques on a multiple-use timber sale. For Ecol Manage 23(4):285–296
Roma P, Perrone G (2016) Cooperation among competitors: a comparison of cost-sharing mechanisms. IJPE 180:172–182
Rosenthal EC (1990) Monotonicity of the core and value in dynamic cooperative games. IJGT 19(1):45–57
Roth AE (ed) (1988a) The Shapley value: essays in honor of Lloyd S. Shapley. Cambridge University Press, Cambridge
Roth AE (1988b) The expected utility of playing a game. In: Roth AE (ed) The Shapley value: essays in honor of Lloyd S. Shapley. Cambridge University Press, Cambridge, pp 51–70
Roth AE, Verrecchia RE (1979) The Shapley value as applied to cost allocation: a reinterpretation. JAR 17(1):295–303
Schmeidler D (1969) The nucleolus of a characteristic function game. SIAP 17(6): 1163–1170
Serrano R (2013) Lloyd Shapley's matching and game theory. ScanJE 115(3):599–618
Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games, vol II. Princeton University Press, Princeton, pp 307–317
Shapley LS (1967) On balanced sets and cores. NRLQ 14(4):453–460
Shapley LS (1971) Cores of convex games. IJGT 1(1):11–26
Shapley LS (1988) Utility comparison and the theory of games. In: Roth AE (ed) The Shapley value: essays in honor of Lloyd S. Shapley. Cambridge University Press, Cambridge, pp 307–319
Shubik M (1962) Incentives, decentralized control, the assignment of joint costs, and internal pricing. MS 8(3):325–343
Sönmez T (1994) Populational-monotonicity of the nucleolus on a class of public good problems. University of Rochester, Rochester
Songhuai D, Xinghua Z, Lu M, Hui X (2006) A novel nucleolus-based loss allocation method in bilateral electricity markets. IEEE Trans. Power Syst 21(1):28–33
Spinetto RD (1975) Fairness in cost allocation and cooperative games. Decis Sci 6(3):482–491
Sprumont Y (1990) Population monotonic allocation schemes for cooperative games with transferable utility. GEB 2(4):378–394
Stamtsis GC, Erlich I (2004) Use of cooperative game theory in power system fixed-cost allocation. IEE Proc 151(3):401–406
Sudhölter P, Peleg B (1998) Nucleoli as maximizers of collective satisfaction functions. SCW 15(3):383–412
Suzuki M, Nakayama M (1976) The cost assignment of the cooperative water resource development: a game theoretical approach. MS 22(10):1081–1086
Thomson W (1983a) The fair division of a fixed supply among a growing population. MOR 8(3):319–326
Thomson W (1983b) Problems of fair division and the egalitarian solution. JET 31(2):211–226
Timmer J, Chessa M, Boucherie RJ (2013) Cooperation and game-theoretic cost allocation in stochastic inventory models with continuous review. EJOR 231(3):567–576
Tijs SH (1981) Bounds for the core of a game and the τ-value. In: Moeschlin O, Pallaschke D (eds) Game theory and mathematical economics. North Holland Publishing Company, Amsterdam, pp 123–132
Tijs SH, Driessen TSH (1986) Game theory and cost allocation problems. MS 32(8):1015–1028
Tripsas M, Schrader S, Sobrero M (1995) Discouraging opportunistic behavior in collaborative R&D: a new role for government. Res. Policy 24(3):367–389

Trost R, Heim S (2017) Setting incentives for managers: incentive compatibility, similarity rule, and goal congruence. In: Mueller D, Trost R. (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp 3–22
Uyar A, Kuzey E (2016) Does management accounting mediate the relationship between cost system design and performance? Adv Account 35:170–176
von Neumann JM, Morgenstern O (1944) Theory of games and economic behavior. Princeton University Press, Princeton
Winter E (2002) The Shapley value. In: Aumann RJ, Hart S (eds) Handbook of game theory with economic applications, vol III. Reprint, 2007, North Holland, Amsterdam, pp 2025–2054
Young HP (1985) Monotonic solutions of cooperative games. IJGT 14(2):65–72
Young HP (1994) Cost allocation. In: Aumann RJ, Hart S (eds) Handbook of game theory with economic applications, vol II. North Holland, Amsterdam, pp 1193–1235
Zelewski S (2009) Faire Verteilung von Effizienzgewinnen in Supply-Webs - ein spieltheoretischer Ansatz auf der Basis des τ-Werts (Fair distribution of efficiency gains in supply-webs - a game-theoretic approach based on the τ-value). Logos, Berlin
Zelewski S (2017) Fair distribution of cooperation gains in supply chains—a justification program from an economic point of view. In: Mueller D, Trost R (eds) Game theory in management accounting—implementing incentives and fairness. Springer, Berlin, pp 233–257
Zelewski S, Heeb T (2017) Characteristics of the τ-value and the χ-value. In: Mueller D, Trost R (eds) Game theory in management accounting – implementing incentives and fairness. Springer, Berlin, pp 407–429

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List of Journal Abbreviations

AMJ	Academy of Management Journal
AOS	Accounting, Organizations and Society
BFuP	Betriebswirtschaftliche Forschung und Praxis
CG	Corporate Governance: An International Review
EER	European Economic Review
EI	Economic Inquiry
EJOR	European Journal of Operational Research
EJPSM	European Journal of Purchasing & Supply Management
EL	Economics Letters
ET	Economic Theory
GEB	Games and Economic Behavior
HBR	Harvard Business Review
HCMS	Health Care Management Science
IGTR	International Game Theory Review
IJGT	International Journal of Game Theory
IJIO	International Journal of Industrial Organization
IJPE	International Journal of Production Economics
IMM	Industrial Marketing Management
JAE	Journal of Accounting & Economics
JAL	Journal of Accounting Literature
JapER	Japanese Economic Review
JAR	Journal of Accounting Research
JB	Journal of Business
JBEM	Journal of Business Economics and Management
JEBO	Journal of Economic Behavior & Organization
JEL	Journal of Economic Literature
JEMS	Journal of Economics & Management Strategy
JEP	Journal of Economic Perspectives
JET	Journal of Economic Theory
JLE	Journal of Labor Economics
JLEO	Journal of Law, Economics & Organization
JMAR	Journal of Management Accounting Research
JME	Journal of Mathematical Economics
JMR	Journal of Marketing Research
JNM	Journal of Nursing Management

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JOB	Journal of Organizational Behavior
JPolE	Journal of Political Economy
JPSM	Journal of Purchasing and Supply Management
JPubE	Journal of Public Economics
JPubET	Journal of Public Economic Theory
JS	Journal of Scheduling
MAR	Management Accounting Research
MDE	Managerial and Decision Economics
MMOR	Mathematical Methods of Operations Research
MOR	Mathematics of Operations Research
MP	Mathematical Programming
MS	Management Science
MSS	Mathematical Social Sciences
NRL	Naval Research Logistics
NRLQ	Naval Research Logistics Quarterly
ORL	Operations Research Letters
POM	Production and Operations Management
RAS	Review of Accounting Studies
RED	Review of Economic Design
RES	Review of Economic Studies
ScanJE	The Scandinavian Journal of Economics
SCW	Social Choice and Welfare
SIAP	Journal on Applied Mathematics of the Society for Industrial and Applied Mathematics (SIAM)
SMJ	Strategic Management Journal
TAER	The American Economic Review
TAR	The Accounting Review
TBJE	The Bell Journal of Economics
TEJ	The Economic Journal
TJF	The Journal of Finance
Top	An Official Journal of the Spanish Society of Statistics and Operations Research
TQJE	The Quarterly Journal of Economics
TRJE	The RAND Journal of Economics
TRPB	Transportation Research Part B: Methodological
TRPE	Transportation Research Part E: Logistics and Transportation Review
ZfB	Zeitschrift für Betriebswirtschaft

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