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Variational Methods for Strongly Indefinite Problems

Yanheng Ding

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Preface

This monograph consists of a series of lectures given partly at the Morningside Center of Mathematics of Chinese Academy of Sciences and the Department of Mathematics of Rutgers University, and entirely at the Department of Mathematics of the University of Franche-Comté in a course of nonlinear analysis in March and April of 2006. The material was mainly taken from some joint work with Thomas Bartsch done while the author as an Alexander von Humboldt fellow visited Giessen University. It presents some results concerning methods in critical point theory oriented towards differential equations which are variational in nature with strongly indefinite Lagrangian functionals. The author thanks greatly T. Bartsch for his kindnesses to him. He would like also to thank H. Brézis for his encouragements and F. H. Lin, Y. Y. Li for the discussions on mathematics of common interest. He also thanks L. Jeanjean for his invitation to come to Besancon and for his suggestions on the content. Finally he thanks the University of Franche-Comté for its optional support.

Yanheng Ding

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Chapter 1

Introduction

The classical Calculus of Variations deals with finding minima of functionals $\Phi : X \to \mathbb{R}$ that are bounded below. The basic idea of the direct method is to consider a minimizing sequence $\Phi(u_n) \to \inf \Phi$, to find a convergent subsequence $u_{n_k} \to u$, and to show that $\Phi(u) = \inf \Phi$. In order to make this work the space X should have a topology which is rather weak for the existence of a convergent subsequence, and rather strong so that Φ is lower semicontinuous. In many applications the functional is not bounded below and instead of a minimizer one is interested in critical points. This is the concern of the Calculus of Variations in the Large or Critical Point Theory, which has undergone an enormous development in the last century due to the work of mathematicians like Morse, Lusternik, Schnirelman, Palais, Smale, Rabinowitz, Ambrosetti, Lions, Struwe, Witten, Floer and many others, with applications to problems from analysis, geometry and mathematical physics. Here one usually requires X to be a Banach manifold and Φ to be differentiable. An essential ingredient is the construction of a flow φ on X so that $\Phi(\varphi(t,u))$ is decreasing in t. This flow is used in the spirit of Morse theory, to construct deformations of sublevel sets $\Phi^c = \{u \in X : \Phi(u) \le c\}$, and to find Palais-Smale sequences $(u_n)_n$, that is: $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$, replacing the minimizing sequences. Typical results are the mountain pass theorem of Ambrosetti and Rabinowitz or various linking theorems. The proofs use in an essential way topological concepts based on the Brouwer or Leray-Schauder degree. The theory has also been extended to deal with (semi-)continuous functions on metric spaces, forced by problems from nonlinear elasticity (see [Degiovanni and Schuricht (1998)]). Another generalization concerns variational methods for functionals on closed convex subsets of Banach spaces developed by Struwe [Struwe (1989)] for Plateau's problem. Such functionals appear also in variational inequalities.

Motivated by several applications, for instance to finite- and infinite-dimensional Hamiltonian systems, nonlinear Schrödinger equations and nonlinear Dirac equations, we were led to consider C^1-functionals $\Phi : E = E^- \oplus E^+ \to \mathbb{R}$ defined on the product $E = E^- \oplus E^+$ of Banach spaces $E^\pm$ with $\dim E^\pm = \infty$ but where one needs to work with the weak topology on E^- in order to gain compactness. The functionals typically have the form

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u) \quad \text{for } u = u^- + u^+ \in E^- \oplus E^+. \tag{1.1}$$

Since $\dim E^\pm = \infty$ the functional is strongly indefinite. Thus all of its critical points have infinite Morse index. Moreover, $\Psi' : E \to E^*$ is not completely continuous and the Palais-Smale condition does not hold in our applications. This makes applications of Leray-Schauder degree type arguments rather subtle. On the other hand the functional $\Psi : E \to \mathbb{R}$ is weakly sequentially lower semicontinuous and $\Psi' : E \to E^*$ is weakly sequentially continuous. It turns out that the product topology

$$\mathcal{T} = (\text{weak topology on } E^-) \times (\text{norm topology on } E^+)$$

is well suited for certain arguments because $\Phi : (E, \mathcal{T}) \to \mathbb{R}$ is sequentially upper semicontinuous, and $\Phi' : (E, \mathcal{T}) \to (E^*, \text{weak}^* \text{ topology})$ is continuous. Given a finite-dimensional subspace $F \subset E^+$ the unit ball of $E^- \oplus F$ is $\mathcal{T}$-compact, and given a bounded sequence $(u_n)_n$ the negative part $(u_n^-)_n$ $\mathcal{T}$-converges (up to a subsequence). When one wants to develop critical point theory with this topology on E one needs to construct deformations on E which are $\mathcal{T}$-continuous. Deformations are usually obtained by integrating vector fields which in turn are constructed with the help of partitions of unity. So one needs to construct these in a $\mathcal{T}$-Lipschitz continuous way. A more difficult situation occurs when one is interested in “normalized solutions”, that is critical points of Φ constrained to the unit sphere $SE = \{u \in E : \|u\| = 1\}$ or to other finite-codimensional submanifolds X of E.

The $\mathcal{T}$-topology on X is not metrizable, therefore the by now well developed critical point theory for (semi-)continuous functions on metric spaces cannot be applied. Instead the $\mathcal{T}$-topology is generated by a family $\mathcal{D}$ of semi-metrics. A pair $(X, \mathcal{D})$ consisting of a set X and a family of semi-metrics is called a *gage space*; see [Kelley (1995)]. The paper [Bartsch and Ding (2006I)] is a first step to develop critical point theory on gage spaces. We begin by settling some basic topological questions. We introduce the concept of a Lipschitz map $(X, \mathcal{D}) \to \mathbb{R}$ and of a Lipschitz normal gage space (disjoint closed sets can be separated by Lipschitz maps). We find conditions on $(X, \mathcal{D})$ so that X is Lipschitz normal and so that Lipschitz partitions of unity (subordinated to a given open cover) exist. In particular, we show that given a Banach space B, an arbitrary subset $B_0 \subset B$, and letting $\mathcal{D}$ be the family of semi-metrics on $X = B^*$ given by $d_b(x, y) := |\langle b, x - y\rangle_{B,B^*}|$, $b \in B_0$, the gage space $(B^*, \mathcal{D})$ is Lipschitz normal. More generally, if (Y, d_Y) is a metric space then the product gage space $(B^*, \mathcal{D}) \times (Y, d_Y)$ is Lipschitz normal and has Lipschitz partitions of unity. In addition, if B is separable and $B_0 \subset B$ is dense then also every locally closed subset (that is, an intersection of an open and a closed subset) of this product gage space is Lipschitz normal and has Lipschitz partitions of unity subordinated to an arbitrary open cover.

We then present some nonlinear problems where the abstract theory developed here can be applied. These problems arise in mechanics, physics, control theory and other topics, which are variational in nature with the feature that their solutions correspond to critical points of certain strongly indefinite functionals of the form (1.1). We are interested in the existence and multiplicity of solutions to these problems. The details are arranged in the last four chapters. In Chapter 5 we study the homoclinic orbits in the classical Hamiltonian systems

$$\begin{cases} \mathcal{J}\dfrac{d}{dt}z + L(t)z = R_z(t,z) & \text{for } t \in \mathbb{R} \\ z(t) \to 0 & \text{as } |t| \to \infty \end{cases}$$

with periodic or non-periodic (with respect to the time t) Hamiltonians. Chapter 6 is devoted to the standing waves of the nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with V and g being periodic in x. We also treat here semiclassical states of a Hamiltonian system of perturbed Schrödinger equations:

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x,\varphi,\psi) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x,\varphi,\psi) \\ (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2) \end{cases}$$

without any periodicity assumption. Chapter 7 deals with localized solutions of the nonlinear Dirac equations with external fields

$$\begin{cases} -i\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + \beta m u + M(x)u = G_u(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with either scale potentials (i.e., $M(x) = \beta V(x)$), or vector potentials (say, the Coulomb-type potentials). We also study semiclassical solutions (as $\hbar \to 0$). Finally, in Chapter 8 we handle solutions of homoclinic type to the systems of diffusion equations

$$\begin{cases} \partial_t u - \Delta_x u + \mathrm{b}(t,x)\cdot\nabla_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v - \mathrm{b}(t,x)\cdot\nabla_x v + V(x)v = H_u(t,x,u,v) \end{cases}$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ with $u(t,x),\ v(t,x) \to 0$ as $|t| + |x| \to \infty$. In all these problems the nonlinear terms are assumed to be either asymptotically linear or super linear. In the arguments certain analytical estimates which are needed to check the assumptions of the abstract results require different techniques. We prove new results extending the previous relative works in the literature.

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Chapter 2

Lipschitz partitions of unity

Let X be a set and $\mathcal{D}$ a family of semi-metrics on X. The pair $(X, \mathcal{D})$ is called a *gage space*. We write $\mathcal{T}_d$ for the topology on X associated to the semi-metric $d : X \times X \to \mathbb{R}$. Let $\mathcal{T}_{\mathcal{D}}$ be the topology on X generated by all $\mathcal{T}_d$, $d \in \mathcal{D}$, that is, the coarsest topology containing all $\mathcal{T}_d$, $d \in \mathcal{D}$. If $\mathcal{D} = \{d_n : n \in \mathbb{N}\}$ is countable then $\mathcal{T}_{\mathcal{D}}$ is semi-metrizable. Namely, setting $\tilde{d_n} := \frac{d_n}{1+d_n}$ and $d := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\tilde{d_n}$ one easily checks that $\mathcal{T}_{\mathcal{D}} = \mathcal{T}_d$. We call $\mathcal{D}$ saturated if $d, d' \in \mathcal{D}$ implies $\max\{d, d'\} \in \mathcal{D}$. Clearly, the family

$$\overline{\mathcal{D}} := \big\{ \max\{d_1, \dots, d_k\} : k \in \mathbb{N},\ d_1, \dots, d_k \in \mathcal{D} \big\}$$

is the smallest saturated family of semi-metrics on X which contains $\mathcal{D}$, the saturation of $\mathcal{D}$. It generates the same topology as $\mathcal{D}$. In this section, all topological notions refer to $\mathcal{T}_{\mathcal{D}} = \mathcal{T}_{\overline{\mathcal{D}}}$.

A basis of this topology is given by the sets

$$U_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}, \quad x \in X,\ d \in \overline{\mathcal{D}},\ \varepsilon > 0.$$

In fact, for $x \in X$ the sets $U_\varepsilon(x, d)$, $d \in \overline{\mathcal{D}}$, $\varepsilon > 0$, form a neighborhood basis because given semi-metrics $d_1, \dots, d_k$, and given $\varepsilon_1, \dots, \varepsilon_k > 0$ we set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_k\}$, $d = \max\{d_1, \dots, d_k\}$ and obtain

$$U_{\varepsilon_1}(x; d_1) \cap \ldots \cap U_{\varepsilon_k}(x; d_k) \supset U_\varepsilon(x; d).$$

Definition 2.1 ([Bartsch and Ding (2006I)]). *A map $f : X \to (M, d_M)$ into a semi-metric space M with semi-metric d_M is said to be Lipschitz (continuous) if there exist $d \in \overline{\mathcal{D}}$ and $\lambda > 0$ such that*

$$d_M(f(x), f(y)) \ \le \lambda d(x, y) \quad \textit{for all } x, y \in X.$$

f is called locally Lipschitz (continuous) if every $x \in X$ has a neighborhood U_x such that the restriction $f|_{U_x}$ is Lipschitz continuous.

Clearly, a (locally) Lipschitz map is continuous. Lipschitz continuity depends of course on $\mathcal{D}$ and not just on the topology $\mathcal{T}_{\mathcal{D}}$. We call two gage spaces $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ equivalent if there exists a homeomorphism $h : X \to Y$ such that for every map $f : (Y, \mathcal{E}) \to (M, d_M)$ into a semi-metric space there holds: f is (locally)

Lipschitz if and only if $f \circ h$ is (locally) Lipschitz. In this sense, $(X, \mathcal{D})$ and $(X, \overline{\mathcal{D}})$ are equivalent.

For $Y \subset X$ and $d \in \mathcal{D}$ we set

$$d(\,.\,, Y) : X \to \mathbb{R}, \quad d(x, Y) := \inf\{d(x, y) : y \in Y\}.$$

Then

$$|d(x_1, Y) - d(x_2, Y)| \leq d(x_1, x_2),$$

so $d(\,.\,, Y)$ is Lipschitz. Clearly, the zero set of $d(\,.\,, Y)$ is the closure of Y with respect to the topology $\mathcal{T}_d$.

If $A \subset X$ is closed and $x \notin A$ then there exists a neighbourhood $U_\varepsilon(x; d) \subset X \setminus A$. The map

$$f : X \to [0, 1], \; f(y) = \min\{1, d(x, y)/\varepsilon\}$$

is Lipschitz and satisfies $f(x) = 0$, $f|_A \equiv 1$. Thus one can separate a point and a disjoint closed set by a Lipschitz map. In particular, X is completely regular. It is easy to see that one can also separate a compact set and a disjoint closed set by a Lipschitz map.

In general, X need not be normal. If X is normal we do not know whether two disjoint closed sets can be separated by a locally Lipschitz map. Similarly, if X is paracompact we do not know whether one can construct locally finite partitions of unity subordinated to an open cover of X and such that the maps in the partition of unity are locally Lipschitz. In this section we shall prove results in this direction.

Lemma 2.1. *$f : X \to M$ is locally Lipschitz if, and only if, for every $x \in X$ there exists $d \in \overline{\mathcal{D}}$, $\varepsilon > 0$, $\lambda > 0$ such that*

$$d_M(f(y), f(z)) \leq \lambda d(y, z) \quad \textit{for all } y, z \in U_\varepsilon(x; d).$$

Proof. Suppose f is locally Lipschitz. Thus there exist $d_1 \in \overline{\mathcal{D}}$, $\varepsilon > 0$ such that $f|_{U_\varepsilon(x; d_1)}$ is Lipschitz, that is, for some $d_2 \in \overline{\mathcal{D}}$, $\lambda > 0$ we have

$$d_M(f(y), f(z)) \leq \lambda d_2(y, z) \quad \text{for all } y, z \in U_\varepsilon(x; d_1).$$

Setting $d := \max\{d_1, d_2\}$ the conclusion follows. The other implication is trivial. □

Lemma 2.2. *Let $f : X \to M$ be locally Lipschitz. Then for $K \subset X$ compact there exists a neighbourhood U of K in X such that $f|_U$ is Lipschitz.*

Proof. For $x \in K$ we choose $d_x \in \overline{\mathcal{D}}$, $\varepsilon_x > 0$, $\lambda_x > 0$ such that

$$d_M(f(y), f(z)) \leq \lambda_x d_x(y, z) \quad \text{for } y, z \in U_{\varepsilon_x}(x; d_x).$$

There exist $x_1, \ldots, x_n \in K$ with $K \subset \bigcup_{j=1}^n U_{\varepsilon_{x_j}/2}(x_j; d_{x_j})$. For $j = 1, \ldots, n$ we set $\varepsilon_j := \varepsilon_{x_j}$, $d_j := d_{x_j}$, $\lambda_j := \lambda_{x_j}$, $U_j := U_{\varepsilon_j/2}(x_j; d_j)$, and $U := \bigcup_{j=1}^n U_j$.

We first show that $f(U)$ is bounded, that is

$$S := \sup\{d_M(f(x), f(y)) : x, y \in U\} < \infty.$$

For $x, y \in U$ there exist i, j with $x \in U_i$, $y \in U_j$. Then we have

$$\begin{aligned} d_M\big(f(x), f(y)\big) &\le d_M(f(x), f(x_i)) + d_M(f(x_i), f(x_j)) + d_M(f(x_j), f(y)) \\ &\le \lambda_i d_i(x, x_i) + d_M(f(x_i), f(x_j)) + \lambda_j d_j(x_j, y) \\ &\le \frac{\lambda_i \varepsilon_i}{2} + d_M(f(x_i), f(x_j)) + \frac{\lambda_j \varepsilon_j}{2} \\ &\le \max_{k,l} \left(\frac{\lambda_k \varepsilon_k}{2} + d_M(f(x_k), f(x_l)) + \frac{\lambda_l \varepsilon_l}{2} \right) \\ &< \infty. \end{aligned}$$

Now we prove that $f|_U$ is Lipschitz. Set $\varepsilon := \frac{1}{2}\min\{\varepsilon_1, \dots, \varepsilon_n\}$, $\lambda := \max\{\lambda_1, \dots, \lambda_n, S/\varepsilon\}$ and $d := \max\{d_1, \dots, d_n\}$. For $x, y \in U$ we choose j with $y \in U_j$. If $d_j(x, y) < \varepsilon_j/2$ then $x \in U_{\varepsilon_j}(x_j; d_j)$ and therefore

$$d_M\big(f(x), f(y)\big) \le \lambda_j d_j(x, y) \le \lambda d(x, y),$$

as required. If on the other hand $d_j(x, y) \ge \varepsilon_j/2 \ge \varepsilon$ then

$$d_M\big(f(x), f(y)\big) \le S \le \lambda d_j(x, y) \le \lambda d(x, y).$$

□

Lemma 2.3. *Let $K \subset X$ be compact and $A \subset X$ be closed such that $A \cap K = \emptyset$. Then there exists $d \in \overline{\mathcal{D}}$ with*

$$d(K, A) = \inf\{d(x, y) : x \in K, y \in A\} > 0.$$

Proof. There exist $x_1, \dots, x_n \in K$, $\varepsilon_1, \dots, \varepsilon_n > 0$ and $d_1, \dots, d_n \in \overline{\mathcal{D}}$ with $K \subset \bigcup_{j=1}^{n} U_{\varepsilon_j}(x_j; d_j)$ and $\bigcup_{j=1}^{n} U_{2\varepsilon_j}(x_j; d_j) \subset X \setminus A$. Then $d := \max\{d_1, \dots, d_n\}$ does the job: $d(K, A) \ge \min\{\varepsilon_1, \dots, \varepsilon_k\}$. □

In the situation of Lemma 2.3 the map

$$f : X \to [0, 1], \quad f(x) := \frac{d(x, K)}{d(x, K) + d(x, A)} \ ,$$

is well defined and Lipschitz, because the maps $d(., K)$, $d(., A)$ are Lipschitz and $d(x, K) + d(x, A) \ge d(K, A) > 0$ for all $x \in X$. Clearly, $f|_K \equiv 0$ and $f|_A \equiv 1$. Thus a compact set K and a disjoint closed set A can be separated by a Lipschitz map.

Definition 2.2 ([Bartsch and Ding (2006I)]). *A gage space $(X, \mathcal{D})$ is said to be Lipschitz normal if X is Hausdorff, (equivalently, $\mathcal{D}$ separates points), and if for any two closed disjoint sets $A, B \subset X$ there exists a locally Lipschitz map $f : X \to [0, 1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.*

If $\mathcal{D} = \{d\}$ and d is a metric then $(X, \mathcal{D})$ is Lipschitz normal.

Lemma 2.4. *Suppose $(X, \mathcal{D})$ is Lipschitz normal and paracompact. Then for every open covering $\mathscr{U}$ of X there exists a subordinated locally finite partition of unity consisting of locally Lipschitz maps.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a locally finite refinement of $\mathscr{U}$ and let $\{V_\lambda : \lambda \in \Lambda\}$ be an open cover of X with $\overline{V}_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$. Let $\rho_\lambda : X \to [0,1]$ be a locally Lipschitz map with $\rho_\lambda|_{\overline{V}_\lambda} \equiv 1$ and $\rho_\lambda|_{X\setminus U_\lambda} \equiv 0$. Then

$$\rho : X \to [1,\infty), \quad \rho(x) = \sum_{\lambda\in\Lambda} \rho_\lambda(x),$$

is well defined and locally Lipschitz because $\overline{V}_\lambda \subset \operatorname{supp}\rho_\lambda \subset \overline{U}_\lambda$, hence each $x \in X$ has a neighbourhood which intersects only finitely many $\operatorname{supp}\rho_\lambda$. The maps $\pi_\lambda := \rho_\lambda/\rho : X \to [0,1]$, $\lambda \in \Lambda$, are also locally Lipschitz and form the required partition of unity. □

We shall now find conditions on the topology of X such that $(X, \mathcal{D})$ is Lipschitz normal. Recall that X is said to be σ-compact if there exists an increasing sequence $X_1 \subset X_2 \subset \dots$ of compact subsets of X whose union is X. If X is σ-compact then it is also paracompact (hence normal) because X is regular.

Theorem 2.1 ([Bartsch and Ding (2006I)]). *If X is σ-compact then $(X, \mathcal{D})$ is Lipschitz normal.*

Proof. Let $\emptyset = X_0 \subset X_1 \subset X_2 \subset \dots$ be compact subsets of X with $X = \bigcup_n X_n$. Let $A, B \subset X$ be disjoint closed subsets. We construct inductively sequences $(V_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$ of open subsets of X such that $V_n \subset V_{n+1}$, $W_n \subset W_{n+1}$, $(X \setminus A) \cup (A \cap X_n) \subset V_n$, $B \cup X_n \subset W_n$, and $\overline{W}_n \cap A \subset V_n$, for all $n \in \mathbb{N}_0$. For $n = 0$ we set $V_0 := X \setminus A$ and choose a neighbourhood W_0 of B with $\overline{W}_0 \subset V_0$. If V_n and W_n have been defined for some $n \ge 0$, observe that

$$A_n := A \cap X_{n+1} \setminus V_n \subset X \setminus \overline{W}_n \quad \text{is compact.} \tag{2.1}$$

According to Lemma 2.3 there exists $d_n \in \overline{\mathcal{D}}$ with

$$\delta_n := \frac{1}{2} d_n(A_n, \overline{W}_n) > 0. \tag{2.2}$$

Now we define

$$V_{n+1} := V_n \cup U_{\delta_n}(A_n; d_n). \tag{2.3}$$

Since $(X \setminus A) \cup (A \cap X_n) \subset V_n$ we have $X_{n+1} \subset (X \setminus A) \cup (A \cap X_{n+1}) \subset V_{n+1}$. By normality there exists an open neighbourhood W'_{n+1} of X_{n+1} with $\overline{W'}_{n+1} \subset V_{n+1}$. Setting $W_{n+1} := W_n \cup W'_{n+1}$ we obtain $B \cup X_{n+1} \subset W_{n+1}$ and $\overline{W}_{n+1} \cap A \subset (\overline{W}_n \cap A) \cup \overline{W'}_{n+1} \subset V_{n+1}$. This finishes the construction of $(V_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$.

For $n \in \mathbb{N}_0$ we now consider the map

$$f_n : X \to [0,1], \quad f_n(x) := \frac{d_n(x, \overline{U}_{\delta_n}(A_n; d_n))}{d_n(x, \overline{U}_{\delta_n}(A_n; d_n)) + d_n(x, X \setminus U_{2\delta_n}(A_n; d_n))} .$$

This map is well defined and locally Lipschitz. Clearly we have

$$f_n(x) = 0 \quad \Leftrightarrow \quad d_n(x, A_n) \le \delta_n$$

and

$$f_n(x) = 1 \quad \Leftrightarrow \quad d_n(x, A_n) \geq 2\delta_n.$$

Since $W_n \subset X \setminus U_{2\delta_n}(A_n; d_n)$ by (2.2) we see that $f_n|_{W_n} \equiv 1$ and therefore $f_m|_{W_n} \equiv 1$ for all $m \geq n$. This implies that the map $f := \inf_{n \in \mathbb{N}_0} f_n$ satisfies $f|_{W_n} = \min_{0 \leq k \leq n} f_k|_{W_n}$. Thus f is locally Lipschitz because $\{W_n : n \in \mathbb{N}_0\}$ is an open cover of X. From $B \subset W_0 \subset W_n$ we deduce $f_n|_B \equiv 1$ for all n, so $f|_B \equiv 1$. Finally, observe that

$$V_n = (X \setminus A) \cup \bigcup_{k=0}^{n-1} U_{\delta_k}(A_k; d_k) \quad \text{for } n \geq 0,$$

hence $f|_{V_n \cap A} \equiv 0$. This yields $f|_{A \cap X_n} \equiv 0$ for all n and thus $f|_A \equiv 0$. □

It is clear that a closed subspace $Y \subset X$ with the induced family $\mathcal{D}_Y$ of semi-metrics $d|_Y : Y \times Y \to \mathbb{R}$ is Lipschitz normal when $(X, \mathcal{D})$ is Lipschitz normal. In [Smirnov (1951)] Smirnov proved that an open F_σ-subspace $Y \subset X$ of a normal space X is normal. Recall that Y is an F_σ-subspace of X if $Y = \bigcup_{n \in \mathbb{N}} Y_n$ is the union of countably many closed subsets Y_n of X. A corresponding result holds for Lipschitz normality.

Theorem 2.2 ([Bartsch and Ding (2006I)]). *Let $(X, \mathcal{D})$ be Lipschitz normal and $Y \subset X$ be an open F_σ-subspace. Then $(Y, \mathcal{D}_Y)$ is Lipschitz normal.*

Proof. Let $Y = \bigcup_{n \in \mathbb{N}} Y_n$ with $Y_n \subset X$ closed and $Y_n \subset Y_{n+1}$ for $n \in \mathbb{N}$. Consider two closed disjoint subsets A, B of Y. We write $\overline{A}$, $\overline{B}$ for the closures of A and B in X. Thus $\overline{A} \cap Y = A$, $\overline{B} \cap Y = B$ and $\overline{A} \cap \overline{B} \cap Y = \emptyset$. As in the proof of Theorem 2.1 we construct inductively open subsets V_n, W_n of Y with $V_n \subset V_{n+1}$, $W_n \subset W_{n+1}$, $(Y \setminus A) \cup (A \cap Y_n) \subset V_n$, $B \cup Y_n \subset W_n$ and $\overline{W}_n \cap A \cap Y \subset V_n$, for all $n \in \mathbb{N}_0$; here $Y_0 := \emptyset$. We set $V_0 := Y \setminus A$ and choose an open neighbourhood $W_0 \subset Y$ of B such that $\overline{W}_0 \cap Y \subset V_0$. This is possible since Y is normal. Suppose V_n, W_n are defined for some $n \geq 0$. Then $A_n := A \cap Y_{n+1} \setminus V_n$ is closed in X and disjoint from the closed subset $\overline{W}_n$ of X. Since X is Lipschitz normal there exists a locally Lipschitz continuous map $f_n : X \to [0, 1]$ with $f_n|_{A_n} \equiv 0$ and $f_n|_{\overline{W}_n} \equiv 1$. We set

$$V_{n+1} := V_n \cup \{x \in Y : f_n(x) < 1/2\}$$

so that

$$Y_{n+1} \subset (Y \setminus A) \cup (A \cap Y_{n+1}) \subset V_{n+1}.$$

As a consequence of the normality of X there exists an open neighbourhood W'_{n+1} of Y_{n+1} with $\overline{W'}_{n+1} \subset V_{n+1}$. We set $W_{n+1} := W_n \cup W'_{n+1}$.

In order to define a Lipschitz map $f : Y \to [0, 1]$ which separates A and B let $\chi : [0, 1] \to [0, 1]$ be defined by $\chi(t) = 0$ for $0 \leq t \leq 1/2$, and $\chi(t) = 2t - 1$ for $1/2 \leq t \leq 1$. Now we define

$$f : Y \to [0, 1], \quad f(x) := \inf_{n \in \mathbb{N}} \chi \circ f_n(x).$$

From $f_n|_{\overline{W}_n} \equiv 1$ we deduce $f_m|_{\overline{W}_n} \equiv 1$ for all $m \geq n$, hence $f|_{\overline{W}_n} = \min_{0\leq k\leq n} \chi\circ f_k|_{\overline{W}_n}$. This implies that $f|_{\overline{W}_n}$ is locally Lipschitz for $n \in \mathbb{N}_0$, and consequently f is locally Lipschitz because $\{W_n : n \in \mathbb{N}\}$ is an open cover of Y. Moreover, $f|_B \equiv 1$ because $B \subset W_0 \subset W_n$ for all $n \in \mathbb{N}_0$. Finally, observe that

$$V_n = (Y \setminus A) \cup \bigcup_{k=0}^{n-1} \{y \in Y : f_k(y) < 1/2\},$$

and that

$$A \cap Y_n \subset A \cap V_n \subset \bigcup_{k=0}^{n-1} \{y \in Y : f_k(y) < 1/2\} \subset \bigcup_{k=0}^{n-1} \{y \in Y : \chi \circ f_k(y) = 0\}.$$

This implies $f|_{A\cap Y_n} \equiv 0$ for all $n \in \mathbb{N}$ and therefore $f|_A \equiv 0$. □

Remark 2.1. From the above proof one sees that each of the locally Lipschitz maps from Y to $[0, 1]$ of Theorem 2.2 can be required to be also a locally Lipschitz map from X to $[0, 1]$.

Next we investigate the behavior of Lipschitz normality with respect to finite products. Recall that the product $X \times Y$ of normal spaces X, Y need not be normal whereas the product of a σ-compact space X and a paracompact space Y is paracompact, hence normal by a result of Michael (see Proposition 4 of [Michael (1953)]). We extend this result to Lipschitz normality. In addition to $(X, \mathcal{D})$ we consider a set Y and a family $\mathcal{E}$ of semi-metrics on Y. Let $\mathcal{T}_\mathcal{E}$ be the associated topology on Y. For $d \in \mathcal{D}$ and $e \in \mathcal{E}$ we have an induced semi-metric $d \times e$ on $Z = X \times Y$ defined by

$$d \times e\big((x_1, y_1), (x_2, y_2)\big) := \max\{d(x_1, x_2), e(y_1, y_2)\}.$$

The topology on $X \times Y$ generated by $\mathcal{D} \times \mathcal{E} = \{d \times e : d \in \mathcal{D}, e \in \mathcal{E}\}$ is the product topology $(X, \mathcal{T}_\mathcal{D}) \times (Y, \mathcal{T}_\mathcal{E})$.

Theorem 2.3 ([Bartsch and Ding (2006I)]). *Let $(X, \mathcal{D})$ be σ-compact and $(Y, \mathcal{E})$ paracompact and Lipschitz normal. Then $(X \times Y, \mathcal{D} \times \mathcal{E})$ is Lipschitz normal.*

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of X with $X = \bigcup_{n\in\mathbb{N}} X_n$ and $X_0 = \emptyset$. We set $Z := X \times Y$ and $Z_n := X_n \times Y$, $n \in \mathbb{N}$. Let A, B be closed subsets of Z and set $A_y := A\cap X \times \{y\}$ for $y \in Y$. We proceed as in the proof of Theorem 2.2 and construct inductively increasing sequences $(V_n)_{n\in\mathbb{N}}$, $(W_n)_{n\in\mathbb{N}}$ of open subsets of Z with $(Z \setminus A) \cup (A \cap Z_n) \subset V_n$, $B \cup Z_n \subset W_n$, $\overline{W}_n \cap A \subset V_n$. The inductive step also leads to a locally Lipschitz map $f_n : X \to [0, 1]$ which will be used later to finish the proof.

We begin with $V_0 := Z \setminus A$ and an open set W_0 satisfying $B \subset W_0$ and $\overline{W}_0 \subset V_0$. Here we used that Z is normal. Suppose V_n, W_n are given for some $n \geq 0$. Then $A_y \cap Z_{n+1} \setminus V_n$ is compact and disjoint from $\overline{W}_n$, for any $y \in Y$. Thus there exist open sets $W_y, V_y \subset X$, and $e_y \in \overline{\mathcal{E}}$, $\varepsilon_y > 0$ such that $\overline{V}_y \subset W_y$, and

$$A_y \cap Z_{n+1} \setminus V_n \subset V_y \times U_{\varepsilon_y/2}(y; e_y) \subset \overline{W}_y \times \overline{U}_{\varepsilon_y}(y; e_y) \subset Z \setminus \overline{W}_n.$$

Let $P_Y : X \times Y \to Y$ be the projection. Since X_n is compact the restriction $P_Y|_{Z_n}$ is closed. Thus $P_Y(A \cap Z_{n+1} \setminus V_n)$ is a closed subset of Y and therefore paracompact. Consequently there exists a locally finite open refinement $\{N_\lambda : \lambda \in \Lambda_n\}$ of the covering $\{U_{\varepsilon_y/2}(y; e_y) : y \in P_Y(A \cap Z_{n+1} \setminus V_n)\}$ of $P_Y(A \cap Z_{n+1} \setminus V_n)$. There also exists an open covering $\{P_\lambda : \lambda \in \Lambda_n\}$ of $P_Y(A \cap Z_{n+1} \setminus V_n)$ satisfying $\overline{P}_\lambda \subset N_\lambda$. For $\lambda \in \Lambda_n$ we choose $y_\lambda = y$ with $N_\lambda \subset U_{\varepsilon_y/2}(y; e_y)$. Then $\{V_{y_\lambda} \times P_\lambda : \lambda \in \Lambda_n\}$ and $\{W_{y_\lambda} \times N_\lambda : \lambda \in \Lambda_n\}$ are locally finite open (in $X \times Y$) covers of $A \cap Z_{n+1} \setminus V_n$ such that

$$\overline{V}_{y_\lambda} \times \overline{P}_\lambda \subset W_{y_\lambda} \times N_\lambda \subset \overline{W}_{y_\lambda} \times \overline{N}_\lambda \subset Z \setminus \overline{W}_n.$$

We set

$$V_{n+1} := V_n \cup \bigcup_{\lambda \in \Lambda_n} (V_{y_\lambda} \times P_\lambda)$$

so that

$$Z_{n+1} \subset (Z \setminus A) \cup (A \cap Z_{n+1}) \subset V_{n+1}.$$

Since $X \times Y$ is normal there exists an open neighbourhood W'_{n+1} of Z_{n+1} in $X \times Y$ with $\overline{W'}_{n+1} \subset V_{n+1}$. Setting $W_{n+1} := W_n \cup W'_{n+1}$ we clearly have $B \cup Z_{n+1} \subset W_{n+1}$ and

$$\overline{W}_{n+1} \cap A \subset (\overline{W}_n \cap A) \cup \overline{W'}_{n+1} \subset V_{n+1}.$$

Now we construct the map $f_n : X \to [0,1]$. For $\lambda \in \Lambda_n$ let $g_\lambda : X \to [0,1]$ be a locally Lipschitz map with $g_\lambda|_{\overline{V}_{y_\lambda}} \equiv 0$ and $g_\lambda|_{X \setminus W_{y_\lambda}} \equiv 1$. It exists because $(X, \mathcal{D})$ is Lipschitz normal by Theorem 2.1. Similarly, let $h_\lambda : Y \to [0,1]$ be locally Lipschitz satisfying $h_\lambda|_{\overline{P}_\lambda} \equiv 0$ and $h_\lambda|_{Y \setminus \overline{N}_\lambda} \equiv 1$. Now we define

$$f_{n+1} : X \times Y \to [0,1], \quad f_{n+1}(x,y) := \inf_{\lambda \in \Lambda_n} \max\{g_\lambda(x), h_\lambda(y)\}.$$

Setting

$$g_\lambda \times h_\lambda : X \times Y \to [0,1], \quad (x,y) \mapsto \max\{g_\lambda(x), h_\lambda(y)\},$$

we see that $g_\lambda \times h_\lambda|_{\overline{V}_{y_\lambda} \times \overline{P}_\lambda} \equiv 0$ and $g_\lambda \times h_\lambda|_{Z \setminus (W_{y_\lambda} \times N_\lambda)} \equiv 1$. Clearly $g_\lambda \times h_\lambda$ is locally Lipschitz because g_λ and h_λ have this property. Since $\{W_{y_\lambda} \times N_\lambda : \lambda \in \Lambda_n\}$ is locally finite it follows that for each $(x,y) \in X \times Y$ there exists a neighbourhood U of (x,y) and a finite set $\Lambda \subset \Lambda_n$ with $f_{n+1}|_U = \min_{\lambda \in \Lambda} g_\lambda \times h_\lambda|_U$. This implies that f_{n+1} is locally Lipschitz. Finally we define the map

$$f := \inf_n f_n : X \times Y \to [0,1], \quad f(x,y) = \inf_{n \in \mathbb{N}} f_n(x,y).$$

By construction we have $f_n|_{\overline{W}_n} \equiv 1$ because $\overline{W}_{y_\lambda} \times \overline{N}_\lambda \subset X \times Y \setminus W_n$ for every λ. This implies the local Lipschitz continuity of f as in the proof of Theorem 2.2. Clearly $f|_B \equiv 1$ because $B \subset W_0 \subset \overline{W}_n$ for every $n \in \mathbb{N}_0$. And $f|_A \equiv 0$ follows inductively from

$$A \cap Z_{n+1} \setminus V_n \subset \bigcup_{\lambda \in \Lambda_n} (V_{y_\lambda} \times P_\lambda)$$

and $f_n|_{V_{y_\lambda} \times P_\lambda} \equiv 0$ for every n. □

Example 2.1. Let B be a Banach space, $X = B^*$ its dual, and $B_0 \subset B$ be arbitrary which separates points. Define $\mathcal{D}_0 = \{d_b : b \in B_0\}$ by $d_b(x,y) = |\langle b, x-y\rangle_{B,B^*}|$ for $x, y \in X$. The topology $\mathcal{T}_0$ generated by $\mathcal{D}_0$ is contained in the weak* topology on B^*, and it coincides with the weak* topology if $B_0 = B$. By the Banach-Alaoglu theorem $(B^*, \mathcal{T}_0)$ is σ-compact, and $(B^*, \mathcal{D}_0)$ is Lipschitz normal as a consequence of Theorem 2.1.

If in addition B_0 is countable then $(B^*, \mathcal{T}_0)$ is perfectly normal, that is, it is normal and every closed subset of $(B^*, \mathcal{T}_0)$ is a G_δ-subset. In fact, it is easy to check that

$$A = \bigcap_{b \in B_0} \bigcap_{m \in \mathbb{N}} \{x \in X : d_b(x, A) < 1/m\}.$$

We proved that every $\mathcal{T}_0$-closed subset is a G_δ-subset, hence every $\mathcal{T}_0$-open subset of $(B^*, \mathcal{T}_0)$ is an F_σ-subset. Thus $(B^*, \mathcal{D}_0)$ is paracompact and Lipschitz normal by Theorem 2.1. Moreover, if $(Y, \mathcal{E})$ is Lipschitz normal and paracompact then $(B^* \times Y, \mathcal{D}_0 \times \mathcal{E})$ is Lipschitz normal and paracompact. If $(Y, \mathcal{E})$ is a metric space then $B^* \times Y$ is perfectly normal; see Proposition 5 of [Michael (1953)].

We remark that if $C \subset B_0$ is a countable subset then any $\mathcal{T}_C$-closed subset A of X is a G_δ-subset of $(B^*, \mathcal{T}_0)$, where $\mathcal{T}_C$ denotes the topology generated by $C_0 := \{d_c : c \in C\}$.

In conclusion: If B is a separable Banach space, $B_0 \subset B$ a countable dense subset, and (Y, d) a metric space then $(B^* \times Y, \mathcal{D}_0 \times \{d\})$ and every open subset of this product gage space is paracompact and Lipschitz normal. Consequently also every locally closed subset (being a closed subset of an open subset) is paracompact and Lipschitz normal.

Appendix

We collect for the reader's convenience some topological concepts which we used previously (see [Kelley (1995)]).

Definition A.1. Let X be a set. A nonnegative real function $d(\cdot,\cdot)$ defined on $X \times X$ is called a *semi-metric* if it satisfies:

(1) $d(x,x) = 0$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \le d(x,z) + d(z,x)$.

In the following let X denote a topological space.

Definition A.2. X is said to be *Hausdorff* if for any $x \neq y \in X$ there exist two disjoint open subsets U and V of X such that $x \in U$ and $y \in V$. It is said to be *regular* if for any closed subset A and any element $x \notin A$ there exist two disjoint open subsets U and V such that $A \subset U$ and $x \in V$. It is said to be *normal* if for any two disjoint closed subsets A and B there exist two disjoint open subsets U and

V such that $A \subset U$ and $B \subset V$.

Theorem A.3. (1) *Assume X is regular. If U is an open subset of X and $x \in U$, then there is an open subset V of X such that $x \in V \subset \overline{V} \subset U$.*

(2) *Assume X is normal. If A is a closed subset and U is an open subset with $A \subset U$ then there exists an open subset V such that $A \subset V \subset \overline{V} \subset U$.*

Theorem A.4.(Urysohn) *X is normal if and only if for any two disjoint closed subsets A and B there is a continuous map $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.*

Definition A.5. X is said to be *completely regular* if for any closed subset A and any element $x \notin A$ there is a continuous map $f : X \to [0, 1]$ satisfying $f(x) = 0$ and $f(y) = 1$ for all $y \in A$.

Definition A.6. X is said to be *paracompact* if any open covering of X possesses an open locally finite refinement.

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Chapter 3

Deformations on locally convex topological vector spaces

Let E be a real vector space and $\mathcal{P}$ a family of semi-norms on E which separates points. To each $p \in \mathcal{P}$ we associate a semi-metric d_p, defined by $d_p(x,y) = p(x-y)$. We write $\overline{\mathcal{P}}$ for the saturation of $\mathcal{P}$ which consists of all finite maxima of elements of $\mathcal{P}$. Then $\overline{\mathcal{D}} = \{d_p : p \in \overline{\mathcal{P}}\}$. The topology $\mathcal{T}_{\mathcal{P}} = \mathcal{T}_{\mathcal{D}}$ induced by $\mathcal{P}$ or $\mathcal{D}$ on E turns E into a locally convex, Hausdorff topological vector space. All topological notions for E refer to this topology, all Lipschitz notions to $\mathcal{D}$, except if explicitly stated otherwise. In our applications, E is a Banach space with respect to a norm $\|\cdot\| \notin \mathcal{P}$, and $\mathcal{T}_{\mathcal{P}}$ is contained in the weak topology.

Consider an open subset $W \subset E$, a locally finite partition $\{\pi_j : j \in J\}$ on W, and a family $\{w_j : j \in J\}$ in E. We assume that the maps $\pi_j : E \to [0,1]$ are locally Lipschitz continuous (cf. Remark 2.1). Setting

$$f : W \to E, \quad f(u) = \sum_{j\in J} \pi_j(u) w_j,$$

it is clear that for $u \in W$ the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\varphi(t,u) = f(\varphi(t,u)) \\ \quad \varphi(0,u) = u \end{cases} \tag{3.1}$$

has a unique solution

$$\varphi(\,\cdot\,,u) : I_u = (T^-(u), T^+(u)) \to W$$

defined on a maximal interval $I_u \subset \mathbb{R}$. In fact, there exists a neighbourhood $U \subset W$ of u so that $J_u := \{j \in J : U \cap \operatorname{supp} \pi_j \neq \emptyset\}$ is finite. Let F_u be the span of u and w_j, $j \in J_u$. Then the Cauchy problem

$$\begin{cases} \dot\eta(t) = f\big(\eta(t)\big) = \displaystyle\sum_{j\in J_u} \pi_j(\eta(t)) w_j \\ \eta(0) = u \end{cases}$$

has a unique solution $\eta_\delta : [-\delta,\delta] \to F_u$ for $\delta > 0$ small enough because $f|_U$ is locally Lipschitz continuous. One can now argue as in the case of ordinary differential equations in order to obtain the maximal solution. Observe that for $I \subset I_u$ compact the set $\varphi(I,u) = \{\varphi(t,u) : t \in I\}$ is contained in a finite-dimensional subspace. This is not the case for the whole trajectory $\varphi(I_u,u)$, in general. Setting

$$\mathcal{O} := \{(t,u) : u \in W, t \in I_u\} \subset \mathbb{R} \times W$$

we have a map $\varphi : \mathcal{O} \to W$ which is a flow on W.

Theorem 3.1. *The following conclusions are true:*

a) $\mathcal{O}$ *is an open subset of* $\mathbb{R} \times W$*;*

b) φ *is locally Lipschitz.*

Proof. a) Let $(t_0,u_0) \in \mathcal{O}$ and suppose without loss of generality that $t_0 = 0$. We choose $t_1, t_2 \in I_{u_0}$ with $t_1 < 0$ and $t_2 > 0$. The set $K := \varphi([t_1,t_2],u_0)$ is compact, so there exists an open neighbourhood U of K with $J_0 := \{j \in J : U \cap \operatorname{supp} \pi_j \neq \emptyset\}$ being finite, and such that $\pi_j|_U$ is Lipschitz for $j \in J_0$. Hence there exists $p \in \overline{\mathcal{P}}$ and $\lambda > 0$ with

$$|\pi_j(u) - \pi_j(v)| \le \lambda p(u-v) \quad \text{for all } u,v \in U$$

and $d(K, E \setminus U) > 0$ where $d = d_p \in \overline{\mathcal{D}}$. We choose $\delta > 0$ with $\delta < d(K, E \setminus U)$, set $M := \sum_{j \in J_0} \lambda p(w_j)$ and choose $\varepsilon > 0$ with $\varepsilon \le \delta / 2e^{M(t_2 - t_1)}$. We claim that for $u \in U_\varepsilon(u_0; d)$ the orbit $\varphi(t,u)$ is defined on $[t_1,t_2]$ and lies in $U_\delta(K;d) \subset U$. Suppose to the contrary that there exists $t_3 \in (0,t_2]$ with $\varphi(t,u) \in U_\delta(K;d)$ for $t \in [0,t_3)$ and $d(\varphi(t_3,u),K) = \delta$. Then

$$\begin{aligned}
&p(\varphi(t,u) - \varphi(t,u_0)) \\
&\qquad \le p(u-u_0) + p\left(\int_0^t \big(f(\varphi(s,u)) - f(\varphi(s,u_0))\big) ds\right) \\
&\qquad \le p(u-u_0) + \sum_{j \in J_0} p(w_j) \int_0^t |\pi_j(\varphi(s,u)) - \pi_j(\varphi(s,u_0))| ds \\
&\qquad \le p(u-u_0) + \sum_{j \in J_0} \lambda p(w_j) \int_0^t p(\varphi(s,u) - \varphi(s,u_0)) ds \\
&\qquad = p(u-u_0) + M \int_0^t p(\varphi(s,u) - \varphi(s,u_0)) ds.
\end{aligned}$$

Now Gronwall's inequality yields

$$p(\varphi(t,u) - \varphi(t,u_0)) \le p(u-u_0)e^{Mt} < \varepsilon e^{Mt} \le \delta/2 \tag{3.2}$$

for $t \in [0,t_2]$ contradicting $d(\varphi(t_3,u),K) = \delta$.

Thus we have shown that $[0,t_2] \subset I_u$ for $u \in U_\varepsilon(u_0,d)$. Similarly one sees that $[t_1,0] \subset I_u$ for $u \in U_\varepsilon(u_0,d)$. It follows that $[t_1,t_2] \times U_\varepsilon(u_0,d) \subset \mathcal{O}$.

b) Since φ is differentiable with respect to t it suffices to show that φ is locally Lipschitz with respect to u. The argument proceeds as in a) and is essentially standard. In fact, given $(t_0,u_0) \in \mathcal{O}$ one can produce a neighbourhood N of (t_0,u_0) in $\mathcal{O}$, $p \in \overline{\mathcal{P}}$ and $M > 0$ so that

$$p(\varphi(t,u) - \varphi(t,v)) \le p(u-v)e^{M|t|} \quad \text{for } (t,u),(t,v) \in N;$$

compare the proof of (3.2) above. □

For the critical point theory which we want to develop now, we suppose there exists a norm $\|\cdot\| : E \to \mathbb{R}$ on E so that $(E, \|\cdot\|)$ is a Banach space, and so that all $p \in \mathcal{P}$ are of the form $p(u) = |u_p^*(u)|$ for some $u_p^* \in E^*$. Thus the topology $\mathcal{T}_\mathcal{P}$ induced by $\mathcal{P}$ is contained in the weak topology of E. We distinguish the topologies by using notions like $\mathcal{P}$-open, $\mathcal{P}$-closed to refer to $\mathcal{T}_\mathcal{P}$, versus norm open, norm closed to refer to the norm topology. Observe that if a map $f : (E, \mathcal{P}) \to (M, d)$ into a metric space is (locally) Lipschitz then also $f : (E, \|\cdot\|) \to (M, d)$ is (locally) Lipschitz. We assume in the remainder of the chapter that every $\mathcal{P}$-open subset of E is paracompact and Lipschitz normal with respect to $\mathcal{P}$.

We consider a functional $\Phi : E \to \mathbb{R}$ which we assume to be C^1 with respect to the norm $\|\cdot\|$. For $a, b \in \mathbb{R}$ we write $\Phi^a := \{u \in E : \Phi(u) \le a\}$, $\Phi_a := \{u \in E : \Phi(u) \ge a\}$, and $\Phi_a^b := \Phi_a \cap \Phi^b$. In our applications the functional Φ is $\mathcal{P}$-upper semicontinuous but not $\mathcal{P}$-continuous. The sets Φ_a have empty $\mathcal{P}$-interior and the sets Φ^a are not $\mathcal{P}$-closed, any $a \in \mathbb{R}$. Moreover, the map $\Phi' : (E, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is not continuous, only its restriction to Φ_a. Here and after $\mathcal{T}_{w^*}$ denotes the weak* topology on E^*. The map

$$\tau(u) := \sup\{t \ge 0 : \varphi(t, u) \in \Phi^a\}$$

is not $\mathcal{P}$-continuous, and there may be no continuous map $r : (\Phi^b, \mathcal{T}_\mathcal{P}) \to (\Phi^a, \mathcal{T}_\mathcal{P})$ which is the identity on Φ^a.

The following theorem is a $\mathcal{P}$-continuous version of the non-critical interval theorem in critical point theory.

Theorem 3.2 ([Bartsch and Ding (2006I)]). *Consider $a, b \in \mathbb{R}$ with $a < b$ so that Φ_a is $\mathcal{P}$-closed and $\Phi' : (\Phi_a^b, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Suppose moreover that*

$$\alpha := \inf\{\|\Phi'(u)\| : u \in \Phi_a^b\} > 0. \tag{3.3}$$

Then there exists a deformation $\eta : [0,1] \times \Phi^b \to \Phi^b$ with the properties:

(i) η is continuous with either the $\mathcal{P}$-topology or the norm topology on Φ^b;
(ii) for each t the map $u \mapsto \eta(t, u)$ is a homeomorphism of Φ^b onto $\eta(t, \Phi^b)$ with the $\mathcal{P}$-topology or with the norm topology;
(iii) $\eta(0, u) = u$ for all $u \in \Phi^b$;
(iv) $\eta(t, \Phi^c) \subset \Phi^c$ for all $c \in [a, b]$ and all $t \in [0, 1]$;
(v) $\eta(1, \Phi^b) \subset \Phi^a$;
(vi) each point $u \in \Phi^b$ has a $\mathcal{P}$-neighbourhood U in Φ^b so that the set $\{v - \eta(t, v) : v \in U, 0 \le t \le 1\}$ is contained in a finite-dimensional subspace of E;
(vii) if a finite group G acts isometrically on E and if Φ is G-invariant, then η is equivariant in u.

Here G acts isometrically on E if each $g \in G$ induces a bounded linear map $R_g \in \mathscr{L}(E)$ which preserves the norm, and such that the unit $e \in G$ induces the identity operator $R_e = Id_E$ and $R_g \circ R_h = R_{gh}$ for any $g, h \in G$. Observe that

$R_g : (E, \mathcal{T}_\mathcal{P}) \to (E, \mathcal{T}_\mathcal{P})$ is also continuous. We simply write $gu := R_g(u)$ as usual. The most important example is the antipodal action of $G = \{1, -1\} \cong \mathbb{Z}/2$ on E.

Proof. For each $u \in \Phi_a^b$ we choose $w(u) \in E$ with $\|w(u)\| \le 2$ and such that $\Phi'(u)w(u) > \|\Phi'(u)\|$. There exists a $\mathcal{P}$-open neighbourhood $N(u)$ of u in E so that

$$\Phi'(v)w(u) > \|\Phi'(u)\| \quad \text{for all } v \in N(u) \cap \Phi_a^b.$$

For $u \in E \setminus \Phi_a$ we set $N(u) := E \setminus \Phi_a$. Then $W := \bigcup_{u \in \Phi^b} N(u)$ is a $\mathcal{P}$-open subset of E containing Φ^b. Let $\{U_j : j \in J\}$ be a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement of the covering $\{N(u) : u \in \Phi^b\}$, and let $\{\pi_j : j \in J\}$ be a $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz partition of unity subordinated to $\{U_j : j \in J\}$. For $j \in J$ with $U_j \cap \Phi_a \neq \emptyset$ we choose $u_j \in \Phi_a^b$ so that $U_j \subset N(u_j)$, and we set $w_j := w(u_j)$. For $j \in J$ with $U_j \cap \Phi_a = \emptyset$ we set $w_j := 0$. Now we define the vector field

$$f : W \to E, \quad f(u) := \frac{a-b}{\alpha} \sum_{j \in J} \pi_j(u) w_j \ ,$$

which is locally Lipschitz with respect to the norm. Let $\varphi(t,u)$ be the associated flow on W which is both norm continuous as well as $\mathcal{P}$-continuous. Since $\|f(u)\| \le 2(b-a)/\alpha$ for every $u \in W$ and since

$$\Phi'(u)f(u) \le a - b < 0 \quad \text{for } u \in \Phi_a^b$$

we see that $\varphi(t,u)$ is defined for all $(t,u) \in [0,\infty) \times \Phi^b$ and that $\eta := \varphi|_{[0,1]\times\Phi^b}$ satisfies (i)–(v). Property (vi) follows from the fact that f is $\mathcal{P}$-locally finite-dimensional. Finally, if Φ is G-invariant we replace $f(u)$ by $\widetilde{f}(u) := \frac{1}{|G|}\sum_{g \in G} gf(g^{-1}u)$. The corresponding flow $\widetilde{\varphi}$ has all properties of φ and is equivariant in u because $\widetilde{f}$ is equivariant. □

Recall that $(u_n)_n$ is a $(PS)_c$-sequence if $\Phi(u_n) \to c$ and $\|\Phi'(u_n)\| \to 0$ as $n \to \infty$. We say that $(u_n)_n$ is a $(C)_c$-sequence if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ as $n \to \infty$. A set $\mathscr{A} \subset E$ is said to be a $(PS)_c$-attractor if for any $\varepsilon, \delta > 0$ and any $(PS)_c$-sequence there exist $n_0 \in \mathbb{N}$ with $u_n \in U_\varepsilon(\mathscr{A} \cap \Phi_{c-\delta}^{c+\delta})$ for $n \ge n_0$. This concept is due to [Bartsch and Ding (1999)]. Similarly we define a $(C)_c$-attractor if this property holds for $(C)_c$-sequences. A $(PS)_c$-attractor is a $(C)_c$-attractor but not vice versa. Given $I \subset \mathbb{R}$ we say $\mathscr{A}$ is a $(PS)_I$-attractor, or $(C)_I$-attractor, if $\mathscr{A}$ is a $(PS)_c$-attractor, or $(C)_c$-attractor, respectively, for every $c \in I$.

Theorem 3.2 results immediately the following consequence.

Corollary 3.1. *Suppose $c \in \mathbb{R}$ is a regular value of Φ. Suppose moreover that there exists $\varepsilon_0 > 0$ so that $\Phi_{c-\varepsilon}$ is $\mathcal{P}$-closed for $0 < \varepsilon \le \varepsilon_0$, and such that $\Phi' : (\mathrm{clos}_\mathcal{P}(\Phi_{c-\varepsilon_0}^{c+\varepsilon_0}), \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Then, if Φ satisfies the $(PS)_c$-condition there exists $\delta > 0$ and a deformation $\eta : [0,1] \times \Phi^{c+\delta} \to \Phi^{c+\delta}$ satisfying the properties (i) - (vii) from Theorem 3.2 with $a := c - \delta$, $b := c + \delta$.*

Motivated by our applications we now consider the following situation. Suppose $E = X \oplus Y$ where X and Y are Banach spaces and X is separable and reflexive. Let $\mathcal{S} \subset X^*$ be a dense subset, let $\mathcal{Q}$ be the corresponding set of semi-norms $q_s(x) := \left|\langle x, s\rangle_{X,X^*}\right|$, $s \in \mathcal{S}$, on X, and $\mathcal{D} = \{d_s : s \in \mathcal{S}\}$ be the associated family of semi-metrics on $X \cong X^{**}$ as defined in Example 2.1. Let $\mathcal{P}$ be the family of semi-norms on E consisting of all semi-norms

$$p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x+y) = q_s(x) + \|y\|, \qquad s \in \mathcal{S}.$$

$\mathcal{P}$ induces the product topology on E given by the $\mathcal{Q}$-topology on X and the norm topology on Y. It is contained in the product topology $(X, \mathcal{T}_w) \times (Y, \|\cdot\|)$ on E. The product $(X \times Y, \mathcal{D} \times \{\|\cdot\|\})$ is a product gage space as described in Example 2.1. Recall that we have assumed that every $\mathcal{P}$-open subset is paracompact and Lipschitz normal in this chapter (this is the case for example if $\mathcal{S}$ is additionally countable). By $P_X : E = X \oplus Y \to X$ we denote the continuous projection onto X along Y, and by $P_Y := I - P_X : E \to Y$ the complementary projection.

Theorem 3.3 ([Bartsch and Ding (2006I)]). *Consider $a, b \in \mathbb{R}$ with $a < b$ so that Φ_a is $\mathcal{P}$-closed and $\Phi' : (\Phi_a^b, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Suppose moreover that*

$$\alpha := \inf\{(1 + \|u\|)\|\Phi'(u)\| : u \in \Phi_a^b\} > 0 \tag{3.4}$$

and

$$\text{there exists } \gamma > 0 \text{ with } \|u\| < \gamma\|P_Y u\| \text{ for all } u \in \Phi_a^b. \tag{3.5}$$

Then there exists a deformation $\eta : [0,1] \times \Phi^b \to \Phi^b$ with the properties (i)–(vii) from Theorem 3.2.

Proof. Observe that, by (3.5), given $u \in \Phi_a^b$, the set

$$\mathscr{E}_u := \{v \in E : \gamma\|P_Y v\| > \|u\|\}$$

is a $\mathcal{P}$-open neighborhood of u.

As before, for each $u \in \Phi_a^b$ we choose $w(u) \in E$ with $\|w(u)\| \le 2$ such that $\Phi'(u)w(u) > \|\Phi'(u)\|$. There is a $\mathcal{P}$-open neighborhood $N(u) \subset \mathscr{E}_u$ of u such that $\Phi'(v)w(u) > \|\Phi'(u)\|$, hence jointly with (3.4),

$$(1 + \|u\|)\Phi'(v)w(u) > (1 + \|u\|)\|\Phi'(u)\| \ge \alpha \quad \text{for } v \in N(u) \cap \Phi_a^b. \tag{3.6}$$

For $u \in E \setminus \Phi_a$ we set $N(u) := E \setminus \Phi_a$. Set $W := \bigcup_{u \in \Phi^b} N(u)$. Let $\{U_j : j \in J\}$ be a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement of $\{N(u) : u \in \Phi^b\}$, and let $\{\pi_j : j \in J\}$ be a $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz partition of unity subordinated to $\{U_j : j \in J\}$. For $j \in J$ with $U_j \cap \Phi_a \neq \emptyset$ we choose $u_j \in \Phi_a^b$ so that $U_j \subset N(u_j)$, and we set $w_j := (1 + \|u_j\|)w(u_j)$. For $j \in J$ with $U_j \cap \Phi_a = \emptyset$ we set $w_j := 0$. Define the vector field

$$f : W \to E, \quad f(u) := \frac{a-b}{\alpha} \sum_{j \in J} \pi_j(u) w_j \ ,$$

which is locally Lipschitz with respect to the norm. Let $\varphi(t,u)$ be the associated flow on W which is both norm continuous as well as $\mathcal{P}$-continuous. Since $\operatorname{supp}\pi_j \subset U_j \subset \mathscr{E}_{u_j}$ if $u_j \in \Phi_a^b$ and $w_j = 0$ if $U_j \cap \Phi_a = \emptyset$, one has

$$\|f(u)\| \le \frac{2(b-a)}{\alpha}(1+\gamma\|u\|) \text{ for every } u \in W.$$

Moreover, by definition and (3.6),

$$\Phi'(u)f(u) \le a-b < 0 \quad \text{for } u \in \Phi_a^b.$$

Thus $\varphi(t,u)$ is defined for all $(t,u) \in [0,\infty) \times \Phi^b$ and that $\eta := \varphi|_{[0,1]\times\Phi^b}$ satisfies (i)–(vi). Finally, if Φ is G-invariant we replace $f(u)$ by $\widetilde{f}(u) := \frac{1}{|G|}\sum_{g\in G} gf(g^{-1}u)$. The corresponding flow $\widetilde{\varphi}$ has all properties of φ and is equivariant in u because $\widetilde{f}$ is equivariant. □

As a consequence we have

Corollary 3.2. *Suppose $c \in \mathbb{R}$ is a regular value of Φ. Suppose moreover that there exists $\varepsilon_0 > 0$ so that $\Phi_{c-\varepsilon}$ is $\mathcal{P}$-closed for $0 < \varepsilon \le \varepsilon_0$, and such that $\Phi' : (\operatorname{clos}_{\mathcal{P}}(\Phi_{c-\varepsilon_0}^{c+\varepsilon_0}), \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Then, if Φ satisfies* (3.5) *and the $(C)_c$-condition there exists $\delta > 0$ and a deformation $\eta : [0,1] \times \Phi^{c+\delta} \to \Phi^{c+\delta}$ satisfying the properties (i)– (vii) from Theorem 3.2 with $a := c-\delta$, $b := c+\delta$.*

Now we treat the case where (3.3) (or (3.4)) does not hold, that is, there exist $(PS)_c$-sequences (or $(C)_c$-sequences) for some $c \in [a,b]$. One can prove various versions of deformation lemmas in the presence of (PS)-sequences or (C)-sequences with $\mathcal{P}$-continuous deformations. The next result is a noncritical interval theorem when Φ' is not bounded away from 0.

Theorem 3.4 ([Bartsch and Ding (2006I)]). *Consider $a,b \in \mathbb{R}$ with $a < b$, $I := [a,b]$, so that Φ_a is $\mathcal{P}$-closed. Suppose $\Phi' : (\operatorname{clos}_{\mathcal{P}}(\Phi_a^b), \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous and*

$$\Phi'(u) \ne 0 \textit{ for all } u \in \operatorname{clos}_{\mathcal{P}}(\Phi_a^b). \tag{3.7}$$

Then the following holds.

a) *If Φ has a $(PS)_I$-attractor $\mathscr{A}$ so that $P_X\mathscr{A} \subset X$ is bounded and*

$$\beta := \inf\{\|P_Y u - P_Y v\| : u,v \in \mathscr{A}, P_Y u \ne P_Y v\} > 0 \tag{3.8}$$

then there exists a deformation $\eta : [0,1]\times\Phi^b \to \Phi^b$ with the properties (i), (iii)–(vii) from Theorem 3.2.

b) *If Φ has a $(C)_I$-attractor $\mathscr{A}$ so that* (3.8) *holds, $P_Y\mathscr{A} \subset Y$ is bounded, and if* (3.5) *holds, then there exists η as in* a).

Proof. We only prove b) which is a bit more difficult and mention the changes for the proof of a) at the end.

We set $B := P_Y\mathscr{A}$ and denote $U_\sigma := X \times U_\sigma(B)$ for $\sigma > 0$; here $U_\sigma(B) = \{y \in Y : \mathrm{dist}_{\|\cdot\|}(y, B) < \sigma\}$. Clearly U_σ is $\mathcal{P}$-open. We fix $\sigma < \beta/2$ and observe that

$$\alpha := \inf\{(1 + \|u\|) \cdot \|\Phi'(u)\| : u \in \Phi_a^b \setminus U_{\sigma/4}\} > 0$$

because $\mathscr{A}$ is a $(C)_I$-attractor. By (3.7) we may choose for every $u \in \Phi_a^b$ a pseudo-gradient vector $w(u) \in E$ with $\|w(u)\| \le 2$ and $\Phi'(u)w(u) > \|\Phi'(u)\|$. For $u \in \Phi_a^b \setminus U_{\sigma/2}$ there exists a $\mathcal{P}$-open neighbourhood $N(u) \subset X \times U_{\sigma/4}(P_Y u)$ in E such that

$$(1 + \|u\|)\Phi'(v)w(u) > (1 + \|u\|)\|\Phi'(u)\| \ge \alpha \quad \text{for all } v \in N(u) \cap \Phi_a^b.$$

For $u \in \Phi_a^b \cap U_{\sigma/2}$ there exists a $\mathcal{P}$-open neighbourhood $N(u) \subset U_{3\sigma/4}$ with

$$\Phi'(v)w(u) > \|\Phi'(u)\| > 0 \quad \text{for all } v \in N(u) \cap \Phi_a^b.$$

Finally, for $u \in E \setminus \Phi_a$ we set $N(u) := E \setminus \Phi_a$ which is also $\mathcal{P}$-open by assumption. Then $W := \bigcup_{u \in \Phi^b} N(u)$ is a $\mathcal{P}$-open subset of E. Let $\{U_j : j \in J\}$ be a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement of the covering $\{N(u) : u \in \Phi^b\}$ of W, and let $\{\pi_j : j \in J\}$ be a $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz continuous partition of unity subordinated to $\{U_j : j \in J\}$. For $j \in J$ with $U_j \cap \Phi_a \ne \emptyset$ we choose $u_j \in \Phi_a^b$ with $U_j \subset N(u_j)$, and we set $w_j := (1 + \|u_j\|)w(u_j)$. If $U_j \cap \Phi_a = \emptyset$ we set $w_j := 0$. We consider the vector field

$$f : W \to E, \quad f(u) := -\sum_{j \in J} \pi_j(u)w_j,$$

and the associated flow $\varphi(t, u)$ on W. As before φ is continuous both with the norm topology on W and with the $\mathcal{P}$-topology on W. We have that $\Phi'(u)f(u) \le -\alpha$ for $u \notin U_{3\sigma/4}$. If $\pi_j(u)w_j \ne 0$ then $u \in N(u_j)$ for $u_j \in \Phi_a^b$. In the case $u_j \in \Phi_a^b \setminus U_{\sigma/2}$ we have $N(u_j) \subset X \times U_{\sigma/2}(P_Y u_j)$, hence $\|P_Y u_j - P_Y u\| < \sigma/2$. In the case $u_j \in \Phi_a^b \cap U_{\sigma/2}$ we have $N(u_j) \subset U_\sigma$, hence $\|P_Y u_j\| \le \sigma + c$ where c is a bound for $B = P_Y\mathscr{A}$ which exists by assumption. In any case it follows from (3.5) that $\|u_j\| \le C(1 + \|u\|)$ for some $C > 0$, provided $\pi_j(u)w_j \ne 0$. From this we obtain

$$\|f(u)\| \le \sum_{j \in J} \pi_j(u)(1 + \|u_j\|)\|w(u_j)\| \le 2C(1 + \|u\|) \tag{3.9}$$

for all $u \in \Phi^b$. This implies that $\varphi(t, u)$ is defined for all $t \ge 0$, $u \in \Phi^b$. By construction we have

$$\Phi'(u)f(u) < 0 \quad \text{for all } u \in \Phi_a^b$$

and

$$\Phi'(u)f(u) \le -\alpha < 0 \quad \text{for all } u \in \Phi_a^b \setminus U_{\sigma/2}.$$

We claim that for $u \in \Phi^b$ there exists $T(u) > 0$ with $\Phi(\varphi(T(u), u)) < a$. Arguing indirectly we assume $\varphi(t, u) \in \Phi_a^b$ for all $t \ge 0$, some $u \in \Phi^b$. A standard argument using $\Phi'(u)f(u) \le -\alpha$ for $u \notin U_{3\sigma/4}$ yields that there exists $T > 0$ with $\varphi(t, u) \in U_\sigma$ for all $t \ge T$. It follows from (3.8) that $\varphi(t, u) \in X \times U_\sigma(w)$ for some $w \in B$, all

$t \ge T$. By construction of the neighbourhoods $N(u_j)$ we obtain $u_j \in X \times U_{3\sigma/2}(w)$ if $\pi_j(\varphi(t,u)) > 0$. Therefore we obtain for $t \ge T$:

$$\begin{aligned} \frac{d}{dt}\Phi(\varphi(t,u)) &\le -\inf\{(1+\|u_j\|)\|\Phi'(u_j)\| : \pi_j(\varphi(t,u)) \ne 0\} \\ &\le -\inf\{(1+\|u_j\|)\|\Phi'(u_j)\| : u_j \in \Phi_a^b \cap (X \times U_{3\sigma/2}(w))\}. \end{aligned}$$

This cannot be bounded away from 0 because $\lim_{t\to\infty} \Phi(\varphi(t,u)) \ge a$. Consequently there exists a sequence $(u_{j_k})_k$ in $\Phi_a^b \cap (X \times U_{3\sigma/2}(w))$ with $(1+\|u_{j_k}\|)\|\Phi'(u_{j_k})\| \to 0$. Since $\mathscr{A}$ is a $(C)_I$-attractor it follows that u_{j_k} converges to $\mathscr{A}$ in norm, that is, $\mathrm{dist}_{\|\cdot\|}(u_{j_k}, \mathscr{A}) \to 0$ as $k \to \infty$. This implies $P_Y u_{j_k} \to w$ because $\sigma < \beta/2$ with β from (3.8). Moreover, $(P_X u_{j_k})_k$ is bounded by (3.5), hence a subsequence $\mathcal{Q}$-converges to some $v \in X$. Therefore $v + w \in \mathrm{clos}_{\mathcal{P}}(\Phi_a^b)$ and $\Phi'(v+w) = 0$, contradicting (3.7).

Since $\varphi(T(u),u) \in E \setminus \Phi_a$ there exists a $\mathcal{P}$-open neighbourhood $V(u)$ of u with $\varphi(T(u),v) \in E \setminus \Phi_a$ for $v \in V(u)$. Set $V := \bigcup_{u\in\Phi^b} V(u)$ and choose a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement $\{W_\lambda : \lambda \in \Lambda\}$ of the covering $\{V(u) : u \in \Phi^b\}$ of V and a $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz continuous partition of unity $\{\pi_\lambda : \lambda \in \Lambda\}$ subordinated to $\{W_\lambda : \lambda \in \Lambda\}$. Setting

$$\tau : \Phi^b \to [0,\infty), \quad \tau(u) := \sum_{\lambda\in\Lambda} \pi_\lambda(u) T(u_\lambda),$$

the map

$$\eta : [0,1] \times \Phi^b \to \Phi^b, \eta(t,u) := \varphi(t\tau(u), u),$$

has the required properties. In the equivariant case we replace f by $\widetilde{f}$ as in the proof of Theorem 3.2 so that φ is equivariant in u. We also replace $\tau(u)$ by $\widetilde{\tau}(u) := \frac{1}{|G|}\sum_{g\in G} \tau(g^{-1}u)$ which is G-invariant. This implies that η is equivariant in u and proves part b).

The proof of a) proceeds as above with $(1+\|u_j\|)w(u_j)$ replaced by $w_j := w(u_j)$. The vector field f is then automatically bounded. The bound for $(P_X u_{j_k})_k$ needed above follows from the boundedness of $P_X\mathscr{A}$. □

Now we prove a deformation theorem in the presence of critical points. Results of this type are needed for the existence of multiple critical points.

Theorem 3.5 ([Bartsch and Ding (2006I)]). *Consider $a, b \in \mathbb{R}$ with $a < b$, $I := [a,b]$, such that $\Phi : (\Phi_a^b, \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semi-continuous, and $\Phi' : (\Phi_a^b, \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous.*

a) *If Φ has a $(PS)_I$-attractor $\mathscr{A}$ then for every $c \in (a,b)$ and every $\sigma > 0$ there exists a deformation $\eta : [0,1] \times \Phi^b \to \Phi^b$ with the properties (i) - (iv), (vi), (vii) from Theorem 3.2, and*

(viii) $\eta(1, \Phi^{c+\delta}) \subset \Phi^{c-\delta} \cup U_\sigma$ and $\eta(1, \Phi^{c+\delta} \setminus U_\sigma) \subset \Phi^{c-\delta}$ for $\delta > 0$ small enough; here $U_\sigma = X \times U_\sigma(P_Y\mathscr{A})$ is as in the proof of Theorem 3.4.

b) *If Φ has a $(C)_I$-attractor $\mathscr{A}$ so that $P_Y\mathscr{A} \subset Y$ is bounded, and such that* (3.5) *is satisfied, then the conclusion as in a) holds.*

Proof. Again we shall only prove b) because the proof of a) is similar and somewhat simpler. Fix $c \in (a,b)$ and $\sigma > 0$. Since $\mathscr{A}$ is a $(C)_I$-attractor there exists $\alpha > 0$ with

$$(1 + \|u\|) \cdot \|\Phi'(u)\| \geq 2\alpha \quad \text{if } u \in \Phi_a^b \setminus U_{\sigma/3}.$$

For $u \in \Phi_a^b \setminus U_{\sigma/3}$ there exists $w(u) \in E$ with $\|w(u)\| \leq 2$ and $\Phi'(u)w(u) \geq \|\Phi'(u)\|$. By the continuity condition on Φ' there exists a $\mathcal{P}$-open neighbourhood $N(u)$ of u such that

$$(1 + \|u\|) \cdot \Phi'(v)w(u) > \alpha \quad \text{for } v \in N(u) \cap \Phi_a^b.$$

We may also assume that $\|u\| < \gamma\|P_Y v\|$ holds for $v \in N(u) \cap \Phi_a^b$. If $u \in \Phi_a^b \cap U_{\sigma/3}$ we define $w(u) := 0$ and $N(u) := U_{\sigma/3}$. Finally, if $\Phi(u) < a$ we set $w(u) := 0$ and $N(u) := E \setminus \Phi_a$. All sets $N(u)$ are $\mathcal{P}$-open, so there exists a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement $\{U_j : j \in J\}$ of $\{N(u) : u \in \Phi^b\}$ together with a subordinated $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz partition of unity $\{\pi_j : j \in J\}$. For $j \in J$ we choose $u_j \in \Phi^b$ with $U_j \subset N(u_j)$, and we define $w_j := (1 + \|u_j\|)w(u_j)$. The vector field

$$f : W := \bigcup_{j \in J} U_j = \bigcup_{u \in \Phi^b} N(u) \to E, \quad f(u) := -\sum_{j \in J} \pi_j(u)w_j,$$

induces a flow $\varphi(t,u)$ on W which is norm continuous and $\mathcal{P}$-continuous. In the equivariant case we replace the vector field f by its symmetrized version as in the proof of Theorem 3.4. Clearly, $\Phi'(u)f(u) \leq 0$ for all $u \in \Phi^b$. If $u \in U_j \subset N(u_j)$ and $w_j \neq 0$ then $\|u_j\| < \gamma\|P_Y u\|$, hence $\|w_j\| \leq 2(1 + \|u_j\|) \leq 2(1 + \gamma\|P_Y u\|)$. This implies

$$\|f(u)\| \leq 2(1 + \gamma\|P_Y u\|) \leq 2(1 + \gamma\|u\|) \tag{3.10}$$

for all $u \in W$ and therefore $\|f(u)\|$ is bounded on U_σ because $P_Y\mathscr{A}$ is bounded. It also follows that $\varphi(t,u)$ is defined for all $t \geq 0$, all $u \in \Phi^b$. We may therefore define $\eta := \varphi|_{[0,1]\times\Phi^b}$. It is easy to check that η satisfies the properties (i)-(iv), (vi) and (vii).

In order to prove (viii) suppose to the contrary that $\eta(1, \Phi^{c+\delta}) \not\subset \Phi^{c-\delta} \cup U_\sigma$ for every $\delta > 0$. Then there exists a sequence $u_n \in \Phi^{c+1/n}$ and a sequence $t_n \in (0,1)$ with $\frac{d}{dt}\Phi(\eta(t,u_n))|_{t=t_n} \to 0$. From (3.10) it follows that $\eta(t_n,u_n)$ is a $(C)_c$-sequence, hence, since $\mathscr{A}$ is a $(C)_I$-attractor, $\eta(t_n,u_n) \in U_{\sigma/3}$ for n large. Consequently, there are $0 \leq r_n < s_n \leq 1$ such that $\eta(r_n,u_n) \in \partial U_{\sigma/3}$, $\eta(s_n,u_n) \in \partial U_\sigma$, and $\eta(t,u_n) \in U_\sigma \setminus U_{\sigma/3}$ for $t \in (r_n,s_n)$. This implies $\|\eta(r_n,u_n) - \eta(t,u_n)\| \geq 2\sigma/3$. Let $M > 0$ be a bound for $\|f(u)\|$ in U_σ. Then $\|\eta(r_n,u_n) - \eta(s_n,u_n)\| \leq M(s_n - r_n)$ and therefore $s_n - r_n \geq 2\sigma/3M$. This however leads to the contradiction:

$$\begin{aligned}\frac{2}{n} &> \Phi(\eta(r_n, u_n)) - \Phi(\eta(s_n, u_n)) = -\int_{r_n}^{s_n} \frac{d}{dt}\Phi(\eta(t,u))\,dt \\ &= -\int_{r_n}^{s_n} \Phi'(\eta(t,u)f(\eta(t,u))\,dt \ge \alpha(s_n - r_n) \ge \alpha\sigma/3M\end{aligned}$$

for all $n \in \mathbb{N}$. In a similar way one proves that $\eta(1, \Phi^{c+\delta} \setminus U_\sigma) \subset \Phi^{c-\delta}$ for $\delta > 0$ small enough. □

Chapter 4

Critical point theorems

Let X, Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. We write $\|\cdot\|$ for the norms on X, Y, and E. Let $\mathcal{S} \subset X^*$ be a dense subset and $\mathcal{D} = \{d_s : s \in \mathcal{S}\}$ be the associated family of semi-metrics on $X \cong X^{**}$ as defined in Example 2.1. Let $\mathcal{P}$ be the family of semi-norms on E consisting of all semi-norms

$$p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x+y) = |s(x)| + \|y\|, \qquad s \in \mathcal{S}.$$

Thus $\mathcal{P}$ induces the product topology on E given by the $\mathcal{D}$-topology on X and the norm topology on Y. It is contained in the product topology $(X, \mathcal{T}_w) \times (Y, \|\cdot\|)$ on E. The product $(X \times Y, \mathcal{D} \times \{\|\cdot\|\})$ is a product gage space as described in Example 2.1. The associated topology is just $\mathcal{T}_{\mathcal{P}}$. Remark that if $\mathcal{S}$ is additionally countable then every open subset is paracompact and Lipschitz normal. Clearly $\mathcal{S}$ is countable if and only if $\mathcal{P}$ is countable. Our basic hypothesis is:

(Φ_0) $\Phi \in C^1(E, \mathbb{R})$; $\Phi : (E, \mathcal{T}_{\mathcal{P}}) \to \mathbb{R}$ is upper semicontinuous, that is, Φ_a is $\mathcal{P}$-closed for every $a \in \mathbb{R}$; and $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{P}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$.

In fact, for our critical point theorems we can weaken the condition on Φ'. It is required only for a in a certain interval, and Φ_a can be replaced by subsets like Φ_a^b, depending on the situation. Similarly, Φ_a needs to be $\mathcal{P}$-closed for certain values of a only. In our applications (Φ_0) holds because the following result applies.

Theorem 4.1 ([Bartsch and Ding (2006I)]). *Consider a functional* $\Phi \in C^1(E, \mathbb{R})$ *of the form*

$$\Phi(u) = \frac{1}{2}\left(\|y\|^2 - \|x\|^2\right) - \Psi(u) \qquad \text{for } u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ *is bounded from below;*

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semicontinuous, that is,* $u_n \rightharpoonup u$ *in* E *implies* $\Psi(u) \le \liminf \Psi(u_n)$*;*

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous.*

(iv) $\nu : E \to \mathbb{R}$, $\nu(u) = \|u\|^2$, *is* C^1 *and* $\nu' : (E, \mathcal{T}_w) \to (E^*, \mathcal{T}_{w^*})$ *is sequentially continuous.*

Then Φ satisfies (Φ_0). Moreover, for any countable dense subset $\mathcal{S}_0 \subset \mathcal{S}$, Φ satisfies (Φ_0) with $\mathcal{P}$ replaced by $\mathcal{P}_0 := \{p_s \in \mathcal{P} : s \in \mathcal{S}_0\}$ and $\mathcal{T}_\mathcal{P}$ by $\mathcal{T}_{\mathcal{P}_0}$ associated to $\mathcal{P}_0$.

Proof. Let $\mathcal{S}_0 \subset \mathcal{S}$ be a countable dense subset of X^* with associated family $\mathcal{P}_0 \subset \mathcal{P}$ of semi-norms on E. The topology $\mathcal{T}_{\mathcal{P}_0}$ is then metrizable. Clearly the identity map $(E, \mathcal{T}_w) \to (E, \mathcal{T}_\mathcal{P}) \to (E, \mathcal{T}_{\mathcal{P}_0})$ are continuous. Moreover, if $(u_n)_n$ is a bounded sequence in E which $\mathcal{P}_0$-converges towards $u \in E$ then it also converges weakly to u. Here we use the fact that $\mathcal{T}_{\mathcal{P}_0}$ is Hausdorff. Now we show that Φ_a is $\mathcal{P}_0$-closed, hence $\mathcal{P}$-closed for every $a \in \mathbb{R}$. Since $\mathcal{T}_{\mathcal{P}_0}$ is metrizable it suffices to show that Φ_a is sequentially $\mathcal{P}_0$-closed. Consider a sequence $(u_n)_n$ in Φ_a which $\mathcal{P}_0$-converges to $u \in E$, and write $u_n = x_n + y_n, u = x + y \in X \oplus Y$. Observe that y_n converges to y in norm. Since Ψ is bounded below it follows from

$$\frac{1}{2}\|x_n\|^2 = \frac{1}{2}\|y_n\|^2 - \Phi(u_n) - \Psi(u_n) \le C$$

that $(x_n)_n$ is bounded, hence it converges weakly towards x and therefore $u_n \rightharpoonup u$. From condition (ii) and the form of Φ it follows that $\Phi(u) \ge \liminf \Phi(u_n) \ge a$, so $u \in \Phi_a$. Next we show that $\Phi' : (\Phi_a, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. It suffices to prove that $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{P}_0}) \to (E^*, \mathcal{T}_{w^*})$ is sequentially continuous because $\mathcal{T}_{\mathcal{P}_0} \subset \mathcal{T}_\mathcal{P}$ and $\mathcal{T}_{\mathcal{P}_0}$ is metrizable. Suppose $(u_n)_n$ $\mathcal{P}_0$-converges towards u in Φ_a. As above it follows that $(u_n)_n$ is bounded and converges weakly towards u. Then $\Phi'(u_n) \overset{w^*}{\to} \Phi'(u)$ by (iii) and (iv). □

Next we introduce a new version of linking in the infinite-dimensional setting. Of course, linking is essentially a finite-dimensional concept depending on degree theory or methods from algebraic topology. Here we extend it in a rather general and simple way. We need some notations. Given a subset $A \subset Z$ of a locally convex topological vector space we write $L(A) := \overline{\text{span}(A)}$ for the smallest closed linear subspace containing A, and we write ∂A for the boundary of A in $L(A)$. For a linear subspace $F \subset Z$ we set $A_F := A \cap F$. Finally let $I = [0, 1]$.

Definition 4.1 ([Bartsch and Ding (2006I)]). *Given two subsets $Q, S \subset Z$ with $S \cap \partial Q = \emptyset$, we say that Q finitely links with S if for any finite-dimensional linear subspace $F \subset Z$ with $F \cap S \ne \emptyset$, and any continuous deformation $h : I \times Q_F \to F + L(S)$ with $h(0, u) = u$ for all u, and $h(I \times \partial Q_F) \cap S = \emptyset$ there holds $h(t, Q_F) \cap S \ne \emptyset$ for all $t \in I$.*

Example 4.1. We present three examples of finite linking. In all cases the proof of the finite linking property is not difficult and based on a Brouwer degree argument.

a) Given an open subset $\mathcal{O} \subset Z$, $u_0 \in \mathcal{O}$, and $u_1 \in Z \setminus \overline{\mathcal{O}}$, then $Q = \{tu_1 + (1-t)u_0 : t \in I\}$ finitely links with $S := \partial\mathcal{O}$.

b) Suppose Z is the topological sum $Z = Z_1 \oplus Z_2$ of two linear subspaces, $\mathcal{O} \subset Z_1$ is open and $u_0 \in \mathcal{O}$. Then $Q = \overline{\mathcal{O}}$ finitely links with $S = \{u_0\} \times Z_2$.

c) Given $Z = Z_1 \oplus Z_2$ as in b), two open subsets $\mathcal{O}_1 \subset Z_1$, $\mathcal{O}_2 \subset Z_2$, and $u_1 \in \mathcal{O}_1$, $u_2 \in Z_2 \setminus \overline{\mathcal{O}_2}$. Then $Q = \overline{\mathcal{O}_1} \times \{tu_2 : t \in I\}$ finitely links with $S = \{u_1\} \times \partial\mathcal{O}_2$.

Now we come back to our functional $\Phi : E \to \mathbb{R}$. If $Q \subset E$ finitely links with $S \subset E$ we set

$$\Gamma_{Q,S} := \{h \in C(I \times Q, E) : h \text{ satisfies } (h_1) - (h_5)\}$$

where

(h_1) $h : I \times (Q, \mathcal{T}_{\mathcal{P}}) \to (E, \mathcal{T}_{\mathcal{P}})$ is continuous;
(h_2) $h(0, u) = u$ for all $u \in Q$;
(h_3) $\Phi(h(t, u)) \le \Phi(u)$ for all $t \in I$, $u \in Q$;
(h_4) $h(I \times \partial Q) \cap S = \emptyset$
(h_5) each $(t, u) \in I \times Q$ has a $\mathcal{P}$-open neighborhood W such that the set $\{v - h(s, v) : (s, v) \in W \cap (I \times Q)\}$ is contained in a finite-dimensional subspace of E.

Theorem 4.2 ([Bartsch and Ding (2006I)]). *Suppose Φ satisfies (Φ_0) with $\mathcal{P}$ countable, and let $Q, S \subset E$ be such that Q is $\mathcal{P}$-compact and Q finitely links with S. If $\sup \Phi(\partial Q) \le \inf \Phi(S)$ then there exists a $(PS)_c$-sequence for*

$$c := \inf_{h \in \Gamma_{Q,S}} \sup_{u \in Q} \Phi(h(1, u)) \in [\inf \Phi(S), \sup \Phi(Q)].$$

If $c = \inf \Phi(S)$ and if for all $\delta > 0$ the set $S^\delta := \{u \in E : \mathrm{dist}_{\|\,.\,\|}(u, S) \le \delta\}$ is $\mathcal{P}$-closed then there exists a $(PS)_c$-sequence $(u_n)_n$ with $u_n \to S$ in norm.

Proof. The inequality $c \le \sup \Phi(Q)$ is obvious. In order to see $c \ge \inf \Phi(S)$ we first observe that $h(I \times \partial Q) \cap S = \emptyset$ for every $h \in \Gamma_{Q,S}$ by (h_3). Since Q is $\mathcal{P}$-compact there exists a finite-dimensional subspace F containing $\{u - h(t, u) : (t, u) \in I \times Q\}$. Consequently $h(I \times Q_F) \subset F$. Since Q finitely links with S there exists $u \in Q$ with $h(1, u) \in S$ which implies $\sup_{u \in Q} \Phi(h(1, u)) \ge \inf \Phi(S)$ as claimed.

Assume that $\|\Phi'(u)\| \ge \alpha$ for all $u \in \Phi_{c-\varepsilon}^{c+\varepsilon}$, some $\alpha, \varepsilon > 0$. Notice that since $\mathcal{P}$ is assumed to be countable, every $\mathcal{P}$-open subset is paracompact and Lipschitz normal (see Example 2.1). We can take η to be the deformation from Theorem 3.2 for $a := c - \varepsilon$, $b := c + \varepsilon$. Now we choose $h \in \Gamma_{Q,S}$ with $\sup \Phi(h(1, Q)) < c + \varepsilon$ and define $g : I \times Q \to E$ by $g(t, u) := \eta(t, h(t, u))$. Then $g(0, u) = u$ for all u and g satisfies $(h_1) - (h_4)$. Moreover, (h_5) follows from the equality $u - g(t, u) = (u - h(t, u)) + (h(t, u) - \eta(t, h(t, u)))$. Thus $g \in \Gamma_{Q,S}$ which leads to the contradiction $c \le \sup_{u \in Q} \Phi(g(1, u)) \le c - \varepsilon$.

We have seen that there exists a $(PS)_c$-sequence. Now suppose $c = \inf \Phi(S)$. If there does not exist a $(PS)_c$-sequence converging to S in norm then there exist $\varepsilon > 0$, $\delta > 0$, and $\alpha > 0$ so that $\|\Phi'(u)\| \ge \alpha$ for all $u \in S^\delta \cap \Phi_{c-\varepsilon}^{c+\varepsilon}$. For such u we choose $w(u) \in E$ with $\|w(u)\| \le 2$ and $\Phi'(u)w(u) > \|\Phi'(u)\|$. We then choose a $\mathcal{P}$-open neighborhood $N(u)$ of u so that $\Phi'(v)w(u) > \|\Phi'(u)\| \ge \alpha$ for all $v \in N(u) \cap \Phi_{c-\varepsilon}$. For $u \in \Phi_{c-\varepsilon}^{c+\varepsilon} \setminus S^\delta$ we put $N(u) := E \setminus S^\delta$ and $w(u) := 0$. Lastly, for $u \in E \setminus \Phi_{c-\varepsilon}$ we set $N(u) := E \setminus \Phi_{c-\varepsilon}$ and $w(u) := 0$. Then $W := \bigcup_{u \in \Phi^{c+\varepsilon}} N(u)$ is $\mathcal{P}$-open. Let $\{U_j : j \in J\}$ be a $\mathcal{P}$-locally finite $\mathcal{P}$-open refinement of $\{N(u) : u \in \Phi^{c+\varepsilon}\}$ and $\{\pi_j : j \in J\}$ a $\mathcal{P}$-locally $\mathcal{P}$-Lipschitz partition of unity subordinated to the covering

$\{U_j : j \in J\}$. For $j \in J$ with $U_j \cap S^\delta \neq \emptyset$ we choose u_j with $U_j \subset N(u_j)$ and define $w_j := w(u_j)$. For $j \in J$ with $U_j \cap S^\delta = \emptyset$ we set $w_j := 0$. Then the vector field $f(u) := -\sum_{j\in J} \pi_j(u)w_j$ satisfies $\|f(u)\| \leq 2$ for all $u \in W$, and it satisfies $\Phi'(u)f(u) \geq \alpha$ for all $u \in S^\delta$. Let $\varphi^t(u)$ be the associated flow as in Chapter 3.

By construction we have that, if $u \in \Phi^{c+\alpha\delta/4} \setminus S^{\delta/2}$ then $\varphi^t(u) \notin S$ for all $t \geq 0$. Moreover, if $u \in S^{\delta/2}$ we have $\varphi^t(u) \in S^\delta$ for $0 \leq t \leq \delta/4$, hence $\Phi(\varphi^{\delta/2}(u)) \leq \Phi(u) - \alpha\delta/4 < c$. Now take any $h \in \Gamma_{Q,S}$ with $\sup \Phi(h(1,u)) \leq c + \alpha\delta/4$. Our considerations yield $\varphi^{\delta/2}(h(1,u)) \notin S$ for all $u \in Q$. This contradicts the linking condition on Q and S because $\varphi^{\delta/2} \circ h(t, .) \in \Gamma_{Q,S}$. □

Similarly, the finitely linking yields also a $(C)_c$-sequence. We need the additional assumption:

(Φ_+) there exists $\zeta > 0$ such that $\|u\| < \zeta\|P_Y u\|$ for all $u \in \Phi_0$.

Remark 4.1. Let $\mathcal{S}_0 \subset \mathcal{S}$ be any countable dense subset with associated family $\mathcal{P}_0$ of semi-norms.

(1) The assumptions (Φ_0) and (Φ_+) imply that Φ_a is $\mathcal{P}_0$-closed and $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{P}_0}) \to (E^*, \mathcal{T}_{w*})$ is continuous for each $a \geq 0$, see the proof of Theorem 4.1. Indeed, let $(u_n)_n$ in Φ_a which $\mathcal{P}_0$-converges to $u \in E$, and write $u_n = x_n + y_n, u = x + y \in X \oplus Y$. Then $\|y_n - y\| \to 0$, hence y_n is bounded. It follows from (Φ_+) that x_n, hence u_n is bounded. This implies that u_n weakly hence $\mathcal{P}$-converges to u. Now (Φ_0) implies that $u \in \Phi_a$, and $\Phi'(u_n)v \to \Phi'(u)v$ for all $v \in E$.

(2) (Φ_+) implies clearly (3.5) with $0 \leq a \leq b$.

(3) Since every $\mathcal{P}_0$-open subset is paracompact and Lipschitz normal, Theorems 3.2, 3.3, 3.4 and 3.5 are applicable with gage topology $\mathcal{T}_{\mathcal{P}_0}$ for $0 \leq a \leq b$. Letting η stand for the deformations given by these theorems, we note that $\eta : [0,1] \times \Phi^b \to \Phi^b$ are $\mathcal{P}$-continuous because of their locally finite-dimensional property *(vi)*.

Theorem 4.3. *Suppose* Φ *satisfies* (Φ_0) *and* (Φ_+). *Let* $Q, S \subset E$ *be such that* Q *is* $\mathcal{P}$*-compact and* Q *finitely links with* S. *If* $\kappa := \inf \Phi(S) > 0$ *and* $\sup \Phi(\partial Q) \leq \kappa$ *then* Φ *has a* $(C)_c$*-sequence with* $\kappa \leq c \leq \sup \Phi(Q)$.

Proof. Repeating the arguments of the first two paragraphs of the proof of Theorem 4.2 with the application of Theorem 3.2 replaced by Theorem 3.3 (see Remark 4.1) yields the desired conclusion. □

As a corollary of Theorems 4.2 we obtain an improvement of a very useful critical point theorem of Kryszewski and Szulkin [Kryszewski and Szulkin (1998)].

Theorem 4.4 ([Bartsch and Ding (2006I)]). *Consider a functional* $\Phi : E \to \mathbb{R}$ *satisfying* (Φ_0) *with* $\mathcal{P}$ *countable, e. g.* Φ *is as in Theorem 4.1. Suppose there exist* $R > r > 0$ *and* $e \in Y$, $\|e\| = 1$ *such that we have for* $S := \{u \in Y : \|u\| = r\}$, $Q = \{v + te \in E : v \in X,\ \|v\| < R, 0 < t < R\}$*:* $\inf \Phi(S) \geq \Phi(0) \geq \sup \Phi(\partial Q)$.

Then there exists a $(PS)_c$-sequence for

$$c := \inf_{h\in\Gamma_{Q,S}} \sup_{u\in Q} \Phi(h(1,u)) \in [\inf \Phi(S), \sup \Phi(Q)].$$

If $c = \inf \Phi(S)$ then there exists a $(PS)_c$-sequence $(u_n)_n$ with $u_n \to S$ in norm.

Proof. By Example 4.1c) Q finitely links with S. Observe that Q is $\mathcal{P}$-compact and that $S^\delta := \{u \in E : \mathrm{dist}_{\|\,.\,\|}(u,S) \le \delta\}$ is $\mathcal{P}$-closed. Therefore the corollary follows from Theorem 4.2. □

The original theorem in [Kryszewski and Szulkin (1998)] deals with the case where E is a Hilbert space, Φ is as in Proposition 4.1, and $\inf \Phi(S) > \Phi(0) \ge \sup \Phi(\partial Q)$. The additional information on the Palais-Smale sequence in the case $c = \inf \Phi(S)$ has not been obtained in [Kryszewski and Szulkin (1998)]. It is however important in applications when $c = \Phi(0)$ in order to construct a nontrivial critical point. If the stronger hypothesis $\inf \Phi(S) > \Phi(0)$ holds then $c > \Phi(0)$. This is sufficient to deduce the existence of a nontrivial critical point.

We have also the following consequence of Theorem 4.3.

Theorem 4.5. *Let Φ satisfy (Φ_0) and (Φ_+), and suppose there exist $R > r > 0$ and $e \in Y$, $\|e\| = 1$ such that for $S := \{u \in Y : \|u\| = r\}$, $Q = \{v + te \in E : v \in X,\ \|v\| < R, 0 < t < R\}$ we have $\kappa := \inf \Phi(S) > 0$ and $\sup \Phi(\partial Q) \le \kappa$ then Φ has a $(C)_c$-sequence with $\kappa \le c \le \sup \Phi(Q)$.*

Next we investigate symmetric functionals. We restrict our attention to the symmetry group $G = \{e^{2k\pi i/p} : 0 \le k < p\} \cong \mathbb{Z}/p$, p a prime number. Using the more elaborate methods from [Bartsch (1993)] we could deal with more general symmetry groups; see Remark 4.2 below. We suppose that G acts linearly and isometrically on X and Y, hence on $E = X \times Y$. We also assume that the action is fixed point free on $E \setminus \{0\}$, that is, the fixed point set $E^G := \{u \in E : gu = u \text{ for all } g \in G\} = \{0\}$ is trivial. If A is a topological space on which G acts continuously (e.g. $A \subset E$ is invariant) then the genus of A, $\mathrm{gen}(A) \in \mathbb{N}_0 \cup \{\infty\}$, is by definition the infimum over all $k \in \mathbb{N}_0$ such that there exist open invariant subsets $U_1, \dots, U_k \subset A$ covering A, and there exist equivariant maps $U_j \to G$, $j = 1, \dots, k$. Here we use the convention $\inf \emptyset = \infty$. In particular, $\mathrm{gen}(A) = \infty$ if $A^G \ne \emptyset$. The genus possesses the following standard properties:

1° Normalization: If $u \notin E^G$, $\mathrm{gen}(Gu) = 1$;
2° Mapping property: If $f \in C(A,B)$ and f is equivariant, i.e. $fg = gf$ for all $g \in G$, then $\mathrm{gen}(A) \le \mathrm{gen}(B)$;
3° Monotonicity: If $A \subset B$, $\mathrm{gen}(A) \le \mathrm{gen}(B)$;
4° Subadditivity: $\mathrm{gen}(A \cup B) \le \mathrm{gen}(A) + \mathrm{gen}(B)$;
5° Continuity: If A is compact and $A \cap E^G = \emptyset$, then $\mathrm{gen}(A) < \infty$ and there is an invariant neighborhood U of A such that $\mathrm{gen}(A) = \mathrm{gen}(U)$.

These properties can be found in [Bartsch (1993)] or [Chang (1993); Rabinowitz (1986)].

In addition to (Φ_0) we require the following conditions:

(Φ_1) Φ is G-invariant;
(Φ_2) there exists $r > 0$ with $\kappa := \inf \Phi(S_r Y) > \Phi(0) = 0$ where $S_r Y := \{y \in Y : \|y\| = r\}$;
(Φ_3) there exist a finite-dimensional G-invariant subspace $Y_0 \subset Y$ and $R > r$ such that we have for $E_0 := X \times Y_0$ and $B_0 := \{u \in E_0 : \|u\| \le R\}$: $b := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_r Y)$.

Now we define a kind of pseudo-index for the topology of the sublevel sets Φ^c for $c \in \mathbb{R}$. For this purpose we consider the set $\mathcal{M}(\Phi^c)$ of maps $g : \Phi^c \to E$ with the properties

(P_1) g is $\mathcal{P}$-continuous and equivariant;
(P_2) $g(\Phi^a) \subset \Phi^a$ for all $a \in [\kappa, b]$;
(P_3) each $u \in \Phi^c$ has a $\mathcal{P}$-open neigbourhood $W \subset E$ such that the set $(id - g)(W \cap \Phi^c)$ is contained in a finite-dimensional linear subspace of E.

Observe that, if $g \in \mathcal{M}(\Phi^a)$, $h \in \mathcal{M}(\Phi^c)$ with $a < c$ and $h(\Phi^c) \subset \Phi^a$ then $g \circ h \in \mathcal{M}(\Phi^c)$. The properties (P_1) and (P_2) are trivially satisfied by $g \circ h$. Property (P_3) follows from the equality $id - g \circ h = id - h + (id - g) \circ h$. The pseudo-index of Φ^c is then defined by

$$\psi(c) := \min\{\operatorname{gen}(g(\Phi^c) \cap S_r Y) : g \in \mathcal{M}(\Phi^c)\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Observe that it does not play a role whether we use the norm topology or the $\mathcal{P}$-topology on Φ^c since both induce the same topology on $S_r Y \subset Y$. As a consequence of the monotonicity of the genus the function $\psi : \mathbb{R} \to \mathbb{N}_0 \cup \{\infty\}$ is nondecreasing. Clearly we have $\psi(c) = 0$ for $c < \kappa$ since then $\Phi^c \cap S_r Y = \emptyset$.

Lemma 4.1. *If Φ satisfies $(\Phi_0) - (\Phi_3)$ then $\psi(c) \ge n := \dim Y_0$ for $c \ge b = \sup \Phi(E_0)$.*

Proof. See Lemma 4.3 of [Bartsch and Ding (1999)]. Fix $c \ge \sup \Phi(E_0) = \sup \Phi(B_0)$. We shall show that $\operatorname{gen}(g(B_0) \cap S_r Y) \ge n$ for any $g \in \mathcal{M}(\Phi^c)$. Then $\psi(c) \ge n$ because $B_0 \subset \Phi^c$ and because the genus is monotone. Fix $g \in \mathcal{M}(\Phi^c)$. Since B_0 is $\mathcal{P}$-compact it follows from (P_3) that $(id - g)(B_0)$ is contained in a finite-dimensional subspace F of E. We may assume that $F_Y := P_Y F \supset Y_0$ and $F = F_X \oplus F_Y$ with $F_X := P_X F \subset X$. Consider the set

$$\mathcal{O} := \{u \in B_0 \cap F : \|g(u)\| < r\} \subset F$$

and the map

$$h : \partial\mathcal{O} \to F_X, \quad h(u) := P_X \circ g(u).$$

We observe that $g(B_0 \cap F) \subset F$ because $(id - g)(B_0) \subset F$. Thus h is well defined. Moreover, $g\colon B_0 \cap F \to F$ is continuous by (P_1) since F is finite-dimensional. In addition, (P_2) implies that $0 \in \mathcal{O}$ and $\overline{\mathcal{O}} \subset int(B_0 \cap F)$. Therefore $\mathcal{O}$ is a bounded open neighborhood of 0 in $F_n := F \cap (X \oplus Y_0)$, hence, $\mathrm{gen}(\partial\mathcal{O}) = \dim F_n$. From the monotonicity of the genus we obtain

$$\mathrm{gen}\left(\partial\mathcal{O} \setminus h^{-1}(0)\right) \le \mathrm{gen}(P_X F_n \setminus \{0\}) = \dim P_X F_n\,.$$

The continuity and the subadditivity yield

$$\mathrm{gen}(\partial\mathcal{O}) \le \mathrm{gen}\left((h^{-1}(0)\right) + \mathrm{gen}(\partial\mathcal{O}\backslash h^{-1}(0)).$$

It follows that

$$\mathrm{gen}(h^{-1}(0)) \ge \dim F_n - \dim P_X F_n = \dim Y_0.$$

Finally, $h(u) = 0$ implies $g(u) \in Y$ and $u \in \partial\mathcal{O}$ implies $\|g(u)\| = r$, thus $g(h^{-1}(0)) \subset g(B_0) \cap S_r Y$. Therefore, using the monotonicity of the genus once more we obtain the desired inequality

$$\mathrm{gen}(g(B_0) \cap S_r Y) \ge \mathrm{gen}(g(h^{-1}(0))) \ge \mathrm{gen}(h^{-1}(0)).$$

□

For later arguments we introduce a comparison function $\psi_d : [0, d] \to \mathbb{N}_0$. For $d > 0$ fixed set

$$\mathcal{M}_0(\Phi^d) := \{g \in \mathcal{M}(\Phi^d) : \ g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d)\}.$$

Then we define for $c \in [0, d]$

$$\psi_d(c) := \min\left\{\mathrm{gen}(g(\Phi^c) \cap S_r Y) : \ g \in \mathcal{M}_0(\Phi^d)\right\}.$$

Note that since $\mathcal{M}_0(\Phi^d) \subset \mathcal{M}(\Phi^d) \hookrightarrow \mathcal{M}(\Phi^c)$ via restriction $g \mapsto g|_{\Phi^c}$ we have $\psi(c) \le \psi_d(c)$ for all $c \in [0, d]$.

Theorem 4.6 ([Bartsch and Ding (2006I)]). *Let $(P_1) - (P_3)$ be satisfied. Assume that Φ satisfies also either (Φ_0) with $\mathcal{P}$ countable and the $(PS)_c$-condition or $(\Phi_0), (\Phi_+)$ and the $(C)_c$-condition for $c \in [\kappa, b]$, then it has at least $n := \dim Y_0$ G-orbits of critical points.*

Proof. We only treat the situation where $(PS)_c$-condition is satisfied because the other situation can be handled similarly.

For $i = 1, \ldots, n$ we set

$$c_i := \inf\{c \ge 0 : \psi(c) \ge i\} \in [\kappa, b].$$

If c_i is not a critical value then there exists $\varepsilon > 0$ so that $\inf\{\|\Phi'(u)\| : u \in \Phi_{c-\varepsilon}^{c+\varepsilon}\} > 0$. Now Theorem 3.2 yields a deformation η such that $h := \eta(1, \cdot) \in \mathcal{M}(\Phi^{c_i+\varepsilon})$ and $h(\Phi^{c_i+\varepsilon}) \subset \Phi^{c_i-\varepsilon}$. This implies the contradiction

$$\begin{aligned}\psi(c_i - \varepsilon) &= \min\{\mathrm{gen}(g(\Phi^{c_i-\varepsilon}) \cap S_r Y) : g \in \mathcal{M}(\Phi^{c_i-\varepsilon})\}\\ &\ge \min\{\mathrm{gen}(g(h(\Phi^{c_i+\varepsilon})) \cap S_r Y) : g \in \mathcal{M}(\Phi^{c_i-\varepsilon})\}\\ &\ge \min\{\mathrm{gen}(g(\Phi^{c_i+\varepsilon}) \cap S_r Y) : g \in \mathcal{M}(\Phi^{c_i+\varepsilon})\}\\ &= \psi(c_i + \varepsilon).\end{aligned}$$

Here we used the monotonicity of the genus and the fact that for $g \in \mathcal{M}(\Phi^{c_i-\varepsilon})$ the composition $g \circ h \in \mathcal{M}(\Phi^{c_i+\varepsilon})$. Thus c_i is a critical value for $i = 1, \dots, n$.

Suppose Φ has only finitely many critical points in Φ^b_κ. Then $\mathscr{A} := \{u \in \Phi^b_\kappa : \Phi'(u) = 0\}$ is a finite $(PS)_I$-attractor, so (3.8) holds trivially true. For $\sigma > 0$ small we then have that $U_\sigma(P_Y\mathscr{A}) \subset Y$ is the disjoint union of the σ-balls around the elements of $P_Y\mathscr{A}$. This implies that $\text{gen}(U_\sigma) = \text{gen}(U_\sigma(P_Y\mathscr{A})) = \text{gen}(P_Y\mathscr{A}) = 1$ where $U_\sigma = X \times U_\sigma(P_Y\mathscr{A})$. Let $\eta : [0,1] \times \Phi^b \to \Phi^b$ be a deformation as in Theorem 3.5a). For $\delta > 0$ small enough the map $h := \eta(1, \cdot)$ satisfies $h(\Phi^{c_i+\delta}) \subset \Phi^{c_i-\delta} \cup U_\sigma$. Let $d = b + 1$ and choose $g_0 \in \mathcal{M}_0(\Phi^d)$ such that $\psi_d(c_i - \delta) = \text{gen}(g_0(\Phi^{c_i-\delta}) \cap S_rY)$. Consequently,

$$
\begin{aligned}
\psi_d(c_i + \delta) &= \min\{\text{gen}(g(\Phi^{c_i+\delta}) \cap S_rY) : g \in \mathcal{M}(\Phi^{c_i+\delta})\} \\
&\le \text{gen}(g_0 \circ h(\Phi^{c_i+\delta}) \cap S_rY) \\
&\le \text{gen}(g_0(\Phi^{c_i-\delta} \cup U_\sigma) \cap S_rY) \\
&\le \text{gen}(g_0(\Phi^{c_i-\delta}) \cap S_rY) + \text{gen}(g_0(U_\sigma)) \\
&\le \psi_d(c_i - \delta) + 1.
\end{aligned}
$$

This implies that $\kappa < c_1 < c_2 < \cdots < c_n \le b$ so we have even n distinct critical values. □

Remark 4.2. Theorem 4.6 holds true for more general classes of symmetries, for instance for the abelian p-group $(\mathbb{Z}/p)^k$ acting without fixed points on $E \setminus \{0\}$, or for any finite group G which acts freely on $E \setminus \{0\}$. If $G = (S^1)^k$ is a torus, or more generally $G = (S^1)^k \times (\mathbb{Z}/p)^l$ is a p-torus then Φ has at least $\frac{1}{2}\dim Y_0$ G-orbits of critical points. For $G = SU(2)$ we obtain at least $\frac{1}{4}\dim Y_0$ G-orbits of critical points. In all these cases there exists an index theory $i : \{A \subset E : A \text{ is invariant}\} \to \mathbb{N}_0 \cup \{\infty\}$ satisfying the monotonicity, continuity, and subadditivity properties as well as a dimension property: $i(F \setminus \{0\}) = c \cdot \dim F$ for a finite-dimensional invariant linear subspace $F \subset E$. We refer the reader to [Bartsch and Ding (1999); Bartsch (1996); Benci (1982); Clapp and Puppe (1991)] for a discussion of group actions, index theories, examples, and applications.

Our last critical point theorem is concerned with the existence of an unbounded sequence of critical values in the presence of symmetries. We stick to the case where $G = \mathbb{Z}/p$ acts linearly, isometrically on E and has no fixed points in $E \setminus \{0\}$. The hypothesis (Φ_3) is replaced by

(Φ_4) there exists an increasing sequence of finite-dimensional G-invariant subspaces $Y_n \subset Y$ and there exist $R_n > r$ such that we have for $B_n := \{u \in X \times Y_n : \|u\| \le R_n\}$: $\sup \Phi(X \times Y_n) < \infty$ and $\sup \Phi(X \times Y_n \setminus B_n) < \beta := \inf \Phi(\{u \in Y : \|u\| \le r\})$.

Here $r > 0$ is from (Φ_2). We also need the following compactness condition:

(Φ_I) One of the following holds:
- $\mathcal{P}$ is countable and Φ satisfies the $(PS)_c$-condition for every $c \in I$;
- $\mathcal{P}$ is countable and Φ has a $(PS)_I$-attractor $\mathscr{A}$ with $P_X\mathscr{A} \subset X \setminus \{0\}$ bounded and satisfying (3.8);
- (Φ_+) holds and Φ has a $(C)_I$-attractor $\mathscr{A}$ with $P_Y\mathscr{A} \subset Y \setminus \{0\}$ bounded and satisfying (3.8).

Theorem 4.7 ([Bartsch and Ding (2006I)]). *If Φ satisfies $(\Phi_0) - (\Phi_2)$, (Φ_4), and (Φ_I) for any compact interval $I \subset (0,\infty)$ then Φ has an unbounded sequence of critical values.*

Proof. Similarly to the proof of Theorem 4.6 we consider the set $\mathcal{M}(\Phi^c)$ of maps $g : \Phi^c \to E$ with the properties $(P_1) - (P_3)$ and the pseudoindex $\psi(c)$. Given a finite-dimensional invariant subspace $Y_n \subset Y$ we claim that $\psi(c) \geq \dim Y_n$ for any $c \geq \sup \Phi(X \times Y_n)$ as in Lemma 4.1. In fact, given $g \in \mathcal{M}(\Phi^c)$ we show that $\mathrm{gen}(g(B_n) \cap S_rY) \geq \dim Y_n$. The claim follows then using the monotonicity of the genus. Since B_n is $\mathcal{P}$-compact there exists a finite-dimensional subspace $F \subset E$ containing $(id - g)(B_n)$. Making F larger if necessary we may assume that $Y_n \subset F$ and $F = P_XF + P_YF$. We define

$$\mathcal{O} := \{u \in B_n \cap F : \|g(u)\| < r\}$$

and

$$h : \partial\mathcal{O} \to P_XF, \quad h(u) := P_X(g(u)).$$

Now one continues as in the proof of Lemma 4.1 in order to prove:

$$\begin{aligned} \mathrm{gen}(g(\Phi^c) \cap S_rY) &\geq \mathrm{gen}(g(B_n) \cap S_rY) \\ &\geq \mathrm{gen}(h^{-1}(0)) \\ &\geq \mathrm{gen}(\partial\mathcal{O}) - \mathrm{gen}(\partial\mathcal{O} \setminus h^{-1}(0)) \\ &\geq \dim(F \cap (X + Y_n)) - \mathrm{gen}(P_XF \setminus \{0\}) \\ &= \dim(F \cap (X + Y_n)) - \dim P_XF \\ &= \dim Y_n. \end{aligned}$$

If the set of critical values of Φ is bounded above by some $m > 0$ then ψ is constant on (m, ∞). This follows immediately from Theorem 3.4. Therefore the theorem is proved if we can show that ψ achieves only finite values. In order to see this we consider the comparison function ψ_d, $d > 0$, defined as before. Recall that $\psi(c) \leq \psi_d(c)$ for $c \in [0, d]$. Therefore it suffices to prove that ψ_d achieves only finite values. Clearly $\psi_d(c) = 0$ for $c < \kappa$ because $id \in \mathcal{M}_0(\Phi^d)$. Thus it suffices to show that for any $c \in (0, d]$ there exists $\delta > 0$ with $\psi_d(c + \delta) \leq \psi_d(c - \delta) + 1$.

Set $I := [\kappa/2, d + 1]$ and let $\mathscr{A}$ be a $(PS)_I$-attractor (or $(C)_I$-attractor) as in (Φ_I). This exists in particular if Φ satisfies the $(PS)_c$-condition for $c \in I$. We shall show that for any $c \in [\kappa, d]$ there exists $\delta > 0$ with $\psi_d(c + \delta) \leq \psi_d(c - \delta) + 1$. Fix $\sigma < \beta/2$ where β is as in (3.8), and let η be a deformation as in Theorem 3.5. Then

$h := \eta(1,\cdot) \in \mathcal{M}_0(\Phi^d)$ and $h(\Phi^{c+\delta}) \subset \Phi^{c-\delta} \cup U_\sigma$ for $\delta > 0$ small. We fix such a δ and choose $g \in \mathcal{M}_0(\Phi^d)$ with $\psi_d(c-\delta) = \text{gen}(g(\Phi^{c-\delta}) \cap S_rY)$. Then $g \circ h \in \mathcal{M}_0(\Phi^d)$ and $g \circ h(\Phi^{c+\delta}) \subset g(\Phi^{c-\delta}) \cup g(U_\sigma)$. Consequently, using the standard properties of the genus we obtain:

$$\begin{aligned}\psi_d(c+\delta) &\leq \text{gen}(g \circ h(\Phi^{c+\delta}) \cap S_rY) \\ &\leq \text{gen}(g(\Phi^{c-\delta}) \cup g(U_\sigma)) \cap S_rY) \\ &\leq \text{gen}((g(\Phi^{c-\delta}) \cap S_rY) \cup g(U_\sigma)) \\ &\leq \text{gen}(g(\Phi^{c-\delta}) \cap S_rY) + \text{gen}(g(U_\sigma)) \\ &\leq \psi_d(c-\delta) + 1.\end{aligned}$$

The equality $\text{gen}(g(U_\sigma)) = \text{gen}(U_\sigma) \leq 1$ follows from the discreteness of $P_Y(U_\sigma)$. $\square$

Remark 4.3. Earlier versions of Theorem 4.7 have been proved in [Bartsch and Ding (1999, 2002)] (see also [Kryszewski and Szulkin (1998)]). As in Remark 4.2 the theorem holds true for more general classes of symmetries; cf. [Bartsch (1993)].

Chapter 5

Homoclinics in Hamiltonian systems

Consider the following Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(t, z), \tag{HS}$$

where $z = (p, q) \in \mathbb{R}^{2N}$, $\mathcal{J}$ denotes the standard symplectic structure in $\mathbb{R}^{2N}$:

$$\mathcal{J} := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ has the form

$$H(t, z) = \frac{1}{2} L(t) z \cdot z + R(t, z)$$

with $L(t)$ being a continuous symmetric $2N \times 2N$-matrix valued function, $R_z(t, z) = o(|z|)$ as $z \to 0$ and being either super linear or asymptotically linear as $|z| \to \infty$. A solution z of (HS) is a homoclinic orbit if $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$. We study the existence and multiplicity of homoclinic orbits. In the first six sections we deal with the case where the Hamiltonian depends periodically on t and in the last section we handle the Hamiltonian without periodicity assumptions.

5.1 Existence and multiplicity results for periodic Hamiltonians

In the last years, existence and multiplicity of homoclinic orbits of (HS) were studied extensively by means of critical point theory, and many results were obtained based on various hypotheses on the functions L and R which we recall firstly below.

On L, it was assumed that either L is constant such that each eigenvalue of the matrix $\mathcal{J}L$ has nonzero real part (see [Arioli and Szulkin (1999); Coti-Zelati, Ekeland and Séré (1990); Hofer and Wysocki (1990); Séré (1992, 1993); Szulkin and Zou (2001); Tanaka (1991)]), or L depends on t such that, more or less abstractly, 0 lies in a gap (at least the boundary) of $\sigma(A)$, the spectrum of the Hamiltonian operator $A := -\left(\mathcal{J}\frac{d}{dt} + L\right)$ (see [Ding and Girardi (1999); Ding and Willem (1999)]).

For the super linear case, it was always assumed that R satisfies a condition of the type of Ambrosetti-Rabinowitz, that is, there is $\mu > 2$ such that

$$0 < \mu R(t, z) \le R_z(t, z) z \quad \text{whenever } z \ne 0, \tag{5.1}$$

together with a technique assumption that there is $\kappa \in (1,2)$ such that

$$|R_z(t,z)|^\kappa \le c\,(1 + R_z(t,z)z) \quad \text{for all } (t,z) \tag{5.2}$$

(here and below c or c_i stands for a generic positive constant). In order to establish the multiplicity, a regularity condition was also required: there are $\delta > 0$ and $\varsigma \ge 1$ such that

$$|R_z(t,z+h) - R_z(t,z)| \le c_0(1 + |z|^\varsigma)|h| \quad \text{if } |h| \le \delta. \tag{5.3}$$

See [Coti-Zelati, Ekeland and Séré (1990); Ding and Willem (1999); Hofer and Wysocki (1990); Tanaka (1991)] for the existence of at least one homoclinic orbit. Infinitely homoclinic orbits were obtained firstly in the striking work [Séré (1992, 1993)] provided moreover that $R(t,z)$ is strictly convex in z, and later in [Ding and Girardi (1999)] and [Arioli and Szulkin (1999)] respectively provided additionally that $R(t,z)$ is even in z and that $R(t,z)$ possesses certain more general symmetries.

In the asymptotically linear case, the existence of one homoclinic orbits was obtained in the paper [Szulkin and Zou (2001)]. As far as we know there were no results of existence of infinitely homoclinic orbits in this case.

The goal of this chapter is to establish the existence and multiplicity of homoclinic orbits of (HS) under different hypotheses via new information in critical point theory for strongly indefinite functionals stated in the previous chapter. In contrast to the works mentioned above, the main contributions here are in three aspects: *firstly we deal with the super linearities more general than the Ambrosetti-Rabinowitz type condition* (5.1)*; secondly we prove that the asymptotically linear system possesses infinitely many homoclinic orbits; and thirdly we establish without the assumption* (5.3) *the existence of infinitely homoclinic orbits.*

For describing our results, we will use the $2N \times 2N$ matrix

$$\mathcal{J}_0 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and the notation

$$\tilde{R}(t,z) := \frac{1}{2}R_z(t,z)z - R(t,z).$$

In the following, for any symmetric matrix value function $M \in C(\mathbb{R}, \mathbb{R}^{2N\times 2N})$, let $\wp(M(t))$ be the set of all eigenvalues of $M(t)$ and set

$$\lambda_M := \inf_{t\in\mathbb{R}} \min \wp(M(t)), \quad \Lambda_M := \sup_{t\in\mathbb{R}} \max \wp(M(t)).$$

In particular, we denote $\lambda_0 := \lambda_{\mathcal{J}_0 L}$ and $\Lambda_0 := \Lambda_{\mathcal{J}_0 L}$ for $M(t) = \mathcal{J}_0 L(t)$.

We make the following hypotheses:

(L_0) $L(t)$ is 1-period in t, and $\mathcal{J}_0 L(t)$ is positive definite;

(R_0) $R(t,z)$ is 1-period in t, $R(t,z) \ge 0$ and $R_z(t,z) = o(|z|)$ as $z \to 0$ uniformly in t.

It is apparent that, under the periodicity condition, if z is a homoclinic orbit then $k * z$ is also a homoclinic orbit for any $k \in \mathbb{Z}$, where $(k * z)(t) = z(t + k)$ for all $t \in \mathbb{R}$. Two homoclinic orbits z_1 and z_2 will be said being geometrically distinct if $k * z_1 \neq z_2$ for all $k \in \mathbb{Z}$.

Firstly we treat the super linear case. Assume

(S_1) $R(t, z)|z|^{-2} \to \infty$ uniformly in t as $|z| \to \infty$;

(S_2) $\tilde{R}(t, z) > 0$ if $z \neq 0$, and there exist $r_1 > 0$ and $\nu > 1$ such that $|R_z(t, z)|^\nu \leq c_1 \tilde{R}(t, z)|z|^\nu$ if $|z| \geq r_1$.

Theorem 5.1 ([Ding (2006)]). *Let* (L_0)*,* (R_0) *and* (S_1)*-*(S_2) *be satisfied. Then* (HS) *has at least one homoclinic orbit. If in addition* $R(t, z)$ *is even in* z *then* (HS) *has infinitely many geometrically distinct homoclinic orbits.*

Remark 5.1. *a)* The following functions satisfy (R_0) and (S_1)-(S_2) but do not verify (5.1):

$Ex1.$ $R(t, z) = a(t)\Big(|z|^2 \ln(1 + |z|) - \frac{1}{2}|z|^2 + |z| - \ln(1 + |z|)\Big)$,

$Ex2.$ $R(t, z) = a(t)\Big(|z|^\mu + (\mu - 2)|z|^{\mu-\epsilon} \sin^2\big(\frac{|z|^\epsilon}{\epsilon}\big)\Big)$, $\mu > 2$, $0 < \epsilon < \mu - 2$,

where $a(t) > 0$ and is 1-periodic in t.

b) If $R(t, z)$ satisfies (5.1) and (5.2), then (S_1)-(S_2) hold. Indeed it is clear that $R(t, z) \geq c_1|z|^\mu$ for z away from 0, $\tilde{R}(t, z) \geq \frac{\mu-2}{2\mu} R_z(t, z)z > 0$ if $z \neq 0$, and

$$\begin{aligned} |R_z(t, z)|^\nu &\leq c_2|R_z(t, z)|^{\nu-\kappa} R_z(t, z)z \leq c_3|z|^{(\nu-\kappa)/(\kappa-1)}\tilde{R}(t, z) \\ &\leq c_4\tilde{R}(t, z)|z|^\nu \end{aligned}$$

for all $|z| \geq 1$ and $1 < \nu \leq \kappa/(2 - \kappa)$.

c) If $|R_z(t, z)||z| \leq c_1 R_z(t, z)z$ for $|z|$ large, say $|z| \geq r_1$, then (S_2) is satisfied provided

$(\hat{S}_2)$ There exist $p > 2$ and $\omega \in (0, 2)$ such that, for all $|z| \geq r_1$, $|R_z(t, z)| \leq c_2|z|^{p-1}$ and

$$R(t, z) \leq \left(\frac{1}{2} - \frac{1}{c_3|z|^\omega}\right) R_z(t, z)z.$$

Indeed, it is easy to check that $(\hat{S}_2)$ implies that $|R_z(t, z)|^\nu \leq c_4\tilde{R}(t, z)|z|^\nu$ for all $|z| \geq r_1$, $1 < \nu \leq (p - \omega)/(p - 2)$.

We now turn to the asymptotically linear case. Let

$$\mathcal{J}_1 := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

We assume besides (L_0) that

(L_1) $L(t)$ and $\mathcal{J}_1$ are anti-commutative: $\mathcal{J}_1 L(t) = -L(t)\mathcal{J}_1$ for all $t \in \mathbb{R}$.

For example, if $B(t)$ is a $N\times N$ symmetric matrix valued function, then the function

$$\begin{pmatrix} 0 & B(t) \\ B(t) & 0 \end{pmatrix}$$

satisfies (L_1). For the nonlinearity we assume

(A_1) $R_z(t,z) - L_\infty(t)z = o(|z|)$ uniformly in t as $|z| \to \infty$, where $L_\infty(t)$ is a symmetric matrix function with $\lambda_{L_\infty} > \Lambda_0$;

(A_2) $\tilde{R}(t,z) \geq 0$, and there is $\delta_0 \in (0, \lambda_0)$ such that if $|R_z(t,z)| \geq (\lambda_0 - \delta_0)|z|$ then $\tilde{R}(t,z) \geq \delta_0$;

We point out that a condition similar to (A_2) was firstly used in Jeanjean [Jeanjean (1999)] for dealing with existence of solutions to certain asymptotically linear problems on $\mathbb{R}^N$. We will prove the following result.

Theorem 5.2 ([Ding (2006)]). *Let* (L_0)-(L_1), (R_0) *and* (A_1)-(A_2) *be satisfied. Then* (HS) *has at least one homoclinic orbit. If moreover* $R(t,z)$ *is even in* z *and satisfies also*

(A_3) *there is* $\delta_1 > 0$ *such that* $\tilde{R}(t,z) \neq 0$ *if* $0 < |z| \leq \delta_1$,

then (HS) *has infinitely many geometrically distinct homoclinic orbits.*

As mentioned before, if L is constant such that 0 lies in a gap (Λ', Λ), $\Lambda' < 0 < \Lambda$, of the spectrum $\sigma(A)$ and (R_0), (A_1)-(A_2) are satisfied, then one homoclinic orbit was obtained in [Szulkin and Zou (2001)]. The most interesting result here, in Theorem 5.2, refers to the multiplicity.

Remark 5.2. The following function satisfies (R_0) and (A_1)-(A_3) provided $a(t) > \Lambda_0$ and is 1-periodic in t:

*Ex*3. $R(t,z) := a(t)|z|^2\left(1 - \frac{1}{\ln(e+|z|)}\right)$.

A more example is the following

*Ex*4. $R_z(t,z) = h(t,|z|)z$, where $h(t,s)$ is 1-periodic in t and increasing for $s \in [0,\infty)$, and $h(t,s) \to 0$ as $s \to 0$, $h(t,s) \to a(t)$ as $s \to \infty$ with $a(t) > \Lambda_0$, uniformly in t.

The following five sections are organized as follows. In next section we study the spectrum of the operator A. We show by (L_0) that $\sigma(A) \subset \mathbb{R} \setminus (-\lambda_0, \lambda_0)$. If (L_1) holds, then $\sigma(A)$ is symmetric with respect to $0 \in \mathbb{R}$. Thus (L_0) and (L_1) imply that $\lambda_0 \leq \inf(\sigma(A) \cap (0,\infty)) \leq \Lambda_0$ which is needed in the asymptotically linear case for getting a linking structure. In Section 5.3, based on the description on $\sigma(A)$, we obtain a proper variational setting for (HS) and represent the associated variational functional in the form $\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \int_{\mathbb{R}} R(t,z)$ defined on a Hilbert space $E = \mathscr{D}(|A|^{1/2}) \cong H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$ with decomposition $E = E^- \oplus E^+, z =$

$z^- + z^+$, $\dim E^{\pm} = \infty$. In Section 5.4 we show the linking structure of Φ, that is, $\inf \Phi(E^+ \cap \partial B_r) > 0$ for some $r > 0$ and there is an increasing sequence $(Y_n) \subset E^+$ of finite dimensional subspaces such that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ in $E_n := E^- \oplus Y_n$. Unlike the so called "Fountain" structure (see [Bartsch (1993); Willem (1996)]) where $\sup \Phi(E_n) \to \infty$ as $n \to \infty$, we have $\sup_n \sup \Phi(E_n) < \infty$. In Section 5.5 we show the boundedness of Cerami sequences for Φ, and then, by establishing without the regularity condition (5.3) a splitting result, prove that for any bounded interval $I \subset \mathbb{R}$, there is a discrete $(C)_I$-attractors consisting of finite sums of critical points of Φ so that any Cerami sequence at level $c \in I$ converges to $\mathscr{A}$. In Section 5.6 we firstly prove Theorem 5.1 by constructing a Cerami sequence at positive level via Theorem 4.5 and applying the concentration principle to get a nontrivial critical point of Φ, and we then apply Theorem 4.7 to prove the existence of infinitely many homoclinic orbits, that is, Theorem 5.2.

5.2 Spectrum of the Hamiltonian operator

In order to establish a variational setting for the system (HS) we study in this section the spectrum of the Hamiltonian operator.

Note that $A = -\left(\mathcal{J}\frac{d}{dt} + L\right)$ is selfadjoint on $L^2(\mathbb{R}, \mathbb{R}^{2N})$ with domain $\mathscr{D}(A) = H^1(\mathbb{R}, \mathbb{R}^{2N})$. Let $\sigma(A)$ and $\sigma_c(A)$ denote, respectively, the spectrum and the continuous spectrum. Set

$$\mu_e := \inf\{\lambda : \ \lambda \in \sigma(A) \cap [0, \infty)\}. \tag{5.4}$$

Throughout the book by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot,\cdot)_{L^2}$ the usual L^2-inner product.

Proposition 5.1. *Assume (L_0) is satisfied. Then*

1° *A has only absolute continuous spectrum* : $\sigma(A) = \sigma_c(A)$;
2° $\sigma(A) \subset \mathbb{R} \setminus (-\lambda_0, \lambda_0)$;
3° *if (L_1) also holds, $\sigma(A)$ is symmetric* : $\sigma(A) \cap (-\infty, 0) = -\sigma(A) \cap (0, \infty)$; *and* $\mu_e \le \Lambda_0$.

Proof. For the proof of 1° we see [Ding and Willem (1999)] where it was proved that, for any periodic symmetric matrix function $M(t)$, the spectrum of the operator $-\left(\mathcal{J}\frac{d}{dt} + M\right)$ is absolute continuous.

In order to show 2°, we consider the operator A^2 with domain $\mathscr{D}(A^2) = H^2(\mathbb{R}, \mathbb{R}^{2N})$. Observe that $\mathcal{J}_0^2 = I$ and $\mathcal{J}_0\mathcal{J} = -\mathcal{J}\mathcal{J}_0$. We have, for $z \in \mathscr{D}(A^2)$,

$$
\begin{aligned}
(A^2z,z)_{L^2} =& |Az|_2^2 = \left|\left(\mathcal{J}\frac{d}{dt}+\mathcal{J}_0(\mathcal{J}_0L-\lambda_0)\right)z+\lambda_0\mathcal{J}_0 z\right|_2^2 \\
=& \left|\left(\mathcal{J}\frac{d}{dt}+\mathcal{J}_0(\mathcal{J}_0L-\lambda_0)\right)z\right|_2^2+\lambda_0^2|\mathcal{J}_0z|_2^2 \\
&+(\mathcal{J}\dot{z},\,\lambda_0\mathcal{J}_0z)_{L^2}+(\lambda_0\mathcal{J}_0z,\,\mathcal{J}\dot{z})_{L^2} \\
&+(\mathcal{J}_0(\mathcal{J}_0L-\lambda_0)z,\,\lambda_0\mathcal{J}_0z)_{L^2}+(\lambda_0\mathcal{J}_0z,\,\mathcal{J}_0(\mathcal{J}_0L-\lambda_0)z)_{L^2} \\
=& \left|\left(\mathcal{J}\frac{d}{dt}+\mathcal{J}_0(\mathcal{J}_0L-\lambda_0)\right)z\right|_2^2+\lambda_0^2|z|_2^2 \\
&+2\lambda_0((\mathcal{J}_0L-\lambda_0)z,\ z)_{L^2} \\
\ge& \lambda_0^2|z|_2^2\,.
\end{aligned}
$$

Thus $\sigma(A^2)\subset[\lambda_0^2,\infty)$. Let $(F_\lambda)_{\lambda\in\mathbb{R}}$ and $(\tilde{F}_\lambda)_{\lambda\ge 0}$ denote the spectral families of A and A^2, respectively. Recall that

$$
\tilde{F}_\lambda = F_{\lambda^{1/2}} - F_{-\lambda^{1/2}-0} = F_{[-\lambda^{1/2},\lambda^{1/2}]} \qquad \text{for all } \lambda\ge 0, \tag{5.5}
$$

see (3.96) in Chapter VIII of [Dautray and Lions (1990)]. We obtain

$$
\dim(F_{[-\lambda^{1/2},\,\lambda^{1/2}]}L^2) = \dim(\tilde{F}_\lambda L^2) = 0 \quad \text{for } 0\le\lambda<\lambda_0^2, \tag{5.6}
$$

hence $\sigma(A)\subset\mathbb{R}\setminus(-\lambda_0,\lambda_0)$ which is 2°.

We now turn to 3°. Let $\lambda\in\sigma(A)\cap(0,\infty)$. Take a sequence $(z_n)\subset\mathscr{D}(A)$ such that $|z_n|_2=1$ and $|(A-\lambda)z_n|_2\to 0$. Set $\tilde{z}_n=\mathcal{J}_1z_n$. Then $|\tilde{z}_n|_2=1$. Since $\mathcal{J}\mathcal{J}_1=-\mathcal{J}_1\mathcal{J}$ and $\mathcal{J}_0\mathcal{J}_1=-\mathcal{J}_1\mathcal{J}_0$, we obtain $A\tilde{z}_n=-\mathcal{J}_1Az_n$ and

$$
|(A-(-\lambda))\tilde{z}_n|_2 = |-\mathcal{J}_1(A-\lambda)z_n|_2\to 0.
$$

This implies that $-\lambda\in\sigma(A)$. Similarly, if $\lambda\in\sigma(A)\cap(-\infty,0)$ then $-\lambda\in\sigma(A)\cap(0,\infty)$. Thus $\sigma(A)$ is symmetric with respect to 0. For showing $\mu_e\le\Lambda_0$ we consider again the operator A^2. Let $\tilde{\mu}_e:=\inf\sigma(A^2)$. Clearly $\tilde{\mu}_e\ge\lambda_0^2$. We claim that $\tilde{\mu}_e\le\Lambda_0^2$. Arguing indirectly, assume $\tilde{\mu}_e>\Lambda_0^2$. Observe that $\mathcal{J}\frac{d}{dt}$ is selfadjoint in L^2 with $0\in\sigma(\mathcal{J}\frac{d}{dt})=\mathbb{R}$, and thus we can take a sequence $z_n\in C_0^\infty(\mathbb{R},\mathbb{R}^{2N})$ with $|z_n|_2=1$ and $|\mathcal{J}\frac{d}{dt}z_n|_2\to 0$. Then

$$
\begin{aligned}
\Lambda_0^2<\tilde{\mu}_e=\tilde{\mu}_e|z_n|_2^2 &\le (A^2z_n,z_n)_{L^2}=(Az_n,Az_n)_{L^2} \\
&=\left|\mathcal{J}\frac{d}{dt}z_n+Lz_n\right|_2^2\le\left(\left|\mathcal{J}\frac{d}{dt}z_n\right|_2+|Lz_n|_2\right)^2 \\
&\le o(1)+\Lambda_0^2,
\end{aligned}
$$

a contradiction. Now using (5.5), for any $\varepsilon>0$,

$$
\dim(F_{[-(\tilde{\mu}_e+\varepsilon)^{1/2},\,(\tilde{\mu}_e+\varepsilon)^{1/2}]}L^2)=\dim(\tilde{F}_{\tilde{\mu}_e+\varepsilon}L^2)=\infty
$$

which, together with (5.6), implies that at least one of $\pm\tilde{\mu}_e^{1/2}$ belongs to $\sigma(A)$, hence by the symmetry $\pm\tilde{\mu}_e^{1/2}\in\sigma(A)$. We get $\mu_e\le\tilde{\mu}_e^{1/2}\le\Lambda_0$, finishing the proof. □

5.3 Variational setting

In virtue of Proposition 5.1, $L^2 = L^2(\mathbb{R}, \mathbb{R}^{2N})$ possesses the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+$$

corresponding to the spectrum decomposition of A such that $(Az, z)_{L^2} \le -\lambda_0 |z|_2^2$ for $z \in L^- \cap \mathscr{D}(A)$ and $(Az, z)_{L^2} \ge \lambda_0 |z|_2^2$ for $z \in L^+ \cap \mathscr{D}(A)$. Denoting by $|A|$ the absolute value, let $E := \mathscr{D}(|A|^{1/2})$ be the Hilbert space equipped with the inner product

$$(z_1, z_2) = \left(|A|^{1/2} z_1,\, |A|^{1/2} z_2\right)_{L^2}$$

and the norm $\|z\| = (z, z)^{1/2}$. E has the orthogonal decomposition

$$E = E^- \oplus E^+ \quad \text{where} \quad E^{\pm} = E \cap L^{\pm}.$$

Observe that, letting $A_0 = \mathcal{J}\frac{d}{dt} + \mathcal{J}_0$, Proposition 5.1 implies that there are $c_1, c_2 > 0$ such that

$$c_1 |A_0 z|_2 \le |Az|_2 \le c_2 |A_0 z|_2$$

for all $z \in H^1(\mathbb{R}, \mathbb{R}^{2N})$. A Fourier analysis shows that $|A_0 z|_2 = \|z\|_{H^1}$, hence $c_1 \|z\|_{H^1} \le |Az|_2 \le c_2 \|z\|_{H^1}$. Thus by interpolation one has $c_1' \|z\|_{H^{1/2}} \le \|z\| \le c_2' \|z\|_{H^{1/2}}$ for all $z \in E$ (cf. [Ding and Willem (1999)]). Using the Sobolev embedding theorem (on $H^{1/2}$) we get directly the following lemma.

Lemma 5.1. *Under (L_0), the space E embeds continuously into $L^p(\mathbb{R}, \mathbb{R}^{2N})$ for any $p \ge 2$, and compactly into $L^p_{loc}(\mathbb{R}, \mathbb{R}^{2N})$ for any $p \in [1, \infty)$.*

Note that, using A, the system (HS) can be rewritten as

$$Az = R_z(t, z). \tag{5.7}$$

On E we define the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z) \quad \text{where} \quad \Psi(z) = \int_{\mathbb{R}} R(t, z). \tag{5.8}$$

Our hypotheses on $H(t, z)$ imply that $\Phi \in C^1(E, \mathbb{R})$ and a standard argument invoking (5.7) shows that critical points of Φ are homoclinic orbits of (HS) (cf. [Ding and Willem (1999)]). We will write Φ' for the derivative of Φ.

Observe that if (S_2) holds, then $|R_z(t, z)|^{\nu} \le c_1 |R_z(t, z)| |z|^{\nu+1}$, hence

$$|R_z(t, z)| \le d_1 |z|^{p-1} \quad \text{if } |z| \ge r_1 \tag{5.9}$$

for $p \ge 2\nu/(\nu - 1)$. Clearly (5.9) remains true for all $p \ge 2$ if (A_1) holds.

Lemma 5.2. *Let (L_0) and (R_0) be satisfied, and assume moreover either (S_1)-(S_2) or (A_1)-(A_2) hold. Then Ψ is non-negative, weakly sequentially lower semicontinuous, and Ψ' is weakly sequentially continuous.*

Proof. By (R_0), $R(t,z)$ is non-negative, so is Ψ. Let $z_j \in E$ with $z_j \rightharpoonup z$ in E. By Lemma 5.1, $z_j(t) \to z(t)$, hence $R(t, z_j(t)) \to R(t, z(t))$ for a.e. $t \in \mathbb{R}$. Thus the Lebesgue theorem implies

$$\begin{aligned}\Psi(z) = \int_{\mathbb{R}} R(t,z) &= \int_{\mathbb{R}} \lim_{j\to\infty} R(t,z_j) \\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}} R(t,z_j) = \liminf_{j\to\infty} \Psi(z_j),\end{aligned}$$

proving that Ψ is weakly sequentially lower semi-continuous.

To show that Ψ' is weakly sequentially continuous, let $z_j \rightharpoonup z$ in E. By Lemma 5.1, $z_j \to z$ in L^p_{loc} for any $p \ge 1$. By (R_0) and (5.9) we can take $p > 2$ so that $|R_z(t,z)| \le c_1(|z| + |z|^{p-1})$. It is clear that, for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\Psi'(z_j)\varphi = \int_{\mathbb{R}} R_z(t,z_j)\varphi \to \int_{\mathbb{R}} R_z(t,z)\varphi = \Psi'(z)\varphi. \tag{5.10}$$

Since C_0^∞ is dense in E , for any $w \in E$ we take $\varphi_n \in C_0^\infty$ such that $\|\varphi_n - w\| \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}|\Psi'(z_j)w - \Psi'(z)w| &\le |(\Psi'(z_j) - \Psi'(z))\varphi_n| + |(\Psi'(z_j) - \Psi'(z))(w - \varphi_n)| \\ &\le |(\Psi'(z_j) - \Psi'(z))\varphi_n| \\ &\quad + c_2 \int_{\mathbb{R}} \left(|z| + |z_j| + |z|^{p-1} + |z_j|^{p-1}\right) |w - \varphi_n| \\ &\le |(\Psi'(z_j) - \Psi'(z))\varphi_n| + c_3\|w - \varphi_n\|.\end{aligned}$$

For any $\varepsilon > 0$, fix n so that $\|w - \varphi_n\| < \varepsilon/2c_3$. By (5.10) there is j_0 so that $|(\Psi'(z_j) - \Psi'(z))\varphi_n| < \varepsilon/2$ for all $j \ge j_0$. Then $|\Psi'(z_j)w - \Psi'(z)w| < \varepsilon$ for all $j \ge j_0$, proving the weakly sequentially continuity. □

5.4 Linking structure

We now study the linking structure of Φ. Remark that (R_0) and (5.9) implies that, given arbitriarily $p \ge 2\nu/(\nu-1)$ in the super linear case, $p \ge 2$ in the asymptotically linear case, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|R_z(t,z)| \le \varepsilon|z| + C_\varepsilon|z|^{p-1} \tag{5.11}$$

and

$$R(t,z) \le \varepsilon|z|^2 + C_\varepsilon|z|^p \tag{5.12}$$

for all (t,z). Firstly we have the following lemma.

Lemma 5.3. *Under the assumptions of Lemma 5.2, there is $r > 0$ such that $\kappa := \inf \Phi(S_r^+) > \Phi(0) = 0$ where $S_r^+ = \partial B_r \cap E^+$.*

Proof. Choose $p > 2$ such that (5.12) holds for any $\varepsilon > 0$. This, jointly with Lemma 5.1, yields

$$\Psi(z) \le \varepsilon |z|_2^2 + C_\varepsilon |z|_p^p \le C(\varepsilon \|z\|^2 + C_\varepsilon \|z\|^p)$$

for all $z \in E$. Now the lemma follows from the form (5.8) of Φ. □

In the following, we fix arbitrarily an $\omega \ge 2\mu_e$ for the super linear case (where μ_e is the number defined by (5.4)), and set $\omega := \lambda_{L_\infty}$ for the asymptotically linear case. Remark that Proposition 5.1 and (A_1) imply that $\lambda_0 \le \mu_e \le \Lambda_0 < \lambda_{L_\infty}$ (this is the only place we use (L_1)). Thus, in both super and asymptotically cases, we can take a number $\bar{\mu}$ satisfying

$$\mu_e < \bar{\mu} < \omega. \tag{5.13}$$

Since $\sigma(A) = \sigma_c(A)$, the subspace $Y_0 := (F_{\bar{\mu}} - F_0)L^2$ is infinite dimensional (recall that $(F_\lambda)_{\lambda \in \mathbb{R}}$ denotes the spectrum family of A). Note that

$$Y_0 \subset E^+ \quad \text{and} \quad \mu_e |w|_2^2 \le \|w\|^2 \le \bar{\mu} |w|_2^2 \quad \text{for all } w \in Y_0. \tag{5.14}$$

For any finite dimensional subspace Y of Y_0 set $E_Y = E^- \oplus Y$.

Lemma 5.4. *Let the assumptions of Lemma 5.2 be satisfied, and assume (L_1) also holds for the asymptotically linear case. Then for any finite dimensional subspace Y of Y_0, $\sup \Phi(E_Y) < \infty$, and there is $R_Y > 0$ such that $\Phi(z) < \inf \Phi(B_r)$ for all $z \in E_Y$ with $\|z\| \ge R_Y$.*

Proof. It is sufficient to show that $\Phi(z) \to -\infty$ as $z \in E_Y, \|z\| \to \infty$. Arguing indirectly, assume that for some sequence $z_j \in E_Y$ with $\|z_j\| \to \infty$, there is $M > 0$ such that $\Phi(z_j) \ge -M$ for all j. Then, setting $w_j = z_j/\|z_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^+ \to w^+ \in Y$ and

$$-\frac{M}{\|z_j\|^2} \le \frac{\Phi(z_j)}{\|z_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}} \frac{R(t, z_j)}{\|z_j\|^2}. \tag{5.15}$$

Remark that $w^+ \ne 0$. Indeed, if not then it follows from (5.15) that

$$0 \le \frac{1}{2}\|w_j^-\|^2 + \int_{\mathbb{R}} \frac{R(t, z_j)}{\|z_j\|^2} \le \frac{1}{2}\|w_j^+\|^2 + \frac{M}{\|z_j\|^2} \to 0,$$

in particular, $\|w_j^-\| \to 0$, hence $1 = \|w_j\| \to 0$, a contradiction.

First, consider the super linear case and suppose $(S_1) - (S_2)$ hold. Then by (S_1) there is $r_0 > 0$ such that $R(t, z) \ge \omega |z|^2$ if $|z| \ge r_0$. Using (5.13)-(5.14),

$$\begin{aligned} \|w^+\|^2 - \|w^-\|^2 - \omega \int_{\mathbb{R}} |w|^2 &\le \bar{\mu}|w^+|_2^2 - \|w^-\|^2 - \omega|w^+|_2^2 - \omega|w^-|_2^2 \\ &\le -\big((\omega - \bar{\mu})|w^+|_2^2 + \|w^-\|^2\big) < 0, \end{aligned}$$

hence, there is $a > 0$ large such that

$$\|w^+\|^2 - \|w^-\|^2 - \omega \int_{-a}^{a} |w|^2 < 0. \tag{5.16}$$

Note that

$$\begin{aligned}\frac{\Phi(z_j)}{\|z_j\|^2} \leq & \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2\right) - \int_{-a}^{a} \frac{R(t,z_j)}{\|z_j\|^2} \\ = & \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2 - \omega\int_{-a}^{a}|w_j|^2\right) - \int_{-a}^{a}\frac{R(t,z_j)-\frac{\omega}{2}|z_j|^2}{\|z_j\|^2} \\ \leq & \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2 - \omega\int_{-a}^{a}|w_j|^2\right) + \frac{a\omega r_0^2}{\|z_j\|^2}.\end{aligned}$$

Thus (5.15) and (5.16) imply that

$$\begin{aligned}0 \leq & \lim_{j\to\infty}\left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{-a}^{a}\frac{R(t,z_j)}{\|z_j\|^2}\right) \\ \leq & \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2 - \omega\int_{-a}^{a}|w|^2\right) \;<\; 0,\end{aligned}$$

a contradiction.

Next consider the asymptotically linear case and assume (A_1) holds. By (5.13)-(5.14) again,

$$\begin{aligned}\|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}} L_\infty(t)ww \leq & \|w^+\|^2 - \|w^-\|^2 - \omega|w|_2^2 \\ \leq & -\left((\omega-\bar{\mu})|w^+|_2^2 + \|w^-\|^2\right) < 0,\end{aligned}$$

hence, for some $a > 0$,

$$\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a} L_\infty(t)ww \;<\; 0. \tag{5.17}$$

Set

$$F(t,z) := R(t,z) - \frac{1}{2}L_\infty(t)zz. \tag{5.18}$$

By (A_1), $|F(t,z)| \leq C|z|^2$ and $F(t,z)/|z|^2 \to 0$ as $|z| \to \infty$ uniformly in t. It follows from Lebesgue's dominated convergence theorem and the fact $|w_j - w|_{L^2(-a,a)} \to 0$ that

$$\lim_{j\to\infty}\int_{-a}^{a}\frac{F(t,z_j)}{\|z_j\|^2} = \lim_{j\to\infty}\int_{-a}^{a}\frac{F(t,z_j)|w_j|^2}{|z_j|^2} = 0.$$

Thus (5.15) and (5.17) imply that

$$\begin{aligned}0 \leq & \lim_{j\to\infty}\left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{-a}^{a}\frac{R(t,z_j)}{\|z_j\|^2}\right) \\ \leq & \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a}L_\infty(t)ww\right) \;<\; 0,\end{aligned}$$

a contradiction. □

As a special case we have

Lemma 5.5. *Under the assumptions of Lemma 5.4, letting $e \in Y_0$ with $\|e\| = 1$, there is $r_0 > 0$ such that $\sup\Phi(\partial Q) = 0$ where $Q := \{u = u^- + se :\ u^- \in E^-, s \geq 0, \|u\| \leq r_0\}$.*

5.5 The (C) sequences

We now study the Cerami sequences.

Lemma 5.6. *Under the assumptions of Lemma 5.2, any $(C)_c$-sequence is bounded.*

Proof. Let $(z_j) \subset E$ be such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1 + \|z_j\|)\Phi'(z_j) \to 0. \tag{5.19}$$

Then

$$C_0 \geq \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}} \tilde{R}(t, z_j)\,. \tag{5.20}$$

Arguing indirectly, assume up to a subsequence $\|z_j\| \to \infty$. Set $v_j = z_j/\|z_j\|$. Then $\|v_j\| = 1$ and $|v_j|_s \leq \gamma_s\|v_j\| = \gamma_s$ for all $s \in [2, \infty)$. Noting that

$$\Phi'(z_j)(z_j^+ - z_j^-) = \|z_j\|^2 \left(1 - \int_{\mathbb{R}} \frac{R_z(t, z_j)(v_j^+ - v_j^-)}{\|z_j\|}\right),$$

it follows from (5.19) that

$$\int_{\mathbb{R}} \frac{R_z(t, z_j)(v_j^+ - v_j^-)}{\|z_j\|} \to 1. \tag{5.21}$$

First we consider the super linear case and suppose $(S_1) - (S_2)$ hold. Set for $r \geq 0$

$$g(r) := \inf \left\{\tilde{R}(t, z) : t \in \mathbb{R} \text{ and } z \in \mathbb{R}^{2N} \text{ with } |z| \geq r\right\}$$

(S_2) implies $g(r) > 0$ for all $r > 0$. Moreover,

$$\begin{aligned} c_1\tilde{R}(t, z) &\geq \left(\frac{|R_z(t, z)|}{|z|}\right)^\nu = \left(\frac{|R_z(t, z)||z|}{|z|^2}\right)^\nu \\ &\geq \left(\frac{R_z(t, z)z}{|z|^2}\right)^\nu \geq \left(\frac{2R(t, z)}{|z|^2}\right)^\nu, \end{aligned}$$

which, jointly with (S_1), implies $\tilde{R}(t, z) \to \infty$ uniformly in t, consequently $g(r) \to \infty$ as $r \to \infty$. Furthermore, set for $0 \leq a < b$

$$\Omega_j(a, b) = \{t \in \mathbb{R} : a \leq |z_j(t)| < b\}$$

and

$$c_a^b := \inf \left\{\frac{\tilde{R}(t, z)}{|z|^2} : t \in \mathbb{R} \text{ and } z \in \mathbb{R}^{2N} \text{ with } a \leq |z| \leq b\right\}.$$

Since $R(t, z)$ depends periodically on t and $\tilde{R}(t, z) > 0$ if $z \neq 0$, one has $c_a^b > 0$ and

$$\tilde{R}(t, z_j(x)) \geq c_a^b|z_j(t)|^2 \quad \text{for all } t \in \Omega_j(a, b).$$

It follows from (5.20) that

$$C_0 \geq \int_{\Omega_j(0,a)} \tilde{R}(t,z_j) + \int_{\Omega_j(a,b)} \tilde{R}(t,z_j) + \int_{\Omega_j(b,\infty)} \tilde{R}(t,z_j)$$
$$\geq \int_{\Omega_j(0,a)} \tilde{R}(t,z_j) + c_a^b \int_{\Omega_j(a,b)} |z_j|^2 + g(b)|\Omega_j(b,\infty)|.$$

Thus

$$|\Omega_j(b,\infty)| \leq \frac{C_0}{g(b)} \to 0$$

as $b \to \infty$ uniformly in j, which implies by Hölder inequality that for any $s \in [2,\infty)$,

$$\int_{\Omega_j(b,\infty)} |v_j|^s \leq \gamma_{2s}^s \, |\Omega_j(b,\infty)|^{1/2} \to 0 \tag{5.22}$$

as $b \to \infty$ uniformly in j. In addition, for any fixed $0 < a < b$,

$$\int_{\Omega_j(a,b)} |v_j|^2 = \frac{1}{\|z_j\|^2} \int_{\Omega_j(a,b)} |z_j|^2 \leq \frac{C_0}{c_a^b \|z_j\|^2} \to 0 \tag{5.23}$$

as $j \to \infty$.

Let $0 < \varepsilon < 1/3$. By (R_0) there is $a_\varepsilon > 0$ such that $|R_z(t,z)| < \frac{\varepsilon}{\gamma_2}|z|$ for all $|z| \leq a_\varepsilon$, consequently,

$$\begin{aligned} &\int_{\Omega_j(0,a_\varepsilon)} \frac{|R_z(t,z_j)|}{|z_j|} \, |v_j| \, |v_j^+ - v_j^-| \\ &\leq \int_{\Omega_j(0,a_\varepsilon)} \frac{\varepsilon}{\gamma_2} |v_j^+ - v_j^-| \, |v_j| \leq \frac{\varepsilon}{\gamma_2} |v_j|_2^2 \leq \varepsilon \end{aligned} \tag{5.24}$$

for all j. By (S_2) and (5.22), setting $\mu = 2\nu/(\nu-1)$ and $\nu' = \mu/2 = \nu/(\nu-1)$, we can take $b_\varepsilon \geq r_1$ large so that

$$\begin{aligned} &\int_{\Omega_j(b_\varepsilon,\infty)} \frac{|R_z(t,z_j)|}{|z_j|} |v_j||v_j^+ - v_j^-| \\ &\leq \left(\int_{\Omega_j(b_\varepsilon,\infty)} \frac{|R_z(t,z_j)|^\nu}{|z_j|^\nu} \right)^{1/\nu} \left(\int_{\Omega_j(b_\varepsilon,\infty)} \left(|v_j^+ - v_j^-| \, |v_j| \right)^{\nu'} \right)^{1/\nu'} \\ &\leq \left(\int_{\mathbb{R}} c_1 \tilde{R}(t,z_j) \right)^{1/\nu} \left(\int_{\mathbb{R}} |v_j^+ - v_j^-|^\mu \right)^{1/\mu} \left(\int_{\Omega_j(b_\varepsilon,\infty)} |v_j|^\mu \right)^{1/\mu} \\ &< \varepsilon \end{aligned} \tag{5.25}$$

for all j. Note that there is $\gamma = \gamma(\varepsilon) > 0$ independent of j such that $|R_z(t,z_j)| \leq \gamma|z_j|$ for $t \in \Omega_j(a_\varepsilon, b_\varepsilon)$. By (5.23) there is j_0 such that

$$\begin{aligned} &\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} \frac{|R_z(t,z_j)|}{|z_j|} \, |v_j| \, |v_j^+ - v_j^-| \\ &\leq \gamma \int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |v_j^+ - v_j^-| \, |v_j| \\ &\leq \gamma |v_j|_2 \left(\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |v_j|^2 \right)^{1/2} < \varepsilon \end{aligned} \tag{5.26}$$

for all $j \geq j_0$. Now the combination of (5.24)-(5.26) implies that for $j \geq j_0$

$$\int_{\mathbb{R}} \frac{R_z(t,z_j)\,(v_j^+ - v_j^-)}{\|z_j\|} \leq \int_{\mathbb{R}} \frac{|R_z(t,z_j)|}{|z_j|}\,|v_j|\,|v_j^+ - v_j^-| \, < 3\varepsilon < 1$$

which contradicts (5.21).

Next we consider the asymptotically linear case, hence assume (A_1)-(A_2) are satisfied. Following the terminology introduced by Lions on the concentration compactness principle [Lions (1984)], observe that either (v_j) is vanishing (in this case $|v_j|_s \to 0$ for all $s > 2$), or it is nonvanishing, that is, there are $r, \eta > 0$ and $(a_j) \subset \mathbb{Z}$ such that $\limsup_{j\to\infty} \int_{a_j-r}^{a_j+r} |v_j|^2 \ \geq \eta$. We show as in [Jeanjean (1999); Szulkin and Zou (2001)] that (v_j) is neither vanishing nor nonvanishing.

Assume (v_j) is vanishing. Set, in virtue of (A_2),

$$I_j := \left\{ t \in \mathbb{R} : \ \frac{|R_z(t,z_j(t))|}{|z_j(t)|} \leq \lambda_0 - \delta_0 \right\}.$$

By Proposition 5.1, $\lambda_0 |v_j|_2^2 \leq \|v_j\|^2 = 1$ and we get

$$\left| \int_{I_j} \frac{R_z(t,z_j)(v_j^+ - v_j^-)}{\|z_j\|} \right| = \left| \int_{I_j} \frac{R_z(t,z_j)(v_j^+ - v_j^-)|v_j|}{|z_j|} \right|$$
$$\leq (\lambda_0 - \delta_0)|v_j|_2^2 \leq \frac{\lambda_0 - \delta_0}{\lambda_0} < 1$$

for all j. This, jointly with (5.21), implies that for $I_j^c := \mathbb{R} \setminus I_j$

$$\lim_{j\to\infty} \int_{I_j^c} \frac{R_z(t,z_j)(v_j^+ - v_j^-)}{\|z_j\|} > 1 - \frac{\lambda_0 - \delta_0}{\lambda_0} = \frac{\delta_0}{\lambda_0}.$$

Recalling that by (R_0) and (A_1)

$$|R_z(t,z)| \leq C|z| \quad \text{for all } (t,z), \tag{5.27}$$

there holds for an arbitrarily fixed $s > 2$

$$\int_{I_j^c} \frac{R_z(t,z_j)(v_j^+ - v_j^-)}{\|z_j\|} \leq C \int_{I_j^c} |v_j^+ - v_j^-||v_j|$$
$$\leq C|v_j|_2 |I_j^c|^{(s-2)/2s} |v_j|_s \leq C\gamma_2 |I_j^c|^{(s-2)/2s} |v_j|_s.$$

Since $|v_j|_s \to 0$, one gets $|I_j^c| \to \infty$. By (A_2), $\tilde{R}(t,z_j) \geq \delta_0$ on I_j^c, hence

$$\int_{\mathbb{R}} \tilde{R}(t,z_j) \geq \int_{I_j^c} \tilde{R}(t,z_j) \geq \delta_0 |I_j^c| \,\to\, \infty,$$

contrary to (5.20).

Assume (v_j) is nonvanishing. Setting $\tilde{z}_j(t) = z_j(t + a_j)$, $\ \tilde{v}_j(t) = v_j(t + a_j)$ and $\varphi_j(t) = \varphi(t - a_j)$ for any $\varphi \in C_0^\infty$ we have by (A_1) (see (5.18) for $F(t,z)$)

$$\Phi'(z_j)\varphi_j = (z_j^+ - z_j^-, \varphi_j) - (L_\infty z_j, \varphi_j)_{L^2} - \int_{\mathbb{R}} F_z(t,z_j)\varphi_j$$
$$= \|z_j\| \left((v_j^+ - v_j^-, \varphi_j) - (L_\infty v_j, \varphi_j)_{L^2} - \int_{\mathbb{R}} F_z(t,z_j)\varphi_j \frac{|v_j|}{|z_j|} \right)$$
$$= \|z_j\| \left((\tilde{v}_j^+ - \tilde{v}_j^-, \varphi) - (L_\infty \tilde{v}_j, \varphi)_{L^2} - \int_{\mathbb{R}} F_z(t,\tilde{z}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{z}_j|} \right).$$

This results

$$(\tilde{v}_j^+ - \tilde{v}_j^-, \varphi) - (L_\infty \tilde{v}_j, \varphi)_{L^2} - \int_{\mathbb{R}} F_z(t, \tilde{z}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{z}_j|} \to 0.$$

Since $\|\tilde{v}_j\| = \|v_j\| = 1$, we can assume that $\tilde{v}_j \rightharpoonup \tilde{v}$ in E, $\tilde{v}_j \to \tilde{v}$ in L^2_{loc} and $\tilde{v}_j(t) \to \tilde{v}(t)$ a.e. in $\mathbb{R}$. Since $\lim_{j\to\infty} \int_{-r}^r |\tilde{v}_j|^2 \geq \eta$, $\tilde{v} \neq 0$. By (5.27)

$$\left| F_z(t, \tilde{z}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{z}_j|} \right| \leq C|\varphi||\tilde{v}_j|,$$

it follows from (A_1) and the dominated convergence theorem that

$$\int_{\mathbb{R}} F_z(t, \tilde{z}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{z}_j|} \to 0,$$

hence

$$(\tilde{v}^+ - \tilde{v}^-, \varphi) - (L_\infty \tilde{v}, \varphi)_{L^2} = 0.$$

Thus $\tilde{v}$ is an eigenfunction of the operator $\tilde{A} := \mathcal{J}\frac{d}{dt} + (L + L_\infty)$ contradicting with the fact that $\tilde{A}$ has only continuous spectrum (since $L(t) + L_\infty(t)$ is 1-periodic, see [Ding and Willem (1999)]). □

In the following lemma we discuss further the $(C)_c$-sequence $(z_j) \subset E$. By Lemma 5.6 it is bounded, hence, we may assume without loss of generality that $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^q_{loc} for $q \geq 1$ and $z_j(t) \to z(t)$ a.e. in t. Plainly z is a critical point of Φ. Set $z_j^1 = z_j - z$.

Lemma 5.7. *Under the assumptions of Lemma 5.2, along a subsequence:*

1) $\Phi(z_j^1) \to c - \Phi(z)$;
2) $\Phi'(z_j^1) \to 0$.

Proof. The verification of 1) is somewhat standard (cf. [Ding and Girardi (1999)]), so we only check 2).

Observe that, for any $\varphi \in E$,

$$\Phi'(z_j^1)\varphi = \Phi'(z_j)\varphi + \int_{\mathbb{R}} \left(R_z(t, z_j) - R_z(t, z_j^1) - R_z(t, z)\right)\varphi.$$

Since $\Phi'(z_j) \to 0$, it suffices to show that

$$\sup_{\|\varphi\|\leq 1} \left| \int_{\mathbb{R}} \left(R_z(t, z_j) - R_z(t, z_j^1) - R_z(t, z)\right)\varphi \right| \to 0. \tag{5.28}$$

Recall that if R satisfies (5.3), then (5.28) follows easily from a standard argument, see e.g. [Arioli and Szulkin (1999); Ding and Girardi (1999)]. However, in our case such a regularity condition is not available and we hence provide another argument. By (5.11) we choose $p \geq 2$ such that $|R_z(t, z)| \leq |z| + C_1|z|^{p-1}$ for all (t, z), and let q stands for either 2 or p. Set $I_a := [-a, a]$ for $a > 0$.

We claim that there is a subsequence (z_{j_n}) such that, for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ satisfying

$$\limsup_{n\to\infty} \int_{I_n\setminus I_r} |z_{j_n}|^q \le \varepsilon \tag{5.29}$$

for all $r \ge r_\varepsilon$. For verifying (5.29) note that, for each $n \in \mathbb{N}$, $\int_{I_n} |z_j|^q \to \int_{I_n} |z|^q$ as $j \to \infty$. There exists $i_n \in \mathbb{N}$ such that

$$\int_{I_n} (|z_j|^q - |z|^q) < \frac{1}{n} \quad \text{for all } j = i_n + m,\ m = 1, 2, 3, \ldots.$$

Without loss of generality we can assume $i_{n+1} \ge i_n$. In particular, for $j_n = i_n + n$ we have

$$\int_{I_n} (|z_{j_n}|^q - |z|^q) < \frac{1}{n}.$$

Observe that there is r_ε satisfying

$$\int_{\mathbb{R}\setminus I_r} |z|^q < \varepsilon \tag{5.30}$$

for all $r \ge r_\varepsilon$. Since

$$\begin{aligned}\int_{I_n\setminus I_r} |z_{j_n}|^q &= \int_{I_n} (|z_{j_n}|^q - |z|^q) + \int_{I_n\setminus I_r} |z|^q + \int_{I_r} (|z|^q - |z_{j_n}|^q) \\ &\le \frac{1}{n} + \int_{\mathbb{R}\setminus I_r} |z|^q + \int_{I_r} (|z|^q - |z_{j_n}|^q),\end{aligned}$$

(5.29) now follows.

As in [Ackermann (2004)] let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{z}_n(t) = \eta(2|t|/n)z(t)$ and set $h_n := z - \tilde{z}_n$. Since z is a homoclinic orbit, we have by definition that $h_n \in H^1$ and

$$\|h_n\| \to 0 \text{ and } |h_n|_\infty \to 0 \text{ as } n \to \infty. \tag{5.31}$$

Observe that for any $\varphi \in E$

$$\begin{aligned}&\int_{\mathbb{R}} \big(R_z(t, z_{j_n}) - R_z(t, z^1_{j_n}) - R_z(t, z)\big)\varphi \\ = &\int_{\mathbb{R}} (R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n))\varphi \\ &+ \int_{\mathbb{R}} \big(R_z(t, z^1_{j_n} + h_n) - R_z(t, z^1_{j_n})\big)\varphi \\ &+ \int_{\mathbb{R}} (R_z(t, \tilde{z}_n) - R_z(t, z))\varphi.\end{aligned}$$

Plainly, by (5.31),

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} (R_z(t, \tilde{z}_n) - R_z(t, z))\varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. It remains for checking (5.28) to show that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}}\left(R_z(t,z_{j_n}) - R_z(t,z_{j_n}-\tilde{z}_n) - R_z(t,\tilde{z}_n)\right)\varphi\right| = 0 \tag{5.32}$$

and

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}}\left(R_z(t,z^1_{j_n}+h_n) - R_z(t,z^1_{j_n})\right)\varphi\right| = 0 \tag{5.33}$$

uniformly in $\|\varphi\| \le 1$.

To check (5.32), note that (5.31) and the compactness of Sobolev embeddings imply that, for any $r > 0$,

$$\lim_{n\to\infty}\left|\int_{I_r}\left(R_z(t,z_{j_n}) - R_z(t,z_{j_n}-\tilde{z}_n) - R_z(t,\tilde{z}_n)\right)\varphi\right| = 0$$

uniformly in $\|\varphi\| \le 1$. For any $\varepsilon > 0$ let $r_\varepsilon > 0$ so large that (5.29) and (5.30) hold. Then

$$\limsup_{n\to\infty}\int_{I_n\setminus I_r}|\tilde{z}_n|^q \le \int_{\mathbb{R}\setminus I_r}|z|^q \le \varepsilon$$

for all $r \ge r_\varepsilon$. Using (5.29) for $q = 2, p$ we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{\mathbb{R}}\left(R_z(t,z_{j_n}) - R_z(t,z_{j_n}-\tilde{z}_n) - R_z(t,\tilde{z}_n)\right)\varphi\right| \\
&= \limsup_{n\to\infty}\left|\int_{I_n\setminus I_r}\left(R_z(t,z_{j_n}) - R_z(t,z_{j_n}-\tilde{z}_n) - R_z(t,\tilde{z}_n)\right)\varphi\right| \\
&\le c_1\limsup_{n\to\infty}\int_{I_n\setminus I_r}\left(|z_{j_n}|+|\tilde{z}_n|\right)|\varphi| \\
&\quad + c_2\limsup_{n\to\infty}\int_{I_n\setminus I_r}\left(|z_{j_n}|^{p-1}+|\tilde{z}_n|^{p-1}\right)|\varphi| \\
&\le c_1\limsup_{n\to\infty}\left(|z_{j_n}|_{L^2(I_n\setminus I_r)}+|\tilde{z}_n|_{L^2(I_n\setminus I_r)}\right)|\varphi|_2 \\
&\quad + c_2\limsup_{n\to\infty}\left(|z_{j_n}|^{p-1}_{L^p(I_n\setminus I_r)}+|\tilde{z}_n|^{p-1}_{L^p(I_n\setminus I_r)}\right)|\varphi|_p \\
&\le c_3\varepsilon^{1/2} + c_4\varepsilon^{(p-1)/p},
\end{aligned}$$

which implies (5.32).

For verifying (5.33), define $g(t,0) = 0$ and

$$g(t,z) = \frac{R_z(t,z)}{|z|} \quad \text{if } z \ne 0.$$

By (R_0), g is continuous at $z = 0$, hence in $\mathbb{R}\times\mathbb{R}^{2N}$. g is 1-periodic in t since R_z is. This, jointly with the uniformly continuity in $[0,1]\times B_a$, implies that g is uniformly continuous in $\mathbb{R}\times B_a$ for any $a > 0$ where $B_a := \{z \in \mathbb{R}^{2N} : |z| \le a\}$. Moreover, it follows from (5.11) that $|g(t,z)| \le c_5(1+|z|^{p-2})$ for all (t,z). Set

$$C^a_n := \{t\in\mathbb{R} : |z^1_{j_n}(t)| \le a\} \quad \text{and} \quad D^a_n := \mathbb{R}\setminus C^a_n.$$

Since the Lebesgue measure

$$|D_n^a| \le \frac{1}{a^p}\int_{D_n^a} |z_{j_n}^1|^p \le \frac{C}{a^p} \to 0 \quad \text{as } a \to \infty,$$

one has, for any $\varepsilon > 0$, there is $\hat{a} > 0$ such that

$$\left| \int_{D_n^a} \left(R_z(t, z_{j_n}^1 + h_n) - R_z(t, z_{j_n}^1)\right)\varphi \right| \le \varepsilon \tag{5.34}$$

uniformly in $\|\varphi\| \le 1$ for all $a \ge \hat{a}$ and all n. By the uniformly continuity of g on $\mathbb{R} \times B_{\hat{a}}$, there is $\delta > 0$ satisfying

$$|g(t, z+h) - g(t,z)| < \varepsilon \;\text{ for all } (t,z) \in \mathbb{R} \times B_{\hat{a}} \text{ and } |h| \le \delta,$$

and by (5.31), there exists n_0 such that $|h_n|_\infty \le \delta$ for all $n \ge n_0$, hence

$$|g(t, z_{j_n}^1 + h_n) - g(t, z_{j_n}^1)| < \varepsilon \;\text{ for all } n \ge n_0 \text{ and } t \in C_n^{\hat{a}}.$$

Note that

$$\begin{aligned}\left(R_z(t, z_{j_n}^1 + h_n) - R_z(t, z_{j_n}^1)\right)\varphi &= g(t, z_{j_n}^1 + h_n)\left(|z_{j_n}^1 + h_n| - |z_{j_n}^1|\right)\varphi \\ &\quad + \left(g(t, z_{j_n}^1 + h_n) - g(t, z_{j_n}^1)\right)|z_{j_n}^1|\varphi\end{aligned}$$

and, by (5.31), $|h_n|_2 < \varepsilon$, $|h_n|_p < \varepsilon$ for all $n \ge n_1$, some $n_1 \ge n_0$. Thus, for all $\|\varphi\| \le 1$ and $n \ge n_1$,

$$\begin{aligned}&\left| \int_{C_n^{\hat{a}}} \left(R_z(t, z_{j_n}^1 + h_n) - R_z(t, z_{j_n}^1)\right)\varphi \right| \\ &= \int_{C_n^{\hat{a}}} c_5\left(1 + |z_{j_n}^1 + h_n|^{p-2}\right)|h_n|\,|\varphi| + \varepsilon \int_{C_n^{\hat{a}}} |z_{j_n}^1|\,|\varphi| \\ &\le c_5|h_n|_2|\varphi|_2 + c_5|z_{j_n}^1 + h_n|_p^{p-2}|h_n|_p|\varphi|_p + \varepsilon|z_{j_n}^1|_2|\varphi|_2 \\ &\le c_6\varepsilon\end{aligned}$$

which, together with (5.34), implies (5.33) ending the proof. □

Let $\mathcal{K} := \{z \in E : \Phi'(z) = 0\}$ denote the critical set of Φ.

Lemma 5.8. *Under the assumptions of Lemma 5.2, there hold*

a) $\theta := \inf\{\|z\| : z \in \mathcal{K} \setminus \{0\}\} > 0$*;*
b) $\hat{c} := \inf\{\Phi(z) : z \in \mathcal{K} \setminus \{0\}\} > 0$ *provided additionally in the asymptotically linear case* (A_3) *also holds.*

Proof. *a)* Assume there is a sequence $(z_j) \subset \mathcal{K} \setminus \{0\}$ with $z_j \to 0$. Then

$$0 = \|z_j\|^2 - \int_{\mathbb{R}} R_z(t, z_j)(z_j^+ - z_j^-).$$

Choose $p > 2$ such that (5.11) holds. Thus for any $\varepsilon > 0$ small,

$$\|z_j\|^2 \le \varepsilon|z_j|_2^2 + C_\varepsilon|z_j|_p^p$$

which implies $\|z_j\|^2 \le c_1 C_\varepsilon \|z_j\|^p$ or equivalently $\|z_j\|^{2-p} \le c_1 C_\varepsilon$, a contradiction.

b) Assume there is a sequence $(z_j) \subset \mathcal{K} \setminus \{0\}$ such that $\Phi(z_j) \to 0$. Then

$$o(1) = \Phi(z_j) = \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}} \tilde{R}(t, z_j) \tag{5.35}$$

and

$$\|z_j\|^2 = \int_{\mathbb{R}} R_z(t, z_j)(z_j^+ - z_j^-). \tag{5.36}$$

Clearly (z_j) is a $(C)_{c=0}$ sequence, hence is bounded by Lemma 5.6.

First consider the super linear case. Using (5.35) and the notations defined in the proof of Lemma 5.6, we see that, for any $0 < a < b$ and $s \ge 2$, $\int_{\Omega_j(a,b)} |z_j|^2 \to 0$ and $\int_{\Omega_j(b,\infty)} |z_j|^s \to 0$ as $j \to \infty$. Therefore, as in the proof of Lemma 5.6, it follows from (5.36) that for any $\varepsilon > 0$

$$\limsup_{j\to\infty} \|z_j\|^2 \le \varepsilon,$$

contradicting to a).

Next consider the asymptotically linear case. Since $\|z_j\| \ge \theta$ by a), (5.36) and (5.11) imply that (z_j) is nonvanishing. By the $\mathbb{Z}$-invariance of Φ, up to a translation, we can assume $z_j \rightharpoonup z \in \mathcal{K} \setminus \{0\}$. Since z is a homoclinic orbit of (HS), $z(t) \to 0$ as $|t| \to \infty$. Thus there is a bounded interval $I \subset \mathbb{R}$ with the measure $|I| > 0$ such that $0 < |z(t)| \le \delta$ for $t \in I$ by (A_3). Now (5.35) implies

$$0 \ge \lim_{j\to\infty} \int_I \tilde{R}(t, z_j) = \int_I \tilde{R}(t, z) > 0,$$

a contradiction. □

Let $[r]$ denote the integer part of $r \in \mathbb{R}$, and $\mathcal{F} := (\mathcal{K} \setminus \{0\})/\mathbb{Z}$, a set consisting of arbitrarily chosen representatives of the $\mathbb{Z}$-orbits. As a consequence of the above lemmas, we have the following result (see [Coti-Zelati and Rabinowitz (1992); Ding and Girardi (1999); Kryszewski and Szulkin (1998); Séré (1992)]).

Lemma 5.9. *Assume that* (L_0) *and* (R_0) *are satisfied, and either* (S_1)*-*(S_2) *or* (A_1)*-*(A_3) *hold. Let* (z_j) *be a* $(C)_c$*-sequence. Then either*

(i) $z_j \to 0$ *(and hence* $c = 0$*), or*

(ii) $c \ge \hat{c}$ *and there exist a positive integer* $\ell \le \left[\frac{c}{\hat{c}}\right]$*, points* $\overline{z}_1, \cdots, \overline{z}_\ell \in \mathcal{F}$*, a subsequence denoted again by* (z_j)*, and sequences* $(a_j^i) \subset \mathbb{Z}$ *such that*

$$\left\| z_j - \sum_{i=1}^{\ell} (a_j^i * \overline{z}_i) \right\| \to 0 \ \textit{as}\ j \to \infty$$

$$|a_j^i - a_j^k| \to \infty \ \ \textit{for}\ i \ne k \ \textit{as}\ j \to \infty$$

and

$$\sum_{i=1}^{\ell} \Phi(\overline{z}_i) = c.$$

Proof. See [Ding and Girardi (1999)]. It can be outlined as follows. Lemma 5.6 shows that (z_j) is bounded: $\|z_j\| \le M$. In addition,

$$c = \lim_{j\to\infty} \left(\Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j \right) = \lim_{j\to\infty} \int_{\mathbb{R}} \tilde{R}(t, z_j) \ge 0, \tag{5.37}$$

and, as the proof of *b*) of Lemma 5.8, $c = 0$ if and only if $z_j \to 0$ in E.

Assume $c > 0$. The concentration principle implies that (z_j) is either vanishing or nonvanishing. By (5.11) and (5.12), choose $p > 2$ such that, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying $\tilde{R}(t, z) \le \varepsilon\lambda_0 M^{-2}|z|^2 + C_\varepsilon|z|^p$. If (z_j) is vanishing, then it follows from (5.37) that, for $\varepsilon < c$,

$$c = \lim_{j\to\infty} \int_{\mathbb{R}} \tilde{R}(t, z_j) \le \lim_{j\to\infty} \int_{\mathbb{R}} \left(\frac{\varepsilon\lambda_0|z_j|^2}{M^2} + C_\varepsilon|z_j|^p \right) \le \varepsilon,$$

a contradiction. Thus (z_j) is nonvanishing and by the $\mathbb{Z}$-invariance of Φ we can find a sequence $(k_j^1) \subset \mathbb{Z}$ such that $k_j^1 * z_j \rightharpoonup z^1 \in \mathcal{K} \setminus \{0\}$. Let $\overline{z}_1 \in \mathcal{F}$ be the representative in which z^1 lies, and let $k^1 \in \mathbb{Z}$ be such that $k^1 * z^1 = \overline{z}_1$. Set $\bar{k}_j^1 = k^1 + k_j^1$ and $z_j^1 := \bar{k}_j^1 * z_j - \overline{z}_1$. By $\mathbb{Z}$-invariance and Lemma 5.7, (z_j^1) is a Cerami sequence at level $c - \Phi(\overline{z}_1)$. By *(i)*, $c - \Phi(\overline{z}_1) \ge 0$ which, jointly with Lemma 5.8-*b*), implies $\hat{c} \le \Phi(\overline{z}_1) \le c$. There are two possibilities: $c = \Phi(\overline{z}_1)$ or $c > \Phi(\overline{z}_1)$.

If $c = \Phi(\overline{z}_1)$, repeating the argument for *(i)* shows that $z_j^1 \to 0$ in E, consequently, the lemma holds with $\ell = 1$ and $a_j^1 = -\bar{k}_j^1$.

If $c > \Phi(\overline{z}_1)$, then we argue again as above with (z_j) and c replaced by (z_j^1) and $c - \Phi(\overline{z}_1)$ respectively, and obtain $\overline{z}_2 \in \mathcal{F}$ with $\hat{c} \le \Phi(\overline{z}_2) \le c - \Phi(\overline{z}_1)$. After at most $\left[\frac{c}{\hat{c}}\right]$ steps we arrive the desired conclusion. □

5.6 Proofs of the main results

We are now in a position to give the proofs of Theorems 5.1 and 5.2. In order to prove the theorems we choose $X = E^-$ and $Y = E^+$ with $E^\pm$ given in Section 5.3. Then $E = X \oplus Y$ and Φ defined by (5.8) fit the general framework of Chapter 4, which suggests the applications of Theorems 4.5 and 4.7.

Proof. [Proofs of Theorems 5.1 and 5.2] (Existence). In virtue of Lemma 5.2 and the form of Φ, an application of Theorem 4.1 shows that Φ satisfies (Φ_0). The expression (5.8) of Φ, together with the nonnegativity of $R(t, z)$, implies the condition (Φ_+). Lemma 5.3 is nothing but (Φ_2), which jointly with Lemma 5.5 gives the linking condition of Theorem 4.5. Therefore, Φ possesses a $(C)_c$-sequence $(z_n)_{n\in\mathbb{N}}$ with $\kappa \le c \le \sup\Phi(Q)$ where $\kappa > 0$ is from Lemma 5.2 and Q is the subset given by Lemma 5.5. By Lemma 5.6, (z_n) is bounded. Consequently, $\Phi'(z_n) \to 0$. A standard argument shows that (z_n) is non-vanishing, that is, there exist $r, \eta > 0$ and $(a_n) \subset \mathbb{Z}$ such that $\limsup_{n\to\infty} \int_{a_n - r}^{a_n + r} |z_n|^2 \ge \eta$. Set $v_n := a_n * z_n$. It follows from the invariance of the norm and of the functional under the $*$-action that $\|v_n\| = \|z_n\| \le C$ and $\Phi(v_n) \to c \ge \kappa, \Phi'(v_n) \to 0$. Therefore $v_n \rightharpoonup v$ in E with

$v \neq 0$ and $\Phi'(v) = 0$, that is, v is a nontrivial solution of (HS), and the existence is proved.

(Multiplicity). We now establish the multiplicity. The proof will be completed in an indirect way, namely, we show that if

$$\mathcal{K}/\mathbb{Z} \text{ is a finite set,} \tag{†}$$

then Φ possesses an unbounded sequence of critical values, a contradiction. We do this by checking that, if (†) is true then (Φ) verifies all the assumptions of Theorem 4.7.

The assumptions (Φ_0) and (Φ_2) have already been verified as above. By assumption $R(t,z)$ is even in z, hence Φ satisfies (Φ_1). Recall that $\dim(Y_0) = \infty$. Let (f_k) be a base of Y_0 and set $Y_n := \text{span}\{f_1, ..., f_n\}$, $E_n := E^- \oplus Y_n$. With such a choice of sequence of subspaces it follows from Lemma 5.4 that (Φ_4) is satisfied. In order to check (Φ_I) assume (†) holds. Given $\ell \in \mathbb{N}$ and a finite set $\mathcal{B} \subset E$, let

$$[\mathcal{B}, \ell] := \left\{ \sum_{i=1}^{j} (a_i * z_i) : \ 1 \leq j \leq \ell, \, a_i \in \mathbb{Z}, \, z_i \in \mathcal{B} \right\}.$$

Following an argument of [Coti-Zelati, Ekeland and Séré (1990); Coti-Zelati and Rabinowitz (1991)] one sees that

$$\inf\{\|z - z'\| : \ z, z' \in [\mathcal{B}, \ell], \ z \neq z'\} > 0. \tag{5.38}$$

Recalling that $\mathcal{F} = (\mathcal{K} \setminus \{0\})/\mathbb{Z}$, (†) implies that $\mathcal{F}$ is a finite set and, since Φ' is odd, we may assume $\mathcal{F}$ is symmetric. For any compact interval $I \subset (0, \infty)$ with $b := \max I$, set $\ell = [b/\hat{c}]$ and take $\mathscr{A} = [\mathcal{F}, \ell]$. Then $P^+\mathscr{A} = [P^+\mathcal{F}, \ell]$ where P^+ stands for the projector onto E^+. By (†), $P^+\mathcal{F}$ is a finite set and

$$\|z\| \leq \ell \max\{\|\bar{z}\| : \ \bar{z} \in \mathcal{F}\} \quad \text{for } z \in \mathscr{A}$$

which implies that $\mathscr{A}$ is bounded. In addition, by Lemma 5.9, $\mathscr{A}$ is a $(C)_I$-attractor, and by (5.38),

$$\begin{aligned} &\inf\{\|z_1^+ - z_2^+\| : \ z_1, z_2 \in \mathscr{A}, \, z_1^+ \neq z_2^+\} \\ &= \inf\{\|z - z'\| : \, z, z' \in P^+\mathscr{A}, \, z \neq z'\} > 0. \end{aligned}$$

This argument shows that Φ verifies (Φ_I), and the proof hereby is complete. □

5.7 Non periodic Hamiltonians

In this section we are interested in the system (HS) without assuming periodicity conditions. The materials are taken from the paper [Ding and Jeanjean (2007)].

Below, For two given symmetric real matrix functions $M_1(t)$ and $M_2(t)$, we say that $M_1(t) \leq M_2(t)$ if

$$\max_{\xi \in \mathbb{R}^{2N}, |\xi|=1} (M_1(t) - M_2(t))\, \xi \cdot \xi \leq 0.$$

For convenience, any real number b will be regarded as the matrix bI_{2N} when matrices are concerned. We make the following assumptions:

(H_0) There is $b > 0$ such that the set $\Lambda^b := \{t \in \mathbb{R} : \mathcal{J}_0 L(t) < b\}$ is nonempty and has finite measure;

(H_1) $R(t,z) \geq 0$ and $R_z(t,z) = o(|z|)$ as $z \to 0$ uniformly in t;

(H_2) $R_z(t,z) = M(t)z + r_z(t,z)$, with M a bounded, continuous symmetric $2N \times 2N$-matrix valued function and $r_z(t,z) = o(|z|)$ uniformly in t as $|z| \to \infty$;

(H_3) $m_0 := \inf_{t\in\mathbb{R}} \left[\inf_{(\xi\in\mathbb{R}^{2N},\,|\xi|=1)} M(t)\xi \cdot \xi\right] > \inf \sigma(A) \cap (0,\infty)$;

(H_4) Either (i) $0 \notin \sigma(A - M)$ or (ii) $\tilde{R}(t,z) \geq 0$ for all (t,z) and $\tilde{R}(t,z) \geq \delta_0$ for some $\delta_0 > 0$ and all (t,z) with $|z|$ large enough;

(H_5) $\gamma < b_{\max}$, where $\gamma := \sup_{|t|\geq t_0, z\neq 0} |R_z(t,z)|/|z|$ for some $t_0 \geq 0$, and $b_{\max} := \sup\{b : |\Lambda^b| < \infty\}$.

We will show that the set $\sigma(A) \cap (0, b_{\max})$ consists only of eigenvalues of finite multiplicity. From the definition of m_0 and γ we have $m_0 < \gamma < b_{max}$. Let ℓ denote the number of eigenfunctions with corresponding eigenvalues lying in $(0, m_0)$.

Theorem 5.3 ([Ding and Jeanjean (2007)]). *Let* $(H_0) - (H_5)$ *be satisfied. Then* (HS) *has at least one homoclinic orbit. If in addition* $R(t,z)$ *is even in* z*, then* (HS) *has at least* ℓ *pairs of homoclinic orbits.*

Remark 5.3. Let $q \in C^1(\mathbb{R}, \mathbb{R})$ satisfy

(q_0) There is $b > 0$ such that $0 < |Q^b| < \infty$ where $Q^b := \{t \in \mathbb{R} : q(t) < b\}$.

Then $L(t) = q(t)\mathcal{J}_0$ satisfies (H_0).

In the works where $H(t,z)$ is periodic the periodicity is used to control the lack of compactness due to the fact that (HS) is set on all $\mathbb{R}$. In our situation we manage to recover sufficient compactness by imposing a control on the *size* of $R(t,z)$ with respect to the behavior of $L(t)$ at infinity in t, see condition (H_5).

The proof of Theorem 5.3 can be outlined as follows. We first study the spectrum of the operator A showing, thanks to (H_0), that the essential spectrum $\sigma_e(A) \subset \mathbb{R} \setminus (-b_{\max}, b_{\max})$. Based on the description on $\sigma(A)$, we derive a variational setting for (HS) and represent the associated functional in the form $\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \int_{\mathbb{R}} R(t,z)$ with Φ being defined on the Hilbert space $E = \mathscr{D}(|A|^{1/2}) \hookrightarrow H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$ with decomposition $E = E^- \oplus E^0 \oplus E^+, z = z^- + z^0 + z^+$, $\dim E^\pm = \infty$. We then show the linking structure of Φ, that is, $\inf \Phi(E^+ \cap \partial B_\rho) > 0$ for some $\rho > 0$ and there are finite dimensional subspaces $Y \subset E^+$ such that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ in $E_Y := E^- \oplus E^0 \oplus Y$. Subsequently we show that the Cerami condition for Φ holds. Since E^0 maybe nontrivial this require some care. Finally, we arrive at the proof of Theorem 5.3.

5.7.1 *Variational setting*

In order to establish a variational setting for the system (HS) we first study the spectrum of the associated Hamiltonian operator.

Recall that $A = -\left(\mathcal{J}\frac{d}{dt} + L\right)$ is selfadjoint on $L^2(\mathbb{R}, \mathbb{R}^{2N})$ with domain $\mathscr{D}(A) = H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is bounded and $\mathscr{D}(A) \subset H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is unbounded. Observe that $\mathscr{D}(A)$ is a Hilbert space with the graph inner product

$$(z, w)_A := (Az, Aw)_{L^2} + (z, w)_{L^2}$$

and the induced norm $|z|_A := (z, z)_A^{1/2}$.

Set $A_0 := \mathcal{J}\frac{d}{dt} + \mathcal{J}_0$ which is a selfadjoint operator acting on $L^2(\mathbb{R}, \mathbb{R}^{2N})$ with $\mathscr{D}(A_0) = H^1(\mathbb{R}, \mathbb{R}^{2N})$ and satisfies $A_0^2 = -\frac{d^2}{dt^2} + 1$. Plainly,

$$||A_0|z|_2 = |A_0 z|_2 = \|z\|_{H^1} \quad \text{for all } z \in H^1 \tag{5.39}$$

where $|A_0|$ denotes the absolute value of A_0 as usual.

Lemma 5.10. *The condition $\mathscr{D}(A) \subset H^1(\mathbb{R}^1, \mathbb{R}^{2N})$ implies that there is $\gamma_1 > 0$ such that*

$$\|z\|_{H^1} = ||A_0|z|_2 \le \gamma_1 |z|_A \quad \textit{for all } z \in \mathscr{D}(A). \tag{5.40}$$

Proof. Let A_r be the restriction of A_0 to $\mathscr{D}(A)$. A_r is a linear operator from $\mathscr{D}(A)$ to L^2. We claim that A_r is closed. Indeed, let $z_n \xrightarrow{|\cdot|_A} z$ and $A_r z_n \xrightarrow{|\cdot|_2} w$. Then $z \in \mathscr{D}(A)$, and since A_0 is closed, $A_r z_n = A_0 z_n \to A_0 z = A_r z$, hence the claim. Now the closed graph theorem implies that $A_r \in \mathscr{L}(\mathscr{D}(A), L^2)$ (the Banach space of bounded linear operators), so $|A_0 z|_2 = |A_r z|_2 \le \gamma_1 |z|_A$ for all $z \in \mathscr{D}(A)$. This, together with (5.39), implies (5.40). □

Let $\sigma(A)$, $\sigma_d(A)$ and $\sigma_e(A)$ denote, respectively, the spectrum, the eigenvalues of finite multiplicity, and the essential spectrum of A. Set

$$\mu_e^- := \sup\left(\sigma_e(A) \cap (-\infty, 0]\right), \quad \mu_e^+ := \inf\left(\sigma_e(A) \cap [0, \infty)\right).$$

Proposition 5.2. *Assume (H_0) is satisfied. Then $\sigma_e(A) \subset \mathbb{R} \setminus (-b_{\max}, b_{\max})$, that is, $\mu_e^- \le -b_{\max}$ and $\mu_e^+ \ge b_{\max}$.*

Proof. Let $b > 0$ be such that $|\Lambda^b| < \infty$. Set

$$(\mathcal{J}_0 L(t) - b)^+ := \begin{cases} \mathcal{J}_0 L(t) - b & \text{if } \mathcal{J}_0 L(t) - b \ge 0 \\ 0 & \text{if } \mathcal{J}_0 L(t) - b < 0 \end{cases}$$

and $(\mathcal{J}_0 L(t) - b)^- := (\mathcal{J}_0 L(t) - b) - (\mathcal{J}_0 L(t) - b)^+$. We have, since $\mathcal{J}_0^2 = I$, $A = A_1 - \mathcal{J}_0(\mathcal{J}_0 L(t) - b)^-$ where

$$A_1 = -\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) - b\mathcal{J}_0.$$

Observe that $\mathcal{J}_0\mathcal{J} = -\mathcal{J}\mathcal{J}_0$. Thus, for $z \in \mathscr{D}(A)$,

$$
\begin{aligned}
(A_1 z, A_1 z)_{L^2} =& |A_1 z|_2^2 = \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z + b\mathcal{J}_0 z\right|_2^2 \\
=& \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z\right|_2^2 + b^2|z|_2^2 \\
& + (\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_{L^2} + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_{L^2} \\
& + \left(\mathcal{J}_0(\mathcal{J}_0 L - b)^+ z,\, b\mathcal{J}_0 z\right)_{L^2} + \left(b\mathcal{J}_0 z,\, \mathcal{J}_0(\mathcal{J}_0 L - b)^+ z\right)_{L^2} \\
=& \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z\right|_2^2 + b^2|z|_2^2 \\
& + 2b((\mathcal{J}_0 L - b)^+ z,\ z)_{L^2} \\
\geq& b^2|z|_2^2 .
\end{aligned}
\tag{5.41}
$$

Here we have used the fact that $(\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_{L^2} + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_{L^2} = 0$. Indeed for $z = (u, v) \in C_0^\infty$ one has

$$
\begin{aligned}
& (\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_{L^2} + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_{L^2} \\
&= 2b\int_{\mathbb{R}} (\dot{u}u - \dot{v}v) = b\int_{\mathbb{R}} \frac{d}{dt}(u^2(t) - v^2(t))) \\
&= b\lim_{t\to\infty}(|u(t)|^2 - |u(-t)|^2 - |v(t)|^2 + |v(-t)|^2) = 0.
\end{aligned}
$$

Thus, since C_0^∞ is dense in E we get the result. Now (5.41) implies that $\sigma(A_1) \subset \mathbb{R}\setminus(-b, b)$.

We claim that $\sigma_e(A) \cap (-b, b) = \emptyset$. Assume by contradiction that there is $\lambda \in \sigma_e(A)$ with $|\lambda| < b$. Let $(z_n) \subset \mathscr{D}(A)$ with $|z_n|_2 = 1$, $z_n \rightharpoonup 0$ in L^2 and $|(A-\lambda)z_n|_2 \to 0$. It follows from (5.40) that

$$\|z_n\|_{H^1} \leq c_1|z_n|_A = c_1(|Az_n|_2^2 + |z_n|_2^2)^{1/2} \leq c_2(|(A-\lambda)z_n|_2^2 + \lambda^2 + 1)^{1/2} \leq c_3,$$

hence $|\mathcal{J}_0(\mathcal{J}_0 L - b)^- z_n|_2 \to 0$. We get

$$
\begin{aligned}
o(1) = |(A-\lambda)z_n|_2 &= |A_1 z_n - \lambda z_n - \mathcal{J}_0(\mathcal{J}_0 L - b)^- z_n|_2 \\
&\geq |A_1 z_n|_2 - |\lambda| - o(1) \\
&\geq b - |\lambda| - o(1)
\end{aligned}
$$

which implies that $0 < b - |\lambda| \leq 0$, a contradiction.

Since the claim is true for any $b > 0$ with $|\Lambda^b| < \infty$, one sees that $\sigma_e(A) \subset \mathbb{R}\setminus(-b_{\max},\ b_{\max})$. □

Remark 5.4. a) If $L(t)$ satisfies: $|\Lambda^b| < \infty$ for any $b > 0$, then, as a consequence of Proposition 5.2, $\mu_e^- = -\infty$ and $\mu_e^+ = \infty$, that is, $\sigma(A) = \sigma_d(A)$.

b) Let $L(t) = q(t)\mathcal{J}_0$ with $q(t)$ satisfying (q_0). Then $\sigma_e(A) \subset \mathbb{R}\setminus(-b_{\max},\ b_{\max})$. Moreover, $\sigma(A)$ is symmetric : $\sigma(A)\cap(-\infty, 0) = -\sigma(A)\cap(0,\infty)$ (see the proof of Proposition 5.1). In particular, letting $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k$ be all the eigenvalues below $\inf\sigma_e(A^2)$ of A^2, $\{\pm\lambda_j^{1/2}\ :\ j = 1, ..., k\}$ are all the eigenvalues in $(\mu_e^-,\ \mu_e^+)$ of A. Therefore, one obtains the eigenvalues of A from those of $A^2 = -\frac{d^2}{dt^2} + q^2 + \dot{q}\mathcal{J}\mathcal{J}_0$ which can be calculated via the minimax principle.

Note that since 0 now may belong to $\sigma(A)$, we need more arguments for getting the suitable variational framwork.

Let $\{F_\lambda : \lambda \in \mathbb{R}\}$ denote the spectral family of A. A has the polar decomposition $A = U|A|$ with $U = 1 - F_0 - F_{-0}$ (see [Kato (1966)]). Proposition 5.2 implies that 0 is at most an isolated eigenvalue of finite multiplicity of A. L^2 has the orthogonal decomposition:

$$L^2 = L^- \oplus L^0 \oplus L^+, \quad z = z^- + z^0 + z^+$$

so that A is negative definite (resp. positive definite) in L^- (resp. L^+) and $L^0 = \ker A$. In fact, $L^\pm = \{z \in L^2 : Uz = \pm z\}$ and $L^0 = \{z \in L^2 : Uz = 0\}$. It follows from

$$\begin{aligned}(z^+, z^-)_{L^2} &= (Uz^+, z^-)_{L^2} = (z^+, Uz^-)_{L^2} \\ &= (z^+, -z^-)_{L^2} = -(z^+, z^-)_{L^2}\end{aligned}$$

that L^+ and L^- are orthogonal with respect to the L^2-inner product. Similarly one sees that $L^\pm$ and L^0 are orthogonal with respect to the L^2-inner product.

Let $P^0 : L^2 \to L^0$ denote the associated projector. P^0 commutes with A and $|A|$. On $\mathscr{D}(A)$ we introduce the inner product

$$\begin{aligned}\langle z, w\rangle_A :=&(Az, Aw)_{L^2} + (P^0 z, P^0 w)_{L^2} \\ =&(|A|z, |A|w)_{L^2} + (P^0 z, w)_{L^2}\end{aligned}$$

whose deduced norm will be denoted by $\|z\|_A$. It is clear that $|\cdot|_A$ and $\|\cdot\|_A$ are equivalent norms on $\mathscr{D}(A)$:

$$\gamma_2|z|_A \le \|z\|_A \le \gamma_3|z|_A \quad \text{for all } z \in \mathscr{D}(A).$$

Define

$$\tilde{A} := |A| + P^0.$$

Then $\mathscr{D}(\tilde{A}) = \mathscr{D}(A)$. Noting that $P^0|A| = |A|P^0 = 0$ we have for $z, w \in \mathscr{D}(A)$,

$$\begin{aligned}(\tilde{A}z, \tilde{A}w)_{L^2} &= (|A|z, |A|w)_{L^2} + (|A|z, P^0 w)_{L^2} + (P^0 z, |A|w)_{L^2} + (P^0 z, P^0 w)_{L^2} \\ &= (|A|z, |A|w)_{L^2} + (P^0 z, P^0 w)_{L^2} = \langle z, w\rangle_A,\end{aligned}$$

hence,

$$\gamma_2|z|_A \le \|z\|_A = |\tilde{A}z|_2 \le \gamma_3|z|_A \quad \text{for all } z \in \mathscr{D}(A). \tag{5.42}$$

Let $E := \mathscr{D}(|A|^{1/2})$ be the domain of the self-adjoint operator $|A|^{1/2}$ which is a Hilbert space equipped with the inner product

$$(z, w) = (|A|^{1/2}z, |A|^{1/2}w)_{L^2} + (P^0 z, P^0 w)_{L^2}$$

and the induced norm $\|z\| = (z, z)^{1/2}$. E possesses the following decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm \text{ and } E^0 = L^0,$$

orthogonal with respect to both the inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Observe that for all $z \in \mathscr{D}(A)$ and $w \in \mathscr{D}(|A|^{1/2})$

$$(\tilde{A}^{1/2}z, \tilde{A}^{1/2}w)_{L^2} = (\tilde{A}z, w)_{L^2} = ((|A| + P^0)z, w)_{L^2} = (|A|z, w)_{L^2} + (P^0 z, w)_{L^2}$$
$$= (|A|^{1/2}z, |A|^{1/2}w)_{L^2} + (P^0 z, P^0 w)_{L^2} = (z, w).$$

Consequently, since $\mathscr{D}(A) = \mathscr{D}(\tilde{A})$ is a core of $\tilde{A}^{1/2}$ we have

$$(z, w) = (\tilde{A}^{1/2}z, \tilde{A}^{1/2}w)_{L^2} \quad \text{for all } z, w \in \mathscr{D}(|A|^{1/2})$$

which induces in particular that

$$\|z\| = |\tilde{A}^{1/2}z|_2 \quad \text{for all } z \in E. \tag{5.43}$$

Lemma 5.11. *E embeds continuously into $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$, hence, E embeds continuously into L^p for all $p \geq 2$ and compactly into L^p_{loc} for all $p \geq 1$.*

Proof. Firstly, by interpolation theory we have that $H^{1/2} = [H^1, L^2]_{1/2}$ (see Theorem 2.4.1 of [Triebel (1978)]). Noting that $\mathscr{D}(|A_0|^0) = L^2$ and by (5.39) one has

$$H^{1/2} = [\mathscr{D}(|A_0|), \mathscr{D}(|A_0|^0)]_{1/2}$$

with equivalent norms. It then follows from Theorem 1.18.10 of [Triebel (1978)] that

$$H^{1/2} = [\mathscr{D}(|A_0|), \mathscr{D}(|A_0|^0)]_{1/2} = \mathscr{D}(|A_0|^{1/2}),$$

hence $\|z\|_{H^{1/2}}$ and $||A_0|^{1/2}z|_2$ are equivalent norms in $H^{1/2}$:

$$\gamma_4\|z\|_{H^{1/2}} \leq ||A_0|^{1/2}z|_2 \leq \gamma_5\|z\|_{H^{1/2}} \quad \text{for all } z \in H^{1/2}. \tag{5.44}$$

By (5.40),

$$||A_0|z|_2 \leq \gamma_1|\tilde{A}z|_2 = |(\gamma_1\tilde{A})z|_2$$

for all $z \in \mathscr{D}(A)$. Thus $(|A_0|z, z)_{L^2} \leq (\gamma_1\tilde{A}z, z)_{L^2}$ for all $z \in \mathscr{D}(A)$ (see Proposition III 8.11 of [Edmunds and Evans (1987)]). This implies

$$||A_0|^{1/2}z|_2^2 = (|A_0|z, z)_{L^2} \leq (\gamma_1\tilde{A}z, z)_{L^2} = \gamma_1|\tilde{A}^{1/2}z|_2^2$$

for all $z \in \mathscr{D}(A)$ (see, Proposition III 8.12 of [Edmunds and Evans (1987)]). Since $\mathscr{D}(A)$ is a core of $\tilde{A}^{1/2}$ we obtain that $||A_0|^{1/2}z|_2^2 \leq \gamma_1|\tilde{A}^{1/2}z|_2^2$ for all $z \in E$. This, jointly with (5.43), shows that

$$||A_0|^{1/2}z|_2^2 \leq \gamma_1\|z\|^2 \quad \text{for all } z \in E$$

which, together with (5.44), implies that

$$\|z\|_{H^{1/2}} \leq \gamma_6\|z\| \quad \text{for all } z \in E$$

ending the proof. □

From now on we fix a number b with

$$\gamma < b < \mathrm{b}_{\max} \tag{5.45}$$

where γ appears in (H_5). Let k be the number of the eigenfunctions with corresponding eigenvalues lying in $[-b, b]$. We write $f_i \, (1 \le i \le k)$ for the eigenfunctions. Setting

$$L^d := \mathrm{span}\{f_1, \cdots, f_k\},$$

we have another orthogonal decomposition

$$L^2 = L^d \oplus L^e, \quad u = u^d + u^e.$$

Correspondingly, E has the decomposition:

$$E = E^d \oplus E^e \ \text{ with } \ E^d = L^d \text{ and } E^e = E \cap L^e, \tag{5.46}$$

orthogonal with respect to both the inner products $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$. Remark that by Proposition 5.2

$$b|z|_2^2 \le \|z\|^2 \quad \text{for all } z \in E^e. \tag{5.47}$$

On E we define the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z) \ \text{ where } \ \Psi(z) = \int_{\mathbb{R}} R(t, z). \tag{5.48}$$

Our hypotheses on $H(t, z)$ imply that $\Phi \in C^1(E, \mathbb{R})$ and a standard argument shows that critical points of Φ are homoclinic orbits of (HS).

Lemma 5.12. *Let $(H_0) - (H_2)$ be satisfied. Then Ψ is non-negative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

Proof. It is similar to that of Lemma 5.2, hence the details are omitted. □

5.7.2 *Linking structure*

We now study the linking structure of Φ. Remark that under $(H_1) - (H_2)$, given $p \ge 2$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|R_z(t, z)| \le \varepsilon|z| + C_\varepsilon |z|^{p-1}$$

and

$$R(t, z) \le \varepsilon|z|^2 + C_\varepsilon |z|^p \tag{5.49}$$

for all (t, z). First we have the following lemma.

Lemma 5.13. *Let $(H_0) - (H_2)$ be satisfied. Then there is $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho^+) > 0$ where $S_\rho^+ = \partial B_\rho \cap E^+$.*

Proof. Choose $p > 2$ such that (5.49) holds for any $\varepsilon > 0$. This yields

$$\Psi(z) \le \varepsilon|z|_2^2 + C_\varepsilon |z|_p^p \le C(\varepsilon\|z\|^2 + C_\varepsilon \|z\|^p)$$

for all $z \in E$. Now the lemma follows from the form of Φ (see (5.48)). □

In the following, we arrange all the eigenvalues (counted with multiplicity) of A in $(0, m_0)$ by $0 < \mu_1 \le \mu_2 \le ... \le \mu_\ell < m_0$ and let e_j denote the corresponding eigenfunctions: $Ae_j = \mu_j e_j$ for $j = 1, ..., \ell$. Set $Y_0 := \mathrm{span}\{e_1, ..., e_\ell\}$. Note that

$$\mu_1 |w|_2^2 \le \|w\|^2 \le \mu_\ell |w|_2^2 \quad \text{for all } w \in Y_0. \tag{5.50}$$

For any finite dimensional subspace W of Y_0 set $E_W = E^- \oplus E^0 \oplus W$.

Lemma 5.14. *Let $(H_0) - (H_3)$ be satisfied and $\rho > 0$ be given by Lemma 5.13. Then for any subspace W of Y_0, $\sup \Phi(E_W) < \infty$, and there is $R_W > 0$ such that $\Phi(z) < \inf \Phi(B_\rho \cap E^+)$ for all $z \in E_W$ with $\|z\| \ge R_W$.*

Proof. It is sufficient to show that $\Phi(z) \to -\infty$ as $z \in E_W, \|z\| \to \infty$. Arguing indirectly we assume that for some sequence $(z_j) \subset E_W$ with $\|z_j\| \to \infty$, there is $c > 0$ such that $\Phi(z_j) \ge -c$ for all j. Then, setting $w_j = z_j/\|z_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^0 \to w^0$, $w_j^+ \to w^+ \in Y$ and

$$-\frac{c}{\|z_j\|^2} \le \frac{\Phi(z_j)}{\|z_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}} \frac{R(t, z_j)}{\|z_j\|^2}. \tag{5.51}$$

We claim that $w^+ \neq 0$. Indeed, if not it follows from (5.51) and (H_1) that $\|w_j^-\| \to 0$ and thus $w_j \to w = w^0$. Also $\int_{\mathbb{R}} \frac{R(t,z_j)}{\|z_j\|^2} \to 0$.

Recall that $R(t, z) = \frac{1}{2} M(t) z \cdot z + r(t, z)$ and $r(t, z)/|z|^2 \to 0$ uniformly in t as $|z| \to \infty$. Thus, since $|z_j(t)| \to \infty$ if $w(t) \neq 0$,

$$\begin{aligned} \int_{\mathbb{R}} \frac{r(t, z_j)}{\|z_j\|^2} &= \int_{\mathbb{R}} \frac{r(t, z_j)}{|z_j|^2} |w_j|^2 \\ &\le \int_{\mathbb{R}} \frac{|r(t, z_j)|}{|z_j|^2} |w_j - w|^2 + \int_{\mathbb{R}} \frac{|r(t, z_j)|}{|z_j|^2} |w|^2 \\ &= o(1) + \int_{w(t)\neq 0} \frac{|r(t, z_j)|}{|z_j|^2} |w|^2 = o(1). \end{aligned} \tag{5.52}$$

Also, by (H_3),

$$\frac{1}{2} \int_{\mathbb{R}} \frac{M(t) z_j \cdot z_j}{\|z_j\|^2} = \frac{1}{2} \int_{\mathbb{R}} \frac{M(t) z_j \cdot z_j}{|z_j|^2} |w_j|^2 \ge \frac{m_0}{2} |w_j|_2^2. \tag{5.53}$$

From (5.52)-(5.53) and since $\int_{\mathbb{R}} \frac{R(t,z_j)}{\|z_j\|^2} \to 0$ it follows that $|w_j|_2 \to 0$. Then $1 = \|w_j\| \to 0$ and this contradiction implies that $w^+ \neq 0$. Now since

$$\begin{aligned} \|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}} M(t) w \cdot w &\le \|w^+\|^2 - \|w^-\|^2 - m_0 |w|_2^2 \\ &\le -\big((m_0 - \mu_\ell)|w^+|_2^2 + \|w^-\|^2 + m_0 |w^0|_2^2\big) < 0 \end{aligned}$$

(see (5.50)), there is $a > 0$ such that

$$\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a} M(t) w \cdot w \, < \, 0. \tag{5.54}$$

As in (5.52) it follows from the fact $|w_j - w|_{L^2(-a,a)} \to 0$ that

$$\lim_{j\to\infty} \int_{-a}^{a} \frac{r(t,z_j)}{\|z_j\|^2} = \lim_{j\to\infty} \int_{-a}^{a} \frac{r(t,z_j)|w_j|^2}{|z_j|^2} = 0.$$

Thus (5.51) and (5.54) imply that

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{-a}^{a} \frac{R(t,z_j)}{\|z_j\|^2} \right) \\ &\le \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a} M(t)w\cdot w \right) < 0, \end{aligned}$$

a contradiction. □

As a special case we have

Lemma 5.15. *Let $(H_0) - (H_3)$ be satisfied and $\kappa > 0$ be given by Lemma 5.13. Then, letting $e \in Y_0$ with $\|e\| = 1$, there is $r_0 > 0$ such that* $\sup \Phi(\partial Q) \le \kappa$ *where* $Q := \{u = u^- + u^0 + se : \ u^- + u^0 \in E^- \oplus E^0, s \ge 0, \|u\| \le r_0\}$.

5.7.3 *The (C)-condition*

Here we discuss the Cerami condition.

Lemma 5.16. *Let $(H_0) - (H_2)$ and $(H_4) - (H_5)$ be satisfied. Then any $(C)_c$-sequence is bounded.*

Proof. Let $(z_j) \subset E$ be such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1 + \|z_j\|)\Phi'(z_j) \to 0. \tag{5.55}$$

Then, for a $C_0 > 0$,

$$C_0 \ge \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}} \tilde{R}(t,z_j)\,. \tag{5.56}$$

To prove that (z_j) is bounded we develop a contradiction argument related to the one introduced in [Jeanjean (1999)]. We assume that, up to a subsequence, $\|z_j\| \to \infty$ and set $v_j = z_j/\|z_j\|$. Then $\|v_j\| = 1$, $|v_j|_s \le \gamma_s\|v_j\| = \gamma_s$ for all $s \in [2,\infty)$, and passing to a subsequence if necessary, $v_j \rightharpoonup v$ in E, $v_j \to v$ in L^s_{loc} for all $s \ge 1$, $v_j(t) \to v(t)$ for a.e. $t \in \mathbb{R}$. Since, by (H_2), $|r_z(t,z)| = o(z)$ as $|z| \to \infty$ uniformly in t and $|z_j(t)| \to \infty$ if $v(t) \ne 0$, it is easy to see that

$$\int_{\mathbb{R}} \frac{R_z(t,z_j(t))\varphi(t)}{||z_j||} \to \int_{\mathbb{R}} M(t)v\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R},\mathbb{R}^{2N})$. From this we deduce, using (5.55), that

$$\mathcal{J}\frac{d}{dt}v + (L(t) + M(t))v = 0. \tag{5.57}$$

Multiplying (5.57) by $J^{-1} = -J$ we also get

$$\frac{d}{dt}v = \mathcal{J}(L(t) + M(t))v. \tag{5.58}$$

We claim that $v \neq 0$. Arguing by contradiction we assume that $v = 0$. Then $v_j^d \to 0$ in E and $v_j \to 0$ in L^s_{loc}. Set $I_0 := (-t_0, t_0)$ and $I_0^c := \mathbb{R} \setminus I_0$ where $t_0 > 0$ is the number given in (H_5). It follows from

$$\frac{\Phi'(z_j)(z_j^{e+} - z_j^{e-})}{\|z_j\|^2} = \|v_j^e\|^2 - \int_{\mathbb{R}} \frac{R_z(t, z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j|$$

that

$$\begin{aligned}
\|v_j^e\|^2 = & \int_{I_0} \frac{R_z(t, z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j| \\
& + \int_{I_0^c} \frac{R_z(t, z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j| + o(1) \\
& \leq c \int_{I_0} |v_j||v_j^{e+} - v_j^{e-}| + \gamma \int_{I_0^c} |v_j||v_j^{e+} - v_j^{e-}| + o(1) \\
& \leq \gamma |v_j^e|_2^2 + o(1).
\end{aligned}$$

By (5.47) one gets

$$\left(1 - \frac{\gamma}{b}\right) \|v_j^e\|^2 \leq o(1),$$

which implies, by (5.45), that $\|v_j^e\|^2 \to 0$. Hence $1 = \|v_j\|^2 = \|v_j^d\|^2 + \|v_j^e\|^2 \to 0$, a contradiction.

Therefore, $v \neq 0$ which is impossible if (i) of (H_4) is satisfied. Thus we assume (ii) of (H_4). Let $\Omega_j(0, r) := \{t \in \mathbb{R} : \ |z_j(t)| < r\}$, $\Omega_j(r, \infty) := \{t \in \mathbb{R} : \ |z_j(t)| \geq r\}$, and set for $r \geq 0$

$$g(r) := \inf \left\{ \tilde{R}(t, z) : \ t \in \mathbb{R} \text{ and } z \in \mathbb{R}^{2N} \text{ with } |z| \geq r \right\}.$$

By assumption there is $r_0 > 0$ such that $g(r_0) > 0$, hence one has by (5.56) that $|\Omega_j(r_0, \infty)| \leq C_0/g(r_0)$. Set $\Omega := \{t : \ v(t) \neq 0\}$. Since v satisfies (5.58) it follows from Cauchy Uniqueness Principle that $\Omega = \mathbb{R}$. Indeed otherwise $v \equiv 0$ on $\mathbb{R}$ contradicting the fact that $v \neq 0$. Now since $|\Omega| = \infty$ there exists $\varepsilon > 0$ and $\omega \subset \Omega$ such that $|v(t)| \geq 2\varepsilon$ for $t \in \omega$ and $2C_0/g(r_0) \leq |\omega| < \infty$. By an Egoroff's theorem we can find a set $\omega' \subset \omega$ with $|\omega'| > C_0/g(r_0)$ such that $v_j \to v$ uniformly on ω'. So for almost all j, $|v_j(t)| \geq \varepsilon$ and $|z_j(t)| \geq r$ in ω'. Then

$$\frac{C_0}{g(r_0)} < |\omega'| \leq |\Omega_j(r, \infty)| \leq \frac{C_0}{g(r_0)},$$

a contradiction. □

In the following lemma we discuss further the $(C)_c$-sequence $(z_j) \subset E$. By Lemma 5.16 it is bounded, hence, we may assume without loss of generality that $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^q_{loc} for $q \geq 1$ and $z_j(t) \to z(t)$ a.e. in t. Plainly z is a critical point of Φ.

Choose $p > 2$ such that $|R_z(t, z)| \le |z| + C_1|z|^{p-1}$ for all (t, z), and let q stands for either 2 or p. Set $I_a := [-a, a]$ for $a > 0$. As (5.29) we see easily that along a subsequence, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \int_{I_n \setminus I_r} |z_{j_n}|^q \le \varepsilon \tag{5.59}$$

for all $r \ge r_\varepsilon$. Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \le 1$, $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{z}_n(t) = \eta(2|t|/n)z(t)$ and set $h_n := z - \tilde{z}_n$. Since z is a homoclinic orbit, we have by definition that $h_n \in H^1$ and

$$\|h_n\| \to 0 \quad \text{and} \quad |h_n|_\infty \to 0 \quad \text{as } n \to \infty. \tag{5.60}$$

Repeating the argument of (5.32) we see that, under $(H_0) - (H_2)$ and $(H_4) - (H_5)$,

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} \left(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\right) \varphi \right| = 0 \tag{5.61}$$

uniformly in $\varphi \in E$ with $\|\varphi\| \le 1$. Then we have

Lemma 5.17. *Let $(H_0) - (H_2)$ and $(H_4) - (H_5)$ be satisfied. Then*

1) $\Phi(z_{j_n} - \tilde{z}_n) \to c - \Phi(z)$;
2) $\Phi'(z_{j_n} - \tilde{z}_n) \to 0$.

Proof. One has

$$\Phi(z_{j_n} - \tilde{z}_n) = \Phi(z_{j_n}) - \Phi(\tilde{z}_n) + \int_{\mathbb{R}} \left(R(t, z_{j_n}) - R(t, z_{j_n} - \tilde{z}_n) - R(t, \tilde{z}_n)\right).$$

Using (5.60) it is not difficult to check that

$$\int_{\mathbb{R}} \left(R(t, z_{j_n}) - R(t, z_{j_n} - \tilde{z}_n) - R(t, \tilde{z}_n)\right) \to 0.$$

This, together with $\Phi(z_{j_n}) \to c$ and $\Phi(\tilde{z}_n) \to \Phi(z)$, gives 1).

To verify 2), observe that, for any $\varphi \in E$,

$$\Phi'(z_{j_n} - \tilde{z}_n)\varphi = \Phi'(z_{j_n})\varphi - \Phi'(\tilde{z}_n)\varphi + \int_{\mathbb{R}} \Big(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\Big)\varphi.$$

By (5.61) we get

$$\lim_{n\to\infty} \int_{\mathbb{R}} \Big(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\Big)\varphi = 0$$

uniformly in $\|\varphi\| \le 1$, proving 2). □

Lemma 5.18. *Let $(H_0) - (H_2)$ and $(H_4) - (H_5)$ be satisfied. Then Φ satisfies the $(C)_c$-condition.*

Proof. In the following we use the decomposition $E = E^d \oplus E^e$ (see (5.46)). Recall that $\dim(E^d) < \infty$. Write

$$y_n := z_{j_n} - \tilde{z}_n = y_n^d + y_n^e.$$

Then $y_n^d = (z_{j_n}^d - z^d) + (z^d - \tilde{z}_n^d) \to 0$ and, by Lemma 5.17, $\Phi(y_n) \to c - \Phi(z)$, $\Phi'(y_n) \to 0$. Set $\bar{y}_n^e = y_n^{e+} - y_n^{e-}$. Observe that

$$o(1) = \Phi'(y_n)\bar{y}_n^e = \|y_n^e\|^2 - \int_{\mathbb{R}} R_z(t, y_n)\bar{y}_n^e.$$

Thus it follows that

$$\begin{aligned}
\|y_n^e\|^2 &\leq o(1) + \int_{I_0} \frac{|R_z(t,y_n)|}{|y_n|}|y_n||\bar{y}_n^e| + \int_{I_0^c} \frac{|R_z(t,y_n)|}{|y_n|}|y_n||\bar{y}_n^e| \\
&\leq o(1) + c\int_{I_0} |y_n||\bar{y}_n^e| + \gamma \int_{I_0^c} |y_n||\bar{y}_n^e| \\
&\leq o(1) + \gamma|y_n^e|_2^2 \leq o(1) + \frac{\gamma}{b}\|y_n^e\|^2.
\end{aligned}$$

Hence $(1 - \frac{\gamma}{b})\|y_n^e\|^2 \to 0$, and so $\|y_n\| \to 0$. Remark that $z_{j_n} - z = y_n + (\tilde{z}_n - z)$, hence $\|z_{j_n} - z\| \to 0$. This ends the proof. □

5.7.4 *Proof of Theorem 5.3*

First we have

Lemma 5.19. Φ *satisfies* (Φ_0).

Proof. We first show that Φ_a is $\mathcal{T}_{\mathcal{S}}$-closed for every $a \in \mathbb{R}$. Consider a sequence (z_n) in Φ_a which $\mathcal{T}_{\mathcal{S}}$-converges to $z \in E$, and write $z_n = z_n^- + z_n^0 + z_n^+$, $z = z^- + z^0 + z^+$. Observe that (z_n^+) converges to z^+ in norm. Since Ψ is bounded from below it follows from

$$\frac{1}{2}\|z_n^-\|^2 = \frac{1}{2}\|z_n^+\|^2 - \Phi(z_n) - \Psi(z_n) \leq C$$

that (z_n^-) is bounded, hence it converges weakly towards z^-. Since $\dim E^0 < \infty$, the $\mathcal{T}_{\mathcal{S}}$-convergence coincides with the weak convergence. Therefore $z_n \rightharpoonup z$. From Lemma 5.12 and the form of Φ it follows that $\Phi(z) \geq \liminf \Phi(z_n) \geq a$, so $z \in \Phi_a$. Next we show that $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{S}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Suppose (z_n) $\mathcal{T}_{\mathcal{S}}$-converges towards z in Φ_a. As above it follows that (z_n) is bounded and converges weakly towards z. Then $\Phi'(z_n) \overset{w^*}{\to} \Phi'(z)$ by Lemma 5.12. □

Also we have

Lemma 5.20. *Under* $(H_0) - (H_2)$*, for any* $c > 0$*, there is* $\zeta > 0$ *such that :*

$$\|z\| < \zeta\|z^+\| \quad \textit{for all } z \in \Phi_c.$$

Proof. We assume by contradiction that for some $c > 0$ there is a sequence (z_n) with $\Phi(z_n) \geq c$ and $\|z_n\|^2 \geq n\|z_n^+\|^2$. The form of Φ implies

$$\|z_n^- + z_n^0\|^2 \geq (n-1)\|z^+\|^2 \geq (n-1)\left(2c + \|z_n^-\|^2 + 2\int_{\mathbb{R}} R(t, z_n)\right),$$

or

$$\|z_n^0\|^2 \geq (n-1)2c + (n-2)\|z_n^-\|^2 + 2(n-1)\int_{\mathbb{R}} R(t, z_n).$$

Since $c > 0$ and $R(t, z) \geq 0$, it follows that $\|z_n^0\| \to \infty$, hence $\|z_n\| \to \infty$. Set $w_n = z_n/\|z_n\|$. We have $\|w_n^+\|^2 \leq 1/n \to 0$. By

$$1 \geq \|w_n^0\|^2 \geq \frac{(n-1)2c}{\|z_n\|^2} + (n-2)\|w_n^-\|^2 + 2(n-1)\int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2},$$

we also have $\|w_n^-\|^2 \leq 1/(n-2) \to 0$. Therefore, $w_n \to w = w^0$ in E and $\|w^0\| = 1$. Recall that $R(t, z) = \frac{1}{2}M(t)z \cdot z + r(t, z)$ with $|r(t, z)|/|z|^2 \to 0$ as $|z| \to \infty$. Therefore, since $|z_n(t)| \to \infty$ for $w(t) \neq 0$,

$$\begin{aligned}\int_{\mathbb{R}} \frac{r(t, z_n)}{\|z_n\|^2} &= \int_{w(t)\neq 0} \frac{r(t, z_n)}{|z_n|^2}|w_n|^2 + \int_{w(t)=0} \frac{r(t, z_n)}{|z_n|^2}|w_n - w|^2 \\ &\leq 2\int_{w(t)\neq 0} \frac{|r(t, z_n)|}{|z_n|^2}|w|^2 + c|w_n - w|_2^2 \to 0.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{2(n-1)} \geq \int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} &= \frac{1}{2}\int_{\mathbb{R}} M(t)w_n \cdot w_n + \int_{\mathbb{R}} \frac{r(t, z_n)}{\|z_n\|^2} \\ &\geq \frac{m_0}{2}|w_n|_2^2 + o(1),\end{aligned}$$

consequently, $w^0 = 0$, a contradiction. □

Proof. [Proof of Theorem 5.3] (Existence). With $X = E_- \oplus E^0$ and $Y = E_+$ the condition (Φ_0) holds by Lemma 5.19 and (Φ_+) holds by Lemma 5.20. Lemma 5.13 implies (Φ_2). Lemma 5.15 shows that Φ possesses the linking structure of Theorem 4.5. Finally, Φ satisfies the $(C)_c$-condition by virtue of Lemma 5.18. Therefore, Φ has at least one critical point z with $\Phi(z) \geq \kappa > 0$.

(Multiplicity). Assume moreover that $R(t, z)$ is even in z. Then Φ is even hence satisfies (Φ_1). Lemma 5.14 says that Φ satisfies (Φ_3) with $\dim Y_0 = \ell$. Therefore, Φ has at least ℓ pairs of nontrivial critical points by Theorem 4.6. □

Chapter 6

Standing waves of nonlinear Schrödinger equations

This chapter is devoted to the study on existence and multiplicity of solutions to the nonlinear Schrödinger equations. In the first five sections we treat standing waves of a single equation with periodic potential and nonlinearity and 0 lying in a gap of spectrum of the Schrödinger operator, and in the last section we handle semiclassical states of a (Hamiltonian) system of perturbed Schrödinger equations. The nonlinear couplings are assumed to be either asymptotically linear or super linear.

6.1 Introduction and results

We consider the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{NS}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a potential and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ a nonlinear coupling which is either asymptotically linear or super linear as $|u| \to \infty$.

The equation (NS) arises when one seeks for the standing wave solutions of the following nonlinear Schrödinger equation

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\varphi + W(x)\varphi - f(x,|\varphi|)\varphi. \tag{6.1}$$

A standing wave solution of (6.1) is a solution of the form $\varphi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$. It is clear that $\varphi(x,t)$ solves (6.1) if and only if $u(x)$ solves (NS) with $V(x) = \frac{2m}{\hbar^2}(W(x) - E)$ and $g(x,u) = \frac{2m}{\hbar^2} f(x,|u|)u$.

The Schrödinger equation with periodic potentials and nonlinearities has found a great deal of interest in last years because not only it is important in applications but it provides a good model for developing mathematical methods, see, e.g., [Alama and Li (1992I); Ackermann (2004); Alama and Li (1992II); Bartsch and Ding (1999); Buffoni, Jeanjean and Stuart (1993); Chabrowski and Szulkin (2002); Costa and Tehrani (2001); Coti-Zelati and Rabinowitz (1992); Ding and Li (1995); Ding and Luan (2004); Heinz, Küpper and Stuart (1992); Jeanjean (1994); Kryszewski and

Szulkin (1998); Li and Szulkin (2002); Troestler and Willem (1996); Van Heerden (2004); Willem and Zou (2003)] and the references therein. It is known that for periodic potentials the spectrum $\sigma(A)$ of the operator $A := -\Delta + V$ selfadjoint on $L^2(\mathbb{R}^N)$ is a union of closed intervals (cf. [Reed and Simon (1978)]). There have been many results on existence and multiplicity of solutions of such an equation depending on the location of 0 relative to $\sigma(A)$, among which we recall the following ones.

Case 1. $0 < \inf \sigma(A)$. In [Coti-Zelati and Rabinowitz (1992)] Coti-Zelati and Rabinowitz proved via a mountain-pass argument that (NS) has infinitely many solutions provided $g \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and satisfies *the superlinear condition: there is* $\mu > 2$ *such that*

$$0 < \mu G(x,u) \le g(x,u)u \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R} \setminus \{0\} \tag{6.2}$$

and the subcritical condition: there is $s \in (2, 2^*)$ *such that*

$$|g_u(x,u)| \le c_1 + c_2 |u|^{s-2} \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{6.3}$$

Here (and in the following) $G(x,u) := \int_0^u g(x,t)\,dt$, $2^* = \infty$ if $N = 1, 2$, $2^* = 2N/(N-2)$ if $N \ge 3$, and c_i denote positive constants. This result was shown recently in [Ding and Luan (2004); Van Heerden (2004)] to remain true for more general nonlinearities, particularly, for asymptotically linear ones.

Case 2. 0 *lies in a gap of* $\sigma(A)$, *that is,*

$$\underline{\Lambda} := \sup\left(\sigma(A) \cap (-\infty, 0)\right) < 0 < \overline{\Lambda} := \inf\left(\sigma(A) \cap (0, \infty)\right) \tag{6.4}$$

Assume again (6.2) and (6.3) are satisfied. If $G(x,u)$ is strictly convex, existence and multiplicity of solutions of (NS) were established in Alama and Li [Alama and Li (1992I)], Alama and Li [Alama and Li (1992II)] and Buffoni et al. [Buffoni, Jeanjean and Stuart (1993)] by virtue of a mountain-pass reduction. Without the convexity, by using a generalized linking argument together with a weaker topology setting, Troestler and Willem [Troestler and Willem (1996)] and Kryszewski and Szulkin [Kryszewski and Szulkin (1998)] obtained the existence, and multiplicity provided $g(x,u)$ is odd in u, of solutions of (NS). See also [Ackermann (2004); Chabrowski and Szulkin (2002); Ding and Li (1995)].

Case 3. 0 *is a boundary point of a gap of* $\sigma(A)$, *precisely,* $0 \in \sigma(A)$ *and* $(0, \overline{\Lambda}) \cap \sigma(A) = \emptyset$. Under (6.2), together with some other conditions, Bartsch and Ding [Bartsch and Ding (1999)] found at least one nontrivial solution, and infinitely many solutions provided moreover $g(x,u)$ is odd in u. The existence result was later extended to a slightly more general superlinear case in Willem and Zou [Willem and Zou (2003)].

Observe that the conditions (6.2)-(6.3) play an important role for showing that any Palais-Smale sequence is bounded in the works.

A case different from the above is that 0 *lies in a gap and neither* $G(x,u)$ *is convex nor* (6.2) *holds.* This case is difficult because the mountain-pass reduction of [Alama and Li (1992I)] is not available on one hand, and it is not known if the

Palais-Smale sequences are bounded on the other hand. We choose this case as the object of the present chapter.

Firstly we handle the asymptotically linear problem. In what follows, $\tilde{G}(x,u) := \frac{1}{2}g(x,u)u - G(x,u)$ and $\lambda_0 := \min\{-\underline{\Lambda}, \overline{\Lambda}\}$ where $\underline{\Lambda}$ and $\overline{\Lambda}$ are the numbers given by (6.4). Assume

(V_0) $V(x)$ is 1-periodic in x_j for $j = 1, ..., N$ such that $0 \notin \sigma(-\Delta + V)$;
(N_0) $g(x,u)$ is 1-periodic in x_j for $j = 1, ..., N$, $G(x,u) \geq 0$ and $g(x,u) = o(|u|)$ as $u \to 0$ uniformly in x.
(N_1) $g(x,u) - V_\infty(x)u = o(|u|)$ as $|u| \to \infty$ uniformly in x with $\inf V_\infty > \overline{\Lambda}$;
(N_2) $\tilde{G}(x,u) \geq 0$, and there is $\delta_0 \in (0, \lambda_0)$ such that $\tilde{G}(x,u) \geq \delta_0$ whenever $g(x,u)/u \geq \lambda_0 - \delta_0$.

In [Li and Szulkin (2002)] it was proved that if (V_0) and (N_0) $-$ (N_2) hold then (NS) has at least one solution. Observe that, due to the periodicity of V and g, if u is a solution of (NS), then so is $k * u$ for each $k = (k_1, ..., k_N) \in \mathbb{Z}^N$ where $(k*u)(x) = u(x+k)$. Two solutions u_1 and u_2 are said to be geometrically distinct if $k * u_1 \neq u_2$ for all $k \in \mathbb{Z}^N$. We will prove the following multiplicity result.

Theorem 6.1 ([Ding and Lee (2006)]). *Let* (V_0) *and* (N_0) $-$ (N_2) *be satisfied. Then* (NS) *has at least one solution. If moreover* $g(x,u)$ *is odd in* u *and, for some* $\delta > 0$, $\tilde{G}(x,u) > 0$ *whenever* $0 < |u| \leq \delta$, *then* (NS) *possesses infinitely many geometrically distinct solutions.*

Next we deal with the superlinear case. Assume

(N_3) $G(x,u)/u^2 \to \infty$ as $|u| \to \infty$uniformly in x;
(N_4) $\tilde{G}(x,u) > 0$ if $u \neq 0$, and there exist $r_0 > 0$ and $\sigma > \max\{1, N/2\}$ such that $|g(x,u)|^\sigma \leq c_0 \tilde{G}(x,u)|u|^\sigma$ if $|u| \geq r_0$.

Theorem 6.2 ([Ding and Lee (2006)]). *Under the conditions* (V_0), (N_0) *and* (N_3) $-$ (N_4), (NS) *has at least one nontrivial solution. If in addition* $g(x,u)$ *is odd in* u *then* (NS) *possesses infinitely many geometrically distinct solutions.*

Before going on some nonlinear examples and comments on the assumptions are in order.

The following function is odd and satisfies all the asymptotically linear conditions (N_0) $-$ (N_2) :

*Ex*1. $g(x,u) = V_\infty(x)u\left(1 - \frac{1}{\ln(e+|u|)}\right)$ where $V_\infty(x)$ is 1-periodic in x_j for $j = 1, ..., N$ with $\inf V_\infty > \overline{\Lambda}$.

Another asymptotically linear example is the following

*Ex*2. $g(x,u) = h(x,|u|)u$, where $h(x,s)$ is 1-periodic in x_j and increasing for $s \in [0,\infty)$, and $h(x,s) \to 0$ as $s \to 0$ and $h(x,s) \to V_\infty(x)$ as $s \to \infty$ with

$V_\infty(x) > \overline{\Lambda}$ uniformly in x.

Clearly, Ex2 satisfies $(N_0) - (N_2)$.

Examples satisfying the superlinear conditions (N_0) and $(N_3) - (N_4)$ are the following functions with $V_\infty(x) > 0$ and being 1-periodic in x_j :

*Ex*3. $g(x,u) = V_\infty(x)u\ln(1+|u|)$,

*Ex*4. $G(x,u) = V_\infty(x)\Big(|u|^\mu + (\mu-2)|u|^{\mu-\epsilon}\sin^2\big(\frac{|u|^\epsilon}{\epsilon}\big)\Big)$ where $\mu > 2$, $0 < \epsilon < \mu - 2$ if $N = 1, 2$ and $0 < \epsilon < \mu + N - N\mu/2$ if $N \geq 3$.

Remark that these functions do not satisfy (6.2). For getting more examples satisfying the superlinear conditions we show the following

Lemma 6.1. *The assumption (N_4) holds provided $g(x,u)$ satisfies :*

(1°) *there exist $r_1 > 0$ and $p \in (2, 2^*)$ such that $|g(x,u)| \leq c_1|u|^{p-1}$ if $|u| \geq r_1$;*

(2°) *$2G(x,u) < g(x,u)u$ if $u \neq 0$, and there exist $r_1 > 0, \nu > 0$ with $\nu < 2$ if $N = 1$, $\nu < N + p - pN/2$ if $N \geq 2$, such that*

$$G(x,u) \leq \left(\frac{1}{2} - \frac{1}{c_2|u|^\nu}\right) g(x,u)u \quad \textit{if } |u| \geq r_1.$$

Proof. By (2°), $\tilde{G}(x,u) > 0$ if $u \neq 0$ which implies $G(x,u) \geq cu^2$, hence $g(x,u)u \geq 2cu^2$, for $|u| \geq 1$. It follows from also (2°) that

$$\frac{g(x,u)u}{c_2|u|^\nu} \leq \tilde{G}(x,u)$$

for $|u|$ large. Consequently

$$\frac{2c|u|^{2-\nu}}{c_2} \leq \frac{g(x,u)u}{c_2|u|^\nu} \leq \tilde{G}(x,u)$$

which implies $\tilde{G}(x,u) \to \infty$ as $|u| \to \infty$ uniformly in x because $\nu < 2$. Observe that, for $|u|$ large,

$$\begin{aligned} |g(x,u)|^\sigma \leq c\,\tilde{G}(x,u)|u|^\sigma &\Longleftrightarrow \frac{(g(x,u)u)^\sigma}{c|u|^{2\sigma}} \leq \tilde{G}(x,u) \\ &\Longleftrightarrow G(x,u) \leq \left(\frac{1}{2} - \frac{(g(x,u)u)^{\sigma-1}}{c|u|^{2\sigma}}\right) g(x,u)u \\ &\Longleftrightarrow \frac{(g(x,u)u)^{\sigma-1}}{c|u|^{2\sigma}} \leq \frac{1}{2} - \frac{G(x,u)}{g(x,u)u}. \end{aligned}$$

Set $\sigma = (p-\nu)/(p-2)$. Then $\sigma > N/2$, and by (1°)

$$\frac{(g(x,u)u)^{\sigma-1}}{c|u|^{2\sigma}} \leq \frac{1}{a_1|u|^{2\sigma-p(\sigma-1)}} = \frac{1}{a_1|u|^\nu},$$

by (2°)

$$\frac{1}{c_2|u|^\nu} \leq \frac{1}{2} - \frac{G(x,u)}{g(x,u)u}.$$

Hence (N_4) holds. □

It is apparent that if $g(x,u)$ satisfies (6.2)-(6.3) than it satisfies $(1^\circ)-(2^\circ)$, hence $(N_3)-(N_4)$. This fact, together with the examples $Ex3$ and $Ex4$, shows that the superlinear assumptions of Theorem 6.2 are indeed more general than (6.2)-(6.3).

6.2 Preliminaries

Assume that (V_0) holds and let as before $A = -\Delta + V$, the selfadjoint operator acting on $L^2(\mathbb{R}^N,\mathbb{R})$ with domain $\mathscr{D}(A) = H^2(\mathbb{R}^N,\mathbb{R})$. Then (NS) can be rewritten as an equation in $L^2(\mathbb{R}^N,\mathbb{R})$

$$Au = g(x,u). \tag{6.5}$$

In virtue of (V_0) we have the orthogonal decomposition

$$L^2 = L^2(\mathbb{R}^N,\mathbb{R}) = L^- \oplus L^+, \quad u = u^- + u^+$$

such that A is negative (resp., positive) in L^- (resp., in L^+).

Let $E = \mathscr{D}(|A|^{1/2})$ be equipped with the inner product

$$(u,v) = (|A|^{1/2}u, |A|^{1/2}v)_{L^2}$$

and norm $\|u\| = ||A|^{1/2}u|_2$ where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. By (V_0), $E = H^1(\mathbb{R}^N,\mathbb{R})$ with equivalent norms. Therefore E embeds continuously in L^p for all $p \ge 2$ with $p \le 2^*$ if $N \ge 3$, and compactly in L^p_{loc} for all $p \in [1,2^*)$. In addition we have the decomposition

$$E = E^- \oplus E^+ \quad \text{where } E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

On E we define the functional

$$\Phi(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u) \quad \text{where} \quad \Psi(u) = \int_{\mathbb{R}^N} G(x,u).$$

Note that

$$-\underline{\Lambda}|u|_2^2 \le \|u\|^2 \ \text{ for } u \in E^- \quad \text{and} \quad \overline{\Lambda}|u|_2^2 \le \|u\|^2 \ \text{ for } u \in E^+ \tag{6.6}$$

(see (6.4)). The hypotheses on g imply that $\Phi \in C^1(E,\mathbb{R})$ and a standard argument invoking the representation (6.5) shows that critical points of Φ are solutions of (NS). We are seeking for critical points of Φ.

Observe that, assuming (N_0) holds and (N_1) or (N_4) is satisfied, given $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|g(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \tag{6.7}$$

and

$$|G(x,u)| \le \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{6.8}$$

for all (x,u), where $p > 2$ in case (N_1), and $p \ge 2\sigma/(\sigma-1)$ in case (N_4). Remark that in case (N_4), $2\sigma/(\sigma-1) < 2^*$. Using this fact and the Sobolev embedding theorem one checks easily the following

Lemma 6.2. *Let (V_0) and (N_0) be satisfied, and assume moreover $(N_1)-(N_2)$ or $(N_3)-(N_4)$ hold. Then Ψ is non-negative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

6.3 The linking structure

In this section we discuss the linking structure of the functional Φ. Firstly we have the following lemma.

Lemma 6.3. *Under the assumptions of Lemma 6.2, there is $r > 0$ such that $\kappa := \inf \Phi(S_\rho^+) > 0$ where $S_r^+ = \partial B_r \cap E^+$.*

Proof. It follows from (6.8) and the Sobolev embedding theorem that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\Psi(u) \le \varepsilon |u|_2^2 + C_\varepsilon |u|_p^p \le C(\varepsilon \|u\|^2 + C_\varepsilon \|u\|^p)$$

for all $u \in E$. This, jointly with the form of Φ, implies the lemma. □

In the following, for the asymptotically quadratic case we set $\omega = \inf V_\infty$, and for the superquadratic case we choose $\omega = 2\overline{\Lambda}$. Take a number $\bar{\mu}$ satisfying

$$\overline{\Lambda} < \bar{\mu} < \omega. \tag{6.9}$$

Since $\sigma(A)$ is absolutely continuous (cf. [Reed and Simon (1978)]), the subspace $Y_0 := (P_{\bar{\mu}} - P_0)L^2$ is infinite dimensional, where $(P_\lambda)_{\lambda \in \mathbb{R}}$ denotes the spectrum family of A. Note that by definition and (6.6)

$$Y_0 \subset E^+ \quad \text{and} \quad \overline{\Lambda} |w|_2^2 \le \|w\|^2 \le \bar{\mu} |w|_2^2 \quad \text{for all } w \in Y_0. \tag{6.10}$$

For any finite dimensional subspace Y of Y_0 set $E_Y = E^- \oplus Y$.

Lemma 6.4. *Let the assumptions of Lemma 6.2 be satisfied. Then for any finite dimensional subspace Y of Y_0, $\sup \Phi(E_Y) < \infty$, and there is $R_Y > r$ such that $\Phi(u) < \inf \Phi(B_r)$ for all $u \in E_Y$ with $\|u\| \ge R_Y$.*

Proof. It is sufficient to show that $\Phi(u) \to -\infty$ as $u \in E_Y, \|u\| \to \infty$. Arguing indirectly, assume that for some sequence $u_j \in E_Y$ with $\|u_j\| \to \infty$, there is $M > 0$ such that $\Phi(u_j) \ge -M$ for all j. Then, setting $w_j = u_j / \|u_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^+ \to w^+ \in Y$ and

$$-\frac{M}{\|u_j\|^2} \le \frac{\Phi(u_j)}{\|u_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}^N} \frac{G(x, u_j)}{\|u_j\|^2}. \tag{6.11}$$

Remark that $w^+ \neq 0$. Indeed, if not then it follows from (6.11) that

$$0 \le \frac{1}{2}\|w_j^-\|^2 + \int_{\mathbb{R}^N} \frac{G(x, u_j)}{\|u_j\|^2} \le \frac{1}{2}\|w_j^+\|^2 + \frac{M}{\|u_j\|^2} \to 0,$$

in particular, $\|w_j^-\| \to 0$, hence $1 = \|w_j\| \to 0$, a contradiction.

First, consider the asymptotically linear case and assume (N_1) holds. By (6.9)-(6.10) again,

$$\begin{aligned} \|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}^N} V_\infty(x) w^2 &\le \|w^+\|^2 - \|w^-\|^2 - \omega |w|_2^2 \\ &\le -\big((\omega - \bar{\mu})|w^+|_2^2 + \|w^-\|^2\big) < 0, \end{aligned}$$

hence, there is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$\|w^+\|^2 - \|w^-\|^2 - \int_\Omega V_\infty(x)w^2 \; < \; 0. \tag{6.12}$$

Let

$$f(x,u) := g(x,u) - V_\infty(x)u \quad \text{and} \quad F(x,u) = \int_0^u f(x,s)ds. \tag{6.13}$$

By (N_1), $|F(x,u)| \le Cu^2$ and $F(x,u)/u^2 \to 0$ as $|u| \to \infty$ uniformly in x. It follows from Lebesgue's dominated convergence theorem and the fact $|w_j - w|_{L^2(\Omega)} \to 0$ that

$$\lim_{j\to\infty} \int_\Omega \frac{F(x,u_j)}{\|u_j\|^2} = \lim_{j\to\infty} \int_\Omega \frac{F(x,u_j)|w_j|^2}{|u_j|^2} = 0.$$

Thus (6.11) and (6.12) imply that

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_\Omega \frac{G(x,u_j)}{\|u_j\|^2} \right) \\ &\le \frac{1}{2} \left(\|w^+\|^2 - \|w^-\|^2 - \int_\Omega V_\infty(x)w^2 \right) \; < \; 0, \end{aligned}$$

a contradiction.

Next consider the superlinear case and so suppose $(N_3)-(N_4)$ hold. Then there is $r > 0$ such that $G(x,u) \ge \omega|u|^2$ if $|u| \ge r$. Using (6.9)-(6.10),

$$\begin{aligned} \|w^+\|^2 - \|w^-\|^2 - \omega \int_{\mathbb{R}^N} w^2 &\le \bar{\mu}|w^+|_2^2 - \|w^-\|^2 - \omega|w^+|_2^2 - \omega|w^-|_2^2 \\ &\le -\left((\omega - \bar{\mu})|w^+|_2^2 + \|w^-\|^2\right) < 0, \end{aligned}$$

hence, there is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$\|w^+\|^2 - \|w^-\|^2 - \omega \int_\Omega w^2 < 0. \tag{6.14}$$

Note that

$$\begin{aligned} \frac{\Phi(u_j)}{\|u_j\|^2} \le& \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2\right) - \int_\Omega \frac{G(x,u_j)}{\|u_j\|^2} \\ =& \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2 - \omega\int_\Omega |w_j|^2\right) - \int_\Omega \frac{G(x,u_j) - \frac{\omega}{2}|u_j|^2}{\|u_j\|^2} \\ \le& \frac{1}{2}\left(\|w_j^+\|^2 - \|w_j^-\|^2 - \omega\int_\Omega |w_j|^2\right) + \frac{\omega r^2|\Omega|}{2\|u_j\|^2} \end{aligned}$$

($|\Omega|$ denotes the Lebesgue measure of Ω). Thus (6.11) and (6.14) imply that

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_\Omega \frac{G(x,u_j)}{\|u_j\|^2} \right) \\ &\le \frac{1}{2} \left(\|w^+\|^2 - \|w^-\|^2 - \omega\int_\Omega w^2 \right) \; < \; 0, \end{aligned}$$

a contradiction. □

As a special case we have

Lemma 6.5. *Under the assumptions of Lemma 6.2, letting* $e \in Y_0$ *with* $\|e\| = 1$*, there is* $r_0 > 0$ *such that* $\sup \Phi(\partial Q) = 0$ *where* $Q := \{u = u^- + se : \; u^- \in E^-, s \ge 0, \|u\| \le r_0\}$*.*

6.4 The (C) sequences

In this section we consider the boundedness of $(C)_c$-sequences. Firstly, we have

Lemma 6.6. *Under the assumptions of Lemma 6.2, any $(C)_c$-sequence is bounded.*

Proof. Let $(u_j) \subset E$ be such that

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|)\Phi'(u_j) \to 0. \tag{6.15}$$

Observe that for j large

$$C_0 \geq \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^N} \tilde{G}(x, u_j)\,. \tag{6.16}$$

Arguing indirectly, assume by contradiction that $\|u_j\| \to \infty$. Set $v_j = u_j/\|u_j\|$. Then $\|v_j\| = 1$ and $|v_j|_s \leq \gamma_s\|v_j\| = \gamma_s$ for $s \in [2,\, 2^*)$. Observe that, from (6.15) and

$$\Phi'(u_j)(u_j^+ - u_j^-) = \|u_j\|^2 \left(1 - \int_{\mathbb{R}^N} \frac{g(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|}\right),$$

it follows that

$$\int_{\mathbb{R}^N} \frac{g(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \to 1. \tag{6.17}$$

First we consider the asymptotically linear case, hence assume $(N_1) - (N_2)$ are satisfied. By Lions' concentration compactness principle [Lions (1984)], either (v_j) is vanishing (in this case $|v_j|_s \to 0$ for all $s \in (2, 2^*)$), or it is nonvanishing, that is, there are $r, \eta > 0$ and $(a_j) \subset \mathbb{Z}^N$ such that $\limsup_{j\to\infty} \int_{B(a_j,r)} |v_j|^2 \geq \eta$. We show that (v_j) is neither vanishing nor nonvanishing.

Assume (v_j) is vanishing. Set, in virtue of (N_2),

$$\Omega_j := \left\{x \in \mathbb{R}^N : \frac{g(x, u_j(x))}{u_j(x)} \leq \lambda_0 - \delta_0\right\}.$$

Then $\lambda_0|v_j|_2^2 \leq \|v_j\|^2 = 1$ and we have

$$\left|\int_{\Omega_j} \frac{g(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|}\right| = \left|\int_{\Omega_j} \frac{g(x, u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|}\right|$$
$$\leq (\lambda_0 - \delta_0)|v_j|_2^2 \leq \frac{\lambda_0 - \delta_0}{\lambda_0} < 1$$

for all j. This, jointly with (6.17), implies that for $\Omega_j^c := \mathbb{R}^N \setminus \Omega_j$

$$\lim_{j\to\infty} \int_{\Omega_j^c} \frac{g(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} > 1 - \frac{\lambda_0 - \delta_0}{\lambda_0} = \frac{\delta_0}{\lambda_0}\,.$$

Recalling that by (N_0) and (N_1)

$$|g(x, u)| \leq C|u| \quad \text{for all } (x, u), \tag{6.18}$$

there holds for an arbitrarily fixed $s \in (2, 2^*)$

$$\begin{aligned}\int_{\Omega_j^c} \frac{g(x,u_j)(v_j^+ - v_j^-)}{\|u_j\|} &\le C \int_{\Omega_j^c} |v_j^+ - v_j^-||v_j| \\ &\le C|v_j|_2 |\Omega_j^c|^{(s-2)/2s} |v_j|_s \le C\gamma_2 |\Omega_j^c|^{(s-2)/2s} |v_j|_s.\end{aligned}$$

Since $|v_j|_s \to 0$, one gets $|\Omega_j^c| \to \infty$. By (N_2), $\tilde{G}(x,u_j) \ge \delta_0$ on Ω_j^c, hence

$$\int_{\mathbb{R}^N} \tilde{G}(x,u_j) \ge \int_{\Omega_j^c} \tilde{G}(x,u_j) \ge \delta_0 |\Omega_j^c| \to \infty,$$

contrary to (6.16).

Assume (v_j) is nonvanishing. Setting $\tilde{u}_j(x) = u_j(x + a_j)$, $\tilde{v}_j(x) = v_j(x + a_j)$ and $\varphi_j(x) = \varphi(x - a_j)$ for any $\varphi \in C_0^\infty$ we have by (N_1) (see (6.13) for $f(x,u)$)

$$\begin{aligned}\Phi'(u_j)\varphi_j &= (u_j^+ - u_j^-, \varphi_j) - (V_\infty u_j, \varphi_j)_{L^2} - \int_{\mathbb{R}^N} f(x,u_j)\varphi_j \\ &= \|u_j\| \left((v_j^+ - v_j^-, \varphi_j) - (V_\infty v_j, \varphi_j)_{L^2} - \int_{\mathbb{R}^N} f(x,u_j)\varphi_j \frac{|v_j|}{|u_j|} \right) \\ &= \|u_j\| \left((\tilde{v}_j^+ - \tilde{v}_j^-, \varphi) - (V_\infty \tilde{v}_j, \varphi)_{L^2} - \int_{\mathbb{R}^N} f(x,\tilde{u}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{u}_j|} \right).\end{aligned}$$

This results

$$(\tilde{v}_j^+ - \tilde{v}_j^-, \varphi) - (V_\infty \tilde{v}_j, \varphi)_{L^2} - \int_{\mathbb{R}^N} f(x,\tilde{u}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{u}_j|} \to 0.$$

Since $\|\tilde{v}_j\| = \|v_j\| = 1$, we can assume that $\tilde{v}_j \rightharpoonup \tilde{v}$ in E, $\tilde{v}_j \to \tilde{v}$ in L^2_{loc} and $\tilde{v}_j(x) \to \tilde{v}(x)$ a.e. in $\mathbb{R}^N$. Since $\lim_{j\to\infty} \int_{B(0,r)} |\tilde{v}_j|^2 \ge \eta$, $\tilde{v} \ne 0$. By (6.18)

$$\left| f(x,\tilde{u}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{u}_j|} \right| \le C|\varphi||\tilde{v}_j|,$$

it follows from (N_1) and the dominated convergence theorem that

$$\int_{\mathbb{R}^N} f(x,\tilde{u}_j)\varphi \frac{|\tilde{v}_j|}{|\tilde{u}_j|} \to 0,$$

hence

$$(\tilde{v}^+ - \tilde{v}^-, \varphi) - (V_\infty \tilde{v}, \varphi)_{L^2} = 0.$$

Thus $\tilde{v}$ is an eigenfunction of the operator $\tilde{A} := -\Delta + (V - V_\infty)$ contradicting with the fact that $\tilde{A}$ has only continuous spectrum.

Next we consider the superlinear case and suppose $(N_3) - (N_4)$ hold. Set for $r \ge 0$

$$h(r) := \inf \left\{ \tilde{G}(x,u) : x \in \mathbb{R}^N \text{ and } u \in \mathbb{R} \text{ with } |u| \ge r \right\}$$

By (N_4), $h(r) > 0$ for all $r > 0$, and $h(r) \to \infty$ as $r \to \infty$. For $0 \le a < b$ let

$$\Omega_j(a,b) = \left\{ x \in \mathbb{R}^N : a \le |u_j(x)| < b \right\}$$

and

$$c_a^b := \inf\left\{\frac{\tilde G(x,u)}{u^2} : x\in\mathbb{R}^N \text{ and } u\in\mathbb{R} \text{ with } a\le |u|\le b\right\}.$$

Since $G(x,u)$ depends periodically on x and $\tilde G(x,u) > 0$ if $u \neq 0$, one has $c_a^b > 0$ and

$$\tilde G(x,u_j(x)) \ge c_a^b |u_j(x)|^2 \quad \text{for all } x\in\Omega_j(a,b).$$

It follows from (6.16) that

$$\begin{aligned} C_0 &\ge \int_{\Omega_j(0,a)} \tilde G(x,u_j) + \int_{\Omega_j(a,b)} \tilde G(x,u_j) + \int_{\Omega_j(b,\infty)} \tilde G(x,u_j) \\ &\ge \int_{\Omega_j(0,a)} \tilde G(x,u_j) + c_a^b \int_{\Omega_j(a,b)} |u_j|^2 + h(b)|\Omega_j(b,\infty)|. \end{aligned} \tag{6.19}$$

Invoking (N_4), set $\tau := 2\sigma/(\sigma-1)$ and $\sigma' = \tau/2$. Since $\sigma > \max\{1, N/2\}$ one sees $\tau\in(2,2^*)$. Fix arbitrarily $\hat\tau \in (\tau, 2^*)$. Using (6.19),

$$|\Omega_j(b,\infty)| \le \frac{C_0}{h(b)} \to 0$$

as $b\to\infty$ uniformly in j, which implies by Hölder inequality that

$$\int_{\Omega_j(b,\infty)} |v_j|^\tau \le \gamma_{\hat\tau}^\tau\, |\Omega_j(b,\infty)|^{1-\tau/\hat\tau} \to 0 \tag{6.20}$$

as $b\to\infty$ uniformly in j. Using (6.19) again, for any fixed $0<a<b$,

$$\int_{\Omega_j(a,b)} |v_j|^2 = \frac{1}{\|u_j\|^2}\int_{\Omega_j(a,b)} |u_j|^2 \le \frac{C_0}{c_a^b\|u_j\|^2} \to 0 \tag{6.21}$$

as $j\to\infty$.

Let $0<\varepsilon<1/3$. By (N_0) there is $a_\varepsilon > 0$ such that $|g(x,u)| < \frac{\varepsilon}{\gamma_2}|u|$ for all $|u|\le a_\varepsilon$, consequently,

$$\begin{aligned} &\int_{\Omega_j(0,a_\varepsilon)} \frac{g(x,u_j)}{|u_j|}\, |v_j|\, |v_j^+ - v_j^-| \\ &\le \int_{\Omega_j(0,a_\varepsilon)} \frac{\varepsilon}{\gamma_2} |v_j^+ - v_j^-|\, |v_j| \le \frac{\varepsilon}{\gamma_2}|v_j|_2^2 \le \varepsilon \end{aligned} \tag{6.22}$$

for all j. By (N_4) and (6.20) we can take $b_\varepsilon \ge r_0$ large so that

$$\begin{aligned} &\int_{\Omega_j(b_\varepsilon,\infty)} \frac{g(x,u_j)}{|u_j|}(v_j^+ - v_j^-)|v_j| \\ &\le \left(\int_{\Omega_j(b_\varepsilon,\infty)} \frac{|g(x,u_j)|^\sigma}{|u_j|^\sigma}\right)^{1/\sigma} \left(\int_{\Omega_j(b_\varepsilon,\infty)} \left(|v_j^+ - v_j^-|\,|v_j|\right)^{\sigma'}\right)^{1/\sigma'} \\ &\le \left(\int_{\mathbb{R}^N} c_0\tilde G(x,u_j)\right)^{1/\sigma} \left(\int_{\mathbb{R}^N} |v_j^+ - v_j^-|^\tau\right)^{1/\tau} \left(\int_{\Omega_j(b_\varepsilon,\infty)} |v_j|^\tau\right)^{1/\tau} \\ &< \varepsilon \end{aligned} \tag{6.23}$$

for all j. Note that there is $\gamma = \gamma(\varepsilon) > 0$ independent of j such that $|g(x, u_j)| \le \gamma |u_j|$ for $x \in \Omega_j(a_\varepsilon, b_\varepsilon)$. By (6.21) there is j_0 such that

$$\begin{aligned}&\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} \frac{g(x,u_j)}{|u_j|}\,|v_j|\,|v_j^+ - v_j^-| \\ \le\,& \gamma \int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |v_j^+ - v_j^-|\,|v_j| \\ \le\,& \gamma |v_j|_2 \left(\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |v_j|^2\right)^{1/2} < \varepsilon\end{aligned} \tag{6.24}$$

for all $j \ge j_0$. Now the combination of (6.22)-(6.24) implies that for $j \ge j_0$

$$\int_{\mathbb{R}^N} \frac{g(x,u_j)\,(u_j^+ - u_j^-)}{\|u_j\|^2} < 3\varepsilon < 1$$

which contradicts (6.17). □

In the following lemma we discuss further the $(C)_c$-sequence $(u_j) \subset E$. By Lemma 6.6 it is bounded, hence, without loss of generality, we may assume that $u_j \rightharpoonup u$. Plainly u is a critical point of Φ. Set $u_j^1 = u_j - u$.

Lemma 6.7. *Under the assumptions of Lemma 6.2, one has, as $j \to \infty$,*

1) $\Phi(u_j^1) \to c - \Phi(u)$;
2) $\Phi'(u_j^1) \to 0$.

Proof. If $g \in C^1$ with $|g_u(x,u)| \le c_1(1 + |u|^{p-2})$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, some $c_1 > 0$ and $p \in (2, 2^*)$, then this lemma follows easily from a standard argument, see e.g. [Coti-Zelati and Rabinowitz (1992)]. However, in our case such a regularity condition is not available and we hence need to provide another argument. The verification of 1) is similar to and simpler than that of 2), so we only check the latter.

Observe that, for any $\varphi \in E$,

$$\Phi'(u_j^1)\varphi = \Phi'(u_j)\varphi + \int_{\mathbb{R}^N} \big(g(x,u_j) - g(x,u_j^1) - g(x,u)\big)\,\varphi.$$

Since $\Phi'(u_j) \to 0$, it suffices to show that

$$\sup_{\|\varphi\|\le 1} \left| \int_{\mathbb{R}^N} \big(g(x,u_j) - g(x,u_j^1) - g(x,u)\big)\,\varphi \right| \to 0. \tag{6.25}$$

We argue as in the proof of Lemma 5.8. By (6.7) we choose $p \ge 2$ such that $|g(x,u)| \le |u| + C_1|u|^{p-1}$ for all (x,u), and let q stand for either 2 or p. Set $B_a := \{x \in \mathbb{R}^N : \ |x| \le a\}$ for $a > 0$. We have similarly to (5.29) that there is a subsequence (u_{j_n}) such that, for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ satisfying

$$\limsup_{n\to\infty} \int_{B_n \setminus B_r} |u_{j_n}|^q \le \varepsilon \tag{6.26}$$

for all $r \ge r_\varepsilon$. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{u}_n(x) = \eta(2|x|/n)u(x)$ and set $h_n := u - \tilde{u}_n$. Then $h_n \in H^2$ and

$$\|h_n\| \to 0 \quad \text{and} \quad |h_n|_q \to 0 \quad \text{as } n \to \infty. \tag{6.27}$$

Observe that for any $\varphi \in E$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(g(x,u_{j_n}) - g(x,u^1_{j_n}) - g(x,u)\big)\,\varphi \\
= &\int_{\mathbb{R}^N} \left(g(x,u_{j_n}) - g(x,u_{j_n} - \tilde{u}_n) - g(x,\tilde{u}_n)\right)\varphi \\
&+ \int_{\mathbb{R}^N} \big(g(x,u^1_{j_n} + h_n) - g(x,u^1_{j_n})\big)\,\varphi \\
&+ \int_{\mathbb{R}^N} \left(g(x,\tilde{u}_n) - g(x,u)\right)\varphi
\end{aligned}$$

and, by (6.27),

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \left(g(x,\tilde{u}_n) - g(x,u)\right)\varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. In order to check (6.25) it remains to show that

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \left(g(x,u_{j_n}) - g(x,u_{j_n} - \tilde{u}_n) - g(x,\tilde{u}_n)\right)\varphi \right| = 0 \tag{6.28}$$

and

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \big(g(x,u^1_{j_n} + h_n) - g(x,u^1_{j_n})\big)\,\varphi \right| = 0 \tag{6.29}$$

uniformly in $\|\varphi\| \le 1$. This can be done along the same lines of (5.32) and (5.33). Here, for the reader's convenience we repeat the arguments for (6.29). Define $f(x,0) = 0$ and

$$f(x,u) = \frac{g(x,u)}{|u|} \quad \text{if } u \ne 0.$$

f is continuous and 1-periodic in x_j. This implies that f is uniformly continuous in $\mathbb{R}^N \times I_a$ for any $a > 0$ where $I_a := \{u \in \mathbb{R} : |u| \le a\}$. Moreover, $|f(x,u)| \le c_1(1 + |u|^{p-2})$ for all (x,u). Set

$$C^a_n := \{x \in \mathbb{R}^N : |u^1_{j_n}(x)| \le a\} \quad \text{and} \quad D^a_n := \mathbb{R}^N \setminus C^a_n.$$

Since (u^1_j) is bounded, $|u^1_j|_2^2 \le C$, the Lebesgue measure

$$|D^a_n| \le \frac{1}{a^p} \int_{D^a_n} |u^1_{j_n}|^p \le \frac{C}{a^p} \to 0 \quad \text{as } a \to \infty.$$

By Hölder inequality

$$
\begin{aligned}
&\left|\int_{D_n^a} \left(g(x,u_{j_n}^1+h_n)-g(x,u_{j_n}^1)\right)\varphi\right| \\
&\le c_1\int_{D_n^a}\left(|u_{j_n}^1|+|u_{j_n}^1|^{p-1}+|h_n|+|h_n|^{p-1}\right)|\varphi| \\
&\le c_1\left(|D_n^a|^{(2^*-2)/2^*}|u_{j_n}^1|_{2^*}|\varphi|_{2^*}+|D_n^a|^{(2^*-p)/2^*}|u_{j_n}^1|_{2^*}^{p-1}|\varphi|_{2^*}\right) \\
&\quad + c_1\left(|D_n^a|^{(2^*-2)/2^*}|h_n|_{2^*}|\varphi|_{2^*}+|D_n^a|^{(2^*-p)/2^*}|h_n|_{2^*}^{p-1}|\varphi|_{2^*}\right) \\
&\le c_2\left(|D_n^a|^{(2^*-2)/2^*}+|D_n^a|^{(2^*-p)/2^*}\right)\|\varphi\|,
\end{aligned}
$$

it follows that, for any $\varepsilon>0$, there is $\hat{a}>0$ such that

$$
\left|\int_{D_n^a} \left(g(x,u_{j_n}^1+h_n)-g(x,u_{j_n}^1)\right)\varphi\right|\le\varepsilon \tag{6.30}
$$

uniformly in $\|\varphi\|\le 1$ and $n\in\mathbb{N}$. By the uniformly continuity of f on $\mathbb{R}^N\times I_{\hat{a}}$, there is $\delta>0$ satisfying

$$
|f(x,u+h)-f(x,u)|<\varepsilon \ \text{ for all } (x,u)\in\mathbb{R}^N\times I_{\hat{a}} \text{ and } |h|\le\delta,
$$

Set

$$
V_n^\delta := \{x\in\mathbb{R}^N : |h_n(x)|\le\delta\} \quad\text{and}\quad W_n^\delta := \mathbb{R}^N\setminus V_n^\delta .
$$

Clearly, the Lebesgue measure

$$
|W_n^\delta|\le\frac{1}{\delta^2}\int_{W_n^\delta}|h_n|^2\le\frac{1}{\delta^2}|h_n|_2^2\ \to\ 0 \quad \text{as } n\to\infty.
$$

Since $|C_n^{\hat{a}}\cap W_n^\delta|\le|W_n^\delta|\to 0$, as before, it follows from the Hölder inequality that there is n_0 such that

$$
\left|\int_{C_n^{\hat{a}}\cap W_n^\delta}\left(g(x,u_{j_n}^1+h_n)-g(x,u_{j_n}^1)\right)\varphi\right|\le\varepsilon \quad\text{for all } n\ge n_0
$$

uniformly in $\|\varphi\|\le 1$ (see the proof of (6.30)). Moreover,

$$
|f(x,u_{j_n}^1+h_n)-f(x,u_{j_n}^1)|<\varepsilon \ \text{ for all } x\in C_n^{\hat{a}}\cap V_n^\delta .
$$

Note that

$$
\begin{aligned}
\left(g(x,u_{j_n}^1+h_n)-g(x,u_{j_n}^1)\right)\varphi &= f(x,u_{j_n}^1+h_n)\left(|u_{j_n}^1+h_n|-|u_{j_n}^1|\right)\varphi \\
&\quad+\left(f(x,u_{j_n}^1+h_n)-f(x,u_{j_n}^1)\right)|u_{j_n}^1|\,\varphi
\end{aligned}
$$

and, by (6.27), $|h_n|_2<\varepsilon,\ |h_n|_p<\varepsilon$ for all $n\ge n_1$, some $n_1\ge n_0$. Thus, for all $\|\varphi\|\le 1$ and $n\ge n_1$,

$$
\begin{aligned}
&\left|\int_{C_n^{\hat{a}}\cap V_n^\delta}\left(g(x,u_{j_n}^1+h_n)-g(x,u_{j_n}^1)\right)\varphi\right| \\
&\le \int_{C_n^{\hat{a}}\cap V_n^\delta} c_1\left(1+|u_{j_n}^1+h_n|^{p-2}\right)|h_n|\,|\varphi|+\varepsilon\int_{C_n^{\hat{a}}\cap V_n^\delta}|u_{j_n}^1|\,|\varphi| \\
&\le c_2|h_n|_2|\varphi|_2+c_2|u_{j_n}^1+h_n|_p^{p-2}|h_n|_p|\varphi|_p+\varepsilon|u_{j_n}^1|_2|\varphi|_2 \\
&\le c_3\varepsilon.
\end{aligned}
$$

Since $C_n^{\hat{a}} = \big(C_n^{\hat{a}} \cap V_n^{\delta}\big) \cup \big(C_n^{\hat{a}} \cap W_n^{\delta}\big)$, the above estimates imply that

$$\left| \int_{C_n^{\hat{a}}} \big(g(x, u_{j_n}^1 + h_n) - g(x, u_{j_n}^1)\big)\, \varphi \right| \le (c_3 + 1)\varepsilon \quad \text{for all } n \ge n_1$$

uniformly in $\|\varphi\| \le 1$, which, together with (6.30), implies that

$$\sup_{\|\varphi\|\le 1} \left| \int_{\mathbb{R}^N} \big(g(x, u_{j_n}^1 + h_n) - g(x, u_{j_n}^1)\big)\, \varphi \right| \le c_4\varepsilon \quad \text{for all } n \ge n_1,$$

and the proof of (6.29) is complete. □

Let $\mathcal{K} := \{u \in E : \ \Phi'(u) = 0\}$, the critical set of Φ.

Lemma 6.8. *Under the assumptions of Lemma 6.2, there hold*

a) $\nu := \inf\{\|u\| : u \in \mathcal{K} \setminus \{0\}\} > 0$*;*

b) $\theta := \inf\{\Phi(u) : u \in \mathcal{K} \setminus \{0\}\} > 0$ *provided, in the asymptotically linear case, for some* $\delta > 0$, $\tilde{G}(x, u) > 0$ *whenever* $0 < |u| \le \delta$*.*

Proof. *a)* Assume there is a sequence $(u_j) \subset \mathcal{K} \setminus \{0\}$ with $u_j \to 0$. Then

$$0 = \|u_j\|^2 - \int_{\mathbb{R}^N} g(x, u_j)(u_j^+ - u_j^-).$$

Using (6.7), for $p > 2$ and $\varepsilon > 0$ small,

$$\|u_j\|^2 \le \varepsilon |u_j|_2^2 + C_\varepsilon |u_j|_p^p$$

which implies $\|u_j\|^2 \le c_1 C_\varepsilon \|u_j\|^p$ or equivalently $\|u_j\|^{2-p} \le c_1 C_\varepsilon$, a contradiction.

b) Assume there is a sequence $(u_j) \subset \mathcal{K} \setminus \{0\}$ such that $\Phi(u_j) \to 0$. Then

$$\|u_j\|^2 = \int_{\mathbb{R}^N} g(x, u_j)(u_j^+ - u_j^-). \tag{6.31}$$

and

$$o(1) = \Phi(u_j) = \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^N} \tilde{G}(x, u_j) \tag{6.32}$$

Clearly (u_j) is a $(C)_{c=0}$ sequence, hence is bounded by Lemma 6.6. By *a)*, $\|u_j\| \ge \nu$.

First consider the asymptotically linear case. It follows from (6.31) and (6.7) that (u_j) is nonvanishing. Since Φ is $\mathbb{Z}^N$-invariant, up to a translation, we can assume $u_j \rightharpoonup u \in \mathcal{K} \setminus \{0\}$. Since, by assumptions on g, $G(x, u) \ge 0$ and $\tilde{G}(x, u) \ge 0$, one has $g(x, u) = 0$. This implies that u is an eigenfunction of the operator A contrary to that $\sigma(A)$ is absolutely continuous.

Next consider the superlinear case. Using (6.32) and the notations introduced in the proof of Lemma 6.6, we see that, for any $0 < a < b$ and $s \in (2, 2^*)$, $\int_{\Omega_j(a,b)} |u_j|^2 \to 0$ and $\int_{\Omega_j(b,\infty)} |u_j|^s \to 0$ as $j \to \infty$. Therefore, it follows from (6.7) and (6.31) that for any $\varepsilon > 0$

$$\limsup_{j\to\infty} \|u_j\|^2 \le \varepsilon,$$

contradicting to a). □

Let $[r]$ denote the integer part of $r \in \mathbb{R}$. As a consequence of Lemmas 6.6-6.8, we have the following result (see [Coti-Zelati and Rabinowitz (1992); Kryszewski and Szulkin (1998)]).

Lemma 6.9. *Under the assumptions of Lemma 6.2, let (u_j) be a $(C)_c$-sequence. Then either*

(i) $u_j \to 0$ *(and hence $c = 0$), or*
(ii) $c \geq \theta$ *and there exist a positive integer $\ell \leq \left[\frac{c}{\theta}\right]$, points $\overline{u}_1, \cdots, \overline{u}_\ell \in \mathcal{K} \setminus \{0\}$, a subsequence denoted again by (u_j), and sequences $(a_j^i) \subset \mathbb{Z}^N$ such that*

$$\left\| u_j - \sum_{i=1}^{\ell} (a_j^i * \overline{u}_i) \right\| \to 0 \text{ as } j \to \infty$$

$$|a_j^i - a_j^k| \to \infty \text{ for } i \neq k \text{ as } j \to \infty$$

and

$$\sum_{i=1}^{\ell} \Phi(\overline{u}_i) = c.$$

6.5 Proofs of the existence and multiplicity

We are now in a position to establish the main results. In order to apply the abstract Theorems 4.5 and 4.7 to Φ, we choose in the following $X = E^-$ and $Y = E^+$. Since X is separable and reflexive, we choose $\mathcal{S}$ to be a countable dense subset of X^*.

Proof. [Proof of Theorems 6.1 and 6.2] (Existence). With $X = E^-$ and $Y = E^+$ the condition (Φ_0) (see Chapter 4) holds by Lemma 6.2 together with an application of Theorem 4.1. The condition (Φ_+) follows obviously from the form of Φ. The combination of Lemmas 6.3 and 6.5 shows that the linking condition of Theorem 4.5 is satisfied. Therefore, Φ has a $(C)_c$-sequence $(u_n)_n$ with $\kappa \leq c \leq \sup \Phi(Q) < \infty$ where Q is defined by Lemma 6.5. By virtue of Lemma 6.6 the sequence $(u_n)_n$ is bounded. Consequently, $\Phi'(u_n) \to 0$. A standard argument shows that (z_n) is a non-vanishing sequence [Lions (1984)], that is, for some $r, \eta > 0$, there is $(a_n) \subset \mathbb{Z}^N$ such that $\limsup_{n\to\infty} \int_{D(a_n,r)} |z_n|^2 \geq \eta$ where $D(a_n, r)$ denotes the ball in $\mathbb{R}^N$ with center a_n and radius r. Set $w_n := a_n * u_n$. It follows from the invariance of the norm and of the functional under the $*$-action that $\|w_n\| = \|u_n\| \leq C$ and $\Phi(w_n) \to c \geq \kappa, \Phi'(w_n) \to 0$. Therefore $w_n \rightharpoonup w$ in E with $w \neq 0$ and $\Phi'(w) = 0$, that is, w is a nontrivial solution of (NS), and the existence part of Theorems 6.1 and 6.2 is proved.

(Multiplicity). We now establish the multiplicity. The proof will be completed in an indirect way. Namely, assuming

$$\mathcal{K}/\mathbb{Z}^N \text{ is a finite set,} \tag{\dagger}$$

we prove that Φ possesses an unbounded sequence of critical values, which is a contradiction.

Assume that $g(x,-u) = -g(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Then $\Phi(0) = 0$ and Φ is even, that is, (Φ_1) is satisfied (see Chapter 4). (Φ_2) is clear by Lemma 6.3. Recall that $\dim(Y_0) = \infty$. Let (f_k) be a base of Y_0 and set $Y_n := \text{span}\{f_1, ..., f_n\}$ and $E_n := E^- \oplus Y_n$. The condition (Φ_4) follows from Lemma 6.4.

Given $\ell \in \mathbb{N}$ and a finite set $\mathcal{B} \subset E$, let

$$[\mathcal{B}, \ell] := \left\{ \sum_{i=1}^{j} (a_i * u_i) : \ 1 \le j \le \ell, \, a_i \in \mathbb{Z}^N, \, u_i \in \mathcal{B} \right\}.$$

Following an argument of [Coti-Zelati and Rabinowitz (1992)] one sees that

$$\inf\{\|u - u'\| : \ u, u' \in [\mathcal{B}, \ell], \ u \ne u'\} > 0. \tag{6.33}$$

Let $\mathcal{F}$ be a set consisting of arbitrarily chosen representatives of the orbits of $\mathcal{K} \setminus \{0\}$. Then (†) implies that $\mathcal{F}$ is a finite set and, since Φ' is odd, we may assume $\mathcal{F}$ is symmetric. Observe that the points $\bar{u}_i$'s in Lemma 6.9 can be chosen to lie in $\mathcal{F}$. For any compact interval $I \subset (0, \infty)$ with $b := \max I$, set $\ell = [b/\theta]$ and take $\mathscr{A} = [\mathcal{F}, \ell]$. Then $P^+\mathscr{A} = [P^+\mathcal{F}, \ell]$. Clearly, $P^+\mathcal{F}$ is a finite set and

$$\|u\| \le \ell \max\{\|\bar{u}\| : \ \bar{u} \in \mathcal{F}\}$$

for all $u \in \mathscr{A}$, i.e., $\mathscr{A}$ is bounded. In addition, by Lemma 6.9, $\mathscr{A}$ is a $(C)_I$-attractor, and using (6.33),

$$\begin{aligned} &\inf\{\|u_1^+ - u_2^+\| : \ u_1, u_2 \in \mathscr{A}, \ u_1^+ \ne u_2^+\} \\ &= \inf\{\|u - u'\| : \ u, u' \in P^+\mathscr{A}, \ u \ne u'\} > 0. \end{aligned}$$

This argument shows that Φ possesses the following property: *If* (†) *is true, then for any compact interval $I \subset (0, \infty)$, there is a $(C)_I$-attractor $\mathscr{A}$ with $P^+(\mathscr{A})$ bounded and* $\inf\{\|u_1^+ - u_2^+\| : u_1, u_2 \in \mathscr{A}, \ u_1^+ \ne u_2^+\} > 0$. Therefore, the condition (Φ_I) is verified. Now Theorem 4.7 applies. □

6.6 Semiclassical states of a system of Schödinger equations

The results of this section are chosen from [Ding and Lin (2006)]. We investigate the existence and multiplicity of semiclassical solutions of the following Hamiltonian system of perturbed Schrödinger equations

$$\begin{cases} -\varepsilon^2 \Delta\varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x, \varphi, \psi) \\ -\varepsilon^2 \Delta\psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x, \varphi, \psi) \\ w := (\varphi, \psi) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \end{cases}$$

where α and β are continuous real functions on $\mathbb{R}^N$, and $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is of class C^1. Setting

$$\mathscr{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{F}(x, w) = \frac{1}{2}\beta(x)|w|^2 + F(x, w),$$

the system presents the form

$$-\varepsilon^2\Delta w + \alpha(x)w = \mathscr{J}\tilde{F}_w(x,w), \quad w \in H^1(\mathbb{R}^N,\mathbb{R}^2)$$

which can be regarded as the stationary system of the nonlinear vector Schrödinger equation

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \gamma(x)\phi - \mathscr{J} f(x,|\phi|)\phi$$

with $\phi(x,t) = w(x)e^{-\frac{iEt}{\hbar}}$, $\alpha(x) = \gamma(x) - E, \varepsilon^2 = \frac{\hbar^2}{2m}$ and $\tilde{F}_w(x,w) = f(x,|w|)w$.

We assume that $\alpha(x)$ and $\beta(x)$ satisfy the following condition

(A_0) $|\beta(x)| \le \alpha(x)$ for all $x \in \mathbb{R}^N$, $\alpha(x_0) = \beta(x_0)$ for some x_0, and there is $b > 0$ such that the set $\{x \in \mathbb{R}^N : \alpha(x) - |\beta(x)| < b\}$ has finite Lebesgue measure.

Concerning the nonlinearities we will consider two cases: *subcritical and critical superlinearities.*

First we consider the subcritical problem. For notational unification we write $G(x,w)$ instead of $F(x,w)$, and read the system as:

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi - \beta(x)\psi = G_\psi(x,w) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi - \beta(x)\varphi = G_\varphi(x,w) \\ w = (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2). \end{cases} \qquad (\mathcal{P}_\varepsilon)$$

We assume

(G_0) $g_1)$ $G \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ and $G_w(x,w) = o(|w|)$ uniformly in x as $w \to 0$;

$g_2)$ there are $c_0 > 0$ and $\nu > 2N/(N+2)$ such that $|G_w(x,w)|^\nu \le c_0\,(1 + G_w(x,w)w))$ for all (x,w);

$g_3)$ there are $a_0 > 0, p > 2$ and $\mu > 2$ such that $G(x,w) \ge a_0|w|^p$ and $\mu G(x,w) \le G_w(x,w)w$ for all (x,w).

Remark that, setting $q := \frac{\nu}{\nu-1}$, one has by (g_2) that $q < 2^*$ and $|G_w(x,w)| \le c_1(1+|w|^{q-1})$, hence $G(x,w)$ is subcritical. For a solution $w_\varepsilon = (\varphi_\varepsilon,\psi_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$ we denote its energy by

$$E(w_\varepsilon) := \int_{\mathbb{R}^N}\Big(\varepsilon^2\nabla\varphi_\varepsilon\nabla\psi_\varepsilon + \alpha(x)\varphi_\varepsilon\psi_\varepsilon\Big) - \int_{\mathbb{R}^N}\Big(\frac{1}{2}\beta(x)|w_\varepsilon|^2 + G(x,w_\varepsilon)\Big).$$

Theorem 6.3 ([Ding and Lin (2006)]). *Let (A_0) and (G_0) be satisfied.*

(1) *For any $\sigma > 0$ there is $\mathcal{E}_\sigma > 0$ such that if $\varepsilon \le \mathcal{E}_\sigma$, $(\mathcal{P}_\varepsilon)$ has at least one nontrivial solution w_ε satisfying (i) $\int_{\mathbb{R}^N} G(x,w_\varepsilon) \le \frac{2\sigma}{\mu-2}\,\varepsilon^N$ and (ii) $0 < E(w_\varepsilon) \le \sigma\,\varepsilon^N$.*

(2) *Assuming additionally that $G(x,w)$ is even in w, for any $m \in \mathbb{N}$ and $\sigma > 0$ there is $\mathcal{E}_{m\sigma} > 0$ such that if $\varepsilon \le \mathcal{E}_{m\sigma}$, $(\mathcal{P}_\varepsilon)$ has at least m pairs solutions w_ε which satisfy the estimates (i) and (ii).*

Next we consider the critical problem:

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi - \beta(x)\psi = G_\psi(x,w) + K(x)|w|^{2^*-2}\psi \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi - \beta(x)\varphi = G_\varphi(x,w) + K(x)|w|^{2^*-2}\varphi \\ w = (\varphi,\psi) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \end{cases} \qquad (\mathcal{Q}_\varepsilon)$$

(where $N \ge 3$). Assume $K(x)$ is bounded, that is,

(K_0) $K \in C(\mathbb{R}^N)$, $0 < \inf K \le \sup K < \infty$.

Denote the energy of a solution $w_\varepsilon = (\varphi_\varepsilon, \psi_\varepsilon)$ of $(\mathcal{Q})_\varepsilon$ by

$$\begin{aligned} E(w_\varepsilon) := &\int_{\mathbb{R}^N} \Big(\varepsilon^2 \nabla\varphi_\varepsilon \nabla\psi_\varepsilon + \alpha(x)\varphi_\varepsilon\psi_\varepsilon\Big) \\ &- \int_{\mathbb{R}^N} \Big(\frac{1}{2}\beta(x)|w_\varepsilon|^2 + G(x,w_\varepsilon) + \frac{1}{2^*}K(x)|w_\varepsilon|^{2^*}\Big). \end{aligned}$$

We have

Theorem 6.4 ([Ding and Lin (2006)]). *Let* $(A_0), (K_0)$ *and* (G_0) *be satisfied. Then both the conclusions* (1) *and* (2) *of Theorem 6.3 are true with* $(\mathcal{P}_\varepsilon)$ *replaced by* $(\mathcal{Q}_\varepsilon)$ *and* (i) *by*

$$\frac{\mu-2}{2}\int_{\mathbb{R}^N} G(x,w_\varepsilon) + \frac{1}{N}\int_{\mathbb{R}^N} K(x)|w_\varepsilon|^{2^*} \le \sigma\,\varepsilon^N.$$

6.6.1 *An equivalent variational problem*

Let

$$u = \frac{\varphi+\psi}{2}, \quad v = \frac{\varphi-\psi}{2}, \quad z = (u,v),$$

$$V(x) = \alpha(x) - \beta(x), \quad W(x) = \alpha(x) + \beta(x)$$

and

$$H(x,z) = H(x,u,v) = \frac{1}{2}G\left(x, \frac{u+v}{2}, \frac{u-v}{2}\right).$$

Then $(\mathcal{P}_\varepsilon)$ reads as

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = H_u(x,z) \\ -\varepsilon^2\Delta v + W(x)v = -H_v(x,z) \\ z = (u,v) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \end{cases} \qquad (\tilde{\mathcal{P}}_\varepsilon)$$

and $(\mathcal{Q}_\varepsilon)$ as

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = H_u(x,z) + K(x)|z|^{2^*-2}u \\ -\varepsilon^2\Delta v + W(x)v = -\Big(H_v(x,z) + K(x)|z|^{2^*-2}v\Big) \\ \ z = (u,v) \in H^1(\mathbb{R}^N, \mathbb{R}^2). \end{cases} \qquad (\tilde{\mathcal{Q}}_\varepsilon)$$

The assumption (A_0) implies that V and W satisfy

(V_0) $V(x_0) = \min V = 0$; and the set $\{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure.

(W_0) $W \geq 0$; and the set $\{x \in \mathbb{R}^N : W(x) < b\}$ has finite Lebesgue measure.

And (G_0) implies that $H(x, z)$ satisfies

(H_0) $h_1)$ $H_z(x, z) = o(|z|)$ uniformly in x as $z \to 0$;

$h_2)$ there are $c_0 > 0$ and $\nu > 2N/(N+2)$ such that $|H_z(x, z)|^\nu \leq c_0 (1 + H_z(x, z)z))$ for all (x, z);

$g_3)$ there are $a_0 > 0, p > 2$ and $\mu > 2$ such that $H(x, z) \geq a_0|z|^p$ and $\mu H(x, z) \leq H_z(x, z)z$ for all (x, z).

Setting $\lambda = \varepsilon^{-2}$, $(\tilde{\mathcal{P}}_\varepsilon)$ is equivalent to

$$\begin{cases} -\Delta u + \lambda V(x)u = \lambda H_u(x, z) \\ -\Delta v + \lambda W(x)v = -\lambda H_v(x, z) \\ z = (u, v) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \end{cases} \tag{$\mathcal{P}_\lambda$}$$

and $(\tilde{\mathcal{Q}}_\varepsilon)$ equivalent to

$$\begin{cases} -\Delta u + \lambda V(x)u = \lambda(H_u(x, z) + K(x)|z|^{2^*-2}u) \\ -\Delta v + \lambda W(x)v = -\lambda(H_v(x, z)v + K(x)|z|^{2^*-2}v) \\ z = (u, v) \in H^1(\mathbb{R}^N, \mathbb{R}^2). \end{cases} \tag{$\mathcal{Q}_\lambda$}$$

Letting

$$E(z_\lambda) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\Big) - \Big(|\nabla v_\lambda|^2 + \lambda W(x)|v_\lambda|^2\Big) \\ - \lambda \int_{\mathbb{R}^N} H(x, z_\lambda)$$

denote the energy of the solution $z_\lambda = (u_\lambda, v_\lambda)$ of $(\mathcal{P}_\lambda)$, and similarly

$$E(z_\lambda) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\Big) - \Big(|\nabla v_\lambda|^2 + \lambda W(x)|v_\lambda|^2\Big) \\ - \lambda \int_{\mathbb{R}^N} \Big(H(x, z_\lambda) + \frac{1}{2^*}K(x)|z_\lambda|^{2^*}\Big)$$

for the solution $z_\lambda = (u_\lambda, v_\lambda)$ of $(\mathcal{Q}_\lambda)$, we are led to prove

Theorem 6.5. *Let (V_0), (W_0) and (H_0) be satisfied.*

(1) *For any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that if $\lambda \geq \Lambda_\sigma$, $(\mathcal{P}_\lambda)$ has at least one nontrivial solution z_λ satisfying (i) $\int_{\mathbb{R}^N} H(x, z_\lambda) \leq \frac{2\sigma}{\mu-2}\lambda^{-\frac{N}{2}}$ and (ii) $0 < E(z_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}$.*

(2) *Assuming additionally that $H(x, z)$ is even in z, for any $m \in \mathbb{N}$ and $\sigma > 0$ there is $\Lambda_{m\sigma} > 0$ such that if $\lambda \geq \Lambda_{m\sigma}$, $(\mathcal{P}_\lambda)$ has at least m pairs solutions z_λ which satisfy the estimates (i) and (ii).*

Theorem 6.6. *Let* $(V_0), (W_0), (H_0)$ *and* (K_0) *be satisfied. Then both the conclusions* (1) *and* (2) *of Theorem 6.5 hold with* $(\mathcal{P}_\lambda)$ *replaced* $(\mathcal{Q}_\lambda)$ *and* (*i*) *by*

$$\frac{\mu-2}{2}\int_{\mathbb{R}^N} H(x,z_\lambda)+\frac{1}{N}\int_{\mathbb{R}^N} K(x)|z_\lambda|^{2^*}\le\sigma\,\lambda^{-\frac{N}{2}}.$$

In order to prove the above theorems we introduce the space

$$E_+:=\left\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2<\infty\right\}$$

which is a Hilbert space equipped with the inner product

$$(u_1,u_2)_+:=\int_{\mathbb{R}^N}(\nabla u_1\nabla u_2+V(x)u_1u_2)$$

and the associated norm $\|u\|_+^2=(u,u)_+$. It follows from (V_0) that E_+ embeds continuously in $H^1(\mathbb{R}^N)$. Note that the norm $\|\cdot\|_+$ is equivalent to the one $\|\cdot\|_{+\lambda}$ deduced by the inner product

$$(u_1,u_2)_{+\lambda}:=\int_{\mathbb{R}^N}(\nabla u_1\nabla u_2+\lambda V(x)u_1u_2)$$

for each $\lambda>0$. It thus is clear that, for each $s\in[2,2^*]$, there is $\gamma_s>0$ (independent of λ) such that if $\lambda\ge1$

$$|u|_s\le\gamma_s\|u\|_+\le\gamma_s\|u\|_{+\lambda}\quad\text{for all } u\in E_+.$$

For convenience we will use certain direct sum decompositions of E_+ described below.

Let $A_\lambda:=-\Delta+\lambda V$ denote the selfadjoint operator in $L^2(\mathbb{R}^N)$. By $\sigma(A_\lambda)$, $\sigma_e(A_\lambda)$ and $\sigma_d(A_\lambda)$ we denote the spectrum, the essential spectrum and the eigenvalues of A_λ below $\lambda_e:=\inf\sigma_e(A_\lambda)$, respectively. Note that it is possible that $\lambda_e=\infty$ (hence $\sigma(A_\lambda)=\sigma_d(A_\lambda)$), for example, this is the case if $V(x)\to\infty$ as $|x|\to\infty$.

Lemma 6.10. *Suppose* (V_0) *holds. Then* $\lambda_e\ge\lambda b$.

Proof. Set $V_\lambda(x)=\lambda(V(x)-b), V_\lambda^\pm=\max\{\pm V_\lambda,0\}$ and $D_\lambda=-\Delta+\lambda b+V_\lambda^+$. By (V_0), the multiplicity operator V_λ^- is compact relative to D_λ, hence

$$\sigma_e(A_\lambda)\subset\sigma_e(D_\lambda)\subset[\lambda b,\infty)$$

as required. □

Let k_λ be the number of the eigenvalues below λb. We write $\eta_{\lambda i}$ and $f_{\lambda i}$ $(1\le i\le k_\lambda)$ for the eigenvalues and eigenfunctions. Setting

$$L_\lambda^d:=\operatorname{span}\{f_{\lambda1},\cdots,f_{\lambda k_\lambda}\},$$

we have the orthogonal decomposition

$$L^2(\mathbb{R}^N)=L_\lambda^d\oplus L_\lambda^e,\quad u=u^d+u^e.$$

Correspondingly, E_+ has the decomposition:

$$E_+ = E^d_{+\lambda} \oplus E^e_{+\lambda} \quad \text{with} \quad E^d_{+\lambda} = L^d_\lambda \text{ and } E^e_{+\lambda} = E_+ \cap L^e_\lambda,$$

orthogonal with respect to both the inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_{+\lambda}$.

Letting S denote the best Sobolev constant: $S|u|^2_{2^*} \le \int_{\mathbb{R}^N} |\nabla u|^2$, it is clear that

$$S|u|^2_{2^*} \le \|u\|^2_{+\lambda} \quad \text{for all } u \in E.$$

It follows from Lemma 6.10 that

$$|u|^2_2 \le \frac{1}{b\lambda}\|u\|^2_{+\lambda} \quad \text{for all } u \in E^e_{+\lambda},$$

which, together with interpolation, shows that for each $s \in [2, 2^*]$,

$$|u|^s_s \le a_s \lambda^{-(2^*-s)/(2^*-2)}\|u\|^s_{+\lambda} \quad \text{for all } u \in E^e_{+\lambda} \tag{6.34}$$

where a_s is a constant independent of λ.

Similarly, with replacing $V(x)$ by $W(x)$, we define the Hilbert space E_-, the inner products $(\cdot,\cdot)_-$ and $(\cdot,\cdot)_{-\lambda}$, and the decomposition $E_- = E^d_{-\lambda} \oplus E^e_{-\lambda}$.

Let

$$E := E_+ \times E_-$$

and write for $z = (u, v) \in E$, $z^+ = (u, 0)$ or simply denote $(u, 0)$ by u, and similarly, $z^- = (0, v)$ or simply $(0, v)$ by v. We denote the inner product on E by

$$(z_1, z_2) = (u_1, u_2)_+ + (v_1, v_2)_-$$

and the induced norm by

$$\|z\|^2 = \|u\|^2_+ + \|v\|^2_-.$$

On E there are the equivalent norms

$$\|z\|^2_\lambda = \|u\|^2_{+\lambda} + \|v\|^2_{-\lambda}.$$

E has the orthogonal decomposition

$$E = E^d_\lambda \oplus E^e_\lambda \quad \text{where } E^d_\lambda = E^d_{+\lambda} \times E^d_{-\lambda} \text{ and } E^e_\lambda = E^e_{+\lambda} \times E^e_{-\lambda}.$$

Accordingly, we write $z = z^d + z^e$ for $z = (u, v) \in E$ with $z^d = (u^d, v^d)$ and $z^e = (u^e, v^e)$. Note that $\dim E^d_\lambda < \infty$. It follows from (6.34) that for each $s \in [2, 2^*]$,

$$|z|^2_2 \le \frac{1}{b\lambda}\|z\|^2_\lambda \quad \text{and} \quad |z|^s_s \le a_s \lambda^{-(2^*-s)/(2^*-2)}\|z\|^s_\lambda \tag{6.35}$$

for all $z \in E^e_\lambda$ where a_s is a constant independent of λ.

Define the functional for $z = (u, v) \in E$

$$\begin{aligned}\Phi_\lambda(z) &= \frac{1}{2}\int_{\mathbb{R}^N} \Big(\big(|\nabla u|^2 + \lambda V(x)u^2\big) - \big(|\nabla v|^2 + \lambda W(x)v^2\big)\Big) \\ &\quad - \lambda \int_{\mathbb{R}^N} H(x, z) \\ &= \frac{1}{2}\|u\|^2_{+\lambda} - \frac{1}{2}\|v\|^2_{-\lambda} - \lambda \int_{\mathbb{R}^N} H(x, z).\end{aligned}$$

Under the assumptions (A_0) and (H_0), $\Phi_\lambda \in C^1(E,\mathbb{R})$ and its critical points are solutions of $(\mathcal{P}_\lambda)$.

Similarly, consider the functional

$$\begin{aligned}\Psi_\lambda(z) &= \frac{1}{2}\int_{\mathbb{R}^N} \Big(\big(|\nabla u|^2 + \lambda V(x)u^2\big) - \big(|\nabla v|^2 + \lambda W(x)v^2\big)\Big) \\ &\quad - \lambda \int_{\mathbb{R}^N} \left(H(x,z) + \frac{K(x)}{2^*}|z|^{2^*}\right) \\ &= \frac{1}{2}\|u\|_{+\lambda}^2 - \frac{1}{2}\|v\|_{-\lambda}^2 - \lambda \int_{\mathbb{R}^N} \left(H(x,z) + \frac{K(x)}{2^*}|z|^{2^*}\right).\end{aligned}$$

Then $\Psi_\lambda \in C^1(E,\mathbb{R})$ and critical points of Ψ_λ correspond to solutions of $(\mathcal{Q}_\lambda)$.

First of all we have plainly the following

Lemma 6.11. *Let f_λ stand for either Φ_λ or Ψ_λ.*

(1°) *f_λ is weakly sequentially upper semicontinuous, and f'_λ is weakly sequentially continuous. Moreover, there is $\zeta > 0$ such that for any $c > 0$, $\|z\|_\lambda < \zeta\|u\|_\lambda$ for all $z \in (f_\lambda)_c$.*

(2°) *For each $\lambda \geq 1$, there exists $\rho_\lambda > 0$ such that $\kappa_\lambda := \inf \Psi_\lambda(S_{\rho_\lambda}E_+) > 0$ where $S_{\rho_\lambda} = \{z \in E_+ : \ \|z\|_\lambda = \rho_\lambda\}$.*

(3°) *For any $e \in E_+$ there is $R > \rho_\lambda$ such that $(\Psi_\lambda)|_{\partial Q} \leq 0$ where $Q := \{z = (se_1, v) : v \in E_-, s \geq 0, \|z\|_\lambda \leq R\}$.*

(4°) *For any finite dimensional subspace $F \subset E_+$, there is $R_F > \rho_\lambda$ such that $\Psi_\lambda(u) < \inf \Psi_\lambda(B_{\rho_\lambda} \cap E_+)$ for all $u \in F \times E_- \setminus B_{R_F}$.*

(5°) *Any $(C)_c$-sequence for f_λ is bounded and $c \geq 0$.*

6.6.2 *Proofs of Theorem 6.5*

In this sub-section we treat the subcritical problem $(\mathcal{P}_\lambda)$, thus consider the functional Φ_λ.

Observe that, by (H_0), $c_1|z|^p \leq H(x,z) \leq c_2|z|^q$ for all $|z|$ large where $q = \nu/(\nu-1)$. Hence $\nu \leq p/(p-1) < 2$ since $p > 2$. Set $\tau = \nu/(2-\nu)$. Then for any $\delta > 0$ there are $\rho_\delta > 0$ and $c_\delta > 0$ such that

$$\frac{|H_z(x,z)|}{|z|} \leq \delta \text{ if } |z| \leq \rho_\delta, \quad \frac{|H_z(x,z)|^\tau}{|z|^\tau} \leq c_\delta H_z(x,z)z \text{ if } |z| \geq \rho_\delta. \tag{6.36}$$

Indeed, for $|z| \geq \rho_\delta$ there holds $|H_z(x,z)|^\nu \leq a_\delta H_z(x,z)z$ and

$$\begin{aligned}|H_z(x,z)|^\tau &= |H_z(x,z)|^{\tau-\nu}|H_z(x,z)|^\nu \leq a'_\delta |z|^{(\tau-\nu)/(\nu-1)} H_z(x,z)z \\ &= a'_\delta |z|^\tau H_z(x,z)z.\end{aligned}$$

In addition, setting

$$\tilde{H}(x,z) := \frac{1}{2}H_z(x,z)z - H(x,z).$$

we have

$$\tilde{H}(x,z) \geq \frac{\mu-2}{2\mu}H_z(x,z)z \geq \frac{\mu-2}{2}H(x,z) \geq \frac{a_0(\mu-2)}{2}|z|^p. \tag{6.37}$$

In the following, let (z_j) denote a $(C)_c$-sequence. By the above Lemma 6.11$-$(5°), it is bounded, hence, without loss of generality, we may assume $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^s_{loc} for $1 \le s < 2^*$, and $z_j(x) \to z(x)$ a.e. for $x \in \mathbb{R}^N$. Plainly, z is a critical point of Φ_λ.

Similarly to (6.26), along a subsequence, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ with

$$\limsup_{n\to\infty} \int_{B_{j_n}\setminus B_r} |z_{j_n}|^s \le \varepsilon \tag{6.38}$$

for all $r \ge r_\varepsilon$ and $s \in [2, 2^*)$. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{z}_n(x) = \eta(2|x|/n)z(x)$. Clearly,

$$\|z - \tilde{z}_n\| \to 0 \quad \text{as } n \to \infty. \tag{6.39}$$

Additionally, we have similarly to (6.28)

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \left(H_z(x, z_{j_n}) - H_z(x, z_{j_n} - \tilde{z}_n) - H_z(x, \tilde{z}_n) \right) \varphi \right| = 0$$

uniformly in $\varphi \in E$ with $\|\varphi\| \le 1$. Then repeating the relative argument of the proof of Lemma 6.7 (see also Lemma 5.17) yields the following

Lemma 6.12. *One has:*

1) $\Phi_\lambda(z_{j_n} - \tilde{z}_n) \to c - \Phi_\lambda(z)$;
2) $\Phi'_\lambda(z_{j_n} - \tilde{z}_n) \to 0$.

We now utilize the decomposition $E = E^d_\lambda \oplus E^e_\lambda$. Recall that $\dim(E^d_\lambda) < \infty$. Write

$$y_n := z_{j_n} - \tilde{z}_n = y^d_n + y^e_n.$$

Then $y^d_n = (z^d_{j_n} - z^d) + (z^d - \tilde{z}^d_n) \to 0$ and, by Lemma 6.12, $\Phi_\lambda(y_n) \to c - \Phi_\lambda(z)$, $\Phi'_\lambda(y_n) \to 0$. It follows from

$$\Phi_\lambda(y_n) - \frac{1}{2}\Phi'_\lambda(y_n)y_n = \lambda \int_{\mathbb{R}^N} \tilde{H}(x, y_n)$$

that

$$\lambda \int_{\mathbb{R}^N} \tilde{H}(x, y_n) \to c - \Phi_\lambda(z).$$

Noting that $y_n = (u_{j_n} - \tilde{u}_n, v_{j_n} - \tilde{v}_n)$ we set $\bar{y}_n = (u_{j_n} - \tilde{u}_n, -v_{j_n} + \tilde{v}_n)$. We have $|y_n| = |\bar{y}_n|$ and

$$\begin{aligned} o(1) = \Phi'_\lambda(y_n)\bar{y}_n &= \|y_n\|^2_\lambda - \lambda \int_{\mathbb{R}^N} H_z(x, y_n)\bar{y}_n \\ &= o(1) + \|y^e_n\|^2_\lambda - \lambda \int_{\mathbb{R}^N} H_z(x, y_n)\bar{y}_n. \end{aligned}$$

By (6.35), (6.36) and (6.37), we have for any $\delta > 0$,

$$
\begin{aligned}
&\|y_n^e\|_\lambda^2 + o(1) \\
&= \lambda \int_{\mathbb{R}^N} H_z(x, y_n)\bar{y}_n \\
&\le \lambda \int_{\mathbb{R}^N} \frac{|H_z(x, y_n)|}{|y_n|}|\bar{y}_n|^2 \\
&\le o(1) + \lambda\delta|y_n|_2^2 + \lambda c'_\delta \left(\int_{|y_n| \ge \rho_\delta} \left(\frac{|H_z(x, y_n)|}{|y_n|} \right)^\tau \right)^{1/\tau} |y_n|_q^2 \qquad (6.40) \\
&\le o(1) + \lambda\delta|y_n^e|_2^2 + \lambda c''_\delta \left(\frac{c - \Phi_\lambda(z) + o(1)}{\lambda} \right)^{1/\tau} |y_n^e|_q^2 \\
&\le o(1) + \frac{\delta}{b}\|y_n^e\|_\lambda^2 + C_\delta \lambda^{1-\frac{1}{\tau}-\frac{2(2^*-q)}{q(2^*-2)}} \left(c - \Phi_\lambda(z)\right)^{1/\tau} \|y_n^e\|_\lambda^2 \\
&= o(1) + \frac{\delta}{b}\|y_n^e\|_\lambda^2 + C_\delta \lambda^{\frac{(N-2)(q-2)}{2q}} \left(c - \Phi_\lambda(z)\right)^{1/\tau} \|y_n^e\|_\lambda^2.
\end{aligned}
$$

Remark that $z_{j_n} - z = y_n + (\tilde{z}_n - z)$, hence by (6.39)

$$z_{j_n} - z \to 0 \ \text{ if and only if } \ y_n^e \to 0.$$

Lemma 6.13. *There is a constant $\alpha_0 > 0$ independent of λ such that, for any $(C)_c$-sequence (z_j) for Φ_λ with $z_j \rightharpoonup z$, either $z_j \to z$ along a subsequence or*

$$c - \Phi_\lambda(z) \ge \alpha_0 \lambda^{1-\frac{N}{2}}.$$

Proof. Assume z_j has no convergence. Then using the above notations $\liminf_{n\to\infty} \|y_n^e\|_\lambda > 0$ and $c - \Phi_\lambda(z) > 0$. Choosing $\delta = b/4$, it follows from (6.40) that

$$\frac{3}{4}\|y_n^e\|_\lambda^2 \le o(1) + c_1 \lambda^{\frac{(N-2)(q-2)}{2q}} \left(c - \Phi_\lambda(z)\right)^{1/\tau} \|y_n^e\|_\lambda^2.$$

This implies that

$$1 \le c_2 \lambda^{\frac{N}{2}-1} \left(c - \Phi_\lambda(z)\right)$$

which proves the lemma. □

In particular, we obtain the following

Lemma 6.14. Φ_λ *satisfies the $(C)_c$ condition for all $c < \alpha_0 \lambda^{1-\frac{N}{2}}$.*

Observe that (H_0) implies

$$
\begin{aligned}
\Phi_\lambda(z) &\le \frac{1}{2}\|u\|_{+\lambda}^2 - \frac{1}{2}\|v\|_{-\lambda}^2 - a_0\lambda \int_{\mathbb{R}^N} |z|^p \\
&\le \frac{1}{2}\|u\|_{+\lambda}^2 - \frac{1}{2}\|v\|_{-\lambda}^2 - a_0\lambda \int_{\mathbb{R}^N} |u|^p.
\end{aligned}
$$

We define the functional $J_\lambda \in C^1(E_+, \mathbb{R})$ by setting

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda V(x)u^2\right) - a_0\lambda \int_{\mathbb{R}^N} |u|^p.$$

Then

$$\Phi_\lambda(z) \le J_\lambda(u) - \frac{1}{2}\|v\|_{-\lambda}^2 \quad \text{for all } z \in E. \tag{6.41}$$

Recall that the assumption (V_0) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$.

It is known that

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^2 : \ \varphi \in C_0^\infty(\mathbb{R}^N),\ |\varphi|_p = 1\right\} = 0.$$

For any $\delta > 0$ one can choose $\varphi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\varphi_\delta|_p = 1$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$ so that $|\nabla\varphi_\delta|_2^2 < \delta$. Set

$$e_\lambda(x) := \varphi_\delta(\lambda^{1/2}x). \tag{6.42}$$

Then

$$\operatorname{supp} e_\lambda \subset B_{\lambda^{-1/2}r_\delta}(0).$$

Remark that for $t \ge 0$,

$$\begin{aligned} J_\lambda(te_\lambda) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla e_\lambda|^2 + \lambda V(x)|e_\lambda|^2 - a_0\lambda t^p \int_{\mathbb{R}^N} |e_\lambda|^p \\ &= \lambda^{1-\frac{N}{2}}\left(\frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\varphi_\delta|^2 + V\left(\lambda^{-1/2}x\right)|\varphi_\delta|^2 - a_0 t^p \int_{\mathbb{R}^N} |\varphi_\delta|^p\right) \\ &= \lambda^{1-\frac{N}{2}} I_\lambda(t\varphi_\delta) \end{aligned}$$

where $I_\lambda \in C^1(E_+, \mathbb{R})$ defined by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V\left(\lambda^{-1/2}x\right)|u|^2 - a_0\int_{\mathbb{R}^N} |u|^p.$$

Plainly,

$$\max_{t\ge 0} I_\lambda(t\varphi_\delta) = \frac{p-2}{2p(pa_0)^{2/(p-2)}}\left(\int_{\mathbb{R}^N} |\nabla\varphi_\delta|^2 + V(\lambda^{-1/2}x)|\varphi_\delta|^2\right)^{p/(p-2)}.$$

Since $V(0) = 0$ and note that $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$, there is $\hat\Lambda_\delta > 0$ such that

$$V(\lambda^{-1/2}x) \le \frac{\delta}{|\varphi_\delta|_2^2} \quad \text{for all } |x| \le r_\delta \text{ and } \lambda \ge \hat\Lambda_\delta.$$

This implies that

$$\max_{t\ge 0} I_\lambda(t\varphi_\delta) \le \frac{p-2}{2p(pa_0)^{2/(p-2)}}\,(2\delta)^{p/(p-2)}.$$

Since $I_\lambda(u)$ is even, we obtain that, for all $\lambda \ge \hat\Lambda_\delta$,

$$\max_{t\in\mathbb{R}} J_\lambda(te_\lambda) \le \frac{p-2}{2p(pa_0)^{2/(p-2)}}\,(2\delta)^{p/(p-2)}\,\lambda^{1-\frac{N}{2}}. \tag{6.43}$$

Therefore, we have

Lemma 6.15. *For any $\sigma > 0$ there exists $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\sigma$, there is $e_\lambda \in E_+ \setminus \{0\}$ such that*

$$\max_{z \in F_{\sigma\lambda}} \Phi_\lambda(z) \leq \sigma\, \lambda^{1-\frac{N}{2}},$$

where $F_{\sigma\lambda} := \mathbb{R}e_\lambda \times E_-$.

Proof. Choose $\delta > 0$ so small that

$$\frac{p-2}{2p(pa_0)^{2/(p-2)}}\,(2\delta)^{p/(p-2)} \leq \sigma,$$

and let $e_\lambda \in E_{+\lambda}$ be the function defined by (6.42). Take $\Lambda_\sigma = \hat{\Lambda}_\delta$. Then by (6.43), for any $z \in F_{\sigma\lambda}$,

$$\Phi_\lambda(z) \leq J_\lambda(re_\lambda) - \frac{1}{2}\|v\|_{-\lambda}^2 \leq \sigma\lambda^{1-\frac{N}{2}}$$

which ends the proof. □

In general, for any $m \in \mathbb{N}$, one can choose m functions $\varphi_\delta^j \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi_\delta^i \cap \operatorname{supp}\varphi_\delta^k = \emptyset$ if $i \neq k$, $|\varphi_\delta^j|_p = 1$ and $|\nabla\varphi_\delta^j|_2^2 < \delta$. Let $r_\delta^m > 0$ be such that $\operatorname{supp}\varphi_\delta^j \subset B_{r_\delta^m}(0)$ for $j = 1, ..., m$. Set

$$e_\lambda^j(x) = \varphi_\delta^j(\lambda^{1/2}x) \quad \text{for } j = 1, ..., m$$

and

$$H_{\lambda\delta}^m = \operatorname{span}\{e_\lambda^1, ..., e_\lambda^m\}.$$

Observe that for each $u = \sum_{j=1}^m c_j e_\lambda^j \in H_{\lambda\delta}^m$,

$$\begin{aligned} J_\lambda(u) &= \sum_{j=1}^m J_\lambda(c_j e_\lambda^j) \\ &= \lambda^{1-\frac{N}{2}} \sum_{j=1}^m I_\lambda(|c_j| e_\lambda^j). \end{aligned}$$

Set

$$\beta_\delta := \max\{|\varphi_\delta^j|_2^2 : \ j = 1, ..., m\},$$

and choose $\hat{\Lambda}_{m\delta}$ so that

$$V(\lambda^{-1/2}x) \leq \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \leq r_\delta^m \text{ and } \lambda \geq \hat{\Lambda}_{m\delta}.$$

As before, one obtains easily the following

$$\sup_{u \in H_{\lambda\delta}^m} J_\lambda(u) \leq \frac{m(p-2)}{2p(pa_0)^{2/(p-2)}}\,(2\delta)^{p/(p-2)}\,\lambda^{1-\frac{N}{2}} \tag{6.44}$$

for all $\lambda \geq \hat{\Lambda}_{m\delta}$.

Using this estimate we can prove easily the following

Lemma 6.16. *For any $m \in \mathbb{N}$ and $\sigma > 0$ there exist $\Lambda_{m\sigma} > 0$, such that, for each $\lambda \geq \Lambda_{m\sigma}$, there exists an m-dimensional subspace $F_{\lambda m} \subset E_+$ satisfying*

$$\sup_{z \in F_{\lambda m} \times E_-} \Phi_\lambda(z) \leq \sigma\, \lambda^{1-\frac{N}{2}}.$$

Proof. Choose $\delta > 0$ small so that

$$\frac{m(p-2)}{2p(pa_0)^{2/(p-2)}}\,(2\delta)^{p/(p-2)} \le \sigma,$$

and take $F_{\lambda m} = H^m_{\lambda\delta}$. Then (6.44) yields the conclusion as required. □

Proof. [Proof of Theorem 6.5] First we prove the existence. With $Y = E_+$ and $X = E_-$ the conditions (Φ_0) and (Φ_+) hold and Φ_λ possesses the linking structure of Theorem 4.5 by Lemma 6.11. This, together with Lemma 6.15, shows that for any $\sigma \in (0, \alpha_0)$ there is $\Lambda_\sigma > 0$ so that, if $\lambda \ge \Lambda_\sigma$, Φ_λ has a $(C)_{c_\lambda}$ sequence with $\kappa_\lambda \le c_\lambda \le \sigma\lambda^{1-\frac{N}{2}}$. Hence, by Lemma 6.14, there exists a critical point z_λ satisfying

$$\kappa_\lambda \le \Phi_\lambda(z_\lambda) \le \sigma\lambda^{1-\frac{N}{2}}. \tag{6.45}$$

Since $E(z_\lambda) = \Phi_\lambda(z_\lambda)$, (6.45) implies the estimate (ii). Moreover, by (H_0)

$$\sigma\lambda^{1-\frac{N}{2}} \ge \Phi_\lambda(z_\lambda) = \Phi_\lambda(z_\lambda) - \frac{1}{2}\Phi'_\lambda(z_\lambda)z_\lambda \ge \lambda\Big(\frac{\mu}{2} - 1\Big)\int_{\mathbb{R}^N} H(x, z_\lambda)$$

and we obtain (i).

We now turn to the multiplicity. Assume $H(x, z)$ is even in z. Then Φ_λ is even hence (Φ_1) holds. (Φ_2) follows from Lemma 6.11. By virtue of Lemma 6.16, for any $m \in \mathbb{N}$ and $\sigma \in (0, \alpha_0)$ there is $\Lambda_{m\sigma}$ such that for each $\lambda \ge \Lambda_{m\sigma}$, we can choose a m-dimensional subspace $F_{\lambda m} \subset E_+$ with $b := \max \Phi_\lambda(F_{\lambda m} \times E_-) < \sigma\lambda^{1-\frac{N}{2}}$. Hence, Φ_λ verifies (Φ_3) with $b < \sigma\lambda^{1-\frac{N}{2}}$ for all $\lambda \ge \Lambda_{m\sigma}$. It follows from Lemma 6.14 that Φ_λ checks the $(C)_c$ condition for all $c \in [0, b]$. Now Theorem 4.6 applies. □

6.6.3 *Proof of Theorem 6.6*

We now turn to the critical case, that is, to prove Theorem 2.2 hence Theorem 1.2. We will consider the functional Ψ_λ along the way as before.

In the following set

$$Q(x, z) = H(x, z) + \frac{1}{2^*}K(x)|z|^{2^*}$$

and

$$\tilde{Q}(x, z) = \frac{1}{2}Q_z(x, z)z - Q(x, z).$$

It follows from (H_0) and (K_0) that, for any $\delta > 0$ there are $\rho_\delta > 0$ and $c_\delta > 0$ such that

$$\frac{|Q_z(x, z)|}{|z|} \le \delta \text{ if } |z| \le \rho_\delta, \quad \frac{|Q_z(x, z)|^{N/2}}{|z|^{N/2}} \le c_\delta\tilde{Q}(x, z) \text{ if } |z| \ge \rho_\delta. \tag{6.46}$$

Lemma 6.17. *There is $\alpha_0 > 0$ independent of λ such that any $(C)_c$ sequence with $c < \alpha_0\lambda^{1-\frac{N}{2}}$ contains a convergent subsequence.*

Proof. Let $z_j = (u_j, v_j)$ be a $(C)_c$ sequence: $\Psi_\lambda(z_j) \to c$ and $(1+\|z_j\|_\lambda)\Psi'_\lambda(z_j) \to 0$. Plainly

$$\Psi_\lambda(z_j) - \frac{1}{2}\Psi'_\lambda(z_j)z_j = \lambda \int_{\mathbb{R}^N} \tilde{Q}(x, z_j), \tag{6.47}$$

and, by Lemma 6.11, $c \geq 0$ and (z_j) is bounded. We can assume without loss of generality that $z_j \rightharpoonup z$ with z solving $(\mathcal{Q}_\lambda)$. In addition, there is a subsequence (z_{j_n}) such that (6.38) holds. Define $\tilde{z}_n(x) = \eta(2|x|/n)z(x)$ where $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leq 1$, $\eta(t) = 0$ if $t \geq 2$. As before it is not difficult to check that

$$\Psi_\lambda(z_{j_n} - \tilde{z}_n) \to c - \Psi_\lambda(z) \quad \text{and} \quad \Psi'_\lambda(z_{j_n} - \tilde{z}_n) \to 0. \tag{6.48}$$

Claim: There is a constant $\alpha_0 > 0$ independent of λ such that either $z_j \to z$ or $c - \Psi_\lambda(z) \geq \alpha_0 \lambda^{1-\frac{N}{2}}$.

Write $y_n := z_{j_n} - \tilde{z}_n = y_n^d + y_n^e \in E_\lambda^d \oplus E_\lambda^e$. Then $\Psi_\lambda(y_n) \to c - \Psi_\lambda(z)$ and $\Psi'_\lambda(y_n) \to 0$ by (6.48). Similarly to (6.47), it follows from (6.48) that

$$\lambda \int_{\mathbb{R}^N} \tilde{Q}(x, y_n) \to c - \Psi_\lambda(z). \tag{6.49}$$

Noting that $y_n = (u_{j_n} - \tilde{u}_n, v_{j_n} - \tilde{v}_n)$ we set $\bar{y}_n = (u_{j_n} - \tilde{u}_n, -v_{j_n} + \tilde{v}_n)$. We have $|y_n| = |\bar{y}_n|$ and, using one after the other the fact $y_n^d \to 0$, (6.46), Hölder inequality, (6.49) and (6.35), we get for any $\delta > 0$,

$$\begin{aligned} \|y_n^e\|_\lambda^2 + o(1) &= \lambda \int_{\mathbb{R}^N} Q_z(x, y_n)\bar{y}_n \leq \lambda \int_{\mathbb{R}^N} \frac{|Q_z(x, y_n)|}{|y_n|}|\bar{y}_n||y_n| \\ &\leq o(1) + \lambda\delta|y_n|_2^2 \\ &\qquad + \lambda c'_\delta \left(\int_{|y_n| \geq \rho_\delta} \left(\frac{|Q_z(x, y_n)|}{|y_n|} \right)^{N/2} \right)^{2/N} |y_n|_{2^*}^2 \\ &\leq o(1) + \lambda\delta|y_n^e|_2^2 + \lambda c''_\delta \left(\frac{c - \Psi_\lambda(z)}{\lambda} \right)^{2/N} |y_n^e|_{2^*}^2 \\ &\leq o(1) + \frac{\delta}{b}\|y_n^e\|_\lambda^2 + C_\delta \lambda^{1-\frac{2}{N}} \left(c - \Psi_\lambda(z) \right)^{2/N} \|y_n^e\|_\lambda^2. \end{aligned} \tag{6.50}$$

Remark that $z_{j_n} - z = y_n + (\tilde{z}_n - z)$, hence $z_{j_n} - z \to 0$ if and only if $y_n^e \to 0$. Assume z_j has no convergent subsequence. Then $\liminf_{n\to\infty} \|y_n^e\|_\lambda > 0$ and $c - \Phi_\lambda(z) > 0$. Choosing $\delta = b/4$, it follows from (6.50) that

$$\frac{3}{4}\|y_n^e\|_\lambda^2 \leq o(1) + c_1 \lambda^{1-\frac{2}{N}} \left(c - \Psi_\lambda(z) \right)^{2/N} \|y_n^e\|_\lambda^2.$$

This implies that

$$1 \leq c_2 \lambda^{\frac{N}{2}-1} \left(c - \Psi_\lambda(z) \right).$$

□

Lemma 6.18. *For any $\sigma > 0$ there exists $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\sigma$, there is $e_\lambda \in E_+ \setminus \{0\}$ such that*

$$\max_{z \in F_{\sigma\lambda}} \Psi_\lambda(z) \leq \sigma \lambda^{1-\frac{N}{2}},$$

where $F_{\sigma\lambda} := \mathbb{R}e_\lambda \times E_-$.

Proof. This follows from (6.43) and that

$$\Psi_\lambda(z) \le J_\lambda(u) - \frac{1}{2}\|v\|_\lambda^2 \tag{6.51}$$

for all $z = (u, v)$. □

Lemma 6.19. *For any $m \in \mathbb{N}$ and $\sigma > 0$ there exist $\Lambda_{m\sigma} > 0$, such that, for each $\lambda \ge \Lambda_{m\sigma}$, there exists an m-dimensional subspace $F_{\lambda m} \subset E_+$ satisfying*

$$\sup_{z \in F_{\lambda m} \times E_-} \Psi_\lambda(z) \le \sigma\, \lambda^{1-\frac{N}{2}}.$$

Proof. It follows from (6.44) and (6.51). □

Proof. [Proof of Theorem 6.6] Repeating the arguments of the proof of Theorem 6.5 with Lemmas 6.14, 6.15 and 6.16 replaced respectively by Lemmas 6.17, 6.18 and 6.19 yields the desired results. □

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Chapter 7

Solutions of nonlinear Dirac equations

In this chapter we study nonlinear Dirac equations in external fields and obtain existence and multiplicity results of stationary solutions for several classes of nonlinearities modelling various types of interaction. A typical result states that if the nonlinearity is even and depends periodically on the spacial variable, the problem has infinitely many geometrically different localized solutions.

The chapter is organized as follows. In the first five sections we deal with the equations with scale potentials which are either periodic or of harmonic oscillator type. In Section 7.2 we first state the hypotheses and our main results, then formulate the variational setting and provide basic estimates on the spectrum of the linearization, and lastly prove the theorems for asymptotically quadratic nonlinearity and for superquadratic nonlinearity respectively. Section 7.6 is devoted to handle more general vector potentials. In the last section we consider existence and multiplicity of semiclassical solutions.

7.1 Relative studies

Nonlinear Dirac equations occur in the attempt to model extended relativistic particles with external fields, see [Bjorken and Drell (1965)], [Ranada (1982)], [Esteban and Séré (2002)]. In a general form, such equations are given by

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + G_\psi(x,\psi)\ ; \tag{7.1}$$

here $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_k = \frac{\partial}{\partial x_k}$, c denotes the speed of light, $m > 0$ the mass of the electron, and $\hbar$ denotes Planck's constant. Furthermore, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices whose standard form (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One verifies that $\beta = \beta^*$, $\alpha_k = \alpha_k^*$, $\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}$ and $\alpha_k\beta + \beta\alpha_k = 0$; due to these relations, the linear operator $\mathscr{H}_0 = -ic\hbar\sum_{k=1}^3 \alpha_k\partial_k\psi + mc^2\beta\psi$ is a symmetric operator, such that

$$\mathscr{H}_0^2 = -c^2\hbar^2\Delta + m^2c^4 \ .$$

A solution $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4$ of (7.1), with $\Psi(t,\cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, is a *wave function* which represents the state of a relativistic electron.

The external fields are given by the matrix potential $M(x)$, and the nonlinearity $G : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ represents a nonlinear self-coupling. We assume throughout the chapter that G satisfies $G(x, e^{i\theta}\psi) = G(x,\psi)$, for all $\theta \in [0, 2\pi]$. We are looking for stationary solutions of (7.1) which may be regarded as "particle-like solutions" (see [Ranada (1982)]): they propagate without changing their shape and thus have a soliton-like behavior.

The stationary solutions of equation (7.1) are found by the Ansatz

$$\psi(t,x) = e^{\frac{i\theta t}{\hbar}} u(x) \ ;$$

then $u : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the equation

$$-ic\hbar\sum_{k=1}^{3} \alpha_k\partial_k u + mc^2\beta\, u + M(x)u = G_u(x,u) - \theta u \ . \tag{7.2}$$

Dividing equation (7.2) by $\hbar\, c$, we are led to study equations of the form

$$-i\sum_{k=1}^{3} \alpha_k\, \partial_k u + a\,\beta\, u + \omega\, u + M(x)u = G_u(x,u) \ , \tag{7.3}$$

where $a > 0$ and $\omega \in \mathbb{R}$. We look for weak solutions which are localized in space; more precisely, the solution we find satisfy $u \in \bigcap_{2\le q<\infty} W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$.

First we consider (7.3) in the form

$$-i\sum_{k=1}^{3} \alpha_k\partial_k u + a\beta u + \omega u = F_u(x,u) \tag{7.4}$$

where $a > 0$ and $\omega \in \mathbb{R}$. In [Ranada (1982)] one can find a discussion of functions F which have been used to model various types of self-coupling. In recent years a number of papers appeared dealing with the existence and multiplicity of stationary solutions. In [Balabane, Cazenave, Douady and Merle (1988); Balabane, Cazenave and Vazquez (1990); Cazenave and Vazquez (1986); Merle (1988)] the model

$$F(u) = \frac{1}{2}H(\tilde{u}u), \quad H \in C^2(\mathbb{R},\mathbb{R}), \quad H(0) = 0 \ \text{ where } \ \tilde{u}u := (\beta u, u)_{\mathbb{C}^4} \tag{7.5}$$

was investigated. In these papers the authors obtained for $\omega \in (-a, 0)$ solutions of (7.4) of the type

$$u(x) = \begin{pmatrix} v(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iw(r)\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \end{pmatrix} . \tag{7.6}$$

This ansatz leads to a system of ODEs for $v(r), w(r)$, $r = |x|$, which can be solved using the shooting method. Of course, suitable hypotheses on H were required, and the approach depends heavily on the special form of F and the ansatz (7.6). Another model nonlinearity studied in [Finkelstein, Lelevier and Ruderman (1951); Ranada (1982)] is

$$F(u) = \frac{1}{2}|\tilde{u}u|^2 + b|\tilde{u}\alpha u|^2 \qquad \text{where } \tilde{u}\alpha u := (\beta u, \alpha u)_{\mathbb{C}^4},\ \alpha := \alpha_1\alpha_2\alpha_3$$

with $b > 0$. In [Esteban and Séré (1995)] variational methods are used for the model (7.5) provided the main additional assumption

$$H'(s) \cdot s \geq \theta H(s) \quad \text{for all } s \in \mathbb{R},\ \text{some } \theta > 1$$

holds. The authors obtain infinitely many solutions for the model (7.5) exploiting the inherent symmetry $F(u) = F(-u)$. They work on the space $E^s \subset H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ of functions of the form (7.6) and perturb the function F appropriately so that the perturbed variational integral satisfies the Palais-Smale condition. Then they apply well known variational methods to the perturbed functional on E^s. Solutions of (7.4) are obtained by carefully controlling the passage to the limit from the perturbed functionals to the unperturbed one.

The paper [Esteban and Séré (1995)] also deals with more general nonlinearities $F(u)$ where (7.5) does not hold and the ansatz (7.6) does not apply. The authors show the existence of one (nontrivial) solution provided $F \in C^2(\mathbb{C}^4, \mathbb{R})$ satisfies various growth and sign conditions. An example of such a general nonlinearity is the function

$$F(u) = \mu\left(|u\tilde{u}|^{\tau_1} + b|\tilde{u}\alpha u|^{\tau_2}\right), \quad \tau_1, \tau_2 \in (1,\ 3/2),\ \mu, b > 0.$$

Here one cannot work on the space E^s and the Palais-Smale condition does not hold even for the perturbations, due to the invariance of (7.4) under translations. The idea of [Esteban and Séré (1995)] is to produce a Palais-Smale sequence by a linking argument and then to use concentration compactness arguments in order to obtain a solution. [Esteban and Séré (1995)] does not contain a multiplicity result in the general case. The problem here is that the solutions are not obtained as strong limits from the Palais-Smale sequence but only as weak limits (after suitable translations). Thus even when one has different linkings producing different Palais-Smale sequences it is not clear how to distinguish the weak limits.

Motivated by [Esteban and Séré (1995)] we investigate the Dirac equation by using some of the critical point theorems from Chapter 4. The class of nonlinearities which we treat differs in two ways from those in the other paper mentioned above. First, $F = F(x, u)$ may depend on x and is periodic in each of the variables x_1, x_2, x_3. Second, $F(x, u)$ is asymptotically quadratic or superquadratic in u as $|u| \to \infty$. Consequently, $F(x, u) \to \infty$ as $|u| \to \infty$ which excludes the Lorentz invariant nonlinearities mentioned above. There $F(u)$ may vanish even for large values of $|u|$.

We obtain infinitely many solutions if F is even, not only for superquadratic F but also in the asymptotically quadratic case. We only require $|\omega| < a$, not $-a < \omega < 0$ as in the other papers. The multiplicity result has to be interpreted carefully. As a consequence of the periodicity of $F(x, u)$ in x_1, x_2, x_3, given a solution u any translation $k * u = u(\cdot + k), k \in \mathbb{Z}^3$, is also a solution. Thus there exists a $\mathbb{Z}^3$-orbit of solutions. The infinitely many solutions which we obtain correspond to different $\mathbb{Z}^3$-orbits. Observe that, when F is independent of x then one solution u generates a 3-dimensional manifold of solutions $y * u = u(\cdot + y), y \in \mathbb{R}^3$, consisting of infinitely many $\mathbb{Z}^3$-orbits. In this case we do not obtain any additional solutions.

The $\mathbb{Z}^3$-periodicity has another effect: the functional associated to the problem does not satisfy the Palais-Smale condition. In [Coti-Zelati, Ekeland and Séré (1990)] a weaker version of the Palais-Smale condition was introduced for a $\mathbb{Z}$-period problem; see also [Séré (1992)]. It was shown that this condition suffices to yield a deformation lemma. However, in these paper the functionals are of mountain pass type which is not the case here. In fact, our functionals are of strongly indefinite.

The above mentioned results also apply to the more general equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + (V(x) + a)\beta u + \omega u = F_u(x, u) \tag{7.7}$$

with a potential V periodic in the x_k-variable. We also have results if neither V nor F are periodic provided there is some control on $V(x)$ as $|x| \to \infty$ which excludes the case that V is constant. Here we obtain infinitely many solutions even if F is independent of x.

7.2 Existence results for scalar potentials

We rewrite the equation (7.4) as

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + \omega u = F_u(x, u) \tag{D}$$

with $a > 0$ and shall always assume

(ω) $\omega \in (-a, a)$.
(F_0) $F \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, [0, \infty))$
(F_1) $F(x, u)$ is 1-periodic in x_k, $k = 1, 2, 3$.

This includes the case where $F \in C^1(\mathbb{C}^4, [0, \infty))$ does not depend on x. For our first results we also require

(F_2) $F_u(x, u) = o(|u|)$ as $u \to 0$ uniformly in $x \in \mathbb{R}^3$.

Concerning the behavior of F as $|u| \to \infty$ we begin with the asymptotically quadratic case. Setting

$$\omega_0 := \min\{a + \omega,\ a - \omega\} \quad \text{and} \quad \hat{F}(x, u) := \frac{1}{2}F_u(x, u) \cdot u - F(x, u).$$

we require:

(F_3) There exists $b > a + \omega$ such that $|F_u(x,u) - bu| \cdot |u|^{-1} \to 0$ as $|u| \to \infty$ uniformly in x.

(F_4) $\hat{F}(x,u) \geq 0$, and there exists $\delta_1 \in (0, \omega_0)$ such that $\hat{F}(x,u) \geq \delta_1$ whenever $|F_u(x,u)| \geq (\omega_0 - \delta_1)|u|$.

Theorem 7.1 ([Bartsch and Ding (2006II)]). *Let (ω) and $(F_0) - (F_4)$ be satisfied. Then (D) has at least one nontrivial solution $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$. (D) has infinitely many geometrically distinct solutions $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$ if in addition to the above assumptions F is even in u.*

Here two solutions u_1 and u_2 are said to be geometrically distinct if $k * u_1 \neq u_2$ for all $k \in \mathbb{Z}^3$ where $(k * u)(x) = u(x + k)$.

Next we consider the super-quadratic case where we assume:

(F_5) $F(x,u)|u|^{-2} \to \infty$ as $|u| \to \infty$ uniformly in x.

(F_6) $\hat{F}(x,u) > 0$ if $u \neq 0$, $\hat{F}(x,u) \to \infty$ as $|u| \to \infty$ uniformly in x, and there are $\sigma > 3$ and $r, c_1 > 0$ such that, $|F_u(x,u)|^\sigma \leq c_1 \hat{F}(x,u)|u|^\sigma$ if $|u| \geq r$.

Theorem 7.2 ([Bartsch and Ding (2006II)]). *Let (ω), $(F_0) - (F_2)$ and (F_5), (F_6) be satisfied. Then (D) has at least one nontrivial solution $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$. If moreover F is even in u, then (D) has infinitely many geometrically distinct solutions $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$.*

Now we re-denote the equation (7.7) by:

$$-i \sum_{k=1}^{3} \alpha_k \partial_k u + (V(x) + a)\beta u + \omega u = F_u(x,u). \qquad (D_V)$$

We are interested in the influence of the potential $V : \mathbb{R}^3 \to \mathbb{R}$ on the existence of solutions. First we consider periodic potentials.

(V_1) $V \in C^1(\mathbb{R}^3, [0, \infty))$, and $V(x)$ is 1-periodic in x_k for $k = 1, 2, 3$.

The hypotheses (F_3) and (F_4) will be replaced by

(F_3') There exists $b \in C^1(\mathbb{R}^3, \mathbb{R})$ with $|F_u(x,u) - b(x)u|\,|u|^{-1} \to 0$ as $|u| \to \infty$ uniformly in x, and $\inf b(\mathbb{R}^3) > \sup V(\mathbb{R}^3) + a + \omega$.

(F_4') $\hat{F}(x,u) > 0$ if $u \neq 0$, and $\hat{F}(x,u) \to \infty$ as $|u| \to \infty$ uniformly in x.

Theorem 7.3 ([Bartsch and Ding (2006II)]). *Let (ω), (V_1) and $(F_0) - (F_2)$, (F_3'), (F_4') be satisfied. Then (D_V) has at least one nontrivial solution $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$. If in addition F is even with respect to u then (D_V) has infinitely many geometrically distinct solutions $u \in \bigcap_{\tau \geq 2} W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$.*

Here are some examples where the assumptions apply.

Example 7.1. a) $F(x,u) = \frac{1}{2}b(x)|u|^2\left(1 - \frac{1}{\ln(e+|u|)}\right)$.

b) $F(x,u) = b(x)\varphi(\frac{1}{2}|u|^2)$ where $\varphi : [0,\infty) \to [0,\infty)$ is of class C^2 with $\varphi(0) = \varphi'(0) = 0$, and $\varphi'(s) \to 1$ as $s \to \infty$, $\varphi''(s) \geq 0$.

c) $F_u(x,u) = f(x,|u|)u$, where $f(x,s)$ is even in s; $f(x,s) \to 0$ as $s \to 0$ uniformly in x; $f(x,s)$ is non-decreasing for $s \in [0,\infty)$; and $f(x,s) \to b(x)$ as $s \to \infty$.

Theorem 7.4 ([Bartsch and Ding (2006II)]). *Let (ω), (V_1) and $(F_0) - (F_2)$, (F_5), (F_6) be satisfied. Then (D_V) has at least one nontrivial solution $u \in \bigcap_{\tau\geq 2} W^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4)$. If F is even in u then (D_V) has infinitely many geometrically distinct solutions $u \in \bigcap_{\tau\geq 2} W^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4)$.*

Theorem 7.2 is a special case of Theorem 7.4. Comparing Theorem 7.1 and Theorem 7.3 one sees that assumption (F_4') is somewhat stronger than (F_4). We also have some explicit examples of possible nonlinearities.

Example 7.2. a) $F(x,u) = a(x)\Big(|u|^2\ln(1+|u|) - \frac{1}{2}|u|^2 + |u| - \ln(1+|u|)\Big)$.

b) $F(x,u) = a(x)\Big(|u|^\mu + (\mu-2)|u|^{\mu-\epsilon}\sin^2\left(\frac{|u|^\epsilon}{\epsilon}\right)\Big)$ where $\mu \in (2,3)$ and $0 < \epsilon < \mu - 2$.

c) (F_5) and (F_6) hold if there are $q > 2$ and $\kappa > 3/2$ such that $0 < qF(x,u) \leq F_u(x,u)\cdot u$ if $u \neq 0$, and $|F_u(x,u)|^\kappa \leq c_1\left(1 + F_u(x,u)\cdot u\right)$.

Next we consider potentials of the harmonic oscillator type:

(V_2) $V \in C^1(\mathbb{R}^3,\mathbb{R})$; for each $b > 0$ the set $V^b := \{x \in \mathbb{R}^3 : V(x) \leq b\}$ has finite Lebesgue measure.

This hypothesis is satisfied if $V(x) \to \infty$ as $|x| \to \infty$, for instance.

Theorem 7.5 ([Bartsch and Ding (2006II)]). *Let (V_2), (F_0) and (F_5), (F_6) be satisfied. Then (D_V) has at least one nontrivial solution $u \in \bigcap_{\tau\geq 2} W^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4)$. If moreover F is also even in u then (D_V) has infinitely many solutions $u \in \bigcap_{\tau\geq 2} W^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4)$.*

Remark 7.1. In Theorem 7.5 we only considered superquadratic nonlinearities. With the methods developed in this chapter it is easily possible to consider asymptotically quadratic nonlinearities, and to obtain multiple solutions if the asymptotic term $b(x)$ is large enough. Observe that in Theorem 7.5 we do not make any restriction on the number ω, and we do not need assumptions like (F_4) except for F being even. Moreover, the proof will show that in the even case there exists a sequence of solutions having the energy unbounded.

7.3 Variational setting

For $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R})$ the operator $A := -i\sum_{k=1}^3 \alpha_k \partial_k + (V(x)+a)\,\beta$ is a selfadjoint operator in $L^2 = L^2(\mathbb{R}^3, \mathbb{C}^4)$ (cf. [Dautray and Lions (1990)]). It is unbounded from above and from below. In order to investigate the spectrum of A we consider

$$A^2 = -\Delta + (V+a)^2 + i\sum_{k=1}^3 \beta\alpha_k\partial_k V.$$

Let $\sigma(S)$, $\sigma_d(S)$, $\sigma_e(S)$ and $\sigma_c(S)$ denote, respectively, the spectrum, the discrete spectrum (i. e. the set of eigenvalues of finite multiplicity), the essential spectrum and the continuous spectrum of a self-adjoint operator S on L^2.

Lemma 7.1. a) *If $V \equiv 0$, then $\sigma(A^2) = [a^2, \infty)$.*

b) *If (V_1) holds then $\sigma(A^2) \subset [a^2, \infty)$.*

c) *If (V_2) holds then $\sigma(A^2) = \sigma_d(A^2) = \{\mu_n : n \in \mathbb{N}\}$ with $0 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots$ and $\mu_n \to \infty$.*

Proof. a) is obvious. b) follows from the inequality

$$\begin{aligned}(A^2u, u)_{L^2} = &\left(\Big(-i\sum_{k=1}^3 \alpha_k\partial_k + V\beta\Big)u,\ \Big(-i\sum_{k=1}^3 \alpha_k\partial_k + V\beta\Big)u\right)_{L^2}\\ &+ a^2(u,u)_{L^2} + 2a(Vu,u)_{L^2}\\ \ge\ & a^2(u,u)_{L^2} + 2a(Vu,u)_{L^2}.\end{aligned}$$

c) Suppose (V_2) holds and define

$$W(x) := (V(x)+a)^2 + i\sum_{k=1}^3 \beta\alpha_k\partial_k V(x).$$

Then we have for any $b > 0$

$$C_b := \Big\{x \in \mathbb{R}^3 : \sup_{|\xi|=1} (W(x)\xi, \bar{\xi})_{\mathbb{C}^4} < b\Big\} \subset V^b.$$

Setting $W_b := W - b$, $W_b^+ = \max\{0, W_b\}$, $W_b^- = \min\{0, W_b\}$ and $S_b = -\Delta + (a^2 + b) + W_b^+$ we have $A^2 = S_b + W_b^-$. Using $C_b \subset V^b$ it is easy to check that W_b^- is compact relative to S_b (cf. [Bartsch, Pankov and Wang (2001)]). Hence, by a theorem of Weyl

$$\sigma_e(A^2) = \sigma_e(S_b) \subset \sigma(S_b) \subset [a^2 + b,\ \infty).$$

Since $b > 0$ is arbitrary it follows that $\sigma(A^2) = \sigma_d(A^2)$. Finally, since A^2 is unbounded from above, $\mu_n \to \infty$. □

The domain $\mathscr{D} = \mathscr{D}(A)$ of A is a Hilbert space with inner product

$$(u,v)_{\mathscr{D}} = (Au, Av)_{L^2} + (u,v)_{L^2}.$$

Lemma 7.2. a) *If (V_1) is satisfied, then $\mathscr{D} = H^1(\mathbb{R}^3, \mathbb{C}^4)$ with equivalent norms.*

b) *If (V_2) is satisfied, then $\mathscr{D}$ embeds continuously into $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and compactly into $L^\tau(\mathbb{R}^3, \mathbb{C}^4)$ for all $\tau \in [2, 6)$.*

Proof. a) is clear. For b) it suffices to prove $\mathscr{D} \hookrightarrow L^2(\mathbb{R}^3, \mathbb{C}^4)$ compactly. Let $(e_n)_{n\in\mathbb{N}}$ be an orthornormal basis of eigenfunctions of A^2 associated to the eigenvalues μ_n, $n \in \mathbb{N}$, and set $L_k = \operatorname{span}\{e_1, \cdots, e_k\}$. Let $P_k : \mathscr{D} \to L_k$ denote the orthogonal projection. Consider a weakly converging sequence $u_n \rightharpoonup u$ in $\mathscr{D}$, and define $w_n = u_n - u$ and $C := \sup_n \|w_n\|^2_{\mathscr{D}}$. Given $\varepsilon > 0$ we choose $k \in \mathbb{N}$ so that $C/\mu_k < \varepsilon/2$. Since $P_k w_n \to 0$ as $n \to \infty$ there exists $n_0 \in \mathbb{N}$ so that $\|P_k w_n\|^2_{\mathscr{D}} < \varepsilon/2$ for $n \geq n_0$. Therefore we have

$$|w_n|_2^2 = |P_k w_n|_2^2 + |(I - P_k)w_n|_2^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \geq n_0$. This proves that $u_n \to u$ in L^2. □

Now we consider the operator A. Let $(E_\gamma)_{\gamma\in\mathbb{R}}$ and $(F_\gamma)_{\gamma\geq 0}$ denote the spectral families of A and A^2, respectively. Recall that

$$F_\gamma = E_{\gamma^{1/2}} - E_{-\gamma^{1/2}-0} = E_{[-\gamma^{1/2},\gamma^{1/2}]} \qquad \text{for all } \gamma \geq 0; \tag{7.8}$$

see (3.96) in Chapter VIII of [Dautray and Lions (1990)].

Lemma 7.3. a) *If* $V \equiv 0$ *then* $\sigma(A) = (-\infty, -a] \cup [a, \infty)$.

b) *If* (V_1) *holds, then* $\sigma(A) = \sigma_c(A) \subset (-\infty, -a] \cup [a, \infty)$ *and* $\inf \sigma(|A|) \leq a + \sup V(\mathbb{R}^3)$.

c) *If* (V_2) *holds then* $\sigma(A) = \sigma_d(A) = \left\{\pm\mu_n^{1/2} :\ n \in \mathbb{N}\right\}$.

Proof. a) can be obtained directly by Fourier analysis (cf. [Esteban and Séré (1995)]).

b) Assume (V_1) holds. Using (7.8) and Lemma 7.1b) we obtain

$$\dim (E_{[-\gamma^{1/2},\, \gamma^{1/2}]}L^2) = \dim (F_\gamma L^2) = 0 \quad \text{for } 0 \leq \gamma < a^2,$$

hence $\sigma(A) \subset \mathbb{R} \setminus (-a, a)$. If A has an eigenvalue η with eigenfunction $u \neq 0$ then $A^2 u = \eta^2 u$, so η^2 is an eigenvalue of A^2 contradicting the well-known fact that $\sigma(A^2) = \sigma_c(A^2)$ (cf. [Reed and Simon (1978)]). It follows that A has only continuous spectrum. Finally, since $\sigma\big(-i\sum_{k=1}^3 \alpha_k\partial_k\big) = \mathbb{R}$ there exists a sequence $u_n \in H^1$ with $|u_n|_2 = 1$ and $\left|-i\sum_{k=1}^3 \alpha_k\partial_k u_n\right|_2 \to 0$. This implies

$$|Au_n|_2 \leq \left|-i\sum_{k=1}^{3} \alpha_k\partial_k u_n\right|_2 + |(V + a)u_n|_2 \leq o(1) + a + \sup V(\mathbb{R}^3)$$

and b) follows.

c) By Lemma 7.1c), for all $\gamma \geq 0$ we have

$$\dim(E_{[-\gamma^{1/2},\gamma^{1/2}]}L^2) = \dim(F_\gamma L^2) < \infty,$$

hence $\sigma(A) = \sigma_d(A) \subset \left\{\pm\mu_n^{1/2} :\ n \in \mathbb{N}\right\}$. For $\gamma = \mu_n$ we have

$$0 \neq F_\gamma - F_{\gamma-0} = (E_{\gamma^{1/2}} - E_{\gamma^{1/2}-0}) + (E_{-\gamma^{1/2}} - E_{-\gamma^{1/2}-0}).$$

Assume $\gamma^{1/2}$ is an eigenvalue of A, so $E_{\gamma^{1/2}} - E_{\gamma^{1/2}-0} \neq 0$. Let u be a corresponding eigenfunction and set

$$\mathscr{J} := \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

where I_2 is the unit matrix in $\mathbb{C}^2$. Then

$$\alpha_k \mathscr{J} = -\mathscr{J}\alpha_k \quad \text{for } k = 1,2,3 \text{ and } \beta\mathscr{J} = -\mathscr{J}\beta.$$

Setting $v = \mathscr{J}u$ one has

$$Av = A\mathscr{J}u = -\mathscr{J}Au = -\mathscr{J}\gamma^{1/2}u = -\gamma^{1/2}v,$$

so $-\gamma^{1/2}$ is also an eigenvalue of A. Similarly, if $-\gamma^{1/2}$ is an eigenvalue of A then $\gamma^{1/2}$ is an eigenvalue of A. □

Observe that we have an orthogonal decomposition

$$L^2 = L^- \oplus L^0 \oplus L^+, \quad u = u^- + u^0 + u^+,$$

such that A is negative definite on L^-, positive definite on L^+, and vanishes on L^0. Clearly, $L^0 = \{0\}$ if $V(x) \equiv 0$ or if (V_1) holds.

Let $E = \mathscr{D}(|A|^{1/2})$ be the Hilbert space equipped with the inner product

$$(u,v) = (|A|^{1/2}u,\, |A|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|u\| = (u,u)^{1/2}$. There is an induced decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad \text{where} \quad E^\pm = E \cap L^\pm, \quad E^0 = E \cap L^0,$$

which is orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

Lemma 7.4. a) *If* (V_1) *holds then* $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ *with equivalent norms, and* $a|u|_2^2 \leq \|u\|^2$*;*

b) *If* (V_2) *holds then* $E \hookrightarrow H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$*, and* E *embeds compactly into* $L^\tau(\mathbb{R}^3, \mathbb{C}^4)$ *for all* $\tau \in [2,3)$.

Proof. The lemma follows easily from Lemma 7.2 and an analysis of interpolation spaces. In fact, using the (complex) interpolation $[\cdot,\cdot]_\theta$ (see [Reed and Simon (1978)]) we have $E = [\mathscr{D}, L^2]_{1/2}$. By Lemma 7.2, if (V_1) holds then

$$[\mathscr{D}, L^2]_{1/2} \cong [H^1, L^2]_{1/2} = H^{1/2},$$

and if (V_2) holds then the embedding

$$[\mathscr{D}, L^2]_{1/2} \hookrightarrow [H^1, L^2]_{1/2} = H^{1/2}$$

is continuous. Moreover in the case of (V_2), using Lemma 7.3c) and the proof of Lemma 7.2b) one sees that E embeds compactly into L^τ for $\tau \in [2,3)$. □

The solutions of the equations (D) and (D_V) will be obtained as critical points of the functional

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2 + \omega|u|_2^2\right) - \int_{\mathbb{R}^3} F(x,u), \tag{7.9}$$

defined on E. Indeed, let $A = U|A|$ denote the polar decomposition of A where $U = (1 - E_0) - E_{-0}$. If $u \in E$ is a critical point of Φ then for all $\varphi \in C_0^\infty$

$$\begin{aligned} 0 &= (u^+ - u^-, \varphi) + \omega(u,\varphi)_{L^2} - \int_{\mathbb{R}^3} F_u(x,u)\varphi \\ &= (u, A\varphi)_{L^2} + \omega(u,\varphi)_{L^2} - \int_{\mathbb{R}^3} F_u(x,u)\varphi \\ &= (u, (A+\omega)\varphi)_{L^2} - \int_{\mathbb{R}^3} F_u(x,u)\varphi, \end{aligned}$$

hence u is a weak solution of (D) or (D_V). Now a bootstrap argument (see [Esteban and Séré (1995)]) yields $u \in W^{1,\tau}(\mathbb{R}^3, \mathbb{C}^4)$ for all $\tau \geq 2$.

7.4 The asymptotically quadratic case

In this section we prove Theorem 7.1 and Theorem 7.3. We begin with the proof of Theorem 7.3. Recall that the functional Φ defined on the space $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4) = E^- \oplus E^+$, given by (7.9):

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2 + \omega|u|_2^2\right) - \Psi(u) \qquad \text{where } \Psi(u) = \int_{\mathbb{R}^3} F(x,u).$$

In order to apply the critical point theorems from Chapter 4 we set $X = E^-$, $Y = E^+$, and $S = X^*$.

First we observe that by (ω) and Lemma 7.4

$$\frac{a - |\omega|}{a}\|u^+\|^2 \leq \left(\|u^+\|^2 \pm \omega|u^+|_2^2\right) \leq \frac{a+|\omega|}{a}\|u^+\|^2 \tag{7.10}$$

and

$$\frac{a - |\omega|}{a}\|u^-\|^2 \leq \left(\|u^-\|^2 \pm \omega|u^-|_2^2\right) \leq \frac{a+|\omega|}{a}\|u^-\|^2. \tag{7.11}$$

Lemma 7.5. Ψ *is weakly sequentially lower semicontinuous and* Φ' *is weakly sequentially continuous. Moreover, there is* $\zeta > 0$ *such that for any* $c > 0$*:*

$$\|u\| < \zeta\|u^+\| \quad \textit{for all} \quad u \in \Phi_c.$$

Proof. The first conclusion follows easily because $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ with equivalent norms, so E embeds continuously into $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [2,3]$ and compactly into $L^q_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $q \in [1,3)$. Since $F \geq 0$, (7.10) and (7.11) imply

$$c \leq \frac{a+|\omega|}{2a}\|u^+\|^2 - \frac{a-|\omega|}{2a}\|u^-\|^2,$$

if $\Phi(u) \geq c$. This yields

$$\frac{a-|\omega|}{2a}\|u\|^2 < \frac{a+|\omega|}{a}\|u^+\|^2,$$

and we obtain the second conclusion. □

Lemma 7.6. *There is* $\rho > 0$ *such that* $\kappa := \inf \Phi(\partial B_\rho \cap E^+) > 0$.

Proof. Choosing $q \in (2, 3)$, it follows from the assumptions that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $F(x,u) \le \varepsilon |u|^2 + C_\varepsilon |u|^q$ for all (x,u). Therefore,

$$\Psi(u) \le \varepsilon |u|_2^2 + C_\varepsilon |u|_q^q \le C(\varepsilon \|u\|^2 + C_\varepsilon \|u\|^q)$$

for all $u \in E$. The desired conclusion now follows easily from (7.10) and (7.11). □

As a consequence of Lemma 7.3 we have

$$a \le \inf \sigma(A) \cap [0,\infty) \le a + \sup V(\mathbb{R}^3).$$

We choose a number γ such that

$$a + \sup V(\mathbb{R}^3) < \gamma < \inf b(\mathbb{R}^3) - \omega. \tag{7.12}$$

Since A is invariant under the action of $\mathbb{Z}^3$ by (V_1), the subspace $Y_0 := (E_\gamma - E_0)L^2$ is infinite-dimensional, and

$$(a+\omega)|u|_2^2 \le \|u\|^2 + \omega |u|_2^2 \le (\gamma + \omega)|u|_2^2 \quad \text{for all } u \in Y_0. \tag{7.13}$$

Let $(\gamma_n)_{n\in\mathbb{N}} \subset \sigma(A)$ satisfy $\gamma_0 := a < \gamma_1 < \gamma_2 < \cdots \le \gamma$. For each $n \in \mathbb{N}$, take an element $e_n \in (E_{\gamma_n} - E_{\gamma_{n-1}})L^2$ with $\|e_n\| = 1$ and define $Y_n := \operatorname{span}\{e_1, \ldots, e_n\}$, $E_n := E^- \oplus Y_n$.

Lemma 7.7. $\sup \Phi(E_n) < \infty$ *for each* $n \in \mathbb{N}$, *and there is a sequence* $R_n > 0$ *such that* $\sup \Phi(E_n \setminus B_n) < \inf \Phi(B_\rho)$ *where* $B_n = \{u \in E_n : \|u\| \le R_n\}$.

Proof. By (7.13) and the form of Φ it is obvious that $\sup \Phi(E_n) < \infty$. For $n \in \mathbb{N}$ fixed we now show that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in E_n$. Suppose to the contrary that there exists $M > 0$ and a sequence $u_j \in E_n$ with $\|u_j\| \to \infty$ and $\Phi(u_j) \ge -M$ for all j. Then the normalized sequence $v_j := u_j / \|u_j\|$ satisfies (up to a subsequence) $v_j \rightharpoonup v$, $v_j^- \rightharpoonup v^-$, $v_j^+ \to v^+ \in Y_n$ and

$$\frac{\Phi(u_j)}{\|u_j\|^2} = \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega |v_j|_2^2\right) - \int_{\mathbb{R}^3} \frac{F(x,u_j)}{\|u_j\|^2} \ge \frac{-M}{\|u_j\|^2} = o(1)\,. \tag{7.14}$$

Using (7.11) we obtain as $j \to \infty$:

$$\begin{aligned} o(1) = \; -\frac{M}{\|u_j\|^2} &\le \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega |v_j|_2^2\right) \\ &= \|v_j^+\|^2 - \frac{1}{2}\|v_j\|^2 + \frac{\omega}{2}|v_j|_2^2 \\ &\le \|v_j^+\|^2 + \frac{1}{2}\frac{|\omega| - a}{a}\|v_j^-\|^2. \end{aligned}$$

Thus v_j^+ is bounded away from 0 and therefore $v^+ \ne 0$. We define

$$R(x,u) := F(x,u) - \frac{1}{2} b(x) u \cdot u \quad \text{and} \quad b_0 := \inf b(\mathbb{R}^3)$$

where $b:\mathbb{R}^3\to\mathbb{R}$ is from (F_3'). Then we have $F(x,u)\le c|u|^2$, $R(x,u)\,|u|^{-2}\to 0$ as $|u|\to\infty$, and

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\left(\|u^+\|^2-\|u^-\|^2+\omega|u|_2^2-\int_{\mathbb{R}^3}b(x)|u|^2\right)-\int_{\mathbb{R}^3}R(x,u)\\ &\le \frac{1}{2}\left(\|u^+\|^2+\omega|u^+|_2^2\right)-\frac{a-|\omega|}{2a}\|u^-\|^2-\frac{b_0}{2}|u|_2^2-\int_{\mathbb{R}^3}R(x,u)\end{aligned}\tag{7.15}$$

for $u\in E$. By (7.13), (7.12) and $v^+\neq 0$ there holds

$$\begin{aligned}&\left(\|v^+\|^2+\omega|v^+|_2^2\right)-\frac{a-|\omega|}{a}\|v^-\|^2-b_0|v|_2^2\\ \le&\ -(b_0-\gamma-\omega)|v^+|_2^2-\frac{a-|\omega|}{a}\|v^-\|^2\\ <&\ 0,\end{aligned}$$

hence, there is a bounded domain $\Omega\subset\mathbb{R}^3$ such that

$$\left(\|v^+\|^2+\omega|v^+|_2^2\right)-\frac{a-|\omega|}{a}\|v^-\|^2-b_0\int_\Omega|v|^2\,<\,0.\tag{7.16}$$

It follows from Lebesgue's dominated convergence theorem that

$$\lim_{j\to\infty}\int_\Omega\frac{R(x,u_j)}{\|u_j\|^2}=\lim_{j\to\infty}\int_\Omega\frac{R(x,u_j)|v_j|^2}{|u_j|^2}=0.$$

Thus, using (7.14)–(7.16) we obtain

$$\begin{aligned}0&\le\liminf_{j\to\infty}\left(\frac{1}{2}\left(\|v_j^+\|^2-\|v_j^-\|^2+\omega|v_j|_2^2\right)-\int_\Omega\frac{F(x,u_j)}{\|u_j\|^2}\right)\\ &\le\frac{1}{2}\left(\|v^+\|^2+\omega|v^+|_2^2\right)-\frac{a-|\omega|}{2a}\|v^-\|^2-\frac{b_0}{2}\int_\Omega|v|^2\\ &<0,\end{aligned}$$

a contradiction. □

As a consequence, we have

Lemma 7.8. $\Phi|_{\partial Q}\le 0$ *where* $Q:=\{u=u^-+se_1:\ u^-\in E^-, s\ge 0, \|u\|\le R_1\}$.

Proof. By our assumptions we have $\Psi(u)\ge 0$. Thus

$$\Phi(u^-)=-\frac{1}{2}\left(\|u^-\|^2-\omega|u^-|_2^2\right)-\Psi(u^-)\le-\frac{1}{2}(a-\omega)|u^-|_2^2-\Psi(u^-)\le 0$$

which, together with Lemma 7.7, implies the lemma. □

Lemma 7.9. *If* (ω), $(F_0)-(F_2)$, (F_3'), *and* (F_4') *hold then any* $(C)_c$*-sequence is bounded.*

Proof. Let $(u_j) \subset E$ be a $(C)_c$-sequence:

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|)\Phi'(u_j) \to 0.$$

It follows from (F_3'), (F_4') that for j large

$$C_0 \geq \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^3} \hat{F}(x, u_j). \tag{7.17}$$

Assume by contradiction that $\|u_j\| \to \infty$ and set $v_j = u_j/\|u_j\|$. Then $|v_j|_s \leq \gamma_s$ for all $s \in [2, 3]$. It follows from (7.10) – (7.11) that

$$\begin{aligned}\Phi'(u_j)(u_j^+ - u_j^-) &= \|u_j\|^2 \left(\|v_j\|^2 - \omega|v_j|_2^2 - \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \right) \\ &\geq \|u_j\|^2 \left(\frac{a - |\omega|}{a} - \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \right).\end{aligned}$$

Thus

$$\liminf_{j\to\infty} \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \ \geq \ \ell := \frac{a - |\omega|}{a}. \tag{7.18}$$

As before we set

$$h(r) := \inf \left\{ \hat{F}(x, u) : x \in \mathbb{R}^3 \text{ and } u \in \mathbb{C}^4 \text{ with } |u| \geq r \right\},$$

$$\Omega_j(\rho, r) = \left\{ x \in \mathbb{R}^3 : \rho \leq |u_j(x)| < r \right\}$$

and

$$c_\rho^r := \inf \left\{ \frac{\hat{F}(x, u)}{|u|^2} : x \in \mathbb{R}^3 \text{ and } u \in \mathbb{C}^4 \text{ with } \rho \leq |u| \leq r \right\}.$$

By (F_4'), $h(r) \to \infty$ as $r \to \infty$ and by definition

$$\hat{F}(x, u_j(x)) \geq c_\rho^r |u_j(x)|^2 \quad \text{for all } x \in \Omega_j(\rho, r).$$

It follows from (7.17) that

$$C_0 \geq \int_{\Omega_j(0,\rho)} \hat{F}(x, u_j) + c_\rho^r \int_{\Omega_j(\rho,r)} |u_j|^2 + h(r)|\Omega_j(r, \infty)|.$$

Observe that $|\Omega_j(b, \infty)| \leq C_0/h(r) \to 0$ as $r \to \infty$ uniformly in j, and, for any fixed $0 < \rho < r$,

$$\int_{\Omega_j(\rho,r)} |v_j|^2 = \frac{1}{\|u_j\|^2} \int_{\Omega_j(\rho,r)} |u_j|^2 \leq \frac{C_0}{c_\rho^r \|u_j\|^2} \to 0$$

as $j \to \infty$.

Let $0 < \varepsilon < \ell/3$. By (F_2) there is $\rho_\varepsilon > 0$ such that $|F_u(x, u)| < \frac{\varepsilon}{\gamma_2}|u|$ for all $|u| \leq \rho_\varepsilon$, consequently,

$$\int_{\Omega_j(0,\rho_\varepsilon)} \frac{|F_u(x, u_j)|}{|u_j|} |v_j| \, |v_j^+ - v_j^-| \leq \frac{\varepsilon}{\gamma_2} |v_j|_2^2 \leq \varepsilon$$

for all j. Recall that, by $(F_3') - (F_4')$, $|F_u(x,u)| \le c_1|u|$ for all (x,u). Using Hölder inequality we can take r_ε large so that

$$\begin{aligned}\int_{\Omega_j(r_\varepsilon,\infty)} \frac{|F_u(x,u_j)|}{|u_j|}|v_j^+ - v_j^-|\,|v_j| &\le c_1 \int_{\Omega_j(r_\varepsilon,\infty)} |v_j^+ - v_j^-|\,|v_j| \\ &\le c_1|\Omega_j(r_\varepsilon,\infty)|^{1/6}|v_j^+ - v_j^-|_2|v_j|_3 \\ &\le c_1|\Omega_j(r_\varepsilon,\infty)|^{1/6}\gamma_2\gamma_3 \\ &< \varepsilon\end{aligned}$$

for all j. Moreover, there is j_0 such that

$$\begin{aligned}\int_{\Omega_j(\rho_\varepsilon,r_\varepsilon)} \frac{|F_u(x,u_j)|}{|u_j|}\,|v_j|\,|v_j^+ - v_j^-| &\le c_1 \int_{\Omega_j(\rho_\varepsilon,r_\varepsilon)} |v_j^+ - v_j^-|\,|v_j| \\ &\le c_1|v_j|_2\left(\int_{\Omega_j(\rho_\varepsilon,r_\varepsilon)} |v_j|^2\right)^{1/2} \\ &< \varepsilon\end{aligned}$$

for all $j \ge j_0$. Therefore, for $j \ge j_0$,

$$\int_{\mathbb{R}^N} \frac{|F_u(x,u_j)\,(v_j^+ - v_j^-)}{\|u_j\|} < 3\varepsilon < \ell$$

which contradicts (7.18). □

Proof. [Proof of Theorem 7.3 (Existence)] With $X = E^-$ and $Y = E^+$ the conditions (Φ_0), (Φ_+) hold by Lemma 7.5. Together with Lemma 7.6 and Lemma 7.8 we have all the assumptions of Theorem 4.5 verified. Therefore, there exists a sequence (u_m) satisfying $\Phi(u_m) \to c \ge \kappa$ and $(1+\|u_m\|)\Phi'(u_m) \to 0$. By Lemma 7.9, (u_m) is bounded, hence $\Phi'(u_m) \to 0$. Now by the concentration compactness principle (cf. [Lions (1984)]) and the $\mathbb{Z}^3$-invariance of Φ, a standard argument shows that there is $u \ne 0$ such that $\Phi'(u) = 0$. □

Now we turn to the multiplicity. We start with to discuss further the $(C)_c$-sequence $(u_j) \subset E$. By Lemma 7.9 it is bounded, hence, without loss of generality, we may assume that $u_j \rightharpoonup u$. Plainly u is a critical point of Φ. Set $u_j^1 = u_j - u$. We have similarly to Lemma 6.7 the following

Lemma 7.10. *Under the assumptions of Lemma 7.9, one has, along a subsequence as $j \to \infty$,*

1) $\Phi(u_j^1) \to c - \Phi(u)$;
2) $\Phi'(u_j^1) \to 0$.

Proof. The verification of 1) is similar to and simpler than that of 2), so we only check the latter.

Observe that, for any $\varphi \in E$,

$$\Phi'(u_j^1)\varphi = \Phi'(u_j)\varphi + \int_{\mathbb{R}^3} \big(F_u(x,u_j) - F_u(x,u_j^1) - F_u(x,u)\big)\,\varphi.$$

Since $\Phi'(u_j) \to 0$, it suffices to show that along a subsequence

$$\sup_{\|\varphi\|\le 1} \left| \int_{\mathbb{R}^3} \big(F_u(x,u_j) - F_u(x,u_j^1) - F_u(x,u)\big)\, \varphi \right| \to 0. \tag{7.19}$$

We argue as in the proof of Lemma 6.7. Set $B_d := \{x \in \mathbb{R}^N : \ |x| \le d\}$ for $d > 0$. We have similarly to (6.26) that there is a subsequence (u_{j_n}) such that, for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ satisfying

$$\limsup_{n\to\infty} \int_{B_n \setminus B_r} |u_{j_n}|^q \le \varepsilon \tag{7.20}$$

for all $r \ge r_\varepsilon$. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{u}_n(x) = \eta(2|x|/n)u(x)$ and set $h_n := u - \tilde{u}_n$. Then $\|h_n\| \to 0$ as $n \to \infty$. Observe that for any $\varphi \in E$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(F_u(x,u_{j_n}) - F_u(x,u_{j_n}^1) - F_u(x,u)\big)\, \varphi \\
&= \int_{\mathbb{R}^3} \left(F_u(x,u_{j_n}) - F_u(x,u_{j_n} - \tilde{u}_n) - F_u(x,\tilde{u}_n)\right) \varphi \\
&\quad + \int_{\mathbb{R}^3} \big(F_u(x,u_{j_n}^1 + h_n) - F_u(x,u_{j_n}^1)\big)\, \varphi \\
&\quad + \int_{\mathbb{R}^3} \left(F_u(x,\tilde{u}_n) - F_u(x,u)\right) \varphi.
\end{aligned}$$

Since $\|h_n\| \to 0$ it is easy to see that

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^3} \left(F_u(x,\tilde{u}_n) - F_u(x,u)\right) \varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. Recalling that the Sobolev embedding is locally compact and using (7.20) one gets, for any $\varepsilon > 0$ and $r \ge r_\varepsilon$,

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^3} \left(F_u(x,u_{j_n}) - F_u(x,u_{j_n} - \tilde{u}_n) - F_u(x,\tilde{u}_n)\right) \varphi \right| \\
&= \limsup_{n\to\infty} \left| \int_{B_n \setminus B_r} \left(F_u(x,u_{j_n}) - F_u(x,u_{j_n} - \tilde{u}_n) - F_u(x,\tilde{u}_n)\right) \varphi \right| \\
&\le c_1 \limsup_{n\to\infty} \int_{B_n \setminus B_r} \left(|u_{j_n}| + |\tilde{u}_n|\right) |\varphi| \\
&\le c_2 \varepsilon^{1/2},
\end{aligned}$$

consequently,

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^3} \left(F_u(x,u_{j_n}) - F_u(x,u_{j_n} - \tilde{u}_n) - F_u(x,\tilde{u}_n)\right) \varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. Finally, along the same lines of (6.29) it is not difficult to show that

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^3} \big(F_u(x,u_{j_n}^1 + h_n) - F_u(x,u_{j_n}^1)\big)\, \varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. (7.19) is hereby verified. □

Let $\mathcal{K} := \{u \in E : \ \Phi'(u) = 0\}$, the critical set of Φ.

Lemma 7.11. *Under the assumptions of Lemma 7.9, there hold*

a) $\nu := \inf\{\|u\| : u \in \mathcal{K} \setminus \{0\}\} > 0$;
b) $\theta := \inf\{\Phi(u) : u \in \mathcal{K} \setminus \{0\}\} > 0$.

Proof. See the proof of Lemma 6.8. □

Let $\mathcal{F}$ be a set consisting of arbitrarily chosen representatives of the $\mathbb{Z}^3$-orbits of $\mathcal{K}$. When Φ' is odd we may assume $\mathcal{F} = -\mathcal{F}$. Let $[r]$ denote the integer part of $r \in \mathbb{R}$.

Lemma 7.12. *Let the assumptions of Lemma 7.9 be satisfied and let (u_m) be a $(C)_c$-sequence. Then either*

(i) $u_m \to 0$ *and* $c = 0$, *or*
(ii) $c \geq \theta$ *and there exist a positive integer* $\ell \leq [c/\theta]$, *points* $\overline{u}_1, \cdots, \overline{u}_\ell \in \mathcal{F}$, *a subsequence denoted again by* (u_m), *and sequences* $(a_m^i) \subset \mathbb{Z}^3$, $i = 1, \ldots, \ell$, *such that*

$$\left\| u_m - \sum_{i=1}^{\ell} (a_m^i * \overline{u}_i) \right\| \to 0$$

and

$$\sum_{i=1}^{\ell} \Phi(\overline{u}_i) = c.$$

Proof. The argument proceeds as in the proof of Lemma 5.9, so we only give a sketch of it. First of all, (u_m) is bounded by Lemma 7.9. It follows that $\Phi'(u_m) \to 0$ and

$$0 \leq \int_{\mathbb{R}^3} \hat{F}(x, u_m) = \Phi(u_m) - \frac{1}{2}\Phi'(u_m)u_m \to c,$$

thus $c \geq 0$. Assume now that (u_m) does not converge to 0. As before, the concentration compactness principle implies that either (u_m) is vanishing in which case $|u_m|_p \to 0$ for all $p \in (2, 3)$, or it is nonvanishing. Fixing a $p \in (2, 3)$, by (F_2) and (F_3'), for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|F_u(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1} \quad \text{for all } (x, u). \tag{7.21}$$

If (u_m) is vanishing one checks easily with the help of (7.21) that

$$\|u_m^+\|^2 = \Phi'(u_m)u_m^+ + \int_{\mathbb{R}^3} F_u(x, u_m)u_m^+ \to 0$$

and similarly $\|u_m^-\| \to 0$, so $\|u_m\| \to 0$. Therefore (u_m) must be nonvanishing. Now since Φ is invariant under the $\mathbb{Z}^3$-action, a standard argument enables us to choose a sequence $(a_m) \subset \mathbb{Z}^3$ such that the sequence $v_m := a_m * u_m$ converges to $v \in \mathcal{K}$ weakly in E and strongly in L^p_{loc} for all $p \in [1, 3)$. Note that $\Phi(v_m) = \Phi(u_m)$,

$\|\Phi'(v_m)\| = \|\Phi'(u_m)\|$ and $\|v_m\| = \|u_m\|$. Setting $w_m = v_m - v$ it follows from Lemma 7.10 that

$$\Phi(w_m) \to c - \Phi(v) \quad \text{and} \quad \Phi'(w_m) \to 0. \tag{7.22}$$

Lemma 7.11 and (7.22) imply $\theta \le \Phi(v) \le c$. There are two possibilities: $c = \Phi(v)$ or $c > \Phi(v)$. If $c = \Phi(v)$ then $w_m \to 0$. If $c > \Phi(v)$, then arguing as above with (u_m) and c replaced by (w_m) and $c' = c - \Phi(v)$, respectively, we obtain $v' \in \mathcal{K}$ with $\theta \le \Phi(v') \le c - \theta$. After at most $\left[\frac{c}{\theta}\right]$ steps we obtain the conclusion. □

The proof of Theorem 7.3 will be completed in an indirect way. Namely, we show that if

$$\mathcal{K}/\mathbb{Z}^3 \text{ is a finite set} \tag{7.23}$$

then condition (Φ_I) is satisfied. Then we apply Theorem 4.7 and obtain an unbounded sequence of critical values which contradicts (7.23). So we now assume (7.23). Then $\mathcal{F}$ is a finite set by (7.23), and since Φ' is odd we may assume $\mathcal{F} = -\mathcal{F}$.

For $\ell \in \mathbb{N}$ and a finite set $\mathcal{B} \subset E$ we define

$$[\mathcal{B}, \ell] := \left\{ \sum_{i=1}^{j} (a_i * u_i) : 1 \le j \le \ell,\, a_i \in \mathbb{Z}^3,\, u_i \in \mathcal{B} \right\}.$$

An argument similar to one from [Coti-Zelati, Ekeland and Séré (1990)] or [Coti-Zelati and Rabinowitz (1992)] shows

$$\inf\{\|u - u'\| : u, u' \in [\mathcal{B}, \ell],\, u \ne u'\} > 0. \tag{7.24}$$

As a consequence of Lemma 7.12 we have the following

Lemma 7.13. *Assume* (7.23). *Then* Φ *satisfies* (Φ_I).

Proof. Given a compact interval $I \subset (0, \infty)$ with $d := \max I$ we set $\ell := [d/\theta]$ and $\mathscr{A} = [\mathcal{F}, \ell]$. Clearly $E^\pm$ are $\mathbb{Z}^3$-invariant because A is $\mathbb{Z}^3$-invariant. We have $P_Y[\mathcal{F}, \ell] = [P_Y\mathcal{F}, \ell]$. Thus it follows from (7.24) that

$$\inf\{\|u_1^+ - u_2^+\| :\; u_1, u_2 \in \mathscr{A},\, u_1^+ \ne u_2^+\} \;>\; 0.$$

In addition, $\mathscr{A}$ is a $(C)_I$-attractor by Lemma 7.12, and $\mathscr{A}$ is bounded because $\|u\| \le \ell \max\{\|\bar{u}\| : \bar{u} \in \mathcal{F}\}$ for all $u \in \mathscr{A}$. □

Proof. [Proof of Theorem 7.3 (Multiplicity)] Assume by contradiction that (D_V) has only finitely many geometrically distinct solutions, that is, (7.23) holds. Then Φ satisfies (Φ_0)–(Φ_I) by Lemmas 7.5–7.7 and 7.13. Therefore Theorem 4.7 yields an unbounded sequence of critical values for Φ which contradicts (7.23). This proves that (D_V) has infinitely many geometrically distinct solutions. □

Now we turn to the

Proof. [Proof of Theorem 7.1] The main difference to the proof of Theorem 7.3 lies in the boundedness of the $(C)_c$-sequences. We choose γ such that $a < \gamma < b - \omega$ where b is from (F_3), and define the finite dimensional subspace $Y_n \subset E^+$ as before. We assume that (ω) and $(F_0) - (F_4)$ are satisfied.

CLAIM 1. The conclusions of Lemmas 7.5–7.8 are true.

This can be proved as before. Next we obtain

CLAIM 2. Any $(C)_c$-sequence is bounded.

In order to see this we introduce the following norm on E:

$$\|u\|_\omega = \left(\|u\|^2 + \omega(|u^+|_2^2 - |u^-|_2^2)\right)^{1/2}.$$

With $\omega_0 = \min\{a - \omega, a + \omega\}$ and using (7.10), (7.11) we have

$$\omega_0 |u|_2^2 \le \|u\|_\omega^2 \qquad \text{and} \qquad \frac{a - |\omega|}{a}\|u\|^2 \le \|u\|_\omega^2 \le \frac{a + |\omega|}{a}\|u\|^2. \tag{7.25}$$

Consider a $(C)_c$-sequence $(u_n) \subset E$:

$$\Phi(u_n) \to c \qquad \text{and} \qquad (1 + \|u_n\|)\Phi'(u_n) \to 0. \tag{7.26}$$

It suffices to show that $(\|u_n\|_\omega)$ is bounded. Arguing indirectly we assume that $\|u_n\|_\omega \to \infty$ and set $v_n = u_n/\|u_n\|_\omega$. Then by the concentration compactness principle [Lions (1984)], (v_n) is either vanishing which implies $|v_n|_p \to 0$ for all $p \in (2, 3)$, or it is nonvanishing. Recall that a sequence $(w_n) \subset E$ is vanishing if, for each $r > 0$, $\lim_{n\to\infty} \sup_{a\in\mathbb{R}^3} \int_{B_r(a)} |w_n|^2 = 0$. It is nonvanishing if there are $r, \eta > 0$ and $(a_n) \subset \mathbb{R}^3$ such that $\limsup_{n\to\infty} \int_{B_r(a_n)} |w_n|^2 \ge \eta$. Clearly, in the nonvanishing case we may assume $(a_n) \subset \mathbb{Z}^3$ by enlarging r if necessary. Therefore the proof of Claim 2 will be completed if we show that (v_n) is neither vanishing nor nonvanishing.

Assume (v_n) is vanishing. By definition

$$\begin{aligned}\Phi'(u_n)(u_n^+ - u_n^-) &= \|u_n\|^2 + \omega(|u_n^+|_2^2 - |u_n^-|_2^2) - \int_{\mathbb{R}^3} F_u(x, u_n)(u_n^+ - u_n^-) \\ &= \|u_n\|_\omega^2 \left(1 - \int_{\mathbb{R}^3} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega}\right),\end{aligned}$$

hence by (7.26):

$$\int_{\mathbb{R}^3} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega} \to 1.$$

We set

$$\Omega_n := \left\{x \in \mathbb{R}^3 : \ \frac{|F_u(x, u_n(x))|}{|u_n(x)|} \le \omega_0 - \delta_1\right\}$$

where δ_1 is the constant from (F_4). By (F_4) and (7.25)

$$\begin{aligned}\left|\int_{\Omega_n} \frac{F_u(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega}\right| &= \left|\int_{\Omega_n} \frac{F_u(x, u_n)(v_n^+ - v_n^-)|v_n|}{|u_n|}\right| \\ &\le (\omega_0 - \delta_1)|v_n|_2^2 \le 1 - \frac{\delta_1}{\omega_0} < 1\end{aligned}$$

for all n. Thus, setting $\Omega_n^c := \mathbb{R}^3 \setminus \Omega_n$ we obtain

$$\lim_{n\to\infty} \int_{\Omega_n^c} \frac{F_u(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega} = 1 - \lim_{n\to\infty} \int_{\Omega_n} \frac{F_u(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega} \geq \frac{\delta_1}{\omega_0}.$$

By (F_2) we have $|F_u(x,u)| \leq C|u|$ for all (x,u), so for $p \in (2, 3)$:

$$\int_{\Omega_n^c} \frac{F_u(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|_\omega} \leq C \int_{\Omega_n^c} |v_n|^2 \leq C|\Omega_n^c|^{(p-2)/p} |v_n|_p^{2/p}.$$

Since $|v_n|_p \to 0$, one gets $|\Omega_n^c| \to \infty$. Recall that $\hat{F}(x,u_n) \geq \delta_1$ on Ω_n^c by (F_4), hence

$$\int_{\mathbb{R}^3} \hat{F}(x,u_n) \geq \int_{\Omega_n^c} \hat{F}(x,u_n) \geq \delta_1 |\Omega_n^c| \to \infty.$$

However, it follows from (7.26) that $\int_{\mathbb{R}^3} \hat{F}(x,u_n) = \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n \to c$, yielding a contradiction.

Assume (v_n) is nonvanishing and set $\tilde{u}_n(x) = u_n(x + a_n)$, $\tilde{v}_n(x) = v_n(x + a_n)$, $\varphi_n(x) = \varphi(x - a_n)$ for any $\varphi \in C_0^\infty$. We then have with $R(x,u) := F(x,u) - \frac{1}{2}b|u|^2$:

$$\begin{aligned}
&\Phi'(u_n)\varphi_n \\
&= (u_n^+ - u_n^-, \varphi_n) + (\omega - b)(u_n, \varphi_n)_{L^2} - \int_{\mathbb{R}^3} R_u(x,u_n)\varphi_n \\
&= \|u_n\|_\omega \left((v_n^+ - v_n^-, \varphi_n) + (\omega - b)(v_n, \varphi_n)_{L^2} - \int_{\mathbb{R}^3} R_u(x,u_n)\varphi_n \frac{|v_n|}{|u_n|} \right) \\
&= \|u_n\|_\omega \left(\tilde{v}_n^+ - \tilde{v}_n^-, \varphi) + (\omega - b)(\tilde{v}_n, \varphi)_{L^2} - \int_{\mathbb{R}^3} R_u(x,\tilde{u}_n)\varphi \frac{|\tilde{v}_n|}{|\tilde{u}_n|} \right).
\end{aligned}$$

This yields

$$(\tilde{v}_n^+ - \tilde{v}_n^-, \varphi) + (\omega - b)(\tilde{v}_n, \varphi)_{L^2} - \int_{\mathbb{R}^3} R_u(x,\tilde{u}_n)\varphi \frac{|\tilde{v}_n|}{|\tilde{u}_n|} \to 0.$$

Since $\|\tilde{v}_n\|_\omega = \|v_n\|_\omega = 1$, we can assume that $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, $\tilde{v}_n \to \tilde{v}$ in L^2_{loc} and $\tilde{v}_n(x) \to \tilde{v}(x)$ a.e. in $\mathbb{R}^3$. Observe that $\tilde{v} \neq 0$ because $\lim_{n\to\infty} \int_{B(0,r)} |\tilde{v}_n|^2 \geq \eta$. Next $|R_u(x,u)| \leq C|u|$ implies

$$\left| R_u(x,\tilde{u}_n)\varphi \frac{|\tilde{v}_n|}{|\tilde{u}_n|} \right| \leq C|\varphi||\tilde{v}_n|,$$

so it follows from (F_3) and the dominated convergence theorem that

$$(\tilde{v}^= - \tilde{v}^-, \varphi) + (\omega - b)(\tilde{v}, \varphi)_{L^2} = 0.$$

This implies that $A\tilde{v} = (b - \omega)\tilde{v}$, hence

$$-\Delta\tilde{v} + a^2\tilde{v} = A^2\tilde{v} = (b - \omega)^2\tilde{v},$$

that is, $\tilde{v}$ is an eigenfunction of the operator $A^2 = -\Delta + a^2$ contradicting the fact that A^2 has only continuous spectrum.

Finally, repeating the arguments of the proof of Theorem 7.3, we obtain the desired results. □

7.5 Super-quadratic case

In this section we prove Theorems 7.2, 7.4, and 7.5. Obviously Theorem 7.2 is a special case of Theorem 7.4 corresponding to $V(x) \equiv 0$. For the proof of Theorem 7.4 we consider as before the functionals

$$\Psi(u) = \int_{\mathbb{R}^3} F(x,u) \quad \text{and} \quad \Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2 + \omega|u|_2^2\right) - \Psi(u)$$

on $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ from (7.9). We choose $\gamma > \gamma_0 := a + |\omega| + \sup V(\mathbb{R}^3)$, and set $Y_0 := (E_\gamma - E_0)L^2$. We also choose a strictly increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\sigma(A) \cap (\gamma_0, \gamma)$ and elements $e_n \in (E_{\gamma_n} - E_{\gamma_{n-1}})L^2$ with $\|e_n\| = 1$, and define $Y_n := \operatorname{span}\{e_1, \ldots, e_n\}$ and $E_n = E^- \oplus Y_n$. Then $(Y_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite dimensional subspaces of E^+ and

$$\gamma_0|u|_2^2 \le \|u\|^2 \le \gamma|u|_2^2 \qquad \text{for all } u \in Y_0. \tag{7.27}$$

Lemma 7.14. *Under* (ω), $(F_0) - (F_2)$ *and* $(F_5) - (F_6)$, *the following conclusions hold:*

a) Ψ *is weakly sequentially lower-semicontinuous and* Φ' *is weakly sequentially continuous. For* $c > 0$ *there exists* $\zeta > 0$ *such that* $\|u\| < \zeta\|u^+\|$ *for all* $u \in \Phi_c$.

b) *There exists* $\rho > 0$ *such that* $\kappa := \inf \Phi(\partial B_\rho \cap E^+) > 0$.

c) $\sup \Phi(E_n) < \infty$, *and there is a sequence* $R_n > 0$ *such that* $\sup \Phi(E_n \setminus B_n) \le \inf \Phi(B_\rho)$, *where* $B_n = \{u \in E_n : \|u\| \le R_n\}$.

Proof. a) is clear because $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^{1/2}}$, and $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ embeds continuously into $L^p(\mathbb{R}^3, \mathbb{C}^4)$ for $p \in [2,3]$, compactly into $L^p_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ for $p \in [1,3)$.

Hypothesis (F_6) yields

$$|F_u(x,u)| \le a_1|u|^{p-1} \quad \text{for all} \quad |u| \ge r,$$

where $p := 2\sigma/(\sigma-1) \in (2,3)$. This together with (F_2) implies that, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$F(x,u) \le \varepsilon|u|^2 + C_\varepsilon|u|^p \qquad \text{for all } (x,u).$$

Therefore $\Psi(u) \le \varepsilon|u|_2^2 + C_\varepsilon|u|_p^p \le C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p)$ for all $u \in E$. b) follows now easily from

$$\Phi(u) \ge \frac{a - |\omega|}{2a}\|u\|^2 - C\varepsilon\|u\|^2 - CC_\varepsilon\|u\|^p$$

for all $u \in E^+$ and ε small.

It remains to check c). Note that, as a consequence of (F_5) there is $R > 0$ such that $F(x,u) \ge \gamma|u|^2$ if $|u| \ge R$. It is clear that $\sup \Phi(E_n) < \infty$. We show that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in E_n$. Assume by contradiction that there is a sequence (u_j) in E_n and $M > 0$ satisfying $\|u_j\| \to \infty$ and $\Phi(u_j) > -M$. Setting

$v_j = u_j/\|u_j\|$ we have (along a subsequence) $v_j^+ \to v^+$ and $v_j^- \rightharpoonup v^-$. Then $v^+ \neq 0$ because otherwise, $\|v_j^+\| \to 0$ and by (7.10)

$$\begin{aligned} o(1) \leq \frac{\Phi(u_j)}{\|u_j\|^2} &\leq \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega |v_j|_2^2\right) \\ &\leq \frac{a+|\omega|}{2a}\|v_j^+\|^2 - \frac{a-|\omega|}{2a}\|v_j^-\|^2, \end{aligned}$$

which implies

$$\frac{a-|\omega|}{2a}\limsup_{j\to\infty}\|v_j^-\|^2 \leq 0,$$

hence $1 = \|v_j\| \to 0$, a contradiction. Observe that by (7.27)

$$\begin{aligned} &\frac{1}{2}\Big(\|v^+\|^2 - \|v^-\|^2 + (\omega - 2\gamma)|v|_2^2\Big) \\ &\leq \frac{1}{2}\left(\|v^+\|^2 - \|v^-\|^2 - \left(2 - \frac{|\omega|}{\gamma}\right)\|v^+\|^2\right) \\ &= -\frac{1}{2}\left(1 + \frac{|\omega|}{\gamma}\right)\|v^+\|^2 - \frac{1}{2}\|v^-\|^2 \\ &\leq -\frac{1}{2}\|v\|^2, \end{aligned}$$

hence, there is a bounded domain $\Omega \subset \mathbb{R}^3$ such that

$$\frac{1}{2}\left(\|v^+\|^2 - \|v^-\|^2 + \omega|v|_2^2 - 2\gamma\int_\Omega |v|^2\right) \leq -\frac{1}{4}\|v\|^2.$$

It follows that

$$\begin{aligned} \frac{\Phi(u_j)}{\|u_j\|^2} &\leq \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega|v_j|_2^2\right) - \int_\Omega \frac{F(x,u_j)}{\|u_j\|^2} \\ &= \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega|v_j|_2^2 - 2\gamma\int_\Omega |v_j|^2\right) - \int_\Omega \frac{F(x,u_j) - \gamma|u_j|^2}{\|u_j\|^2} \\ &\leq \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega|v_j|_2^2 - 2\gamma\int_\Omega |v_j|^2\right) \\ &\quad - \int_{\Omega\cap\{|u_j|\leq R\}} \frac{F(x,u_j) - \gamma|u_j|^2}{\|u_j\|^2} \\ &\leq \frac{1}{2}\left(\|v_j^+\|^2 - \|v_j^-\|^2 + \omega|v_j|_2^2 - 2\gamma\int_\Omega |v_j|^2\right) + \frac{C_R|\Omega|}{\|u_j\|^2}, \end{aligned}$$

where $C_R = \max\{F(x,u) : x \in \Omega, |u| \leq R\}$. Consequently,

$$0 \leq -\frac{1}{4}\|v\|^2 - \liminf_{j\to\infty}\frac{C_R|\Omega|}{\|u_j\|^2} = -\frac{1}{4}\|v\|^2,$$

a contradiction. □

As a consequence of Lemma 7.14 c) we have

Lemma 7.15. *Under the assumptions of Lemma 7.14,* $\Phi|_{\partial Q} \leq 0$ *where* $Q := \{u = u^- + se_1 : \|u\| \leq R_1, s \geq 0\}$.

Lemma 7.16. *Under the assumptions of Lemma 7.14, $(C)_c$-sequences are bounded.*

Proof. See the proof of Lemma 7.9. Let $(u_j) \subset E$ be such that $\Phi(u_j) \to c$ and $(1 + \|u_j\|)\Phi'(u_j) \to 0$. We then have

$$C_0 \geq \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^3} \hat{F}(x, u_j). \tag{7.28}$$

Assume by contradiction that $\|u_j\| \to \infty$ and set $v_j = u_j/\|u_j\|$. Then $|v_j|_s \leq \gamma_s$ for all $s \in [2, 3]$. By definition

$$\Phi'(u_j)(u_j^+ - u_j^-) \geq \|u_j\|^2 \left(\frac{a - |\omega|}{a} - \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \right),$$

hence,

$$\liminf_{j\to\infty} \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} \ \geq \ \ell := \frac{a - |\omega|}{a}. \tag{7.29}$$

Let $h(r)$, $\Omega_j(\rho, r)$ and c_ρ^r be as before. Then $h(r) \to \infty$ as $r \to \infty$ and

$$\hat{F}(x, u_j(x)) \geq c_\rho^r |u_j(x)|^2 \quad \text{for all } x \in \Omega_j(\rho, r).$$

It follows from (7.28) that $|\Omega_j(b, \infty)| \leq C_0/h(r) \to 0$ as $r \to \infty$ uniformly in j, and, for any fixed $0 < \rho < r$,

$$\int_{\Omega_j(\rho,r)} |v_j|^2 = \frac{1}{\|u_j\|^2} \int_{\Omega_j(\rho,r)} |u_j|^2 \leq \frac{C_0}{c_\rho^r \|u_j\|^2} \to 0$$

as $j \to \infty$.

Let $0 < \varepsilon < \ell/3$. Firstly by (F_2) take $\rho_\varepsilon > 0$ small such that

$$\int_{\Omega_j(0,\rho_\varepsilon)} \frac{|F_u(x, u_j)|}{|u_j|} |v_j|\, |v_j^+ - v_j^-| \leq \varepsilon,$$

then by (F_6) and Hölder inequality take r_ε large so that

$$\begin{aligned}
&\int_{\Omega_j(r_\varepsilon,\infty)} \frac{|F_u(x, u_j)|}{|u_j|} |v_j^+ - v_j^-|\, |v_j| \\
&\leq \left(\int_{\Omega_j(r_\varepsilon,\infty)} \frac{|F_u(x, u_j)|^\sigma}{|u_j|^\sigma} \right)^{1/\sigma} \left(\int_{\Omega_j(r_\varepsilon,\infty)} \left(|v_j^+ - v_j^-|\, |v_j|\right)^{3/2} \right)^{2/3} |\Omega_j(r_\varepsilon, \infty)|^{(\sigma-3)/3\sigma} \\
&\leq \left(\int_{\mathbb{R}^3} c_1 \hat{F}(x, u_j) \right)^{1/\sigma} \left(\int_{\mathbb{R}^3} \left(|v_j^+ - v_j^-|\, |v_j|\right)^{3/2} \right)^{2/3} |\Omega_j(r_\varepsilon, \infty)|^{(\sigma-3)/3\sigma} \\
&< \varepsilon
\end{aligned}$$

uniformly in j. Finally choose j_0 so that

$$\int_{\Omega_j(\rho_\varepsilon,r_\varepsilon)} \frac{|F_u(x, u_j)|}{|u_j|} |v_j^+ - v_j^-|\, |v_j| < \varepsilon$$

for all $j \geq j_0$. Thus

$$\liminf_{j\to\infty} \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)}{\|u_j\|} < 3\varepsilon < \ell,$$

which however, contradicts (7.29). $\square$

Repeating the arguments of Lemmas 7.10, 7.11, 7.12 and 7.13 gives the following

Lemma 7.17. *Let* (ω), $(F_0)-(F_2)$ *and* $(F_5)-(F_6)$ *be satisfied. Assume* Φ *has only finitely many geometrically distinct critical points. Then for any interval* $I=[c,d]\subset(0,\infty)$, Φ *has a* $(C)_I$*-attractor* $\mathscr{A}$ *with* $P^+\mathscr{A}\subset E^+$ *bounded and* $\inf\{\|u^+-v^+\| : u,v\in\mathscr{A},\ u^+\neq v^+\}>0$.

Proof. [Proof of Theorem 7.4] With $X=E^-$ and $Y=E^+$ all conditions of Theorem 4.3 are satisfied as a consequence of Lemmas 7.14–7.16. Therefore Φ possesses a $(C)_c$-sequence (u_m) with $\kappa\le c\le\sup\Phi(Q)$. Using the concentration compactness principle the invariance of Φ with respect to the $\mathbb{Z}^3$-action yields a critical point $u\neq 0$.

Furthermore, assume $F(x,u)$ is also even in u. If (D_V) has only finitely many geometrically distinct solutions, then with Lemma 7.17 we see that Φ satisfies all hypotheses of Theorem 4.7, hence it has an unbounded sequence of positive critical values. □

Next we turn to the

Proof. [Proof of Theorem 7.5] As before we look for critical points of the functional Φ on E. According to Lemma 7.3c) the spectrum of A is purely discrete: $\sigma(A)=\sigma_d(A)=\{\pm\mu_n^{1/2}: n\in\mathbb{N}\}$. We arrange the eigenvalues of A less than $-\omega$ as

$$-\infty<\cdots\le\eta_2^-\le\eta_1^-<-\omega\qquad\text{with eigenfunctions } e_j^- :\ Ae_j^-=\eta_j^-e_j^-,$$

and those larger than $-\omega$ as

$$-\omega<\eta_1^+\le\eta_2^+\le\cdots\qquad\text{with eigenfunctions } e_j^+ :\ Ae_j^+=\eta_j^+e_j^+.$$

Setting

$$E_\omega^\pm=\operatorname{clos\,span}\{e_j^\pm :\ j\in\mathbb{N}\}\qquad\text{and}\qquad E_\omega^0=\ker(A+\omega)$$

we then have the decomposition

$$E=E_\omega^-\oplus E_\omega^0\oplus E_\omega^+,\quad u=u^-+u^0+u^+.$$

We define a new inner product on E by

$$(u,v)_\omega=(|A+\omega|^{1/2}u,|A+\omega|^{1/2}v)_{L^2}+(u^0,v^0)_{L^2}$$

with associated norm $\|u\|_\omega$. Note that $\|\cdot\|_\omega$ is equivalent to $\|\cdot\|$. It is obvious that

$$\omega_0|u|_2^2\le\|u\|_\omega^2\ \text{ for } u\in E_\omega^-\oplus E_\omega^+,\quad \omega_0:=\min\{\eta_1^++\omega,\ -(\eta_1^-+\omega)\}\tag{7.30}$$

and that the functional Φ can be written as

$$\Phi(u)=\frac{1}{2}\left(\|u^+\|_\omega^2-\|u^-\|_\omega^2\right)-\Psi(u)\quad\text{with } \Psi(u)=\int_{\mathbb{R}^3}F(x,u).$$

For $u=\sum_{j\in\mathbb{N}}(c_j^-e_j^-+c_j^+e_j^-)+u^0\in E$ we have:

$$\|u\|_\omega^2=\sum_{j\in\mathbb{N}}\left((\eta_j^++\omega)|c_j^+|^2-(\eta_j^-+\omega)|c_j^-|^2\right)+|u^0|_2^2.$$

In order to apply Theorems 4.3 and 4.7 we set $X = E_\omega^- \oplus E_\omega^0$ and $Y = E_\omega^+$.

Claim 1. Φ satisfies (Φ_0), (Φ_+) and (Φ_2).

(Φ_0) follows easily from the continuity of embedding $E \hookrightarrow H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, (Φ_+) from the form of Φ, and (Φ_2) from Lemma 7.6.

Claim 2. Φ verifies (Φ_4).

For $n \in \mathbb{N}$, we define $Y_n := \operatorname{span}\{e_1^+, \dots, e_n^+\}$. Then (7.30) implies

$$\omega_0 |u|_2^2 \le \|u\|_\omega^2 \le \eta_n^+ |u|_2^2 \qquad \text{for } u \in Y_n.$$

Repeating the argument of the proof of Lemma 7.14c) yields (Φ_4).

Claim 3. Φ satisfies the $(C)_c$ condition for all $c \ge 0$.

Let $(u_j) \subset E$ be a $(C)_c$-sequence. Then (7.28) remains true in the present case. We first verify the boundedness of $(\|u_j\|_\omega)$. Assume by contradiction that $\|u_j\|_\omega \to \infty$ and set $v_j = u_j/\|u_j\|_\omega^2$ as before. After passing to a subsequence we have: $v_j \rightharpoonup v$, $v_j^0 \to v^0$, $\wp := \lim_{j\to\infty} \|v_j^- + v_j^+\|_\omega$ exists. We distinguish the two cases: $\wp = 0$ or $\wp > 0$, and we write $\tilde u_j = u_j^- + u_j^+$, $\tilde v_j = v_j^- + v_j^+$. If $\wp = 0$ then $\|v_j^0\|_\omega = |v_j^0|_2 \to 1 = |v^0|_2$. For $\delta > 0$ we consider the sets $\Omega_\delta = \{x \in \mathbb{R}^3 : \ |v^0(x)| \ge 2\delta\}$ and $\Omega_{j\delta} = \{x \in \mathbb{R}^3 : \ |\tilde v_j(x)| \ge \delta\}$. Since $v^0 \in C(\mathbb{R}^3)$ and $|v^0|_2 = 1$, $|\Omega_\delta| > 0$ for all δ small. By (7.30)

$$|\Omega_{j\delta}| \le \frac{1}{\delta^2} \int_{\mathbb{R}^3} |\tilde v_j|^2 \le \frac{1}{\delta^2 \omega_0} \|\tilde v_j\|_\omega^2 \to 0,$$

hence, $|\Omega_\delta \setminus \Omega_{j\delta}| \to |\Omega_\delta|$ as $j \to \infty$. Now for $x \in \Omega_\delta \setminus \Omega_{j\delta}$ there holds $|v_j(x)| \ge \delta/2$, hence $|u_j(x)| \ge \frac{\delta}{2} \|u_j\|_\omega$ for $j \ge j_\delta$. From this and the definition of $h(r)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \hat F(x, u_j) &\ge \int_{\Omega_\delta \setminus \Omega_{j\delta}} \hat F(x, u_j) \\ &\ge h\Big(\frac{\delta}{2} \|u_j\|_\omega\Big) |\Omega_\delta \setminus \Omega_{j\delta}| \\ &\to \infty \end{aligned}$$

contradicting (7.28). Next assume $\wp > 0$ and observe that

$$\Phi'(u_j)(u_j^+ - u_j^-) = \|u_j\|_\omega^2 \left(\|\tilde v_j\|_\omega^2 - \int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|^2} \right),$$

hence

$$\int_{\mathbb{R}^3} \frac{F_u(x, u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|} \to \wp^2.$$

Set

$$Q_j := \left\{ x \in \mathbb{R}^3 : \ \frac{|F_u(x, u_j(x))|}{|u_j(x)|} \le \frac{\omega_0 \wp^2}{2} \right\} \qquad \text{and} \qquad Q_j^c := \mathbb{R}^3 \setminus Q_j.$$

Then we have

$$\left| \int_{Q_j} \frac{F_u(x, u_j)(v_j^+ - v_j^-)|v_j|}{|u_j|} \right| \le \frac{\wp^2 \omega_0}{2} |v_j|_2^2 \le \frac{\wp^2}{2},$$

and therefore

$$\lim_{j\to\infty}\int_{Q_j^c}\frac{F_u(x,u_j)(v_j^+-v_j^-)|v_j|}{|u_j|}\ge\frac{\wp^2}{2}\,. \tag{7.31}$$

Now repeating the arguments of the last part of the proof of Lemma 7.16 it is not difficult to see that

$$\lim_{j\to\infty}\int_{Q_j^c}\frac{F_u(x,u_j)(v_j^+-v_j^-)|v_j|}{|u_j|}=0,$$

contradicting (7.31). Therefore, (u_j) must be bounded in E, and a standard argument (using the fact that $E\hookrightarrow L^\tau(\mathbb{R}^3,\mathbb{C}^4)$ embeds compactly for $\tau\in[2,3)$) shows that (u_j) has a convergent subsequence in E.

In conclusion, Φ satisfies the conditions of Theorem 4.3. If F is even in $u\in\mathbb{C}^4$ then it satisfies the conditions of Theorem 4.7. This completes the proof of Theorem 7.5. □

7.6 More general external fields

The contents of this and the next section are chosen from the work of [Ding and Ruf (2006)]. In the present section we consider the equation (7.3) with more general vector potentials. We rewrite for convenience (7.3) in the form

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u+a\beta u+M(x)u=R_u(x,u), \tag{$\mathcal{P}$}$$

where $a=mc>0$ and $M(x)=\big(m_{jk}(x)\big)$ is a 4×4 symmetric real matrix function defined almost everywhere on $\mathbb{R}^3$, that is, $m_{jk}(x)=m_{kj}(x)\in\mathbb{R}$ for $j,k=1,2,3,4$ and a.e. $x\in\mathbb{R}^3$, such that

$$A:=H_0+M\quad\text{with}\quad H_0:=-i\sum_{k=1}^{3}\alpha_k\partial_k+a\beta$$

is a selfadjoint operator in $L^2(\mathbb{R}^3,\mathbb{C}^4)$.

To treat the nonlinear problem, it is crucial to have information about the spectrum of the linearized operator A in the origin. Our assumptions will guarantee that A has a spectral gap around the origin, and that there exist a finite number (or infinitely many) eigenvalues in the spectral gap. We are mainly interested in the potentials $M(x)$ which either are of Coulomb-type, i.e. tend to 0 as $|x|\to\infty$ and are singular at the origin (e.g. the Coulomb potential $\kappa/|x|$), or have the property that for some $b>0$ the measure of the sublevel set Ω_b of $\beta M(x)$ is finite (i.e $|\Omega_b|=|\{x\in\mathbb{R}^3:\beta M(x)<b\}|<\infty$).

We will consider nonlinearities $R_u(x,u)$ which are asymptotically linear, i.e. $R_u(x,u)=Q(x)u+o(|u|)$ for $|u|\to\infty$, where $Q(x)$ is a continuous and symmetric 4×4-matrix-function. We assume that $q_0:=\inf_x Q_{\min}(x)>0$ where $Q_{\min}(x)$

denotes the minimal eigenvalue of $Q(x)$. Furthermore, we assume that $R_u(x,u) = o(|u|)$ for u near 0, that $q_\infty := \limsup_{|x|\to\infty} Q_{\max}(x)$ lies in the spectral gap where $Q_{\max}(x)$ denotes the maximal eigenvalue of $Q(x)$, and that between 0 and q_0 lie some eigenvalues of A. We recall that such type of nonlinearities have been introduced by [Amann and Zehnder (1980)] in other contexts.

7.6.1 *Main results*

Precisely we suppose that $R(x,u)$ satisfies

(R_1) $R(x,u) \geq 0$ and $R_u(x,u) = o(|u|)$ as $u \to 0$ uniformly in x;
(R_2) $R_u(x,u) - Q(x)u = o(|u|)$ uniformly in x as $|u| \to \infty$, where Q is a continuous symmetric 4×4 real matrix function;
(R_3) Either (i) $0 \notin \sigma(A - Q)$, or (ii) $\tilde{R}(x,u) \geq 0$ and there exist δ_0, $\nu_0 > 0$ such that $\tilde{R}(x,u) \geq \delta_0$ if $|u| \geq \nu_0$;
(R_4) $q_0 := \inf Q_{\min}(x) > \inf \sigma(A) \cap (0,\infty)$.

Here (and below) we denote by $\sigma(B)$ the spectrum of an operator B, and we write

$$\tilde{R}(x,u) := \frac{1}{2} R_u(x,u) \cdot u - R(x,u)$$

($u \cdot v$ or uv denotes the scalar product of $\mathbb{C}^4$). For convenience any real function $U(x)$ will be regarded as the symmetric matrix $U(x)I_4$ where I_4 denotes the 4×4 identity matrix. For two given symmetric 4×4 real matrix functions $L_1(x)$ and $L_2(x)$, we write that $L_1(x) \leq L_2(x)$ if and only if

$$\max_{\xi \in \mathbb{C}^4, |\xi|=1} (L_1(x) - L_2(x))\, \xi \cdot \bar{\xi} \leq 0.$$

Set

$$q_\infty := \limsup_{|x|\to\infty} \Big(\sup_u \frac{|R_u(x,u)|}{|u|} \Big).$$

First we consider the Coulomb type potential

(M_1) M is a continuous symmetric real 4×4-matrix function on $\mathbb{R}^3 \setminus \{0\}$, and $0 \geq M(x) \geq -\frac{\kappa}{|x|}$ where $\kappa < \frac{\sqrt{3}}{2}$.

It is known that the corresponding operator A is selfadjoint with domain $\mathscr{D}(A) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ and $\sigma_e(A) = \mathbb{R} \setminus (-a,a)$, $\sigma_d(A) \cap (0,a) \neq \emptyset$ where $\sigma_e(A)$ denotes the essential spectrum and $\sigma_d(A)$ the eigenvalues of finite multiplicity (cf. [Griesemer and Siedentop (1999)], [Thaller (1992)]). We assume in addition to $(R_1)-(R_4)$ that

(R_5) $q_\infty < a$.

Involving (R_4) let ℓ be the number of elements of $(0,q_0) \cap \sigma(A)$. We are going to prove the following result.

Theorem 7.6 ([Ding and Ruf (2006)]). *Assume that* (M_1) *and* (R_1)*-*(R_5) *hold. Then* $(\mathcal{P})$ *has at least one solution. If additionally* $R_u(x,u)$ *is odd in* $u \in \mathbb{C}^4$ *then* $(\mathcal{P})$ *has* ℓ *pairs of solutions.*

Next we consider the problem $(\mathcal{P})$ with the matrix potential $M(x)$ satisfying

(M_2) $M \in L^\infty(\mathbb{R}^3, \mathbb{R}^{4\times 4})$, and there is $b > 0$ such that $|\Omega_b| < \infty$ where $\Omega_b := \{x \in \mathbb{R}^3 : \beta M(x) < b\}$.

Here we write $|S|$ for the Lebesgue measure of $S \subset \mathbb{R}^3$. We define the number $b_{\max} := \sup\{b : |\Omega_b| < \infty\}$. Assume instead of (R_5) that

$(\hat{R}_5)$ $q_\infty < a + b_{\max}$.

Theorem 7.7 ([Ding and Ruf (2006)]). *Assume that* (M_2)*,* (R_1)*-*(R_4) *and* $(\hat{R}_5)$ *hold. Then* $(\mathcal{P})$ *has at least one solution. If additionally* $R_u(x,u)$ *is odd in* $u \in \mathbb{C}^4$ *then* $(\mathcal{P})$ *has* ℓ *pairs of solutions.*

7.6.2 *Variational arguments*

We begin with a slight general situation. Throughout the subsection we always assume that the matrix $M(x)$ is such that $A = H_0 + M$ is a self-adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathscr{D}(A) \subset H^1(\mathbb{R}^3, \mathbb{C}^4)$, and consider the equation $(\mathcal{P})$ with $R(x,u)$ satisfying (R_1)-(R_4).

Let

$$\mu_e^- := \sup\left(\sigma_e(A) \cap (-\infty, 0)\right), \quad \mu_e^+ := \inf\left(\sigma_e(A) \cap (0, \infty)\right),$$

and $\mu_e := \min\{-\mu_e^-, \ \mu_e^+\}$. We assume

(A_0) $\mu_e^- < 0 < \mu_e^+$;
(R_0) $q_\infty < \mu_e$.

We are going to prove the following result.

Theorem 7.8. *Assume that* (R_1)*-*(R_4)*,* (A_0) *and* (R_0) *hold. Then* $(\mathcal{P})$ *has at least one solution. If additionally* $R_u(x,u)$ *is odd in* $u \in \mathbb{C}^4$ *then* $(\mathcal{P})$ *has* ℓ *pairs of solutions.*

The assumption (A_0) induces an orthogonal decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^4)$:

$$L^2 = L^- \oplus L^0 \oplus L^+, \quad u = u^- + u^0 + u^+$$

so that A is negative definite (resp. positive definite) in L^- (resp. L^+) and $L^0 = \ker A$. Let $P^\pm : L^2 \to L^\pm$ and $P^0 : L^2 \to L^0$ denote the associated projectors.

Let $E := \mathscr{D}(|A|^{1/2})$ be the domain of the self-adjoint operator $|A|^{1/2}$ which is a Hilbert space equipped with the inner product

$$(u, v) := (|A|^{1/2}u, |A|^{1/2}v)_{L^2} + (P^0 u, P^0 v)_{L^2}$$

and the induced norm $\|u\| = (u,u)^{1/2}$. E possesses the following decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad \text{with} \quad E^\pm = E \cap L^\pm \text{ and } E^0 = L^0,$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$ inner products.

Lemma 7.18. *E embeds continuously in $H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, hence it embeds continuously in $L^p(\mathbb{R}^3,\mathbb{C}^4)$ for all $p \in [2,3]$ and compactly in $L^p_{loc}(\mathbb{R}^3,\mathbb{C}^4)$ for all $p \in [1,3)$.*

Proof. See the proof of Lemma 5.11. Observe that the norm $\|u\|_{H^1}$ of H^1 is equivalent to the one given by $||H_0|u|_2$ where as usual $|H_0|$ denotes the absolute value of H_0. Hence by interpolation theory the norm $\|u\|_{H^{1/2}}$ of $H^{1/2}$ is equivalent to the one given by $||H_0|^{1/2}u|_2$.

Remark that the assumption (A_0) implies that 0 is at most an isolate eigenvalue of finite multiplicity of A. Define the (strictly) positive selfadjoint operator acting in L^2:

$$\tilde{A} = |A| + P^0 \quad \text{with } \mathscr{D}(\tilde{A}) = \mathscr{D}(A).$$

$\mathscr{D}(A)$ is a Hilbert space with the norm

$$\|u\|_A := |\tilde{A}u|_2 = \left(||A|u|_2^2 + |P^0u|_2^2\right)^{1/2}$$

and, as in the proof of Lemma 5.10 it is easy to check that, since $\mathscr{D}(A) \subset H^1$,

$$\|u\|_{H^1} \le c_1\|u\|_A \quad \text{for all } u \in \mathscr{D}(A).$$

Therefore, by interpolation theory (cf. [Triebel (1978)]),

$$\|u\|_{1/2} \le c_2||H_0|^{1/2}u|_2 \le c_3|\tilde{A}^{1/2}u|_2 = c_3\|u\|$$

for all $u \in E$. □

For further requirements we fix arbitrarily a positive number γ with

$$q_\infty < \gamma < \mu_e. \tag{7.32}$$

Let n be the number of the eigenvalues in the interval $[-\gamma,\gamma]$. We write η_j and f_j $(1 \le i \le n)$ for the eigenvalues and eigenfunctions. Setting

$$L^d := \operatorname{span}\{f_1,\cdots,f_n\},$$

we have another orthogonal decomposition

$$L^2 = L^d \oplus L^e, \quad u = u^d + u^e.$$

Correspondingly, E has the decomposition:

$$E = E^d \oplus E^e \ \text{ with } \ E^d = L^d \text{ and } E^e = E \cap L^e,$$

orthogonal with respect to both the inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

We define on E the following functional

$$\Phi(u) := \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u) \ \text{ with } \ \Psi(u) := \int_{\mathbb{R}^3} R(x,u).$$

Remark that by assumptions (R_1)-(R_2) and (R_0), given $p \in (2,3]$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|R_u(x,u)| \le \varepsilon|u| + C_\varepsilon |u|^{p-1} \tag{7.33}$$

and

$$R(x,u) \le \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{7.34}$$

for all (x,u). Thus $\Phi \in C^1(E,\mathbb{R})$ and a standard argument shows that critical points of Φ are weak solutions of $(\mathcal{P})$. Moreover, by [Esteban and Séré (1995)], such solutions are in $W^{1,s}(\mathbb{R}^3,\mathbb{C}^4)$ for all $s \ge 2$ (see also [Bartsch and Ding (2006II)]).

Lemma 7.19. *Let (R_1)-(R_2), (A_0) and (R_0) be satisfied. Then Ψ is weakly sequentially lower semicontinuous and Φ' is weakly sequentially continuous. Moreover, there is $\zeta > 0$ such that for any $c > 0$:*

$$\|u\| < \zeta\|u^+\| \quad \textit{for all} \;\; u \in \Phi_c. \tag{7.35}$$

Proof. The first conclusion follows easily because $E \hookrightarrow H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, so E embeds continuously into $L^q(\mathbb{R}^3,\mathbb{C}^4)$ for $q \in [2,3]$ and compactly into $L^q_{loc}(\mathbb{R}^3,\mathbb{C}^4)$ for $q \in [1, 3)$. For showing (7.35) we adopt an argument of [Ding and Jeanjean (2007)]. Arguing indirectly assume by contradiction that for some $c > 0$ there is a sequence $u_n \in \Phi_c$ and $\|u_n\|^2 \ge n\|u^+\|^2$. This, jointly with the form of Φ, yields that

$$\|u_n^- + u_n^0\|^2 \ge (n-1)\|u^+\|^2 \ge (n-1)\left(2c + \|u_n^-\|^2 + 2\int_{\mathbb{R}^3} R(x,u_n)\right),$$

or

$$\|u_n^0\|^2 \ge (n-1)2c + (n-2)\|u_n^-\|^2 + 2(n-1)\int_{\mathbb{R}^3} R(x,u_n).$$

Since $c > 0$ and $R(x,u) \ge 0$, it follows that $\|u_n^0\| \to \infty$, hence $\|u_n\| \to \infty$. Set $w_n = u_n/\|u_n\|$. We have $\|w_n^+\|^2 \le 1/n \to 0$. By

$$1 \ge \|w_n^0\|^2 \ge \frac{(n-1)2c}{\|u_n\|^2} + (n-2)\|w_n^-\|^2 + 2(n-1)\int_{\mathbb{R}^3} \frac{R(x,u_n)}{\|u_n\|^2},$$

we also have $\|w_n^-\|^2 \le 1/(n-2) \to 0$. Therefore, $w_n \to w = w^0$ in E and $\|w^0\| = 1$. By (R_2) we set

$$r(x,u) := R(x,u) - \frac{1}{2}Q(x)u\cdot u. \tag{7.36}$$

Then $|r(x,u)|/|u|^2 \to 0$ as $|u| \to \infty$ uniformly in x. Particularly $|r(x,u)| \le c_1|u|^2$. Observe that $|u_n(x)| \to \infty$ for $w(x) \ne 0$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{r(x,u_n)}{\|u_n\|^2} &= \int_{w(x)\ne 0} \frac{r(x,u_n)}{|u_n|^2}|w_n|^2 + \int_{w(x)=0} \frac{r(x,u_n)}{|u_n|^2}|w_n - w|^2 \\ &\le 2\int_{w(x)\ne 0} \frac{|r(x,u_n)|}{|u_n|^2}|w|^2 + 2c_1|w_n - w|_2^2 \to 0.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{2(n-1)} &\ge \int_{\mathbb{R}^3} \frac{R(x,u_n)}{\|u_n\|^2} = \frac{1}{2}\int_{\mathbb{R}^3} Q(x)w_n\cdot w_n + \int_{\mathbb{R}^3} \frac{r(x,u_n)}{\|u_n\|^2} \\ &\ge \frac{q_0}{2}|w_n|_2^2 + o(1),\end{aligned}$$

consequently, $w^0 = 0$, a contradiction. □

Lemma 7.20. *Under the assumptions of Lemma 7.19, there is $\rho > 0$ such that $\kappa := \inf \Phi(\partial B_\rho \cap E^+) > 0$.*

Proof. Choosing $p \in (2, 3)$, it follows from (7.34),

$$\Psi(u) \le \varepsilon |u|_2^2 + C_\varepsilon |u|_p^p \le C(\varepsilon \|u\|^2 + C_\varepsilon \|u\|^p)$$

for all $u \in E$. The desired conclusion now follows easily. □

In the following, we arrange all the eigenvalues (counted in multiplicity) of A in $(0, q_0)$ by $0 < \mu_1 \le \mu_2 \le ... \le \mu_\ell < q_0$ and let e_j denote the corresponding eigenfunctions: $Ae_j = \mu_j e_j$ for $j = 1, ..., \ell$. Set $Y_0 := \text{span}\{e_1, ..., e_\ell\}$. Note that

$$\mu_1 |w|_2^2 \le \|w\|^2 \le \mu_\ell |w|_2^2 \quad \text{for all } w \in Y_0. \tag{7.37}$$

For any subspace F of Y_0 set $E_F = E^- \oplus E^0 \oplus F$.

Lemma 7.21. *Let (R_1), (R_2), (R_4), (A_0) and (R_0) be satisfied. Then for any subspace F of Y_0, $\sup \Phi(E_F) < \infty$, and there is $R_F > 0$ such that $\Phi(u) < \inf \Phi(B_\rho)$ for all $u \in E_F$ with $\|u\| \ge R_F$.*

Proof. Clearly, it is sufficient to check that $\Phi(u) \to -\infty$ as $u \in E_F, \|u\| \to \infty$. Arguing indirectly, assume that for some sequence $u_j \in E_F$ with $\|u_j\| \to \infty$, there is $c > 0$ such that $\Phi(u_j) \ge -c$ for all j. Then, setting $w_j = u_j/\|u_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^0 \to w^0$, $w_j^+ \to w^+ \in Y_0$ and

$$-\frac{c}{\|u_j\|^2} \le \frac{\Phi(u_j)}{\|u_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}^3} \frac{R(x, u_j)}{\|u_j\|^2}. \tag{7.38}$$

Remark that $w^+ \ne 0$. Indeed, if not then it follows from (7.38) that

$$0 \le \frac{1}{2}\|w_j^-\|^2 + \int_{\mathbb{R}^3} \frac{R(x, u_j)}{\|u_j\|^2} \le \frac{1}{2}\|w_j^+\|^2 + \frac{c}{\|u_j\|^2} \to 0,$$

in particular, $\|w_j^-\| \to 0$, hence $w_j \to w = w^0$. Since $r(x, u)/|u|^2 \to 0$ uniformly in x as $|u| \to \infty$ and $|u_j(x)| \to \infty$ if $w(x) \ne 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{r(x, u_j)}{\|u_j\|^2} &= \int_{\mathbb{R}^3} \frac{r(x, u_j)}{|u_j|^2}|w_j|^2 \\ &\le 2\int_{\mathbb{R}^3} \frac{|r(x, u_j)|}{|u_j|^2}|w_j - w|^2 + 2\int_{\mathbb{R}^3} \frac{|r(x, u_j)|}{|u_j|^2}|w|^2 \\ &= o(1) + 2\int_{w(x)\ne 0} \frac{|r(x, u_j)|}{|u_j|^2}|w|^2 = o(1)\end{aligned}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^3} \frac{Q(x)u_j \cdot u_j}{\|u_j\|^2} = \frac{1}{2}\int_{\mathbb{R}^3} \frac{Q(x)u_j \cdot u_j}{|u_j|^2}|w_j|^2 \ge \frac{q_0}{2}|w_j|_2^2$$

It then follows from $\int_{\mathbb{R}^3} \frac{R(x,u_j)}{\|u_j\|^2} \to 0$ that $|w_j|_2 \to 0$, consequently $1 = \|w_j\| \to 0$, a contradiction. Now since

$$\begin{aligned}\|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}^3} Q(x)w \cdot w &\le \|w^+\|^2 - \|w^-\|^2 - q_0|w|_2^2 \\ &\le -\left((q_0 - \mu_\ell)|w^+|_2^2 + \|w^-\|^2 + q_0|w^0|_2^2\right) < 0,\end{aligned}$$

there is $d > 0$ such that

$$\|w^+\|^2 - \|w^-\|^2 - \int_{B_d} Q(x)w \cdot w \, < \, 0. \tag{7.39}$$

Since $|r(x,u)| \le c_1|u|^2$ it follows from the fact $|w_j - w|_{L^2(B_d)} \to 0$ that

$$\lim_{j\to\infty} \int_{B_d} \frac{r(x,u_j)}{\|u_j\|^2} = \lim_{j\to\infty} \int_{B_d} \frac{r(x,u_j)|w_j|^2}{|u_j|^2} = 0.$$

Thus (7.38) and (7.39) imply that

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{B_d} \frac{R(x,u_j)}{\|u_j\|^2} \right) \\ &\le \frac{1}{2} \left(\|w^+\|^2 - \|w^-\|^2 - \int_{B_d} Q(x)w \cdot w \right) \, < \, 0, \end{aligned}$$

a contradiction. □

As a special case we have

Lemma 7.22. *Under the conditions of Lemma 7.21, letting $e \in Y_0$ with $\|e\| = 1$, there is $r_0 > 0$ such that $\sup \Phi(\partial Q) = 0$ where $Q := \{u = u^- + u^0 + se : \ u^- + u^0 \in E^- \oplus E^0, s \ge 0, \|u\| \le r_0\}$.*

We now discuss the Cerami condition. We adapt an argument of [Ding and Jeanjean (2007)] (see also [Ding and Szulkin (2007)]). Remark that by (R_0) and (7.36), given $\gamma_0 \in (q_\infty, \ \gamma)$, there exists $t_0 > 0$ large so that

$$\sup_u \frac{|R_u(x,u)|}{|u|} < \gamma_0 \quad \text{if } |x| \ge t_0. \tag{7.40}$$

Set

$$I_0 := \{x \in \mathbb{R}^3 : \ |x| < t_0\} \ \text{ and } \ I_0^c := \mathbb{R}^3 \setminus I_0.$$

Lemma 7.23. *Let (R_1)-(R_4), (R_0) and (A_0) be satisfied. Then any $(C)_c$-sequence is bounded.*

Proof. Let $(u_j) \subset E$ be such that

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|)\Phi'(u_j) \to 0.$$

Then

$$C_0 \ge \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}^3} \tilde{R}(x,u_j)\,. \tag{7.41}$$

Arguing indirectly we assume that, up to a subsequence, $\|u_j\| \to \infty$ and set $v_j = u_j/\|u_j\|$. Then $\|v_j\| = 1$, $|v_j|_s \le C_s\|v_j\| = C_s$ for all $s \in [2,3]$, and passing to a subsequence if necessary, $v_j \rightharpoonup v$ in E, $v_j \to v$ in L^s_{loc} for all $s \in [1,3)$, $v_j(x) \to v(x)$ for a.e. $x \in \mathbb{R}^3$. Since, by (R_2), $|R_u(x,u)| \le c_1|u|$ and $|u_j(x)| \to \infty$ if $v(x) \ne 0$, it is easy to see that

$$\int_{\mathbb{R}^3} \frac{R_u(x,u_j(x))v_j\varphi(x)}{|u_j(x)|} \to \int_{\mathbb{R}^3} Q(x)v\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, hence

$$Av = Q(x)v. \tag{7.42}$$

We claim that $v \neq 0$. Arguing by contradiction assume $v = 0$. Then $v_j^d \to 0$ in E and $v_j \to 0$ in L^s_{loc}. Observe that

$$\frac{\Phi'(u_j)(u_j^{e+} - u_j^{e-})}{\|u_j\|^2} = \|v_j^e\|^2 - \int_{\mathbb{R}^3} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j|. \tag{7.43}$$

It follows from (7.43) and (7.40) that

$$\begin{aligned}
\|v_j^e\|^2 &= \int_{I_0} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j| \\
&\quad + \int_{I_0^c} \frac{R_u(x,u_j)}{|u_j|}(v_j^{e+} - v_j^{e-})|v_j| + o(1) \\
&\le c_1 \int_{I_0} |v_j||v_j^{e+} - v_j^{e-}| + \gamma_0 \int_{I^c} |v_j||v_j^{e+} - v_j^{e-}| + o(1) \\
&\le o(1) + \gamma_0 |v_j^e|_2^2 \\
&\le o(1) + \frac{\gamma_0}{\gamma}\|v_j^e\|^2
\end{aligned}$$

hence $\left(1 - \frac{\gamma_0}{\gamma}\right)\|v_j^e\|^2 \to 0$, which implies that $1 = \|v_j\|^2 = \|v_j^d\|^2 + \|v_j^e\|^2 \to 0$, a contradiction.

Therefore, $v \neq 0$, This is a contradiction if (i) of (R_4) is satisfied.

Assume (ii) of (R_4) is satisfied. Set $\Omega_j(r,\infty) := \{x \in \mathbb{R}^3 : \ |u_j(x)| \ge r\}$ for $r \ge 0$. By assumption $\tilde{R}(x,u) \ge \delta_0$ if $|u| \ge \nu_0$, hence, $|\Omega_j(\nu_0,\infty)| \le C_0/\delta_0$ by (7.41). Note that v is a solution of (7.42). Set $\Omega := \{x : \ v(x) \neq 0\}$. By the weak unique continuation property for Dirac operator one has $|\Omega| = \infty$ (cf. [Booss-Bavnbej (2000)]). There exist $\varepsilon > 0$ and $\omega \subset \Omega$ such that $|v(x)| \ge 2\varepsilon$ for $x \in \omega$ and $2C_0/\delta_0 \le |\omega| < \infty$. By an Egoroff's theorem we can find a set $\omega' \subset \omega$ with $|\omega'| > C_0/\delta_0$ such that $v_j \to v$ uniformly on ω'. So for almost all j, $|v_j(x)| \ge \varepsilon$ and $|u_j(x)| \ge \nu_0$ in ω'. Then

$$\frac{C_0}{\nu_0} < |\omega'| \le |\Omega_j(\nu_0,\infty)| \le \frac{C_0}{\nu_0},$$

a contradiction. The proof hereby is completed. □

In the following lemma we discuss further the $(C)_c$-sequence $(u_j) \subset E$. By Lemma 7.22 it is bounded, hence, we may assume without loss of generality that $u_j \rightharpoonup u$ in E, $u_j \to u$ in L^q_{loc} for $q \in [1,3)$ and $u_j(x) \to u(x)$ a.e. in x. Plainly u is a critical point of Φ.

Choose $p \in (2,3)$ such that $|R_u(x,u)| \le |u| + C_1|u|^{p-1}$ for all (x,u), and let q stands for either 2 or p. Set $B_d := \{x \in \mathbb{R}^3 : |x| \le d\}$ for $d > 0$. As (7.20) we have: along a subsequence, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \int_{B_n \setminus B_r} |u_{j_n}|^q \le \varepsilon \tag{7.44}$$

for all $r \geq r_\varepsilon$. Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \leq 1$, $\eta(s) = 0$ if $s \geq 2$. Define $\tilde{u}_n(x) = \eta(2|x|/n)u(x)$ and set $h_n := u - \tilde{u}_n$. Since u solves $(\mathcal{P})$, we have by definition that $h_n \in H^1$ and

$$\|h_n\| \to 0 \quad \text{and} \quad |h_n|_p \to 0 \quad \text{as } n \to \infty \tag{7.45}$$

for $p \in [2,3]$. In addition we have

Lemma 7.24. *Under the conditions of Lemma 7.23 we have*

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^3} \left(R_u(x, u_{j_n}) - R_u(x, u_{j_n} - \tilde{u}_n) - R_u(x, \tilde{u}_n) \right) \varphi \right| = 0$$

uniformly in $\varphi \in E$ with $\|\varphi\| \leq 1$.

Proof. Note that (7.44), (7.45) and the compactness of Sobolev embeddings imply that, for any $r > 0$,

$$\lim_{n\to\infty} \left| \int_{B_r} \left(R_u(x, u_{j_n}) - R_u(x, u_{j_n} - \tilde{u}_n) - R_u(x, \tilde{u}_n) \right) \varphi \right| = 0$$

uniformly in $\|\varphi\| \leq 1$. For any $\varepsilon > 0$ let $r_\varepsilon > 0$ so large that (7.44) holds. Then

$$\limsup_{n\to\infty} \int_{B_n \setminus B_r} |\tilde{u}_n|^q \leq \int_{\mathbb{R}^3 \setminus B_r} |u|^q \leq \varepsilon$$

for all $r \geq r_\varepsilon$. Using (7.44) for $q = 2, p$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_{\mathbb{R}^3} \left(R_u(x, u_{j_n}) - R_u(x, u_{j_n} - \tilde{u}_n) - R_u(x, \tilde{u}_n) \right) \varphi \right| \\
&= \limsup_{n\to\infty} \left| \int_{B_n \setminus B_r} \left(R_u(x, u_{j_n}) - R_u(x, u_{j_n} - \tilde{u}_n) - R_u(x, \tilde{u}_n) \right) \varphi \right| \\
&\leq c_1 \limsup_{n\to\infty} \int_{B_n \setminus B_r} \left(|u_{j_n}| + |\tilde{u}_n| \right) |\varphi| \\
&\quad + c_2 \limsup_{n\to\infty} \int_{B_n \setminus B_r} \left(|u_{j_n}|^{p-1} + |\tilde{u}_n|^{p-1} \right) |\varphi| \\
&\leq c_1 \limsup_{n\to\infty} \left(|u_{j_n}|_{L^2(B_n \setminus B_r)} + |\tilde{u}_n|_{L^2(B_n \setminus B_r)} \right) |\varphi|_2 \\
&\quad + c_2 \limsup_{n\to\infty} \left(|u_{j_n}|^{p-1}_{L^p(B_n \setminus B_r)} + |\tilde{u}_n|^{p-1}_{L^p(B_n \setminus B_r)} \right) |\varphi|_p \\
&\leq c_3 \varepsilon^{1/2} + c_4 \varepsilon^{(p-1)/p},
\end{aligned}$$

which implies the conclusion as required. □

Lemma 7.25. *Under the conditions of Lemma 7.23, one has along a subsequence:*

1) $\Phi(u_{j_n} - \tilde{u}_n) \to c - \Phi(u)$;
2) $\Phi'(u_{j_n} - \tilde{u}_n) \to 0$.

Proof. One has

$$\Phi(u_{j_n}-\tilde u_n)=\Phi(u_{j_n})-\Phi(\tilde u_n) + \int_{\mathbb{R}^3}\left(R(x,u_{j_n})-R(x,u_{j_n}-\tilde u_n)-R(x,\tilde u_n)\right).$$

Using (7.44) it is not difficult to check that

$$\int_{\mathbb{R}^3}\left(R(x,u_{j_n})-R(x,u_{j_n}-\tilde u_n)-R(x,\tilde u_n)\right)\to 0.$$

This, together with the facts $\Phi(u_{j_n})\to c$ and $\Phi(\tilde u_n)\to\Phi(u)$, gives 1).

To verify 2), observe that, for any $\varphi\in E$,

$$\Phi'(u_{j_n}-\tilde u_n)\varphi=\Phi'(u_{j_n})\varphi-\Phi'(\tilde u_n)\varphi + \int_{\mathbb{R}^3}\Big(R_u(x,u_{j_n})-R_u(x,u_{j_n}-\tilde u_n)-R_u(x,\tilde u_n)\Big)\varphi.$$

By Lemma 7.24 we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\Big(R_u(x,u_{j_n})-R_u(x,u_{j_n}-\tilde u_n)-R_u(x,\tilde u_n)\Big)\varphi=0$$

uniformly in $\|\varphi\|\le 1$, proving 2). □

Lemma 7.26. *Under the conditions of Lemma 7.23, Φ satisfies the $(C)_c$ condition.*

Proof. In the following we will utilize the decomposition $E=E^d\oplus E^e$. Recall that $\dim(E^d)<\infty$. Write

$$y_n:=u_{j_n}-\tilde u_n=y_n^d+y_n^e.$$

Then $y_n^d=(u_{j_n}^d-u^d)+(u^d-\tilde u_n^d)\to 0$ and, by Lemma 7.25, $\Phi(y_n)\to c-\Phi(u)$, $\Phi'(y_n)\to 0$. Set $\bar y_n^e=y_n^{e+}-y^{e-}$. Observe that

$$o(1)=\Phi'(y_n)\bar y_n^e=\|y_n^e\|^2-\int_{\mathbb{R}^3}R_u(x,y_n)\bar y_n^e. \tag{7.46}$$

It follows from (7.46) that

$$\begin{aligned}\|y_n^e\|^2&\le o(1)+\int_{I_0}\frac{|R_u(x,y_n)|}{|y_n|}|y_n||\bar y_n^e|+\int_{I_0^c}\frac{|R_u(x,y_n)|}{|y_n|}|y_n||\bar y_n^e|\\&\le o(1)+c_1\int_{I_0}|y_n||\bar y_n^e|+\gamma_0\int_{I_0^c}|y_n||\bar y_n^e|\\&\le o(1)+\gamma_0|y_n^e|_2^2\le o(1)+\frac{\gamma_0}{\gamma}\|y_n^e\|^2,\end{aligned}$$

hence $(1-\gamma_0/\gamma)\|y_n\|\le o(1)$, i.e., $y_n\to 0$, finishing the proof. □

7.6.3 *Proof of Theorem 7.8*

In order to prove Theorem 7.8 we apply Theorems 4.5 and 4.6. Set $X = E^- \oplus E^0$ and $Y = E^+$ with $u = x + y$, $x = u^- + u^0$, $y = u^+$ for $u \in E$. Then X is separable and reflexive and so is X^*. We may assume $\mathcal{S}$ is countable and dense in X^*. Therefore, $\mathcal{T}_\mathcal{S}$ is metrizable so its convergence is equivalent to sequentially convergence.

Proof. [Proof of Theorem 7.8] (Existence). Observe that if $c > 0$ and $u_n \in \Phi_c$ with $u_n = x_n + y_n \to u = x + y$ in $\mathcal{T}_\mathcal{S}$ then $y_n \to y$ in norm. (7.35) then implies $\|u_n\|$ is bounded, consequently, $u_n \rightharpoonup u$. Thus by Lemma 7.19

$$c \leq \lim_{n\to\infty} \Phi(u_n) \leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u) = \Phi(u)$$

which proves that Φ_c is $\mathcal{T}_\mathcal{S}$ closed. Lemma 7.19 implies also that $\Phi'(u_n)v \to \Phi'(u)v$ for all $v \in E$, that is, $\Phi' : (\Phi_c, \mathcal{T}_\mathcal{S}) \to (E^*, w^*)$ is continuous. Thus Φ verifies (Φ_0). Lemma 7.19 implies also (Φ_+). Lemmas 7.20 and 7.21 show that Φ possesses the linking structure of Theorem 4.5. Final, Φ satisfies the $(C)_c$-condition by virtue of Lemma 7.26. Therefore, Φ has at least one critical point u with $\Phi(u) \geq \kappa > 0$.

(Multiplicity). Assume moreover $R(x, u)$ is even in u. Then Φ is even, hence satisfies (Φ_1). Lemma 7.20 is nothing but (Φ_2). Lemma 7.21 says that Φ satisfies (Φ_3) with $\dim Y_0 = \ell$. Therefore, Φ has at least ℓ pairs of nontrivial critical points by Theorem 4.6. □

7.6.4 *Proofs of Theorems 7.6 and 7.7*

We now turn to the proofs of Theorems 7.6 and 7.7.

Proof. [Proof of Theorem 7.6] Assume (M_1) holds. Then one has $\mu_e^- = -a$ and $\mu_e^+ = a$. Now Theorem 7.8 applies. □

Remark 7.2. Similarly, one can get existence and multiplicity results of solutions to $(\mathcal{P})$ if the Coulomb potential is replaced by the electrostatic potential $M(x) = \gamma\phi_{el}I_4$ where γ is a positive constant and ϕ_{el} is a real function satisfying, e.g.,

$(\hat{M}_1)$ $\phi_{el} \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, $\phi_{el}(x) \leq 0$,

see [Thaller (1992)]. Another typical example is

$$H = H_0 + \frac{\gamma}{1 + |x|^2}$$

which has finitely many eigenvalues in $(-mc^2, mc^2)$ if $\gamma < 1/8m$ and infinitely many eigenvalues for $\gamma > 1/8m$.

For proving Theorem 7.7 we first establish the following result.

Lemma 7.27. *Assume that* (M_2) *is satisfied. Then*

$$\sigma_e(A) \subset \mathbb{R} \setminus (-(a + b_{\max}),\ (a + b_{\max})),$$

that is, $\mu_e^- \leq -(a + b_{\max})$ *and* $\mu_e^+ \geq (a + b_{\max})$.

Proof. Let $b > 0$ be such that $|\Omega_b| < \infty$. Set

$$(\beta M(x) - b)^+ := \begin{cases} \beta M(x) - b & \text{if } \beta M(x) - b \geq 0 \\ 0 & \text{if } \beta M(x) - b < 0 \end{cases}$$

and $(\beta M(x)-b)^- := (\beta M(x)-b)-(\beta M(x)-b)^+$. We have $A = A_1+\beta(\beta M(x)-b)^-$ where

$$A_1 = -i\sum_{k=1}^{3} \alpha_k\partial_k + (a+b)\beta + \beta(\beta M(x) - b)^+.$$

Since $\beta^2 = I$ and $\beta\alpha_j = -\alpha_j\beta$, we have, for $u \in \mathscr{D}(A)$,

$$\begin{aligned}
&(A_1u, A_1u)_{L^2} \\
&= \left|\left(-i\sum \alpha_k\partial_k + \beta(\beta M - b)^+ + (a+b)\beta\right) u\right|_2^2 \\
&= \left|\left(-i\sum \alpha_k\partial_k + \beta(\beta M - b)^+\right) u\right|_2^2 + (a+b)^2|u|_2^2 \\
&\quad + \left(-i\sum \alpha_k\partial_k u,\ (a+b)\beta u\right)_{L^2} + \left((a+b)\beta u,\ -i\sum \alpha_k\partial_k u\right)_{L^2} \\
&\quad + \left(\beta(\beta M - b)^+u,\ (a+b)\beta u\right)_{L^2} + \left((a+b)\beta u,\ \beta(\beta M - b)^+u\right)_{L^2} \\
&= \left|\left(-i\sum \alpha_k\partial_k + \beta(\beta M - b)^+)u\right) u\right|_2^2 + (a+b)^2|u|_2^2 \\
&\quad + 2(a+b)\left((\beta M - b)^+u,\ u\right)_{L^2} \\
&\geq (a+b)^2|u|_2^2 .
\end{aligned}$$

Thus $\sigma(A_1) \subset \mathbb{R} \setminus (-(a+b), (a+b))$.

We claim that $\sigma_e(A) \cap (-(a+b), (a+b)) = \emptyset$. Assume by contradiction that there is $\mu \in \sigma_e(A)$ with $|\mu| < a+b$. Let $u_n \in \mathscr{D}(A)$ with $|u_n|_2 = 1$, $u_n \rightharpoonup 0$ in L^2 and $|(A-\mu)u_n|_2 \to 0$. Then $\|u_n\|_{H^1}$ is bounded and hence $|\beta(\beta M - b)^-u_n|_2 \to 0$. We get

$$\begin{aligned}
o(1) = |(A-\mu)u_n|_2 &= |A_1u_n - \mu u_n + \beta(\beta M - b)^-u_n|_2 \\
&\geq |A_1u_n|_2 - |\mu| - o(1) \\
&\geq (a+b) - |\mu| - o(1)
\end{aligned}$$

which implies that $0 < (a+b) - |\mu| \leq 0$, a contradiction.

Since the claim keeps true for any $b > 0$ with $|\Omega^b| < \infty$, one sees that $\sigma_e(A) \subset \mathbb{R} \setminus (-(a+b_{\max}),\ (a+b_{\max}))$. □

Remark 7.3. Form the proof of Lemma 7.27 one sees that if (M_2) is replaced by the stronger one

$(\hat{M}_2)$ $|\Omega_b| < \infty$ for any $b > 0$,

then $\sigma(A) = \sigma_d(A)$, that is, the Dirac operator A has only eigenvalues of finite multiplicity.

It follows from Lemma 7.27 that 0 is at most an isolated eigenvalue of finite multiplicity of A. Letting $\tilde{A} = |A| + P^0$ as before one sees that $|u|_2 \le c_1|\tilde{A}u|_2$, consequently, jointly with the assumption that $M \in L^\infty$ by (M_2),

$$|H_0u|_2 \le |Au|_2 + |Mu|_2 \le |Au|_2 + |M|_\infty|u|_2 \le |\tilde{A}u|_2.$$

This implies that $\mathscr{D}(A) \subset H^1$. On the other hand, it follows from

$$|\tilde{A}u|_2 \le |H_0u|_2 + |Mu|_2 + |P^0u|_2 \le c_2|H_0u|_2$$

that $H^1 \subset \mathscr{D}(A)$. Therefore, $\mathscr{D}(A) = H^1$.

We now can give the following

Proof. [Proof of Theorem 7.7] Lemma 7.27 implies (A_0), hence Theorem 7.8 applies and yields the desired conclusions. □

7.7 Semiclassical solutions

Finally we consider the Dirac equation (7.2). A family $u_\hbar$, $\hbar \to 0$, of solutions of (7.2) will be called semiclassical solutions. The semiclassical point of view is important for studying Dirac operators and the semiclassical methods are employed in treating Dirac equation problems, see p. 308 in [Thaller (1992)] and the references therein. We are interested in the potential of the type $M(x) = V(x)\beta$ (i.e., the scalar potential, cf. [Thaller (1992)]). For convenience we rewrite the equation in the form

$$-\varepsilon^2 \sum_{k=1}^{3} i\alpha_k\partial_k u + (a + V(x))\beta u = R_u(x,u) \qquad (\mathcal{P}_\varepsilon)$$

($\varepsilon^2 := \hbar$) where V is a real function satisfying

(V) $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R})$, and there are $x_0 \in \mathbb{R}^3$ and $b > 0$ such that $V(x_0) \le 0$ and $|\Omega_b| < \infty$ where $\Omega_b := \{x \in \mathbb{R}^3 : \ V(x) < b\}$.

Assume the nonlinearity $R(x,u)$ satisfies (R_1)-(R_3) and $(\hat{R}_5)$. We are going to establish the following result:

Theorem 7.9 ([Ding and Ruf (2006)]). *Let (V), (R_1)-(R_3) and $(\hat{R}_5)$ be satisfied. Assume $q_0 > a$. Then there is $\mathcal{E}_0 > 0$ such that $(\mathcal{P}_\varepsilon)$ has at least one solution for each $\varepsilon \in (0, \mathcal{E}_0)$. If additionally $R_u(x,u)$ is odd in $u \in \mathbb{C}^4$ then for each $m \in \mathbb{N}$ there is $\mathcal{E}_m > 0$ such that $(\mathcal{P}_\varepsilon)$ has m solutions for each $\varepsilon \in (0, \mathcal{E}_m)$.*

We note that in this theorem we assume only that $q_0 > a$ which is weaker than (R_4).

Obverse that, by dividing ε^2 and setting $\lambda = 1/\varepsilon^2$ in the equation $(\mathcal{P}_\varepsilon)$, we have the following equivalent problem:

$$-i \sum_{k=1}^{3} \alpha_k\partial_k u + \lambda(a + V(x))\beta u = \lambda R_u(x,u). \qquad (\mathcal{P}_\lambda)$$

We are led to study the existence and multiplicity of solutions of $(\mathcal{P}_\lambda)$ for $\lambda \to \infty$. Therefore, we will prove the following theorem.

Theorem 7.10. *Let (V), (R_1)-(R_3) and $(\hat{R}_5)$ be satisfied. Assume $q_0 > a$. Then there is $\Lambda_0 > 0$ such that $(\mathcal{P}_\lambda)$ has at least one solution for each $\lambda \geq \Lambda_0$. If additionally $R_u(x,u)$ is odd in $u \in \mathbb{C}^4$ then for each $m \in \mathbb{N}$ there is $\Lambda_m > 0$ such that $(\mathcal{P}_\lambda)$ has m solutions for each $\lambda \geq \Lambda_m$.*

For getting this result we will apply Theorem 7.8. For distinguishability we write $A_\lambda = -i\sum_{k=1}^3 \alpha_k \partial_k + \lambda(a+V)\beta$ instead of A. Note that the assumption (V) implies that the matrix $\lambda\beta V$ satisfies (M_2). Therefore by Lemma 7.27 we have the following result.

Lemma 7.28. *Assume that (V) holds. Then*

$$\sigma_e(A_\lambda) \subset \mathbb{R} \setminus (-\lambda(a+b_{\max}),\ \lambda(a+b_{\max}))\,.$$

By virtue of this lemma the space L^2 has the orthogonal decomposition: $L^2 = L_\lambda^- \oplus L_\lambda^0 \oplus L_\lambda^+$ such that A_λ is negative (resp. positive) definite on L_λ^- (resp. on L_λ^+), and $L_\lambda^0 = \ker A_\lambda$. We can define $E_\lambda = \mathscr{D}(|A_\lambda|^{1/2})$ equipped with the inner product

$$(u,v)_\lambda := (|A|_\lambda^{1/2}u, |A|_\lambda^{1/2}v)_{L^2} + (P_\lambda^0 u, P_\lambda^0 v)_{L^2}$$

and the induced norm $\|u\|_\lambda = (u,u)_\lambda^{1/2}$, where $P_\lambda^0 : L^2 \to L_\lambda^0$ denotes the orthogonal projector. E_λ embeds continuously into $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. Hence E_λ embeds continuously into L^p for all $p \in [2,3]$ and compactly into L^p_{loc} for all $p \in [1,3)$. Moreover, E_λ possesses the following decomposition

$$E_\lambda = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+,$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_\lambda$ inner products. On E_λ we define the functional

$$\Phi_\lambda(u) := \frac{1}{2}\|u^+\|_\lambda^2 - \frac{1}{2}\|u^-\|_\lambda^2 - \lambda\int_{\mathbb{R}^3} R(x,u).$$

Then $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its critical points are solutions of $(\mathcal{P}_\lambda)$.

We now prove

Lemma 7.29. *Assume that (V) holds. Then for any $m \in \mathbb{N}$ there is $\Lambda_m > 0$ such that A_λ has at least m eigenvalues (counted in multiplicity) lying in $(0, \lambda q_0)$ for each $\lambda \in [\Lambda_m, \infty)$.*

We will establish this lemma constructively. Observe that since $\sigma_e(A_\lambda) \subset \mathbb{R} \setminus (-\lambda(a+b_{\max}),\ \lambda(a+b_{\max}))$, it is sufficient to show that there exist m linearly independent elements $\varphi \in E_\lambda^+$ with $|\varphi|_2 = 1$ and $\|\varphi\|_\lambda < \lambda q_0$. By assumption, $q_0 > a$. Given

$$0 < \theta < \min\left\{\frac{q_0 - a}{2q_0},\ \frac{1}{2}\right\},$$

set

$$D_\theta := \{x \in \mathbb{R}^3 : \frac{\theta q_0}{2} \le V(x) \le \theta q_0\} \quad \text{and} \quad \Omega_\theta := \text{int } D_\theta.$$

For each $m \in \mathbb{N}$, we choose m real functions $\omega^j \in C_0^\infty(\Omega_\theta, \mathbb{R})$, $j = 1, \dots, m$, satisfying

$$|\omega^j|_2 = 1 \quad \text{and} \quad \text{supp } \omega^j \cap \text{supp } \omega^k = \emptyset \text{ if } j \neq k.$$

Set

$$\varphi_j = (\omega^j, 0, 0, 0) \in C_0^\infty(\Omega_\theta, \mathbb{C}^4) \quad \text{for } j = 1, \dots, m.$$

Clearly $\varphi_1, \dots, \varphi_m$ are linearly independent,

$$\begin{aligned} A_\lambda \varphi_j &= (0, 0, -i\partial_3 \omega^j, -i\partial_1 \omega^j + \partial_2 \omega^j) + (\lambda(a+V)\omega^j, 0, 0, 0) \\ &= (\lambda(a+V)\omega^j, 0, -i\partial_3\omega^j, -i\partial_1\omega^j + \partial_2\omega^j), \end{aligned}$$

$$\Big(-i \sum_{k=1}^3 \alpha_k \partial_k \varphi_j, \ \varphi_j \Big)_{L^2} = 0,$$

and

$$\lambda \left(a + \frac{\theta q_0}{2} \right) \le (A_\lambda \varphi_j, \varphi_j)_{L^2} = \lambda \int_{\mathbb{R}^3} (a+V)|\omega^j|^2 \le \lambda \left(a + \theta q_0 \right), \tag{7.47}$$

$$|A_\lambda \varphi_j|_2^2 = (A_\lambda^2 \varphi_j, \varphi_j)_{L^2} = |\nabla \omega^j|_2^2 + \lambda^2 \int_{\mathbb{R}^3} (a+V)^2 |\omega^j|^2,$$

so

$$|\nabla \omega^j|_2^2 + \lambda^2 \left(a + \frac{\theta q_0}{2} \right)^2 \le |A_\lambda \varphi_j|_2^2 \le |\nabla \omega^j|_2^2 + \lambda^2 \left(a + \theta q_0 \right)^2.$$

For each $\lambda > 0$ we have the representation $\varphi_j = \varphi_{\lambda j}^- + \varphi_{\lambda j}^0 + \varphi_{\lambda j}^+$ $(j = 1, \dots, m)$. Set

$$Z_m := \text{span}\{\varphi_1, \dots, \varphi_m\}, \quad Z_{\lambda m} := \text{span}\{\varphi_{\lambda 1}^+, \dots, \varphi_{\lambda m}^+\}.$$

Lemma 7.30. *For each $\lambda > 0$ and $m \in \mathbb{N}$, $\dim(Z_{\lambda m}) = m$.*

Proof. It suffices to show that $\varphi_{\lambda 1}^+, \dots, \varphi_{\lambda m}^+$ are linearly independent. Suppose that $\sum_{j=1}^m a_j \varphi_{\lambda j}^+ = 0$ with $a_j \in \mathbb{R}$, $j = 1, \dots, m$. Then

$$\sum_{j=1}^m a_j \varphi_j = \sum_{j=1}^m a_j \varphi_{\lambda j}^- + \sum_{j=1}^m a_j \varphi_{\lambda j}^+ = \sum_{j=1}^m a_j \varphi_{\lambda j}^- \in E_\lambda^-,$$

so

$$\begin{aligned} -\left\| \sum_{j=1}^m a_j \varphi_{\lambda j}^- \right\|_\lambda^2 &= \left(A_\lambda \left(\sum_{j=1}^m a_j \varphi_{\lambda j}^- \right), \sum_{j=1}^m a_j \varphi_{\lambda j}^- \right)_{L^2} \\ &= \left(A_\lambda \left(\sum_{j=1}^m a_j \varphi_j \right), \sum_{j=1}^m a_j \varphi_j \right)_{L^2} \\ &= \sum_{j=1}^m |a_j|^2 (A_\lambda \varphi_j, \varphi_j)_{L^2}. \end{aligned}$$

This implies $a_j = 0$ for $j = 1, \dots, m$ because $(A_\lambda \varphi_j, \varphi_j)_{L^2} > 0$ by (7.47). □

In the following we set

$$\alpha := \max\left\{|\nabla\omega^j|_2^2 : j = 1, \ldots, m\right\}$$

which depends on m and the choice of ω^j, but is independent of λ. Denote

$$\hat{u} := \sum_{j=1}^{m} c_j\varphi_j \in Z_m \quad \text{for} \quad u = \sum_{j=1}^{m} c_j\varphi_{\lambda j}^{+} \in Z_{\lambda m}.$$

It is clear that

$$\hat{u}^{+} = u \quad \text{and} \quad |\hat{u}|_2^2 = \sum_{j=1}^{m} c_j^2.$$

Lemma 7.31. *We have:*

i) for each $\lambda \geq 1$, $\zeta|\hat{u}|_2 \leq |u|_2 \leq |\hat{u}|_2$ for all $u \in Z_{\lambda m}$, where $\zeta > 0$ is independent of λ;

ii) for each $\lambda \geq 1$ and all $u \in Z_{\lambda m}$,

$$\lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2 \leq \|u\|_\lambda^2 \leq \lambda\left(\frac{\alpha}{\lambda^2} + (a + \theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2\,;$$

(iii) there is $\Lambda_m > 0$ such that for each $\lambda \geq \Lambda_m$ and all $u \in Z_{\lambda m}$,

$$\|u\|_\lambda^2 - \lambda q_0|u|_2^2 \leq -\lambda q_0\xi_\theta|\hat{u}|_2|u|_2$$

where

$$\xi_\theta = \frac{2a\,(q_0 - a - 2\theta q_0) + (1 - 2\theta)\theta q_0^2}{4q_0(a + \theta q_0)}.$$

Proof. Let $u \in Z_{\lambda m}$. Observe that

$$\begin{aligned}\|u\|_\lambda^2 - \|\hat{u}^{-}\|_\lambda^2 = (A_\lambda\hat{u}, \hat{u})_{L^2} &= \sum_{j=1}^{m}|c_j|^2(A_\lambda\varphi_j, \varphi_j)_{L^2} \\ &\geq \lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2,\end{aligned}$$

$$\begin{aligned}|A_\lambda\hat{u}|_2^2 = \sum_{j=1}^{m}|c_j|^2|A_\lambda\varphi_j|_2^2 &\leq \sum_{j=1}^{m}|c_j|^2\left(|\nabla\omega^j|_2^2 + \lambda^2\,(a + \theta q_0)^2\right) \\ &\leq \left(\alpha + \lambda^2\,(a + \theta q_0)^2\right)|\hat{u}|_2^2,\end{aligned}$$

$$\|u\|_\lambda^2 = (A_\lambda\hat{u}, u)_{L^2} \leq |A_\lambda\hat{u}|_2|u|_2 \leq \left(\alpha + \lambda^2\,(a + \theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2.$$

Hence

$$\lambda\left(a + \frac{\theta q_0}{2}\right)|\hat{u}|_2^2 \leq \|u\|_\lambda^2 \leq \lambda\left(\frac{\alpha}{\lambda^2} + (a + \theta q_0)^2\right)^{1/2}|\hat{u}|_2|u|_2 \tag{7.48}$$

which is the *ii*).

Obviously, $|u|_2 \le |\hat u|_2$. In order to check the first inequality of *i)*, we note that by (7.48)

$$|u|_2 \ge f(\lambda)|\hat u|_2 \quad \text{where} \quad f(\lambda) := \frac{\lambda(2a+\theta q_0)}{2\left(\lambda^2(a+\theta q_0)^2+\alpha\right)^{1/2}}. \tag{7.49}$$

It is clear that $f(\lambda)$ is strictly increasing and

$$\lim_{\lambda\to\infty} f(\lambda) = \frac{2a+\theta q_0}{2(a+\theta q_0)}\,.$$

Hence

$$\frac{2a+\theta q_0}{2\left(\alpha+(a+\theta q_0)^2\right)^{1/2}} \le f(\lambda) < \frac{2a+\theta q_0}{2(a+\theta q_0)} \quad \text{for all } \lambda \ge 1$$

and *i)* follows.

Using (7.48) and (7.49) one sees

$$\begin{aligned}
&\|u\|_\lambda^2 - \lambda q_0|u|_2^2 = (A_\lambda \hat u, u)_{L^2} - \lambda q_0 |u|_2^2 \le \left(|A_\lambda \hat u|_2 - \lambda q_0|u|_2\right)|u|_2 \\
&\le \left(\left(\alpha+\lambda^2(a+\theta q_0)^2\right)^{1/2} - \lambda q_0 \frac{\lambda(2a+\theta q_0)}{2\left(\lambda^2(a+\theta q_0)^2+\alpha\right)^{1/2}}\right)|\hat u|_2|u|_2 \\
&= -\,\lambda q_0 h(\lambda)|\hat u|_2|u|_2
\end{aligned} \tag{7.50}$$

where

$$h(\lambda) = \frac{2a+\theta q_0}{2\left(\frac{\alpha}{\lambda^2}+(a+\theta q_0)^2\right)^{1/2}} - \frac{\left(\frac{\alpha}{\lambda^2}+(a+\theta q_0)^2\right)^{1/2}}{q_0}.$$

Note that

$$\begin{aligned}
\lim_{\lambda\to\infty} h(\lambda) &= \frac{2a+\theta q_0}{2(a+\theta q_0)} - \frac{a+\theta q_0}{q_0} \\
&= \frac{2a\left(q_0-a-2\theta q_0\right)+(1-2\theta)\theta q_0^2}{2q_0(a+\theta q_0)} \\
&= 2\xi_\theta\,.
\end{aligned} \tag{7.51}$$

Now *iii)* follows from (7.50) and (7.51). □

Proof. [Proof of Lemma 7.29] From (*iii*) of Lemma 5.4 we obtain for $\lambda \ge \Lambda_m$

$$\begin{aligned}
\mu_m\left(A_\lambda|_{L_\lambda^+}\right) :&= \inf_{\substack{F\subset E_\lambda^+ \\ \dim(F)=m}} \; \sup_{\substack{\varphi\in E_\lambda^-\oplus F \\ |\varphi|_2=1}} (A_\lambda\varphi,\varphi)_{L^2} \\
&\le \sup_{\substack{u\in Z_{\lambda m} \\ |u|_2=1}} (A_\lambda u, u)_{L^2} \\
&\le \sup_{\substack{u\in Z_{\lambda m} \\ |u|_2=1}} \lambda q_0\left(1-\xi_\theta|\hat u|_2\right) \\
&< \lambda q_0
\end{aligned}$$

as required. □

Proof. [Proof of Theorem 7.10] By Lemma 7.28, we see that (A_0) is satisfied, and we have additionally $\mu_e \geq \lambda(a + b_{\max})$ which, jointly with $(\hat{R}_5)$, implies $\lambda q_\infty < \mu_e$, i.e., (R_0) holds. By Lemma 7.29, for any $m \in \mathbb{N}$, there is $\Lambda_m > 0$ such that the number $\#[(0, \lambda q_0) \cap \sigma(A_\lambda)] \geq m$ for all $\lambda \geq \Lambda_m$. This implies particularly that (R_4) holds, therefore, Theorem 7.8 applies. □

Remark 7.4. Let $\gamma > 0$ be a parameter and consider the supersymmetric Dirac operator $H_\gamma := H_0 + \gamma V \beta$ where H_0 is the free Dirac operator and the scalar field $\gamma V(x)\beta$ satisfies the condition (V). Checking the proof of Lemma 7.29, we have, as a by-product, the following asymptotic estimate on the number of eigenvalues of H_γ.

Lemma 7.32. *Let (V) be satisfied. Then*

$$\sigma_e(H_\gamma) \subset \mathbb{R} \setminus \big(-(a + \gamma b_{\max}),\ a + \gamma b_{\max} \big)$$

and the number $\mathscr{N}(\gamma) := \#[(0,\ a + \gamma b_{\max}) \cap \sigma_d(H_\gamma)] \to \infty$ as $\gamma \to \infty$.

Chapter 8

Solutions of a system of diffusion equations

In this chapter we consider the system

$$\begin{cases} \partial_t u - \Delta_x u + \mathrm{b}(t,x)\cdot\nabla_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v - \mathrm{b}(t,x)\cdot\nabla_x v + V(x)v = H_u(t,x,u,v) \end{cases}$$

for $(t,x) \in \mathbb{R}\times\Omega$, where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $z = (u,v) : \mathbb{R}\times\Omega \to \mathbb{R}^M \times \mathbb{R}^M$, $\mathrm{b} \in C^1(\mathbb{R}\times\overline{\Omega}, \mathbb{R}^N)$, $V \in C(\overline{\Omega}, \mathbb{R})$ and $H \in C^1(\mathbb{R}\times\overline{\Omega}\times\mathbb{R}^{2M}, \mathbb{R})$ depending periodically on t and x. We assume that $H(t,x,0) \equiv 0$ and look for solutions homoclinic to $z = 0$. We deal with the case of $\mathrm{b} = 0$ in the first five sections and the general case in the last section.

8.1 Reviews

We consider firstly the following system:

$$\begin{cases} \partial_t u - \Delta_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v + V(x)v = H_u(t,x,u,v) \end{cases} \quad \text{for } (t,x) \in \mathbb{R}\times\Omega. \tag{FS}$$

Setting

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \mathcal{J}_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \text{and} \qquad A = \mathcal{J}_0(-\Delta_x + V),$$

(FS) reads as

$$\mathcal{J}\partial_t z = -Az + H_z(t,x,z).$$

Thus (FS) can be regarded as an unbounded infinite-dimensional Hamiltonian system in $L^2(\Omega, \mathbb{R}^{2M})$. Our hypotheses on $V : \Omega \to \mathbb{R}$ and $H : \mathbb{R}\times\Omega\times\mathbb{R}^{2M} \to \mathbb{R}$ will be stated below. It follows from these assumptions that $H_u(t,x,0,0) = 0 = H_v(t,x,0,0)$ for all $(t,x) \in \mathbb{R}\times\Omega$. So the constant function $(u_0, v_0) \equiv (0,0)$ is a stationary solution of (FS). We seek solutions $z = (u,v) : \mathbb{R}\times\overline{\Omega} \to \mathbb{R}^{2M}$ of (FS) satisfying the boundary conditions

$$z(t,x) \to 0 \quad \text{as } |t| + |x| \to \infty$$

if $\Omega = \mathbb{R}^N$ or

$$z(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \partial\Omega \qquad \text{and} \qquad z(t,x) \to 0 \quad \text{as } |t| \to \infty$$

when Ω is a bounded smooth domain. So these solutions are homoclinic to the stationary solution $(0,0)$.

For finite-dimensional Hamiltonian systems there are many papers concerning the existence of homoclinic solutions and the dynamics around them. The classical Poincaré-Melnikov method (see [Melnikov (1963)]) is of a perturbative nature. It consists of investigating the intersection of the stable and unstable manifolds of the equilibrium. A bifurcation approach can be found in [Stuart (1989)], for instance. Dynamical systems methods have been extended to deal with various infinite-dimensional Hamiltonian systems, e. g. the KdV-equation (cf. [Kuksin (1993)]). These methods are however not applicable to (FS) simply because the initial value problem for (FS) is not well posed. In the 1990s a variational approach to the existence of homoclinics in finite-dimensional Hamiltonian systems was developed and successfully applied; see [Ambrosetti and Badiale (1998)], [Ambrosetti and Badiale (1998)], [Coti-Zelati, Ekeland and Séré (1990)], [Coti-Zelati and Rabinowitz (1991)], [Ding and Girardi (1999)], [Ding and Willem (1999)], [Hofer and Wysocki (1990)], [Rabinowitz (1990)], [Séré (1992)], [Séré (1993)], [Tanaka (1991)]. With the variational methods it became possible to obtain homoclinics under quite general assumptions on the Hamiltonian. The main technical difficulty is the lack of compactness due to the fact that one has to work on $H^1(\mathbb{R}, \mathbb{R}^{2M})$, and there are no compact embeddings into L^p-spaces. This problem is of course also present when dealing with (FS).

If Ω is a smoothly bounded domain and H is independent of t with $H(x, e^{\theta\mathcal{J}}z) = H(x,z)$ for all $\theta \in \mathbb{R}$, there is a lot of recent work on standing wave solutions to (FS), i. e. solutions of the form $z(t,x) = e^{-t\lambda\mathcal{J}}w(x)$ with w solving the associated stationary Hamiltonian type system of elliptic equations:

$$\begin{cases} Aw + \lambda w = H_w(x,w) & \text{in } \Omega, \\ \qquad w = 0 & \text{on } \partial\Omega; \end{cases}$$

see [Bartsch and de Figueiredo (1999)], [de Figueiredo (1998)] and the references there. In [Bartsch and de Figueiredo (1999)] the case $\Omega = \mathbb{R}^N$ was also treated although only in a setting where $-\Delta_x + V$ has pure point spectrum if restricted to a certain space of symmetric functions.

There is not much work on nonstationary solutions of systems like (FS). Brézis and Nirenberg [Brézis and Nirenberg (1978)] considered the system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \qquad \text{in } (0,T) \times \Omega$$

on a bounded domain where $f, g \in L^\infty(\Omega)$, subject to the boundary conditions $u = v = 0$ on $(0,T) \times \partial\Omega$ and $u(0,x) = v(T,x) = 0$ on Ω. Using Schauder's fixed point theorem they obtained a (generalized) solution (u, v) with $u \in L^4$ and $v \in L^6$ (Theorem V.4 of [Brézis and Nirenberg (1978)]).

In their paper [Clément, Felmer and Mitidieri (1997)] Clément, Felmer and Mitidieri considered the problem

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2} v \\ -\partial_t v - \Delta_x v = |u|^{p-2} u \end{cases} \quad \text{in } (-T,\ T) \times \Omega \tag{8.1}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$, and

$$\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1. \tag{8.2}$$

They proved that there exists $T_0 > 0$ such that for each $T > T_0$, (8.1) has at least one positive solution satisfying the boundary condition

$$u(t, \cdot)|_{\partial\Omega} = 0 = v(t, \cdot)|_{\partial\Omega} \quad \text{for all } t \in (-T, T) \tag{8.3}$$

and the periodicity condition

$$u(-T, \cdot) = u(T, \cdot) \quad \text{and} \quad v(-T, \cdot) = v(T, \cdot).$$

Using the special structure of (8.1) Clément et al. were able to obtain this solution via the mountain pass theorem. Moreover, by passing to the limit as $T \to \infty$, they showed that (8.1) has at least one positive solution defined on $\mathbb{R} \times \Omega$ satisfying (8.3) for all $t \in \mathbb{R}$, and

$$\lim_{|t| \to \infty} u(t, x) = 0 = \lim_{|t| \to \infty} v(t, x) \quad \text{uniformly in } x \in \Omega.$$

Our study of (FS) is motivated by [Clément, Felmer and Mitidieri (1997)]. One of our goals is to develop a variational setting in order to obtain a homoclinic solution of (FS) directly. In addition we can treat nonlinearities depending on both time and space variables. Finally, we are also able to treat the case where $-\Delta_x + V$ has essential spectrum below and above 0. The associated functional will be strongly indefinite and a reduction to the mountain pass theorem is not possible. Moreover, the Palais-Smale condition does not hold. The proof is based on critical point theorems of linking type for strongly indefinite functionals stated previously. The difficulty in applying these theorems to (FS) is to find the proper functional analytic setting. We use the concentration-compactness method in order to control weak limits of Palais-Smale sequences. Applied to the explicit system (8.1) our result is weaker than the one in [Clément, Felmer and Mitidieri (1997)] in the sense that we require $2 < p, q < 2(N+2)/N$ instead of (8.2). On the other hand, we obtain even infinitely many geometrically distinct homoclinic solutions in this case.

The remainder of the chapter is organized as follows. The main results are formulated in the next section. In Section 8.3 we discuss the operators $A = \mathcal{J}_0(-\Delta_x + V)$, $\mathcal{J}A$ and $\mathcal{J}\partial_t + A$. This will be done in an abstract setting which can also be applied to prove the existence of periodic or heteroclinic solutions of (FS). Moreover, it seems to be applicable to other infinite-dimensional Hamiltonian systems. In

Section 8.4 we establish the functional setting for the variational approach to (FS), including, in particular, embedding properties between certain function spaces and the regularity theory which we need. These results are also useful when one wants to treat other types of functions H. Then, in Section 8.5, we prove the main results. Finally, in the last section we discuss some extensions of the results.

8.2 Main results

We treat the two cases where $\Omega = \mathbb{R}^N$, or $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain simultaneously. First we formulate the hypotheses on the potential V.

(V_1) $V \in C(\overline{\Omega}, \mathbb{R})$; if $\Omega = \mathbb{R}^N$ then V is T_j-periodic in x_j for $j = 1, \cdots, N$.

As a consequence of (V_1) the operator $S = -\Delta_x + V$ is a selfadjoint operator on $L^2(\Omega)$. The domain of S is $\mathscr{D}(S) = W^{2,2} \cap W_0^{1,2}(\Omega, \mathbb{R}^{2M})$. By $\sigma(S)$ we denote the spectrum of S. Our second assumption on V is

(V_2) $0 \notin \sigma(S)$

Observe that $\sigma(S) \subset \mathbb{R}$ is bounded below. If $\Omega = \mathbb{R}^N$ then $\sigma(S)$ is purely continuous. It is allowed that S has essential spectrum below 0.

The general assumptions on the Hamiltonian H are:

(H_1) $H \in C^1(\mathbb{R} \times \Omega \times \mathbb{R}^{2M}, \mathbb{R})$ is T_0-periodic in t; if $\Omega = \mathbb{R}^N$ then H is T_j-periodic in x_j for $j = 1, \cdots, N$;

(H_2) there is $\beta > 2$ such that
$$0 < \beta H(t,x,z) \le H_z(t,x,z)z \quad \text{for all } t \in \mathbb{R},\ x \in \Omega, z \neq 0;$$

(H_3) there are $\alpha \in (2, 2(N+2)/N)$ and $a_1 > 0$ such that
$$|H_z(t,x,z)|^{\alpha'} \le a_1 H_z(t,x,z)z \quad \text{for all } t \in \mathbb{R},\ x \in \Omega,\ |z| \ge 1;$$
where $\alpha' := \alpha/(\alpha-1)$ is the dual exponent;

(H_4) $H_z(t,x,z) = o(|z|)$ as $z \to 0$ uniformly in t and x.

The model nonlinearity is
$$H(t,x,u,v) = a(t,x)|u|^p + b(t,x)|v|^q \tag{8.4}$$
with $2 < p, q < 2(N+2)/N$; $a, b : \mathbb{R} \times \Omega \to (0, \infty)$ are required to be T_0-periodic in the t-variable, and T_j-periodic in x_j if $\Omega = \mathbb{R}^N$.

In order to state our results we introduce for $r \ge 1$ the Banach space
$$\begin{aligned} B_r &= B_r(\mathbb{R} \times \Omega, \mathbb{R}^{2M}) \\ &:= W^{1,r}\big(\mathbb{R}, L^r(\Omega, \mathbb{R}^{2M})\big) \cap L^r\big(\mathbb{R}, W^{2,r} \cap W_0^{1,r}(\Omega, \mathbb{R}^{2M})\big) \end{aligned}$$
equipped with the norm
$$\|z\|_{B_r} = \left(\int_{\mathbb{R}\times\Omega} \left(|z|^r + |\partial_t z|^r + \sum_{j=1}^N \left| \partial_{x_j}^2 z \right|^r \right) \right)^{1/r}.$$
B_r is sometimes called anisotropic space. Clearly B_2 is a Hilbert space.

Theorem 8.1 ([Bartsch and Ding (2002)]). *Suppose* (V_1)*,* (V_2) *and* $(H_1) - (H_4)$ *hold. Then* (FS) *has at least one nontrivial solution* z *which lies in* $B_r(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$ *for* $2 \le r < \infty$*.*

In order to state a multiplicity result we require moreover the following assumptions:

(H_5) there are $p \in (2, 2(N+2)/N)$ and $\delta, a_2 > 0$ such that

$$|H_z(t,x,z+w) - H_z(t,x,z)| \le a_2(1+|z|^{p-1})|w|$$

for all $(t,x,z) \in \mathbb{R}\times\Omega\times\mathbb{R}^{2M}$ and $|w| \le \delta$;

(H_6) H is even in z: $H(t,x,-z) = H(t,x,z)$ for all $(t,x,z) \in \mathbb{R}\times\Omega\times\mathbb{R}^{2M}$.

The model nonlinearity (8.4) satisfies $(H_1) - (H_6)$.

In the case $\Omega = \mathbb{R}^N$ two solutions z_1 and z_2 of (FS) are said to be geometrically distinct if $z_1 \ne k * z_2$ for all $0 \ne k = (k_0, k_1, \cdots, k_N) \in \mathbb{Z}^{1+N}$; here

$$k * z(t,x) := z(t + k_0T_0, x_1 + k_1T_1, \cdots, x_N + k_NT_N).$$

For Ω bounded, two solutions z_1 and z_2 of (FS) are said to be geometrically distinct if $z_1 \ne k * z_0$ for all $0 \ne k \in \mathbb{Z}$ where

$$k * z(t,x) := z(t + kT_0, x).$$

Theorem 8.2 ([Bartsch and Ding (2002)]). *Suppose* (V_1)*,* (V_2) *and* $(H_1) - (H_6)$ *hold. Then* (FS) *has infinitely many geometrically distinct solutions* z *which lie in* $B_r(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$ *for* $2 \le r < \infty$*.*

We shall only give the details of the proofs in the case $\Omega = \mathbb{R}^N$. If $\Omega \subset \mathbb{R}^N$ is bounded the theorems can be proved similarly and are somewhat easier.

8.3 Linear preliminaries

In this section we discuss the operators $A = \mathcal{J}_0(-\Delta_x + V)$, $\mathcal{J}A$ and $\mathcal{J}\partial_t + A$ in a more general abstract setting. Let $\mathcal{H}_0$ be e (strong) symplectic Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{H}_0}$, the norm $\|\cdot\|_{\mathcal{H}_0}$ and the symplectic form $\omega(\cdot,\cdot)$. This induces the symplectic structure $\mathcal{J} \in \mathcal{L}(\mathcal{H}_0)$ in the usual way defined by: $\omega(w,z) = (\mathcal{J}w,z)_{\mathcal{H}_0}$ for all $w, z \in \mathcal{H}_0$. It follows that $\mathcal{J}^* = -\mathcal{J}$ but not necessarily $\mathcal{J}^2 = -I$. In order to achieve this we replace the inner product $\langle w,z\rangle$ on $\mathcal{H}_0$ by the (equivalent) one $\langle |\mathcal{J}|^{1/2}w, |\mathcal{J}|^{1/2}z\rangle$ where $|\mathcal{J}| = \sqrt{\mathcal{J}^*\mathcal{J}} = \sqrt{-\mathcal{J}^2}$. Thus we may assume that $\mathcal{J}$ satisfies $\mathcal{J}^* = -\mathcal{J}$ and $\mathcal{J}^2 = -\mathcal{J}^*\mathcal{J} = -I$. Now we consider an operator A defined on $\mathscr{D}(A) \subset \mathcal{H}_0$ and such that

(A_1) A is selfadjoint and $0 \notin \sigma(A)$;

(A_2) $\mathcal{J}A + A\mathcal{J} = 0$.

By $(A_1)-(A_2)$, the operator $\mathcal{J}A$ with $\mathscr{D}(\mathcal{J}A)=\mathscr{D}(A)$ is also selfadjoint such that $0\notin\sigma(\mathcal{J}A)$, and thus there are $\alpha<0<\beta$ with $(\alpha,\beta)\cap\sigma(\mathcal{J}A)=\emptyset$. Therefore we have an orthogonal decomposition

$$\mathcal{H}_0=\mathcal{H}_0^-\oplus\mathcal{H}_0^+,\ z=z^-+z^+$$

corresponding to the negative and the positive spectrum of $\mathcal{J}A$. Let $P^\pm:\mathcal{H}_0\to\mathcal{H}_0^\pm$ denote the orthogonal projections, and $\{E(\lambda):\ \lambda\in\mathbb{R}\}$ the spectral family of $\mathcal{J}A$. We have

$$\mathcal{J}A=\int_{-\infty}^{\infty}\lambda\,dE(\lambda)=\int_{-\infty}^{\alpha}\lambda\,dE(\lambda)+\int_{\beta}^{\infty}\lambda dE(\lambda)$$

and

$$P^-=\int_{-\infty}^{\alpha}dE(\lambda)\quad\text{and}\quad P^+=\int_{\beta}^{\infty}dE(\lambda).$$

Setting

$$U(t)=e^{t\mathcal{J}A}=\int_{-\infty}^{\infty}e^{t\lambda}dE(\lambda)$$

we obtain

$$\begin{cases}\|U(t)P^-U(s)^{-1}\|_{\mathcal{H}_0}\le e^{-a(t-s)} & \text{if } t\ge s;\\ \|U(t)P^+U(s)^{-1}\|_{\mathcal{H}_0}\le e^{-a(s-t)} & \text{if } t\le s;\end{cases}\tag{8.5}$$

here $a=\min\{-\alpha,\beta\}>0$. Set $\mathcal{H}:=L^2(\mathbb{R},\mathcal{H}_0)$ with the inner product and norm denoted by $(\cdot,\cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ respectively. Let $L:=(\mathcal{J}\partial_t+A)$ be the selfadjoint operator acting in $\mathcal{H}$ with domain

$$\mathscr{D}(L)=\left\{z\in W^{1,2}(\mathbb{R},\mathcal{H}_0)\colon z(t)\in\mathscr{D}(A)\ a.\ e.,\ \int_{\mathbb{R}}\|Az(t)\|^2_{\mathcal{H}_0}\,dt<\infty\right\}.$$

Proposition 8.1. *If $(A_1)-(A_2)$ hold then $0\notin\sigma(L)$.*

Proof. Arguing indirectly we assume $0\in\sigma(L)$. Then there exists a sequence (z_n) in $\mathscr{D}(L)$ with $\|z_n\|_{\mathcal{H}}=1$ and $\|Lz_n\|_{\mathcal{H}}\to 0$. Setting $w_n:=Lz_n\in L^2(\mathbb{R},\mathcal{H}_0)$ we observe that $\partial_t z_n=\mathcal{J}Az_n-\mathcal{J}w_n$ and

$$z_n(t)=-\int_{-\infty}^{t}U(t)P^-U(s)^{-1}\mathcal{J}w_n(s)ds+\int_{t}^{\infty}U(t)P^+U(s)^{-1}\mathcal{J}w_n(s)ds.$$

Let $\chi^\pm:\mathbb{R}\to\mathbb{R}$ be the characteristic function of $\mathbb{R}_0^\pm$ where $\mathbb{R}_0^-:=(-\infty,0]$ and $\mathbb{R}_0^+:=[0,\infty)$. Then we have

$$\begin{aligned}z_n(t)&=-\int_{\mathbb{R}}U(t)P^-U(s)^{-1}\chi^+(t-s)\mathcal{J}w_n(s)ds\\&\qquad+\int_{\mathbb{R}}U(t)P^+U(s)^{-1}\chi^-(t-s)\mathcal{J}w_n(s)ds\\&=:z_n^-(t)+z_n^+(t)\end{aligned}$$

Now (8.5) implies

$$\|z_n^-(t)\|_{\mathcal{H}_0} \le \int_{\mathbb{R}} e^{-a(t-s)}\chi^+(t-s)\|w_n(s)\|_{\mathcal{H}_0} ds$$

and

$$\|z_n^+(t)\|_{\mathcal{H}_0} \le \int_{\mathbb{R}} e^{-a(s-t)}\chi^-(t-s)\|w_n(s)\|_{\mathcal{H}_0} ds.$$

Setting $g^+(\tau) = e^{-a\tau}\chi^+(\tau)$ and $g^-(\tau) = e^{a\tau}\chi^-(\tau)$ we obtain

$$\|z_n^-(t)\|_{\mathcal{H}_0} \le (g^+ * \|w_n\|_{\mathcal{H}_0})(t)$$

and

$$\|z_n^+(t)\|_{\mathcal{H}_0} \le (g^- * \|w_n\|_{\mathcal{H}_0})(t)$$

where $*$ denotes the convolution. Observe that

$$\int_{\mathbb{R}} g^+ = \int_{\mathbb{R}} g^- = \frac{1}{a}.$$

By the convolution inequality

$$\|z_n^\pm\|_{\mathcal{H}} \le \frac{1}{a}\|w_n\|_{\mathcal{H}} \to 0 \qquad \text{as } n \to \infty,$$

a contradiction. □

By Proposition 8.1 there is an orthogonal decomposition

$$\mathcal{H} = L^2(\mathbb{R}, \mathcal{H}_0) = \mathcal{H}^- \oplus \mathcal{H}^+, \quad z = z^- + z^+,$$

such that L is negative in $\mathcal{H}^-$ and positive in $\mathcal{H}^+$. Let $E = \mathscr{D}(|L|^{1/2})$ be the Hilbert space with the inner product

$$(w, z)_E = (|L|^{1/2}w, |L|^{1/2}z)_{\mathcal{H}}$$

and the norm

$$\|z\|_E = (z, z)_E^{1/2}.$$

Then we have

$$E = E^- \oplus E^+ \quad \text{with} \quad E^\pm = E \cap \mathcal{H}^\pm.$$

Remark 8.1. We point out that the conclusion of Proposition 8.1 remains true if the conditions (A_1) and (A_2) are replaced by

(A_3) A is a bounded and selfadjoint operator with $\sigma(\mathcal{J}A) \cap i\mathbb{R} = \emptyset$.

If A is a bounded selfadjoint operator then the selfadjoint operator L acting on $L^2(\mathbb{R},\mathcal{H}_0)$ has purely continuous spectrum. Indeed, if there is $\lambda\in\mathbb{R}$ and $0\neq z\in L^2(\mathbb{R},\mathcal{H}_0)$ satisfying $Lz=\lambda z$, then $z(t)=e^{t\mathcal{J}(A-\lambda)}z(0)$ for all $t\in\mathbb{R}$. Since $z\in L^2(\mathbb{R},\mathcal{H}_0)$ this yields $z(0)=0$ and therefore $z=0$, a contradiction. If in addition $\sigma(\mathcal{J}A)\cap i\mathbb{R}=\emptyset$ then it is not difficult to verify that $0\notin\sigma(L)$ via an analysis of dichotomy.

More generally, consider a continuous and T-periodic map $A:\mathbb{R}\to\mathscr{L}(\mathcal{H}_0,\mathcal{H}_0)$ with $A(t)$ selfadjoint for $t\in\mathbb{R}$. The monodromy operator $U(T)$ associated to the differential equation $\dot z(t)=\mathcal{J}A(t)z(t)$ is by definition the value at $t=T$ of the solution of the Cauchy problem

$$\dot U(t)=\mathcal{J}A(t)U(t),\quad U(0)=I.$$

If $U(T)$ has a logarithm (this is the case, in particular, if $\sigma(U(T))$ does not contain a closed curve surrounding the origin), then $\sigma(L)$ consists of continuous spectrum. If, in addition, the mean value $\overline{A}:=T^{-1}\int_0^T A(t)dt$ satisfies $\sigma(\mathcal{J}\overline{A})\cap i\mathbb{R}=\emptyset$, then $0\notin\sigma(L)$. For details we refer to [Ding and Willem (1999)].

8.4 Functional setting

We return to the system (FS) and observe that both operators $A=\mathcal{J}_0S=\mathcal{J}_0(-\Delta+V)$ and $\mathcal{J}A$ acting on $\mathcal{H}_0=L^2(\Omega,\mathbb{R}^{2M})$ are selfadjoint with domains $\mathscr{D}(A)=\mathscr{D}(\mathcal{J}A)=W^{2,2}\cap W_0^{1,2}(\Omega,\mathbb{R}^{2M})$.

Lemma 8.1. *If $0\notin\sigma(S)$ then $0\notin\sigma(A)\cup\sigma(\mathcal{J}A)$.*

Proof. We only show that $0\notin\sigma(\mathcal{J}A)$ since $0\notin\sigma(A)$ can be proved similarly. Arguing indirectly assume $0\in\sigma(\mathcal{J}A)$. Then there exist elements $z_n=(u_n,v_n)\in\mathscr{D}(\mathcal{J}A)$ with $|z_n|_2^2=|u_n|_2^2+|v_n|_2^2=1$ and $|\mathcal{J}Az_n|_2^2=|Su_n|_2^2+|Sv_n|_2^2\to0$. Without loss of generality we may assume that $|u_n|_2\geq\delta$ (where $\delta>0$ is a constant). Then, setting $\widetilde{u}_n:=u_n/|u_n|_2$ we have $\widetilde{u}_n\in\mathscr{D}(S)$, $|\widetilde{u}_n|_2=1$ and $|S\widetilde{u}_n|_2=|Su_n|_2/|u_n|_2\leq|Su_n|_2/\delta\to0$ as $n\to\infty$. This implies $0\in\sigma(S)$, a contradiction. □

As a consequence of Lemma 8.1, we have

$$d_1\|z\|_{W^{2,2}}^2\leq|Az|_2^2=\int_\Omega|Az|^2\leq d_2\|z\|_{W^{2,2}}^2 \tag{8.6}$$

for all $z\in W^{2,2}\cap W^{1,2}(\Omega,\mathbb{R}^{2M})$, where d_1,d_0 denote generic positive constants.

As in Section 8.3 let $\mathcal{H}:=L^2(\mathbb{R},\mathcal{H}_0)$ with its inner product denoted again by $(\cdot,\cdot)_{L^2}$. Then

$$\mathcal{H}\cong L^2(\mathbb{R}\times\Omega,\mathbb{R}^{2M})\cong\left[L^2(\mathbb{R}\times\Omega)\right]^{2M}\cong\left[L^2(\mathbb{R})\otimes L^2(\Omega)\right]^{2M}$$

with equivalent norms, where $\otimes$ is the tensor product. Recall that the set

$$\begin{aligned}&C_0^\infty(\mathbb{R})\otimes C_0^\infty(\Omega,\mathbb{R}^{2M})\\&=\left\{\sum_{i=1}^n f_ig_i:n\in\mathbb{N},\ f_i\in C_0^\infty(\mathbb{R}),\ g_i\in C_0^\infty(\Omega,\mathbb{R}^{2M})\right\}\end{aligned}$$

is dense in both $\mathcal{H}$ and $B_r(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$ for all $r \geq 1$. Let $L := \mathcal{J}\partial_t + A$ be the selfadjoint operator acting in $\mathcal{H}$ with $\mathscr{D}(L) = B_2(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$. That the norms in $\mathscr{D}(L)$ and B_2 are equivalent is a consequence of Lemma 8.3 below. It is clear that the assumption (A_2) of the previous section holds. In addition, Lemma 8.1 implies that (A_1) is also satisfied provided $0 \notin \sigma(S)$. Therefore Proposition 8.1 yields the following lemma.

Lemma 8.2. *If* $0 \notin \sigma(S)$ *then* $0 \notin \sigma(L)$.

Now we consider the operator $L_0 := \mathcal{J}\partial_t + \mathcal{J}_0(-\Delta+1)$. This is a selfadjoint operator in $\mathcal{H}$ with domain $\mathscr{D}(L_0) = \mathscr{D}(L)$. Since $-\Delta + 1 \geq 1$ Lemma 8.2 implies $0 \notin \sigma(L_0)$. Note that $L = L_0 + \mathcal{J}_0(V-1)$.

Lemma 8.3. *For every* $r \geq 1$ *there exist constants* $d_1, d_2 > 0$ *such that*

$$d_1\|z\|_{B_r}^r \leq |L_0 z|_r^r = \int_{\mathbb{R}\times\Omega} |L_0 z|^r \leq d_2\|z\|_{B_r}^r \quad \text{for all } z \in B_r.$$

Consequently, $L_0 : B_r \to L^r$ *is an isomorphism,* $r \geq 1$.

Proof. We consider first the case $\Omega = \mathbb{R}^N$. Let $\mathcal{F}_t$ and $\mathcal{F}_x$ be the Fourier transforms in t and x respectively, and $\mathcal{F} := \mathcal{F}_t \circ \mathcal{F}_x$ the Fourier transform in (t,x). Recall that $z \in B_r(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M})$ if and only if $(1+\tau^2+|y|^4)^{1/2}|(\mathcal{F}z)(\tau,y)| \in L^r(\mathbb{R}\times\mathbb{R}^N)$. This in turn is equivalent to the statement that both $(1+\tau^2)^{1/2}|(\mathcal{F}_t z)(\tau,x)|$ and $|(1+|y|^4)^{1/2}|(\mathcal{F}_x z)(t,y)|$ are in $L^r(\mathbb{R}\times\mathbb{R}^N)$. Next we observe that the following norms are equivalent:

$$\begin{aligned}\|z\|_{B_r} &\sim \left|\left(1+\tau^2+|y|^4\right)^{1/2}(\mathcal{F}z)(\tau,y)\right|_r \\ &\sim \left|\left(1+\tau^2\right)^{1/2}(\mathcal{F}_t z)(\tau,x)\right|_r + \left|\left(1+|y|^4\right)^{1/2}(\mathcal{F}_x z)(t,y)\right|_r\end{aligned}$$

By a direct calculation we get

$$|(\mathcal{F}(L_0 z))(\tau,y)| = \left(\tau^2 + (1+|y|^2)^2\right)^{1/2}|(\mathcal{F}z)(\tau,y)|$$

and the desired result for $\Omega = \mathbb{R}^N$ follows. The case that Ω is bounded can be dealt with similarly by noting that $z \in B_r(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$ if and only if $\phi z \in B_r(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}^{2M})$ for all $\phi \in C_0^\infty(\mathbb{R}\times\Omega,\mathbb{R})$. □

Now we turn to the selfadjoint operator L. By Lemma 8.2 there exists $b > 0$ such that $[-b,b]\cap\sigma(L) = \emptyset$. Let $\{F(\lambda) : \lambda \in \mathbb{R}\}$ be the spectral family of L and $U = 1 - 2F(0)$. Then U is a unitary isomorphism of $\mathcal{H}$ and $L = U|L| = |L|U$. There is an associated orthogonal decomposition

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+, \quad z = z^- + z^+,$$

where $\mathcal{H}^\pm = \{z \in \mathcal{H} : Uz = \pm z\}$. From

$$|Lz|_2^2 = \int_{-\infty}^{-b} \lambda^2 d(F(\lambda)z,z)_{L^2} + \int_b^\infty \lambda^2 d(F(\lambda)z,z)_{L^2} \geq b^2|z|_2^2$$

it follows that

$$|Lz|_2^2 \le |z|_2^2 + |Lz|_2^2 \le (1+b^{-2})|Lz|_2^2. \tag{8.7}$$

Therefore $\mathscr{D}(L)$ equipped with the the inner product

$$(z_1, z_2)_L = (Lz_1, Lz_2)_{L^2}$$

is a Hilbert space.

Lemma 8.4. *If $0 \notin \sigma(S)$ then for all $z \in \mathscr{D}(L)$*

$$d_1\|z\|_{B_2} \le \|z\|_L \le d_2\|z\|_{B_2}.$$

Proof. Given $f_1, f_2 \in C_0^\infty(\mathbb{R})$ and $g_1, g_2 \in C_0^\infty(\Omega, \mathbb{R}^{2M})$ integration by parts yields

$$\begin{aligned}
&\int_{\mathbb{R}\times\Omega} \Big(\langle (\partial_t f_1) J g_1, f_2 \cdot A g_2 \rangle + \langle f_1 \cdot A g_1, (\partial_t f_2) J g_2 \rangle \Big) \\
&= \Big(\int_{\mathbb{R}} (\partial_t f_1) f_2 \Big) \cdot \Big(\int_\Omega \langle J g_1, A g_2 \rangle \Big) + \Big(\int_{\mathbb{R}} f_1 \partial_t f_2 \Big) \cdot \Big(\int_\Omega \langle A g_1, J g_2 \rangle \Big) \\
&= -\Big(\int_{\mathbb{R}} f_1 \partial_t f_2 \Big) \cdot \Big(\int_\Omega \langle J g_1, A g_2 \rangle \Big) + \Big(\int_{\mathbb{R}} f_1 \partial_t f_2 \Big) \cdot \Big(\int_\Omega \langle J g_1, A g_2 \rangle \Big) \\
&= 0.
\end{aligned}$$

Here we also used that $J^T A = A^T J$. It follows that we have for $z = \sum_{i=1}^n f_i g_i \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\Omega, \mathbb{R}^{2M})$:

$$\begin{aligned}
\|z\|_L^2 &= \int_{\mathbb{R}\times\Omega} |Lz|^2 \\
&= \int_{\mathbb{R}\times\Omega} \left| \sum_{i=1}^n \big(J\partial_t(f_i g_i) + A(f_i g_i) \big) \right|^2 \\
&= \int_{\mathbb{R}\times\Omega} \big(|\partial_t z|^2 + |Az|^2 \big) \\
&= |\partial_t z|_2^2 + |Az|_2^2.
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\Omega, \mathbb{R}^{2M})$ is dense in $\mathscr{D}(L) = B_2(\mathbb{R}\times\Omega, \mathbb{R}^{2M})$ the equality $\|z\|_L^2 = |\partial_t z|_2^2 + |Az|_2^2$ holds for all $z \in \mathscr{D}(L)$. The lemma follows. □

Remark 8.2. For $\Omega = \mathbb{R}^N$ Lemma 8.4 implies that $\mathscr{D}(L)$ is continuously embedded in $L^r(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ for r satisfying $2 \le r < \infty$ if $N = 1$, and $0 \le \big(\frac{1}{2} - \frac{1}{r}\big)\big(1 + \frac{N}{2}\big) \le 1$ if $N \ge 2$. $\mathscr{D}(L)$ embeds compactly in $L^r_{loc}(\mathbb{R}\times\Omega, \mathbb{R}^{2M})$ for all $r \ge 2$ if $N = 1$, and if $N \ge 2$ for all $r \ge 2$ satisfying $\big(\frac{1}{2} - \frac{1}{r}\big)\big(1 + \frac{N}{2}\big) < 1$ (see [Besov, Il'in and Nikol'skii (1975)]). In the case where Ω is smoothly bounded recall that

$$\|u\|_{W^{s,r}(\Omega,\mathbb{R}^{2M})} = \inf_{\substack{g \in W^{k,r}(\mathbb{R}^N, \mathbb{R}^{2M}) \\ g|_\Omega = u}} \|g\|_{W^{k,r}(\mathbb{R}^N,\mathbb{R}^{2M})} \tag{8.8}$$

(see [Triebel (1978)], for instance). It follows that the above embedding results also hold when Ω is bounded. Here "compactly in L^r_{loc}" means that the embedding $\mathscr{D}(L) \to L^r((a,b)\times\Omega, \mathbb{R}^{2M})$ is compact for all $-\infty < a < b < \infty$ (see also the proof of Lemma A.1 in [Clément, Felmer and Mitidieri (1997)]).

In the following let $E = \mathscr{D}(|L|^{1/2})$ be equipped with the inner product

$$(z_1, z_2) = (|L|^{1/2} z_1, |L|^{1/2} z_2)_{L^2}$$

and the norm $\|z\| = (z,z)^{1/2}$ as in Section 8.3. We have the decomposition

$$E = E^- \oplus E^+, \qquad \text{where } E^\pm = E \cap \mathcal{H}^\pm$$

which is orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. We write $z = z^- + z^+$ for $z \in E$ according to this decomposition.

Lemma 8.5. *E is continuously embedded in $L^r(\mathbb{R}\times\Omega, \mathbb{R}^{2M})$ for any $r \geq 2$ if $N = 1$, and for $r \in [2, 2(N+2)/N]$ if $N \geq 2$. E is compactly embedded in $L^r_{loc}(\mathbb{R}\times\Omega, \mathbb{R}^{2M})$ for any $r \geq 2$ if $N = 1$, and for $r \in [2, 2(N+2)/N)$ if $N \geq 2$.*

Proof. We only consider the case $N \geq 2$ and $\Omega = \mathbb{R}^N$ since the other cases can be handled similarly. Going to the complexification $\mathcal{H}\times\mathcal{H} \cong \mathcal{H} + i\mathcal{H}$ and using the (complex) interpolation $[\cdot,\cdot]_\theta$ (see [Triebel (1978)]) one sees that

$$E = \mathscr{D}(|L|^{1/2}) \cong [\mathscr{D}(L), L^2]_{1/2}$$

(see also example 3 in Appendix IX.4 of [Reed and Simon (1978)]). By Remark 8.2, the embeddings

$$E \cong [\mathscr{D}(L), L^2]_{1/2} \hookrightarrow [L^r, L^2]_{1/2} \hookrightarrow L^q$$

are continuous for $r = \infty$ if $N = 2$, and $r = 2(N+2)/(N-2)$ if $N \geq 3$, and if q satisfies $\frac{1}{q} = \frac{1}{2}\big(\frac{1}{2} + \frac{1}{r}\big)$, that is, if $q = 2(N+2)/N$. For $r \in (2,q)$, the Hölder inequality implies

$$|z|_r \leq |z|_2^{1-\theta} |z|_q^\theta \quad \text{with } \theta = \frac{q(r-2)}{r(q-2)}.$$

Therefore E is continuously embedded in L^r for $r \in [2, 2(N+2)/N]$. Similarly, using again Remark 8.2 we see that E is compactly embedded in L^r_{loc} for $r \in [1, 2(N+2)/N)$. □

Lemma 8.6. *Under the assumptions of Theorem 8.1 the functional $\Phi : E \to \mathbb{R}$ defined by*

$$\Phi(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) - \int_{\mathbb{R}\times\Omega} H(t,x,z)$$

lies in $C^1(E,\mathbb{R})$. Critical points of Φ are weak solutions of (FS) *and are elements of $B_r(\mathbb{R}\times\Omega, \mathbb{R}^{2M})$ for $2 \leq r < \infty$.*

Proof. From (H_3) and (H_4) it follows that

$$|H_z(t,x,z)| \leq |z| + c|z|^{\alpha-1} \tag{8.9}$$

with $2 < \alpha < 2(N+2)/N$. Using Lemma 8.5 this implies $\Phi \in C^1(E,\mathbb{R})$ in a standard way.

In order to prove the regularity result we need the following embedding theorem from [Besov, Il'in and Nikol'skii (1975)]:

$$B_q \hookrightarrow L^r \quad \text{is continuous for } q > 1,\ 0 \le \frac{1}{q} - \frac{1}{r} \le \frac{2}{N+2}. \tag{8.10}$$

Set

$$\varphi(q) := \begin{cases} (N+2)q/(N+2-2q) & \text{if } 0 < q < (N+2)/2; \\ \infty & \text{if } q \ge (N+2)/2. \end{cases}$$

So $B_q \hookrightarrow L^r$ is continuous for $1 < q \le r < \varphi(q)$ and also for $r = \varphi(q)$ if $\varphi(q) < \infty$.

Now let $z \in E$ be a weak solution of (FS). We set $w = \mathcal{J}_0(1-V)z + H_z(\,.\,,\,.\,,z)$ so that z is a weak solution of $L_0 z = w$, hence

$$z = L_0^{-1} w = L_0^{-1}\big(\mathcal{J}_0(1-V)z + H_z(\,.\,,\,.\,,z)\big).$$

Now we define $\chi_z : \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\chi_z(t,x) = \begin{cases} 1 & \text{if } |z(t,x)| < 1, \\ 0 & \text{if } |z(t,x)| \ge 1, \end{cases}$$

and set

$$w_1(t,x) = \mathcal{J}_0\big(1 - V(x)\big)z(t,x) + H_z(t,x,\chi_z(t,x)z(t,x))$$

and

$$w_2(t,x) = H_z(t,x,(1-\chi_z(t,x))z(t,x)).$$

Then we have $w(t,x) = w_1(t,x) + w_2(t,x)$. From our assumptions on V and H it follows that

$$|w_1(t,x)| \le d|z(t,x)| \tag{8.11}$$

and

$$|w_2(t,x)| \le \begin{cases} 0 & \text{if } \ |z(t,x)| < 1; \\ d|z(t,x)|^{\alpha-1} & \text{if } \ |z(t,x)| \ge 1. \end{cases} \tag{8.12}$$

Thus $w_1 \in L^r$ for $r \in [2, r_1]$ where $r_1 := 2(N+2)/N$, and $w_2 \in L^r$ for $r \in [1, q_1]$ where $q_1 = r_1/(\alpha - 1)$. Here we used that

$$\operatorname{meas}(\{(t,x) \in \mathbb{R} \times \Omega : |z(t,x)| \ge 1\}) \le \int_{\mathbb{R}\times\Omega} |z|^2 < \infty.$$

Now we obtain

$$z_1 := A_0^{-1} w_1 \in B_r \quad \text{ for } r \in [2, r_1] \tag{8.13}$$

and

$$z_2 = L_0^{-1} w_2 \in B_r \quad \text{ for } r \in [1, q_1]. \tag{8.14}$$

We distinguish two cases.

Case 1: $q_1 \geq (N+2)/2$.
Then $z_2 \in L^r$ for all $r \in [q_1, \infty)$ as a consequence of (8.13). By interpolation we get $z_2 \in L^r$ for $r \geq 2$. Since $r_1 > q_1 \geq (N+2)/2$ we similarly obtain $z_1 \in L^r$ for $r \geq 2$.

Case 2: $q_1 < (N+2)/2$.
In this case we define inductively $r_{k+1} := \varphi(q_k)$ and $q_{k+1} := r_{k+1}/(\alpha-1) < r_{k+1}$. Suppose $z_1 \in B_r$ for $r \in [2, r_k]$, and $z_2 \in B_r$ for $r \in [2, q_k]$. Then $z_1 \in L^r$ for $r \in [2, \varphi(r_k)]$, and $z_2 \in L^r$ for $r \in [2, \varphi(q_k)]$, so $z \in L^r$ for $r \in [2, r_{k+1}]$ because $\varphi(r_k) > r_{k+1}$. This implies $w_1 \in L^r$ for $r \in [2, r_{k+1}]$, and $w_2 \in L^r$ for $r \in [2, q_{k+1}]$. We claim that there exists $k_0 \geq 1$ with $q_{k_0} \geq (N+2)/2$. Then we are back in case 1 and therefore done.

By induction one proves that

$$r_k = \frac{2(N+2)}{N(\alpha-1)^{k-1} - 4\sum_{i=1}^{k-2}(\alpha-1)^i} = \frac{2(N+2)(\alpha-2)}{(\alpha-1)^k(N(\alpha-2)-4)+4}.$$

Since $2 < \alpha < 2(N+2)/N = 2 + 4/N$ we see that $\varphi(q_{k-1}) < 0$ for k large enough. This implies $q_{k-1} \geq (N+2)/2$ as required. □

8.5 Solutions to (FS)

As a consequence of Lemma 8.6 it suffices to show the existence of critical points of Φ defined on $E = X \oplus Y$ with $X = E^-$ and $Y = E^+$. Theorem 8.1 will be proved with the help of the critical point Theorem 4.4.

Proof. [Proof of Theorem 8.1] First we verify that the conditions of Theorem 4.4 are satisfied for our functional Φ from Section 8.4 on E.

Since $H(t,x,z) \geq 0$ the functional $\Psi(z) = \int_{\mathbb{R}\times\Omega} H(t,x,z)$ is bounded from below. Let $z_n \rightharpoonup z$. Then Lemma 8.5 implies $z_n \to z$ in L^2_{loc}, hence $z_n \to z$ for a.e. $(t,x) \in \mathbb{R} \times \Omega$. By Fatou's lemma we obtain

$$\liminf_{n\to\infty} \int_{\mathbb{R}\times\Omega} H(t,x,z_n) \geq \int_{\mathbb{R}\times\Omega} \lim_{n\to\infty} H(t,x,z_n) = \int_{\mathbb{R}\times\Omega} H(t,x,z)$$

which proves the lower semicontinuity of Ψ. For any $w \in C_0^\infty$ the dominated convergence theorem yields

$$\Psi'(z_n)w = \int_{\mathbb{R}\times\Omega} H_z(t,x,z_n)w \to \Psi'(z)w \quad \text{as } n \to \infty.$$

This, together with (8.10) implies that Ψ' is weakly sequentially continuous. An application of Theorem 4.1 shows that Φ verifies (Φ_0).

Observe that (H_3) and (H_4) imply that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ with

$$H(t,x,z) \leq \varepsilon|z|^2 + c_\varepsilon|z|^\alpha \quad \text{for all } (t,x,z). \tag{8.15}$$

Thus we have

$$\Phi(z) \geq \frac{1}{2}\|z\|^2 - \varepsilon|z|_3^2 - c_\varepsilon|z|_\alpha^\alpha \quad \text{for every } z \in E^+.$$

Now because $\alpha > 2$ it is easy to see that Φ checks (Φ_2): there exists $r > 0$ with $\kappa := \inf \Phi(S_r Y) > \Phi(0) = 0$.

Consider $e \in E^+$ with $\|e\| = 1$. (H_2) and (H_3) yield that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$H(t,x,z) \geq c_\varepsilon |z|^\beta - \varepsilon |z|^2 \quad \text{for all } (t,x,z). \tag{8.16}$$

Therefore, for $z = z^- + \zeta e$ we have

$$\Phi(z) \leq \frac{1}{2}\left(\zeta^2 - \|z^-\|^2\right) + \varepsilon |z|_2^2 - c_\varepsilon |z|_\beta^\beta$$

hence there is $R > r$ such that $\sup \Phi(\partial Q) = 0$ where $Q := \{z + \zeta e : \ z \in E^-, \|z\| < R, 0 < \zeta < R\}$.

Now Theorem 4.4 yields a sequence $(z_k)_k$ such that $\Phi'(z_k) \to 0$ and $\Phi(z_k) \to c$ with $\kappa \leq c \leq \sup \Phi(\overline{Q})$. A standard computation using $(H_2) - (H_4)$ shows that $(z_k)_k$ is bounded. We claim that there exist $a > 0$ and a sequence $(y_k)_k$ in $\mathbb{R} \times \Omega$ such that (possibly after passing to a subsequence)

$$\lim_{k \to \infty} \int_{B(y_k,1)} |z_k|^2 \geq a. \tag{8.17}$$

Indeed, if not, then by a variation of Lions' concentration compactness lemma [Lions (1984)] we have $z_k \to 0$ in L^s for any $s \in (2, (2N+4)/N)$. Now from (H_3) and (H_4) it follows that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$|H_z(t,x,z)| \leq \varepsilon |z| + c_\varepsilon |z|^{\alpha - 1} \quad \text{for all } (t,x,z).$$

Therefore, using the Hölder inequality we obtain

$$\lim_{k \to \infty} \int_{\mathbb{R}^{1+N}} H_z(t,x,z_k) z_k^\pm = 0$$

which yields

$$\|z_k^+\|^2 = \Phi'(z_k) z_k^+ + \int_{\mathbb{R} \times \Omega} H_z(t,x,z_k) z_k^+ \to 0.$$

This implies $\lim_{k \to \infty} \Phi(z_k) \leq 0$, a contradiction. Now by (8.17) we may assume that there exist $\rho > 0$ independent of k and $y_k' \in T_0 \mathbb{Z}$ if Ω is bounded, $y_k' \in T_0 \mathbb{Z} \times \cdots \times T_N \mathbb{Z}$ if $\Omega = \mathbb{R}^N$ satisfying

$$\int_{B(y_k', \rho)} |z_k|^2 > a/2. \tag{8.18}$$

We shift z_k by y_k' and obtain $\bar{z}_k(t,x) := y_k' * z_k$. Clearly $\|\bar{z}_k\| = \|z_k\|$ and we may suppose that $\bar{z}_k \to z$ weakly in E and strongly in $L^2_{loc}(\mathbb{R} \times \Omega, \mathbb{R}^{2M})$. By (8.18) and the periodicity of H we obtain $z \neq 0$ and $\Phi'(z) = 0$. □

We now turn to the multiplicity result Theorem 8.2.

Proof. [Proof of Theorem 8.2] We will apply Theorem 4.7. (Φ_0) and (Φ_2) have already been verified above. Clearly (Φ_1) is satisfied since H is even in z and $H(t,x,0) = 0$. (Φ_4) can be shown as the verification of linking structure in the proof of Theorem 8.1. The proof will be completed in an indirect way. Namely, we show that if

$$\text{(FS) has only finitely many geometrically distinct solutions} \tag{8.19}$$

then condition (Φ_I) is satisfied. Then we apply Theorem 4.7 and obtain an unbounded sequence of critical values which contradicts (8.19). Consequently, (8.19) is wrong and (FS) has infinitely many geometrically distinct solutions. It does not follow that these solutions have unbounded energy. A similar argument has been used in [Bartsch and Ding (1999)], [Séré (1992)]. So we assume (8.19). There is $\alpha > 0$ satisfying

$$\inf \Phi\left(\mathcal{K} \setminus \{0\}\right) > \alpha$$

where $\mathcal{K} := \{z \in E : \ \Phi'(z) = 0\}$. Let $\mathcal{F} \subset \mathcal{K}$ consist of arbitrarily chosen representatives of the orbits of $\mathcal{K}$ under the action of $\mathbb{Z}^{1+N}$. By the evenness of H with respect to z we may assume that $\mathcal{F} = -\mathcal{F}$. Let $[r]$ denote the integer part of r for any $r \in \mathbb{R}$. A standard concentration-compactness argument as, for example, in [Coti-Zelati and Rabinowitz (1992)] or [Kryszewski and Szulkin (1998)] yields the following claim:

($\star$) Let $(z_n)_n$ be a $(PS)_c$-sequence for Φ. Then $c \geq 0$, (z_n) is bounded, and either $z_n \to 0$ (corresponding to $c = 0$); or $c \geq \alpha$ and there are $\ell \leq [c/\alpha]$, $w_i \in \mathcal{F}\setminus\{0\}$, $i = 1, ..., \ell$, a subsequence denoted again by (z_n), and ℓ sequences $(a_{in})_n \subset \mathbb{Z}$ if Ω is bounded, $(a_{in})_n \subset \mathbb{Z}^{1+N}$ if $\Omega = \mathbb{R}^N$, $i = 1, \cdots, \ell$ such that

$$\left\| z_n - \sum_{i=1}^{\ell} a_{in} * w_i \right\| \to 0 \text{ as } \ n \to \infty,$$

$$|a_{in} - a_{jn}| \to \infty \text{ as } n \to \infty, \text{if } i \neq j,$$

and

$$\sum_{i=1}^{\ell} \Phi(w_i) = c.$$

It is only in the proof of ($\star$) that the hypothesis (H_5) is being used.

Given a compact interval $J \subset (0, \infty)$ with $d := \max J$ we set $\ell := [d/\alpha]$ and

$$[\mathcal{F}, \ell] := \left\{ \sum_{i=1}^{j} k_i * w_i : \ 1 \leq j \leq \ell, k_i \in \mathbb{Z}, w_i \in \mathcal{F} \right\}$$

if Ω is bounded,

$$[\mathcal{F}, \ell] := \left\{ \sum_{i=1}^{j} k_i * w_i : \ 1 \leq j \leq \ell, k_i \in \mathbb{Z}^{1+N}, w_i \in \mathcal{F} \right\}$$

if $\Omega = \mathbb{R}^N$. As a consequence of ($\star$) we see that $[\mathcal{F}, \ell]$ is a $(PS)_J$-attractor. It is not difficult to check that

$$\inf \left\{ \|u^+ - v^+\| : \ u, v \in [\mathcal{F}, \ell], u^+ \neq v^+ \right\} > 0$$

(see e.g. [Coti-Zelati and Rabinowitz (1992)]). Therefore (Φ_I) is satisfied and Theorem 8.2 is proved. □

8.6 Some extensions

In this section we present some extensions which are motivated by earlier work on Schrödinger equations or on homoclinic solutions of finite-dimensional Hamiltonian systems.

8.6.1 0 *is a boundary point of* $\sigma(S)$

Recall that since the potential V depends periodically on x the spectrum of S is purely continuous and a union of disjoint closed intervals. Thus we are interested in the case where 0 is a boundary point of $\sigma_{\rm ess}(S)$. For notational convenience assume the Hamiltonian $H(t,x,z), z := (u,v)$ is of the form

(h_0) $H(t,x,z) = h(t,x)|z|^p$ where $p \in (2,\ 2(N+2)/N)$, $h \in C(\mathbb{R}\times\overline{\Omega},\mathbb{R})$, $h(t,x) > 0$ and is T_0-periodic in t, T_j-periodic in x_j, $j = 1, ..., N$.

Theorem 8.3. *Assume* (V_1) *and* $0 \in \sigma(S)$ *with* $(0,a)\cap\sigma(S) = \emptyset$ *for some* $a > 0$. *Let* $H(t,x,z)$ *satisfy* (h_0). *Then* (FS) *has infinitely many geometrically distinct solutions which lie in* $B_r(\mathbb{R}\times\Omega,\mathbb{R}^{2M})$ *for any* $r \in [p,\infty)$.

The proof of this theorem proceeds along the way of the argument of Theorem 1.2 of [Bartsch and Ding (1999)] where we considered the equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & x \in \mathbb{R}^N, \\ \qquad\qquad u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Let E be the space of B_2 under the norm

$$\|z\|_p := (||L|^{1/2}z|_2^2 + |z|_p^2)^{1/2}.$$

The space L^2 has the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+$$

such that L is negative (resp. positive) definite on L^- (resp. on L^+). This deduces the direct sum

$$E = E^- \oplus E^+$$

with $E^+ = L^+ \cap B_2$ and E^- being the completion of $L^- \cap B_2$ under the norm $\|\cdot\|_p$. Define on E the functional

$$\Phi(z) := \frac{1}{2}||L|^{1/2}z^+|_2^2 - \frac{1}{2}||L|^{1/2}z^-|_2^2 - \int_{\mathbb{R}\times\Omega} h(t,x)|z|^p.$$

Then $\Phi \in C^1(E,\mathbb{R})$ and critical points of Φ are weak solutions of (FS). The regularity of the solutions may be established similarly to the proof of Lemma 8.6. Now one checks that Φ verifies the assumptions of Theorem 4.7 and completes the proof.

It would be interesting to investigate whether the solutions from Theorem 8.3 are limits of solutions u_λ of (FS) with V replaced by $V+\lambda$, $\lambda \to 0$.

8.6.2 *More general symmetries*

The multiplicity result of Theorem 8.2 remains true if the evenness (H_5) is replaced by more general symmetries. Let $\rho : G \to GL(2M, \mathbb{R})$ be a symplectic representation of the compact Lie group G on $V = \mathbb{R}^{2M}$. Thus $\rho(g)^*\mathcal{J}\rho(g) = \mathcal{J}$ and $\rho(g)^*\mathcal{J}_0\rho(g) = \mathcal{J}_0$ for all $g \in G$. For example, letting $\rho_0 : G \to O(M)$ be an orthogonal representation of G on $\mathbb{R}^M$, the representation

$$\rho(g) := \begin{pmatrix} \rho_0(g) & 0 \\ 0 & \rho_0(g) \end{pmatrix}$$

is a symplectic representation of G on $\mathbb{R}^{2M}$.

The representation ρ is said to be admissible if every continuous equivariant map $\partial\mathcal{O} \to V^{k-1}$, where $\mathcal{O}$ is an open bounded invariant neighbourhood of 0 in V^k, $k \geq 2$, has a zero; see [Bartsch (1993)] for an investigation of admissible representations.

Theorem 8.4. *Suppose* (V_1), (V_2) *and* $(H_1)-(H_5)$ *are satisfied. Suppose moreover that* ρ *is an admissible symplectic representation of a compact Lie group* G *on* $\mathbb{R}^{2M}$ *such that* $H(t, x, \rho(g)z) = H(t, x, z)$ *for all* (t, x, z) *and* $g \in G$. *Then* (FS) *has infinitely many geometrically distinct solutions* z *which lie in* $B_r(\mathbb{R}^{1+N}, \mathbb{R}^{2M})$ *for any* $r \in [2, \infty)$.

For the proof one proceeds as in the proof of Theorem 8.2. Instead of considering an even functional Φ one has to deal with a functional which is invariant with respect to the induced action of G on $\mathscr{D}(L) \subset L^2\left(\mathbb{R}, L^2\left(\mathbb{R}, \mathbb{R}^{2M}\right)\right)$. Checking the proof of Theorem 5.2 in [Bartsch and Ding (1999)] one sees that the admissibility condition is precisely the version of the Borsuk-Ulam theorem which is needed; cf. also [Arioli and Szulkin (1999)].

The extension also holds in the case where Ω is a smoothly bounded domain in $\mathbb{R}^N$.

8.6.3 *More general nonlinearities*

The results of Theorems 8.1 and 8.2 remain true if the Hamiltonian satisfies more general nonlinear assumptions. For simplicity we only consider the situation where $\Omega = \mathbb{R}^N$. Setting

$$\tilde{H}(t, x, z) := \frac{1}{2} H_z(t, x, z)z - H(t, x, z),$$

the conditions (H_2) and (H_3) can be replaced by the following asymptotically linearities

(A_1) $H_z(t, x, z) - V_\infty(t, x)z = o(|z|)$ uniformly in (t, x) as $|z| \to \infty$ with $\inf V_\infty > \sup V$;

(A_2) $\tilde{H}(t, x, z) > 0$ if $z \neq 0$, and $\tilde{H}(t, x, z) \to \infty$ uniformly in (t, x) as $|z| \to \infty$;

or by the more general super linearities

(S_1) $H(t,x,z)/|z|^2 \to \infty$ uniformly in (t,x) as $|z| \to \infty$;
(S_2) $\tilde{H}(t,x,z) > 0$ if $z \neq 0$, and there exist $r > 0$ and $\sigma > 1$ if $N = 1$, $\sigma > 1 + \frac{N}{2}$ if $N \geq 2$ such that $|H_z(t,x,z)|^\sigma \leq c_1 \tilde{H}(t,x,z)|z|^\sigma$ for $|z| \geq r$.

Theorem 8.5. *Let (V_1), (V_2), (H_1) and (H_4) be satisfied. Assume either $(A_1) - (A_2)$ or $(S_1) - (S_2)$ hold. Then* (FS) *has at least one nontrivial solution $z \in B_r$ for all $r \geq 2$. If moreover $H(t,x,z)$ is even in z, then* (FS) *has infinitely many geometrically distinct solutions $z \in B_r$ for all $r \geq 2$.*

The main difference to the proof of Theorems 8.1 and 8.2 lies in the study on $(PS)_c$-sequences replaced by $(C)_c$-sequences. However, this can be carried out along the lines of Chapter 6 for the Schrödinger equations.

8.6.4 *More general systems*

We consider existence and multiplicity of homoclinic type solutions of the following system of diffusion equations on $\mathbb{R} \times \mathbb{R}^N$

$$\begin{cases} \partial_t u - \Delta_x u + \mathrm{b}(t,x)\cdot\nabla_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v - \mathrm{b}(t,x)\cdot\nabla_x v + V(x)v = H_u(t,x,u,v) \end{cases} \qquad (\widehat{\mathrm{FS}})$$

where $\mathrm{b} \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $H \in C^1(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{2M}, \mathbb{R})$. We make the following assumptions on V and b:

(V_0) $a := \min V > 0$, and V is T_j-periodic in x_j for $j = 1, \cdots, N$;
(B_0) $\mathrm{b} \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^N)$, $\mathrm{div}\,\mathrm{b}(t,x) = 0$ and b is T_0-periodic in t and T_j-periodic in x_j for $j = 1, \cdots, N$.

The assumption (B_0) is a gauge condition which according to [Nagasawa (1993)] is harmless but technically necessary. The following result is from [Ding, Luan and Willem (2007)].

Theorem 8.6 ([Ding, Luan and Willem (2007)]). *Let (V_0), (B_0), (H_1) and (H_4) be satisfied. Assume either $(A_1) - (A_2)$ or $(S_1) - (S_2)$ hold. Then* $(\widehat{\mathrm{FS}})$ *has at least one nontrivial solution $z \in B_r$ for all $r \geq 2$. If moreover $H(t,x,z)$ is even in z, then* $(\widehat{\mathrm{FS}})$ *has infinitely many geometrically distinct solutions $z \in B_r$ for all $r \geq 2$.*

The main difference between the proofs of Theorem 8.5 and Theorem 8.6 lies in the establishment of variational frameworks. We outline this as follows.

Let $L := \mathcal{J}(\partial_t + \mathrm{b}\cdot\nabla_x) + A$. With the condition (B_0), L is a selfadjoint operator acting in $L^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ with domain $\mathscr{D}(L) = B_2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$. Let $\sigma(L)$ and $\sigma_e(L)$ denote respectively the spectrum and essential spectrum of L. Let $\underline{\lambda} := \inf\,(\sigma(L)\cap(0,\infty))$.

Recall that the operator $\mathcal{S} = -\Delta_x + V$ is self-adjoint on $L^2(\mathbb{R}^N, \mathbb{R})$. It follows from (V_0) that $\sigma(\mathcal{S}) \subset [a, \infty)$.

Lemma 8.7. *Let* (V_0) *and* (B_0) *be satisfied. Then*

1° $\sigma(L) = \sigma_e(L)$, *i.e., L has only essential spectrum;*
2° $\sigma(L) \subset \mathbb{R} \setminus (-a, a)$*;*
3° $\sigma(L)$ *is symmetric with respect to* 0*, that is,* $\sigma(L) \cap (-\infty, 0) = -\sigma(L) \cap (0, \infty)$*;*
4° $a \le \underline{\lambda} \le \max V$*.*

Proof. Since, by (V_0) and (B_0), L commutes with the $\mathbb{Z}$-action $*$, it is evident that $\sigma(L) = \sigma_e(L)$, hence 1° is true.

In order to show 2°, assume by contradiction that there is $\mu \in (-a, a) \cap \sigma(L)$. Let $z_n = (u_n, v_n) \in \mathscr{D}(L)$ with $|z_n|_2 = 1$ such that $|(L - \mu)z_n|_2 \to 0$. Denoting

$$\bar{z}_n = \mathcal{J}_0 z_n = (v_n, u_n)$$

we get

$$\begin{aligned} ((L - \mu)z_n, \bar{z}_n)_{L^2} &= (\mathcal{J}(\partial_t + \mathrm{b}\cdot\nabla_x)z_n,\, \bar{z}_n)_{L^2} + (\mathcal{S}z_n,\, z_n)_{L^2} - \mu(z_n, \bar{z}_n)_{L^2} \\ &= (\mathcal{S}z_n,\, z_n)_{L^2} - \mu(z_n, \bar{z}_n)_{L^2} \ge a - |\mu|, \end{aligned}$$

that is, $a - |\mu| \to 0$ which is a contradiction. 2° is proved.

In order to check 3° let $\lambda \in \sigma(L) \cap (0, \infty)$ and $z_n \in \mathscr{D}(L)$ with $|z_n|_2 = 1$ and $z_n \rightharpoonup 0$ in L^2 such that $|(L - \lambda)z_n|_2 \to 0$. We will show that $-\lambda \in \sigma(L)$. Define $\hat{z}_n = \mathscr{J}_1 z_n$ where

$$\mathscr{J}_1 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

Then $|\hat{z}_n|_2 = 1$ and $\hat{z}_n \rightharpoonup 0$ in L^2. Observe that $\mathscr{J}_1\mathcal{J} = -\mathcal{J}\mathscr{J}_1$, $\mathscr{J}_1\mathcal{J}_0 = -\mathcal{J}_0\mathcal{J}$ and

$$L\hat{z}_n = -\mathscr{J}_1 L z_n.$$

We get

$$|(L - (-\lambda))\hat{z}_n|_2 = |\mathscr{J}_1(L - \lambda)z_n|_2 = |(L - \lambda)z_n|_2 \to 0.$$

This implies that $-\lambda \in \sigma(L)$. Similarly, it is easy to show that if $\lambda \in \sigma(L) \cap (-\infty, 0)$ then $-\lambda \in \sigma(L)$. This proves 3°.

By 2°, $\underline{\lambda} \ge a$. For further discussion, we regard $\mathcal{J}\partial_t$ as a self-adjoint operator on $L^2(\mathbb{R}, \mathbb{R}^{2m})$, and similarly $-\Delta_x$ as a self-adjoint operator on $L^2(\mathbb{R}^N, \mathbb{R})$. By the Fourier transform, one sees $\sigma(\mathcal{J}\partial_t) = \mathbb{R}$. Take $f_n \in \mathscr{D}(\mathcal{J}\partial_t)$ with $|f_n|_2^2 = \int_{\mathbb{R}} |f_n|^2 dt = 1$ and $|\mathcal{J}\partial_t f_n|_2 \to 0$. Since $\sigma(-\Delta_x) = [0, \infty)$ we can choose $g_n \in \mathscr{D}(-\Delta_x)$ with $|g_n|_2^2 = \int_{\mathbb{R}^N} |g_n|^2 dx = 1$ and $|\Delta_x g_n|_2 \to 0$. Set $z_n = f_n g_n$. Then $|z_n|_2 = 1$ and

$$|Lz_n|_2 \le |\mathcal{J}\partial_t f_n|_2 + |\mathrm{b}|_\infty |\nabla_x g_n|_2 + |\Delta_x g_n|_2 + \max V \;\to\; \max V.$$

This implies that there is $\lambda \in \sigma(L)$ with $a \le |\lambda| \le \max V$. By 3° one has $\pm\lambda \in \sigma(L)$. Hence $\underline{\lambda} \le \max V$, ending the proof of 4°. □

Recall that $L_0 := \mathcal{J}\partial_t + \mathcal{J}_0(-\Delta_x + 1)$ and

$$d_1\|z\|_{B_r}^r \le |L_0 z|_r^r \le d_2\|z\|_{B_r}^r \tag{8.20}$$

for all $z \in B_r$ (see Lemma 8.3).

Lemma 8.8. *Assume that* (V_0) *and* (B_0) *are satisfied. Then*

$$c_1|L_0 z|_2^2 \le |Lz|_2^2 \le c_2|L_0 z|_2^2$$

for all $z \in B_2$*. Consequently,*

$$c_1'\|z\|_{B_2}^2 \le |Lz|_2^2 \le c_2'\|z\|_{B_2}^2$$

for all $z \in B_2$*.*

Proof. The right inequality follows from (8.20) and the relationship

$$Lz = L_0 z + \mathcal{J}_0(V-1)z + \mathcal{J}\mathrm{b}\cdot\nabla_x z$$

which implies

$$|Lz|_2^2 \le |L_0 z|_2^2 + d_3(|z|_2^2 + |\nabla_x z|_2^2) \le c_2|L_0 z|_2^2.$$

We now prove the left inequality. Assume by contradiction that there is a sequence $(z_n)_n \subset B_2$ with $|L_0 z_n|_2 = 1$ and $|Lz_n|_2 \to 0$. Then as before, setting $\bar{z}_n = \mathcal{J}_0 z_n$ one has

$$(Lz_n, \bar{z}_n)_{L^2} = (\mathcal{S}z_n, z_n)_{L^2} = \int_{\mathbb{R}\times\mathbb{R}^N} (|\nabla_x z_n|^2 + V|z_n|^2),$$

hence $\int_{\mathbb{R}\times\mathbb{R}^N}(|\nabla_x z_n|^2 + V|z_n|^2) \le |Lz_n|_2|\bar{z}_n|_2 = |Lz_n|_2 \to 0$. In particular, $|z_n|_2 \to 0$ and $|\mathcal{J}\mathrm{b}\cdot\nabla_x z_n|_2 \to 0$. Observe that

$$\begin{aligned}(\mathcal{J}_0\mathcal{S}z_n, \mathcal{J}\partial_t z_n)_{L^2} &= (\mathcal{J}\partial_t\mathcal{J}_0\mathcal{S}z_n, z_n)_{L^2} = -(\mathcal{J}_0\mathcal{S}\mathcal{J}\partial_t z_n, z_n)_{L^2} \\ &= -(\mathcal{J}\partial_t z_n, \mathcal{J}_0\mathcal{S}z_n)_{L^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}|Lz_n|_2^2 &= |(\mathcal{J}(\partial_t + \mathrm{b}\cdot\nabla_x)z_n + \mathcal{J}_0\mathcal{S}z_n|_2^2 \\ &= |(\partial_t + \mathrm{b}\cdot\nabla_x)z_n|_2^2 + |\mathcal{S}z_n|_2^2 \\ &\quad + (\mathcal{J}(\partial_t + \mathrm{b}\cdot\nabla_x)z_n,\ \mathcal{J}_0\mathcal{S}z_n)_{L^2} + (\mathcal{J}_0\mathcal{S}z_n,\ \mathcal{J}(\partial_t + \mathrm{b}\cdot\nabla_x)z_n)_{L^2} \\ &= |\partial_t z_n|_2^2 + |\mathcal{S}z_n|_2^2 + (\mathcal{J}\partial_t z_n,\ \mathcal{J}_0\mathcal{S}z_n)_{L^2} + (\mathcal{J}_0\mathcal{S}z_n,\ \mathcal{J}\partial_t z_n)_{L^2} + o(1) \\ &= |L_0 z_n|_2^2 + o(1),\end{aligned}$$

that is, $1 = |L_0 z_n|_2^2 = |Lz_n|_2^2 + o(1) \to 0$, a contradiction. Therefore, $c_1|L_0 z|_2^2 \le |Lz|_2^2$ for all $z \in B_2$. □

It follows from Lemma 8.7, that $L^2 = L^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}^{2M})$ possesses the orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+$$

such that L is negative (resp. positive) definite in L^- (resp. L^+).

Let $E := \mathscr{D}(|L|^{1/2})$, the Hilbert space with the inner product

$$(z_1, z_2) = \left(|L|^{1/2} z_1,\, |L|^{1/2} z_2\right)_{L^2}$$

and the norm $\|z\| = (z, z)^{1/2}$. E has the orthogonal decomposition

$$E = E^- \oplus E^+ \quad \text{where} \quad E^\pm = E \cap L^\pm .$$

It is clear that $\|z\|^2 \geq a|z|_2^2$ for all $z \in E$.

Let below $N^* := \infty$ if $N = 1$ and $N^* := 2(N+2)/N$ if $N \geq 2$. As a consequence of Lemma 8.8 we have

Lemma 8.9. *E is continuously embedded in L^r for any $r \geq 2$ if $N = 1$, and for $r \in [2, N^*]$ if $N \geq 2$. E is compactly embedded in L^r_{loc} for all $r \in [1, N^*)$.*

Proof. See Lemma 8.5. □

On E we define the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z) \;\; \text{where} \;\; \int_{\mathbb{R}\times\mathbb{R}^N} H(t, x, z).$$

By assumptions $\Phi \in C^1(E, \mathbb{R})$ and its critical points give rise to solutions of $(\widehat{\mathrm{FS}})$.

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