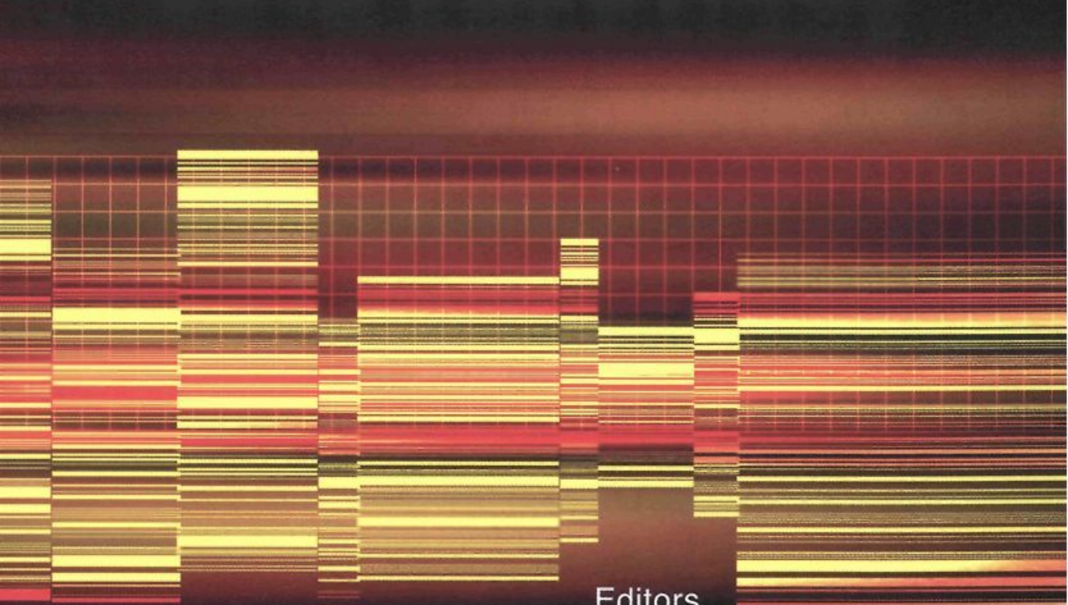
Proceedings of the Conference

in Memory of Tsuneo Arakawa

AUTOMORPHIC FORMS AND ZETA FUNCTIONS



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Rikkyo University, Japan 4 – 7 September 2004

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Proceedings of the Conference in Memory of Tsuneo Arakawa

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In memory of
Tsuneo Arakawa
(1949–2003)

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PREFACE

Tsuneo Arakawa, an eminent researcher in modular forms in several variables and zeta functions, passed away suddenly on October 3, 2003 by rupture of a cerebral aneurysm. This is the proceedings of the "Conference on Automorphic Forms and Zeta Functions" in memory of Tsuneo Arakawa, which was held at Rikkyo University, Tokyo, on September 4–7, 2004. This volume is dedicated to his memory. Most of the papers are based on the lectures given at the conference. Some of the authors, such as Don Zagier and Aloys Krieg, who could not take part in the conference, contributed their papers at our solicitation.

Arakawa's works are reviewed mainly in the first article of this volume. The articles of S. Hayashida and H. Narita may also serve as reviews of Arakawa's achievements on skew holomorphic Jacobi forms and automorphic forms on $Sp(1,q)$, respectively, since they report new results of their own obtained on the basis of Arakawa's results. Many of the other papers are also more or less related to Arakawa's works. The example most notable in this respect is the article of Ibukiyama and Katsurada. The Koecher-Maass type Dirichlet series considered in that paper was first introduced by Arakawa almost 30 years ago, and attracted attention only very recently. We believe that this collection of papers illustrates both the fruitfulness of Arakawa's works and current trends in modular forms in several variables and related zeta functions.

The conference was supported financially by the Grant in Aid for Scientific Research from JSPS (Japan Society of Promotion of Science) (B) No. 16340012 (Principal Researcher F. Sato) and partly by the same JSPS Grant (A) No. 13304002 (Principal Researcher T. Ibukiyama). We thank JSPS for the support. We also thank all the speakers at the conference and the more than 100 participants.

September, 2005

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CONTENTS

TSUNEO ARAKAWA AND HIS WORKS

1. Tsuneo Arakawa (1949 – 2003)

Tsuneo Arakawa was born on April 14, 1949 in Yokohama, Japan. In March 1968, he graduated from the senior high school at Komaba that is affiliated with the Tokyo University of Education (currently University of Tsukuba). He then entered the University of Tokyo. There he finished his bachelor's degree as a mathematics major in 1973 and stayed there for his graduate study. He specialized in Siegel modular forms under the supervision of Yasutaka Ihara. In 1975 he completed his master's thesis, which was later published as [3], and obtained his master's degree. Then immediately he was appointed as a lecturer at the Department of Mathematics, Rikkyo University. He was promoted to the rank of assistant professor in 1984, to associate professor in 1986 and then to professor in 1993. He remained there till his untimely death in 2003. He was awarded the doctorate of science from the University of Tokyo in 1982 for his work on the analytic continuation of the Koecher-Maaß zeta functions. In 1978 he married Miyuki Makiyama and later they had two children.

His first mathematical achievement after his appointment at Rikkyo was the analytic continuation and the determination of the principal part of the Koecher-Maaß zeta functions ([2], [11]). Some of us still remember the day when he reported this result at the Algebra Colloquium of the University of Tokyo. The audience was awed by the great skill with which he carried out the computation, which seemed quite complicated to them. It is well known that Professor Klingen devoted the final chapter of his book *Introductory lectures on Siegel modular forms*, which is a standard reference in the field, to this fundamental result. With this result Arakawa established himself as a member of the group of mathematicians at the forefront of work on modular forms of several variables.

From January 1980 to March 1981 he stayed in Bonn as a visiting member of the SFB Theoretische Mathematik of Bonn University, which later became the Max-Planck-Institut für Mathematik. His second long term stay in Germany was from April 1988 to March 1990, first in Göttingen as a visiting member of SFB 170 and then in Bonn at the Max-Planck-Institut. He enjoyed his several short term visits to Germany, especially to Oberwolfach and Mannheim.

Through his graduate study he became familiar with the works of German mathematicians such as Siegel and Maaß. In fact, the first paper his advisor suggested that he study was Siegel's famous paper "Einführung in die Theorie der Modulfunktionen n-ten Grades". His mathematics follows the tradition of German mathematics. Through his stays in Germany he became acquainted with most of the German mathematicians working in the field. The friendships he nurtured with them are clearly reflected in this memorial volume.

Another mathematician who was quite influential to him was Takuro Shintani, who was an assistant professor at the University of Tokyo when Arakawa was a graduate student. One can see Shintani's influence on him in many of his research papers. One of the notable features of Shintani's works was that he reached the deepest mathematical truth through hard but skillful computations. Arakawa's works share this feature with Shintani's.

As is seen from the list of publications at the end of this article, his works are approximately classified into three subjects:

(1) Siegel modular forms and Jacobi forms,
(2) Selberg zeta functions, and
(3) special values of zeta and L-functions.

His achievements in each subject will be reviewed in each of the following sections.

He was very modest and generous. The articles of Hayashida and Narita in this volume clearly show his generosity in sharing his ideas with young mathematicians. He was also very kind to the students in his classes. He carefully prepared his lectures and he was willing to help his students, spending much time outside the classroom. They appreciated his generosity immensely.

We were deeply dismayed by the news of his abrupt death on October 3, 2003, only a short time after he talked to us at the beginning of the winter semester about his future plans. His death was a great loss for mathematics and for all of us who knew him.

The Editors

2. Arakawa's works on Siegel and Jacobi modular forms

Arakawa worked widely in the area of modular forms :

(i) Dirichlet series associated with modular forms: [1], [2], [6], [11], [21], [25], [26].
(ii) Dimension formula for modular forms: [3], [4], [7], [12], [14], [15].
(iii) Jacobi forms: [13], [19], [21], [17], [20], [24], [28].

(i) Koecher introduced Dirichlet series corresponding to holomorphic Siegel modular forms and Maass obtained their analytic continuations and functional equations using the method of invariant differential operators. The Dirichlet series are now called the Koecher-Maass series.

In his first paper [1], Arakawa investigated several Dirichlet series related to the Fourier coefficients of the Eisenstein series on the Siegel upper half-plane, which can be viewed as an analogue of the Koecher-Maass series for the real analytic Siegel Eisenstein series (cf. the article of Ibukiyama and Katsurada in this volume). He proved their analytic continuations and functional equations using Shintani's method and results on zeta functions associated with quadratic forms.

Arakawa investigated the Koecher-Maass series for holomorphic forms very precisely and gave residue formulas. In [2], he treated the case of Siegel modular forms of level one. His method is based on the Klingen Eisenstein series and the structure theorem of the space of Siegel modular forms. In the Procceeding of Taniguchi Symposium in Katata 1983, Klingen suggested the probelm to prove the theorem without help of Klingen Eisenstein series. Arakawa's first proof, which was different from the one in [2], of the residue formula meeted the demand of this problem. He published it later in [11] and extended the residue formulas to the case of Siegel modular forms with arbitrary level. Furthermore he obtained explicit functional equations and residue formulas for Epstein-Koecher zeta functions. After 8 years, Arakawa returned to this theme again in [21]. He introduced Koecher-Maass series for a Jacobi form of degree n, weight k and index S and obtained a meromorphic continuation and a functional equation following Maass' original method. Moreover under the assumption on the maximality of S, he gave the residue formula.

Arakawa was also interested in L-functions associated with Hecke eigen modular forms. In [6], along the lines of Andrianov, he succeeded to obtain analytic continuations and functional equations of the spinor L-functions associated with vector-valued Siegel modular forms of degree two (under a certain technical assumption on Fourier coefficients). He also considered the spinor L-function associated with automorphic forms on $Sp(1,1)$ belonging to a non-holomophic discrete series (unpublished work).

In [25], Arakawa constructed Saito-Kurokawa lifting from cusp forms of weight $k-1/2$ (k is an odd positive integer) and level 4 to Siegel modular forms of weight k and level 4 with non-trivial character. He used the method of Duke-Imomoḡlu, the converse theorem of Imai and some results of Katok-Sarnak. He also gave another construction of the lifting by means of Eichler-Zagier, where Jacobi forms were effectively used. In [26], Arakawa, Makino and Sato extended Imai's converse theorem to non-cuspidal case. Consequently, another proof of the Saito-Kurokawa lifting for not necessarily cuspidal Siegel modular forms was given.

(ii) In [3], Arakawa gave explicit dimension formulas of the spaces of cusp forms on the Siegel upper half-plane of degree two with respect to some arithmetic groups having only zero dimensional cusps. Such groups are defined from quaternion unitary groups of degree two. The same results had been obtained by Yamaguchi by a different method (using the Hirzeburch-Riemann-Roch theorem). His calculation is based on the Selberg trace formula.

He wrote two papers [4] and [7] on dimension formulas of certain non-holomorphic cusp forms on $Sp(1,q)$ (cf. Narita's article in this volume).

The papers [12], [14], [15] treat Selberg zeta functions and dimension formulas of lower weights. The results in these papers will be discussed in the next section.

(iii) In [13], real analytic Jacobi Eisenstein series $E_{k,m}((\tau,z),s)$ of weight k and index m with respect to the Jacobi group were studied. These coincide with the holomorphic Jacobi Eisenstein series introduced by Eichler-Zagier at $s=(k-1/2)/2$. Arakawa proved that $E_{k,m}((\tau,z),s)$ was analytically continued to a meromorphic function in the whole s-plane and satisfied a functional equation between s and $1-s$.

Calculating Rankin-Selberg convolution of Siegel Eisenstein series and Siegel cusp forms was developed by Garrett and Böcherer. It was a powerful technique for investigating Siegel modular forms and their L-functions. In [19], Arakawa showed that this technique was applicable to the case of Jacobi forms of degree n. He established Garrett-Böcherer decomposition of real analytic Jacobi Eisenstein series and obtained an integral representation of the standard L-function of a Hecke eigen Jacobi cusp form.

In [17] and [20], Arakawa studied Siegel formula for Jacobi forms. Let S be a fixed positive definite half-integral symmetric matrix of size l. Denote by $\mathrm{Sym}^*_{m+l}(S : \mathbf{Z})^+$ the set of positive definite half-integral symmetric matrices with lower right $l \times l$ submatrix S. Then

$$B_{m.l}(\mathbf{Z}) := \left\{ \begin{pmatrix} a & 0 \\ x & 1_l \end{pmatrix} \,\middle|\, a \in SL_m(\mathbf{Z}),\ x \in M_{l,m}(\mathbf{Z}) \right\}$$

acts on $\mathrm{Sym}^*_{m+l}(S : \mathbf{Z})^+$ naturally. He introduced the notion of S-calss and S-genus through this action. Each S-genus consits of finitely many S-classes. Assume that $m > n$ and let $Q \in \mathrm{Sym}^*_{m+l}(S : \mathbf{Z})^+$ and $T \in \mathrm{Sym}^*_{n+l}(S : \mathbf{Z})^+$. Denote by $A(Q,T)$ the number of solutions of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $(x_1 \in M_{m,n}(\mathbf{Z}), x_2 \in M_{l,n}(\mathbf{Z}))$ of the equation $Q[\begin{pmatrix} x_1 & 0 \\ x_2 & 1_l \end{pmatrix}] = T$. Let $Q_1, \ldots, Q_H$ be a complete set of representatives of the S-classes in the S-gunus of Q. Then Arakawa proved the "arithmetic Siegel formula"

$$\left(\sum_{j=1}^{H} \frac{A(Q_j,T)}{E(Q_j)} \right) \left(\sum_{j=1}^{H} \frac{1}{E(Q_j)} \right)^{-1} = \varepsilon \prod_v \alpha_v(Q,T).$$

Here $E(Q_j) = \#O(Q_j) \cap B_{m,l}(\mathbf{Z})$, $\alpha_v(Q,T)$ are local densities and $\varepsilon = 1$ or $1/2$. As in the original case, this can be reformulated to the "analytic Siegel formula". He proved that the Jacobi Eisenstein seies coincided with a finite linear sum of theta sereis. In [20], Arakawa obtained a Minkowski-Siegel formula (mass formula) for $Q \in \mathrm{Sym}^*_{m+1}(1, \mathbf{Z})^+$ when $\det(2Q) = 1$.

Jacobi forms of weight k give rise to modular forms of weight k by restriction $z = 0$. More generally, Eichler-Zagier defined a linear mapping $D_\nu : J_{k,1}(\Gamma_0(N), \chi) \longrightarrow M_{k+\nu}(\Gamma_0(N), \chi)$ by using differential operators. Arakawa and Böcherer investigated the kernel of D_0 precisely in [24] and [28]. First they showed that Ker D_0 was isomorphic to $M_{k-1}(\Gamma_0(N), \chi\overline{\omega})$, where ω is the character of $SL_2(\mathbf{Z})$ occuring in the transformation law of η^6. Second, they characterized $D_2(\mathrm{Ker}\ D_0)$ in terms of vanishing orders at cusps. In [28], they showed that the restriction map $J_{2,1}(\Gamma_0(N)) \longrightarrow M_2(\Gamma_0(N))$ was injective for any square free N (This was first observed by Kramer for prime levels). As an application of their results, a conjecture of Hashimoto on theta series was proved.

Takashi Sugano (Kanazawa University)

3. Arakawa's works on Selberg zeta functions

3.1. Tsuneo Arakawa wrote several articles with focus on the relation between the dimension of the space of various automorphic forms and the vanishing order of Selberg zeta functions. In this review, we will pick up some of his results and will give a general framework in which we can understand Arakawa's achievements more clearly. The topics we will discuss in this section are the followings. Let $S_k(\Gamma,\chi)$ be the space of the cusp forms of weight k with respect to a subgroup Γ of $SL_2(\mathbb{Z})$ of finite index with a finite dimensional unitary representation (or a finite dimensional unitary multiplier system) χ. On the other hand let $Z_\Gamma(s,\chi)$ be the Selberg zeta function defined by

$$Z_\Gamma(s,\chi)=\prod_{\{\gamma\},\mathrm{tr}\gamma>0}\prod_{m=0}^{\infty}\det\left(1-\chi(\gamma)N(\gamma)^{-s-m}\right)$$

where $\prod_{\{\gamma\},\mathrm{tr}\gamma>0}$ is the product over the primitive hyperbolic Γ-conjugacy classes of positive trace. Here we assume that Γ contains -1_2. Then the first result of Arakawa's which we will discuss is

Theorem 3.1. ([12])

$$\dim S_2(\Gamma,\chi)=ord_{s=1}Z_\Gamma(s,\chi)+\dim\chi\cdot vol(\Gamma\backslash G)\cdot\frac{1}{4}+[elliptic]+[parabolic],$$

and

$$\dim S_1(\Gamma,\chi)=\frac{1}{2}ord_{s=1/2}Z_\Gamma(s,\chi)+[elliptic]+[parabolic]$$

where [elliptic] and [parabolic] are terms coming from the elliptic conjugacy classes and the parabolic conjugacy classes respectively.

Here $G=SL_2(\mathbb{R})$ and $\mathrm{vol}(\Gamma\backslash G)$ is the volume of the quotient space $\Gamma\backslash G$ with respect to the Haar measure on G which induces the G-invariant measure

$$\frac{1}{\pi}\frac{dxdy}{y^2}\qquad(x+\sqrt{-1}y\in\mathfrak{H})\tag{1}$$

on the complex upper half plane $\mathfrak{H}$ (the Haar measure on the maximal compact subgroup $K=SO(2)$ is normalized so that the volume of K is equal to 1).

We will also discuss the dimension of the space of Jacobi forms. Let S be a positive definite half-integral matrix of odd size l and $J_{k,S}(\Gamma)$ the space of the Jacobi forms of weight k and index S with respect to $\Gamma=SL_2(\mathbb{Z})$. On the other hand the transformation formula of the theta series

$$\vartheta_{S,r}(z,w)=\sum_{q\in\mathbb{Z}^l}\mathbf{e}\left(z\cdot S[q+r]+2S(q+r,w)\right)\quad(z\in\mathfrak{H},w\in\mathbb{C}^l)\tag{2}$$

$(r\in(2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l)$ with respect to Γ produces a multiplier system χ which is canonically decomposed into two sub-multiplier system $\chi_\pm$ (see §3.5 for the precise definition). Here $\mathbf{e}(t) = \exp(2\pi\sqrt{-1}t)$ and $S(X,Y) = XS{}^tY$ and $S[X] = S(X,X)$ as usual. Then the Selberg zeta function $Z_{\Gamma,S,\varepsilon}(s)$ is defined by

$$Z_{\Gamma,S,\varepsilon}(s) = \prod_{\{\gamma\},\mathrm{tr}(\gamma)>0} \prod_{m=0}^{\infty} \det\left(1 - \chi_\varepsilon(\gamma)N(\gamma)^{-s-m}\right).$$

Then Arakawa gives the following dimension formula;

Theorem 3.2. ([14], [15])

$$\dim J^{cusp}_{(l+3)/2,S}(\Gamma) = \mathrm{ord}_{s=3/4} Z_{\Gamma,S,\varepsilon}(s) + \dim \chi_\varepsilon \cdot vol(\Gamma\backslash G) \cdot \frac{1}{8} + [elliptic] + [parabolic],$$

and

$$\dim J_{(l+1)/2,S}(\Gamma) = \mathrm{ord}_{s=3/4} Z_{\Gamma,S,-\varepsilon}(s)$$

with

$$\varepsilon = (-1)^{(l+3)/2} = \begin{cases} 1 & l \equiv 1 \pmod 4 \\ -1 & l \equiv 3 \pmod 4 \end{cases}.$$

Here $J^{cusp}_{(l+3)/2,S}(\Gamma)$ denotes the space of cuspidal Jacobi forms and [elliptic] and [parabolic] are the contributions from the elliptic conjugacy classes and the parabolic conjugacy classes of Γ respectively.

Actually, as we will see in §3.5, the space of Jacobi forms appearing in the theorem above are closely related with the space of the cusp forms of weight 3/2 or 1/2 with respect to Γ with multiplier system χ_ε. So these results (Theorem 3.1 and Theorem 3.2) of Arakawa's are to give dimension formulae of the space of the cusp forms of low weight by means of the Selberg zeta functions.

In the proof of these results, Arakawa used Fischer's Selberg trace formula [F] which involves multiplier systems. In the rest of this section, we will review these results from representation theoretic point of view using a standard trace formula.

3.2. Let us start with an abstract trace formula. Let G be a locally compact unimodular group, Γ a discrete subgroup of G, χ a finite dimensional unitary representation of Γ. For the sake of simplicity, we will assume that $\Gamma\backslash G$ is compact. Let us denote by π_Γ^χ the unitarily induced representation $\mathrm{Ind}_\Gamma^G \chi$ of G. Taking a $f \in L^1(G)$ such that $\pi_\Gamma^\chi(f)$ is trace class operator, we have a trace formula

$$\sum_{\pi\in\widehat{G}} m(\pi,\pi_\Gamma^\chi)\operatorname{tr}\pi(f) = \dim\chi\cdot\operatorname{vol}(\Gamma\backslash G)\cdot f(1)$$
$$+\sum_{\{\gamma\}_\Gamma\neq 1}\operatorname{tr}\chi(\gamma)\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\cdot\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)d\dot{x}. \qquad (3)$$

Here $\widehat{G}$ denotes the set of the unitary equivalence classes of the irreducible unitary representations of G and $m(\pi,\pi_\Gamma^\chi)$ denotes the multiplicity of π in π_Γ^χ which is finite because $\Gamma\backslash G$ is compact. $\sum_{\{\gamma\}_\Gamma\neq 1}$ is the summation over the non-trivial conjugacy classes of Γ. G_γ denotes the centralizer in G of $\gamma\in\Gamma$ and $\Gamma_\gamma=\Gamma\cap G_\gamma$. Since $\Gamma\backslash G$ is compact, the centralizer G_γ is unimodular, and $G_\gamma\backslash G$ has a right G-invariant measure $d\dot{x}$.

Now let K be a compact subgroup of G and δ an irreducible unitary representation of K such that $m(\delta,\pi|_K)<\infty$ for all $\pi\in\widehat{G}$ (for example, if G is a connected semi-simple real Lie group of finite center and K a maximal compact subgroup of G, then $m(\delta,\pi|_K)\leq\dim\delta$ for all $\delta\in\widehat{K}$ and $\pi\in\widehat{G}$). Let us denote by $\widehat{G}(\delta)$ the set of $\pi\in\widehat{G}$ such that $m(\delta,\pi|_K)>0$ which we will call the class-δ representations of G. Assume that $f\in L^1(G)$ satisfies the conditions

(1) K-central, that is, $f(k^{-1}xk)=f(x)$ for all $k\in K$,
(2) $e_\delta * f=f$, where $e_\delta(k)=\dim\delta\cdot\operatorname{tr}\delta(k^{-1})$ and $*$ denotes the convolution product over K.

Then we have $\operatorname{tr}\pi(f)=0$ for all non-class-δ representations π. So, putting such f in the abstract trace formula (3), we will pick up only the class-δ representations on the left hand side of the trace formula which we will call the class-δ trace formula.

3.3. Now let $G=SL_2(\mathbb{R})$, $K=SO(2)$ and $\delta_n\in\widehat{K}$ such that $\delta_n\begin{bmatrix} d & -c\\ c & d\end{bmatrix}=$ $(\sqrt{-1}c+d)^n$ with $n\in\mathbb{Z}$. Let Γ be a discrete subgroup of G. For the sake of simplicity, we assume that Γ is torsion-free and $\Gamma\backslash G$ is compact. The unitary dual of $SL_2(\mathbb{R})$ consists of the following representations:

π	$\pi\|_K$	$\lambda(\pi)$	name
$\pi^{0,s}\quad (s \in \sqrt{-1}\mathbb{R})$	$\bigoplus_{l\in 2\mathbb{Z}} \delta_l$	$s/2$	principal series
$\pi^{1,s}\quad (0 \neq s \in \sqrt{-1}\mathbb{R})$	$\bigoplus_{l\in 2\mathbb{Z}+1} \delta_l$	$s/2$	
$\pi_+^{1,0}$	$\bigoplus_{l\in 2\mathbb{Z}+1, l>0} \delta_l$	0	limit of discrete series
$\pi_-^{1,0}$	$\bigoplus_{l\in 2\mathbb{Z}+1, l<0} \delta_l$	0	
$\pi_n\quad (1 < n \in \mathbb{Z})$	$\bigoplus_{0\leq l\in \mathbb{Z}} \delta_{n+2l}$	$\frac{n-1}{2}$	holomorphic discrete series
$\pi_n\quad (-1 > n \in \mathbb{Z})$	$\bigoplus_{0\leq l\in \mathbb{Z}} \delta_{n-2l}$	$\frac{\|n\|-1}{2}$	anti-holomorphic discrete series
$\pi^\sigma\quad (0 < \sigma < 1)$	$\bigoplus_{l\in \mathbb{Z}} \delta_{2l}$	$\sigma/2$	complementary series
$\mathbf{1}_G$	δ_0	$1/2$	trivial representation

Let Ω be the Casimir element of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. Then, for any $\pi \in \widehat{G}$, the operator $\pi(\Omega) = \frac{1}{2}\left(\lambda(\pi)^2 - \frac{1}{4}\right)$ is a scalar multiplication. The formal degree of a discrete series representation $\pi_n \in \widehat{G}$ ($n \in \mathbb{Z}$, $|n| > 1$) is $(|n|-1)/4$ if the Haar measure on G is fixed so that it induces the G-invariant measure on $\mathfrak{H} = G/K$ given by (1). The square-integrable representation π_n is integrable if and only if $|n| > 2$. Now the Plancherel formula gives

$$f(1) = \sum_{\pi\in\widehat{G}_d} d_\pi \cdot \operatorname{tr}\pi(f) + \frac{1}{4\pi}\sum_{j=0,1}\int_{\mathbb{R}} \operatorname{tr}\pi^{j,2\sqrt{-1}r}(f)\cdot \mu_j(r)dr$$

for $f \in L^1(G)$ satisfying a suitable growth condition. Here $\widehat{G}_d$ is the subset of $\widehat{G}$ consisting of the discrete series and d_π is the formal degree of $\pi \in \widehat{G}_d$. The density function $\mu_j(r)$ is given by

$$\mu_j(r) = \begin{cases} \pi\cdot\tanh(\pi r) & : j = 0 \\ \pi\cdot\coth(\pi r) & : j = 1 \end{cases}.$$

If $\pi \in \widehat{G}_d$ is integrable, then we have the multiplicity formula

$$m(\pi, \pi_\Gamma^\chi) = \dim\chi\cdot \operatorname{vol}(\Gamma\backslash G)\cdot d_\pi.$$

Then, for an $f \in L^1(G)$ which is K-central and $e_{\delta_n} * f = f$ and satisfies a suitable growth condition, we have the class-δ_n trace formula

$$\begin{aligned}\sum_{\pi \in \widehat{G}(\delta_n) \setminus \widehat{G}_d} & m(\pi, \pi_\Gamma^\chi) \cdot \operatorname{tr} \pi(f) + [*] \\ &= \dim \chi \cdot \operatorname{vol}(\Gamma \backslash G) \cdot \frac{1}{4\pi} \int_{\mathbb{R}} \operatorname{tr} \pi^{j, 2\sqrt{-1}r}(f) \mu_j(r) dr \\ &\quad + \sum_{\{\gamma\}_\Gamma \neq 1} \operatorname{tr}\chi(\gamma) \cdot \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \cdot \int_{G_\gamma \backslash G} f(x^{-1} \gamma x) d\dot{x}. \end{aligned} \tag{4}$$

Here $j = 0, 1$ is fixed so that $n \equiv j(2)$. The term

$$[*] = \left\{ m(\pi_{\pm 2}, \pi_\Gamma^\chi) - \dim \chi \cdot \operatorname{vol}(\Gamma \backslash G) \cdot d_{\pi_{\pm 2}} \right\} \cdot \operatorname{tr} \pi_{\pm 2}(f)$$

appears only if $n \neq 0$ is even, and $\pm$ is the signature of n. Now the Selberg zeta function of class-δ_n with representation χ is defined by

$$Z_{\Gamma, n}(s, \chi) = \prod_{\{\gamma\}_\Gamma} \prod_{m=0}^{\infty} \det \left(1 - \chi(\gamma) \cdot \varepsilon(\gamma)^n \cdot N(\gamma)^{-s-m}\right).$$

Here $\prod_{\{\gamma\}_\Gamma}$ is the product over the primitive hyperbolic Γ-conjugacy classes. A hyperbolic element $\gamma \in \Gamma$ is conjugate with respect to $SL_2(\mathbb{R})$ to an element $\begin{bmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{bmatrix} \in SL_2(\mathbb{R})$ with $|a(\gamma)| > 1$, and we put $N(\gamma) = a(\gamma)^2$ and $\varepsilon(\gamma) = a(\gamma)/|a(\gamma)|$. Choosing a suitable test function $f \in L^1(G)$, the class-δ_n trace formula (4) gives

$$\begin{aligned}\sum_{\pi \in \widehat{G}(\delta_n) \setminus \widehat{G}_d} & m(\pi, \pi_\Gamma^\chi) \left\{ \frac{H(s - \frac{1}{2} + \lambda(\pi))}{s - \frac{1}{2} + \lambda(\pi)} + \frac{H(s - \frac{1}{2} - \lambda(\pi))}{s - \frac{1}{2} - \lambda(\pi)} \right\} + [**] \\ &= -2 \dim \chi \cdot \operatorname{vol}(\Gamma \backslash G) \sum_{2 \leq n \in \mathbb{Z}, n \equiv n(2)} \frac{H(s + \frac{k}{2} - 1)}{s + \frac{k}{2} - 1} \cdot \frac{k-1}{4} \\ &\quad + \frac{d}{ds} \log Z_{\Gamma, n}(s, \chi). \end{aligned} \tag{5}$$

Here the function $H(z)$ is a holomorphic function of $z \in \mathbb{C}$ and the term

$$[**] = \left\{ m(\pi_{\pm 2}, \pi_\Gamma^\chi) - \dim \chi \cdot \operatorname{vol}(\Gamma \backslash G) \cdot d_{\pi_{\pm 2}} \right\} \left\{ \frac{H(s-1)}{s-1} + \frac{H(s)}{s} \right\}$$

appears only if $n \neq 0$ is even, and $\pm$ is the signature of n. The expansion (5) gives the analytic continuation of $Z_{\Gamma, n}(s, \chi)$ to the entire complex plane, and we can read out the following relations between vanishing order of Selberg zeta functions and the multiplicities of representations:

(i) the vanishing order of $Z_{\Gamma,n}(s,\chi)$ with even n at $s = 1$ is equal to $m(\mathbf{1}_G, \chi)$ due to the class-δ_0 trace formula and Frobenius reciprocity. It is also equal to

$$m(\pi_{\pm 2}, \pi_\Gamma^\chi) - \dim\chi \cdot \mathrm{vol}(\Gamma\backslash G) \cdot \frac{1}{4} \quad (\pm = \text{the signature of } n)$$

due to the class δ_n-trace formula with even $n \neq 0$.

(ii) the vanishing order of $Z_{\Gamma,n}(s,\chi)$ with odd n at $s = 1/2$ is equal to

$$2 \cdot m(\pi_\pm^{1,0}, \pi_\Gamma^\chi) \qquad (\pm = \text{the signature of } n).$$

Now the discrete series π_2 has the minimal K-type δ_2, and the $\check{\delta}_2$-isotypic component in the $\check{\pi}_2$-isotypic component of $\pi_\Gamma^{\check{\chi}}$ corresponds bijectively to the space $S_2(\Gamma,\chi)$ of the cusp forms of weight 2 and representation χ with respect to Γ (here the check marks denote the contragredient representations). On the other hand the representation $\pi_+^{1,0}$ is not a discrete series but a so called highest weight module. It has the minimal K-type δ_1, and the $\check{\delta}_1$-isotypic component in the $\check{\pi}_+^{1,0}$-isotypic component of $\pi_\Gamma^{\check{\chi}}$ corresponds bijectively to the space $S_1(\Gamma,\chi)$ of the cusp forms of weight 1 and representation χ with respect to Γ. So we have

$$\dim S_2(\Gamma,\chi) = \mathrm{ord}_{s=1} Z_{\Gamma,n}(s,\chi) + \dim\chi \cdot \mathrm{vol}(\Gamma\backslash G) \cdot \frac{1}{4}$$

for even n and

$$\dim S_1(\Gamma,\chi) = \frac{1}{2} \cdot \mathrm{ord}_{s=1/2} Z_{\Gamma,n}(s,\chi)$$

with odd n. So far we assume that Γ is torsion-free. But if Γ contains $-1_2 \in SL_2(\mathbb{Z})$ and $\chi(-1_2) = (-1)^n$, then the definition of the Selberg zeta function should be

$$\begin{aligned} Z_\Gamma(s,\chi) &= \prod_{\{\gamma\}_\Gamma}{}' \prod_{m=0}^{\infty} \det\left(1 - \chi(\gamma)\varepsilon(\gamma)^n N(\gamma)^{-s-m}\right) \\ &= \prod_{\{\gamma\}_\Gamma, \mathrm{tr}\,\gamma>0} \prod_{m=0}^{\infty} \det\left(1 - \chi(\gamma)\varepsilon(\gamma)^n N(\gamma)^{-s-m}\right) \end{aligned}$$

where $\prod'_{\{\gamma\}_\Gamma}$ is the product over the primitive hyperbolic Γ-conjugacy classes modulo the multiplication by -1_2, and $\prod_{\{\gamma\}_\Gamma, \mathrm{tr}\,\gamma>0}$ is the product over the primitive hyperbolic Γ-conjugacy classes with positive trace. This is the definition of the Selberg zeta functions given in the papers of Arakawa.

3.4. Because the fundamental group of $SL_2(\mathbb{R})$ is the infinite cyclic group, there exists uniquely a connected real Lie group $\widetilde{SL}_2(\mathbb{R})$ and a continuous group homomorphism $p : \widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ such that the order of the kernel $\operatorname{Ker} p$ is two. Such a two-fold covering group of $SL_2(\mathbb{R})$ is constructed explicitly in [T2] so that

$$\widetilde{K} = p^{-1}(K) = \{(\varepsilon, k) \in \mathbb{C}^\times \times K \mid \varepsilon^2 = J(k, \sqrt{-1})^{-1}\},$$

which is a connected subgroup of the direct product $\mathbb{C}^\times \times K$. Here we put $J(\sigma, z) = cz + d$ for $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$ as usual. The maximal compact subgroup $\widetilde{K}$ of $\widetilde{SL}_2(\mathbb{R})$ has the irreducible unitary representations $\delta_{n/2}$ for $n \in \mathbb{Z}$ defined by $\delta_{n/2}(\varepsilon, k) = \varepsilon^{-n}$.

Now let ω be the Weil representation of $\widetilde{SL}_2(\mathbb{R})$ which has irreducible decomposition $\omega = \omega_+ \oplus \omega_-$ with respect to the dual pair $(SL_2(\mathbb{R}), O(2))$. The irreducible representation $\omega_\pm$ has the following $\widetilde{K}$-type decompositions

$$\check{\omega}_+|_{\widetilde{K}} = \bigoplus_{0 \le k \in \mathbb{Z}} \delta_{\frac{1}{2}+2k}, \quad \check{\omega}_-|_{\widetilde{K}} = \bigoplus_{0 \le k \in \mathbb{Z}} \delta_{\frac{1}{2}+2k+1}.$$

Then ω_- is a discrete series representation of $\widetilde{SL}_2(\mathbb{R})$ of formal degree $\frac{1}{4}\left(\frac{3}{2} - 1\right) = \frac{1}{8}$. On the other hand ω_+ is not a discrete series but a highest weight module of $\widetilde{SL}_2(\mathbb{R})$. The action of the Casimir operator Ω of $\widetilde{SL}_2(\mathbb{R})$ on $\omega_\pm$ is defined by $\lambda(\omega_\pm) = 1/4$. Now we have the class-$\delta_{n/2}$ trace formula for $\widetilde{SL}_2(\mathbb{R})$, $\widetilde{K}$ and $\widetilde{\Gamma} = p^{-1}(\Gamma)$. Then the same arguments as in §3.3 gives the definition of the Selberg zeta functions $Z_{\widetilde{\Gamma}, \frac{1}{2}+n}(s, \chi)$ with $n \in \mathbb{Z}$ and a finite dimensional unitary representation χ of $\widetilde{\Gamma}$ and the relations between the vanishing order of Selberg zeta functions and the multiplicities of representations;

(1) the vanishing order of $Z_{\widetilde{\Gamma}, \frac{1}{2}+n}(s, \chi)$ with even n at $s = 3/4$ is equal to the multiplicity $m(\omega_+, \pi_{\widetilde{\Gamma}}^{\chi})$,
(2) the vanishing order of $Z_{\widetilde{\Gamma}, \frac{1}{2}+n}(s, \chi)$ with odd n at $s = 3/4$ is equal to

$$m(\omega_-, \pi_{\widetilde{\Gamma}}^{\chi}) - \dim \chi \cdot \operatorname{vol}(\widetilde{\Gamma} \backslash \widetilde{SL}_2(\mathbb{R})) \cdot \frac{1}{8}.$$

Now the contragredient representation $\check{\omega}_+$ has the minimal $\widetilde{K}$-type $\delta_{1/2}$, and the $\delta_{1/2}$-isotypic component in the $\check{\omega}_+$-isotypic component of $\pi_{\widetilde{\Gamma}}^{\chi}$ corresponds bijectively to the space $S_{1/2}(\widetilde{\Gamma}, \chi)$ of the cusp forms of weight $1/2$ with representation χ with respect to $\widetilde{\Gamma}$. On the other hand $\check{\omega}_-$ has the minimal $\widetilde{K}$-type $\delta_{3/2}$, and the $\delta_{3/2}$-isotypic component in the $\check{\omega}_-$-isotypic component of $\pi_{\widetilde{\Gamma}}^{\chi}$ corresponds bijectively to the space $S_{3/2}(\widetilde{\Gamma}, \chi)$ of the cusp form of weight 3/2 with representation χ with respect to $\widetilde{\Gamma}$. So we have the following dimension formulae

$$\dim S_{1/2}(\widetilde{\Gamma}, \chi) = \mathrm{ord}_{s=3/4} Z_{\widetilde{\Gamma}, \frac{1}{2}+n}(s, \chi)$$

with even n and

$$\dim S_{3/2}(\widetilde{\Gamma}, \chi) = \mathrm{ord}_{s=3/4} Z_{\widetilde{\Gamma}, \frac{1}{2}+n}(s, \chi) + \dim \chi \cdot \mathrm{vol}(\widetilde{\Gamma} \backslash \widetilde{SL}_2(\mathbb{R})) \cdot \frac{1}{8}$$

with odd n.

3.5. Finally we will consider the theta multiplier system and Jacobi forms. Put $\Gamma = SL_2(\mathbb{Z})$ and $\widetilde{\Gamma} = p^{-1}(\Gamma)$. Let $\Theta_S(z, w) = (\vartheta_{S,r}(z, w))_{r \in ({}^{"}S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l}$ be the system of theta series (2). Then, for any $\widetilde{\gamma} \in \widetilde{\Gamma}$, we have

$$\Theta_S\left(\gamma z, \frac{w}{J(\gamma, z)}\right) = J_{\frac{1}{2}}(\widetilde{\gamma}, z)^l \mathbf{e}\left(\frac{c}{J(\gamma, z)} \cdot S[w]\right) U_S(\widetilde{\gamma}) \Theta_S(z, w).$$

Here $\gamma = p(\widetilde{\gamma})$ and $J_{\frac{1}{2}}(\widetilde{\gamma}, z)$ is the factor of automorphy of weight 1/2 defined in [T2], and U_S is a unitary representation of $\widetilde{\Gamma}$ of dimension $\sharp((2S)^{-1}\mathbb{Z}^l/\mathbb{Z}^l)$ (see a general theory given in [T3]). Let us denote by χ the contragredient representation of U_S. There exists a unitary matrix L such that

$$\Theta_S(z, -w) = L \Theta_S(z, w).$$

Because χ commutes with L, the unitary representation χ decomposes into the ± 1-eigen spaces of L which are denoted by $\chi_\pm$. Put $\widetilde{\gamma}_0 = (\varepsilon, -1_2) \in \widetilde{\Gamma}$ which is an element of the center of $\widetilde{SL}_2(\mathbb{R})$. Then we have

$$\chi_\pm(\widetilde{\gamma}_0) = \pm(-1)^{(l+1)/2}\varepsilon, \quad \omega_\pm(\widetilde{\gamma}_0) = \pm\varepsilon$$

which implies that

(1) $m(\omega_+, \mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})} \chi_\varepsilon) > 0$ only if $\varepsilon = (-1)^{(l+1)/2}$,

(2) $m(\omega_-, \mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})} \chi_\varepsilon) > 0$ only if $\varepsilon = (-1)^{(l+3)/2}$.

Put $Q = 2S$. Let us consider a $\mathbb{R}$-vector space $V = M_{l,2}(\mathbb{R})$ endowed with a symplectic $\mathbb{R}$-form $D_Q(x, y) = \mathrm{tr}(QxJ_n{}^t y)$ ($J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$). Then

$$\Lambda = \{(x, Q^{-1}y) \in V \mid x, y \in \mathbb{Z}^l\}$$

is a $\mathbb{Z}$-lattice in V which is self-dual with respect to the symplectic form D_Q containing a $\mathbb{Z}$-lattice $L = M_{l,2}(\mathbb{Z})$. The dual lattice of L with respect to D_Q is $L^* = Q^{-1}L$, and we have $L \subset \Lambda \subset L^*$. The Weil representation Ω_Q of $\widetilde{Sp}(V)$ (the two-fold-covering group of $Sp(V)$) is realized on the representation space of the induced representation $\mathrm{Ind}_{H[\Lambda]}^{H[V]}\xi$ of the Heisenberg group $H[V] = V \times \mathbb{R}$ associated with the symplectic space (V, D_Q). Here ξ is a character of the subgroup $H[\Lambda] = \Lambda \times \mathbb{R}$ of $H[V]$ defined by $\xi(h) = \mathbf{e}\left(t + \frac{1}{2}\langle x, y\rangle_Q\right)$ for $h = ((x, y), t) \in H[\Lambda]$ where the element of V is denoted by (x, y) with $x, y \in \mathbb{R}^l$, and $\langle x, y\rangle_Q = D_Q((x, 0), (0, y))$. There exists a canonical injection of $SL_2(\mathbb{R})$ into $Sp(V)$ which is extended to an injective continuous group homomorphism $\widetilde{SL}_2(\mathbb{R}) \to \widetilde{Sp}(V)$. It is injective because l is odd. Let us denote by ω_Q the restriction of Ω_Q to $\widetilde{SL}_2(\mathbb{R})$, which is unitarily equivalent to the l-fold tensor product of ω. Then $\omega_{Q,J} = \omega_Q \otimes \mathrm{Ind}_{H[\Lambda]}^{H[V]}\xi$ is an irreducible unitary representation of the semi-direct product $\widetilde{SL}_2(\mathbb{R}) \ltimes H[V]$. On the other hand the unitary representation U_S of $\widetilde{\Gamma}$ is realized on the representation space of the induced representation $\mathrm{Ind}_{H[\Lambda]}^{H[L^*]}\xi$ (see [T3] for the details). Then, by means of the canonical isomorphism $\mathrm{Ind}_{H[\Lambda]}^{H[V]}\xi \simeq \mathrm{Ind}_{H[L^*]}^{H[V]}\left(\mathrm{Ind}_{H[\Lambda]}^{H[L^*]}\xi\right)$, we have an isomorphism

$$\mathrm{Ind}_{\Gamma \ltimes H[\Lambda]}^{SL_2(\mathbb{R}) \ltimes H[V]}\xi \simeq \left(\mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})}\check{U}_S\right) \otimes \omega_{Q,J}. \tag{6}$$

Here the unitary representations of $SL_2(\mathbb{R}) \times H[V]$ is identified with unitary representations of its two-fold covering group $\widetilde{SL}_2(\mathbb{R}) \ltimes H[V]$ via covering mapping. Finally define a group structure on $H[V, D] = V \times \mathrm{Sym}_l(\mathbb{R})$ by

$$(x, s) \cdot (y, t) = (x + y, s + t + \frac{1}{2}D(x, y))$$

with $D(x, y) = \frac{1}{2}(xJ_n{}^t y - yJ_n{}^t x)$. Then $(x, s) \mapsto (x, \mathrm{tr}(Qs))$ is a surjective group homomorphism of $H[V, D]$ onto $H[V]$. So the unitary representations of $SL_2(\mathbb{R}) \ltimes H[V]$ is identified with unitary representations of $SL_2(\mathbb{R}) \ltimes H[V, D]$ via this canonical surjection. Also $\xi_Q(x, s) = \xi(x, \mathrm{tr}(Qs))$ defines a character of $H[\Lambda, D] = \Lambda \times \mathrm{Sym}_l(\mathbb{R})$, and we have an equivariant unitary isomorphism

$$\mathrm{Ind}_{\Gamma \ltimes H[\Lambda, D]}^{SL_2(\mathbb{R}) \ltimes H[V, D]}\xi_Q \simeq \mathrm{Ind}_{\Gamma \ltimes H[\Lambda]}^{SL_2(\mathbb{R}) \ltimes H[V]}\xi. \tag{7}$$

Then the canonical isomorphisms (6) and (7) imply a canonical isomorphism

$$\begin{aligned}\mathrm{Ind}_{\Gamma \ltimes H[\Lambda, D]}^{SL_2(\mathbb{R}) \ltimes H[V, D]}\check{\xi}_Q &\simeq \left(\mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})}U_S\right) \otimes \check{\omega}_{Q,J} \\ &\simeq \left(\mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})}\check{\chi}_+ \otimes \check{\omega}_{Q,J}\right) \oplus \left(\mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})}\check{\chi}_- \otimes \check{\omega}_{Q,J}\right).\end{aligned} \tag{8}$$

Put $\pi_\pm = \omega_\pm \otimes \omega_{Q,J}$, which is an irreducible unitary representation of $SL_2(\mathbb{R}) \ltimes H[V]$, or equivalently an irreducible unitary representation of $SL_2(\mathbb{R}) \ltimes H[V,D]$. The contragredient representations $\check{\pi}_+$ and $\check{\pi}_-$ have the minimal K-types $\delta_{(l+1)/2}$ and $\delta_{(l+3)/2}$ respectively with multiplicity one. Then the $\delta_{(l+1)/2}$-isotypic component of the $\check{\pi}_+$ in $\mathrm{Ind}_{\Gamma\ltimes H[\Lambda,D]}^{SL_2(\mathbb{R})\ltimes H[V,D]}\check{\xi}_Q$ corresponds bijectively to the space of the Jacobi forms of weight $(l+1)/2$ and index S (see [T1]). So the multiplicity of $\check{\pi}_+$ in $\mathrm{Ind}_{\Gamma\ltimes H[\Lambda,D]}^{SL_2(\mathbb{R})\ltimes H[V,D]}\check{\xi}_Q$ is equal to the dimension of the space of the Jacobi forms of weight $(l+1)/2$ and index S. On the other hand, the canonical isomorphism (8) implies that the multiplicity of $\check{\pi}_+$ in $\mathrm{Ind}_{\Gamma\ltimes H[\Lambda,D]}^{SL_2(\mathbb{R})\ltimes H[V,D]}\check{\xi}_Q$ is equal to the multiplicity of $\check{\omega}_+$ in $\mathrm{Ind}_{\tilde{\Gamma}}^{\widetilde{SL}_2(\mathbb{R})}\check{\chi}_\varepsilon$ $(\varepsilon = (-1)^{(l+1)/2})$ which is equal to the dimension of the modular forms of weight $1/2$ and representation χ_ε with respect to Γ as we have seen in §3.4. We have a similar relation between the space of Jacobi forms of weight $(l+3)/2$ index S and the space of the modular forms of weight $3/2$ with representation χ_ε $(\varepsilon = (-1)^{(l+3)/2})$.

3.6. We have considered Arakawa's works under the assumption that $\Gamma\backslash SL_2(\mathbb{R})$ is compact. Under this assumption, the problems are simplified in the following points

(1) there is no need to consider the parabolic or elliptic conjugacy classes in Γ,
(2) there is no need to consider the cuspidality of the automorphic forms.

The achievements of the works of Arakawa's are to overcome all these difficulties by his ingeniously powerful calculations.

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4. Arakawa's works on special values of zeta and L-functions

Arakawa's papers related to the special values of zeta and L-functions are grouped into three categories:

(i) Zeta functions related to real quadratic fields, [5], [8], [9], [16], [18].
(ii) L-functions attached to the ternary zero form $x_1x_2 - x_{12}^2$, [10].
(iii) Multiple zeta and L-values, [22], [23], [29].

Here we briefly describe each work.

(i) Topics covered in these papers are the ray-class invariants and the construction of class fields of real quadratic fields, the Stark-Shintani conjecture, and the Dedekind and the Hirzebruch sums.

In [5], Arakawa investigates a series $H(\alpha, s)$ defined for a real quadratic number α and converges for $\Re s < 0$. This series was originally introduced by Lewittes and Berndt in 70's when α is a number in the upper half-pane, as a certain generalization of the Dedekind eta function and the Eisenstein series. Arakawa studied this for *real* α, and proved the meromorphic continuation of $H(\alpha, s)$ to the whole s-plane, with a possible simple pole at $s = 0$. Arakawa defined a certain quantity using the coefficient of the Laurant expansion of $H(\alpha, s)$ at $s = 0$, and proved the "modular" property of this quantity, i.e., a certain invariance property with respect to the linear fractional transformations $\alpha \to (a\alpha + b)/(c\alpha + d)$ for elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Owing to this modular property, he is able to construct ray-class invariants of real quadratic fields, which turn out to be closely related to the Stark-Shintani invariant, the invariant conjecturally yields a unit in a ray class field. Arakawa also writes down his invariants in terms of the double gamma function of Barnes. In [8], he expresses a relative class number by unit index, assuming the Stark-Shintani conjecture. In [9] and [16], Arakawa studies the Dirichlet series

$$\xi(s, \alpha) := \sum_{n=1}^{\infty} \frac{\cot(\pi n \alpha)}{n^s},$$

defined for a real quadratic number α and converges for $\Re s > 1$. He gives an expression of this series in terms of the Barnes double zeta function, thereby proving the analytic continuation of $\xi(s, \alpha)$ with enough information on poles. He extracts a certain invariant, as he did in [5] for $H(\alpha, s)$, from the expansion of $\xi(s, \alpha)$ at the pole $s = 1$, and proves the modular invariance property of that invariant as well as the "Hecke eigen" property and a connection to the Dedekind sum. As a corollary, an intriguing identity of the Dedekind sum is obtained. He also obtains a formula for his invariant in terms of the Hirzebruch sum, which is defined by the continued fraction expansion of α and has a nice connection to the class number of *imaginary* quadratic fields. Furthermore, Arakawa expresses special values of Hecke Grössencharacter L-series of real quadratic fields in terms of residues of $\xi(s, \alpha)$ at some poles. The paper [18] establishes an expression of the value at $s = 0$ of the partial zeta function of real quadratic fields in terms of the invariants in [5], and describes explicit Galois action, thus giving a certain reciprocity law.

(ii) The paper [10] concerns the special values of the "Hashimoto L-function" attached to the ternary zero form $x_1x_2 - x_{12}^2$. Arakawa evaluates some of these special values by the generalized Bernoulli numbers, and for some others he poses formulas as conjectures. The method is a very hard calculation of contour integrals, showing his utter ability of manipulating integrals and series. However, a great progress toward the understanding of this subject has been made by work of T. Ibukiyama and H. Saito, who showed the Hashimoto L-function could be written explicitly in terms of the Riemann zeta and the Dirichlet L- functions, thus deducing Arakawa's results and conjectures in a much simpler and transparent way.

(iii) For these works, let me allow myself to be a bit personal. On July 28, 1995, I (Kaneko) wrote a letter to Arakawa in which I reported my recent observation on a formal similarity between formulas of multiple zeta values and poly-Bernoulli numbers. Arakawa wrote back less than two months later, informing me of his discovery of a certain zeta function, whose values at positive integers give multiple zeta values (certain combinations of them to be precise) and at negative integers poly-Bernoulli numbers. This was the start of our collaboration to remain until his passing. We wrote three papers jointly, [22], [23], and [29]. In [22] we studied a certain type of zeta functions in connection with the multiple zeta values and the poly-Bernoulli numbers. A Clausen–von Staudt type theorem of poly-Bernoulli numbers was proved in [23]. In our final joint paper [29], we studied derivation and regularized double shuffle relations of multiple L-values as well as basic analytic properties of a one variable function related to multiple L-values.

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List of Publications of Tsuneo Arakawa

Research papers

1. Dirichlet series related to the Eisenstein series on the Siegel upper half-plane. *Comment. Math. Univ. St. Paul.* **27** (1978/79), no. 1, 29–42.
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ESTIMATE OF THE DIMENSIONS OF HILBERT MODULAR FORMS BY MEANS OF DIFFERENTIAL OPERATORS

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Dedicated to the memory of Tsuneo Arakawa

The purpose of this exposition is to estimate the dimensions of Hilbert modular forms by means of differential operators. In this exposition, we define two differential operators. One gives a directional derivative to a diagonal part, which is useful for estimating the dimensions of Hilbert modular forms. And the other is a Rankin-Cohen-Ibukiyama type differential operator, which is useful for making a new modular form from known modular forms. In several cases, using these differential operators, we can determine the dimensions of Hilbert modular forms. For the sake of simplicity, in this exposition, we treat only the case of discriminant 12. However, we remark that the same method is available for some other cases.

1. Definitions and notations

1.1. *Hilbert modular forms*

We denote the totally real quadratic field with discriminant 12 by K, its integer ring by $\mathcal{O}$, and its fundamental unit by ε:

$$K := \mathbb{Q}(\sqrt{3}),\ \mathcal{O} := \mathbb{Z}(\sqrt{3}) \ \text{ and } \ \varepsilon := 2+\sqrt{3}.$$

For $x = u + v\sqrt{3} \in K$, we write its conjugate number by $x' = u - v\sqrt{3} \in K$ and its trace by $\mathrm{tr}(x) = x + x' = 2u \in \mathbb{Q}$. We put the dual of $\mathcal{O}$ by

$$\begin{aligned}\mathcal{O}^* &:= \{\nu \in K \mid \mathrm{tr}(\mu\nu) \in \mathbb{Z} \text{ for any } \mu \in \mathcal{O}\} \\ &= \left\{ \nu = \frac{1}{2}u + \frac{\sqrt{3}}{6}v \in K \;\middle|\; u, v \in \mathbb{Z} \right\}.\end{aligned}$$

We denote the complex upper half plane by

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}.$$

For $k \in \mathbb{Z}$, we say that a holomorphic function F on $\mathbb{H}^2$ is a Hilbert modular form of weight k if

$$F(\tau_1, \tau_2) = (c\tau_1 + d)^{-k} (c'\tau_2 + d')^{-k} F\left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'}\right)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O})$. We denote the $\mathbb{C}$-vector space of all Hilbert modular forms of weight k by A_k. From Köcher principle, $F \in A_k$ has a Fourier expansion

$$F(\tau_1, \tau_2) = \sum_{\substack{\nu \in \mathcal{O}^* \\ \nu \geq 0,\ \nu' \geq 0}} c(\nu)\, \mathbf{e}(\nu\tau_1 + \nu'\tau_2),$$

where we put $\mathbf{e}(w) = \exp(2\pi\sqrt{-1}w)$. We denote the $\mathbb{C}$-vector space of all symmetric Hilbert modular forms of weight k by

$$A_k^+ = \left\{F \in A_k \,\middle|\, F(\tau_1, \tau_2) = F(\tau_2, \tau_1)\right\}$$

and the $\mathbb{C}$-vector space of all skew-symmetric Hilbert modular forms of weight k by

$$A_k^- = \left\{F \in A_k \,\middle|\, F(\tau_1, \tau_2) = -F(\tau_2, \tau_1)\right\}.$$

If $F(\tau_1, \tau_2) \in A_k$, then

$$\frac{F(\tau_1, \tau_2) + F(\tau_2, \tau_1)}{2} \in A_k^+ \quad \text{and} \quad \frac{F(\tau_1, \tau_2) - F(\tau_2, \tau_1)}{2} \in A_k^-.$$

Hence we have

$$A_k = A_k^+ \oplus A_k^-,$$

where the symbol $\oplus$ means the direct sum as $\mathbb{C}$-vector spaces. Put

$$\begin{aligned}
A_k^{++} &:= \left\{F \in A_k^+ \,\middle|\, F(\tau_1, \tau_2) = F(\varepsilon\tau_1, \varepsilon'\tau_2)\right\}, \\
A_k^{+-} &:= \left\{F \in A_k^+ \,\middle|\, F(\tau_1, \tau_2) = -F(\varepsilon\tau_1, \varepsilon'\tau_2)\right\}, \\
A_k^{-+} &:= \left\{F \in A_k^- \,\middle|\, F(\tau_1, \tau_2) = F(\varepsilon\tau_1, \varepsilon'\tau_2)\right\}, \\
A_k^{--} &:= \left\{F \in A_k^- \,\middle|\, F(\tau_1, \tau_2) = -F(\varepsilon\tau_1, \varepsilon'\tau_2)\right\}.
\end{aligned}$$

If $F(\tau_1, \tau_2) \in A_k^+$ (resp. A_k^-), then

$$\frac{F(\tau_1, \tau_2) + F(\varepsilon\tau_1, \varepsilon'\tau_2)}{2} \in A_k^+ \quad (\text{resp. } A_k^-)$$

and

$$\frac{F(\tau_1,\tau_2) - F(\varepsilon\tau_1,\varepsilon'\tau_2)}{2} \in A_k^+ \quad \text{(resp. } A_k^- \text{).}$$

Consequently, we have

$$A_k = A_k^{++} \oplus A_k^{+-} \oplus A_k^{-+} \oplus A_k^{--}.$$

1.2. *Elliptic modular forms*

For $k \in \mathbb{Z}$, we say that a holomorphic function f on $\mathbb{H}$ is an elliptic modular form of weight k if

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n)\, \mathbf{e}\,(n\tau).$$

We denote the $\mathbb{C}$-vector space of all elliptic modular forms of weight k by M_k. Put

$$M_k(n) := \left\{ f \in M_k \;\middle|\; a_f(r) = 0 \text{ if } r < n \right\}.$$

It is well-known that there are two algebraically independent elliptic modular forms

$$e_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\mathbf{e}(n\tau) \in M_4$$

and

$$e_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\mathbf{e}(n\tau) \in M_6.$$

The graded ring of all elliptic modular forms are generated by e_4 and e_6 :

$$\bigoplus_{k \in \mathbb{Z}} M_k = \mathbb{C}[e_4, e_6].$$

Because the Ramanujan Delta function

$$\Delta(\tau) = \frac{1}{1728}\left(e_4(\tau)^3 - e_6(\tau)^2\right) = \mathbf{e}(\tau) \prod_{n=1}^{\infty} \{1 - \mathbf{e}(n\tau)\}^{24} \in M_{12}(1)$$

has no zero on $\mathbb{H}$, we have $M_k(n) = \Delta^n M_{k-12n}$. Hence we have

$$\dim M_k(n) = \dim M_{k-12n}.$$

2. Estimate of the dimension

2.1. *Directional differential operators*

Put

$$D := \frac{\sqrt{3}}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial\tau_1} - \frac{\partial}{\partial\tau_2}\right).$$

For a non-negative integer n, define an operator D_n, which maps from a holomorphic function on $\mathbb{H}^2$ to a holomorphic function on $\mathbb{H}$ by

$$(D_n F)(\tau) := (D^n F)(\tau, \tau).$$

Put

$$\begin{aligned}
A_k^{++}(n) &:= \left\{F \in A_k^{++} \mid D_r(F) = 0 \text{ if } r < n\right\},\\
A_k^{+-}(n) &:= \left\{F \in A_k^{+-} \mid D_r(F) = 0 \text{ if } r < n\right\},\\
A_k^{-+}(n) &:= \left\{F \in A_k^{-+} \mid D_r(F) = 0 \text{ if } r < n\right\},\\
A_k^{--}(n) &:= \left\{F \in A_k^{--} \mid D_r(F) = 0 \text{ if } r < n\right\}.
\end{aligned}$$

Lemma 2.1. *If $F \in A_k^{++}(n)$, then $D_n F \in M_{2k+2n}$.*

Proof. For $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F(\tau_1, \tau_2) = (c\tau_1 + d)^{-k}(c\tau_2 + d)^{-k} F\left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a\tau_2 + b}{c\tau_2 + d}\right). \tag{1}$$

For $0 \le s \le r < n$,

$$\left(\left(\frac{\partial}{\partial\tau_1} + \frac{\partial}{\partial\tau_2}\right)^s D^{r-s} F\right)(\tau, \tau) = \frac{d^s}{d\tau^s}\left(D_{r-s}F\right) = 0.$$

Hence, for $0 \le t \le r < n$,

$$\left(\frac{\partial^r}{\partial\tau_1^t \partial\tau_2^{r-t}} F\right)(\tau, \tau) = 0.$$

Therefore, from the equation (1), we have $D_n F \in M_{2k+2n}$. □

The similar lemma holds for $A_k^{+-}(n)$, $A_k^{-+}(n)$ or $A_k^{--}(n)$. From the definition of symmetric or skew-symmetric forms, we have $D_{2n+1}\left(A_k^{+}\right) = \{0\}$ or $D_{2n+2}\left(A_k^{-}\right) = \{0\}$. Hence we have the following proposition.

Proposition 2.1. *For any non-negative integer n, there exist four exact sequences*

$$
\begin{array}{l}
0 \longrightarrow A_k^{++}(2n+2) \longrightarrow A_k^{++}(2n) \xrightarrow{D_{2n}} M_{2k+4n}, \\
0 \longrightarrow A_k^{+-}(2n+2) \longrightarrow A_k^{+-}(2n) \xrightarrow{D_{2n}} M_{2k+4n}, \\
0 \longrightarrow A_k^{-+}(2n+3) \longrightarrow A_k^{-+}(2n+1) \xrightarrow{D_{2n+1}} M_{2k+4n+2}, \\
0 \longrightarrow A_k^{--}(2n+3) \longrightarrow A_k^{--}(2n+1) \xrightarrow{D_{2n+1}} M_{2k+4n+2}.
\end{array}
$$

2.2. *Fourier coefficients of Hilbert modular forms*

Now we investigate these exact sequences more precisely. Assume $F \in A_k^{++}$ and set its Fourier coefficients by

$$F(\tau_1, \tau_2) = \sum_{\substack{\nu \in \mathcal{O}^* \\ \nu \geq 0,\ \nu' \geq 0}} c(\nu)\, \mathbf{e}\left(\nu\tau_1 + \nu'\tau_2\right).$$

From the transformation formula of $F \in A_k^{++}$, we have $c(\nu) = c(\nu')$ and $c(\nu) = c(\varepsilon\nu)$. Now, for the sake of simplicity, we put $c(u,v) := c(\frac{1}{2}u + \frac{\sqrt{3}}{6}v)$. Put $\Lambda := \left\{(u,v) \in \mathbb{Z}^2 \mid |v| \leqq \sqrt{3}u\right\}$. Then F has a Fourier expansion

$$F(\tau_1, \tau_2) = \sum_{(u,v) \in \Lambda} c(u,v)\, \mathbf{e}\left(\left(\frac{1}{2}u + \frac{\sqrt{3}}{6}v\right)\tau_1 + \left(\frac{1}{2}u - \frac{\sqrt{3}}{6}v\right)\tau_2\right).$$

Thus, easily we have

$$\left(D_{2r}F\right)(\tau) = \sum_{(u,v) \in \Lambda} v^{2r} c(u,v)\, \mathbf{e}(u\tau).$$

Lemma 2.2. *If $F \in A_k^{++}(2n)$, then $c(u,v) = 0$ for any $u < n$.*

Proof. We show this lemma by induction on n. If $n = 0$, this lemma is trivial. Now we assume that this lemma holds for $n \leq r$. Let $F \in A_k^{++}(2(r+1))$. From the assumption, $c(u,v) = 0$ for any $u < r$. If $v > r$, then

$$c(r,v) = c\left(\varepsilon^{-1}\left(\frac{1}{2}r + \frac{\sqrt{3}}{6}v\right)\right) = c(2r - v, -3r + 2v) = 0$$

because $2r - v < r$. If $v < -r$, then $c(r,v) = c(r,-v) = 0$. Hence $F \in A_k^{++}(2(r+1))$ means

$$\sum_{v=-r}^{r} v^{2j} c(r,v) = 0 \quad (j = 0, 1, \ldots, r)$$

and $c(r,v) = c(r,-v)$. Then we have

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1^2 & \dots & r^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1^{2r} & \dots & r^{2r} \end{pmatrix} \begin{pmatrix} c(r,0) \\ 2c(r,1) \\ \vdots \\ 2c(r,r) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By the Vandermonde formula, we have $c(r,v) = 0$. □

From this lemma, we have $D_{2n}\left(A_k^{++}(2n)\right) \subset M_{2k+4n}(n)$. The similar lemma holds for A_k^{+-}, A_k^{-+} or A_k^{--}. Consequently, we have the following proposition.

Proposition 2.2. *For any non-negative integer n, there exist four exact sequences*

$$\begin{array}{l} 0 \longrightarrow A_k^{++}(2n+2) \longrightarrow A_k^{++}(2n) \xrightarrow{D_{2n}} M_{2k+4n}(n), \\ 0 \longrightarrow A_k^{+-}(2n+2) \longrightarrow A_k^{+-}(2n) \xrightarrow{D_{2n}} M_{2k+4n}(n+1), \\ 0 \longrightarrow A_k^{-+}(2n+3) \longrightarrow A_k^{-+}(2n+1) \xrightarrow{D_{2n+1}} M_{2k+4n+2}(n+2), \\ 0 \longrightarrow A_k^{--}(2n+3) \longrightarrow A_k^{--}(2n+1) \xrightarrow{D_{2n+1}} M_{2k+4n+2}(n+1). \end{array}$$

From this proposition, we have the upper bounds for the dimensions of A_k^{++}, A_k^{+-}, A_k^{-+} and A_k^{--}.

Theorem 2.1. *In the following table, each dimension of the left hand side is not greater than the coefficient of x^k on the formal power series development of the right hand side.*

$$\begin{array}{ll} A_k^{++} \cdots & \dfrac{1}{(1-x^2)(1-x^3)(1-x^4)} \\ A_k^{+-} \cdots & \dfrac{x^6}{(1-x^2)(1-x^3)(1-x^4)} \\ A_k^{-+} \cdots & \dfrac{x^{11}}{(1-x^2)(1-x^3)(1-x^4)} \\ A_k^{--} \cdots & \dfrac{x^5}{(1-x^2)(1-x^3)(1-x^4)} \end{array}$$

Proof. From the previous proposition, the dimension of A_k^{++} is not greater than

$$\sum_{n=0}^{\infty} \dim M_{2k+4n}(n) = \sum_{n=0}^{\infty} \dim M_{2k-8n} = \sum_{n=0}^{\infty} \dim M_{2(k-4n)}.$$

This is the coefficient of x^k on the formal power series development of

$$\sum_{n=0}^{\infty} \frac{x^{4n}}{(1-x^2)(1-x^3)} = \frac{1}{(1-x^2)(1-x^3)(1-x^4)}.$$

We can prove the other three cases in an analogous way. □

Hence, if we construct the algebraically independent modular forms $G_2 \in A_2^{++}$, $G_3 \in A_3^{++}$ and $G_4 \in A_4^{++}$, the upper bound of the dimension of A_k^{++} in Theorem 2.1 equals to the true dimension of A_k^{++}. And additionally, if we construct the non-zero modular forms $G_6 \in A_6^{+-}$ and $G_5 \in A_5^{--}$, all upper bounds in Theorem 2.1 equal to the true dimensions of A_k^{++}, A_k^{+-}, A_k^{-+} and A_k^{--}. Hence, if we assume the existence of these forms, we have given a new method of the determination of the dimension of Hilbert modular forms on $\mathbb{Q}(\sqrt{3})$. In fact, Gundlach [4] constructed these forms G_2, G_3, G_4, G_5 and G_6. We remark that he also determined all generators of the ring of $\bigoplus_{k \in \mathbb{Z}} A_k$, using dimension formula of Hilbert modular forms.

Theorem 2.2. *We have*

$$\begin{aligned} A := \bigoplus_{k \in \mathbb{Z}} A_k^{++} =& \mathbb{C}[G_2, G_3, G_4], \\ \bigoplus_{k \in \mathbb{Z}} A_k^{+-} =& G_6 A, \\ \bigoplus_{k \in \mathbb{Z}} A_k^{-+} =& G_5 G_6 A, \\ \bigoplus_{k \in \mathbb{Z}} A_k^{++} =& G_5 A. \end{aligned}$$

Gundlach did not refer to the structures of A_k^{++}, A_k^{+-}, A_k^{-+} and A_k^{--}. But the author believes that he knew these structures, because he constructed G_5 and G_6 in his paper [4].

3. A differential operator of Rankin-Cohen-Ibukiyama type

Rankin-Cohen type differential operators were extended to the Siegel modular forms of general degree by Ibukiyama [6]. Especially, Aoki and Ibukiyama [1] showed that this differential operator gives a very simple relation among the generators of Siegel modular forms of degree 2. In this section, we prove the analogous result on Hilbert modular forms.

For $F_1 \in A_{k_1}$, $F_2 \in A_{k_2}$ and $F_3 \in A_{k_3}$, put

$$[F_1, F_2, F_3] := \det \begin{pmatrix} k_1 F_1 & k_2 F_2 & k_3 F_3 \\ \frac{\partial}{\partial \tau_1} F_1 & \frac{\partial}{\partial \tau_1} F_2 & \frac{\partial}{\partial \tau_1} F_3 \\ \frac{\partial}{\partial \tau_2} F_1 & \frac{\partial}{\partial \tau_2} F_2 & \frac{\partial}{\partial \tau_2} F_3 \end{pmatrix}.$$

By direct calculation, we have $[F_1, F_2, F_3] \in A_{k_1+k_2+k_3+2}$. Especially, $[G_2, G_3, G_4] \in A_{11}^{-+}$.

Theorem 3.1. *There exists a non-zero constant c such that*

$$[G_2, G_3, G_4] = cG_5 G_6$$

Proof. Because $\dim A_{11}^{-+} = 1$, there exists $c \in \mathbb{C}$ such that $[G_2, G_3, G_4] = cG_5 G_6$. By direct calculation, we have $[G_2, G_3, G_4] \neq 0$. □

4. Remark

In several cases, we can determine the dimensions of Hilbert modular forms by analogous way. In each case, the Rankin-Cohen-Ibukiyama type differential operator gives a simple relation between the generators of the ring of Hilbert modular forms.

4.1. *In the case of discriminant* 12, *another type*

Hilbert modular forms on $\mathbb{H} \times \mathbb{H}^-$ are defined in an analogous way, where $\mathbb{H}^-$ is a complex lower half plane. Gundlach [4] determined all generators and the dimension of Hilbert modular forms of this type. There exist four generators W_1, W_3, W_4 and W_{10} of weight $1, 3, 4$ and 10. The dimension of weight k is the coefficient of x^k on the formal power series development of

$$\frac{1 + x^{10}}{(1-x)(1-x^3)(1-x^4)}.$$

There exists a non-zero constant w such that

$$[W_1, W_3, W_4] = wW_{10}.$$

4.2. *In the case of discriminant* 8

Müller [7] determined all generators and the dimension of Hilbert modular forms on $\mathbb{Q}(\sqrt{2})$. There exist five generators Y_2, Y_4, Y_5, Y_6 and Y_9 of weight

2, 4, 5, 6 and 9. The dimension of weight k is the coefficient of x^k on the formal power series development of

$$\frac{(1+x^5)(1+x^9)}{(1-x^2)(1-x^4)(1-x^6)}.$$

There exists a non-zero constant y such that

$$[Y_2, Y_4, Y_6] = yY_5Y_9.$$

4.3. *In the case of discriminant* 5

Gundlach [3] determined the generators and the dimension of Hilbert modular forms on $\mathbb{Q}(\sqrt{5})$. There exist four generators Z_2, Z_5, Z_6 and Z_{15} of weight 2, 5, 6 and 15. The dimension of weight k is the coefficient of x^k on the formal power series development of

$$\frac{1+x^{15}}{(1-x^2)(1-x^5)(1-x^6)}.$$

There exists a non-zero constant z such that

$$[Z_2, Z_5, Z_6] = zZ_{15}.$$

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MARSDEN–WEINSTEIN REDUCTION, ORBITS AND REPRESENTATIONS OF THE JACOBI GROUP

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Dedicated to the memory of Tsuneo Arakawa

Guillemin and Sternberg started a method to give a geometric meaning to multiplicities of representations via the Marsden-Weinstein reduction of appropriate coadjoint orbits carrying the representations. This method is applied here to the case of discrete series representations of the Jacobi group. We get some explicit formulae for the orbits and their symplectic forms and arrive at a volume formula, which, as we hope, may be seen as a first approximation to the desired result.

While trying to get a geometric understanding of certain lifts in the theory of automorphic forms, in particular the Maass lift from automorphic representations of the Jacobi group to those of the symplectic group (see [2] and [6]), I got fascinated by the *orbit method.* This is a very old theme, initiated by Kirillov for nilpotent Lie groups ([10]) and then propagated and extended to more general cases by him, Kostant, Duflo and many others (for an overview see for instance the articles [12], [14], or one of the books [11] or [13]). It allows for a geometric construction of certain representations of a given linear group. The question which representations can be constructed by the orbit method and which not has found new interest in the last years, as to be seen from the reports of for instance Vogan [16] or Vergne [15]. Moreover the possibility to determine multiplicities as symplectic invariants has been exploited successfully by several autors (see the report by Guillemin, Lerman and Sternberg [8]). Encouraged by this and based on calculations by Yang [17], I studied the question which representations of the Jacobi group G^J are carried by coadjoint orbits ([3],[4]). From here, there is a natural way to the next question, namely whether more information on the structure of these representations is obtainable by studying more intensely the geometry of the orbits. Here we have to comply with the problem that the orbits are in general not compact. So, difficulties and perhaps interesting new phenomenae are to be expected when in the following we venture a first step in this direction.

As Tsuneo Arakawa had a strong interest in the Jacobi group and worked successfully with it (see [1]), I hope he would have appreciated the following material, though part of it still is preliminary. But I also hope that this note will stimulate further work in this direction.

1. Some General Remarks on the Orbit Method and Marsden-Weinstein Reduction

Here, at first, we reproduce some elements from the general situation as depicted with more details for instance in [11], [12], [13] or [8].

Let π and χ be irreducible representations of a real linear group G and a closed subgroup $K \subset G$. Then we are interested in the problem to determine the multiplicity of χ in π (as a representation of K), i.e. $\mathrm{mult}(\chi, \pi\mid_K)$, resp. the space of intertwining operators $\mathrm{Hom}_K(\chi, \pi)$.

We take the (not too special) case that χ and π are carried by coadjoint orbits:

The group G acts on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ by the coadjoint action Ad^*, i.e. for $\eta \in \mathfrak{g}^*$ $(g, \eta) \longmapsto (\mathrm{Ad}^* g)\,\eta =: g \cdot \eta$.

Let

$$M := G \cdot \eta$$

be the orbit of $\eta \in \mathfrak{g}*$, s.t. π has a representation space $\mathcal{Q}(M) = \mathcal{H}_\pi$ consisting of certain polarized sections of a metrized hermitian line bundle with curvature given by the (integral) symplectic form ω_M associated to M by the Kirillov-Kostant-Souriau form B_{U_*} with

$$B_{U_*}(\hat{X}, \hat{Y}) = \eta([\hat{X}, \hat{Y}]) \ \text{ for } \ \hat{X}, \hat{Y} \in \mathfrak{g}.$$

Similarly, let χ be carried by the K-orbit

$$O := K \cdot \eta \subset \mathfrak{k}^* \subset \mathfrak{g}^*$$

with symplectic form ω_O. Then we have a symplectic product manifold $M \times O^-$ where ω_O is changed to $-\omega_O$. And we have a moment map

$$\Psi : M \times O^- \longrightarrow \mathfrak{k}^*,\ (m, \eta) \mapsto \Phi(m) - \eta$$

where Φ is the K-moment map coming from the injection $\iota : M \hookrightarrow \mathfrak{g}^*$ composed with the restriction from $\mathfrak{g}^*$ to $\mathfrak{k}^*$. Then one has the *Marsden-Weinstein reduction*

$$M_O := M_{\text{red}} = \Psi^{-1}(0)/\,K$$

which will be identfied with $\Phi^{-1}(\eta)/K_\eta, K_\eta$ the stabilizing group of $\eta \in \mathfrak{k}^*$. If everything is nice, M_O is a symplectic manifold allowing again a quantization procedure $\mathcal{Q}$ and the associated space $\mathcal{Q}(M_O)$ is the geometric analogue of $\operatorname{Hom}_K(\chi, \pi)$ (as explained in [9] 6.). This way, at least in some special cases one has precise information: χ does not appear in π if M_O is empty. And the multiplicity is one if M_O is just a point. There are a lot of more refined results measuring higher multiplicities by the symplectic volume of M_O resp. its Riemann-Roch number $\operatorname{RR}(M_O)$ (see [15] and [8]).

2. Discrete Series Representations of SL(2, R) and the Jacobi Group

We introduce the Jacobi group

$$G = G^J := G_1 \ltimes H,\ G_1 = SL(2, \mathbf{R}),\ H = \operatorname{Heis}(\mathbf{R})$$

as the subgroup of $\operatorname{Sp}(2, \mathbf{R})$ generated by G_1 and H via the usual embedding

$$\operatorname{Heis}(\mathbf{R}) \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\operatorname{SL}(2, \mathbf{R}) \ni M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We write

$$g = M(\lambda, \mu, \kappa) = (p, q, r)M \in G^J$$

so that $(\lambda, \mu) = (p, q)M$. As in [5] or [17], we describe the Lie algebra $\mathfrak{g}^J$ as a subalgebra of $\mathfrak{sp}(2, \mathbf{R})$ by

$$G(x, y, z, p, q, r) = \begin{pmatrix} x & 0 & y & q \\ p & 0 & q & r \\ z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and denote

$$X = G(1,0,\dots,0),\dots,R = G(0,\dots,0,1).$$

We get the commutators

$$\begin{aligned}&[X,Y] = 2Y,\ [X,Z] = -2Z,\ [Y,Z] = X,\\ &[X,P] = -P,\ [X,Q] = Q,\quad [P,Q] = 2R,\\ &[Y,P] = -Q,\ [Z,Q] = -P,\end{aligned}$$

all others are zero. Hence, we have the complexified Lie algebras given by

$$\mathfrak{sl}(2,\mathbf{R})_c =< Z_1, X_\pm >,\ \ \mathfrak{g}_c^J =< Z_1, X_\pm, Y_\pm, Z_0 >$$

with

$$Z_1 = -i(Y - Z),\quad Z_0 = -iR.$$

$$X_\pm = (1/2)(X \pm i(Y+Z)),\quad Y_\pm = (1/2)(P \pm iQ).$$

In this text we want to look at the following discrete series representations, which are most intimately tied to modular forms, i.e. the representation π_k^+ of G_1, $\pi_{m,k}^+$ of G^J and the characters χ_k resp. $\chi_{m,k}$ of $K_1 = \mathrm{SO}(2)$ resp. $K^J := \mathrm{SO}(2) \times \mathbf{R}$. For $k \in \mathbf{Z}$ and $m \in \mathbf{R}, m > 0$, these characters are simply given by

$$\chi_k((r(\vartheta)) = e^{ik\vartheta},\quad r(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in \mathrm{SO}(2)$$

and

$$\chi_{m,k}((r(\vartheta),\kappa) = e^{2\pi i m\kappa} e^{ik\vartheta},\quad r(\vartheta) = \mathrm{SO}(2),\ \kappa \in \mathbf{R} \cong Z(G^J).$$

The discrete series representations are most easily fixed by their infinitesimal versions: $d\pi_k^+$ is given on the space

$$V_k^+ :=< w_\ell >_{\ell \in 2\mathbf{N}_0}$$

by

$$X_- w_0 = 0,\quad Z_1 w_\ell = (k+2\ell)\, w_\ell,\quad X_+ w_\ell = w_{\ell+2},$$

where we use the abbreviations $X_- w_0 = d\pi_k^+(X_-)w_0$ etc. We see immediately

Remark 2.1. *We have*

$$\begin{aligned}\mathrm{mult}(\chi_{k'}, \pi_k^+) &= 1 \text{ for } \quad k' = k+2\ell,\ \ell \in \mathbf{N}_0\\ &= 0 \text{ else.}\end{aligned}$$

The representation $\pi^+_{m,k}$ of G^J has the infinitesimal $d\pi^+_{m,k}$ given on the space

$$V^+_{m,k} =< v_j \otimes w_\ell >_{j\in\mathbf{N}_0,\ell\in 2\mathbf{N}_0}$$

by

$$\begin{aligned}
Z_0(v_j \otimes w_\ell) &= 2\pi m(v_j \otimes w_\ell), \quad Z_1(v_j \otimes w_\ell) = (k+j+\ell)(v_j \otimes w_\ell),\\
Y_-(v_0 \otimes w_0) &= X_-(v_0 \otimes w_0) = 0,\\
Y_+(v_j \otimes w_\ell) &= v_{j+1} \otimes w_\ell,\\
X_+(v_j \otimes w_\ell) &= -(1/(2\pi m))(v_{j+2} \otimes w_\ell) + (v_j \otimes w_{\ell+2}).
\end{aligned}$$

We put

$$\mu(t) := \#\{\ (j,\ell) \in \mathbf{N}_0^2, j+2\ell = t\ \},\ t \in \mathbf{N}_0,$$

i.e. $\mu(t) = 1$ for $t = 0$ and 1, $\mu(t) = 2$ for $t = 2$ and 3, etc.
Then we can write

$$\mu(t) = (1/2)(1 + t + (\cos(\pi t/2))^2)$$

and hence

Remark 2.2. *We have*

$$\mathrm{mult}(\chi_{m,k'}, \pi^+_{m,k}) = \mu(k'-k) \text{ for } k' \geq k.$$

There are several ways to realize these discrete series representations. They are discussed for instance in [5] p.48ff. As elaborated in [3] and [4], these representations are also associated to coadjoint orbits of G_1 resp. G^J.

3. Coadjoint Orbits of SL(2, R) and G^J

For a semisimple Lie algebra $\mathfrak{g}$ realized as a subalgebra of $M_n(\mathbf{R})$, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the map

$$\mathfrak{g}^* \longrightarrow \mathfrak{g}, \quad \eta \longmapsto U_\eta$$

defined by

$$\eta(\hat{Y}) = \mathrm{tr}(U_\eta \hat{Y}) =:< U_\eta,\ \hat{Y} > \text{ for all } \hat{Y} \in \mathfrak{g}.$$

Following closely Yang, we realize $(\mathfrak{g}^J)^*$ as a subspace of $\mathfrak{sp}(2,\mathbf{R})$ by the matrices

$$M(x_*, y_*, z_*, p_*, q_*, r_*) = \begin{pmatrix} x_* & p_* & z_* & 0\\ 0 & 0 & 0 & 0\\ y_* & q_* & -x_* & 0\\ q_* & r_* & -p_* & 0 \end{pmatrix}, x_*, \ldots, r_* \in \mathbf{R},$$

and put

$$X_* = M(1,0,\dots,0),\dots,R_* = M(0,\dots,0,1).$$

Then $X_*,\dots,R_*$ is a basis of $(\mathfrak{g}^J)^*$ and we have with the notation from Section 2

$$< X_*, X >= 2,\ < Y_*, Y >= 1,\ < Z_*, Z >= 1$$
$$< P_*, P >= 2,\ < Q_*, Q >= 2,\ < R_*, R >= 1$$

By a straightforward computation one obtains

Lemma 3.1. *For*

$$g = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the coadjoint action of g on $M(x,\dots,r)$ is given by

$$\mathrm{Ad}^*(g)M(x_*,\dots,r_*) = g\cdot M(x_*,\dots,r_*) = M(\tilde{x}_*,\dots,\tilde{r}_*)$$

with

$$\tilde{x}_* = (ad+bc)x_* + bdy_* - acz_* + (2ac\mu - (ad+bc)\lambda)p_*$$
$$+ ((ad+bc)\mu - 2bd\lambda)q_* + r_*(a\mu - b\lambda)(c\mu - d\lambda),$$

$$\tilde{y}_* = 2dcx_* + d^2y_* - c^2z_* + 2(c\mu - d\lambda)(cp_* + dq_*) + r_*(c\mu - d\lambda)^2,$$

$$\tilde{z}_* = -2abx_* - b^2y_* + a^2z_* - 2(a\mu - b\lambda)(ap_* + bq_*) - r_*(a\mu - b\lambda)^2,$$

$$\tilde{p}_* = ap_* + bq_* + r_*(a\mu - b\lambda),$$

$$\tilde{q}_* = cp_* + dq_* + r_*(c\mu - d\lambda),$$

$$\tilde{r}_* = r_*.$$

By an adequate specialization one has the following *elliptic* orbits.

Case 1. $G = G_1$

For

$$U_* = k(Y_* - Z_*),\quad k \geq 0$$

we get the two-dimensional G_1-orbit

$$\begin{aligned} M_k^+ &= G_1 \cdot U_* \simeq G_1/G_{1U_*}, \quad G_{1U_*} = \mathrm{SO}(2), \\ &= \{(x_*, y_*, z_*) \in \mathbf{R}^3, f(x_*, y_*, z_*) = x_*^2 + y_* z_* + k^2 = 0, y_* > z_*\}, \\ &= \{(x_*, \check{y}_*, \check{z}_*) \in \mathbf{R}^3, \check{f}(x_*, \check{y}_*, \check{z}_*) = x_*^2 + \check{y}_*^2 - \check{z}_*^2 + k^2 = 0, \check{z}_* > 0\}, \end{aligned}$$

where we introduced $y_* = \check{y}_* + \check{z}_*, z_* = \check{y}_* - \check{z}_*$ to show the standard form of the upper half of the two-sheeted hyperboloid.

Case 2. $G = G^J$
For

$$U_* = k(Y_* - Z_*) + mR_*, \quad k \geq 0$$

we get the four-dimensional G^J-orbit

$$\begin{aligned} M_{m,k}^+ &= G^J \cdot U_* \simeq G^J/G^J{}_{U_*}, \quad G^J{}_{U_*} = K^J = \mathrm{SO}(2) \times \mathbf{R}, \\ &= \{(x_*, y_*, z_*, p_*, q_*) \in \mathbf{R}^5, f(x_*, y_*, z_*, p_*, q_*) = 0, y_* > z_*\}, \\ &= \{(x_*, \check{y}_*, \check{z}_*, p_*, q_*) \in \mathbf{R}^5, \check{f}(x_*, \check{y}_*, \check{z}_*, p_*, q_*) = 0, \check{z}_* > 0\}, \end{aligned}$$

where

$$f(x_*, y_*, z_*, p_*, q_*) = m(x_*^2 + y_* z_* + k^2) - 2p_* q_* x_* + p_*^2 y_* - q_*^2 z_*$$

and

$$\check{f}(x_*, \check{y}_*, \check{z}_*, p_*, q_*) = m(x_*^2 + \check{y}_*^2 - \check{z}_*^2 + k^2) - 2p_* q_* x_* + (p_*^2 - q_*^2)\check{y}_* + (p_*^2 + q_*^2)\check{z}_*.$$

By the general theory, these orbits are *integral* or *prequantizable*, iff k is (up to a factor 2π) an integer. And with the aid of a (complex) polarization, these orbits are associated to the discrete series representations π_k^+ resp. $\pi_{m,k}^+$ of G_1 and G^J (see [4]).

The same way, the characters $\chi_{k'}$ resp. $\chi_{m,k'}$ belong to K_1- resp. K^J-orbits being here simply points, namely

$$\begin{aligned} K_1 \cdot U_* &= \{k'(Y_* - Z_*)\} && \text{for} \quad U_* = k'(Y_* - Z_*), \\ K^J \cdot U_* &= \{k'(Y_* - Z_*) + mR_*\} && \text{for} \quad U_* = k'(Y_* - Z_*) + mR_*. \end{aligned}$$

4. Marsden-Weinstein Reduction and Symplectic Volumes

The general procedure from Section 1 proposes to take in Case 1 the moment map Φ with

$$\Phi(x_*, \check{y}_*, \check{z}_*) = \check{z}_*$$

to produce the reduced space

$$M_{O_{k'}} := \Phi^{-1}(k')/K_1.$$

This means that here we have

$$M_k^{k'} := \Phi^{-1}(k') = \{(x_*, \check{y}_*) \in \mathbf{R}^2, \check{f}_{k'}(x_*, \check{y}_*) = x_*^2 + \check{y}_*^2 - k'^2 + k^2 = 0\}.$$

We see from Lemma 3.1 that $r(\vartheta) \in K_1 = \mathrm{SO}(2)$ acts by

$$(r(\vartheta), \begin{pmatrix} x_* \\ \check{y}_* \end{pmatrix}) \longmapsto r(2\vartheta)\begin{pmatrix} x_* \\ \check{y}_* \end{pmatrix}.$$

Hence $M_{O_{k'}}$ is just a point for $k' \geq k$ and empty else. This reflects the multiplicity statement from Remark 2.1 if one remembers the fact that for odd $k' - k$ there is no equivariant line bundle on $M_{O_{k'}}$ giving rise to a representation of G_1.

Now we indicate the outcome of what happens when one tries to carry over the same procedure to Case 2:
Here we have the moment map $\Phi : M^+_{m,k} \longrightarrow (\mathfrak{k}^J)^*$ with

$$\Phi(x_*, \check{y}_*, \check{z}_*, p_*, q_*, r_*) = (\check{z}_*, r_*),$$

This leads to a subspace of $M^+_{m,k}$ given by

$$\begin{aligned} M^{k'}_{m,k} &:= \Phi^{-1}(k', m) \\ &= \{(x_*, \check{y}_*, p_*, q_*) \in \mathbf{R}^4, \check{f}_{k',m}(x_*, \check{y}_*, p_*, q_*) = 0\}, \end{aligned}$$

where

$$\check{f}_{k',m}(x_*, \check{y}_*, p_*, q_*) = m(x_*^2 + \check{y}_*^2 - k'^2 + k^2) - 2p_*q_*x_* + (p_*^2 - q^2*)\check{y}_* + (p_*^2 + q_*^2)k'.$$

It is useful to write this a bit differently: We introduce

$$\tau = x_* + iy_*,\ z = p_* + iq_*,\ \rho^2 = k'^2 - k^2$$

to get

$$M^{k'}_{m,k} = \{(\tau, z) \in \mathbf{C}^2,\ m(\mid \tau \mid^2 - \rho^2) + (z^2\bar{\tau} - \bar{z}^2\tau)i/2 + \mid z \mid^2 k' = 0\},$$

and

$$x_* = \tilde{r}\cos\tilde{\varphi},\ y_* = \tilde{r}\sin\tilde{\varphi},\ p_* = r\cos\varphi,\ q_* = r\sin\varphi$$

to get

$$M^{k'}_{m,k} = \{(\tilde{r}, \tilde{\varphi}, r, \varphi),\ m(\tilde{r}^2 - \rho^2) + r^2(\tilde{r}\sin(\tilde{\varphi} - 2\varphi) + k') = 0\}.$$

We see by Lemma 3.1 that the action of the stabilizing group $G^J_{U_*} = K^J$ is given here by

$$(r(\vartheta), (\tau, z)) \longmapsto (e^{i2\vartheta}\tau, e^{i\vartheta}z)$$

resp.

$$(r(\vartheta), (\tilde{r}, r, \varphi_1, \varphi_2)) \longmapsto (\tilde{r}, r, \varphi_1, \varphi_2 + 4\vartheta)$$

for $\varphi_1 := \tilde{\varphi} - 2\varphi, \varphi_2 := \tilde{\varphi} + 2\varphi$. Then we see

Remark 4.1. *The reduced space $M_{O_{k'}} = \Phi^{-1}(k', m)/\mathrm{SO}(2)$ may be written as*

$$M_{O_{k'}} = \{(\tilde{r}, r, \varphi_1), m(\tilde{r}^2 - \rho^2) + r^2(\tilde{r}\sin\varphi_1 + k') = 0\}.$$

If we use $(\tilde{r}, \varphi_1)$ as parameters, we have

$$r^2 = \frac{m(\rho^2 - \tilde{r}^2)}{\tilde{r}\sin\varphi_1 + k'}\,.$$

$M_{O'_k}$ has a compact subspace M_ρ. Introducing

$$u = \tilde{r}\cos\varphi_1, v = \tilde{r}\sin\varphi_1,$$

this is written as

$$M_\rho = \{(u, v, r), r^2 = \frac{m(\rho^2 - (u^2 + v^2))}{v + k'}\,,\ u^2 + v^2 \leq \rho^2\}.$$

Now we would like to have a symplectic invariant associated to M_ρ which captures information on the multiplicity of $\chi_{m,k'}$ in $\pi^+_{m,k}$. Following the standard procedure, one is led to determine the symplectic volume $\mathrm{vol}(M_\rho)$ of M_ρ. To do this, we take a G^J-invariant symplectic form ω on $M^+_{m,k}$ and use it to construct a form $\tilde{\omega}$ for the reduced space $M_{O_{k'}}$. Our ω is the realization of the Kirillov-Kostant-Souriau form B_{U_*} for $M^+_{m,k}$. We assemble the calculations leading to an explicit form in an Appendix and state here simply our result:

Proposition. *The Kirillov-Kostant-Souriau form B_{U_*} of $M^+_{m,k}$ induces the form $\tilde{\omega}$ living on the reduced space $M_{O_{k'}}$ which is given by*

$$\tilde{\omega} = \frac{\tilde{r}\,d\tilde{r} \wedge d\varphi_1}{\tilde{r}\sin\varphi_1 + k'} = \frac{du \wedge dv}{v + k'}\,, u = \tilde{r}\cos\varphi_1, v = \tilde{r}\sin\varphi_1.$$

Integration over the compact part M_ρ of $M_{O_{k'}}$ leads to

Corollary. *We have*

$$\mathrm{vol}\,(M_\rho) = \int_{u^2+v^2 \leq \rho^2 = k'^2 - k^2} \frac{du \wedge dv}{v + k'} = 2\pi(k' - k)\,.$$

This is not more than a qualitative statement in the direction of the multiplicity formula from Remark 2.2. We can only hope that someone will come up with a prescription leading exactly to the formula

$$\mu(k'-k) = (1/2)(1 + (k'-k) + (\cos(\pi(k'-k)/2))^2).$$

5. Appendix: Explicit Expressions of Symplectic Forms for Elliptic Coadjoint Obits of G^J

The KKS-form B_{U_*} for $M^+_{k,m}$ is given by

$$B_{U_*}(\widehat{X},\widehat{Y}) = \langle U_*, [\widehat{X},\widehat{Y}]\rangle\,,\ \widehat{X} = xX + \ldots + rR\,,\ \widehat{Y} = x'X + \ldots + r'R\,.$$

Using the formulae from Section 3, one obtains

Remark 5.1. *We have for* $U_* = k\,(Y_* - Z_*) + mR_*$

$$B_{U_*}(\widehat{X},\widehat{Y}) = 2k\,(x\,(y'+z') - x'\,(y+z)) + 2m\,(p\,q' - p'\,q)\,.$$

The associated symplectic form ω on $M := M^+_{k,m}$ is defined by

$$\omega\,(\widehat{X}_M,\widehat{Y}_M) = B_{U_*}(\widehat{X},\widehat{Y})\,.$$

As usual, $\widehat{X}_M$ and $\widehat{Y}_M$ denote the vector fields associated to $\widehat{X}$ and $\widehat{Y}$: If $h = h\,(x_*,\check{y}_*,p_*,q_*)$ is a function on M expressed in the parameters $x_*,\check{y}_*,p_*,q_*$, s. t. $\check{z}_* = \check{z}_*\,(x_*,\check{y}_*,p_*,q_*)$ is fixed by the equation

$$\check{f} = m\,(x_*^2 + \check{y}_*^2 - \check{z}_*^2 + k^2) - 2p_*q_*x_* + (p_*^2 - q_*^2)\,\check{y}_* + (p_*^2 + q_*^2)\,\check{z}_* = 0\,,$$

one has

$$X_M\,h\,(x_*,\check{y}_*,p_*,q_*) = \frac{d}{dt}\,h\,((\exp tX)\cdot(x_*,\check{y}_*,p_*,q_*))\,|_{t=0}\ .$$

Remark 5.2. *By a standard computation we have*

$$\begin{aligned}
X_M &= -2\check{z}_*\,\partial_{\check{y}_*} + p_*\,\partial_{p_*} - q_*\,\partial_{q_*}\,,\\
Y_M &= (\check{y}_* + \check{z}_*)\,\partial_{x_*} + q_*\,\partial_{p_*} - x_*\,\partial_{\check{y}_*}\,,\\
Z_M &= (-\check{y}_* + \check{z}_*)\,\partial_{x_*} + p_*\,\partial_{q_*} + x_*\,\partial_{\check{y}_*}\,,\\
P_M &= -p_*\,\partial_{x_*} - q_*\,\partial_{\check{y}_*} - r_*\,\partial_{q_*}\,,\\
Q_M &= q_*\,\partial_{x_*} - q_*\,\partial_{\check{y}_*} + r_*\,\partial_{p_*}\,.
\end{aligned}$$

In principle, one should be able to give an explicit formula for ω using these ingredients. Intending to avoid nasty computations, we take here another route which should have some interest in its own.

$M^+_{k,m}$ is diffeomorphic to $G^J / K^J = \mathfrak{H} \times \mathbf{C}$: G^J acts on $\mathfrak{H} \times \mathbf{C}$ as usual (see [7]) by

$$(g, (\dot{\tau}, \dot{z})) \longmapsto g\,(\dot{\tau}, \dot{z}) = (g(\dot{\tau}), (\dot{z} + \lambda\dot{\tau} + \mu)/(c\dot{\tau} + d)),$$

where $(\dot{\tau}, \dot{z}) \in \mathfrak{H} \times \mathbf{C}$ and $g = M\,(\lambda, \mu, \kappa)$. For $g = (\dot{p}, \dot{q}, \dot{k})\,M$ we have

$$g\,(i, 0) = (M(i) = \dot{\tau}, \dot{p}\,\dot{\tau} + \dot{q}) ,$$

where $(\dot{p}, \dot{q})$ and (λ, μ) are related by

$$(\dot{p}, \dot{q}) = (\lambda, \mu) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (-\mathrm{II}, \mathrm{I}) , \ \mathrm{I} = \mu a - \lambda b , \ \mathrm{II} = \mu c - \lambda d .$$

Hence, we have an identification $\varphi : \mathfrak{H} \times \mathbf{C} \rightarrow M^+_{k,m}$ which is G^J-equivariant for this operation of G^J on $\mathfrak{H} \times \mathbf{C}$ and the coadjoint action on $M^+_{k,m}$, namely φ is given by

$$\varphi\,(\dot{x}, \dot{y}, \dot{p}, \dot{q}) = (x_*, y_*, z_*, p_*, q_*)$$

with

$$\begin{aligned} x_* &= k\,\dot{x}\,/\,\dot{y}, \\ y_* &= k\,/\,\dot{y} + m\dot{p}^2, \\ z_* &= -k\,(\dot{x}^2 + \dot{y}^2)\,/\,\dot{y} - m\,\dot{q}^2, \\ p_* &= m\,\dot{q}, \\ q_* &= -m\,\dot{p}\,. \end{aligned}$$

The inverse map $\psi = \varphi^{-1}$ is given by

$$\dot{x} = (mx_* - p_*q_*)/(my_* - q_*^2) , \ \dot{y} = mk\,/\,(my_* - q_*^2),$$

$$\dot{p} = -q_*\,/\,m , \quad \dot{q} = p_*\,/\,m .$$

It is well-known and easy to see that

$$\omega_1 = \frac{d\dot{x} \wedge d\dot{y}}{\dot{y}^2} \quad \text{and} \quad \omega_2 = d\dot{p} \wedge d\,\dot{q}$$

are G^J-invariant symplectic forms on $\mathfrak{H} \times \mathbf{C}$. We transfer these forms via ψ to $M^+_{k,m}$ and get the following forms in the parameters $(x_*, \check{y}_*, p_*, q_*)$

used above:

Remark 5.3. *We have with* $\check{f}_{\check{z}_*} = -2m\,\check{z}_* + p_*^2 + q_*^2$

$$\begin{aligned}\omega_1 \circ \psi = 2\,/\,(k\check{f}_{\check{z}_*}))\ (m\,dx_* \wedge d\check{y}_* + p_*\,dx_* \wedge dp_* - q_*\,dx_* \wedge dq_* \\ + q_*\,d\check{y}_* \wedge dp_* + p_*\,d\check{y}_* \wedge dq_* + (1/m)\,(p_*^2 + q_*^2)\,dp_* \wedge dq_*\big)\ .\end{aligned}$$

Changing to the coordinates

$$\tau = x_* + iy_*\,,\ z = p_* + iq_*$$

we come to

$$\omega_1 \circ \psi = (1/(k\check{f}_{\check{z}_*}))\,(im\,d\tau \wedge d\,\bar{\tau} + \bar{z}\,d\tau \wedge d\,\bar{z} + z\,d\,\bar{\tau} \wedge dz + (i/m)\,dz \wedge d\,\bar{z})\ ,$$

and for

$$x_* = \widetilde{r}\,\cos\varphi\,,\ y_* = \widetilde{r}\,\sin\varphi\,,\ p_* = r\,\cos\varphi\,,\ q_* = r\,\sin\varphi\,,\ \varphi_1 := \widetilde{\varphi} - 2\,\varphi$$

we have

$$\begin{aligned}\omega_1 \circ \psi = 2/(k(r^2 - 2m\check{z}_*))\,(m\,\widetilde{r}d\widetilde{r} \wedge d\widetilde{\varphi} + r\,\cos\varphi_1\,d\widetilde{r} \wedge dr + r\,\widetilde{r}\sin\varphi_1\,dr \wedge d\widetilde{\varphi} \\ + r^2\,\sin\varphi_1\,d\widetilde{r} \wedge d\varphi + r^2\,\widetilde{r}\cos\varphi_1\,d\widetilde{\varphi} \wedge d\varphi + (1/m)\,r^3\,dr \wedge d\varphi)\ .\end{aligned}$$

Remark 5.4. *We have*

$$\omega_2 \circ \psi = (1/m^2)\,dp \wedge dq = (i/(2m^2))\,dz \wedge d\,\bar{z} = (1/m^2)\,r\,dr \wedge d\varphi\ .$$

We compare this to the prescription for the KKS-form given in Remark 5.1: From Remark 5.1 we get for $U_* = k\,(Y_* - Z_*) + mR_*$

$$\begin{aligned}\widehat{X}_M\,(U_*) &= (y + z)\,k\,\partial_{x_*} - 2xk\,\partial_{\check{y}_*} + qm\,\partial_{p_*} - pm\,\partial_{q_*} \\ \widehat{Y}_M\,(U_*) &= (y' + z')\,k\,\partial_{x_*} - 2x'\,k\,\partial_{\check{y}_*} + q'm\,\partial_{p_*} - p'\,m\,\partial_{q_*}\end{aligned}$$

and hence from Remark 5.3 and 5.4 for $\omega := \alpha\,\omega_1 \circ \psi + \beta\,\omega_2 \circ \psi$ evaluated at U_*

$$\omega\,(\widehat{X}_M, \widehat{Y}_M)\,(U_*) = 2\alpha\,((x + y)\,x' - (x' + y')\,x) + \beta\,(pq' - p'q)\ .$$

So, we have to take $\alpha = -k$ and $\beta = 2m$ to get

Remark 5.5. *The KKS-form for* $M^+_{k,m} = G^J \cdot (k(Y_* - Z_*) + m\,R_*)$ *is given in the parameters* $x_*, \check{y}_* = y_* + z_*, p_*, q_*$ *by the symplectic form*

$$\begin{aligned}\omega \circ \psi = -(2/(-2m\,\check{z}_* + p_*^2 + q_*^2))\,(mdx_* \wedge dy_* + p_*dx_* \wedge dp_* \\ - q_*\,dx_* \wedge dq_* + q_*d\,\check{y}_* \wedge dp_* + p_*\,d\,\check{y}_*dq_* + 2\,\check{z}_*\,dp_* \wedge dq_*)\ .\end{aligned}$$

Here $\check{z}_*$ is a function of the parameters $x_*, \check{y}_*, p_*, q_*$ fixed by the equation $\check{f}\,(x_* \check{y}_*, \check{z}_*, p_*, q_*) = 0$.

ω restricts to a form on the subspace $\phi^{-1}\,(k', m)$ of $M^+_{k,m}$ by putting $\check{z}_* = k'$ and then induces a form $\tilde{\omega}$ on the space $M_{O_{k'}} = \phi^{-1}\,(k', m)\,/\,\mathrm{SO}(2)$ in the following way. From the proof of Remark 4.1 we know that $\phi^{-1}\,(k', m)$ is described by the equation

$$\check{f}_{k',m} \;=\; m\,(\widetilde{r}^2 - \rho^2) + r^2\,(\widetilde{r}\,\sin\,\varphi_1 + k') = 0\,, \quad \rho^2 = k'^2 - k^2$$

with $\varphi_1 = \widetilde{\varphi} - 2\varphi$. Here $\widetilde{r}, r$ and φ_1 are SO(2)-invariant and as in Remark 4.1 will be used for the description

$$M_{O_{k'}} \;=\; \{(\widetilde{r}, r, \varphi_1)\;,\quad \check{f}_{k',m} = 0\}\,.$$

As in the discussion of Remark 4.1, we take $\widetilde{r}, \varphi_1$ as parameters and write

$$r^2 \;=\; \frac{m(\rho^2 - \widetilde{r}^2)}{\widetilde{r}\,\sin\,\varphi_1 + k'}\,.$$

The form from Remark 5.5 is expressed in the variables $(r, \widetilde{r}, \varphi, \widetilde{\varphi})$ restricted to $\check{z} = k'$

$$\begin{aligned}\omega \circ \psi\,|_{\check{z}=k} = -(2\,/\,(r^2\; - \;2mk'))\,&(m\widetilde{r} d\widetilde{r} \wedge d\,\widetilde{\varphi} + r\,\cos\,\varphi_1\,d\widetilde{r} \wedge dr \\ &+ r\widetilde{r}\sin\,\varphi_1\,dr \wedge d\,\widetilde{\varphi} + r^2\,\sin\,\varphi_1\,d\widetilde{r} \wedge d\,\varphi \\ &+ \widetilde{r}\,r^2\,\cos\,\varphi_1\,d\widetilde{\varphi} \wedge d\,\varphi + 2k' r dr \wedge d\,\varphi)\,.\end{aligned}$$

If we introduce $\widetilde{\varphi} = (1/2)\,(\varphi_1 + \varphi_2)\,,\ \varphi = (1/4)\,(\varphi_2 - \varphi_1)$ and restrict to $\varphi_2 = 0$ we get

$$\begin{aligned}\omega \circ \psi\,|_{\check{z}=k',\varphi_2=0} = -(2\,/\,(r^2 - 2mk'))\,&((m\widetilde{r}/2 - (r^2/4)\sin\,\varphi_1) d\widetilde{r} \wedge d\,\varphi_1 \\ &+ r\,(\,(\widetilde{r}/2)\sin\,\varphi_1 - k'/2) dr \wedge d\,\varphi_1 + r\,\cos\,\varphi_1\,d\widetilde{r} \wedge dr\,)\,.\end{aligned}$$

By

$$d\check{f}_{\check{k}} \;=\; (2m\,\widetilde{r} + r^2\,\sin\,\varphi_1)\,d\widetilde{r} + 2r\,(\widetilde{r}\,\sin\,\varphi_1 + k')\,dr + r^2\,\widetilde{r}\,\cos\,\varphi_1\,d\varphi_1 = 0$$

we replace dr and get the form $\tilde{\omega}$ living on $M_{O_{k'}}$ expressed in the parameters $\widetilde{r}, \varphi_1$ as in the Proposition in Section 4

$$\tilde{\omega} \;=\; \frac{\widetilde{r}\,d\widetilde{r} \wedge d\varphi_1}{\widetilde{r}\,\sin\,\varphi_1 + k'}\,.$$

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ON EISENSTEIN SERIES OF DEGREE TWO FOR SQUAREFREE LEVELS AND THE GENUS VERSION OF THE BASIS PROBLEM I

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Dedicated to the memory of Tsuneo Arakawa

We apply the doubling method to elliptic modular forms of squarefree level N using the Siegel Eisenstein series of degree two attached to an *arbitrary cusp*. Combining these computations with Siegel's theorem, we can consider the basis problem for any given genus of positive quadratic forms of level N; we generalize some results of Waldspurger [23].

1. Introduction

This paper can be viewed as a commentary on the the famous work of Waldspurger [23] concerning the basis problem for elliptic modular forms. Waldspurger's paper was written before the doubling method was available; implicitly he used already some features of that method, e.g. a genus theta series of degree two appears in his calculations (to be more precise: only its first Fourier-Jacobi-coefficient $\varphi_1(\tau, z)$ is considered after restriction to $z = 0$). To compute the Petersson product of a cusp form against $\varphi_1(\tau, 0)$, he compares the Fourier coefficients of $\varphi_1(\tau, 0)$ with those of a construction by Zagier [24]. Thanks to the works of Garrett [13] (and his followers) we know today how to avoid the laborious calculus of comparing Fourier coefficients by investigating directly the integral of a cusp form (of arbitrary degree n) against an Eisenstein series of double degree $2n$; this computation is nowadays standard for unramified primes, but less clear for primes dividing the level.

The main purpose of the present paper is to understand some of Waldspurger's results from the point of view of pullbacks of Eisenstein series of degree two, in particular, we want to shed some light on the following beautiful (but mysterious) result [23, Théorème 3]:

> *Let $m = 2k$ be divisible by 4 and let $D > 1$ be a squarefree integer with $D \equiv 1 \bmod 4$. Then all cusp forms of weight k, level D and primitive nebentypus are linear combinations of theta series attached to positive definite quadratic forms of level D and discriminant D or discriminant D^{m-1} if and only if the Hecke operator $U(D)$ does not have a real eigenvalue on the space of cusp forms in question.*

For some attempts to understand this condition on the eigenvalues of $U(D)$ see the papers of Ponomarev [18, 19].

In turns out (this is perhaps the only merit of the approach presented here) that the result above is specific for the very special genera chosen by Waldspurger: For all other genera of the same level and primitive nebentypus the basis problem has an unconditional affirmative answer (this is at least true for prime levels).

It seems to us that our method is more flexible: One can (at least in principle) handle the basis problem for any genus of squarefree level in this way; the bulk of calculation is done independently of choosing a genus to be considered.

We do not expect that our method could possibly be extendend to higher degree (or levels which are not squarefree). In the higher degree case a more modest (but tractable) aim along these lines is the consideration of the basis problem for a given level (not fixing a specific genus), see [7] for squarefree levels.

In principle, the application of the doubling method to the basis problem is well understood. One starts with a genus theta series of degree $2n$, which by Siegel's result is an Eisenstein series; we just quote Siegel without explaining all the notations:

$$\frac{1}{m(\mathfrak{S})}\sum_i \frac{1}{\epsilon(S_i)}\theta^{2n}(Z,S_i) = E^{2n}(Z,\mathfrak{S}) = \sum_{C,D}\gamma(C,D)\det(CZ+D)^{-\frac{m}{2}}.$$

Here $\mathfrak{S}$ denotes a genus of positive definite quadratic forms of (even) rank m and the S_i are representatives of the classes in $\mathfrak{S}$. For a given cusp form

f of degree n, one has then to study the map

$$f \longmapsto \frac{1}{m(\mathfrak{S})}\sum_i \frac{1}{\epsilon(S_i)} < f, \theta^n(*, S_i) > \theta^n(S_i)$$
$$= \int_{\Gamma\backslash\mathbb{H}_n} f(z)\overline{E^{2n}(\begin{pmatrix} -\bar{z} & 0 \\ 0 & w \end{pmatrix}, \mathfrak{S})}d^*w.$$

One has to understand the integral over the genus Eisenstein series in some way. Here we propose a somewhat brutal strategy: We write the genus Eisenstein series as a linear combination of Eisenstein series attached to the inequivalent cusps. Then one tries to compute the same kind of integral for each of these group-theoretical Eisenstein series individually (by considerations completely independent of theta series). This seems to be a somewhat painful task in general. Indeed it is the second main topic of this paper to work this out explicitly for the case $n = 1$, squarefree levels and *all* Eisenstein series. It turns out that in this case (say for newforms of haupttypus) the contribution from the ramified primes is a rational function with denominator independent of the cusp in question; the numerator has a simple structure (for nebentypus the situation is somewhat more complicated). It is this numerator, which – in some cases – may create an obstruction against a positive solution of the basis problem.

Of course there is also a "global obstruction" coming from zeroes or poles of L-functions (in the case of low weights). In this paper however we deal only with large weights and our main focus is the structure of numerators (at bad primes) and its relevance for the basis problem.

There are not many papers dealing with similiar questions (explicit pullbacks of Eisenstein series with levels); typically one picks out a favourite cusp, for which the compution works smoothly, see e.g. [9], [21].

Our paper is organized as follows: In section 3 we describe from the viewpoint of group theory the Siegel-Eisenstein series of squarefree level. In section 4 some coset decompositions (with level) are studied; these results are then used in sections 5-7 to do the unfolding: For each Siegel-Eisenstein series of degree 2 we determine explicitly the pullback integral. To our knowledge, this has never been done for *all* Eisenstein series and it may (perhaps) be of some interest to see explicitly the "numerators" showing up. In the final section 8 we apply our results to the basis problem, emphasizing the case of primitive (quadratic) nebentypus of prime level. We rediscover here some of Waldspurger's results and add some cases not covered in [23].

Arakawa liked the pullback machinery of Garrett and its application to the basis problem. He himself wrote a paper on this topic for the case of Jacobi forms [1]. Arakawa and I often made jokes about our different attitudes towards Jacobi forms. It is very sad that the present paper cannot be used for such joyful disputes among us.

2. Preliminaries

For most standard notations concerning modular forms we refer to the literature [12, 15, 20].

We use $\exp(z)$ for $e^{2\pi i z}$; let $\mathbb{H}_n$ be Siegel's upper half space with the usual action of the proper symplectic similitude group $GSp^+(n,\mathbb{R})$; this group also acts on functions f on $\mathbb{H}_n$ by the slash operator:

$$(f \mid_k g)(Z) = \det(g)^{\frac{k}{2}} j(g,Z)^{-k} f(g < Z >)$$

with $j(g,Z) = \det(CZ+D)$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We denote by $M_k^n(N,\chi)$ and $S_k^n(N,\chi)$ respectively the space of Siegel modular forms of degree n and weight k for the ususal congruence subgroup $\Gamma_0^n(N)$ with nebentypus χ. We omit the superscript n if $n=1$. We often use (for $w = u + iv \in \mathbb{H}_n$) the symbol

$$d^*w = \det(v)^{k-(n+1)} du\, dv$$

in integrals describing a Petersson scalar product (the "weight" k should always be clear from the context).

For any subgroup G of $Sp(n,\mathbb{R})$ we define G_∞ to consist of those elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in G with $C = 0$.

We will often use a standard embedding

$$\iota_{n,n} : Sp(n) \times Sp(n) \hookrightarrow Sp(2n),$$

defined by

$$\iota_{n,n} : (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

Sometimes we write $g^\downarrow$ instead of $\iota_{n,n}(1_{2n}, g)$. Tacitly we use the embedding $\iota_{n,n}$ also for symplectic similitudes, if the similitude factors agree in both components.

3. Cusps and Eisenstein series for $\Gamma_0^n(N)$

We start in a more general setting. Let $N = p_1 \cdots p_r$ be a squarefree positive integer and let χ be a Dirichlet character mod N; we decompose this Dirichlet character as

$$\chi = \prod_{p|N} \chi_p.$$

For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^n(N)$ we put $\chi(\gamma) := \chi(\det(D_\gamma))$.

For any $R \in Sp(n, \mathbb{Z})$ we can define the Eisenstein series of weight k by

$$E_k^n(Z, \chi, R, s) := \sum_{\gamma \in (R\Gamma_0^n(N)R^{-1})_\infty \backslash R\Gamma_0^n(N)} \overline{\chi}(R^{-1}\gamma) j(\gamma, Z)^{-k} \det(\Im(\gamma < Z >)^s.$$

This is well-defined under the condition

$$\chi(-1) = (-1)^k,$$

which we always assume to hold (we also assume that $k + 2\Re(s) > n + 1$ holds, therefore we have no problems of convergence).

It is easily seen that for $\gamma_\infty = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_0^n(N)_\infty$ and $\gamma \in \Gamma_0^n(N)$ we have

$$E_k^n(Z, \chi, \gamma_\infty \cdot R \cdot \gamma, s) = \det(D)^{-k} \chi(\gamma) \cdot E_k^n(Z, \chi, R, s).$$

Therefore these Eisenstein series depend (essentially) only on the double cosets

$$\Gamma_0^n(N)_\infty \cdot R \cdot \Gamma_0^n(N),$$

moreover this series is zero unless the important *compatibilty condition*

$$\chi(R^{-1}\gamma_\infty R) = \det(D)^{-k}$$

is satified for all $\gamma_\infty \in (R\Gamma_0^n(N)R^{-1}) \cap \Gamma_0^n(N)_\infty$.

We can parametrize the double cosets of $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by the p_i-rank of C $(1 \leq i \leq r)$. There are $(n+1)^r$ many linearly independent such Eisenstein series (this is certainly true if χ is trivial; for nontrivial character one has to take the compatibility condition into account).

Remark 3.1. It is also possible, to define in the same way as above Eisenstein series $E_k^n(Z, \psi, R, s)$ for any $R \in GSp^+(n, \mathbb{Q})$ and any character ψ of $\Gamma_0^n(N)$; occasionally we use such a more general Eisenstein series.

For many purposes it is enough to know these Eisenstein series only up to the action of elements of the normalizer of $\Gamma_0^n(N)$: Let $W \in GSp^+(n,\mathbb{Q})$ normalize $\Gamma_0^n(N)$, then we have, for $R \in Sp(n,\mathbb{Z})$ by a simple calculation

$$E_k^n(Z,\chi,R,s) \mid_k W = E_k^n(Z,\psi,RW,s),$$

where the character ψ of $\Gamma_0^n(N)$ is defined by

$$\psi(\gamma) = \chi(W\gamma W^{-1}).$$

We analyse this more explicitly for degree $n=2$. Then we can characterize an Eisenstein series attached to $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$ essentially by the $\mathbb{F}_p$-rank of C; therefore we define a decomposition

$$N = N_0 \cdot N_1 \cdot N_2$$

attached to R by

$$\operatorname{rank}_{\mathbb{F}_p}(C) = \begin{cases} 2 & p \mid N_2, \\ 1 & p \mid N_1, \\ 0 & p \mid N_0. \end{cases}$$

Occasionally we write $N_i(R)$ to emphasize that the N_i depend on the matrix R describing a cusp.

The main purpose of our considerations here is to see that we may restrict ourselves to the case $N_0 = 1$: Assume therefore that $N_0 > 1$; then we choose an integral 2×2-matrix

$$W_{N_0} = \begin{pmatrix} aN_0 & b \\ N & N_0 \end{pmatrix}$$

of determinant N_0 (involution of Atkin-Lehner-type). Then

$$\tilde{W}_{N_0} := \iota_{1,1}(W_{N_0}, W_{N_0})$$

is normalizing $\Gamma_0^2(N)$ and

$$R \cdot \tilde{W}_{N_0} = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & N_0 \cdot 1_2 \end{pmatrix} \cdot \tilde{R}$$

with

$$\tilde{R} = \begin{pmatrix} aN_0A + NB & bA + N_0B \\ aC + N'D & b(\frac{C}{N_0}) + D \end{pmatrix} \in Sp(2,\mathbb{Z}).$$

Here we put

$$N' := N_1 \cdot N_2.$$

We easily see that

$$\begin{aligned} N_1(\tilde{R}) &= N_1, \\ N_2(\tilde{R}) &= N_0 \cdot N_2, \\ N_0(\tilde{R}) &= 1. \end{aligned}$$

For the Eisenstein series this means

$$E_k^2(Z,\chi,R,s) \mid_k \tilde{W}_{N_0} = N_0^{-k-2s} E_k^2(Z,\psi,\tilde{R},s)$$

with

$$\psi(\gamma) = \chi(\tilde{W}_{N_0}\gamma\tilde{W}_{N_0}^{-1}) = (\chi_{N'}\overline{\chi_{N_0}})(\gamma).$$

Now we fix a decomposition

$$N = N_1 \cdot N_2 \qquad (N_0 = 1)$$

and a matrix

$$\mathcal{R} = \mathcal{R}_{N_1,N_2} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

such that

$$\mathcal{R} \equiv \begin{cases} \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix} \bmod N_2, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\downarrow} \bmod N_1. \end{cases}$$

Then $\mathcal{R}\Gamma_0^2(N)$ consists of all $\mathcal{M} \in Sp(2,\mathbb{Z})$ such that

$$\mathcal{M} \equiv \begin{cases} \begin{pmatrix} -C & -D \\ A & B \end{pmatrix} & \bmod N_2, \\ \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ c_3 & c_4 & d_3 & d_4 \\ c_1 & c_2 & d_1 & d_2 \\ a_3 & a_4 & b_3 & b_4 \end{pmatrix} & \bmod N_1 \end{cases}$$

with

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(N).$$

Here we used the decomposition

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

(and similarly for B, C, D).

The following observations are useful for an explict description of our Eisenstein series:

- $\det(D) \cdot (a_3, a_4) \equiv (-d_2, d_1) \bmod N$;
  by abuse of notation, we write this as
$$\det(D) \equiv \frac{(-d_2, d_1)}{(a_3, a_4)} \bmod N.$$
- $$(\mathcal{R}\Gamma_0^2(N)\mathcal{R}^{-1})_\infty = \left\{ \begin{pmatrix} A & B \\ 0_2 & D \end{pmatrix} \middle| \begin{array}{ll} a_3 \equiv d_2 \equiv b_4 \equiv 0 & \bmod N_1 \\ B \equiv 0 & \bmod N_2 \end{array} \right\}.$$
- In this setting, the compatibility condition means
$$\chi(R^{-1}\gamma R) = 1 \quad \text{for all} \quad \gamma \in \Gamma_0^2(N)_\infty \cap (R\Gamma_0^2(N)R^{-1}).$$
  For such $\gamma = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ we have
$$a_3 \equiv d_2 \equiv b_4 \equiv 0(N_1) \quad \text{and} \quad B \equiv 0(N_2)$$
  and
$$\begin{aligned} \chi(R^{-1}\gamma R) &= \prod_{p|N} \chi_p(R^{-1}\gamma R) \\ &= \chi_{N_0}(\det D)\chi_{N_2}(\det A)\chi_{N_1}(a_1 \cdot d_4). \end{aligned}$$
  The compatibility condition is then
$$\chi_{N_1}^2 = 1.$$

Using the information above, we can write down our Eisenstein series very explicitly as

$$\begin{aligned} &E_k^2(Z, \chi, \mathcal{R}, s) \\ &= \sum_{C,D} \chi_{N_2}(\det(C))\overline{\chi_{N_1}}(\frac{(-d_2, d_1)}{(c_3, c_4)}) \det(CZ + D)^{-k} \frac{\det(Y)^s}{|\det(CZ + D)|^{2s}}. \end{aligned}$$

We often write this Eisenstein series as

$$E_{N_1, N_2, k}^2(Z, \chi, s).$$

We can describe the pairs (C, D) in two ways:

- "group-theoretic description":
  $\begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over $(\mathcal{R}\Gamma_0^2(N)\mathcal{R}^{-1})_\infty \backslash \mathcal{R}\Gamma_0^2(N)$.

- "arithmetic description":
  (C, D) runs over "non-associated coprime symmetric pairs" satisfying the congruences

$$(*)\quad \begin{cases} \det(C) \text{ coprime to } N_2, \\ (c_1, c_2) \equiv 0 \bmod N_1, \\ (c_3, c_4) \text{ and } (d_1, d_2) \text{ both primitive mod } N_1. \end{cases}$$

  Here two such pairs (C, D) and (C', D') are called associated, if there exists $U \in GL(2,\mathbb{Z})^0(N_1)$ such that

$$(C', D') = (UC, UD).$$

  The congruence group $GL(2,\mathbb{Z})^0(N_1)$ consists of those $\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in GL(2,\mathbb{Z})$ satisfying

$$u_2 \equiv 0 \mod N_1.$$

4. Coset decompositions for $\Gamma_0^2(N)$

Our aim is to decompose the Eisenstein series $E^2_{(N_1,N_2),k}(Z, \chi, s)$ in a way appropriate for the computation of the Petersson product of an elliptic cusp form against the restricted Eisenstein series. This can be achieved in two different ways:

First, one can follow the strategy of Garrett (see [13] for level one) and study the double cosets

$$(\mathcal{R}\Gamma_0^2(N)\mathcal{R}^{-1})_\infty \backslash \mathcal{R}\Gamma_0^2(N)/\iota_{1,1}(\Gamma_0(N) \times \Gamma_0(N)).$$

Secondly, one can look at

$$(\mathcal{R}\Gamma_0^2(N)\mathcal{R}^{-1})_\infty \backslash \mathcal{R}\Gamma_0^2(N)/\Gamma_0(N)^{\downarrow}.$$

We use the latter method, which was described for level one in [2] and (for a quite different purpose) in [8] for the case $N = N_2$. We freely switch between the group-theoretic and arithmetic description of the cosets defining the Eisenstein series.

We first remark that for our purpose (evaluating a Petersson product) we only have to consider those (C, D) with

$$0 \neq \begin{pmatrix} c_1 \\ c_3 \end{pmatrix}.$$

We consider

$$\left\{ (C,D) \in \mathbb{Z}^{(2,4)} \;\middle|\; (C,D) \text{ coprime, symmetric, satisfying } (*),\ 0 \neq \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} \right\}.$$

On this set, there is an action of $GL(2,\mathbb{Z})^0(N_1)$ from the left and of $\Gamma_0(N)^{\downarrow}$ from the right. We recall from [2, Proposition 5] –with a minor reformulation– the following decomposition for $Sp(2,\mathbb{Z})_\infty\backslash Sp(2,\mathbb{Z})^*$, where the upper star indicates the condition that the first column of C is different form zero:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \omega_1 & \omega_3 & & \\ \omega_2 & \omega_4 & & \\ & & \omega_4 & -\omega_2 \\ & & -\omega_3 & \omega_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}$$

$$= \begin{pmatrix} a\omega_1 & a\omega_3\alpha - b\omega_2\gamma & b\omega_4 & a\omega_3\beta - b\omega_2\delta \\ \omega_2 & \omega_4\alpha & 0 & \omega_4\beta \\ c\omega_1 & c\omega_3\alpha - d\omega_2\gamma & d\omega_4 & c\omega_3\beta - d\omega_2\delta \\ 0 & \omega_1\gamma & -\omega_3 & \omega_1\delta \end{pmatrix}$$

where

- $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over $SL(2,\mathbb{Z})_\infty\backslash SL(2,\mathbb{Z})$ with the extra condition "$c \neq 0$",
- $R := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ runs over $SL(2,\mathbb{Z})_\infty\backslash SL(2,\mathbb{Z})$,
- $W = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}$ runs over $GL(2,\mathbb{Z})/GL(2,\mathbb{Z}_\infty)$ with $\omega_1 \neq 0$.

We have to adopt this decomposition to the level N: For this purpose we remark

- $SL(2,\mathbb{Z})_\infty\backslash SL(2,\mathbb{Z}) = \dot{\bigcup\limits_{l|N}} (R(l)\Gamma_0(N)R(l)^{-1})_\infty\backslash R(l)\Gamma_0(N)$,

  where $R(l)$ is some fixed element in $SL(2,\mathbb{Z})$ satisfying

$$R(l) \equiv \begin{cases} 1_2 & \bmod p \ \ (p \mid l), \\ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} & \bmod p \ (p \mid \frac{N}{l}). \end{cases}$$

- As a set of representatives for

$$GL(2,\mathbb{Z})^0(N_1)\backslash GL(2,\mathbb{Z})$$

we can choose

$$U = \begin{pmatrix} r_1 & r_2 \\ * & * \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

with

$$r_1 \cdot r_2 = N_1, \qquad t \bmod r_1.$$

For a fixed $U \in GL(2,\mathbb{Z})^0(N_1)\backslash GL(2,\mathbb{Z})$ and a fixed $l \mid N$ and $R \in R(l)\Gamma_0(N)$ we check now, which

$$(C,D) = U \cdot \begin{pmatrix} c\omega_1 & c\omega_3\alpha - d\omega_2\gamma & d\omega_4 & c\omega_3\beta - d\omega_2\delta \\ 0 & \omega_1\gamma & -\omega_3 & \omega_1\delta \end{pmatrix}$$

do actually satisfy the conditions $(*)$; we check the congruence conditions $(*)$ separately for all $p \mid N$ in several steps:

- The case $p \mid N_2$ (see also [8])
  The condition $\det(C)$ coprime to N_2 implies

  $$\boxed{c, \omega_1, \gamma \quad \text{coprime to} \quad p}$$

  in particular, p is coprime to l, in other words, the condition $l \mid N_1$ always holds.
- The case $p \mid N_1$:
  We first observe that C has rank 1 over $\mathbb{F}_p$, therefore

  $$p \mid c\omega_1\gamma.$$

  - The subcase $p \mid N_1$, $p \mid l$ (i.e. $p \mid \gamma$).
    Here c has to be coprime to p (because of the $\mathbb{F}_p$-rank of C) and

    $$C \equiv \begin{pmatrix} u_1c\omega_1 & u_1c\omega_3 \\ u_3c\omega_1 & u_3c\omega_3 \end{pmatrix} \mod p$$

    and from this

    $$p \mid u_1 = r_1 + tr_2$$

    The only possibility here is

    $$p \mid r_1, \qquad t = 0.$$

    Then the conditions $(*)$ are satisfied for (C,D) at least locally at p. Therefore the conditions are in this case

    $$\boxed{p \nmid c, \quad U \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod p}$$

- The subcase $p \mid N_1$, l coprime to p.
  This is the most complicated case. We may assume that $p \mid \alpha$. We also have the condition $p \mid c\omega_1$.
  - * We assume that $p \mid r_1$:
    Then
    $$U \equiv \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \mod p$$
    and
    $$C \equiv \begin{pmatrix} 0 & (-td\omega_2 + \omega_1)\gamma \\ 0 & -d\omega_2\gamma \end{pmatrix} \mod p.$$
    If ω_1 is coprime to p, then we may assume that $p \mid \omega_2$, which means that the second row of C is divisible by p, hence there is no solution at all !
    If $p \mid \omega_1$, then there is one solution mod p, namely
    $$p \mid t, \quad p \mid \omega_1, \qquad d \text{ coprime to } p$$
    This gives the conditions
    $$\boxed{p \mid \omega_1\,, \quad p \nmid d, \quad U \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod p}$$
  - * We assume $p \mid r_2$
    This means $U \equiv 1_2 \bmod p$ and therefore
    $$C \equiv \begin{pmatrix} c\omega_1 & -d\omega_2\gamma \\ o & \omega_1\gamma \end{pmatrix}$$
    and hence the conditions (taking ω_2 divisible by p)
    $$\boxed{p \mid c\,, \quad p \nmid \omega_1, \quad U \equiv 1_2(p)}$$

This finishes our "local" investigation of the congruence conditions $(*)$.

Summarizing the information above, we see that for a given decomposition $N = N_1 \cdot N_2$ and a given divisor $l \mid N_1$ we have to collect those

$$\begin{aligned} (\omega_1, \omega_3) &= 1 \qquad (\omega_1 \neq 0) \\ (c, d) &= 1 \qquad (c \neq 0) \end{aligned}$$

which satisfy the additional conditions $\sharp = \sharp(l)$:

$$\sharp(l) \quad \begin{cases} c, \omega_1 & \text{both coprime to } N_2 \\ c & \text{coprime to } l \\ d & \text{coprime to} \frac{N_1}{l} \\ c \cdot \omega_1 & \text{divisible by } \frac{N_1}{l} \end{cases}$$

5. Unfolding I

Now we decompose our Eisenstein series (or rather its "essential part")

$$'E^2_{(N_1,N_2),k}(Z,\chi,s)^* = \sum_{l|N_1} E^2_{(N_1,N_2),k}(Z,\chi,s)^*_l$$

into the parts belonging to a fixed l; we recall that we use the upper star to pick out those (C, D), for which the first column of C is different from zero. By an elementary calculation (or just quoting from similar calculations done e.g. in [2, 8]) we see that

$$E^2_{(N_1,N_2),k}(\iota(z,w),\chi,s)^*_l = \sum_{g\in} j(g,w)^{-k}\ \overline{\chi}(R(l)^{-1}g)\mathcal{H}_{N_1,N_2,l}(z, g < w >, \chi, s$$

with

$$\mathcal{H}_{N_1,N_2,l}(z,w,\chi,s) = \sum_{\omega_1,\omega_3,c,d} \overline{\chi}(...)\frac{y^s v^s}{(c(z\omega_1^2 + w\omega_3^2) + d)^k \mid c(z\omega_1^2 + w\omega_3^2) + d\mid^{2s}},$$

where the ω_i and c, d have to satisfy the conditions $\sharp(l)$, and $(...)$ is some expression depending (possibly) on the ω_i and c, d. By the standard procedure ("unfolding"), we can now compute the Petersson product of these functions $\mathcal{H}$ against a cusp form $f \in S_k(N, \chi)$:

$$I_{N_1,N_2,l}(s)(f) := \int_{\Gamma_0(N)\backslash\mathbb{H}} f(w)\overline{E^2_{N_1,N_2,k}(\iota(-\bar{z},w),\chi,\bar{s})_l}d^*w =$$

$$\int_{R(l)\Gamma_0(N)R(l)^{-1}\backslash\mathbb{H}} (f \mid_k R(l)^{-1})(w)\overline{\sum_{g\in}\bar{\chi}(g)j(g,w)^{-k}\mathcal{H}_{N_1,N_2,l}(z, g < w >)}d^*w,$$

where g runs now over $(R(l)\Gamma_0(N)R(l)^{-1})_\infty\backslash(R(l)\Gamma_0(N)R(l)^{-1}$. We recall that the width of the cusp corresponding to $R(l)$ is $\frac{N}{l}$, therefore, by unfolding the summation over g, the integral above equals

$$\int f\mid_k R(l)^{-1}(w)\overline{\mathcal{H}_{N_1,N_2,l}(-\bar{z},w,\chi,\bar{s})}d^*w,$$

where the integration is over

$$(R(l)\Gamma_0(N)R(l)^{-1})_\infty\backslash\mathbb{H} = \left\{ w = u+iv \in \mathbb{H} \;\middle|\; v>0, u \bmod \frac{N}{l} \right\}.$$

Writing

$$d = d_0 + t\cdot c\omega_3^2 \cdot \frac{N}{l}$$

with $t \in \mathbb{Z}$, and observing that

$$c(z\omega_1^2 + (w + t\frac{N}{l})\omega_3^2 + d_0 = c(z\omega_1^2 + w\omega_3^2) + (d_0 + tc\omega_3^2\frac{N}{l}),$$

we may now write our integral as

$$\sum_{c,d_0,\omega_1,\omega_2} \int_{\mathbb{H}} f\mid_k R(l)^{-1}(w)\overline{\frac{(vy)^{\bar{s}}}{(c(-\bar{z}\omega_1^2 + w\omega_3^2)+d_0)^k \mid c(-\bar{z}\omega_1^2 + w\omega_3^2)+d_0\mid^{2\bar{s}}}}d^*w,$$

where d_0 is now subject to the condition (in addition to $\sharp(l)$)

$$d_0 \quad \bmod \quad c\omega_3^2\frac{N}{l}.$$

The reproducing formula for holomorphic functions (see e.g. [4, 15]) gives

$$\int_{\mathbb{H}} f(w)\overline{\frac{1}{(-\bar{z}+w)^k}} \mid -\bar{z}+w\mid^{2\bar{s}} d^*w = W_k(s)f(z)$$

with

$$W_k(s) = (-1)^{\frac{k}{2}-k}\pi 2^{-2k-2s+4}\frac{1}{k+s-1},$$

which allows us finally to write our integral as

$$I_{N_1,N_2,l}(s) = W_k(s)\sum_{c,d_0,\omega_1,\omega_3} \chi(\ldots)(c\omega_1\omega_3)^{-k-2s} f\mid_k R(l)^{-1}\mid_k \begin{pmatrix} \frac{\omega_1}{\omega_3} & \frac{d_0}{c\omega_1\omega_3} \\ 0 & \frac{\omega_3}{\omega_1}\end{pmatrix},$$

where c, d_0, ω_i are subject to the conditons $\sharp(l)$. To analyse this via Hecke operators, we express $R(l)^{-1}$ in terms of the Atkin-Lehner involution $W_{\frac{N}{l}}$: We choose $R(l)$ in the special form

$$R(l) = \begin{pmatrix} \frac{N}{l} & -b \\ -l & a\frac{N}{l}\end{pmatrix},$$

therefore

$$R(l)^{-1} = \begin{pmatrix} a\frac{N}{l} & b \\ l & \frac{N}{l}\end{pmatrix} = \underbrace{\begin{pmatrix} a(\frac{N}{l})^2 & b \\ N & \frac{N}{l}\end{pmatrix}}_{W_{\frac{N}{l}}} \cdot \begin{pmatrix} \frac{l}{N} & 0 \\ 0 & 1\end{pmatrix}$$

We recall that elements of $W_{\frac{N}{l}}$ normalize $\Gamma_0(N)$, therefore we study in the next section the double cosets

$$\Gamma_0(N)\begin{pmatrix} \frac{l}{N}\frac{\omega_1}{\omega_3} & \frac{l}{N}\frac{d_0}{c\omega_1\omega_3} \\ 0 & \frac{\omega_3}{\omega_1} \end{pmatrix}\Gamma_0(N).$$

6. Double cosets for $\Gamma_0(N)$

From the theory of elementary divisors (or see [5] for a more general statement for the Siegel modular group) it follows that

$$SL(2,\mathbb{Z})\begin{pmatrix} \frac{l}{N}\frac{\omega_1}{\omega_3} & \frac{l}{N}\frac{d_0}{c\omega_1\omega_3} \\ 0 & \frac{\omega_3}{\omega_1} \end{pmatrix}SL(2,\mathbb{Z}) = SL(2,\mathbb{Z})\begin{pmatrix} \frac{l}{N}D^{-1} & 0 \\ 0 & D \end{pmatrix}SL(2,\mathbb{Z})$$

with

$$D = c\omega_1\omega_3.$$

We want to determine the exact double cosets over $\Gamma_0(N)$. For this purpose, we need some properties of the Hecke algebras associated to the pair $(\Gamma_0(N), GL^+(2,\mathbb{Q}))$. We found that the exposition by Krieg [16] fits very well to our framework (one could also use more abstract expositions from the point of view of Iwahori Hecke algebras, for $GL(2)$ see e.g. [10]). Our main tool is [16, Theorem 2], which allows to write all primitive double cosets $\Gamma_0(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_0(N)$ in a canonical form (here a double coset is called primitive, if the entries are integral with gcd equal to 1):

$$\begin{aligned}
&\Gamma_0(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_0(N) \\
&= \Gamma_0(N)A_t\Gamma_0(N)\cdot\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N)\cdot\Gamma_0(N)W_n\Gamma_0(N)
\end{aligned}$$

with $A_t \in SL(2,\mathbb{Z})$ such that

$$A_t \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \bmod\ t^2 \\ \quad 1_2 & \bmod\ (\frac{N}{t})^2 \end{cases}$$

and

$$W_n := A_n \cdot \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

Krieg gives a quite explicit recipe how to get the data t, n, m from the entries a, b, c, d; we will use this recipe tacitly, adopting it to our case of rational matrices; for details we refer to loc.cit. Also, the Hecke algebra can be decomposed into its "p-components", therefore it is sufficient to determine the double cosets in question "locally", i.e. we fix a prime and consider cases, where ω_1, ω_3, c are powers of p, subject to the condition $\sharp$. Our aim is to express all double cosets by those of the form

$$\Gamma_0(N)\begin{pmatrix} p^{-\alpha} & 0 \\ 0 & p^{\beta}\end{pmatrix}\Gamma_0(N) \qquad (\alpha, \beta \geq 0).$$

Viewed as operators on modular forms, they represent (up to normalization) the well-known operators $U(p^{\alpha+\beta})$, if $p \mid N$; note however that for p coprime to N the corresponding operators are not quite proportional to the $T(p^{\alpha+\beta})$.

- "The good primes", i.e. p coprime to N:
  Here it is sufficient to state a slightly more general property: If ω_1, ω_3 and c are all coprime to N, then (for all d coprime to c)

  $$\Gamma_0(N)\begin{pmatrix} \frac{\omega_1}{\omega_3} & \frac{d}{c\omega_1\omega_3} \\ 0 & \frac{\omega_3}{\omega_1}\end{pmatrix}\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} D^{-1} & 0 \\ 0 & D\end{pmatrix}\Gamma_0(N)$$

  with $D = c\omega_1\omega_3$ (just like in the case of level one).
- $p \mid N_2$:
  We have to consider $c = \omega_1 = l = 1$ and $\omega_3 = p^{\nu}$ with d_0 modulo $p^{2\nu+1}$:

  $$\Gamma_0(N)\begin{pmatrix} p^{-1-\nu} & \frac{d_0}{p^{1+\nu}} \\ 0 & p^{\nu}\end{pmatrix}\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} p^{-1-\nu} & 0 \\ 0 & p^{\nu}\end{pmatrix}\Gamma_0(N).$$

  Moreover, we remark here that the d_0 mod $p^{2\nu+1}$ exhaust the full double coset.
- $p \mid N_1, p \mid l$:
  Here, due to the condition $\sharp$ we have $c = 1$; there are two cases:
  - First case: $\omega_1 = 1, \omega_3 = p^{\nu}$ with $\nu \geq 0$:
    Here d_0 runs modulo $p^{2\nu}$; as before we get

    $$\Gamma_0(p)\begin{pmatrix} p^{-\nu} & \frac{d_0}{p^{\nu}} \\ 0 & p^{\nu}\end{pmatrix}\Gamma_0(p) = \Gamma_0(p)\begin{pmatrix} p^{-\nu} & 0 \\ 0 & p^{\nu}\end{pmatrix}\Gamma_0(p).$$

    Here the d_0 also exhaust the full double coset.

– Second case: $\omega_3 = 1, \quad \omega_1 = p^\nu$, $\nu \geq 1$:
Here $d_0 = 0$, the double coset is then

$$\Gamma_0(N) \begin{pmatrix} p^\nu & 0 \\ 0 & p^{-\nu} \end{pmatrix} \Gamma_0(N)$$
$$= \Gamma_0(N) A_p \Gamma_0(N) \Gamma_0(N) \begin{pmatrix} p^{-\nu} & 0 \\ 0 & p^{\nu-1} \end{pmatrix} \Gamma_0(N) \Gamma_0 W_p \Gamma_0(N).$$

- $p \mid N_1$ and p coprime to l:
We have two cases
  – $p \mid \omega_1$:
  This means $\omega_1 = p^\nu$ with $\nu \geq 1$, $\omega_3 = 1$, $c = p^\kappa$, $\kappa \geq 0$, d_0 is coprime to p and runs modulo $p^{\kappa+1}$.
  Then

$$\Gamma_0(N) \begin{pmatrix} p^{\nu-1} & \frac{d_0}{p^{\nu+1+\kappa}} \\ 0 & p^{-\nu} \end{pmatrix} \Gamma_0(N)$$
$$= \Gamma_0(N) \begin{pmatrix} p^{-\nu-1-\kappa} & 0 \\ 0 & p^{\nu+\kappa-1} \end{pmatrix} \Gamma_0(N) \cdot \Gamma_0(N) W_p \Gamma_0(N).$$

  – $\omega_1 = 1$, $\omega_3 = p^\nu$, $\nu \geq 0$, $c = p^\kappa$, $\kappa \geq 1$,
  d_0 is coprime to p runs modulo $p^{2\nu+\kappa+1}$.
  Then

$$\Gamma_0(N) \begin{pmatrix} p^{-\nu-1} & \frac{d_0}{p^{\nu+1+\kappa}} \\ 0 & p^{\nu} \end{pmatrix} \Gamma_0(N)$$
$$= \Gamma_0(N) \begin{pmatrix} p^{-\nu-\kappa-1} & 0 \\ 0 & p^{\nu+\kappa-1} \end{pmatrix} \Gamma_0(N) \cdot \Gamma_0(N) W_p \Gamma_0(N).$$

7. Unfolding II

We assume throughout that χ is a quadratic character mod N and that $f \in S_k(N, \chi)$ is an eigenform of all the Hecke operators $T(n)$ with n coprime to N. Both these assumptions are not essential, but they simplify the formulation considerably.

We will give (under these assumptions) a formula for the Petersson product of f against the restriction of any of the Eisenstein series introduced earlier. The result will involve an L-function attached to f (in the context of $GL(2)$ it is usually called "symmetric square" L-function)

$$L_N(f, s) := \prod_{p \nmid N} \frac{1}{(1 - \alpha_p^2 p^{-s})(1 - \beta_p^2 p^{-s})(1 - \chi(p) p^{-s+k-1})}.$$

It is important to remark that this L-function is the same for f and $f \mid W_{N'}$ for any divisor N' of N. It is also convenient to define (for $M \mid N$ and $\Re(s)$ large enough) the following endomorphism of $S_k(N,\chi)$:

$$Sym^M(s) : f \longmapsto f \mid Sym^M(s) := \sum_{d \mid M^\infty} f \mid \Gamma_0(N) \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \Gamma_0(N) d^{-s}.$$

Here the action of the double coset $\Gamma_0(N) \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \Gamma_0(N) = \cup \Gamma_0(N)\gamma_i$ is normalized to be

$$f \mid \Gamma_0(N) \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \Gamma_0(N) = \sum_\gamma \chi(\gamma_i) f \mid_k \gamma_i,$$

where for $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ as above we put

$$\chi(\gamma_i) := \chi(d \cdot a_i).$$

An easy calculation shows that (for $d \mid N^\infty$)

$$f \mid \Gamma_0(N) \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \Gamma_0(N) = d^{-k+2} f \mid U(d^2)$$

with the standard Hecke operator $U(d^2)$.

If $f = \sum a(n) \exp(nz)$ is a normalized newform (eigenform of all Hecke operators) of level N, then

$$f \mid Sym^M(s) = L^M(f, s+k-2) \cdot f$$

with

$$L^M(f,s) = \prod_{p \mid M} \frac{1}{1 - a(p)^2 p^{-s}}.$$

For such newforms we also define the complete L-function by

$$L(f,s) := L_N(s) \cdot L^N(f,s).$$

The considerations of the previous sections allow us (this is in some sense the main result of this paper!) to give an explicit expression for the integral of f against *all* of the (restricted) Eisenstein series introduced earlier. We put (using a matrix $\mathcal{R} \in SL(2,\mathbb{Z})$ which fits to the decomposition $N = N_0 \cdot N_1 \cdot N_2$)

$$I_{N_0,N_1,N_2)}(s)(f) := \int_{\Gamma_0(N)\backslash\mathbb{H}} f(w) \overline{E_k^2(\iota(-\bar{z},w),\mathcal{R},\chi,\bar{s})} d^*w.$$

From the previous calculations we first obtain a result for $N_0 = 1$; one has to collect the individual contributions in the right way to get the action of the full double cosets described in section 6; we omit the details here. The result is

Proposition 7.1. *Under the assumptions above,*

$$I_{1,N1,N_2}(s)(f) = W_k(s)L_N(f, 2s + 2k - 2) \times \Lambda_{N_1,N_2}(s)(f)$$

with

$$\Lambda_{N_1,N_2}(s)(f) = N_2^{1-\frac{k}{2}} \cdot f\Big|\Big(W_{N_2}U(N_2)Sym^{N_2}(2s+k)$$
$$\times \prod_{p|N_1}(Sym^p(2s+k) + p^{2-2k-2s}W_pU(p^2)Sym^p(2s+k)W_p)\Big).$$

The considerations about the action of the Atkin–Lehner-involutions on Eisenstein series allows us to remove the condition $N_0 = 1$ from the proposition above:

Theorem 7.1. *Let f be as before, then*

$$I_{N_0,N_1,N_2}(f) = W_k(s)L_N(f, 2s + 2k - 2) \times \Lambda_{N_0,N_1,N_2}(s)(f)$$

with

$$\Lambda_{N_0,N_1,N_2}(s)(f) = N_0^{1-2s-\frac{3k}{2}} N_2^{1-\frac{k}{2}}$$
$$\times f\Big|\Big(U(N_0)Sym^{N_0}(2s+k)W_{N_0}W_{N_2}U(N_2)Sym^{N_2}(2s+k)$$
$$\times \prod_{p|N_1}(Sym^p(2s+k) + p^{2-2k-2s}W_pU(p^2)Sym^p(2s+k)W_p\Big).$$

Remark 7.1. If the character is not quadratic, a more complicated formulation occurs, in particular, we cannot put the symmetric square L-function in front; moreover, the condition $\chi_{N_1}^2 = 1$ must be taken into account.

8. The basis problem for squarefree level

We recall that an even integral quadratic form S of rank m is said to be of level N, if $N \cdot S^{-1}$ is again even integral; if N is the smallest such positive integer, we call it the *exact* level of S; if m is even, we define the discriminant of S by $(-1)^{\frac{m}{2}} \det(S)$.

We start now from any genus $\mathfrak{S}$ of positive definite even integral quadratic forms of even rank $m = 2k$, squarefree level N and discriminant $D \equiv 1 \bmod 4$; then the relation

$$N \mid D \mid N^m$$

holds, if N is the exact level; otherwise only $D \mid N^m$ holds in general. For $S \in \mathfrak{S}$ we consider the degree n theta series

$$\theta^n(S,Z) := \sum_{R \in \mathbb{Z}^{(m,n)}} \exp\left(\frac{1}{2}\mathrm{tr}(R^t SRZ)\right);$$

it is well known that this theta series belongs to $M_k^n(N,\chi)$ with

$$\chi = \chi_S = \left(\frac{(-1)^k \det(S)}{\cdot}\right).$$

The subspace generated by all the $\theta^n(S)$ with $S \in \mathfrak{S}$ will be denoted by $\Theta^n_{\mathfrak{S}}$. The genus version of the *basis problem* asks whether all cusp forms can be obtained by such theta series, i.e.:

Does the inclusion $S_k^n(N,\chi) \subset \Theta^n_{\mathfrak{S}}$ hold true ?

An important tool for the investigation is the genus theta series, defined by

$$\theta^n(\mathfrak{S},Z) := \frac{1}{m(\mathfrak{S})}\sum_i \frac{1}{\epsilon(S_i)}\theta^n(S_i,Z),$$

where the S_i run over representatives of the $GL(m,\mathbb{Z})$-classes in $\mathfrak{S}$, the $\epsilon(S_i)$ denote the order of the group of integral units of S_i and $m(\mathfrak{S}) = \sum_i \frac{1}{\epsilon(S_i)}$ is the mass of $\mathfrak{S}$.
By Siegel's theorem, this genus theta series is a linear combination of our Eisenstein series, for degree 2 we can therefore write it as

$$\theta^2(\mathfrak{S},Z) = \sum_{N_0N_1N_2=N} a_{\mathfrak{S}}(N_0,N_1,N_2)E^2_{N_0,N_1,N_2,k}(Z,\chi).$$

This is true at least for $k > 3$; the coefficients can be computed by comparing the constant terms in the Fourier expansions in the cusps, to which the Eisenstein series are attached. For such computations we may refer to [14, 9, 6]. The coefficients are in any case only depending on the genus.

To study the basis problem, we want then, for a given cusp form $f \in S_k(N, \chi)$, to consider the map

$$f \longmapsto \Lambda_{\mathfrak{S}}(f) := \frac{1}{m(\mathfrak{S})} \sum_i < f, \theta(S_i) > \theta(S_i)$$
$$= \sum_{N_0 N_1 N_2 = N} a_{\mathfrak{S}}(N_0, N_1, N_2) \Lambda_{N_0 N_1, N_2}(0)(f).$$

The theory of newforms for trivial and nontrivial character differs somewhat and we have to treat these cases separately.

8.1. *Case I: Haupttypus*

Note that m is divisible by 4 in this case. Let

$$D = \prod D_{p|N} \qquad \text{with} \qquad D_p = p^{2t_p}$$

be a given discriminant where

$$0 \leq t_p \leq \frac{m}{2}.$$

Furthermore let

$$f = \sum_{n=1}^{\infty} a(n) \exp(nz) \in S_k(N)$$

be a normalized newform (eigenform of all Hecke operators). We mention that for all $p \mid N$ we have

$$f \mid U(p) = -p^{\frac{k}{2}-1} f \mid W_p.$$

Then it is easily seen that

$$\Lambda_{N_0, N_1, N_2}(0)(f) = W_k(0) L(f, 2k-2) \mu(N_0 N_2) N_0^{-k} \prod_{p|N_1} (1 + p^{-k}).$$

Furthermore, in this case we can write

$$a_{\mathfrak{S}}(N_0, N_1, N_2) = \pm D_{N_2}^{-1} D_{N_1}^{-\frac{1}{2}}.$$

We do not need to know the exact nature of this sign $\kappa = \kappa(N_0, N_1, N_2, \mathfrak{S})$ (it comes from the Hasse symbol), but it is usefull to know that it is essentially something coming from local data:

$$\kappa = \prod_{p|N} \kappa_p((N_0)_p, (N_1)_p, (N_2)_p, \mathfrak{S}).$$

Therefore we obtain

$$\Lambda_{\mathfrak{S}}(f) = W_k(0) L(f, 2k-2) \times A_N$$

with the crucial "numerator" A_N

$$A_N = \sum_{N_0 N_1 N_2 = N} \kappa(N_0, N_1, N_2, \mathfrak{S}) D_{N_2}^{-1} D_{N_1}^{-\frac{1}{2}} \mu(N_0 N_2) N_0^{-k} \prod_{p \mid N_1} (1 - p^{-k}).$$

This is clearly multiplicative with

$$A_N = \prod_{p \mid N} A_p$$

and

$$A_p = -\kappa(p, 1, 1, \mathfrak{S}) p^{-k} - \kappa(1, 1, p, \mathfrak{S}) p^{-2t_p} + \kappa(1, p, 1, \mathfrak{S}) p^{-t_p} (1 + p^{-k}).$$

These A_p are then different from zero unless

$$D_p = 1 \quad \text{or} \quad D_p = p^{2m}.$$

In both these cases the theta series in question are "old forms", because the quadratic foms involved are of level $\frac{N}{p}$ or scaled from level $\frac{N}{p}$ by a factor p. We obtain

Theorem 8.1. *Assume that N is squarefree and $\mathfrak{S}$ is any genus of even integral positive definite quadratic forms of rank $m = 2k$ (divisible by 4 and different from 4) of level N and with discriminant D, D being a perfect square such that for all $p \mid N$ we have $p^2 \mid D$ and $p^m \nmid D$. Then the full space $S_k(N)^{new}$ of newforms consists of linear combinations of theta series attached to quadratic forms belonging to $\mathfrak{S}$. For a Hecke eigen form f our calculation above even gives an explicit expression of f in terms of the theta series.*

In some sense this is exactly the same kind of result as obtained earlier for level one (without trouble concerning the ramified primes); the level one case was done even for Siegel modular forms [2]; we have not much hope to treat the higher degree case along the lines of the present paper; if we allow all genera of level N, a simpler procedure is possible, see [7].

We remark here that the theorem above goes (in the case of squarefree levels) beyond the theorems stated by Waldspurger, because he considered for each N only one special choice of genus (convenient for his calculation). The case $m = 4$ had to be excluded here because of problems of convergence of our Eisenstein series and L-series.

8.2. *Case II: primitive nebentypus*

Here we stick to the case of prime level $N = p$ with $p \neq 2$; the most general case (including "mixed" cases of nontrivial imprimitive characters) will be treated elsewhere.

We recall the properties of newforms needed here: Let χ be the nontrivial quadratic character mod p defined by the Legendre symbol and k a positive integer satisfying $(-1)^k\chi(-1) = 1$ (later on we also need the condition $k > 2$). A normalized newform

$$f(z) = \sum_{n=1}^{\infty} a(n)\exp(nz) \in S_k(p, \chi)$$

always comes up together with its "companion"

$$f^\rho(z) = \sum_{n=1}^{\infty} \overline{a(n)}\exp(nz) \in S_k(p, \chi).$$

For all such eigenforms f we assume throughout this section that f and f^ρ generate a two-dimensional subspace in $S_k(p, \chi)$; if this assumption does not hold, some (minor) modifications of our statements are necessary. We have the property

$$f \mid W_p = p^{-\frac{k}{2}} W(\chi)\chi(-1) \cdot f^\rho \mid U(p)$$

with the usual Gauß sum $W(\chi) = \sum_{r=1}^{p-1} \chi(r)\exp(\frac{r}{p})$; we write its explicit value in the form

$$\chi(-1)W(\chi) = \varepsilon_p \cdot \sqrt{p} \quad \text{with} \quad \varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1\ (4), \\ -i & \text{if } p \equiv 3\ (4). \end{cases}$$

We use these properties in the form

$$\begin{aligned} f \mid U(p)W_p &= \varepsilon_p p^{\frac{k-1}{2}} f^\rho, \\ f \mid W_p U(p) &= \varepsilon_p p^{\frac{1-k}{2}} \overline{a(p)^2} f^\rho, \\ \mid a(p) \mid^2 &= p^{k-1} \end{aligned}$$

to compute the explicit form of $\Lambda_{N_0,N_1,N_2}(0)(f)$ in this case:

$$\begin{aligned} \Lambda_{p,1,1}(0)(f) &= W_k(0)L_p(f, 2k-2)L^p(f, 2k-2)p^{-k+\frac{1}{2}}\varepsilon_p f^\rho, \\ \Lambda_{1,1,p}(0)(f) &= W_k(0)L_p(f, 2k-2)L^p(f^\rho, 2k-2)p^{\frac{3}{2}-k}\varepsilon_p \overline{a(p)^2} f^\rho, \\ \Lambda_{1,p,1}(0)(f) &= W_k(0)L_p(f, 2k-2) \times \\ &\quad \left\{ L^p(f, 2k-2)f + \chi(-1)L^p(f^\rho, 2k-2)p^{2-2k}\overline{a(p)^2} f \right\}. \end{aligned}$$

Now we consider a genus $\mathfrak{S}$ of even integral quadratic forms, rank $m = 2k$, level p and discriminant $D = (-1)p^t \equiv 1(4)$ with $1 \leq t \leq m-1$. The data $a_{\mathfrak{S}}(...)$ are in this case

$$\begin{aligned} a_{\mathfrak{S}}(p,1,1) &= 1, \\ a_{\mathfrak{S}}(1,p,1) &= i^k p^{-\frac{t}{2}}, \\ a_{\mathfrak{S}}(1,1,p) &= (-1)^k p^{-t}. \end{aligned}$$

To formulate our formulas smoothly, we introduce a root of unity ξ by

$$\xi := \frac{L^p(f^\rho, 2k-2)}{L^p(f, 2k-2)}.$$

Then

$$\begin{aligned} \Lambda_{\mathfrak{S}}(f) = {} & W_k(0)L_p(f,2k-2)L^p(f,2k-2) \\ & \times \left\{ i^k(p^{-\frac{t}{2}} + (-1)^k p^{2-2k-\frac{t}{2}} \xi \overline{a(p)^2}) \cdot f \right. \\ & \left. + \left(p^{-k+\frac{1}{2}} \varepsilon_p + (-1)^k p^{\frac{3}{2}-k-t} \xi \varepsilon_p \overline{a(p)^2} \right) \cdot f^\rho \right\}. \end{aligned}$$

Evidently, the two-dimensional space generated by $\{f, f^\rho\}$ is invariant under the map $\Lambda_{\mathfrak{S}}$. Up to a non-zero constant a coefficient matrix for this map is given by

$$\begin{pmatrix} i^k(p^{-\frac{t}{2}} + (-1)^k p^{2-2k-\frac{t}{2}} \xi\, \overline{a(p)^2}) & p^{-k+\frac{1}{2}} \varepsilon_p + (-1)^k p^{\frac{3}{2}-k-t} \bar{\xi} \varepsilon_p a(p)^2 \\ p^{-k+\frac{1}{2}} \varepsilon_p + (-1)^k p^{\frac{3}{2}-k-t} \xi \varepsilon_p \overline{a(p)^2} & i^k(p^{-\frac{t}{2}} + (-1)^k p^{2-2k-\frac{t}{2}} \bar{\xi} a(p)^2) \end{pmatrix}$$

This matrix is of the form

$$\begin{pmatrix} i^k\alpha & \varepsilon_p\bar{\beta} \\ \varepsilon_p\beta & i^k\bar{\alpha} \end{pmatrix}$$

Observing that $i^{2k} = \varepsilon_p^2$, we see that its determinant is zero iff $| \alpha | = | \beta |$. The equation $| \alpha |^2 = | \beta |^2$ comes down to a quadratic equation for $X := p^t$, namely

$$X^2 - (p^{2k-1}+1)X + p^{2k} = (X-p)(X-p^{2k-1}) = 0.$$

From this observation we get a main result of this paper

Theorem 8.2. *Let p be a prime number, $m = 2k > 6$ even with $(-1)^k p \equiv 1\ (4)$; furthermore let $\mathfrak{S}$ be a genus of even integral positive quadratic forms of rank m, (exact) level p and discriminant $(-1)^k p^t$ with t odd and $1 < t < m-1$. Then all cusp forms in $S_k(p, \left(\frac{\cdot}{p}\right))$ are linear combinations of theta series $\theta(S)$ with $S \in \mathfrak{S}$.*

Waldspurger only treats (for arbitrary squarefree levels $N \equiv 1$ (4), not just primes) the cases $t = 1$ and $t = m - 1$ excluded above. We briefly describe, how these cases can be understood from the point of view of the present paper (including positive and negative discriminants). We may (just as in [23]) concentrate on the case $t = m - 1$, the other case $t = 1$ is just adjoint (from the point of view of modular forms one has just to apply the Fricke involution W_p). In this case, the linear combination of f and f^ρ describing $\Lambda_{\mathfrak{S}}(f)$ takes a very special form, namely (with a factor $C(f)$ different from zero)

$$\Lambda_{\mathfrak{S}}(f) = C(f) \cdot (f + \eta f^\rho)$$

with

$$\eta := \frac{i^k}{\epsilon_p} = \begin{cases} (-1)^{\frac{k}{2}} = (-1)^{\frac{m}{4}} & \text{if } p \equiv 1 \ (4) \ (\text{k even}), \\ (-1)^{\frac{k-1}{2}} = (-1)^{\frac{m-2}{4}} & \text{if } p \equiv 3 \ (4) \ (\text{k odd}). \end{cases}$$

Therefore $\Lambda_{\mathfrak{S}}$ maps $S_k(p, \left(\frac{\cdot}{p}\right))$ into $S_k(p, \left(\frac{\cdot}{p}\right))^\eta$ with

$$S_k(p, \left(\frac{\cdot}{p}\right))^\eta := \left\{ f \in S_k(p, \left(\frac{\cdot}{p}\right)) \,\middle|\, f^\rho = \eta f \right\}.$$

Evidently the map is surjective and we get

Theorem 8.3. (Waldspurger) *Let $\mathfrak{S}$ be a genus of positive definite even quadratic forms of rank $m = 2k$, level p and discriminant $(-1)^{\frac{m}{2}} p^{m-1}$. Then (at least for $k > 3$) we have*

$$S_k(p, \left(\frac{\cdot}{p}\right))^\eta \subset \Theta_{\mathfrak{S}}.$$

To get the "mysterious" result of Waldspurger mentioned in the introduction, we study the genus $\mathfrak{S}$ together with its adjoint $\mathfrak{S}^*$. Then, for a given normalized newform f, we already know that $f + \eta f^\rho \in \Theta_{\mathfrak{S}}$ and therefore

$$\begin{aligned}(f + \eta f^\rho) \mid W_p &= p^{-\frac{k-1}{2}} \epsilon_p \left(f^\rho \mid U(p) + \eta f \mid U(p)\right) \\ &= p^{-\frac{k-1}{2}} \epsilon_p \left(\overline{a(p)} f^\rho + \eta a(p) f\right)\end{aligned}$$

is in $\Theta_{\mathfrak{S}^*}$. Clearly $f + \eta f^\rho$ and $(f + \eta f^\rho) \mid W_p$ generate the two-dimensional space spanned by f and f^ρ unless $a(p)$ is real. If $a(p)$ is real, then $a(p)^2 =$

$\overline{a(p)^2} = p^{k-1}$. In that case we easily see from the explict shape of $\Lambda_{\mathfrak{G}^*}$, that (if $a(p)$ is real)

$$\Lambda_{\mathfrak{G}^*}(f) \in S_k(p, \left(\frac{\cdot}{p}\right))^\eta.$$

Now we consider a (non-zero) linear combination g of f and f^ρ, which is orthogonal to $f + \eta f^\rho$. Then

$$(\Lambda_{\mathfrak{G}} + \Lambda_{\mathfrak{G}^*})(g) \perp g$$

and on the other hand, by expressiong this Petersson product in terms of theta series, we see that

$$g \perp (\Theta_{\mathfrak{G}} + \Theta_{\mathfrak{G}^*}).$$

We obtain in this way the following version of [23, Théorème 3].

Theorem 8.4. *Let $\mathfrak{G}$ be a genus of positive definite even quadratic forms of rank $m = 2k$, level p and discriminant $(-1)^{\frac{m}{2}} p^{m-1}$. Then (at least for $k > 3$) we have, for any normalized newform $f = \sum a(n) \exp(nz) \in S_k(p, \left(\frac{\cdot}{p}\right))$ the following statement:*

$$\mathbb{C}\{f, f^\rho\} \subset \Theta_{\mathfrak{G}} + \Theta_{\mathfrak{G}^*} \iff a(p) \notin \mathbb{R}$$

Moreover, if $a(p)$ is real, then

$$\mathbb{C}\{f, f^\rho\} \cap (\Theta_{\mathfrak{G}} + \Theta_{\mathfrak{G}^*}) = \mathbb{C}\{f + \eta f^\rho\}.$$

In particular, the basis problem has a positive answer here (in the sense that $S_k(p, \left(\frac{\cdot}{p}\right)) \subset \Theta_{\mathfrak{G}} + \Theta_{\mathfrak{G}^*}$) iff the operator $U(p)$ does not have real eigenvalues on $S_k(p, \left(\frac{\cdot}{p}\right))$.

Remark 8.1. As mentioned at the beginning of this subsection, all the statements are made under the assumption, that f and f^ρ generate a two-dimensional space. If one wants to have smooth statements without this assumption, one should restrict oneself to the subspace $S_k(p, \left(\frac{\cdot}{p}\right))_0$ generated by those eigenforms f, for which indeed f, f^ρ are linearly independent. In this way, one can avoid problems with CM forms (which occur for odd weights !). Our main statement (Theorem 8.2) however does not depend on such an assumption. In Theorems 8.3 and 8.4 one should use $S_k(p, \left(\frac{\cdot}{p}\right))_0$ instead of the full space of cusp forms. Waldspurger [23] did not need to worry about such a complication because he made statements only for $p \equiv 1$ (4), where such CM forms do not occur.

Remark 8.2. All the statements of this paper can be generalized to the case of theta series with harmonic polynomials by using differential operators on $\mathbb{H}_2$ with equivariance properties for

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \hookrightarrow Sp(2,\mathbb{R});$$

The application of these differential operators for the basis problem is almost formal; the reader should consult [3,11].

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DOUBLE ZETA VALUES AND MODULAR FORMS

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Dedicated to the memory of Tsuneo Arakawa

1. Introduction and main results

The *double zeta values*, which are defined for integers $r \geqslant 2$, $s \geqslant 1$, by

$$\zeta(r,s) = \sum_{m>n>0} \frac{1}{m^r n^s}, \tag{1}$$

are subject to numerous relations. Already Euler found that when the weight $k = r + s$ is odd the double zeta values can be reduced to products of usual zeta values. Furthermore, he gave the sum formula

$$\sum_{r=2}^{k-1} \zeta(r, k-r) = \zeta(k) \qquad (k > 2). \tag{2}$$

The aims of the present paper are:

- to give other interesting relations among double zeta values,

- to show that the structure of the $\mathbb{Q}$-vector space of all relations among double zeta values of weight k is connected in many different ways with the structure of the space of modular forms M_k of weight k on the full modular group $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, and
- to introduce and study both transcendental and combinatorial "double Eisenstein series" which explain the relation between double zeta values and modular forms and provide new realizations of the space of double zeta relations.

Double zeta values are a special case of *multiple zeta values*, defined by sums like (1) but with longer decreasing sequences of integers, which are known to satisfy a collection of relations called the *double shuffle relations* (cf., e.g., [3], [5], [12]). The specialization of these relations to the double zeta case is given by the following two sets of easily proved relations (see Section 2):

$$\begin{aligned} &\zeta(r,s)+\zeta(s,r)=\zeta(r)\,\zeta(s)-\zeta(k) \qquad (r+s=k;\ r,s\geqslant 2)\,,\\ &\sum_{r=2}^{k-1}\left[\binom{r-1}{j-1}+\binom{r-1}{k-j-1}\right]\zeta(r,k-r)\\ &\qquad\qquad\qquad =\zeta(j)\,\zeta(k-j) \qquad \left(2\leqslant j\leqslant \frac{k}{2}\right). \end{aligned} \tag{3}$$

We wish to study the relations which can be deduced from (3). Since we want to do this algebraically, it is useful to work, not with the double zeta values themselves, which for all we know may satisfy other relations than (3) (it is not even known that any $\zeta(r,s)/\pi^{r+s}$ is irrational), but with the *formal double zeta space* $\mathcal{D}_k$, generated by formal symbols $Z_{r,s}$, $P_{r,s}$ and Z_k subject to the relations (3), with $Z_{r,s}$, $P_{r,s}$ and Z_k taking the role of $\zeta(r,s)$, $\zeta(r)\zeta(s)$ and $\zeta(k)$, respectively, and where r and s are allowed to assume the value 1.

In $\mathcal{D}_k$ we can prove a number of explicit relations. In particular, Euler's result that all $Z_{r,s}$ are rational linear combinations of the $P_{r,s}$ when the weight k is odd holds in the formal double zeta space $\mathcal{D}_k$, so that we can (and usually will) assume that k is even. Similarly, the formal analogue of Euler's sum formula (2) holds in $\mathcal{D}_k$, and in fact (for k even) has a refinement giving the sums of the even- and odd-argument double zeta values of weight k separately. Surprisingly, they are always in the ratio 3:1, independently of k:

Theorem 1. *For even $k > 2$, one has*

$$\sum_{\substack{r=2\\ r\text{ even}}}^{k-1} Z_{r,k-r} = \frac{3}{4}\,Z_k\,, \qquad \sum_{\substack{r=2\\ r\text{ odd}}}^{k-1} Z_{r,k-r} = \frac{1}{4}\,Z_k\,. \tag{4}$$

As an example of a more complicated identity, we show that, for $m,n \geqslant 1$ odd, $m+n=k>2$,

$$2\sum_{\nu=0}^{n-1}\binom{-m}{\nu}B_\nu\,Z_{n-\nu,m+\nu} = \sum_{r+s=k}(-1)^{s-1}\lambda_{m,n}(r,s)P_{r,s}\,, \tag{5}$$

where B_ν is the νth Bernoulli number and

$$\lambda_{m,n}(r,s) = \sum_{\nu=0}^{n-1}\binom{m+\nu-1}{\nu}\binom{r-1}{n-\nu-1}B_\nu \tag{6}$$

(which despite appearances is symmetric in r and s). Since $B_\nu = 0$ for all odd ν except $\nu = 1$, this implies that any $Z_{\mathrm{ev,ev}}$ can be written in terms of $Z_{\mathrm{od,od}}$'s and $P_{r,s}$'s. But in fact only $Z_{\mathrm{od,od}}$'s are required:

Theorem 2. *Let $k > 2$ be even. Then the $Z_{r,k-r}$ with $0 < r < k$ odd are a basis of $\mathcal{D}_k$. There are explicit representations of the elements of various bases of $\mathcal{D}_k$ as linear combinations of the $Z_{\mathrm{od,od}}$'s.*

Theorem 2 will be proved in Section 4 by rewriting the defining relations (3) of $\mathcal{D}_k$ algebraically in terms of the action of the group ring $\mathbb{Z}[\Gamma_1]$ on a space of polynomials. This leads to both a simple proof of the first statement and to several concrete versions of the second. One of these, a variant of (5), is

$$Z_{m+1,k-m-1} + \frac{1}{2}Z_k \;=\; -\frac{2}{m}\sum_{\substack{r+s=k\\ r,\,s\geqslant 1\text{ odd}}}\lambda^0_{m,k-m}(r,s)\big(Z_{r,s}+\frac{1}{2}Z_k\big) \tag{7}$$

for $m = 1,3,\ldots,k-3$, where

$$\lambda^0_{m,n}(r,s) = \lambda_{m,n}(r,s) - \binom{s-1}{m-1}B_{s-m} = \sum_{\ell=0}^{r-2}\binom{k-2-\ell}{m-1}\binom{r-1}{\ell}B_{n-\ell-1}$$

(with $B_\nu = 0$ for $\nu < 0$). Since Z_k equals $4\sum_{r>1\text{ odd}} Z_{r,k-r}$ by Theorem 1, this expresses all even-argument double zeta values in terms of odd-argument ones.

Theorem 2 is false for double zeta values. Instead we have the following result, which gives the first connection with modular forms:

Theorem 3. (Rough statement) *The values* $\zeta(\mathrm{od},\mathrm{od})$ *of weight* k *satisfy at least* $\dim S_k$ *linearly independent relations, where* S_k *denotes the space of cusp forms of weight* k *on* Γ_1.

Example. For $k=12$ and $k=16$, the first weights for which there are non-zero cusp forms on Γ_1, we have the identities

$$\begin{aligned} 28\,\zeta(9,3)+150\,\zeta(7,5)+168\,\zeta(5,7) &= \frac{5197}{691}\,\zeta(12) \qquad (8)\\ 66\,\zeta(13,3)+375\,\zeta(11,5)+686\,\zeta(9,7)+675\,\zeta(7,9) &+ 396\,\zeta(5,11)\\ &= \frac{78967}{3617}\,\zeta(16)\,, \end{aligned}$$

which can be written in terms only of $\zeta(\mathrm{od},\mathrm{od})$'s using Theorem 1. Conjecturally (and numerically), these are the only relations over $\mathbb{Q}$ among odd-argument double zeta values up to weight $\leqslant 16$, and more generally we expect that there are no further relations among the $\zeta(\mathrm{od},\mathrm{od})$ except the ones predicted by Theorem 3.

Although Theorem 3 holds for the "true" double zeta world and is false in the formal one, it is in fact a consequence of a result in the formal space. In fact, it follows from two different—though complementary—results. Both of them involve period polynomials. We recall the definition of these polynomials. (A more detailed review will be given in Section 5.) For each even k we consider the space V_k of homogeneous polynomials of degree $k-2$ in two variables and the subspace $W_k \subset V_k$ of polynomials satisfying the relations $P(X,Y)+P(-Y,X)=0$, $P(X,Y)+P(X-Y,X)+P(Y,Y-X)=0$. It splits as the direct sum of subspaces W_k^+ and W_k^- of polynomials which are symmetric and antisymmetric with respect to $X \leftrightarrow Y$, with the former being odd and the latter even with respect to $X \mapsto -X$. The Eichler-Shimura-Manin theory tells us that there are canonical isomorphisms over $\mathbb{C}$ between S_k and W_k^+ and between M_k and W_k^-. The full statement of Theorem 3, given in Section 5, associates to any polynomial in W_k^-, in an injective way, an explicit relation among the numbers $Z_{\mathrm{od},\mathrm{od}}$ and $P_{\mathrm{ev},\mathrm{ev}}$ (and Z_k). For the above example (8), for instance, the polynomial $X^2Y^2(X^2-Y^2)^3$ in W_{12}^- leads to the relation

$$28\,Z_{9,3}+150\,Z_{7,5}+168\,Z_{5,7} \;=\; 28P_{4,8}+\frac{95}{3}P_{6,6}-\frac{167}{3}Z_{12}\,, \qquad (9)$$

which by Euler's theorem agrees with (8) modulo $\mathbb{Q}\pi^{12}$, and similarly the complete version of the relation given above between odd double zeta values in weight 16 is

$$\begin{aligned} 66\, Z_{13,3} + 375\, Z_{11,5} + 686\, Z_{9,7} + 675\, Z_{7,9} + 396\, Z_{5,11} \\ = 66P_{4,12} + 185P_{6,10} + \frac{364}{3}P_{8,8} - \frac{1081}{3}Z_{16}\,. \end{aligned}$$

The other result about formal double zeta values which implies Theorem 3 involves the space W_k^+ rather than W_k^-. More precisely, it involves a certain 1-dimensional extension $\widehat{W}_k^+ \subset V_k + \mathbb{C}\cdot\left(X^{k-1}Y^{-1} + X^{-1}Y^{k-1}\right)$ (see Section 6 for details) which is isomorphic to M_k rather than S_k:

Theorem 4. *If $\{Z_{r,s},\, P_{r,s},\, Z_k\}$ is a collection of numbers satisfying the double shuffle relations in weight k, then the polynomial*

$$\sum_{\substack{r+s=k \\ r,\ s \text{ even}}} P_{r,s}\, X^{r-1}Y^{s-1} \;-\; \frac{Z_k}{2}\left(X^{k-1}Y^{-1} + X^{-1}Y^{k-1}\right)$$

belongs to $\widehat{W}_k^+$ (and to W_k^+ if $Z_k = 0$). Every element of $\widehat{W}_k^+$ arises in this way.

From one point of view, this says that the subspace $\mathcal{P}_k^{\mathrm{ev}}$ of $\mathcal{D}_k$ spanned by the $P_{r,s}$ with r and s even is canonically dual to $\widehat{W}_k^+$. From another, it says that there are $k/6 + \mathrm{O}(1)$ relations among the $P_{\mathrm{ev},\mathrm{ev}}$, these relations being the same as the relations satisfied by the coefficients of period polynomials in W_k^+. In fact, we will prove Theorem 4 in this form. It is this point of view which leads to the most direct connection with modular forms, because it is known (as a consequence of the so-called Rankin-Selberg or unfolding method) that the coefficients of (extended) symmetric period polynomials satisfy the same linear relations as the products $G_rG_s \in M_k$ $(r+s=k)$, where G_r denotes the Eisenstein series of weight r on $\mathrm{PSL}_2(\mathbb{Z})$. (When r or s is equal to 2, the product G_2G_{k-2} must be modified slightly by adding an appropriate multiple of G'_{k-2} to compensate for the non-modularity of G_2.) Thus the proof of Theorem 4, combined with the known facts that the products G_rG_s span M_k and, after dividing by π^k, have rational Fourier coefficients, also leads to the following, more intuitive, statement:

Theorem 5. *The space $\mathcal{P}_k^{\mathrm{ev}}$ is canonically isomorphic to $M_k^{\mathbb{Q}}$, by a map which sends $P_{r,s}$ to $(2\pi i)^{-k}G_rG_s$ (plus a multiple of G'_{k-2} if r or $s = 2$) and Z_k to $(2\pi i)^{-k}G_k$.*

Theorem 5 tells us that there is a realization of the symmetric ($\mathcal{P}$-) part of the double shuffle relations given by products of Eisenstein series. This implies by linear algebra that there must be a realization (and in fact, infinitely many realizations) of the full space $\mathcal{D}_k$ having these products as its symmetric part. It is then natural to ask whether there is a natural choice of such a realization. In the last part of the paper we show, following an idea already adumbrated in [15], that there are in fact two such choices. More precisely, we show that one can extend the map $\mathcal{P}_k^{\mathrm{ev}} \to M_k$ in two different ways to a map from $\mathcal{D}_k$ to a larger space of functions, by finding "double Eisenstein series" which are related to products of Eisenstein series in exactly the same way as double zeta values are related to products of Riemann zeta values. One of these ways is transcendental, in terms of holomorphic functions in the upper half plane, and the other combinatorial, in terms of formal power series in q with rational coefficients. Both ways are interesting, and they also turn out to be related: the Fourier expansion of the transcendental double Eisenstein series splits up into three terms, the most complicated of which is (a multiple of) the combinatorial double Eisenstein series. We now explain this in more detail.

The transcendental version of the double Eisenstein series $G_{r,s}(\tau)$ is defined, in complete analogy with (1), as

$$G_{r,s}(\tau) = \sum_{\substack{\mathbf{m},\,\mathbf{n}\in\mathbb{Z}\tau+\mathbb{Z} \\ \mathbf{m}\succ\mathbf{n}\succ 0}} \frac{1}{\mathbf{m}^r\mathbf{n}^s} \qquad (\tau\in\mathfrak{H} = \text{upper half-plane}),$$

where $\mathbf{n}\succ 0$ means $\mathbf{n} = n\tau + b$ with $n > 0$ or $n = 0, b > 0$ and $\mathbf{m}\succ\mathbf{n}$ means $\mathbf{m}-\mathbf{n}\succ 0$. The series converges absolutely for $r\geqslant 3$, $s\geqslant 2$, and also makes sense for $s = 1$ if the sum over $\mathbf{n}$ (for $\mathbf{m}$ fixed) is interpreted as a Cauchy principal value. The same combinatorial proof that establishes (3) shows that, at least in the convergent cases, the corresponding equations still hold with $\zeta(r,s)$ replaced by $G_{r,s}(\tau)$ and with (each) $\zeta(k)$ replaced by the function

$$G_k(\tau) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}\tau+\mathbb{Z} \\ \mathbf{m}\succ 0}} \frac{1}{\mathbf{m}^k} \qquad (k\geqslant 2)$$

(again to be interpreted as a Cauchy principal value if $k = 2$), which equals the previously mentioned Eisenstein series if k is even. In other words, at least for the cases of absolute convergence, we have a realization of the double shuffle relations on the space of holomorphic functions in $\mathfrak{H}$ given by

$$Z_{r,s} \mapsto G_{r,s}(\tau), \qquad P_{r,s} \mapsto G_r(\tau)G_s(\tau), \qquad Z_k \mapsto G_k(\tau)\,.$$

The combinatorial/arithmetic aspect emerges when we study the Fourier expansions of the single and double Eisenstein series. The former are given by the well-known formula

$$(2\pi i)^{-k} G_k(\tau) \;=\; \widetilde{\zeta}(k) \,+\, g_k(q)\,, \tag{10}$$

where $q = e^{2\pi i\tau}$, $\widetilde{\zeta}(k) = (2\pi i)^{-k}\zeta(k)$, and

$$g_k(q) \;=\; \frac{(-1)^k}{(k-1)!} \sum_{u,n>0} u^{k-1}\, q^{un} \qquad (k \geqslant 2)\,. \tag{11}$$

The corresponding result for $G_{r,s}(\tau)$ is given by

Theorem 6. *The Fourier expansion of $G_{r,s}(\tau)$ for $r \geqslant 3$, $s \geqslant 2$ is given by*

$$(2\pi i)^{-r-s}\, G_{r,s}(\tau) \;=\; \widetilde{\zeta}(r,s) \,+ \sum_{\substack{h+p=r+s\\ h,\,p>1}} C^p_{r,s}\, g_h(q)\, \widetilde{\zeta}(p) \,+\, g_{r,s}(q) \tag{12}$$

with $q = e^{2\pi i\tau}$, $\widetilde{\zeta}(r,s) = (2\pi i)^{-r-s}\zeta(r,s)$,

$$C^p_{r,s} \;=\; \delta_{s,p} \,+\, (-1)^s \binom{p-1}{s-1} \,+\, (-1)^{p-r} \binom{p-1}{r-1} \quad \in \mathbb{Z} \tag{13}$$

and

$$g_{r,s}(q) \;=\; \frac{(-1)^{r+s}}{(r-1)!(s-1)!} \sum_{\substack{m>n>0\\ u,\,v>0}} u^{r-1} v^{s-1} q^{um+vn} \qquad \in \mathbb{Q}[[q]]\,. \tag{14}$$

We can reinterpret this theorem in the light of the following considerations. If k is even, the case when $G_k(\tau)$ is modular (or quasi-modular if $k=2$), then by Euler's theorem the number $\widetilde{\zeta}(k)$ occurring on the right-hand side of (10) is the rational number $-B_k/2k!$, which we denote by β_k. Hence this right-hand side can be replaced by the expression

$$Z_k(q) \;=\; g_k(q) \,+\, \beta_k \qquad (k \geqslant 2)\,, \tag{15}$$

which we call the *combinatorial Eisenstein series* because it is purely combinatorially defined as an element of $\mathbb{Q}[[q]]$ and is proportional to the usual Eisenstein series (and hence modular) when k is even and $\geqslant 4$. In the same way, we define

$$\beta_{r,s}(q) = \sum_{h+p=r+s} C^p_{r,s}\, \beta_p\, g_h(q) \qquad (r,s \geqslant 2) \tag{16}$$

and set

$$Z_{r,s}(q) \;=\; g_{r,s}(q) \,+\, \beta_{r,s}(q) \qquad (r \geqslant 3,\; s \geqslant 2)\,, \tag{17}$$

the *combinatorial double Eisenstein series*. Then the right-hand side of (12) can be rewritten as

$$\widetilde{\zeta}(r,s) + \sum_{\substack{h+p=r+s \\ h,p>1,\, p \text{ odd}}} C^p_{r,s}\, \widetilde{\zeta}(p)\, g_h(q) + Z_{r,s}(q)\,. \tag{18}$$

The three pieces in (18) lie in three non-intersecting $\mathbb{Q}$-subspaces of $\mathbb{C}[[q]]$: the first term is in $\mathbb{C}$ (more precisely, in $\mathbb{R}$ or $i\mathbb{R}$ depending on the parity of $r+s$), the second term in $i\mathbb{R}[[q]]^0$, and the third in $\mathbb{Q}[[q]]^0$, where $A[[q]]^0 = qA[[q]]$ denotes the space of power series without constant term with coefficients in a vector space A. The first term is our familiar double zeta value realization of the double shuffle relations. The second also fulfils the double shuffle relations, independently of the arithmetic natures of $\widetilde{\zeta}(p)$ and $g_h(q)$, because by a simple result which will be proved in Section 2 (Corollary 2.1) the numbers $C^p_{r,s}$ for any odd value of p less than $r+s$ already satisfy these relations. The following theorem, which we will prove in Section 7, says that the combinatorial double Eisenstein series, suitably extended to the missing cases $r = 1$, 2 and $s = 1$, also satisfies the double shuffle relations.

Theorem 7. (Rough statement) *There is a realization of the double shuffle relations in* $\mathbb{Q}[[q]]^0$ *which in the region corresponding to absolute convergence agrees with* (17) *and* (15) *and sends* $P_{r,s}$ *to* $Z_r(q)Z_s(q) - \beta_r\beta_s$ *for* $r,\ s > 2$.

If we now use (18) with the extended definition of $Z_{r,s}(q)$ to *define* the double Eisenstein series $G_{r,s}(\tau)$ in the previously undefined cases $r = 1$, $r = 2$, and $s = 1$, then we find that there is also a realization $\{Z_{r,s}, P_{r,s}, Z_k\}$ of the double shuffle relations in the space of holomorphic functions in the upper half-plane which maps $Z_{r,s}$ to $G_{r,s}(\tau)$ for $r \geqslant 3$, $s \geqslant 2$, $P_{r,s}$ to $G_r(\tau)G_s(\tau)$ for $r,\ s > 2$ and maps Z_k to $G_k(\tau)$ for all $k > 2$.

Remark. Some of the ideas developed in this paper were already mentioned, in a very preliminary form, in [15] and [16]. The discovery that there are unexpected relations among $\zeta(\mathrm{od},\mathrm{od})$'s starting in weight 12 originated with a question posed by T. Terasoma about the linear independence, modulo π^{12}, of $\zeta(r,s)$ with $r,\ s > 1$ odd, $r+s = 12$. We also mention that there is a related phenomenon for the "stable derivation algebra" of Y. Ihara [4] inside the Lie algebra of derivations of the free Lie algebra on two generators. The recent paper of L. Schneps [14] should have a close connection to our present work. Also related are several results of A.B. Goncharov, who defined a coproduct structure on (formal) multiple zeta values in [6] and described relations between double zeta values and the cohomology of $\mathrm{PSL}_2(\mathbb{Z})$ in [5].

2. The formal double zeta space

We begin by discussing the double shuffle relations (3). The first follows from the obvious decomposition of lattice points in $\mathbb{N} \times \mathbb{N}$ into the three disjoint subsets $\{(m,n) \mid m > n\}$, $\{(m,n) \mid m < n\}$ and $\{(m,n) \mid m = n\}$, giving the identity

$$\Bigl(\sum_{m>n} + \sum_{m<n} + \sum_{m=n}\Bigr) \frac{1}{m^r n^s} = \sum_{m\geqslant 1} \frac{1}{m^r} \sum_{n \geqslant 1} \frac{1}{n^s},$$

which is precisely the first equation in (3). For the second, we can use the partial fraction expansion

$$\frac{1}{m^i n^j} = \sum_{r+s=k} \left[\frac{\binom{r-1}{i-1}}{(m+n)^r n^s} + \frac{\binom{r-1}{j-1}}{(m+n)^r m^s} \right] \qquad (i+j=k). \tag{19}$$

(*Proof:* Compute the poles of both sides as rational functions of n, with m fixed.)

In the formal setting, it is convenient to extend the set of generators and relations in (3) slightly by including the case $r = 1$ (in the case of double zeta values, this would give a non-convergent series): we introduce formal variables $Z_{r,s}$, $P_{r,s}$ and Z_k and impose the relations

$$\begin{aligned} Z_{r,s} + Z_{s,r} &= P_{r,s} - Z_k \qquad (r+s=k), \\ \sum_{r+s=k} \left[\binom{r-1}{i-1} + \binom{r-1}{j-1} \right] Z_{r,s} &= P_{i,j} \qquad (i+j=k). \end{aligned} \tag{20}$$

(From now on, whenever we write $r+s=k$ or $i+j=k$ without comment, it is assumed that the variables are integers $\geqslant 1$.)

The *formal double zeta space* is now defined as the $\mathbb{Q}$-vector space

$$\mathcal{D}_k = \frac{\{\mathbb{Q}\text{-linear combinations of formal symbols } Z_{r,s}, P_{r,s}, Z_k\}}{\langle \text{relations (20)} \rangle}.$$

Alternatively, since Eqs. (20) express the $P_{i,j}$ in terms of the $Z_{r,s}$, we can define $\mathcal{D}_k$ as

$$\mathcal{D}_k = \frac{\{\mathbb{Q}\text{-linear combinations of formal symbols } Z_{r,s}, Z_k\}}{\langle \text{relation (22)} \rangle}, \tag{21}$$

where relation (22) is given by taking the difference of Eqs. (20):

$$\sum_{r+s=k}\left[\binom{r-1}{i-1}+\binom{r-1}{j-1}\right]Z_{r,s}=Z_{i,j}+Z_{j,i}+Z_k \qquad (i+j=k)\,. \quad (22)$$

Of course, since both sides of (22) are symmetric in i and j, it is enough to take (22) for $i\leqslant j$. We thus have (for k even) k generators and $k/2$ relations, so

$$\dim \mathcal{D}_k \geqslant \frac{k}{2} \qquad (k \text{ even})\,. \quad (23)$$

(We will see below that in fact equality holds.) Finally, we define the *A-valued points* $\mathcal{D}_k(A)$ of $\mathcal{D}_k$ for any $\mathbb{Q}$-vector space A by

$$\mathcal{D}_k(A)=\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k,A)=\{(Z_{r,s},Z_k)_{r+s=k}\in A^k\,,\ \text{satisfying (22)}\}\,;$$

this can also be represented as the set of $(2k-1)$-tuples $(Z_{r,s},P_{r,s},Z_k)$ satisfying (20), and we will use both forms. An element of $\mathcal{D}_k(A)$ will be called a *realization* of the double zeta space in A. For example, with $A=\mathbb{R}$ and any $\kappa\in\mathbb{R}$ we have an $\mathbb{R}$-realization of $\mathcal{D}_k$ (for $k>2$) given by

$$\begin{aligned}
&Z_{r,s}\mapsto\begin{cases}\zeta(r,s), & \text{if } r>1,\\ \kappa, & \text{if } r=1,\end{cases}\\
&P_{r,s}\mapsto\begin{cases}\zeta(r)\zeta(s), & \text{if } r,s>1,\\ \kappa+\zeta(k-1,1)+\zeta(k), & \text{if } r=1 \text{ or } s=1,\end{cases} \qquad (24)\\
&Z_k\mapsto\zeta(k)\,.
\end{aligned}$$

Here we could also treat κ as a variable and consider this as a realization in $\mathbb{R}+\mathbb{Q}\cdot\kappa$ or $\mathbb{R}[\kappa]$.

We now introduce two convenient ways to work with $\mathcal{D}_k$. The first is by generating functions. Let $(Z_{r,s},P_{r,s},Z_k)_{r+s=k}\in\mathcal{D}_k(A)$ be a realization of $\mathcal{D}_k$ in A. Then we can see easily that the identities (20) are equivalent to the relations

$$\begin{aligned}
\mathfrak{Z}_k(X,Y)+\mathfrak{Z}_k(Y,X) &= \mathfrak{P}_k(X,Y)-Z_k\cdot\frac{X^{k-1}-Y^{k-1}}{X-Y}\,,\\
\mathfrak{Z}_k(X+Y,Y)+\mathfrak{Z}_k(X+Y,X) &= \mathfrak{P}_k(X,Y)
\end{aligned} \qquad (25)$$

for the generating functions

$$\mathfrak{Z}_k(X,Y)=\sum_{r+s=k}Z_{r,s}X^{r-1}Y^{s-1}\,, \qquad \mathfrak{P}_k(X,Y)=\sum_{r+s=k}P_{r,s}X^{r-1}Y^{s-1}$$

of the $Z_{r,s}$ and $P_{r,s}$, respectively, in $A[X,Y]$. (Equations (20) just express the equality of the coefficient of $X^{r-1}Y^{s-1}$ in (25).) Similarly, (22) is equivalent to the single relation

$$\mathfrak{Z}_k(X+Y,Y)+\mathfrak{Z}_k(X+Y,X)-\mathfrak{Z}_k(X,Y)-\mathfrak{Z}_k(Y,X) \\ = Z_k\cdot\frac{X^{k-1}-Y^{k-1}}{X-Y} \tag{26}$$

for the polynomial $\mathfrak{Z}_k$.

As an example of the use of these equations, we will prove the first two identities mentioned in the Introduction, namely the fact that all $Z_{r,s}$'s are combinations of $P_{r,s}$'s and of Z_k if k is odd, and the separate even and odd sum formulas as given in Theorem 1 if k is even. For the first, we can work with (20) with the right-hand sides both replaced by 0 (because we want to work *modulo* all $P_{r,s}$'s and Z_k). Then (25) become simply $\mathfrak{Z}_k(X,Y)+\mathfrak{Z}_k(Y,X)=0$ and $\mathfrak{Z}_k(X+Y,Y)+\mathfrak{Z}_k(X+Y,X)=0$. Rewriting the latter equation as $\mathfrak{Z}_k(X,Y)+\mathfrak{Z}_k(X,X-Y)=0$, we see that $\mathfrak{Z}_k$ is anti-invariant under the two involutions $\varepsilon:(X,Y)\mapsto(Y,X)$ and $\tau:(X,Y)\mapsto(X,X-Y)$. Since $(\varepsilon\tau)^3$ maps (X,Y) to $(-X,-Y)$ and $\mathfrak{Z}_k$ is homogeneous of degree $k-2$, these two relations imply $\mathfrak{Z}_k(X,Y)=(-1)^k\mathfrak{Z}_k(X,Y)$, so $\mathfrak{Z}_k=0$ if k is odd, proving the first identity. (One can refine this proof to give an explicit formula for $\mathfrak{Z}_k(X,Y)$ as $A(X,Y)-A(X,X-Y)+A(Y,Y-X)$, where $A(X,Y)=\sum_{2|r}P_{r,s}X^{r-1}Y^{s-1}-\frac{Z_k}{2}\frac{X^{k-1}-Y^{k-1}}{X-Y}$.) For Theorem 1, it suffices to apply (26) with $(X,Y)=(1,0)$ and $(1,-1)$. This gives (for even k)

$$\mathfrak{Z}_k(1,1)-\mathfrak{Z}_k(0,1)=Z_k\,,\qquad \mathfrak{Z}_k(1,-1)-\mathfrak{Z}_k(0,1)=-\frac{1}{2}Z_k\,,$$

and Theorem 1 follows by adding and subtracting the equations.

We remark that it is occasionally convenient to work with the infinite product $\mathcal{D}=\prod_k\mathcal{D}_k$ consisting of infinite collections of numbers $\big\{\{Z_{r,s}\}_{r,s\geqslant 1},\{P_{r,s}\}_{r,s\geqslant 1},\{Z_k\}_{k\geqslant 1}\big\}$ satisfying (20) for all k. Then the corresponding generating functions $\mathfrak{Z}(X,Y)$, $\mathfrak{P}(X,Y)$ and $\mathfrak{z}(T)=\sum_{k\geqslant 1}Z_kT^{k-1}$ satisfy

$$\begin{aligned}\mathfrak{Z}(X,Y)+\mathfrak{Z}(Y,X) &= \mathfrak{P}(X,Y)-\frac{\mathfrak{z}(X)-\mathfrak{z}(Y)}{X-Y}\,,\\ \mathfrak{Z}(X+Y,Y)+\mathfrak{Z}(X+Y,Y) &= \mathfrak{P}(X,Y)\,,\end{aligned}\tag{27}$$

and similarly for (26). For example, the reader may want to verify that the function $\mathfrak{Z}(X,Y)-\mathfrak{Z}(0,Y)$ is equal to $\sum_{m>n>0}X/m(m-X)(n-Y)$ for the realization (24) and to use this to verify the k-less version of (26) directly for this generating function. (The calculation—which requires some work—gives the result only up to an additive constant, corresponding to the fact that (24) holds only for $k > 2$.)

The following proposition, which will be used in Section 7, gives some easy solutions of relations (26) (with $Z_k = 0$).

Proposition 2.1. *Let $A(X,Y) \in V_k$ be a polynomial which is even with respect to Y. Then the function*

$$\mathfrak{Z}_k(X,Y) = A(X,Y) - A(X,X-Y) + A(Y,Y-X) \tag{28}$$

gives a realization of Equation (26) *with $Z_k = 0$.*

Proof. One checks by direct calculation that if $\mathfrak{Z}_k(X,Y)$ is defined by (28) then both $\mathfrak{Z}_k(X,Y) + \mathfrak{Z}_k(Y,X)$ and $\mathfrak{Z}_k(X+Y,Y) + \mathfrak{Z}_k(X+Y,X)$ equal $A(X,Y) + A(Y,X)$. Note that the assertion of the proposition also holds if $A(X,Y) = A(Y,-X)$ or if A is anti-symmetric (with $\mathfrak{P}_k \equiv 0$ in the latter case). □

Corollary 2.1. *Let $0 < p < k$ be two integers with p odd. Then the numbers $Z_{r,s} = C^p_{r,s}$ $(r+s=k)$ with $C^p_{r,s}$ defined by Equation* (13) *satisfy* (22) *with $Z_k = 0$.*

Proof. This is simply Proposition 2.1 applied to $A(X,Y) = X^{k-p-1}Y^{p-1}$. The corresponding numbers $P_{r,s}$ in (20) are equal to $\delta_{r,p} + \delta_{s,p}$. □

The second way of working with $\mathcal{D}_k$ is by studying the *relations* among the $Z_{r,s}$ (or $Z_{r,s}$, $P_{r,s}$ and Z_k). The following result gives a useful description of them. We introduce the notation

$$V_k = \langle X^{r-1}Y^{s-1} \,|\, r+s=k \rangle, \qquad V_k^* = \langle \frac{1}{m^r n^s} \,|\, r+s=k \rangle.$$

We define an isomorphism $V_k \to V_k^*$ by

$$F(X,Y) = \sum_{r+s=k} \binom{k-2}{r-1} f_{r,s} X^{r-1}Y^{s-1} \mapsto F^*(m,n) = \sum_{r+s=k} \frac{f_{r,s}}{m^r n^s}.$$

Then we have the following

Lemma 2.1. *Let F, G, $H \in V_k$ and F^*, G^*, H^* the corresponding elements of V_k^*. Then the following two statements are equivalent:*

(i) $H^*(m,n) = F^*(m+n,n) + G^*(m,m+n)$,
(ii) $F(X,Y) = H(X,X+Y)$, $\quad G(X,Y) = H(X+Y,Y)$.

Proof. Equation (19) implies that any element $h \in V_k^*$ can be decomposed as $f(m+n,n)+g(m,m+n)$ for some f and g in V_k^*, and this decomposition is obviously unique since $f(1,x)$ has poles only at $x = 0$ and $g(1-x,1)$ only at $x = 1$. If $f = F^*$ etc., then an inspection of (19) shows that the coefficients $f_{r,s}$ and $g_{r,s}$ of F and G are related to the coefficients $h_{r,s}$ of H by

$$f_{r,s} = \sum_{i+j=k} \binom{r-1}{i-1} h_{i,j}\,, \qquad g_{r,s} = \sum_{i+j=k} \binom{s-1}{j-1} h_{i,j}\,.$$

Using the binomial coefficient identity $\binom{k-2}{r-1}\binom{r-1}{i-1} = \binom{k-2}{j-1}\binom{j-1}{s-1}$ $(r+s = i+j = k)$, we find that these formulas are equivalent to (ii). $\square$

Proposition 2.2. *Let $a_{r,s}$ and λ be rational numbers. Then the following three statements are equivalent:*

(i) *The relation*

$$\sum_{r+s=k} a_{r,s} Z_{r,s} = \lambda Z_k \tag{29}$$

holds in $\mathcal{D}_k$.

(ii) *The generating function*

$$A(X,Y) = \sum_{r+s=k} \binom{k-2}{r-1} a_{r,s}\, X^{r-1} Y^{s-1} \quad \in V_k \tag{30}$$

can be written as $H(X,X+Y) - H(X,Y)$ for some symmetric homogeneous polynomial $H \in \mathbb{Q}[X,Y]$ of degree $k-2$, and

$$\lambda = \frac{k-1}{2} \int_0^1 H(t,1-t)dt\,. \tag{31}$$

(iii) *The generating function*

$$A^*(m,n) = \sum_{r+s=k} \frac{a_{r,s}}{m^r n^s} \quad \in V_k^* \tag{32}$$

can be written as $f(m,n) - f(m+n,m) - f(m+n,n)$ for some $f \in V_k^*$, and

$$\lambda = \frac{f(1,1) - A^*(1,1)}{2} = f(2,1)\,. \tag{33}$$

Proof. If we choose the symmetric polynomial $H(X,Y) = X^{i-1}Y^{j-1} + X^{j-1}Y^{i-1}$ and use the binomial theorem to compute the $a_{r,s}$ in (30) and the beta integral to compute $\lambda = (i-1)!(j-1)!/(k-2)!$, then we find that (29) reduces to (22). Since these H's span the space of symmetric polynomials in V_k, this proves the equivalence of the first two statements.

The equivalence of (ii) and (iii) follows by applying the lemma with $F = A + H$, $G(X,Y) = F(Y,X)$ and $f = F^*$, $g(m,n) = f(n,m)$. To check that the values of λ in (ii) and (iii) agree, we again use the beta integral $\int_0^1 t^{r-1}(1-t)^{s-1}dt = \frac{(r-1)!(s-1)!}{(k-1)!}$ to get $(k-1)\int_0^1 H(t,1-t)dt = \sum h_{r,s} = H^*(1,1)$. $\square$

Remark. We can also write Equation (31) as $\lambda = \frac{1}{2}\sum h_{r,s}$, where $H = \sum \binom{k-2}{r-1} h_{r,s} X^{r-1}Y^{s-1}$.

The two approaches outlined above are equivalent by a duality which we will discuss below, but it is very convenient to have both. As an example of the use of the proposition, we give a second quick proof of Theorem 1 from the Introduction. Taking $H = X^{k-2} + Y^{k-2}$ in the proposition gives $a_{r,s} = 1$ $(r \neq 1)$, $a_{1,k-1} = 0$, $\lambda = 1$, while taking $H = (X-Y)^{k-2}$ gives $a_{r,s} = (-1)^r$ $(r \neq 1)$, $a_{1,k-1} = 0$, $\lambda = \frac{1}{2}$. Again adding and subtracting the two relations thus obtained gives (4).

As a second example, we observe that Eq. (22) contains no $Z_{1,k-1}$, so that $Z_{1,k-1}$ is a free variable (as we already saw in the realization (24)). Thus $a_{1,k-1}$ must vanish in any relation of the form (29), and we can also see this in the proposition by setting $X = 0$.

3. Using the action of $\mathbf{PGL_2}(\mathbb{Z})$

We have already repeatedly used the space V_k of homogeneous polynomials of degree $k-2$ in X and Y. We now make this approach more systematic by exploiting two further structures on V_k: the action of the group $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ and the Γ-invariant scalar product. The former is defined in the obvious way by $(F|\gamma)(X,Y) = F(aX+bY, cX+dY)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (we suppose throughout that k is even) and the latter by

$$\langle X^{r-1}Y^{s-1}, X^{m-1}Y^{n-1}\rangle = \frac{(-1)^r}{\binom{k-2}{m-1}}\delta_{(r,s),(n,m)} \tag{34}$$

for $r, s, m, n \geqslant 1$, $r+s = m+n = k$. The invariance property $\langle F|\gamma, G|\gamma\rangle = \langle F, G\rangle$ is easily checked. We extend the action of Γ on V_k to an action of the group ring $R = \mathbb{Z}[\Gamma]$ by linearity. Then $\langle F|\xi, G\rangle = \langle F, G|\xi^*\rangle$, where $\xi \mapsto \xi^*$ is the anti-automorphism of R induced by $\gamma \mapsto \gamma^{-1}$. We occasionally work with the model of V_k consisting of polynomials $f(x)$ of one variable of degree $\leqslant k-2$, corresponding to the homogeneous model via $f(x) = F(x,1)$,

$F(X,Y) = Y^{k-2}f(X/Y)$. The group operation in this version takes the form $(f|\gamma)(x) = (cx+d)^{k-2}f((ax+b)/(cx+d))$.

The group Γ contains distinguished elements. First there are the commuting involutions

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

sending $F(X,Y)$ to $F(Y,X)$ and $F(-X,Y)$, respectively. The (± 1)-eigenspaces of ε will be denoted by $V_k^{\pm}$ and the (± 1)-eigenspaces of δ by V_k^{ev} and V_k^{od}; we also write $V_k^{+,\mathrm{ev}}$ for the space of even symmetric polynomials and similarly for the other three double eigenspaces of dimension $k/4 + \mathrm{O}(1)$. In $\mathrm{PSL}_2(\mathbb{Z})$, we have the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = US = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = U^2S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

with the relations $S^2 = U^3 = 1$, $S = \varepsilon\delta$ and

$$\varepsilon U \varepsilon = U^2, \qquad \varepsilon T \varepsilon = T', \qquad \delta T \delta = T^{-1}, \qquad STS = T'^{-1}.$$

We will also consider various special elements of the group ring $\mathbb{Z}[\Gamma]$. First, we have the projections

$$\pi^+ = \frac{1}{2}(\varepsilon + 1), \quad \pi^{\mathrm{od}} = \frac{1}{2}(1-\delta), \quad \pi^{+,\mathrm{od}} = \pi^+\pi^{\mathrm{od}} = \pi^{\mathrm{od}}\pi^+, \text{ etc.}$$

onto V^+, V^{od}, $V^{+,\mathrm{od}}$ etc. Next, we have the element

$$\Delta = (T-1)(\varepsilon+1)$$

which by (26) essentially characterizes $\mathcal{D}_k(\mathbb{Q})$: the codimension 1 subspace $\mathcal{D}_k^0(\mathbb{Q})$ of realizations with $Z_k = 0$ is identified precisely with $\mathrm{Ker}(\Delta)$, and the full space $\mathcal{D}_k(\mathbb{Q})$ corresponds to the space of $\mathfrak{Z} \in V_k$ such that $\mathfrak{Z}|\Delta \in \mathbb{Q} \cdot \frac{X^{k-1}-Y^{k-1}}{X-Y}$. We can now interpret part of Proposition 2.2 (the equivalence of (i) and (ii) when $Z_k = 0$) as the dual statement of this with respect to the non-degenerate scalar product (34): a relation (29) with $Z_k = 0$ is reformulated equivalently as $\langle A|S, \mathfrak{Z}\rangle = 0$ (compare equations (29) and (30); the extra "S" comes from the interchange of r and s and the sign $(-1)^r$ in (34)). So this holds for all $\mathfrak{Z} \in \mathrm{Ker}(\Delta)$ if and only if $A|S \in \mathrm{Im}(\Delta^*) = V_k^+|(T^{-1}-1)$, i.e., if and only if $A \in V_k^+|(T^{-1}-1)S = V_k^+|(T'-1)$ (for the last step, use $V_k^+ = V_k^+|S$ and $ST^{-1}S = T'$), and this is just (ii) of Proposition 2.2. Finally, we have the element

$$\Lambda = 1 - \varepsilon U + U^2 \quad \in \quad \mathbb{Z}[\Gamma]. \tag{35}$$

It is related to the above elements by

$$\Lambda\,\Delta \;=\; -4\,\pi^{\mathrm{od}}\,\pi^{+} \quad\in\quad \mathbb{Z}[\Gamma]\,, \tag{36}$$

as we see by the calculation

$$\begin{aligned}\Lambda\,(T-1) &= (1-\varepsilon U+U^2)(US-1)\\ &= \bigl[1-U(\varepsilon-1)\bigr]\,S \,+\, U^2(\varepsilon-1) \,-\, 1\\ &= \delta-1 \,+\, (\delta-US+U^2)(\varepsilon-1)\,,\end{aligned}$$

followed by multiplying both sides on the right by $\varepsilon+1$. It follows from (36) that for any polynomial $A\in V_k$ which is even *or* antisymmetric *or* S-invariant, the coefficients of $A|\Lambda$ give a realization of $\mathcal{D}_k$ with $Z_k=0$ by taking $\mathfrak{Z}_k = A|\Lambda$ and $\mathfrak{P}_k = 2\pi^+(A)$. This is equivalent to Proposition 2.1 and the remark in its proof.

As an example for how to work with the structures just introduced, we prove Eq. (5) from the Introduction. To do this, we define

$$B_{m,n}(X,Y) \;=\; \binom{k-2}{m-1} Y^{k-2} B_{n-1}(X/Y) \qquad (m+n=k), \tag{37}$$

where $B_\nu(x) \;=\; \sum_{\mu=0}^{\nu}\binom{\nu}{\mu} B_\mu\, x^{\nu-\mu}$ denotes the νth Bernoulli polynomial. The numbers $\lambda_{m,n}(r,s)$ defined in (6) are the coefficients of the generating series

$$\sum_{r+s=k}\binom{k-2}{r-1}\,\lambda_{m,n}(r,s)\,X^{r-1}Y^{s-1} \;=\; B_{m,n}(X,X+Y)\,. \tag{38}$$

The symmetry $\lambda_{m,n}(r,s)=(-1)^{m-1}\lambda_{m,n}(s,r)$ mentioned for m odd in the Introduction follows from this formula together with the standard property $B_\nu(1-x)=(-1)^\nu B_\nu(x)$ of Bernoulli polynomials (cf., e.g., [2]). Set $a_{r,s}=(-1)^{s-1}\bigl[\binom{s-1}{m-1}B_{s-m}-\lambda_{m,n}(r,s)\bigr]$. Then we see that the polynomial (30) has the form $H(X,X+Y)-H(X,Y)$ with $H(X,Y)=B_{m,n}(X,X-Y)$. The symmetry property just mentioned implies that $H(X,Y)=H(Y,X)$, so we can apply Proposition 2.2 to get (29) with $\lambda=\frac{1}{2}\sum(-1)^{s-1}\lambda_{m,n}(r,s)$. This is (5).

4. Representing even double zeta values in terms of odd ones

In this section, we prove Theorem 2. Since we already know that $\dim\mathcal{D}_k \geqslant k/2$ (cf. (23)), we have only to show that any $Z_{\mathrm{ev,ev}}$ is a linear combination of $Z_{\mathrm{od,od}}$'s. This means that any collection of numbers $\{a_{r,s}\mid$

$r+s=k$; r,s even$\}$ can be completed to a collection $\{a_{r,s}\ (r+s=k),\ \lambda\}$ satisfying (29) in $\mathcal{D}_k$. By Proposition 2.2, this is equivalent to showing that any polynomial $F \in V_k^{\mathrm{od}}$ is the odd part of a polynomial of the form $H|(T-1)$ with $H \in V_k^{+}$ (since then $F|\varepsilon$ is the odd part of $H|(T'-1)$). Thus the result to be proved is:

Proposition 4.1. *The space V_k ($k>2$ even) has the decomposition*

$$V_k^{\mathrm{ev}} + V_k^{+}|(T-1) = V_k\,.$$

Proof. Here it is more convenient to use the 1-variable model. Let $V_k^{T\delta} \subset V_k$ be the fixed point set of the involution $g(x) \mapsto g(1-x)$. Then we have the following commutative diagram with the top row exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}\cdot 1 \arrow[r] & V_k^{T\delta} \arrow[r, "T-1"] \arrow[d, "1+\varepsilon-\varepsilon U"] & V_k^{\mathrm{od}} \arrow[r] & 0 \\
 & & V_k^{+} \arrow[r, "T-1"] & V_k^{+}|(T-1) \arrow[u, "\pi^{\mathrm{od}}"'] &
\end{tikzcd}$$

To see the exactness, we first observe that if $g \in V_k^{T\delta}$ then the polynomial $f = g|(T-1)$ is odd because, from $T\delta T = \delta$ and $g|T\delta = g$, we deduce $g|T = g|\delta$. Conversely, an odd polynomial $f(x) \in V_k$ has degree $\leqslant k-3$ (since k is even) and hence can be written as $g(x+1)-g(x)$ for some $g \in V_k$. But then $g|(T\delta-1)(1-T) = g|(T-1)(1+\delta) = f|(1+\delta) = 0$, so $g|(T\delta-1)$ is a constant and hence zero since it vanishes at $x = 1/2$. (One can also argue that the map $V_k^{T\delta}/\mathbb{Q}\cdot 1 \xrightarrow{T-1} V_k^{\mathrm{od}}$ which is obviously injective, must be an isomorphism because both sides have dimension $k/2-1$.) It is clear that the kernel of $V_k^{T\delta} \xrightarrow{T-1} V_k^{\mathrm{od}}$ is $\mathbb{Q}\cdot 1$. Next, we have to show that $h = g|(1+\varepsilon-\varepsilon U)$ is symmetric for $g \in V_k^{T\delta}$. This follows from $\varepsilon U \varepsilon = U^2 = TSU$ and thus $g|\varepsilon U\varepsilon = g|TSU = g|\delta SU = g|\varepsilon U$. The commutativity of the square now follows from the calculation

$$(\varepsilon-\varepsilon U)(T-1) = (\varepsilon T-1)\varepsilon(1+\delta) + (1-T\delta)\varepsilon(1-U)T\,,$$

which implies that $(h-g)|(T-1) = g|(\varepsilon-\varepsilon U)(T-1) = g|(\varepsilon T-1)\varepsilon(1+\delta)$ which vanishes under $|(1-\delta)$.

It follows from the diagram that the map $\pi^{\mathrm{od}} : V_k^{+}|(T-1) \to V_k^{\mathrm{od}}$ is surjective which is equivalent to the statement of the theorem. □

The proposition and its proof give us an explicit way to realize the asserted decomposition by starting with any basis of $V_k^{T\delta}$. To obtain a relation (29) with prescribed values $a_{r,s} = f_{r,s}$ for r and s even, we write the generating function $f|\varepsilon \in V_k^{\mathrm{od}}$ as $g|(T-1)$ with $g \in V_k^{T\delta}$, then $A = g|(1+\varepsilon-\varepsilon U)(T-1)\varepsilon = g|(1+\varepsilon-\varepsilon U)(T'-1)$ has odd part f and belongs to $V_k^{+}|(T'-1)$, so that Proposition 2.2 applies. To obtain explicit relations of this decomposition, we can choose any basis of the space of functions symmetric about $x = 1/2$. In particular, from the three bases $g(x) = (2x-1)^{k-2-2\nu}$, $(x^2-x)^\nu$ and $B_{2\nu}(x)$, where $0 \leqslant \nu \leqslant (k-2)/2$, we get three explicit collections of relations. For the first one, suitably normalized, we find that the coefficients of the associated relation (29) are

$$a_{r,s} = \begin{cases} 2^{s-1}\binom{r-1}{2\nu} & (r,s \text{ even}) \\ -2^{r-1}\binom{s-1}{2\nu} + \sum\limits_{\alpha+\beta=2\nu} (-1)^\alpha \binom{r-1}{\alpha}\binom{s-1}{\beta} & (r,s \text{ odd}), \end{cases}$$

$$\lambda = \frac{k-1}{2}\binom{k-2}{2\nu}\left[\int_0^1 (2-3t)^{k-2\nu-2} t^{2\nu}\, dt - \frac{1}{2(2\nu+1)}\right],$$

and for the second basis we find

$$\frac{1}{2}\binom{k-2}{r-1} a_{r,s} = \begin{cases} \binom{\nu}{s-\nu-1} & (r,s \text{ even}), \\ (-1)^\nu \binom{k-2-2\nu}{r-\nu-1} - \binom{\nu}{r-\nu-1} & (r,s \text{ odd}) \end{cases}$$

(we omit the value of λ in this case). In both cases, the coefficients for r, s even form a triangular matrix. The third family $g(x) = B_{2\nu}(x)$ yields Eq. (7), as the reader can check as an exercise by imitating the proof of Eq. (5) which was given at the end of Section 3. The following table gives these three collections of relations for the case $k = 12$.

5. Double zeta values and period polynomials

In this section we describe various connections between period polynomials and the (formal) double zeta space, and prove Theorems 3 and 4. Since both of them involve period polynomials, we begin by reviewing these. The definition of period polynomials was already given briefly in the Introduction. The motivation comes from the connection with modular forms, which we will review in the next section. Here we discuss only the algebraic properties.

The space W_k is defined as

$$W_k = \operatorname{Ker}(1+S) \cap \operatorname{Ker}(1+U+U^2) \quad \subset V_k, \tag{39}$$

Table 1: Relations among double zeta values of weight 12 coming from Proposition 4.1. Each row of the table gives a relation among the (formal) double zeta values displayed in the top line. The $Z_{1,11}$ column has been omitted, since all of its entries would be zero.

$g(x)$	$Z_{2,10}$	$Z_{4,8}$	$Z_{6,6}$	$Z_{8,4}$	$Z_{10,2}$	Z_{12}	$Z_{3,9}$	$Z_{5,7}$	$Z_{7,5}$	$Z_{9,3}$	$Z_{11,1}$
$\frac{1}{2}\binom{10}{10}(2x-1)^{10}$	512	128	32	8	2	$-\frac{1355}{4}$	-3	-15	-63	-255	-1023
$\frac{1}{2}\binom{10}{8}(2x-1)^{8}$	0	384	320	168	72	$-\frac{21}{4}$	-99	-243	-387	-243	45
$\frac{1}{2}\binom{10}{6}(2x-1)^{6}$	0	0	160	280	252	$\frac{129}{2}$	-294	-238	-62	-14	210
$\frac{1}{2}\binom{10}{4}(2x-1)^{4}$	0	0	0	56	168	$\frac{51}{2}$	-126	-14	2	-14	210
$\frac{1}{2}\binom{10}{2}(2x-1)^{2}$	0	0	0	0	18	$-\frac{7}{4}$	9	-3	-3	13	45
$\binom{10}{10}(x^2-x)^5$	1	$\frac{1}{6}$	$\frac{1}{126}$	0	0	$-\frac{331}{504}$	0	0	$-\frac{1}{21}$	$-\frac{4}{9}$	-2
$\binom{10}{8}(x^2-x)^4$	0	3	$\frac{10}{7}$	0	0	$-\frac{1}{4}$	0	0	$-\frac{15}{7}$	-2	0
$\binom{10}{6}(x^2-x)^3$	0	0	5	$\frac{7}{2}$	0	$\frac{1}{2}$	0	-14	-10	0	0
$\binom{10}{4}(x^2-x)^2$	0	0	0	7	0	$-\frac{4}{3}$	0	28	30	$\frac{28}{3}$	0
$\binom{10}{2}(x^2-x)$	0	0	0	0	9	$\frac{19}{4}$	-18	-24	-24	-16	0
$\binom{10}{10}B_{10}(x)$	1	0	0	0	0	$-\frac{767}{1155}$	$\frac{5}{33}$	$-\frac{41}{165}$	$-\frac{31}{231}$	$-\frac{41}{165}$	$-\frac{61}{33}$
$\binom{10}{8}B_{8}(x)$	0	3	0	0	0	$\frac{1}{6}$	-3	5	2	$-\frac{11}{3}$	-3
$\binom{10}{6}B_{6}(x)$	0	0	5	0	0	$-\frac{4}{3}$	10	-18	-15	$\frac{16}{3}$	10
$\binom{10}{4}B_{4}(x)$	0	0	0	7	0	$\frac{13}{6}$	-14	14	16	$-\frac{14}{3}$	-14
$\binom{10}{2}B_{2}(x)$	0	0	0	0	9	1	-3	-9	-9	-1	15
$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$	1	1	1	1	1

i.e. as the intersection of the (-1)-eigenspace of the involution S and the sum of the $\left(\frac{-1\pm\sqrt{-3}}{2}\right)$-eigenspaces of the element U of order 3. Since $\varepsilon S\varepsilon = S$ and $\varepsilon U\varepsilon = U^2$, the involution ε acts on W_k and splits it as the direct sum of subspaces $W_k^{\pm} = W_k \cap V_k^{\pm}$ of symmetric and antisymmetric polynomials. Since elements in W_k are also (-1)-eigenfunctions of S and since $S\varepsilon = \varepsilon S = \delta$, we also have $W_k^{+} = W_k^{\mathrm{od}} \subset V_k^{+,\mathrm{od}}$ and $W_k^{-} = W_k^{\mathrm{ev}} \subset V_k^{-,\mathrm{ev}}$. Another important property of period polynomials is given by the following lemma.

Lemma 5.1. *Let $k > 2$ be even. Then*

$$W_k \;=\; \mathrm{Ker}\big(1 - T - T',\, V_k\big)$$

and

$$W_k^{\pm} \;=\; \mathrm{Ker}\big(1 - T \mp T\varepsilon,\, V_k\big)\,.$$

Proof. It is equivalent for $f \in V_k$ to be in $\mathrm{Ker}(1 - T - T')$ or to satisfy $f|(1+S) = f|(1+U+U^2)$, since $(1-T-T')S = (1+S)-(1+U+U^2)$. But a polynomial which is fixed by both S and U is fixed by the full modular group and thus vanishes. The second assertion of the lemma follows from the first, because if f is annihilated by $1-T-T' = 1-T-\varepsilon T\varepsilon$ and $f|\varepsilon = \pm f$ then f is also annihilated by $1-T\mp T\varepsilon$, and conversely if f is annihilated by $1-T\mp T\varepsilon$ then $f = f|T(1\pm\varepsilon) \in V_k^{\pm}$ and hence $f|(1-T-T') = f|(1-T-\varepsilon T\varepsilon) = 0$. $\square$

Remark. The operator $\mathcal{L} = 1-T-T'$ plays a key role in the discovery by J. Lewis that there are holomorphic functions annihilated by this operator which have the same relation to the so-called Maass wave forms as period polynomials have to holomorphic modular forms ([10], [11]). We call the equation $f|\mathcal{L} = 0$ the *Lewis equation.*

As in the Introduction, we denote by $\mathcal{P}_k$ the subspace of $\mathcal{D}_k$ spanned by the $P_{r,s}$ (and Z_k, but it can be omitted by virtue of Theorem 1), and by $\mathcal{P}_k^{\mathrm{ev}}$ the subspace spanned by the $P_{\mathrm{ev},\mathrm{ev}}$. Note that $\mathcal{P}_k^{\mathrm{ev}}$ corresponds to generating functions in V_k^{od} because of the shift by 1 in the exponents of X and Y.

Theorem 3. *The spaces $\mathcal{P}_k^{\mathrm{ev}}$ and W_k^- are canonically isomorphic to each other. More precisely, to each $p \in W_k^-$ we associate the coefficients $p_{r,s}$ and $q_{r,s}$ $(r+s=k)$ which are defined by $p(X,Y) = \sum \binom{k-2}{r-1} p_{r,s} X^{r-1}Y^{s-1}$ and $p(X+Y,Y) = \sum \binom{k-2}{r-1} q_{r,s} X^{r-1}Y^{s-1}$. Then $q_{r,s} - q_{s,r} = p_{r,s}$ (in particular $q_{r,s} = q_{s,r}$ for r, s even) and*

$$\sum_{\substack{r+s=k\\ r,\,s\ \mathrm{even}}} q_{r,s} Z_{r,s} \;\equiv\; 3 \sum_{\substack{r+s=k\\ r,\,s\ \mathrm{odd}}} q_{r,s} Z_{r,s} \pmod{Z_k}, \tag{40}$$

and conversely, an element $\sum_{r,\,s\ \mathrm{odd}} c_{r,s} Z_{r,s} \in \mathcal{D}_k$ belongs to $\mathcal{P}_k^{\mathrm{ev}}$ if and only if $c_{r,s} = q_{r,s}$ arising in this way.

Remarks. 1. The equivalence of the first and last statements of the theorem follows from Theorem 2: since the $Z_{\mathrm{od},\mathrm{od}}$ form a basis of $\mathcal{D}_k$, it is equivalent to speak of elements of $\mathcal{P}_k^{\mathrm{ev}}$ or of relations of the form $\sum(*)P_{\mathrm{ev},\mathrm{ev}} = \sum(*)Z_{\mathrm{od},\mathrm{od}}$.

2. Since the double zeta realizations $\zeta(r)\zeta(s)$ and $\zeta(k)$ of $P_{r,s}$ (r, s even) and Z_k are rational multiples of π^k, and since π^k is a $\mathbb{Q}$-linear combination of $Z_{\mathrm{od},\mathrm{od}}$'s by Theorem 1, Theorem 3 as stated here contains the "rough statement" given in the Introduction. (The number of relations drops from $\dim W_k^- = \dim M_k$ to $\dim S_k = \dim M_k - 1$ because one relation gets used up to eliminate $\zeta(k)$.)

Example 1. For every even $k>2$, the space W_k^- contains the polynomial $p(x) = x^{k-2}-1$ (in the inhomogeneous notation). Here $p(x+1) =$

$\sum_{r\neq 1}\binom{k-2}{r-1}x^{r-1}$, i.e., $q_{1,k-1} = 0$ and all other $q_{r,s}$ are equal to 1, and Theorem 3 reduces to a weaker version of Theorem 1.

Example 2. The space W_{12}^- is 2-dimensional, spanned by the two polynomials $p(x) = x^{10} - 1$ and $x^2(x^2-1)^3$. For the latter, we have $p(x+1) = x^8 + 8x^7 + 25x^6 + 38x^5 + 28x^4 + 8x^3$, so the $p_{r,s}$ and $q_{r,s}$ of the theorem are given (after multiplication by 1260) by the table

Table 2: The coefficients $p_{r,s}$ and $q_{r,s}$ for $p(x) = x^2(x^2-1)^3 \in W_{12}^-$.

r	1	2	3	4	5	6	7	8	9	10	11
s	11	10	9	8	7	6	5	4	3	2	1
$1260\,p_{r,s}$	0	0	−28	0	18	0	−18	0	28	0	0
$1260\,q_{r,s}$	0	$\underline{0}$	0	$\underline{84}$	168	$\underline{190}$	150	$\underline{84}$	28	$\underline{0}$	0

The $q_{r,s}$ with r and s even (underlined) are symmetric and the relation (40), divided by 3, becomes

$$28Z_{8,4} + \frac{190}{3}Z_{6,6} + 28Z_{4,8} \equiv 28Z_{9,3} + 150Z_{7,5} + 168Z_{5,7} \qquad (\mathrm{mod}\ Z_{12}),$$

in agreement with Eq. (9) of the Introduction. The example for $k = 16$ given there arises in the same way from the even period polynomial $p(x) = x^2(x^2-1)^3(2x^4 - x^2 + 2) \in W_{16}^-$.

Proof. The function $q = p|T$ satisfies $q|(1-\varepsilon) = p|(T - T\varepsilon) = p|(T + \varepsilon T\varepsilon) = p$ because p is antisymmetric and satisfies the Lewis equation. This shows that $q_{r,s} - q_{s,r} = p_{r,s}$ and also means that if we decompose q in the obvious way as $q = q^{\mathrm{ev},+} + q^{\mathrm{ev},-} + q^{\mathrm{od},+} + q^{\mathrm{od},-}$, then $q^{\mathrm{ev},-} = \frac{1}{2}p$ and $q^{\mathrm{od},-} = 0$. Write $[a,b,c]$ to denote $aq^{\mathrm{ev},+} + bq^{\mathrm{ev},-} + cq^{\mathrm{od},+}$. Then

$$\begin{aligned} [0,\,2,\,0]\,\big|\,T' &= p|T' = p|\varepsilon T\varepsilon = -p|T\varepsilon = -q|\varepsilon = [-1,\,1,\,-1]\,, \\ [1,\,-1,\,-1]\,\big|\,T' &= q|ST' = p|TST' = p|S = -p = [0,\,-2,\,0]\,, \end{aligned}$$

and hence

$$[2,\,0,\,-2]\,\big|\,(T'-1) = [-1,\,-3,\,-1] - [2,\,0,\,-2] = [-3,\,-3,\,1]\,.$$

This says that $q^{\mathrm{od}} - 3q^{\mathrm{ev}}$ is the image under $T' - 1$ of the symmetric polynomial $2\big(q^{\mathrm{ev},+} - q^{\mathrm{od},+}\big)$, so Proposition 2.2 implies Eq. (40). (The omitted coefficient of Z_k in (40) is easily determined, by computing the integral in (31), as $\sum(-1)^{r-1}q_{r,s}$.) This gives a map $W_k^- \longrightarrow \mathcal{P}_k^{\mathrm{ev}}$ which is obviously injective since the antisymmetrization of the coefficients on the right-hand side of (40) are the coefficients of p itself. We omit the proof of surjectivity since it will follow from Theorem 4 below that $\dim W_k^- = \dim \mathcal{P}_k^{\mathrm{ev}}$, so that injectivity suffices. $\square$

Theorem 3 tells us that, given *any* relation of the form (29) in $\mathcal{D}_k$ with $a_{r,s} = a_{s,r}$ for r and s even, there exists a unique element $p \in W_k^-$ such that the $a_{r,s}$ with r, s odd are equal to the numbers $q_{r,s}$ in the theorem. On the other hand, given an element of $\mathcal{P}_k^{\mathrm{ev}}$, its representation as a linear relation of the generators $P_{\mathrm{ev,ev}}$ is not unique, because these generators are not linearly independent. The next proposition, which is a first form of Theorem 4 of the Introduction, describes the relations among them, i.e., all relations of the form (29) with $a_{r,s} = a_{s,r}$ for *all* r and s. It turns out that in any such relation the odd-index $a_{r,s}$ all vanish (in accordance with the widely believed and numerically verified statement that there are no relations over $\mathbb{Q}$ among products of values of the Riemann zeta function at odd arguments), while the even-index $a_{r,s}$ are related to the space W_k^+.

Proposition 5.1. *Let $a_{r,s}$ and μ be numbers with $a_{s,r} = a_{r,s}$. Then the following three statements are equivalent:*

(i) *The relation*

$$\sum_{r+s=k} a_{r,s} P_{r,s} = \mu Z_k \tag{41}$$

holds in $\mathcal{D}_k$.

(ii) *The generating function*

$$A(X,Y) = \sum_{r+s=k} \binom{k-2}{r-1} a_{r,s} X^{r-1} Y^{s-1} \quad \in V_k \tag{42}$$

can be written as $A = H|(1-S)$ for some $H \in V_k^U \cap V_k^+$, and

$$\mu = \langle H, \frac{X^{k-1} - Y^{k-1}}{X-Y} \rangle . \tag{43}$$

(iii) *The generating function*

$$A^*(m,n) = \sum_{r+s=k} \frac{a_{r,s}}{m^r n^s} \quad \in V_k^* \tag{44}$$

can be written as $f(m,n) - f(m+n,n) - f(m,m+n)$ for some symmetric $f \in V_k^*$, and

$$\mu = f(1,1) . \tag{45}$$

If these statements hold, then $a_{r,s} = 0$ for odd r and s.

Proof. Except for the assertions about the value of the constant μ, which we will leave to the reader, each part of this proposition is equivalent to the corresponding part of Proposition 2.2 of Section 2 with the extra condition that $a_{r,s} = a_{s,r}$. For (i) this is obvious; for (iii) it follows because $f(m,n) - A^*(m,n) = f(m+n,m)+f(m+n,n)$ in Proposition 2.2 is always symmetric, so that A is symmetric if and only if f is; and for (ii) it follows because if $H \in V_k^+$, then the element $H|(T'-1)$ is symmetric if and only if $H = H|U$ (because $H|T'(\varepsilon - 1) = H|(1-U)T$), in which case $H|(T'-1) = H|(T-1) = H|(S-1)$. The last assertion of the proposition is then clear since $A \in V_k^+|(1-S) \subset V_k^{\mathrm{od}}$. $\square$

Proposition 5.1, without the statements concerning μ, says that the following are equivalent for symmetric A:

(i′) $\sum a_{r,s} P_{r,s} \equiv 0 \pmod{Z_k}$;
(ii′) $A \in V_k^{U,+}|(1-S)$;
(iii′) $A^* \in V_k^*|(1-T-T')$ (in the obvious notation).

Statement (ii′) in turn is equivalent to

(iv′) $A \in V_k^{\mathrm{od}}$ and $A \perp W_k^+$ with respect to the scalar product (34),

because for $v \in V$ we have:

$$\begin{aligned} v \text{ is orthogonal to } (V_k^{U,+})|(1-S) &= V_k|(1+U+U^2)(1+\varepsilon)(1-S) \\ \Leftrightarrow \quad & v|(1-S)(1+\varepsilon)(1+U+U^2) = 0 \\ \Leftrightarrow \quad & v|(1-S)(1+\varepsilon) \in \mathrm{Ker}(1+U+U^2) \cap \mathrm{Ker}(1+S) \cap \mathrm{Ker}(1-\varepsilon) \\ \Leftrightarrow \quad & v|(1-S)(1+\varepsilon) \in W_k^+ \\ \Leftrightarrow \quad & v \in W_k^+ + V_k^- + V_k^{\mathrm{ev}}\,. \end{aligned}$$

On the other hand, (i′) is equivalent to the condition that $\sum a_{r,s} P_{r,s} = 0$ for any realization $\{Z_{r,s},\, P_{r,s},\, Z_k\}$ of $\mathcal{D}_k$ with $Z_k = 0$, while (iv′) says that all $a_{\mathrm{od,od}}$ are zero and $\sum a_{r,s} p_{r,s} = 0$ for any $p = \sum p_{r,s} X^{r-1} Y^{s-1} \in W_k^+$. The equivalence of (i′) and (iv′) therefore says that *any* symmetric collection of numbers $\{P_{r,s}\ (r,\ s \text{ odd})\}$ can be extended to a realization of $\mathcal{D}_k$ with $Z_k = 0$, while a symmetric collection of numbers $\{P_{r,s}\ (r,\ s \text{ even})\}$ can be extended to a realization of $\mathcal{D}_k$ with $Z_k = 0$ if and only if the corresponding generating function belongs to W_k^+. This is precisely the statement of Theorem 4 as given in the Introduction in the case $Z_k = 0$. (For the full statement we need the extended period polynomial space $\widehat{W}_k^+$, which will be discussed in the next section in the context of modular forms.) Before doing that, we give a slight improvement of the result just stated.

This result says that, if P is any polynomial belonging to $V_k^{+,\mathrm{ev}}$ or to $W_k^+ \subset V_k^{+,\mathrm{od}}$, then there exists a $Z \in V_k$ with

$$Z\,|\,(1+\varepsilon) \;=\; P\,, \qquad Z\,|\,T(1+\varepsilon) \;=\; P\,. \tag{46}$$

The following proposition makes this explicit:

Proposition 5.2. *(i) Let* $P \in V_k^{+,\mathrm{ev}}$. *Then* $Z = \frac{1}{2}\,P|\Lambda$ *with* Λ *as in* (35) *gives a solution of Eqs.* (46).

(ii) Let $P \in W_k^+$. *Then* $Z = \frac{1}{3}\,P|(T^{-1}+1)$ *gives a solution of Eqs.* (46).

Proof. (i) The identities $\Lambda(1+\varepsilon) \;=\; 1+\varepsilon$ and $\Lambda T \;=\; S + T(1-\varepsilon)$ together with $P|\varepsilon = P|\delta = P$ immediately imply (46).

(ii) By Lemma 5.1, P satisfies the Lewis equation $P|(1-T-T') = 0$. Hence, using $P|\varepsilon = P$ and $P|\delta = -P$, we find

$$P|(1+T^{-1})(1+\varepsilon) = 2P + P|(T^{-1} - \delta T^{-1}\varepsilon) = 3P + P|(1-T-T')T^{-1} = 3P$$

and $P|(T^{-1}+1)T(1+\varepsilon) = 2P \,+\, P|(T+T') = 3P\,.$ □

6. Double zeta values and modular forms

In this section we reinterpret the results of Section 5 from the modular point of view. To do this, we begin by reviewing the theory of period polynomials of modular forms on $\mathrm{PSL}_2(\mathbb{Z})$, including various supplementary results which are less well-known and which are needed here.

The *period polynomial associated to a cusp form* $f \in S_k$ can be defined by

$$P_f(X,Y) \;=\; \int_0^\infty (X - Y\tau)^{k-2}\, f(\tau)\, d\tau\,. \tag{47}$$

The identity $(aX+bY) - (cX+dY)\tau = (a-c\tau)(X - \gamma^{-1}(\tau)Y)$ together with the modularity of f shows that

$$P_f\,|\gamma(X,Y) \;=\; \int_{\gamma^{-1}(0)}^{\gamma^{-1}(\infty)} (X - Y\tau)^{k-2}\, f(\tau)\, d\tau$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. In particular,

$$P_f|(1-T-T') = \left(\int_0^\infty - \int_{-1}^\infty - \int_0^{-1}\right)(X - Y\tau)^{k-2}\, f(\tau)\, d\tau \;=\; 0\,,$$

so $P_f \in W_k$ by Lemma 5.1. (One can also verify that $P_f|(1+S) = P_f|(1+U+U^2) = 0$ directly by a similar calculation.) This gives the basic connection between cusp forms and period polynomials. The complete Eichler-Shimura-Manin theory, of which summaries can be found in many places (e.g. [9]), tells us that the maps assigning to f the symmetric (odd) and antisymmetric (even) parts P_f^+ and P_f^- of P_f give isomorphisms from the space S_k of cusp forms of weight k on Γ_1 onto W_k^+ and a codimension 1 subspace of W_k^-. (The latter was determined in [8].) The theory is also "defined over $\mathbb{Q}$" in the sense that the even and odd period polynomials of a normalized Hecke eigenform f are proportional to polynomials with coefficients in the number field generated by the Fourier coefficients of f, and transform properly under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For instance, the even and odd period polynomials of the modular form $\Delta \in S_{12}$, in the inhomogeneous version, are multiples of $\frac{36}{691}(x^{10}-1) - x^2(x^2-1)^3$ and $x(x^2-1)^2(x^2-4)(4x^2-1)$, respectively.

For $f \in M_k$ with $f(\infty) = a_0 \neq 0$ the integral (47) diverges, but the modified definition [17]

$$\begin{aligned}\widehat{P}_f(X,Y) &= \int_{\tau_0}^{\infty} (X-Y\tau)^{k-2}\big(f(\tau)-a_0\big)d\tau + \int_0^{\tau_0} (X-Y\tau)^{k-2}\Big(f(\tau)-\frac{a_0}{\tau^k}\Big)d\tau \\ &\quad + \frac{a_0}{k-1}\left(\frac{1}{Y} - \frac{\tau_0^{1-k}}{X}\right)(X-Y\tau_0)^{k-1} \qquad (\text{any } \tau_0 \in \mathfrak{H})\end{aligned}$$

makes sense, since the integrals converge and the derivative of the right-hand side with respect to τ_0 vanishes. We call this function the *extended period polynomial* of f. For instance, the extended period polynomial of the Eisenstein series G_k is

$$\frac{2\pi i\,\zeta(k-1)}{2k-2}\big(X^{k-2}-Y^{k-2}\big) - \frac{(2\pi i)^k}{2k-2}\sum_{r=0}^{k}\frac{B_r}{r!}\frac{B_{k-r}}{(k-r)!}X^{r-1}Y^{k-r-1},$$

where B_r denotes the rth Bernoulli number.

The function $\widehat{P}_f(X,Y)$ no longer lies in V_k, but in the larger space $\widehat{V}_k = \bigoplus_{\substack{r+s=k\\ r,s\geqslant 0}} \mathbb{C}\cdot X^{r-1}Y^{s-1}$. On the other hand, the same calculation as before shows that it is again annihilated by $1+S$, $1+U+U^2$ and $1-T-T'$. (The group Γ_1 does not act on $\widehat{V}_k$, but it acts on the larger space $\mathbb{C}(X,Y)$ of rational functions in two variables, so this statement makes sense.) In other words, $\widehat{P}_f$ belongs to the space

$$\widehat{W}_k := \mathrm{Ker}(1+S,\,\widehat{V}_k)\cap\mathrm{Ker}(1+U+U^2,\,\widehat{V}_k) = \mathrm{Ker}(1-T-T',\,\widehat{V}_k),$$

which we call the space of *extended period polynomials*. This space fits into a short exact sequence $0 \longrightarrow W_k \longrightarrow \widehat{W}_k \xrightarrow{\lambda} \mathbb{C} \longrightarrow 0$, where λ associates to $\widehat{P} \in \widehat{W}_k$ the coefficient of X^{k-1}/Y in $\widehat{P}$. By writing $\widehat{P} = P + \lambda(\widehat{P})(X^{k-1}/Y + Y^{k-1}/X)$ we can identify $\widehat{W}_k$ with the space

$$\widehat{\widehat{W}}_k \;=\; \{(P,\lambda) \in V_k \times \mathbb{C} \mid P|(1+S) = P|(1+U+U^2) + \lambda\Phi_k = 0\}\,,$$

where $\Phi_k \in V_k$ is the polynomial

$$\begin{aligned}\Phi_k(X,Y) \;&=\; \Big(\frac{X^{k-1}}{Y} + \frac{Y^{k-1}}{X}\Big)\Big|(1+U+U^2)\\ &=\; \Big(\frac{X^{k-1}-(X-Y)^{k-1}}{Y} + \frac{Y^{k-1}+(X-Y)^{k-1}}{X} + \frac{X^{k-1}-Y^{k-1}}{X-Y}\Big)\,.\end{aligned}$$

Corresponding to the canonical splitting $M_k = S_k \oplus \mathbb{C}\cdot G_k$ of modular forms into cusp forms and Eisenstein series, we have the splittings $\widehat{W}_k = W_k \oplus \mathbb{C}\cdot\widehat{\mathcal{E}}_k$, $\widehat{\widehat{W}}_k = W_k \oplus \mathbb{C}\cdot(\mathcal{E}_k, B_k)$, where

$$\begin{aligned}\widehat{\mathcal{E}}_k(X,Y) &= \sum_{\substack{r+s=k\\ r,\,s\geqslant 0}} \binom{k}{r} B_r B_s X^{r-1}Y^{s-1}\\ &= \mathcal{E}_k(X,Y) + B_k\Big(\frac{Y^{k-1}}{X} + \frac{X^{k-1}}{Y}\Big)\,.\end{aligned} \tag{48}$$

The space $\widehat{W}_k$ also splits into symmetric and antisymmetric parts $\widehat{W}_k^{\pm}$. Since $\widehat{\mathcal{E}}_k$ is symmetric, we have $\widehat{W}_k^{+} = W_k^{+} \oplus \mathbb{C}\cdot\widehat{\mathcal{E}}_k$ and $\widehat{W}_k^{-} = W_k^{-}$.

If f and g are two cusp forms, then the Petersson scalar product (f,g) is proportional to the pairing $\langle P_f^{+}|(T-T^{-1}),\, P_g^{-}\rangle$ ([7], [8]). (It is essential that odd period polynomials are always paired with even ones, because if f is a normalized Hecke eigenform then the coefficients of $P_f^{\pm}$ belong to $\omega_{\pm}(f)\,\overline{\mathbb{Q}}[X,Y]$ with some constants $\omega_{\pm}(f)$ whose product is essentially (f,f). But the "straight" pairing $\langle P_f^{+},\, P_g^{-}\rangle$ would vanish since P_f^{+} and P_g^{-} have opposite symmetry properties and also opposite parity. The effect of $|(T-T^{-1})$ is to change odd polynomials to even ones and vice versa.) This pairing is Hecke invariant, but we do not explain this here since we have not discussed the action of Hecke operators on period polynomials. It extends in a natural way to a pairing $\widehat{W}_k^{+} \times W_k^{-} \to \mathbb{C}$ in such a way that the codimension 1 subspace of W_k^{-} corresponding to period polynomials of cusp forms is precisely the space of polynomials whose pairing with $\widehat{\mathcal{E}}_k$ vanishes (cf. [8]). The result (Theorem 9 of [8]) is

$$\{P_f^{-} \mid f \in S_k\} = \Big\{ \sum_{r+s=k} \tbinom{k-2}{r-1} a_{r,s} X^{r-1}Y^{s-1} \,\Big|\, \sum_{r+s=k} (-1)^{\frac{r-1}{2}} \kappa_{r,s} a_{r,s} = 0 \Big\}\,,$$

where

$$\kappa_{r,s} = -\kappa_{s,r} = 2 \sum_{\substack{0<j\leqslant k \\ j \text{ even}}} \binom{j-1}{r-1}\binom{k}{j} B_j B_{k-j} - \binom{k}{r} B_r B_s \\ + \left[(-1)^{r-1} + \binom{k-1}{r-1} - \binom{k-1}{r}\right] B_k . \tag{49}$$

Equation (49) says that $\kappa_{r,s}$ is essentially the coefficient of $X^{r-1}Y^{s-1}$ in $\mathcal{E}_k|T$ with $\mathcal{E}_k$ as in (48).

We can now easily extend Proposition 5.1 (and hence the preliminary version of Theorem 4 mentioned in the previous section) to a statement concerning $\widehat{W}_k^+$ rather than W_k^+. Given any symmetric extended period polynomial

$$\widehat{P} = \sum_{\substack{r+s=k \\ r,s\geqslant 0 \\ r,\, s \text{ even}}} p_{r,s} X^{r-1}Y^{s-1} \in \widehat{W}_k^+$$

(so that $\lambda(\widehat{P}) = p_{k,0} = p_{0,k}$), there is a realization of $\mathcal{P}_k$ with

$$P_{r,s} \mapsto \begin{cases} p_{r,s} & (r,\, s \text{ even}), \\ 0 & (r,\, s \text{ odd}), \end{cases} \qquad Z_k \mapsto -2\, p_{k,0} .$$

To see this, instead of going through the whole proof of Proposition 5.2, keeping careful track of the constant μ, it is sufficient (since W_k^+ has codimension 1 in $\widehat{W}_k^+$) to check this for one single extended "polynomial" $\widehat{P}$ which belongs to $\widehat{W}_k^+$ but not to W_k^+, and this is easy: the function $\widehat{\mathcal{E}}_k$ defined in (48) has coefficients $P_{r,s} = \binom{k}{r}B_rB_s = 4k!\,\beta_r\beta_s$, where $\beta_r = \zeta(r)/(2\pi i)^r = -B_r/2r!$ ($r\geqslant 0$ even), and on the other hand the original double zeta realization of $\mathcal{D}_k$ has $P_{r,s} = \zeta(r)\zeta(s) = (2\pi i)^k\beta_r\beta_s$ and $Z_k = \zeta(k) = (2\pi i)^k\beta_k$. This completes the discussion of extended period polynomials and the proof of Theorem 4.

We also mention the corresponding extension of Proposition 5.2:

Supplement to Proposition 5.2: *If* $\widehat{P} = P + \lambda(X^{k-1}Y^{-1} + X^{-1}Y^{k-1})$, *then we have a solution of* (25) *with* $\mathfrak{P}_k = P$ *and* $Z_k = -2\lambda$ *given by*

$$\begin{aligned} \mathfrak{Z}_k &= \frac{1}{3}P\big|(T^{-1}+1) + \frac{\lambda}{6}\frac{X^{k-1}}{Y}\bigg|\, U^2(1+\varepsilon)(5-3U+U\varepsilon) \\ &= \frac{1}{3}P\big|(T^{-1}+1) + \frac{\lambda}{6}\frac{X^{k-1}-Y^{k-1}}{X-Y}\bigg|\,(5-3U+U\varepsilon) . \end{aligned}$$

Proof. Just apply the calculation of the proof of part (ii) of Proposition 5.2 to $\widehat{P}$ with $\widehat{\mathfrak{Z}}_k = \frac{1}{3}\widehat{P}|(1+T^{-1})$ since $\widehat{P}$ satisfies the same relations as P. □

If we apply this result to the special case $\widehat{P} = \frac{1}{4k!}\widehat{\mathcal{E}}_k$ then we find that, as well as the "Euler realization" of $\mathcal{D}_k$ in $\mathbb{R}$ with $Z^E_{r,s} = \zeta(r,s)$, $P^E_{r,s} = \zeta(r)\zeta(s)$ and $Z^E_k = \zeta(k)$, we also have a "Bernoulli realization" with $Z^B_k = \beta_k$, $P^B_{r,s} = \beta_r\beta_s$ (so that $P^B_{r,s} = 0$ if r and s are odd and $k > 2$), and with $Z^B_{r,s}$ equal to the "double Bernoulli number"

$$\begin{aligned} Z^B_{r,s} &= \frac{1}{3}\sum_{m+n=k}\binom{m-1}{r-1}\beta_m\beta_n + \frac{\beta_r\beta_s}{3} - \frac{\beta_k}{12}\left[5+3\binom{k-1}{r-1}-\binom{k-1}{r}\right] \\ &= \frac{1}{3}\sum_{\substack{m+n=k\\ m,n\geqslant 0}}\binom{m-1}{r-1}\beta_m\beta_n - \frac{\beta_k}{12}\left[1+\binom{k-1}{r-1}-\binom{k-1}{r}\right] \\ &\quad + \frac{1}{3}(\beta_r\beta_s - \beta_k)\,. \end{aligned}$$

These are almost exactly the same as the numbers (49) occurring in the result on the image of cusp forms in W^-_k cited above! Also note that the number $Z^B_{r,s}$ is essentially the double zeta value $\zeta(1-r,1-s)/(r-1)!(s-1)!$ at negative arguments, which has been studied in [1].

The Bernoulli realization has the same even-index $p_{r,s}$ as the Euler realization (up to a factor $(2\pi i)^k$), but 0 instead of the presumably transcendental values $\zeta(\text{od})\zeta(\text{od})/(2\pi i)^k$; the fact that both the original $P_{r,s}$ and the new ones can be realized in $\mathcal{D}_k$ is an illustration of the fact that the numbers $P_{\text{od},\text{od}}$ in $\mathcal{P}_k$ are completely unconstrained (this corresponds to the vanishing of $a_{\text{od},\text{od}}$ in Proposition 5.1).

The results of Section 5 were formulated purely algebraically, but we can now easily relate them to the theory of modular forms. A result of Rankin ([13]; see also [8]) says that, for a normalized Hecke eigenform $f \in S_k$, one has

$$\sum_{\substack{r+s=k\\ r,\, s\ \text{even}}} (f,\, G_rG_s)\, X^{r-1}Y^{s-1} = c_f\, P^+_f(X,Y)\,,$$

where c_f is essentially $P^-_f(1,0)$.

In particular, the cusp forms $G_rG_s - \frac{\beta_r\beta_s}{\beta_k}G_k$ satisfy the same linear relations as elements of W^+_k, so they form the coefficients of an element

$$\mathfrak{G}_k(\tau;X,Y) = \sum_{\substack{r,\,s\geqslant 1\\ r+s=k}} \Big(G_r(\tau)G_s(\tau) - \frac{\beta_r\beta_s}{\beta_k}G_k(\tau)\Big)X^{r-1}Y^{s-1} \in W^+_k \otimes S_k\,.$$

But then setting $\widehat{\mathfrak{G}}_k = \mathfrak{G}_k - \frac{1}{2B_k}G_k(\tau)\widehat{\mathcal{E}}_k$ gives the much simpler statement that the polynomial

$$\widehat{\mathfrak{G}}_k(\tau; X, Y) = \sum_{\substack{r,\, s \geqslant 0 \\ r+s=k}} G_r(\tau) G_s(\tau) X^{r-1} Y^{s-1}$$

belongs to $\widehat{W}_k^+ \otimes M_k$. Theorem 5 follows immediately.

7. Double Eisenstein series

At the end of the last section, we applied Rankin's result relating products of Eisenstein series to period polynomials of cusp forms to show that there is a realization of the double shuffle space $\mathcal{D}_k$ sending $P_{r,s}$ to $G_r(\tau)G_s(\tau)$ and Z_k to $G_k(\tau)$ for all r, s and k. This proof, however, is very indirect, and in view of the simplicity of the final statement one would expect that there should be a simpler and more natural argument. This is indeed the case, as we now explain. This alternative approach also leads directly to the double Eisenstein series which are the final topic of this paper.

We present the argument in a more abstract form than we will use here. Let A be a discrete subgroup of $\mathbb{C}$ (or possibly some more general commutative topological field) which is totally ordered, i.e. can be decomposed into a disjoint union $A^+ \cup \{0\} \cup (-A^+)$ with A^+ closed under addition. For $\mathbf{m}, \mathbf{n} \in A$ we write $\mathbf{n} \succ 0$ to mean $\mathbf{n} \in A^+$ and $\mathbf{m} \succ \mathbf{n}$ to mean $\mathbf{m} - \mathbf{n} \succ 0$. Then we claim that, at least in the range for which the sums $Z(r) = \sum_{\mathbf{m} \succ 0} \mathbf{m}^{-r}$ converge, the numbers $P_{r,s} = Z(r)Z(s)$ and $Z_k = Z(k)$ give a realization of $\mathcal{P}_k$. Indeed, by (iii) of Proposition 2.2 we can write $A^*(\mathbf{m}, \mathbf{n}) = f(\mathbf{m}, \mathbf{n}) - f(\mathbf{m}+\mathbf{n}, \mathbf{n}) - f(\mathbf{m}, \mathbf{m}+\mathbf{n})$ for some $f \in V_k^* \otimes \mathbb{C}$ and

$$\begin{aligned}
\sum_{r+s=k} a_{r,s} Z(r) Z(s) &= \sum_{\mathbf{m},\, \mathbf{n} \succ 0} A^*(\mathbf{m}, \mathbf{n}) \\
&= \Bigg(\sum_{\mathbf{m},\, \mathbf{n} \succ 0} - \sum_{\mathbf{m} \succ \mathbf{n} \succ 0} - \sum_{\mathbf{n} \succ \mathbf{m} \succ 0} \Bigg) f(\mathbf{m}, \mathbf{n}) \\
&= \sum_{\mathbf{m} = \mathbf{n} \succ 0} A^*(\mathbf{m}, \mathbf{n}) \;=\; f(1,1)\, Z(k)\,.
\end{aligned}$$

Applying this concept to the special case where $A = \mathbb{Z}\tau + \mathbb{Z}$ for some number τ in the upper half plane $\mathfrak{H}$ with the ordering described in the Introduction, we find that indeed the functions $\{G_r(\tau)\, G_s(\tau),\, G_k(\tau)\}$ give a realization of the $\{P_{r,s}, Z_k\}$-part of $\mathcal{D}_k$. The corresponding realization of the $\{Z_{r,s}\}$-part is given by the double Eisenstein series $G_{r,s}$ as defined in the Introduction. In this section, we study these in more detail.

A simple estimate shows that the series defining G_k converges absolutely if and only if $k > 2$ and the series defining $G_{r,s}$ if and only if $s > 1$ and $r > 2$. Our first object is to compute the Fourier expansion of $G_{r,s}$ (Theorem 6). We begin by recalling the corresponding computation for G_k, which is of course well-known. We define power series $\varphi_k^0(q)$ and $\varphi_k(q)$ in $\mathbb{Q}[[q]]$ — both actually polynomials of degree k in $q/(1-q)$ — by

$$\varphi_k^0(q) \;=\; \frac{(-1)^k}{(k-1)!}\sum_{u=1}^{\infty} u^{k-1}\,q^u\,, \quad \varphi_k(q) \;=\; -\frac{1}{2}\delta_{k,1}+\varphi_k^0(q) \qquad (k\geqslant 1).$$

The Lipschitz formula says that

$$\sum_{a\in\mathbb{Z}}\frac{1}{(\tau+a)^k} \;=\; (2\pi i)^k\varphi_k(q) \tag{50}$$

for $\tau \in \mathfrak{H}$ and all $k\geqslant 1$, where $q = e^{2\pi i\tau}$ as usual and where the sum on the left-hand side has to be interpreted as a Cauchy principal value if $k = 1$. Applying this to $G_k(\tau)$ with $k\geqslant 2$, where the summation in the non-absolutely convergent case $k = 2$ is to be carried out in the order $\succ$, we find

$$G_k(\tau) \;=\; \sum_{a>0}\frac{1}{a^k} + \sum_{m>0}\sum_{a\in\mathbb{Z}}\frac{1}{(m\tau+a)^k} \;=\; \zeta(k) \;+\; (2\pi i)^k\,g_k(q),$$

where

$$g_k(q) \;=\; \sum_{m>0}\varphi_k^0(q^m) \;=\; -\sum_{m,\,u>0}\frac{(-u)^{k-1}}{(k-1)!}\,q^{mu}\,. \tag{51}$$

The statement of Theorem 6, which we repeat here for convenience, was that the Fourier expansion of $G_{r,s}(\tau)$ in the convergent case is given by the analogous formula

$$G_{r,s}(\tau) \;=\; \zeta(r,s) + \sum_{h+p=k} C_{r,s}^p\,(2\pi i)^h g_h(q)\zeta(p) \;+\; (2\pi i)^k g_{r,s}(q)\,, \tag{52}$$

where $k = r+s$, $C_{r,s}^p$ is a simple numerical coefficient given by (13), and

$$g_{r,s}(q) = \sum_{m>n>0}\varphi_r^0(q^m)\varphi_s^0(q^n) = \sum_{\substack{m>n>0\\ u,\,v>0}}\frac{(-u)^{r-1}}{(r-1)!}\frac{(-v)^{s-1}}{(s-1)!}\,q^{mu+nv}. \tag{53}$$

(The condition $h > 1$ and $p > 1$ in (12) can be dropped, even though the definitions of $\widetilde{\zeta}(p)$ and $g_h(q)$ are problematic in these cases, because $C_{r,s}^p$ vanishes when $p = 1$ or $p = r+s-1$ unless $r = 1$.)

Proof of Theorem 6. We divide the sum of the defining series of

$$G_{r,s}(\tau) = \sum_{m\tau+a \succ n\tau+b \succ 0} \frac{1}{(m\tau+a)^r(n\tau+b)^s}$$

into four terms, according as $m = n = 0$, $m > n = 0$, $m = n > 0$, or $m > n > 0$. It is obvious that the terms of the first type give the double zeta value $\zeta(r,s)$ and that those of the second type give $(G_r(\tau) - \zeta(r))\zeta(s) = (2\pi i)^r g_r(q)\zeta(s)$, while those of the fourth type, again by virtue of the Lipschitz formula (50), give $(2\pi i)^k \sum_{m>n>0} \varphi_r(m\tau)\varphi_s(n\tau)$. (Note that it does not matter here whether we write $\varphi_r\varphi_s$ or $\varphi_r^0\varphi_s^0$ since we are assuming that both r and s are greater than 1.) Finally, the sum of the terms of the third type can be written as $\sum_{m>0} \Psi_{r,s}(m\tau)$, where

$$\Psi_{r,s}(\tau) = \sum_{a>b} \frac{1}{(\tau+a)^r(\tau+b)^s}\,.$$

This sum converges absolutely because we are assuming that $r,\ s \geqslant 2$, and is obviously periodic. To calculate its Fourier development, we use the partial fraction decomposition

$$\frac{1}{(\tau+a)^r(\tau+b)^s} = \sum_{h+p=r+s} \left[\frac{(-1)^s \binom{p-1}{s-1}}{c^p\,(\tau+a)^h} + \frac{(-1)^{p-r} \binom{p-1}{r-1}}{c^p\,(\tau+b)^h} \right]$$

(compare (19)), where $c = a - b > 0$ and where we use our usual convention that the condition "$h + p = k$" tacitly includes "$h \geqslant 1,\ p \geqslant 1$." Using (50) yet again, we obtain

$$\Psi_{r,s}(\tau) = \sum_{h+p=r+s} \left[(-1)^s \binom{p-1}{s-1} + (-1)^{p-r} \binom{p-1}{r-1} \right] \zeta(p)\,(2\pi i)^h \varphi_h(\tau)\,,$$

where the implied interchange of order of summation is justified because the expression in square brackets vanishes if $p = 1$ (because r and s are > 1) or $h = 1$ (because the binomial coefficient $\binom{r+s-2}{r-1}$ is symmetric in r and s). Replacing τ by $m\tau$ and summing over $m \geqslant 1$ replaces φ_h by g_h in this expression, and combining with the terms already computed, we obtain the desired formula (52). $\square$

As explained in the Introduction, we want to do two things: find the "right" definition of the double Eisenstein series $G_{r,s}(\tau)$ in the cases when the original series defining it does not converge absolutely (i.e., if $r = 1$ or 2 or if $s = 1$), and give a purely combinatorial proof that the extended function satisfy the double shuffle relations. As also already explained, for the latter purpose we can ignore the term $\zeta(r,s)$ and the terms with p

odd in the middle sum in (52), because they individually satisfy the double shuffle relations (the latter because of the corollary to Proposition 2.1 in Section 2). If we remove these terms, then what is left, after division by $(2\pi i)^k$ (where $k = r + s$ is the total weight as usual) is the power series $Z_{r,s}(q)$ defined in (17). We now extend this definition to all values of r and s by setting

$$Z_{r,s}(q) \;=\; g_{r,s}(q) \;+\; \beta_{r,s}(q) \;+\; \frac{1}{2}\,\varepsilon_{r,s}(q) \qquad (r,\ s \geqslant 1), \tag{54}$$

where $g_{r,s}(q)$ is defined by (53), $\beta_{r,s}(q)$ by

$$\beta_{r,s}(q) \;=\; \sum_{h+p=k} C^p_{r,s}\,\beta_p\,g_h(q) \qquad (r,\ s \geqslant 1,\ \ r+s=k) \tag{55}$$

with $\beta_p = -\dfrac{B_p}{2\,p!}$ (B_p = pth Bernoulli number), and

$$\varepsilon_{r,s}(q) = \delta_{r,2}\,g^*_s(q) - \delta_{r,1}\,g^*_{s-1}(q) + \delta_{s,1}\big(g^*_{r-1}(q) + g_r(q)\big) + \delta_{r,1}\delta_{s,1}g_2(q) \tag{56}$$

with

$$g^*_k(q) = -\sum_{m>0} m\varphi^0_{k+1}(q^m) = \frac{(-1)^k}{k!}\sum_{m,\,u>0} mu^k q^{um} = \frac{1}{k}\,q\,\frac{d}{dq}\,g_k(q)\,. \tag{57}$$

Notice that β_p was previously used only for p even, where it was defined by $\beta_p = (2\pi i)^{-p}\zeta(p)$ and hence equal to $-B_p/2p!$ by Euler's theorem; now we take the latter formula as the definition for all values of p. This does not affect the definition when r and s are larger than 1, so that (55) agrees with (16) in these cases, because β_p is 0 for odd $p > 1$ and $C^1_{r,s} = 0$ for $r,\ s > 1$ (as we already used in the above proof). Since the further correction terms (56) are non-zero only if $r \leq 2$ or $s = 1$, they do not occur in the region of convergence of $G_{r,s}$, so that (54) agrees with the earlier definition (17) in the cases where it was applicable. The reason for the various terms in (56) will become clear in the course of the proof of Theorem 7, but one term has a clear explanation which we can mention now: in the above derivation, the only place where it mattered that s was strictly greater than 1 was for the absolute convergence of the inner sum in $G_{r,s}(\tau) = \sum_{m,a}(m\tau+a)^{-r}\sum_n\sum_b(n\tau+b)^{-s}$, and if we interpret this sum as a Cauchy principal value and use (50) here too, then we see that the only effect on the final calculation is to replace the factor $\varphi^0_s(q)$ in (53) by $\varphi_s(q)$. Since they differ only by the constant $-1/2$ when $s = 1$, and not at all otherwise, this adds $-\frac{1}{2}\delta_{s,1}\sum_{m>0}(m-1)\varphi^0_r(q^m) = \frac{1}{2}\delta_{s,1}\big(g^*_{r-1}(q) + g_r(q)\big)$ to $g_{r,s}(q)$, and this accounts for the third term in the definition of $\varepsilon_{r,s}(q)$. Finally, we mention that (56) apparently contains the term $g_0^*(q)$ when $r=1$ or $s=1$, and this is not defined by the last formula in (57) (although the other two formulas do still make sense and lead to the definition $g_0^*(q)=g_2(q)$), but this is not important because the two terms in (56) that potentially involve $g_0^*(q)$ occur only when $r=s=1$ and then cancel.

After these long preliminaries we can finally state and prove the full version of Theorem 7 from the Introduction.

Theorem 7. *There is a realization in* $\mathbb{Q}[[q]]^0$ *of the double shuffle relations* (20) *for all weights with* $Z_{r,s}(q)$ *defined by* (54),

$$P_{r,s}(q)=g_r(q)g_s(q)+\beta_r g_s(q)+\beta_s g_r(q)+\frac{1}{2}(\delta_{r,2}\,g_s^*(q)+\delta_{s,2}\,g_r^*(q)) \quad (58)$$

for all $r,\ s\geqslant 1$, *and with* $Z_k(q)=g_k(q)$ *for* $k>2$, $Z_2(q)=0$.

Proof. The proof will be shorter than the discussion leading up to the statement. Of course we use generating functions. We drop the "(q)" in the names of elements of $\mathbb{Q}[[q]]$ and systematically write $\gamma(X)$ and $\gamma(X,Y)$ for the generating functions $\sum_{k\geqslant 1}\gamma_k X^{k-1}$ and $\sum_{r,\,s\geqslant 1}\gamma_{r,s}X^{r-1}Y^{s-1}$ associated to sequences $\{\gamma_k\}$ or $\{\gamma_{r,s}\}$ indexed by one or two integers, respectively. Then from the definitions (51), (57), (53), (55) and (56) we have

$$\begin{aligned}
\beta(X) &= \sum_{k\geqslant 1}\beta_k\,X^{k-1} = \frac{1}{2}\left(\frac{1}{X}-\frac{1}{e^X-1}\right),\\
g(X) &= \sum_{k\geqslant 1} g_k\,X^{k-1} = -\sum_{u>0} e^{-uX}\,\frac{q^u}{1-q^u},\\
g^*(X) &= \sum_{k\geqslant 1} g_k^*\,X^{k-1} = \frac{1}{X}\left(\sum_{u>0} e^{-uX}\,\frac{q^u}{(1-q^u)^2}-g_2\right),\\
g(X,Y) &= \sum_{r,\,s\geqslant 1} g_{r,s}\,X^{r-1}Y^{s-1} = \sum_{\substack{m>n>0\\ u,\,v>0}} e^{-uX-vY}\,q^{mu+nv}\\
&= \sum_{u,\,v>0} e^{-uX-vY}\,\frac{q^u}{1-q^u}\,\frac{q^{u+v}}{1-q^{u+v}},\\
\beta(X,Y) &= \sum_{r,\,s\geqslant 1}\beta_{r,s}\,X^{r-1}Y^{s-1} = \sum_{h,\,p\geqslant 1}\beta_p\,g_h\left(\sum_{r+s=k} C_{r,s}^p X^{r-1}Y^{s-1}\right)\\
&= \sum_{h,\,p\geqslant 1}\beta_p\,g_h\left(Y^{p-1}X^{h-1}-(X-Y)^{p-1}(X^{h-1}-Y^{h-1})\right)\\
&= \beta(Y)\,g(X)\;-\;\beta(X-Y)\left(g(X)-g(Y)\right),
\end{aligned}$$

$$\varepsilon(X,Y) \;=\; \sum_{r,\,s\geqslant 1} \varepsilon_{r,s}\, X^{r-1}Y^{s-1} = (X-Y)g^*(Y) + Xg^*(X) + g(X) + g_2,$$

and we want to show that the generating functions

$$\begin{aligned}
\mathfrak{Z}(X,Y) &= g(X,Y) + \beta(X,Y) + \frac{1}{2}\varepsilon(X,Y)\,,\\
\mathfrak{P}(X,Y) &= g(X)g(Y) + \beta(X)g(Y) + \beta(Y)g(X)\\
&\quad + \frac{1}{2}\big(Xg^*(Y) + Yg^*(X)\big)\,,\\
\mathfrak{z}(X) &= g(X) - g_2
\end{aligned}$$

satisfy (27). So we must calculate $Z(X,Y) + Z(Y,X)$ and $Z(X+Y,Y) + Z(X+Y,X)$ for each of the three pieces $Z = g,\ \beta$, and ε constituting $\mathfrak{Z}$.

From the above formulas for the generating functions we find

$$\begin{aligned}
&g(X,Y) + g(Y,X)\\
&\quad= \sum_{u,\,v>0} e^{-uX-vY}\left(\frac{q^u}{1-q^u} + \frac{q^v}{1-q^v}\right)\frac{q^{u+v}}{1-q^{u+v}}\\
&\quad= \sum_{u,\,v>0} e^{-uX-vY}\left(\frac{q^u}{1-q^u}\,\frac{q^v}{1-q^v} - \frac{q^{u+v}}{1-q^{u+v}}\right)\\
&\quad= g(X)g(Y) - \sum_{w>0} \frac{e^{(1-w)Y} - e^{(1-w)X}}{e^X - e^Y}\,\frac{q^w}{1-q^w}\\
&\quad= g(X)g(Y) + \frac{e^Y}{e^X - e^Y}\,g(Y) - \frac{e^X}{e^X - e^Y}\,g(X)\\
&\quad= g(X)g(Y) - \frac{g(X)+g(Y)}{2} - \coth\Big(\frac{X-Y}{2}\Big)\,\frac{g(X)-g(Y)}{2}\,,\\
&\beta(X,Y) + \beta(Y,X) = \beta(Y)g(X) + \beta(X)g(Y)\\
&\qquad\qquad - \frac{g(X)-g(Y)}{X-Y} + \coth\Big(\frac{X-Y}{2}\Big)\,\frac{g(X)-g(Y)}{2}\\
&\varepsilon(X,Y) + \varepsilon(Y,X) = Xg^*(Y) + Yg^*(X) + g(X) + g(Y) + 2g_2\,,
\end{aligned}$$

and adding up these three equations (the last with a coefficient 1/2) we obtain the first of equations (27) for $\mathfrak{Z}$, $\mathfrak{P}$ and $\mathfrak{z}$ as defined above. Similarly, we have

$$\begin{aligned}
&g(X+Y,Y) + g(X+Y,X)\\
&\qquad= \left(\sum_{v>u>0} + \sum_{u>v>0}\right) e^{-uX-vY}\,\frac{q^u}{1-q^u}\,\frac{q^v}{1-q^v}
\end{aligned}$$

$$= \Big(\sum_{u,\,v>0} - \sum_{u=v>0}\Big) e^{-uX-vY}\,\frac{q^u}{1-q^u}\,\frac{q^v}{1-q^v}$$
$$= g(X)g(Y) - \sum_{u>0} e^{-u(X+Y)}\Big(\frac{q^u}{(1-q^u)^2} - \frac{q^u}{1-q^u}\Big)$$
$$= g(X)g(Y) - (X+Y)g^*(X+Y) - g_2 - g(X+Y)\,,$$
$$\beta(X+Y,Y) + \beta(X+Y,X) = \beta(X)g(Y) + \beta(Y)g(X)\,,$$
$$\varepsilon(X+Y,Y) + \varepsilon(X+Y,X)$$
$$= Xg^*(Y) + Yg^*(X) + 2(X+Y)g^*(X+Y) + 2g(X+Y) + 2g_2\,,$$

and combining these gives the second of equations (27). □

Remarks. 1. It is notable that the power series defined by (58), with the constant term $\beta_r\beta_s$ added, is a modular form of weight $r+s$ for all even r, $s>0$, the correction terms $\delta_{r,2}g_s^*(q)$ and $\delta_{s,2}g_r^*(q)$ being just what is needed to compensate for the non-modularity of $(\beta_r + g_r(q))(\beta_s + g_s(q))$ when r or s is equal to 2.

2. One can also ask whether it is possible to lift Theorem 7 from $\mathbb{Q}[[q]]^0$ to all of $\mathbb{Q}[[q]]$ by adding a suitable constant term to $Z_{r,s}(q)$ in such a way that the relations (20) still hold when we add $\beta_r\beta_s$ to $P_{r,s}(q)$ in order to make it modular for r and s even. This is equivalent to finding a realization of $\mathcal{D}_k$ in $\mathbb{Q}$ with $P_{r,s} = \beta_r\beta_s$ for r and s even. One such realization is provided by the "Bernoulli realization" given in Section 6, but there may be other ones which are more naturally related to the combinatorial double Eisenstein series $Z_{r,s}(q)$.

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TYPE NUMBERS AND LINEAR RELATIONS OF THETA SERIES FOR SOME GENERAL ORDERS OF QUATERNION ALGEBRAS

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To the memory of Prof. Tsuneo Arakawa

1. Introduction

We discuss again the relation of the type numbers of certain class of orders of definite quaternion algebras over $\boldsymbol{Q}$ and dimensions of spaces of weight 2 cusp forms, as well as the linear relations of their theta series.

Let $\boldsymbol{B}$ be a definite quaternion algebra over $\boldsymbol{Q}$, and $q_0 = q_1 \dots q_t$ be the product of primes at which $\boldsymbol{B}$ ramifies. One can then define, for each positive integer N such that $(q_0, N) = 1$, and a product $q = q_1^{2e_1+1} \dots q_t^{2e_t+1}$ of odd powers of $q_1, \dots, q_t$ a class of orders $\mathcal{O}$ of level qN (see Section 1 for precise definition). It is called an *Eichler order* (resp. *split order*) if qN is square free (resp. $q = q_0$). The number $H(q, N)$ of left or right ideal classes of $\mathcal{O}$ is determined by (q, N) and is called the class number of $\mathcal{O}$. This is well understood, since it is a basic invariant in the correspondence of Eichler and Jaquet-Langlands which plays a central role in the basis problem for quaternary theta series (c.f. [Ei2], [HS], [Pi4], [Pi5]).

On the other hand, the number $T(q, N)$ of isomorphism classes, called *type number*, of orders of level (q, N) has been paid less attention. The only exception was the special case $T(q, 1)$ with q a prime, which plays a role in the arithmetic of super singular elliptic curves over fields of characteristic q.

In late 80's a paper of B.Gross [Gr], followed by a series of papers (e.g. [BS1], [BS2], [BS3]) of S.Böcherer and R.Schulze-Pillot, developed new aspects of the arithmetic of quaternion algebras including the significance of the type numbers. They showed, among others, that the dimension of the space spanned by the theta series of ternary lattices $T_2(\mathcal{O}) := (\boldsymbol{Z} + 2\mathcal{O}) \cap \{\mathrm{Tr}(x) = 0\}$, attached to the *Eichler orders* $\mathcal{O}$, is equal to the number of Hecke eigen forms f in a subspace of weight 2 cusp forms $S_2(q_0 N)$ such that $L(f, 1) \neq 0$.

Inspired by [Gr], the author began a computational research on the arithmetic of quaternion algebras. Especially, he tried to enumerate in a systematic way the set of representatives of isomorphism classes of orders of given level (q_0, N), and compute their theta seris. He then found numerically that the above connection is more direct; namely whenever we have a linear relation among the ternary theta series for $T_2(\mathcal{O}_j)$, $(1 \leqslant j \leqslant T(q_0, N))$, we always have the same linear relation among the quaternary theta series for $\mathcal{O}_j$. This fact, stated as a conjecture in [Ha2], was proved by Arakawa and Böcherer in [AB].

Now this note is an extension of the work [Ha2]. We extend the relation between the type numbers and certain spaces of weight 2 cusp forms, and study the same problem as above for the class of general orders of level (q, N). In contrast with the case of Eichler orders, the situation becomes very much more complicated, partly due to the appearance of old forms. Correspondingly, we observe that there are abundance of linear relations among the ternary theta series for $\mathcal{O}_j$ $(1 \leqslant j \leqslant T(q_0, N))$ when N is not square free. After the numerical computations which cover all possible values of (q, N) with $qN < 10^4$, however, we are able to formulate some conjectures, which include the relation between the dimension of the space spanned by the theta series of $T_2(\mathcal{O})$ and the number of certain Hecke eigen forms $f \in S_2(qN)$ of weight 2 such that $L(f, 1) \neq 0$. The calculation has been done by PC using UBASIC.

2. Type numbers of split and non-split orders

Let $\boldsymbol{B}$ be a definite quaternion algebra over $\boldsymbol{Q}$, and let $q_0 = q_1 \dots q_t$ be the product of primes at which $\boldsymbol{B}$ ramifies, so that t is a positive odd integer. Let $q = q_1^{2e_1+1} \dots q_t^{2e_t+1}$ be a product of odd powers of $q_1 \dots q_t$ and let N be a positive integer prime to q. Then one can define orders of level (q, N) which are called *Eichler order* (resp. *split order*) if qN is square free (resp. $q = q_0$), and *non-split order* in the remaining case. They are orders $\mathcal{O}$ of $\boldsymbol{B}$ satisfying the following local condition at each prime p:

(1) For $p = q_i$, $\mathcal{O}_p := \mathcal{O} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ is isomorphic to $O_p + p^{e_i} u O_p$, where O_p

is the ring of integers in the unramified quadratic extension L_p of $\boldsymbol{Q}_p$ and we write $\boldsymbol{B}_p = L_p + uL_p$ with $u^2 = p,\ u\alpha = \overline{\alpha}u,\ \forall \alpha \in L_p$.

(2) For $p \neq q_i$, $\mathcal{O}_p := \mathcal{O} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ is isomorphic to $R_p(m) := \begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p^m\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix} = \begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ N\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix}$, where we have $\boldsymbol{B}_p := \boldsymbol{B} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p \cong \mathrm{M}_2(\boldsymbol{Q}_p)$ and $p^m || N$.

These are the most general class of orders treated in the literature (cf. [Pi4], [Pi5]) in order to establish the representability of modular forms by quaternary theta series.

We denote by $T(q,N)$ the number of isomorphism classes and call it the *type number* of orders of level (q,N), It also depends only on (q,N) and in fact is an important family of arithmetic invariants of $\boldsymbol{B}$. In 1941, Deuring studied it in his theory on super singular elliptic curves and found the following remarkable relation with the dimension of a space of cusp forms, in the case q is a prime and $N = 1$.

$$T(q,1) = 1 + \dim S_2(q)^{(-)}. \tag{1}$$

where $(-)$ indicates the (-1)-eigen space of the Fricke involution. Although an explixit formula for $T(q,N)$ in the case q is square-free (split order) has been given by Pizer [Pi1], [Pi3], the significance of this result seems to have been overlooked. It is in fact an interesting question to find an analogous relation with modular forms as (1). This was found by the author in [Ha1], and generalized in [HH] for split orders of level (q_0,N):

$$T(q_0,N) = 1 + \dim S_2(q;N)^{(-,+)}, \tag{2}$$

where $S_2(q;N)^{(-,+)}$ is the subspace of $S_2(qN)$ such that

$$S_2(q_0;N) \cong \bigoplus_{m|N} \bigoplus_{d|\frac{N}{m}} S_2^0(q_0 m)^{[d]} \qquad \subseteq S_2(q_0 N).$$

and $(-,+)$ indicates the (-1) (resp. $(+1)$) -eigen space of the Atkin-Lehner involution W_p for $p|q_0$ (resp. $p|N$). As usual $S_2^0(M)$ denotes the space of new forms in $S_2(M)$, and $S_2^0(M)^{[d]}$ is the space spanned by $f(d\tau)$ for $f(\tau) \in S_2^0(M)$.

Writing $N = {p_1}^{r_1}\ldots{p_s}^{r_s}$, we can derive the following expression from (2):

$$T(q_0,N) = 1 + \sum_{k_1=0}^{r_1} \cdots \sum_{k_s=0}^{r_s} \sum_{\varepsilon_1,\ldots,\varepsilon_s \in\{0,1\}} \left[\frac{k_1+2-\varepsilon_1}{2}\right] \cdots \left[\frac{k_s+2-\varepsilon_s}{2}\right]$$
$$\times \dim S_2^0(q_0 {p_1}^{r_1-k_1}\ldots{p_s}^{r_s-k_s})^{(-,(-1)^{\varepsilon_1},\ldots,(-1)^{\varepsilon_s})}. \tag{3}$$

In particular, for $N = p^r$,

$$T(q_0, p^r) = 1 + \sum_{k=0}^{r} \left[\frac{k+2}{2}\right] \dim S_2^0(q_0 p^{r-k})^{(-,+)} + \sum_{k=0}^{r-1} \left[\frac{k+1}{2}\right] \dim S_2^0(q_0 p^{r-k})^{(-,-)}.$$

$$T(q_0, p^2) = 1 + \dim S_2^0(q_0 p^2)^{(-,+)} + \dim S_2^0(q_0 p)^{(-,+)} + \dim S_2^0(q_0 p)^{(-,-)} + 2 \dim S_2^0(q_0)^{(-)}.$$

$$T(q_0, p^3) = 1 + \dim S_2^0(q_0 p^3)^{(-,+)} + \dim S_2^0(q_0 p^2)^{(-,-)} + \dim S_2^0(q_0 p^2)^{(-,+)} + \dim S_2^0(q_0 p)^{(-,-)} + 2 \dim S_2^0(q_0 p)^{(-,+)} + 2 \dim S_2^0(q_0)^{(-)}.$$

Now our first result in this note is the following expression of $T(q, N)$ which extends (3).

Theorem 2.1. *For $q = q_1^{2e_1+1} \dots q_t^{2e_t+1}$ and $N = p_1{}^{r_1} \dots p_s{}^{r_s}$, $(q, N) = 1$, we have the following equality*

$$T(q, N) = 1 + \sum_{q^* | q} \sum_{k_1=0}^{r_1} \cdots \sum_{k_s=0}^{r_s} \sum_{\varepsilon_1, \dots, \varepsilon_s \in \{0,1\}} \left[\frac{k_1 + 2 - \varepsilon_1}{2}\right] \cdots \left[\frac{k_s + 2 - \varepsilon_s}{2}\right] \times \dim S_2^0(q^* p_1{}^{r_1 - k_1} \dots p_s{}^{r_s - k_s})^{(-,(-1)^{\varepsilon_1}, \dots, (-1)^{\varepsilon_s})}. \tag{4}$$

where q^* is a product of odd powers of $q_1, \dots, q_t$.

We shall not give the proof of this result here, since it is proved by some long and tedious computations. The deteail will appear elsewhere.

If $q = q_1{}^{2e+1}$ and $N = p^r$ are both prime powers, we have

$$T(q_1{}^{2e+1}, p^r) = 1 + \sum_{h=0}^{e} \sum_{k=0}^{r} \left[\frac{k+2}{2}\right] \dim S_2^0(q_1{}^{2h+1} p^{r-k})^{(-,+)} + \sum_{h=0}^{e} \sum_{k=0}^{r-1} \left[\frac{k+1}{2}\right] \dim S_2^0(q_1{}^{2h+1} p^{r-k})^{(-,-)}. \tag{5}$$

3. Construction of orders of level (q, N)

We choose, to each (q, N) as in Theorem 2.1, the parameter (p, s) of our family which varies according to the following conditions:

(1) p is a positive integer prime to q such that $p \equiv 3 \pmod 4$, with

$$\begin{cases} (i_a) & \left(\dfrac{-p}{q_i}\right) = -1 \text{ for } 1 \leqslant i \leqslant r,\ q_i \neq 2, \\ (i_b) & \left(\dfrac{-p}{\ell}\right) = +1 \text{ for any prime divisor } \ell \mid N, \ell \neq 2, \\ (ii_a) & p \equiv 3 \pmod 8 \text{ if } q_i = 2 \text{ for some i}, \\ (ii_b) & p \equiv 7 \pmod 8 \text{ if } 2 \mid N. \end{cases} \tag{6}$$

(2) s is a positive divisor of $\frac{p+1}{4}$, prime to q.
(3) There exists $a \in \boldsymbol{Z}$ such that $a^2 qN + s \equiv 0 \pmod p$.

If (p, s) satisfies the above conditions, $\boldsymbol{B}$ is expressed as

$$\boldsymbol{B} = \boldsymbol{Q} + \boldsymbol{Q}i + \boldsymbol{Q}j + \boldsymbol{Q}ij, \quad \text{with} \quad i^2 = -sqN,\ j^2 = -p,\ ij = -ji.$$

Proposition 3.1. *The following set $\mathcal{O}(q, N; p, s)$ is an order of level (q, N).*

$$\mathcal{O}(q, N; p, s) := \boldsymbol{Z}e_1 + \boldsymbol{Z}e_2 + \boldsymbol{Z}e_3 + \boldsymbol{Z}e_4,$$
$$e_1 = 1, \quad e_2 = \frac{1+j}{2}, \quad e_3 = \frac{i+ij}{2s}, \quad e_4 = \frac{aqNj + ij}{p}.$$

Proof. We give a sketch of the proof, since it is the same as the proof of Proposition 3.1, [Ha2]. That the above $\mathcal{O} = \mathcal{O}(q, N; p, s)$ forms a $\boldsymbol{Z}$-order is proved by expressing the products $e_h e_k$ as $\boldsymbol{Z}$-linear combinations. Then we have

$$\det(\mathrm{Tr}(e_h, \bar{e}_k)) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{p+1}{2} & 0 & aqN \\ 0 & 0 & \frac{(p+1)qN}{2} & qN \\ 0 & aqN & qN & \frac{2qN(a^2qN+s)}{p} \end{pmatrix} = (qN)^2.$$

Next we note that $\mathcal{O}$ contains a subring $\boldsymbol{Z}[e_2] \cong \boldsymbol{Z}[\frac{1+\sqrt{-p}}{2}]$ which splits, by ℓ-adic completion, as

$$\boldsymbol{Z}_\ell[e_2] \cong \boldsymbol{Z}_\ell[\frac{1+\sqrt{-p}}{2}] = \boldsymbol{Z}_\ell \oplus \boldsymbol{Z}_\ell$$

for any prime divisor ℓ of N. Then for the prime $\ell | N$ we apply the following lemma of Hijikata, from which it follows that $\mathcal{O}_\ell$ is conjugate in $\boldsymbol{B}_\ell \cong \mathrm{GL}_2(\boldsymbol{Q}_\ell)$ to a split order. Namely we have

$$\mathcal{O}_\ell \cong R_0(\ell^n)_\ell := \begin{pmatrix} \boldsymbol{Z}_\ell & \boldsymbol{Z}_\ell \\ \ell^n \boldsymbol{Z}_\ell & \boldsymbol{Z}_\ell \end{pmatrix} \quad (n \in \boldsymbol{Z}, n \geqslant 0).$$

Lemma 3.1. (Hijikata) *For a Z_ℓ-order $\mathcal{O}_\ell \subset \mathrm{M}_2(Q_\ell)$, the following conditions are equivalent:*

(1) $\mathcal{O}_\ell \cong R_0(\ell^n)_\ell := \begin{pmatrix} Z_\ell & Z_\ell \\ \ell^n Z_\ell & Z_\ell \end{pmatrix} \quad (n \in Z, n \geqslant 0)$.

(2) $\mathcal{O}_\ell = \mathcal{O}_{1,\ell} \cap \mathcal{O}_{2,\ell}$, *where* $\mathcal{O}_{1,\ell}$, $\mathcal{O}_{2,\ell}$ *are maximal orders of* $\mathrm{M}_2(Q_\ell)$.

(3) $Z_\ell \oplus Z_\ell \subset \mathcal{O}_\ell$.

Taking the standard Z_ℓ-basis of $R_0(\ell^n)_\ell$, we immediately have

$$\det(\mathrm{Tr}(e'_h, \bar{e}'_k)) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\ell^n & 0 \\ 0 & -\ell^n & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \ell^{2n}.$$

Comparing the above calculations, we obtain $\ell^n || N$.

As for the prime $\ell | q$ we apply

Lemma 3.2. (Pizer) *Let L_ℓ be the unramified quadratic extension of Q_ℓ and O_ℓ be the ring of integers of L_ℓ. For a Z_ℓ-order $\mathcal{O}_\ell$ of the division quaternion algebra B_ℓ over Q_ℓ, the following conditions are equivalent:*

(1) $\mathcal{O}_\ell$ *is a non-split order of level* ℓ^{2e+1}.

(2) $\mathcal{O}_\ell \otimes O_\ell$ *is a split* O_ℓ*-order of level* ℓ^{2e+1} *in* $\mathrm{M}_2(L_\ell)$.

(3) $O_\ell \subset \mathcal{O}_\ell$.

Indeed by our condition (ii-a),(ii-b) we have

$$Z_\ell[e_2] \cong Z_\ell[\frac{1+\sqrt{-p}}{2}] \cong O_\ell,$$

for any prime divisor ℓ of q. This completes the proof of Proposition 3.1.□

It can be shown that the isomorphism class of $\mathcal{O}(q, N; p, s)$ is independent of the choice of a. From Proposition 3.1 we have the following *simultaneously* parametrized families of quadratic forms:

$$F_Q(q, N; p, s; X) = X_1{}^2 + X_1X_2 + \frac{p+1}{4}X_2{}^2 + \frac{qN(p+1)}{4s}X_3{}^2 + qN(aX_2 + X_3)X_4 + \frac{qN(a^2qN+s)}{p}X_4{}^2 \qquad (7)$$

with reduced discriminant $(qN)^2$, which we call an *order form*. Let

$$\begin{aligned} T_1(\mathcal{O}(q,N;p,s)) &:= \mathcal{O}(q,N;p,s) \cap \{x \in \boldsymbol{B}; x+\bar{x}=0\} \\ &= \boldsymbol{Z}j + \boldsymbol{Z}\frac{i+ij}{2} + \boldsymbol{Z}\frac{(aqNj+ij)}{p}, \end{aligned}$$

be the first ternary lattice attached to $\mathcal{O}(q,N;p,s)$. Then the corresponding ternary form is given explicitly as

$$\begin{aligned} &F_{\mathrm{T}_1}(q,N;p,s;X) \\ &= pX_1^2 + \frac{qN(a^2q+s)}{p}X_2^2 + qN(X_3-2aX_1)X_4 + \frac{qN(p+1)}{4s}{X_3}^2, \end{aligned}$$

which has reduced discriminant $2(qN)^2$. Also from the second ternary lattice

$$\begin{aligned} T_2(\mathcal{O}(q,N;p,s)) &:= (\boldsymbol{Z}+2\mathcal{O}(q,N;p,s)) \cap \{x \in \boldsymbol{B}; x+\bar{x}=0\} \\ &= \boldsymbol{Z}j + \boldsymbol{Z}\frac{i+ij}{s} + \boldsymbol{Z}\frac{2(aqNj+ij)}{p}, \end{aligned}$$

we obtain the second ternary form (T_2)

$$\begin{aligned} &F_{\mathrm{T}_2}(q,N;p,s;X) \\ &= p{X_2}^2 + \frac{4qN(a^2qN+s)}{p}{X_4}^2 + 4qN(aX_2+X_3)X_4 + \frac{qN(p+1)}{s}{X_3}^2. \end{aligned}$$

This has reduced discriminant $32(qN)^2$.

Our computations for theta series of level (q,N) are based on the following working hypothesis:

Conjecture 3.1. *Any order $\mathcal{O}$ of level (q,N) is isomorphic to some $\mathcal{O}(q,N;p,s)$.*

4. Theta series

Let $\mathcal{O}_j$ $(1 \leqslant j \leqslant T(q,N))$ be a complete set of representatives of isomorphic classes of orders of level (q,N). We study the linear (in)dependence of the following three kinds of theta series attached to them.

$$\begin{aligned} \vartheta_j^{(Q)}(\tau) &:= \sum_{a \in \mathcal{O}_j} e[\mathrm{Nr}(a)\tau], \\ \vartheta_j^{(T_1)}(\tau) &:= \sum_{a \in T_1(\mathcal{O}_j)} e[\mathrm{Nr}(a)\tau], \\ \vartheta_j^{(T_2)}(\tau) &:= \sum_{a \in T_1(\mathcal{O}_j)} e[\mathrm{Nr}(a)\tau]. \end{aligned}$$

Note that this is not a basis broblem, since the theta series $\vartheta_j^{(Q)}(\tau)$ does not necessarily belong to $S_2^0(qm)^{(-,+)}$. The significance of this problem was first given by B.Gross(1985) [Gr], who showed that a linear relation of $\{\vartheta_j^{(T_2)}(\tau)\}$ implies the existence of an eigen form $f \in S_2^0(q)$ with $L(f,1)=0$. This remarkable result was generalized to for the case of maximal orders ($N=1$) and q is a prime, and generalized by Böcherer, Schulze-Pillot [BS1], [BS2] for arbitrary Eichler orders (i.e., qN is square free). To state their results, put

$$\Theta^{(T_2)}(q,N) := < \vartheta_j^{(T_2)}(\tau) | 1 \leqslant j \leqslant T(q,N) > . \tag{8}$$

Theorem 4.1. (Gross [Gr], Böcherer, Schulze-Pillot [BS1], [BS2]) *Suppose qN is square free, and let $g(\tau) \in S_{3/2}^0(q,N)$ be a new form, and $f(\tau)$ be the normalized new form obtained from g by the Shimura correspondence. Then we have*

$$g \in \Theta^{(T_2)}(q,N) \quad \Leftrightarrow \quad L(f,1) \neq 0,$$

$$L(f,1)g(\tau) = c \sum_{j=1}^{H(q,N)} \frac{< g, \vartheta_j^{(T_2)} >}{e_j} \cdot \vartheta_j^{(T_2)}(\tau) \tag{9}$$

where $e_j = \#(\mathcal{O}_j^\times)$ and $c \neq 0$ depends only on qN.

Here we note that if qN is square free, then q is a product of odd number of distinct primes so that any eigen form $f \in S_2^0(qN)^{(-,+)}$ has root number $+1$, i.e., $L(f,s) = L(f,2-s)$. Hence $L(f,1)=0$ implies $L'(f,1)=0$. Thus when qN is square free, the eigen forms in $S_2^0(qN)$ having root number -1 whose L-functions vanish automatically at $s=1$, do not play any role in our problem. We refer to Kramer [Kr] for the first numerical computation of the linear relation of theta series which was for $q=389, N=1, T(389,1)=22$. In [Ha2], we made a systematic computation for all possible levels (q,N) with $qN < 10^4$, qN=square free. Among 17445 such pairs (q,N) there exist 1646 pairs for which the theta series are *not* linearly independent. The total number of independent linear relations is 2466. Based on this computation we made in [Ha2] the following conjecture:

Conjecture 4.1. *The linear relations for $\vartheta_j^{(Q)}(\tau)$, $\vartheta_j^{(T_2)}(\tau)$ hold simultaneously with the same coefficients. Namely we have*

$$\sum_{j=1}^{T(q,N)} c_j \, \vartheta_j^{(T_2)}(\tau) = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{T(q,N)} c_j \, \vartheta_j^{(Q)}(\tau) = 0. \tag{10}$$

Furthermore, the same linear relation holds for $\vartheta_j^{(T_1)}(\tau)$ if N is not divisible by 4.

Now it is a great pleasure for the author to remark that, the above conjecture 4.1 attracted the attention of T. Arakawa and S. Böcherer, who proved it by using highly sophisticated arguments (see [AB]). However, since the proof given in [AB] seems to depend heavily on the assumption that qN is square-free, one can still ask what happens if this condition is removed.

We remark also that the above conjecture has been checked to be true, by our numerical computations, for all possible (q, N) with $qN < 10^4$ such that qN is *not* square-free (The square free case was treated already in [Ha2]). Indeed we observed that, among 6565 such pairs (q, N), there exist 201835 linear relations all of which satisfy the above conjecture 4.1, *except for the theta series $\vartheta_j^{(T_1)}(\tau)$ for N divisible by* 4. Moreover, the number of linear relations increases rapidly when N is a multiple of high power of a prime. This is not very surprising, since, as the formula (3) suggests, the same eigen form would corresponds to several independent linear relations even if we assume that the theorem of Gross, and Böcherer, Schulze-Pillot remains true. However, from the naive point of view on the arithmetic of integral quadratic forms, abundance of such linear relations in theta series is itself an interesting phenomenon.

We shall make another conjecture based on our computations. First we observe the following equation which is derived from (3). Let q, p be distinct primes. Then we have

$$T(q,p^r) - T(q,p^{r-1}) = \sum_{k=0}^{r} \dim S_2^0(qp^{r-k})^{(-,(-1)^k)}. \tag{11}$$

Conjecture 4.2. *For distinct primes q, p and nonnegative integer r we have*

$$\begin{aligned}&\dim \Theta^{(T_2)}(q,p^r)\\ &= \sum_{k=0}^{r} \left[\frac{k+2}{2}\right] \#\left\{ f \in S_2^0(qp^{r-k})^{(-,+)} \,\middle|\, L(f,1) \neq 0 \right\}\end{aligned} \tag{12}$$

Let $G(q,p^r) = T(q,p^r) - \dim \Theta^{(T_2)}(q,p^r)$ be the number of linear relations in $\{\vartheta_j^{(T_2)}(\tau)\}$. Then the above conjecture 4.2 is equivalent to

$$G(q,p^r) - G(q,p^{r-1}) \\ = \sum_{k=0}^{r} \#\left\{ f \in S_2^0(qp^{r-k})^{(-,(-1)^k)} \,\middle|\, L(f,1) = 0 \right\}. \tag{13}$$

Note that the right hand side is expressed as

$$\sum_{k=0;\ \text{even}}^{r} \#\left\{ f \in S_2^0(qp^{r-k})^{(-,+)} \mid L(f,1) = 0 \right\} \\ + \sum_{k=0;\ \text{odd}}^{r} \dim S_2^0(qp^{r-k})^{(-,-)}.$$

Under Conjecture 4.2, the vanishing of L-functions at $s = 1$ for $f \in S_2^0(qp^n)^{(-,\pm)}$ will now be controlled by the linear relations of theta series of various levels (q,p^r). This can naturally be extended to the arbitrary level (q,N).

Finally we remark that Conjecture 3.1 is confirmed if one finds $T(q,N)$ values of (p_j,s_j,a_j) for which the theta series $\vartheta_j^{(Q)}(\tau)$ are distinct. In fact our numerical computation which covers all possible values of (q,N) supports the following:

Conjecture 4.3. *The theta series of non isomorphic orders are distinct.*

We should note that some examples of non isometric pairs of quaternary lattices having the same theta series are known which, however, are not attached to orders of quaternion algebras. On the other hand, it has been proved by A.Schiemann [Sch] that non isometric ternary lattices have *distinct* theta series. Hence we have

$$\mathcal{O}_i \not\cong \mathcal{O}_j \quad \Rightarrow \quad \vartheta_i^{(T_1)}(\tau) \neq \vartheta_j^{(T_1)}(\tau), \quad \vartheta_i^{(T_2)}(\tau) \neq \vartheta_j^{(T_2)}(\tau)$$

As in [Ha2], it turns out that the above conjectures are all true for each (q,N) with $qN < 10^4$. The actual computations were done by calculating first the representation numbers of positive integers by $F_{\mathrm{T}_2}(q,N;p,s;X)$ for various (p,s,a), to obtain as many nonisomorphic orders as $T(q,N)$, which give a complete set of representatives of orders of level (q,N). Then we check the other conjectures by computing $\vartheta_j^{(T_1)}(\tau)$, $\vartheta_j^{(T_2)}(\tau)$ and $\vartheta_j^{(Q)}(\tau)$.

5. Examples for split orders of high power levels

Here we assume that q, p are distinct primes. We put

$$V(-,-) := \#\left\{ f \in S_2^0(qp^r)^{(-,-)} \,\middle|\, L(f,1)=0 \right\} \quad (= \dim S_2(q,p^r)^{(-,-)}),$$

$$V(-,+) := \#\left\{ f \in S_2^0(qp^r)^{(-,+)} \,\middle|\, L(f,1)=0 \right\},$$

$$\text{diff} := G(q,p^r) - G(q,p^{r-1}),$$

$$d(-,+) := \dim S_2^0(qp^r)^{(-,+)}.$$

$q = 19 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
19	0	2	0	0	0	0	1
38	1	3	0	0	0	0	1
76	2	4	0	0	0	0	0
152	3	6	0	0	0	0	1
304	4	11	0	0	0	2	4
608	5	21	2	2	0	3	6
1216	6	40	5	3	0	7	11
2432	7	78	15	10	1	15	21
4864	8	152	33	18	0	32	40
9728	9	300	77	44	2	66	78

$q = 41 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
41	0	4	0	0	0	0	3
82	1	4	0	0	0	0	0
164	2	7	0	0	0	0	0
328	3	11	0	0	0	3	4
656	4	24	3	3	0	3	7
1312	5	42	6	3	0	9	11
2624	6	88	19	13	1	16	24
5248	7	164	39	20	1	38	42

$q = 53 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
53	0	4	0	0	0	0	3
106	1	5	0	0	0	0	1
212	2	8	0	0	0	1	0
424	3	15	1	1	0	3	5
848	4	29	4	3	0	5	8
1696	5	56	11	7	1	11	15
3392	6	110	25	14	0	23	29
6784	7	216	57	32	2	48	56

$q = 59 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
59	0	6	0	0	0	0	5
118	1	7	0	0	0	0	1
236	2	12	0	0	0	1	0
472	3	16	1	1	0	1	2
944	4	34	3	2	1	7	12
1888	5	61	11	8	0	13	16
3776	6	126	27	16	1	24	34
7552	7	238	59	32	0	55	61

$q = 107 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
107	0	8	0	0	0	0	7
214	1	10	0	0	0	1	2
428	2	18	1	1	0	3	0
856	3	28	4	3	0	4	5
1712	4	58	10	6	1	13	18
3424	5	109	26	16	0	25	28
6848	6	222	58	32	1	48	58

$q = 223 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
223	0	13	0	0	0	0	12
446	1	22	1	1	1	1	9
892	2	35	2	1	0	9	0
1784	3	71	12	10	0	10	18
3568	4	122	24	12	1	24	28
7136	5	246	58	34	0	47	64

$q = 563 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
563	0	33	1	1	1	0	32
1126	1	45	2	1	1	7	12
2252	2	84	10	8	0	19	0
4504	3	146	31	21	1	26	31
9008	4	295	68	37	3	70	84

$q = 1567 \quad N = p^r,\ p = 2$

qN	r	$T(q,p^r)$	$G(q,p^r)$	diff	$V(-,+)$	$V(-,-)$	$d(-,+)$
1567	0	73	3	3	3	0	72
3134	1	117	4	1	1	22	44
6268	2	211	29	25	0	65	0

6. Linear relations for split orders with $T \leqslant 12$

In what follows, we shall give some tables of the results of our computation. Firstly we present, for small levels qN and small type numbers such that $T(q,N) \leqslant 12$ at which we find a linear relation of theta series. The symbol G denotes the number of independent relations. We tabulate j $(1 \leqslant j \leqslant T(q,N))$, and the values of the parameters p_j, s_j, a_j of a complete set of representatives of isomorphic classes of split orders of level (q,N), followed by the coefficients c_j so that we have simultaneously a linear relations

$$\sum_{j=1}^{T(q,N)} c_j \vartheta_j^{(T_2)}(\tau) = \sum_{j=1}^{T(q,N)} c_j \vartheta_j^{(Q)}(\tau) = \sum_{j=1}^{T(q,N)} c_j \vartheta_j^{(T_1)}(\tau) = 0,$$

where the last equation holds under the assumption that N is not divisible by 4.

$q{*}N = 184 = 23{*}2^3$ $T = 10,\ G = 1$

j	p	s	a	c_j
1	31	1	4	1
2	31	2	1	-2
3	39	2	5	-1
4	47	1	23	0
5	47	2	20	2
6	47	3	6	-1
7	71	3	17	0
8	151	1	33	-1
9	239	4	90	1
10	271	2	51	1

$q{*}N = 272 = 17{*}2^4$ $T = 10,\ G = 1$

j	p	s	a	c_j
1	7	1	1	0
2	23	1	11	1
3	23	2	9	-1
4	31	1	3	-1
5	31	2	7	1
6	39	1	1	0
7	63	4	5	-1
8	71	3	35	0
9	79	2	27	0
10	159	4	22	1

$q{*}N = 312 = 13{*}2^3{\cdot}3$ $T = 11,\ G = 1$

j	p	s	a	c_j
1	47	1	6	-1
2	47	2	5	0
3	47	3	22	2
4	71	1	31	1
5	71	3	16	-2
6	119	5	23	0
7	167	2	57	0
8	167	3	75	0
9	239	5	53	0
10	479	4	90	-1
11	791	6	17	1

$q{*}N = 352 = 11{*}2^5$ $T = 11,\ G = 1$

j	p	s	a	c_j
1	15	2	2	0
2	23	1	6	-1
3	23	2	7	1
4	31	1	13	1
5	31	2	11	-1
6	47	2	2	0
7	47	3	10	-1
8	71	2	30	0
9	71	3	1	0
10	223	4	22	1
11	279	5	16	0

$q*N = 396 = 11*2^2 \cdot 3^2 \quad T = 11,\ G = 1$

j	p	s	a	c_j
1	23	1	3	1
2	23	2	8	-1
3	47	1	17	-1
4	47	2	22	1
5	47	3	16	1
6	71	3	8	-1
7	119	5	30	-1
8	191	4	72	0
9	191	6	30	-1
10	383	3	181	1
11	599	6	269	1

$q*N = 440 = 5*2^3 \cdot 11 \quad T = 11,\ G = 1$

j	p	s	a	c_j
1	7	1	1	0
2	63	1	1	0
3	63	4	2	-2
4	127	1	37	1
5	127	2	43	0
6	167	3	41	0
7	167	6	32	0
8	183	1	40	-1
9	263	6	7	2
10	743	6	367	-1
11	1487	12	739	1

$q*N = 296 = 37*2^3 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	15	1	2	1
2	23	1	10	-1
3	23	2	4	1
4	31	1	12	1
5	31	2	3	-1
6	39	1	10	0
7	79	4	4	0
8	103	1	27	-1
9	119	3	11	-1
10	143	3	27	0
11	167	6	74	0
12	239	5	28	1

$q*N = 351 = 13*3^3 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	11	1	1	0
2	47	1	19	1
3	47	2	8	0
4	47	3	7	-1
5	59	3	1	-1
6	71	1	35	-1
7	71	3	14	1
8	119	3	37	1
9	239	5	87	0
10	359	2	179	0
11	383	8	191	-1
12	527	6	69	1

$q*N = 456 = 19*2^3 \cdot 3 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	23	1	11	-1
2	23	2	9	1
3	47	1	15	1
4	47	2	11	-1
5	47	3	8	0
6	119	3	16	-1
7	119	5	25	0
8	191	6	17	1
9	215	6	23	0
10	239	4	37	-1
11	263	6	84	0
12	359	9	127	1

$q*N = 476 = 7*2^2 \cdot 17 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	15	1	2	-1
2	127	2	32	1
3	127	4	4	-1
4	135	1	17	0
5	151	1	16	1
6	151	2	19	-1
7	191	3	57	0
8	239	2	1	0
9	239	4	99	1
10	239	5	19	0
11	767	8	200	-1
12	1327	4	249	1

$q{*}N = 616 = 11{*}2^3{\cdot}7 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	31	1	15	1
2	31	2	4	-1
3	47	1	13	-1
4	47	2	3	1
5	47	3	15	0
6	103	2	1	0
7	111	2	22	0
8	159	5	26	1
9	199	5	33	0
10	335	4	79	-1
11	551	6	74	-1
12	719	4	260	1

$q{*}N = 624 = 3{*}2^4{\cdot}13 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	55	1	9	0
2	79	2	39	2
3	79	4	35	-2
4	103	1	29	1
5	103	2	31	-2
6	199	5	74	0
7	367	1	69	-1
8	367	4	138	2
9	391	7	75	0
10	607	4	57	-1
11	727	7	77	0
12	1951	8	5	1

$q{*}N = 1242 = 2{*}3^3{\cdot}23 \quad T = 12,\ G = 1$

j	p	s	a	c_j
1	11	1	1	0
2	83	1	32	-1
3	83	3	1	1
4	107	1	20	1
5	107	3	39	-1
6	155	3	44	1

$q{*}N = 1242 = 2{*}3^3{\cdot}23 \quad T = 12,\ G = 1$

j	p	s	a	c_j
7	227	3	94	-1
8	251	7	54	0
9	419	5	130	0
10	827	9	289	-1
11	971	9	226	1
12	1019	3	55	0

7. Linear relations for non-split orders with $T \leqslant 20$

$qN = 664 = 8{*}83 \qquad T = 13 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	19	1	1	0	2	35	1	1	−2	3	43	1	3	0
4	91	1	8	2	5	107	3	40	1	6	115	1	36	1
7	139	5	36	−1	8	179	3	56	−1	9	179	5	79	2
10	251	3	62	0	11	283	1	92	−1	12	323	9	22	−2
13	379	5	25	1										

$qN = 864 = 32{*}27 \qquad T = 17 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	3	0	2	35	1	4	−1	3	59	1	24	1
4	59	3	28	−2	5	83	1	12	−1	6	83	3	10	1
7	107	3	21	1	8	131	1	58	0	9	131	3	23	−1
10	179	5	9	0	11	227	1	105	1	12	251	3	77	1
13	251	7	102	−1	14	371	1	87	0	15	587	7	6	1
16	755	9	112	−1	17	899	5	34	1					

$qN = 936 = 8{*}117 \qquad T = 10 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	35	1	2	−1	2	107	3	36	1	3	131	1	18	1
4	131	3	29	−1	5	155	1	37	0	6	179	5	47	−1
7	251	7	33	−1	8	419	3	139	0	9	659	5	148	1
10	659	11	274	1										

$qN = 1000 = 8{*}125 \qquad T = 15 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	1	0	2	19	1	7	0	3	59	1	16	1
4	59	3	1	−1	5	91	1	1	−1	6	99	5	7	0
7	131	3	33	0	8	139	5	67	1	9	179	5	14	−1
10	219	5	14	0	11	251	1	125	0	12	299	3	35	1
13	419	5	79	0	14	419	7	189	−1	15	499	5	178	1

$qN = 1152 = 128{*}9 \qquad T = 20 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	2	0	2	35	3	1	0	3	59	1	14	1
4	59	3	23	−1	5	83	3	18	−1	6	107	1	43	−1
7	107	3	25	1	8	131	3	44	1	9	155	3	24	2
10	179	1	24	0	11	179	5	4	−1	12	227	3	44	−2
13	251	7	111	0	14	347	3	128	1	15	371	3	55	−1
16	419	7	191	−1	17	515	3	171	0	18	587	7	126	1
19	1019	5	179	0	20	1091	13	41	1					

$qN = 1080 = 8{*}135 \qquad T = 12 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	4	0	2	59	1	6	−1	3	59	3	7	1
4	131	1	62	1	5	131	3	2	−1	6	179	3	39	1
7	179	5	71	0	8	251	7	50	−1	9	371	3	73	0
10	419	3	102	−1	11	419	5	202	0	12	1259	15	491	1

$qN = 1080 = 27{*}40 \qquad T = 20 \qquad G = 2$

j	p	s	a	c_j	c'_j	j	p	s	a	c_j	c'_j
1	31	1	5	0	1	2	31	2	9	0	−1
3	79	1	32	0	−1	4	79	2	28	0	1
5	79	4	15	0	0	6	151	2	19	−1	0
7	199	2	87	1	0	8	199	5	52	−2	0
9	271	1	135	0	0	10	271	4	1	−1	0
11	319	5	94	2	0	12	439	10	141	0	−1
13	751	4	252	0	1	14	991	8	475	0	−1
15	1039	10	273	1	0	16	1279	8	480	0	0
17	1351	13	250	−1	0	18	1519	19	482	1	0
19	1759	5	411	0	1	20	2479	10	328	0	0

$qN = 1512 = 8{*}189 \qquad T = 17 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	59	1	13	−1	2	59	3	25	1	3	83	1	40	1
4	83	3	22	−2	5	131	3	37	1	6	227	1	25	0
7	227	3	112	−1	8	251	1	65	0	9	251	3	80	1
10	251	7	31	1	11	419	5	80	0	12	419	7	202	−1
13	467	9	121	−1	14	587	3	72	0	15	1091	13	474	1
16	1979	5	565	−1	17	2099	15	661	1					

$qN = 1656 = 8{*}207 \qquad T = 14 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	3	0	2	83	1	41	1	3	83	3	35	−1
4	107	1	34	0	5	107	3	30	0	6	155	1	22	−1
7	203	3	41	0	8	227	3	80	0	9	251	7	44	1
10	419	5	209	−1	11	419	7	203	0	12	467	3	108	1
13	467	9	182	0	14	779	5	145	0					

$qN = 1800 = 8{*}225 \qquad T = 18 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	5	0	2	59	1	23	−1	3	59	3	17	1
4	131	3	9	0	5	179	1	55	1	6	179	5	39	−1
7	251	3	119	0	8	251	7	112	0	9	299	5	7	1
10	371	3	9	0	11	419	5	102	−1	12	419	7	69	1
13	491	3	3	−1	14	659	5	262	0	15	659	11	263	−1
16	1091	7	139	−1	17	1091	13	251	1	18	1259	5	330	1

$qN = 1992 = 8{*}249 \qquad T = 19 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	35	3	1	0	2	107	1	49	−1	3	107	3	26	2
4	155	3	9	−2	5	179	1	41	1	6	179	3	63	−2
7	179	5	23	0	8	251	1	63	0	9	251	7	115	1
10	299	3	125	2	11	371	3	85	−1	12	467	3	8	−2
13	587	7	244	0	14	755	3	94	0	15	755	7	22	2
16	779	13	89	1	17	827	9	304	−1	18	1499	5	259	0
19	2243	17	923	0										

$qN = 2016 = 32{*}63 \qquad T = 20 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	59	1	17	−1	2	59	3	10	1	3	83	1	11	1
4	83	3	23	−1	5	131	3	48	0	6	227	3	76	1
7	251	3	28	−1	8	251	7	77	0	9	299	1	99	0
10	419	5	97	−1	11	419	7	4	0	12	467	1	77	0
13	587	7	82	−1	14	755	9	204	0	15	899	15	128	1
16	971	3	453	0	17	971	9	187	0	18	1091	13	440	−1
19	1139	5	284	1	20	1427	7	645	1					

$qN = 2080 = 32{*}65 \qquad T = 19 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	131	1	33	−1	2	131	3	56	1	3	139	1	58	1
4	139	5	1	−1	5	179	3	71	−1	6	179	5	65	2
7	251	7	58	0	8	259	5	39	−2	9	339	5	19	0
10	491	3	163	0	11	571	11	36	1	12	659	5	275	0
13	731	3	29	0	14	971	9	293	−1	15	1091	13	442	0
16	1979	5	34	0	17	2219	5	4	−1	18	2259	5	205	1
19	3779	7	1511	1										

$qN = 2088 = 8{*}261 \qquad T = 16 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	35	3	3	0	2	59	1	10	−1	3	59	3	8	1
4	83	1	36	1	5	83	3	30	−1	6	107	3	31	−1
7	179	3	3	1	8	179	5	33	0	9	227	3	11	0
10	299	5	1	−1	11	347	3	120	−1	12	371	3	11	1
13	1019	5	44	0	14	1499	15	232	0	15	1619	9	187	0
16	2507	11	441	1										

$qN = 2232 = 8{*}279 \qquad T = 20 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	1	0	2	83	1	28	1	3	83	3	32	−1
4	179	3	46	0	5	179	5	31	−1	6	203	1	1	−1
7	251	3	84	1	8	251	7	20	1	9	275	3	39	2
10	347	1	66	0	11	347	3	24	−2	12	491	3	200	−1
13	587	7	157	−1	14	611	9	21	−1	15	827	3	405	1
16	1019	3	113	0	17	1091	7	257	1	18	2099	5	675	0
19	2123	9	327	0	20	4091	31	25	1					

$qN = 2520 = 8{*}315 \qquad T = 18 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	59	1	19	−1	2	59	3	27	1	3	131	1	13	1
4	131	3	30	0	5	251	3	122	−1	6	251	7	41	1
7	299	5	55	0	8	419	5	169	−1	9	419	7	102	0
10	899	3	231	0	11	899	15	40	−1	12	1259	9	25	−1
13	1259	15	534	0	14	1931	21	125	1	15	2771	9	708	1
16	2939	7	1190	−1	17	4091	31	1244	1	18	6299	35	2820	0

$qN = 3960 = 440{*}9 \qquad T = 14 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	203	1	26	−1	2	323	1	84	1	3	323	9	14	0
4	443	3	148	−1	5	467	3	52	1	6	587	1	195	0
7	587	7	45	0	8	683	9	201	−1	9	1763	7	175	1
10	1787	3	794	0	11	1907	9	496	−1	12	2267	21	704	1
13	2963	13	1467	−1	14	7307	9	3077	1					

$qN = 5304 = 1768*3 \qquad T = 20 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	11	1	4	0	2	227	1	19	0	3	227	3	42	−1
4	275	1	59	1	5	539	1	51	0	6	539	9	43	0
7	635	1	31	−1	8	683	3	130	0	9	683	9	29	1
10	827	9	117	−1	11	1331	9	155	0	12	1451	3	415	1
13	1763	7	151	−1	14	1931	7	865	0	15	2411	9	110	−1
16	2459	15	11	0	17	2579	15	110	0	18	3947	21	719	0
19	4019	15	964	1	20	6131	7	2095	1					

$qN = 6120 = 680*9 \qquad T = 20 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	107	1	22	−1	2	107	3	32	1	3	227	1	76	1
4	227	3	59	−1	5	683	3	228	1	6	683	9	240	−1
7	827	9	170	0	8	923	1	88	0	9	923	3	92	0
10	1187	9	41	−1	11	1187	11	93	0	12	1763	7	51	−1
13	2003	3	922	−1	14	2267	9	733	1	15	2747	3	305	0
16	2963	19	164	0	17	3947	21	285	1	18	7643	13	72	−1
19	9323	7	2277	1	20	14507	9	8	1					

$qN = 6600 = 264*25 \qquad T = 18 \qquad G = 1$

j	p	s	a	c_j	j	p	s	a	c_j	j	p	s	a	c_j
1	91	1	6	−1	2	379	1	59	0	3	379	5	27	0
4	619	1	161	1	5	619	5	91	−2	6	1291	17	268	−1
7	1699	1	364	0	8	1699	17	716	1	9	2491	7	651	−1
10	3019	5	1078	1	11	3331	7	405	0	12	3499	7	1678	1
13	4339	7	567	0	14	6499	25	637	−1	15	7219	19	1690	1
16	11251	29	4425	1	17	11971	41	4742	−1	18	14611	13	3151	1

8. Table of $T(q, N)$ with $qN < 1000$, $G > 0$: split orders

Here we shall give a table of split levels (q, N), and $T(q, N)$ with $qN < 1000$, for which the number G of (independent) linear relations is positive.

qN	q	N	T	G	qN	q	N	T	G	qN	q	N	T	G
184	23	8	10	1	248	31	8	13	1	272	17	16	10	1
284	71	4	16	1	296	37	8	12	1	312	13	24	11	1
316	79	4	15	1	344	43	8	13	1	351	13	27	12	1
352	11	32	11	1	368	23	16	16	2	369	41	9	14	1
376	47	8	19	2	380	19	20	13	1	396	11	36	11	1
416	13	32	16	1	423	47	9	17	1	424	53	8	15	1
428	107	4	18	1	436	109	4	15	1	440	5	88	11	1
452	113	4	16	1	456	19	24	12	1	459	17	27	17	1
464	29	16	17	2	472	59	8	16	1	476	7	68	12	1
488	61	8	18	1	496	31	16	20	2	504	7	72	14	1

qN	q	N	T	G	qN	q	N	T	G	qN	q	N	T	G
508	127	4	21	1	513	19	27	15	1	524	131	4	24	1
531	59	9	21	2	536	67	8	20	2	540	5	108	14	1
544	17	32	18	2	548	137	4	19	1	552	23	24	16	1
556	139	4	22	1	564	47	12	18	1	567	7	81	18	1
568	71	8	26	4	572	13	44	16	1	584	73	8	22	2
592	37	16	19	2	594	11	54	15	1	596	149	4	22	1
603	67	9	19	1	604	151	4	26	2	608	19	32	21	2
612	17	36	14	1	616	11	56	12	1	620	31	20	18	1
621	23	27	22	1	624	3	208	12	1	624	13	48	15	1
632	79	8	27	4	636	53	12	20	1	637	13	49	16	1
639	71	9	25	2	640	5	128	20	1	644	23	28	13	1
650	13	50	15	1	652	163	4	22	1	656	41	16	24	3
657	73	9	20	1	664	83	8	23	2	668	167	4	32	2
672	7	96	16	1	680	17	40	15	1	684	19	36	17	1
688	43	16	23	2	692	173	4	25	1	696	3	232	13	1
696	29	24	19	1	702	13	54	19	1	704	11	64	24	3
708	59	12	20	1	712	89	8	24	1	716	179	4	30	2
720	5	144	14	1	724	181	4	25	1	725	29	25	22	2
728	7	104	13	1	728	13	56	16	1	732	61	12	19	1
736	23	32	28	4	738	41	18	18	1	744	3	248	13	1
744	31	24	19	2	747	83	9	27	2	752	47	16	31	4
756	7	108	15	1	760	5	152	17	2	760	19	40	18	2
764	191	4	37	3	768	3	256	20	1	772	193	4	25	2
774	43	18	20	1	775	31	25	24	1	776	97	8	29	3
783	29	27	28	2	788	197	4	27	2	792	11	72	16	2
796	199	4	34	2	801	89	9	28	3	808	101	8	27	1
812	29	28	17	1	816	17	48	20	1	820	41	20	18	1
824	103	8	36	4	828	23	36	23	2	832	13	64	26	2
836	19	44	19	1	837	31	27	24	2	844	211	4	31	3
846	47	18	25	1	848	53	16	29	4	850	17	50	18	1
852	71	12	26	1	855	19	45	21	1	856	107	8	28	4
860	5	172	20	1	860	43	20	23	1	868	7	124	13	1
872	109	8	30	4	873	97	9	26	2	875	7	125	24	1
876	73	12	21	1	880	5	176	15	1	880	11	80	21	1
884	17	52	17	1	888	3	296	15	1	888	37	24	27	3
891	11	81	28	2	892	223	4	35	2	896	7	128	31	2
904	113	8	30	4	908	227	4	36	2	909	101	9	32	2
912	19	48	19	2	916	229	4	31	1	918	17	54	23	2
920	5	184	18	1	920	23	40	20	2	924	7	132	15	1
925	37	25	24	1	927	103	9	31	1	928	29	32	30	4
931	19	49	26	2	932	233	4	32	3	932	233	4	32	3
936	13	72	22	2	940	5	188	19	1	940	47	20	19	1
944	59	16	34	3	945	7	135	19	1	948	79	12	21	1
952	7	136	20	3	952	17	56	25	2	956	239	4	45	4
960	5	192	21	1	963	107	9	33	2	964	241	4	33	2
976	61	16	33	4	981	109	9	30	1	984	3	328	13	1
984	41	24	22	1	988	13	76	20	1	992	31	32	38	5
996	83	12	27	2	999	37	27	33	4					

9. Table of $T(q, N)$ with $qN < 5000$, $G > 0$: non-split orders

Here we shall give a table of the levels (q, N) of non-split orders with $qN < 5000$, for which the number G of (independent) linear relations is positive.

qN	q	N	T	G	qN	q	N	T	G	qN	q	N	T	G
664	8	83	13	1	864	27	32	21	2	864	32	27	17	1
936	8	117	10	1	972	243	4	25	1	1000	8	125	15	1
1000	125	8	28	2	1080	8	135	12	1	1080	27	40	20	2
1125	125	9	30	2	1152	128	9	20	1	1323	27	49	26	1
1372	343	4	44	3	1404	27	52	24	2	1431	27	53	28	1
1500	125	12	33	2	1512	8	189	17	1	1512	27	56	22	1
1656	8	207	14	1	1664	128	13	25	1	1688	8	211	24	1
1696	32	53	21	1	1701	243	7	35	1	1728	27	64	40	5
1752	8	219	21	1	1800	8	225	18	1	1836	27	68	25	2
1888	32	59	27	1	1944	8	243	35	3	1944	243	8	45	6
1952	32	61	26	1	1992	8	249	19	1	2000	125	16	55	8
2016	32	63	20	1	2052	27	76	27	2	2056	8	257	26	1
2080	32	65	19	1	2088	8	261	16	1	2160	27	80	32	4
2232	8	279	20	1	2250	125	18	43	2	2264	8	283	35	2
2312	8	289	30	1	2376	8	297	24	1	2376	27	88	32	3
2400	32	75	27	1	2484	27	92	30	2	2520	8	315	18	1
2528	32	79	31	1	2592	32	81	40	4	2600	8	325	22	2
2646	27	98	38	2	2656	32	83	37	1	2664	8	333	24	2
2688	128	21	30	1	2700	27	100	39	3	2728	8	341	21	1
2744	8	343	40	2	2744	343	8	88	15	2776	8	347	37	2
2808	8	351	25	3	2808	27	104	39	6	2848	32	89	35	1
2862	27	106	42	1	2952	8	369	24	1	3000	8	375	28	1
3000	125	24	58	8	3024	27	112	42	4	3032	8	379	41	1
3087	343	9	81	6	3096	8	387	30	2	3112	8	389	36	1
3132	27	116	43	3	3168	32	99	30	1	3200	128	25	46	2
3240	8	405	30	3	3267	27	121	57	2	3320	8	415	26	1
3336	8	417	30	1	3348	27	124	42	3	3375	27	125	64	5
3375	125	27	86	12	3384	8	423	28	2	3400	8	425	27	2
3402	243	14	50	1	3424	32	107	43	1	3448	8	431	38	1
3456	27	128	78	14	3456	128	27	57	8	3500	125	28	61	4
3528	8	441	32	3	3560	8	445	26	1	3672	8	459	32	4
3672	27	136	43	7	3712	128	29	42	1	3744	32	117	34	4
3752	8	469	25	1	3780	27	140	30	2	3800	8	475	34	2
3816	8	477	30	2	3834	27	142	47	1	3872	32	121	50	2
3888	243	16	86	14	3960	8	495	24	1	3960	440	9	14	1
3996	27	148	57	5	4000	32	125	55	5	4000	125	32	106	20
4024	8	503	47	1	4064	32	127	51	1	4104	8	513	36	2

qN	q	N	T	G	qN	q	N	T	G	qN	q	N	T	G
4104	27	152	48	7	4116	343	12	77	6	4185	27	155	42	1
4200	8	525	27	1	4232	8	529	52	1	4248	8	531	40	2
4312	8	539	36	1	4320	27	160	64	11	4320	32	135	42	5
4392	8	549	36	4	4428	27	164	52	5	4500	125	36	80	12
4504	8	563	62	2	4536	8	567	40	4	4563	27	169	77	2
4600	8	575	33	1	4608	512	9	72	7	4617	243	19	76	1
4632	4632	1	23	1	4644	27	172	57	5	4648	8	581	36	1
4680	8	585	26	1	4704	32	147	47	1	4725	27	175	54	4
4752	27	176	57	8	4824	8	603	44	3	4840	8	605	36	1
4860	243	20	70	6	4860	243	20	70	6	4896	32	153	42	3
4920	8	615	26	1	4960	32	155	36	1	4968	8	621	39	7
4968	27	184	58	8	4992	128	39	46	1	5000	8	625	70	7

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SKEW-HOLOMORPHIC JACOBI FORMS OF HIGHER DEGREE

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Dedicated to the memory of Prof. Tsuneo Arakawa

The holomorphic Jacobi forms of higher degree have similar properties like Siegel modular forms. On the other hand, the skew-holomorphic Jacobi forms are not holomorphic functions but vanish under a certain differential operator, and have transformation formula like holomorphic Jacobi forms. The purpose of this exposition is to show some properties of skew-holomorphic Jacobi forms of higher degree like holomorphic Jacobi forms.

1. Introduction

The notion of *skew-holomorphic* Jacobi forms was first introduced by Skoruppa [9] in the case of degree 1, and generalized for higher degree by Arakawa [1]. Skew-holomorphic Jacobi forms are not holomorphic, but still satisfy transformation formula like *holomorphic* Jacobi forms, are annihilated by a certain differential operator, the so called Heat operator. Because of this fact and because of the transformation formula, skew-holomorphic Jacobi forms have similar properties as holomorphic Jacobi forms. Moreover, it is known that the space of holomorphic Jacobi forms of index 1 and the space of skew-holomorphic Jacobi forms of index 1 are linearly isomorphic to a certain subspace of Siegel modular forms of half-integral weight. Hence, the theory of skew-holomorphic Jacobi forms is also for the study of the Siegel modular forms of *half-integral* weight.

The purpose of this exposition is to show some properties of skew-holomorphic Jacobi forms of higher degree: an isomorphism between the space of skew-holomorphic Jacobi forms and a certain subspace of Siegel modular forms of half-integral weight, the (analytic) Siegel formula, Klingen type Eisenstein series, and a structure theorem for the space of skew-holomorphic Jacobi forms and for the plus space. In particular, the Siegel formula for skew-holomorphic Jacobi is already shown in Arakawa's paper [1].

2. Holomorphic Jacobi forms and skew-holomorphic Jacobi forms of higher degree

In this section we recall the definition of holomorphic and skew-holomorphic Jacobi forms of higher degree.

We denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n, and by $Sp_n(\mathbb{R})$ the symplectic group of size $2n$. Let l be a natural number and let $G^J_{n,l}$ be the subgroup of $Sp_{n+l}(\mathbb{R})$ consisting of all elements of the form

$$(g,[(\lambda,\mu),\kappa]) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_l & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_l \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1_l & {}^t\mu & \kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1_l \end{pmatrix},$$

where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$, $\lambda,\mu \in M_{n,l}(\mathbb{R})$, and $\kappa \in M_l(\mathbb{R})$ satisfy $\kappa + {}^t\mu\lambda \in \mathrm{Sym}(n,\mathbb{R})$. We put $\Gamma^J_{n,l} := G^J_{n,l} \cap Sp_{n+l}(\mathbb{Z})$.

Let Sym^*_l be the set of all half-integral symmetric matrices of size l, i.e.

$$Sym^*_l := \{(a_{i,j}) \in M_l(\mathbb{Q}) \mid 2a_{i,j} = 2a_{j,i} \in \mathbb{Z}\ ,\ a_{i,i} \in \mathbb{Z} \text{ for all } i \text{ and } j\}.$$

Fix an $S \in Sym^*_l$ be a half-integral symmetric matrix, and assume $S > 0$. We use the symbol $e(x)$ ($x \in M_n(\mathbb{C})$) as an abbreviation for $\exp(2\pi\sqrt{-1}\,\mathrm{tr}(x))$.

We define the holomorphic Jacobi forms as follows.

Definition 1. Let $\phi(\tau,z)$ be a holomorphic function on $\mathfrak{D}_{n,l} := \mathfrak{H}_n \times M_{n,l}(\mathbb{C})$. Set $F(\tau,z) := \phi(\tau,z)\,e(S\tau')$, where $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{n+l}$, $\tau \in \mathfrak{H}_n$, and $\tau' \in \mathfrak{H}_l$. We say ϕ is *a holomorphic Jacobi form* of weight k, of index S and of degree n, if F satisfies the following two conditions.

(1) $F|_k\gamma = F$ for any $\gamma \in \Gamma^J_{n,l}$, where $|_k$ is the usual slash operator, i.e. $(F|_k\gamma)(Z) := \det(CZ+D)^{-k}F(\gamma\cdot Z)$, and where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^J_n$, and $\gamma\cdot Z := (AZ+B)(CZ+D)^{-1}$.
(2) The function F has Fourier expansion of the form

$$F(Z) = \sum_{N=\left(\begin{smallmatrix} M & \frac{1}{2}r \\ \frac{1}{2}{}^tr & S \end{smallmatrix}\right) \geqslant 0} A(N)\,e(NZ),$$

where M and r run over all elements of Sym^*_n and $M_{l,n}(\mathbb{Z})$ respectively.

Moreover we say ϕ is *a holomorphic Jacobi cusp form*, if the Fourier coefficients satisfy $A(N) = 0$ unless $N > 0$.

We denote by $J_{k,S}^{-(n)}$ (resp. $J_{k,S}^{-cusp(n)}$) the space of holomorphic Jacobi form (resp. holomorphic Jacobi cusp form) of weight k, index S and degree n.

Next, we define skew-holomorphic Jacobi forms as follows.

Definition 2. Let $\phi(\tau, z)$ be a real analytic function on $\mathfrak{H}_n \times M_{n,l}(\mathbb{C})$. Set $F(\tau, z) := \phi(\tau, z)\, e(S\tau')$, where $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{n+l}$, $\tau \in \mathfrak{H}_n$, and $\tau' \in \mathfrak{H}_l$. We say ϕ is *a skew-holomorphic Jacobi form* of weight k, of index S and of degree n, if ϕ satisfies the following two conditions.

(i) $F|_k^{sk}\gamma = F$ for any $\gamma \in \Gamma_{n,l}^J$, where we define the slash operator $|_k^{sk}$ as follows, $(F|_k\gamma)(Z) := |\det(CZ+D)|^{-l}\overline{\det(CZ+D)}^{l-k} F(\gamma \cdot Z)$.

(ii) The function ϕ satisfies the cusp condition, namely, ϕ has the Fourier expansion of the form ;

$$\phi(\tau, z) = \sum_{\substack{M \in Sym_n^*, r \in M_{l,n}(\mathbb{Z}) \\ 4M - {}^t r S^{-1} r \leqslant 0}} C(M, r)\, e(M\tau - \frac{1}{2} i(4M - {}^t r S^{-1} r) + r^t z),$$

where M and r run over all element of Sym_n^* and $M_{l,n}(\mathbb{Z})$, respectively.

Moreover we say ϕ is *a skew-holomorphic Jacobi cusp form*, if the Fourier coefficients satisfy $C(M, r) = 0$ unless $4M - {}^t r S^{-1} r < 0$.

The above condition *(ii)* is equivalent to the following condition,

(ii′) $\Delta_S \phi = 0$, where Δ_s is a differential operator defined by

$$\Delta_S := \frac{\partial}{\partial \tau} - \frac{1}{8\pi i}\left(\frac{\partial}{\partial z}\right) S^{-1t}\left(\frac{\partial}{\partial z}\right),$$

and where $\frac{\partial}{\partial \tau} := \left(\frac{1+\delta_{i,j}}{2}\frac{\partial}{\partial \tau_{i,j}}\right)$, $\frac{\partial}{\partial z} := \left(\frac{\partial}{\partial z_{i,j}}\right)$, and $\delta_{i,j}$ is the Kronecker delta.

We denote by $J_{k,S}^{+(n)}$ (resp. $J_{k,S}^{+cusp(n)}$) the space of skew-holomorphic Jacobi form (resp. skew-holomorphic Jacobi cusp form) of weight k, index S and degree n.

In the following sections we summarize some results for the skew-holomorphic Jacobi forms.

3. Siegel modular form of half-integral weight and generalized plus space

It is known the space of holomorphic Jacobi forms (resp. skew-holomorphic Jacobi forms) of index 1 is linearly isomorphic to the generalized plus space.

(cf. Eichler-Zagier [3], Ibukiyama [7], Hayashida [4]). The Kohnen plus space is a certain subspace of the space of modular forms of half-integral weight introduced by Kohnen [8]. It is known that the Kohnen plus space of weight $k+1/2$ corresponds to the space of elliptic modular form of weight $2k$ with belonging to $SL(2,\mathbb{Z})$. The notion of Kohnen plus space is generalized by Ibukiyama [7] to higher degree. In this section we introduce Siegel modular forms of half-integral weight and the generalized plus space, and explain the above isomorphism.

For a natural number N, define a congruence subgroup of $Sp_n(\mathbb{Z})$ by $\Gamma_0^{(n)}(N) := \{(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in Sp_n(\mathbb{Z}) \mid C \equiv 0 \mod N\}$. In order to introduce a factor of automorphy of half integral weight, we put

$$\theta(\tau) := \sum_{p \in M_{1,n}(\mathbb{Z})} e({}^t p \tau p) \ .$$

We define the Dirichlet character ψ by $\psi(t) = \left(\frac{-4}{t}\right)$, where $\left(\frac{*}{*}\right)$ is the Legendre symbol. We consider the character of $\Gamma_0^{(n)}(4)$ defined by $\psi(\det D)$ for any $M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \Gamma_0^{(n)}(4)$. By abuse of notation, we denote this character also by ψ. The following transformation formula is known:

$$\theta(M\tau)^2/\theta(\tau)^2 \ = \ \psi(M)\det(C\tau + D) \quad \text{for every} \quad M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \Gamma_0^{(n)}(4).$$

By virtue of the above formula, we can define by $\left(\frac{\theta(M\tau)}{\theta(\tau)}\right)^{2k-1}$ a factor of automorphy of weight $k-1/2$. We define the Siegel modular form of half-integral weight as follows.

Definition 3. Let k be an integer, and let χ be a character of $\Gamma_0^{(n)}(4)$. We say that a holomorphic function h on $\mathfrak{H}_n$ is a *Siegel modular form of weight $k-1/2$ of degree n with character χ* if h satisfies the following two conditions:

(1) $h(M\cdot\tau) = \chi(M)\left(\frac{\theta(M\cdot\tau)}{\theta(\tau)}\right)^{2k-1} h(\tau)$, for all $M \in \Gamma_0^{(n)}(4)$.

(2) h is holomorphic at all cusps (This condition is satisfied automatically when $n \geqslant 2$ by the Köcher principle).

Moreover, if h satisfies the following condition (3) we say h is a cusp form.

(3) The function $\det(\mathrm{Im}\tau)^{\frac{1}{2}(k-\frac{1}{2})}|h(\tau)|$ is bounded on $\mathfrak{H}_n$.

We denote by $M_{k-1/2}(\Gamma_0^{(n)}(4),\chi)$ (resp. $S_{k-1/2}(\Gamma_0^{(n)}(4),\chi)$) the space of Siegel modular forms (resp. Siegel cusp forms) of weight $k-1/2$ with character χ of degree n.

Let l be an integer and let $h \in M_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l)$, then the function h has the Fourier expansion $h(\tau) = \sum_T c_h(T)\, e(T\tau)$, where T runs over all symmetric half integral matrices. The above Fourier coefficients satisfy $c(T) = 0$, unless T is positive semi-definite. We define the subspace $M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l)$ of $M_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l)$ by

$$\begin{aligned} &M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l) \\ &:= \Big\{ h(\tau) \in M_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l) \text{ ; the coefficients satisfy } c_h(T) = 0 \text{ ,} \\ &\qquad \text{unless } T \equiv (-1)^{k+l+1} \mu^t \mu \mod 4\, Sym_n^* \text{ for some } \mu \in M_{1,n}(\mathbb{Z}) \Big\} . \end{aligned}$$

Moreover we put

$$S^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l) := M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l) \cap S_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l) .$$

We call $M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^l)$ *the plus space*. This is the notion of the "plus space" for general degree n with character ψ^l. This "plus space" was first defined for $n = 1$, $l = 0$ and $k \in \mathbb{Z}$ by Kohnen [8], and was generalized for $n > 1$, $l = 0$, and $k \in 2\mathbb{Z}$ by Ibukiyama [7], for $n > 1$, $l \equiv k \pmod 2$ by Hayashida-Ibukiyama [6].

The following Theorem is known (cf. Eichler-Zagier [3] in the case $n = 1$, $k \equiv 0 \mod 2$, by Ibukiyama [7] in the case $n > 1$, $k \equiv 0 \mod 2$, by Skoruppa [9] in the case $n = 1$, $k \equiv 1 \mod 2$, by Hayashida-Ibukiyama [6] in the other cases.)

Theorem 3.1. *The plus space is linearly isomorphic to the space of holomorphic Jacobi forms of index* 1 *and the space of skew-holomorphic Jacobi forms of index* 1*, respectively. More precisely*

$$J_{k,1}^{-(n)} \cong M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^k) \ , \ J_{k,1}^{+(n)} \cong M^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^{k-1}) \ ,$$

$$J_{k,1}^{-\,cusp(n)} \cong S^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^k) \ , \ J_{k,1}^{+\,cusp(n)} \cong S^+_{k-1/2}(\Gamma_0^{(n)}(4), \psi^{k-1}) \ .$$

Moreover these isomorphisms are compatible with the action of Hecke operators respective spaces.

The linear decomposition of Jacobi forms with theta series played an important rules to prove this theorem.

The explicit structure of the plus space was obtained by Kohnen [8] in the case of degree $n = 1$, and Hayashida-Ibukiyama [6] in the case of degree $n = 2$.

4. Siegel's formula

In this section, we recall theta series and Siegel's formula for skew-holomorphic Jacobi forms. The result in this section is part of Arakawa's work [1].

We fix a half-integral symmetric matrix $S \in Sym_l^*$, and assume $S > 0$. Let m, n and l be integers satisfying $m \geqslant n$ and $m \geqslant l$, and let $Q = \begin{pmatrix} M & \frac{1}{2}q \\ \frac{1}{2}{}^tq & S \end{pmatrix} \in Sym_{m+l}^*$ be a half-integral symmetric matrix such that $\tilde{Q} := M - \frac{1}{4}qS^{-1}{}^tq < 0$ with $\det(2Q) = (-1)^m$.

We define the following series

$$\begin{aligned}\theta_{Q,n}^{skew}(\tau, z) &:= \sum_{G=\binom{G_1}{G_2}\in M_{m+l,n}(\mathbb{Z})} e(Q[G]\tau - 2i\tilde{Q}[G_1]Im\tau + z({}^tq\,2S)G) \\ &= \sum_{\substack{N\in Sym_n^*, r\in M_{n,l}(\mathbb{Z}) \\ 4N - rS^{-1}{}^tr \leqslant 0}} A(Q;T)\, e(N\tau - \frac{1}{2}i(4N - rS^{-1}{}^tr)Im\tau + r^z),\end{aligned}$$

where $A(Q;T) := \#\left\{\binom{x_1}{x_2} \in M_{m+l,n}(\mathbb{Z}) \mid Q[\begin{pmatrix} x_1 & 0 \\ x_2 & 1_l \end{pmatrix}] = T\right\}$, and $T = \begin{pmatrix} N & \frac{1}{2}r \\ \frac{1}{2}{}^tr & S \end{pmatrix}$.

By using the well-known theta transformation formulas, we obtain $\theta_{Q,n}^{skew}(\tau, z) \in J_{\frac{m+l}{2},S}^{+(n)}$.

We explain the notions S-class and S-genus. We say two elements $Q = \begin{pmatrix} N & \frac{1}{2}r \\ \frac{1}{2}{}^tr & S \end{pmatrix}$, $Q' = \begin{pmatrix} N' & \frac{1}{2}r' \\ \frac{1}{2}{}^tr' & S \end{pmatrix} \in Sym_{m+l}^*$ are in the same *S-class*, if there exists $\gamma = \begin{pmatrix} a & 0 \\ x & 1_l \end{pmatrix} \in SL_{m+l}(\mathbb{Z})$ with ${}^t\gamma Q\gamma = Q'$. Similarly, if, for any prime p, there exists $\gamma_p = \begin{pmatrix} a & 0 \\ x & 1_l \end{pmatrix} \in SL_{m+l}(\mathbb{Z}_p)$ with ${}^t\gamma_p Q\gamma = Q'_p$ and moreover if Q and Q' have the same signature, we say Q and Q' are in the same *S-genus.*

We denote by $E_{k,S}^{sk}(\tau, z)$ the holomorphic-Jacobi Eisenstein series of $J_{k,S}^{+(n)}$. For the definition of this $E_{k,S}^{sk}(\tau, z)$ see Eq. (1) in the next section. This is defined by $E_{k,S}^{sk}(\tau, z) := E_{n,0,S}^{sk}(1; (\tau, z))$, where we regard the constant function 1 as a skew-holomorphic Jacobi form of weight k of index S, and of degree 0.

Arakawa [1] showed the following theorem for skew-holomorphic Jacobi forms.

Theorem 4.1. *Let Q be as above, and let $m > 2n + l + 2$. Then we have*

$$\left(\sum_{i=1}^{H} \frac{\theta_{Q_i,n}^{skew}(\tau, z)}{E(Q_i)}\right) \bigg/ \left(\sum_{i=1}^{H} \frac{1}{E(Q_i)}\right) = E_{\frac{m+l}{2},S}^{skew}(\tau, z),$$

where $E(Q_i) := \#\{(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}) \in M_{m+l,m}(\mathbb{Z}) \mid Q_i[(\begin{smallmatrix} x_1 & 0 \\ x_2 & 1_l \end{smallmatrix})] = Q_i\}$, *and where* $Q_1, \ldots, Q_H$ *are complete set of representatives of the* S*-classes in the given* S*-genus of* Q.

Moreover Arakawa [1] and Ziegler [10] independently showed similar results in the case of holomorphic Jacobi forms using different methods.

5. Klingen type Eisenstein series

In this section we introduce the Klingen type Eisenstein series for skew-holomorphic Jacobi forms and an application to the plus space.

Let r be an integer $(0 \leqslant r \leqslant n)$. We define the following subgroups,

$$\Gamma_{n,r} := \left\{ g \in Sp_n(\mathbb{Z}) \mid g = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix} , A_1, B_1, C_1, D_1 \in M_r(\mathbb{Z}) \right\},$$

and

$$\Gamma^J_{n,r} := \left\{ (g, [(\lambda, \mu), \kappa]) \in \Gamma^J_n \,|\, g \in \Gamma_{n,r}, \lambda = \left(\begin{smallmatrix} \lambda_1 \\ 0 \end{smallmatrix}\right) \in M_{n,l}(\mathbb{Z}), \lambda_1 \in M_{r,l}(\mathbb{Z}) \right\},$$

with $(g, [(\lambda, \mu), \kappa])$ as in Section 2. Let $\phi(\tau_1, z_1)$ be a skew-holomorphic Jacobi form in $J^{+cusp(r)}_{k,S}$ and let k be an integer satisfies $k \equiv l \mod 2$, where l is the size of S. We define a function ϕ^* on $\mathfrak{D}_{n,l}$ by

$$\phi^*(\tau, z) := \phi(\tau_1, z_1) \ ,$$

where $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $(\tau_1, z_1) \in \mathfrak{D}_{n,l}$. We define *the Klingen type Eisenstein series* associated to a skew holomorphic Jacobi form of ϕ by

$$E^{sk}_{n,r,S}(\phi; (\tau, z)) := \sum_{\gamma \in \Gamma^J_{n,r} \backslash \Gamma^J_n} (\phi^* |^{sk}_k \gamma)(\tau, z) \ , \qquad (\tau, z) \in \mathfrak{D}_{n.l}. \qquad (1)$$

The above sum does not depend on the choice of the representatives, because ϕ satisfies the transform formula and the condition that $k \equiv l \mod 2$.

It is not difficult to see that this $E^{sk}_{n,r,S}(\phi; (\tau, z))$ satisfies the transformation formula of the definition of skew holomorphic Jacobi forms in $J^{+(n)}_{k,S}$. Because ϕ is a cusp form, there exists a constant C satisfying

$$|\phi(\tau_1, z_1)| \det(Y_1)^{\frac{k}{2}} e(-S^t\beta_1(iY_1)^{-1}\beta_1) \ < \ C \, , \text{ for every } (\tau_1, z_1) \in \mathfrak{D}_{r,l},$$

where β_1 and Y_1 are the imaginary part of z_1 and τ_1 respectively. Hence, by the same calculation as Ziegler [10] Theorem 2.5, we can show the following fact, if the weight k satisfies $k > n+l+r+1$ then the sum of right hand side of $E^{sk}_{n,r,S}(\phi;(\tau,z))$ is uniformly absolutely convergent in the wider sense on $\mathfrak{D}_{n,l}$.

For a function $F(\tau,z)$ on $\mathfrak{D}_{n,l}$, we define the Siegel operator Φ^n_r by

$$\Phi^n_r(F)(\tau_1,z_1) := \lim_{t\to+\infty} F\left(\left(\begin{smallmatrix}\tau_1 & 0\\ 0 & it1_{n-r}\end{smallmatrix}\right),\left(\begin{smallmatrix}z_1\\ 0\end{smallmatrix}\right)\right), \quad (\tau_1,z_1)\in\mathfrak{D}_{n,r}.$$

Theorem 5.1. *Let $\phi \in J^{+cusp(r)}_{k,S}$. If $k > n+l+r+1$ satisfies $k \equiv l \mod 2$, then we have the followings,*

(1) *$E^{sk}_{n,r,S}(\phi;(\tau,z))$ is an element of $J^{+(n)}_{k,S}$.*
(2) *$\Phi^n_r(E^{sk}_{n,r,S}(\phi;(\tau,z))) = \phi(\tau_1,z_1)$ for every $\phi(\tau_1,z_1) \in J^{+cusp(r)}_{k,S}$. Hence, the Siegel operator Φ^n_r induces a surjective map from $J^{+(n)}_{k,S}$ to $J^{+cusp(r)}_{k,S}$.*

This theorem is also true for $S \geqslant 0$. (see Hayashida [5])

Here, we follow Arakawa's work [2]. We impose the following condition on the index $S > 0$.

(C1) If $S[x] \in \mathbb{Z}$ for $x \in (2S)^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x \in M_{l,1}(\mathbb{Z})$.

By the same argument as in Arakawa [2] (Proposition 4.1, Theorem 4.2), we deduce the following Proposition 5.1 and Theorem 5.2.

Proposition 5.1. *Let $F \in J^{+(n)}_{k,S}$. Under the condition $(C1)$ on S, we have $F \in J^{+cusp(n)}_{k,S}$ if and only if $\Phi^n_{n-1}(F) = 0$.*

Theorem 5.2. *Assume that S satisfies the condition $(C1)$. Let k be a positive integer with $k > 2n+l+1$, $k \equiv l \mod 2$. Then we have the direct sum decomposition $J^{+(n)}_{k,S} = \bigoplus_{r=0}^{n} J^{+(n)}_{k,S,r}$, where $J^{+(n)}_{k,S,r} := \{E^{sk}_{n,r,S}(F;(\tau,z)) | F \in J^{+cusp(r)}_{k,S}\}$.*

From now on we consider the case $S = 1$. Here the condition $(C1)$ is obviously satisfied. We consider the space of skew-holomorphic Jacobi forms of index 1. By virtue of Theorem 3.1, we can view the plus space as the space of holomorphic Jacobi forms of index 1 or as the space of skew-holomorphic Jacobi forms of index 1. Hence, by using theorem 5.2, if k is an odd integer satisfies $k > 2n+2$, we can also obtain a similar decomposition for the plus space of degree n of weight $k-1/2$ with trivial character. Namely, if k is odd integer, we can deduce that the plus space of weight $k-1/2$ is spanned by so called *Klingen-Cohen type Eisenstein series* which corresponds to the Klingen type Eisenstein series of skew-holomorphic Jacobi forms of index 1. This decomposition first pointed out by Arakawa [2] by using holomorphic Jacobi forms. Namely if k is even, he showed a decomposition of the plus space by using Klingen-Cohen type Eisenstein series which correspond to the holomorphic Jacobi forms.

If k is odd with $k > 2n+2$, then we have the following map $\hat{E}_{n,r}$:

$$\begin{array}{rcl} M^{+}_{k-1/2}(\Gamma_0^{(n)}(4)) & \cong & J^{+(n)}_{k,1} \\ & \uparrow \hat{E}_{n,r} \circlearrowleft \uparrow E_{n,r,1} & \\ S^{+}_{k-1/2}(\Gamma_0^{(r)}(4)) & \cong & J^{+cusp(r)}_{k,1}. \end{array}$$

Moreover we have the following theorem.

Theorem 5.3. *Let k be an odd integer larger than $2n+2$. Then we have a decomposition:*

$$M^{+}_{k-1/2}(\Gamma_0^{(n)}((4)) = \bigoplus_{r=0}^{n} \hat{E}_{n,r}(S^{+}_{k-1/2}(\Gamma_0^{(r)}(4))),$$

where $\hat{E}_{n,r}(S^{+}_{k-1/2}(\Gamma_0^{(r)}(4))) := \left\{\hat{E}_{n,r}(F) \mid F \in S^{+}_{k-1/2}(\Gamma_0^{(r)}(4))\right\}$.

Finally Arakawa [2] also solved a basis problem for the space of holomorphic Jacobi forms of index 1 by generalizing the pullback formula in the framework of holomorphic Jacobi forms and also solved a certain basis problem for the plus space. Similar results are expected to the case of skew-holomorphic Jacobi forms.

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A HERMITIAN ANALOG OF THE SCHOTTKY FORM

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Dedicated to the memory of Professor Tsuneo Arakawa

We compute the filtration of the Hermitian modular forms arising from the theta series associated with even unimodular Gaussian $\mathbb{Z}[i]$-lattices of rank 8. In this filtration a Hermitian analog of the Schottky form appears. Moreover we compare our results with Ikeda's lifts and consider the analogous problem for quaternionic modular forms over the Hurwitz order.

1. Introduction

It is well-known (cf. Conway and Sloane [3]) that there exists only one isometry class of even unimodular $\mathbb{Z}$-lattices in dimension 8, resp. 2 classes in dimension 16 resp. 24 classes in dimension 24. Considering the associated Siegel modular forms it was shown by Kneser [16] and Igusa [8] (cf. also Poor and Yuen [23]) that in dimension 16 one obtains linear independent theta series of degree $n \geqslant 4$. If $n = 4$ the difference of the two theta series is equal to a multiple of the Schottky form (Igusa [9], Freitag [7]).

Considering even unimodular $\mathbb{Z}[i]$-lattices it was shown by Iyanaga [13] that there is only one isometry class of rank 4. In the case of rank 8 resp. rank 12 a classification is due to Schiemann [25] as well as Kitazume and Munemasa [14], where one gets 3 resp. 28 isometry classes.

In this paper we consider the space of Hermitian modular forms associated with the even unimodular $\mathbb{Z}[i]$-lattices of rank 8 similar to the investigation by Nebe and Venkov [22] on the Siegel modular forms arising from the Niemeier lattices. Our corresponding filtration yields Hermitian modular forms of weight 8, namely the Siegel-Eisenstein series as well as non-trivial cusp forms of degree 2 and degree 4. Thus we construct a Hermitian analog of the Schottky form. Moreover we compare our results with Ikeda's lifts (Ikeda [10], [11], [12]) and consider the analogous problem for quaternionic modular forms over the Hurwitz order, which surprisingly turns out to be much simpler.

2. Hermitian modular forms

Let

$$\mathcal{H}_n := \{Z \in \mathbb{C}^{n\times n};\ \tfrac{1}{2i}(Z - \bar{Z}^{tr}) > 0\}$$

denote the *Hermitian half-space* of degree n. The Hermitian modular group of degree n over the Gaussian number field

$$\Gamma_n = \{M \in \mathbb{Z}[i]^{2n\times 2n};\ J[M] := \bar{M}^{tr} J M = J\},\quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I stands for the identity matrix, acts on $\mathcal{H}_n$. The space $\mathcal{M}_k(\Gamma_n)$ of *Hermitian modular forms* of degree n and weight k consists of all holomorphic functions $f : \mathcal{H}_n \to \mathbb{C}$ satisfying

$$f\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k f(Z) \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$$

as well as the usual condition of boundedness if $n = 1$. The subspace $\mathcal{M}_k(\Gamma_n)^{sym}$ is characterized by the additional invariance under the transpose mapping

$$f(Z^{tr}) = f(Z).$$

Each $f \in \mathcal{M}_k(\Gamma_n)$ possesses a Fourier expansion of the form

$$f(Z) = \sum_{T \geqslant 0} \alpha_f(T)\, e^{2\pi i \operatorname{trace}(TZ)}$$

where $T = (t_{\nu\mu})$ runs through the set of half-integral matrices, i.e.

$$T = \bar{T}^{tr}, \quad t_{\nu\nu}, 2t_{\nu\mu} \in \mathbb{Z}[i] \quad \text{for all } \nu, \mu.$$

If $n > 1$ we have got a Fourier-Jacobi expansion of the form

$$f(Z) = \sum_{m=0}^{\infty} f_m(Z), \quad f_m(Z) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & m \end{smallmatrix}\right) \geqslant 0} \alpha_f(T)\, e^{2\pi i \operatorname{trace}(TZ)}.$$

Clearly $f_0 = f \mid \phi$ holds, where ϕ denotes the Siegel ϕ-operator (cf. Krieg [17] for details). Denote the kernel of the ϕ-operator, which is the subspace of *cusp forms* in $\mathcal{M}_k(\Gamma_n)$, by $\mathcal{S}_k(\Gamma_n)$.

Lemma 2.1. *Each $f \in \mathcal{M}_k(\Gamma_n)$ is uniquely determined by its Fourier-Jacobi coefficients $f_0, \ldots, f_m$, where*

$$m \leqslant \frac{k}{\pi\sqrt{3}}, \quad \text{if } n = 2,$$

$$m \leqslant \frac{k\sqrt[3]{2}}{\pi\sqrt{3}}, \quad \text{if } n = 3,$$

$$m \leqslant \frac{2k}{\pi\sqrt{3}}, \quad \text{if } n = 4.$$

Proof. One can directly follow the proof of Satz 1 in Eichler [5]. It says that f is uniquely determined by $f_0, \ldots, f_m$, where

$$m \leqslant \frac{k \cdot \mu_n^2}{2\pi\sqrt{3}},$$

and μ_n denotes the corresponding Hermite constant, i.e.

$$\min\{Y[c];\ 0 \neq c \in \mathbb{Z}[i]^n\} \leqslant \mu_n \cdot (\det Y)^{1/n} \quad \text{for all } Y = \bar{Y}^{tr} > 0. \qquad (*)$$

We have $\mu_2 = \sqrt{2}$ due to Krieg [17], I Section 4. Moreover note that according to Voronoi's result the optimal constant μ_n in $(*)$ is attained whenever Y is perfect. It follows from Staffeldt [26] that there is only one isometry class of perfect matrices for $n = 3$ given by

$$Y = \lambda \begin{pmatrix} 1 & 1/2 & (1+i)/2 \\ 1/2 & 1 & (1+i)/2 \\ (1-i)/2 & (1-i)/2 & 1 \end{pmatrix}, \quad \lambda > 0.$$

Thus we have $\mu_3 = \sqrt[3]{4}$. Moreover we conclude $\mu_4 = 2$ from the estimation of the Hermite constant on $\mathbb{Z}^8$ by Blichfeldt [2] as well as the existence of the matrix S in Iyanaga [13], Dern and Krieg [4] resp. section 3. Hence the claim follows. $\square$

In the next step we want to derive an analogous result for Jacobi forms.

Lemma 2.2. *Let f_m, $m \in \mathbb{N}_0$, be the m-th Fourier-Jacobi coefficient of $f \in \mathcal{M}_k(\Gamma_3)$. Suppose that*

$$\alpha_f(T) = 0 \quad \text{for all } T = \begin{pmatrix} * & * \\ * & m \end{pmatrix} \text{ with } \operatorname{trace}(T) \leqslant 2m + \frac{16+2k}{\pi\sqrt{3}},$$

then

$$f_m \equiv 0.$$

Proof. One can follow the proof of Theorem 1 in Klingen [15]. Note that there exists a non-trivial Hermitian cusp form in $\mathcal{S}_8(\Gamma_2)$ due to Freitag [6] resp. Dern and Krieg [4] as well as

$$\mathrm{trace}(Y^{-1}) \leqslant \frac{8}{\sqrt{3}} \quad \text{for all } Z = X + iY \in \mathcal{F}(2;\mathbb{C})$$

due to Krieg [17], p.66. Thus we get

$$\mathrm{trace}(Y_0^{-1}) \leqslant \frac{8\pi m}{8+k} + \frac{8}{\sqrt{3}} \quad \text{for} \quad Y_0 = \begin{pmatrix} Y & 0 \\ 0 & \widetilde{y}_4 \end{pmatrix} \begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix}, X + iY \in \mathcal{F}(2;\mathbb{C})$$

$$\widetilde{y}_4 = \frac{8+k}{4\pi m}, \quad y \in \mathbb{C}^2, \quad \mathrm{Re}(y),\ \mathrm{Im}(y) \in [-\tfrac{1}{2}, \tfrac{1}{2}]^2.$$

Then the claim follows in exactly the same way as in Klingen [15], Theorem 1. □

We denote the Siegel half-space of degree n by

$$\mathcal{S}_n = \{Z \in \mathcal{H}_n;\ Z = Z^{tr}\}.$$

Remark. a) A similar procedure in the case of Siegel modular forms is due to Poor and Yuen [23].

b) Clearly Lemma 2 holds for all Hermitian Jacobi forms of weight k and index m on $\mathcal{H}_2 \times \mathbb{C}^4$ with the same proof.

3. Even unimodular Gaussian lattices

An even unimodular *Gaussian lattice* Λ of rank m is given by a $\mathbb{C}$-basis of vectors $\lambda_1, \dots, \lambda_m \in \mathbb{C}^m$ such that

$$\Lambda = \mathbb{Z}[i]\lambda_1 + \dots + \mathbb{Z}[i]\lambda_m, \quad \bar{\lambda}^{tr}\lambda \in 2\mathbb{Z} \quad \text{for all } \lambda \in \Lambda,$$
$$S = \left(\bar{\lambda}_\nu^{tr}\lambda_\mu\right) \quad \text{satisfies} \quad \det S = 1.$$

It is well-known that such lattices exist if and only if $m \equiv 0 \pmod 4$ (cf. Krieg [17], Kitazume and Munemasa [14]). If $m = 4$ there is just one isometry class given by

$$S = e_8(i) = \begin{pmatrix} 2I & B \\ \bar{B}^{tr} & 2I \end{pmatrix}, \qquad B = \begin{pmatrix} 1+i & i \\ i & 1-i \end{pmatrix},$$
$$\sharp\, \mathrm{Aut}\, S = \sharp\, \{U \in \mathrm{Gl}_4(\mathbb{Z}[i]);\ S[U] = S\} = 2^{10} \cdot 3^2 \cdot 5$$

(cf. Dern and Krieg [4], Iyanaga [13], Schiemann [25]). If $m = 8$ the results by Schiemann [25] as well as Kitazume and Munemasa [14] say that there are exactly three isometry classes of lattices given by the Gram matrices

$$S_1 = E_8 = \widehat{S} = \begin{pmatrix} 2I^{(4)} & \widehat{B} \\ \widehat{B}^{tr} & 2I^{(4)} \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix},$$

$$\sharp \operatorname{Aut} S_1 = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7,$$

$$S_2 = d_{16}^{+}(i) = \begin{pmatrix} 2 & -1 & 0 & -1 & -1 & -1 & -i & 1+i \\ -1 & 2 & 1-i & 0 & 0 & 0 & 0 & -i \\ 0 & 1+i & 2 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 & 1 & 1 & i & -1 \\ -1 & 0 & 0 & 1 & 2 & 1 & i & -1 \\ -1 & 0 & 0 & 1 & 1 & 2 & i & -1 \\ i & 0 & 0 & -i & -i & -i & 2 & i \\ 1-i & i & 1 & -1 & -1 & -1 & -i & 4 \end{pmatrix},$$

$$\sharp \operatorname{Aut} S_2 = 2^{22} \cdot 3^2 \cdot 5 \cdot 7,$$

$$S_3 = e_8(i) \oplus e_8(i) = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix},$$

$$\sharp \operatorname{Aut} S_3 = 2^{21} \cdot 3^4 \cdot 5^2.$$

Let L_1 be the Gram matrix of the root $\mathbb{Z}$-lattice of E_8 and L_2 of D_{16}^{+} which represent the indecomposable even unimodular $\mathbb{Z}$-lattices in dimension 8 resp. 16. We consider the Hermitian and Siegel theta series given by

$$\Theta^{(n)}(Z,T) := \sum_{G \in \mathbb{Z}[i]^{m\times n}} e^{\pi i \operatorname{trace}(T[G]\cdot Z)}, \quad Z \in \mathcal{H}_n, \ 0 < T = \overline{T}^{tr} \in \mathbb{C}^{m\times m},$$

$$\vartheta^{(n)}(Z,T) := \sum_{G \in \mathbb{Z}^{m\times n}} e^{\pi i \operatorname{trace}(T[G]\cdot Z)}, \quad Z \in \mathcal{S}_n, \ 0 < T = T^{tr} \in \mathbb{R}^{m\times m}.$$

Their properties as modular forms and their relations are described in

Lemma 3.3. *One has*

a) $\Theta^{(n)}(\,\cdot\,, S_\nu) \in \mathcal{M}_8(\Gamma_n)^{sym}$, $\nu = 1,2,3$.
b) $\Theta^{(n)}(\,\cdot\,, S_3) = \Theta^{(n)}(\,\cdot\,, S)^2$.
c) $\Theta^{(n)}(\,\cdot\,, S_1)|_{\mathcal{S}_n} = \Theta^{(n)}(\,\cdot\,, S_3)|_{\mathcal{S}_n} = \vartheta^{(n)}(\,\cdot\,, L_1)^2$,

$\Theta^{(n)}(\,\cdot\,, S_2)|_{\mathcal{S}_n} = \vartheta^{(n)}(\,\cdot\,, L_2)$.

Proof. a) In view of Krieg [17], IV.2.6, it remains to be shown that the theta series are symmetric. This follows from Krieg [17], IV.1.13, because S_ν and $\bar{S}_\nu = S_\nu^{tr}$ are isometric, which is clear from the classification, as the orders of the automorphism groups coincide.

b) Use Krieg [17], IV.1.14.

c) This follows from Schiemann [25]. □

At first we consider small weights.

Corollary 3.1. *One has*

$$\mathcal{M}_k(\Gamma_n) = \{0\} \quad \textit{for } k = 1, 2, 3 \quad \textit{and all } n \in \mathbb{N},$$
$$\mathcal{M}_4(\Gamma_n) = \mathbb{C} \cdot \Theta^{(n)}(\,\cdot\,, S)$$

for all $n \in \mathbb{N}$ with the possible exception of $n = 4$.

Proof. Apply the results on singular modular forms for $k < n$ in Krieg [17] resp. Vasudevan [27] as well as Lemma 1 and Lemma 3. □

The adjacency matrix A of S_1, S_2, S_3, which contains the number of even 2-neighbors of the lattice S_ν, which are isometric to S_μ, as (ν, μ)-entry, was also computed by Schiemann [25]:

$$A = \begin{pmatrix} 42\,360 & 43\,200 & 12\,096 \\ 40\,960 & 42\,360 & 14\,336 \\ 36\,864 & 46\,080 & 14\,712 \end{pmatrix}.$$

The operator K defined by A just as in Nebe and Venkov [22] or in Nebe and Teider [21] has got three eigenvectors

$$v_1 = \frac{1}{305}\begin{pmatrix} 128 \\ 135 \\ 42 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -8 \\ 3 \\ 5 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 8 \\ -15 \\ 7 \end{pmatrix}$$

with respect to the eigenvalues

$$\lambda_1 = 2^3 \cdot 3 \cdot 13 \cdot 313, \qquad \lambda_2 = 2^3 \cdot 3^2 \cdot 5 \cdot 11, \qquad \lambda_3 = -2^3 \cdot 3 \cdot 7 \cdot 13.$$

Therefore we define $F_\nu^{(n)} \in \mathcal{M}_8(\Gamma_n)^{sym}$ by

$$F_1^{(n)} = \tfrac{1}{305}\left(128 \cdot \Theta^{(n)}(\,\cdot\,, S_1) + 135 \cdot \Theta^{(n)}(\,\cdot\,, S_2) + 42 \cdot \Theta^{(n)}(\,\cdot\,, S_3)\right),$$
$$F_2^{(n)} = -8 \cdot \Theta^{(n)}(\,\cdot\,, S_1) + 3 \cdot \Theta^{(n)}(\,\cdot\,, S_2) + 5 \cdot \Theta^{(n)}(\,\cdot\,, S_3),$$
$$F_3^{(n)} = 8 \cdot \Theta^{(n)}(\,\cdot\,, S_1) - 15 \cdot \Theta^{(n)}(\,\cdot\,, S_2) + 7 \cdot \Theta^{(n)}(\,\cdot\,, S_3).$$

Theorem 3.1. *One has*

a) $0 \neq F_1^{(n)} \in \mathcal{M}_8(\Gamma_n)^{sym}$ *for all* $n \in \mathbb{N}$. $F_1^{(n)}$ *is equal to the normalized Siegel-Eisenstein series of degree* n *and weight* 8 *if* $n < 4$.
b) $0 \neq F_2^{(2)} \in \mathcal{S}_8(\Gamma_2)^{sym}$.
c) $0 \neq F_3^{(4)} \in \mathcal{S}_8(\Gamma_4)^{sym}$.

Proof. a) One has

$$F_1^{(n)} = \sum_{\nu=1}^{3} m_\nu \cdot \Theta^{(n)}(\,\cdot\,, S_\nu), \qquad m_\nu = \frac{\frac{1}{\sharp\,\mathrm{Aut}\,S_\nu}}{\frac{1}{\sharp\,\mathrm{Aut}\,S_1} + \frac{1}{\sharp\,\mathrm{Aut}\,S_2} + \frac{1}{\sharp\,\mathrm{Aut}\,S_3}}.$$

Thus the claim follows from Krieg [19].

b) The constant Fourier coefficient of $F_2^{(n)}$ is 0. One computes

$$\alpha_{F_2^{(2)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 49\,152.$$

Thus the claim follows from Freitag [6] resp. Dern and Krieg [4].

c) $F_3^{(4)}$ does not vanish identically because this is already true for the restriction to the Siegel half-space due to Lemma 3 as well as Kneser [16] resp. Igusa [8]. Thus it suffices to show that $F_3^{(3)} = F_3^{(4)} \mid \phi \equiv 0$. In view of Lemma 1 we have to prove that the Fourier-Jacobi coefficients f_0, f_1 vanish. At first $f_0 = F_3^{(2)} = 0$ follows from

$$\alpha_{F_2^{(2)}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \alpha_{F_2^{(2)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

and Freitag [6] resp. Dern and Krieg [4]. In order to derive $f_1 \equiv 0$ we apply Lemma 2 and show that

$$\alpha_{F_3^{(3)}}(T) = 0 \quad \text{for all } T = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}, \ \operatorname{trace}(T) < 8.$$

This is done by explicit calculations using MAGMA. $\square$

The filtration, i.e. the dimensions of the subspaces $\mathcal{S}_8(\Gamma_n)_\Theta$ of cusp forms spanned by theta series, is given by

Corollary 3.2.

a) *One has*

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n\geqslant 5$
$\dim \mathcal{S}_8(\Gamma_n)_\Theta$	1	0	1	0	1	0

b) *The theta series $\Theta^{(n)}(\,\cdot\,, S_\nu)$, $\nu = 1,2,3$, are linearly independent if and only if $n \geqslant 4$.*

Concerning the action of the Hecke algebra we obtain

Corollary 3.3. *The Hermitian modular forms $F_\nu^{(n)}$, $\nu = 1,2,3$, are eigenforms under all Hecke operators.*

Proof. The subspace spanned by theta series is invariant under all Hecke operators due to the description of singular modular forms (cf. Vasudevan [27], Krieg [17]) and the commutation relation of Hecke operators with the Siegel ϕ-operator (cf. Krieg [18]). For $F_1^{(n)}$ the result follows from Krieg [19]. Considering $F_3^{(n)}$ the result is a consequence of the Theorem because a Hecke operator maps $F_3^{(4)}$ onto a cusp form, hence a multiple of $F_3^{(4)}$. Dealing with $F_2^{(n)}$ one may use the same arguments as in Nebe and Venkov [22]. □

The relation mentioned in the title is derived in

Corollary 3.4. *$F_3^{(4)}|_{\mathcal{S}_4}$ is equal to a multiple of the Schottky form.*

Proof. $F_3^{(4)}|_{\mathcal{S}_4} = 15 \cdot \left(\vartheta^{(4)}(\,\cdot\,, L_2) - \vartheta^{(4)}(\,\cdot\,, L_1)^2\right)$ holds due to Lemma 2. Now use Kneser [16] or Igusa [8]. □

Remark. a) The Schottky form is not an Ikeda lift (cf. Ikeda [10]) because there is no non-trivial elliptic cusp form of weight 8.

b) The Schottky form is not a Miyawaki lift (cf. Ikeda [11]) because there is no non-trivial Siegel cusp form of degree 2 and weight 8.

c) The cusp form $F_2^{(2)}$ belongs to the Maaß space (cf. Krieg [19]) — which is true for all modular forms of degree 2 and weight 8 — and can therefore be considered as a Hermitian Ikeda lift (cf. Ikeda [12]). Moreover there exists $0 \neq f \in \mathcal{S}_8(\Gamma_4)$, which is a Hermitian Ikeda lift of the elliptic modular form $\eta(\tau)^4 \cdot \eta(4\tau)^4 \cdot \eta(2\tau)^2$ of weight 5 with respect to $\Gamma_0(4)$, where $\eta(\tau)$ denotes the Dedekind eta-function. We conjecture that f is a multiple of $F_3^{(4)}$.

d) Consider quaternionic modular forms over the Hurwitz order just as in Krieg [17]. Quebbemann [24] showed that there is only one isometry class of stable lattices whose theta series are quaternionic modular forms of weight 4 resp. weight 8. Hence there is no quaternionic analog of the Schottky form in this sense.

Moreover there are only three isometry classes of stable lattices of rank 6 (cf. Quebbemann [24], Bachoc and Nebe [1]), one of them being the Leech lattice, which yield quaternionic modular forms of weight 12. These three quaternionic theta series are linearly independent if and only if the degree is $n \geqslant 2$. This follows from the description of the root lattices in Quebbemann [24] resp. Bachoc and Nebe [1] and Krieg [20] and already holds for the restrictions of the quaternionic theta series to the Siegel half-space of degree 2. Note that all quaternionic modular forms of degree 2 and weight $4, 8, 12$ are therefore linear combinations of theta series.

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THE SIEGEL SERIES AND SPHERICAL FUNCTIONS ON $O(2n)/(O(n) \times O(n))$

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To the memory of Tsuneo Arakawa

1. Introduction

Fix a rational prime p and let $Sym_n(\mathbb{Q}_p)$ be the space of symmetric matices of size n with entries in the p-adic number field $\mathbb{Q}_p$. A symmetric matrix $T = (t_{ij}) \in Sym_n(\mathbb{Q}_p)$ is called *half-integral* if $t_{ii} \in \mathbb{Z}_p$ and $t_{ij} \in \frac{1}{2}\mathbb{Z}_p$. Denote by $\mathcal{H}_n(\mathbb{Z}_p)$ the space of half-integral symmetric matices of size n. Let ψ be an additive character of $\mathbb{Q}_p$ with conductor $\mathbb{Z}_p$. For $T \in Sym_n(\mathbb{Q}_p)$, the Siegel series $b_p(T; s)$ is defined by

$$b_p(T; s) = \int_{Sym_n(\mathbb{Q}_p)} \nu_p(R)^{-s} \psi(\mathrm{tr}(TR))\, dR \quad (s \in \mathbb{C}),$$

where $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R. The integral $b_p(T; s)$ converges absolutely for $\mathrm{Re}(s) > n$ and represents a rational function of p^{-s} (for more precise information, see Section 3). It is easy to see that $b_p(T; s)$ vanishes unless T is half-integral.

The Siegel series appear as the p-factors of the Fourier coefficients of the Siegel Eisenstein series and are important arithmetic invariants for integral quadratic forms. The purpose of the present paper is to give a new integral expression of $b_p(T; s)$ and relate it to a spherical function on the symmetric space $O(2n)/(O(T)\times O(T))$. We also discuss the functional equation satisfied by $b_p(T;s)$ from the view point of the harmonic analysis on $O(2n)/(O(T)\times O(T))$.

Let us explain our results in some detail. In the following we always assume that T is nondegenerate, since the properties of $b_p(T;s)$ can be reduced to the nondegenerate case. Recall that the value of $b_p(T;s)$ at a positive integer $s=k\geqslant n$ is equal to the so-called local density

$$\alpha_p(H_k,T)=\lim_{\ell\to\infty}p^{-\ell(2kn-n(n+1)/2)}N_{p^\ell}(H_k,T),$$

where

$$H_k=\overbrace{\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\perp\cdots\perp\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}}^{k}.$$

and

$$N_{p^\ell}(H_k,T)=\sharp\left\{v\in M_{2k,n}(\mathbb{Z}_p/p^\ell\mathbb{Z}_p)\;\middle|\;{}^tvH_kv\equiv T\pmod{p^\ell\mathcal{H}_n(\mathbb{Z}_p)}\right\}.$$

If we consider the polynomial mapping

$$f_k:M_{2k,n}(\mathbb{Q}_p)\longrightarrow Sym_n(\mathbb{Q}_p),\quad f_k(x)={}^txH_kx,$$

then it is also known that $\alpha_p(H_k,T)$ (and hence $b_p(T;k)$) is given by an integral over the fibre $f_k^{-1}(T)$ (see, e.g., [17]). Note here that the integrals that express $b_p(T;s)$ are taken over different domains for different k. Put $f=f_n:M_{2n,n}\to Sym_n$. Then our first result is the following integral representation of $b_p(T;s)$ valid for any s (not only for integer arguments):

$$b_p(T;s)=\prod_{i=1}^{n}\frac{1-p^{-s+i-1}}{1-p^{-i}}\cdot\int_{f^{-1}(T)\cap M_{2n,n}(\mathbb{Z}_p)}|\det x_2|_p^{s-n}\,|\Theta_T|_p\,,$$

where x_2 denotes the lower n by n matrix of $x\in M_{2n,n}(\mathbb{Q}_p)$. The construction of the measure $|\Theta_T|_p$ and the proof of this identity are given in Section 2.

Put $x_T=\begin{pmatrix}T\\E_n\end{pmatrix}$. Here E_n denotes the identity matrix of size n. Let $O(H_n)$ be the orthogonal group of H_n. Then, x_T belongs to $f^{-1}(T)$ and the function $g\mapsto|\det(gx_T)_2|^{s-n}$ on $O(H_n)$ defines a meromorphic section of the degenerate principal series representation of $O(H_n)$ with repsect to the Siegel parabolic subgroup. The Poisson transform of this function gives a kind of spherical functions on the symmetric space $O(H_n)/(O(T)\times O(T))$. As will be shown in Section 3, the integral representation of $b_p(T;s)$ enables us to express it as a linear combination of values of this spherical function.

In [10], Katsurada proved a functional equation of $b_p(T; s)$ in a quite explicit form, which had earier been proved by Karel [8] in an abstract form as a functional equation of the Whittaker function of a p-adic group (in the present case $Sp_{2n}(\mathbb{Q}_p)$) (see also [3], [13], [14]). Katsurada's proof of the functional equation (as well as the one given by Böcherer and Kohnen [3]) is based on the (global) functional equation of the real analytic Siegel Eisenstein series. On the other hand Karel's proof is purely local and based on harmonic analysis on p-adic groups. The relation given in Section 3 between the Siegel series and the spherical function on $O(H_n)/(O(T) \times O(T))$ provides us another local approach to the functional equation. In Section 4, we formulate the functional equation as the one for the spherical function. The functional equation of the spherical function will be proved in Section 6 after some preliminaries given in Section 5 on degenerate principal series representations of $O(H_n)$.

Notation. We denote by E_n the identity matrix of size n and by $0_{m,n}$ the m by n zero matrix. We put $0_{n,n} = 0_n$. The diagonal matrix with diagonal entries $a_1, \ldots, a_n$ is denoted by $\mathrm{diag}(a_1, \ldots, a_n)$. For a symmetric matrix A of size m and a matrix $v \in M_{m,n}$, we denote $A[v] = {}^t vAv$, which is a symmetric matrix of size n.

2. An integral representation of the Siegel series

We keep the notation introduced in Section 1. In particular we let $f : M_{2n,n} \to Sym_n$ be the polynomial mapping defined by $f(x) = H_n[x]$. For $T \in Sym_n(\mathbb{Q}_p) \cap GL_n(\mathbb{Q}_p)$, we put $X_T := f^{-1}(T)$ and consider it as an affine algebraic variety defined over $\mathbb{Q}_p$.

Choose a rational differential form ω on $M_{2n,n}$ of degree $n(3n-1)/2$ satisfying

$$\omega \wedge f^*(dT) = dx, \quad dT = \bigwedge_{1 \leqslant i \leqslant j \leqslant n} dt_{ij}, \quad dx = \bigwedge_{\substack{1 \leqslant i \leqslant 2n \\ 1 \leqslant j \leqslant n}} dx_{ij}.$$

Here dT is the canonical gauge form on Sym_n and dx is the canonical gauge form on $M_{2n,n}$. Then the restriction $\omega|_{X_T}$ defines a differential form on X_T and is independent of the choice of ω. We denote by $|\Theta_T|_p$ the measure on $X_T(\mathbb{Q}_p)$ induced by $\omega|_{X_T}$.

Let $\zeta_n(s)$ be the p-adic local zeta function of the matrix algebra M_n:

$$\zeta_n(s) = \int_{M_n(\mathbb{Z}_p)} |\det x|_p^{s-n}\, dx.$$

The following explicit formula is well-konwn:

$$\zeta_n(s) = \prod_{i=1}^{n} \frac{1-p^{-i}}{1-p^{-s+i-1}}.$$

Theorem 2.1. *If* $\mathrm{Re}(s) > n$, *then we have*

$$b_p(T;s) = \zeta_n(s)^{-1} \cdot \int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, |\Theta_T|_p,$$

where x_2 *denotes the lower* n *by* n *matrix of* $x \in M_{2n,n}(\mathbb{Q}_p)$.

Proof. The key to the proof is the following identity, which holds if $\mathrm{Re}(s)$ is sufficiently large:

$$\begin{aligned} &\int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, |\Theta_T|_p \qquad (1) \\ &= \lim_{e\to\infty} \int_{p^{-e}Sym_n(\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\, dR \int_{M_{2n,n}(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, \psi(\mathrm{tr}(H_n[x]R))\, dx. \end{aligned}$$

We admit this identity for the moment and prove the theorem. Since $\mathrm{tr}(H_n[x]R) = \mathrm{tr}(R\,{}^t x_2 x_1)$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M_{2n,n}(\mathbb{Q}_p)$, the integral on the right hand side of (1) is equal to

$$\begin{aligned} &\int_{p^{-e}Sym_n(\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\, dR \int_{M_{2n,n}(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, \psi(\mathrm{tr}(R\,{}^t x_2 x_1))\, dx \\ &= \int_{p^{-e}Sym_n(\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\, dR \int_{M_n(\mathbb{Z}_p)\cap M_n(\mathbb{Z}_p)R^{-1}} |\det x_2|_p^{s-n}\, dx_2. \end{aligned}$$

By the theory of elementary divisors, there exist $k_1, k_2 \in GL_n(\mathbb{Z}_p)$ and integers $\lambda_1, \dots, \lambda_i \geqslant 0$, $\lambda_{i+1}, \dots, \lambda_n > 0$ such that

$$R = k_1 \mathrm{diag}(p^{\lambda_1}, \cdots, p^{\lambda_i}, p^{-\lambda_{i+1}}, \cdots, p^{-\lambda_n}) k_2.$$

Then, putting $D_R = \mathrm{diag}(1, \cdots, 1, p^{\lambda_{i+1}}, \cdots, p^{\lambda_n})$, we have

$$\begin{aligned} \int_{M_n(\mathbb{Z}_p)\cap M_n(\mathbb{Z}_p)R^{-1}} |\det x_2|_p^{s-n}\, dx_2 &= \int_{M_n(\mathbb{Z}_p)D_R} |\det x_2|_p^{s-n}\, dx_2 \\ &= |\det D_R|_p^{s} \int_{M_n(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, dx_2 = \nu_p(X)^{-s}\zeta_n(s). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\, |\Theta_T|_p &= \zeta_n(s) \cdot \lim_{e\to\infty} \int_{p^{-e}Sym_n(\mathbb{Z}_p)} \nu_p(X)^{-s} \psi(-\mathrm{tr}(TR))\, dR \\ &= \zeta_n(s) \cdot b_p(T;s). \end{aligned}$$

By analytic continuation, this identity holds for $\mathrm{Re}(s) > n$. □

Now we prove the identity (1).

Proof of (1). For $T \in Sym_n(\mathbb{Q}_p) \cap GL_n(\mathbb{Q}_p)$ and for any locally constant function ϕ on $M_{2n,n}(\mathbb{Q}_p)$ of compact support, the identity

$$\begin{aligned}&\int_{X_T(\mathbb{Q}_p)} \phi(x)\,|\Theta_T|_p \\ &= \lim_{e\to\infty} \int_{Sym_n(\frac{1}{p^e}\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\,dR \int_{M_{2n,n}(\mathbb{Q}_p)} \phi(x)\psi(\mathrm{tr}(H_n[x]R))\,dx\end{aligned}$$

can be proved by a similar argument to that in [17] (see also Theorem 8.3.1 of [7]). Moreover there exists a positive integer r depending on T satisfying the property that

(*) if $e > r$, ϕ vanishes outside $M_{2n,n}(\mathbb{Z}_p)$ and the value $\phi(x)$ is determined by the residue class of $x \bmod p^e$, then the integral on the right hand side is independent of e.

Note that the integral with respect to R does not necessarily converge absolutely. For a positive integer ℓ, we denote by χ_ℓ (resp. ϕ_0) the characteristic function of the set $\{x \in M_{2n,n}(\mathbb{Q}_p) \mid \det x_2 \notin p^\ell\mathbb{Z}_p\}$ (resp. $M_{2n,n}(\mathbb{Z}_p)$). Then $|\det x_2|_p^{s-n}\chi_\ell(x)\phi_0(x)$ is locally constant and of compact support and we have

$$\begin{aligned}&\int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\,\chi_\ell(x)\,|\Theta_T|_p \\ &= \lim_{e\to\infty}\left(\int_{Sym_n(\frac{1}{p^e}\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\,dR \right.\\ &\qquad\left.\times \int_{M_{2n,n}(\mathbb{Z}_p)} |\det x_2|_p^{s-n}\,\chi_\ell(x)\psi(\mathrm{tr}(H_n[x]R))\,dx\right). \qquad (2)\end{aligned}$$

By the definition of χ_ℓ, the value of the function $|\det x_2|_p^{s-n}\chi_\ell(x)$ on $M_{2n,n}(\mathbb{Z}_p)$ is determined by the residue class of $x \bmod p^\ell$. Hence, by (*), the integral on the right hand side of (2) does not depend on e if $e \geqslant \ell > r$. Therefore, if $\mathrm{Re}(s) \geqslant n$, then it follows from the Lebesgue convergence theorem that

$$\int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n} |\Theta_T|_p = \lim_{\ell\to\infty} \int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n} \chi_\ell(x) |\Theta_T|_p$$
$$= \lim_{\ell\to\infty} \left(\int_{Sym_n(\frac{1}{p^\ell}\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\, dR \right.$$
$$\left. \times \int_{M_{2n,n}(\mathbb{Z}_p)} |\det x_2|_p^{s-n} \chi_\ell(x)\psi(\mathrm{tr}(H_n[x]R))\, dx \right).$$

To finish the proof, we need an estimate of

$$\left| \int_{Sym_n(\frac{1}{p^\ell}\mathbb{Z}_p)} \psi(-\mathrm{tr}(TR))\, dR \right.$$
$$\left. \times \int_{M_{2n,n}(\mathbb{Z}_p)} |\det x_2|_p^{s-n} (\chi_\ell(x)-1)\psi(\mathrm{tr}(H_n[x]R))\, dx \right|. \tag{3}$$

It is obvious that

$$(3) \leqslant \int_{Sym_n(\frac{1}{p^\ell}\mathbb{Z}_p)} dR \int_{\{x\in M_{2n,n}(\mathbb{Z}_p)\,|\,\det x_2\in p^\ell\mathbb{Z}_p\}} |\det x_2|_p^{\mathrm{Re}(s)-n}\, dx$$
$$= p^{\ell n(n+1)/2} \int_{\{x_2\in M_n(\mathbb{Z}_p)\,|\,\det x_2\in p^\ell\mathbb{Z}_p\}} |\det x_2|_p^{\mathrm{Re}(s)-n}\, dx_2.$$

We expand the zeta function $\zeta_n(s)$ of matrix algebra as follows:

$$\zeta_n(s) = \sum_{k=0}^{\infty} v_k p^{-ks}, \quad v_k = \int_{\{x_2\in M_n(\mathbb{Z}_p)\,|\,\det x_2\in p^k\mathbb{Z}_p^\times\}} \frac{dx_2}{|\det x_2|_p^n}.$$

Then, since $\zeta_n(s)$ converges when $\mathrm{Re}(s) > n-1$, for any $\epsilon > 0$ there exists a constant C_ϵ satisfying $v_k < C_\epsilon p^{k(n-1+\epsilon)}$ for every k. Hence we have

$$\int_{\{x_2\in M_n(\mathbb{Z}_p)\,|\,\det x_2\in p^\ell\mathbb{Z}_p\}} |\det x_2|_p^{\mathrm{Re}(s)-n}\, dx_2 = \sum_{k=\ell}^{\infty} v_k p^{-k\mathrm{Re}(s)}$$
$$< C_\epsilon \sum_{k=\ell}^{\infty} p^{-k(\mathrm{Re}(s)-n+1-\epsilon)} = \frac{C_\epsilon}{1-p^{-(\mathrm{Re}(s)-n+1-\epsilon)}} \cdot p^{-\ell(\mathrm{Re}(s)-n+1-\epsilon)}.$$

Thus, if $\mathrm{Re}(s) > n(n+3)/2 - 1 + \epsilon$, we obtain

$$(3) < \frac{C_\epsilon}{1-p^{-(\mathrm{Re}(s)-n+1-\epsilon)}} \cdot p^{-\ell(\mathrm{Re}(s)-n(n+3)/2+1-\epsilon)} \longrightarrow 0 \quad (\ell\to\infty).$$

This implies that the identity (1) holds for $\mathrm{Re}(s) > n(n+3)/2 - 1$. $\square$

3. Spherical functions on $O(H_n)/(O(T) \times O(T))$ and the relation to the Siegel series

Let G be the orthogonal grouop of H_n:

$$G = O(H_n) = \{g \in GL_{2n}(\mathbb{Q}_p) \mid H_n[g] = H_n\}.$$

Then, by Witt's theorem, the left action of G on $X_T(\mathbb{Q}_p)$ is transitive. We choose $x_T = \begin{pmatrix} T \\ E_n \end{pmatrix}$ as a representative point. Consider the subgroup H of G given by

$$H = \{g \in G \mid gx_T = x_T h \quad \text{for some } h \in GL_n(\mathbb{Q}_p)\}. \tag{4}$$

For any $g \in H$, the h satisfying $gx_T = x_T h$ is necessarily in $O(T) = \{h \in GL_n(\mathbb{Q}_p) \mid T[h] = T\}$. If we put

$$H_1 = \left\{ \begin{pmatrix} {}^t h^{-1} & 0 \\ 0 & h \end{pmatrix} \,\middle|\, h \in O(T) \right\}, \quad H_2 = \{g \in G \mid gx_T = x_T\},$$

then it is easy to see that H_1, H_2 are contained in H and $H = H_1 H_2$. The group H is isomorphic to $O(T) \times O(T)$. The isomorphism is given explicitly by

$$O(T) \times O(T) \ni (h_1, h_2) \longmapsto \tilde{T}^{-1} \begin{pmatrix} {}^t h_1^{-1} & 0 \\ 0 & {}^t h_2^{-1} \end{pmatrix} \tilde{T} \in H, \quad \tilde{T} = \begin{pmatrix} E_n & -T \\ E_n & T \end{pmatrix}.$$

Denote by P the Siegel parabolic subgroup of G, namely,

$$P = \left\{ q = \begin{pmatrix} m & 0 \\ 0 & {}^t m^{-1} \end{pmatrix} \begin{pmatrix} E_n & A \\ 0 & E_n \end{pmatrix} \,\middle|\, m \in GL_n(\mathbb{Q}_p),\ A \in Alt_n(\mathbb{Q}_p) \right\}.$$

In the following we denote an element in P by q (not by p) to distiguish it from the fixed prime p, and the symbol $m(q)$ stands for the element in $GL_n(\mathbb{Q}_p)$ on the diagonal block of $q \in P$. The group P acts on the open set $X'_T(\mathbb{Q}_p) = \{x \in X_T(\mathbb{Q}_p) \mid \det x_2 \neq 0\}$ transitively. We define a function $\Psi_{T,s}(g)$ on G by setting

$$\Psi_{T,s}(g) = |\det(gx_T)_2|_p^{s-n} \quad (g \in G).$$

As is easily seen, the function $\Psi_{T,s}$ satisfies the property

$$\Psi_{T,s}(qgh) = |\det m(q)|_p^{-(s-n)}\, \Psi_{T,s}(g) \quad (q \in P,\ h \in H,\ g \in G). \tag{5}$$

We consider the maximal compact subgroup $K = G \cap GL_{2n}(\mathbb{Z}_p)$ of G. We define the spherical function $\omega_T(g; s)$ on G/H (with respect to the parabolic subgroup P) by

$$\omega_T(g; s) = \int_K \Psi_{T,s}(kg)\, dk, \tag{6}$$

where dk is the normalized Haar measure on K. The integral defining ω_T is absolutely convergent for $\mathrm{Re}(s) \geqslant n$ and represents a rational function of p^{-s}. Moreover the function ω_T is an eigenfunction on G/H under the natural action of the Hecke algebra of G with respect to K.

The following theorem gives a relation between the Siegel series $b_p(T;s)$ and the spherical function ω_T.

Theorem 3.1. *Decompose the set $X_T(\mathbb{Z}_p)$ into K-orbits and write*

$$X_T(\mathbb{Z}_p) = Kg_1x_T \cup \cdots \cup Kg_rx_T \quad (g_1, \ldots, g_r \in G).$$

Then we have

$$b_p(T;s) = \zeta_n(s)^{-1} \sum_{i=1}^{r} c_i \cdot \omega_T(g_i;s), \quad c_i = \int_{Kg_ix_T} |\Theta_T|_p .$$

Proof. Note that the number of K-orbits in $X_T(\mathbb{Z}_p)$ is finite, since $X_T(\mathbb{Z}_p)$ is compact and K-orbits are open. For $\mathrm{Re}(s) > n$, we have

$$\begin{aligned}\int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n} |\Theta_T|_p &= \sum_{i=1}^{r} \int_{Kg_ix_T} |\det x_2|_p^{s-n} |\Theta_T|_p \\ &= \sum_{i=1}^{r} \int_{Kg_ix_T} \left(\int_K |\det(kx)_2|_p^{s-n}\, dk \right) |\Theta_T|_p .\end{aligned}$$

The integral over K depends only on the K-orbit to which x belongs and is equal to $\omega_T(g_i;s)$. Hence we obtain

$$\int_{X_T(\mathbb{Z}_p)} |\det x_2|_p^{s-n} |\Theta_T|_p = \sum_{i=1}^{r} c_i \cdot \omega_T(g_i;s).$$

Now the theorem follows immediately from Theorem 2.1. □

4. Functional equation of the Siegel series

Now we discuss the functional equations satisfied by the Siegel sereis $b_p(T;s)$ and the spherical functions ω_T.

For $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$, put

$$\gamma_p(T;X) = \begin{cases} (1-X)(1-p^{n/2}\xi_p(T)X)^{-1} \cdot \prod_{j=1}^{n/2}(1-p^{2j}X^2) & (n \equiv 0 \bmod 2), \\ (1-X) \cdot \prod_{j=1}^{(n-1)/2}(1-p^{2j}X^2) & (n \equiv 1 \bmod 2), \end{cases}$$

where, for even n,

$$\xi_p(T) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(\sqrt{d(T)}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{d(T)}) \text{ is unramified}, \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{d(T)}) \text{ is ramified}, \end{cases} \quad d(T) = (-1)^{n/2} \det T.$$

Then $b_p(T;s)/\gamma_p(T;p^{-s})$ is a polynomial of p^{-s}. Namely $b_p(T;s) = \gamma_p(T;p^{-s})F_p(T;p^{-s})$ for some polynomial $F_p(T;X)$ ([11]) and $F_p(T;X)$ satisfies the following functional equation.

Theorem 4.1. (Katsurada [10], Theorem 3.2) *The function $F_p(T;p^{-s})$ satisfies the following functional equation:*

$$F_p(T;p^{-(n+1-s)}) = \zeta_p(T)\,(p^{(n+1)/2-s})^{-\eta_p(T)}\,|\det T|_p^{-s+(n+1)/2}\,F_p(T;p^{-s}).$$

Here

$$\zeta_p(T) = \begin{cases} 1 & (n \equiv 0 \bmod 2) \\ h_p(T)\,(\det T, (-1)^{(n-1)/2}\det T)_p\,(-1,-1)_p^{(n^2-1)/8} & (n \equiv 1 \bmod 2) \end{cases}$$

where $h_p(T)$ is the Hasse invariant of T and $(\cdot,\cdot)_p$ is the Hilbert symbol, and

$$\eta_p(T) = 2\left[\frac{n}{2}\right]\delta_{2p} + \begin{cases} (-1)^{\delta_{2p}}\,r(\det T) & (n \equiv 0 \bmod 2), \\ 0 & (n \equiv 1 \bmod 2), \end{cases}$$

where $\delta_{2p} = 0$ or 1 according as $p > 2$ or $= 2$, and $r(\det T) = 0$ or 1 according as $\mathrm{ord}_p(\det T)$ *is even or odd.*

Katsurada's proof of this functional equation (as well as the one given by Böcherer and Kohnen [3]) is based on the (global) functional equation of the real analytic Siegel Eisenstein series. A local proof can be given on the basis of the result of Karel [8] (see also [13]). Here we give another local proof based on Theorem 3.1 and the representation theory of the p-adic group $G = O(H_n)$. By Theorem 3.1, what we have to do is to prove the following theorem.

Theorem 4.2. *The spherical function ω_T satisfies the functional equation*

$$|\det T|_p^{(s-1)/2}\,\omega_T(g; n+1-s) = c(T;s)\,|\det T|_p^{(n-s)/2}\,\omega_T(g;s), \qquad (7)$$

where

$$c(T;s) = \zeta_p(T)(p^{(n+1)/2-s})^{-\eta_p(T)} \cdot \frac{\zeta_n(s)\gamma_p(T;p^{-s})}{\zeta_n(n+1-s)\gamma_p(T;p^{-(n+1-s)})}. \qquad (8)$$

Note that, once the functional equation (7) is established, then we can see immediately (without using the explicit formula (8)) that $c(T,s)$ depends only on the $GL_n(\mathbb{Q}_p)$-equivalence class of T. Indeed, for $h \in GL_n(\mathbb{Q}_p)$, we have

$$x_{T[h]} = \tilde{h} x_T {}^t h, \quad \tilde{h} = \begin{pmatrix} {}^t h & 0 \\ 0 & h^{-1} \end{pmatrix}.$$

Hence, we obtain

$$|\det(T[h])|_p^{(n-s)/2}\,\omega_{T[h]}(g;s) = |\det T|_p^{(n-s)/2}\,\omega_T\left(g\tilde{h};s\right).$$

This implies that $c(T[h];s) = c(T;s)$.

Recall that a half-integral symmetric matrix $T \in \mathcal{H}_n(\mathbb{Z}_p)$ is called *maximal* if there exist no matrices $V \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$ such that $\det V \in p\mathbb{Z}_p$ and $T[V^{-1}] \in \mathcal{H}_n(\mathbb{Z}_p)$. Each $GL_n(\mathbb{Q}_p)$-equivalence class in $\mathcal{H}_n(\mathbb{Z}_p)$ contains a maximal element. Therefore, by the $GL_n(\mathbb{Q}_p)$-invariance of $c(T;s)$, it is sufficient to determine $c(T;s)$ for maximal half-integral symmetric matrices. Maximal half-integral symmetric matrices are classifed and the Siegel series for them are already known ([10] Lemma 3.3, [11]; see also [2]). In these cases it is easy to check that $c(T;s)$ coincides with the right hand side of (8).

Thus Theorem 4.2 (and hence Theorem 4.1) follows from the following weaker version of the functional equation.

Proposition 4.1. *The spherical function ω_T satisfies the functional equation*

$$\omega_T(g; n+1-s) = d(T;s)\omega_T(g;s), \tag{9}$$

where $d(T;s)$ is a rational function of p^{-s} independent of $g \in G$.

Proposition 4.1 will be proved in Section 6 after some preparations on degenerate principal series for $O(H_n)$ given in the next section.

5. Degenerate principal series representation for $O(H_n)$

As in the previous sections, we put $G = O(H_n)$, $K = G \cap GL_{2n}(\mathbb{Z}_p)$ and denote by P the Siegel parabolic subgroup of G. Let $\mathcal{C}^\infty(G)$ be the space of locally constant functions on G and $\mathcal{S}(G)$ the subspace of $\mathcal{C}^\infty(G)$ consisting of functions with compact support. The space of left K-invariant functions in $\mathcal{S}(G)$ is denoted by $\mathcal{S}(K\backslash G)$. For $f_1, f_2 \in \mathcal{C}^\infty(G)$, if one of f_1 and f_2 is in $\mathcal{S}(G)$, then the convolution product

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g)\,dx$$

defines a function in $\mathcal{C}^\infty(G)$, where dx is the Haar measure on G normalized by $\int_K dx = 1$.

For $\lambda \in \mathbb{C}$, we define $I(P,\lambda)$ by

$$I(P,\lambda) = \left\{\phi \in \mathcal{C}^\infty(G) \;\middle|\; \phi(qg) = |\det m(q)|_p^{\lambda+(n-1)/2}\,\phi(g)\ (g \in G,\ q \in P)\right\}.$$

The right translation of G on $I(P,\lambda)$ gives an admissible representation π_λ of G, the degenerate principal series representation:

$$\pi_\lambda(x)\phi(g) = \phi(gx) \quad (g, x \in G).$$

The representation π_λ is irreducible if

$$\prod_{i=1}^{n}\left(1 - p^{-(2\lambda+n-2i+1)}\right) \neq 0. \tag{10}$$

(The corresponding irreducibility criterion is proved for $Sp(2n)$ by Gustafson [4]. His proof can be transfered to the present case almost word by word. See also [1].) The parameter λ is said to be *generic*, if it satisfies the condition (10).

Let $\mathrm{pr}_\lambda : \mathcal{S}(G) \to I(P,\lambda)$ be the canonical surjection defined by

$$\mathrm{pr}_\lambda(\tilde{\phi})(g) = \int_P \tilde{\phi}(qg)\,|\det m(q)|_p^{-\lambda+\frac{n-1}{2}}\,dq,$$

where dq is the left invariant measure on P normalized by $\int_{P\cap K} dp = 1$. We denote by $\mathbf{1}_\lambda$ the function in $I(P,\lambda)$ that is identically equal to 1 on K. Then, we have

$$\mathrm{pr}_\lambda(\tilde{\phi}) = \mathbf{1}_\lambda * \tilde{\phi} \quad (\tilde{\phi} \in \mathcal{S}(K\backslash G)). \tag{11}$$

This implies that $\mathrm{pr}_\lambda(\mathcal{S}(K\backslash G)) = \mathbf{1}_\lambda * \mathcal{S}(K\backslash G)$ is G-invariant. Hence the restriction of pr_λ to $\mathcal{S}(K\backslash G)$ is surjective, if λ is generic.

Let $\mathcal{D}(P,-\lambda)$ be the (algebraic) dual of $I(P,\lambda)$ and $\mathcal{D}_c(P,-\lambda)$ the space of continuous functions ψ on G satisfying $\psi(qg) = |\det m(q)|^{-\lambda+\frac{n-1}{2}}\,\psi(g)$ $(q \in P,\ g \in G)$. Then $\mathcal{D}_c(P,-\lambda)$ can be regarded as a subspace of $\mathcal{D}(P,-\lambda)$ by the pairing

$$\langle \psi, \phi \rangle = \int_K \phi(k)\psi(k)\,dk \quad (\psi \in \mathcal{D}_c(P,-\lambda),\ \phi \in I(P,\lambda)).$$

Let $\mathcal{C}(K\backslash G)$ be the space of all left K-invariant fucntions on G. Then, $\mathcal{C}(K\backslash G)$ can be identified with the dual space of $\mathcal{S}(K\backslash G)$ by the pairing

$$(\omega, \tilde{\phi}) = \int_G \omega(x)\tilde{\phi}(x)\,dx \quad (\omega \in \mathcal{C}(K\backslash G),\ \tilde{\phi} \in \mathcal{S}(K\backslash G)).$$

Define the mapping $\mathcal{P}_{-\lambda} : \mathcal{D}(P,-\lambda) \to \mathcal{C}(K\backslash G)$ to be the dual of $\mathrm{pr}_\lambda : \mathcal{S}(K\backslash G) \to I(P,\lambda)$. The mapping $\mathcal{P}_{-\lambda}$ is a special case of the Poisson transformation studied by Kato in [9] in much greater generality. The

Poisson transformation $\mathcal{P}_{-\lambda}$ is injective if $-\lambda$ is generic (equivalently, if λ is generic). For a $\psi \in \mathcal{D}_c(P, -\lambda)$ and $\tilde{\phi} \in \mathcal{S}(K\backslash G)$, we have

$$\begin{aligned}\mathcal{P}_{-\lambda}\psi(\tilde{\phi}) = \langle \psi, \mathrm{pr}_\lambda(\tilde{\phi})\rangle &= \int_K \psi(k) \left(\int_P \tilde{\phi}(qk) \left|\det m(q)\right|_p^{-\lambda + \frac{n-1}{2}} dq \right) dk \\ &= \int_K \int_P \psi(qk)\tilde{\phi}(qk)\, dq\, dk = \int_G \psi(x)\tilde{\phi}(x)\, dx \\ &= \int_G \psi(x) \int_K \tilde{\phi}(kx)\, dk\, dx = \int_G \left(\int_K \psi(kx)\, dk \right) \tilde{\phi}(x)\, dx.\end{aligned}$$

Thus the Poisson transform of a continuous function $\psi \in \mathcal{D}_c(P, -\lambda)$ is given by

$$\mathcal{P}_{-\lambda}\psi(g) = \int_K \psi(kg)\, dk.$$

Let $\mathcal{H}(G, K)$ be the Hecke algebra of G with repsect to K. Namely $\mathcal{H}(G, K)$ is the space of K-biinvariant functions in $\mathcal{S}(G)$ equipped with the convolution product. Since the space $I(P, \lambda)^K$ of K-invariant functions in $I(P, \lambda)$ is 1-dimensional and is spanned by $\mathbf{1}_\lambda$, the Hecke algebra $\mathcal{H}(G, K)$ acts on $I(P, \lambda)^K$ as scalar. Hence we have

$$\mathbf{1}_\lambda * f = \hat{f}(\lambda)\mathbf{1}_\lambda \quad (f \in \mathcal{H}(G, K)).$$

The scalar $\hat{f}(\lambda)$ is given by

$$\hat{f}(\lambda) = \int_P \left|\det m(q)\right|_p^{-\lambda + \frac{n-1}{2}} f(q)\, dq.$$

Note that $\hat{f}(-\lambda) = \hat{f}(\lambda)$. To see this, we consider the involution $q \mapsto q^* := w\,{}^t q w^{-1}$ of P, where $w = \begin{pmatrix} 0_n & E_n \\ E_n & 0_n \end{pmatrix} \in K$. Then we have $f(q^*) = f(q)$ for any $f \in \mathcal{H}(G, K)$ and $dq^* = \left|\det m(q)\right|^{n-1} dq$. Hence we have

$$\begin{aligned}\hat{f}(-\lambda) &= \int_P \left|\det m(q)\right|_p^{\lambda + \frac{n-1}{2}} f(q)\, dq = \int_P \left|\det m(q)\right|_p^{\lambda + \frac{n-1}{2}} f(q^*)\, dq \\ &= \int_P \left|\det m(q)\right|_p^{-\lambda + \frac{n-1}{2}} f(q)\, dq = \hat{f}(\lambda).\end{aligned}$$

Lemma 5.1. *Put*

$$\mathcal{A}(K\backslash G, \lambda) = \left\{ \omega \in \mathcal{C}(K\backslash G) \,\middle|\, f * \omega = \hat{f}(\lambda)\omega \ (f \in \mathcal{H}(G, K)) \right\}.$$

Then the Poisson transformation $\mathcal{P}_{-\lambda}$ defines a G-morphism of $\mathcal{D}(P, -\lambda)$ to $\mathcal{A}(K\backslash G, -\lambda) = \mathcal{A}(K\backslash G, \lambda)$, which is injective if λ is generic.

Proof. It is enough to prove that the image is included by $\mathcal{A}(K\backslash G,-\lambda)$. For $\psi \in \mathcal{D}(P,-\lambda)$, $f \in \mathcal{H}(G,K)$ and $\tilde{\phi} \in \mathcal{S}(K\backslash G)$, we have

$$\begin{aligned}(f * \mathcal{P}_{-\lambda}(\psi), \tilde{\phi}) &= \int_G \left(\int_G f(x)\mathcal{P}_{-\lambda}(\psi)(x^{-1}g)\, dx \right) \tilde{\phi}(g)\, dg \\ &= \int_G \mathcal{P}_{-\lambda}(\psi)(g) \left(\int_G f(x)\tilde{\phi}(xg)\, dx \right) dg.\end{aligned}$$

This shows that $(f * \mathcal{P}_{-\lambda}(\psi), \tilde{\phi}) = \psi(\mathrm{pr}_\lambda(\check{f} * \tilde{\phi}))$, where $\check{f}(x) = f(x^{-1})$. By (11), we have

$$\mathrm{pr}_\lambda(\check{f} * \tilde{\phi}) = \mathbf{1}_\lambda * (\check{f} * \tilde{\phi}) = (\mathbf{1}_\lambda * \check{f}) * \tilde{\phi} = \hat{\check{f}}(\lambda)\mathbf{1}_\lambda * \tilde{\phi} = \hat{f}(-\lambda)\mathrm{pr}_\lambda(\tilde{\phi}).$$

Hence

$$(f * \mathcal{P}_{-\lambda}(\psi), \tilde{\phi}) = \hat{f}(-\lambda)\psi(\mathrm{pr}_\lambda(\tilde{\phi})) = (\hat{f}(-\lambda)\mathcal{P}_{-\lambda}(\psi), \tilde{\phi}).$$

This proves that $\mathcal{P}_{-\lambda}(\psi)$ is in $\mathcal{A}(K\backslash G,-\lambda)$ for any $\psi \in \mathcal{D}(P,-\lambda)$. □

Let $T_\lambda : I(P,\lambda) \to I(P,-\lambda)$ be the intertwining operator given by the analytic continuation of the integral

$$T_\lambda(\phi)(g) = \int_{Alt_n(\mathbb{Q}_p)} \phi\left(\begin{pmatrix} 0_n & E_n \\ E_n & A \end{pmatrix} g \right) dA \quad (g \in G,\ \phi \in I(P,\lambda)), \tag{12}$$

where dA is the Haar measure of the space $Alt_n(\mathbb{Q}_p)$ of alternating matrices of size n normalized by $\int_{Alt_n(\mathbb{Z}_p)} dA = 1$. The integral defining T_λ is absolutely convergent for $\mathrm{Re}\,\lambda > \frac{n-1}{2}$, and the analytic continuation of T_λ multiplied by

$$\prod_{i=1}^{[n/2]} (1 - p^{-(2\lambda - n + 2i)})$$

is entire and gives a non-trivial intertwining operator. This can be proved by the method developed in [15], Section 4. What we have to take into account is the fact that $G = O(H_n)$ is not connected and there exist two open Bruhat cells $Pw_{n-1}P$ and Pw_nP, where

$$w_r = \begin{pmatrix} 0 & 0 & E_r & 0 \\ 0 & E_{n-r} & 0 & 0 \\ E_r & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{n-r} \end{pmatrix}$$

for $r = n-1, n$. Anyway, the analysis of the regularity of T_λ can be reduced to the p-adic local zeta function of the Pfaffian studied in [5].

The intertwining operator T_λ maps a K-invariant vector to a K-invariant one. In particular we have

$$T_\lambda \mathbf{1}_\lambda = c(\lambda)\mathbf{1}_{-\lambda}, \quad c(\lambda) = \prod_{i=1}^{n} \frac{1-p^{-(2\lambda+n-2i+1)}}{1-p^{-(2\lambda-n+i)}}.$$

This explicit expression of $c(\lambda)$ can be proved by the method of Kitaoka. (See the proof of Theorem 1 and Corollaries to it in [11]. The function $c(\lambda)$ is the alternating analogue of $b_p(s, O^{(n)})$ in [11].)

Lemma 5.2. *Let $T_\lambda^* : \mathcal{D}(P,\lambda) \to \mathcal{D}(P,-\lambda)$ be the dual mapping of $T_\lambda : I(P,\lambda) \to I(P,-\lambda)$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{D}(P,\lambda) & \xrightarrow{\mathcal{P}_\lambda} & \mathcal{A}(K\backslash G,\lambda) \\ \big\downarrow{\scriptstyle T_\lambda^*} & & \big\downarrow{\scriptstyle \times c(\lambda)} \\ \mathcal{D}(P,-\lambda) & \xrightarrow{\mathcal{P}_{-\lambda}} & \mathcal{A}(K\backslash G,-\lambda), \end{array}$$

where the right vertical arrow is the multiplication by $c(\lambda)$.

Proof. For $\psi \in \mathcal{D}(P,\lambda)$ and $\tilde{\phi} \in \mathcal{S}(K\backslash G)$, we have

$$\begin{aligned} (\mathcal{P}_{-\lambda} \circ T_\lambda^*(\psi), \tilde{\phi}) &= \langle T_\lambda^*(\psi), \mathrm{pr}_\lambda(\tilde{\phi})\rangle = \langle T_\lambda^*(\psi), \mathbf{1}_\lambda * \tilde{\phi}\rangle \\ &= \langle \psi, T_\lambda(\mathbf{1}_\lambda * \tilde{\phi})\rangle = \langle \psi, T_\lambda(\mathbf{1}_\lambda) * \tilde{\phi}\rangle \\ &= \langle \psi, c(\lambda)\mathbf{1}_{-\lambda} * \tilde{\phi}\rangle = c(\lambda)\langle \psi, \mathbf{1}_{-\lambda} * \tilde{\phi}\rangle \\ &= c(\lambda)\langle \psi, \mathrm{pr}_{-\lambda}(\tilde{\phi})\rangle = (c(\lambda)\mathcal{P}_\lambda(\psi), \tilde{\phi}). \end{aligned}$$

This proves the commutativity of the diagram in the lemma. □

6. Proof of the functional equation of spherical functions

Now we are in a position to prove Proposition 4.1. Let $\Psi_{T,s}(g)$ be the function defined in Section 3. Then, by (5), if $\mathrm{Re}\, s \geqslant n$, then $\Psi_{T,s}$ belongs to $\mathcal{D}_c(P, -s + \frac{n+1}{2})$. The analytic continuation of $\Psi_{T,s}$ gives a distribution in $\mathcal{D}(P, -s + \frac{n+1}{2})$ depending meromorphically on $s \in \mathbb{C}$. The spherical function $\omega_T(g; s)$ defined by (6) is nothing but the Poisson transform of $\Psi_{T,s}$:

$$\omega_T(g;s) = \mathcal{P}_{-s+\frac{n+1}{2}}(\Psi_{T,s})(g).$$

Let H be the subgroup of G defined by (4) and denote by $\mathcal{D}(P, -s + \frac{n+1}{2})^H$ the space of H-invariant distributions in $\mathcal{D}(P, -s + \frac{n+1}{2})$. It is obvious that $\Psi_{T,s}$ is in $\mathcal{D}(P, -s + \frac{n+1}{2})^H$.

Lemma 6.1. *If $s \notin \{0,1,\ldots,n-1\} + \frac{2\pi i}{\log p}\mathbb{Z}$, then $\Psi_{T,s}$ is holomophic at s and $\mathcal{D}(P, -s + \frac{n+1}{2})^H$ coincides with the 1-dimensional vector space spanned by $\Psi_{T,s}$.*

Admitting Lemma 6.1 for the moment, we continue the proof of Proposition 4.1. Set $\lambda = -s + \frac{n-1}{2}$ with s satisfying the condition of Lemma 6.1. Since T_λ^* maps $\mathcal{D}(P,\lambda)^H$ to $\mathcal{D}(P,-\lambda)^H$, Lemma 6.1 implies that there exists a rational function $a(T;s)$ of p^{-s} satisfying $T_\lambda^*(\Psi_{T,s}) = a(T;s)\Psi_{T,n+1-s}$. Applying the Poisson transformation, we have by Lemma 5.2

$$\begin{aligned} a(T;s)\,\omega_T(g,n+1-s) &= \mathcal{P}_{-\lambda}\circ T_\lambda^*(\Psi_{T,s})(g) \\ &= c(\lambda)\,\mathcal{P}_\lambda(\Psi_{T,s})(g) \\ &= c\left(-s+\frac{n-1}{2}\right)\,\omega_T(g,s). \end{aligned}$$

This completes the proof of Proposition 4.1. □

For a subset U of G, we denote by $\mathcal{D}(P,-s+\frac{n+1}{2})_U^H$ the space of distributions in $\mathcal{D}(P,-s+\frac{n+1}{2})^H$ with support in U. For a (P,H)-double coset $\mathcal{O} = Pg_0H$ in G, put $G_\mathcal{O} = \{(q,h)\in P\times H \mid qg_0h^{-1} = g_0\}$, which is determined up to conjugate, and denote by $\delta_\mathcal{O}$ the modulus character of $G_\mathcal{O}$.

Proof of Lemma 6.1. It is known that the group G has a finite number of (P,H)-double cosets (see Lemma 6.2 below). As is noted in Section 3, $X_T'(\mathbb{Q}_p)$ is a single P-orbit. This implies that PH is the unique open dense (P,H)-double coset. Therefore, for any distribution ψ in $\mathcal{D}(P,-s+\frac{n+1}{2})^H$, the restriction $\psi|_{PH}$ is a constant multiple of $\Psi_{T,s}$ (by the uniqueness of relatively invariant distributions on a homogeneous space). Hence what we have to prove is that

$$\mathcal{D}(P,-s+\frac{n+1}{2})_{G-PH}^H = \{0\} \quad \text{unless} \quad s\in\{0,1,\ldots,n-1\}+\frac{2\pi i}{\log p}\mathbb{Z}. \tag{13}$$

Note that this yields also the holomorphy statement on $\Psi_{T,s}$ in the lemma, since the first non-zero coefficient of the Laurent expansion of $\langle\Psi_{T,s},\phi\rangle$ ($\phi\in I(P,s-\frac{n+1}{2})$) at a pole defines a distribution in $\mathcal{D}(P,-s+\frac{n+1}{2})_{G-PH}^H$. By the same argument as in [6], Section 2, we see that, if $\mathcal{D}(P,-s+\frac{n+1}{2})_{G-PH}^H \neq \{0\}$, then there exists a double coset $\mathcal{O}$ contained in $G-PH$ such that $\mathcal{D}(P,-s+\frac{n+1}{2})_\mathcal{O}^H \neq 0$. The modulus character of $P\times H$ is given by $\Delta(q,h) = |\det m(q)|_p^{-(n-1)}$. Hence, if $\mathcal{D}(P,-s+\frac{n+1}{2})_\mathcal{O}^H \neq \{0\}$, then, the character identity

$$\delta_\mathcal{O}(q,h) = |\det m(q)|_p^{-s+1} \;\; (= |\det m(q)|_p^{-s+n}\cdot\Delta(q,h))$$

should hold for any $(q,h) \in G_{\mathcal{O}}$. Put $G_r = \{g \in G \mid \operatorname{rank}(gx_T)_2 = r\}$ $(0 \leqslant r \leqslant n)$. Then G_r are $P \times H$-stable and G_n is the open (P,H)-double coset. By Corollary to Lemma 6.2 below, we have $\delta_{\mathcal{O}}(q,h) = |\det m(q)|_p^{-r+1}$ if $\mathcal{O} \subset G_r$ for $r < n$. This proves (13). $\square$

Lemma 6.2. *Let F be the algebraic closure of $\mathbb{Q}_p$ and assume that $T = E_n$. For a non-negative integer r, put*

$$G_r(F) = \{g \in G(F) \mid \operatorname{rank}(gx_T)_2 = r\}.$$

Then, $G_r(F)$ is non-empty if and only if $n \geqslant r \geqslant n/2$, and the $(P(F),H(F))$-double coset decomposition of $G(F)$ is given by

$$G(F) = \bigcup_{\substack{r \\ n \geqslant r \geqslant n/2}} G_r(F), \quad G_r(F) = P(F) g_r H(F),$$

where g_r is an element in $G(F)$ such that

$$g_r x_T = x^{(r)} := \left(\begin{array}{c|c} E_r & -\sqrt{-1} J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \\ \hline E_r & \sqrt{-1} J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \end{array}\right), \quad J_{r,n-r} = \begin{pmatrix} 0_{2r-n,n-r} \\ E_{n-r} \end{pmatrix}.$$

The double coset $G_n(F)$ is the unique open double coset.

Corollary. *Let $r < n$ and $\mathcal{O}$ be a (P,H)-double coset contained in G_r. Then the identity component of $G_{\mathcal{O}}$ is isomorphic to*

$$(SO_{2r-n} \times GL_{n-r} \times GL_{n-r}) \ltimes (M_{2n-2r,2r-n} \oplus Alt_{n-r} \oplus Alt_{n-r})$$

over the algebraic closure of $\mathbb{Q}_p$, and $\delta_{\mathcal{O}}(q,h) = |\det m(q)|_p^{-(r-1)}$.

Proof. The group $G_{\mathcal{O}}$ is isomorphic to the isotropy subgroup of $P \times O_n$ at $x^{(r)}$ over the algebraic closure of $\mathbb{Q}_p$. Let us calculate its Lie algebra. An element

$$\left(\left(\begin{array}{cc|cc} B_1 & B_2 & A_1 & A_2 \\ B_3 & B_4 & -{}^tA_2 & A_3 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_n}} & -{}^tB_1 & -{}^tB_3 \\ & & -{}^tB_2 & -{}^tB_4 \end{array}\right), \begin{pmatrix} K_1 & K_2 \\ -{}^tK_2 & K_3 \end{pmatrix} \right)$$

of the Lie algebra of $P \times O_n$ is in the isotropy subalgebra at $x^{(r)}$ if and only if

$$\begin{pmatrix} B_1 & A_1 \\ B_3 & -{}^tA_2 \\ 0 & -{}^tB_1 \\ 0 & -{}^tB_2 \end{pmatrix} \begin{pmatrix} E_r & -\sqrt{-1}J_{r,n-r} \\ E_r & \sqrt{-1}J_{r,n-r} \end{pmatrix} = \begin{pmatrix} E_r & -\sqrt{-1}J_{r,n-r} \\ 0 & 0 \\ E_r & \sqrt{-1}J_{r,n-r} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K_1 & K_2 \\ -{}^tK_2 & K_3 \end{pmatrix}.$$

Solving this equation, we see that the isotropy subalgebra at $x^{(r)}$ is the collection of elements of the form

$$\left(\begin{array}{c} \begin{array}{cc} & \begin{array}{cccccc} 2r-n & n-r & n-r & 2r-n & n-r & n-r \end{array} \\ \begin{array}{c} 2r-n \\ n-r \\ n-r \\ 2r-n \\ n-r \\ n-r \end{array} & \begin{pmatrix} A_1 & 0 & 0 & 0 & -{}^tC_1 & {}^tC_3 \\ C_1 & C_2 & 0 & C_1 & A_2 & 0 \\ C_3 & 0 & C_4 & -C_3 & 0 & A_3 \\ & & & A_1 & -{}^tC_1 & -{}^tC_3 \\ & 0 & & 0 & -{}^tC_2 & 0 \\ & & & 0 & 0 & -{}^tC_4 \end{pmatrix} \end{array}, \\ \begin{array}{cc} & \begin{array}{ccc} 2r-n & n-r & n-r \end{array} \\ \begin{array}{c} 2r-n \\ n-r \\ n-r \end{array} & \begin{pmatrix} A_1 & -{}^tC_1 & -\sqrt{-1}\,{}^tC_1 \\ C_1 & \frac{1}{2}(C_2 - {}^tC_2 + A_2) & \frac{\sqrt{-1}}{2}(C_2 + {}^tC_2 - A_2) \\ \sqrt{-1}\,C_1 & -\frac{\sqrt{-1}}{2}(C_2 + {}^tC_2 + A_2) & \frac{1}{2}(C_2 - {}^tC_2 - A_2) \end{pmatrix} \end{array} \end{array} \right),$$

where A_1, A_2, A_3 are alternating and C_1, C_2, C_3, C_4 are arbitrary. Therefore the identity component of $G_{\mathcal{O}}$ is isomorphic to

$$(SO_{2r-n} \times GL_{n-r} \times GL_{n-r}) \ltimes (M_{2n-2r,2r-n} \oplus Alt_{n-r} \oplus Alt_{n-r})$$

over the algebraic closure of $\mathbb{Q}_p$. The modulus character $\delta_{\mathcal{O}}(q,h)$ of $G_{\mathcal{O}}$ is equal to $|\det Ad(q,h)|_{\mathfrak{u}}|_p^{-1}$, where $\mathfrak{u}$ is the unipotent radical of the isotropy subalgebra, which is isomorphic to the abelian Lie algebra $M_{2n-2r,2r-n} \oplus Alt_{n-r} \oplus Alt_{n-r}$. Now it is easy to see that $\delta_{\mathcal{O}}(q,h) = |\det m(q)|_p^{-(r-1)}$. □

For the proof of Lemma 6.2, we need the following.

Lemma 6.3. *Let F be an algebraically closed field of characteristic* 0. *Consider the prehomogeneous vector space $(GL_n \times O_m, M_{n,m})$ $(1 \leqslant n \leqslant m)$, where the action of $GL_n \times O_m$ on $M_{n,m}$ is given by*

$$v \longmapsto gvk^{-1} \quad (g \in GL_n,\ k \in O_m, v \in M_{n,m}).$$

For any non-negative integers r, ℓ, put

$$V_{r,\ell} = \left\{ v \in M_{n,m}(F) \,\middle|\, \operatorname{rank} v = r,\ \operatorname{rank} v\,{}^t v = \ell \right\}.$$

Then, $V_{r,\ell}$ is non-empty if and only if $n \geqslant r \geqslant l \geqslant \max\{0, 2r-m\}$, and the $GL_n(F) \times O_m(F)$-orbit decomposition of $M_{n,m}(F)$ is given by

$$M_{n,m}(F) = \bigcup_{\substack{r,\ell \\ n \geqslant r \geqslant l \geqslant \max\{0,2r-m\}}} V_{r,\ell}.$$

Further, a complete set of representatives of the $GL_n(F) \times O_m(F)$-orbits is given by

$$\{ v_{r,\ell} \mid n \geqslant r \geqslant l \geqslant \max\{0, 2r-m\} \}, \quad v_{r,\ell} = \left(\begin{array}{c|c|c} E_r & \sqrt{-1} J_{r,r-\ell} & 0 \\ \hline 0 & 0 & 0 \end{array} \right).$$

A proof of the above lemma (under the condition $n \leqslant m/2$) is found in [16], Example 9.2. The case $n > m/2$ can be proved similarly.

Proof of Lemma 6.2. The $(P(F), H(F))$-double coset decomposition of $G(F)$ is equivalent to the $P(F) \times O_n(F)$-orbit decomposition of $X_T(F)$ under the action $(q,h)x = qxh^{-1}$ ($q \in P(F)$, $h \in O_n(F)$). We prove that, if $\operatorname{rank} x_2 = r$ for $x \in X_T(F)$, then $r \geqslant n/2$ and there exists an $(q,h) \in P(F) \times O_n(F)$ such that $(q,h)x = x^{(r)}$. Since $\operatorname{rank} x_2 = r$, by Lemma 6.3, x is sent by the action of an element in $P(F) \times O_n(F)$ to an element x' in $X_T(F)$ whose x_2 part is of the form

$$\left(\begin{array}{c|c|c} E_r & \sqrt{-1} J_{r,r-\ell} & 0 \\ \hline 0 & 0 & 0_{n-r,n-2r+\ell} \end{array} \right).$$

If $n - 2r - \ell > 0$, then the rank of $H_n[x']$ is less that n and $x' \notin X_T(F)$. Hence $\ell = 2r - n$. Therefore we may assume that x is of the form

$$x = \left(\begin{array}{c|c} y_1 & y_2 \\ \hline y_3 & y_4 \\ \hline E_r & \sqrt{-1} J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \end{array} \right).$$

Multiply x by

$$\left(\begin{array}{c|c} E_n & \begin{matrix} -\frac{1}{2}(y_1 - {}^t y_1) & {}^t y_3 \\ -y_3 & 0_{n-r} \end{matrix} \\ \hline 0_n & E_n \end{array}\right) \in P(F).$$

Then x becomes

$$x = \left(\begin{array}{c|c} y_1' & y_2' \\ \hline 0 & y_4' \\ \hline E_r & \sqrt{-1}J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \end{array}\right),$$

where y_1' is a symmetric matrix. The condition $H_n[x] = T = E_n$ implies that $y_1' = E_r$ and $y_2' = -\sqrt{-1}J_{r,n-r}$. Hence,

$$x = \left(\begin{array}{c|c} E_r & -\sqrt{-1}J_{r,n-r} \\ \hline 0 & y_4' \\ \hline E_r & \sqrt{-1}J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \end{array}\right).$$

Multiply x now by

$$\left(\begin{array}{cc|cc} E_r & 0 & 0 & A_2 \\ {}^t A_2 & E_{n-r} & -{}^t A_2 & {}^t A_2 A_2 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0_n}} & E_r & -A_2 \\ & & 0 & E_{n-r} \end{array}\right) \in P(F), \quad A_2 = \begin{pmatrix} 0_{2r-n,n-r} \\ -\sqrt{-1}\,{}^t y_4' \end{pmatrix}.$$

Then we finally obtain

$$x = x^{(r)} = \left(\begin{array}{c|c} E_r & -\sqrt{-1}J_{r,n-r} \\ \hline 0 & 0_{n-r} \\ \hline E_r & \sqrt{-1}J_{r,n-r} \\ \hline 0_{n-r,r} & 0_{n-r} \end{array}\right).$$

It is obvious that $G_n(F)$ is open. $\square$

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KOECHER–MAAß SERIES FOR REAL ANALYTIC SIEGEL EISENSTEIN SERIES

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Dedicated to the memory of Tsuneo Arakawa

1. Introduction

Koecher and Maaß defined a Dirichlet series $L(s, F)$, which is called the Koecher-Maaß series, as the Mellin transform of a Siegel modular form F of degree n and proved the functional equation (cf. Koecher [Ko1], [Ko2], Maaß [Ma]). The Koecher-Maaß series coincides with the usual Dirichlet series defined by Hecke when $n = 1$, and if F is a common eigenfunction, then $L(s, F)$ has Euler product. When $n \geqslant 2$, this Dirichlet series does not have Euler product in general, and is very different from automorphic L-functions, which have Euler product by definition. However, still, it is unexpectedly easily described for some modular forms. In [I-K1], we announced an explicit formula for $L(s, E_{n,k})$ for the holomorphic Siegel-Eisenstein series $E_{n,k}$ of weight k and degree n. We also gave an explicit formula of $L(s, F)$ when F is the Klingen Eisenstein lifting or the Ikeda lifting of an elliptic cusp form (cf. [I-K3], [I-K4]). We note that $L(s, E_{n,k})$ can also be regarded as the zeta function associated to a certain prehomo-

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geneous vector space. From this point of view, Saito gave a generalization of our result in [I-K1] (cf. [Sa]). On the other hand, in [A], Arakawa defined the Koecher-Maaß series associated with the real analytic Siegel Eisenstein series, and gave a functional equation of it, which is rather complicated.

In this paper, we shall give an explicit formula for the Koecher-Maaß series associated with the real analytic Siegel Eisenstein series, including a precise proof of the main results announced in [I-K1] (cf. Theorems 1.1 and 1.2 in Section 1). The formula depends heavily on the parity of n, but it is given as a sum of products of shifts of Riemann zeta functions and the convolution product of zeta functions associated with modular forms of half integral weights. By using such a formula combined with the result in Mizuno [Mi], we get a *functional equation for the Koecher-Maaß series for* the real analytic Eisenstein series, which is far simpler than those obtained by Arakawa [A] (cf. Theorems 4.1 and 4.2). This is a variant of the simple functional equation in [I-S2].

To state our main result explicitly, first we review the definition of the Koecher-Maaß series of a holomorphic modular form and the real analytic Siegel Eisenstein series.

Let G be a group acting on a set Y. We then denote by Y/G the set of equivalence classes of Y with respect to G. We sometimes use the same notation Y/G to denote a complete set of representatives of it. For any ring R, we denote by $S_n(R)$ the set of $n \times n$ symmetric matrices with entries in R. Let $G = GL_n(R)$ or $SL_n(R)$. Then G acts on $S_n(R)$ as usual by $G \times S_n(R) \ni (\gamma, T) \mapsto {}^t\gamma T \gamma \in S_n(R)$. Let R be an integral domain of characteristic different from 2, and K its quotient field. We call an element $A = (a_{ij})$ of $S_n(K)$ half integral if $a_{ij} \in \frac{1}{2}R$, and $a_{ii} \in R$. For the field $\mathbb{R}$ of real numbers, we denote by $S_n(\mathbb{R})^+$ the set of positive definite matrices in $S_n(\mathbb{R})$. We denote by $\mathcal{L}_n$ the submodule of $S_n(\mathbb{R})$ of $n \times n$ half integral matrices and put $\mathcal{L}_n^+ = \mathcal{L}_n \cap S_n(\mathbb{R})^+$. For a holomorphic Siegel modular form $f(Z)$ with respect to the full modular group $Sp(n, \mathbb{Z})$, we have the usual Fourier expansion

$$f(Z) = \sum_T a(T) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)},$$

where T runs over all semi-positive definite half-integral $n \times n$ matrices, and tr denotes the trace of a matrix. For any $T \in \mathcal{L}_n^+$, we denote by $e(T)$ the order of the finite group $\{\gamma \in SL_n(\mathbb{Z}); {}^t\gamma T \gamma = T\}$. We define the Koecher-

Maaß series as

$$L(s,f) = \sum_{\mathcal{L}_n^+/SL_n(\mathbb{Z})} \frac{a(T)}{e(T)\det(T)^s}.$$

This series converges for a complex number s with sufficiently large real part, and have a meromorphic continuation to the whole s plane. This is based on an integral expression of this series together with Gamma factors as a kind of Mellin transform of $f(Z)$ (cf. [Ma]). However, in case of general real analytic automorphic forms, we cannot define such Mellin transform, because of some difficult pathology of the convergence (cf. [Ma] p.307). All we know is that we can give a definition of the Koecher-Maaß series for real analytic Eisenstein series reasonably well by using its Fourier coefficients (see [A]). Now, we take a real analytic Siegel Eisenstein

$$E_{n,k}(Z,\sigma) = \sum_{C,D} \det(CZ+D)^{-k} |\det(CZ+D)|^{-2\sigma},$$

where k is an even integer and σ is a complex number. This series converges if $2\mathrm{Re}(\sigma)+k > n+1$. In order to explain the Fourier expansion, we introduce several notations. For complex numbers α, β and $Y \in S_n(\mathbb{R})^+$, $T \in S_n(\mathbb{R})$, we put

$$\begin{aligned}&\xi_n(Y,T,\alpha,\beta)\\ &= \int_{S_n(\mathbb{R})} e^{-2\pi\sqrt{-1}\mathrm{tr}(TX)} \det(X+\sqrt{-1}Y)^{-\alpha} \det(X-\sqrt{-1}Y)^{-\beta} dx,\end{aligned}$$

where $dX = \prod_{1 \leqslant i \leqslant j \leqslant n} dX_{ij}$. For a special case where $\beta = 0$ and $\alpha = k$ is a positive integer, we get

$$\xi_n(Y,T,k,0) = \frac{(-1)^{nk/2} 2^{n(k-(n-1)/2)} \pi^{nk}}{\pi^{n(n-1)/4} \prod_{j=1}^n \Gamma(k-(j-1)/2)} \det(T)^{k-(n+1)/2} e^{-2\pi \mathrm{tr}(TY)}.$$

For $T \in S_n(\mathbb{Q})$, we define the Siegel series by

$$b(T,\sigma) = \sum_{R \in S_n(\mathbb{Q})/S_n(\mathbb{Z})} \nu(R)^{-\sigma} e^{2\pi\sqrt{-1}\mathrm{tr}(TR)},$$

where $\nu(R) = [R\mathbb{Z}^n + \mathbb{Z}^n : \mathbb{Z}^n]$. Since $E_{n,k}(Z+S,\sigma) = E_{n,k}(Z,\sigma)$ for $S \in S_n(\mathbb{Z})$, we have the Fourier expansion

$$E_{n,k}(Z) = \sum_{T \in \mathcal{L}_n} c_{n,k}(T,\sigma,Y) e^{2\pi\sqrt{-1}\mathrm{tr}(TX)}$$

where we denote $Z = X + \sqrt{-1}Y$. The formula for the coefficients $c_{n,k}(T,\sigma,Y)$ is known and can be written by using the above confluent hypergeometric function and the Siegel series. We need here the formula only for $T \in \mathcal{L}_n$ with $\det(T) \neq 0$ (but not necessarily positive definite), which is given by

$$c_{n,k}(T,\sigma,Y) = b(T,k+2\sigma)\xi_n(Y,T,\sigma+k,\sigma).$$

For the sake of simplicity, we write

$$\gamma_n(\sigma) = e^{\pi\sqrt{-1}n\sigma/2}\prod_{i=0}^{n-1}\frac{\pi^{\sigma-i/2}}{\Gamma(\sigma-i/2)},$$

where $\Gamma(s)$ is the Gamma function. In order to define the Koecher-Maaß series consistently with the holomorphic case, for any $T \in \mathcal{L}_n$ with $\det(T) \neq 0$, we put

$$a_{n,k}(T,\sigma) = \gamma_n(k+2\sigma)|\det(2T)|^{k+2\sigma-(n+1)/2}2^n b(T,k+2\sigma).$$

(e.g. see [Ma] p.306). We denote by $S_n(\mathbb{R})^{(i)}$ the set of elements of $S_n(\mathbb{R})$ with signature $(i, n-i)$, and for a subset S of $S_n(\mathbb{R})$, we write $S^{(i)} = S \cap S_n(\mathbb{R})^{(i)}$. We define the volume $\mu(T)$ of $T \in S_n(\mathbb{Z})^{(i)}$ as follows. Let $GL_n(\mathbb{R})^+ = \{g \in GL_n(\mathbb{R}); \det g > 0\}$, and dg the measure on $GL_n(\mathbb{R})^+$ defined by $dg = (\det g)^{-n}\prod_{1\leq\alpha,\beta\leq n} dg_{\alpha\beta}$ for $g = (g_{\alpha\beta})$. Furthermore, we define the measure dy on $S_n(\mathbb{R})^{(i)}$ as $dy = \prod_{1\leq\alpha\leq\beta\leq n} dy_{\alpha\beta}$ for $y = (y_{\alpha\beta})$. Fix $T \in S_n(\mathbb{Q})^{(i)}$, and let $\Phi_T : GL_n(\mathbb{R})^+ \to S_n(\mathbb{R})^{(i)}$ be the mapping defined by $\Phi_T(g) = {}^tgTg$ for $g \in GL_n(\mathbb{R})^+$. We define the stabilizer subgroup Γ_T of T as $\Gamma_T = \{g \in SL_n(\mathbb{Z}); {}^tgTg = T\}$. Let $\mathcal{T}$ be a relatively compact open set in $S_n(\mathbb{R})^{(i)}$, and $Y = \phi_T^{-1}(\mathcal{T})$. Then Γ_T acts on Y by the right multiplication. Let Y_0 be a fundamental domain with respect to this action of Γ_T. Then the ratio

$$\mu(T) = \int_{Y_0} dg \Big/ \int_{\mathcal{T}} |\det y|^{-(n+1)/2} dy$$

is finite and independent of the choice of $\mathcal{T}$ except for the case $(n,i) = (2,1)$. When $T \in L_n^{(n)}$, we have $c_n\mu(T) = e(T)^{-1}$, where $c_n = 2\pi^{-n(n+1)/4}\prod_{j=1}^{n}\Gamma(j/2)$. We note that we have $\mu(cT) = \mu(T)$ for positive rational number c. For each i with $0 \leqslant i \leqslant n$, we define the Koecher-Maaß series for $E_{k,n}(Z,\sigma)$ by

$$L_{n,k}^{(i)}(s,\sigma) = c_n \sum_{\mathcal{L}_n^{(i)}/SL_n(\mathbb{Z})} \frac{a_{n,k}(T,\sigma)\mu(T)}{|\det(T)|^s}.$$

Here we note the following. If $\sigma = 0$ and T is not positive definite, we get $\xi(Y,T,\sigma+k,0) = 0$. Even in this case, we get non zero coefficients $a_{n,k}(T,0)$

in general, and non zero Dirichlet series $L_{n,k}^{(i)}(s,0)$. Even in that case, we can get a unified formula for $L_{n,k}^{(i)}(s,0)$, so we don't exclude the case. Our first main result is as follows:

Theorem 1.1. *For $j = 0,\dots,n$ put $\epsilon_j = (-1)^{(n-j)(n-j-1)/2}$ and $\delta_j = (-1)^j$. Let $n \geqslant 3$ is odd. Then we get*

$$
\begin{aligned}
&L_{n,k}^{(j)}(s,\sigma)\\
&= 2^{(n-1)s} \frac{\prod_{i=1}^{(n-1)/2} \zeta(1-2i)}{\zeta(1-k-2\sigma)\prod_{i=1}^{(n-1)/2}\zeta(1-2k-4\sigma+2i)}\\
&\quad \times \Bigg\{ \zeta(s)\zeta(s-k-2\sigma+1) \prod_{i=1}^{(n-1)/2} \zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1)\\
&\qquad + (-1)^{(n^2-1)/8}\epsilon_j\delta_j^{(n+1)/2}\zeta(s-\frac{n-1}{2})\zeta(s-k-2\sigma+\frac{n+1}{2})\\
&\qquad \times \prod_{i=1}^{(n-1)/2} \zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i) \Bigg\}.
\end{aligned}
$$

Next, we fix even n. For a fundamental discriminant d, let χ_d be the Kronecker character corresponding to the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. For any complex number t with positive real part, and $\delta = \pm 1$, we define the Dirichlet series

$$
\tilde{D}^*(s,t;\delta) = \sum_{d \in \mathcal{D}_\delta} d^{-s} L(1-\frac{t}{2},\chi_{\delta d}) \cdot \frac{\zeta(2s)\zeta(2s-t+1)}{L(2s-t/2+1,\chi_{\delta d})},
$$

where $\mathcal{D}_\delta$ is the set of positive integers such that δd is a fundamental discriminant. We note that $\tilde{D}^*(s,n;(-1)^{n/2}\delta)$ coincides with $D_n^*(s,\delta)$ in [I-S]. Now, for $i = 1,2$, write

$$
\tilde{D}^*(s,\sigma_i;\delta) = \sum_{m=1}^{\infty} a_i(m) m^{-s}
$$

and define the convolution product $\tilde{D}^*(s,\sigma_1;\delta) \otimes \tilde{D}^*(s,\sigma_2;\delta)$ as

$$
\tilde{D}^*(s,\sigma_1;\delta) \otimes \tilde{D}^*(s,\sigma_2;\delta)) = \zeta(2s-\sigma_1/2-\sigma_2/2+1) \sum_{m=1} a_1(m)a_2(m)m^{-s}.
$$

Then we have

Theorem 1.2. *Let ϵ_j and δ_j be as in Theorem 1.1. Let $n \geqslant 4$ is even. Then we get*

$$
L_{n,k}^{(j)}(s,\sigma) = 2^{ns} \frac{\prod_{i=1}^{n/2-1}\zeta(1-2i)}{\zeta(1-k-2\sigma)\prod_{i=1}^{n/2}\zeta(1-2k-4\sigma+2i)}
$$

$$\times \left\{ \tilde{D}^*(s,n;(-1)^{n/2}\delta_j) \otimes \tilde{D}^*(s,2k+4\sigma-n;(-1)^{n/2}\delta_j) \right.$$
$$\times \prod_{i=1}^{n/2-1} \zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1)$$
$$+ \frac{1+(-1)^{n/2}\delta_j}{2}\epsilon_j(-1)^{n(n+2)/8}\zeta(1-n/2)\zeta(1-k-2\sigma+n/2)$$
$$\left. \times \prod_{i=1}^{n/2} \zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i) \right\}.$$

We prove these two theorems by using Siegel's formula on quadratic forms, and by calculating a certain power series attached to local densities and local Siegel series. An explicit closed formula for local Siegel series was obtained by the second named author (cf. [Ka]), but we don't use any such precise result in this paper. We note that Theorems 1.1 and 1.2 can also be proved in case $j = n$ by using the method as in [I-K4]. But we don't know whether they can be proved in the case $j < n$ in such a way.

2. Siegel's formula

In this section, first we review on Siegel's formula which expresses the Fourier coefficients $a_{n,k}(T,\sigma)$ and the volume $\mu(T)$ for $T \in \mathcal{L}$ in terms of local densities, and rewrite it in a form suitable for our calculation. For a symmetric matrix S and a matrix X for which the product ${}^tXSX$ is defined, we use the notation $S[X] = {}^tXSX$ as usual. Put $S_n(\mathbb{Z})_e = 2\mathcal{L}_n$, and $S_n(\mathbb{Z})_o = S_n(\mathbb{Z})\backslash S_n(\mathbb{Z})_e$. We call a matrix in $S_n(\mathbb{Z})_e$ (resp. $S_n(\mathbb{Z})_o$) an even integral (resp. an odd integral) matrix. For a prime number p, we denote by $\mathcal{L}_{n,p}$ the set of $n \times n$ half-integral matrices over $\mathbb{Z}_p$. We put $S_n(\mathbb{Z}_p)_e = 2\mathcal{L}_{n,p}$, and $S_n(\mathbb{Z}_p)_o = S_n(\mathbb{Z}_p)\backslash S_n(\mathbb{Z}_p)_e$. We take two positive integers m and n and assume that $m \geqslant n$. For $S \in S_m(\mathbb{Z}_p)$ and $T \in S_n(\mathbb{Q}_p)$, we define the local density representing T by S by

$$\alpha_p(S,T) = 2^{\delta_{m,n}} \lim_{\nu\to\infty} p^{(n(n+1)/2-mn)\nu}|A_{p^\nu}(S,T)|,$$

where $\delta_{m,n}$ is Kronecker's delta, and

$$A_{p^\nu}(S,T) = \{X \in M_{mn}(\mathbb{Z}_p/p^\nu\mathbb{Z}_p);\ S[X]-T \in p^\nu S_n(\mathbb{Z}_p)_e\}.$$

Now we define the primitive Fourier coefficient of the Siegel Eisenstein series. For a non-degenerate $n\times n$ matrix D with entries in $\mathbb{Z}_p$, we put $\pi_p(D) =$

$(-1)^i p^{<i-1>}$ or 0 according as D belongs to $GL_n(\mathbb{Z}_p)(E_{n-i} \perp pE_i)GL_n(\mathbb{Z}_p)$ for some $0 \le i \le n$, or not. Here we write $< j >= j(j+1)/2$ for an integer j. Furthermore, for a non-degenerate $n \times n$ matrix D with entries in $\mathbb{Z}$, put $\pi(D) = \prod_p \pi_p(D)$. This is a certain generalization of the Möbius function and we call this the generalized Möbius function (cf. [I-K2]). Then for an element $T \in \mathcal{L}_n^{(i)}$, we define the *modified* T-th *primitive Fourier coefficient* $a^*_{n,k}(T,\sigma)$ of $E_{n,k}(Z,\sigma)$ by

$$a^*_{n,k}(T,\sigma) = \sum_D \pi(D) a_{n,k}(T[D^{-1}],\sigma),$$

where D runs over a complete set of representatives of left $GL_n(\mathbb{Z})$-equivalence classes of non-degenerate $n \times n$ matrices with entries in $\mathbb{Z}$. We note that $a^*_{n,k}(T,\sigma)$ is the primitive Fourier coefficient introduced by Böcherer and Raghavan [B-R] if T is positive definite and $\sigma = 0$. By the inversion formula for the generalized Möbius function we have

$$a_{n,k}(T,\sigma) = \sum_D a^*(T[D^{-1}],\sigma),$$

where D runs over a complete set of representatives of left $GL_n(\mathbb{Z})$-equivalence classes of non-degenerate $n \times n$ matrices with entries in $\mathbb{Z}$ (cf. [I-K2]). Thus we have the following:

Proposition 2.1. *Put*

$$\zeta(GL_n, s) = \prod_{i=1}^{n} \zeta(s-i+1),$$

and

$$\tilde{L}^{(i)}_{n,k}(s,\sigma) = c_n \sum_{\mathcal{L}_n^{(i)}/SL_n(\mathbb{Z})} \frac{a^*_{n,k}(T,\sigma)\mu(T)}{|\det(T)|^s}.$$

Then

$$L^{(i)}_{n,k}(s,\sigma) = \zeta(GL_n, 2s)\tilde{L}^{(i)}_{n,k}(s,\sigma).$$

Proof. The assertion is an easy consequence of Lemma 3.2 of [I-K2]. □

We define the primitive Siegel series $b^*(T,\sigma)$ for $T \in S_n(\mathbb{Q})$ by

$$b^*(T,\sigma) = \sum_D \pi(D)|\det D|^{-2\sigma+n+1} b(T[D^{-1}],\sigma),$$

where D runs over a complete set of representatives of left $GL_n(\mathbb{Z})$-equivalence classes of non-degenerate $n \times n$ matrices with entries in $\mathbb{Z}$. Then we have

$$a^*_{n,k}(T,\sigma) = \gamma_n(k+2\sigma)|\det(2T)|^{k+2\sigma-(n+1)/2}2^n b^*(T,k+2\sigma).$$

To reduce our problem to the local computation, for $T \in S_n(\mathbb{Q}_p)$, we define the local Siegel series by

$$b_p(T,\sigma) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \nu_p(R)^{-\sigma} \mathbf{e}_p^{\mathrm{tr}(TR)},$$

where $\nu_p(R) = [R\mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$, and $\mathbf{e}_p$ denotes the additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(u) = e^{2\pi\sqrt{-1}u}$ for $u \in \mathbb{Q}$. Furthermore, we define the primitive local Siegel series $b_p^*(T,\sigma)$ by

$$b_p^*(T,\sigma) = \sum_D \pi_p(D) p^{(-2\sigma+n+1)\mathrm{ord}_p(\det D)} a(T[D^{-1}]),$$

where D runs over a complete set of representatives of left $GL_n(\mathbb{Z}_p)$-equivalence classes of non-degenerate $n \times n$ matrices with entries in $\mathbb{Z}_p$. Then for $T \in S_n(\mathbb{Q})$ we have

$$b(T,\sigma) = \prod_p b_p(T,\sigma), \quad \text{and} \quad b^*(T,\sigma) = \prod_p b_p^*(T,\sigma),$$

if the real part of σ is large enough.

Let T_1 and T_2 be elements of $\mathcal{L}_n \cap GL_n(\mathbb{Q})$. Then we say T_2 belongs to the genus of T_1 if there exists $g_v \in GL_n(\mathbb{Z}_v)$ such that $T_1[g_v] = T_2$ for each place v, where we put $\mathbb{Z}_\infty = \mathbb{R}$. For an element $T \in \mathcal{L}_n$, let $((T))$ denote the set of matrices belonging to the genus of T. Obviously, the above formula for $a_{n,k}(T,\sigma)$ implies that $a_{n,k}(T,\sigma)$ depends only on the genus $((T))$. Now we shall rewrite Siegel's formula. Let $\mathcal{G}_n^{(i)}$ denote the set of all genera of $n \times n$ even-integral matrices of signature $(i, n-i)$. Then, by the Minkowski-Siegel Mass formula, for any $T \in \mathcal{L}_n^{(i)}$, we get

$$\sum_{A \in \overline{((T))}} \mu(A) = \frac{2^{1-\delta_{n,1}-n}|\det(T)|^{(n+1)/2}}{\prod_{p<\infty} \alpha_p(T,T)},$$

where $\overline{((T))}$ denotes the set of all $SL_n(\mathbb{Z})$-equivalence classes of matrices belonging to the genus of T. Notation being the same as above, we easily get the following:

$$L_{n,k}^{(i)}(s,\sigma) = e^{\sqrt{-1}n(k+2\sigma)/2}\, 2^{ns-\delta_{n,1}} \cdot \frac{\gamma(k+2\sigma)}{c_n}$$

$$\times \sum_{((T))\in \mathcal{G}_n^{(i)}} \prod_{p<\infty} \frac{b_p^*(2^{-1}T, k+2\sigma)}{\alpha_p(T,T)} |\det T|^{k+2\sigma-s}.$$

To reduce this to local calculation, we introduce several notation. For any $T \in S_n(\mathbb{Q}_p)$, there exists a diagonal matrix with diagonal components a_1, ..., a_n which is $GL_n(\mathbb{Q}_p)$ equivalent to T. We define the Hasse invariant $\varepsilon(T)$ of T by

$$\varepsilon(T) = \prod_{1\leqslant i\leqslant j\leqslant n} (a_i, a_j)_p,$$

where $(x, y)_p$ is the Hilbert symbol at p (cf. Kitaoka [Ki2]). For each $d \in \mathbb{Z}_p$, and for each prime p, we put

$$S_n(\mathbb{Z}_p, d)_e = \{T \in S_n(\mathbb{Z}_p)_e;\ \det T = d\}.$$

For $\sigma \in \mathbb{C}$, a function ϕ on $S_n(\mathbb{Q}_p)$ which takes the same value on each $GL_n(\mathbb{Z}_p)$-equivalence class, and a p-adic integer $d \neq 0$, put

$$\gamma_p(d, \phi, \sigma) = \sum_{T\in S_n(\mathbb{Z}_p,d)_e/GL_n(\mathbb{Z}_p)} \frac{b_p^*(2^{-1}T, \sigma)\phi(T)}{\alpha_p(T,T)}.$$

For integers $k > 0$, $d \neq 0$ and a family $\phi = \{\phi_p\}_p$ of $GL_n(\mathbb{Z}_p)$ invariant functions, put

$$\lambda(d, \phi, \sigma) = \prod_p \gamma_p(d, \phi_p, \sigma).$$

For each $i = 0, 1, ..., n$ define the partial zeta function $\tilde{L}_{n,k}^{(i)}(s, \sigma, \phi)$ by

$$\tilde{L}_{n,k}^{(i)}(s, \sigma, \phi) = \sum_{d\in\mathcal{D}_{\delta_i}} \lambda(\delta_i d, \phi, k+2\sigma) d^{k+2\sigma-s}.$$

We denote by ι the trivial function such that $\iota(T) = 1$ for any $T \in S_n(\mathbb{Q}_p)$ and p. By abuse of language, we denote by ε the family of local Hasse invariants and by ι the family of the trivial functions. Then by using the same argument as in Ibukiyama and Saito [I-S], we get:

Proposition 2.2. *Notation being as above, we have*

$$\tilde{L}_{n,k}^{(i)}(s, \sigma) = e^{\sqrt{-1}n(k+2\sigma)/2} 2^{ns-\delta_{1,n}-1} \frac{\gamma(k+2\sigma)}{c_n} \left(\tilde{L}_{n,k}^{(i)}(s, \sigma, \iota) + \tilde{L}_{n,k}^{(i)}(s, \sigma, \varepsilon)\right).$$

3. Proof of main theorems

We prove our main results. In order to calculate $\tilde{L}_{n,k}^{(i)}(s,\sigma,\phi)$, we define some series. Let ω be a function on $S_n(\mathbb{Q}_p)$ such that $\omega(T) = \omega(T[g])$ for every $T \in S_n(\mathbb{Z}_p)$ and $g \in GL_n(\mathbb{Z}_p)$. For such a function ω and for each $d_0 \in \mathbb{Z}_p^*$, we define a formal power series $D(X,\sigma,d_0,\omega)$ by

$$D(X,\sigma,d_0,\omega) = \sum_{e=0}^{\infty} \gamma_p(p^e d_0,\omega,\sigma)X^e.$$

To describe $D(p^{-s},\sigma,d_0,\omega)$, we introduce some more notation. For each $d \in \mathbb{Z}_p$, and for each prime p, we put

$$S_n(\mathbb{Z}_p,d)_o = \{T \in S_n(\mathbb{Z}_p)_o;\ \det T = d\},$$

and

$$S_n(\mathbb{Z}_p,d) = \{T \in S_n(\mathbb{Z}_p)\ ;\ \det T = d\}.$$

For each integer $r \geqslant 0$, each $d_0 \in \mathbb{Z}_p^*$ and each function f on $S_n(\mathbb{Q}_p)$ which takes the same value on each $GL_r(\mathbb{Z}_p)$ orbit, we put

$$\zeta_r(u,f,d_0) = \sum_{m=0}^{\infty} \sum_{T \in S_r(\mathbb{Z}_p,p^m d_0)/GL_r(\mathbb{Z}_p)} \frac{f(T)}{\alpha_p(T,T)} u^m,$$

and

$$\zeta_r^*(u,f,d_0) = \sum_{m=0}^{\infty} \sum_{T \in S_r(\mathbb{Z}_p,p^m d_0)_e/GL_r(\mathbb{Z}_p)} \frac{f(T)}{\alpha_p(T,T)} u^m.$$

When p is odd, we have $\zeta_r = \zeta_r^*$. We regard $\zeta_0 = \zeta_0^* = 1$ for $r = 0$ in the above definition. Let ω is the trivial function ι on $S_n(\mathbb{Q}_p)$ or the Hasse invariant ε on $S_n(\mathbb{Q}_p)$. Let $Z_n(u,\omega,d_0)$ and $Z_n^*(u,\omega,d_0)$ be the formal power series in Theorems 5.1, 5.2, and 5.3 of [I-S]. Then we note that

$$\zeta_n(u,\omega,d_0) = 2^{\delta_{2,p}n} Z_n(p^{-(n+1)/2}((-1)^{(n+1)/2},p)_p u,\omega,d_0)$$

or

$$\zeta_n(u,\omega,d_0) = 2^{\delta_{2,p}n} Z_n(p^{-(n+1)/2}u,\omega,d_0)$$

according as n is odd and $\omega = \varepsilon$, or not. Here we recall that the definition of local density in our paper is a little bit different from that in [I-S]. Furthermore, we note that

$$D(u,\sigma,d_0,\omega) = \zeta_n^*(u,\phi,d_0)$$

for $\phi(T) = \omega(T)b_p(2^{-1}T, \sigma)$. Let $\mathcal{U}_p$ be a complete set of representatives of $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2}$ if p is an odd prime number, and $\mathcal{U}_{m,2} = \{(-1)^{m/2}, (-1)^{m/2}5\}$ if m is an even integer. Let $p \neq 2$. Then, for each positive integer m and $d_1 \in \mathcal{U}_p$, there exists a unique, up to $\mathbb{Z}_p$-equivalence, element of $S_m(\mathbb{Z}_p) \cap GL_m(\mathbb{Z}_p)$ with determinant d_1, which will be denoted by Θ_{m,d_1}. Furthermore, for each even positive integer m and $d_1 \in \mathcal{U}_{m,2}$ there exists a unique, up to $\mathbb{Z}_2$-equivalence, element of $S_m(\mathbb{Z}_2)_e \cap GL_m(\mathbb{Z}_2)$ with determinant d_1, which will be also denoted by Θ_{m,d_1}.

Lemma 3.1. *Notation being as above, we have*
(1) *Let $p \neq 2$. Then we have*

$$\zeta_n(u, f, d_0) = \sum_{r=0}^{n-1} \sum_{d_1 \in \mathcal{U}_p} \sum_{m=0}^{\infty} \sum_{T' \in S_r(\mathbb{Z}_p, d_0 d_1^{-1} p^m)/GL_r(\mathbb{Z}_p)} u^{r+m} \times \frac{f(\Theta_{n-r,d_1} \perp pT')}{\alpha_p(\Theta_{n-r,d_1} \perp pT', \Theta_{n-r,d_1} \perp pT')} + \sum_{m=0}^{\infty} \sum_{T' \in S_n(\mathbb{Z}_p, d_0 p^m)/GL_n(\mathbb{Z}_p)} u^{n+m} \frac{f(pT')}{\alpha_p(pT', \perp pT')}.$$

(2) *Let $p = 2$. Then we have*

$$\begin{aligned}\zeta_n^*(u, f, d_0) = &\sum_{\substack{0 \le r \le n-1 \\ n \equiv r \bmod 2}} \sum_{d_1 \in \mathcal{U}_{n-r,2}} \sum_{m=0}^{\infty} \sum_{T' \in S_r(\mathbb{Z}_2, d_0 d_1^{-1} 2^m)_e/GL_r(\mathbb{Z}_2)} u^{r+m} \\ &\times \frac{f(\Theta_{n-r,d_1} \perp 2T')}{\alpha_2(\Theta_{n-r,d_1} \perp 2T', \Theta_{n-r,d_1} \perp 2T')} \\ &+ \sum_{m=0}^{\infty} \sum_{T' \in S_n(\mathbb{Z}_2, d_0 2^m)_e/GL_n(\mathbb{Z}_2)} u^{n+m} \frac{f(2T')}{\alpha_2(2T', 2T')} \\ &+ \sum_{\substack{0 \le r \le n-1 \\ n \equiv r \bmod 2}} \sum_{m=0}^{\infty} \sum_{T' \in S_r(\mathbb{Z}_2, d_0(-1)^{(n-r)/2} 2^m)_o/GL_r(\mathbb{Z}_2)} u^{r+m} \\ &\times \frac{f(\Theta_{n-r,(-1)^{(n-r)/2}} \perp 2T')}{\alpha_2(\Theta_{n-r,(-1)^{(n-r)/2}} \perp 2T', \Theta_{n-r,(-1)^{(n-r)/2}} \perp 2T')}. \\ &+ \sum_{m=0}^{\infty} \sum_{T' \in S_n(\mathbb{Z}_2, d_0 2^m)_o/GL_n(\mathbb{Z}_2)} u^{n+m} \frac{f(2T')}{\alpha_2(2T', 2T')}.\end{aligned}$$

Proof. First let $p \neq 2$. Then by the theory of Jordan decomposition, any $T \in S_n(\mathbb{Z}_p)$ is equivalent, over $\mathbb{Z}_p$, to $\Theta_{n-r,d_1} \perp pT'$ with some $0 \le r \le n, d \in$

$\mathbb{Z}_p^*$ and $T' \in S_r(\mathbb{Z}_p)$. Here we understand Θ_{n-r,d_1} is the empty matrix if $r = n$. Furthermore, by the uniqueness of the Jordan decomposition, $r, d_1 \bmod \mathbb{Z}_p^{*2}$ and the $\mathbb{Z}_p$-equivalence class of T' are uniquely determined by T. Thus the assertion (1) is proved.

Next let $p = 2$. Then, by the theory of canonical decomposition due to Watson [W], any $T \in S_n(\mathbb{Z}_2)_e$ is equivalent, over $\mathbb{Z}_2$, to $\Theta_{n-r,d_1} \perp 2\tilde{T}$ with some $0 \leq r \leq n$ such that $n \equiv r \bmod 2, d_1 \in \mathbb{Z}_p^*$ such that $(-1)^{(n-r)/2} d_1 \equiv 1 \bmod 4$, and $\tilde{T} \in S_r(\mathbb{Z}_2)$. Here we understand Θ_{n-r,d_1} is the empty matrix if $r = n$. The integer r is uniquely determined by T. Furthermore, if $\tilde{T} \in S_r(\mathbb{Z}_2)_e$, then d_1 and the $\mathbb{Z}_2$-equivalence class of $\tilde{T}$ are uniquely determined by T. On the other hand, if $\tilde{T} \in S_r(\mathbb{Z}_2)_o$, then there exists a $T' \in 2S_r(\mathbb{Z}_2)_o$ such that T is equivalent, over $\mathbb{Z}_2$, to $\Theta_{n-r,(-1)^{(n-r)/2}} \perp T'$, and T' is uniquely determined, up to $\mathbb{Z}_2$-equivalence, by T. This proves the assertion (2). □

Let p be a prime. For a non-zero element $a \in \mathbb{Q}_p$ we put $\chi_p(a) = 1, -1$, or 0 according as $\mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \mathbb{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbb{Q}_p$. For a non-degenerate half-integral matrix B of even degree n, put $\xi_p(B) = \chi_p((-1)^{n/2} \det B)$. For an $n \times n$ half-integral matrix B over $\mathbb{Z}_p$, let $(\bar{W}, \bar{q})$ denote the quadratic space over $\mathbb{Z}_p/p\mathbb{Z}_p$ defined by the quadratic form $\bar{q}(\mathbf{x}) = B[\mathbf{x}] \bmod p$, and define the radical $R(\bar{W})$ of $\bar{W}$ by

$$R(\bar{W}) = \{\mathbf{x} \in \bar{W}; \bar{B}(\mathbf{x}, \mathbf{y}) = 0 \text{ for any } \mathbf{y} \in \bar{W}\},$$

where $\bar{B}$ denotes the associated symmetric bilinear form of $\bar{q}$. We then put $l_p(B) = \operatorname{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p} R(\bar{W})^{\perp}$, where $R(\bar{W})^{\perp}$ is the orthogonal complement of $R(\bar{W})$ in $\bar{W}$. Furthermore, in case $l_p(B)$ is even, put $\bar{\xi}_p(B) = 1$ or -1 according as $R(\bar{W})^{\perp}$ is hyperbolic or not. Here we make the convention that $\xi_p(B) = 1$ if $l_p(B) = 0$. We note that $\bar{\xi}_p(B)$ is different from the $\xi_p(B)$.

Lemma 3.2. *Let T be an $n \times n$ half-integral matrix over $\mathbb{Z}_p$. Put $l = l_p(T)$. Then we have*

$$b_p^*(T, \sigma) = (1 - p^{-\sigma}) \times \begin{cases} (1 + \bar{\xi}_p(T) p^{n-l/2-\sigma}) \prod_{j=0}^{n-l/2-1} (1 - p^{2j-2\sigma}) & \text{if } l \text{ is even} \\ \prod_{j=0}^{n-(l+1)/2} (1 - p^{2j-2\sigma}) & \text{if } l \text{ is odd.} \end{cases}$$

Proof. The assertion follows from Lemma 9 of Kitaoka [Ki1]. □

Lemma 3.3. *Let $n = n_0 + r$. Let $\Theta \in S_{n_0}(\mathbb{Z}_p)_e \cap GL_{n_0}(\mathbb{Z}_p)$.*

(1) *Let* $T' \in S_r(\mathbb{Z}_p)_e$. *Then we have*

$$\alpha_p(\Theta \perp pT', \Theta \perp pT') = 2p^{r(r+1)/2}\alpha_p(T',T') \prod_{i=1}^{[n_0/2]} (1-p^{-2i})$$
$$\times \begin{cases} (1+\xi_p(\Theta)p^{-n_0/2})^{-1} & \text{if } n_0 \text{ is even} \\ 1 & \text{if } n_0 \text{ is odd.} \end{cases}$$

(2) *Let* $p=2$, *and let* $T' \in S_r(\mathbb{Z}_p)_o$. *Then* n_0 *is even, and we have*

$$\alpha_p(\Theta \perp 2T', \Theta \perp 2T') = 2^{r(r+1)/2+2}\alpha_p(T',T') \prod_{i=1}^{n_0/2-1} (1-p^{-2i}).$$

Proof. The assertion follows from p.110 and p.111 of Kitaoka [Ki2]. □

For a non-negative integer l, a prime number p, and $d \in \mathbb{Z}_p$, put

$$\psi_{l,p}(d) = \begin{cases} \chi_p((-1)^{l/2}d) & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

Here we understand that we have $\psi_{0,p}(d) = 1$. Furthermore, we define $(q)_m = \prod_{i=1}^m (1-q^i)$.

Proposition 3.1. *For a non-negative integer* l *let* $\iota^{(l)}$ *be the constant function on* $S_l(\mathbb{Z}_p)$ *taking the value 1, and* $\varepsilon^{(l)}$ *the Hasse invariant on* $S_l(\mathbb{Z}_p)$. *Let* k *be a complex number. For* $d \in \mathbb{Z}_p^*$ *put*

$$f(r,d) = 1 + \psi_{n-r,p}(d)p^{(n+r)/2-k},$$
$$c(r) = (1-p^{-k}) \prod_{i=1}^{n-[(n-r)/2]-1} (1-p^{2i-2k}),$$

and

$$h(n_1,d) = (p^{-2})^{-1}_{[n_1/2]}(1+\psi_{n_1,p}(d)p^{-n_1/2}) \times \begin{cases} 1 & \text{if } p \neq 2 \\ 2 & \text{if } p = 2. \end{cases}$$

(1) *Let* $p \neq 2$. *Then we have*

$$D(u,k,d_0,\iota^{(n)})$$
$$= 2^{-1}\sum_{r=0}^{n-1}\sum_{d_1 \in \mathcal{U}_p} p^{-r(r+1)/2}u^r c(r)h(n-r,d_1)f(r,d_1)\zeta_r(u,\iota^{(r)},d_0d_1^{-1})$$
$$+ 2^{-1}u^n p^{-n(n+1)/2}c(n)h(0,1)f(n,1)\zeta_n(u,\iota^{(n)},d_0),$$

and

$$
\begin{aligned}
&D(u,k,d_0,\varepsilon^{(n)}) \\
&= 2^{-1}\sum_{r=1}^{n-1}\sum_{d_1\in\mathcal{U}_p}\Big\{p^{-r(r+1)/2}u^r c(r)h(n-r,d_1)e(r,d_0,d_1) \\
&\qquad\qquad \times f(r,d_1)\zeta_r((p^{r+1}d_1,p)_p u,\varepsilon^{(r)},d_0d_1^{-1})\Big\} \\
&\qquad + 2^{-1}u^n p^{-n(n+1)/2}c(n)h(0,1)e(n,d_0,1)f(n,1)\zeta_n((p^{n+1},p)_p u,\varepsilon^{(n)},d_0),
\end{aligned}
$$

where $e(r,d_0,d_1)=(-1,p)_p^{r(r+1)/2}(d_0,p)_p^{r+1}(d_1,p)_p$.

(2) *Let* $p=2$. *Then we have*

$$
\begin{aligned}
&D(u,k,d_0,\iota^{(n)}) \\
&= 4^{-1}\sum_{\substack{0\le r\le n-1\\ n\equiv r \bmod 2}}\ \sum_{d_1\in\mathcal{U}_{n-r,2}}\Big\{u^r 2^{-r(r+1)/2}c(r)h(n-r,d_1) \\
&\qquad \times (1+2^{(n+r)/2-k}\psi_{n-r,p}(d_1))\zeta_r^*(u,\iota^{(r)},d_0d_1^{-1})\Big\} \\
&\qquad + 2^{-1}u^n 2^{-n(n+1)/2}c(n)h(0,1)\zeta_n^*(u,\iota^{(n)},d_0d_1^{-1}) \\
&\qquad + 4^{-1}\sum_{\substack{0\le r\le n\\ n\equiv r \bmod 2}} u^r 2^{-r(r+1)/2}c(r)h(n-r,(-1)^{(n-r)/2}) \\
&\qquad \times \Big\{\zeta_r(u,\iota^{(r)},(-1)^{(n-r)/2}d_0)-\zeta_r^*(u,\iota^{(r)},(-1)^{(n-r)/2}d_0)\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&D(u,k,d_0,\varepsilon^{(n)}) \\
&= 4^{-1}\sum_{\substack{0\le r\le n-1\\ n\equiv r \bmod 2}}\ \sum_{d_1\in\mathcal{U}_{n-r,2}}\Big\{u^r 2^{-r(r+1)/2}c(r)h(n-r,d_1)e(r,d_0,d_1) \\
&\qquad\qquad \times (1+2^{(n+r)/2-k}\psi_{n-r,p}(d_1))\zeta_r^*((2^{r+1}d_1,2)_2u,\varepsilon^{(r)},d_0d_1^{-1})\Big\} \\
&\quad + 2^{-1}u^n 2^{-n(n+1)/2}c(n)h(0,1)e(n,d_0,1)\zeta_n^*((2^{n+1}d_1,2)_2u,\varepsilon^{(n)},d_0d_1^{-1}) \\
&\quad + 4^{-1}\sum_{\substack{0\le r\le n\\ n\equiv r \bmod 2}} u^r 2^{-r(r+1)/2}c(r)h(n-r,(-1)^{(n-r)/2})e(r,d_0,(-1)^{(n-r)/2}) \\
&\qquad\qquad \times \Big\{\zeta_r(((-1)^{(n-r)/2}2^{r+1},2)_2u,\omega,(-1)^{(n-r)/2}d_0) \\
&\qquad\qquad\qquad - \zeta_r^*(((-1)^{(n-r)/2}2^{r+1},2)_2u,\omega,(-1)^{(n-r)/2}d_0)\Big\},
\end{aligned}
$$

where

$e(r,d_0,d_1)$

$$= \begin{cases} (-1)^{(n(n+2)+r(r+2))/8}\psi_{n,p}(d_0)((-1)^{(n-r)/2},(-1)^{n/2}d_0)_2 & \text{if } n,r \text{ are even} \\ (-1)^{(n^2+r^2-2)/8}((-1)^{(n+1)/2},d_0)_2((-1)^{(r+1)/2},d_0d_1^{-1})_2 & \text{if } n,r \text{ are odd.} \end{cases}$$

Proof. (1) The first assertion directly follows from Lemmas 3.1, 3.2, and 3.3. To prove the second assertion, let $T = \Theta \perp pT'$ with $\Theta \in S_{n-r}(\mathbb{Z}_p, d_1) \cap GL_{n-r}(\mathbb{Z}_p)$ and $T' \in S_r(\mathbb{Z}_p, p^m d_0 d_1^{-1})$. Then we have

$$\begin{aligned} \varepsilon(T) &= \varepsilon(\Theta)(\det\Theta, \det(pT'))\varepsilon(pT') \\ &= \varepsilon(\Theta)(\det\Theta, \det T')_p(\det\Theta, p^r)_p(-1,p)_p^{r(r+1)/2}(p,\det T')_p^{r+1}\varepsilon(T') \\ &= \varepsilon(\Theta)(d_1, p^m d_0 d_1^{-1})_p(d_1,p^r)_p(-1,p)_p^{r(r+1)/2}(p^{r+1}, p^m d_0 d_1^{-1})_p\varepsilon(T'). \end{aligned}$$

Then we have $\varepsilon(U) = 1$ for any unimodular matrix U. Thus we have

$$\varepsilon(T) = (-1,p)_p^{r(r+1)/2}(d_1,p^r)_p(d_1p^{r+1}, d_0d_1^{-1})_p(p^{r+1}d_1,p)_p^m\varepsilon(T').$$

Thus the second assertion follows from Lemmas 3.1, 3.2, and 3.3.

(2) The first assertion can be proved in the same way as above. Now we have

$$S_{n-r}(\mathbb{Z}_2, d_1)_e \cap GL_{n-r}(\mathbb{Z}_2) \neq \phi$$

if and only if $n-r$ is even and $(-1)^{(n-r)/2}d_1 \equiv 1 \bmod 4$, and further in this case, for $\Theta \in S_{n-r}(\mathbb{Z}_2, d_1)_e \cap GL_{n-r}(\mathbb{Z}_2)$ we have

$$\varepsilon(\Theta) = (-1)^{(n-r)(n-r+2)/8}\psi_{n-r,p}(d_1).$$

Thus, by a computation similar to the above, we have

$$\varepsilon(T) = (-1)^{n(n+2)/8}\psi_{n,p}(d_0)(-1)^{r(r+2)/8}$$

$$\times((-1)^{(n-r)/2},(-1)^{n/2}d_0)_2(2^{r+1}d_1,2)_2^m\varepsilon(T')$$

for n and r even, and

$$\begin{aligned} \varepsilon(T) = (-1)^{(n^2-1)/8}&(d_0,(-1)^{(n+1)/2})_2(-1)^{(r^2-1)/8} \\ &\times (d_0d_1^{-1},(-1)^{(r+1)/2})_2(2^{r+1}d_1,2)_2^m\varepsilon(T') \end{aligned}$$

for n and r odd. Thus the assertion holds by Lemmas 3.1, 3.2, and 3.3. □

Lemma 3.4. (1) *Let l be a positive integer. Then we have the following identity on the three variables q, U and Q :*

$$\begin{aligned} &(1-UQ)(1-UQq)\cdots(1-UQq^{l-1}) \\ &= \sum_{r=0}^{l}\frac{(q^{-1})_l}{(q^{-1})_r(q^{-1})_{l-r}}\prod_{i=0}^{r-1}(1-Qq^i)\prod_{i=1}^{l-r}(1-Uq^{i-1})q^{r(l-r)}U^r. \end{aligned}$$

(2) *Let k be any complex number. Then we have the following identity on the variables p and u :*

$$(1-p^{-2k}u^2)(1-p^{-2k+2}u^2)\cdots(1-p^{-2k+2l-2}u^2)$$
$$=\sum_{r=0}^{l}\frac{(p^{-2})_l}{(p^{-2})_r(p^{-2})_{l-r}}\prod_{i=0}^{r-1}(1-p^{-2k+2i+2l})\prod_{i=1}^{l-r}(1-p^{2i-2l-2}u^2)p^{-2r^2}u^{2r}.$$

Proof. The assertion (1) can easily be proved by using the same argument as in the proof of Lemma 5.5 of [I-S]. The assertion (2) holds by putting $Q=p^{-2k+2l}, U=p^{-2l}u^2$, and $q=p^2$ in (1). □

Now let $\tilde{D}^*(s,n;(-1)^{n/2}\delta_j)\otimes\tilde{D}^*(s,2k+4\sigma-n;(-1)^{n/2}\delta_j)$ be the convolution product in Theorem 1.2. Then by a simple calculation we have

$$\tilde{D}^*(s,n;(-1)^{n/2}\delta_j)\otimes\tilde{D}^*(s,2k+4\sigma-n;(-1)^{n/2}\delta_j)$$
$$=\zeta(2s)\zeta(2s-n+1)\zeta(2s-2k-4\sigma+2)\zeta(2s-2k-4\sigma+n+1)$$
$$\times\sum_{d\in\mathcal{D}_{(-1)^{n/2}\delta_j}} d^{-s}L(1-n/2,\chi_{(-1)^{n/2}\delta_j d})L(1-k-2\sigma+n/2,\chi_{(-1)^{n/2}\delta_j d})$$
$$\times\prod_p\Big\{(1+p^{k+2\sigma-1-2s})(1+p^{k+2\sigma-2s-2}\chi_{(-1)^{n/2}\delta_j d}(p)^2)$$
$$-\chi_{(-1)^{n/2}\delta_j d}(p)p^{n/2-2s-1}(1+p^{k+\sigma-1})(1+p^{k+\sigma-n})\Big\}.$$

Then we have

Theorem 3.1. *Let k be a complex number. Let n be an even integer and $d_0\in\mathbb{Z}_p^*$. Put*

$$D(u,k,d_0,\iota)_e=\frac{1}{2}(D(u,k,d_0,\iota)+D(-u,k,d_0,\iota)),$$

and

$$D(u,k,d_0,\iota)_o=\frac{1}{2}(D(u,k,d_0,\iota)-D(-u,k,d_0,\iota)).$$

(1) *We have*

$$D(p^{-s},k,d_0,\iota)_e=\frac{\kappa_{n,p}(s)(1-p^{-k})}{(p^{-2})_{n/2-1}(1-\psi_{n,p}(d_0)p^{-n/2})(1-\psi_{n,p}(d_0)p^{n/2-k})}$$
$$\times\frac{\prod_{i=1}^{n/2}(1-p^{2i-2k})\prod_{i=1}^{n/2}(1-p^{2i-1-2k-2s})}{(1-p^{-2-2s})\prod_{i=0}^{n/2-1}(1-p^{-2i-3-2s})}$$
$$\times\big\{(1+p^{-1-k-2s})(1+p^{-2-k-2s})$$

$$- p^{-n/2-2-2s}\psi_{n,p}(d_0)(1+p^{1-k})(1+p^{n-k})\Big\},$$

and

$$D(p^{-s},k,d_0,\iota)_o = \frac{\kappa_{n,p}(s)(1-p^{-k})p^{-s}(1+p^{-1-k-2s})}{(p^{-2})_{n/2-1}} \times \frac{\prod_{i=1}^{n/2}(1-p^{2i-2k})\prod_{i=1}^{n/2}(1-p^{2i-1-2k-2s})}{(1-p^{-2-2s})\prod_{i=0}^{n/2-1}(1-p^{-2i-3-2s})},$$

where $\kappa_{n,p}(s) = 2^{(-1+((-1)^{n/2}d_0,-1)_2)s}$ *or* 1 *according as* $p=2$ *or not.*

(2) *We have*

$$D(p^{-s},k,d_0,\varepsilon)_o = 0,$$

and

$$\begin{aligned} D(p^{-s},k,d_0,\varepsilon)_e &= D(p^{-s},k,d_0,\varepsilon) \\ &= \frac{\lambda_{n,p}(s)(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{2i-2k})}{(p^{-2})_{n/2-1}(1-p^{n/2-k}\psi_{n,p}(d_0))(1-p^{-n/2}\psi_{n,p}(d_0))} \\ &\quad \times \frac{\prod_{i=1}^{n/2}(1-p^{-2k+2i-2-2s})}{\prod_{i=1}^{n/2}(1-p^{-2i-2s})}, \end{aligned}$$

where $\lambda_{n,p}(s) = 2^{-1}(-1)^{n(n+2)/8}\psi_{n,2}(d_0)(((-1)^{n/2}d_0,-1)_2+1)$ *or* 1 *according as* $p=2$ *or not.*

Proof. (1) Put $u=p^{-s}, \psi_r = \psi_{r,p}$, and

$$c_n = \frac{(1-p^{-k})\prod_{i=1}^{n/2}(1-p^{2i-2k})}{\prod_{i=1}^{n/2-1}(1-p^{-2i})(1-\psi_n p^{-n/2})(1-\psi_n p^{n/2-k})}.$$

Let $p \neq 2$. Then by (1) of Proposition 3.1, we have

$$\begin{aligned} &(1-p^{-k})^{-1}D(u,k,d,\iota) \\ &= 2^{-1}\sum_{r=0}^{n-1}\sum_{d_1\in\mathcal{U}_p}\Bigg\{p^{-r(r+1)/2}u^r\prod_{i=1}^{n-[(n-r)/2]-1}(1-p^{2i-2k})(p^{-2})^{-1}_{[(n-r)/2]} \\ &\quad \times (1+\psi_{n-r}(d_1)p^{(-n+r)/2})(1+\psi_{n-r}(d_1)p^{(n+r)/2-k})\zeta_r(u,\iota,d_0d_1^{-1})\Bigg\} \\ &\quad + u^n p^{-n(n+1)/2}\prod_{i=1}^{n-1}(1-p^{2i-2k})(1+p^{n-k})\zeta_n(u,\iota,d_0). \end{aligned}$$

Thus we have

$$(1-p^{-k})^{-1}D(u,k,d,\iota)_e$$

$$= 2^{-1}\sum_{r=0}^{n/2-1}\sum_{d_1\in\mathcal{U}_p}\Biggl\{p^{-(2r+1)(2r+2)/2}u^{2r+1}\frac{\prod_{i=1}^{n/2+r}(1-p^{2i-2k})}{(p^{-2})_{n/2-r-1}}$$

$$\times\,\zeta_{2r+1,o}(u,\iota,d_0d_1^{-1})\Biggr\}$$

$$+\,2^{-1}\sum_{r=0}^{n/2-1}\sum_{d_1\in\mathcal{U}_p}\Biggl\{p^{-2r(2r+1)/2}u^{2r}\frac{\prod_{i=1}^{n/2+r-1}(1-p^{2i-2k})}{(p^{-2})_{n/2-r}}$$

$$\times(1+\psi_{n-2r}(d_1)p^{(-n+2r)/2})(1+\psi_{n-2r}(d_1)p^{(n+2r)/2-k})$$

$$\times\,\zeta_{2r,e}(u,\iota,d_0d_1^{-1})\Biggr\}$$

$$+\,u^np^{-n(n+1)/2}\prod_{i=1}^{n-1}(1-p^{2i-2k})(1+p^{n-k})\zeta_{n,e}(u,\iota,d_0),$$

where

$$\zeta_{r,e}(u,\omega,d)=\frac{1}{2}(\zeta_r(u,\omega,d)+\zeta_r(-u,\omega,d)),$$

and

$$\zeta_{r,o}(u,\omega,d)=\frac{1}{2}(\zeta_r(u,\omega,d)-\zeta_r(-u,\omega,d))$$

for $\omega=\iota,\varepsilon$ and $d\in\mathbb{Z}_p^*$. By Theorem 5.1 of [I-S], we have

$$\zeta_{2r+1,o}(u,\iota,d_0d_1^{-1})=\frac{p^{-1}u}{(p^{-2})_r(1-p^{-2}u^2)\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)},$$

and

$$\zeta_{2r,e}(u,\iota,d_0d_1^{-1})=\frac{(1+\psi_{2r}(d_0d_1^{-1})p^{-r})(1-\psi_{2r}(d_0d_1^{-1})p^{-r-2}u^2)}{(p^{-2})_r(1-p^{-2}u^2)\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)}.$$

For $r\geqslant 2$ we have

$$2^{-1}\sum_{d_1\in\mathcal{U}_p}(1+\psi_{n-2r}(d_1)p^{-n/2+r})(1+\psi_{n-2r}(d_1)p^{n/2+r-k})$$

$$\times\,(1+\psi_{2r}(d_0d_1^{-1})p^{-r})(1-\psi_{2r}(d_0d_1^{-1})p^{-r-2}u^2)$$

$$=(1-\psi_n(d_0)p^{-n/2})^{-1}(1-\psi_n(d_0)p^{n/2-k})^{-1}$$

$$\times\,\bigl\{p^{2r-k}(1+p^{-k})(1-p^{-2r})(1+p^{k-2r-2}u^2)$$

$$+\,(1-p^{-n})(1-p^{n-2k})(1-p^{-2}u^2)$$

$$-\,\psi_n(d_0)p^{-n/2-k+2r}(1+p^{n-k})(1-p^{-2r})(1+p^{k-2r-2}u^2)\bigr\}.$$

Thus we have

$c_n^{-1}D(u,k,d,\iota)_e$

$$
\begin{aligned}
&= \left\{1+p^{-k}-\psi_n(d_0)p^{-n/2}(1+p^{n-k})\right\} \\
&\quad\times \sum_{r=0}^{n/2-1} \frac{(p^{-2})_{n/2-1}}{(p^{-2})_r(p^{-2})_{n/2-r-1}} \frac{p^{-(r+1)(2r+1)-1}u^{2r+2}\prod_{i=n/2+1}^{n/2+r}(1-p^{2i-2k})}{(1-p^{-2}u^2)\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)} \\
&\quad+\sum_{r=0}^{n/2} \frac{(p^{-2})_{n/2-1}}{(p^{-2})_r(p^{-2})_{n/2-r}} \frac{p^{-r(2r+1)}u^{2r}\prod_{i=n/2+1}^{n/2+r-1}(1-p^{2i-2k})}{(1-p^{-2}u^2)\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)} \\
&\quad\times \{p^{2r-k}(1+p^{-k})(1-p^{-2r})(1+p^{k-2r-2}u^2) \\
&\qquad+(1-p^{-n})(1-p^{n-2k})(1-p^{-2}u^2) \\
&\qquad-\psi_n(d_0)p^{-n/2-k+2r}(1+p^{n-k})(1-p^{-2r})(1+p^{k-2r-2}u^2)\Big\} \\
&= p^{-2}(1+p^{-k})(1+p^{1-k})u^2 \\
&\quad\times \sum_{r=0}^{n/2-1} \frac{(p^{-2})_{n/2-1}\prod_{i=n/2+1}^{n/2+r}(1-p^{2i-2k})}{(p^{-2})_r(p^{-2})_{n/2-1-r}\prod_{i=1}^{r+1}(1-p^{2i-3-2r}u^2)(1-p^{-2}u^2)p^{2r^2+3r}}u^{2r} \\
&\quad+\sum_{r=0}^{n/2} \frac{(p^{-2})_{n/2}\prod_{i=n/2}^{n/2+r}(1-p^{2i-2k})}{(p^{-2})_r(p^{-2})_{n/2-r}\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)p^{2r^2+r}}u^{2r} \\
&\quad-\psi_n(d_0)p^{-n/2-2}u^2(1+p^{1-k})(1+p^{n-k}) \\
&\quad\times \sum_{r=0}^{n/2-1} \frac{(p^{-2})_{n/2-1}\prod_{i=n/2+1}^{n/2+r}(1-p^{2i-2k})}{(p^{-2})_r(p^{-2})_{n/2-1-r}\prod_{i=1}^{r}(1-p^{2i-3-2r}u^2)(1-p^{-2}u^2)p^{2r^2+3r}}u^{2r}.
\end{aligned}
$$

Thus by (2) of Lemma 3.4, we have

$$
\begin{aligned}
&D(u,k,d,\iota)_e \\
&\quad= p^{-2}(1+p^{-k})(1+p^{1-k})\frac{u^2\prod_{i=1}^{n/2-1}(1-p^{2i-1-2k}u^2)}{(1-p^{-2}u^2)\prod_{i=0}^{n/2-1}(1-p^{-2i-3}u^2)} \\
&\qquad+\frac{(1-p^{-1-2k}u^2)\prod_{i=1}^{n/2-1}(1-p^{2i-1-2k}u^2)}{\prod_{i=0}^{n/2-1}(1-p^{-2i-3}u^2)} \\
&\qquad-\frac{\psi_n(d_0)p^{-n/2-2}u^2(1+p^{1-k})(1+p^{n-k})\prod_{i=1}^{n/2-1}(1-p^{2i-1-2k}u^2)}{(1-p^{-2}u^2)\prod_{i=0}^{n/2-1}(1-p^{-2i-3}u^2)}.
\end{aligned}
$$

Thus by a simple computation, we have

$$
\begin{aligned}
c_n^{-1}D(p^{-s},k,d_0,\iota)_e &= \frac{\prod_{i=1}^{n/2-1}(1-p^{2i-1-2k}u^2)}{(1-p^{-2}u^2)\prod_{i=0}^{n/2-1}(1-p^{-2i-3}u^2)} \\
&\quad\times\left\{(1+p^{-1-k}u^2)(1+p^{-2-k}u^2)\right. \\
&\qquad\left.-p^{-n/2-2}\psi_n(d_0)u^2(1+p^{1-k})(1+p^{n-k})\right\}.
\end{aligned}
$$

This completes the assertion for an odd p and $D(p^{-s}, k, d_0, \iota)_e$. Similarly, we have

$$\begin{aligned}
&c_n'^{-1} D(p^{-s}, k, d_0, \iota)_o \\
&= \sum_{r=0}^{n/2-1} c_{n,r} \cdot \frac{p^{-(r+1)(2r+1)} u^{2r+1} \prod_{i=n/2+1}^{n/2+r} (1 - p^{2i-2k})}{(1 - p^{-2}u^2) \prod_{i=1}^{r} (1 - p^{2i-3-2r} u^2)} \\
&\quad + \sum_{r=0}^{n/2} c_{n,r} \cdot \frac{p^{-(r+1)(2r+3)-1} u^{2r+3} (1 + p^{2r+2-k}) \prod_{i=n/2+1}^{n/2+r} (1 - p^{2i-2k})}{(1 - p^{-2}u^2) \prod_{i=1}^{r+1} (1 - p^{2i-5-2r} u^2)} \\
&= \sum_{r=0}^{n/2-1} c_{n,r} \cdot \frac{u(1 + p^{-1-k}u^2) p^{-(r+1)(2r+1)} u^{2r} \prod_{i=1}^{r} (1 - p^{n+2i-2k})}{(1 - p^{-2}u^2)(1 - p^{-3}u^2) \prod_{i=1}^{r} (1 - p^{2i-5-2r} u^2)},
\end{aligned}$$

where

$$c_n' = \frac{(1 - p^{-k}) \prod_{i=1}^{n/2} (1 - p^{2i-2k})}{\prod_{i=1}^{n/2-1} (1 - p^{-2i})}, \quad c_{n,r} = \frac{(p^{-2})_{n/2-1}}{(p^{-2})_r (p^{-2})_{n/2-1-r}}.$$

Thus by (2) of Lemma 3.4, we have

$$\begin{aligned}
&c_n'^{-1} D(p^{-s}, k, d_0, \iota)_o \\
&= p^{-1} u (1 + p^{-1-k} u^2) \frac{\prod_{i=1}^{n/2-1} (1 - p^{2i-1-2k} u^2)}{(1 - p^{-2}u^2) \prod_{i=0}^{n/2-1} (1 - p^{-2i-3} u^2)}.
\end{aligned}$$

This completes the assertion for an odd p and $D(p^{-s}, k, d_0, \iota)$.

Next let $p = 2$. Then by Theorems 5.1 and 5.3 of [I-S], we have

$$\zeta_{2r,o}(u, \iota, d_0 d_1^{-1}) = \frac{2^{2r-2} u}{(2^{-2})_r (1 - 2^{-2}u^2) \prod_{i=1}^{r} (1 - 2^{2i-2r-3} u^2)}$$

and

$$\zeta_{2r,o}^*(u, \iota, d_0 d_1^{-1}) = \frac{2^{-3} u^3}{(2^{-2})_r (1 - 2^{-2}u^2) \prod_{i=1}^{r} (1 - 2^{2i-2r-3} u^2)}$$

for any $d_1 \in \mathbb{Z}_2^*$. Furthermore, we have

$$\begin{aligned}
&\zeta_{2r,e}(u, \iota, d_0 d_1^{-1}) \\
&= \frac{2^{2r-1}}{(2^{-2})_{r-1} (1 - \psi_{2r}(d_0 d_1^{-1}) 2^{-r})(1 - 2^{-2}u^2) \prod_{i=1}^{r} (1 - 2^{2i-2r-3} u^2)} \\
&\quad \times \begin{cases} 4u^{-2} \{1 - \psi_{2r}(d_0 d_1^{-1}) 2^{-r-2} u^2 \\ \qquad - (1 - p^{-2r-1} u^2)(1 - p^{-2} u^2)\} & \text{if } (-1)^r d_0 d_1^{-1} \equiv 1 \bmod 4, \\ 1 & \text{if } (-1)^r d_0 d_1^{-1} \equiv 3 \bmod 4, \end{cases}
\end{aligned}$$

and

$$\zeta^*_{2r,e}(u,\iota,d_0d_1^{-1})$$
$$= \frac{1}{(2^{-2})_{r-1}(1-\psi_{2r}(d_0d_1^{-1})2^{-r})(1-2^{-2}u^2)\prod_{i=1}^{r}(1-2^{2i-2r-3}u^2)}$$
$$\times \begin{cases} (1-\psi_{2r}(d_0d_1^{-1})2^{-r-2}u^2) & \text{if } (-1)^r d_0d_1^{-1} \equiv 1 \bmod 4 \\ 2^{-2}u^2 & \text{if } (-1)^r d_0d_1^{-1} \equiv 3 \bmod 4. \end{cases}$$

Thus the assertion can be proved similarly to the case $p \neq 2$.

(2) Let $p \neq 2$. Then by (1) of Proposition 3.1 we have

$$c_n''^{-1} D(u,k,d_0,\varepsilon) = (1+p^{-n/2}\psi_n(d_0))(1+p^{-k+n/2}\psi_n(d_0))$$
$$+ 2^{-1}\sum_{r=1}^{n/2-1} \frac{p^{-r(2r+1)}u^{2r}\prod_{i=0}^{r-1}(1-p^{2i+n-2k})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}}$$
$$\times \sum_{d_1\in\mathcal{U}_p} (1+p^{r-n/2}\psi_{n-2r}(d_1))(1+p^{-k+n/2+r}\psi_{n-2r}(d_1))\psi_{2r}(d_0d_1^{-1})$$
$$\times \zeta_{2r}((p^{2r+1}d_1,p)_p u,\varepsilon,d_0d_1^{-1})$$
$$+ \frac{p^{-n(n+1)/2}u^n\prod_{i=0}^{n/2-1}(1-p^{2i+n-2k})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}}$$
$$\times (1+p^{-k+n})\psi_n(d_0)\zeta_n((p^{n+1}d_1,p)_p u,\varepsilon,d_0)$$
$$+ 2^{-1}\sum_{r=1}^{n/2} \frac{p^{-r(2r-1)}u^{2r-1}\prod_{i=0}^{r-1}(1-p^{2i+n-2k})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}}$$
$$\times \sum_{d_1\in\mathcal{U}_p} (d_1(-1)^r,p)_p\zeta_{2r-1}((p^{2r}d_1,p)_p u,\varepsilon,d_0d_1^{-1}),$$

where

$$c_n'' = \frac{(1-p^{-k})\prod_{i=1}^{n/2-1}(1-p^{2i-2k})}{\prod_{i=1}^{n/2}(1-p^{-2i})}.$$

By Theorem 5.2 of [I-S], $\zeta_{2r}((p^{2r+1}d_1,p)_p u,\varepsilon,d)$ is an even function for any $d\in\mathbb{Z}_p^*$ and

$$\zeta_{2r}((p^{2r+1}d_1,p)_p u,\varepsilon,d_0d_1^{-1}) = \frac{1+\psi_{2r}(d_0d_1^{-1})p^{-r}}{(p^{-2})_r\prod_{i=1}^{r}(1-p^{-2i}u^2)}.$$

Further we have

$$\zeta_{2r-1}((p^{2r}d_1,p)_p u,\varepsilon,d_0d_1^{-1}) = \frac{1+p^{-r}((-1)^r d_1,p)_p u}{(p^{-2})_{r-1}\prod_{i=1}^{r}(1-p^{-2i}u^2)}.$$

Thus we have

$$\begin{aligned}
c_n''^{-1} D(u,k,d_0,\varepsilon) &= \sum_{r=1}^{n/2} \frac{p^{-r(2r+1)}u^{2r}\prod_{i=0}^{r-1}(1-p^{2i-2k+n})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}(p^{-2})_r\prod_{i=1}^{r}(1-p^{-2i}u^2)} \\
&\quad \times \left\{p^{-r}(1+p^{-k+2r}) + p^{r-n/2}\psi_n(d_0)(1+p^{n-k})\right\} \\
&\quad + \sum_{r=1}^{n/2} \frac{p^{-2r^2}u^{2r}\prod_{i=0}^{r-1}(1-p^{2i-2k+n})(1-p^{-2r})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}(p^{-2})_r\prod_{i=1}^{r}(1-p^{-2i}u^2)} \\
&= (1+p^{-n/2}\psi_n(d_0))(1+p^{-k+n/2}\psi_n(d_0)) \\
&\quad \times \sum_{r=0}^{n/2} \frac{p^{-2r^2}u^{2r}\prod_{i=0}^{r-1}(1-p^{2i-2k+n})(p^{-2})_{n/2}}{(p^{-2})_{n/2-r}(p^{-2})_r\prod_{i=1}^{r}(1-p^{-2i}u^2)}.
\end{aligned}$$

Thus the assertion follows from (2) of Lemma 3.4.

Next let $p = 2$. First assume that $((-1)^{n/2}d_0, -1)_2 = -1$. Then by Theorems 5.2 and 5.3 of [I-S], we have

$$\zeta_{2r}(u,\varepsilon,d_0d_1^{-1}) = \zeta_{2r}^*(u,\varepsilon,d_0d_1^{-1}) = 0$$

for any $d_1 \in \mathcal{U}_{n-r,2}$. Thus by (2) of Proposition 3.1 we have

$$D(u,k,d_0,\varepsilon) = 0.$$

Next assume that $((-1)^{n/2}d_0, -1)_2 = 1$. Then, again by Theorems 5.2 and 5.3 of [I-S] we have

$$\zeta_{2r}(u,\varepsilon,d_0d_1^{-1}) = \frac{2^r(-1)^{r(r+1)/2}(1+2^{-r}\psi_{2r}(d_0d_1^{-1}))}{(2^{-2})_r\prod_{i=1}^{r}(1-2^{-2i}u^2)}$$

and

$$\zeta_{2r}^*(u,\varepsilon,d_0d_1^{-1}) = \frac{(-1)^{r(r+1)/2}\psi_{2r}(d_0d_1^{-1})(1+2^{-r}\psi_{2r}(d_0d_1^{-1}))}{(2^{-2})_r\prod_{i=1}^{r}(1-2^{-2i}u^2)}$$

for any $d_1 \in \mathcal{U}_{n-r,2}$. Thus, again by using (2) of Proposition 3.1, the assertion can be proved similarly to the case $p \neq 2$. □

The following theorem can be proved in the same manner as above.

Theorem 3.2. *Let $n \geqslant 3$ be an odd integer and $d_0 \in \mathbb{Z}_p^*$. Let k be a complex number.*

(1) *We have*

$$D(p^{-s},k,d_0,\iota) = \frac{2^{-\delta_{2,p}s}\prod_{i=1}^{(n-1)/2}(1-p^{2i-2k})}{(p^{-2})_{(n-1)/2}}$$

$$\times \frac{\prod_{i=1}^{(n-1)/2}(1-p^{2i-1-2k-2s})(1+p^{-k-s})}{(1-p^{-1-s})\prod_{i=1}^{(n-1)/2}(1-p^{-2i-1-2s})}.$$

(2) *We have*

$$D(p^{-s},k,d_0,\varepsilon)$$
$$= \frac{2^{-\delta_{2,p}s}(d_0,(-1)^{(n+1)/2})_p(1-p^{-k})\prod_{i=1}^{(n-1)/2}(1-p^{2i-2k})}{(p^{-2})_{(n-1)/2}}$$
$$\times \frac{(1+p^{-k-(n-1)/2-s})\prod_{i=1}^{(n-1)/2}(1-p^{-2k+2i-2-2s})}{\prod_{i=1}^{(n-1)/2}(1-p^{-2i-2s})}.$$

Theorem 3.3. *Let $n \geqslant 4$ be an even integer.*
(1) *We have*

$$\tilde{L}^{(j)}_{n,k}(s,\sigma,\iota)\zeta(GL_n,2s)$$
$$= \frac{\prod_{i=1}^{n/2-1}\zeta(2i)}{\zeta(k+2\sigma)\prod_{i=1}^{n/2}\zeta(2k+4\sigma-2i)}$$
$$\times \tilde{D}^*(s,n;(-1)^{n/2}\delta_j)\otimes\tilde{D}^*(s,2k+4\sigma-n;(-1)^{n/2}\delta_j)$$
$$\times \prod_{i=1}^{n/2-1}\zeta(2s-2i)\zeta(2s-2k+2i+1).$$

(2) *We have*

$$\tilde{L}^{(j)}_{n,k}(s,\sigma,\varepsilon)\zeta(GL_n,2s)$$
$$= \frac{\epsilon_j(-1)^{n(n+2)/8}\zeta(k+2\sigma-n/2)\prod_{i=1}^{n/2-1}\zeta(2i)\zeta(n/2)}{\zeta(k+2\sigma)\prod_{i=1}^{n/2}\zeta(2k+4\sigma-2i)}$$
$$\times \prod_{i=1}^{n/2}\zeta(2s-2i+1)\zeta(2s+2i-2k-4\sigma)$$

or 0 according as $n \equiv 0 \bmod 4$ or not.

Proof. For each $d \in \mathcal{D}_{(-1)^{n/2}\delta_j}$, put

$$M(s,\sigma,d,\omega) = \sum_{m=1}^{\infty}\lambda((-1)^{n/2}\delta_j dm^2,\omega,k+2\sigma)m^{2k+4\sigma-2s}.$$

Then we have

$$\tilde{L}^{(j)}_{n,k}(s,\sigma,\omega) = \sum_{d\in\mathcal{D}_{(-1)^{n/2}\delta_j}} d^{k+2\sigma-s}M(s,\sigma,d,\omega).$$

By (1) of Theorem 3.1, we have

$$d^{k+2\sigma-s}M(s,\sigma,d,\iota)=\prod_{p|d}D(p^{k+2\sigma-s},k+2\sigma,d,\iota)_o\prod_{p\nmid d}D(p^{k+2\sigma-s},k+2\sigma,d,\iota)_e.$$

Furthermore, by (2) of Theorem 3.1,

$$d^{k+2\sigma-s}M(s,\sigma,d,\varepsilon)=\prod_{p|d}D(p^{k+2\sigma-s},k+2\sigma,d,\varepsilon)_o\prod_{p\nmid d}D(p^{k+2\sigma-s},k+2\sigma,d,\varepsilon)_e.$$

Again by (2) of Theorem 3.1, $D(p^{k+2\sigma-s},k+2\sigma,d_0,\varepsilon)_o=0$. Thus,

$$d^{k+2\sigma-s}M(s,\sigma,d,\varepsilon)=0$$

unless $d=1$. Thus, if $n\equiv 2 \bmod 4$, we have

$$\tilde{L}_{n,k}^{(j)}(s,\sigma,\varepsilon)=0.$$

On the other hand, if $n\equiv 0 \bmod 4$,

$$\tilde{L}_{n,k}^{(j)}(s,\sigma,\varepsilon)=M(s,\sigma,1,\varepsilon).$$

Thus the assertion can be proved. □

Theorem 3.4. *Let $n\geq 3$ be an odd integer.*
(1) *We have*

$$\tilde{L}_{n,k}^{(j)}(s,\sigma,\iota)\zeta(GL_n,2s)=\frac{2^{-s}\prod_{i=1}^{(n-1)/2}\zeta(2i)}{\zeta(k+2\sigma)\prod_{i=1}^{(n-1)/2}\zeta(2k+4\sigma-2i)}$$
$$\times\,\zeta(s)\zeta(s-k-2\sigma+1)\prod_{i=1}^{(n-1)/2}\zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1).$$

(2) *We have*

$$\tilde{L}_{n,k}^{(j)}(s,\sigma,\varepsilon)\zeta(GL_n,2s)=\frac{2^{-s}\prod_{i=1}^{(n-1)/2}\zeta(2i)}{\zeta(k+2\sigma)\prod_{i=1}^{(n-1)/2}\zeta(2k+4\sigma-2i)}$$
$$\times\,(-1)^{(n^2-1)/8}\epsilon_j\delta_j^{(n+1)/2}\zeta(s-\frac{n-1}{2})\zeta(s-k-2\sigma+\frac{n+1}{2})$$
$$\times\prod_{i=1}^{(n-1)/2}\zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i).$$

4. Functional equations and special values of Koecher-Maaß series

Finally we consider the functional equation for the Koecher-Maaß series. Put $\Gamma_n(s) = \prod_{i=1}^{n} \Gamma(s-(i-1)/2)$. First we consider the case n is odd.

Theorem 4.1. *Let $n \geq 3$ be odd. Then $L_{n,k}^{(j)}(s,\sigma)$ depends only on $\epsilon_j \delta_j^{(n+1)/2}$, and thus for $l = 1, 2$ we put*

$$\eta_l(s,\sigma) = (2\pi)^{-ns}\Gamma_n(s)L_{n,k}^{(j)}(s,\sigma)$$

if $(-1)^{(n^2-1)/8}\epsilon_j\delta_j^{(n+1)/2} = (-1)^{l-1}$. Furthermore, put

$$\zeta_1(s,\sigma) = \frac{\cos(\pi s/2)}{\cos(\pi(s/2-\sigma))}\left(\frac{\cos(\pi s)}{\cos(\pi(s-2\sigma))}\right)^{(n-1)/2},$$

and

$$\zeta_2(s,\sigma) = \frac{\sin(\pi(s/2+(-n+3)/4))}{\sin(\pi(s/2-\sigma+(-n+3)/4))}\left(\frac{\sin(\pi s)}{\sin(\pi(s-2\sigma))}\right)^{(n-1)/2}.$$

Then we have

$$\begin{pmatrix} \eta_1(k+2\sigma-s,\sigma) \\ \eta_2(k+2\sigma-s,\sigma) \end{pmatrix}$$
$$= (-1)^{k/2}\begin{pmatrix} \frac{\zeta_1(s,\sigma)+\zeta_2(s,\sigma)}{2} & \frac{\zeta_1(s,\sigma)-\zeta_2(s,\sigma)}{2} \\ \frac{\zeta_1(s,\sigma)-\zeta_2(s,\sigma)}{2} & \frac{\zeta_1(s,\sigma)+\zeta_2(s,\sigma)}{2} \end{pmatrix}\begin{pmatrix} \eta_1(s,\sigma) \\ \eta_2(s,\sigma) \end{pmatrix}.$$

Proof. Put

$$A_1(s,\sigma) = \zeta(s)\zeta(s-k-2\sigma+1) \times \prod_{i=1}^{(n-1)/2} \zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1),$$

and

$$A_2(s,\sigma) = \zeta(s-\frac{n-1}{2})\zeta(s-k-2\sigma+\frac{n+1}{2}) \times \prod_{i=1}^{(n-1)/2} \zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i).$$

Then by the functional equation of Riemann's zeta function, we have

$$A_1(k+2\sigma-s,\sigma) = (-1)^{k/2}\zeta_1(s,\sigma)A_1(s,\sigma),$$

and

$$A_2(k+2\sigma-s,\sigma)=(-1)^{k/2}\zeta_2(s,\sigma)A_2(s,\sigma).$$

Thus the assertion holds. □

Theorem 4.2. *Let $n \geq 4$ be even. Then $L_{n,k}^{(j)}(s,\sigma)$ depends only on δ_j and ϵ_j, and in particular, it does not depend on ϵ_j if $\delta_j=(-1)^{n/2+1}$. Thus for $l=1,2$ we put*

$$\eta_l(s,\sigma)=(2\pi)^{-ns}\Gamma_n(s)L_{n,k}^{(j)}(s,\sigma)$$

if $\delta_j=(-1)^{n/2}$ and $(-1)^{n(n+2)/8}\epsilon_j=(-1)^{l-1}$ and

$$\eta_3(s,\sigma)=(2\pi)^{-ns}\Gamma_n(s)L_{n,k}^{(j)}(s,\sigma)$$

if $\delta_j=(-1)^{n/2+1}$. Furthermore, for $\sigma\neq 1/4+m$ $(m=0,\pm1,\pm2,\ldots)$ put

$$\zeta_1(s,\sigma)=\left(\frac{\cos(\pi s)}{\cos(\pi(s-2\sigma))}\right)^{n/2},$$

$$\zeta_2(s,\sigma)=(-1)^{n/2}\left(\frac{\sin(\pi s)}{\sin(\pi(s-2\sigma))}\right)^{n/2},$$

$$\begin{aligned}\zeta_3(s,\sigma)=&\frac{\cos(\pi\sigma)}{\cos(2\pi\sigma)}\frac{\sin(2\pi s)}{\cos(\pi(s-\sigma))}\\&\times\left(\frac{1}{\sin(2\pi s)}+\frac{1}{\sin(2\pi(s-2\sigma))}\right)\left(\frac{\cos(\pi s)}{\cos(\pi(s-2\sigma))}\right)^{n/2-1},\end{aligned}$$

and

$$\zeta_4(s,\sigma)=-\left(\frac{\cos(\pi s)}{\cos(\pi(s-2\sigma))}\right)^{n/2-1}\frac{\sin(\pi s)}{\sin(\pi(s-2\sigma))}.$$

(1) *Let $n\equiv 0 \bmod 4$. Then we have*

$$\begin{pmatrix}\eta_1(k+2\sigma-s,\sigma)\\ \eta_2(k+2\sigma-s,\sigma)\\ \eta_3(k+2\sigma-s,\sigma)\end{pmatrix}=\begin{pmatrix}\frac{\zeta_1(s,\sigma)+\zeta_2(s,\sigma)}{2} & \frac{\zeta_1(s,\sigma)-\zeta_2(s,\sigma)}{2} & 0\\ \frac{\zeta_1(s,\sigma)-\zeta_2(s,\sigma)}{2} & \frac{\zeta_1(s,\sigma)+\zeta_2(s,\sigma)}{2} & 0\\ \frac{\zeta_3(s,\sigma)}{2} & \frac{\zeta_3(s,\sigma)}{2} & \zeta_4(s,\sigma)\end{pmatrix}\begin{pmatrix}\eta_1(s,\sigma)\\ \eta_2(s,\sigma)\\ \eta_3(s,\sigma)\end{pmatrix}.$$

(2) *Let $n\equiv 2 \bmod 4$. Then we have*

$$\begin{pmatrix}\eta_1(k+2\sigma-s,\sigma)\\ \eta_2(k+2\sigma-s,\sigma)\\ \eta_3(k+2\sigma-s,\sigma)\end{pmatrix}=\begin{pmatrix}\frac{\zeta_2(s,\sigma)+\zeta_4(s,\sigma)}{2} & \frac{\zeta_2(s,\sigma)-\zeta_4(s,\sigma)}{2} & \zeta_3(s,\sigma)\\ \frac{\zeta_2(s,\sigma)-\zeta_4(s,\sigma)}{2} & \frac{\zeta_2(s,\sigma)+\zeta_4(s,\sigma)}{2} & \zeta_3(s,\sigma)\\ 0 & 0 & \zeta_1(s,\sigma)\end{pmatrix}\begin{pmatrix}\eta_1(s,\sigma)\\ \eta_2(s,\sigma)\\ \eta_3(s,\sigma)\end{pmatrix}.$$

Proof. Put

$$\Omega(s,1)=\pi^{-2s}\Gamma(s)\Gamma(s-k-2\sigma+n/2+1/2)\tilde{D}^*(s,n,1)\otimes\tilde{D}^*(s,k+2\sigma-n,1)$$

and

$$\Omega(s,-1)=\pi^{-2s}\Gamma(s-n/2+1/2)\Gamma(s-k-2\sigma+1)\tilde{D}^*(s,n,-1)\otimes\tilde{D}^*(s,k+2\sigma-n,-1).$$

Then by Mizuno[Mi], we have

$$\begin{aligned}\Omega(k+2\sigma-s,1)&=\Omega(s,1),\\ \Omega(k+2\sigma-s,-1)&=\Omega(s,-1)-c(s,\sigma,k)\Omega(s,1),\end{aligned}$$

where

$$\begin{aligned}c(s,\sigma,k)=&\frac{\cos\pi\sigma}{\cos 2\pi\sigma}\frac{\sin\pi(s-2\sigma)\cos\pi s}{\cos\pi(s-\sigma)}\left(\frac{2}{\sin 2\pi s}+\frac{2}{\sin 2\pi(s-2\sigma)}\right)\\ &\times\frac{\Gamma(s-n/2+1/2)\Gamma(s-k-2\sigma+1)}{\Gamma(s)\Gamma(s-k-2\sigma+n/2+1/2)}.\end{aligned}$$

Thus the assertion holds. □

As for the special values of the Koecher-Maaß series, we have the following.

Theorem 4.3. *Let $n\geq 3$ be odd and $k\geq 2n$. Then for any integer $n-1\leq r\leq k-n-1$ we have*

$$\pi^{-rn+(n^2-1)/4}L_{n,k}^{(i)}(r,0)\in\mathbb{Q}.$$

Remark. A result similar to the above is expected in the case n is even.

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A SHORT HISTORY ON INVESTIGATION OF THE SPECIAL VALUES OF ZETA AND L-FUNCTIONS OF TOTALLY REAL NUMBER FIELDS

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Dedicated to the memory of Tsuneo Arakawa

Introduction

Even the senior author of this article considers that it seems too early for him to talk about the history of mathematics. This would be more case for the younger. But this workshop is for the memory of Arakawa, so we consider it makes sense to give a talk on some theme which was a favorite one for him (cf. [Ar82], [Ar85], [Ar88], [Ar89], [Ar93], [Ar94]). Since Masanobu Kaneko talked about these papers of Arakawa directly, we are going to discuss the *circumstances* of these papers.

We can start our talk from the well-known formula of Euler in 1735 on the values of the Riemann zeta function at even positive integers:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{\pi^{2k}}{(2k)!} B_{2k},$$

where B_m is the m-th Bernoulli number. Probably one can find some analogous formulae proved after this result, say, for the Gaussian number field $\mathbf{Q}(\sqrt{-1})$. But we do not dig out possible special results here.

The early history of modern investigation of special values of zeta functions over number fields is shortly reviewed in the introductions of Klingen's paper [Kl62a] or of Barner's paper [Ba69]. A most extensive history from the dawn of the study of algebraic number field is found in the introduction of Meyer's book [Me57a].

In 20's Hecke [He24, p.219] conjectured the corresponding statement for arbitrary totally real algebraic number fields K, namely

$$\zeta_K(2k) = \pi^{2kn} d_K^{1/2} r \qquad (k = 1, 2, \dots),$$

where d_K the discriminant, n the degree of K and r a rational number depending on K and k. For quadratic fields there is an elementary proof via Dirichlet L-function shown by Siegel [Si22]. This was later generalized by Leopoldt [Le58] for abelian totally real number fields, introducing generalized Bernoulli numbers.

Meyer [Me57a], [Me57b] and Siegel [Si61] developed the idea of Hecke for real quadratic fields K to have certain "analytic class number formula" for abelian extensions L of K which has "isobaric" ramification at archimedean places. After that this method was extended to the evaluation of certain Hecke L-functions $L_K(s, \chi)$ at positive integers $s = n$ by Meyer [Me66], Lang [La68], Barner [Ba68] and Siegel [Si68]. Among others, Meyer always pursued *elementary method* to compute these arithmetic invariants by investigating the properties of generalized Dedekind sums (section 2).

The construction of p-adic L-functions, initiated by Kubota-Leopldt in 60's, became a trendy theme of 70's together with the theory of modular symbols (cf. Manin [Ma72], Mazur and Swinnerton-Dyer [MSD74]), and the people's interest was oriented toward the geometric interpretation of the known analytic results than to get new analytic expression. However in mid-70's, Shintani's paper [Shin76] brought a paradigm change in the analytic aspect. This method works for arbitrary totally real number fields K of degree n, and gives an effective method of computations of the class numbers of certain abelian extensions L of K and the values of the Hecke L-function $L_K(s, \chi)$ at positive integers s for certain characters χ, by linear combinations of the n products of the values of Bernoulli polynomials at rational numbers. Note that here appeared the proto-type of higher Dedekind sums. This result is also applied to the construction of p-adic L-functions by Cassou-Noguès [CN79]. There are some papers of *revisionism* in 70's and 80's (section 3).

In 90's new development appeared. In [Sc92], [Sc93], Sczech found an ingenious way to generalize the method of Hecke, Meyer and Siegel to arbitrary totally real number fields by analyzing a conditionally converging Eisenstein series on $GL(n,\mathbf{Z})\backslash GL(n,\mathbf{R})$, which is named Eisenstein cocycles. Meanwhile Solomon [So98], [So99] began to study another cocycles called Shintani cocycles, which is cohomologous to Eisenstein cocycles (section 4).

In spite of these efforts mentioned above, there seems to be still unsettled problems (section 5).

1. Before 1950: Hecke, Siegel, and others

It would be wise to leave aside the question whether the unrealized "plan" of Eisenstein or the "dream" of Kronecker covered more general subjects than just the theory of classical complex multiplication, to a serious historian or to an eternal mystery. We begin with the work of Erich Hecke here.

1.1. *Hecke*

To discuss the invariants of abelian extensions of a real quadratic field K, we should start from Hecke's paper [He17], [He21]. In [He17], he gave a "plan" of this research. It consists of 3 sections: (1) Kronecker limit formula and the so-called Hecke's integration formula for real quadratic fields, (2) to write (partial) zeta functions of general algebraic number fields K as pull-back integrations of the Epstein zeta function of degree $[K:\mathbf{Q}]$, (3) the announcement of the relative class number formulae for abelian extensions of number fields. Apart from the arrangement of the original paper, we start from Epstein zeta function.

1.1.1. *The integral expression of Hecke*

(A) *The Epstein zeta functions*

Given $g \in GL(n,\mathbf{R})$ or $g \in SL(n,\mathbf{R})$, we associate a symmetric positive-definite matrix $Y_g = g \cdot {}^t g$ of size n. The whole set of such matrices are denoted by $\mathcal{P}$ or by $\mathcal{P}_1$, respectively, which is isomorphic to a symmetric spaces $GL(n,\mathbf{R})/O(n)$ or $SL(n,\mathbf{R})/SO(n)$, respectively. Given a nonzero row vector $\mathbf{m} \in \mathbf{Z}^n$ of size n, the value of the real quadratic form $\mathbf{m}\, Y\, {}^t\mathbf{m}$ ($Y \in \mathcal{P}$ or $\mathcal{P}_1$) is nonzero, and we can define a series

$$Z(s,Y) := \sum_{\mathbf{m}\in\mathbf{Z}^n\backslash\{0\}} (\mathbf{m}\, Y\, {}^t\mathbf{m})^{-s}$$

for $s \in \mathbf{C}$, which is called an Epstein zeta function. It converges absolutely for $Re(s) > n/2$ and is invariant under $GL(n,\mathbf{Z})$ or $SL(n,\mathbf{Z})$ with respect to the natural action of $GL(n,\mathbf{Z})$ (resp. $SL(n,\mathbf{Z})$) on $\mathcal{P}$ (resp. $\mathcal{P}_1$) given by

$$Y \mapsto \gamma Y^t\gamma \qquad (\gamma \in GL(n,\mathbf{Z}) \text{ or } SL(n,\mathbf{Z})).$$

Epstein showed the analytic continuation in s and the functional equation of $Z(s,Y)$. It is a kind of Eisenstein series on $GL(n,\mathbf{Z})\backslash GL(n,\mathbf{R})/O(n)$ or on $SL(n,\mathbf{Z})\backslash SL(n,\mathbf{R})/SO(n)$, belonging to a very small degenerate principal series representation of $GL(n,\mathbf{R})$ or $SL(n,\mathbf{R})$.

(B) *The pull-back to maximal tori*

Let K be an algebraic extension of degree n over $\mathbf{Q}$, then we have a natural embedding of the algebraic torus $T_K := Res_{K/\mathbf{Q}}\mathbb{G}_m$ to $GL(n)$, where $Res_{K/\mathbf{Q}}$ is the restriction of scalars of Weil.

In the down-to-earth way, this is defined as follows. Fix an integral basis of the integer ring $\mathcal{O}_K$ of K:

$$\mathcal{O}_K = \mathbf{Z}\omega_1 + \cdots + \mathbf{Z}\omega_n.$$

Consider the norm form of the linear form $x = \sum_{i=1}^n x_i\omega_i$ with n variables $x_1,\ldots,x_n$:

$$N(x;\omega) := \prod_{j=1}^n \Big(\sum_{i=1}^n x_i\omega_i^{(j)}\Big).$$

Here

$$\mathcal{O}_K \ni \alpha \mapsto (\alpha^{(1)},\ldots,\alpha^{(n)}) \in \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^n$$

is the image of the canonical ring homomorphism. The action of the multiplicative group $\mathcal{O}_K^\times$ on $\mathcal{O}_K$ is extended to the action of $T_K(\mathbf{R}) = (\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{R})^\times$ on $\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^n$. Thus we have compatible groups homomorphisms

$$i_{\mathbf{Z}} : \mathcal{O}_K^\times \to GL(n,\mathbf{Z}), \quad i_{\mathbf{R}} : T_K(\mathbf{R}) \to GL(n,\mathbf{R}).$$

Take the norm 1 part $T_K^{(1)}$ in T_K:

$$T_K^{(1)} := \{(x_1,\ldots,x_n) \mid N(x;\omega) = 1\}.$$

Then we have $i_{\mathbf{Z}} : \mathcal{O}_K^{(1)} \to SL(n,\mathbf{Z})$.

Choose a point Y_0 in $\mathcal{P}_1$, and restrict $Z(s,Y)$ to the $T_K^{(1)}(\mathbf{R})$-orbit $\mathcal{Q}_{K,\omega}$ of Y_0. Then the function $Z(s,Y)$ $(Y \in \mathcal{Q}_{K,\omega})$ is periodic with respect to $\mathcal{O}_K^{(1)}$, and define a function on the compact double coset:

$$\mathcal{O}_K^{(1)}\backslash T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R}) \cap SO(Y_0))$$

of real dimension $r = r_1 + r_2 - 1$, which is a finite extension of a compact real torus of dimension r. Here $SO(Y_0)$ is the stabilizer of Y_0 in $SL(n, \mathbf{R})$ which is isomorphic to $SO(n)$, and r_1, r_2 are the numbers of real places and complex places of K, respectively.

Now we can consider the Fourier expansion of the pull-back $Z(s,*)|_{Q_{K,\omega}}$:

$$Z(s,Y) = \sum_{\psi \in \mathcal{O}_K^{(1)} \backslash \widehat{T_K^{(1)}(\mathbf{R})}} a_\psi(s)\psi(Y).$$

Choose a fundamental domain $D(Y_0)$ in $T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R}) \cap SO(Y_0))$ with respect to $\mathcal{O}_K^{(1)}$. Then the constant term a_0 with respect to the trivial character "$\psi = 0$" is the average

$$a_0 = \int_{D(Y_0)} Z(s,Y)dv(Y)$$

with an adequate normalization of the invariant integral dv of $v \in T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R}) \cap SO(Y_0))$. Meanwhile it is given by the Dedekind zeta function:

$$a_0 = w\frac{2^{-r_2 s}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}}{2^{r_1-1}nR \cdot \Gamma(ns/2)} \cdot \zeta_K(s),$$

where R is the regulator of K and w the number of roots of unity in K. Similarly other terms a_ψ, which are twisted integral of $Z(s,Y)$ in one side, are also expressed by appropriate zeta functions with Grössencharacter. This is the content of Section 2 of [He17].

(C) *The case of real quadratic fields*

Let K be a real quadratic field. For a non-zero real number a, we have

$$\Gamma\left(\frac{s}{2}\right)|a|^{-s} = \int_0^\infty e^{-a^2 t} t^{s/2}\frac{dt}{t}.$$

Apply this formula for $a = \mu \in K$ and its conjugate $a = \mu'$, then form the product of the two integrals to obtain a double integral

$$\Gamma\left(\frac{s}{2}\right)^2 |\mu\mu'|^{-s} = \int_0^\infty\int_0^\infty e^{-(\mu^2 t + \mu'^2 t')}(tt')^{s/2}\frac{dt}{t}\frac{dt'}{t'}.$$

This is, in turn, rewritten by the change of variables

$$t = uv^2, \quad t' = uv^{-2}$$

to have

$$\Gamma\Big(\frac{s}{2}\Big)^2 |\mu\mu'|^{-s} = 4\int_0^\infty\int_0^\infty e^{-u(\mu^2v^2+\mu'^2v^{-2})}u^s\frac{du}{u}\frac{dv}{v}$$
$$= 4\,\Gamma(s)\int_0^\infty (\mu^2v^2+{\mu'}^2v^{-2})^{-s}\frac{dv}{v}.$$

Choose an (absolute) ideal class A of K, then the partial zeta function

$$\zeta_K(s,A) := \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}$$

associated with A is written as

$$\zeta_K(s,A) = N(\mathfrak{b})^s \sum_{\mu\in\mathfrak{b}/E_K}{}' |N(\mu)|^{-s},$$

if one picks up a fixed ideal $\mathfrak{b}$ belonging to the inverse class A^{-1}. Here E_K is the group of units in $\mathcal{O}_K$. Let $d_K > 0$ be the discriminant of K and ε the fundamental unit satisfying $\varepsilon > 1$. Then

$$\zeta_K(s,A) = \frac{1}{2}N(\mathfrak{b})^s \sum_{\mu\in\mathfrak{b}/<\varepsilon>}{}' |\mu\mu'|^{-s}$$
$$= \frac{2\,\Gamma(s)}{\Gamma(s/2)^2}N(\mathfrak{b})^s\int_0^\infty \sum_{\mu\in\mathfrak{b}/<\varepsilon>}{}' (\mu^2v^2+{\mu'}^2v^{-2})^{-s}\frac{dv}{v}$$
$$= \frac{2\,\Gamma(s)}{\Gamma(s/2)^2}N(\mathfrak{b})^s\int_1^\varepsilon \sum_{\mu\in\mathfrak{b}}{}'(\mu^2v^2+{\mu'}^2v^{-2})^{-s}\frac{dv}{v}$$
$$= \frac{\Gamma(s)}{\Gamma(s/2)^2}N(\mathfrak{b})^s\int_1^{\varepsilon^2} \sum_{\mu\in\mathfrak{b}}{}'\Big(\frac{v}{\mu^2v^2+{\mu'}^2}\Big)^s\frac{dv}{v}.$$

Choose an integral basis $[\beta_1,\beta_2]$ of $\mathfrak{b}$ such that

$$\beta_1\beta_2' - \beta_2\beta_1' = N(\mathfrak{b})\sqrt{d_K} > 0$$

and set

$$\tau = \frac{\beta_2 v\sqrt{-1}+\beta_2'}{\beta_1 v\sqrt{-1}+\beta_1'}.$$

Then the last integrand is an Epstein zeta function on $SL(2,\mathbf{R})$, which is identified with the usual real analytic Eisenstein series

$$f(\tau,s) := Z(s,Y_\tau) = \sum_{(m,n)\in\mathbf{Z}^2\backslash\{0\}} \frac{y^s}{|m\tau+n|^{2s}}$$

on the complex upper half plane $\mathfrak{H}$ by associating $Y_\tau = \dfrac{1}{y}\begin{pmatrix} 1 & x \\ x & x^2+y^2 \end{pmatrix} \in \mathcal{P}_1$ for $\tau = x + \sqrt{-1}\,y \in \mathfrak{H}$. Thus summing up the above equalities, we have the following:

Theorem 1.1. (Hecke's integral expression) *Let K be a real quadratic field. Then, under the above notation,*

$$\zeta_K(s,A) = d_K^{-s/2}\frac{\Gamma(s)}{\Gamma(s/2)^2}\int_1^{\varepsilon^2} f(\tau,s)\frac{dv}{v}.$$

Remark. We formulate the above formula only for absolute ideal class (in the wide sense). The cases of ring classes and ray classes are quite analogous. See Barner [Ba68, Korollar (3.21)] and Siegel [Si61], [Si68].

1.1.2. *Kronecker limit formula*

When $n = 2$ we have the following:

Theorem 1.2. (Kronecker limit formula) $f(\tau,s) = Z(s,Y_\tau)$ *has a simple pole at $s = 1$ with residue π and has the Laurent expansion:*

$$f(\tau,s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{y}\,|\eta(\tau)|^2)) + O(s-1).$$

Here γ is the Euler constant, η is the Dedekind eta function, and $O(s-1)$ is the Landau symbol.

Combined with Hecke's integral expression, we have the following formula to study the behavior of $\zeta_K(s,A)$ at $s = 1$:

Theorem 1.3. (Hecke's integral expression + Kronecker limit formula) *Under the notation in Section 1.1.1, we have the Laurent expansion at $s = 1$:*

$$\zeta_K(s,A) = \frac{2\log\varepsilon}{\sqrt{d_K}}\frac{1}{s-1} + \frac{2}{\sqrt{d_K}}\varphi(\mathfrak{b}) + O(s-1),$$

where the constant term $\varphi(\mathfrak{b})$ has an integral expression:

$$\varphi(\mathfrak{b}) = 2\gamma\log\varepsilon - \int_1^{\varepsilon^2}\log\Big\{d_K^{1/4}\Big(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}\Big)^{1/2}\eta(\tau)\eta(-\bar{\tau})\Big\}\frac{dv}{v}.$$

Here $\bar{\tau}$ means complex conjugate of τ.

This, together with its variations, is fundamental to investigate the L-functions on a real quadratic field K associated with ring class groups or ray class groups. This formula was the starting point of Meyer's research.

1.1.3. *Holomorphic Eisenstein series in the Hilbert modular case*

In his paper [He24], Hecke considered Eisenstein series of two complex variables belonging to the Hilbert modular group associated with a real quadratic field K. Let A be an (absolute) ideal class of K. Fix an ideal $\mathfrak{a} \in A$ and an ideal class character χ_k such that for the principal ideal (λ) its value is given by the signature $\chi_k((\lambda)) = sgn(\lambda\lambda')^k$. He considered an Eisenstein series

$$G_k(\tau, \tau'; A, \chi_k) = \sum_{\substack{(c,d)_1 \\ c \equiv d \equiv 0\,(\mathfrak{a})}}' \frac{N(\mathfrak{a})^k \chi_k(\mathfrak{a})}{(c\tau + d)^k (c'\tau' + d')^k},$$

which defines a (holomorphic) Hilbert modular form of dimension $-k$ (i.e., of weight k) if $k \geqslant 3$. Here $(c,d)_1$ means (c,d) runs over a complete system of representative of non-associated pairs ((c,d) and (c',d') are called associated if $(c,d) = \varepsilon(c',d')$ with some totally positive unit ε). And when $k = 2$ we need the method of regularization (Hecke's trick). This Eisenstein series has the Fourier expansion:

$$\begin{aligned} G_k(\tau, \tau'; A, \chi_k) &= A_k(A, \chi_k) \\ &\quad + B_k \sum_{\nu > 0} c_k(\nu, A, \chi_k) \exp\{2\pi\sqrt{-1}(\tau\nu - \tau'\nu')/\sqrt{d_K}\}, \end{aligned}$$

with

$$A_k(A, \chi_k) = N(\mathfrak{a})^k \chi_k(\mathfrak{a}) \sum_{(\mu)_1,\, \mu \equiv 0(\mathfrak{a})}' N(\mu)^{-k} = c\,\zeta(k, A, \chi_k),$$

$$B_k = \frac{(2\pi)^{2k}}{\Gamma(k)^2} \sqrt{d_K}^{\,1-2k}.$$

Here $c(= 2, 4)$ is the number of non-associated units. Since, at least in retrospect, it is not difficult to see that the coefficients $c_k(\nu, A, \chi_k)$ are rational numbers, it is natural to expect that the ratio $A_k(A, \chi_k)/B_k\chi_k(\mathfrak{a})$ is also a rational number. (And this is the case for quadratic field by using Dirichlet L-function.)

Anyway, Hecke did not seem to publish any proof of "Satz 3" in [He24]. However this paper contains two ideas: (1) to investigate the values of Dedekind zeta functions or their variants at positive integers, one should investigate the constant terms of appropriate Eisenstein series, and (2) write the Dedekind zeta functions and related L-functions (or their values at natural numbers) as the Fourier coefficients of the *hyperbolic* Fourier expansion of Eisenstein series along tori in $SL(2)$ or $GL(2)$.

Note that these are both Fourier expansions: one is parabolic, the other is hyperbolic. These appear repeatedly in the later papers by other mathematicians.

1.1.4. *A generalization of complex multiplication theory*

In retrospect, we find that there is a more important paper [He21], which inspired the subsequent investigations by Meyer and other people. This discusses the relative class number H/h of a totally imaginary quadratic extension K over a totally real number field k. But he gave the proof only for a real quadratic field k. The contents of [He21] belongs to a generalization of complex multiplication theory.

1.2. *Siegel*

Some authors quoted the Siegel's paper [Si22] as another source of the problem. But the theme of [Si22] was to consider Waring's problem for real quadratic fields, and his interest was concentrated into "singular series" in the sense of Hardy-Littlewood. One finds a comment in the last few pages (pp. 152–153). Here he wanted to describe the asymptotic behavior of the singular series, to show that the leading coefficient is a rational number. To have this he needed a finite sum expression of the values of the Dirichlet L-function associated with the quadratic character at the positive integers, in terms of the values of Bernoulli polynomials at rational numbers. This should be considered as a proto-type of generalized Bernoulli numbers later developed by Leopoldt [Le58].

This kind of results appeared sometimes in the later papers on quadratic forms by Siegel, which had been the main theme for him from mid-20's until mid-50's. As far as we can see from his publications, there seems to be no systematic discussion about this kind of problem up to the time of Tata Lecture Note [Si61].

1.3. *Herglotz*

The first response to the paper [He21] of Hecke came from Gustav Herglotz [Her23], whose main concerns were analysis and differential geometry but also wrote several interesting papers on number theory. He tried to rewrite Hecke's integral (Theorem 1.1) in an elementary forms. The Dedekind sums show up already in his paper as a summand in the sum expression of Hecke's integral. His method is to approach the two ends of Hecke's integral to the cusps of $SL(2,\mathbf{Z})$, which was later used by Siegel [Si61]. Other important summand is written in terms of Gauss function $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. Almost the same result together with a certain function $F(x)$ (by the same symbol as that of Herglotz by chance! or by mediocre?) was reproduced by Zagier as a part of his paper [Za75].

1.4. *Dedekind sums by Rademacher and others*

Richard Dedekind (1831-1916) began the study of *Dedekind sums* which appears in the transformation formula of the Dedekind eta function $\eta(\tau)$ with respect to $SL(2,\mathbf{Z})$. Hans Rademacher developed the investigation of Dedekind sums, as found in his Collected Papers [Ra74] (the papers 26-29, 31, 44, 53, 56, 57, 59, 67) and in the monograph [RG72] of Rademacher-Grosswald. The first motivation for him to investigate this seemed to be the study of the partition numbers $p(n)$ (or its generating function).

For our purpose, it suffices to review the part which is closely related to the transformation formula of $\log\eta$:

$$\begin{aligned}\log\eta(\tau) &= \frac{\pi\sqrt{-1}\tau}{12} + \sum_{m=1}^{\infty}\log(1-q^m)\\ &= \frac{\pi\sqrt{-1}\tau}{12} - \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}q^{mr} \quad (q = e^{2\pi\sqrt{-1}\tau}).\end{aligned}$$

The classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\mu=1}^{k-1}\left(\left(\frac{h\mu}{k}\right)\right)\cdot\left(\left(\frac{\mu}{k}\right)\right) = \sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\left(\frac{h\mu}{k}\right)\right)$$

with

$$((x)) := \begin{cases} x - [x] - 1/2, & \text{if } x \notin \mathbf{Z},\\ 0 & \text{if } x \in \mathbf{Z}.\end{cases}$$

For an element $M = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,\mathbf{Z})$, we define

$$\Phi(M) := \begin{cases} b/d & \text{for } c = 0,\\ (a+d)/c - 12(\operatorname{sgn} c)s(d,|c|) & \text{for } c \neq 0.\end{cases}$$

Then we can write the transformation formula of $\log\eta$ as

$$\log\eta\Big(\frac{a\tau+b}{c\tau+d}\Big) = \log\eta(\tau) + \frac{1}{2}(\operatorname{sgn} c)^2\log\Big(\frac{c\tau+d}{\sqrt{-1}\operatorname{sgn} c}\Big) + \frac{\pi\sqrt{-1}}{12}\Phi(M)$$

(cf. [RG72, Formula(60), p. 49]). An important result is the composition law, given as follows ([RG72, Formula(62), p. 51]):

Theorem 1.4. **([Ra31])** *If $M'' = M'M \in SL(2, \mathbf{Z})$, that is,*

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we have

$$\Phi(M'') = \Phi(M) + \Phi(M) - 3\,\mathrm{sign}(cc'c'').$$

This property of Φ, together with the reciprocity law of the Dedekind sum:

$$s(k,h) + s(h,k) = -\frac{1}{4} + \frac{1}{12}\Big(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\Big),$$

is considered essential in *elementary* computation of the Dedekind sum, and which is later generalized by Meyer [Me57b].

A start of this kind generalization is found in the paper of Apostol [Ap50]. Here the function $\log \eta$ is replaced by a Lambert series:

$$G_p(\tau) := \sum_{n=1}^{\infty} n^{-p} q^n/(1-q^n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{-p} q^{mn} \quad (p \geqslant 1).$$

When $p = 1$, this is $-\log$ of the generating function of the partition $p(n)$. Here a generalized Dedekind sum:

$$s_p(h,k) := \sum_{mu=1}^{k-1} \frac{mu}{k} \bar{B}_p(h\mu/k)$$

is introduced, where $\bar{B}_p(x)$ is the p-the Bernoulli function, i.e., $B_p(x - [x])$. The reciprocity law of $s_p(h,k)$ is given in [Ap50, Theorem 1] and the transformation formula of $G_p(\tau)$ is in [Ap50, Theorem 2]. There was a missing term in the transformation formula, and this was corrected by Iseki [Is57].

Remark. We try to pin-point the papers of Leonard Carlitz, which are related to the Dedekind sums. His papers are sometimes quoted more often than the papers of the initiative authors on the same particular themes. We are at a loss before the numbers of his papers, and are forced to leave the question for the readers themselves to find the adequate references among his publications.

2. From 1951 untill 1969

In 50's there was a development to have class number formula of abelian extension of real quadratic fields, mainly by Meyer. He evaluated the values at $s = 1$ of the Hecke L-series $L_K(s, \chi)$ for real quadratic fields K, starting from Hecke's integration formula. The final result is to have an expression of $L_K(1, \chi)$ as a product of power of π and a rational number which is effectively computable by finite steps by utilizing Dedekind sums. A similar result is discussed by Siegel in Chapter 2 of his Tata Lecture Note [Si61].

In 60's this method was extended to the problem of the evaluation of $L_K(s, \chi)$ at positive integer $s = m$ with an appropriate parity. This also was initiated by Meyer [Me66] for the case of $s = 2$, and later extended to general m by Lang [La68], Barner [Ba69] and Siegel [Si68].

2.1. *Class number formula by Meyer and Siegel*

2.1.1. *Meyer's monograph, [Me57a]*

The study of Curt Meyer on the class number formula is mainly found in [Me57a] and [Me57b]. The former one, the book [Me57a] is an improved version of his dissertation at Berlin in 1950, written under the guidance of Helmut Hasse. The second one [Me57b] is his "Habilitationsschrift" at Hamburg in 1955. These are very elaborated and important papers, but it is not so kind about the explanation of the organization of its contents.

The book [Me57a] consists three chapters, but the main body is chapter 2 (Section 4–Section 13) in the total 15 sections of the title: "Kroneckersche Grenzformeln für die L-Funktionen der Ringklassen und der Strahlklassen in quadratischen Zahlkörpern und ihre Anwendung auf die Summation für L-Reihen".

Though it is true that he proved a number of applications of the Kronecker limit formula to express the values of L-functions $L_K(s, \psi)$ or $L_K(s, \chi)$ at $s = 1$, associated with an imaginary or real quadratic field K and for a ringclass character ψ or a rayclass character χ, a bit confusing point is that all the titles of the sections from Section 4 to Section 13 have the same word "Kroneckersche Grenzformeln," and do not indicate the contents properly.

In the first two sections (Section 4, Section 5), he considered the case of imaginary quadratic fields. Here are variants of the Kronecker limit formula. The Laurent expansion at $s = 1$ modulo $O(s - 1)$ of the partial zeta functions associated with ring classes and ray classes are described by using the "singular values" of certain elliptic modular invariants introduced by

Hasse. This is a part of the classical theory of complex multiplication and is known from the time of Kronecker, more or less.

We remark that a most substantial result from slightly different view point is found in the paper [Ram64], which is the source of *elliptic units.*

From Section 6 to Section 13, the case of real quadratic fields is investigated. But as we see soon Section 6 and Section 7 have no essential results. To explain this it is better to use the table of the (infinite types) of the ray class characters in Siegel [Si61, Chapter II, Section 5, p. 115]:

Type	(i)	(ii)	(iii)	(iv)
Conductor at infinity	1	p_∞	$\mathfrak{p}_\infty$	$\mathfrak{p}'_\infty$
The associated sign character	1	$\frac{N(\lambda)}{\lvert N(\lambda)\rvert}$	$\frac{\lambda}{\lvert\lambda\rvert}$	$\frac{\lambda'}{\lvert\lambda'\rvert}$

The characters of type (i) is discussed in Section 6 and Section 7. But he had just introduced the notations of the integrations of elliptic modular forms and pointed out that they have nice formal properties as "arithmetic invariants" (pp. 48–50 in Section 6 and pp. 54–56 in Section 7). His original and substantial results are Section 9–Section 13, the last 5 sections in Chapter II of [Me57a]. This is the characters of type (ii) in the terminology of Siegel [Si61].

Meyer's finial end was to write the results in *elementary ways.* His investigation proceeded in 3 steps:

(1) Apply the integration formula of Hecke, to write the difference $\zeta_K(1, A) - \zeta_K(1, A^*)$ as a finite sum of the form $\log\{f(M(\tau))/f(\tau)\}$ with some modular form f and with some element $M \in SL(2, \mathbf{Z})$ representing units in $\mathcal{O}_K$. Here A and A^* defines the same ideal class in the wide sense, but distinct in the narrow sense (impizite Grenzformel, Section 9, Section 10).
(2) The term are written in terms of some (generalized) Dedekind sums (explizite Grenzeformel, Section 12, Section 13).
(3) These Dedekind sums are deeply investigated, among others their composition rules are studied [Me57b].

Siegel never discussed the last step (the (generalized) Dedekind sums) in his writings. He also could not get essential results for the characters of type (i), for which Meyer had no results.

To handle the ray classes, he needed the so-called "second Kronecker limit formula" formulated by using σ functions of Weierstrass or Fricke-Klein. The transformation formula of these σ functions was discussed in Section 11, and the multipliers of the transformation are generalized Dedekind sums.

Remark 1. We said nothing about Section 8 of [Me57a]. This is the cases of characters of types (iii) and (iv). We shall remark again in the subsection after the next.

Remark 2. Thus the essence of Meyer's results is to give the relative class number of the extension L/L_0 where L/K is an abelian extension and L_0 totally real and L its CM extension. Probably it is not unexpected, such problem is described by generalized Dedekind sums.

Remark 3. The paper of Zagier [Za75] is quite instructive to understand the essence of Meyer's work (cf. [Za75, Section 4]). But the part of the continued fraction (Section 5) is probably strongly related to the composition rule of the generalized Dedekind sums of [Me57b].

Remark 4. The book [Me57a] of Meyer is quite unreadable. Other people around us (e.g., Shintani, Arakawa) had the same opinion. There are a few reasons of this difficulty: say, it uses the symbols in the papers of Hasse; and as a whole the organization of book is good, because significant results are collected to the single chapter; but the most serious fact is that there is no statement written as Theorem (Satz), Proposition, and Lemma (Hilfassatz). This is also the case for other papers of him and his student Lang. The readers have to find the important statements by themselves. And those who want to quote them, they have the challenging jobs to point out the important statements by the formula numbers.

2.1.2. *Generalized Dedekind sums by Meyer [Me57b]*

To treat the L-functions associated with the characters of the ray class group, Meyer had to handle the transformation formula of elliptic modular forms of higher level. Then we have a new multiplier in the transformation formula different from Dedekind sums: this is *generalized Dedekind sums* and the reciprocity law and *the composition law* are the theme of Meyer's paper [Me57b].

2.1.3. *Class number formula by Siegel [Si61]*

This is the theme of Chapter II, Section 5 of the famous Tata Lecture Note [Si61] by him. In Chapter I of [Si61], he discussed not only the first Kronecker limit formula for Eisenstein series $f(\tau, s)$ of level 1, but also the second Kronecker limit formula for the Eisenstein series for principal congruence subgroups of level $f > 1$. Here in place of the Dedekind η function, there appears the f-division values of the (odd) theta function $\vartheta_1(\tau, u)$.

After recalling the classical results on Pell equation by Kronecker in Section 1 of Chapter II, Siegel's lecture proceeds in an almost parallel way to that of the book of Meyer [My57a]: application of Kronecker limit formula to imaginary quadratic fields (including the comments to the results of generalized Gauss sums by Hasse) in Section 2, an review of Hecke's integral expression in Section 3, real quadratic fields and ray class characters χ of type (i) in Section 4, and real quadratic fields and ray class characters χ of type (ii) in Section 5; and gives essentially the same results or *non-results.* But his presentation and the arguments of the proofs are very lucid.

(A) *The case of ray class characters χ of infinite type (i)*

The main *result* of this case is the following:

Theorem 2.1. **([Si61, Theorem 11])** *Let K be a real quadratic field with discriminant d_K and $\mathfrak{f}$ an integral ideal. Fix a number $\gamma \in K$ such that $(\gamma\sqrt{d_K})$ has exact denominator $\mathfrak{f}$, that is, $(\gamma) = \mathfrak{q}/\mathfrak{f}(\sqrt{d_K})$ for $(\mathfrak{q}, \mathfrak{f}) = 1$. For a ray class character χ with conductor $\mathfrak{f} \neq (1)$ and infinite type (i), the value $L_K(1, \chi)$ is given by*

$$L_K(1,\chi) = \frac{1}{2T}\sum_B \bar{\chi}(B)\int_{\tau_0}^{\tau_0^*} \log|\varphi(u_B, v_B, \tau)|^2 \frac{d\tau}{F_B(\tau)},$$

where B runs over all the ray classes modulo $\mathfrak{f}$ and for each B, we choose $\mathfrak{b}_B = \mathbf{Z}\beta_1 + \mathbf{Z}\beta_2 \in B$ and put $u_B = \mathrm{tr}(\beta_1\gamma)$, $v_B = \mathrm{tr}(\beta_2\gamma)$, and $w = \beta_2/\beta_1$. Here τ_0 is any fixed point in $\mathfrak{H}$ and $\tau_0^* = (a\tau_0 + b)(c\tau_0 + d)^{-1}$ is the modular transform of τ_0 determined by $\mathfrak{b}_B$: $\varepsilon\beta_2 = a\beta_2 + b\beta_1$, $\varepsilon\beta_1 = c\beta_2 + d\beta_1$. T is certain exponential sum and the functions $\varphi(u, v, \tau)$ and $F_B(\tau)$ $(\tau \in \mathfrak{H})$ are given as

$$\varphi(u, v, \tau) = \exp\{\pi\sqrt{-1}u(u\tau - v)\}\frac{\vartheta_1(v - u\tau, \tau)}{\eta(\tau)},$$

$$F_B(\tau) = \frac{\sqrt{d_K}}{\omega - \omega'}(\tau - \omega)(\tau - \omega') = a_1\tau^2 + b_1\tau + c_1$$

with a primitive form $a_1\tau^2 + b_1\tau + c_1$ satisfying $a_1 > 0$, $b_1^2 - 4a_1c_1 = d_K$.

The point here is that there still remains the integration of $\log|\varphi(v_B, u_B, \tau)|$ which is a transcendent and is not evaluated in an *elementary way.* So it gives no essential new results, because this is simply a paraphrase of Hecke's integral formula (the same as [Me57a, Section 6, Section 7]). The difference is that Siegel mentioned the method of Herglotz.

Remark. This direction seems to be still a dead-end until the present, as far as we know. The point $s = 1$ is not critical value in the sense of Deligne (Corvalis, 1979).

(B) *The case of ray class characters χ of infinite type (ii)*

In this case, Hecke's integral expression leads the following:

Theorem 2.2. **([Si61, Theorem 12])** *Let χ be a ray classes character of conductor $\mathfrak{f}$ with infinite type (ii). Under the same notation in Theorem 2.1 we have*

$$L(1,\chi) = \frac{\pi^2}{T\sqrt{d_K}} \sum_B \bar{\chi}(B) G(B),$$

where the summation is over all ray classes B modulo $\mathfrak{f}$ and

$$G(B) = \frac{v(\beta_1)}{2\pi i} \times \begin{cases} [\log \varphi(u_B, v_B, \tau)]_{\tau_0}^{\tau_0^*}, & \text{for } \mathfrak{f} \neq (1), \\ [\log(\sqrt{(\tau-\omega)(\tau-\omega')}\,\eta^2(\tau_0))]_{\tau_0}^{\tau_0^*}, & \text{for } \mathfrak{f} = (1). \end{cases}$$

Here $v(\beta_1) = N(\beta_1)/|N(\beta_1)|$.

The next step is to compute the value $[\log \varphi(u_B, v_B, \tau)]_{\tau_0}^{\tau_0^*}$ which is a rational number $\in \frac{1}{12f}\mathbf{Z}$ with $f = N(\mathfrak{f})$.

Theorem 2.3. *Under the same hypothesis and notation as above,*

$$G(B) = v(\beta_1)\Big\{\mathcal{P}_2(u_B)\frac{a+d}{c} - \sum_{k=0}^{c-1} \mathcal{P}_1\Big(\frac{k+u_B}{c}\Big)\mathcal{P}_1\Big(\frac{k+u_B}{c} - v_B\Big) - \nu(\mathfrak{f})\Big\}$$

where $u_B = v_B = 0$ for $\mathfrak{f} = (1)$. Here $\mathcal{P}_1(x)$ is the Bernoulli function periodic modulo $\mathbf{Z}$ such that $\mathcal{P}_1(x) = x - [x] - \frac{1}{2}$ if $0 \leqslant x < 1$, and the constant $\nu(\mathfrak{f})$ is equal to $\frac{1}{4}$ if $\mathfrak{f} = (1)$ or to 0 otherwise.

Siegel's proof basically uses the idea of Herglotz, to approach z_0 to ∞ and z_0^* to another cusp. We do not write here the exact formula, but his computation is used again in [Si68] in the computation of the special values, and here appears a "Lambert series" at least implicitly, which is an iterated indefinite integral of a holomorphic Eisenstein series.

2.1.4. *The case of the ray characters of infinite type (iii), (iv): An unsolved problem by Meyer and Siegel*

If one reads the arguments of Meyer and Siegel, one finds that there is one difficult case that is not completely solved either by Meyer or by Siegel. This is the cases of character type (iii) and (iv). In Meyer [Me57a], this case is handled in Section 8 (pp. 56–66), however the formula involves the terms which are expressed by modified Bessel functions. As far as we know no one gave any algebraic expression of $L_K(1,\chi)$ in this case, and probably it may not have such expression. There is a paper by Shintani [Shin77a] to write the special values $L_K(1,\chi)$ for this type of χ by using Barnes' double gamma function Γ_2.

2.2. *Leopoldt, 1958, 1962*

We cannot say much about the work of Leopoldt here. But in his survey article [Le62], he discussed *generalized Bernoulli numbers*, p-adic L-functions and their values at $s = 1$, and integral basis of the integer ring of the abelian extensions over $\mathbf{Q}$ in terms of Gaussian sums. These become the proto-type of subsequent generalization for other algebraic number fields. In the introduction of this paper, the author, quoting the paper of Hasse at 1952, mentioned the so-called "Hasse's program," which seems to mean the attempt and effort to have effectively computable way to have arithmetic invariants of algebraic number fields.

2.3. *Klingen's papers, 1962*

There are two papers [Kl62a] and [Kl62b] related to our theme. The main theorem of the first paper [Kl62a] is the following:

Theorem 2.4. *Let K be a totally real algebraic number field of degree n with discriminant d_K. For a natural number k let χ_k be a character of the ideal class group of K in the narrow sense such that its infinite type is given by $\chi_k((\alpha)) = N(\alpha)^k$ for principal ideals (α). Then for the partial Dedekind zeta function of any given ideal class A, we have*

$$\zeta_K(k, A, \chi_k) = \pi^{kn}\sqrt{d_K}\,\chi_k(\mathfrak{a})r \qquad (\mathfrak{a} \in A)$$

with a rational number r, if either of the following two conditions is satisfied:

(1) *k is an even natural number,*
(2) *k is an odd natural number > 1, and any unit of K has positive norm.*

He used Eisenstein series which are Hilbert modular forms over K defined as follows. Let $K = K^{(1)}, \ldots, K^{(n)}$ be conjugations of K and $\tau = (\tau^{(1)}, \ldots, \tau^{(n)})$ be n independent complex numbers in the upper half plane. As in Section 1.1.3, we define an Eisenstein series

$$G_k(\tau, A, \chi_k) = \sum_{\substack{(c,d)_1 \\ c \equiv d \equiv 0\,(\mathfrak{a})}}{}' \frac{N(\mathfrak{a})^k \chi_k(\mathfrak{a})}{N(c\tau + d)^k} \qquad (\mathfrak{a} \in A).$$

If $k > 2$, it converges uniformly in any compact subset of τ's, and defines a holomorphic Hilbert modular form of weight k. When $k = 2$ we can use the regularization procedure of Hecke [He24] to start with

$$G_k(\tau, A, \chi_k, s) = \sum_{\substack{(c,d)_1 \\ c \equiv d \equiv 0(\mathfrak{a})}}{}' \frac{N(\mathfrak{a})^{k+2s} \chi_k(\mathfrak{a})}{N(c\tau + d)^k |N(c\tau + d)|^{2s}} \qquad (\mathfrak{a} \in A)$$

for $Re(s) > 1 - k/2$. Then the constant term of this is given by $e\,\zeta_K(k, A, \chi_k)$ with the index e of the subgroup of totally positive units in K in the whole unit group, and other Fourier coefficients are the common constant

$$\frac{(-2\pi\sqrt{-1})^{kn} N(\mathfrak{a})^{k-1} \chi_k(\mathfrak{a})}{\sqrt{d_K}((k-1)!)^n}$$

times natural numbers which are generalized sum of divisors. If we normalize $G_k(\tau, A, \chi_k, s)$ by the last constant, the new series has rational Fourier coefficients except for the constant coefficient. Klingen showed this remaining coefficient is also rational by elimination method, since the graded ring of Hilbert modular forms is finitely generated by Hans Maass's result.

Later Siegel [Si69] gave a modified proof, reducing the problem to the structure of the graded ring of elliptic modular forms over $\mathbf{Z}$, by pulling-back Hilbert modular forms to elliptic modular forms by utilizing the diagonal modular embedding.

Remark. Klingen suggested yet another argument to use the volume formula of the fundamental domain of Siegel modular groups and the Gauss-Bonnet formula for V-manifolds in the sense of Satake, to settle the case of Dedekind zeta functions. This method gives a shaper result to control the denominator of the rational factor r.

2.4. *On the values of L-functions by Meyer, Lang, Barner, and Siegel*

The special values of ring class L-functions for real quadratic fields of certain infinite types were discussed by Meyer [Me66], Heinrich Lang [La68], and

Kurt Barner [Ba69]. The strategy of them are almost the same as that for the class number formula, and the results are described by the similar invariants.

Siegel [Si68] gave the explicit formula for the ray class L-function for the same infinite type by different method. Katayama [Ka76] also treated the same problem by the method related to Barner [Ba69].

2.4.1. *Meyer, Lang, and Barner*

Meyer [Me66] discussed the values $L_K(2, \chi)$ for χ of type (i). His student Lang [La68] considered the case of absolute class group in the narrow sense, and Barner [Ba69] the case of ring class group and they expressed the values $L_K(2k, \chi)$ for χ of type (i) and $L_K(2k+1, \chi)$ for χ of type (ii) in terms of generalized Dedekind sums.

The method is in common with these three papers. They start from Hecke's integral expression and the explicit calculation of the iterated primitives (i.e., the iterated indefinite integrals) plays the key role technically. These are given the name *Lambert series.* Among others the paper [Ba69] is written in the natural order to make the pass-way of the logic very clear, so that it is most readable, in spite that it is quite computational. The latter half of [La68] is devoted to give many examples of computation of $L_K(2, \chi)$.

Remark. As far as we can see, there is no discussion about an *elementary way* to compute their new kinds of generalized Dedekind sums.

2.4.2. *Siegel*

To state the main result of Siegel [Si68], we recall the definition of partial zeta functions.

Let $\mathfrak{b}$ and $\mathfrak{f}$ be two relatively prime integral ideals in $\mathcal{O}_K$. The *partial zeta function* to the ray class $\mathfrak{b}$ modulo $\mathfrak{f}$ is defined by

$$\zeta_K(s, \mathfrak{b}, \mathfrak{f}) := \sum_{\mathfrak{a} \equiv \mathfrak{b} \bmod \mathfrak{f}} N(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ runs over all integral ideals in $\mathcal{O}_K$ which are in the same narrow ray class as $\mathfrak{b}$ modulo $\mathfrak{f}$.

Let K be a real quadratic field. Put $\mathfrak{a} := \mathfrak{b}(\mathfrak{d}\mathfrak{f})^{-1}$, where $\mathfrak{d}$ is the different of K and choose an integral basis w_1, w_2 of $\mathfrak{a}$ such that $-w_2/w_1 := w > w'$. Let $\varepsilon_{\mathfrak{f}}(> 1)$ be the generator of $E_{\mathfrak{f}}^{+}$, the set of totally positive units in K congruent to 1 modulo $\mathfrak{f}$. Since $\varepsilon_{\mathfrak{f}}\mathfrak{a} \subset \mathfrak{a}$, there exists $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$

such that $\varepsilon_{\mathfrak{f}} w_1 = dw_1 - cw_2$ and $\varepsilon_{\mathfrak{f}} w_2 = -bw_1 + aw_2$.

Then the functional equation for the partial zeta function implies that, for $k = 1, 2, \ldots$,

$$\begin{aligned}\zeta_K(1-k, \mathfrak{b}, \mathfrak{f}) &= L_k \sum_{\substack{\mu \in \mathfrak{a} \\ \mu \bmod E_{\mathfrak{f}}^+}} \frac{\exp 2\pi\sqrt{-1}\mathrm{tr}(\mu)}{N(\mu)^k} \\ &= L_k \sum_{(m,n)\in \mathbf{Z}^2}{}' \frac{\exp 2\pi\sqrt{-1}\{m\,\mathrm{tr}(w_1) + n\,\mathrm{tr}(w_2)\}}{(m-nw)^k (m-nw')^k}\end{aligned}$$

with some constant L_k depending on k.

As in the proof of Hecke's integral expression, Siegel expressed it as an *Eichler integral* (he did not use the word) in [Si68, Hilfssatz 1],

$$\zeta_K(1-k, \mathfrak{b}, \mathfrak{f}) = L_k' \int_{z_0}^{z_0^*} E_k(z; \mathrm{tr}(w_1), \mathrm{tr}(w_2))\, Q(z)^{k-1}\, dz,$$

where z_0 is any point in the upper half plane, $z_0^* = \sigma(z_0)$,

$$E_k(z; u, v) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\exp 2\pi\sqrt{-1}(mu + nv)}{(nz - m)^{2k}}$$

is the Eisenstein series of weight $2k$ for the congruence subgroup $\Gamma(N(\mathfrak{f}))$, and $Q(z) = (w_1 z + w_2)(w_1' z + w_2')$. By evaluating the above integral, Siegel obtained the following:

Theorem 2.5. **([Si61])** *Under the above notation, for* $k = 1, 2, \ldots$,

$$\begin{aligned}\zeta_K(1-k, \mathfrak{b}, \mathfrak{f}) &= (-1)^k \left(\frac{N(\mathfrak{f})}{N(\mathfrak{a})}\right)^{k-1} \mathrm{sgn}(w_1 w_1') \\ &\quad \times \sum_{i=0}^{2k-1} (-1)^i \frac{c^{2k-i-1}}{i!(2k-i)} R_k^{(i)}\left(\frac{a}{c}\right) \\ &\quad \times \sum_{l \,(\mathrm{mod}\, c)} P_i\left(a\frac{u+l}{c} + v\right) P_{2k-i}\left(\frac{u+l}{c}\right),\end{aligned}$$

where

$$R_k(z) = \int_{-d/c}^{z} Q(z)^{k-1} dz$$

and $u = \mathrm{tr}(w_1)$, $v = \mathrm{tr}(w_2)$. *When* $k = 1$ *and* $\mathfrak{f} = (1)$, *the correction term* $-1/4$ *should be added.*

Since $Q(z)$ is a quadratic polynomial in z, by the integration by part, the *Eichler integral* is written in terms of $2k-1$ times iterated indefinite integral (i.e., the Lambert series of Meyer, Lang, and Barner). It becomes a Dirichlet series $c_0 + \sum_{n=1}^{\infty} c_n n^{1-2k}$ with coefficients c_n, each of which is a finite sum

$$\sum_{m=1}^{2k-1} (-1)^m [R_k^{(2k-m)}(z)\, q_n^{(m-1)}(\tau)]_{z_0}^{z_0^*},$$

with

$$q_n(z) = \frac{e^{2\pi\sqrt{-1}n(v+(u-1/2)z)} + e^{-2\pi\sqrt{-1}n(v+(u-1/2)z)}}{e^{-\pi\sqrt{-1}nz} - e^{\pi\sqrt{-1}nz}}.$$

Then the similar proof of the formula of $L_K(1,\chi)$ is applied to get Theorem 2.5.

3. From 70's to 80's

3.1. *Shintani*

After the work of Meyer and Siegel, the next breakthrough was brought by Takuro Shintani [Shin76], [Shin77a], [Shin77b]. In his famous paper [Shin76], Shintani suggested an ingenious idea to evaluate the special values of partial zeta functions at nonpositive integers for arbitrary totally real number fields.

Further, in the next papers [Shin77a], [Shin77b], ([Shin76]) he applied the method developed in [Shin76] to study $L_K(1,\chi)$ in the case where χ splits at most one real prime, and expressed it in terms of Barnes' multiple gamma functions (multiple sine functions). Especially in [Shin77a], *unsolved* cases (Section 2.1.4) of $L_K(1,\chi)$ for real quadratic fields K are settled.

3.1.1. *Explicit formula of $\zeta_K(1-m,\mathfrak{b},\mathfrak{f})$ by Shintani*

The evaluation of $\zeta_K(1-m,\mathfrak{b},\mathfrak{f})$ consists of two steps. In the first Shintani introduced a new zeta function $\zeta(s,A,x)$ which can be regarded as a generalization of Hurwitz zeta function. He proved the analytic continuation and expressed its special values in terms of certain generating function which is a generalization of generating function of Bernoulli numbers.

In the second step, which is the core of his idea, along the cone decomposition of $\mathbf{R}_+^n$ (n is the degree of K) he wrote the partial zeta function $\zeta_K(s,\mathfrak{b},\mathfrak{f})$ as a finite sum of his (sector) zeta function $\zeta(s,A,x)$, and thus arrived at the explicit formula.

We review these arguments more precisely. Let $A = (a_{ij})$ be an $r \times m$ matrix ($r \leq m$) with positive entries. For $x = (x_1, \ldots, x_r) \in \mathbf{C}^r$ and $s \in \mathbf{C}$, Shintani defined a zeta function:

$$\zeta(s, A, x) := \sum_{n_1, \ldots, n_r = 0}^{\infty} \prod_{j=1}^{m} \Big\{ \sum_{i=1}^{r} a_{ij}(n_i + x_i) \Big\}^{-s}.$$

When $r = m = 1$ and $A = 1$, it coincides with the Hurwitz zeta function $\sum_{n=0}^{\infty}(n+x)^{-s}$, equivalently a partial zeta function of $\mathbf{Q}$. As in the classical argument for Hurwitz zeta functions, he proved the following:

Proposition 3.1. (**[Shin76, Proposition 1]**) *The zeta function* $\zeta(s, A, x)$ *converges absolutely for* $\mathrm{Re}(s) > r/m$ *and is continued to a meromorphic function on the whole s-plane. Moreover the special value at* $s = 1 - k$ ($k = 1, 2, \ldots$) *is evaluated as*

$$\zeta(1 - k, A, x) = (-1)^{m(k-1)} k^{-m} \sum_{l=1}^{m} \frac{B_k(A, 1 - x)^{(l)}}{m},$$

where $B_k(A, y)^{(l)}/(k!)^m$ *is the coefficient of* $u^{(k-1)m}(t_1 \cdots t_{l-1} t_{l+1} \cdots t_m)^{k-1}$ *in the Laurent expansion at the origin of the generating function*

$$\prod_{i=1}^{r} \frac{\exp(u y_i \sum_{j=1}^{m} a_{ij} t_j)}{\exp(u \sum_{j=1}^{m} a_{ij} t_j) - 1} \bigg|_{t_l = 1}.$$

Let us relate $\zeta_K(s, \mathfrak{b}, \mathfrak{f})$ with $\zeta(s, A, x)$. We immediately have

$$\zeta_K(s, \mathfrak{b}, \mathfrak{f}) = N(\mathfrak{b})^{-s} \sum_{\mu} N(\mu)^{-s},$$

where the summation is over all totally positive numbers in K which satisfy $\mu - 1 \in \mathfrak{b}^{-1}\mathfrak{f}$ and are not associated with each other under the action of the group $E_{\mathfrak{f}}^{+}$, totally positive units in K congruent to 1 modulo $\mathfrak{f}$. Keeping this in mind, we decompose the set of totally positive elements V_+ of K which can be seen a subset of $\mathbf{R}_{+}^{n}$, under the action of $E_{\mathfrak{f}}^{+}$. Then

$$V_+ = \bigsqcup_{u \in E_{\mathfrak{f}}^{+}} \bigsqcup_{j \in J} u\, C_j$$

where $C_j = C(v_{j_1}, \ldots, v_{j_{r(j)}}) = \{\sum_{k=1}^{r(j)} t_j v_{j_k} \mid t_j > 0\}$ is an open simplicial cone with generators $v_{j_1}, \ldots, v_{j_{r(j)}} \in \mathfrak{f}$ and $\sharp J < \infty$. According as the decomposition above, Shintani obtained the expression

$$\zeta_K(s, \mathfrak{b}, \mathfrak{f}) = N(\mathfrak{b})^{-s} \sum_{j \in J} \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f} + 1)} \zeta(s, A_j, x).$$

Here $A_j \in M(r(j), n)$ whose (l, m)-th entry is the m-th conjugate $v_{j_l}^{(m)}$ and for each subset S of F, $R(j, S)$ is the set of $x = (x_1, \ldots, x_{r(j)}) \in \mathbf{Q}^{r(j)}$ satisfying the conditions (i) $0 < x_k \le 1$ and (ii) $\sum_{k=1}^{r(j)} x_k v_{j_k} \in S$.

Combined with Proposition 3.1, he reached the following:

Theorem 3.1. **([Shin76, Theorem 1])** *Under the above notation,*

$$\zeta_K(1-m, \mathfrak{b}, \mathfrak{f}) = m^{-n} N(\mathfrak{b})^{m-1} \sum_{j\in J} (-1)^{r(j)} \sum_{x\in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} B_m(A_j, x).$$

As a corollary to Theorem 3.1, Shintani reproved the rationality of $\zeta_K(1-m, \mathfrak{b}, \mathfrak{f})$ and also applied it to real quadratic fields, to derive the formula of Siegel (Theorem 2.5).

3.1.2. *Kronecker limit formula*

Now we explain Shintani's contributions to Kronecker limit formula. Let χ be a character of the narrow ideal class group $H_K(\mathfrak{f})$ modulo $\mathfrak{f}$ of K. For $x \in \mathcal{O}_K$ satisfying $x \equiv 1 \pmod{\mathfrak{f}}$, we have

$$\chi((x)) = \prod_{i=1}^{r_1} \operatorname{sgn}(x^{(i)})^{a_i}.$$

Here $x^{(i)}$ $(1 \le i \le n)$ is the i-th conjugate of x. Denote by $a(\chi) = \sum a_i$ and $b(\chi) = n - a(\chi)$. Hecke's functional equation for $L_K(s, \chi)$ is

$$\xi_K(s, \chi) = W(\chi)\xi_K(1-s, \chi^{-1}),$$

where

$$\xi_K(s, \chi) = A^s \Gamma(s/2)^{b(\chi)} \Gamma((s+1)/2)^{a(\chi)} L_K(s, \chi)$$

with $A = \sqrt{|d_K| N(\mathfrak{f})/\pi^n}$ and $W(\chi) \in \mathbf{C}^{(1)}$. It implies that the study of $L_K(1, \chi)$ is reduced to the $b(\chi)$-th derivative of $L_K(s, \chi)$ at $s = 0$, hence, that of $\zeta(s, A, x)$ at $s = 0$.

To describe Shintani's result, we recall Barnes' multiple gamma functions [Bar04]. For $\omega = (\omega_1, \ldots, \omega_r) \in \mathbf{R}_+^r$ and $x > 0$, define the *multiple zeta function* $\zeta_r(s, \omega, x)$ by

$$\zeta_r(s, \omega, x) = \sum_{m_1, \ldots, m_r = 0}^{\infty} (x + m_1\omega_1 + \cdots + m_r\omega_r)^{-s}.$$

It is known that $\zeta_r(s, \omega, x)$ converges absolutely for $\mathrm{Re}(s) > r$ and is continued to a holomorphic function on whole s-plane except for the simple poles at $s = 1, 2, \ldots, r$. Define the function $\rho_r(\omega)$ and the *multiple gamma function* $\Gamma_r(x, \omega)$ by

$$-\log \rho_r(\omega) = \lim_{x \to +0} \left\{ \frac{d}{ds} \zeta_r(s, \omega, x) \Big|_{s=0} + \log x \right\},$$
$$\frac{d}{ds} \zeta_r(s, \omega, x) \Big|_{s=0} = \log \left(\frac{\Gamma_r(x, \omega)}{\rho_r(\omega)} \right).$$

When $r = 1$, it is essentially the usual gamma function: $\Gamma_1(x, w)/\rho_1(\omega) = (\sqrt{2\pi})^{-1} \Gamma(x/\omega) \exp\{(x/\omega - 1/2) \log \omega)\}$. Shintani represented the derivative $\zeta'(0, A, x)$ at $s = 0$ in terms of $\Gamma_r(x, \omega)$. Therefore combined with the decomposition of $L_K(s, \chi)$ along the cone decomposition, he obtained the following:

Theorem 3.2. **([Shin77b, Theorem 1])** *Let χ be a primitive character of $\mathfrak{f}$ which splits only one of n real primes (i.e., $b(\chi) = 1$). Then*

$$\begin{aligned} L_K(1, \chi) = \frac{2W(\chi)\pi^{n-1}}{\sqrt{|d_K| N(\mathfrak{f})}} & \sum_{c \in H_K(\mathfrak{f})} \sum_{j \in J} \sum_{x \in R(C_j, c)} \chi^{-1}(c) \\ & \times \left\{ \log \left(\prod_{m=1}^{n} \frac{\Gamma_{r(j)}(x_m, A_j^{(m)})}{\rho(A_j^{(m)})} \right) \right. \\ & \left. \quad + \frac{1}{n}(-1)^{r(j)} \sum_{l=(l_1, \ldots, l_r)} C_l(A_j) \left(\prod_{k=1}^{r(j)} \frac{B_{l_k}(x_k)}{l_k!} \right) \right\}, \end{aligned}$$

where $A_j^{(m)}$ denote the m-th row vector of A_j, $l_1, \ldots, l_r$ run through the nonnegative integers satisfying $l_1 + \cdots + l_r = r$ and

$$C_l(A) = \sum_{1 \le j,\, k \le n,\, j \ne k} \int_0^1 \{ \prod_{i=1}^{r} (a_{ij} + a_{ik} u)^{l_j - 1} - \prod_{i=1}^{r} a_{ij}^{l_i - 1} \} \frac{du}{u}$$

for $A = (a_{ij})$.

Shintani [Shin76, Theorem 3] also find an explicit formula in the case $b(\chi) = 0$, and which can be thought as an answer to Hecke's conjecture on relative class number of totally complex quadratic extension of totally real number field ([He21]). Further, the case of $n = 2$ with $b(\chi) = 1$ was deeply investigated in [Shin77a] and [Shin78] with numerical examples. He considered double gamma functions (double sine functions) play an important role in the construction of class field and proposed an conjecture related to Stark's conjecture.

3.2. *Revision in terms of Eichler integrals*

Larry J. Goldstein [Go80] re-proved the formula of Siegel for partial zeta function (Theorem 2.5) in the framework of Eichler integral which was studied by Eichler [Ei57] and Shimura [Shim59]. The technique used in Goldstein's paper are familiar ones, however, his argument is clear and conceptual.

Let us recall some fundamental facts about Eichler integrals [Ei57], [Shim59], [Go80]. Let h be an automorphic form on $\mathfrak{H}$ of weight $n+2$ for a Fuchsian group Γ (assumed to have the cusp $i\infty$). The *Eichler integral* $H(z)$ of h is an $(n+1)$-fold iterated integral of h. There are many choices for such integrals, in particular, we may choose

$$H_0(z) = \frac{1}{(n+1)!}a_0 z^{n+1} + \left(\frac{\lambda}{2\pi\sqrt{-1}}\right)^{n+1} \sum_{m=1}^{\infty} \frac{a_m}{m^{n+1}} e^{2\pi\sqrt{-1}mz/\lambda},$$

where $h(\tau) = \sum_{m=0}^{\infty} a_m e^{2\pi\sqrt{-1}m\tau/\lambda}$ is the Fourier expansion at the cusp $i\infty$.

For any Eichler integral $H(z)$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, set

$$S_\sigma(z) := H(\sigma(z))j(\sigma,z)^n - H(z),$$

with $j(\sigma,z) = (cz+d)$. It is known that $S_\sigma(z) \in \mathbf{C}_n[z]$ (=polynomials in z of degree $\le n$), which is called the period polynomial. Moreover, if we denote by $S_\sigma(z) = (z^n, z^{n-1}, \dots, 1)S(\sigma)$ with $S(\sigma) \in M(n+1,1,\mathbf{C})$, the cocycle relation holds:

$$S(\sigma\tau) = {}^tM_n(\tau)S(\sigma) + S(\tau) \quad (\sigma,\tau \in \Gamma).$$

Here $M_n(\sigma) \in SL(n+1,\mathbf{R})$ is the matrix of the n-th symmetric tensor representation of $SL(2,\mathbf{R})$. This relation is a consequence of the automorphy of h.

The target in the paper [Go80] is (transformation formula of) "generalized Eichler integral"

$$G(z,p) = \int_{z_0}^{z} h(\tau)p(\tau)d\tau \quad (p(z) \in \mathbf{C}_n[z]).$$

Proposition 3.2. ([Go80]) *If $p(\sigma(z))j(\sigma,z)^n = p(z)$, the quantity ${}^tPP_n^{-1}S(\sigma)$ does not depend on the choice of Eichler integral and*

$$G(\sigma(z),p) - G(z,p) = n!\,{}^tPP_n^{-1}S(\sigma).$$

Here $p(z) = (z^n, z^{n-1}, \ldots, 1)P$ *and* $P_n = \begin{pmatrix} & & p_1 \\ & \iddots & \\ p_{n+1} & & \end{pmatrix}$ *with* $p_j = (-1)^{j-1}\binom{n}{j-1}$.

Therefore the task is reduced to compute the period polynomial for some Eichler integral of the Eisenstein series $E_k(z; u, v)$ (see Section 2.4.2), that is,

$$H_0(\sigma(z))j(\sigma, z)^{2k-2} - H_0(z),$$

which is essentially the same as the transformation of Lambert series in the work of Meyer, Lang and Barner.

Goldstein expressed the constant term of $H_0(z)$ by inverse Mellin transformation of finite sum of certain zeta functions (the product of partial L-function of $\mathbf{Q}$). By shifting the line of integration and applying the functional equation of the above zeta function, the period polynomial is computed by the residue calculation. We remark that this "Mellin transform technique" was also used in the proof of the transformation formula for $\log(\eta(z))$ by Apostol [Ap50] and Goldstein and de la Torre [GT74].

Remark 1. Here we remark that constructions of p-adic L-function over number fields. Coates and Sinott [CS74] considered this problem based on Siegel's formula for real quadratic fields. For totally real number fields, it was established by Deligne and Ribet [DR80] (see also [Ri79]) by using Hilbert modular forms, and by Barsky [Bars77] and Cassou-Noguès [CN79] by utilizing Shintani's results.

Remark 2. Halbritter gave an explicit formula for $\zeta_K(k, A)$ with $k \geq 2$ for real quadratic fields [Ha85a], and totally real cubic fields [Ha85b], [Ha88]. His method is elementary and related to Siegel's approach [Si75] and uses some reciprocity laws of generalized Dedekind sums (cf. [Ha85c]).

Remark 3. Here we can not afford to describe the work of Hirzebruch and Zagier [HZ74] on a relation between Dedekind sums and geometry.

4. After 1990, cocycles on $GL(n, \mathbf{Q})$

Around 1990, new approach was introduced by Glenn Stevens [St89] and Robert Sczech [Sc92], [Sc93]. Both of them and later David Solomon [So98], [So99] constructed certain *universal* 1-cocycles on the group $GL(2, \mathbf{Q})$ by different methods and showed that the special values $\zeta_K(s,\mathfrak{b},\mathfrak{f})$ at integers admit a cohomological interpretation. Roughly speaking, these author's work consists of the following contents:

(1) construct a cocycle,
(2) describe the link between the cocycle and the special value of partial zeta function,
(3) express the cocycle in terms of generalized Dedekind sums.

When (1) and (3) are established, the cocycle property implies the reciprocity law of the resulting generalized Dedekind sum, for example, the classical reciprocity law or its generalization by Rademacher and others (cf. [RG72]), and more generalized ones are re-proved in these papers.

Moreover, with the aid of the cocycle relation, one can design an efficient algorithm for the computation of special values. Some of numerical examples are given in [CGS00] and [GS03].

4.1. *Stevens*

Stevens [St89] proposed to investigate the Eichler integrals similar to Goldstein from the viewpoint of modular symbols in the sense of Manin [Ma72], Mazur-Swinnwerton-Dyer [MSD74]. Among others Stevens defined certain cocycle on $GL_2(\mathbf{Q})$.

4.2. *Eisenstein cocycle by Sczech*

Sczech's starting point was to focus the period

$$\int_{\tau}^{A\tau} \sum_{m,n}{}' (mz+n)^{-2}dz, \qquad (A \in SL(2,\mathbf{Z}),\ \tau \in \mathfrak{H}) \tag{$\sharp$}$$

of an Eisenstein series of weight 2. The integrand of ($\sharp$) does not converge absolutely, and the handling of this kind sums is attributed to Eisenstein by Weil [We76]. We are ignorant who justified firmly the argument to evaluate ($\sharp$) as $\log\eta(\tau) - \log\eta(A\tau)$, which is heuristically natural, that can be expressed by (classical) Dedekind sum. On the other hand, according to "Hecke's trick," ($\sharp$) is related to $\zeta_K(1,\mathfrak{b},\mathfrak{f})$.

To avoid some difficulties in treating the period ($\sharp$), Sczech considered the series

$$\sum_{m,n}{}' \frac{A\tau - \tau}{(mA\tau+n)(m\tau+n)} \tag{$\natural$}$$

which arises from (♯) by termwise integration.

This series is more manageable than the integral (♮), and especially it can be discussed with a real analytic method, though still it converges conditionally. Then we should specify a limit process, which Sczech called "Q-limit." Here Q is some fixed nondegenerate binary form. In the case of (♮), it means that the summation is taken so that $\lim_{t\to\infty}\sum_{|Q(m,n)|<t}$.

The *cocycle* (♮) itself was already well-known (see Schoenberg's book [Sch74]). But Sczech emphasized the advantage of its extension to $GL(n)$ $(n > 2)$ and in his paper [Sc93], which was ingeniously achieved while cocycles of Stevens and Solomon are not at hand now.

Following [Sc93] (see also [GS03]), we briefly recall the definition of Eisenstein cocycle for $GL(n,\mathbf{Q})$. Let $\mathcal{A} = (A_1,\ldots,A_n)$ be an n-tuple of matrices $A_i \in GL(n,\mathbf{R})$ and A_{ij} the j-th column of A_i. Then for nonzero $x \in \mathbf{R}^n$ and each A_i, there exists at least one A_{ij} such that $\langle x, A_{ij}\rangle \neq 0$ and denoted by A_{i,j_i} the first column in A_i with this property. Define

$$\psi(\mathcal{A})(x) := \frac{\det(A_{1,j_1},\ldots,A_{n,j_n})}{\langle x, A_{1,j_1}\rangle\cdots\langle x, A_{n,j_n}\rangle}.$$

To evaluate the special values $\zeta_K(s,\mathfrak{b},\mathfrak{f})$ at non-positive integers, we consider an action of a differential operator $P(-\partial/\partial x_1,\ldots,-\partial/\partial x_n)$ with a homogeneous polynomial $P(X_1,\ldots,X_n)$ on $\psi(\mathcal{A})(x)$:

$$\psi(\mathcal{A})(P,x) := P\Big(-\frac{\partial}{\partial x_1},\ldots,-\frac{\partial}{\partial x_n}\Big)\psi(\mathcal{A})(x).$$

Now we can define the *Eisenstein cocycle* Ψ by averaging them over the lattice $\mathbf{Z}^n$:

$$\Psi(\mathcal{A})(P,Q,v) := (2\pi\sqrt{-1})^{-n-\deg P}\sum_{x\in\mathbf{Z}^n}\exp(2\pi\sqrt{-1}\langle x,v\rangle)\psi(\mathcal{A})(P,x)\Big|_Q.$$

Here $v \in \mathbf{R}^n$ and $|_Q$ means the Q-limit.

Since we are now interested in the special values at non-positive integers, the exponential function is multiplied. When we treat the values at positive integers, it does not appear and the sum is taken over $\mathbf{Z}^n + u$, and the resulting cocycle is called *trigonometric cocycle*. This cocycle was introduced in [MS79] and discussed in [Sc92] for $n = 2$. The results of [Sc92] and [Sc93] are summarized as follows:

Theorem 4.1. **([Sc92], [Sc93])** (i) *Ψ represents a nontrivial cohomology class in $H^{n-1}(GL(n,\mathbf{Z}),M)$ where M is the set of all $\mathbf{C}$-valued functions $f(P,Q,v)$ and on which $GL(n,\mathbf{Z})$ acts by $Xf(P,Q,v) = \det(X)f({}^tXP,X^{-1}Q,X^{-1}v)$ for $X \in GL(n,\mathbf{Z})$.*

(ii) *For suitable choices of* $(\mathcal{A}, P, Q, v)$, *the Eisenstein cocycle* Ψ *represents the special values of partial zeta function at non-positive integers. More precisely, we denote* $\mathfrak{f}\mathfrak{b}^{-1} = \sum \mathbf{Z} w_j$ *with the dual basis* $\{w_j^*\}$ *determined by* $\mathrm{tr}(w_i^* w_j) = 1$ *and define* $P(X) = N(\mathfrak{b}) \cdot N(\sum_j X_j w_j)$, $Q(X) = N(\sum_j X_j w_j^*)$ *and* $v = (v_j) \in \mathbf{Q}^n$ *with* $v_j = \mathrm{tr}(w_j^*)$. *Let* $\rho : E_{\mathfrak{f}}^+ \to SL(n, \mathbf{Z})$ *be the map defined by* $\rho(\varepsilon) = W^* \mathrm{diag}(\varepsilon^{(1)}, \ldots, \varepsilon^{(n)})(W^*)^{-1}$ *with* $W^* = ((w_i^*)^{(j)})$ *and set* $A_i = \rho(\varepsilon_i)$ *with* $E_{\mathfrak{f}}^+ = \langle \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle$. *Then*

$$\zeta_K(1-s, \mathfrak{b}, \mathfrak{f}) = \eta \sum_{\varepsilon \in E^+/E_{\mathfrak{f}}^+} \sum_{\pi \in \mathfrak{S}_{n-1}} \mathrm{sgn}(\pi) \times \Psi((1, A_1, A_1 A_2, \ldots, A_1 \cdots A_{\pi(n-1)}))(P^{s-1}, Q, \rho(\varepsilon) v).$$

Here $\eta \in \{\pm 1\}$ *is determined by* $\eta = (-1)^{n-1} \mathrm{sgn}(\det W) \mathrm{sgn}(\det(\log \varepsilon_i^{(j)}))$.
(iii) Ψ *can be expressed by finite sum of generalized Dedekind sums, which can be explicitly given, if the datum* $(\mathcal{A}, P, Q, v)$ *is specified.*

Roughly speaking (i) and (ii) follows from the way of construction of Ψ. In the proof of (iii), the most important and difficult part is to control the Q-limit which is necessary (only) for the special value at $s = 0$. The "Q-limit formula" is studied in Sczech's dissertation [Sc82] for $n = 2$ and the latter part of [Sc93] for general n. For example,

$$\lim_{t \to \infty} \sum_{\substack{(p_1, p_2) \in \mathbf{Z}^2 \\ |Q(p_1, p_2)| < t}} \frac{\exp 2\pi\sqrt{-1}(p_1 v_1 + p_2 v_2)}{p_1^k p_2^l} = \frac{(2\pi\sqrt{-1})^{k+l}}{k!\, l!} \mathcal{P}_k(v_1) \mathcal{P}_l(v_2) + S(Q)$$

where

$$\mathcal{P}_k(y) = -\frac{k!}{(2\pi\sqrt{-1})^k} \sum_m{}' \frac{e^{2\pi\sqrt{-1} m y}}{m^k}$$

is the periodical Bernoulli function and the error term

$$S(Q) = \begin{cases} \pi^2 - \frac{2\pi}{m} \sum_{i=1}^m |\arg \frac{\alpha_i}{\beta_i}| & \text{if } (v_1, v_2) \in \mathbf{Z}^2 \text{ and } k = l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

with $Q(p_1, p_2) = \prod_{i=1}^m (\alpha_i p_1 - \beta_i p_2)$.

The explicit formulas given in [Sc93, Theorem 6.7] are certainly effective, though, are rather complicated to perform practical computations for given datum. He emphasizes that the reason is the cocycle property is not used to find them.

Let us observe how the cocycle relation reduce the task in the case of $\zeta_K(0,\mathfrak{b},\mathfrak{f})$ for a real quadratic field K. For the purpose, we should compute

$$\begin{aligned}&\Psi\left(1,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)(1,Q,v)\\&=\frac{1}{2c}\mathcal{P}_2(v_2)+\frac{d}{2c}\mathcal{P}_2(cv_1-av_2)-\sum_{j(\mathrm{mod}\ c)}\mathcal{P}_1\Big(\frac{j+v_2}{|c|}\Big)\mathcal{P}_1\Big(a\frac{j+v_2}{c}-v_1\Big).\end{aligned}$$

If $|c|$ is not small, the number of the last sum is not suitable to evaluate. Then we apply the Euclidean algorithm to the first column of $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and obtain a decomposition

$$A=B_1\cdots B_N,\qquad B_i=\begin{pmatrix}b_i&-1\\1&0\end{pmatrix}.$$

Thus the cocycle relation $\Psi(AB)=\Psi(A)+A\Psi(B)$ leads to

$$\Psi(1,A)=\sum_{i=0}^{N-1}(B_1\cdots B_i)\Psi(1,B_{i+1}).$$

At a rough estimate, the order of N is $\log|c|$, therefore, the number of terms is reduced from $|c|$ to $\log|c|$.

In the recent paper [GS03], Gunnells and Sczech introduce a higher dimensional Dedekind sum $D(L,\sigma,e,v)$ inspired by the definition of the Eisenstein cocycle. For a lattice L of rank l with $L^*=\mathrm{Hom}_{\mathbf{Z}}(L,\mathbf{Z})$, $\sigma=(\sigma_1,\ldots,\sigma_r)\in(L^*)^r$ $(r\geq l)$, a tuple $e=(e_1,\ldots,e_r)$ of positive integers and $v\in L^*\otimes\mathbf{R}$, define

$$D(L,\sigma,e,v):=(2\pi\sqrt{-1})^{-\sum e_i}\sum_{x\in L}{}'\frac{\exp(2\pi\sqrt{-1}\langle x,v\rangle)}{\langle x,\sigma_1\rangle^{e_1}\cdots\langle x,\sigma_r\rangle^{e_r}},$$

where $\langle\ ,\ \rangle:L\times L^*\to\mathbf{Z}$ is the canonical paring. If $e_j=1$ for some j, the above series converges conditionally, then the Q-limit is applied to define D. The reciprocity law of the new Dedekind sum is derived and combined with a refinement of the modular algorithm of Ash and Rudolph [AR79], they estimate the number of terms. As an application of their effective algorithm, some of numerical examples of $\zeta_K(0,\mathfrak{b},\mathfrak{f})$ are computed in [CGS00], where $\mathfrak{f}=N\mathcal{O}_K$ with various integers N and $\mathfrak{b}=\mathcal{O}_K$ for cubic or quartic fields K.

4.3. *Shintani cocycle by Solomon*

Solomon defined another cocycle on $(P)GL(2, \mathbf{Q})$ which he called Shintani cocycle. His method is algebro-combinatorial and is closely related to the work of Shintani [Shin76] as the name indicates. The starting point of Solomon's work is the modification of the generating functions which appear in Shintani's explicit formula.

Let Λ be a rank 2 lattice in $\mathbf{R}^2$ and $\mathbf{x} \in \mathbf{R}^2/\Lambda$. The symbols $\mathfrak{r}$ and $\mathfrak{s}$ denote Λ-rational rays emananting from the origin in $\mathbf{R}^2$, that is, equivalence classes for the multiplicative action of $\mathbf{Q}_+^\times$ on $\mathbf{Q}\,\Lambda\backslash\{\mathbf{0}\}$. For the above datum with $\mathfrak{r} \neq \pm\mathfrak{s}$ and $\mathbf{z} = (z_1, z_2)$, define

$$\widetilde{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) := \frac{\sum_{\mathbf{a}\in\mathbf{x}\cap P(\mathbf{r},\mathbf{s})} e^{\mathbf{z}.\mathbf{a}}}{(1 - e^{\mathbf{z}.\mathbf{r}})(1 - e^{\mathbf{z}.\mathbf{s}})}.$$

Here we have chosen $\mathbf{r} \in \mathfrak{r} \cap \Lambda$ and $\mathbf{s} \in \mathfrak{s} \cap \Lambda$ and $P(\mathbf{r}, \mathbf{s})$ denotes the half-open parallelogram

$$P(\mathbf{r}, \mathbf{s}) := \{\mu\mathbf{r} + \nu\mathbf{s} \mid \mu, \nu \in \mathbf{R}, 0 < \mu \leq 1, 0 \leq \nu < 1\}.$$

The *Shintani function* P is defined to be the mean of $\widetilde{P}$:

$$P(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) := \frac{1}{2}\mathrm{sgn}(r_1 s_2 - r_2 s_1)(\tilde{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) + \tilde{P}(\Lambda, \mathbf{x}, \mathfrak{s}, \mathfrak{r}; \mathbf{z})).$$

Here $(r_1, r_2) \in \mathfrak{r}$ and $(s_1, s_2) \in \mathfrak{s}$. Let $P(\Lambda, \mathfrak{r}, \mathfrak{s})$ be a map $\mathbf{R}^2/\Lambda \ni \mathbf{x} \mapsto P(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z})$ and define the *Shintani cocycle* $\Psi_\mathfrak{r}$ by

$$\Psi_\mathfrak{r}(M) := P(\mathbf{Z}^2, \mathfrak{r}, M\mathfrak{r})$$

for $M \in GL(2, \mathbf{Q})$ and the fixed ray $\mathfrak{r}$.

The cocycle relation follows from the "Juxtaposition Lemma" ([So98, Lemma 2.2]) and the connection between the Shintani cocycle $\Psi_\mathfrak{r}(M)$ and the special values of zeta function is essentially guaranteed by the work of Shintani. The procedure of finding an explicit form of Shintani function in terms of generalized Dedekind sum is relatively easy thanks to the way of definition of it, and we need no difficulty such as the Q-limit argument in the paper of Sczech.

As an application, essentially due to the cocycle properties of Shintani functions, Solomon discusses a new proof of Halbritter's reciprocity law for generalized Dedekind sum [Ha85c].

Extension to general n is attempted by Hu and Solomon [HS01]. They introduce a new parametrization of simplicial cones instead of "generating rays" in 2-dimensional case, and succeed the constructions of cocycles in case of $n = 3$, and $n \geq 4$ for *generic* cones. They also evaluated the cocyles in terms of generalized Dedekind sums, though, degenerate cones for $n \geq 4$ is not achieved obstructed by the combinatorial difficulties.

5. Further problems

Summing up the known results up to the present time, we have a table:

$n = [K : \mathbf{Q}]$	$n = 2$	$n \geq 3$
class number formula	Meyer [Me57], Siegel [Si61], Shintani [Shin77a]	Shintani [Shin77b], ♠
special values at $k \geq 2$	Meyer [Me66], Lang [La68], Barner [Ba69], Siegel [Si68], Sczech [Sc92], Solomon [So98]	Shintani [Shin76], Sczech [Sc93]

Note here that the above problem is concerned with the case of *critical values* in the sense of Deligne. There seems to be a big problem at ♠, and the method of Sczech and Solomon seems to be difficult to apply: for arbitrary totally real number field K of degree $n \geq 3$ and pair of abelian extensions L/L_0 over K such that L_0 is totally real and L a CM-extension of L_0, find an *effectively computable formula* of the relative class number h_L/h_{L_0} in terms of something like *hyper Dedekind sums.* The method of Shintani is universal and quite powerful even for numerical computation, but the multiple gamma functions would not be elementary functions.

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GENUS THETA SERIES, HECKE OPERATORS AND THE BASIS PROBLEM FOR EISENSTEIN SERIES

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Dedicated to the memory of Tsuneo Arakawa

We derive explicit formulas for the action of the Hecke operator $T(p)$ on the genus theta series of a positive definite integral quadratic form and prove a theorem on the generation of spaces of Eisenstein series by genus theta series. We also discuss connections of our results with Kudla's matching principle for theta integrals.

1. Introduction

In the theory of theta series of positive definite quadratic forms the problem of giving explicit formulas for the action of Hecke operators on theta series has received some attention [1, 18].

If p is prime to the level N of the quadratic form q of rank m in question, the action of the usual generators $T(p), T_i(p^2)$ of the p-part of the Hecke algebra for the group $\Gamma_0^{(n)}(N) \subseteq Sp_n(\mathbb{Z})$ is known [1, 18] except for the case that $n < \frac{m}{2}$ and $\chi(p) = -1$, where χ is the nebentype character of the degree n theta series of q. In this last case it is unknown whether $T(p)$ leaves the space of cusp forms generated by the theta series of positive definite quadratic forms of the same level and rational square class of the discriminant invariant. Some deep results concerning this question have been obtained by Waldspurger [17].

To our surprise, there seem to be no results available even for the question how to describe the action of $T(p)$ on the genus theta series of q, i.e., Siegel's weighted average over the theta series of the quadratic forms q' in the genus of q.

The present note intends to fill this gap. It turns out that we have different methods available to express the image of the genus theta series under the operator $T(p)$ in terms of theta series: Using results of Freitag [4], Salvati Manni [13] and Chiera [3] one obtains an expression as a linear combination of theta series of positive definite quadratic forms of level $\mathrm{lcm}(N, 4)$.

We show in Section 5 that this result can be improved to an (explicit) expression as a linear combination of genus theta series of positive definite quadratic forms of level N if N is an odd prime. In fact we prove in that case that any $n + 1$ of the genera of quadratic forms that are rationally equivalent to the given genus and have level dividing N yield a basis of the relevant space of holomorphic Eisenstein series.

This can be generalized to arbitrary square free level under a slightly technical condition on the degree n depending on the $\mathbb{Q}_p$-equivalence class for p dividing N of the given genus of quadratic forms; generalizations to arbitrary level will be the subject of future work.

On the other hand, using the explicit expression for the action of Hecke operators on Fourier coefficients of modular forms given in [1], Siegel's mass formula and relations between the local densities of quadratic forms we find a much simpler expression: The genus theta series is transformed into a multiple of the genus theta series of a different genus of quadratic forms. If $\chi(p) = -1$, the genus involved turns out to be indefinite, and the theta series is the one defined by Siegel ($n = 1$) and Maaß [16,12]. This phenomenon is an instance (with quite explicit data) of the matching principle for Siegel-Weil integrals attached to different quadratic spaces that has been observed by Kudla in [10], we discuss this in Section 6.

As a consequence of our work we are able to give a positive solution to the basis problem for modular forms in a number of new cases; this will be done in joint work with S. Böcherer.

2. Preliminaries

Let L be a lattice of full rank on the m-dimensional vector space V over $\mathbb{Q}$, $q :\,:= \;V \longrightarrow \mathbb{Q}$ a positive definite quadratic form with $q(L) \subseteq \mathbb{Z}$, $B(x, y) = q(x + y) - q(x) - q(y)$ the associated symmetric bilinear form, $N = N(L)$ the level of q (i.e., $N^{-1}\mathbb{Z} = q(L^{\#})\mathbb{Z}$, where $L^{\#}$ is the dual lattice of L with respect to B); we assume $m = 2k$ to be even.

Let R be $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p and let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices (a_{ij}) of degree n with entries in $\frac{1}{2}R$ such that a_{ii} $(i = 1, ..., n)$ and $2a_{ij}$ $(1 \leq i \neq j \leq n)$ belong to R.

We note that for $\mathbf{x} = (x_1, \ldots, x_n) \in L^n$ the matrix $q(\mathbf{x}) := (\frac{1}{2}B(x_i, x_j))$ is in the set $\mathcal{H}_n(\mathbb{Z})$; we also note that $\mathcal{H}_n(\mathbb{Z}_p)$ is equal to the set $M_n^{\mathrm{sym}}(\mathbb{Z}_p)$ of symmetric $n \times n$ matrices over $\mathbb{Z}_p$ for $p \neq 2$. For two square matrices T_1 and T_2 we write $T_1 \perp T_2 = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right)$.

We often write $a \perp T$ instead of $(a) \perp T$ if (a) is a matrix of degree 1. If $K = (K, q')$ is a quadratic $\mathbb{Z}_p$-lattice with Gram matrix T with respect to some basis we will freely switch notation between T and K, so for example if K is a one-dimensional lattice with basis vector of squared length a and M a quadratic lattice with Gram matrix T we write as above $a \perp T = (a) \perp T = K \perp T = K \perp M$.

The theta series

$$\vartheta^{(n)}(L, Z) = \sum_{\mathbf{x}=(x_1,\ldots,x_n)\in L^n} \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z)$$

of degree n of (L, q) is well-known to be in the space $M_k^{(n)}(\Gamma_0^{(n)}(N), \chi)$ of Siegel modular forms of weight $k = \frac{m}{2}$ and character χ, where χ is the character of $\Gamma_0^{(n)}(N)$ given by $\chi\left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\right) = \tilde{\chi}(\det D)$, $\tilde{\chi}$ is the Dirichlet character modulo N given by $\tilde{\chi}(d) = \left(\frac{(-1)^k \det L}{d}\right)$ for $d > 0$ and $\det L$ is the determinant of the Gram matrix of L with respect to some basis [1].

For definitions and notations concerning modular forms we refer again to [1], we recall that the Hecke operator associated to the double coset

$$\Gamma_0(N) \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & p & & \\ & & & & \ddots & \\ & & & & & p \end{pmatrix} \Gamma_0(N)$$

is as usual denoted by $T(p)$.

We let $\{L_1, \ldots, L_h\}$ be a set of representatives of the classes of lattices in the genus of L, put $w = \sum_{i=1}^h \frac{1}{|O(L_i)|}$ (where $O(L_i)$ is the group of isometries of L onto itself with respect to q) and write

$$\vartheta^{(n)}(\text{gen } L, Z) = \frac{1}{w}\sum_{i=1}^{h} \frac{\vartheta^{(n)}(L_i, Z)}{|O(L_i)|}$$

for Siegel's weighted average over the genus.

By Siegel's theorem (see [9]) the Fourier coefficient $r(\text{gen } L, A)$ at a positive semidefinite half integral symmetric matrix A can be expressed as a product of local densities,

$$r(\text{gen } L, A) = c \cdot (\det A)^{\frac{m-n-1}{2}} (\det L)^{\frac{n}{2}} \prod_{\ell \text{ prime}} \alpha_\ell(L, A) \tag{1}$$

with some constant c. Here the local density $\alpha_\ell(L, A)$ is given as

$$\begin{aligned}\alpha_\ell(L, A) &= \alpha_\ell(S, A)\\ &= \ell^{j\cdot(\frac{n\cdot(n+1)}{2}-mn)} \cdot \#\{\mathbf{x} \in L^n/\ell^j L^n \mid q(\mathbf{x}) \equiv A \bmod \ell^j \mathcal{H}_n(\mathbb{Z}_\ell)\}\\ &= \ell^{j\cdot(\frac{n\cdot(n+1)}{2}-mn)} \# \mathcal{A}_j(S, A),\end{aligned}$$

for sufficiently large j with an additional factor $\frac{1}{2}$ if $m = n$ where S denotes a Gram matrix of L and where we write

$$\begin{aligned}\mathcal{A}_j(L, A) &= \mathcal{A}_j(S, A)\\ &= \{\mathbf{x} \in L^n/\ell^j L^n \mid q(\mathbf{x}) \equiv A \bmod \ell^j \mathcal{H}_n(\mathbb{Z}_\ell)\}\\ &= \{X = (x_{ij}) \in M_{m,n}(\mathbb{Z}_\ell)/\ell^j M_{m,n}(\mathbb{Z}_\ell) \mid A[X] - B \in \ell^j \mathcal{H}_n(\mathbb{Z}_\ell)\}\end{aligned}$$

3. Eisenstein series and theta series

Proposition 3.1. *Let L be a lattice of rank $m = 2k$ with positive definite quadratic form q of square free level N as above, let $n < k - 1$ and let $F = \vartheta^{(n)}(gen(L))$ denote the genus theta series of L of degree n. Then for any prime $p \nmid N$ the modular form $F|_k T(p)$ is a linear combination of genus theta series of genera of lattices with positive definite quadratic form of level $N' = lcm(N, 4)$.*

Proof. By [2] $G := F|_k T(p)$ is an eigenfunction of infinitely many Hecke operators $T(\ell)$ for the primes $\ell \nmid pN$ with $\chi(\ell) = 1$ (where χ is the nebentyp character for $\vartheta^{(n)}(L)$). Proposition 4.3 of [4] implies then that G is in the space that is generated by Eisenstein series for the principal congruence subgroup of level N; this can also be obtained from Siegel's main theorem if one uses that this space is Hecke invariant. Theorem 6.9 of [4] (see also [13]) then implies that G is a linear combination of theta series with characteristic for the principal congruence subgroup of level $N' = \mathrm{lcm}(N,4)$. Since G is in fact a modular form for $\Gamma_0(N')$, Chiera's Theorem 4.4 [3] implies that G is a linear combination of theta series $\vartheta^{(n)}(K_j)$ attached to full lattices K_j with quadratic form of level dividing N'. It is well known that the values of the theta series of lattices in the same genus at zero dimensional cusps are the same. From Proposition 3.3 of [4] we can then conclude that G is in fact a linear combination of the $\vartheta^{(n)}(\mathrm{gen}(K_j))$ as asserted. □

4. Action of $T(p)$ and local densities

The action of the Hecke operator $T(p)$ on the Fourier coefficients at non-degenerate matrices A has been described explicitly by Maaß [11] and by Andrianov (Ex. 4. 2. 10 of [1]).

Let K be a $\mathbb{Z}$-lattice with quadratic form of rank n that has Gram matrix $p \cdot A$ with respect to some basis and write $\mathcal{M}_i$ for the set of lattices $M \supset K$ for which K has elementary divisors $(1,\dots,1,p,\dots,p)$ with $(n-i)$ entries p.

Then if $F(Z) \in M_k^{(n)}(\Gamma_0^{(n)}(N),\chi)$,

$$F(Z) = \textstyle\sum_{A\geq 0} f(A)\exp(2\pi i \mathrm{tr}(AZ)),$$

$$G(Z) = (F|_k T(p))(Z) = \textstyle\sum g_p(A)\exp(2\pi i \mathrm{tr}(AZ)),$$

one has for non-degenerate A:

$$g_p(A) = \chi(p)^n p^{nk-\frac{n(n+1)}{2}} \sum_{i=0}^{n} (\bar{\chi}(p)p^{-k})^i p^{i\frac{i+1}{2}} \sum_{M\in\mathcal{M}_i} f(M), \tag{2}$$

where by $f(M)$ we denote the Fourier coefficient at an arbitrary Gram matrix of the lattice M (the coefficient $f(A)$ depends only on the integral equivalence class of A). Here by convention $f(M)$ is zero if the Gram matrix of M is not half integral.

Proposition 4.1. *Let*

$$F(Z) := \vartheta^{(n)}(\mathrm{gen}\ L, Z) = \sum_{A\geqslant 0} f(A)\exp(2\pi i\ \mathrm{tr}(Z)),$$

$$G(Z) := (F|_k T(p))(Z) = \sum_{A\geqslant 0} g_p(A)\exp(2\pi i\ \mathrm{tr}(AZ)).$$

Then $g_p(A) = \lambda_p(L)\prod_{\ell\ \mathrm{prime}} \alpha_\ell(\tilde{L}_\ell, A)$, *where*

$$\lambda_p(L) = p^{nk-\frac{n(n+1)}{2}} \prod_{j=1}^{n}(1+\chi(p)p^{j-k})$$

and the $\mathbb{Z}_p$-lattice $\tilde{L}_\ell$ is given by

$$\tilde{L}_\ell = \begin{cases} L_\ell & \text{if } p = \ell \\ {}^pL_\ell & \text{otherwise.} \end{cases}$$

Here ${}^pL_\ell$ denotes the lattice L_ℓ with quadratic form scaled by p.

Proof. It is (by induction) enough to consider nondegenerate A. We write the total factor in front of $f(M)$ for $M \in \mathcal{M}_i$ in as γ_i and rewrite (2) in the present situation as

$$g_p(A) = c \cdot (\det L)^{\frac{n}{2}} \sum_{i=0}^{n} \gamma_i \sum_{M \in \mathcal{M}_i} (\det M)^{\frac{m-n-1}{2}} \prod_\ell \alpha_\ell(L_\ell, M) \tag{3}$$

by inserting the expression for $f(M)$ from (1) (Siegel's theorem). Since $\det M = p^{2i-n} \det A$ for $M \in \mathcal{M}_i$ this becomes

$$\begin{aligned} g_p(A) =& c \cdot (\det L)^{\frac{n}{2}} (\det A)^{\frac{m-n-1}{2}} p^{-n(\frac{m-n-1}{2})} \\ & \cdot \sum_{i=0}^{n} \gamma_i \sum_{M \in \mathcal{M}_i} p^{(\frac{m-n-1}{2})\cdot 2i} \prod_\ell \alpha_\ell(L_\ell, M_\ell). \end{aligned} \tag{4}$$

Now for $\ell \neq p$ we have $M_\ell = K_\ell$ for all M occurring, hence $\alpha_\ell(L_\ell, M_\ell) = \alpha_\ell(L_\ell, pA) = \alpha_\ell({}^pL_\ell, A) = \alpha_\ell(\tilde{L}_\ell, A)$, for all $\ell \neq p$. So it remains to prove

$$p^{-n\frac{m-n-1}{2}} \sum_{i=0}^{n} \gamma_i \sum_{M \in \mathcal{M}_i} p^{(\frac{m-n-1}{2})\cdot 2i} \alpha_p(L_p, M_p) = \lambda_p(L) \alpha_p(L_p, A). \tag{5}$$

We insert $\gamma_i = \chi(p)^n p^{nk - \frac{n(n+1)}{2}} (\chi(p)p^{-k})^i p^{i\frac{i+1}{2}}$ and divide both sides of (5) by $p^{nk-\frac{n(n+1)}{2}}$ to get

$$\begin{aligned} &\sum_{i=1}^{n} (\chi(p)p^{-k})^{n-i} p^{-i(n+1)} p^{\frac{i(i+1)}{2}} \sum_{M \in \mathcal{M}_i} \alpha_p(L, M) \\ &\qquad = \prod_{j=1}^{n} (1 + \chi(p)p^{j-k}) \alpha_p(L, A) \end{aligned} \tag{6}$$

as the assertion that we have to prove.

For $\chi(p) = 1$ this is proved in [18] (see also [2]), where it is also proved for $\chi(p) = -1$ and $n \geqslant k$ (in which case the factor $\lambda_p(L)$ is zero). To prove it for $\chi(p) = -1$ notice that L_p is unimodular even by assumption. By Lemma 3.5 of [14] there exists a polynomial $G_p(M; X)$ such that $\alpha_p(\hat{L}_p, M) =$

$G_p(M; \chi_{\hat{L}_p}(p)p^{-\hat{k}})$ is true for all (even) unimodular $\mathbb{Z}_p$-lattices $\hat{L}_p$ of even rank $2\hat{k}$ with $\hat{k} \in \mathbb{N}$ and with

$$\chi_{\hat{L}_p}(p) := \begin{cases} 1 & \text{if } (-1)^{\hat{k}} \det \hat{L}_p \text{ is a square in } \mathbb{Q}_p \\ -1 & \text{otherwise.} \end{cases}$$

Hence both sides of our assertion (6) are polynomials in $X = \chi(p)p^{-\hat{k}}$ as $\hat{L}_p$ varies over (even) unimodular $\mathbb{Z}_p$-lattices of (varying) rank $2\hat{k}$. The truth of the assertion for $\hat{L}_p$ with $\chi_{\hat{L}_p}(p) = 1$ and $\hat{k}$ arbitrary shows that these polynomials take the same value at infinitely many places, hence must be identical. The assertion is therefore true for all even unimodular L_p of even rank. □

Lemma 4.2. *There is a unique isometry class of rational quadratic spaces $\tilde{V} = (\tilde{V}, \tilde{q})$ of dimension m, such that*

$$\tilde{V}_\ell \cong V'_\ell := \begin{cases} {}^pV_\ell & \text{if } p \neq \ell \\ V_p & \text{if } p = \ell \end{cases} \tag{7}$$

for finite primes ℓ and $\tilde{V}_\infty = \tilde{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is either positive definite or of signature $(m-2, 2)$.
$\tilde{V}$ carries a lattice $\tilde{L}$ such that

$$\tilde{L}_\ell \cong \begin{cases} {}^pL_\ell & \text{if } p \neq \ell \\ L_p & \text{if } p = \ell. \end{cases} \tag{8}$$

$\tilde{V}_\infty$ is indefinite if and only if $\chi(p) = -1$. The same assertion is true if one requires $\tilde{V}_\infty$ to be of signature $(m-2-4j, 2+4j)$ instead of $(m-2, 2)$ for some $1 \leq j \leq \frac{m-2}{4}$.

Proof. If $s_\ell V_\ell$ denotes the Hasse symbol of the quadratic space V_ℓ and V'_ℓ is the quadratic $\mathbb{Q}_\ell$-space as in (7), the discriminant of V'_ℓ is that of V_ℓ and the product of the Hasse symbols $s_\ell V'_\ell$ over the finite primes ℓ is the Hilbert symbol

$$(p, (-1)^{\frac{m}{2}} \det L)_p \cdot \prod_{\ell \text{ prime}} s_\ell V_\ell$$

by Hilbert's reciprocity law, with $(p, (-1)^{\frac{m}{2}} \det L)_p = \chi(p)$.

If V'_∞ is positive definite for $\chi(p) = 1$ and of signature $(m-2, 2)$ if $\chi(p) = -1$ one sees therefore that disc $V'_\ell =$ disc V_ℓ for all ℓ (including ∞) and $\prod_{\ell,\infty} s_\ell V'_\ell = 1$, hence there is a rational quadratic space $\tilde{V}$ such that $\tilde{V}_\ell \cong V'_\ell$ for all ℓ including ∞. The uniqueness of $\tilde{V}$ is clear from the Hasse-Minkowski theorem, and that $\tilde{L}$ as in (8) exists on $\tilde{V}$ is obvious. □

We recall that for an integral lattice of positive determinant and even rank Siegel [16] for degree one and Maaß [12] for arbitrary degree defined a holomorphic theta series in the indefinite case whose Fourier coefficients at positive definite A are proportional to the product of the local densities of that lattice, subject to the restriction that the signature (m_+, m_-) satisfies the condition $\min(\frac{m_+ + m_- - 3}{2}, m_+, m_-) \geq n$. Denote this theta series (if it is defined) for $\tilde{L}$, normalized such that its Fourier coefficient at A is equal to

$$c \cdot (\det A)^{\frac{m-n-1}{2}} (\det \tilde{L})^{\frac{n}{2}} \prod_{\ell \text{ prime}} \alpha_\ell(\tilde{L}, A),$$

by $\vartheta(\tilde{L}, z)$. The signature condition is in our situation always satisfied if $n = 1$, for bigger n it can be satisfied by choosing j in 4.2 appropriately if $n \leq k - 2$ (with $k = m/2$). If the signature condition is not satisfied, we use the same notation for the series with these Fourier coefficients (without knowing a priori whether this series defines a modular form).

Then we arrive at the following final result:

Theorem 4.3. *Let L be as above, p a prime with $p \nmid \det L$, $\tilde{L}$ the quadratic lattice with*

$$\tilde{L}_\ell = \begin{cases} {}^{p}L_\ell & \text{if } p \neq \ell \\ L_p & \text{if } p = \ell. \end{cases}$$

and of signature $(m, 0)$ if $\chi(p) = 1$, of signature $(m-2, 2)$ if $\chi(p) = -1$. Then

$$\vartheta^{(n)}(\text{gen } L, z) \mid T(p) = \lambda_p(L) \vartheta^{(n)}(\text{gen } \tilde{L}, z)$$

with

$$\lambda_p(L) = p^{nk - \frac{n(n+1)}{2}} \prod_{j=1}^{n} (1 + \chi(p) p^{j-k}).$$

In particular, the series $\vartheta^{(n)}(\text{gen } \tilde{L}, z)$ defines a modular form of the same level as L for all $n < k$.

Remark 4.4. a) $\lambda_p(L) = 0$ if $n \geq k$ holds with $\chi(p) = -1$, which agrees with Andrianov's result [2] for this case.

b) In the introduction we mentioned the question whether the space of cusp forms generated by the theta series of positive definite lattices of fixed level and rational square class of the discriminant is invariant under the action of the Hecke operators. In view of our theorem we might reformulate this question by substituting "modular forms" for "cusp forms" and omitting the restriction to positive definite lattices. Since the indefinite theta series of Siegel and Maaß don't contribute to the space of cusp forms, this doesn't change the problem with regard to the subspace of cusp forms.

c) Of course the same result holds true when we take an indefinite lattice $\tilde{L}$ of signature $(m-2,2)$ as above as our starting point. The lattices appearing in $\vartheta(\text{gen } \tilde{L}, z) \mid T(p)$ are then positive definite if $\chi(p) = -1$, indefinite if $\chi(p) = +1$.

5. Spaces of genus theta series for odd prime level

We will need some additional notations in this section.

Let p be an odd prime. For a non-zero element $a \in \mathbb{Q}_p$ we put $\chi_p(a) = 1, -1$, or 0 according as $\mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p$, $\mathbb{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbb{Q}_p$. For a non-degenerate half-integral matrix B of even degree n, put $\xi_p(B) = \chi_p((-1)^{n/2} \det B)$.

Further for non-negative integers l, e and matrices $A \in \mathcal{H}_m(\mathbb{Z}_p), B \in \mathcal{H}_n(\mathbb{Z}_p)$ define

$$\mathcal{B}_e(A,B)^{(l)} = \{X = (x_{ij}) \in \mathcal{A}_e(A,B);\ \operatorname{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p}(x_{i,j})_{1\le i\le m, 1\le j\le l} = l\}$$

(with $\mathcal{A}_e(A,B)$ as in Section 2) and

$$\beta_p(A,B)^{(l)} = \lim_{e\to\infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{B}_e(A,B)^{(l)}.$$

We note that

$$\beta_p(A,B)^{(0)} = \alpha_p(A,B).$$

In particular put

$$\beta_p(A,B) = \beta_p(A,B)^{(n)},$$

and call it (as usual) the primitive density. Further for $0 \le i \le m$ put

$$\pi_{m,i} = GL_m(\mathbb{Z}_p)(pE_i \perp E_{m-i})GL_m(\mathbb{Z}_p)$$

Furthermore let $H_k = \overbrace{H \perp \ldots \perp H}^{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

Our goal in this section is to prove the following theorem:

Theorem 5.1. *Let p be an odd prime, $k, n \in \mathbf{N}$ with $n \le k-1$ and $p \equiv (-1)^k \bmod 4$. Then the space of modular forms for $\Gamma_0^{(n)}(p)$ spanned by* *the genus theta series of degree n attached to the genus of positive definite integral quadratic lattices of rank $2k$, level p and discriminant p^{2r+1} for some $0 \leq r < k$ and the space spanned by the genus theta series of degree n (in the sense of Theorem 4.3) attached to the genus of integral quadratic lattices of signature $(2k-2-4j, 2+4j)$ $(1 \leq j \leq \frac{2k-2}{4})$, level p and discriminant p^{2r+1} for some $0 \leq r < k$ coincide. This space has dimension $n+1$ and is equal to the space of holomorphic Eisenstein series for the group $\Gamma_0^{(n)}(p)$ of weight k and nontrivial quadratic character.*

For each of these signatures the theta series of any $n+1$ of the k genera of level dividing p and having this signature form a basis of this space of modular forms.

The proof of this theorem will require a few intermediate results which may be of independent interest. A half-integral matrix S_0 over $\mathbb{Z}_p$ is called $\mathbb{Z}_p$-maximal if it is the empty matrix or a matrix corresponding to a $\mathbb{Z}_p$-maximal lattice. The main result we need is the following theorem, whose proof again is broken up into several steps:

Theorem 5.2. *Let p be an odd prime, let $T \in \mathcal{H}_n(\mathbb{Z}_p)$. Let k be a positive integer, and S_0 be a $\mathbb{Z}_p$-maximal half-integral matrix of degree not greater than 2. Then there exist rational numbers $a_i = a_i(k, S_0, T)$ $(i = 0, 1, 2, ..., n)$ such that*

$$\alpha_p(H_{k-l-1} \perp pH_l \perp S_0, T) = a_0 + a_1 p^l + ... + a_n p^{nl}$$

for any $l = 0, 1, ..., k-1$.

To prove the theorem, first we remark that for $p \neq 2$ a $\mathbb{Z}_p$-maximal matrix S_0 of degree not greater than 2 is equivalent over $\mathbb{Z}_p$ to one of the following matrices:

(M-1) ϕ (empty matrix),
(M-2) u_1 with $u_1 \in \mathbb{Z}_p^*$,
(M-3) pu_1 with $u_1 \in \mathbb{Z}_p^*$,
(M-4) $u_1 \perp u_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$,
(M-5) $u_1 \perp pu_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$,
(M-6) $pu_1 \perp pu_2$ with $u_1, u_2 \in \mathbb{Z}_p^*$ such that $-u_1u_2 \notin (\mathbb{Z}_p^*)^2$.

Lemma 5.3. *Let S_0 be the matrix in Theorem 5.2. For a non-negative integer l put $B_l = B_{S_0,l} = pH_l \perp S_0$ and $\tilde{B}_{l,i} = \tilde{B}_{S_0,l,i} = H_i \perp pH_{l-i} \perp S_0$. Let $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$.*

(1) *Let S_0 be of type (M-3) or (M-5). Then for any $k \geq n$ we have*

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T) = \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+1)} C_{2l+1,i}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T),$$

where $C_{m,i} = \frac{\prod_{j=1}^{i}(p^{m+1-2j}-1)}{\prod_{j=1}^{i}(p^j-1)}$ for an odd positive integer m and an integer i such that $i \leq (m-1)/2$.

(2) *Let S_0 be of type (M-1),(M-2),(M-4), or (M-6). Put $\epsilon = \epsilon(S_0) = -1$ or 1 according as S_0 is of type (M-6) or not. Then for any $k \geq n$ we have*

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T) = \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+1)} C_{2l,i,\epsilon}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T),$$

where $C_{m,i,\epsilon} = \frac{(p^{m/2}-\epsilon)(p^{m/2-i}+\epsilon)\prod_{j=1}^{i-1}(p^{m-2j}-1)}{\prod_{j=1}^{i}(p^j-1)}$ for an even positive integer m and an integer i such that $i \leq m/2$, and $\epsilon = \pm 1$.

(3)

$$\alpha_p(H_{k+l}, -H \perp T) = (1-p^{-(k+l)})(1+p^{-(k+l-1)})\alpha_p(H_{k+l-1}, T)$$

Proof. By Proposition 2.2 of [7], we have

$$\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T) = \sum_{i=0}^{2l+2}(-1)^i p^{i(i-1)/2+i((n+2l+2)+1-(2k+2l+2))} \times \sum_{G \in GL_{2l+2}(\mathbb{Z}_p)\backslash \pi_{2l+2,i}} \alpha_p(H_{k+l+1}, -B_l[G^{-1}] \perp T).$$

We note that $\alpha_p(H_{k+l+1}, -B_l[G^{-1}] \perp T) = 0$ if $G \in \pi_{2l+2,i}$ with $i \geq l+1$. Fix $i = 0, 1, ..., l$. Then by Lemma 2.3 of [6], we have $-B_l[G^{-1}] \perp T \sim -\tilde{B}_{l,i} \perp T$ if $G \in \pi_{l,i}$ and $B_l[G^{-1}] \in \mathcal{H}_{2l+2}(\mathbb{Z}_p)$. Furthermore, by Proposition 2.8 of [6] we have

$$\#(GL_{2l+2}(\mathbb{Z}_p)\backslash\{G \in \pi_{2l+2,i}\,;\, B_l[G^{-1}] \in \mathcal{H}_{2l+2}(\mathbb{Z}_p)\}) = \frac{\prod_{j=1}^{i}(p^{2l+2-2j}-1)}{\prod_{j=1}^{i}(p^j-1)}.$$

This proves the assertion (1). Similarly, the assertion (2) can be proved. Now again by Proposition 2.2 of [7] we have

$$\beta_p(H_{k+l}, H)\alpha_p(H_{k+l-1}, T) = \alpha_p(H_{k+l}, -H \perp T).$$

On the other hand, we have

$$\beta_p(H_{k+l}, H) = (1 - p^{-(k+l)})(1 + p^{-(k+l-1)})$$

(e.g. Lemma 9, [8].) Thus the assertion (3) holds. □

Now for a non-degenerate half-integral matrix B of degree n over $\mathbb{Z}_p$ define a polynomial $\gamma_p(B; X)$ in X by

$$\gamma_p(B;X) = \begin{cases} (1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases}$$

For a half-integral matrix B of degree over $\mathbb{Z}_p$, let $(\bar{W}, \bar{q})$ denote the quadratic space over $\mathbb{Z}_p/p\mathbb{Z}_p$ defined by the quadratic form $\bar{q}(\mathbf{x}) = B[\mathbf{x}] \bmod p$, and define the radical $R(\bar{W})$ of $\bar{W}$ by

$$R(\bar{W}) = \{\mathbf{x} \in \bar{W}; \bar{B}(\mathbf{x}, \mathbf{y}) = 0 \text{ for any } \mathbf{y} \in \bar{W}\},$$

where $\bar{B}$ denotes the associated symmetric bilinear form of $\bar{q}$. We then put $l_p(B) = \operatorname{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p} R(\bar{W})^{\perp}$, where $R(\bar{W})^{\perp}$ is the orthogonal complement of $R(\bar{W})$ in $\bar{W}$. Furthermore, in case $l_p(B)$ is even, put $\bar{\xi}_p(B) = 1$ or -1 according as $R(\bar{W})^{\perp}$ is hyperbolic or not. Here we make the convention that $\xi_p(B) = 1$ if $l_p(B) = 0$. We note that $\bar{\xi}_p(B)$ is different from $\xi_p(B)$.

Lemma 5.4.

(1) *Let B be a half-integral matrix of degree n over $\mathbb{Z}_p$. Put $l = l_p(B)$. Then we have*

$$\beta_p(H_m, B) = (1 - p^{-m})(1 + \bar{\xi}_p(B)p^{n-l/2-m}) \prod_{j=0}^{n-l/2-1} (1 - p^{2j-2m})$$

if l is even,

$$\beta_p(H_m, B) = \prod_{j=0}^{n-(l+1)/2} (1 - p^{2j-2m})$$

if l is odd.

(2) *Let $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$. Then there exists a polynomial $F_p(T, X)$ such that $\alpha_p(H_m, T) = \gamma_p(T; p^{-m})F_p(T, p^{-m})$.*

Proof. The assertion (1) follows from Lemma 9, [8]. The assertion (2) is well known (cf. [8]). □

Let $(\ ,\ )_p$ be the Hilbert symbol over $\mathbb{Q}_p$ and h_p the Hasse invariant (for the definition of the Hasse invariant, see [9]). Let B be a non-degenerate symmetric matrix of degree n with entries in $\mathbb{Q}_p$. We define

$$\begin{cases} \eta_p(B) = h_p(B)(\det B, (-1)^{(n-1)/2}\det B)_p & \text{if } n \text{ is odd} \\ \xi_p(B) = \chi_p((-1)^{n/2}\det B) & \text{if } n \text{ is even.} \end{cases}$$

From now on we often write $\xi(B)$ instead of $\xi_p(B)$ and so on if there is no fear of confusion. For a non-degenerate half-integral matrix B of degree n over $\mathbb{Z}_p$ put $D(B) = \det B$ and $d(B) = \mathrm{ord}_p(D(B))$. Further, put

$$\delta(B) = \begin{cases} 2[(d(B)+1)/2] & \text{if } n \text{ is even} \\ d(B) & \text{if } n \text{ is odd} \end{cases}.$$

Let $\nu(B)$ be the least integer l such that $p^l B^{-1} \in \mathcal{H}_n(\mathbb{Z}_p)$. Further put $\xi'(B) = 1 + \xi(B) - \xi(B)^2$ for a matrix B of even degree. Then we have

Proposition 5.5. *Let $B_1 = (b_1)$ and B_2 be non-degenerate half-integral matrices of degree 1 and $n-1$, respectively over $\mathbb{Z}_p$, and put $B = B_1 \perp B_2$. Assume that $\mathrm{ord}_p(b_1) \geq \nu(B_2) - 1$.*

(1) *Let n be even. Then we have*

$$\begin{aligned} &F_p(-(H_i \perp pH_{l-i}) \perp B, p^{-(k+l)}) \\ &\quad = \frac{1-\xi p^{n/2-k}}{1-p^{n-2k+1}} F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l-1)}) \\ &\qquad + K(B)p^{l-i}F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l)}), \end{aligned}$$

where $\xi = \xi(B)$, and $K(B)$ is a rational number depending only on B.

(2) *Let n be odd. Then we have*

$$\begin{aligned} &F_p(-(H_i \perp pH_{l-i}) \perp B, p^{-(k+l)}) \\ &\quad = \frac{1}{1-\tilde{\xi} p^{(n+1)/2-k}} F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l-1)}) \\ &\qquad + K(B)p^{l-i}F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l)}), \end{aligned}$$

where $\tilde{\xi} = \xi(B_2)$, and $K(B)$ is a rational number depending only on B. Here we understand that B_2 is the empty matrix and that we have $\tilde{\xi} = 1$ if $n = 1$.

Proof. (1) Let n be even. We have $\mathrm{ord}_p(b_1) \geq \nu(-(H_i \perp pH_{l-i}) \perp B_2)-1$. Thus by Theorem 4.1 of [7], we have

$$\begin{aligned}
&F_p(-(H_i \perp pH_{l-i}) \perp B, p^{-(k+l)}) \\
&\quad = \frac{1-\xi(l,i)p^{(n+2l)/2-(k+l)}}{1-p^{n+2l+1-2(k+l)}} F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l-1)}) \\
&\qquad + (-1)^{\xi(l,i)+1}\xi(l,i)'\tilde{\eta}(l,i)\frac{1-\xi(l,i)p^{(n+2l)/2+1-(k+l)}}{1-p^{n+2l+1-2(k+l)}} \\
&\qquad \times (p^{(n+2l)/2-(k+l)})^{\delta(l,i)-\tilde{\delta}(l,i)+\xi(l,i)^2} p^{\delta(l,i)/2} \\
&\qquad \times F_p(-(H_i \perp pH_{l-i}) \perp B_2, p^{-(k+l)}),
\end{aligned}$$

where $\xi(l,i) = \xi(-(H_i \perp pH_{l-i}) \perp B), \xi(l,i)' = \xi'(-(H_i \perp pH_{l-i}) \perp B), \tilde{\eta}(l,i) = \eta(-(H_i \perp pH_{l-i}) \perp B_2), \delta(l,i) = \delta(-(H_i \perp pH_{l-i}) \perp B)$, and $\tilde{\delta}(l,i) = \delta(-(H_i \perp pH_{l-i}) \perp B_2)$. We note that $\xi(l,i), \xi(l,i)'$ and $\tilde{\eta}(l,i)$ are independent of l and i, and they are equal to ξ, ξ', and $\eta(B_2)$, respectively. Furthermore, we have $\delta(l,i) = 2l-2i+2[(\mathrm{ord}_p(\det T)+1)/2]$ and $\tilde{\delta}(l,i) = 2l-2i+\mathrm{ord}_p(\det \hat{T})$. Thus the assertion holds. Similarly, the assertion holds in case n is odd. □

Proposition 5.6. *Let S_0 and the others be as in Lemma 5.3. $T = b_1 \perp b_2 \perp \ldots \perp b_n$ with $\mathrm{ord}_p(b_1) \geq \mathrm{ord}_p(b_2) \geq \ldots \geq \mathrm{ord}_p(b_n)$. Put $\hat{T} = b_2 \perp \ldots \perp b_n$.*

(1) *Assume that $n+\deg S_0$ is even. Put $K(S_0,T) = \frac{1-p^{n-2k}}{1-p^{n/2-k}\xi}K(-S_0 \perp T)$, where $\xi = \xi(-S_0 \perp T)$, and $K(-S_0 \perp T)$ is the rational number in Proposition 5.5 . Then we have*

$$\begin{aligned}
&\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T) \\
&\quad = \frac{(1-p^{-(k+l+1)})(1+p^{-(k+l)})}{1-p^{n-2k+1}}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\qquad + p^{l-i}K(S_0,T)\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T}).
\end{aligned}$$

(2) *Assume that $n+\deg S_0$ is odd. Put $K(S_0, T) = (1-p^{(n-1)/2-k}\tilde{\xi})K(-S_0 \perp T)$, where $\tilde{\xi} = \xi(-S_0 \perp \hat{T})$, and $K(-S_0 \perp T)$ is the rational number in Proposition 5.5. Then we have*

$$\begin{aligned}
&\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T) \\
&\quad = \frac{(1-p^{-(k+l+1)})(1+p^{-(k+l)})}{1-p^{n-2k+1}}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\qquad + p^{l-i}K(S_0,T)\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T}).
\end{aligned}$$

Proof. By (1) of Proposition 5.5 and (2) of Lemma 5.4, we have

$$\begin{aligned}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T) &= \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)})F_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}) \\ &= \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}) \\ &\quad \times [\frac{1-\xi p^{n/2-k}}{1-p^{n-2k+1}} F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l)}) \\ &\quad + p^{l-i} K(-S_0 \perp T) F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l+1)})].\end{aligned}$$

We note that

$$\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l)}) = \frac{1-p^{n/2-k}\xi}{(1-p^{-(k+l+1)})(1+p^{-(k+l)})} \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}),$$

and

$$\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l+1)}) = \frac{1-p^{n/2-k}\xi}{1-p^{n-2k}} \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}).$$

Thus the assertion (1) holds.

Now by (2) of Proposition 5.5 and (2) of Lemma 5.4, we have

$$\begin{aligned}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp T) &= \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)})F_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}) \\ &= \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}) \\ &\quad \times [\frac{1}{1-\tilde{\xi} p^{(n+1)/2-k}} F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l)}) \\ &\quad + p^{l-i} K(-S_0 \perp T) F_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l+1)})].\end{aligned}$$

We note that

$$\begin{aligned}\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l)}) &= \frac{1-p^{n+1-2k}}{(1-p^{-(k+l+1)})(1+p^{-(k+l)})(1-\tilde{\xi} p^{(n+1)/2-k})} \\ &\quad \times \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}),\end{aligned}$$

and

$$\gamma_p(-\tilde{B}_{l,i} \perp \hat{T}, p^{-(k+l+1)}) = \frac{1}{1-p^{(n-1)/2-k}\tilde{\xi}} \gamma_p(-\tilde{B}_{l,i} \perp T, p^{-(k+l+1)}).$$

Thus the assertion (2) holds. □

Remark 5.7. In the above theorem, $K(S_0, T)$ can be expressed explicitly in terms of the invariants of T.

Proposition 5.8. *Let S_0, T and $\hat{T}$ and the others be as in Proposition 5.6.*

(1) *Assume that S_0 is of type (M-3) or (M-5). Then for any non-negative integer $l \le k-1$ we have*

$$\begin{aligned}\alpha_p(H_{k-l-1} \perp B_l, T) = (1-p^{n-2k+1})^{-1} \\ \times \{(1-p^{-2k+2l+2})\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) \\ + p^{n-2k+1}(p^{2l}-1)\alpha_p(H_{k-l-1} \perp B_{l-1}, \hat{T})\} \\ + p^l K(S_0, T)\alpha_p(H_{k-l-1} \perp B_l, \hat{T}),\end{aligned}$$

where $K(S_0,T)$ is the rational number in Proposition 5.5. In particular, if $n=1$, for a non-zero element T of $\mathbb{Z}_p$, we have

$$\alpha_p(H_{k-l-1} \perp B_l, T) = 1 + cp^l,$$

where $c = c(S_0,T)$ is the rational number determined by T and S_0.

(2) *Assume that S_0 is of type (M-1),(M-2),(M-4) or (M-6). Put $l' = l+1$ or l according as S_0 is of type (M-6) or not. Let $\epsilon = \epsilon(S_0)$ be as in Lemma 5.3, and $\bar{\xi} = \bar{\xi}(S_0)$. Put $\epsilon = -1$ or 1 according as S_0 is of type (M-6) or not. Then for non-negative integer $l \le k-1$ we have*

$$\begin{aligned}\alpha_p(H_{k-l-1} \perp B_l, T) = (1-p^{n-2k+1})^{-1} \\ \times \{(1-p^{-k+l'+1}\bar{\xi})(1+p^{-k+l'}\bar{\xi})\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) \\ + p^{n-2k+1}(p^{l'}-\epsilon)(p^{l'-1}+\epsilon)\alpha_p(H_{k-l-1} \perp B_{l-1}, \hat{T})\} \\ + K(S_0,T)p^l\alpha_p(H_{k-l-1} \perp B_l, \hat{T}),\end{aligned}$$

where $K(S_0,T)$ is the rational number in Proposition 5.5. In particular, if $n=1$, for a non-zero element T of $\mathbb{Z}_p$, we have

$$\alpha_p(H_{k-l-1} \perp B_l, T) = 1 + cp^l,$$

where $c = c(S_0,T)$ is a rational number determined by T and S_0. Throughout (1) and (2), we understand $\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) = 1$ if $l = k-1$.

Proof. (1) First let $n + \deg S_0$ be even. Then by (1) of Proposition 5.6 and

(1) of Lemma 5.3, we have

$$\begin{aligned}
&\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, T) \\
&\quad = \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+1)} C_{2l+1,i} \\
&\qquad \times \{\frac{(1-p^{-(k+l+1)})(1+p^{-(k+l)})}{1-p^{n-2k+1}}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\qquad\quad + \alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T})p^{l-i}K(S_0, T)\} \\
&\quad = \frac{(1-p^{-(k+l+1)})(1+p^{-(k+l)})}{1-p^{n-2k+1}} \\
&\qquad \times [\sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l+1,i}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\qquad\quad + \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+2)}(p^{-i}-1) C_{2l+1,i}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T})] \\
&\qquad + \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+1)} C_{2l+1,i}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T})p^l K(S_0, T).
\end{aligned}$$

By (1) of Lemma 5.3 and (1) of Lemma 5.4, we have

$$\begin{aligned}
&(1-p^{-(k+l+1)})(1+p^{-(k+l)}) \\
&\qquad \times \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l+1,i}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\quad = (1-p^{-(k+l+1)})(1+p^{-(k+l)})\beta_p(H_{k+l}, -B_l)\alpha_p(H_{k-l-2} \perp B_l, \hat{T}) \\
&\quad = (1-p^{2l+2-2k})\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-2} \perp B_l, \hat{T})
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k)} C_{2l+1,i}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T}) \\
&\qquad\qquad = \beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, \hat{T}).
\end{aligned}$$

Furthermore, again by (1) and (3) of Lemma 5.3, and (1) of Lemma 5.4, we have

$$
\begin{aligned}
&(1-p^{-(k+l+1)})(1+p^{-(k+l)})\sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+2)}(p^{-i}-1)\\
&\qquad\qquad\qquad \times C_{2l+1,i}\alpha_p(H_{k+l},-\tilde{B}_{l,i}\perp\hat{T})\\
&\quad=(1-p^{-(k+l+1)})(1+p^{-(k+l)})p^{n-2k+1}(p^{2l}-1)\\
&\qquad\times\sum_{i=1}^{l}(-1)^{i-1}p^{(i-2)(i-1)/2+(i-1)(n-2k+2)}C_{2l-1,i-1}\\
&\qquad\qquad\times\alpha_p(H_{k+l},-\tilde{B}_{l-1,i-1}\perp -H\perp\hat{T})\\
&\quad=(1-p^{-(k+l+1)})(1+p^{-(k+l)})p^{n-2k+1}(p^{2l}-1)\\
&\qquad\times\sum_{i=1}^{l}(-1)^{i-1}p^{(i-2)(i-1)/2+(i-1)(n-2k+2)}\\
&\qquad\times C_{2l-1,i-1}(1-p^{-(k+l)})(1+p^{-(k+l-1)})\\
&\qquad\times\alpha_p(H_{k+l-1},-\tilde{B}_{l-1,i-1}\perp\hat{T})\\
&\quad=p^{n-2k+1}(p^{2l}-1)(1-p^{-(k+l+1)})(1-p^{-2(k+l)})(1+p^{-(k+l-1)})\\
&\qquad\qquad\times\beta_p(H_{k+l-1},-B_{l-1})\alpha_p(H_{k-l-1}\perp B_{l-1},\hat{T})\\
&\quad=p^{n-2k+1}(p^{2l}-1)\beta_p(H_{k+l+1},-B_{l-1})\alpha_p(H_{k-l-1}\perp B_{l-1},\hat{T}).
\end{aligned}
$$

This proves the assertion (1) in case $n+\deg S_0$ is odd. Next again by (2) of Proposition 5.6 and (1) of Lemma 5.3, the assertion (1) can be proved in case $n+\deg S_0$ is odd.

(2) First let $n+\deg S_0$ be even. Then by (1) of Proposition 5.6 and (2) of Lemma 5.3, we have

$$
\begin{aligned}
&\beta_p(H_{k+l+1},-B_l)\alpha_p(H_{k-l-1}\perp B_l,T)\\
&\quad=\sum_{i=0}^{l'}(-1)^i p^{i(i-1)/2+i(n-2k+1)}C_{2l',i,\epsilon}\\
&\qquad\times\{\frac{(1-p^{-(k+l'+1)})(1+p^{-(k+l')})}{1-p^{n-2k+1}}\alpha_p(H_{k+l},-\tilde{B}_{l,i}\perp\hat{T})\\
&\qquad\qquad+\alpha_p(H_{k+l+1},-\tilde{B}_{l,i}\perp\hat{T})p^{l-i}K(S_0,T)\}.
\end{aligned}
$$

We evaluate this further as

$$\frac{(1-p^{-(k+l'+1)})(1+p^{-(k+l')})}{1-p^{n-2k+1}}$$
$$\times [\sum_{i=0}^{l'}(-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l',i,\epsilon}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T})$$
$$+ \sum_{i=0}^{l'}(-1)^i p^{i(i-1)/2+i(n-2k+2)}(p^{-i}-1) C_{2l',i,\epsilon}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T})]$$
$$+ \sum_{i=0}^{l'}(-1)^i p^{i(i-1)/2+i(n-2k)} C_{2l',i,\epsilon}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T}) p^l K(S_0, T).$$

By (1) and (3) of Lemma 5.3 and (1) of Lemma 5.4, we have

$$(1-p^{-(k+l'+1)})(1+p^{-(k+l')})$$
$$\times \sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k+2)} C_{2l',i,\epsilon}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T})$$
$$= (1-p^{-(k+l'+1)})(1+p^{-(k+l')})\beta_p(H_{k+l}, -B_l)\alpha_p(H_{k-l-2} \perp B_l, \hat{T})$$
$$= (1-\bar{\xi}p^{l'+1-k})(1+\bar{\xi}p^{l'-k})\beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-2} \perp B_l, \hat{T})$$

and

$$\sum_{i=0}^{l}(-1)^i p^{i(i-1)/2+i(n-2k)} C_{2l',i,\epsilon}\alpha_p(H_{k+l+1}, -\tilde{B}_{l,i} \perp \hat{T})$$
$$= \beta_p(H_{k+l+1}, -B_l)\alpha_p(H_{k-l-1} \perp B_l, \hat{T}).$$

Furthermore, again by (1) of Lemma 5.3, and (1) of Lemma 5.4, we have

$$(1-p^{-(k+l'+1)})(1+p^{-(k+l')})\sum_{i=0}^{l'}(-1)^i p^{i(i-1)/2+i(n-2k+2)}(p^{-i}-1)$$
$$\times C_{2l',i,\epsilon}\alpha_p(H_{k+l}, -\tilde{B}_{l,i} \perp \hat{T})$$
$$= (1-p^{-(k+l'+1)})(1+p^{-(k+l')})p^{n-2k+1}(p^{l'}-\epsilon)(p^{l'-1}+\epsilon)$$
$$\times \sum_{i=1}^{l'}(-1)^{i-1} p^{(i-2)(i-1)/2+(i-1)(n-2k+2)}$$
$$\times C_{2l'-2,i-1,\epsilon}\alpha_p(H_{k+l}, -\tilde{B}_{l-1,i-1} \perp -H \perp \hat{T}),$$

which can be transformed into

$$\begin{aligned}
&(1-p^{-(k+l'+1)})(1+p^{-(k+l')})p^{n-2k+1}(p^{l'}-\epsilon)(p^{l'-1}+\epsilon)\\
&\quad\times\sum_{i=1}^{l}(-1)^{i-1}p^{(i-2)(i-1)/2+(i-1)(n-2k+2)}C_{2l'-2,i-1,\epsilon}\\
&\quad\times(1-p^{-k+l'})(1+p^{-(k+l'-1)})\alpha_p(H_{k+l-1},-\tilde{B}_{l-1,i-1}\perp\hat{T})\\
&=p^{n-2k+1}(p^{l'}-\epsilon)(p^{l'-1}+\epsilon)(1-p^{-(k+l'+1)})(1-p^{-2(k+l')})\\
&\quad\times(1+p^{-(k+l'-1)})\beta_p(H_{k+l-1},-B_{l-1})\alpha_p(H_{k-l-1}\perp B_{l-1},\hat{T})\\
&=p^{n-2k+1}(p^{l'}-\epsilon)(p^{l'-1}+\epsilon)\\
&\quad\times\beta_p(H_{k+l+1},-B_{l-1})\alpha_p(H_{k-l-1}\perp B_{l-1},\hat{T}).
\end{aligned}$$

This proves the assertion (2) in case $n+\deg S_0$ is odd. Next again by (2) of Proposition 5.6 and Lemma 5.3, the assertion (2) can be proved in case $n+\deg S_0$ is odd. □

Proof of Theorem 5.2. We prove the assertion by induction on n. The assertion for $n=1$ follows from (2) of Proposition5.8. Let $n\geq 2$ and assume that the assertion holds for $n-1$. Then by the induction assumption we have

$$\alpha_p(H_{s-t-1}\perp B_t,\hat{T})=\sum_{j=0}^{n-1}a_jp^{tj},$$

and

$$\alpha_p(H_{s-t-2}\perp B_t,\hat{T})=\sum_{j=0}^{n-1}a'_jp^{tj},$$

where $a_j=a_j(s,S_0,\hat{T})$ and $a'_j(s-1,S_0,\hat{T})$ in Theorem 5.2. We may assume that $T=b_1\perp b_2\perp\ldots\perp b_n$ with $\mathrm{ord}_p(b_1)\geq\mathrm{ord}_p(b_2)\geq\ldots\geq\mathrm{ord}_p(b_n)$. First assume that S_0 is of type (M-3) or (M-5). Thus by Proposition 5.8 we have

$$\begin{aligned}
\alpha_p(H_{k-l-1}\perp B_l,T)&=\frac{1-p^{-2k+2l+2}}{1-p^{n-2k+1}}\alpha_p(H_{k-l-2}\perp B_l,\hat{T})\\
&\quad+\frac{p^{n-2k+1}(p^{2l}-1)}{1-p^{n-2k+1}}\alpha_p(H_{k-l-1}\perp B_{l-1},\hat{T})\\
&\quad+p^lK(S_0,T)\alpha_p(H_{k-l-1}\perp B_l,\hat{T})
\end{aligned}$$

which is equal to

$$\frac{1-p^{-2k+2l+2}}{1-p^{n-2k+1}}\sum_{j=0}^{n-1} a'_j p^{lj} + \frac{p^{n-2k+1}(p^{2l}-1)}{1-p^{n-2k+1}}\sum_{j=0}^{n-1} a'_j p^{(l-1)j}$$
$$+ p^l K(S_0,T)\sum_{j=0}^{n-1} a_j p^{lj}.$$

For $0 \le j \le n-1$ put

$$M(j) = \frac{1-p^{-2k+2l+2}}{1-p^{n-2k+1}} a'_j p^{lj} + \frac{p^{n-2k+1}(p^{2l}-1)}{1-p^{n-2k+1}} a'_j p^{(l-1)j} + p^l K(S_0,T) a_j p^{lj}.$$

Then for $j \le n-2, M(j)$ is a polynomial in p^l of degree at most $n-1$. On the other hand,

$$\begin{aligned} M(n-1) &= \frac{1-p^{-2k+2l+2}}{1-p^{n-2k+1}} a'_{n-1} p^{l(n-1)} + \frac{p^{n-2k+1}(p^{2l}-1)}{1-p^{n-2k+1}} a'_{n-1} p^{(l-1)(n-1)} \\ &\quad + a_{n-1} p^l K(S_0,T) p^{l(n-1)} \\ &= a'_{n-1} \frac{1-p^{-2k+2}}{1-p^{n-2k+1}} p^{l(n-1)} + a_{n-1} K(T) p^{ln}. \end{aligned}$$

Thus $\alpha_p(H_{k-l-1} \perp B_l, T)$ is a polynomial in p^l of degree at most n. This proves the assertion in case (M-3) or (M-5). Similarly, the assertion can be proved in the remaining case. □

Remark 5.9. A more careful analysis shows that we have $a_0(k, S_0, T) = 1$ in the above theorem.

Corollary to Theorem 5.2. *Let the notation be as above. For any n-tuple $(l_1, l_2, ..., l_n)$ of complex numbers, put $\mu(l_1, ..., l_n) = \prod_{1 \le j \le i \le n}(p^{l_i} - p^{l_j})$. Then for any integers $0 \le l_1 < ... < l_{n+2} \le k$ and $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$ we have*

$$\sum_{j=1}^{n+2} (-1)^{j-1} \mu(l_1, ..., l_{j-1}, l_{j+1}, ..., l_{n+2}) \alpha_p(H_{k-l_j-1} \perp B_{l_j}, T) = 0.$$

Theorem 5.10. *Let $k \ge n+1$. Let $n+1$ integers $0 \le l_1 ... < l_{n+2} \le k$ be given, let $\lambda_1, ..., \lambda_{n+1}$ be rational numbers such that*

$$\sum_{j=1}^{n+1} \lambda_j \alpha_p(H_{n-l_j+1} \perp B_{l_j+k-n-2}, T) = 0$$

for any $T \in \mathcal{H}_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$. Then we have $\lambda_1 = ... = \lambda_{n+1} = 0$.

Proof. We prove the assertion by induction on n. The assertion for $n = 1$ follows from Proposition 5.8. Let $n \geq 2$, and assume that the assertion holds for $n-1$. The above relation holds for $T = p^{2r} \perp \hat{T}$ with any integer r and $\hat{T} \in \mathcal{H}_{n-1}(\mathbb{Z}_p) \cap GL_{n-1}(\mathbb{Q}_p)$. Then by Proposition 5.8,

$$\sum_{l=1}^{n+1} \lambda_l \{(1-p^{2l-2n-2})\alpha_p(H_{n-l} \perp B_{l+k-n-2}, \hat{T}) \\ + p^{n-2k+1}(p^{2l+2k-2n-4}-1)\alpha_p(H_{n-l+1} \perp B_{l+k-n-3}, \hat{T})\} \\ + p^{(n-2k+1)r} w(\hat{T}) \sum_{l=1}^{n+1} \lambda_l p^{l+k-n-2} \alpha_p(H_{n-l+1} \perp B_{l+k-n-2}, \hat{T}) = 0,$$

where $w(\hat{T})$ is a certain rational number depending only on $\hat{T}$. Thus by taking the limit $r \to \infty$ we obtain

$$\sum_{l=1}^{n+1} \lambda_l \{(1-p^{2l-2n-2})\alpha_p(H_{n-l} \perp B_{l+k-n-2}, \hat{T}) \\ + p^{n-2k+1}(p^{2l+2k-2n-4}-1)\}\alpha_p(H_{n-l+1} \perp B_{l+k-n-3}, \hat{T}) = 0 \qquad (*)$$

and

$$\sum_{l=1}^{n+1} \lambda_l p^{l+k-n-2} \alpha_p(H_{n-l+1} \perp B_{l+k-n-2}, \hat{T}) = 0 \qquad (**).$$

Rewriting (*) we have

$$\sum_{l=1}^{n} (\lambda_l(1-p^{2l-2n-2}) + \lambda_{l+1}p^{n-2k+1}(p^{2l+2k-2n-2}-1)) \\ \times \alpha_p(H_{n-l} \perp B_{l+k-n-2}, \hat{T})\} = 0.$$

Thus by the induction hypothesis, we have

$$\lambda_l(1-p^{2l-2n-2}) + \lambda_{l+1}p^{n-2k+1}(p^{2l+2k-2n-2}-1) = 0$$

for any $l = 0, 1, ..., n$. In particular

$$\lambda_n = -\frac{p^{n-2k+3}(p^{2k-2}-1)}{p^2-1}\lambda_{n+1} \qquad (***).$$

On the other hand, by the Corollary to Theorem 5.2 we have

$$\sum_{l=1}^{n+1} (-1)^{l-1} \mu_l \alpha_p(H_{n-l+1} \perp B_{l+k-n-2}, \hat{T}) = 0 \qquad (****),$$

where $\mu_l = \mu(k-n-1, ..., l+k-n-3, l+k-n-1, ..., k-1)$. By (**) and (****), and the induction hypothesis, we have

$$\lambda_l = (-1)^{l-n-1} \frac{\mu_l}{\mu_{n+1}} p^{-l+n+1} \lambda_{n+1},$$

and in particular

$$\lambda_n = -\frac{\mu_n}{\mu_{n+1}} p \lambda_{n+1} = -\frac{p^n - 1}{p-1} \lambda_{n+1}. \qquad (*****)$$

If $\lambda_{n+1} \neq 0$, (*****) contradicts (***), since $n \geq 2$ and $k \geq n+1$. Thus we have $\lambda_{n+1} = 0$ and therefore $\lambda_l = 0$ for any $l = 1, ..., n+1$. This completes the induction. □

We can now prove Theorem 5.1: We notice first that the genera of lattices of level p on the space of the given lattice are represented by lattices $L^{(i)}$ whose p-adic completions have a Gram matrix that is $\mathbb{Z}_p$-equivalent to $H_{k-i-1} \perp pH_i \perp S_0$ with a fixed S_0 of degree 2 as in Theorem 5.2. Altogether there are $k \geq n+1$ such genera.

As a consequence of Siegel's theorem one sees that the linear independence of any $n+1$ of the degree n theta series of the genera of the $L^{(i)}$ is implied by the linear independence of the corresponding p-adic local density functions $T \mapsto \alpha_p(L^{(i)}, T)$ stated in Theorem 5.10.

Since all the genus theta series are (by Siegel's theorem) in the space of Eisenstein series associated to zerodimensional boundary components (cusps) and since there are $n+1$ such cusps in the case of prime level, it is clear that both types of genus theta series generate the full space of Eisenstein series.

Corollary 5.11. *Let L be a lattice on the quadratic space V over q of level p as in Theorem 5.1 and put $F = \vartheta^{(n)}(gen(L))$. Then the modular form $F|_k T(\ell)$ can be expressed as a linear combination of theta series of positive definite lattices of level p on V for all primes $\ell \neq p$.*

Proof. This is clear from Theorem 5.10 and Theorem 4.3. □

Remark 5.12. a) The result of Theorem 5.1 is more generally true in the case of square free level N, in which case the dimension of the space spanned by the genus theta series becomes $(n+1)^{\omega(N)}$ where $\omega(N)$ is the number of primes dividing N; one has then a basis of genus theta series if one considers $(n+1)^{\omega(N)}$ genera of lattices on the same quadratic space V such that for each p dividing n one has $n+1$ local integral equivalence classes. In that case our proof given above requires the restriction that the anisotropic kernel of the quadratic space under consideration has dimension at most 2. Moreover we can not guarantee the holomorphy of the indefinite genus theta series if the character is trivial (i.e., if the underlying quadratic space has square discriminant). One proceeds in the proof as above, adding an induction on the number of primes $\omega(N)$ dividing N.

b) A different (and much shorter) proof of Theorems 5.2 and 5.10 has been communicated to us by Y. Hironaka and F. Sato [5]. The proof given here gives a little more information (e.g. explicit recursion relations) than theirs. The proof of Hironaka and Sato removes the restriction on the anisotropic kernel mentioned above (if one strengthens the condition on n to $n+1<k$ in the new cases) and provides also a version for levels that are not square free. The application of that version to the study of the space of Eisenstein series generated by the genus theta series in the case of arbitrary level will be the subject of future work.

6. Connection with Kudla's matching principle

In Section 4 we have seen that the Hecke operator $T(p)$ can provide a connection between theta series for lattices in positive definite quadratic space (V_1, q_1) and in a related indefinite quadratic space (V_2, q_2). Such a connection has recently been observed in a different setup by Kudla [10]. We sketch his approach briefly in order to study the relation to our construction, for details we refer to [10], Section 4.1.

Let (V_1, q_1) be a positive definite quadratic space over $\mathbb{Q}$ of dimension m and discriminant d, let (V_2, q_2) be a space of the same dimension m and discriminant d, but of signature $(m-2, 2)$. We fix $n>0$ and an additive character ψ of $\mathbb{Q}_{\mathbb{A}}$. Consider the oscillator representations $\omega_1 = \omega_{1,\psi}$ of $\widetilde{\mathrm{Sp}}_n(\mathbb{A}) \times O_{(V_1,q_1)}(\mathbb{A})$ on the Schwartz space $S((V_1(\mathbb{A}))^n)$ and $\tilde{\omega}$ of $\widetilde{\mathrm{Sp}}_n(\mathbb{A}) \times O_{(V_2,q_2)}(\mathbb{A})$ on $S((V_2(\mathbb{A}))^n)$, where $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ denotes the usual metaplectic double cover of the adelic symplectic group $\mathrm{Sp}_n(\mathbb{A})$.

For $j=1,2$ we have then for $\varphi \in S((V_j(\mathbb{A}))^n)$ the theta kernel

$$\theta(\tilde{g}, h_j; \varphi_j) = \sum_{x \in V_j(\mathbb{Q})} \omega_j(\tilde{g}) \varphi_j(h_j^{-1}x)$$

$$(\tilde{g} \in \widetilde{\mathrm{Sp}}_n(\mathbb{A}), := \ h_j \in O_{(V_j,q)}(\mathbb{A})).$$

and the theta integral

$$I(\tilde{g}; \varphi_j) = \int_{O_{(V_j,q_j)}(\mathbb{Q}) \backslash O_{(V_j,q_j)}(\mathbb{A})} \theta(\tilde{g}, h_j, \varphi_j) dh_j$$

which (under our conditions) is absolutely convergent for $j = 1$ and for $j = 2$ if V_2 is anisotropic or $m > n + 2$.

Let now L_j be a lattice on V_j and assume φ_j to be factored as $\varphi_j = \prod_v \varphi_{j,v}$ over all places v of $\mathbb{Q}$, where $\varphi_{j,p} = \mathbf{1}_{L_{j,p}}$ is the characteristic function of the lattice $L_{j,p}$ in the $\mathbb{Q}_p$-space $V_{j,p}$ for all finite primes p. Then for $\varphi_{1,\infty}(\mathbf{x}) = \exp(-2\pi \operatorname{tr}(q(\mathbf{x})))$ for $\mathbf{x} \in (V_1 \otimes \mathbb{R})^n$ (the Gaussian vector) the intgral $I(\tilde{g}; \varphi_1)$ is the adelic function corresponding to the Siegel modular form

$$\vartheta^{(n)}\ (\operatorname{gen}(L_j), Z)$$

in the usual way.

For the space V_2 we consider two different test functions at infinity: If we choose a fixed majorant ξ of q and put

$$\varphi_{2,\infty,\xi}(\mathbf{x}) = \exp(-2\pi \operatorname{tr}(\xi(\mathbf{x}))) \quad \text{for } \mathbf{x} \in (V_2 \otimes \mathbb{R})^n,$$

the value of the theta kernel

$$\theta(\tilde{g}, \mathbf{1}_{V_2}, \varphi_{2,\infty,\xi} \otimes \prod_{p \neq \infty} \varphi_{2,p})$$

at $h_2 = \mathbf{1}_{V_2}$ corresponds to the theta function

$$\vartheta^{(n)}(L_2, \xi, Z) = \sum_{\mathbf{x} \in L_2^n} \exp(2\pi i \operatorname{tr}(q(\mathbf{x})X)) \exp(-2\pi \operatorname{tr}(\xi(\mathbf{x})Y))$$

(with $Z = X + iY \in \mathfrak{H}_n$) considered by Siegel, and its integral over $O_{(V_{2,q})}(\mathbb{Q}) \setminus O_{(V_{2,q})}(\mathbb{A})$ corresponds to the integral of this theta function over the space of majorants ξ; this is a nonholomorphic modular form in the space of Eisenstein series by Siegel's theorem (or its extension to the Siegel-Weil-Theorem).

Applying a certain differential operator as outlined in [10] to $\varphi_{2,\infty,\xi}$, we obtain a different test function $\varphi'_{2,\infty,\xi}$, and the integral of the theta kernel $\theta(\tilde{g}, h, \varphi'_{2,\infty,\xi} \otimes \prod_{p \neq \infty} \varphi_{2,p})$ over $O_{(V_{2,q})}(\mathbb{Q}) \setminus O_{(V_{2,q})}(\mathbb{A})$ corresponds to the holomorphic theta series of the indefinite lattice L_2 considered by Siegel in [16] and by Maaß in [12] whenever the latter is defined.

To simplify the discussion, we restrict now (following [10]) to $n = 1$. We denote by χ the quadratic character of $\mathbb{Q}_{\mathbb{A}}^\times / \mathbb{Q}^\times$ defined by

$$\chi_v(x) = (x, (-1)^{\frac{m(m-1)}{2}} d)_v$$

for all places v, where $(\ ,\ )_v$ is the Hilbert symbol. Then associated to φ_j there is a unique standard section $\Phi_j\ :=\ \tilde{G}(\mathbb{A}) \times \mathbb{C} \longrightarrow \mathbb{C}$ with

$\Phi_j(\cdot, s) \in I(s, \chi)$, (where $I(s,\chi)$ is the principal series representation of $\tilde{G}(\mathbb{A})$ with parameter s and character χ) such that for $s_0 = \frac{m}{2} - 1$ one has

$$\Phi_j(\tilde{g}, s_0) = (\omega_j(\tilde{g})\varphi_j)(0) =: \lambda_j(\varphi_j).$$

With the Eisenstein series

$$E(\tilde{g}, s; \varphi_j) := \sum_{\gamma \in \tilde{P}_{\mathbb{Q}} \backslash \tilde{G}_{\mathbb{Q}}} \Phi_j(\gamma \tilde{g}, s)$$

associated to Φ_j, the Siegel-Weil theorem asserts that $E(\tilde{g}, s; \varphi_j)$ is holomorphic at $s = s_0$ and that one has the identities

$$E(\tilde{g}, s_0; \varphi_j) = \kappa \cdot I(\tilde{g}; \varphi_j)$$

where $\kappa = 2$ if $m \leqslant 2$ and $\kappa = 1$ otherwise.

The above maps $\lambda_j ::= S(V(\mathbb{A})) \longrightarrow I(s_0, \chi)$ factor into a product $\lambda_j = \prod_v \lambda_{j,v}$ over all places v of $\mathbb{Q}$ and Kudla gives the following definition.

Definition 6.1. (Kudla)

(a) Let v be a (finite or infinite) place of $\mathbb{Q}$, let $V_{1,v}$ and $V_{2,v}$ be quadratic spaces over $\mathbb{Q}_v$ of dimension m and discriminant d. Then functions $\varphi_{1v} \in S(V_{1,v})$ and $\varphi_{2,v} \in S(V_{2,v})$ are said to match if $\lambda_{1,v}(\varphi_{1,v}) = \lambda_{2,v}(\varphi_{2,v})$.

(b) Let V_1, V_2 be quadratic spaces over $\mathbb{Q}$ of the same dimension m and discriminant d. Then two test functions $\varphi_1 \in S(V_1(\mathbb{A}))$ and $\varphi_2 \in S(V_2(\mathbb{A}))$ match, if $\lambda_1(\varphi_1) = \lambda_2(\varphi_2)$. Equivalently, two factorisable test functions $\varphi_1 = \bigotimes_v \varphi_{1,v}, \varphi_2 = \bigotimes_v \varphi_{2,v}$ match if $\varphi_{1,v}$ and $\varphi_{2,v}$ match for all places v.

The matching principle observed by Kudla in [10] then states that for matching test functions $\varphi_1 \in S(V_1(\mathbb{A}))$, $\varphi_2 \in S(V_2(\mathbb{A}))$ one has with $\Phi(\cdot, s_0) = \lambda_1(\varphi_1) = \lambda_2(\varphi_2)$:

$$I(\tilde{g}; \varphi_1) = E(\tilde{g}, s_0, \Phi) = I(\tilde{g}; \varphi_2).$$

Although this identity is a trivial corollary of the Siegel-Weil theorem, the matching principle gives highly nontrivial arithmetical identities since the integrals $I(\tilde{g}, \varphi_1)$ and $I(\tilde{g}, \varphi_2)$ carry completely different arithmetic information; in [10] the principle is exploited to give identities between degrees of certain special cycles on modular varieties and linear combinations of representation numbers of positive definite quadratic forms. Kudla gives in [10] explicit local matching functions at the infinite place and asserts the existence of local matching functions at the finite places for $m > 4$ and for $m = 4$ if $\chi_p \neq 1$.

We can now state the contribution of our computations from the previous sections to this matching principle:

Proposition 6.2. *Let L, V, q be as in the previous sections, let $n = 1$ and let $\varphi_1 = \prod_v \varphi_{1,v} \in S(V(\mathbb{A}))$ be the test function for the positive definite lattice L as described above. Assume that L is of square free odd level N and that all $p|N$ divide the discriminant of L to an odd power. Let χ be the (primitive) quadratic character mod N with $\vartheta(L,q) \in M_k(\Gamma_0(N),\chi)$ and let p be a prime with $\chi(p) = -1$.*

Let $\vartheta(\mathrm{gen}(L))|_{T(p)} = \sum c_i \vartheta(\mathrm{gen}(L_i))$ be the explicit linear combination of theta series of all the positive definite genera of lattices of level N and discriminant in $d \cdot (\mathbb{Q}^\times)^2$ given by the results of Section 5, let ψ_i be the test function attached to the positive definite lattice L_i as above. Let (V_2, q_2) be the quadratic space $\tilde{V}$ of signature $(m-2,2)$ from Lemma 4.2 in Section 4, let $L_2 = \tilde{L}$ in the notation of Lemma 4.2 and let

$$\varphi_2' = \varphi_{2,\infty,\xi}' \otimes \prod_{p \neq 0} \varphi_{2,p}$$

be the test function attached to L_2 as described above.

Then the test functions

$$\psi := \sum_i c_i \psi_i \in S(V_1(\mathbb{A}))$$

and

$$\varphi_2' \in S(V_2(\mathbb{A}))$$

match and we have

$$I(\tilde{g}, \psi) = I(\tilde{g}, \varphi_2').$$

Proof. This is clear from the discussion above and Theorem 4.3. □

Remark 6.3. As already stated in [10] the matching principle can easily be generalized to arbitrary $\widetilde{Sp}_n$. In the range of our results in Sections 4 and 5 we have then examples for the matching principle for general n in the same way as described above.

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THE QUADRATC MEAN OF AUTOMORPHIC L-FUNCTIONS

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Dedicated to the memory of Tsuneo Arakawa

Quadratic mean value theorems of automorphic L-functions on the critical line are proved under very general conditions and several interesting examples are presented. In particular, we establish the expected upper bound for the quadratic mean of the automorphic L-functions attached to Maass cusp forms on the critical line by a simple approach using only the functional equation.

1. Introduction

Mean-value theorems of automorphic L-functions are of great interest and they play an important role in number Theory. Recently, some mean-value theorems (quadratic mean) have been established under very general conditions for a class of general Dirichlet series by Kanemitsu, Sankaranarayanan and Tanigawa (see Theorems 1, 3 and 4 of [15]). The important aspect of these Theorems 3 and 4 is that they give a better error term whenever $\Re s$ is close to 1 with $\Re s < 1$. However (i) and (ii) of Theorem 1 in [15] give just an upper bound for $0 < \Re s < 1$. Improving the quadratic mean error terms whenever $\Re s$ is away from the line $\frac{1}{2}$ for a general Dirichlet series and liftings of automorphic L-functions have been further proved by Ivic and Matsumoto respectively in [8] and [18]. For other interesting mean-value theorems (with upper bounds and some times even an asymptotic formula with an error term) we refer to [6], [7], [9], [12], [13], [14], [17], [22], [23],

[26], [27], [28], and [29].

The aim of this note is to obtain upper bounds for the quadratic mean of automorphic L-functions on the critical line. The approach here is elementary but it includes several interesting examples which will be given later. We remark that the conditions imposed on the Dirichlet series considered in [15] force $\alpha = 0$ there since our functional equation is under $s \mapsto 1 - s$. We *do not* assume $\alpha = 0$ and therefore we have to prove an analogue of Lemma 3 of [15] in our setup i.e. when α may be positive. We give this result in Lemma 3.2 below.

Our setting is the following.

1. Let $a_n \in \mathbb{C}$, $a_n = O(n^{\alpha+\epsilon})$ $\forall$ $\epsilon > 0$, $\alpha \geq 0$ is fixed, be a sequence of complex numbers. Let $Z(s)$ be the corresponding Dirichlet series i.e. $Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, which is absolutely convergent in the region $\Re s > 1 + \alpha$.
2. We suppose that $Z(s)$ has meromorphic continuation to $\mathbb{C}$, has only real poles and satisfies

$$Z(s) = O\left(e^{\gamma|\mathrm{Im} s|}\right) \tag{1.1}$$

in any finite strip $\sigma_1 \leq \Re s \leq \sigma_2$, $\sigma_2 \geq 1 + \alpha$ where $\gamma = \gamma(\sigma_1, \sigma_2)$ is a positive constant.
3. We suppose that $Z(s)$ satisfies a functional equation of the Riemann zeta-type , $s \mapsto 1 - s$, i.e. we have

$$Z(s)\Delta(s) = A_1 A_2^{-s} Z(1-s)\Delta(1-s) \tag{1.2}$$

where A_1 and A_2 are constants, $A_2 > 0$ and $\Delta(s) = \prod_{i=1}^{\mu} \Gamma(\alpha_i + \beta_i s)$ is the gamma factor accompanying $Z(s)$, each β_i is a positive real number and $\alpha_i \in \mathbb{R} \cup i\mathbb{R}$, $\alpha_i \geq 0$ if $\alpha_i \in \mathbb{R}$.
4. Let

$$\eta = \sum_{i=1}^{\mu} \beta_i, \quad H = 2\eta. \tag{1.3}$$

We will assume throughout that $\eta \geq 1$. We will now state our results. Unless otherwise specified δ and ϵ will denote arbitrarily small positive constant which are not necessarily the same at each occurrence. The letter c with or without any suffix denotes a positive constant not necessarily the same at each occurrence. We prove

Theorem 1. *Let $Z(s)$ be as described above. Then*

$$\int_1^T \left| Z\left(\frac{1}{2}+it\right)\right|^2 dt \ll \left(1+|A_1|^2\left(A_2+\frac{1}{A_2}\right)^{1+2\alpha+\delta}\right) T^{\eta(1+2\alpha)+\delta}, \tag{1.4}$$

where the implied constant depends only on δ and η.

Theorem 2. *Let $Z(s)$ be as described above. Furthermore if $Z(s)$ has the property*

$$\sum_{n\le X} |a_n|^2 \le \kappa X(\log X)^B \tag{1.5}$$

where κ is any arbitrary positive constant and B is any arbitrary non-negative integer, then we have

$$\begin{aligned} Q =: &\int_1^T \left| Z\left(\frac{1}{2}+it\right)\right|^2 dt \\ &\ll \left(1+|A_1|^2\left(A_2+\frac{1}{A_2}\right)^2\right)\kappa\left(4^B\right)((B+1)!)\,\eta^{2B+2}T^{\eta}(\log T)^{B+1}. \end{aligned} \tag{1.6}$$

Remark 1. The implied constant in Theorem 2 is effective and absolute.

Remark 2. Our first task is to obtain an approximate equation for $Z(s)$ using only the functional equation. We do this by appropriately modifying the proof of Lemma 3 of [15]. These kind of arguments have already been developed first by Ramachandra and he has used these ideas in many of his papers (see for example [22] and [23]).

Remark 3. In a recent paper (see [28]), Sankaranarayanan has proved an upper bound uniformly in the parameters T and the conductor N for the quantity

$$\sum_f \frac{1}{< f,f >}\int_T^{2T} \left| L\left(\frac{k}{2}+it, f\right)\right|^2 dt \tag{1.7}$$

where f is a holomorphic cusp form (Hecke eigenform) of even integral weight $k>2$ with level N (i.e on the congruence subgroup $\Gamma_0(N)$) having the first Fourier coefficient $a_f(1)=1$. The sum in (1.7) runs over an orthogonal Hecke basis set and the notation $< f,g >$ denotes the Petersson inner product. We remark here if the $L(s,f)$ is the automorphic L-function attached to a Maass form with respect to the congruence subgroup $\Gamma_0(N)$, then following the ideas of the present paper as well as the paper [28], it is possible to prove an upper bound result uniformly in the parameters T and N for the quantity like (1.7) in this situation. However, we do not carryout the details here.

2. Notations and preliminaries

As is customary for complex variables s (resp. w) we write $s = \sigma + it$ (resp. $w = u + iv$) where σ (resp. u) is the real part of s (resp. w) and t (resp. v) is the imaginary part of s (resp. w). If we write the functional equation (1.2) for $Z(s)$ in the form $Z(s) = \chi(s)Z(1-s)$, then using Stirling's formula for the gamma function we have $\mid \chi(s) \mid \asymp \mid t \mid^{\eta(1-2\sigma)} \mid A_1 \mid A_2^{-\sigma}$ as $\mid t \mid \to \infty$ in any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$ (the notation $f \asymp g$ means $f \ll g$ and $g \ll f$). The parameter T will always be chosen to exceed T_0 where T_0 is a sufficiently large number. As in [15] we repeatedly use the well-known result of Montgomery-Vaughan in our estimates. We record this result here as our Lemma 1. We also give the proof of Lemma 2 in this section for the sake of completeness.

3. Some Lemmas

Lemma 3.1. *(Montgomery-Vaughan's mean value theorem). Let $\{h_n\}$ be an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty} n \mid h_n \mid^2$ is convergent. Then*

$$\int_T^{T+H_1} \left| \sum_{n=1}^{\infty} h_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 \left(H_1 + O(n)\right)$$

where $c \leq H_1 \leq T$.

Proof. See for example [21] or [24]. □

Lemma 3.2. *Let h be a fixed positive constant such that $1 - \frac{1+\alpha}{h} > 0$ and let Y and M be positive parameters with $Y \ll \mid t \mid^c, M \ll \mid t \mid^c$ for some positive constant c. Then for $s = \sigma + it$ with $0 < \sigma < 1$, we have,*

$$Z(s) = S - I_1 - I_2 + O\left(\mid t \mid^{-A}\right) \tag{3.1}$$

where

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-\left(\frac{n}{Y}\right)^h}, \tag{3.2}$$

$$I_1 = \frac{1}{2\pi i} \int_{\substack{\Re w = -\epsilon \\ |\mathrm{Im} w| \le \log^2 |t|}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s+w) \left(\sum_{n \le M} a_n n^{-1+s+w} \right) \frac{dw}{w}, \tag{3.3}$$

$$I_2 = \frac{1}{2\pi i} \int_{\substack{\Re w = -\sigma-\alpha-2\epsilon \\ |\mathrm{Im} w| \le \log^2 |t|}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s+w) \left(\sum_{n > M} a_n n^{-1+s+w} \right) \frac{dw}{w}, \tag{3.4}$$

$\epsilon > 0$ being any real number such that $1 - \frac{(\sigma+\alpha+\epsilon)}{h} > 0$ and $A > 0$ is an arbitrarily large constant.

Proof. We have for any $c' > 1 + \alpha$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-(\frac{n}{Y})^h} = \frac{1}{2\pi i} \int_{(c')} \Gamma(1 + \frac{w}{h}) Y^w Z(s+w) \frac{dw}{w}. \tag{3.5}$$

We truncate the line of integration at $| v | = | \mathrm{Im} w | = \log^2 | t |$ and using Stirling's formula and shifting the contour of integration to the line $\Re w = -\sigma - \alpha - 2\epsilon$ we obtain

$$S = Z(s) + I + O\left(| t |^{-A}\right) \tag{3.6}$$

where

$$I = \frac{1}{2\pi i} \int_{\substack{\Re w = -\sigma-\alpha-2\epsilon \\ |\mathrm{Im} w| \le \log^2 |t|}} \Gamma(1 + \frac{w}{h}) Y^w Z(s+w) \frac{dw}{w}$$

and $A > 0$ is an arbitrarily large constant. Using the functional equation of $Z(s)$ we get

$$I = I_1' + I_2$$

where

$$I_1' = \frac{1}{2\pi i} \int_{\substack{\Re w = -\sigma-\alpha-2\epsilon \\ |v| \le \log^2 |t|}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s+w) \left(\sum_{n \le M} a_n n^{-1+s+w} \right) \frac{dw}{w}.$$

We now shift the contour of integration in I_1' to the line $\Re w = -\epsilon$ and this yields

$$I_1' = I_1 + O\left(| t |^{-A}\right).$$

This proves the lemma. □

4. Proof of the Theorems

Proof of Theorem 1. The proof proceeds along the same lines as that of Theorem 1 in [15] i.e., by using the dyadic partition of the interval $[1,T]$ it suffices to obtain upper bounds for the integral of $\mid S\left(\frac{1}{2}+it\right)\mid^2$, $\mid I_1\left(\frac{1}{2}+it\right)\mid^2$ and $\mid I_2\left(\frac{1}{2}+it\right)\mid^2$ over the segment $T\leqslant t\leqslant 2T$. We choose the free parameters Y and M such that $Y=M\geq T$.

Using Lemma 1 we get

$$\begin{aligned}
Q_1 =: & \int_T^{2T}\left|S\left(\frac{1}{2}+it\right)\right|^2 dt \\
\ll & \sum_{n\leqslant Y}|a_n|^2 n^{-1}(T+n)+\sum_{n>Y}|a_n|^2 n^{-1}\left(\frac{Y}{n}\right)^{lh}(T+n) \\
\ll & \sum_{n\leqslant Y}|a_n|^2 n^{-1}(T+n)+\sum_{n>Y}|a_n|^2\left(\frac{Y}{n}\right)^{lh} \\
\ll & T\sum_{n\leq Y}|a_n|^2 n^{-1}+\sum_{n\leq Y}|a_n|^2+\sum_{n>Y}|a_n|^2\left(\frac{Y}{n}\right)^{lh} \qquad (4.1)
\end{aligned}$$

where ℓ is any positive integer. We observe that we can choose h sufficiently large so that $2\alpha+2\epsilon-lh+1<0$ in the third sum of the right-hand side of (4.1). Therefore the right-hand side in the above is

$$\ll TY^{2\alpha+\delta}+Y^{2\alpha+1+\delta}\ll Y^{2\alpha+1+\delta}, \qquad (4.2)$$

since $Y\geqslant T$. Now using the Cauchy-Schwarz inequality we get as in [15] that

$$\begin{aligned}
Q_2 =: & \int_T^{2T}\left|I_1\left(\frac{1}{2}+it\right)\right|^2 \\
\ll & |A_1|^2 A_2^{-1+2\epsilon}Y^{-2\epsilon}T^{H(1-2(\frac{1}{2}-\epsilon))}\int\limits_{\substack{\Re w=-\epsilon\\ |\mathrm{Im}W|\leqslant\log^2|t|}}\int_T^{2T}\left|\sum_{n\leqslant M}a_n n^{-1+s+w}\right|^2 dtdv \\
\ll & |A_1|^2 A_2^{-1+2\epsilon}\frac{T^{3H\epsilon}}{Y^{2\epsilon}}\sum_{n\leq M}|a_n|^2 n^{-1-2\epsilon}(T+n) \\
\ll & |A_1|^2 A_2^{-1+2\epsilon}\frac{T^{3H\epsilon}}{Y^{2\epsilon}}\sum_{n\leq M}n^{2\alpha+2\epsilon-1-2\epsilon}(T+n) \\
\ll & |A_1|^2 A_2^{-1+2\epsilon}\left\{\frac{T^{3H\epsilon+1}}{Y^{2\epsilon}}\sum_{n\leq M}n^{2\alpha-1}+\frac{T^{3H\epsilon}}{Y^{2\epsilon}}\sum_{n\leq M}n^{2\alpha}\right\}
\end{aligned}$$

$$\ll |A_1|^2 A_2^{-1+2\epsilon} \begin{cases} \frac{T^{3H\epsilon+1}}{Y^{2\epsilon}} \log M + \frac{T^{3H\epsilon}}{Y^{2\epsilon}} M \text{ if } \alpha = 0 \\ \frac{T^{3H\epsilon+1}}{Y^{2\epsilon}} M^{2\alpha} + \frac{T^{3H\epsilon}}{Y^{2\epsilon}} M^{2\alpha+1} \text{ if } \alpha > 0 \end{cases}$$

$$\ll |A_1|^2 A_2^{-1+2\epsilon} \begin{cases} \frac{T^{3H\epsilon}}{Y^{2\epsilon}} (T \log M + M) \text{ if } \alpha = 0 \\ \frac{T^{3H\epsilon}}{Y^{2\epsilon}} (TM^{2\alpha} + M^{2\alpha+1}) \text{ if } \alpha > 0. \end{cases}$$

$$\ll |A_1|^2 A_2^{-1+2\epsilon} \begin{cases} T^{3H\epsilon} M \text{ if } \alpha = 0 \\ T^{3H\epsilon} M^{2\alpha+1} \text{ if } \alpha > 0. \end{cases}$$

$$\ll |A_1|^2 A_2^{-1+2\epsilon} Y^{1+2\alpha+\delta}, \tag{4.3}$$

since $Y = M \geq T$. Similarly we obtain,

$$\int_T^{2T} \left| I_2 \left(\frac{1}{2} + it \right) \right|^2 \ll |A_1|^2 A_2^{2\alpha+4\epsilon} \left(\frac{T^H}{Y} \right)^{1+2\alpha} \cdot T^\delta. \tag{4.4}$$

Note that $Y = M$ and we choose $YM = T^H$ i.e., $M^2 = T^H$ or $M = T^{\frac{H}{2}} = T^\eta$. We therefore finally obtain

$$\int_T^{2T} \left| Z \left(\frac{1}{2} + it \right) \right|^2 dt \ll \left(1 + |A_1|^2 \left(A_2^{-1+2\epsilon} + A_2^{2\alpha+4\epsilon} \right) \right) T^{\eta(2\alpha+1)+\delta}. \tag{4.5}$$

We note that (for $A_2 > 0$)

$$\max \left(A_2, \frac{1}{A_2} \right) \leq A_2 + \frac{1}{A_2}.$$

This concludes the proof of Theorem 1.

Proof of Theorem 2. We observe that the condition imposed on the coefficients $\{a_n\}$ in Theorem 2 guarantees that both the series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}$ and $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = Z(s)$ are absolutely convergent for $\sigma > 1$. Utilising the latter we get the following modification of Lemma 2.

Lemma 4.1. *Let $Z(s)$ be as above and retain all the notations of Lemma 2. For $s = \sigma + it$, with $0 < \sigma < 1$, we have*

$$Z = S - I_1'' - I_2'' + O\left(|t|^{-A} \right)$$

where

$$I_1'' = \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -\epsilon_1 \\ |\mathrm{Im} w| \leqslant \log^2 |t|}} \Gamma \left(1 + \frac{w}{h} \right) Y^w \chi(s+w) \left(\sum_{n \leq M} a_n n^{-1+s+w} \right) \frac{dw}{w},$$

$$I_2'' = \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -\sigma - 2\epsilon_1 \\ |\mathrm{Im} w| \leqslant \log^2 |t|}} \Gamma\left(1 + \frac{w}{h}\right) Y^w \chi(s+w) \left(\sum_{n>M} a_n n^{-1+s+w}\right) \frac{dw}{w}.$$

Here $\epsilon_1 = \frac{10}{\eta \log T} > 0$ *is a real number such that* $1 - \frac{(\sigma + 2\epsilon_1)}{h} > 0$ *and* $A > 0$ *is an arbitrarily large positive constant.*

Proof. The proof is entirely similar to the proof of Lemma 2 and is therefore omitted. □

We continue with the proof of Theorem 2. The idea behind the proof of this theorem is to obtain improved estimates for the integral of $\left|S\left(\frac{1}{2} + it\right)\right|^2$, $\left|I_1\left(\frac{1}{2} + it\right)\right|^2$ etc. by utilising the condition above of the asymptotic behaviour of the sum $\sum_{n \leqslant X} |a_n|^2$. The condition implies that the Dirichlet series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}$ is absolutely convergent in the region $\Re s > 1$. Here also, we choose our free parameters Y and M such that $Y = M \geq T$. Note that $\epsilon_1 = \frac{10}{\eta \log T} > 0$ and B is any arbitrary non-negative integer. As before, we observe that

$$\begin{aligned}
Q_3 =: & \int_T^{2T} \left|S\left(\frac{1}{2} + it\right)\right|^2 dt \\
&= \sum_{n=1}^{\infty} \frac{|a_n|^2}{n} e^{-2\left(\frac{n}{Y}\right)^{lh}} (T + O(n)) \\
&= \sum_{n \leq Y} \cdots + \sum_{n > Y} \cdots \\
&\ll T \sum_{n \leq Y} \frac{|a_n|^2}{n} + \sum_{n \leq Y} |a_n|^2 + Y^{lh} \sum_{n > Y} \frac{|a_n|^2}{n^{lh}} \\
&\ll T \int_1^Y u^{-1}\, d\left(\sum_{n \leq u} |a_n|^2\right) + Y^{lh} \int_Y^{\infty} u^{-lh}\, d\left(\sum_{n \leq u} |a_n|^2\right) \\
&\quad + \kappa Y (\log Y)^B \\
&\ll \kappa \left\{T (\log Y)^{B+1} + Y^{lh} \left(Y^{1-lh} (\log Y)^B\right) + Y (\log Y)^B\right\} \\
&\ll \kappa \left\{T (\log Y)^{B+1} + Y (\log Y)^B\right\} \\
&\ll \kappa Y (\log Y)^B, \qquad (4.6)
\end{aligned}$$

by choosing h sufficiently large such that $lh > 10$ and using integration by parts. Now

$$\begin{aligned}
Q_4 =: &\int_T^{2T} \left| I_1''\left(\frac{1}{2}+it\right)\right|^2 dt \\
&\ll |A_1|^2 A_2^{-1+2\epsilon_1} T^{10\eta\epsilon_1} \sum_{n\leqslant M} |a_n|^2 n^{-1-2\epsilon_1}(T+n) \\
&\ll |A_1|^2 A_2^{-1+2\epsilon_1} T^{10\eta\epsilon_1} \\
&\quad \times \left\{ T\int_1^M u^{-1-2\epsilon_1}\, d\left(\sum_{n\leq u}|a_n|^2\right) + \int_1^M u^{-2\epsilon_1} d\left(\sum_{n\leq u}|a_n|^2\right)\right\} \\
&\ll |A_1|^2 A_2^{-1+2\epsilon_1} \kappa T^{10\eta\epsilon_1} \left\{ \frac{T(\log M)^B}{M^{2\epsilon_1}} + (1+2\epsilon_1)\, T(\log M)^{B+1}\right\} \\
&\quad + |A_1|^2 A_2^{-1+2\epsilon_1} \kappa T^{10\eta\epsilon_1} \left\{ M^{1-2\epsilon_1}(\log M)^B + 2\epsilon_1 M^{1-2\epsilon_1}(\log M)^B\right\} \\
&\ll |A_1|^2 A_2^{-1+2\epsilon_1} \kappa M(\log M)^{B+1} \\
&\ll |A_1|^2 A_2^{-1+2\epsilon_1} \kappa Y(\log Y)^{B+1}, \qquad (4.7)
\end{aligned}$$

by using integration by parts and since $Y = M \geqslant T$. We observe the simple inequality (for $u \geq 10$)

$$\frac{(\log u)^B}{u^{2\epsilon_1}} \leq \frac{(B+1)!}{(2\epsilon_1)^{B+1}} \leq \frac{(B+1)!}{(\epsilon_1)^{B+1}}.$$

Therefore, we have (on using integration by parts)

$$\begin{aligned}
\sum_{n>M} |a_n|^2 n^{-2-4\epsilon_1} &= \int_M^\infty u^{-2-4\epsilon_1}\, d\left(\sum_{n\leq u}|a_n|^2\right) \\
&\ll \kappa \frac{(B+1)!}{(\epsilon_1)^{B+1}} \left(\frac{1}{M^{1+2\epsilon_1}} + (2+4\epsilon_1)\,\frac{1}{M^{1+2\epsilon_1}}\right) \\
&\ll \kappa((B+1)!)\eta^{B+1} \frac{(\log T)^{B+1}}{M} \\
&\ll \kappa((B+1)!)\eta^{B+1} \frac{(\log M)^{B+1}}{M}. \qquad (4.8)
\end{aligned}$$

We also observe that (on using integration by parts again)

$$\begin{aligned}
\sum_{U<n\leq 2U} |a_n|^2 n^{-1-4\epsilon_1} &= \int_U^{2U} u^{-1-4\epsilon_1}\, d\left(\sum_{n\leq u}|a_n|^2\right) \\
&\ll \kappa \frac{(\log(2U))^B}{U^{4\epsilon_1}}
\end{aligned}$$

$$\ll 2^B \kappa \frac{(\log(U))^B}{U^{4\epsilon_1}}. \tag{4.9}$$

We note the following simple inequalities. If a and b are any two non-negative real numbers such that $a+b \geq 10$ and B is any arbitrary non-negative integer, then we have

$$(a+b)^B \leq \max\left((2a)^B, (2b)^B\right) \leq 2^B\left(a^B + b^B\right) \tag{4.10}$$

and also for any integer $j \geq 1$,

$$2^{2j\epsilon_1} \geq e^{j\epsilon_1} \geq \frac{(j\epsilon_1)^B}{(B)!} = \frac{j^B (\epsilon_1)^B}{B!}. \tag{4.11}$$

Therefore, from (4.9), (4.10) and (4.11), we infer that

$$\begin{aligned}
Q_5 &=: \sum_{n>M} |a_n|^2 n^{-1-4\epsilon_1} \\
&= \sum_{\substack{U=2^j M \\ j=0 \text{ to } \infty}} \sum_{U<n\leq 2U} |a_n|^2 n^{-1-4\epsilon_1} \\
&\ll 2^B \kappa \sum_{\substack{U=2^j M \\ j=0 \text{ to } \infty}} \frac{(\log(U))^B}{U^{4\epsilon_1}} \\
&\ll 2^B \kappa \sum_{j=0}^{\infty} \frac{\left(\log\left(2^j M\right)\right)^B}{\left(2^j M\right)^{4\epsilon_1}} \\
&\ll 4^B \kappa \sum_{j=0}^{\infty} \frac{\left((j\log 2)^B + (\log M)^B\right)}{\left(2^j M\right)^{4\epsilon_1}} \\
&\ll 4^B \kappa \left\{ \frac{(\log M)^B}{M^{4\epsilon_1}} + \frac{1}{M^{4\epsilon_1}} \sum_{j=1}^{\infty} \frac{j^B}{2^{4j\epsilon_1}} + \frac{(\log M)^B}{M^{4\epsilon_1}} \sum_{j=1}^{\infty} \frac{1}{2^{4j\epsilon_1}} \right\} \\
&\ll 4^B \kappa \left\{ \frac{(\log M)^B}{M^{4\epsilon_1}} + \frac{1}{M^{4\epsilon_1}} \frac{(B!)}{(\epsilon_1)^B} \sum_{j=0}^{\infty} \frac{1}{2^{2j\epsilon_1}} + \frac{(\log M)^B}{M^{4\epsilon_1}} \sum_{j=0}^{\infty} \frac{1}{2^{4j\epsilon_1}} \right\} \\
&\ll 4^B \kappa \left\{ \frac{(\log M)^B}{M^{4\epsilon_1}} + \frac{1}{M^{4\epsilon_1}} \frac{(B!)}{(\epsilon_1)^B} \frac{2^{2\epsilon_1}}{\epsilon_1} + \frac{(\log M)^B}{M^{4\epsilon_1}} \frac{2^{4\epsilon_1}}{\epsilon_1} \right\} \\
&\ll 4^B \kappa \ (B!)\, \eta^{B+1} (\log M)^{B+1}.
\end{aligned} \tag{4.12}$$

Hence, from (4.8) and (4.12), we deduce

$$Q_6 =: \int_T^{2T} \left| I_2''\left(\frac{1}{2} + it\right) \right|^2 dt$$

$$
\begin{aligned}
&\ll |A_1|^2 A_2^{4\epsilon_1} Y^{-1-4\epsilon_1}\, T^{2\eta+4\epsilon_1 H} \times \int\limits_{\substack{u=-\frac{1}{2}-2\epsilon_1 \\ |v|\leqslant \log^2|t|}} \int_T^{2T} \left| \sum_{n>M} a_n n^{-1+s+w} \right|^2 dt\, dv \\
&\ll |A_1|^2 A_2^{4\epsilon_1} Y^{-1-4\epsilon_1}\, T^{2\eta+6\epsilon_1 H} \sum_{n>M} |a_n|^2\, n^{-2-4\epsilon_1}(T+n) \\
&\ll |A_1|^2 A_2^{4\epsilon_1} \frac{T^{2\eta+12\eta\epsilon_1}}{Y^{1+4\epsilon_1}} \\
&\quad \times \left\{ T\kappa\eta^{B+1}(B+1)!\frac{(\log M)^{B+1}}{M} + 4^B\kappa\eta^{B+1}(B!)(\log M)^{B+1} \right\} \\
&\ll |A_1|^2 A_2^{4\epsilon_1} \kappa\eta^{B+1} 4^B\, ((B+1)!)\, (\log M)^{B+1} \frac{T^{2\eta}}{Y} \\
&\ll |A_1|^2 A_2^{4\epsilon_1} \kappa\eta^{B+1} 4^B\, ((B+1)!)\, \frac{T^{2\eta}}{Y} (\log Y)^{B+1}.
\end{aligned} \tag{4.13}
$$

From (4.6), (4.7) and (4.13), we obtain

$$
\begin{aligned}
Q &=: \int_T^{2T} \left| Z\left(\frac{1}{2}+it\right) \right|^2 dt \\
&\ll \left(1+|A_1|^2\left(A_2^{-1+2\epsilon_1} + A_2^{4\epsilon_1}\right)\right) \kappa\eta^{B+1}\left(4^B\right)((B+1)!)\left\{ Y + \frac{T^{2\eta}}{Y} \right\} (\log Y)^{B+1}.
\end{aligned}
$$

We now set $Y = T^\eta$ (note that $\eta \geq 1$) and finally get

$$
\begin{aligned}
Q &=: \int_T^{2T} \left| Z\left(\frac{1}{2}+it\right) \right|^2 dt \\
&\ll \left(1+|A_1|^2\left(A_2+\frac{1}{A_2}\right)^2\right) \kappa\left(4^B\right)((B+1)!)\,\eta^{2B+2} T^\eta (\log T)^{B+1}.
\end{aligned}
$$

This completes the proof of Theorem 2.

5. Examples

(1) Let f be a Maass cusp form for $\mathrm{SL}_2(\mathbb{Z})$ which is a normalised eigenfunction of all the Hecke operators $T(n)$ $(n \in \mathbb{N})$ as well as the reflection operator T_{-1} where $T_{-1}(z) = -\bar{z}$ for all z in the upper half-plane and let $L(f,s)$ be its standard L function. We have $\eta = 1$ in this case. The best known value of α is $\frac{7}{64}$ which is due to Kim and Sarnak (cf. [16]). Thus Theorem 1 implies that

$$
\int_1^T \left| L\left(f, \frac{1}{2}+it\right) \right|^2 dt \ll T^{\frac{39}{32}+\delta}. \tag{5.1}
$$

We write the Fourier expansion of f as

$$f(z) = \sum_{n \neq 0} \rho(n) W_{\frac{1}{2}+i\chi}(nz). \tag{5.2}$$

Here $W_s(z)$ is the standard Whittakar function and $\rho(n)$ is the n^{th} Fourier coefficient. We normalise $\rho(n)$ by letting

$$\nu(n) = \left(\frac{4\pi|n|}{\cosh \pi\chi}\right)^{\frac{1}{2}} \rho(n). \tag{5.3}$$

We have

$$\nu(n) = \lambda(n)\nu(1) \quad \text{and} \quad T(n)f = \lambda(n)f, \tag{5.4}$$

where $\lambda(n)$ is the n^{th} Hecke eigenvalue. From the asymptotic formula (see (8.17) page 110 of [11]) we deduce that

$$\sum_{n \leq X} |\nu(n)|^2 = |\nu(1)|^2 \sum_{n \leq X} |\lambda(n)|^2 = CX + \left(X^{\frac{7}{8}}\right) \tag{5.5}$$

and hence it follows that

$$\sum_{n \leq X} |\lambda(n)|^2 = C_f X + \left(X^{\frac{7}{8}}\right). \tag{5.6}$$

where $C_f = \frac{C}{|\nu(1)|^2} > 0$. From (5.6), we note that we can take $B = 0$ in Theorem 2 in this situation. Thus Theorem 2 gives that

$$\int_1^T \left| L\left(f, \frac{1}{2} + it\right) \right|^2 dt \ll T\,(\log T), \tag{5.7}$$

for a normalised Maass cusp form f. Kuznetsov [17] has proved an asymptotic formula for the above second moment. We are here interested in getting an upper bound and up to the effective implied constant our upper bound is the same as that obtained by Kuznetsov. Our proof is elementary and simple in contradistinction to that of Kuznetsov which is long and complicated.

(2) Let f be a holomorphic cusp form of half-integral weight $k + \frac{1}{2}$ of level N with $4|N$ in the sense of Shimura [30]. Let us assume that f is an eigenfunction of the usual Fricke involution induced by the action of $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Let

$$L(f, s) = \sum_{n \geqslant 1} a_n n^{-s} \qquad (\Re(s) \gg 0) \tag{5.8}$$

be the Hecke L-function of f, where a_n is the n^{th} Fourier coefficient of f. Note that $L(f,s)$ in general does not have an Euler product, even if f is a Hecke eigenform. We put

$$\tilde{L}(f,s) := L\left(f, s + \frac{k}{2} - \frac{1}{4}\right). \tag{5.9}$$

According to [30] (see p. 481), $\tilde{L}(f,s)$ when completed with one Γ-factor has holomorphic continuation to the entire complex plane, satisfies condition (1.1) and is invariant under $s \mapsto 1-s$ (the proof works in the same way as in the integral weight case). We have $\eta = 1$.

The Ramanujan-Petersson conjecture is not known for the individual coefficients a_n (the best bound known so far seems to be $a_n \ll_{f,\epsilon} n^{\frac{k}{2}-\frac{1}{28}+\epsilon}$ in case $k \geqslant 2$ (cf. [10]). However, by the Rankin-Selberg method it is true on average, i.e. we have

$$\sum_{n \leqslant X} |a_n|^2 \ll X^{1+\epsilon}. \tag{5.10}$$

Therefore, by an argument analogous to the argument leading to Theorem 2 (with the log-power replaced by an X^{ϵ}), we deduce that

$$\int_1^T \left|\tilde{L}\left(f, \frac{1}{2} + it\right)\right|^2 dt \ll T^{1+\epsilon}. \tag{5.11}$$

(3) Let f be a cuspidal Hecke eigenform of integral weight k with respect to the full Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{GL}_{2g}(\mathbb{Z})$ of genus g. We will suppose that $k \geqslant g$. For a prime number p, we denote by $\alpha_{0,p}, \ldots, \alpha_{g,p}$ the Satake p-parameters attached to f (determined up to the action of the Weyl group). We let

$$L_{st}(f,s) = \zeta(s) \prod_p \prod_{j=1}^{g} \left(1 - \alpha_{j,p} p^{-s}\right)^{-1} \left(1 - \alpha_{j,p}^{-1} p^{-s}\right)^{-1} \qquad (\Re(s) > g+1) \tag{5.12}$$

the standard zeta function associated to f. (Note that in the case $g = 1$, one has $L_{st}(f,s) = L(\mathrm{sym}^2 f, s + k - 1)$.)

Put

$$\gamma_g(s) := \prod_{j=0}^{g-1} \Gamma\left(s - \frac{j}{2}\right) \tag{5.13}$$

and define

$$L_{st}^{*}(f,s) := \pi^{-(g+1)s}\Gamma\left(\frac{s+\xi}{2}\right)\gamma_g\left(\frac{s+k-1}{2}\right)\gamma_g\left(\frac{s+k}{2}\right)L_{st}(f,s) \tag{5.14}$$

where

$$\xi = \begin{cases} 0, & \text{if } g \equiv 0 \pmod 2 \\ 1, & \text{if } g \equiv 1 \pmod 2). \end{cases} \tag{5.15}$$

Then it is well-known (cf. e.g. [19] and the literature given there) that $L_{st}^{*}(f,s)$ has meromorphic continuation to the entire complex plane with poles occurring at most at $s = 0, 1$, is invariant under $s \mapsto 1-s$ and that the function $L_{st}(f,s)$ satisfies the growth condition imposed in (1.1).

Clearly we have $\eta = g + \frac{1}{2}$.

Write a_n $(n \in \mathbb{N})$ for the general coefficient of the Dirichlet series $L_{st}(f,s)$, $(\Re(s) > g+1)$. It was proved in [3] (cf. Lemma 3.1 and the proof of Propos. 5.6) that $a_n = O\left(n^{\frac{2}{3}g+1+\epsilon}\right)$ for all g and in fact $a_n = O\left(n^{\frac{g}{2}+\epsilon}\right)$ if $g = 2^r$ $(r \in \mathbb{N})$ is a 2-power.

Therefore Theorem 1 implies that

$$\int_1^T \left|L_{st}\left(f, \frac{1}{2}+it\right)\right|^2 dt \ll T^{(g+\frac{1}{2})(1+2\alpha)+\delta} \tag{5.16}$$

where one can take

$$\alpha = \begin{cases} \frac{2}{3}g, & \text{if } g \geqslant 1 \\ \frac{g}{2}, & \text{if } g = 2^r,\ r \in \mathbb{N}. \end{cases} \tag{5.17}$$

(4) Now suppose that f is a cuspidal Hecke eigenform of integral weight k of genus 2 (cf. above) and for $\Re(s) > k+1$ denote by

$$L_{\text{spin}}(f,s) = \prod_p \left(\left(1-\frac{\alpha_{0,p}}{p^s}\right)\left(1-\frac{\alpha_{0,p}\alpha_{1,p}}{p^s}\right)\left(1-\frac{\alpha_{0,p}\alpha_{2,p}}{p^s}\right)\left(1-\frac{\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}}{p^s}\right)\right)^{-1} \tag{5.18}$$

the spinor zeta function associated to f. Put

$$L_{\text{spin}}^{*}(f,s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L_{\text{spin}}(f,s). \tag{5.19}$$

We normalize by setting

$$\tilde{L}_{\text{spin}}(f,s) := L_{\text{spin}}\left(f, s+k-\frac{3}{2}\right) \tag{5.20}$$

and

$$\tilde{L}^*_{\mathrm{spin}}(f,s) := L^*_{\mathrm{spin}}\left(f, s+k-\frac{3}{2}\right). \tag{5.21}$$

By [1] then the latter function has meromorphic continuation to the entire complex plane with poles at most occurring at $s = \frac{3}{2}, -\frac{1}{2}$, is $(-1)^k$-invariant under $s \mapsto 1-s$ and $\tilde{L}_{\mathrm{spin}}(f,s)$ satisfies (1.1). We have $\eta = 2$.
According to [3] (cf. Propos. 5.4) one has $a_n = O\left(n^{k-1+\epsilon}\right)$ where a_n is the general coefficient of $L_{\mathrm{spin}}(f,s)$. We therefore can take $\alpha = \frac{1}{2}$ and thus from Theorem 1 deduce the bound

$$\int_1^T \left|\tilde{L}_{\mathrm{spin}}\left(f, \frac{1}{2}+it\right)\right|^2 dt \ll T^{4+\delta}. \tag{5.22}$$

We remark that a result of Weissauer asserts the truth of the Ramanujan-Petersson conjecture for f (i.e. $\alpha = 0$) in case f is not a Maass lift. However, no complete proof so far has appeared in the literature.

(5) Let f be a Maass cusp form on $\mathrm{SL}(2,\mathbb{Z})$ which is a normalised Hecke eigenform as in Example 1. Let (α_p, β_p) be the p-Satake parameters of f, i.e. $\alpha_p + \beta_p = \lambda(p)$ and $\alpha_p\beta_p = 1$ where $T(p)f = \lambda(p)f$, p a prime (the pair (α_p, β_p) is unique upto permutation). The symmetric square L-function of f is defined by the Euler product

$$L\left(\mathrm{sym}^2 f, s\right) = \prod_p \left\{\left(1-\alpha_p^2 p^{-s}\right)\left(1-p^{-s}\right)\left(1-\beta_p^2 p^{-s}\right)\right\}^{-1}. \tag{5.23}$$

This product converges absolutely for $\Re(s)$ large and the resulting function has analytic continuation to the entire complex plane and satisfies the functional equation (cf. [25]),

$$L_\infty\left(\mathrm{sym}^2 f, s\right) L\left(\mathrm{sym}^2 f, s\right) = \pm L_\infty\left(\mathrm{sym}^2 f, 1-s\right) L\left(\mathrm{sym}^2 f, 1-s\right) \tag{5.24}$$

where

$$L_\infty\left(\mathrm{sym}^2 f, s\right) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s}{2}+i\chi\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-i\chi\right) \tag{5.25}$$

and $\frac{1}{4} + \chi^2 = \lambda$, λ being the Laplace eigenvalue of f. Note that $\lambda > \frac{1}{4}$ and therefore χ is real and without loss of generality can be taken to be positive. We remark that there is a typographical error in the definition of χ (denoted as t in [25]) in [25]. The t there should be defined as $\lambda = \frac{1}{4} + t^2$ and *not as* $\lambda = \frac{1-t^2}{4}$ as given in [25]. In this situation, we have $\eta = \frac{3}{2}$.

It is well known (see [5]) that f has a lift to a cusp form F on GL(3) whose standard L-function is $L(s,F) = L\left(\mathrm{sym}^2 f, s\right)$. Letting $a(m,n)$ denote the Fourier coefficient of F (vide [25] and [2]) as is customary, we have (see [4]) that the Rankin-Selberg convolution of F with itself, namely $L(s, F \otimes F)$ is given by

$$L(s, F \otimes F) = \zeta(3s) \left(\sum_{m,n=1}^{\infty} \frac{a(m,n)\overline{a(m,n)}}{(m^2 n)^s} \right),$$
$$= \zeta(3s) \left(\sum_{m,n=1}^{\infty} \frac{|a(m,n)|^2}{(m^2 n)^s} \right), \quad \Re(s) \gg 1. \tag{5.26}$$

Note that in this notation we have,

$$L\left(\mathrm{sym}^2 f, s\right) = \sum_{n=1}^{\infty} \frac{a(1,n)}{n^s}. \tag{5.27}$$

$L(s, F \otimes F)$ has meromorphic continuation to the entire complex plane (see [4] and [20]) and satisfies a Riemann type functional equation. The only singularity of $L(s, F \otimes F)$ in the half-plane $\Re(s) > 0$ is a simple pole at $s = 1$. Standard number theoretic techniques then show that

$$\sum_{m^2 n \le X} |a(m,n)|^2 \ll X \quad \text{as} \quad X \to \infty, \tag{5.28}$$

which implies that

$$\sum_{n \le X} |a(1,n)|^2 \ll X. \tag{5.29}$$

Therefore, we can take $B = 0$ in (1.5) in the case of $L\left(\mathrm{sym}^2 f, s\right)$ and hence from Theorem 2, we obtain

$$\int_1^T \left| L\left(\mathrm{sym}^2 f, \frac{1}{2} + it\right) \right|^2 dt \ll T^{\frac{3}{2}} (\log T). \tag{5.30}$$

It should be mentioned here that a similar result was proved in [27] when f is a holomorphic cusp form for the full modular group.

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INNER PRODUCT FORMULA FOR KUDLA LIFT

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To the memory of Tsuneo Arakawa

Let f be a cuspidal Hecke eigenform on $U(1,1)$ and $\mathcal{L}f$ the Kudla lift of f (a theta lift from $U(1,1)$ to $U(2,1)$). In this paper, we study the Petersson norm of $\mathcal{L}f$. As an application, we give a criterion for the non-vanishing of $\mathcal{L}f$.

0. Introduction

0.1.

Let K be an imaginary quadratic field of discriminant D with integer ring $\mathcal{O}_K$. For simplicity, we assume that the class number of K is one in the introduction. Let f be a holomorphic cusp form on $\Gamma_0(D)$ of weight $l-1$ and character $\left(\dfrac{D}{*}\right)$. Let $U(S)$ be the unitary group of a hermitian matrix

$$S = \begin{pmatrix} & & 1/\sqrt{D} \\ & 1 & \\ -1/\sqrt{D} & & \end{pmatrix}$$

of signature $(2,1)$. Kudla ([6]) constructed a holomorphic cusp form $\mathcal{L}f$ on $\Gamma = U(S) \cap \mathrm{GL}_3(\mathcal{O}_K)$ of weight l as a theta lift of f. He also showed that $\mathcal{L}f$ is a Hecke eigenform if so is f.

The object of the paper is to show that, for a Hecke eigenform f, the square of the Petersson norm of $\mathcal{L}f$ is expressed essentially in terms of

$L(f;1)$, where $L(f;s)$ is the standard L-function of f. As an application, we obtain a criterion for the non-vanishing of $\mathcal{L}f$.

0.2.

The paper is organized as follows. In Section 1, after preparing several notations, we state the main results of the paper; an inner product formula for Kudla lift and a non-vanishing criterion (Theorem 1.1). In Section 2, we briefly recall the theory of metaplectic representations of unitary groups. The Kudla lift is introduced in Section 3. By using the see-saw dual reductive pair and the Siegel-Weil formula proved in [14], we reduce the proof of the inner product formula to the calculation of certain zeta integral introduced in Section 4. In Section 5 and Section 6, we recall the basic identity and several basic facts about local spherical functions. These enable us to decompose the zeta integral into a product of local integrals of local spherical functions in Section 7. Section 8 and Section 9 are devoted to the local calculations in the non-archimedean case. In Section 10, we calculate the local integral at the archimedean prime.

1. Main results

1.1.

Let K be an imaginary quadratic field of discriminant D with integer ring $\mathcal{O}_K$. We fix an embedding of K into $\mathbf{C}$ and let $\kappa = \sqrt{D}$ be the square root of D with positive imaginary part. Denote by σ the nontrivial automorphism of $K/\mathbf{Q}$. For $x \in K$, we put $\mathrm{Tr}(x) = x + x^\sigma$ and $\mathrm{N}(x) = xx^\sigma$. When $x \neq 0$, we write $x^{-\sigma}$ for $(x^\sigma)^{-1}$. Set $K^1 = \{t \in K^\times \mid tt^\sigma = 1\}$. For a prime v of K, let $K_v = K \otimes_{\mathbf{Q}} \mathbf{Q}_v$. If $v = p$ is finite, we put $\mathcal{O}_{K,p} = \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_p \subset K_p$. When p ramifies in $K/\mathbf{Q}$, we fix a prime element Π of K_p. When p splits in $K/\mathbf{Q}$, we fix an identification K_p with $\mathbf{Q}_p \oplus \mathbf{Q}_p$ and put $\Pi_1 = (p,1)$ and $\Pi_2 = (1,p)$. We denote by $\mathbf{Q}_\mathbf{A}$ and $K_\mathbf{A}$ the adele rings of $\mathbf{Q}$ and K respectively. Let $K_{\mathbf{A},f}$ (resp. K_∞) be the finite (resp. the infinite) part of the adele ring $K_\mathbf{A}$, and put $\mathcal{O}_{K,f} = \prod_{p<\infty} \mathcal{O}_{K,p} \subset K_{\mathbf{A},f}$. For $a \in K_v^\times$, we put $||a||_v = |\mathrm{N}(a)|_v$, where $|\cdot|_v$ stands for the absolute value of $\mathbf{Q}_v$. For $a = (a_v) \in K_\mathbf{A}^\times$, set $||a|| = \prod_v ||a_v||_v$.

By an ideal of K, we always mean a nonzero fractional ideal of K. Denote by h_K and w_K the class number of K and the number of roots of unity in K respectively. Let ω be the quadratic Hecke character of $\mathbf{Q}$ corresponding to $K/\mathbf{Q}$. Denote by $\mathcal{X}$ the set of unitary Hecke characters χ

of K with $\chi|_{\mathbf{Q}_\mathbf{A}^\times} = \omega$. For $\chi \in \mathcal{X}$, define an integer $w_\infty(\chi)$ to be $\chi(z_\infty) = (z_\infty/|z_\infty|)^{w_\infty(\chi)}$ for $z_\infty \in K_\infty^\times$.

Let ψ be the additive character of $\mathbf{Q}_\mathbf{A}/\mathbf{Q}$ with $\psi(x_\infty) = \mathbf{e}[x_\infty] := \exp(2\pi i x_\infty)$ for $x_\infty \in \mathbf{R}$. For each prime v of $\mathbf{Q}$, we write ψ_v for the restriction of ψ to $\mathbf{Q}_v$. A diagonal matrix of degree n with the i-th diagonal component a_i $(1 \leq i \leq n)$ is denoted by $\mathrm{diag}(a_1, \ldots, a_n)$.

1.2.

In this subsection, we normalize various Haar measures. Let v be a prime of $\mathbf{Q}$. Let dx_v (resp. dz_v) be the Haar measure on $\mathbf{Q}_v$ (resp. K_v) self-dual with respect to the pairing $(x, y) \mapsto \psi_v(xy)$ (resp. $(z, w) \mapsto \psi_v(\mathrm{Tr}(z^\sigma w))$). We normalize a Haar measure $d^\times x_v$ (resp. $d^\times z_v$) on $F_v^\times$ (resp. $K_v^\times$) by

$$\int_{\mathbf{Z}_p^\times} d^\times x_p = \int_{\mathcal{O}_{K,p}^\times} d^\times z_p = 1$$

if $v = p < \infty$, and

$$d^\times x_\infty = \frac{dx_\infty}{|x_\infty|_\infty}, \; d^\times z_\infty = \frac{dz_\infty}{||z_\infty||_\infty}.$$

Finally let

$$dx = \prod_v dx_v, \; dz = \prod_v dz_v, \; d^\times x = \prod_v d^\times x_v, \; d^\times z = \prod_v d^\times z_v,$$

where v runs over the primes of $\mathbf{Q}$.

1.3.

For $A \in M_{m,n}(K)$, we put $A^* = {}^tA^\sigma$. Let $H = U(T)$ be the unitary group of an anti-hermitian matrix $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$$H_\mathbf{Q} = \{h \in GL_2(\mathbf{Q}) \mid h^*Th = T\}.$$

Define elements of H by

$$\mathbf{d}(a) = \begin{pmatrix} a^\sigma & 0 \\ 0 & a^{-1} \end{pmatrix}, \; \mathbf{n}(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \; \bar{\mathbf{n}}(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

for $a \in K^\times, b \in \mathbf{Q}$. For a finite prime p, let $\mathcal{U}_p = H_p \cap GL_2(\mathcal{O}_{K,p})$ and

$$\mathcal{U}_0(D)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}_p \mid c \in D \cdot \mathcal{O}_{K,p} \right\}.$$

Let $\chi \in \mathcal{X}$. For $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}_0(D)_p$, set

$$\widetilde{\chi}_p(u) = \begin{cases} \chi_p(a) & \cdots\ c \in p\mathcal{O}_{K,p} \\ \chi_p(c) & \cdots\ c \in \mathcal{O}_{K,p} - p\mathcal{O}_{K,p}. \end{cases}$$

Then $\widetilde{\chi} = \prod_{p<\infty} \widetilde{\chi}_p$ defines a unitary character of $\mathcal{U}_0(D)_f = \prod_{p<\infty} \mathcal{U}_0(D)_p$ (cf. [13], Section 5).

For $h_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_\infty$ and $z \in \mathfrak{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, put

$$h_\infty\langle z\rangle = \frac{az+b}{cz+d} \in \mathfrak{H}, \quad j'(h_\infty, z) = (\det h_\infty)^{-1}(cz+d) \in \mathbf{C}^\times.$$

Denote by $\mathcal{U}_\infty$ the stabilizer of $z_0 = \sqrt{-1} \in \mathfrak{H}$ in H_∞.

We normalize the Haar measure dh_v on H_v by

$$\int_{H_v} f(h_v)dh_v = \int_{\mathbf{Q}_v} dx \int_{K_v^\times} d^\times y \int_{\mathcal{U}_v} du\, ||y||_v^{-1} f(\mathbf{n}(x)\mathbf{d}(y)u),$$

where du is normalized so that $\mathrm{vol}(\mathcal{U}_v) = 1$. Let dh be the Haar measure on $H_{\mathbf{A}}$ given as the product measure of dh_v over the primes v of $\mathbf{Q}$.

1.4.

In what follows, we fix a positive integer $l > 2$ divisible by w_K and $\chi \in \mathcal{X}$ with $w_\infty(\chi) = -1$. Let S_{l-1} be the space of smooth functions f on $H_{\mathbf{Q}}\backslash H_{\mathbf{A}}$ satisfying the following:

(i) For $h \in H_{\mathbf{A}}, u_f \in \mathcal{U}_0(D)_f$ and $u_\infty \in \mathcal{U}_\infty$, we have

$$f(hu_fu_\infty) = j'(u_\infty, z_0)^{-(l-1)}\, \widetilde{\chi}(u_f)\, f(h).$$

(ii) For any $h_f \in H_f$, $h_\infty\langle z_0\rangle \mapsto j'(h_\infty, z_0)^{l-1} f(h_f h_\infty)$ is holomorphic on $\mathfrak{H}$.

(iii) $\displaystyle\int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} f(\mathbf{n}(x)h)dx = 0 \qquad (h \in H_{\mathbf{A}}).$

We call S_{l-1} the space of holomorphic cusp form on $\mathcal{U}_0(D)_f$ of weight $l-1$ and character $\widetilde{\chi}$. We define the Petersson inner product on S_{l-1} by

$$\langle f, f'\rangle_H = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h)\overline{f'(h)}dh \qquad (f, f' \in S_{l-1})$$

and put $||f||_H = \sqrt{\langle f, f\rangle_H}$.

Let $\mathcal{Y}_l$ be the set of unitary characters Ω of $K^1_{\mathbf{A}}/K^1$ with $\Omega|_{\mathcal{O}^\times_{\mathbf{A},f}} = 1$ and $\Omega(t_\infty) = t^l_\infty$ for $t_\infty \in K^1_\infty$. We have an orthogonal decomposition

$$S_{l-1} = \bigoplus_{\Omega \in \mathcal{Y}_l} S_{l-1}(\chi\Omega)$$

with $S_{l-1}(\chi\Omega) = \{f \in S_{l-1} \mid f(th) = (\chi\Omega)(t)f(h) \quad (t \in K^1_{\mathbf{A}}, h \in H_{\mathbf{A}})\}$. Here we use the same letter χ to denote the restriction of χ to $K^1_{\mathbf{A}}$.

1.5.

For each finite prime p, we define Hecke operators acting on S_{l-1} as follows. Let $f \in S_{l-1}$.

(i) If p is inert in $K/\mathbf{Q}$, put

$$\begin{aligned} \mathcal{T}_p f(h) = -f(h\mathbf{d}(\pi^{-1})) - \sum_{x \in \mathbf{Z}_p^\times/p\mathbf{Z}_p} f(h\mathbf{n}(\pi^{-1}x)) \\ - \sum_{x \in \mathbf{Z}_p/p^2\mathbf{Z}_p} f(h\mathbf{n}(x)\mathbf{d}(\pi)), \end{aligned}$$

where π is a prime element of $\mathbf{Q}_p$.

(ii) If p ramifies in $K/\mathbf{Q}$, put

$$\begin{aligned} \mathcal{T}_p f(h) = \chi_p(\Pi) \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\mathbf{n}(x)\mathbf{d}(\Pi)) \\ + \chi_p^{-1}(\Pi) \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f\left(h\,\overline{\mathbf{n}}(Dx)\,\mathbf{d}(\Pi^{-1})\right). \end{aligned}$$

(iii) If p splits in $K/\mathbf{Q}$, put

$$\mathcal{T}_{p,1} f(h) = \chi_p^{-1}(\Pi_1)\{f(h\mathbf{d}(\Pi_1^{-1})) + \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\mathbf{n}(x)\mathbf{d}(\Pi_2))\},$$

$$\mathcal{T}_{p,2} f(h) = \chi_p^{-1}(\Pi_2)\{f(h\mathbf{d}(\Pi_2^{-1})) + \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} f(h\mathbf{n}(x)\mathbf{d}(\Pi_1))\}.$$

Note that $S_{l-1}(\chi\Omega)$ is invariant under Hecke operators, and that $\mathcal{T}_{p,2}f = \Omega_p(\Pi_2/\Pi_1)\mathcal{T}_{p,1}f$ for $f \in S_{l-1}(\chi\Omega)$ if p splits in $K/\mathbf{Q}$.

1.6.

We say that $f \in S_{l-1}(\chi\Omega)$ is a Hecke eigenform of eigenvalues $\{\lambda_p\}$ ($\lambda_p \in \mathbf{C}$ if p does not split in $K/\mathbf{Q}$ and $\lambda_p = (\lambda_{p,1}, \lambda_{p,2}) \in \mathbf{C}^2$ if p splits in $K/\mathbf{Q}$) if, for every $p < \infty$, $\mathcal{T}_p f = \lambda_p f$ in the non-split case and $\mathcal{T}_{p,i} f = \lambda_{p,i} f$ $(i = 1, 2)$

in the split case. Note that $\lambda_{p,2} = \Omega_p(\Pi_2/\Pi_1)\lambda_{p,1}$ in the split case. For a Hecke eigenform f of eigenvalues $\{\lambda_p\}$, we define the automorphic L-function $L(f;s)$ as follows:

$$L(f;s) = \prod_{p<\infty} L_p(f;s),$$

$$L_p(f;s) = L_p(\lambda_p;s)$$

$$= \begin{cases} \left(1-(p^{-1}\lambda_p+1-p^{-1})p^{-2s}+p^{-4s}\right)^{-1} & \cdots p \text{ is inert in } K/\mathbf{Q} \\ \left(1-p^{-1/2}\lambda_p p^{-s}+p^{-2s}\right)^{-1} & \cdots p \text{ ramifies in } K/\mathbf{Q} \\ \prod_{i=1}^{2}(1-p^{-1/2}\lambda_{p,i}p^{-s}+\Omega_p(\Pi_i/\Pi_i^\sigma)p^{-2s})^{-1} & \cdots p \text{ splits in } K/\mathbf{Q} \end{cases}$$

Remark. Suppose that the class number of K is one. Then f corresponds to a holomorphic cusp form f_{dm} of weight $l-1$ and character $\left(\frac{D}{*}\right)$ on $\Gamma_0(D)$, where $f_{dm}(h_\infty\langle z_0\rangle) = j'(h_\infty, z_0)^{l-1}f(h_\infty)$ $(h_\infty \in H_\infty)$. Let $f_{dm}(z) = \sum_{m=1}^{\infty} c(m)\mathbf{e}[mz]$ be the Fourier expansion of f_{dm}. Then $L(f;s)$ coincides with the Rankin L-function

$$\zeta(2s)\sum_{\mathfrak{a}} c(\mathrm{N}(\mathfrak{a}))\alpha^l\,\mathrm{N}(\mathfrak{a})^{-(s+l-1)}$$

except local factors at p ramified in $K/\mathbf{Q}$. Here $\mathfrak{a} = (\alpha)$ runs over the nonzero integral ideals of K.

1.7.

Let $G = U(S)$ be the unitary group of a hermitian matrix

$$S = \begin{pmatrix} & & \kappa^{-1} \\ & 1 & \\ -\kappa^{-1} & & \end{pmatrix}.$$

Note that the signature of S is $(2,1)$. For $a \in K^\times$ and $t \in K^1$, we write $d_G(a,t)$ for $\mathrm{diag}(a^\sigma, t, a^{-1}) \in G_{\mathbf{Q}}$. We also set

$$(w,x) = \begin{pmatrix} 1 & \kappa w^\sigma & x+\frac{\kappa}{2}ww^\sigma \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}$$

for $w \in K$ and $x \in \mathbf{Q}$.

Put $\mathcal{K}_p = G_p \cap GL_3(\mathcal{O}_{K,p})$ and $\mathcal{K}_f = G_{\mathbf{A},f} \cap GL_3(\mathcal{O}_{K,f}) = \prod_{p<\infty} \mathcal{K}_p$. Then $\mathcal{K}_p$ (resp. $\mathcal{K}_f$) is a maximal open compact subgroup of G_p (resp. $G_{\mathbf{A},f}$). We define an action of $G_\infty = G(\mathbf{R})$ on $\mathcal{D} = \{{}^t(z,w) \in \mathbf{C}^2 \mid (z - \overline{z})/\kappa - w\overline{w} > 0\}$ and a holomorphic automorphic factor $J\colon G_\infty \times \mathcal{D} \to \mathbf{C}^\times$ by

$$g_\infty\langle Z\rangle = {}^t\!\left(\frac{a_1 z + b_1 w + c_1}{J(g_\infty, Z)},\ \frac{a_2 z + b_2 w + c_2}{J(g_\infty, Z)}\right),$$

$$J(g_\infty, Z) = a_3 z + b_3 w + c_3$$

for

$$g_\infty = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in G_\infty,\ Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{D}.$$

Let $\mathcal{K}_\infty = \{g \in G_\infty \mid g\langle Z_0\rangle = Z_0\}$ with $Z_0 = {}^t(\kappa/2, 0) \in \mathcal{D}$. It is known that $\mathcal{K}_\infty = G_\infty \cap U(R)$, where $R = \mathrm{diag}(-2/D, 1, 1/2)$. We put $\mathcal{K} = \mathcal{K}_f\mathcal{K}_\infty$.

We normalize the Haar measure dg on $G_\mathbf{A}$ by

$$\int_{G_\mathbf{A}} f(g)dg$$
$$= \int_{\mathbf{Q}_\mathbf{A}} dx \int_{K_\mathbf{A}} dw \int_{K_\mathbf{A}^\times} d^\times a \int_{K_\mathbf{A}^1} d^1 t \int_{\mathcal{K}} dk\ ||a||^{-2} f((w,x)d_G(a,t)k)$$

for $f \in L^1(G_\mathbf{A})$. Here the Haar measure d^1t on $K_\mathbf{A}^1$ is normalized by $\mathrm{vol}(K^1\backslash K_\mathbf{A}^1) = 1$, and dk is normalized by $\mathrm{vol}(\mathcal{K}) = 1$.

Let $\mathfrak{S}_l$ be the space of smooth functions F on $G_\mathbf{Q}\backslash G_\mathbf{A}$ satisfying the following three conditions:

(i) $F(gk_f k_\infty) = J(k_\infty, Z_0)^{-l} F(g) \quad (g \in G_\mathbf{A}, k_f \in \mathcal{K}_f, k_\infty \in \mathcal{K}_f)$.

(ii) For any $g_f \in G_{\mathbf{A},f}$, $g_\infty\langle Z_0\rangle \mapsto J(g_\infty, Z_0)^l F(g_\infty g_f)$ is holomorphic on $\mathcal{D}$.

(iii) $\displaystyle\int_{\mathbf{Q}\backslash\mathbf{Q}_\mathbf{A}} F((0,x)g)dx = 0 \quad (g \in G_\mathbf{A})$.

We call $\mathfrak{S}_l$ the space of cusp forms on $\mathcal{K}_f$ of weight l. Define the Petersson inner product on $\mathfrak{S}_l$ by

$$\langle F, F'\rangle_G = \int_{G_\mathbf{Q}\backslash G_\mathbf{A}} F(g)\overline{F'(g)}dg \qquad (F, F' \in \mathfrak{S}_l)$$

and put $||F||_G = \sqrt{\langle F, F\rangle_G}$. We have an orthogonal decomposition

$$\mathfrak{S}_l = \bigoplus_{\Omega \in \mathcal{Y}_l} \mathfrak{S}_l(\Omega^{-1})$$

with $\mathfrak{S}_l(\Omega^{-1}) = \{F \in \mathfrak{S}_l \mid F(t1_3 \cdot g) = \Omega^{-1}(t)F(g) \quad (t \in K^1_{\mathbf{A}})\}$. Shintani defined the action of Hecke operators on $\mathfrak{S}_l$ and the automorphic L-function $L(F;s)$ for a Hecke eigenform $F \in \mathfrak{S}_l$ ([16]; see also [6] and [13]).

1.8.

Define a unitary representation $\mathcal{M}'_{\chi,v}$ of $G_v \times H_v$ on $\mathcal{S}(K_v^3)$ by

$$\begin{aligned}
\mathcal{M}'_{\chi,v}(g \times 1_2)\Psi(z) &= \chi_v(\det g)\Psi(g^{-1}z) \qquad (g \in G_v),\\
\mathcal{M}'_{\chi,v}(1_3 \times \mathbf{d}(a))\Psi(z) &= \chi_v^{-3}(a)\|a\|_v^{3/2}\Psi(az) \qquad (a \in K_v^\times),\\
\mathcal{M}'_{\chi,v}(1_3 \times \mathbf{n}(b))\Psi(z) &= \psi_v(bz^*Sz)\Psi(z) \qquad (b \in \mathbf{Q}_v),\\
\mathcal{M}'_{\chi,v}(1_3 \times w_v)\Psi(z) &= \lambda_{K_v}(\psi_v)\int_{K_v^3} \psi_v(\mathrm{Tr}(z^*Sz'))\Psi(z')d_Sz'
\end{aligned}$$

for $\Psi \in \mathcal{S}(K_v^3)$ and $z \in K_v^3$. Here $w_v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H_v$, d_Sz' is the Haar measure on K_v^3 self-dual with respect to the pairing $(z,z') \mapsto \psi_v(\mathrm{Tr}(z^*Sz'))$, and $\lambda_{K_v}(\psi_v)$ denotes the Weil constant (for the precise definition, see [11], Section 3.3). Set $\mathcal{M}'_\chi(g) = \bigotimes_v \mathcal{M}'_{\chi,v}(g_v)$ for $g = (g_v) \in G_{\mathbf{A}}$. Then $\mathcal{M}'_\chi$ defines a unitary representation of $G_{\mathbf{A}} \times H_{\mathbf{A}}$ on $\mathcal{S}(K^3_{\mathbf{A}})$.

1.9.

We define a test function $\Psi_0 \in \mathcal{S}(K^3_{\mathbf{A}})$ by

$$\begin{aligned}
\Psi_0(X) = 2i^l \left(\frac{2}{\sqrt{D}}x_{1,\infty} + x_{3,\infty}\right)^l \exp\left(-2\pi\left\{-\frac{2}{D}|x_{1,\infty}|^2 + |x_{2,\infty}|^2 + \frac{1}{2}|x_{3,\infty}|^2\right\}\right)\\
\times \Psi_{0,f}(X_f),
\end{aligned}$$

where $X = X_\infty X_f \in K^3_{\mathbf{A}}, X_\infty = {}^t(x_{1,\infty}, x_{2,\infty}, x_{3,\infty}) \in \mathbf{C}^3$ and $\Psi_{0,f}$ is the characteristic function of $\mathcal{O}^3_{K,f}$. Define a theta kernel $\theta'_\chi \colon G_{\mathbf{Q}}\backslash G_{\mathbf{A}} \times H_{\mathbf{Q}}\backslash H_{\mathbf{A}} \to \mathbf{C}$ by

$$\begin{aligned}
&\theta'_\chi(g,h)\\
&= \chi^{-1}(\det g)\chi^{-2}(\det h)\sum_{X \in K^3} \mathcal{M}'_\chi(g \times h)\Psi_0(X) \qquad (g \in G_{\mathbf{A}}, h \in H_{\mathbf{A}}).
\end{aligned}$$

For $f \in \mathcal{S}_{l-1}(\chi\Omega)$, we set

$$\mathcal{L}_\chi f(g) = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} \theta'_\chi(g,h)f(h)dh \qquad (g \in G_{\mathbf{A}}).$$

Then $\mathcal{L}_\chi f \in \mathfrak{S}_l(\Omega^{-1})$ (cf. [5], [13]). We call $\mathcal{L}_\chi f$ the *Kudla lift* of f.

1.10.

Let $f \in S_{l-1}(\chi\Omega)$. For each prime factor p of D, put $W_{D,p}f(h) = f(hw_{D,p})$ $(h \in H_{\mathbf{A}})$, where $w_{D,p} = \begin{pmatrix} & \sqrt{D}^{-1} \\ \sqrt{D} & \end{pmatrix} \in H_p$. Then $f \mapsto W_{D,p}f$ defines an involution of $S_{l-1}(\chi\Omega)$ commuting with Hecke operators. Suppose that f satisfies the following conditions:

(1.1) f is a Hecke eigenform of eigenvalues $\{\lambda_p\}$.
(1.2) For each prime factor p of D, we have $W_{D,p}f = \epsilon_p f$ $(\epsilon_p = \pm 1)$.

Set

$$C_p(f,\chi) = 1 + \epsilon_p \lambda_{K_p}(\psi_p)\chi_p(\sqrt{D})\,\frac{\nu_p^{\delta_p} + \nu_p^{-\delta_p}}{2},$$

where $\delta_p = \mathrm{ord}_p D$ and $\nu_p^{\pm 1} \in \mathbf{C}^\times$ is given by $\lambda_p = p^{1/2}(\nu_p + \nu_p^{-1})$.

We are now able to state the main result of the paper.

Theorem 1.1. *Suppose that $f \in S_{l-1}(\chi\Omega)$ satisfies (1.1) and (1.2).*

(i) *We have*

$$||\mathcal{L}_\chi f||_G^2 = \gamma(f,\chi)||f||_H^2,$$

where

$$\gamma(f,\chi) = w_K^{-1}\pi^{-l}|D|^{3/2}(l-1)!\prod_{p|D}\frac{C_p(f,\chi)}{1+p^{-1}}\cdot L(f;1).$$

(ii) *We have $\mathcal{L}f \neq 0$ if and only if*

$$\prod_{p|D} C_p(f,\chi)\cdot L(f;1) \neq 0.$$

2. Metaplectic representations

2.1.

Let K^m be the K-vector space of column vectors in K of degree m. Let $Q \in GL_m(K)$ with $Q^* = -Q$. Define a nondegenerate alternating form $\langle\, ,\rangle_Q$ on K^m by $\langle w, w'\rangle_Q = \mathrm{Tr}(w^*Qw')$ $(w, w' \in K^m)$. Let $\mathcal{N}_Q$ be the Heisenberg group attached to $(K^m, \langle\, ,\rangle_Q)$: By definition, $\mathcal{N}_Q = K^m \times \mathbf{Q}$ as a set and the multiplication law is given by

$$(w,x)(w',x') = (w+w', x+x'+\frac{1}{2}\langle w, w'\rangle_Q).$$

For $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$, we set

$$A_\epsilon = \operatorname{diag}(\epsilon_1, \dots, \epsilon_n), \quad T_\epsilon = \begin{pmatrix} 0 & A_\epsilon \\ -A_\epsilon & 0 \end{pmatrix}, \quad H_\epsilon = U(T_\epsilon).$$

The group $H_{\epsilon,\infty} = H_\epsilon(\mathbf{R})$ acts on the bounded symmetric domain $\mathbf{D}_n = \{Z \in M_n(\mathbf{C}) \mid \frac{1}{i}(Z - {}^t\overline{Z}) > 0\}$ by $h \cdot Z^{\tilde{}} = h\langle Z\rangle^{\tilde{}} \cdot \mathcal{J}_\epsilon(h, Z)$, where $Z^{\tilde{}} = \begin{pmatrix} Z \\ A_\epsilon \end{pmatrix}$ and $\mathcal{J}_\epsilon(h, Z) \in GL_n(\mathbf{C})$. Put $\mathcal{U}_{\epsilon,\infty} = \{h \in H_{\epsilon,\infty} \mid h\langle Z_0\rangle = Z_0\}$ $(Z_0 = i1_n \in \mathbf{D}_n)$. Then $\mathcal{U}_{\epsilon,\infty}$ is a maximal compact subgroup of $H_{\epsilon,\infty}$ and

$$\mathcal{U}_{\epsilon,\infty} = \{\delta(u_1, u_2) := C_\epsilon^{-1} \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} C_\epsilon \mid u_1, u_2 \in U(n)\},$$

where

$$C_\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} A_\epsilon & i1_n \\ A_\epsilon & -i1_n \end{pmatrix}.$$

It follows that $\mathcal{U}_{\epsilon,\infty} \subset U(2n)$. We also have $\mathcal{J}_\epsilon(\delta(u_1, u_2), Z_0) = A_\epsilon u_1 A_\epsilon$.

Put

$$\mathbf{T}_\epsilon = \begin{pmatrix} & & \kappa^{-1}T_\epsilon \\ & T_\epsilon & \\ -\kappa^{-1}T_\epsilon & & \end{pmatrix}, \quad \mathbf{H}_\epsilon = U(\mathbf{T}_\epsilon).$$

Define a homomorphism from $G \times H_\epsilon$ into $\mathbf{H}_\epsilon$ by

$$(g, h) \mapsto g \otimes h := (g_{ij} 1_{2n})_{1 \le i,j \le 3} \cdot \begin{pmatrix} h & & \\ & h & \\ & & h \end{pmatrix} \qquad (g = (g_{ij}) \in G, h \in H_\epsilon).$$

2.2.

Let ρ^{T_ϵ} be a smooth representation of $\mathcal{N}_{T_\epsilon}(\mathbf{A})$ on $\mathcal{S}(K_{\mathbf{A}}^n)$ defined by

$$\begin{aligned} &\rho^{T_\epsilon}\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, x\right)\varphi(z) \\ &= \psi\left(-\operatorname{Tr}(w_1^* A_\epsilon z) - \frac{1}{2}\operatorname{Tr}(w_1^* A_\epsilon w_2) + x\right)\varphi(z + w_2) \end{aligned}$$

for $w_1, w_2, z \in K_{\mathbf{A}}^n, x \in \mathbf{Q}_{\mathbf{A}}, \varphi \in \mathcal{S}(K_{\mathbf{A}}^n)$. We also define a smooth representation $\rho^{\mathbf{T}_\epsilon}$ of $\mathcal{N}_{\mathbf{T}_\epsilon}$ on $\mathcal{S}(K_{\mathbf{A}}^n) \otimes \mathcal{S}(K_{\mathbf{A}}^{2n})$ by

$$\begin{aligned} &\rho^{\mathbf{T}_\epsilon}\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, x\right)(\varphi \otimes \Phi)(z, Z) \\ &= \rho^{T_\epsilon}(w_2, 0)\varphi(z) \cdot \psi\left(-\operatorname{Tr}(\kappa^{-1} w_1^* T_\epsilon Z) - \frac{1}{2}\operatorname{Tr}(\kappa^{-1} w_1^* T_\epsilon w_3) + x\right)\Phi(Z + w_3) \end{aligned}$$

for $z \in K_{\mathbf{A}}^n, w_1, w_2, w_3, Z \in K_{\mathbf{A}}^{2n}, x \in F_{\mathbf{A}}, \varphi \in \mathcal{S}(K_{\mathbf{A}}^n), \Phi \in \mathcal{S}(K_{\mathbf{A}}^{2n})$.

For $\chi \in \mathcal{X}$, we denote by $\mathcal{M}_\chi^{T_\epsilon}$ and $\mathcal{M}_\chi^{\mathbf{T}_\epsilon}$ the metaplectic representations of $H_{\epsilon,\mathbf{A}}$ and $\mathbf{H}_{\epsilon,\mathbf{A}}$ attached to $(\rho^{T_\epsilon}, \chi)$ and $(\rho^{\mathbf{T}_\epsilon}, \chi)$ resepectively (cf. [10]). The following two results are proved by straightforward calculations.

Lemma 2.1. *Let $\varphi \in \mathcal{S}(K_{\mathbf{A}}^n)$ and $z \in K_{\mathbf{A}}^n$.*

(i) *For $a \in GL_n(K_{\mathbf{A}})$, we have*

$$\mathcal{M}_\chi^{T_\epsilon}\left(\begin{pmatrix} a & 0 \\ 0 & A_\epsilon^{-1}(a^*)^{-1}A_\epsilon \end{pmatrix}\right)\varphi(z) = \chi(\det a)||\det a||^{1/2}\varphi(A_\epsilon^{-1}a^*A_\epsilon z).$$

(ii) *For $b \in M_n(K_{\mathbf{A}})$ with $(A_\epsilon b)^* = A_\epsilon b$, we have*

$$\mathcal{M}_\chi^{T_\epsilon}\left(\begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix}\right)\varphi(z) = \psi(z^*A_\epsilon bz)\varphi(z).$$

Lemma 2.2. *Let $\varphi \otimes \Phi \in \mathcal{S}(K_{\mathbf{A}}^n) \otimes \mathcal{S}(K_{\mathbf{A}}^{2n})$ and $z \in K_{\mathbf{A}}^n, Z \in K_{\mathbf{A}}^{2n}$.*

(i) *For $h \in H_{\mathbf{A}}$, we have*

$$\mathcal{M}_\chi^{\mathbf{T}_\epsilon}(1_3 \otimes h)(\varphi \otimes \Phi)(z, Z) = \chi(\det h)\mathcal{M}_\chi^{T_\epsilon}(h)\varphi(z) \cdot \Phi(h^{-1}Z).$$

(ii) *For $a \in K_{\mathbf{A}}^\times, t \in K_{\mathbf{A}}^1$, we have*

$$\mathcal{M}_\chi^{\mathbf{T}_\epsilon}(d_G(a,t)\otimes 1_{2n})(\varphi\otimes\Phi)(z, Z) = \chi^{-2n}(a)||a||^n\, \mathcal{M}_\chi^{T_\epsilon}(t1_{2n})\varphi(z)\Phi(aZ).$$

(iii) *For $w \in K_{\mathbf{A}}, x \in F_{\mathbf{A}}$, we have*

$$\mathcal{M}_\chi^{\mathbf{T}_\epsilon}((w,x)\otimes 1_{2n})(\varphi\otimes\Phi)(z, Z) = \psi(\kappa^{-1}Z^*T_\epsilon Z\cdot x)(\rho^{T_\epsilon}(-wZ, 0)\varphi)(z)\,\Phi(Z).$$

3. Kudla lift

3.1.

In what follows, we write $\mathbf{T}$ for

$$\mathbf{T}_{(1)} = \begin{pmatrix} & & \kappa^{-1}T \\ & T & \\ -\kappa^{-1}T & & \end{pmatrix}$$

and put $\mathbf{H} = U(\mathbf{T})$. Let $\varphi_0 = \varphi_{0,f} \otimes \varphi_{0,\infty} \in \mathcal{S}(K_{\mathbf{A}})$, where $\varphi_{0,f}$ is the characteristic function of $\mathcal{O}_{K,f}$ and $\varphi_{0,\infty}(z) = \mathbf{e}[iN(z)]$ $(z \in \mathbf{C})$. We also let $\Phi_0 = \Phi_{0,f} \otimes \Phi_{0,\infty} \in \mathcal{S}(K_{\mathbf{A}}^2)$, where $\Phi_{0,f}$ is the characteristic function of $\mathcal{O}_{K,f}^2$ and

$$\Phi_{0,\infty}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1 - iz_2)^l \mathbf{e}\left[\frac{i}{2}(\mathrm{N}(z_1) + \mathrm{N}(z_2))\right] \qquad \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{C}^2\right).$$

Note that $\mathcal{M}_\chi^T(u_f)\varphi_{0,f} = \widetilde{\chi}(u_f)\varphi_{0,f}$ holds for $u_f \in \mathcal{U}_0(D)_f$ (see [13], Lemma 6.3). It is proved that

$$\begin{aligned}&\mathcal{M}_\chi^{\mathbf{T}}(k \otimes u)(\varphi_0 \otimes \Phi_0)\\ &\quad = \chi(\det k)\chi^2(\det u)\widetilde{\chi}(u_f)^{-1} J(k_\infty, Z_0)^{-l} j'(u_\infty, z_0)^{l-1} \cdot \varphi_0 \otimes \Phi_0\end{aligned}$$

for $k = k_f k_\infty \in \mathcal{K}_f\mathcal{K}_\infty$ and $u = u_f u_\infty \in \mathcal{U}_0(D)_f\mathcal{U}_\infty$.

Remark. We recall a relation between two realizations $\mathcal{M}_\chi^{\mathbf{T}}$ and $\mathcal{M}'_\chi$ given in Section 1 (for detail, see [13], Section 6.5). Define an intertwining operator $I\colon \mathcal{S}(K_{\mathbf{A}}) \otimes \mathcal{S}(K_{\mathbf{A}}^2) \to \mathcal{S}(K_{\mathbf{A}}^3)$ by

$$I(\varphi\otimes\Phi)(z) = \varphi(z_2)\cdot \int_{K_{\mathbf{A}}} \psi(\mathrm{Tr}(\kappa^{-1} z_1^\sigma u))\Phi\left(\begin{pmatrix} u \\ z_3 \end{pmatrix}\right) du \qquad \left(z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in K_{\mathbf{A}}^3\right).$$

We then have $I \circ \mathcal{M}_\chi^{\mathbf{T}}(g \otimes h) = \mathcal{M}'_\chi(g \times h) \circ I \quad (g \in G_{\mathbf{A}}, h \in H_{\mathbf{A}})$ and $I(\varphi_0 \otimes \Phi_0) = \Psi_0$.

3.2.

For $g \in G_{\mathbf{A}}, h \in H_{\mathbf{A}}$, set

$$\begin{aligned}&\theta_\chi(g,h)\\ &\quad = \chi^{-1}(\det g)\chi^{-2}(\det h) \sum_{\xi \in K, X \in K^2} \mathcal{M}_\chi^{\mathbf{T}}(g \otimes h)(\varphi_0 \otimes \Phi_0)(\xi, X).\end{aligned}$$

Note that $\theta_\chi = \theta'_\chi$ by Poisson summation formula and the above remark. It follows that, for $f \in S_{l-1}$,

$$\mathcal{L}_\chi f(g) = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h)\theta_\chi(g,h)dh \qquad (g \in G_{\mathbf{A}}).$$

The following fact is proved by Kudla ([5], [6]; see also [13]).

Theorem 3.1. *Let* $f \in S_{l-1}(\chi\Omega)$.

(i) *We have* $\mathcal{L}_\chi f \in \mathfrak{S}_l(\Omega^{-1})$.

(ii) *Assume that* f *is a Hecke eigenform. Then so is* $\mathcal{L}_\chi f$ *and* $L(\mathcal{L}_\chi f; s) = \zeta_K(s)L(f;s)$.

4. Inner product formula

4.1.

We prove Theorem 1.1 by invoking the machinery of see-saw dual reductive pair (cf. [7]). Put

$$A' = A_{(1,-1)} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ T' = T_{(1,-1)} = \begin{pmatrix} & A' \\ -A' & \end{pmatrix},\ H' = H_{(1,-1)} = U(T'),$$

$$\mathbf{T}' = \mathbf{T}_{(1,-1)} = \begin{pmatrix} & & \kappa^{-1}T' \\ & T' & \\ -\kappa^{-1}T' & & \end{pmatrix}, \mathbf{H}' = \mathbf{H}_{(1,-1)} = U(\mathbf{T}').$$

Define an embedding $\iota\colon H \times H \to H'$ by

$$\iota(h_1, h_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix} \qquad \left(h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in H \right).$$

We also define a homomorphism $G \times H'$ to $\mathbf{H}'$ as in Section 2.1:

$$(g, h') \mapsto g \otimes h' = (g_{ij} \cdot 1_4)_{1 \le i,j \le 3} \begin{pmatrix} h' & & \\ & h' & \\ & & h' \end{pmatrix}.$$

Let $\varphi_0' = \varphi_{0,f}' \otimes \varphi_{0,\infty}' \in \mathcal{S}(K_{\mathbf{A}}^2)$, where $\varphi_{0,f}'$ is the characteristic function of $\mathcal{O}_{K,f}^2$ and $\varphi_{0,\infty}'(z) = \mathbf{e}[i\, {}^t\bar{z}z]$ $(z \in \mathbf{C}^2)$. We also let $\Phi_0' = \Phi_{0,f}' \otimes \Phi_{0,\infty}' \in \mathcal{S}(K_{\mathbf{A}}^4)$, where $\Phi_{0,f}'$ is the characteristic function of $\mathcal{O}_{K,f}^4$ and

$$\Phi_{0,\infty}'(Z) = (z_1 - iz_3)^l \overline{(z_2 - iz_4)^l}\, \mathbf{e}\left[\frac{i}{2} Z^* Z\right] \quad (Z = {}^t(z_1, z_2, z_3, z_4) \in \mathbf{C}^4).$$

For $g \in G_{\mathbf{A}}$ and $h_1, h_2 \in H_{\mathbf{A}}$, we define

$$\begin{aligned} &\Theta_\chi(g, h_1, h_2) \\ &= \chi^{-2}(\det g)\chi^{-2}(\det h_1)\chi^{-1}(\det h_2) \sum_{\xi \in K^2, X \in K^4} \mathcal{M}_\chi^{\mathbf{T}'}(g \otimes \iota(h_1, h_2))(\varphi_0' \otimes \Phi_0')(\xi, X). \end{aligned}$$

The following is easily verified.

Lemma 4.1. *For fixed $h_1, h_2 \in H_{\mathbf{A}}$, the function $g \mapsto \Theta_\chi(g, h_1, h_2)$ is rapidly decreasing on $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$.*

Lemma 4.2. *For $f, f' \in S_{l-1}$, we have*

$$\int_{(H_{\mathbf{Q}} \backslash H_{\mathbf{A}})^2} f(h_1)\overline{f'(h_2)}\, \Theta_\chi(g, h_1, h_2)\, dh_1 dh_2 = \mathcal{L}_\chi f(g)\, \overline{\mathcal{L}_\chi f'(g)} \qquad (g \in G_{\mathbf{A}}).$$

Proof. For $\xi = {}^t(\xi^+, \xi^-) \in K^2$ and $X = {}^t(x_1, x_2, x_3, x_4) \in K^4$, we have

$$\begin{aligned}&\mathcal{M}_\chi^{\mathbf{T}'}(g \otimes \iota(h_1, h_2))(\varphi_0' \otimes \Phi_0')(\xi, X)\\ &= \mathcal{M}_\chi^{\mathbf{T}}(g \otimes h_1)(\varphi_0 \otimes \Phi_0)(\xi^+, X^+)\mathcal{M}_\chi^{-\mathbf{T}}(g \otimes h_2)\left(\overline{\varphi_0 \otimes \Phi_0}\right)(\xi^-, X^-),\end{aligned}$$

where $X^+ = {}^t(x_1, x_3),\ X^- = {}^t(x_2, x_4)$. It follows that

$$\begin{aligned}&\Theta_\chi(g, h_1, h_2)\\ &= \chi^{-2}(\det g)\chi^{-2}(\det h_1)\chi^{-1}(\det h_2) \sum_{\xi^+ \in K, X^+ \in K^2} \mathcal{M}_\chi^{\mathbf{T}}(g \otimes h_1)(\varphi_0 \otimes \Phi_0)(\xi^+, X^+)\\ &\times \sum_{\xi^- \in K, X^- \in K^2} \mathcal{M}_\chi^{-\mathbf{T}}(g \otimes h_2)\left(\overline{\varphi_0 \otimes \Phi_0}\right)(\xi^-, X^-).\end{aligned}$$

It now remains to show that

$$\begin{aligned}&\chi^{-1}(\det g)\chi^{-1}(\det h)\mathcal{M}_\chi^{-\mathbf{T}}(g \otimes h)\left(\overline{\varphi \otimes \Phi}\right)\\ &= \overline{\chi^{-1}(\det g)\chi^{-2}(\det h)\mathcal{M}_\chi^{\mathbf{T}}(g \otimes h)(\varphi \otimes \Phi)}\end{aligned}$$

holds for $g \in G_{\mathbf{A}}, h \in H_{\mathbf{A}}, \varphi \in \mathcal{S}(K_{\mathbf{A}}), \Phi \in \mathcal{S}(K_{\mathbf{A}}^2)$. This follows from the definition of $\mathcal{M}_\chi^{\pm\mathbf{T}}$ (cf. [10]) and the fact that $\overline{\rho^{\mathbf{T}}(w,0)(\varphi \otimes \Phi)} = \rho^{-\mathbf{T}}(w,0)\left(\overline{\varphi \otimes \Phi}\right)$. □

We thus have proved the following:

Proposition 4.1. *For $f \in S_{l-1}$, we have*

$$||\mathcal{L}f||_G^2 = \int_{(H_{\mathbf{Q}}\backslash H_{\mathbf{A}})^2} f(h_1)\overline{f(h_2)}\,dh_1dh_2 \int_{G_{\mathbf{Q}}\backslash G_{\mathbf{A}}} \Theta_\chi(g, h_1, h_2)\,dg.$$

4.2.

We now recall the Siegel-Weil formula for $(U(2,1), U(2,2))$ after [14]. Let $P' = N'M'$ be the Siegel parabolic subgroup of H' with

$$N' = \left\{n'(b) := \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} \mid b \in M_2(K), (A'b)^* = A'b\right\}$$

and

$$M' = \left\{d'(a) := \begin{pmatrix} a & 0 \\ 0 & (A')^{-1}(a^*)^{-1}A' \end{pmatrix} \mid a \in GL_2(K)\right\}.$$

We have the Iwasawa decomposition $H'_{\mathbf{A}} = P'_{\mathbf{A}}\mathcal{U}'_f\mathcal{U}'_\infty$, where $\mathcal{U}'_f = H'_{\mathbf{A},f} \cap GL_4(\mathcal{O}_{K,f})$ and $\mathcal{U}'_\infty = \mathcal{U}_{(1,-1),\infty} = \{h' \in H'_\infty \mid h'\langle i1_2\rangle = i1_2\}$. For $h' \in H'_{\mathbf{A}}$

and $s \in \mathbf{C}$, put

$$\begin{aligned}&\phi(h', s; \chi)\\ &= \chi^{-1}(\det h')\,(\mathcal{M}_\chi^{T'}(h')\varphi_0')(0)\,||\det a(h')||^s \int_{K_{\mathbf{A}}^2} \Phi_0'\left((h')^{-1}\begin{pmatrix} X \\ 0 \end{pmatrix}\right) dX,\end{aligned}$$

where we choose $a(h') \in GL_2(K_{\mathbf{A}})$ so that $h' \in N_{\mathbf{A}}' d'(a(h'))\mathcal{U}_f'\mathcal{U}_\infty'$. Then we see that $\phi(n'(b)d'(a)h', s; \chi) = \chi^{-1}(\det a)||\det a||^{s+3/2}\phi(h', s; \chi)$. Set

$$E(h', s; \chi) = \sum_{\gamma' \in P_{\mathbf{Q}}' \backslash H_{\mathbf{Q}}'} \phi(\gamma' h', s; \chi).$$

The Eisenstein series $E(h', s; \chi)$ is continued to a meromorphic function of s on $\mathbf{C}$, and holomorphic at $s = 0$. The following Siegel-Weil formula is proved in [14].

Theorem 4.1. *For $h_1, h_2 \in H_{\mathbf{A}}$, we have*

$$\int_{G_{\mathbf{A}}} \Theta_\chi(g, h_1, h_2)\, dg = c_{SW} \cdot \chi(\det h_2)\, E(\iota(h_1, h_2), 0; \chi),$$

where

$$c_{SW} = \frac{|D|^3}{4\pi^2 w_K}\, \zeta(2)\, L(\omega; 3).$$

4.3.

For $f \in S_{l-1}$, we define the zeta integral

$$Z(f, s; \chi) = \int_{(H_{\mathbf{Q}} \backslash H_{\mathbf{A}})^2} f(h_1)\, \overline{f(h_2)}\, \chi(\det h_2)\, E(\iota(h_1, h_2), s; \chi)\, dh_1 dh_2.$$

The integral is continued to a meromorphic function of s on $\mathbf{C}$, and holomorphic at $s = 0$. In view of Proposition 4.1 and Theorem 4.1, we have proved

Proposition 4.2. *For $f \in S_{l-1}$, we have*

$$||\mathcal{L}_\chi f||_G^2 = c_{SW}\, Z(f, 0; \chi).$$

4.4.

Theorem 1.1 is a direct consequence of Proposition 4.2 and the following result, which will be proved in the remaining part of the paper.

Theorem 4.2. *Let $f \in S_{l-1}(\chi\Omega)$. Assume that f is a Hecke eigenform and that, for every $p|D$, we have $W_{D,p}f = \epsilon_p f$ with $\epsilon_p \in \{\pm 1\}$. Then we have*

$$Z(f,s;\chi) \\ = l!\,|D|^{-3/2}\pi^{2-l}2^{-2s+2}\prod_{p|D}\frac{C_p(f,\chi)}{1+p^{-1}}\,||f||_H^2\,\frac{L(f;s+1)}{(s+l)\,\zeta(2s+2)L(\omega;2s+3)},$$

where $C_p(f,\chi)$ is defined in Section 1.10.

5. The basic identity

5.1.

For $f \in S_{l-1}(\chi\Omega)$, we set

$$W_f(h) = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h_1 h)\,\overline{f(h_1)}\,dh_1 \qquad (h \in H_{\mathbf{A}}).$$

The object of this section is to show the following *basic identity.*

Proposition 5.1. *For $f \in S_{l-1}(\chi\Omega)$, we have*

$$Z(f,s;\chi) = \int_{H_{\mathbf{A}}} \phi(\Upsilon_0\iota(h,1_2),s;\chi)\,W_f(h)\,dh,$$

where

$$\Upsilon_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in H'_{\mathbf{Q}}.$$

Proof. Though this is a standard fact (for example, see [1]), we give a sketch of proof for completeness. We first observe

(i) $H'_{\mathbf{Q}} = P'_{\mathbf{Q}}\cdot\Upsilon_0\cdot\iota(H_{\mathbf{Q}}\times H_{\mathbf{Q}}) \cup P'_{\mathbf{Q}}\cdot 1_4\cdot\iota(H_{\mathbf{Q}}\times H_{\mathbf{Q}})$ (a disjoint union).

(ii) $\Upsilon_0^{-1}P'\Upsilon_0 \cap \iota(H\times H) = \iota(H^d)$, where $H^d = \{(h,h) \mid h \in H\}$.

(iii) $P' \cap \iota(H\times H) = \iota(P\times P)$, where $P = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in H\right\}$.

It follows that

$$Z(f,s;\chi) = Z_0(f,s;\chi) + Z_1(f,s;\chi),$$

where

$$\begin{aligned} &Z_0(f,s;\chi) \\ &= \int_{(H_{\mathbf{Q}}\backslash H_{\mathbf{A}})^2} f(h_1)\,\overline{f(h_2)}\,\chi(\det h_2) \sum_{\gamma\in H_{\mathbf{Q}}} \phi(\Upsilon_0\iota(\gamma h_1,h_2),s;\chi)\,dh_1dh_2 \end{aligned}$$

and

$$\begin{aligned} &Z_1(f,s;\chi) \\ &= \int_{(H_{\mathbf{Q}}\backslash H_{\mathbf{A}})^2} f(h_1)\,\overline{f(h_2)}\,\chi(\det h_2) \sum_{\gamma_1,\gamma_2\in P_{\mathbf{Q}}\backslash H_{\mathbf{Q}}} \phi(\iota(\gamma_1 h_1,\gamma_2 h_2),s;\chi)\,dh_1dh_2. \end{aligned}$$

Since f is cuspidal, we have

$$Z_1(f,s;\chi) = \int_{(P_{\mathbf{Q}}\backslash H_{\mathbf{A}})^2} f(h_1)\,\overline{f(h_2)}\,\chi(\det h_2)\,\phi(\iota(h_1,h_2),s;\chi)\,dh_1dh_2 = 0.$$

Next we have

$$\begin{aligned} &Z_0(f,s;\chi) \\ &= \int_{H_{\mathbf{A}}} dh_1 \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} dh_2\,\phi(\Upsilon_0\iota(h_1,h_2),s;\chi)\,\chi(\det h_2)\,f(h_1)\,\overline{f(h_2)} \\ &= \int_{H_{\mathbf{A}}} dh_1 \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} dh_2\,\phi(\Upsilon_0\iota(h_2,h_2)\iota(h_1,1_2),s;\chi)\,\chi(\det h_2)\,f(h_2h_1)\,\overline{f(h_2)}. \end{aligned}$$

Observe that, for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_{\mathbf{A}}$ and $h' \in H'_{\mathbf{A}}$, we have

$$\begin{aligned} \phi(\Upsilon_0\iota(h,h)h',s;\chi) &= \phi\left(\begin{pmatrix} d & c & c & 0 \\ b & a & 0 & -b \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}\Upsilon_0 h',s;\chi\right) \\ &= \chi^{-1}(\det\begin{pmatrix} d & c \\ b & a \end{pmatrix})||\det\begin{pmatrix} d & c \\ b & a \end{pmatrix}||^{s+1/2}\,\phi(\Upsilon_0 h',s;\chi) \\ &= \chi^{-1}(\det h)\phi(\Upsilon_0 h',s;\chi). \end{aligned}$$

This implies that

$$\begin{aligned} &Z_0(f,s;\chi) \\ &= \int_{H_{\mathbf{A}}} dh_1 \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h_2h_1)\,\overline{f(h_2)}dh_2\,\phi(\Upsilon_0\iota(h_1,1_2),s;\chi) \\ &= \int_{H_{\mathbf{A}}} \phi(\Upsilon_0\iota(h_1,1_2),s;\chi))W_f(h_1)dh_1, \end{aligned}$$

which completes the proof of the proposition. □

6. Local spherical function

6.1.

In this section, we fix a finite prime p and suppress it from the notation. The object of this section is to summarize several results of local spherical functions, whose proofs we omit. We write F and K for $\mathbf{Q}_p$ and K_p respectively. Let $\mathcal{O}_F = \mathbf{Z}_p$ be the integer ring of F and $\mathrm{ord}_F\colon F^\times \to \mathbf{Z}$ the normalized additive valuation of F. Put $\delta = \mathrm{ord}_F D$. Fix a prime element π of F and put $\mathfrak{p}_F = \pi\mathcal{O}_F$. Let $X_{unr}(K^\times) = \mathrm{Hom}(K^\times/\mathcal{O}_K^\times, \mathbf{C}^\times)$ and $X_{unr}(K^1) = \mathrm{Hom}(K^1/\mathcal{O}_K^1, \mathbf{C}^\times)$. Throughout this section, we fix $\chi \in \mathcal{X}$ and $\Omega \in X_{unr}(K^1)$. Let $\widetilde{\chi}$ be the character of $\mathcal{U}_0(D)$ given in Section 1.3.

6.2.

Let $\mathcal{W}$ be the space of functions W on H satisfying

(6.1) $W(uhu') = \widetilde{\chi}(uu')W(h) \quad (u, u' \in \mathcal{U}_0(D), h \in H)$.
(6.2) $W(th) = (\chi\Omega)(t)W(h) \quad (t \in K^1, h \in H)$.

Define Hecke operators on $\mathcal{W}$ as follows:

(i) If K/F is inert, put

$$\mathcal{T}W(h) = -W(h\mathbf{d}(\pi^{-1})) - \sum_{x\in\mathcal{O}_F^\times/\mathfrak{p}_F} W(h\mathbf{n}(\pi^{-1}x)) - \sum_{x\in\mathcal{O}_F/\mathfrak{p}_F^2} W(h\mathbf{n}(x)\mathbf{d}(\pi)).$$

(ii) If K/F ramifies, put

$$\mathcal{T}W(h) = \chi(\Pi) \sum_{x\in\mathcal{O}_F/\mathfrak{p}_F} W(h\mathbf{n}(x)\mathbf{d}(\Pi)) + \chi^{-1}(\Pi) \sum_{x\in\mathcal{O}_F/\mathfrak{p}_F} W\left(h\,\overline{\mathbf{n}}(Dx)\,\mathbf{d}(\Pi^{-1})\right),$$

where Π is a prime element of K.

(iii) If K/F splits, put

$$\mathcal{T}_1W(h) = \chi^{-1}(\Pi_1)\{W(h\mathbf{d}(\Pi_1^{-1})) + \sum_{x\in\mathcal{O}_F/\mathfrak{p}_F} W(h\mathbf{n}(x)\mathbf{d}(\Pi_2))\},$$

$$\mathcal{T}_2W(h) = \chi^{-1}(\Pi_2)\{W(h\mathbf{d}(\Pi_2^{-1})) + \sum_{x\in\mathcal{O}_F/\mathfrak{p}_F} W(h\mathbf{n}(x)\mathbf{d}(\Pi_1))\},$$

where $\Pi_1 = (\pi, 1), \Pi_2 = (1, \pi)$.

It is noted that, for $f \in S_{l-1}(\chi\Omega)$, we have $W_p = W_f|_{H_p}$ is in $\mathcal{W}$ and $\mathcal{T}W_p = W_{\mathcal{T}_p f}|_{H_p}$ ($\mathcal{T}_i W_p = W_{\mathcal{T}_{p,i} f}|_{H_p}$ if p splits in $K/\mathbf{Q}$).

6.3.

First consider the case of $\delta = 0$. Note that $\mathcal{U} = \mathcal{U}_0(D)$ in this case. When K/F is inert, set $\mathcal{W}(\lambda) = \{W \in \mathcal{W} \mid \mathcal{T}W = \lambda W\}$ for $\lambda \in \mathbf{C}$. When K/F splits, set $\mathcal{W}(\lambda) = \{W \in \mathcal{W} \mid \mathcal{T}_i W = \lambda_i W \quad (i = 1, 2)\}$ for $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$. Define $\eta_\lambda \colon H \to \mathbf{C}$ by

$$\eta_\lambda(\mathbf{n}(b)\mathbf{d}(a)u) = (\chi^{-1}\nu)(a)||a||^{1/2}\widetilde{\chi}(u) \qquad (b \in F, a \in K^\times, u \in \mathcal{U}).$$

Here we choose $\nu \in X_{unr}(K^\times)$ as follows:

(i) If K/F is inert,

$$\lambda = p\left(\nu(\pi) + \nu^{-1}(\pi)\right) - p + 1.$$

(ii) If K/F splits,

$$\begin{cases} \lambda_1 = p^{1/2}\left\{\nu(\Pi_1^{-1}) + \nu(\Pi_2)\right\}, \\ \Omega(\Pi_1/\Pi_2) = \nu(\Pi_1^{-1}\Pi_2). \end{cases}$$

(Note that $\lambda_2 = p^{1/2}\left\{\nu(\Pi_2^{-1}) + \nu(\Pi_1)\right\}$ follows.)

We set

$$W_\lambda(h) = \mathrm{vol}(\mathcal{U}_0(D))^{-1} \int_{\mathcal{U}_0(D)} \widetilde{\chi}(u^{-1})\eta_\lambda(uh)du.$$

Note that W_λ does not depend on the choice of ν.

Proposition 6.1. *Suppose that $\delta = 0$.*

(i) *We have $W_\lambda \in \mathcal{W}(\lambda)$ and $W_\lambda(1) = 1$.*
(ii) *We have $\mathcal{W}(\lambda) = \mathbf{C} \cdot W_\lambda$.*

6.4.

We next consider the case of $\delta > 0$. Define operators $\mathcal{Q}_i$ $(i = 1, 2)$ on $\mathcal{W}$ by

$$\mathcal{Q}_1 W(h) = W(hw_D), \ \mathcal{Q}_2 W(h) = W(w_D^{-1}h),$$

where $w_D = \begin{pmatrix} & \sqrt{D}^{-1} \\ \sqrt{D} & \end{pmatrix}$. Note that the operators $\mathcal{T}, \mathcal{Q}_1, \mathcal{Q}_2$ commute each other. Put $h_k = \mathbf{d}(\Pi^k)$ for $k \in \mathbf{Z}$.

Lemma 6.1. *Let $W \in \mathcal{W}$.*

(i) *We have*

$$\mathrm{Supp}W \subset \bigcup_{k\in\mathbf{Z}} \mathcal{U}_0(D)\, h_k\, \mathcal{U}_0(D) \cup \bigcup_{k\in\mathbf{Z}} \mathcal{U}_0(D)\, h_k w_D\, \mathcal{U}_0(D).$$

(ii) *Suppose that* $Q_1 W = Q_2 W = \epsilon W$ *with* $\epsilon \in \{\pm 1\}$. *Then we have* $W(h_{-k}) = \chi^{2k}(\Pi)\cdot W(h_k)$ *for* $k \in \mathbf{Z}$.

6.5.

For $\lambda \in \mathbf{C}$ and $\epsilon \in \{\pm 1\}$, we set

$$\mathcal{W}(\lambda;\epsilon) = \{W \in \mathcal{W} \mid \mathcal{T}W = \lambda W, Q_1 W = Q_2 W = \epsilon W, \}.$$

Let $W \in \mathcal{W}(\lambda;\epsilon)$. In view of the above lemma, W is determined by the values $\{W(h_k) \mid k \in \mathbf{Z}, k \geq 0\}$. We define $\nu^{\pm 1} \in \mathbf{C}^\times$ by the condition $\lambda = p^{1/2}\left(\nu + \nu^{-1}\right)$.

Proposition 6.2. *Suppose that* $\delta > 0$.

(i) *We have* $\dim \mathcal{W}(\lambda;\epsilon) = 1$.

(ii) *There exists a unique element* $W_{\lambda,\epsilon}$ *of* $\mathcal{W}(\lambda;\epsilon)$ *with* $W_{\lambda,\epsilon}(h_0) = 1$ *and we have*

$$W_{\lambda,\epsilon}(h_k) = p^{-k/2}\chi^{-k}(\Pi)\,\frac{\nu^k + \nu^{-k}}{2} \qquad (k \geq 0).$$

7. Local zeta integral

7.1.

Let $f \in S_{l-1}(\chi\Omega)$. Assume that f is a Hecke eigenforms with eigenvalues $\{\lambda_p\}_{p<\infty}$ and that, for any $p|D$, we have $W_{D,p}f = \epsilon_p f$ with $\epsilon_p \in \{\pm 1\}$. We put

$$W_\infty(h_\infty) = \left\{\frac{1}{2}\det h_\infty^{-1}\cdot(-i,1)h_\infty\begin{pmatrix} i \\ 1\end{pmatrix}\right\}^{1-l} \qquad (h_\infty \in H_\infty).$$

Lemma 7.1. *Let* f *be as above. Then, for* $h = (h_v) \in H_{\mathbf{A}}$, *we have*

$$W_f(h) = W_\infty(h_\infty)\cdot \prod_{p\nmid D} W_{\lambda_p}(h_p) \prod_{p|D} W_{\lambda_p,\epsilon_p}(h_p)\, ||f||_H^2.$$

Proof. By Proposition 6.1 and Proposition 6.2, we have

$$W_f(h) = \prod_{p\nmid D} W_{\lambda_p}(h_p) \prod_{p|D} W_{\lambda_p,\epsilon_p}(h_p) \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} f(h_1 h_\infty)\overline{f(h_1)}dh_1.$$

The last integral is equal to

$$\int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} dh_1 \int_{\mathcal{U}_\infty} du_\infty f(h_1 u_\infty h_\infty) j'(u_\infty, z_0)^{l-1} \overline{f(h_1)} dh_1.$$

A holomorphy of f implies that

$$\int_{\mathcal{U}_\infty} f(hu_\infty h_\infty) j'(u_\infty, z_0)^{l-1} du_\infty = W_\infty(h_\infty) f(h) \qquad (h \in H_{\mathbf{A}}, h_\infty \in H_\infty)$$

and we are done. □

7.2.

Let v be a prime of $\mathbf{Q}$. For $h \in H_v$ and $s \in \mathbf{C}$, we put

$$\Lambda_v(h, s) = \phi_v(\Upsilon_0 \iota(h, 1_2), s; \chi).$$

If $p \not| D$, set

$$Z_p(\lambda_p; s) = \int_{H_p} \Lambda_p(h, s) W_{\lambda_p}(h) dh.$$

If $p|D$, set

$$Z_p(\lambda_p, \epsilon_p; s) = \int_{H_p} \Lambda_p(h, s) W_{\lambda_p, \epsilon_p}(h) dh.$$

Finally set

$$Z_\infty(s) = \int_{H_\infty} \Lambda_\infty(h, s) W_\infty(h) dh.$$

The following result is a direct consequence of Proposition 5.1 and Lemma 7.1.

Proposition 7.1. *Let f be as in Section 7.1. Then we have*

$$Z(f, s; \chi) = Z_\infty(s) \cdot \prod_{p \not| D} Z_p(\lambda_p; s) \prod_{p|D} Z_p(\lambda_p, \epsilon_p; s) \, ||f||_H^2.$$

7.3.

To complete the proof of Theorem 4.2, it now remains to show the following results, whose proofs are given in Sections 8–10. Recall that $L_p(\lambda_p; s)$ is defined in Section 1.6.

Proposition 7.2. *Suppose that p does not divide D. Then we have*

$$Z_p(\lambda_p; s) = \lambda_{K_p}(\psi_p)^{-1} \frac{L_p(\lambda_p; s+1)}{\zeta_{\mathbf{Q}_p}(2s+2) L_p(\omega; 2s+3)}.$$

Proposition 7.3. *Suppose that p divides D. Define $\nu_p^{\pm 1} \in \mathbf{C}^\times$ by $\lambda_p = p^{1/2}\left(\nu_p + \nu_p^{-1}\right)$. Then we have*

$$Z_p(\lambda_p, \epsilon_p; s) = \frac{\lambda_{K_p}(\psi_p)^{-1}}{p^{3\delta/2}(1+p^{-1})} \frac{L_p(\lambda_p; s+1)}{\zeta_{\mathbf{Q}_p}(2s+2)L(\omega_p; 2s+3)} \times \left\{1 + \epsilon_p \lambda_{K_p}(\psi_p)\chi_p(\sqrt{D}) \frac{\nu_p^{\delta_p} + \nu_p^{-\delta_p}}{2}\right\}.$$

Proposition 7.4. *We have*

$$Z_\infty(s) = \lambda_{K_\infty}(\psi_\infty)^{-1} l!\, \pi^{2-l}\, 2^{-2s+2} \frac{1}{s+l}.$$

8. Local calculation (I)

8.1.

In this section, we use the notation of Section 6. The object of this section is to study $\Lambda(h,s)$.

Lemma 8.1. *For $h \in H$ and $u, u' \in \mathcal{U}_0(D)$, we have*

$$\Lambda(uhu', s) = \widetilde{\chi}(uu')^{-1}\Lambda(h,s).$$

Proof. First observe that we have

$$\phi(h'\iota(u_1,u_2), s; \chi) = \widetilde{\chi}(u_1u_2)^{-1}\phi(h', s; \chi)$$

for $u_1, u_2 \in \mathcal{U}_0(D)$, since $\mathcal{M}_\chi^{T'}(\iota(u_1,u_2))\varphi_0' = \widetilde{\chi}(u_1u_2)^{-1}\varphi_0'$. Let $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}_0(D)$. Then

$$\begin{aligned}
\Lambda(uh, s) &= \phi(\Upsilon_0 \iota(u,u)\iota(h,1_2)\iota(1_2,u^{-1}), s; \chi) \\
&= \widetilde{\chi}(u^{-1})^{-1}\phi\left(d_{H'}\left(\begin{pmatrix} d & b \\ c & a \end{pmatrix}\right)\Upsilon_0\iota(h,1_2), s; \chi\right) \\
&= \widetilde{\chi}(u)\chi^{-1}(\det u)\phi(\Upsilon_0\iota(h,1_2), s; \chi) \\
&= \widetilde{\chi}(u)^{-1}\Lambda(h,s),
\end{aligned}$$

which completes the proof of the lemma. □

8.2.

For $s \in \mathbf{C}$, we define a function $A_s \colon H \to \mathbf{C}$ by

$$A_s(u\mathbf{d}(y)u') = \alpha_s(y) \qquad (u, u' \in \mathcal{U}, y \in K^\times),$$

where $\alpha_s \colon K^\times \to \mathbf{C}$ is given as follows:

(i) If K is a field,

$$\alpha_s(y) = p^{-|\operatorname{ord}_F \mathrm{N}(y)|s} \qquad (y \in K^\times).$$

(ii) If $K = F \oplus F$,

$$\alpha_s(y) = p^{-(|\operatorname{ord}_F y_1| + |\operatorname{ord}_F y_2|)s} \qquad (y = (y_1, y_2) \in K^\times).$$

Recall that, for $h' \in H'$, we take an $a(h') \in GL_2(K)$ so that $h' \in N' d_{H'}(a(h'))\mathcal{U}'$.

Lemma 8.2. *For $h \in H$, we have*

$$||\det a(\Upsilon_0\iota(h, 1_2))||^s = A_s(h).$$

Proof. We write $\tau_{m,n}$ and τ_m for the characteristic functions of $M_{m,n}(\mathcal{O}_K)$ and $M_m(\mathcal{O}_K)$ respectively. By an argument similar to that of the proof of Lemma 8.1, we see that $h \mapsto ||\det a(\Upsilon_0\iota(h, 1_2))||^s$ is bi-$\mathcal{U}$ invariant. For $h' \in H'$ and $s \in \mathbf{C}$, put

$$I(h', s) = \int_{GL_2(K)} \tau_{4,2}\left((h')^{-1}\begin{pmatrix} X \\ 0 \end{pmatrix}\right) ||\det X||^s dX,$$

where dX is the Haar measure on $GL_2(K)$ with $\mathrm{vol}(GL_2(\mathcal{O}_K)) = 1$. It is easily verified that $I(h', s) = \zeta_K(s-1)\zeta_K(s)||\det a(h')||^s$, where

$$\zeta_K(s) = \begin{cases} (1-p^{-2s})^{-1} & \cdots\ K/F \text{ is inert} \\ (1-p^{-s})^{-1} & \cdots\ K/F \text{ ramifies} \\ (1-p^{-s})^{-2} & \cdots\ K/F \text{ splits.} \end{cases}$$

On the other hand, for $y \in K^\times$, we have

$$\begin{aligned} &I(\Upsilon_0\iota(\mathbf{d}(y), 1_2), s) \\ &= \int_{(K^\times)^2} d^\times a_1 d^\times a_2 \int_K d_1 b\, \tau_{4,2}\left(\begin{matrix} \begin{pmatrix} 0 & -y^{-\sigma} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \\ \begin{pmatrix} -y & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \end{matrix}\right) ||a_1||^{s-1}||a_2||^s \\ &= \int_{K^\times} \tau_1(a_1 y)\tau_1(a_1)||a_1||^{s-1} d^\times a_1 \\ &\qquad \int_{K^\times} \tau_1(a_2 y^{-1})\tau_1(a_2)||a_2||^s d^\times a_2 \int_K \tau_1(by)\tau_1(b) db. \end{aligned}$$

The lemma is now derived from the following elementary formulas:

(i) When K is a field,

$$\int_{K^\times} \tau_1(ay)\tau_1(a)||a||^s d^\times a = \zeta_K(s) \times \begin{cases} 1 & \cdots y \in \mathcal{O}_K \\ ||y||^{-s} & \cdots \text{otherwise}, \end{cases}$$

$$\int_K \tau_1(by)\tau_1(b)db = \begin{cases} 1 & \cdots y \in \mathcal{O}_K \\ ||y||^{-1} & \cdots \text{otherwise}. \end{cases}$$

(ii) When $K = F \oplus F$ and $y = (y_1, y_2) \in K^\times, k_i = \operatorname{ord}_F y_i$, we have

$$\int_{K^\times} \tau_1(ay)\tau_1(a)||a||^s d^\times a = \zeta_K(s) \cdot p^{(\mathrm{Min}(k_1,0)+\mathrm{Min}(k_2,0))s},$$

$$\int_K \tau_1(by)\tau_1(b)db = p^{\mathrm{Min}(k_1,0)+\mathrm{Min}(k_2,0)}.$$

□

Lemma 8.3. *For $h \in H$, we have*

$$\Lambda(h,s) = \chi^{-1}(\det h)\left(\mathcal{M}_\chi^{T'}(\Upsilon_0\iota(h,1_2))\varphi_0'\right)(0)\, A_{s+1}(h).$$

Proof. This follows from Lemma 8.2 and the fact that

$$\int_{K^2} \Phi_0'\left((h')^{-1}\begin{pmatrix} X \\ 0 \end{pmatrix}\right) dX = ||\det a(h')|| \qquad (h' \in H').$$

□

Lemma 8.4. *Let $\varphi' \in \mathcal{S}(K^2)$. Then we have*

$$\begin{aligned}&\left(\mathcal{M}_\chi^{T'}(\Upsilon_0\iota(\mathbf{n}(x)\mathbf{d}(y),1_2))\varphi'\right)(0)\\ &\quad = \lambda_K(\psi)^{-1}\chi(y)^{-1}||y||^{1/2}\int_K \psi\left(x\,\mathrm{N}(w)\right)\varphi'\begin{pmatrix} yw \\ -w \end{pmatrix} dw\end{aligned}$$

for $x \in F$ and $y \in K^\times$.

Proof. Put

$$h' = \Upsilon_0\iota(\mathbf{n}(x)\mathbf{d}(y),1_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -y^\sigma & 0 & -y^{-1}x & 0 \\ y^\sigma & 1 & y^{-1}x & 0 \\ 0 & 0 & y^{-1} & 1 \end{pmatrix}.$$

Then $\mathrm{Ker}(h' - 1_4) = K \cdot {}^t(1, -y^\sigma, 0, 1)$. In view of [10], we obtain

$$\begin{aligned}
&\mathcal{M}_\chi^{T'}(h')\varphi'(0) \\
&= \lambda_K(\psi)^{-3}\chi(-y^{-1})||y||^{-1/2} \int_{K^3} dw_1 dw_2 dw_3\, \psi\left(\frac{1}{2}\langle \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 0 \end{pmatrix}, h' \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 0 \end{pmatrix} \rangle_{T'}\right) \\
&\qquad \rho^{T'}\left((1_4 - h')\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 0 \end{pmatrix}, 0\right)\varphi'(0) \\
&= \lambda_K(\psi)^{-1}\chi^{-1}(y)||y||^{-1/2} \\
&\qquad \int_{K^3} \psi\left(\mathrm{Tr}\left(y^\sigma w_1^\sigma w_1 + w_1^\sigma w_2 + (y^{-1}x - 1)w_1^\sigma w_3 - y^{-1}w_2^\sigma w_3 - \frac{1}{2}x\,\mathrm{N}(y)^{-1}w_3^\sigma w_3\right)\right) \\
&\qquad \varphi'\begin{pmatrix} -y^\sigma w_1 - w_2 + (1 - y^{-1}x)w_3 \\ -y^{-1}w_3 \end{pmatrix} dw_1 dw_2 dw_3.
\end{aligned}$$

Changing the variable w_2 into $w_2 - y^\sigma w_1 - y^{-1}xw_3$, we obtain

$$\begin{aligned}
&\mathcal{M}_\chi^{T'}(h')\varphi'(0) \\
&= \lambda_K(\psi)^{-1}\chi^{-1}(y)||y||^{-1/2} \\
&\qquad \int_{K^3} \psi\left(\mathrm{Tr}\left(w_1^\sigma w_2 - y^{-1}w_2^\sigma w_3 + \frac{1}{2}\,x\,\mathrm{N}(y^{-1}w_3)\right)\right) \\
&\qquad \varphi'\begin{pmatrix} -w_2 + w_3 \\ -y^{-1}w_3 \end{pmatrix} dw_1 dw_2 dw_3 \\
&= \lambda_K(\psi)^{-1}\chi^{-1}(y)||y||^{-1/2} \int_K \psi(x\,\mathrm{N}(y^{-1}w_3))\varphi'\begin{pmatrix} w_3 \\ -y^{-1}w_3 \end{pmatrix} dw_3. \\
&= \lambda_K(\psi)^{-1}\chi(y)^{-1}||y||^{1/2} \int_K \psi\left(x\,\mathrm{N}(w)\right)\varphi'\begin{pmatrix} yw \\ -w \end{pmatrix} dw,
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 8.5.

$$\mathcal{M}_\chi^{T'}(\iota(w_D, 1_2))\varphi_0' = \lambda_K(\psi)^{-1}\chi(\sqrt{D})\varphi_0'.$$

Proof. Since $w_D = \begin{pmatrix} \sqrt{D}^{-1} & \\ & -\sqrt{D} \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, we have

$$\begin{aligned} \mathcal{M}_\chi(w_D)\varphi_0(z) &= \chi(\sqrt{D}^{-1})||\sqrt{D}^{-1}||^{1/2}\mathcal{M}_\chi\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\varphi_0(-\sqrt{D}^{-1}z) \\ &= \chi(\sqrt{D}^{-1})\lambda_K(\psi)\mathrm{char}_{\sqrt{D}^{-1}\mathcal{O}_K}(-\sqrt{D}^{-1}z) \\ &= \lambda_K(\psi)^{-1}\chi(\sqrt{D})\varphi_0(z), \end{aligned}$$

from which the lemma follows. □

Proposition 8.1. *For $x \in F, y \in K^\times$ and $c \in \{0,1\}$, we have*

$$\begin{aligned} \Lambda(\mathbf{n}(x)\mathbf{d}(y)w_D^c, s) &= \lambda_K(\psi)^{c-1}\chi(\sqrt{D}^c y)||y||^{1/2} \\ &\quad \times A_{s+1}(\mathbf{n}(x)\mathbf{d}(\sqrt{D}^{-c}y))\int_K \psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw)dw. \end{aligned}$$

Proof. The proposition is a direct consequence of Lemma 8.3, Lemma 8.4 and Lemma 8.5. □

8.3.

We next calculate the integral

$$\int_F \Lambda(\mathbf{n}(x)\mathbf{d}(y), s)dx.$$

By Proposition 8.1, the integral is equal to $\lambda_K(\psi)^{-1}\chi(y)||y||^{1/2}J(y, s+1)$, where

$$J(y,s) = \int_F dx \int_K dw\, A_s(\mathbf{n}(x)\mathbf{d}(y))\psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw).$$

Lemma 8.6. *For $y \in K^\times$, we have $J(y^{-1}, s) = \overline{J(y, \overline{s})}$.*

Proof. Since $A_s(h^{-1}) = A_s(h)$, we have

$$\begin{aligned} &J(y^{-1}, s) \\ &= \int_F dx \int_K dw\, A_s(\mathbf{d}(y)\mathbf{n}(-x))\psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(y^{-1}w) \\ &= \int_F dx \int_K dw\, A_s(\mathbf{n}(-\mathrm{N}(y)x)\mathbf{d}(y))\psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(y^{-1}w). \end{aligned}$$

Changing the variables $x \mapsto -\mathrm{N}(y)^{-1}x$ and then $w \mapsto yw$, we obtain

$$\begin{aligned} J(y^{-1}, s) &= \int_F dx \int_K dw\, A_s(\mathbf{n}(x)\mathbf{d}(y))\psi(-x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw) \\ &= \overline{J(y, \overline{s})}. \end{aligned}$$

□

The following formula for Gauss sum is well-known.

Lemma 8.7. *For $x \in F^\times$, we have*

$$\int_{\mathcal{O}_K} \psi(x\,\mathrm{N}(w))dw = \begin{cases} p^{-\delta/2} & \cdots \mathrm{ord}_F\, x \geq 0 \\ \lambda_K(\psi)\omega(x)|x|_F^{-1} & \cdots \mathrm{ord}_F\, x \leq -\delta \\ 0 & \cdots \textit{otherwise}, \end{cases}$$

where $\delta = \mathrm{ord}_F\, D$.

The next two results are straightforward consequences of Lemma 8.7.

Lemma 8.8. *Suppose that K is a field and let $y \in K^\times$ with $\mathrm{ord}_F\, \mathrm{N}(y) \geq 0$ (and hence $y \in \mathcal{O}_K$).*

(i) *We have*

$$\int_K \varphi_0(w)\varphi_0(yw)\, dw = p^{-\delta/2}.$$

(ii) *For $k \geq 1$, we have*

$$\int_{\pi^{-k}\mathcal{O}_F^\times} dx \int_K dw\, \psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw) = \begin{cases} (-1)^k(1-p^{-1}) & \cdots \delta = 0 \\ 0 & \cdots \delta > 0. \end{cases}$$

Lemma 8.9. *Suppose that $K = F \oplus F$. Let $y \in \pi^{l_1}\mathcal{O}_F^\times \oplus \pi^{l_2}\mathcal{O}_F^\times$ with $\mathrm{ord}_F\, \mathrm{N}(y) (= l_1 + l_2) \geq 0$.*

(i) *We have*

$$\int_K \varphi_0(w)\varphi_0(yw)\, dw = p^{\mathrm{Min}(l_1,0)+\mathrm{Min}(l_2,0)}.$$

(ii) *For $k \geq 1$, we have*

$$\begin{aligned} &\int_{\pi^{-k}\mathcal{O}_F^\times} dx \int_K dw\, \psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw) \\ &= (1-p^{-1}) \times \begin{cases} 1 & \cdots l_1, l_2 \geq 0 \\ p^{\mathrm{Min}(k+l_2,0)} & \cdots l_1 > 0, l_2 < 0 \\ p^{\mathrm{Min}(k+l_1,0)} & \cdots l_1 < 0, l_2 > 0. \end{cases} \end{aligned}$$

Proposition 8.2. *We have*

$$\int_F \Lambda(\mathbf{n}(x)\mathbf{d}(y), s)dx = p^{-\delta/2}\,\lambda_K(\psi)^{-1}\frac{L(\omega; 2s+2)}{L(\omega; 2s+3)}\,\chi(y)||y||^{1/2}\alpha_{s+1}(y).$$

Proof. It is sufficient to show that

$$J(y, s) = p^{-\delta/2}\frac{L(\omega; 2s)}{L(\omega; 2s+1)}\alpha_s(y). \tag{8.1}$$

In view of Lemma 8.6, to prove (8.1), we may (and do) assume that $\mathrm{ord}_F\,\mathrm{N}(y) \geq 0$. Then we have

$$\begin{aligned}
&J(y, s)\\
&= \int_{\mathcal{O}_F} dx \int_K dw\, \alpha_s(y)\psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw)\\
&\quad + \sum_{k=1}^{\infty}\int_{\pi^{-k}\mathcal{O}_F^\times} dx \int_K dw\ \psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw)\\
&\quad A_s\left(\begin{pmatrix} 1 & 0\\ x^{-1} & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix} 1 & 0\\ x^{-1} & 1\end{pmatrix}\mathbf{d}(y)\right)\\
&= \alpha_s(y)\int_K \varphi_0(w)\varphi_0(yw)\,dw\\
&\quad + \sum_{k=1}^{\infty}\alpha_s(\pi^k y)\int_{\pi^{-k}\mathcal{O}_F^\times} dx \int_K dw\,\psi(x\,\mathrm{N}(w))\varphi_0(w)\varphi_0(yw).
\end{aligned}$$

Suppose that K is a field. Then, by Lemma 8.8, we have

$$\begin{aligned}
J(y, s) &= \alpha_s(y)\,p^{-\delta/2} + \begin{cases} \sum_{k=1}^{\infty}\alpha_s(\pi^k y)(-1)^k(1-p^{-1}) & \cdots\ \delta = 0\\ 0 & \cdots\ \delta > 0\end{cases}\\
&= \alpha_s(y)\,p^{-\delta/2}\times\begin{cases} 1 + (1-p^{-1})\sum_{k=1}^{\infty}(-1)^k p^{-2ks} & \cdots\ \delta = 0\\ 1 & \cdots\ \delta > 0\end{cases}\\
&= \alpha_s(y)\,p^{-\delta/2}\cdot\frac{L(\omega; 2s)}{L(\omega; 2s+1)},
\end{aligned}$$

which proves (8.1) in this case. We can prove (8.1) in the split case in a similar manner. □

9. Local calculation (II)

9.1.

We keep the notation of Section 8. The object of this section is to prove Proposition 7.2 and Proposition 7.3.

9.2. *Proof of Proposition 7.2*

Assume that $\delta = 0$. By Lemma 8.1 and the definition of W_λ (Section 6.3), we obtain

$$\begin{aligned} Z(\lambda, s) &= \int_H \Lambda(h, s)\eta_\lambda(h)dh \\ &= \int_{K^\times} (\chi^{-1}\nu)(y)\|y\|^{-1/2}d^\times y \int_F \Lambda(\mathbf{n}(x)\mathbf{d}(y), s)dx. \end{aligned}$$

By Proposition 8.2, we have

$$\begin{aligned} Z(\lambda, s) &= \lambda_K(\psi)^{-1}\frac{L(\omega; 2s+2)}{L(\omega; 2s+3)} \int_{K^\times} \nu(y)\alpha_{s+1}(y)d^\times y \\ &= \lambda_K(\psi)^{-1}\frac{L(\omega; 2s+2)}{L(\omega; 2s+3)} \cdot \frac{L(\nu; s+1)L(\nu^{-1}; s+1)}{\zeta_K(2s+2)} \\ &= \lambda_K(\psi)^{-1}\frac{L(\lambda; s+1)}{\zeta_F(2s+2)L(\omega; 2s+3)}, \end{aligned}$$

which proves Proposition 7.2.

9.3. *Proof of Proposition 7.3*

Assume that $\delta > 0$. It follows from Proposition 8.1 that

$$\begin{aligned} \Lambda(h_k, s) &= \lambda_K(\psi)^{-1}\chi(\Pi^k)p^{-\delta/2-|k|(s+3/2)}, \\ \Lambda(h_k w_D, s) &= \chi(\sqrt{D}\Pi^k)p^{-\delta/2-|k|/2-|k-\delta|(s+1)} \end{aligned}$$

for $k \in \mathbf{Z}$. It is easily verified that, for $k \in \mathbf{Z}$,

$$\mathrm{vol}(\mathcal{U}_0(D)h_k\mathcal{U}_0(D)) = p^{|k|}\mathrm{vol}(\mathcal{U}_0(D)) = \frac{p^{|k|}}{p^\delta(1+p^{-1})}.$$

Let $W = W_{\lambda,\epsilon}$. By Lemma 6.1, Lemma 8.1 and the above facts, we have

$$\begin{aligned} &Z(\lambda, \epsilon; s) \\ &= \sum_{k\in\mathbf{Z}} \mathrm{vol}(\mathcal{U}_0(D)h_k\mathcal{U}_0(D))\Lambda(h_k, s)W(h_k) \\ &\quad + \sum_{k\in\mathbf{Z}} \mathrm{vol}(\mathcal{U}_0(D)h_k w_D\mathcal{U}_0(D))\Lambda(h_k w_D, s)W(h_k w_D) \\ &= \frac{1}{p^\delta(1+p^{-1})}\sum_{k\in\mathbf{Z}} p^{|k|}W(h_k)\left\{\Lambda(h_k, s) + \epsilon\Lambda(h_k w_D, s)\right\} \\ &= \frac{\lambda_K(\psi)^{-1}}{p^{3\delta/2}(1+p^{-1})}\sum_{k\in\mathbf{Z}} \chi^k(\Pi)W(h_k)\left\{p^{-|k|(s+1/2)} + \epsilon\lambda_K(\psi)\chi(\sqrt{D})p^{|k|/2-|k-\delta|(s+1)}\right\}. \end{aligned}$$

Proposition 6.2 and Lemma 6.1 (ii) imply that

$$W(h_k) = p^{-|k|/2}\chi^{-k}(\Pi)\frac{\nu^k+\nu^{-k}}{2}.$$

A straightforward calculation shows that $Z(\lambda,\epsilon;s)$ is equal to

$$\frac{\lambda_K(\psi)^{-1}}{p^{3\delta/2}(1+p^{-1})}\frac{1-p^{-2s-2}}{(1-\nu p^{-s-1})(1-\nu^{-1}p^{-s-1})}\left\{1+\epsilon\lambda_K(\psi)\chi(\sqrt{D})\,\frac{\nu^\delta+\nu^{-\delta}}{2}\right\}$$

which completes the proof of Proposition 7.3.

10. Local calculation (III)

10.1.

In this section, we calculate

$$Z_\infty(s) = \int_{H_\infty}\Lambda_\infty(h,s)W_\infty(h)dh.$$

We often suppress ∞ from the notation. Recall that

$$\begin{aligned}\Lambda(h,s) &= \chi^{-1}(\det h)\left(\mathcal{M}_\chi^{T'}(\Upsilon_0\iota(h,1_2))\varphi_0'\right)(0)\,||\det a(\Upsilon_0\iota(h,1_2))||^s\\ &\quad\times\int_{\mathbf{C}^2}\Phi_0'\left((\Upsilon_0\iota(h,1_2))^{-1}\begin{pmatrix}X\\0\end{pmatrix}\right)dX,\\ \varphi_0'(z) &= \mathbf{e}[iz^*z] \qquad (z\in\mathbf{C}^2),\\ \Phi_0'(Z) &= (z_1-iz_3)^l\,\overline{(z_2-iz_4)}^l\,\mathbf{e}\left[\frac{i}{2}Z^*Z\right] \qquad (Z={}^t(z_1,z_2,z_3,z_4)\in\mathbf{C}^4),\\ W(h) &= (2\det h)^{l-1}\Delta(h)^{1-l},\end{aligned}$$

where

$$\Delta(h) = (-i,1)h\begin{pmatrix}i\\1\end{pmatrix} = a-ib+ic+d \qquad \left(h=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in H\right).$$

Since $h\mapsto\Lambda(h,s)W(h)$ is right $\mathcal{U}$-invariant, we have

$$Z_\infty(s) = 4\pi\int_{\mathbf{R}}dx\int_0^\infty d^\times y\,y^{-2}\,\Lambda\left(\mathbf{n}(x)\mathbf{d}(y),s\right)W(\mathbf{n}(x)\mathbf{d}(y)).$$

In this section, we frequently use the following elementary formulas.

Lemma 10.1.

(i) *For* $\tau\in\mathbf{C}$ *with* $\operatorname{Re}\tau>0$, $\alpha,\beta,\gamma\in\mathbf{C}$ *and* $l\in\mathbf{Z}, l\geq 0$, *we have*

$$\int_{\mathbf{C}}\mathbf{e}\left[i\tau\,\mathrm{N}(w)+\alpha w+\beta\overline{w}\right](w+\gamma)^l dw = \frac{1}{\tau}\left(\gamma+\frac{i\beta}{\tau}\right)^l\mathbf{e}\left[i\frac{\alpha\beta}{\tau}\right].$$

(ii) *For* $s \in \mathbf{C}$ *with* $\operatorname{Re} s > \frac{1}{2}$, *we have*

$$\int_{\mathbf{R}} (1+x^2)^{-s} dx = \sqrt{\pi}\, \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}.$$

(iii) *For* $s, s' \in \mathbf{C}$ *with* $\operatorname{Re} s > 0, \operatorname{Re} s' > 0, \operatorname{Re}(s'-s) > 0$, *we have*

$$\int_0^\infty y^s (1+y)^{-s'} d^\times y = \frac{\Gamma(s)\Gamma(s'-s)}{\Gamma(s')}.$$

Lemma 10.2. *For* $h \in H$, *we have*

$$\| \det a(\Upsilon_0 \iota(h, 1_2)) \| = |\Delta(h)|^{-2}.$$

Proof. Recall that H' acts on $\mathbf{D}_2 = \{Z \in M_2(\mathbf{C}) \mid \frac{1}{i}(Z - {}^t\overline{Z}) > 0\}$ via

$$h' \begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} h'\langle Z \rangle \\ A \end{pmatrix} j(h', Z) \qquad \left(h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in H' \right),$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $j(h', Z) = A^{-1}c'Z + A^{-1}d'A \in GL_2(\mathbf{C})$. It is easily verified that $\| \det a(h') \| = |\det j(h', i1_2)|^{-2} \quad (h' \in H')$ and $\det j(\Upsilon_0 \iota(h, 1_2), i1_2) = i\Delta(h) \ \ (h \in H)$, from which the lemma follows. □

Lemma 10.3. *For* $h = \mathbf{n}(x)\mathbf{d}(y) \ \ (x \in \mathbf{R}, y > 0)$, *we have*

$$\left(\mathcal{M}_\chi^{T'}(\Upsilon_0 \iota(h, 1_2))\varphi_0' \right)(0) = \frac{1}{i\Delta(h)}.$$

Proof. By an argument similar to that of the proof of Lemma 8.4, we have

$$\left(\mathcal{M}_\chi^{T'}(\Upsilon_0 \iota(\mathbf{n}(x)\mathbf{d}(y), 1_2))\varphi_0' \right)(0) = i^{-1}y^{-1}I,$$

where

$$\begin{aligned} I &= \int_{\mathbf{C}^3} \mathbf{e}\left[\operatorname{Tr}(\overline{w_1}w_2 - y^{-1}\overline{w_2}w_3 + \frac{1}{2}xy^{-2}\operatorname{N}(w_3)) \right] \varphi_0' \begin{pmatrix} -w_2 + w_3 \\ -y^{-1}w_3 \end{pmatrix} dw_1 dw_2 dw_3 \\ &= \int_{\mathbf{C}} \mathbf{e}\left[(i + iy^{-2} + xy^{-2})\operatorname{N}(w_3) \right] dw_3 \int_{\mathbf{C}} dw_1 \\ &\quad \int_{\mathbf{C}} \mathbf{e}\left[i\operatorname{N}(w_2) + (-y^{-1} - i)\operatorname{Tr}(\overline{w_3}w_2) \right] \mathbf{e}[\operatorname{Tr}(\overline{w_1}w_2)] dw_2. \end{aligned}$$

In view of Lemma 10.1 (i) and (ii), I is equal to

$$\begin{aligned}
&\int_{\mathbf{C}} \mathbf{e}\left[(i + iy^{-2} + xy^{-2})\,\mathrm{N}(w_3)\right] dw_3 \\
&\qquad \int_{\mathbf{C}} \mathbf{e}\left[i(-y^{-1}w_3 - iw_3 + w_1)(-y^{-1}\overline{w_3} - i\overline{w_3} + \overline{w_1})\right] dw_1 \\
&= \int_{\mathbf{C}} \mathbf{e}\left[(i + iy^{-2} + xy^{-2})\,\mathrm{N}(w_3)\right] dw_3 \\
&= (1 + y^{-2} - ixy^{-2})^{-1} \\
&= y\Delta(h)^{-1}
\end{aligned}$$

and we are done. □

Lemma 10.4. *For $h = \mathbf{n}(x)\mathbf{d}(y)$ $(x \in \mathbf{R}, y > 0)$, we have*

$$\int_{\mathbf{C}^2} \Phi_0'\left((\Upsilon_0\iota(h, 1_2))^{-1}\begin{pmatrix} X \\ 0 \end{pmatrix}\right) dX = 2^{l+2}\pi^{-l} l! \cdot \Delta(h)^l |\Delta(h)|^{-2l-2}.$$

Proof. We write J for the integral of the lemma, and Δ for $\Delta(h)$ to simplify the notation. Then

$$\begin{aligned}
J &= \int_{\mathbf{C}^2} \Phi_0'\left({}^t(-y^{-1}x_2 + y^{-1}xx_1, x_2, -yx_1, x_1)\right) dx_1 dx_2 \\
&= \int_{\mathbf{C}^2} (-y^{-1}x_2 + y^{-1}xx_1 + iyx_1)^l\, \overline{(x_2 - ix_1)^l} \\
&\qquad \mathbf{e}\left[\frac{i}{2}\left\{\mathrm{N}(-y^{-1}x_2 + y^{-1}xx_1) + \mathrm{N}(x_2) + \mathrm{N}(-yx_1) + \mathrm{N}(x_1)\right\}\right] dx_1 dx_2.
\end{aligned}$$

Changing the variable x_2 into $x_2 + ix_1$ and using Lemma 10.1 (ii), we see that J is equal to

$$\begin{aligned}
&(iy - iy^{-1} + y^{-1}x)^l \int_{\mathbf{C}} \overline{x_2}^l\, \mathbf{e}\left[\frac{i}{2}(1 + y^{-2})\,\mathrm{N}(x_2)\right] dx_2 \\
&\qquad \int_{\mathbf{C}} \mathbf{e}\left[\frac{i|\Delta|^2}{2}\mathrm{N}(x_1) - \frac{\overline{\Delta}}{2y}\overline{x_2}x_1 + \frac{\Delta}{2y}x_2\overline{x_1}\right]\left(x_1 - \frac{1}{iy^2 - i + x}x_2\right)^l dx_1 \\
&= (iy - iy^{-1} + y^{-1}x)^l \frac{2}{|\Delta|^2} \int_{\mathbf{C}} \overline{x_2}^l\, \mathbf{e}\left[\frac{i}{2}(1 + y^{-2})\,\mathrm{N}(x_2)\right] \\
&\qquad \left(-\frac{1}{y(iy - iy^{-1} + y^{-1}x)}x_2 + i\frac{2}{|\Delta|^2}\frac{\Delta}{2y}x_2\right)^l \mathbf{e}\left[i\frac{2}{|\Delta|^2}\left(-\frac{\overline{\Delta}}{2y}\frac{\Delta}{2y}\mathrm{N}(x_2)\right)\right] dx_2 \\
&= 2^{l+1}|\Delta|^{-2}\overline{\Delta}^{-l} \int_{\mathbf{C}} \mathrm{N}(x_2)^l\, \mathbf{e}\left[\frac{i}{2}\mathrm{N}(x_2)\right] dx_2 \\
&= 2^{l+2}\pi^{-l} l! \cdot \Delta^l |\Delta|^{-2l-2},
\end{aligned}$$

which completes the proof of the lemma. □

Proposition 10.1. *For $h = \mathbf{n}(x)\mathbf{d}(y)$ $(x \in \mathbf{R}, y > 0)$, we have*

$$\Lambda(h,s)W(h) = \frac{2^{2l+1}\pi^{-l}l!}{i}\,|\Delta(h)|^{-2s-2l-2}.$$

Proof. By Lemma 10.2, Lemma 10.3 and Lemma 10.4, we have

$$\Lambda(h,s) = \frac{2^{l+2}\pi^{-l}l!}{i}\,\Delta(h)^{l-1}|\Delta(h)|^{-2s-2l-2}.$$

On the other hand, we have

$$W(h) = 2^{l-1}\Delta(h)^{1-l}$$

in view of the definition of W (cf. Section 7.1). The proposition immediately follows from these. □

10.2. *Proof of Proposition 7.4*

By Section 10.1 and Proposition 10.1, we have

$$Z_\infty(s) = \frac{2^{2l+3}\pi^{1-l}l!}{i}\int_{\mathbf{R}} dx \int_0^\infty d^\times y\, y^{-2}|y + y^{-1} - iy^{-1}x|^{-2(s+l+1)}.$$

Changing the variables x into $(1+y^2)x$ and then y into $\sqrt{y}$, we obtain

$$\begin{aligned} Z_\infty(s) &= \frac{2^{2l+2}\pi^{1-l}l!}{i}\int_{\mathbf{R}}(1+x^2)^{-(s+l+1)}dx \int_0^\infty y^{s+l}(1+y)^{-2s-2l-1}\,d^\times y \\ &= \frac{2^{2l+2}\pi^{1-l}l!}{i}\sqrt{\pi}\,\frac{\Gamma\left(s+l+\frac{1}{2}\right)}{\Gamma(s+l+1)}\cdot\frac{\Gamma(s+l)\Gamma(s+l+1)}{\Gamma(2s+2l+1)} \\ &= i^{-1}l!\,\pi^{2-l}\,2^{-2s+2}\,\frac{1}{s+l}. \end{aligned}$$

We have now completed the proof of Proposition 7.4 since $\lambda_{K_\infty}(\psi_\infty) = i$.

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ON CERTAIN AUTOMORPHIC FORMS OF $Sp(1, q)$ (ARAKAWA'S RESULTS AND RECENT PROGRESS)

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Dedicated to the memory of Prof. Tsuneo Arakawa

0. Introduction

Around the beginning of the 1980s, Professor Tsuneo Arakawa initiated the research of certain non-holomorphic automorphic forms on the real symplectic group $Sp(1, q)$ of signature $(1+, q-)$. In [14] we understood them as automorphic forms on $Sp(1, q)$ generating quaternionic discrete series (for the definition of this discrete series, see Gross-Wallach [8]). Arakawa gave two published works [2] and [3] for the research. They deal with an explicit dimension formula for the spaces of such automorphic forms with respect to neat non-uniform lattice subgroups. On the other hand, Arakawa left us several unpublished notes on the study of such forms. In one of them he formulated a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$, which is inspired by Kudla's work [13] on a theta lifting from elliptic modular forms to holomorphic automorphic forms on $SU(1, q)$. In addition to this, there is his unpublished work on the spinor L-function attached to the automorphic forms on $Sp(1, 1)$ above. He considered it by following the method of Andrianov [1] (see also [4]).

The aim of this note is two-fold. One aim is to survey Arakawa's result on the dimension formula and the other to explain our recent result on Arakawa's theta lifting together with his result. As for the latter work, we proved that the images of the lifting are bounded automorphic forms on

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$Sp(1,q)$ generating quaternionic discrete series for an arbitrary q. This kind of result is already done by Arakawa for the case of $q = 1$ but our method of proof is different. To be precise we use the theory of Fourier expansion developed in [14] for these automorphic forms.

We explain the plan of this note. In Section 1 we describe the structure of $Sp(1,q)$ and its subgroups etc. In Section 2 we recall Arakawa's definition of the automorphic forms on $Sp(1,q)$ and the results on the Fourier expansion by Arakawa [3] and [14]. The section 3 is devoted to the survey on Arakawa's works on the dimension formula. In Section 4 we review Arakawa's formulation of the theta lifting and overview the proof of our result on the lifting. A detail of it will appear elsewhere.

Notations

For a ring R, R^n (resp. $M_m(R)$) denotes the set of row vectors with its length n and entries in R (resp. the set of $m \times m$-matrices with coefficients in R). Given a set S of integers, $\mathrm{l.c.m.}\, S$ means the least common multiple of S.

1. Structure of $Sp(1,q)$

Throughout this paper $\mathbb{H}$ denotes the Hamilton quaternion algebra with the standard basis $\{1, i, j, k\}$. We can embed $\mathbb{H}$ into $M_2(\mathbb{C})$ by

$$\varphi : \mathbb{H} \ni x_1 + x_2 i + x_3 j + x_4 k \mapsto \begin{pmatrix} x_1 + \sqrt{-1}x_2 & x_3 + \sqrt{-1}x_4 \\ -(x_3 - \sqrt{-1}x_4) & x_1 - \sqrt{-1}x_2 \end{pmatrix} \in M_2(\mathbb{C})$$

for $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Its reduced trace tr and reduced norm ν are given by

$$\mathbb{H} \ni a \mapsto \mathrm{tr}(a) := a + \bar{a} \in \mathbb{R},$$
$$\mathbb{H} \ni a \mapsto \nu(a) := a\bar{a} \in \mathbb{R}_{\geq 0},$$

where $\mathbb{H} \ni a \mapsto \bar{a} \in \mathbb{H}$ is the main involution of $\mathbb{H}$. Via φ these tr and ν correspond to the trace and the determinant of $M_2(\mathbb{C})$, respectively. For an element $a \in \mathbb{H}$ we will often use $\sqrt{\nu(a)}$. Thus we denote it simply by $d(a)$.

With a fixed positive definite quaternion-Hermitian matrix $S \in M_{q-1}(\mathbb{H})$, we set

$$Q := \begin{pmatrix} -S & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

When $q = 1$ we understand Q as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the symplectic group $G = Sp(1, q)$ of signature $(1+, q-)$ is defined as

$$G = Sp(1, q) := \{g \in M_{q+1}(\mathbb{H}) \mid {}^t\bar{g}Qg = Q\}.$$

For a description of G it is convenient to write g in the block decomposition $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ (resp. $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$) with $a_1 \in M_{q-1}(\mathbb{H})$, $b_1, c_1, {}^ta_2, {}^ta_3 \in {}^t\mathbb{H}^{q-1}$ and $b_2, c_2, b_3, c_3 \in \mathbb{H}$ when $q > 1$ (resp. $a_1, b_1, a_2, b_2 \in \mathbb{H}$ when $q = 1$). The Riemannian symmetric space corresponding to G is the quaternion hyperbolic space (cf. [10, Chap.X, Section 2]), which is realized as

$$H := \begin{cases} \{z = (w, \tau) \in {}^t\mathbb{H}^{q-1} \times \mathbb{H} \mid \mathrm{tr}(\tau) > {}^t\bar{w}Sw\} & (q > 1) \\ \{z \in \mathbb{H} \mid \mathrm{tr}(z) > 0\} & (q = 1) \end{cases}$$

(cf. [2, Section 1], [3, (0.3)]). The group G acts on H via the linear fractional transformation

$$g(z) := \begin{cases} ((a_1 w + b_1 \tau + c_1)\mu(g, z)^{-1}, (a_2 w + b_2 \tau + c_2)\mu(g, z)^{-1}) & (q > 1) \\ (a_1 z + b_1)\mu(g, z)^{-1} & (q = 1) \end{cases},$$

where we write g in the block decomposition above and where $\mu(g, z)$ denotes the automorphic factor for $G \times H$ given by

$$\mu(g, z) := \begin{cases} a_3 w + b_3 \tau + c_3 & (q > 1) \\ a_2 z + b_2 & (q = 1) \end{cases}.$$

Let $z_0 := \begin{cases} (0, 1) & (q > 1) \\ 1 & (q = 1) \end{cases}$. Then $K := \{g \in G \mid g(z_0) = z_0\}$ forms a maximal compact subgroup of G. This can be expressed as the isometry subgroup for the majorant $R := \begin{pmatrix} S & \\ & 1_2 \end{pmatrix}$ of Q;

$$K = \{g \in G \mid {}^t\bar{g}Rg = R\},$$

where $R := 1_2$ when $q = 1$.

Moreover a maximal unipotent subgroup N and a maximal split torus

A of G are given as follows:

$$N := \begin{cases} \left\{ n(w,x) := \begin{pmatrix} 1_{q-1} & 0_{q-1,1} & w \\ {}^t\bar{w}S & 1 & \frac{1}{2}{}^t\bar{w}Sw + x \\ 0_{1,q-1} & 0 & 1 \end{pmatrix} \;\middle|\; w \in {}^t\mathbb{H}^{q-1},\ x \in X_{\mathbb{R}} \right\} & (q > 1), \\ \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in X_{\mathbb{R}} \right\} & (q = 1), \end{cases}$$

$$A := \begin{cases} \left\{ a = a_y := \begin{pmatrix} 1_{q-1} & & \\ & \sqrt{y} & \\ & & \sqrt{y}^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_+ \right\} & (q > 1), \\ \left\{ a = a_y := \begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_+ \right\} & (q = 1). \end{cases}$$

Here

$$X_{\mathbb{R}} := \{x \in \mathbb{H} \mid \mathrm{tr}(x) = 0\}$$

is the set of pure quaternions. Then G has an Iwasawa decomposition $G = NAK$.

2. Reviews on automorphic forms of $Sp(1,q)$ introduced by Arakawa

In this section we recall the automorphic forms on G defined by Arakawa [2] and [3], and collect the results in [2], [3] and [14] necessary for the later discussion. Let B denote a fixed definite quaternion algebra over $\mathbb{Q}$. From now on, we assume $S \in M_{q-1}(B)$ for Q and fix a $\mathbb{Q}$-structure $G(\mathbb{Q})$ of G by setting $G(\mathbb{Q}) := G \cap M_{q+1}(B)$. Moreover we note that almost all results in this section are dealt with in [14] for the case $S = 1_{q-1}$. They remain valid for general S since we can reduce the problems to the case of $S = 1_{q-1}$ by suitable transformation of variables.

For a positive integer κ let σ_κ denote the pull-back of the κ-th symmetric tensor representation of $GL_2(\mathbb{C})$ to $\mathbb{H}^\times$ via φ. This defines an irreducible representation (τ_κ, V_κ) of K by

$$\tau_\kappa(k) := \sigma_\kappa(\mu(k, z_0)) \quad (k \in K).$$

Let π_κ be the discrete series representation of G with minimal K-type τ_κ. Due to the regularity of the Harish-Chandra parameter of π_κ, its existence is justified when $\kappa > 2q-1$ (for details of discrete series see [12, Chap.IX, Section 7, Theorem 9.20, Chap.XII, Section 5, Theorem 12.21]). This discrete series is a quaternionic discrete series in the sense of Gross and Wallach [8, Proposition 5.7, Note 5.9]. Inspired by Takahashi [17], Arakawa introduced in [3] some wider class of discrete series representations of G containing π_κ. This π_κ correponds to the case of "$m = 0$" in [3].

For this π_κ we recall its matrix coefficient $\omega_\kappa : G \to \mathrm{End}(V_\kappa)$ explicitly given by

$$\omega_\kappa(g) := \sigma_\kappa(D(g))^{-1}\nu(D(g))^{-1} \quad (g \in G),$$

where

$$D(g) := \frac{1}{2}(\tau(g(z_0)) + 1)\mu(g, z_0)$$

with

$$\tau(z) := \begin{cases} z & (q = 1) \\ \text{the second entry of } z & (q > 1) \end{cases}$$

for $z \in H$ (cf. [2, Section 1], [3, (3.5)]). When $\kappa > 4q$, ω_κ is integrable on G (cf. [3, Lemma 2.10 (ii)]), which implies that π_κ is an integrable representation.

Let $dg := y^{-2(q+1)}dwdxdydk$ be the invariant measure determined by the Iwasawa decomposition of G, where dw, dx and dy denote the Euclidean measure on ${}^t\mathbb{H}^{q-1}$, $X_\mathbb{R}$ and $\mathbb{R}$ respectively, and dk is the invariant measure of K such that $\int_K dk = 1$ (cf. [3, Section 1.2]). We denote by d_κ the formal degree of π_κ with respect to dg (cf. [3, (2.4)]) and by $\Delta(S)$ the positive real number such that $d(Sw) = \Delta(S)^2 dw$. We set $c_\kappa := 2^{-2(q+1)}\pi^{-2q}\Delta(S)\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-2q)}$ (cf. [3, Proposition 2.9]), which turns out to be $\frac{d_\kappa}{\dim V_\kappa}$ in [3, Section 2.6].

Using these notations, Arakawa gave a definition of the automorphic forms on G as follows (cf. [3, Section 3.2]):

Definition 2.1. Let $\kappa > 4q$. For a lattice subgroup Γ of G let $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ be the space of V_κ-valued continuous functions f on G satisfying
(1) f is bounded on G,
(2) $f(\gamma g k) = \tau_\kappa(k)^{-1}f(g), \quad \forall(\gamma, g, k) \in \Gamma \times G \times K$,
(3) $c_\kappa \int_G \omega_\kappa(g^{-1}h)f(g)dg = f(h)$.

Remark 2.2. (1) This automorphic form is checked to be cuspidal when Γ is non-uniform (cf. [3, Proposition 3.1]).
(2) With representation theoretic terminology we can formulate the notion

of these automorphic forms without assuming their boundedness or the integrability of π_κ when $\Gamma \subset G(\mathbb{Q})$. Actually we can understand them as automorphic forms generating π_κ (cf. [14, Definition 6.1, Definition 8.1]).

For our study of the theta lifting to $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ the Fourier expansion provides an important tool. Thus we recall several results in [3] and [14] on the expansion. To this end we introduce some notations.

We assume $\Gamma \subset G(\mathbb{Q})$. Let Ξ be a complete set of representatives for the set of Γ-cusps $\Gamma\backslash G(\mathbb{Q})/P(\mathbb{Q})$, where $P(\mathbb{Q}) := P \cap G(\mathbb{Q})$ with the standard proper parabolic subgroup P of G. For each $c \in \Xi$ we set

$$\begin{aligned} N_{\Gamma,c} &:= N \cap c^{-1}\Gamma c, \\ X_{\Gamma,c} &:= \{x \in X_{\mathbb{R}} \mid n(0,x) \in N_{\Gamma,c}\}, \\ X^*_{\Gamma,c} &:= \text{dual lattice of } X_{\Gamma,c} \text{ with respect to tr}, \\ \Lambda_c &:= \{\lambda \in {}^t\mathbb{H}^{q-1} \mid n(\lambda, x_\lambda) \in N_{\Gamma,c} \ \exists x_\lambda \in X_{\mathbb{R}}\}, \end{aligned}$$

where Λ_c is defined when $q > 1$. Here we note that x_λ in the definition of Λ_c is unique modulo $X_{\Gamma,c}$ for each $\lambda \in \Lambda_c$.

Let $q > 1$. For $\xi \in X^*_{\Gamma,c} \setminus \{0\}$ we introduce a space of theta functions

$$\Theta_{\xi,c} := \left\{ \theta \in C({}^t\mathbb{H}^{q-1}) \;\middle|\; \begin{array}{l} \theta(w+\lambda) = \mathbf{e}(\mathrm{tr}(\xi({}^t\bar{w}S\lambda - x_\lambda)))\theta(w) \ \forall \lambda \in \Lambda_c \\ \int_{{}^t\mathbb{H}^{q-1}} k_\xi(w', w)\theta(w')dw' = \theta(w) \end{array} \right\},$$

where $C({}^t\mathbb{H}^{q-1})$ is the space of continuous functions on ${}^t\mathbb{H}^{q-1}$ and

$$\begin{aligned} k_\xi(w', w) :=& \Delta(S)2^{4(q-1)}\nu(\xi)^{q-1}\exp(-2\pi d(\xi){}^t\overline{(w-w')}S(w-w')) \\ &\times \mathbf{e}(-\mathrm{tr}(\xi{}^t\bar{w}'Sw)). \end{aligned}$$

For each $\xi \in X^*_{\Gamma,c} \setminus \{0\}$ we fix $u_\xi \in \{x \in \mathbb{H} \mid \nu(x) = 1\}$ such that $u_\xi i \bar{u}_\xi = \xi/d(\xi)$. We denote by v_κ a fixed highest weight vector of V_κ. Then the Fourier expansion of $f \in \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ at a cusp $c \in \Xi$ can be written as follows (cf. [3, Theorem 6.1], [14, Theorem 6.3, Section 9]):

Theorem 2.3. *Let $f \in \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$. When $q > 1$ the Fourier expansion of f at a cusp $c \in \Xi$ is written as*

$$f(cn(w,x)a_y) = \sum_{\xi \in X^*_{\Gamma,c}\setminus\{0\}} a^f_\xi(w) y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\mathbf{e}(\mathrm{tr}(\xi x))\sigma(u_\xi)v_\kappa$$

and when $q = 1$ it is written as

$$f(cn(x)a_y) = \sum_{\xi \in X^*_{\Gamma,c}\setminus\{0\}} C^f_\xi y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\mathbf{e}(\mathrm{tr}(\xi x))\sigma(u_\xi)v_\kappa,$$

where $a^f_\xi \in \Theta_{\xi,c}$ and C^f_ξ denotes a constant dependent only on ξ and f.

For this theorem we remark that this expansion does not depend on the choices of u_ξ's (cf. [14, Remark 6.4 (1)]) and that our work [14] provides Fourier expansion valid also for unbounded forms.

Now we state a proposition crucial for our result on the theta lifting.

Proposition 2.4. *Let f' be a V_κ-valued continuous function on G satisfying the condition* (2) *in Definition* 2.1. *If the Fourier expansion f' is of the form in Theorem* 2.3, *then $f' \in \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$.*

Proof. When $q > 1$ this is deduced from the following two:

(i) the Fourier series in Theorem 2.3 converges uniformly and absolutely on $G_\epsilon := \{g \in G \mid g = n(w,x)a_y k \text{ with } y > \epsilon\}$ for each $\epsilon > 0$ (cf. [14, Corollary 7.4]),

(ii) the functions appearing in the Fourier expanison fulfill the condition in Definition 2.1 (3) (cf. [14, Lemma 8.6]).

The proof for the case of $q = 1$ is parallel. □

3. Dimension formula of $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$

In this section we survey Arakawa's results [2] and [3] on the dimension formula of $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$. For this section we do not assume that a discrete subgroup Γ to define $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ is contained in $G(\mathbb{Q})$. Arakawa's study on the dimension formula starts from [2], which deals with the case of $q = 1$. In [3] he generalizes it to the case of an arbitray q. Now recall that we remarked in Section 2 that some wide class of discrete series representations of G containing π_κ is introduced in [3]. To be precise Arakawa [3] established a dimension formula for such class of discrete series. However, for simplicity, we consider only the case of π_κ here.

First we state his result for the case of $q = 1$. Let $\mathcal{O}$ be a fixed maximal order of B and $d(B)$ be a product of primes at which B is ramified. For a positive integer N we set

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \cap M_2(B) \;\middle|\; a-1,\ b,\ c,\ d-1 \in N\mathcal{O} \right\},$$

i.e. a principal congruence subgroup of G. Then Arakawa's result for the case of $q = 1$ is given as

Theorem 3.1. *Suppose $\kappa > 4$ and $N \geqslant 3$. Then*

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{A}_0(\Gamma(N)\backslash G, \omega_\kappa) =& 2^{-8}3^{-3}5^{-1}[\Gamma(1):\Gamma(N)]\kappa(\kappa-1)(\kappa+1) \\ & \times \prod_{p|d(B)} (p-1)(p^2+1) \\ & - 2^{-3}3^{-1}[\Gamma(1):\Gamma(N)]N^{-3} \prod_{p|d(B)} (p-1).\end{aligned}$$

For this theorem see [2, Theorem 2].

This result is extended to the case of an arbitrary q. It is formulated for a general neat non-uniform lattice subgroup Γ of G, which means that $\Gamma\backslash G$ is not compact but its volume is finite and that if some power of $\gamma \in \Gamma$ is unipotent, γ is necessarily unipotent. We note that $\Gamma(N)$ in Theorem 3.1 is an example of such Γ.

Let $t(\Gamma)$ be the number of Γ-cusps. Then Arakawa's dimension formula for the case of any q is stated as

Theorem 3.2. *Suppose $\kappa > 4q$. Let Γ be a neat non-uniform lattice of G. Then*

$$\dim_{\mathbb{C}} \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa) = \begin{cases} d_\kappa \int\limits_{\Gamma\backslash G} dg & (q>1) \\ d_\kappa \int\limits_{\Gamma\backslash G} dg - t(\Gamma) & (q=1) \end{cases}.$$

For this theorem see [3, Theorem 2].

We explain the outline of the proof for these two theorems. The proof begins with

Theorem 3.3. *Suppose $\kappa > 4q$. Then*

$$\dim_{\mathbb{C}} \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa) = \int_G \operatorname{Tr} K_\kappa^\Gamma(g,g)dg,$$

where

$$K_\kappa^\Gamma(g,h) := c_\kappa \sum_{\gamma\in\Gamma} \omega_\kappa(g^{-1}\gamma h) \quad \textit{for } g,h \in G.$$

For this theorem see [2, Theorem 1] and [3, Theorem 1, Theorem 4.2]. Then the steps to deduce Theorem 3.1 and Theorem 3.2 consist of

(i) classification of elements in Γ,
(ii) evaluation of the contribution of each class in (1) to the formula in Theorem 3.3.

As for the step (1), the neatness of Γ implies that Γ is divided into the following three classes:

(i) central elements, (ii) hyperbolic elements, (iii) unipotent elements.

For this step see [3, Section 5.1 Definition, Lemma 5.5]. In particular, the set of central elements consists only of $\{1\}$.

Then it suffices to evaluate the contributions of the classes (i), (ii) and (iii). Among the three contributions, the contribution of the central element, i.e. $\{1\}$ is the easiest to handle. It is nothing but $d_\kappa \int_{\Gamma\backslash G} dg$. Recalling the definition of c_κ and d_κ in Section 2, we see

$$d_\kappa = 2^{-2(q+1)}\pi^{-2q}\Delta(S)\frac{\Gamma(\kappa+2)}{\Gamma(\kappa+1-2q)}.$$

When $q = 1$ Arakawa also calculates the volume of $\Gamma(N)\backslash G$ explicitly, which is equal to

$$2^{-4}3^{-3}5^{-1}\pi^2[\Gamma(1):\Gamma(N)]\prod_{p|d(B)}(p-1)(p^2+1)$$

(cf. [2, Section 3]). Thus, when $q = 1$ and $\Gamma = \Gamma(N)$, the contribution of the central element is

$$2^{-8}3^{-3}5^{-1}[\Gamma(1):\Gamma(N)]\kappa(\kappa-1)(\kappa+1)\prod_{p|d(B)}(p-1)(p^2+1).$$

Next we evaluate the contribution of hyperbolic elements in Γ. The absolute convergence of the contribution is stated in [2, Lemma 3.5] and [3, Lemma 5.1]. Arakawa essentially uses the convergence of Eisenstein series in order to justify such convergence. In fact, he estimates such contribution by a certain partial sum of Eisenstein series. Then, by formal computation, we can write the hyperbolic contribution as a sum of orbital integrals

$$\int_{C_\gamma\backslash G} \operatorname{Tr}\omega_\kappa(g^{-1}\gamma g)dg$$

for hyperbolic $\gamma \in \Gamma$, where C_γ denotes the centralizer of γ in Γ. Now we note that the Selberg principle (cf. [9, Theorem 11]) asserts that this integral is equal to zero. Therefore there is no contribution of hyperbolic elements to $\dim_{\mathbb{C}} \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$.

As the last step, it remains to calculate the unipotent contribution. For details on this see [2, Section 3] and [3, Section 5.2]. We divide the unipotent elements in Γ into two sets as follows:

$$\Pi_1(\Gamma) := \{\gamma \in \Gamma \mid \gamma \text{ is } G\text{-conj. to some } n(0,x) \in N \text{ with } x \in X_{\mathbb{R}} \setminus \{0\}\},$$
$$\Pi_2(\Gamma) := \{\gamma \in \Gamma \mid \gamma \text{ is } G\text{-conj. to some } n(w,x) \in N \text{ with } w \in \mathbb{H}^{q-1} \setminus \{0\}\}.$$

When $q = 1$ the set $\Pi_2(\Gamma)$ never appears. We call the contribution by $\Pi_1(\Gamma)$ (resp. $\Pi_2(\Gamma)$) the central unipotent contribution (resp. the non-central unipotent contribution).

We first consider the non-central unipotent contribution, which is proved to vanish. As for its convergence, Arakawa uses the Fourier transformation formula of ω_κ (cf. [2, Lemma 1.2]) and the Poisson summation formula. Actually these two formulas imply that the contribution is estimated by a sum of certain convergent integrals over $c^{-1}\Gamma c \cap N\backslash G$, where the summation runs over representatives c's of Γ-cusps. More precisely the integrand for each c is bounded by a sum of rapidly decreasing functions (more specifically, some negative power of the exponential function) over some lattice L_c of $\mathbb{H}^{q-1}$. Then, exchanging formally some integral and sum involved in the contribution, the problem turns out to be reduced to vanishing of period integrals of non-trivial additive characters on $\mathbb{H}^{q-1}/L_c$. Since such integrals are actually proved to vanish, we see the vanishing of the non-central unipotent contribution.

Next we deal with the central unipotent contribution, which is evaluated as

$$\begin{cases} -t(\Gamma) & (q = 1) \\ 0 & (q > 1) \end{cases}.$$

The method to deduce this is to understand this contribution as a sum of special values of some zeta integrals representing the Epstein zeta function attached to the quadratic form on $X_{\mathbb{R}}$ defined by the reduced norm ν. Arakawa proved that the contribution is equal to a finite sum of special values of such integrals in the convergence range. Hence the understanding above is justified. Furthermore he related such special values to the evaluation of the Epstein zeta function above at $1 - q$. In fact, he showed that the finite parts of the zeta integrals coincide with such zeta function. In view of the vanishing of that Epstein zeta function at negative integers we know that there is no central unipotent contribution for $q > 1$. For the case of $q = 1$ Arakawa carried out an explicit computation of the infinite component of the zeta-integral. With the help of the fact that the special value of our Epstein zeta function at 0 is equal to -1, such calculation leads to the evaluation of the central unipotent contribution above for the case of $q = 1$. Moreover, when $\Gamma = \Gamma(N)$ and $q = 1$, Arakawa calculated such contribution more explicitly in [2, Section 3], given as

$$-2^{-3}3^{-1}[\Gamma(1) : \Gamma(N)]N^{-3} \prod_{p \mid d(B)} (p-1).$$

The deduction of this starts from writing the contribution as

$$-[\Gamma(1) : \Gamma(N)]N^{-3} \sum_{c \in \Xi} \frac{1}{[\Gamma(1) \cap cPc^{-1} : \Gamma(1) \cap cNc^{-1}]}$$

(cf. [2, (3.6)]), which the argument above yields. To make the summation with respect to Ξ more explicit Arakawa used a Mass formula for unit groups of maximal orders in B (cf. [2, (3.8)]). This shows the explicit evaluation just above. As a result, all the theorems above are settled.

4. Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$

This section is devoted to the explanation of Arakawa's works and our recent progress on a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$. We divide this section into 6 subsections. In Section 4.1 we introduce a theta series to formulate the lifting and state our theorem. In Section 4.2 we deduce a transformation formula of the theta series. Then, in Section 4.3, we show that the theta series lifts elliptic cusp forms to some automorphic forms on $Sp(1, q)$ via the convolution. In Section 4.4 and Section 4.5 we consider the liftings of elliptic Poincaré series. In order to verify that such liftings belong to $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ (with some fixed arithmetic subgroup Γ), we study their Fourier expansion in Section 4.5. Then the proof of the theorem is completed in Section 4.6. For this section we remark that Section 4.2, Section 4.3 and Section 4.4 are based on Arakawa's unpublished notes.

4.1. *Statement of Theorem*

We set $V := \mathbb{H}^{q+1}$. As we did in Section 3, we let $\mathcal{O}$ be a fixed maximal order of B and the product $d(B)$ of ramified non-Archimedean primes of B. An isomorphism $\mathbb{H} \simeq \mathbb{R}^4$ induces $V \simeq \mathbb{R}^{4(q+1)}$. The quaternion Hermitian matrices Q and R define two Hermitian forms on V:

$$(*,*)_Q : V \times V \ni (x, y) \mapsto (x, y)_Q := \operatorname{tr} xQ^t\bar{y} \in \mathbb{R},$$
$$(*,*)_R : V \times V \ni (x, y) \mapsto (x, y)_R := \operatorname{tr} xR^t\bar{y} \in \mathbb{R}.$$

Via the isomorphism $V \simeq \mathbb{R}^{4(q+1)}$ the form $(*,*)_Q$ (resp. $(*,*)_R$) is identified with a symmetric bilinear form on $\mathbb{R}^{4(q+1)}$ of signature $(4+,4q-)$ (resp. $(4(q+1)+,0-)$). In view of this identification we can regard G as a subgroup of the special orthogonal group $SO(4,4q)$ of signature $(4+,4q-)$.

Let $\mathfrak{h}$ denote the complex upper half plane. For $x \in V$ and $z = s + \sqrt{-1}t \in \mathfrak{h}$ we set

$$Q_z(x) := s(x,x)_Q + \sqrt{-1}t(x,x)_R,$$
$$F_z(x) := \sigma_\kappa(x_q + x_{q+1})\mathrm{e}\left(\frac{1}{2}Q_z(x)\right),$$

where $x = (x_1, x_2, \dots, x_{q+1}) \in V$ and we regard σ_κ as an $\mathrm{End}(V_\kappa)$-valued function on $\mathbb{H}$. Then we define the theta series on $\mathfrak{h} \times G$ as follows:

$$\theta(z,g) := t^{2q} \sum_{l \in L} F_z(l^t \bar{g}^{-1}) \quad ((z,g) \in \mathfrak{h} \times G),$$

where $L := \mathcal{O}^{q+1}$.

¿From now on, we assume

$$S = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_{q-1} \end{pmatrix}$$

with $\alpha_i \in \mathbb{Z}_{>0}$ for $1 \leqslant i \leqslant q-1$ when $q > 1$. Moreover let N be an integer divisible by

$$\begin{cases} \mathrm{l.\,c.\,m.}\{2, d(B), \alpha_1, \alpha_2, \dots, \alpha_{q-1}\} & (q > 1) \\ \mathrm{l.\,c.\,m.}\{2, d(B)\} & (q = 1) \end{cases}$$

and let

$$\Gamma := \{\gamma \in G \cap M_{q+1}(B) \mid L^t\bar{\gamma} = L\}.$$

We denote by $\mathcal{S}_{\kappa-2q+2}(\Gamma_0(N))$ the space of elliptic cusp forms of weight $\kappa - 2q + 2$ with respect to $\Gamma_0(N)$. For $f \in \mathcal{S}_{\kappa-2q+2}(\Gamma_0(N))$ we set

$$\Phi(g,f) := \int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\theta(z,g)^* t^{\kappa-2q} ds dt \quad (g \in G),$$

where $\theta(z,g)^*$ means the contragredient of $\theta(z,g)$. Here we note that (τ_κ, V_κ) is self dual and that we can identify $\mathrm{End}(V_\kappa)$ with $\mathrm{End}(V_\kappa^*)$.

Then we are ready to state our result for the theta lifting:

Theorem 4.1. *Let $\kappa > 4q+2$. For any $v \in V_\kappa$, $\Phi(g,f) \cdot v$ belongs to $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$. Namely the mapping*

$$S_{\kappa-2q+2}(\Gamma_0(N)) \ni f \mapsto \Phi(g,f) \cdot v \in \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$$

gives a theta lifting to $\mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$.

For this theorem we should note that $(SL_2(\mathbb{R}), Sp(1,q))$ does not form a reductive dual pair unless $q = 1$ (for the definition of a reductive dual pair see [11, Section 5]). Hence the usual formulation of the theta lifting is impossible for this pair when $q > 1$. However, we recall that $Sp(1,q)$ can be regarded as a subgroup of $SO(4,4q)$ and note that $(SL_2(\mathbb{R}), SO(4,4q))$ forms a reducitve dual pair. In view of this Arakawa regards the theta series $\theta(z,g)$ as the restriction of a theta series on $\mathfrak{h} \times SO(4,4q)$ in order to formulate the theta lifting to $Sp(1,q)$. In other words, the lifting can be understood as the restriction of a theta correspondence for a reductive dual pair $(SL_2(\mathbb{R}), SO(4,4q))$ to the pair $(SL_2(\mathbb{R}), Sp(1,q))$.

4.2. *Transformation formula of $\theta(z,g)$*

As the first step for our theorem, we give a transformation formula of $\theta(z,g)$:

Proposition 4.2. *For $(\delta,\gamma,k) \in \Gamma_0(N) \times \Gamma \times K$,*

$$\theta(\delta(z), \gamma g k) = J(\delta, z)^{\kappa-2q+2}\theta(z,g)\tau_\kappa(k),$$

where $J(h,z) := cz+d$ is the automorphic factor for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and $z \in \mathfrak{h}$.

Proof. The left Γ-invariance of this theta series follows from its definition. The transformation law with respect to $k \in K$ is confirmed by direct calculation. The most difficult step is to deduce the transformation formula with respect to $\Gamma_0(N)$.

In order to obtain such transformation formula, we need two formulas. One formula is Shintani's transformation formula [16, Proposition 1.6] of theta series with respect to $SL_2(\mathbb{Z})$. To be more precise the formula is induced by the action of $SL_2(\mathbb{Z})$ on theta series via the restriction of the Weil representation r of $Sp(V \times V)$ to $SL_2(\mathbb{R})$, where $Sp(V \times V)$ denotes the symplectic group attached to the alternating form

$$(V \times V) \times (V \times V) \ni (v_1, v_2) \times (w_1, w_2) \mapsto (v_1, w_2)_Q - (w_1, v_2)_Q.$$

Another necessary formula is

Lemma 4.3. *For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,*

$$r(\sigma)F_z(x) = \epsilon(\sigma)^{q+1}J(\sigma,z)^{2q-\kappa-2}|J(\sigma,z)|^{-4q}F_{\sigma(z)}(x),$$

where $\epsilon(\sigma) := \begin{cases} -1 & (c \neq 0) \\ 1 & (c = 0) \end{cases}.$

The hardest step to deduce this formula is to calculate the transformation formula with respect to $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For this we note that, by changing of variables $y = (y_1, y_2, \ldots, y_{q+1}) := xU^{-1}$, $F_z(x)$ becomes

$$\sigma(y_{q+1})\mathbf{e}\left(\frac{1}{2}(s(-\sum_{i=1}^{q}\nu(y_i) + \nu(y_{q+1})) + \sqrt{-1}t\sum_{i=1}^{q+1}\nu(y_i))\right),$$

where

$$U := \begin{pmatrix} T^{-1} & 0 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

with $2S = T^t\bar{T}$. Moreover we remark that the matrix coefficients of σ_κ are proved to be harmonic polynomials on $\mathbb{H}$. Then we see that the transformation with respect to w is verified by following the technique to deduce the transformation formula of theta series associated with harmonic polynomials (cf. [6, Chap.I, Section 2]). With the help of such transformation formula, we obtain the transformation formula for general elements in $SL_2(\mathbb{R})$ from the formula of $r(\sigma)$ for $\sigma \in SL_2(\mathbb{R})$ in [16, Section 1, 4].

Then the two formulas mentioned above prove our desired transformation formula of the theta series with respect to $\Gamma_0(N)$. □

4.3. *Construction of the lifting*

Hereafter let $\| * \|_\kappa$ and $\| * \|_R$ be norms induced by a fixed K-invariant inner product of V_κ with respect to τ_κ and the inner product $(*,*)_R$ of $\mathbb{H}^{q+1}$, respectively. We shall prove that $\Phi(g,f)$ defines a certain automorphic form on G for $f \in \mathcal{S}_{\kappa-2q+2}(\Gamma_0(N))$. We state

Proposition 4.4. (1) *Let $\kappa > 2q-2$. For $f \in \mathcal{S}_{\kappa-2q+2}(\Gamma_0(N))$, $\Phi(g,f)$ converges absolutely and uniformly on any compact subset of G. More precisely, $\Phi(g,f)$ is of moderate growth.*
(2) *For any $(\gamma,k) \in \Gamma \times K$, $\Phi(g,f)$ satisfies*

$$\Phi(\gamma gk, f) = \tau_\kappa(k)^{-1}\Phi(g,f).$$

Proof. The second assertion follows immediately from Proposition 4.2. The proof of (1) needs

Lemma 4.5. (1) *Let a be a fixed positive real number. For any $\alpha \in \mathbb{R}_{>0}$ there exists a positive constant C_α depending only on α such that*

$$\exp(-ab) < C_\alpha b^{-\alpha} \quad \textit{for any } b > 0.$$

(2) *With a suitable choice of a finite subset Ξ_0 of $SL_2(\mathbb{Z})$,*

$$\mathfrak{h} = \cup_{\sigma \in \Xi_0} \Gamma_0(N) \sigma \mathcal{S}_{M,u},$$

where

$$\mathcal{S}_{M,u} := \{ s + \sqrt{-1}t \in \mathfrak{h} \mid -M \leqslant s \leqslant M, \ t > u \}$$

with some positive real number M and u.
(3) *Let L^* be the dual lattice of L with respect to $(*,*)_Q$. For each $h \in L^*/L$ we set*

$$\theta(z, g, h) := \sum_{\substack{l \in L^* \\ l \equiv h \mod L}} t^{2q} F_z(l^t \bar{g}^{-1}).$$

Then, for $v \in V_\kappa$, we have

$$\|\theta(z, g, h)v\|_\kappa < C_\alpha C(g) t^{2q-\alpha} \left(\sum_{l \in L \setminus \{0\}} \|l\|_R^{\kappa - 2\alpha} \right) \|v\|_\kappa,$$

where $\alpha > 0$ with $2\alpha - \kappa > 4(q+1)$, and $C(g) \in \mathbb{R}_{>0}$ is of polynomial order with respect to g.

Proof. The first assertion is an elementary fact, which is also given in Oda [15, (5.31)]. The second one is a very special case of the reduction theory (cf. [5, Proposition 15.6]). The estimation of the theta series in the third assertion is obtained by using (1). It is similar to Oda [15, (5.32)]. The condition on α is due to the convergence range of the Epstein zeta function attached to $(*,*)_R$, which appears on the right hand side of the inequality in the assertion (3). □

By virtue of this lemma and Shintani's transformation formula [16, Proposition 1.6] of theta series we see that the norm of $\Phi(g, f) \cdot v$ is estimated by

$$C_\alpha C(g)' \|v\|_\kappa \int_{-M}^{M} \int_{u}^{\infty} t^{2q-\alpha-2} ds dt,$$

where $C(g)'$ is a positive number of polynomial order with respect to g. Since $\kappa > 2q-2$ by our assumption and we take $2\alpha > \kappa+4(q+1)$, we have

$$2q-\alpha-2 < -q-3.$$

Thus $\Phi(g,f)$ is convergent uniformly and absolutely on any compact subset of G, and actually is of moderate growth. Therefore the proof of the proposition is finished.

4.4. *Lifting of elliptic Poincaré series*

Let us consider a lifting of an elliptic Poincaré series

$$G_m(z) := \sum_{\delta\in\Gamma_\infty\backslash\Gamma_0(N)} J(\delta,z)^{-(\kappa+2-2q)}\mathbf{e}(m\delta(z)) \quad (z\in\mathfrak{h})$$

for a positive integer m, where

$$\Gamma_\infty := \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \;\middle|\; n\in\mathbb{Z}\right\}\subset\Gamma_0(N).$$

Here we quote a well-known fact as follows (cf. [7, Chap.III, Section 11, Theorem 6]):

Lemma 4.6. *The space $\mathcal{S}_{\kappa-2q+2}(\Gamma_0(N))$ is spanned by $\{G_m(z)\mid m\in\mathbb{Z}_{>0}\}$.*

By virtue of this lemma it suffices to deal with the lifting $\Phi(g,G_m)$ of G_m for each $m\in\mathbb{Z}_{>0}$ in order to prove our theorem. We write $l\in L$ as $l=(\tilde{l},l_q,l_{q+1})\in L$ with $(l_q,l_{q+1})\in\mathcal{O}^2$ and $\tilde{l}\in\mathcal{O}^{q-1}$ (when $q=1$ we write this as $l=(l_q,l_{q+1})$). For $l\in\mathbb{H}^{q+1}$ with a positive $(l,l)_Q$ we can take a unique element p_l in NA such that $p_l(z_0)=({}^t(\overline{l_{q+1}^{-1}\tilde{l}}),\overline{l_{q+1}^{-1}l_q})\in H$ (when $q=1$ we define p_l by $p_l(z_0)=\overline{l_{q+1}^{-1}l_q}$). Then we have

Proposition 4.7. *Let $\kappa>4q+2$. For a positive integer m*

$$\Phi(g,G_m)=(2m)^{-\frac{\kappa+2}{2}}\frac{\Gamma(\kappa+1)}{(2\pi)^{\kappa+1}}\Omega_m(g)$$

with

$$\Omega_m(g) := \sum_{\substack{l\in L\\ (l,l)_Q=2m}} \omega_\kappa(p_l^{-1}g)\sigma_\kappa(l_{q+1}/d(l_{q+1}))^{-1}.$$

Proof. If we ignore the problem on exchanging the summation and the integration we obtain this equality by formal calculation. In order to justify the calculation to give $\Phi(g, G_m) = (2m)^{-\frac{\kappa+2}{2}} \frac{\Gamma(\kappa+1)}{(2\pi)^{\kappa+1}} \Omega_m(g)$ we follow the argument similar to Oda [15, Section 5, 2]. Due to Lemma 4.5 (3) we show

$$\begin{aligned}
&\|\Phi(g, G_m) \cdot v\|_\kappa \\
&\leqslant \int_{\Gamma_\infty \backslash \mathfrak{h}} C_\alpha C(g) t^{2q-\alpha} \exp(-2\pi m t) \Big(\sum_{l \in L \backslash \{0\}} \|l\|_R^{\kappa - 2\alpha} \Big) \|v\|_\kappa t^{\kappa - 2q} ds dt \\
&\leqslant C_\alpha C\text{"}(g) \|v\|_\kappa \int_0^\infty \int_0^1 t^{\kappa-\alpha} \exp(-2\pi m t) ds dt \\
&= C_\alpha C\text{"}(g) \|v\|_\kappa (2\pi m)^{-(\kappa+1-\alpha)} \Gamma(\kappa + 1 - \alpha)
\end{aligned}$$

for each $v \in V_\kappa$, where $C\text{"}(g) \in \mathbb{R}_{>0}$ is of polynomial order with respect to g. Take α so that $\alpha < \kappa + 1$. This choice of α is possible since $\kappa + 1 > \alpha > \frac{\kappa}{2} + 2(q+1)$ is meaningful when $\kappa > 4q + 2$. Then we see that the expression of $\Phi(g, G_m)$ in the assertion converges absolutely and uniformly on any compact subset of G. Thus the proposition is proved. □

4.5. *Fourier expansion of $\Omega_m(g)$*

In this subsection we study the Fourier expansion of $\Omega_m(g)$. We introduce some notations. Let $c \in \Xi$ and $\xi \in X^*_{\Gamma,c} \setminus \{0\}$. We write the center of N as $Z(N)$. The set $L_c(m)/N_{\Gamma,c} \cap Z(N)$ denotes the quotient of $L_c(m) := \{l \in L^t\bar{c}^{-1} \mid (l,l)_Q = 2m\}$ by $N_{\Gamma,c} \cap Z(N)$-action induced by

$$L_c(m) \ni l \mapsto l^t\bar{\gamma}^{-1} \in L_c(m) \quad (\gamma \in c^{-1}\Gamma c).$$

When $q > 1$ we provide an $\mathrm{End}(V_\kappa)$-valued theta function

$$\theta_\xi(w) := \sum_{l \in L_c(m)/N_{\Gamma,c} \cap Z(N)} d(\xi)^{\kappa-1} \delta_\xi(l, m, \kappa) k^0_\xi(w_l, w) \cdot U(\xi) \cdot \sigma_\kappa(l_{q+1}/d(l_{q+1}))^{-1}$$

on ${}^t\mathbb{H}^{q-1}$ and when $q = 1$ we put

$$C^m_\xi := \sum_{l \in L_c(m)/N_{\Gamma,c} \cap N_Z} d(\xi)^{\kappa-1} \delta_\xi(l, m, \kappa) \cdot U(\xi) \cdot \sigma_\kappa(l_{q+1}/d(l_{q+1}))^{-1}.$$

Here

- $(w_l, \tau_l) := p_l(z_0) \in H$,
- $k^0_\xi(w', w) := (\Delta(S) 2^{4(q-1)} \nu(\xi)^{q-1})^{-1} k_\xi(w', w)$,
- $U(\xi) \in End(V_\kappa)$ is the projection from V_κ onto $\mathbb{C} \cdot \sigma_\kappa(u_\xi) v_\kappa$, where recall that v_κ and $u_\xi \in \{x \in \mathbb{H} \mid \nu(x) = 1\}$ are given just before Theorem 2.3,

- $\delta_\xi(l,m,\kappa) := \dfrac{2\pi^2(4\pi)^{\kappa-1}}{\kappa!}\left(\dfrac{2m}{\nu(l_{q+1})}\right)^{\frac{\kappa}{2}+1}\exp\left(-\dfrac{2\pi d(\xi)m}{\nu(l_{q+1})}\right)$
$\times\, \mathbf{e}\left(-\dfrac{1}{2}\operatorname{tr}\xi(\tau_l-\bar{\tau}_l)\right).$

We note that the summation of $\theta_\xi(w)$ and C_ξ^m is well-defined since $\delta_\xi(l,m,\kappa)\cdot U(\xi)\cdot\sigma_\kappa(l_{q+1}/d(l_{q+1}))^{-1}$ is invariant under $l\mapsto l^t\bar{\gamma}^{-1}$ for $\gamma\in Z(N)\cap N_{\Gamma,c}$.

Using basically Arakawa's Fourier transformation formula of ω_κ in [2, Lemma 1.2], we have

Proposition 4.8. (1) *Let $q>1$. The theta function $\theta_\xi(w)$ above is uniformly bounded on ${}^t\mathbb{H}^{q-1}$ and satisfies*

$$(i)\ \theta_\xi(w+\lambda)=\mathbf{e}(\operatorname{tr}(\xi({}^t\bar{w}S\lambda-x_\lambda)))\theta_\xi(w),\ \forall\lambda\in\Lambda_c,$$

$$(ii)\ \int_{{}^t\mathbb{H}^{q-1}}k_\xi(w',w)\theta_\xi(w')dw'=\theta_\xi(w).$$

(2) *When $q>1$ the Fourier expansion of Ω_m at each cusp $c\in\Xi$ can be written as*

$$\Omega_m(cn(w,x)a_y)=\frac{1}{\operatorname{vol}(X_{\mathbb{R}}/X_{\Gamma,c})}\sum_{\xi\in X^*_{\Gamma,c}\setminus\{0\}}\theta_\xi(w)y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\mathbf{e}(\operatorname{tr}\xi x)$$

and when $q=1$ such expansion can be written as

$$\Omega_m(cn(x)a_y)=\frac{1}{\operatorname{vol}(X_{\mathbb{R}}/X_{\Gamma,c})}\sum_{\xi\in X^*_{\Gamma,c}\setminus\{0\}}C_\xi^m y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\mathbf{e}(\operatorname{tr}\xi x).$$

Here $\operatorname{vol}(X_{\mathbb{R}}/X_{\Gamma,c})$ *denotes the volume of the quotient* $X_{\mathbb{R}}/X_{\Gamma,c}$.

4.6. *Proof of the theorem*

Now we are ready to complete the final step for the proof of Theorem 4.1. Proposition 4.8 (1) means that the coefficients of $\theta_\xi(w)$ belong to $\Theta_{\xi,c}$ when $q>1$. Hence, Proposition 4.8 (2) tells us that $\Omega_m(g)\cdot v$ with $v\in V_\kappa$ has a Fourier expansion of the form in Theorem 2.3 for any q. Then Proposition 2.4 implies $\Omega_m(g)\cdot v\in\mathcal{A}_0(\Gamma\backslash G,\omega_\kappa)$. Therefore we have proved Theorem 4.1.

Remark 4.9. We explain Arakawa's method to prove Theorem 4.1 for the case of $q=1$. Arakawa proved it by writing Ω_m as a finite sum of "Godement kernel function"

$$K_\kappa^\Gamma(g_1,g_2):=\sum_{\gamma\in\Gamma}\omega_\kappa(g_1^{-1}\gamma g_2)\quad(g_1,g_2\in G)$$

(for this function see [2] and [3]). More precisely Arakawa gave a fundamental domain for $\Gamma\backslash H$ explicitly and considered the embedding of $L(m)/\Gamma$ ($L(m) := \{l \in L \mid (l,l)_Q = 2m\}$) into such domain, induced by

$$L(m) \ni (l_1, l_2) \mapsto \overline{l_2^{-1} l_1} \in H.$$

From these he deduced the finiteness of $L(m)/\Gamma$. Here the action of Γ on $L(m)$ is similar to that of $c^{-1}\Gamma c$ on $L_c(m)$ given in Section 4.5. This implies that $\Omega_m(g)$ is a finite sum of $K_\kappa^\Gamma(h,g)$'s with h's ranging over the image of $L(m)/\Gamma$ in $\Gamma\backslash H$. Since $K_\kappa^\Gamma(g_0,g)\cdot v \in \mathcal{A}_0(\Gamma\backslash G, \omega_\kappa)$ for a fixed $(g_0, v) \in G \times V_\kappa$, that leads to the theorem for the case of $q = 1$.

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ON MODULAR FORMS FOR THE PARAMODULAR GROUPS

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Dedicated to the memory of Tsuneo Arakawa

We present several results on Siegel modular forms of degree 2 with respect to the paramodular group. We propose a theory of new- and oldforms for such modular forms and show that such a theory follows from an analogous local theory, which is available, and several conjectural results on the global spectrum of GSp(4). Examples for paramodular cusp forms are obtained as Saito–Korukawa liftings from elliptic cusp forms for $\Gamma_0(N)$.

1. Introduction

Let F be a $\mathfrak{p}$–adic field, and let G be the algebraic F–group GSp(4). In our paper [RS1] we presented a conjectural theory of local newforms for irreducible, admissible, generic representations of $G(F)$ with trivial central character. The main feature of this theory is that it considers fixed vectors under the *paramodular groups* $\mathrm{K}(\mathfrak{p}^n)$, a certain family of compact-open subgroups. The group $\mathrm{K}(\mathfrak{p}^0)$ is equal to the standard maximal compact subgroup $G(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of F. In fact, $\mathrm{K}(\mathfrak{p}^0)$ and $\mathrm{K}(\mathfrak{p}^1)$ represent the two conjugacy classes of maximal compact subgroups of $G(F)$. In general $\mathrm{K}(\mathfrak{p}^n)$ can be conjugated into $\mathrm{K}(\mathfrak{p}^0)$ if n is even, and into $\mathrm{K}(\mathfrak{p}^1)$ if n is odd. Our theory is analogous to CASSELMAN's well-known theory for representations of $\mathrm{GL}(2,F)$; see [Cas]. The main conjecture made in [RS1] states that for each irreducible, admissible, generic representation (π, V) of

$\mathrm{PGSp}(4,F)$ there exists an n such that the space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$ invariant vectors is non-zero; if n_0 is the minimal such n then $\dim_{\mathbb{C}}(V(n_0)) = 1$; and the Novodvorski zeta integral of a suitably normalized vector in $V(n_0)$ computes the L–factor $L(s,\pi)$ (for this last statement we assume that V is the Whittaker model of π).

We recently proved all parts of this conjecture; it is now a theorem*. Parts of the main theorem have been generalized to include non-generic representations. In addition, there is a description of *oldforms*, that is, the spaces $V(n)$ for $n > n_0$. This description is based on certain linear operators $\theta, \theta' ::= V(n) \to V(n+1)$ and $\eta ::= V(n) \to V(n+2)$, which we call *level raising operators* and which play a prominent role in our theory. Complete proofs of the results mentioned above will be provided in [RS2].

Now $G = \mathrm{GSp}(4)$ is the group behind classical Siegel modular forms of degree 2, in the sense that such a modular form can be considered as a function on the adelic group $G(\mathbb{A}_{\mathbb{Q}})$, where it generates an automorphic representation of this group. Exploiting this link between modular forms and representations, we shall explore in this paper the consequences of our local newform theory for Siegel modular forms of degree 2 with respect to paramodular groups. We shall explain how our local theory will imply a global Atkin–Lehner style theory of old– and newforms for paramodular cusp forms, provided we accept some global results on the discrete spectrum of $G(\mathbb{A})$, which have been announced but not yet published.

We shall start in a classical setting, defining the paramodular groups $\Gamma^{\mathrm{para}}(N)$ for positive integers N, and the corresponding spaces $S_k(N)$ of cusp forms of degree 2. We shall then define, for a prime number p, level raising operators θ_p and θ'_p, which multiply the level by p, and η_p, which multiplies the level by p^2. These operators are compatible with the local operators mentioned above, and the connection will be explained. Perhaps surprisingly, the η_p and θ_p operator are compatible, via the Fourier–Jacobi expansion, with the well-known U_p and V_p operators from the theory of Jacobi forms. Paramodular oldforms will be defined, roughly speaking, as those modular forms that can be obtained by repeatedly applying the three level raising operators and taking linear combinations. The space of newforms is defined as the orthogonal complement of the oldforms with respect to the Petersson inner product. We shall formulate conjectural Atkin–Lehner type results for the newforms thus defined, and explain how these results would follow from our local theory together with some plausible

*It was still a conjecture at the time of the Arakawa conference.

global results that are not yet fully available.

Examples of paramodular cusp forms are provided by the Saito–Kurokawa lifting. There is a classical construction available, combining results of SKORUPPA, ZAGIER and GRITSENKO, which produces elements of $S_k(N)$ from elliptic modular forms of level N and weight $2k-2$. However, we propose an alternative group theoretic construction, which gives the additional information that the Saito–Kurokawa liftings we obtain from elliptic newforms are paramodular newforms as defined above. In other words, there is a level-preserving Hecke-equivariant Saito–Kurokawa lifting from cuspidal elliptic newforms (with a "–" sign in the functional equation of the L–function) to cuspidal paramodular newforms of degree 2. We shall explain how this map can be extended to the "certain space" of modular forms defined by SKORUPPA and ZAGIER in [SZ].

In the final section of this paper we will consider two seemingly unrelated theorems on paramodular cusp forms. One says that the θ operator defined before is injective. The other one says that paramodular cusp forms of weight 1 do not exist. We shall translate these theorems into group theoretic statements, where it turns out that the second one is the exact archimedean analogue of the first one.

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2. Definitions

Paramodular groups

In the following we let G be the algebraic $\mathbb{Q}$–group $\mathrm{GSp}(4)$, realized as the set of all $g \in \mathrm{GL}(4)$ such that ${}^tgJg = xJ$ for some $x \in \mathrm{GL}(1)$, where $J = \begin{bmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{bmatrix}$. The element x is called the multiplier of g and denoted by $\lambda(g)$. The kernel of the homomorphism $\lambda : := \mathrm{GSp}(4) \to \mathrm{GL}(1)$ is the symplectic group $\mathrm{Sp}(4)$.

Let N be a positive integer. The *Klingen congruence subgroup* of level N is the set of all $\gamma \in \mathrm{Sp}(4,\mathbb{Z})$ such that

$$\gamma \in \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

(That this is a subgroup becomes obvious by switching the first two rows and first two columns, which amounts to an isomorphism with a more symmetric version of the symplectic group.) This group can be enlarged to the *paramodular group* of level N by allowing certain denominators. Namely, we define

$$\Gamma^{\mathrm{para}}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}).$$

Note that $\Gamma^{\mathrm{para}}(N)$ is not contained in $\Gamma^{\mathrm{para}}(M)$ if $M|N$. In fact, no paramodular group contains any other paramodular group, since the element

$$\begin{bmatrix} 1 & & & \\ & & & N^{-1} \\ & & -1 & \\ & N & & \end{bmatrix}$$

is contained in $\Gamma^{\mathrm{para}}(N)$ only. We also define local paramodular groups. Let F be a non-archimedean local field, $\mathfrak{o}$ its ring of integers and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. We define $\mathrm{K}(\mathfrak{p}^n)$ as the group of all $g \in \mathrm{GSp}(4, F)$ such that

$$g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{o} & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \qquad \text{and} \qquad \det(g) \in \mathfrak{o}^*. \tag{1}$$

These are the local analogues of the groups $\Gamma^{\mathrm{para}}(N)$. In fact, if $F = \mathbb{Q}$, then

$$\Gamma^{\mathrm{para}}(N) = G(\mathbb{Q}) \cap G(\mathbb{R})^+ \prod_p \mathrm{K}(p^{v_p(N)}), \tag{2}$$

where $p^{v_p(N)}$ is the exact power of p dividing N (if $p \nmid N$, then we understand $\mathrm{K}(p^{v_p(N)}) = G(\mathbb{Z}_p)$).

Modular forms

Let $\mathbb{H}_2$ be the Siegel upper half plane of degree 2. The group $G(\mathbb{R})^+ = \{g \in \mathrm{GSp}(4, \mathbb{R}) : := \lambda(g) > 0\}$, which is the identity component of $G(\mathbb{R})$, acts on $\mathbb{H}_2$ by linear fractional transformations $Z \mapsto g\langle Z \rangle$. We define the usual modular factor

$$j(g, Z) = \det(CZ + D) \qquad \text{for } Z \in \mathbb{H}_2 \text{ and } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbb{R})^+.$$

We fix a weight k, which is a positive integer. The slash operator $\big|_k$ or simply $\big|$ on functions $F : := \mathbb{H}_2 \to \mathbb{C}$ is defined as

$$(F\big|g)(Z) = \lambda(g)^k j(g,Z)^{-k} F(g\langle Z\rangle) \qquad \text{for } g \in G(\mathbb{R})^+.$$

The factor $\lambda(g)^k = \det(g)^{k/2}$ ensures that the center of $G(\mathbb{R})^+$ acts trivially. A *modular form* F (always of degree 2) of weight k with respect to $\Gamma^{\rm para}(N)$ is a holomorphic function on $\mathbb{H}_2$ such that $F\big|\gamma = F$ for all $\gamma \in \Gamma^{\rm para}(N)$. We denote the space of such modular forms by $M_k(N)$, and the subspace of cusp forms by $S_k(N)$. Modular forms for the paramodular group have been considered by various authors; see, for example, [IO] and the references therein. In this paper we shall fix the weight k and vary the level N.

We shall often write modular forms as $F(\tau,z,\tau')$, where $\begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix} \in \mathbb{H}_2$. Note that elements $F \in M_k(N)$ have the invariance property $F(\tau,z,\tau'+t) = F(\tau,z,\tau')$ for $t \in N^{-1}\mathbb{Z}$. In particular, F has a Fourier–Jacobi expansion

$$F(\tau,z,\tau') = \sum_{m=0}^{\infty} f_m(\tau,z)e^{2\pi i m\tau'}. \tag{3}$$

Here $f_m \in J_{k,m}$ is a Jacobi form of weight k and index m, as in [EZ]. Since F depends only on τ' modulo $N^{-1}\mathbb{Z}$, we have $f_m = 0$ for $N \nmid m$.

We shall attach to a given $F \in M_k(N)$ an adelic function $\Phi : := G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ in the following way. Let K_N be the compact group $\prod_{p<\infty} \mathrm{K}(p^{v_p(N)})$. Since the local multiplier maps $\mathrm{K}(p^{v_p(N)}) \to \mathbb{Z}_p^*$ are all surjective, it follows from strong approximation for $\mathrm{Sp}(4)$ that $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K_N$. Decomposing a given $g \in G(\mathbb{A})$ accordingly as $g = \rho h \kappa$, we define

$$\Phi(g) = (F\big|_k h)(I), \quad g = \rho h\kappa \text{ with } \rho \in G(\mathbb{Q}),\ :=\ h \in G(\mathbb{R})^+,\ :=\ \kappa \in K_N. \tag{4}$$

Here I is the element $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ of $\mathbb{H}_2$. In view of (2), the function Φ is well-defined. It obviously has the invariance properties

$$\Phi(\rho g\kappa z) = \Phi(g) \qquad \text{for all } g \in G(\mathbb{A}),\ :=\ \rho \in G(\mathbb{Q}),\ :=\ \kappa \in K_N,\ :=\ z \in Z(\mathbb{A}),$$

where Z is the center of $\mathrm{GSp}(4)$. In fact, Φ is an automorphic form on $\mathrm{PGSp}(4,\mathbb{A})$. One can show that Φ is a cuspidal automorphic form if and only if $F \in S_k(N)$. Assuming this is the case, we consider the cuspidal automorphic representation $\pi = \pi_F$ generated by Φ. This representation may not be irreducible, but it always decomposes as a finite direct sum $\pi = \oplus_i \pi_i$ with irreducible automorphic representations π_i.

Atkin–Lehner involutions

We first consider local Atkin–Lehner involutions. Let again F be a non-archimedean local field, and let $\mathfrak{o}$ and $\mathfrak{p}$ be as above. Let ϖ be a generator of $\mathfrak{p}$. The element

$$u_n = \begin{bmatrix} &&&1\\ &&1&\\ &\varpi^n&&\\ \varpi^n&&& \end{bmatrix} \tag{5}$$

is called the *Atkin–Lehner element* of level n. It is easily checked that u_n normalizes the local paramodular group $\mathrm{K}(\mathfrak{p}^n)$. Therefore, if (π, V) is an admissible representation of $G(F)$, the operator $\pi(u_n)$ induces an endomorphism of the (finite-dimensional) space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$–invariant vectors. Assume in addition that π has trivial central character. Then, since u_n^2 is central, this endomorphism on $V(n)$ is an involution, the *Atkin–Lehner involution* of level n (or $\mathfrak{p}^n$). It splits the space $V(n)$ into ± 1 eigenspaces.

To define the global involutions, let N be a positive integer and let p be a prime dividing N. Let p^j be the exact power of p dividing N. Choose a matrix $\gamma_p \in \mathrm{Sp}(4, \mathbb{Z})$ such that

$$\gamma_p \equiv \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix} \bmod p^j\mathbb{Z} \qquad \text{and} \qquad \gamma_p \equiv \begin{bmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{bmatrix} \bmod Np^{-j}\mathbb{Z},$$

and let

$$u_p := \gamma_p \begin{bmatrix} p^j&&&\\ &p^j&&\\ &&1&\\ &&&1 \end{bmatrix}.$$

We call u_p an *Atkin–Lehner element.* A different choice of γ_p results in multiplying u_p from the left with an element of the principal congruence subgroup $\Gamma(N)$. Therefore the action of u_p on modular forms for $\Gamma(N)$ is unambiguously defined. It is easily checked using (2) that u_p normalizes $\Gamma^{\mathrm{para}}(N)$. Consequently the map $F \mapsto F\big|u_p$ defines an endomorphism of $M_k(N)$. Its restriction to cusp forms defines an endomorphism of $S_k(N)$. These endomorphisms are involutions since $u_p^2 \in p^j\Gamma^{\mathrm{para}}(N)$, as is easily checked. To summarize, for a given level N, we can define *Atkin–Lehner involutions* $u_p(F) := F\big|_k u_p$ on $M_k(N)$ and $S_k(N)$ for each $p|N$.

The relation between the local and global Atkin–Lehner involutions is as follows. Let $F \in M_k(N)$ and Φ the corresponding adelic function defined above. Then $u_p(F)$ corresponds to the right translate of Φ by the local Atkin–Lehner element $\underline{u}_{p^j} \in G(\mathbb{Q}_p)$, where p^j is the exact power of p dividing N:

$$(u_p(F)\big|g)(I) = \Phi(g\underline{u}_{p^j}), \qquad g \in G(\mathbb{R})^+,\ \underline{u}_{p^j} = \begin{bmatrix} &&&1\\ &&1&\\ &p^j&&\\ p^j&&& \end{bmatrix}. \tag{6}$$

3. Linear independence at different levels

We shall prove an easy but useful result on modular forms for the paramodular group, starting with an analogous local statement. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$. Let ϖ be a generator of $\mathfrak{p}$. We define

$$t_n := \begin{bmatrix} 1&&&\\ &&&-\varpi^{-n}\\ &&1&\\ &\varpi^n&& \end{bmatrix}.$$

Lemma 3.1. *Let $0 \leqslant n_1 < \cdots < n_r$ be integers. Let $m \geqslant 0$ be an integer such that $m < n_1$. Then the subgroup H generated by $\mathrm{K}(\mathfrak{p}^{n_1}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_r})$ and t_m contains $\mathrm{Sp}(4, F)$.*

Proof. The proof will be easy once we can show that H contains all elements

$$\begin{bmatrix} 1&&&\\ &a&&b\\ &&1&\\ &c&&d \end{bmatrix}, \qquad \text{where } \begin{bmatrix} a&b\\ c&d \end{bmatrix} \in \mathrm{SL}(2, F).$$

By hypothesis the group H contains the elements

$$\begin{bmatrix} 1&&&\\ &a&&b\varpi^{-n_1}\\ &&1&\\ &c\varpi^{n_r}&&d \end{bmatrix}$$

such that $a, b, c, d \in \mathfrak{o}$ and $\begin{bmatrix} a & b\varpi^{-n_1} \\ c\varpi^{n_r} & d \end{bmatrix} \in \mathrm{SL}(2, F)$. Since H also contains t_m, it will suffice to show that the subgroup H' of $\mathrm{SL}(2, F)$ generated by

$$\begin{bmatrix} & -\varpi^{-m} \\ \varpi^m & \end{bmatrix}, \quad \begin{bmatrix} a & b\varpi^{-n_1} \\ c\varpi^{n_r} & d \end{bmatrix}, \quad a, b, c, d \in \mathfrak{o}, := \ ad - bc\varpi^{n_r - n_1} = 1$$

is $\mathrm{SL}(2, F)$. We shall show that the conjugate subgroup $H'' := \begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} H' \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}$ is equal to $\mathrm{SL}(2, F)$, which is equivalent. This subgroup H'' is generated by

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad \begin{bmatrix} a & b\varpi^{m-n_1} \\ c\varpi^{n_r - m} & d \end{bmatrix}, \quad a, b, c, d \in \mathfrak{o}, := \ ad - bc\varpi^{n_r - n_1} = 1.$$

In particular, H'' contains $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ and $\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix}$, and therefore $\mathrm{SL}(2, \mathfrak{o})$. It is not hard to show that the group generated by $\mathrm{SL}(2, \mathfrak{o})$ and $\begin{bmatrix} 1 & \mathfrak{p}^{-1} \\ & 1 \end{bmatrix}$ is all of $\mathrm{SL}(2, F)$. $\square$

Proposition 3.1. *Let F be a non-archimedean local field, and let (π, V) be an admissible representation of $G(F)$ with trivial central character that has no non-zero* $\mathrm{Sp}(4, F)$ *invariant vectors.*† *Then paramodular vectors in V of different levels are linearly independent. More precisely, for $i = 1, \ldots, r$ let $v_i \in V$ be fixed by the paramodular group* $\mathrm{K}(\mathfrak{p}^{n_i})$, *where $n_i \neq n_j$ for $i \neq j$. Then $v_1 + \ldots + v_r = 0$ implies $v_1 = \ldots = v_r = 0$.*

Proof. We may assume that $n_1 < \ldots < n_r$. From $v_1 + \ldots + v_r = 0$ we obtain $-v_1 = v_2 + \cdots + v_r$. This element is invariant under t_{n_1} and $\mathrm{K}(\mathfrak{p}^{n_2}) \cap \ldots \cap \mathrm{K}(\mathfrak{p}^{n_r})$. Since $n_1 < n_2$, by Lemma 3.1, it is invariant under $\mathrm{Sp}(4, F)$; hence, $v_1 = v_2 + \ldots + v_r = 0$. Applying the same argument successively gives $v_2 = \ldots = v_r = 0$. $\square$

This local result has the following global analogue. Note that the corresponding statement for $\Gamma_0(N)$ congruence subgroups is obviously wrong.

Proposition 3.2. *Modular forms for the paramodular group of different levels are linearly independent. More precisely, for $i = 1, \ldots, r$ let $F_i \in M_k(N_i)$, where $N_i \neq N_j$ for $i \neq j$. Then $F_1 + \ldots + F_r = 0$ implies $F_1 = \ldots = F_r = 0$.*

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†For example, π could be an irreducible, infinite-dimensional representation.

Proof. One can either exploit the relationship between modular forms and representations and use Proposition 3.1, or one can give a direct proof along the lines of the local proofs. □

An important consequence of Proposition 3.2 is the following. Soon we will have reason to consider the spaces $M_k(\Gamma^{\rm para}) := \bigoplus_{N=1}^\infty M_k(N)$, see (18). Proposition 3.2 implies that this abstract direct sum is the same as the sum of the spaces $M_k(N)$ taken inside the vector space of all complex-valued functions on $\mathbb{H}_2$.

4. The level raising operators

As before let $M_k(N)$ be the space of modular forms of weight k with respect to the paramodular group of level N. Since no $\Gamma^{\rm para}(N)$ is contained in any other $\Gamma^{\rm para}(M)$ ($M \neq N$), there are no inclusions between (the non-zero ones of) the spaces $M_k(N)$. In particular, for $N|M$, the space $M_k(N)$, if not zero, is not a subspace of $M_k(M)$. However, we shall see that there are natural operators raising the level. For a prime number p, which may or may not divide N, we shall define linear operators θ_p and θ'_p from $M_k(N)$ to $M_k(Np)$. We shall also define an operator η_p from $M_k(N)$ to $M_k(Np^2)$.

The η operator

We begin by defining η_p, since this is easiest. For $F \in M_k(N)$ let

$$\eta_p F := F|\underline{\eta}_p^{-1}, \qquad \text{where} \quad \underline{\eta}_p = \begin{bmatrix} 1 & & & \\ & p^{-1} & & \\ & & 1 & \\ & & & p \end{bmatrix}. \tag{7}$$

One easily checks that $\underline{\eta}_p\Gamma^{\rm para}(N)\underline{\eta}_p^{-1} \supset \Gamma^{\rm para}(Np^2)$. Hence $\eta_p(F) \in M_k(Np^2)$, and we get linear operators

$$\eta_p:\ M_k(N) \longrightarrow M_k(Np^2) \qquad \text{and} \qquad \eta_p:\ S_k(N) \longrightarrow S_k(Np^2).$$

Explicitly, we have $(\eta_pF)(\tau,z,\tau') = p^kF(\tau,pz,p^2\tau')$. If the Fourier–Jacobi expansion of F is written as in (3), then the Fourier–Jacobi expansion of η_pF is given by

$$\begin{aligned}(\eta_pF)(\tau,z,\tau') &= p^k\sum_{m=0}^\infty f_m(\tau,pz)e^{2\pi imp^2\tau'}\\ &= p^k\sum_{m=0}^\infty (U_pf_m)(\tau,z)e^{2\pi imp^2\tau'}.\end{aligned} \tag{8}$$

Here $(U_pf_m)(\tau,z)=f_m(\tau,pz)$ is the operator from $J_{k,m}$ to J_{k,mp^2} defined in section I.4 of [EZ]. If Φ is the adelic function corresponding to F defined in (4), then a straightforward calculation shows that

$$((\eta_pF)\big|g)(I)=\Phi(g\underline{\eta}_p),\qquad g\in G(\mathbb{R})^+,\ \underline{\eta}_p=\begin{bmatrix}1&&&\\&p^{-1}&&\\&&1&\\&&&p\end{bmatrix}\in G(\mathbb{Q}_p).\quad (9)$$

In other words, the adelic function corresponding to η_pF is the right translate of Φ by the p–adic matrix η_p. From the local descriptions (6) and (9) and the matrix identity

$$p\begin{bmatrix}1&&&\\&p^{-1}&&\\&&1&\\&&&p\end{bmatrix}\begin{bmatrix}&&&1\\&&1&\\&p^n&&\\p^n&&&\end{bmatrix}=\begin{bmatrix}&&&1\\&&1&\\&p^{n+2}&&\\p^{n+2}&&&\end{bmatrix}\begin{bmatrix}1&&&\\&p^{-1}&&\\&&1&\\&&&p\end{bmatrix}$$

it is immediate that the η operator commutes with Atkin–Lehner involutions: $u_p\circ\eta_p=\eta_p\circ u_p$. Note that the u_p on the right acts on $M_k(N)$, and the u_p on the left acts on $M_k(Np^2)$.

The θ operator

It is not possible to conjugate the group $\Gamma^{\mathrm{para}}(Np)$ into $\Gamma^{\mathrm{para}}(N)$. Consequently there is no simple operator from $M_k(N)$ to $M_k(Np)$ given by applying a single matrix as in the case of η_p. We can however define an operator by applying $\mathrm{diag}(1,1,p^{-1},p^{-1})$ and then average to restore the paramodular invariance. More precisely, for $F\in M_k(N)$ we define

$$\theta_pF=\sum_{\gamma\in\Gamma_0(p)\backslash SL(2,\mathbb{Z})}F\big|_k\Big(\begin{bmatrix}1&&&\\&1&&\\&&p^{-1}&\\&&&p^{-1}\end{bmatrix}\begin{bmatrix}a&&b&\\&1&&\\c&&d&\\&&&1\end{bmatrix}\Big)\qquad\Big(\gamma=\begin{bmatrix}a&b\\c&d\end{bmatrix}\Big).\tag{10}$$

It is easy to check that θ_pF is well-defined and indeed is an element of $M_k(Np)$. Hence we get linear operators

$$\theta_p:\ M_k(N)\longrightarrow M_k(Np)\qquad\text{and}\qquad\theta_p:\ S_k(N)\longrightarrow S_k(Np).$$

Assume that the Fourier–Jacobi expansion of $F\in M_k(N)$ is given by (3). Then a straightforward calculation shows that

$$(\theta_pF)(\tau,z,\tau')=p\sum_{m=0}^{\infty}(V_pf_m)(\tau,z)e^{2\pi imp\tau'}\tag{11}$$

with

$$(V_p f_m)(\tau, z) = p^{k-1} \sum_{\gamma \in \Gamma_0(p) \backslash \mathrm{SL}(2,\mathbb{Z})} (c\tau + d)^{-k} e^{-2\pi i m p \frac{cz^2}{c\tau+d}} \times f_m\Big(p\frac{a\tau+b}{c\tau+d}, \frac{pz}{c\tau+d}\Big).$$

Note that there is a bijection $\Gamma_0(p)\backslash\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z})\backslash\{A \in M_2(\mathbb{Z}) : := \det(A) = p\}$ given by $\gamma \mapsto \mathrm{diag}(p,1)\gamma$. Hence $V_p f_m$ is exactly the function defined in (2) of section I.4 of [EZ]. The V_p operator is a linear map from $J_{k,m}$ to $J_{k,mp}$. To summarize equations (8) and (11), the operator η_p on $M_k(N)$ is compatible with the operator U_p on Jacobi forms, and θ_p is compatible with the operator V_p.

We now define an adelic version of the θ_p operator. If Φ is a function on $G(\mathbb{A}_\mathbb{Q})$ that is right invariant under the paramodular group $\mathrm{K}(p^j) \subset G(\mathbb{Q}_p)$ of some level p^j, we define a new function $\theta_p\Phi$ by

$$(\theta_p\Phi)(g) = \sum_{\gamma \in \mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)} \Phi\Big(g \underbrace{\begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix}}_{\in G(\mathbb{Q}_p)}\Big), \tag{12}$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $g \in G(\mathbb{A})$. Then $\theta_p\Phi$ is right invariant under $\mathrm{K}(p^{j+1})$. A standard calculation shows that if Φ corresponds to F as in (4), then $\theta_p\Phi$ corresponds to $\theta_p F$. In other words,

$$((\theta_p F)\big|g)(I) = (\theta_p\Phi)(g), \qquad g \in G(\mathbb{R})^+. \tag{13}$$

The θ' operator

While the η operator commutes with Atkin–Lehner involutions, this is no longer true for the θ operator. We use this fact to define a new operator θ_p' from $M_k(N)$ to $M_k(Np)$ by

$$\theta_p' := u_p \circ \theta_p \circ u_p \qquad \text{(the } u_p \text{ are Atkin–Lehner involutions).} \tag{14}$$

Note that the u_p on the right acts on $M_k(N)$, and the u_p on the left acts on $M_k(Np)$. We obtain linear operators

$$\theta_p' : M_k(N) \longrightarrow M_k(Np) \qquad \text{and} \qquad \theta_p' : S_k(N) \longrightarrow S_k(Np).$$

It is clear from (13) and (6) that if $F \in M_k(N)$ corresponds to the adelic function Φ, then $\theta'_p F$ corresponds to the function

$$(\theta'_p\Phi)(g) := \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)} \Phi\big(\underbrace{g\,\underline{u}_{p^{j+1}} \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \underline{u}_{p^j}}_{\in G(\mathbb{Q}_p)}\big), \tag{15}$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $g \in G(\mathbb{A})$. Here p^j is the exact power of p dividing N, and $\underline{u}_{p^j}$ is as in (6). The operator θ'_p has the following simple description on an element $F \in M_k(N)$:

$$\theta'_p F = \eta_p F + \sum_{c\in\mathbb{Z}/p\mathbb{Z}} F\big|_k \begin{bmatrix} 1 & & & \\ & 1 & & cp^{-1}N^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{16}$$

To prove this formula, consider the local description (15) and the matrix identity

$$\underline{u}_{p^{j+1}} \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \underline{u}_{p^j} = p^{j+1} \begin{bmatrix} 1 & & & \\ & d & & cp^{-j-1} \\ & & 1 & \\ & bp^{j+1} & & a \end{bmatrix}.$$

As a system of representatives for $\mathrm{SL}(2,\mathbb{Z}_p)/\Gamma_0(p)$ we can choose $\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}$, $c \in \mathbb{Z}/p\mathbb{Z}$, together with the matrix $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. The first type of representatives leads to the summation in (16). As for $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, note that

$$\begin{bmatrix} 1 & & & \\ & & & -p^{-j-1} \\ & & 1 & \\ & p^{j+1} & & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & p^{-1} & & \\ & & 1 & \\ & & & p \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & & -p^{-j} \\ & & 1 & \\ & p^j & & \end{bmatrix}$$

and that Φ is invariant under the rightmost matrix. In view of (9), this proves (16). Actually, the matrices in (16) are a system of representatives for $\Gamma^{\mathrm{para}}(Np) \cap \Gamma^{\mathrm{para}}(N) \backslash \Gamma^{\mathrm{para}}(Np)$. Hence the θ'_p operator is nothing but the natural trace operator from $M_k(N)$ to $M_k(Np)$. Using formula (16), it is now easy to compute the Fourier–Jacobi expansion of θ'_pF. If that of F is given by (3), then

$$(\theta'_pF)(\tau, z, \tau') = \sum_{m=0}^{\infty} \big(p^k\,(U_pf_{m/p})(\tau, z) + p\,\tilde{f}_{mp}(\tau, z)\big)e^{2\pi i m p\tau'}. \tag{17}$$

Here we understand that $f_{m/p} = 0$ if $p \nmid m$, and

$$\tilde{f}_{mp}(\tau, z) := \begin{cases} f_{mp}(\tau, z) & \text{if } N \mid m, \\ 0 & \text{if } N \nmid m. \end{cases}$$

The algebra of operators

For each prime number p we have now defined operators θ_p, θ'_p and η_p on $M_k(N)$ multiplying the level by p and p^2, respectively. Let us put

$$M_k(\Gamma^{\mathrm{para}}) := \bigoplus_{N=1}^{\infty} M_k(N), \qquad S_k(\Gamma^{\mathrm{para}}) := \bigoplus_{N=1}^{\infty} S_k(N). \tag{18}$$

By definition these are abstract direct sums, but see Proposition 3.2 and the remark thereafter. The collection of operators θ_p for different levels N define endomorphisms of $M_k(\Gamma^{\mathrm{para}})$ and $S_k(\Gamma^{\mathrm{para}})$, and similarly for θ'_p and η_p.

Lemma 4.1. *The operators θ_p, θ'_p and η_p commute pairwise.*

Proof. The matrix $\underline{\eta}_p$ in (7) used to define η_p commutes with the matrices in (10). Hence η_p commutes with θ_p. We already noted before that η_p commutes with Atkin–Lehner involutions. By the definition in (14) it follows that η_p commutes with θ'_p (this can also be seen from (16)). That θ_p commutes with θ'_p is easily proved using (16). □

The lemma states that the algebra $\mathcal{A}_p$ generated by the endomorphisms θ_p, θ'_p and η_p of $M_k(N)$ is commutative. Moreover, it is clear by the local descriptions we have given that for different prime numbers p and q the p operators commute with the q operators. Hence the algebra $\mathcal{A}$ generated by all these operators acting on $M_k(\Gamma^{\mathrm{para}})$ and $S_k(\Gamma^{\mathrm{para}})$ is commutative.

Local representations

Let F be a local non-archimedean field and $\mathfrak{o}$, $\mathfrak{p}$ and ϖ as before. Let (π, V) be an irreducible, admissible representation of $G(F)$ ($G = \mathrm{GSp}(4)$) with trivial central character. Let $V(n) \subset V$ be the subspace of vectors fixed by the paramodular group $\mathrm{K}(\mathfrak{p}^n)$ as defined in (1). We already defined the local Atkin–Lehner involutions on $V(n)$, see (5). We further define:

- An operator η : := $V(n) \to V(n+2)$. It is defined by applying $\pi(\mathrm{diag}(1, \varpi^{-1}, 1, \varpi))$.
- An operator θ : := $V(n) \to V(n+1)$. It is defined by a similar summation as in (12).
- An operator θ' : := $V(n) \to V(n+1)$. It is defined as in (14) or, alternatively, by a formula as in (16).

Just as in the global case these operators generate a commutative algebra of linear operators on the space of paramodular vectors. Now assume that the modular form $F \in M_k(N)$ corresponds to the adelic function Φ, and that Φ generates an irreducible, automorphic representation $\pi = \otimes_{p \leqslant \infty} \pi_p$ of $G(\mathbb{A})$. Then it is clear that each π_p $(p < \infty)$ contains paramodular invariant vectors. It is further clear that the local operators η, θ and θ' are compatible with the global operators. It is our intention to use local representation theoretic results on paramodular vectors to obtain results on classical modular forms.

5. Oldforms and newforms

The main purpose of the operators introduced in the previous section is to define oldforms and newforms. Roughly speaking, all the modular forms in the images of our operators should be considered "old". A modular form that is orthogonal to all the oldforms is "new". Recall the definition (18) of the space $M_k(\Gamma^{\mathrm{para}})$ and the algebra $\mathcal{A}$ acting on it. Let $\mathcal{I} \subset \mathcal{A}$ be the ideal generated by η_p, θ_p and θ'_p, where p runs through all prime numbers. Then we define

$$M_k^{\mathrm{old}}(\Gamma^{\mathrm{para}}) := \mathcal{I}\, M_k(\Gamma^{\mathrm{para}}), \qquad M_k^{\mathrm{old}}(N) := M_k^{\mathrm{old}}(\Gamma^{\mathrm{para}}) \cap M_k(N).$$

Similar definitions are made for cusp forms. Elements of these spaces are called *oldforms.* On the spaces $S_k(N)$ we have the Petersson scalar product, which allows us to define the subspace of *newforms* as the orthogonal complement of the oldforms:

$$S_k^{\mathrm{new}}(N) := S_k^{\mathrm{old}}(N)^{\perp}.$$

We conjecture that paramodular cusp forms have a newform theory that is as nice as the well-known newform theory for elliptic modular forms:

Conjecture 5.1. (Newforms Conjecture) *Let N be a nonnegative integer.*

i) *Assume that* $F \in S_k^{\mathrm{new}}(N)$ *is an eigenform for the unramified local Hecke algebra* $\mathcal{H}_p$ *for almost all* p *not dividing* N. *Then* F *is an eigenform for* $\mathcal{H}_p$ *for* <u>*all*</u> $p \nmid N$.
ii) *Let* $F_i \in S_k^{\mathrm{new}}(N_i)$, $i = 1, 2$, *be two non-zero cusp forms. Assume that* F_1 *and* F_2 *are both eigenforms for the unramified local Hecke algebra* $\mathcal{H}_p$ *for almost all* p. *Assume further that for almost all* p *the Hecke eigenvalues of* F_1 *and* F_2 *coincide. Then* $N_1 = N_2$, *and* F_1 *is a multiple of* F_2.

Our belief in the Newforms Conjecture is based on an analogous local statement and the conjectural structure of the discrete spectrum of PGSp(4). The local statement, whose proof will appear in [RS2], is as follows.

Theorem 5.1. (Local New- and Oldforms Theorem) *Let* F *be a non-archimedean local field of characteristic zero,* $\mathfrak{o}$ *its ring of integers, and* $\mathfrak{p}$ *the maximal ideal of* $\mathfrak{o}$. *Let* (π, V) *be an irreducible, admissible representation of* $G(F)$ *with trivial central character. For* n *a nonnegative integer, let* $V(n)$ *be the space of vectors fixed by the local paramodular group* $\mathrm{K}(\mathfrak{p}^n)$. *Assume that for some* n *we have* $V(n) \neq 0$.

i) *(Local multiplicity one) If* n_0 *is the minimal* n *such that* $V(n) \neq 0$, *then* $\dim(V(n_0)) = 1$.
ii) *(Local oldforms theorem) For any* $n > n_0$, *the space* $V(n)$ *is spanned by vectors obtained by repeatedly applying the operators* θ, θ' *and* η *to the elements of* $V(n_0)$.

Part i) of this theorem states that there is always a *local newform* that is unique up to scalars, provided there are paramodular vectors at all. It can be proved that every *generic* irreducible representation has non-zero paramodular invariant vectors, and that for *tempered* representations this condition is also necessary.

We shall indicate further below how the (global) Newforms Conjecture follows from the local Theorem 5.1 and the following two global statements.

Conjecture 5.2. (Weak Multiplicity One) *If* π *is an irreducible admissible representation of* $\mathrm{PGSp}(4, \mathbb{A}_F)$, *where* F *is any number field, then* π *occurs with multiplicity at most one in the discrete spectrum of* $\mathrm{PGSp}(4, \mathbb{A}_F)$.

Proofs of this conjecture have been announced by several authors, but currently there is no published proof. Note that while this conjecture is assumed to be true over any number field, the following conjecture depends on the arithmetic of $\mathbb{Q}$ and is in general wrong over other number fields.

Conjecture 5.3. (Paramodular Strong Multiplicity One) *If $\pi \cong \otimes_{p\leqslant\infty}\pi_p$ is an irredu-cible discrete automorphic representation of* $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$ *and π is paramodular, i.e., π_p admits a nonzero vector invariant under some paramodular group for all finite p, then π is determined, up to equivalence, by π_∞ and all but finitely many of the π_p for finite p.*

Generally speaking, strong multiplicity one should not hold for irreducible discrete automorphic representations of a connected reductive algebraic group over a number field, and it does not hold for PGSp(4). To explain our reasoning as to why it should hold for the smaller class of paramodular irreducible discrete automorphic representations of $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$, let $\pi \cong \otimes_{p\leqslant\infty}\pi_p$ be such a representation. Let $[\pi]_{\mathrm{near}}$ be the discrete near equivalence class of π, i.e., the set of all irreducible admissible representations $\pi' \cong \otimes_{p\leqslant\infty}\pi'_p$ of $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$ that occur in the discrete spectrum of $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$ and for which $\pi'_p \cong \pi_p$ for almost all p. To verify the conjecture it would suffice to prove that $[\pi]_{\mathrm{near}}$ contains exactly one paramodular element, namely π. Conjecturally, $[\pi]_{\mathrm{near}}$ is the set of automorphic elements of a conjectural Arthur packet $\Pi(\phi)$ corresponding to an Arthur parameter $\phi : \mathrm{L}_{\mathbb{Q}}\times\mathrm{SL}(2,\mathbb{C}) \to {}^L\mathrm{PGSp}(4)$, where $\mathrm{L}_{\mathbb{Q}}$ is the conjectural Langlands group of $\mathbb{Q}$. First, assume ϕ is tempered. Then, conjecturally, all the elements of $[\pi]_{\mathrm{near}}$ are tempered. We can prove that if $p < \infty$, then an irreducible tempered admissible representation of $\mathrm{PGSp}(4,\mathbb{Q}_p)$ is paramodular if and only if it is generic. It follows that the only paramodular element of $[\pi]_{\mathrm{near}}$ with infinity type π_∞ is π (π_p is the generic base point of the local tempered Arthur packet $\Pi(\phi_p)$ for all $p < \infty$). Next, assume ϕ is not tempered. Then, by the Ramanujan conjecture (see 6.1 further below), π is CAP (cuspidal associated to parabolic) with respect to the Borel subgroup B, the Klingen parabolic subgroup Q or the Siegel parabolic subgroup P of PGSp(4). We can prove that no paramodular irreducible discrete automorphic representation of $\mathrm{PGSp}(4,\mathbb{A}_{\mathbb{Q}})$ is CAP with respect to B or Q (it is here that we need the assumption that we are working over $\mathbb{Q}$). Hence, π is CAP with respect to P. Conjecturally the elements of $[\pi]_{\mathrm{near}}$ form what is called a Saito–Kurokawa packet, and we can prove that the only paramodular element of $[\pi]_{\mathrm{near}}$ with infinity type π_∞ is π (as in the tempered case, π_p is the base point of the local nontempered Arthur packet $\Pi(\phi_p)$ for all $p < \infty$). See the next section for more on Saito–Kurokawa packets.

"Proof" of the Newforms Conjecture

We shall now indicate how to obtain a proof of Conjecture 5.1 from Theorem 5.1 and the Conjectures 5.2 and 5.3. Let $F \in S_k^{\text{new}}(N)$ be an eigenform for almost all of the unramified local Hecke algebras. Let Φ be the corresponding adelic function $G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$, and let π be the representation generated by Φ. Then π is a finite direct sum of irreducible cuspidal automorphic representations π_i. Let $\pi_i = \otimes \pi_{i,p}$ with $\pi_{i,p}$ an irreducible, admissible representation of $G(\mathbb{Q}_p)$. The archimedean representations $\pi_{i,\infty}$ all have scalar minimal K-type of weight k and are therefore isomorphic. Since F is $\Gamma^{\text{para}}(N)$ invariant, each $\pi_{i,p}$ for $p < \infty$ has non-zero paramodular vectors. The eigenform condition implies that the local representations $\pi_{i,p}$ and $\pi_{j,p}$ are isomorphic for almost all p and all i, j. By Conjecture 5.3 the representations π_i are all isomorphic. But then, by Conjecture 5.2, there can be only one i; in other words, π is irreducible. This implies part i) of Conjecture 5.1.

Now let F_1 and F_2 be as in ii) of Conjecture 5.1. We just proved that F_1 and F_2 generate irreducible cuspidal automorphic representations $\pi_1 = \otimes \pi_{1,p}$ and $\pi_2 = \otimes \pi_{2,p}$. The condition of F_1 and F_2 having the same Hecke eigenvalues almost everywhere translates into $\pi_{1,p} \cong \pi_{2,p}$ for almost all p. By Conjecture 5.3 it follows that $\pi_1 \cong \pi_2$, and then $\pi_1 = \pi_2$ as spaces of automorphic forms by Conjecture 5.2. We shall write π for $\pi_1 = \pi_2$ and π_p for $\pi_{1,p} = \pi_{2,p}$. Let V_p be a model of π_p. Let $v_\infty \in V_\infty$ be a lowest weight vector (generating the scalar minimal K-type). For $p < \infty$ let $v_p \in V_p$ be the essentially unique local newform according to Theorem 5.1 i). Let F be the function on the upper half plane corresponding to the vector $\otimes v_p \in \otimes \pi_p$. Then F is a paramodular cusp form of weight k. Since v_p is the local newform at every place, the level of F is at least as "good" as the level of F_1, in the sense that $F \in S_k(N)$ with $N|N_1$.

Part ii) of Theorem 5.1 says that every paramodular vector in V_p can be obtained from the local newform v_p by repeatedly applying the local level raising operators and taking linear combinations. Since local and global level raising operators are compatible, this implies that $F_1 = \Theta F$, where Θ is an element of the algebra $\mathcal{A}$ introduced in the previous section. This element Θ cannot be in the ideal $\mathcal{I}$ generated by $\theta_p, \theta'_p, \eta_p$ for all primes p, since otherwise F_1 would be an oldform. Hence Θ is a scalar and F_1 is a multiple of F. The same argument applies to F_2, proving that F_1 and F_2 are multiplies of each other and that $N_1 = N_2 = N$.

6. Saito–Kurokawa liftings

Examples of modular forms for the paramodular group are obtained by the Saito–Kurokawa lifting. Let k be a positive integer. Let $f \in S_{2k-2}(\Gamma_0(N))$ be an elliptic cusp form, which we also assume to be a newform. We also assume that the sign in the functional equation of $L(s, f)$ is -1. Then f corresponds to a cuspidal Jacobi form $\tilde{f} \in J_{k,N}$ via the Skoruppa–Zagier lifting; see [SZ]. From $\tilde{f}$ we can construct a Siegel modular form $F \in S_k(N)$ via GRITSENKO's "arithmetical lifting", which is a generalization of the Maaß construction; see [Gri]. The map $f \mapsto F$ extends to an injective linear map

$$\mathrm{SK} : := S_{2k-2}^{\mathrm{new},-}(\Gamma_0(N)) \longrightarrow S_k(N).$$

Here, the "$-$" indicates the subspace of cusp forms for which the sign in the functional equation is -1. We propose an alternative group theoretic construction of this lifting which gives a little bit more information.

Theorem 6.1. *Let k and N be positive integers. Let $f \in S_{2k-2}^{\mathrm{new}}(\Gamma_0(N))$ be an elliptic cuspidal newform, assumed to be an eigenform for almost all Hecke operators. We assume that the sign in the functional equation of $L(s, f)$ is -1. Then there exists a paramodular Siegel cusp form $F \in S_k^{\mathrm{new}}(N)$ such that the incomplete spin L-function of F is given by*

$$L^S(s, F) = L^S(s, f)\, Z^S(s - 1/2)\, Z^S(s + 1/2). \tag{19}$$

Such an F is unique up to scalars.

We give an outline of the proof, whose details will appear elsewhere. Let π be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ associated to the modular form f (it is generated by an adelic function on $\mathrm{GL}(2, \mathbb{A})$ constructed from f by a similar formula as in (4)). Our hypothesis on $L(s, f)$ assures that there exists a Saito–Kurokawa lifting to $\mathrm{GSp}(4)$, meaning a cuspidal automorphic representation Π on $G(\mathbb{A})$ with trivial central character such that $L^S(s, \Pi) = L^S(s, \pi) Z^S(s - 1/2) Z^S(s + 1/2)$. The construction of Π is carried out in [Sch3] and further investigated in [Sch4]. In fact, there may exist a whole (finite) packet of such Π, but exactly one element in the packet is distinguished in the sense that each of its local components Π_p $(p < \infty)$ contains non-zero paramodular vectors. Hence we can extract a Siegel modular form $F \in S_k(N')$ from Π for some level N' (the archimedean component Π_∞ is such that F is holomorphic of weight k). Further analysis of the Π_p shows that they have a unique paramodular vector at the "right" level and at no better level; see Theorem 6.2 below for more details. In other words, we can actually extract an $F \in S_k(N)$ from Π, which is unique up to scalars, and since the local representations contain no paramodular vectors at lower levels, this F must be a newform.

Theorem 6.1 can be reformulated by saying that there is a Hecke-equivariant injection

$$\mathrm{SK} : := \; S_{2k-2}^{\mathrm{new},-}(\Gamma_0(N)) \longrightarrow S_k^{\mathrm{new}}(N). \tag{20}$$

Here "Hecke-equivariant" has the following meaning. Let $T(p)$ be the usual Hecke operator on $S_{2k-2}(\Gamma_0(N))$. Let $T_S(p)$ and $T'_S(p)$ be the generators for the local Hecke algebra $\mathcal{H}_p$ for Siegel modular forms as in [EZ], Section 6. We define a homomorphism ι of local Hecke algebras by

$$\iota(T_S(p)) = T_J(p) + p^{k-1} + p^{k-2},$$
$$\iota(T'_S(p)) = (p^{k-1} + p^{k-2})T_J(p) + 2p^{2k-3} + p^{2k-4}.$$

Then the map (20) is Hecke-equivariant in the sense that $T(\mathrm{SK}(f)) = \mathrm{SK}(\iota(T)f)$ for all elements $T \in \mathcal{H}_p$, for any $p \nmid N$.

Local liftings

We now present the local ingredient to Theorem 6.1 in more detail. Let F be a p–adic field, and let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character. In [Sch3], a local Saito–Kurokawa lifting $\mathrm{SK}(\pi)$ has been attached to π. It can be constructed as the unique irreducible quotient of the induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}$; we refer to [ST] for the notation and the fact that this induced representation has exactly two irreducible constituents. The representation $\mathrm{SK}(\pi)$ thus constructed is a non-generic, non-tempered, irreducible, admissible representation of $\mathrm{PGSp}(4,F)$. In [RS2] we will give the proof of the following result, which is a local version of the Saito–Kurokawa lifting. The symbol $\mathrm{K}(\mathfrak{p}^n)$ stands for the local paramodular group; see (1).

Theorem 6.2. *Let (π, V) be an irreducible, admissible, infinite-dimensional representation of* $\mathrm{PGL}(2,F)$*, and let $(\mathrm{SK}(\pi), W)$ be the local Saito–Kurokawa lifting of π as explained above. Let $V(n) \subset V$ be the subspace of $\Gamma_0(\mathfrak{p}^n)$–invariant vectors, and let $W(n) \subset W$ be the subspace of* $\mathrm{K}(\mathfrak{p}^n)$ *invariant vectors. Let n_0 be the minimal n such that $V(n) \neq 0$.*

i) *The integer n_0 is also the minimal n such that $W(n) \neq 0$.*
ii) $\dim(W(n_0)) = 1$.

iii) *For any $n \geqslant n_0$, we have*

$$W(n) = \bigoplus_{\substack{d,e \geqslant 0 \\ d+2e=n-n_0}} \theta^d \eta^e W(n_0),$$

where θ and η are the local level raising operators defined earlier.

iv) *All paramodular vectors $w \in W(n)$ are Atkin–Lehner eigenvectors with the same eigenvalue (as some $w_0 \in W(n_0)$).*

Thus, the local lifting Π has a unique newform at the same level as π. This explains the existence and part of the uniqueness assertion in Theorem 6.1 (one also needs to know global multiplicity one in the Saito–Kurokawa space), and the assertion that the lifting is a newform.

Extension to oldforms

We would like to extend the map (20) to include oldforms. However, part iii) of Theorem 6.2 shows that the structure of oldforms in a local representation (π, V) and in its lifting $(\mathrm{SK}(\pi), W)$ is different. While in V the dimensions of the spaces $V(n)$, $n \geqslant n_0$, grow (by [Cas]) like $1, 2, 3, \ldots$, the dimensions of the spaces $W(n)$ grow like $1, 1, 2, 2, 3, 3, \ldots$ (see Table 1 in the appendix, where we can observe these dimensions in the representations IIb, Vb and VIc, which are local Saito–Kurokawa liftings). This suggests that only a *subspace* of the space of oldforms in V can be matched to the oldforms in W. Part iv) of Theorem 6.2 provides the clue that this subspace should consist of the newforms and all oldforms with the same Atkin–Lehner eigenvalue as the newform. This local situation is compatible with the work of Skoruppa and Zagier, which shows that the map (20) can be extended to the "certain space" in the title of [SZ]; see further below for a precise definition.

We shall first describe the local analogue of the "certain space" more precisely. As above let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{PGL}(2, F)$, where F is a non-archimedean local field, and let $V(n) \subset V$ be the subspace of vectors fixed under the local congruence subgroup $\Gamma_0(\mathfrak{p}^n)$. Let n_0 be the minimal n such that $V(n) \neq 0$. Then, by Casselman's theory, $\dim(V(n)) = n - n_0 + 1$ for $n \geqslant n_0$. The local Atkin–Lehner involution

$$u_n = \begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix} \qquad (\varpi \text{ a uniformizer})$$

splits $V(n)$ into ± 1 eigenspaces $V(n)^{\pm}$. The eigenvalue ε at the minimal level n_0 coincides with the value $\varepsilon(1/2, \pi)$ of the ε–factor at $1/2$. Locally, the "certain space" is $\oplus_{n=n_0}^{\infty} V(n)^{\varepsilon}$, i.e., it consists of the newform and all those oldforms with the same Atkin–Lehner eigenvalue as the newform. These oldforms are obtained by repeated application of the operators

$$\alpha : := V(n) \longrightarrow V(n+1), \qquad v \longmapsto v + \pi(\begin{bmatrix} \varpi^{-1} & \\ & 1 \end{bmatrix})v \tag{21}$$

and

$$\beta : := V(n) \longrightarrow V(n+2), \qquad v \longmapsto \pi(\begin{bmatrix} \varpi^{-1} & \\ & 1 \end{bmatrix})v \tag{22}$$

to $V(n_0)$ (it is immediately verified that α and β commute with Atkin–Lehner involutions). One can check that $\alpha^2 v$ is not a multiple of βv, and more generally that

$$V(n)^{\varepsilon} = \bigoplus_{\substack{d,e \geqslant 0 \\ d+2e=n-n_0}} \alpha^d \beta^e V(n_0). \tag{23}$$

We see that the "certain space" can be matched exactly with the complete space of paramodular vectors in $(\mathrm{SK}(\pi), W)$, whose structure is given in Theorem 6.2 iii). More precisely, we can define a local Saito–Kurokawa map

$$\mathrm{SK} : := \bigoplus_{n=n_0}^{\infty} V(n)^{\varepsilon} \longrightarrow \bigoplus_{n=n_0}^{\infty} W(n) \tag{24}$$

by mapping a non-zero vector $v \in V(n_0)$ to a non-zero vector $w \in W(n_0)$ and requiring that $\mathrm{SK} \circ \alpha = \theta \circ \mathrm{SK}$ and $\mathrm{SK} \circ \beta = \eta \circ \mathrm{SK}$. Then SK is a linear isomorphism. We note that SK is *not* canonically defined: Not only do we have a freedom in the normalization of the newforms and the operators, but we could also replace the operator β by a linear combination of β and α^2.

The global version of the "certain space" is defined as follows. On the spaces $S_k(\Gamma_0(N))$ (elliptic modular forms) we have, for any prime number p, the Atkin–Lehner involutions u_p, defined analogously as above in the degree 2 case. If $p \nmid N$, we let u_p be the identity. We are looking for level raising operators $S_k(\Gamma_0(N)) \to S_k(\Gamma_0(Np))$ commuting with Atkin–Lehner involutions; here the prime number p may or may not divide N. The two natural operators $f(z) \mapsto f(z)$ and $f(z) \mapsto f(pz)$ do not have this property, but a computation shows that a certain linear combination has. More precisely, put

$$\alpha_p : := S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(Np)), \qquad f(z) \longmapsto f(z) + p^{k/2} f(pz). \tag{25}$$

Then $\alpha_p \circ u_p = u_p \circ \alpha_p$. Furthermore, another computation shows that $f \mapsto f(pz)$ *does* commute with Atkin–Lehner involutions if we consider $f(pz)$ an element of $S_k(\Gamma_0(Np^2))$. We therefore define

$$\beta_p : := S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(Np^2)), \qquad f(z) \longmapsto p^{k/2} f(pz). \tag{26}$$

Then $\beta_p \circ u_p = u_p \circ \beta_p$. If Φ is the adelic function on $\mathrm{GL}(2, \mathbb{A})$ corresponding to f, then the adelic functions corresponding to $\alpha_p f$ and $\beta_p f$ are

$$g \longmapsto \Phi(g) + \Phi\left(g \begin{bmatrix} p^{-1} & \\ & 1 \end{bmatrix}\right) \qquad \text{and} \qquad g \longmapsto \Phi\left(g \begin{bmatrix} p^{-1} & \\ & 1 \end{bmatrix}\right),$$

respectively. In other words, α_p and β_p are compatible with the local operators (21) and (22). We shall now define the "certain space", which will be denoted by $\mathcal{S}_k(\Gamma_0(N))$. Let

$$S_k^{\mathrm{new}}(\Gamma_0) := \bigoplus_{N=1}^{\infty} S_k^{\mathrm{new}}(\Gamma_0(N)), \qquad S_k(\Gamma_0) := \bigoplus_{N=1}^{\infty} S_k(\Gamma_0(N)).$$

We consider the operators α_p and β_p as endomorphisms of $S_k(\Gamma_0)$. They obviously commute, and operators for different p also commute. Hence we get a commutative algebra $\mathcal{B}$ of endomorphisms of $S_k(\Gamma_0)$ generated by all these operators for all prime numbers p. We define the "certain space" as the image of $S_k^{\mathrm{new}}(\Gamma_0)$ under $\mathcal{B}$,

$$\mathcal{S}_k(\Gamma_0) := \mathcal{B}\, S_k^{\mathrm{new}}(\Gamma_0), \qquad \mathcal{S}_k(\Gamma_0(N)) := \mathcal{S}_k(\Gamma_0) \cap S_k(\Gamma_0(N)).$$

Hence $\mathcal{S}_k(\Gamma_0(N))$ consists of all the newforms of level N plus those oldforms that can be obtained by repeated application of α_p and β_p operators to newforms of lower levels. Those oldforms have the same Atkin–Lehner eigenvalues as the newforms from which they come. If in the above definitions we allow only newforms with a certain sign in the functional equation of their L–function, we obtain the spaces $\mathcal{S}_k^{\pm}(\Gamma_0(N))$

From the local linear independence (23) we derive the global result that

$$\mathcal{S}_k(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{\substack{d,e \geqslant 1 \\ de^2 = N/M}} \alpha_d \beta_e S_k^{\mathrm{new}}(\Gamma_0(M)). \tag{27}$$

Here, $\alpha_d = \prod_i \alpha_{p_i}^{\nu_i}$ if $d = \prod_i p_i^{\nu_i}$, and similarly for β_e. Restricting to newforms with a fixed sign in the functional equation, we get

$$\mathcal{S}_k^{\pm}(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{\substack{d,e \geqslant 1 \\ de^2 = N/M}} \alpha_d \beta_e S_k^{\mathrm{new},\pm}(\Gamma_0(M)). \tag{28}$$

Now for each M we have the maps (20) from $S_{2k-2}^{\mathrm{new},-}(\Gamma_0(M))$ to $S_k^{\mathrm{new}}(M)$. We put them all together to define a linear map

$$S_{2k-2}^{\mathrm{new},-}(\Gamma_0) \longrightarrow S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}})$$

The direct sum decomposition (28) shows that this linear map can be extended to a linear map

$$\mathrm{SK} : := \mathcal{S}_{2k-2}^{-}(\Gamma_0) \longrightarrow S_k(\Gamma^{\mathrm{para}})$$

in such a way that

$$\mathrm{SK} \circ \alpha_p = \theta_p \circ \mathrm{SK} \qquad \text{and} \qquad \mathrm{SK} \circ \beta_p = \eta_p \circ \mathrm{SK}.$$

The image of SK is called the *Maaß space* and denoted by $\mathcal{S}_k(\Gamma^{\mathrm{para}})$. Restricting to a fixed level we get a Saito–Kurokawa lifting SK from $\mathcal{S}_{2k-2}^{-}(\Gamma_0(N))$ to $\mathcal{S}_k(N) = \mathcal{S}_k(\Gamma^{\mathrm{para}}) \cap S_k(N)$. Since SK is compatible with the local isomorphisms (24), we obtain the following result.

Theorem 6.3. *The Saito–Kurokawa lifting (20) can be extended to a Hecke–equivariant isomorphism*

$$\mathrm{SK} : := \mathcal{S}_{2k-2}^{-}(\Gamma_0(N)) \longrightarrow \mathcal{S}_k(N).$$

Characterizations of the Maaß space

The following version of the Ramanujan conjecture is believed to be true, but currently there is no published proof.

Conjecture 6.1. (Ramanujan Conjecture for GSp(4)) *Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of* $\mathrm{GSp}(4, \mathbb{A}_F)$, *where F is any number field. If π is not a CAP representation, then each π_v is tempered.*

The Ramanujan conjecture has the following relevance for the characterization of the Maaß space. We note that in the classical case the characterization of eigenforms in the Maaß space by their spin L–functions having poles was obtained by ODA [Oda] and EVDOKIMOV [Ev].

Theorem 6.4. *Write $F \in S_k(N)$ as a sum $F = \sum_i F_i$, where each $F_i \in S_k(N)$ is a Hecke eigenform for almost all Hecke operators. Then the following statements are equivalent.*

i) *F is an element of the Maaß space $\mathcal{S}_k(N)$.*
ii) *Each of the incomplete spin L–functions $L(s, F_i)$ has a pole at $s = 3/2$.*

iii) *Each F_i corresponds to a vector in an irreducible cuspidal automorphic representation of* $\mathrm{PGSp}(4,\mathbb{A}_\mathbb{Q})$ *that is CAP with respect to the Siegel parabolic subgroup.*

Each of these conditions implies the following.

iv) $\theta_p F = \theta'_p F$ *for each prime number* p.

If the Ramanujan conjecture holds, then the following condition implies the others.

v) *There exists a prime number* p *such that* $\theta_p F$ *is a multiple of* $\theta'_p F$.

Sketch of proof: i) $\Rightarrow$ ii) follows from the shape of the L–function in (19). ii) $\Rightarrow$ iii) follows from the characterization of CAP automorphic representations in [PS] and local results showing that a) global Saito–Kurokawa packets contain at most one element that is paramodular at every finite place, and b) Borel–CAP representations do not have paramodular vectors at every finite place. iii) $\Rightarrow$ i) follows from the fact that the local lifting (24) is onto, meaning SK exhausts the space of paramodular vectors.

i) $\Rightarrow$ iv): Let $\Pi = \otimes\Pi_p$ be an irreducible constituent of the space of cusp forms on $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ generated by the adelic function Φ attached to F. Then, by the group theoretic construction of Saito–Kurokawa liftings indicated after Theorem 6.1, each Π_p for $p<\infty$ is of the form $\mathrm{SK}(\pi)$ for an irreducible, admissible, infinite-dimensional representations π of $\mathrm{PGL}(2,\mathbb{Q}_p)$. One can prove by local computations that $\theta-\theta'$ annihilates the space of paramodular vectors in $\mathrm{SK}(\pi)$. This implies iv) since the local and global operators are compatible.

v) $\Rightarrow$ i): Let $\Pi = \otimes\Pi_p$ be as in the previous paragraph. The hypothesis implies that for some p the local representation Π_p contains a paramodular vector that is annihilated by $\theta-\theta'$. One can prove by local methods that Π_p must be one of the representations in the following list (see the table in the appendix):

- An unramified twist of the trivial representation.
- A representation of type IVb.
- A representation of type Vd or VId.
- A representation of the form $\mathrm{SK}(\pi)$ as in Theorem 6.2.

It is known that one-dimensional representations do not occur in global cusp forms. Representations of type IVb do also not occur in global cusp forms because they are not unitarizable. Representations of type Vd and

VId are not tempered. Hence, if Π_p is one of these representations, and if the Ramanujan conjecture is true, then Π must be a CAP representation. One can give a complete description of the local components of CAP representations, and this description shows that Π must be CAP with respect to B, the minimal parabolic subgroup, or Q, the Klingen parabolic subgroup. But one can further show that in such CAP representations there is always at least one place for which the local component has no paramodular vectors (it is essential here that the ground field is $\mathbb{Q}$; the statement is wrong for other number fields). This proves that Π_p cannot be of type Vd or VId, and must consequently be a local Saito–Kurokawa representation of the form $\mathrm{SK}(\pi)$. These representations are also non-tempered, so that by the Ramanujan conjecture Π must be a CAP representation. By the above mentioned explicit description of CAP representation, Π must indeed be CAP with respect to P, the Siegel parabolic subgroup. In other words, Π lies in the Saito–Kurokawa space. □

In view of the definition (14) of the θ_p' operator we see that, at least if the Ramanujan conjecture is true, the Maaß space can be characterized as the subspace of $S_k(N)$ where θ_p commutes with Atkin–Lehner involutions. For a classical Saito–Kurokawa lifting $F \in S_k(1)$ (full modular group) the condition v) in Theorem 6.4 means

$$V_p f_m = p^{k-1} U_p f_{m/p} + f_{mp} \qquad \text{for } m \geqslant 1,$$

where we understand $f_{m/p} = 0$ for $p \nmid m$; see (11) and (17). In terms of the Fourier expansion $F(\tau, z, \tau') = \sum_{n,r,m} a(n,r,m) e^{2\pi i(n\tau + rz + m\tau')}$ this translates into the conditions

$$a(np, r, m) + p^{k-1} a\Big(\frac{n}{p}, \frac{r}{p}, m\Big) = p^{k-1} a\Big(n, \frac{r}{p}, \frac{m}{p}\Big) + a(n, r, mp) \tag{29}$$

for $n, r, m \in \mathbb{Z}$, with the convention that $a(\alpha, \beta, \gamma) = 0$ if $(\alpha, \beta, \gamma) \notin \mathbb{Z}^3$. The Maaß space for the full modular group is defined by the more general relations

$$a(n, r, m) = \sum_{d | (n,r,m)} d^{k-1} a\Big(\frac{nm}{d^2}, \frac{r}{d}, 1\Big) \qquad \text{for } n, r, m \in \mathbb{Z}; \tag{30}$$

see [Ma] or [EZ] Section 6. To see that the Maaß relations (30) are indeed more general, substitute (30) into (29). Conversely, for m a power of p, the condition (30) is implied by (29). Theorem 6.4 says that if the Ramanujan conjecture holds, then (29) and (30) are actually equivalent:

Corollary 6.1. *Suppose that the Ramanujan conjecture 6.1 holds. Let $F \in S_k(1)$ be a cusp form for* $\mathrm{Sp}(4, \mathbb{Z})$ *with Fourier expansion $F(\tau, z, \tau') =$*

$\sum_{n,r,m} a(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$. *Then F is in the Maaß space if and only if there is a prime number p such that (29) holds.*

We stress that the prime number p in this corollary is completely arbitrary.

7. Two theorems

As before, let $S_k(N)$ be the space of cusp forms of weight k with respect to the paramodular group $\Gamma^{\mathrm{para}}(N)$. In this section we shall elaborate on the representation theoretic meaning of the following two theorems.

Theorem 7.1. *There are no paramodular cusp forms of weight* 1*: The spaces $S_1(N)$ are zero for any N.*

Theorem 7.2. *The operators θ_p and θ_p' from $S_k(N)$ to $S_k(Np)$ are injective for any N and any prime p.*

Both theorems are quickly proved using results on Jacobi forms. Theorem 7.1 follows immediately from the Fourier–Jacobi expansion and a result of SKORUPPA stating that $J_{1,m} = 0$ for any m; see Theorem 5.7 in [EZ]. For Theorem 7.2, note that in view of the definition (14) it is enough to prove the result for θ_p. By (11), the θ_p operator is compatible with the operator V_p on Jacobi forms. But it is a consequence of the results of SKORUPPA and ZAGIER [SZ] that the operator $V_p :{:=}\ J_{k,m}^{\mathrm{cusp}} \to J_{k,mp}^{\mathrm{cusp}}$ on cuspidal Jacobi forms is injective (see Lemma 1.10 of [Sch2] for a corresponding local statement). Theorem 7.2 follows.

We shall now reformulate Theorem 7.1 in terms of representations. For representations of $G(F)$, where F is a local field, we shall employ the notation of [ST] (this paper treats non-archimedean representations, but the notation can also be used for $F = \mathbb{R}$). The symbol ν stands for the normalized absolute value, which in the case $F = \mathbb{R}$ is the usual absolute value $|\,|$. Let ξ_0 be a character of F^* of order 2. Then, by [ST] Lemma 3.6, the induced representation $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}$ has four irreducible constituents. We are interested in the Langlands quotient $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$. In the archimedean case $F = \mathbb{R}$, where ξ_0 is the sign character, it can be shown that $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ has a minimal K–type of weight $(1,1)$. In other words, this is the archimedean representation underlying Siegel modular forms of weight one. Theorem 7.1 is therefore equivalent to the archimedean part of the following statement.

Corollary 7.1. *Let $F \in S_k(N)$, and let Φ be the adelic function corresponding to F. Let $\pi = \oplus \pi_i$ be the cuspidal automorphic representation of* $\mathrm{PGSp}(4, \mathbb{A})$ *generated by Φ, and let $\pi_i = \otimes \pi_{i,p}$ be the tensor product decomposition of the irreducible component π_i of π. Then no $\pi_{i,p}$ $(p \leqslant \infty)$ is equal to $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$, where ξ_0 is a local character of order* 2.

Let us now focus on the non-archimedean content of Corollary 7.1. The appendix contains a table with the complete list of Iwahori–spherical representations of $\mathrm{GSp}(4, F)$ and the dimensions of their spaces of fixed vectors under the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$ for any level $\mathfrak{p}^n$. We see that the dimensions of these spaces are always growing with growing n, except for the representation $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ of type Vd; here ξ_0 is the unique non-trivial unramified quadratic character of F^*. For this representation the dimensions are $1, 0, 1, 0, \ldots$ (one can show that if ξ_0 is ramified, then Vd contains no paramodular vectors at all). Hence the corresponding local statement to Theorem 7.2 is not always true: In the Vd type representations, the local θ and θ' operators from $V(n)$ to $V(n+1)$ are zero (here $V(n)$ is the space of vectors fixed under $\mathrm{K}(\mathfrak{p}^n)$). It follows that these representations cannot occur as local components in automorphic representations generated by elements of $S_k(N)$, which is exactly the statement of Corollary 7.1 for $p < \infty$.

We mention the following local result from [RS2], which says that the representations of type Vd are the *only* counterexamples to the injectivity of θ and θ'.

Theorem 7.3. *Let F be a p–adic field. Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let $V(n)$ be the space of vectors fixed under the paramodular group* $\mathrm{K}(\mathfrak{p}^n)$ *as in (1). Assume that π is not isomorphic to a representation $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma)$ of type Vd, where ξ_0 is the unramified character of order* 2. *Then the operators θ and θ' from $V(n)$ to $V(n+1)$ are injective, for any n.*

This theorem is analogous to Lemma 1.10 of [Sch2], which says that certain Weil representations are the only counterexamples to the injectivity of a local V operator on representations of the Jacobi group. The two results are actually related since these Weil representations are Fourier–Jacobi models of the Vd type representations. Theorem 7.3 says that Vd type representations are the *only* local representations excluded by Theorem 7.2 (in automorphic representations generated by elements of $S_k(N)$). Therefore Theorem 7.2 is the *exact* non-archimedean analogue of Theorem 7.1.

Note that Corollary 7.1 is a Ramanujan type result: The representations of type Vd are non-tempered, and the corollary says that they do not occur in certain cuspidal automorphic representations of $\mathrm{PGSp}(4,\mathbb{A})$. There *are* actually cuspidal automorphic representations of this group, namely certain CAP representations, that contain

- $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ as archimedean component, and moreover
- $L(\nu\xi_0, \xi_0 \rtimes \nu^{-1/2})$ at almost every place.

But one can show that there is always at least one non-archimedean place where the local representation has no paramodular vectors. In other words, cusp forms of weight 1 *do* exist, but not for the paramodular group.

Appendix. Paramodular vectors in Iwahori–spherical representations

Table 1 below lists the dimensions of the spaces of paramodular vectors of any level for each irreducible, admissible representation of $\mathrm{PGSp}(4,F)$ which admits a nonzero vector fixed by the Iwahori subgroup I. We shall explain the contents of the table in detail.

The first column. By [Bo], these representations are exactly the irreducible subquotients of the representations of $\mathrm{PGSp}(4,F)$ induced from unramified quasi-characters of the Borel subgroup. The basic reference on representations of $\mathrm{GSp}(4,F)$ induced from a quasi-character of the Borel subgroup is section 3 of [ST], and we will use the notation of that paper. Thus, St is the Steinberg representation, $\mathbf{1}$ is the trivial representation, and $\nu = |\cdot|$. It is also useful to consult section 4.1 of [T-B]. Let χ_1, χ_2 and σ be unramified quasi-characters of $F^\times$ with $\chi_1\chi_2\sigma^2 = 1$, so that the representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}(4,F)$ induced from the quasi-character $\chi_1 \otimes \chi_2 \otimes \sigma$ has trivial central character. Of course, $\chi_1 \times \chi_2 \rtimes \sigma$ may be reducible. It turns out that by section 3 of [ST], there are six types of $\chi_1 \times \chi_2 \rtimes \sigma$ such that every irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character which contains a nonzero vector fixed by I is an irreducible subquotient of a representative of one of these six types, and that no two representatives of two different types share a common irreducible subquotient. The first column gives the name of the type. In the table we choose a representative for a type with the notation as below, and in subsequent columns we give information about the irreducible subquotients of that representative. The types are described as follows. *Type I:* These are the $\chi_1 \times \chi_2 \rtimes \sigma$ where χ_1, χ_2 and σ are unramified quasi-characters of $F^\times$

Table 1. Paramodular dimensions in Iwahori-spherical representations.

		representation	N	$\varepsilon(1/2,\pi)$	$K(0)$	$K(1)$	$K(2)$	$K(3)$	$K(n)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	0	1	1 $+$	2 $+-$	4 $+++-$	$:=$ 6 $+++$ $---$	$[\frac{(n+2)^2}{4}]$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	1	$-(\sigma\chi)(\varpi)$	0	1 $-$	2 $+-$	$:=$ 4 $++--$	$[\frac{(n+1)^2}{4}]$
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	0	1	1 $+$	1 $+$	2 $++$	2 $++$	$[\frac{n+2}{2}]$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	1 $+$	2 $+-$	$[\frac{n^2}{4}]$
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	0	1	1 $+$	2 $+-$	3 $++-$	4 $++--$	$n+1$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	1 $-$	$[\frac{(n-1)^2}{4}]$
	b	$L((\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}))$	2	1	0	0	1 $+$	1 $+$	$[\frac{n}{2}]$
	c	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma))$	1	$-\sigma(\varpi)$	0	1 $-$	2 $+-$	3 $+--$	n
	d	$\sigma 1_{\mathrm{GSp}(4)}$	0	1	1 $+$	1 $+$	1 $+$	1 $+$	1
V	a	$\delta([\xi_0,\nu\xi_0],\nu^{-1/2}\sigma)$	2	-1	0	0	1 $-$	2 $+-$	$[\frac{n^2}{4}]$
	b	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma))$	1	$\sigma(\varpi)$	0	1 $+$	1 $+$	2 $++$	$[\frac{n+1}{2}]$
	c	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)},\xi_0\nu^{-1/2}\sigma))$	1	$-\sigma(\varpi)$	0	1 $-$	1 $+$	2 $--$	$[\frac{n+1}{2}]$
	d	$L((\nu\xi_0,\xi_0\rtimes\nu^{-1/2}\sigma))$	0	1	1 $+$	0	1 $+$	0	$\frac{1+(-1)^n}{2}$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	2	1	0	0	1 $+$	2 $+-$	$[\frac{n^2}{4}]$
	b	$\tau(T,\nu^{-1/2}\sigma)$	2	1	0	0	0	0	0
	c	$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma))$	1	$-\sigma(\varpi)$	0	1 $-$	1 $-$	2 $--$	$[\frac{n+1}{2}]$
	d	$L((\nu,1_{F^*}\rtimes\nu^{-1/2}\sigma))$	0	1	1 $+$	1 $+$	2 $++$	2 $++$	$[\frac{n+2}{2}]$

such that $\chi_1\chi_2\sigma^2=1$ and $\chi_1\times\chi_2\rtimes\sigma$ is irreducible. See Lemma 3.2 of [ST]. *Type II:* These are the $\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi^2\sigma^2=1$. See Lemmas 3.3 and 3.7 of [ST]. *Type III:* These are the $\chi\times\nu\rtimes\nu^{-1/2}\sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi\sigma^2=1$. See Lemmas 3.4 and 3.9 of [ST]. *Type IV:* These are the $\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2=1$. See Lemma 3.5 of [ST]. *Type V:* These are the $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ where ξ_0 and σ are unramified quasi-characters of $F^\times$ such that ξ_0 has order two and $\sigma^2 = 1$. See Lemma 3.6 of [ST]. *Type VI:* These are the $\nu \times 1 \rtimes \nu^{-1/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.8 of [ST].

The second column. Choose a type as in the first column, and choose a representative $\chi_1 \times \chi_2 \rtimes \sigma$ of that type. Then $\chi_1 \times \chi_2 \rtimes \sigma$ admits a finite number of irreducible subquotients, and this number depends only on the type of $\chi_1 \times \chi_2 \rtimes \sigma$. We index the irreducible subquotients by lower case Roman letters. The letter "a" is reserved for the generic irreducible subquotient.

The representation column. This column lists the irreducible subquotients of the representative of the type of the first column. We use the specific notation as in the discussion of the first column.

The N and $\epsilon(1/2,\pi)$ columns. Suppose π is an entry of the third column, and let φ be the L-parameter associated to π by [KL]. We define N and $\epsilon(1/2,\pi)$ by the equation $\epsilon(s,\varphi,\psi,dx_\psi) = \epsilon(1/2,\pi)q^{-N(s-1/2)}$.

The $\mathrm{K}(0)$, $\mathrm{K}(1)$, $\mathrm{K}(2)$, $\mathrm{K}(3)$ **and** $\mathrm{K}(n)$ **columns.** The numbers in the columns give the dimensions of the spaces of $\mathrm{K}(\mathfrak{p}^n)$ fixed vectors for $n = 0, 1, 2, 3$ and arbitrary $n \geqslant 0$. Note that to save space we have abbreviated $\mathrm{K}(\mathfrak{p}^n)$ by $\mathrm{K}(n)$. The signs under the numbers in the $\mathrm{K}(0)$, $\mathrm{K}(1)$, $\mathrm{K}(2)$ and $\mathrm{K}(3)$ columns indicate how these spaces of $\mathrm{K}(\mathfrak{p}^n)$ fixed vectors split under the action of the Atkin-Lehner operator $\pi(u_n)$. The signs are correct if in the type II case, where the central character of π is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial, and in the type IV, V, and IV cases, where the central character of π is σ^2, the character σ is trivial. If these assumptions are not met, then the plus and minus signs must be interchanged to obtain the correct signs.

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SL(2, $\mathbb{Z}$)–INVARIANT SPACES SPANNED BY MODULAR UNITS

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Dedicated to the memory of Tsuneo Arakawa

Characters of rational vertex operator algebras (RVOAs) arising, *e.g.*, in 2 - dimensional conformal field theories often belong (after suitable normalization) to the (multiplicative) semi-group E^+ of modular units whose Fourier expansions are in $1 + q\,\mathbb{Z}_{\geq 0}[\![q]\!]$, up to a fractional power of q. If, furthermore, all characters of a RVOA share this property then we have an example of what we call modular sets, *i.e.* finite subsets of E^+ whose elements (additively) span a vector space which is invariant under the usual action of SL(2, $\mathbb{Z}$). The appearance of modular sets is always linked to the appearance of other interesting phenomena. The first nontrivial example is provided by the functions appearing in the two classical Rogers-Ramanujan identities, and generalizations of these identities known from combinatorial theory yield further examples. The classification of modular sets and RVOAs seems to be related. This article is a first step towards the understanding of modular sets. We give an explicit description of the group of modular units generated by E^+, we prove a certain finiteness result for modular sets contained in a natural semi-subgroup E_* of E^+, and we discuss consequences, in particular a method for effectively enumerating all modular sets in E_*.

1. Introduction

Two famous identities were discovered 1894 by Rogers [R] and rediscovered 1913 by Ramanujan and 1917 by Schur, and since then have been cited as

Rogers-Ramanujan identities:

$$\prod_{\substack{n\equiv\pm1 \bmod 5\\ n>0}} (1-q^n)^{-1} = \sum_{n\geq0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

$$\prod_{\substack{n\equiv\pm2 \bmod 5\\ n>0}} (1-q^n)^{-1} = \sum_{n\geq0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Apart from their combinatorial meaning concerning partitions, the Rogers-Ramanujan identities encode the following surprising fact. If we set $q = e^{2\pi iz}$ for z in the complex upper half plane, and if we multiply the two identities by $e^{-\pi iz/30}$ and $e^{11\pi iz/30}$, respectively, then the functions involved in these identities become modular functions. As well-known and well-understood this statement appears for the products, which are, via the Jacobi triple product identities, quotients of elementary theta series, as remarkable this fact appears for the theta-like infinite series occurring in these identities. There is no known conceptual method in the theory of modular forms which produces modular functions of this shape*.

The Rogers-Ramanujan identities are the first ones of an infinite series of identities of this kind, namely, of the Andrews-Gordon identities (see Section 2). The infinite series occurring in the Andrews-Gordon identities are more generally of the form

$$f_{A,b,c} = \sum_{n=(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq0}} \frac{q^{nAn^t+bn^t+c}}{(q)_{n_1}\cdots(q)_{n_r}}, \quad (q)_k = (1-q)(1-q^2)\cdots(1-q^k),$$

where A is a symmetric positive definite rational $r\times r$-matrix, where b is a rational row vector of length r and c is a rational number. Again, the $f_{A,b,c}$ occurring in the Andrews-Gordon identities are modular functions (since the products occurring in these identities are).

One may consider the following problem†: *For what A, b and c is $f_{A,b,c}$ a modular function ?*

As it turns out this seems to be a hard question: the answer is not known and the known instances of modular $f_{A,b,c}$ are very exceptional. For $r=1$ the problem was completely solved by Zagier [Z]: there are precisely 7 triples of rational numbers $A>0$, b and c such that $f_{A,b,c}$ is modular (see Table 1).

*At least no such method is known to the authors.

†The first one who mentioned this problem to the authors was Werner Nahm

Table 1. All rational numbers $A > 0, b, c$ such that $f_{A,b,c}$ is modular.

A	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
b	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
c	$-\frac{1}{60}$	$\frac{11}{60}$	$-\frac{1}{48}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{40}$	$\frac{1}{40}$

There are indications that a complete answer to this question would involve $K_3(\overline{Q})$ [N; Section 4], and might be related to the problem of classifying vertex operator algebras or two-dimensional quantum field theories [N; Section 4], [E-S].

However, the mentioned identities exhibit another remarkable fact. Namely, the space of linear combinations of the two products in the Rogers-Ramanujan identities is invariant under the natural action of SL(2, $\mathbb{Z}$) on functions defined on the upper half plane. Moreover, the two products are modular units, they have non-negative integral Fourier coefficients and they are eigenfunctions under $z \mapsto z + 1$. These properties hold also true for the products in the Andrews-Gordon identities (see section 2). More generally, such sets of products arise naturally as conformal characters of various (rational) vertex operator algebras [E-S]. The question of finding all such sets of modular units, the question about the modularity of the $f_{A,b,c}$, and the problem of classifying vertex operator algebras seem to be interwoven.

Hence, instead of trying to investigate directly the functions $f_{A,b,c}$ for modularity, one may hope to come closer to an answer to this problem by seeking first of all for a description of all finite sets of modular units of the indicated shape which span SL(2, $\mathbb{Z}$)-invariant spaces. We shall call such sets modular (see Section 2 for a precise definition). As it turns out, modular sets are indeed very exceptional and their description is a non-trivial task.

This article is first step towards the understanding of modular sets. As a byproduct we shall show that an important subclass of modular sets can be algorithmically enumerated.

2. Statement of results

A modular unit is a modular function on some congruence subgroup of Γ := SL(2, $\mathbb{Z}$) which has no poles or zeros in the upper half plane $\mathbb{H}$. Thus it takes on all its poles and zeros in the cusps. The set U of all modular units is obviously a group with respect to the usual multiplication of modular functions.

In this note we are interested in modular units f whose Fourier coefficients are non-negative integers, which satisfy $f(z + 1) = c\,f(z)$ with a suitable constant c, and whose first Fourier coefficients are 1. Denote by E^+ the semi-group of all such units. In other words, E^+ is the semi-group of all modular units whose Fourier expansion is in $q^s(1 + q\mathbb{Z}_{\geq 0}[\![q]\!])$ for some rational number s. Here q^s, for any real s, denotes the function $q^s(z) = \exp(2\pi i s z)$ with z a variable in $\mathbb{H}$.

Special instances of E^+ are the units

$$[r]_l = q^{-l\mathbb{B}_2(\frac{r}{l})/2} \prod_{\substack{n \equiv r \bmod l \\ n>0}} (1-q^n)^{-1} \prod_{\substack{n \equiv -r \bmod l \\ n>0}} (1-q^n)^{-1}, \qquad (1)$$

where $l \geq 1$ and r are integers such that l does not divide r (*cf.* Lemma 6.1 in Section 6). Here we use $\mathbb{B}_2(x) = y^2 - y + \frac{1}{6}$ with $y = x - \lfloor x \rfloor$ as the fractional part of x.

In particular, we are interested in **modular sets**, by what we mean finite and non-empty subsets S of E^+ such that the subspace (of the complex vector space of all functions on $\mathbb{H}$) which is spanned by the units in S is invariant under Γ. Note that the group U is invariant under Γ: if $f(z)$ is a unit and $A \in \Gamma$ then $f(Az)$ is again a unit. Thus it is easy to write down finite subsets of U whose span is Γ-invariant. In contrast to this, E^+ is not invariant under Γ, and, indeed, as we shall explain in a moment, modular sets seem to be quite exceptional. We call a modular set nontrivial if it contains more units than merely the constant function 1.

An infinite series of examples for nontrivial modular sets is provided by the following. Let l be an odd natural number and set

$$\phi_r = \prod_{\substack{1 \leq j \leq \frac{l-1}{2} \\ j \neq r}} [j]_l \qquad (1 \leq r \leq \tfrac{l-1}{2}).$$

Then, for each l, the set AG_l of all ϕ_r with r in the given range is modular [E-S]. (See also [C-I-Z; Eq. (23)], where, however, the ϕ_r are not given as products, but as quotients of theta functions and the Dedekind η-function. Both expressions for the ϕ_r are easily identified on using the Jacobi triple product identity; *cf.* [E-S] for details.)

The existence of (nontrivial) modular sets is a somewhat remarkable fact. First of all, the notion of modular sets itself is bizarre: the action of Γ on modular units defines automorphisms of the group of modular units, whereas a modular set requires the linear subspace, and not the subgroup, generated by its elements, to be Γ-invariant. More striking, modular sets seem to be bound to other remarkable phenomena. The functions ϕ_r occur as the one side of the Andrews-Gordon identities (see, *e.g.*, [5; Eq. (3.2),

p. 15]):

$$\phi_r = q^{-l\mathbb{B}_2(\frac{r}{l})/2} \sum_n \frac{q^{nAn^t + b_r n^t}}{(q)_{n_1} \cdots (q)_{n_{k-1}}}.$$

Here $k = (l-1)/2$ and $n = (n_1, \ldots, n_{k-1})$ runs over all vectors with non-negative integral entries, A is the matrix $A = (\min(i,j))$, and b_r is the vector with $\min(i+1-r, 0)$ as i-th entry, and finally

$$(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$$

(with the convention $(q)_0 = 1$). The two identities for $l = 5$ are the classical Rogers-Ramanujan identities.

Finally, modular sets show up as sets of conformal characters of certain rational vertex operator algebras [E-S]. In fact, the modular sets AG_l provide also examples of this [K-R-V; Eq. (2.1)-(2.3)], and we do not know any modular set which is not the set of conformal characters of a rational vertex operator algebra [E-S].

The ultimate goal would be a classification of all modular sets. As indicated in [E-S] this is related to the open problem of the classification of a certain class of rational vertex operator algebras arising in 2-dimensional conformal field theories.

The first natural step in the study of modular sets is to ask for a more explicit description of the semi-group E^+. We shall prove the following structure theorem.

Theorem 1. *Let E be the group of units generated by the $[r]_l$ (defined in (1)). Then $\mathbb{Q}^* \cdot E$ coincides with the group of all modular units whose Fourier expansions are in $q^s\,\mathbb{Q}[\![q]\!]$ for suitable rational numbers s.*

In particular, the group E is identical with the group of modular units whose Fourier expansion is in $q^s(1 + q\mathbb{Z}[\![q]\!])$ for some rational number s. Thus, the group of modular units generated by E^+ is obviously contained in E. Since it contains on the other side the generators $[r]_l$ of E, we conclude

Corollary to Theorem 1. *The group of modular units generated by the elements of E^+ coincides with the one generated by the $[r]_l$. In particular, each element of E^+ is a product of integral, though not necessarily positive, powers of the special units $[r]_l$.*

There is another remarkable consequence of Theorem 1. Namely, the first Fourier coefficient of a conformal character needs not to be 1. Thus, with regard to applications to conformal characters, it would be more natural to study the semi-group of modular units with Fourier expansions in $q^s\,\mathbb{Z}_{\geq 0}[\![q]\!]$ for some s. However, by the theorem this semi-group equals $\mathbb{Z}_{>0}\cdot E^+$, which shows that one does not loose any generality by restricting to E^+, as we *a priori* did in this article.

It is worthwhile to describe the structure of the group E, *i.e.* the (multiplicative) relations satisfied by the generators $[r]_l$ of E. For each l we have the obvious homomorphism $\mathbb{Z}[\mathbb{Z}/l\mathbb{Z}] \to E$, which associates to a $\mathbb{Z}$-valued map f on $\mathbb{Z}/l\mathbb{Z}$ the product of all $[r]_l^{f(r)}$, where r runs through a complete set of representatives for the nonzero residue classes modulo l. Moreover, one easily verifies the the distribution relations

$$[r]_l = \prod_{\substack{s \bmod m\\ s\equiv r \bmod l}} [s]_m,$$

valid for all l and m such that $l|m$. We may thus combine the above homomorphisms by setting, for any locally constant $f\colon \widehat{\mathbb{Z}} \to \mathbb{Z}$ with $f(0) = 0$,

$$[\,]^f := \prod_{r \bmod l} [r]_l^{f(r)}.$$

Here $\widehat{\mathbb{Z}}$ denotes the Pruefer ring, (*i.e.* $\widehat{\mathbb{Z}} = \operatorname{proj\,lim} \mathbb{Z}/l\mathbb{Z}$, equipped with the topology generated by the cosets $\widehat{\mathbb{Z}}/l\widehat{\mathbb{Z}}$), and l is any positive integer such that f is constant on the cosets modulo $l\widehat{\mathbb{Z}}$. By the distribution relations $[\,]^f$ does not depend on a particular choice of l. One has:

Supplement to Theorem 1. *The map $f \mapsto [\,]^f$ induces an isomorphism of $\mathrm{L}(\widehat{\mathbb{Z}})/\mathrm{L}(\widehat{\mathbb{Z}})^-$ and the group E of modular units generated by the $[r]_l$ (defined in* (1)*). Here $\mathrm{L}(\widehat{\mathbb{Z}})$ is the group of $\mathbb{Z}$-valued, locally constant maps on $\widehat{\mathbb{Z}}$ vanishing at 0, and $\mathrm{L}(\widehat{\mathbb{Z}})^-$ is the subgroup of odd maps.*

The division by $\mathrm{L}(\widehat{\mathbb{Z}})^-$ is due to the (obvious) relations $[r]_l = [-r]_l$.

Denote by E_* the semi-group of products of non-negative powers of the special functions $[r]_l$. Clearly, E^+ contains the semi-subgroup E_*, and, by the corollary to Theorem 1, E^+ and E_* generate the same group. However, E^+ is strictly larger than E_*; *e.g.*, the function $[1]_4^3/[2]_4 = \eta^{-1}\sum_n q^{n^2}$ (with η denoting the Dedekind eta-function) is in E^+, but not in E_*. Understanding the last example and giving a complete description of E^+ seems to be difficult.

Therefore, we shall consider in the following only modular subsets which are contained in the the semi-subgroup E_* of products of non-negative powers of the $[r]_l$. This restriction seems to be not too serious: in fact, the only examples of modular sets not contained in $\mathbb{Z}_{>0} \cdot E_*$ which we know are in a certain sense trivial (*cf.* [E-S]).

As the second main result of the present article, we shall prove a certain finiteness property for modular subsets of E_*, which will in particular imply a method to systematically enumerate them. Namely, for fixed positive integers n and l, let $E_n(l)$ be the set of all products of the form

$$[r_1, \ldots, r_k]_l := \prod_{j=1}^{k} [r_j]_l,$$

with $k \leq n$, and arbitrary integers r_j which are not divisible by l. The sets $E_n(l)$ are clearly finite. Using the distribution relations it is clear that *any modular subset of E_* is contained in some $E_n(l)$ with suitable n and l*. We shall prove:

Theorem 2. *For each n the number of l such that $E_n(l)$ contains a nontrivial modular set is finite. More precisely, if $E_n(l)$ contains a nontrivial modular set, then $l \leq 13.7^n$.*‡

Our proof will exhibit a method to compute, for a given n, all modular subsets of $E_n(l)$ for all l. This method, however, becomes quickly non-realistic for growing n.

In Table 2 we listed all modular subsets of $E_n(l)$ for $n \leq 3$ and $l \geq 1$. For each n, we listed only those modular sets which do not already belong to some $E_k(l)$ with $k < n$, and which cannot be decomposed into a disjoint union of smaller modular sets. By S^n, for a modular set S and a positive integer n, we denote the set of all n-fold products of functions in S. Obviously, S^n is again modular. Note that, for $n \leq 3$, there is exactly one 'new' modular set, which we called W_7. More examples of modular sets can be found in [E-S].

Table 2. All modular subsets of $E_n(l)$ for $n \leq 3$ and arbitrary l.

	$l = 5$	7	9
$n = 1$	AG_5		
2	AG_5^2	AG_7	
3	AG_5^3	$\mathrm{W}_7 := \{[1,2,3]_7\} \cup \{[r,r,3r]_7 : r = 1,2,3\}$	AG_9

‡Actually the existence of a nontrivial modular set in $E_n(l)$ implies $l \leq 5n$. However, this sharper result relies on a deep analysis of the (projective) SL(2, $\mathbb{Z}$)-module of all modular forms of weight $\frac{1}{2}$, and will be published elsewhere.

The plan of the rest of this article is as follows: In Section 3 we shall prove Theorem 1 and its supplement, and in Section 4 we shall prove Theorem 2. The auxiliary results derived in Section4 have some interest, independent of the proof of Theorem 2, in connection with the question of searching for modular sets. In Section 5 we shall briefly indicate how to use these auxiliary results for calculating, *e.g.*, the above table.

In the proofs of the two theorems we need certain properties of the $[r]_l$'s, which we derive in Section 6 by rewriting $[r]_l$ in terms of l-division values of the Weierstrass σ-function and using some of their basic properties. Since we did not find any convenient reference to cite these properties directly we decided to develop quickly from scratch the corresponding theory in form of a short Appendix and part of Section 6. In particular, we emphasize in the Appendix that the Weierstrass σ-function and its l-division values are best understood by viewing the Weierstrass σ-function as a Jacobi form on the full modular group of weight and index equal to $\frac{1}{2}$ (see Theorem 7.1).

3. The group of units generated by the $[r]_l$

In this section we prove Theorem 1 and its supplement. We shall actually prove the slightly stronger Theorem 3.2. Its proof depends on two well-known facts: first, that the group of all modular units modulo the so-called Siegel units is a torsion group, and, secondly, that modular forms on congruence subgroups with rational Fourier coefficients have bounded denominators. The short proof of the first one is given in Section 6, for the second, deeper one, we refer to the literature.

We precede the proof of Theorem 3.2 by three lemmas. The first one, which we actually call theorem to emphasize its more general usefulness, is a general statement about product expansions of holomorphic and periodic functions in the upper half plane. It is important for the proof of the third lemma, but it also implies directly the supplement to Theorem 1 of Section 2.

Theorem 3.1. *Let f be a holomorphic and periodic function on the upper half plane whose Fourier expansion is in $1 + q\mathbb{Z}[\![q]\!]$. Then there exists a unique sequence $\{a(n)\}$ of integers such that*

$$f = \prod_{n\geq 1} (1 - q^n)^{a(n)},$$

for sufficiently small $|q|$.

Remark. As can be read off from the proof the lemma actually holds true with $\mathbb{Z}$ replaced by an arbitrary subring of $\mathbb{C}$.

Proof. The existence of the sequence $a(n)$ follows by induction on n. Namely, assume that one has already found integers $a(n)$ $(1 \leq n < N)$ such that

$$g := f / \prod_{n=1}^{N-1} (1 - q^n)^{a(n)} = 1 + \mathcal{O}(q^N).$$

Let $-a(N)$ be the Fourier coefficient of g in front of q^N. Clearly $a(N)$ is integral. One has

$$f / \prod_{n=1}^{N} (1 - q^n)^{a(n)} = g/(1 - q^N)^{a(N)} = 1 + \mathcal{O}(q^{N+1}).$$

The uniqueness of the $a(n)$ follows from the uniqueness of the Fourier expansion of $q\frac{d}{dq} \log f$. □

Proof of Supplement to Theorem 1. That the kernel of the map $\mathrm{L}(\widehat{\mathbb{Z}}) \mapsto E$ equals $\mathrm{L}(\widehat{\mathbb{Z}})^-$ follows from the uniqueness of the product expansion in the preceding proposition and on writing

$$[\;]^f = \prod_{r \bmod l} [r]_l^{f(r)} = q^c \prod_{n \geq 1} (1 - q^n)^{-f(n) - f(-n)}$$

with a suitable constant c. The surjectivity is obvious from the definition of E. □

Lemma 3.1. *Let $f \in \frac{1}{D}\mathbb{Z}[\![q]\!]$ for some positive integer D. If some positive integral power of f has integral coefficients, then f has integral coefficients.*

Proof. By assumption about the coefficients of f we can write $f = \gamma \cdot h$ with a suitable rational number γ and with a primitive h. Here primitive means that h is a power series in q with integral coefficients $a(l)$ which are relatively prime. By assumption, $\gamma^N \cdot h^N$, for some integer $N \geq 1$, has integral coefficients. We shall show in a moment that h^N is primitive. From this we deduce that γ^N is integral. Hence γ is integral, which proves the lemma.

It remains to show that h^N is primitive. Let p be a prime. Since h is primitive, there exists an l such that $p | a(j)$ for $j < l$ and $p \nmid a(l)$. But then the q^{Nl}-coefficient of h^N satisfies

$$\sum_{i_1 + \cdots + i_N = Nl} a(i_1) \cdots a(i_N) \equiv a(l)^N \bmod p,$$

and whence is not divisible by p. $\square$

For the following, let $E(l)$, for fixed l, denote the group generated by the $[r]_l$ with $1 \leq r \leq \lfloor l/2 \rfloor$.

Lemma 3.2. *Let f be a modular unit with rational Fourier coefficients. Assume that a positive integral power of f lies in $E(l)$. Then f is in $E(2l)$ (and even in $E(l)$ for odd l).*

Proof. Since f is invariant under a congruence subgroup it has bounded denominators, *i.e.* there exist an integer $D > 0$ such that $D \cdot f$ has integral Fourier coefficients. This well-known fact follows, *e.g.*, on writing $f\eta^{24N}$, with a suitable integer $N > 0$ (and with η denoting the Dedekind eta-function), as linear combination of modular forms with integral Fourier coefficients (which is possible by Theorem 3.52 in [Sh]), deducing from this that $f\eta^{24N}$ has bounded denominators, which in turn implies that f has bounded denominators since η^{-1} has integral Fourier coefficients.

Combining the latter with the fact that some positive integral power of f lies in $E(l)$, we see that, for some rational number s, the function $q^{-s}f$ satisfies the assumption of Lemma 3.1, and hence is in $\mathbb{Z}[\![q]\!]$. Moreover, by assumption, its first Fourier coefficient is 1.

But then $q^{-s}f$ possesses a product expansion as in the Theorem 3.1. By the uniqueness of the $a(n)$, and since a nonzero integral power f^N of f is a product of $[r]_l$'s, we conclude that $Na(n) = Na(m)$ for $n \equiv \pm m \bmod l$, and that $Na(0) = 0$. Since $N \neq 0$ the same holds true with $\{Na(n)\}$ replaced by $\{a(n)\}$. Thus we find, on re-ordering the product expansion of f according to the residue classes of n modulo l, that

$$f^{-1} = [1]_l^{a(1)}[2]_l^{a(2)} \cdots [m-1]_l^{a(m-1)}[m]_l^{\nu a(m)}$$

where $m = \lfloor l/2 \rfloor$, and where $\nu = 1$ for odd l, and $\nu = 1/2$ for even l. Hence, if l is odd, then $f \in E(l)$. If l is even, one uses the distribution relations (in particular, $[l/2]_l^{1/2} = [l/2]_{2l}$) to deduce $f \in E(2l)$. $\square$

Theorem 3.2. *The group of modular units on $\Gamma(l)$ $(= \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv 1 \bmod l\})$ with Fourier expansions in $q^s\,\mathbb{Q}[\![q]\!]$ for suitable rational numbers s is a subgroup of $\mathbb{Q}^* \cdot E(2l)$ (and even of $\mathbb{Q}^* \cdot E(l)$, for odd l).*

Proof. Let f be unit on $\Gamma(l)$ such that, for some rational number s, the function $q^{-s}f$ has rational Fourier coefficients. For showing that f is contained in $\mathbb{Q}^* \cdot E(l)$ or $\mathbb{Q}^* \cdot E(2l)$, respectively, we may assume that f is normalized, *i.e.* that its first Fourier coefficient is 1. By Lemma 3.2 it then suffices to show that a positive integral power of f lies in $E(l)$.

By Theorem 6.2 of Section 6 we know that some nontrivial power of f can be written as product of Siegel units s_α, which are defined by Eq. (2) of Section 6. More precisely, there exists integers $a > 0$, $b(\alpha)$, and a constant c such that

$$f^a = c \cdot \prod_{\alpha \in I} s_\alpha^{b(\alpha)},$$

where I is a finite set of pairs of rational numbers of the form $(\frac{r}{l}, \frac{s}{l})$ with integers r, s such that $\gcd(r, s, l) = 1$.

By replacing a and the $b(\alpha)$ by suitable positive integral multiples we may assume that f^a is invariant under $T = (1,1;1,0)$. On the other hand, by Theorem 6.1 in Section 6 we have that $s_\alpha \circ T$ equals $s_{\alpha T}$, up to multiplication by a constant. Let K denote the field of l-th roots of unity. For an integer y relatively prime to l denote by σ_y the automorphism of K which maps an l-th root of unity ζ to ζ^y. We extend σ_y to an automorphism of the ring $R = \bigoplus_{s \in \mathbb{Q}} q^s\, K[\![q]\!]$ by letting it act on coefficients. Since f has rational coefficients, it is invariant under σ_y. From the formula for s_α in Section 6 it is immediate that, for $\alpha \in I$, one has $s_\alpha \in R$ and that $\sigma_y s_\alpha$ equals $s_{\alpha D(y)}$, up to multiplication by a constant and with $D(y) = (1,0;0,y)$.

Using these properties we can write

$$f^{al\varphi(l)} = \prod_{y \bmod {}^*l} \prod_{h=0}^{l-1} \sigma_y(f^a \circ T^h) = d \prod_{\alpha \in I} \Big(\prod_{y \bmod {}^*l} \prod_{h=0}^{l-1} s_{\alpha T^h D(y)} \Big)$$

with a suitable constant d, where the asterisk indicates that y runs through a complete set of primitive residue classes modulo l, and $\varphi(l)$ denotes as usual the number of such classes.

It remains to show that the expressions t_α in the rightmost parenthesis are in $E(l)$, up to multiplication by constants (whose product then equals d^{-1}, since f is normalized). Write $\alpha = (r, s)/l$ as above. Clearly $\alpha T^h D(y) = (r, t)/l$ with a suitable integer t. If h and y run through the given range, then t runs through a complete set of representatives for the residue classes modulo l which are relatively prime to $\gcd(r, l)$, and each such t is taken on the same number of times, say p (look at the action of the subgroup of GL$(2, \mathbb{Z}/l\mathbb{Z})$ of matrices of the form $(1, x; 0, y)$ on pairs of residue classes (u, v) in $(\mathbb{Z}/l\mathbb{Z})^2$ with $\gcd(u, v, l) = 1$).

Thus t_α is the p-th power of

$$\prod_{\substack{t \bmod l\\ \gcd(t,r,l)=1}} s_{(r,t)/l} = \prod_{t \bmod l}\prod_{d|r,l,t} s_{(r,t)/l}^{\mu(d)} = \prod_{d|r,l}\prod_{u \bmod l/d} s_{(\frac{r}{d},u)/\frac{l}{d}}^{\mu(d)} = \prod_{d|r,l}\left[\frac{r}{d}\right]_{\frac{l}{d}}^{\mu(d)}.$$

Here we used the Moebius function $\mu(d)$, and, for the last identity, Lemma 6.1 of Section 6; moreover, we have to assume that l does not divide r (since s_α, for $\alpha \in \mathbb{Z}^2$, is not defined). On using the distribution relations in E we can rewrite the right hand side as power products of $[r]_l$'s. If l divides r, then we leave it to the reader to verify by a similar calculation (using directly the definition (2) of s_α) that the left hand side of the last identity equals $\prod_r [r]_l$, where r runs through a complete system of representatives for the primitive residue classes modulo l. □

Proof of Theorem 1. This is clearly a consequence of Theorem 3.2. □

4. Properties of modular sets

In this section we shall prove Theorem 2. Actually, we shall prove the slightly stronger Theorem 4.2 below. Its proof will mainly depend on two results: the first one concerns a sort of measure on the projective space over $\mathbb{Z}/l\mathbb{Z}$ (Theorem 4.1; see also the beginning of Section 5). The second result (Lemma 4.2) uses information about the action of Γ on the $[r]_l$, and will not be completely proved before Section 6.

The first lemma gives a necessary criterion for a set $S \subset E^+$ to be modular in terms of the vanishing or pole orders of the functions in S. Let $f \not\equiv 0$ be a modular function on some subgroup of Γ, and let $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ be any cusp. Then there exists a $A \in \Gamma$ such that $s = A\infty$, and a real number α such that $f(Az)q^{-\alpha}(z)$ tends to a non-zero constant for $z = it$ with real $t \to \infty$. The number α does not depend on the choice of A. We set

$$\mathrm{ord}_s(f) = \alpha.$$

Lemma 4.1. *Let S be a finite set of modular functions such that the space spanned by its elements is invariant under* $\mathrm{SL}(2,\mathbb{Z})$. *Then the map*

$$\nu : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}, \quad \nu(s) = \min_{f\in S} \mathrm{ord}_s(f)$$

is constant.

Proof. Indeed, for any fixed $A, B \in \mathrm{SL}(2,\mathbb{Z})$ and any $f \in S$ the function $f \circ A$ is a linear combination of the functions $g \circ B$ with $g \in S$. In particular, comparing the leading terms of the Fourier expansions of these functions, we conclude

$$\mathrm{ord}_\infty(f \circ A) \geq \min_{g \in S} \mathrm{ord}_\infty(g \circ B).$$

Since this is true for any f, and on using $\mathrm{ord}_\infty(f \circ A) = \mathrm{ord}_{A\infty}(f)$ we obtain $\nu(A\infty) \geq \nu(B\infty)$. Interchanging the role of A and B we see that here we actually have an equality. This proves the lemma. □

Lemma 4.2. *Let $s \in \mathbb{P}^1(\mathbb{Q})$. Then*

$$\mathrm{ord}_s([r]_l) = -\frac{t^2}{2l}\mathbb{B}_2\left(\frac{ar}{t}\right),$$

where $s = \frac{a}{c}$ with relatively prime integers a and c (in particular, $a = \pm 1$ and $c = 0$, if $s = \infty$), and where $t = \gcd(c, l)$.

Proof. Let $A \in \Gamma$ be a matrix with first row equal to $(a, c)^t$, *i.e.* such that $A\infty = s$. Then $\mathrm{ord}_s([r]_l) = \mathrm{ord}_\infty([r]_l \circ A)$, and the right hand side is given in Lemma 6.1 of Section 6. □

Combining Lemma 4.1 and Lemma 4.2 we obtain the following necessary criterion for a set $S \subset E_n(l)$ to be modular. This criterion is the key for the proof of Theorem 2. We remark that Lemma 4.3 is actually the only instance in the proof of Theorem 2, where we use that S is contained in E_*, rather than only in E^+

Lemma 4.3. *Let $S \subset E_n(l)$ be modular, and assume that S contains at least one n-fold product (i.e. an element in $E_n(l) \setminus E_{n-1}(l)$). Then, for all divisors t of l, one has*

$$\max_{[a_1,\ldots,a_k]_l \in S} \ \max_{a \bmod {}^* t} \sum_{j=1}^{k} \mathbb{B}_2\left(\frac{a_j a}{t}\right) = \frac{n}{6t^2}.$$

Here the asterisk indicates that a runs through a complete set of representatives for the primitive residue classes modulo t.

Remark. Note that the lemma implies that $\gcd(a_1, \ldots, a_n, l) = 1$ for all n-fold products $\pi = [a_1, \ldots, a_n]_l \in S$. Indeed, if d denotes this gcd, then the lemma applied to $t = d$ becomes $\frac{n}{6} \leq \frac{n}{6d^2}$, whence $d = 1$.

Proof. If $f \in S$ is an k-fold product, then $\mathrm{ord}_0(f) = -\frac{k}{12l}$ by the preceding lemma. Since S contains an n-fold product, we conclude that

$$\min_{f \in S} \mathrm{ord}_0(f) = -\frac{n}{12l}.$$

The claimed inequality is now an immediate consequence of the first two lemmas. □

We call a point $P = (\overline{a}_1, \ldots, \overline{a}_n) \in (\mathbb{Z}/l\mathbb{Z})^n$ *special* if

$$\sum_{j=1}^{n} \mathbb{B}_2\left(\frac{a_j a}{t}\right) \leq \frac{n}{6t^2}$$

for all divisors t of l and all integers a relatively prime to t. Here the bar denotes reduction modulo l.

Theorem 2 will now be a consequence of Lemma 4.3 and the following theorem, whose proof will take the rest of this section.

Theorem 4.1. *For a given n there exist only a finite number of l such that $(\mathbb{Z}/l\mathbb{Z})^n$ contains a special point. More precisely, if $(\mathbb{Z}/l\mathbb{Z})^n$ with $l > 1$ contains a special point, then*

$$l \leq B := \left(\frac{2\,(1 + l^{\frac{1}{n}-1})}{1 - \sqrt{\frac{1}{3} + \frac{2}{3p^2}}} \right)^n,$$

where p is the smallest prime divisor of l.

Theorem 4.2. *If $E_n(l)$ contains a nontrivial modular set, then $l \leq B$ with B as in Theorem 4.1.*

Proof. Let S be a modular subset of $E_n(l)$, and let k be minimal such that S is contained in $E_k(l)$. Let $\pi = [r_1, \ldots, r_k] \in S$. By Lemma 4.3 π yields a special point $(\overline{r}_1, \ldots, \overline{r}_k) \in (\mathbb{Z}/l\mathbb{Z})^k$. Hence, by Theorem 4.1, l is bounded from above by the right hand side of the claimed inequality, but with n replaced by k. Since $k \leq n$, the theorem then follows. □

Proof of Theorem 2. This is an immediate consequence of the preceding theorem. The bound in Theorem 2 is obtained from the bound of Theorem 4.2 by estimating p to below by 2 and on using $1 + l^{1/n-1} \leq 2$. □

It remains to prove Theorem 4.1 on special points. For its proof we use

Lemma 4.4. *Let $P \in (\mathbb{Z}/l\mathbb{Z})^n$. Then there exists an integer b not divisible by l such that $b \cdot P = (\overline{b}_1, \ldots, \overline{b}_n)$ with integers b_j (and where the bar denotes reduction modulo l) satisfying*

$$|b_j| \leq l^{1-\frac{1}{n}} + 1.$$

Remark. Note that the inequality is, for fixed n and asymptotically in growing primes l, best possible, apart from a constant. Indeed, the number of points in $(\mathbb{Z}/l\mathbb{Z})^n$ described by homogeneous coordinates satisfying the above inequality is

$$\leq (2l/l^{1/n}+3)^n \approx 2^n l^{n-1}.$$

But, for growing primes l, this is up to factor 2^n asymptotically equal to the number of orbits of $(\mathbb{Z}/l\mathbb{Z})^n$ modulo multiplication by non-zero elements of $\mathbb{Z}/l\mathbb{Z}$, which is

$$\frac{(l^n-1)}{l-1}+1.$$

Proof. For an integer r, set $B_r = [-r,r]^n \cap \mathbb{Z}^n$, and let C_r denote the reduction of B_r modulo l. Assume $r < \frac{l}{2}$. Then C_r contains exactly $(2r+1)^n$ elements. Note that the sum of two points of C_r always lies in C_{2r}.

Consider the sets $x\cdot P+C_r$, where x runs through $\mathbb{Z}/l\mathbb{Z}$. If the sum of the cardinalities of these sets is strictly greater than l^n, *i.e.* if $l\cdot(2r+1)^n > l^n$, then there exist at least two which have non-empty intersection.

Assume that there exists an integer r satisfying the inequalities of the two preceding paragraphs, *i.e.* satisfying

$$\frac{l}{2} > r > \frac{l^{1-1/n}}{2} - \frac{1}{2} =: \rho.$$

Pick $x \not\equiv x' \bmod l$ such that $x\cdot P + C_r$ and $x'\cdot P + C_r$ contain a common point Q. Then $xP-Q$ and $Q-x'P$ both lie in C_r, and hence their sum $(x-x')P$ is in C_{2r}, whence can be represented by a point in B_{2r}.

If $\frac{l}{2} > \rho+1$ we may take $r=\lfloor \rho+1\rfloor$ to fulfill the above two inequalities. Since then $2r \leq 2\rho+2 = l^{1-1/n}+1$, the lemma follows. Otherwise $\frac{l}{2} \leq \rho+1 \leq 2\rho+2 = l^{1-1/n}+1$, and then the lemma is trivial. □

Proof of Theorem 4.1. Let $P \in (\mathbb{Z}/l\mathbb{Z})^n$. Choose b as in the last lemma. Write $\frac{b}{l} = \frac{a}{t}$ with a divisor t of l and $\gcd(a,t)=1$. Note that $t \neq 1$ (since b is not divisible by l), and hence $t \geq p$ with the smallest prime divisor p of l. Thus,

$$\frac{aP}{t} \equiv \frac{1}{l}(b_1,\dots,b_n) \bmod \mathbb{Z}^n,$$

with integers b_j satisfying

$$|b_l/l| \leq l^{-\frac{1}{n}} + l^{-1} =: s.$$

Since $\mathbb{B}_2$ is decreasing in $[0, \frac{1}{2}]$, we find, for $s \le \frac{1}{2}$, *i.e.* for $l \ge (2(1+l^{\frac{1}{n}-1})^n$, the inequality

$$\sum_{j=1}^{n} \mathbb{B}_2\big(\frac{ab_j}{t}\big) \ge n\mathbb{B}_2(s).$$

Thus, if s satisfies

$$\mathbb{B}_2(s) > \frac{1}{6p^2}\big(\ge \frac{1}{6t^2}\big),$$

then $(\mathbb{Z}/l\mathbb{Z})^n$ can never contain a special point. It is easily checked that the last inequality, together with $l^{-\frac{1}{n}} + l^{-1} = s \le \frac{1}{2}$, is equivalent to

$$s < \frac{1 - \sqrt{\frac{1}{3} + \frac{2}{3p^2}}}{2}.$$

From this the theorem becomes obvious. □

5. Computing modular sets

We explain how we computed Table 2 in Section 2. The remarks of this section can actually be used to find for arbitrary n all modular sets in $E_n(l)$ for all l, though, for growing n, the required computational resources become soon unrealistic§.

The algorithm to enumerate all modular sets is based on the simple observation, that a subset of $E_n(l)$ is modular if and only if the space of functions which it spans is invariant under $z \mapsto -1/z$. This characterization follows easily from the fact that $\mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$ is generated by $z \mapsto -1/z$ and $z \mapsto z+1$. The invariance of a space spanned by a subset of $E_n(l)$ can be checked using explicit transformation formulas for the $[r]_l$ under Γ; *cf.* Theorem 6.1 and Lemma 6.1. (Note that by standard arguments from the theory of modular forms it suffices to check $N = N(l,n)$ many Fourier coefficients only to decide whether $\pi(-1/z)$, for π in $E_n(l)$, is a linear combination of products in $E_n(l)$, where $N(l,n)$ is a constant depending only on l and n, and which can be determined explicitly.)

Now, to find the maximal modular subset of $E_n(l)$ one could proceed as follows: Compute the set S_1 of all π in $E_n(l)$ such that $\pi(-1/z)$ is a linear combination of functions in $E_n(l)$. Next one computes the set S_2 of all functions π in S_1 such that $\pi(-1/z)$ is a linear combination of functions in

§Using a cluster all modular sets in $E_n(l)$ for all l and $n \le 13$ could be determined; the results of this computation will be published elsewhere.

S_1. Continuing like this one obtains a decreasing sequence of sets S_k. Either at some point S_k is empty, and then $E_n(l)$ contains no modular subset, or else $S_k = S_{k+1}$ for some k, and then S_k is the maximal modular subset of $E_n(l)$. However, since the number of products in $E_n(l)$ grows exponentially in l it is necessary to look theoretical means to reduce the computational complexity. We indicate two such means.

Assume $S \subset E_n(l)$ is modular. In this paper we are only interested in $n < 6$. This simplifies the computations a bit since then, for each $0 \leq k \leq n$, the subset $S(k)$ of all products of length k in S is already modular. Indeed, by the very definition of $[r]_l$ the elements of $S(k)$ have a Fourier expansion in powers of $q^{n/12l}$, where $n \in -kl^2 + 6\mathbb{Z}$. Furthermore, the $\pi(-1/z)$, for $\pi \in S(k)$, have a Fourier expansion in powers of $q^{n/12l}$, where $n \in -k + 12\mathbb{Z}$ (*cf.* Lemma 6.1). From this our proposition follows immediately if 3 does not divide l (and since $k < 6$). If 3 divides l, then our argument shows that S can only contain products of length 3 (again using $k < 6$), and our claim follows in this case too.

Hence, for verifying our table we can restrict our search for modular subsets of $E_n(l)$ to modular subsets of $F_n(l) := E_n(l) \setminus E_{n-1}(l)$.

Next, it is not at all necessary to consider all functions in F_n. Namely, let us call a subset T of $\mathbb{P}^{n-1}(\mathbb{Z}/l\mathbb{Z})$ premodular if

$$\max_{P \in T} \beta_t(P) = \frac{n}{6t^2}$$

for all divisors t of l. Here we use

$$\beta_t([\overline{a}_1 : \cdots : \overline{a}_n]) = \max_{a \bmod {}^* t} \sum_{j=1}^{n} \mathbb{B}_2\left(\frac{aa_j}{t}\right)$$

(with the asterisk as in Lemma 4.3 and the bar denoting reduction modulo l). Let $C_n(l)$ be the union of all premodular subsets in $\mathbb{P}^{n-1}(\mathbb{Z}/l\mathbb{Z})$, if there are any, and $C_n(l) = \emptyset$ otherwise.

If $S \subset F_n(l)$ is modular, then, by Lemma 4.3, the set $\overline{S}$ of all points $[\overline{a}_1 : \cdots : \overline{a}_n] \in \mathbb{P}^{n-1}(\mathbb{Z}/l\mathbb{Z})$ such that $\pi = [a_1, \ldots, a_n]_l \in S$ is premodular.

Thus to find the maximal modular subset of $F_n(l)$ one computes first of all $C_n(l)$. If it is non-empty, let S_0 be the set of all products in $F_n(l)$ such that $\overline{S}_0 = C_n(l)$. If it is not clear by other means whether S_0 actually contains a modular subset, then we now proceed as indicated at the beginning of this section:. Let S_1 be the set of all $\pi \in S_0$ such that $\pi(-1/z)$ is a linear combination of functions in S_0. Similarly, construct S_2 from S_1, S_3 from S_2 and so forth. Either some S_k is empty, and then $F_n(l)$ contains no modular set, or $S_k = S_{k+1} \neq \emptyset$ for some k, and then S_k is the maximal modular set in $F_n(l)$.

Assume now $n = 1$ and $l > 1$. Then $\mathbb{P}^{n-1}(\mathbb{Z}/l\mathbb{Z})$ contains only one point $[a]$. If this point yields a premodular set, one has

$$\beta_l([a]) = \mathbb{B}_2\left(\frac{1}{l}\right) = \frac{1}{6l^2}.$$

For the first equality we used that $\mathbb{B}_2(x)$ is even, decreasing between 0 and $\frac{1}{2}$, and that $\gcd(a, l) = 1$. Rewriting this identity as $5 - 6l + l^2 = 0$ we find $l = 5$ as the only solution > 1. And indeed, $F_1(5)$ equals AG_5.

Let $n = 2$ or $n = 3$. We determine, for all l, all non-empty $C_n(l)$. If $C_n(l)$ is non-empty, then $C_n(t)$ is non-empty for all divisors t of l. Theorem 4.1 applied with l equal to a prime p shows that $C_2(p) = \emptyset$ for $p > 37$, and $C_3(p) = \emptyset$ for $p > 113$. A computer search shows that actually $C_2(p) \neq \emptyset$ only for $p = 2, 5, 7$, and $C_3(p) \neq \emptyset$ only for $p = 3, 5, 7$. Next, for each of these primes p, we look for powers p^r such that $C_n(p^r)$ is nonempty. The possible values of r are bounded by Theorem 4.1. Again by a computer search, we find that $C_2(l) \neq \emptyset$ implies $l \mid 2 \cdot 5 \cdot 7$, and that $C_3(l) \neq \emptyset$ implies $l \mid 3^2 \cdot 5 \cdot 7$. A final computer search yields then the Table 2. The above procedure to pass from premodular sets to the maximal modular one, i.e. to descend to S_1, S_2 etc., had actually only been applied twice in the course of our computations: to rule out certain functions for $n = 3$ and $l = 15$, and to prove that W_7 is modular.

6. The $[r]_l$ in terms of l-division values of the Weierstrass σ-function

Problems involving the action of Γ on modular units are most conveniently studied using l-th division values of the Weierstrass σ-function (or Siegel units, as they are called in the literature). This relies on the following two facts: Firstly, the action Γ on a Siegel unit is given by an explicit formula (Theorem 6.1). Secondly, if S denotes the group generated by the Siegel units, then U/S has exponent 2.

The transformation formulas are most naturally and easily derived by using the Jacobi group and considering the Weierstrass σ-function as Jacobi form. Since this approach cannot be found in the literature we present it here in form of an appendix. The resulting formulas, however, are classical and well-known.

For the complicated proof of the second fact, namely that U/S has exponent 2, see the book [K-L] and papers cited therein. For us, fortunately, it suffices to know the considerably simpler fact that U/S is a torsion group (Theorem 6.2). Since we do not know any reference to an easy and direct proof of this, we shall give such a proof here. Finally, we shall describe below the relation between Siegel units and the functions $[r]_l$ and we shall deduce from this and the two theorems on Siegel units the facts (Lemma 6.1) which were used in the preceding paragraphs without proofs.

For a row vector $\alpha = (a_1, a_2) \in \mathbb{Q}^2$, $\alpha \notin \mathbb{Z}^2$, set

$$s_\alpha = q^{-\mathbb{B}_2(a_1)/2} \prod_{\substack{n \equiv a_1\,(\mathbb{Z}) \\ n \geq 0}} (1 - q^n e(a_2))^{-1} \prod_{\substack{n \equiv -a_1\,(\mathbb{Z}) \\ n > 0}} (1 - q^n e(-a_2))^{-1}. \quad (2)$$

Here $e(\dots) = \exp(2\pi i \dots)$. Note that the first product has to be taken over all non-negative n, whereas the second one is over strictly positive n only. Moreover, s_α depends only on α mod $\mathbb{Z}^2$. Finally, s_α has clearly no zeros or poles in the upper half plane. The functions s_α^{-1} are known in the literature as Siegel units.

The following theorem will be proved in the Appendix.

Theorem 6.1. *For all non-integral $\alpha \in \mathbb{Q}^2$ and all $A \in \mathrm{SL}(2,\mathbb{Z})$ there exist a root of unity $c(\alpha, A)$ such that*

$$s_\alpha \circ A = c(\alpha, A)\, s_{\alpha A}.$$

(Here aA means the usual action of A on the row vector a, and $(s_\alpha \circ A)(z) = s_\alpha(Az)$.) Moreover, for a given α, the group of A such that $c(\alpha, A) = 1$ is a congruence subgroup. In particular, s_α is a modular unit.

Remark. The numbers $c(\alpha, A)$ define obviously a cocycle of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, *i.e.* one always has $c(\alpha, AB) = c(\alpha, A) \cdot c(\alpha A, B)$. The actual values of $c(\alpha, A)$ will drop out automatically of the proof given below though we do not need them. In particular, $c(\alpha, A)^{12l} = 1$ for all A, where l is the common denominator of a_1 and a_2.

Theorem 6.2. *Let f be a modular unit on the principal congruence subgroup $\Gamma(l)$. Then a suitable positive integral power of f is, up to multiplication by a constant, contained in the group generated by the Siegel units s_α, where α runs through all pairs of rational numbers of the form $(\frac{r}{l}, \frac{s}{l})$ with integers r, s such that $\gcd(r, s, l) = 1$.*

Proof. Let $U[\Gamma(l)]/\mathbb{C}^*$ be the group of modular units on $\Gamma(l)$ modulo multiplication by constants. Since the map

$$U[\Gamma(l)]/\mathbb{C}^* \to \mathbb{Z}[\Gamma(l)\backslash\mathbb{P}(\mathbb{Q})], \quad \pi \mapsto \text{divisor of } \pi$$

is injective and takes its image in the subgroup of divisors of degree 0, we conclude that the rank of $U[\Gamma(l)]/\mathbb{C}^*$ is $\leq R-1$, where $R = |\Gamma(l)\backslash\mathbb{P}(\mathbb{Q})|$ is the number of cusps of $\Gamma(l)$. It is well-known that R equals the cardinality of I, where I is a complete set of representatives for

$$\{(\frac{r}{l}, \frac{s}{l}) \in \frac{1}{l}\mathbb{Z}^2 \colon \gcd(r,s,l) = 1\}$$

modulo $\mathbb{Z}^2$ and modulo multiplication by ± 1.

On the other hand, for suitable large integers N the powers s_α^N with $\alpha \in I$ are elements in $U[\Gamma(l)]$ (in fact, one may take $N = 12l$). This is an immediate consequence of Theorem 6.1 and the remark subsequent to it.

Moreover, a relation

$$\prod_{\alpha\in I} s_\alpha^{c(\alpha)} = \text{const.}$$

holds true if and only if $c(\alpha)$, as function of α, is constant. Indeed, a constant function $c(\alpha)$ yields a modular unit a power of which is invariant under $\mathrm{SL}(2,\mathbb{Z})$ by Theorem 6.1. Since $\mathrm{SL}(2,\mathbb{Z})$ has only one cusp, this unit must be a constant. That there is no other relation can, *e.g.*, be verified by looking at the logarithmic derivatives of the s_α, which, by well-known theorems, span the space of Eisenstein series on $\Gamma(l)$ [H; pp. 468]. But the dimension of this space is $R-1$. Hence the rank of the subgroup of $U[\Gamma(l)]/\mathbb{C}^*$ generated by the $\mathbb{C}^* \cdot s_\alpha^N$ ($\alpha \in I$) equals $R-1$. We deduce from this that $U[\Gamma(l)]/\mathbb{C}^*$ has full rank $R-1$, and that the $\mathbb{C}^* \cdot s_\alpha^N$ ($\alpha \in I$) generate a subgroup of finite index. □

Lemma 6.1. *Let $l \geq 1$ be an integer. For each integer r not divisible by l one has*

$$[r]_l = \prod_{s \bmod l} s_{(r,s)/l}.$$

In particular, $[r]_l$ is a modular unit. For each $A = (a,b;c,d) \in \Gamma$ one has

$$[r]_l \circ A \in c\, q^{-\frac{t^2}{2l}\mathbb{B}_2(\frac{ar}{t})}(1 + q^{1/l}\, K[\![q^{1/l}]\!]),$$

where $t = \gcd(c,l)$, where K denotes the field of l-th roots of unity, and where c is a constant.

Proof. The formula expressing $[r]_l$ in terms of the s_α is a simple consequence of the polynomial identity

$$\prod_{k \bmod l} (1 - e(k/l)\, Z) = 1 - Z^l.$$

By Theorem 6.1 the function $[r]_l$ is then a modular unit. The last assertion follows from the given formula, Theorem 6.1, and on using

$$-\frac{1}{2}\sum_{s \bmod l} \mathbb{B}_2\Big(\frac{ra+cs}{l}\Big) = -\frac{t}{2}\sum_{\substack{y \bmod l\\ y\equiv ar \bmod t}} \mathbb{B}_2\big(\tfrac{y}{l}\big) = -\frac{t^2}{2l}\mathbb{B}_2\big(\tfrac{ar}{l}\big).$$

Here the second identity is the well-known distribution property of the Bernoulli polynomial $\mathbb{B}_2(x)$. □

7. Appendix: The Weierstrass σ-function as Jacobi form

For $z \in \mathbb{H}$ and $x \in \mathbb{C}$ let

$$\begin{aligned}\phi(z,x) &= 2\pi i\,\eta(z)^2\, e\left(\frac{\eta'}{\eta}(z)\,x^2\right) x \prod_{\substack{l\in\mathbb{Z}z+\mathbb{Z}\\ l\neq 0}} \left(1-\frac{x}{l}\right)\exp\left(\frac{x}{l}+\frac{1}{2}\left(\frac{x}{l}\right)^2\right)\\ &= q^{\frac{1}{12}}\left(\zeta^{1/2}-\zeta^{-1/2}\right)\prod_{n\geq 1}\left(1-q^n\zeta\right)\left(1-q^n\zeta^{-1}\right),\end{aligned}$$

with $\eta(z) = q^{1/24}\prod_{n\geq1}(1-q^n)$ denoting the Dedekind η-function, and with $\zeta^r(x) = \exp(2\pi i r x)$ (see, *e.g.*, [S; pp. 143] for a proof of the equality of the two expressions for ϕ). Note that $\phi(z,x)$ is, up to the factors involving η, the Weierstrass σ-function.

Theorem 7.1. *For* $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and* $\alpha = (a_1,a_2) \in \mathbb{Z}^2$ *one has*

$$\phi\left(Az, \frac{x}{cz+d}\right) e\left(-\frac{cx^2}{2(cz+d)}\right) = \epsilon(A)\phi(z,x),$$
$$\phi(z, x+a_1z+a_2)\, q^{a_1^2/2}\zeta^{a_1}\,(-1)^{a_1+a_2} = \phi(z,x),$$

where $\epsilon(A) = \eta^2(Az)/\big((cz+d)\eta^2(z)\big)$.

Proof. The first formula is an immediate consequence of the first definition for ϕ on using the identity $\mathbb{Z}Az+\mathbb{Z} = \frac{1}{cz+d}(\mathbb{Z}z+\mathbb{Z})$ and

$$\frac{\eta'}{\eta}(Az)\,\frac{1}{(cz+d)^2} = \frac{c}{2(cz+d)} + \frac{\eta'}{\eta}(z),$$

which in turn follows immediately from

$$\eta^2(Az) = \epsilon(A)\eta^2(z)\,(cz+d)$$

with a certain constant $\epsilon(A)$. The second transformation formula can be directly checked using the second formula for ϕ. □

It is convenient to interpret these transformation laws as an invariance property with respect to a certain group, namely the Jacobi group $\mathbb{J}(\mathbb{Z})$. For a ring R (commutative, with 1) denote by $\mathbb{J}(R)$ the group of all triples (A, α, n) of matrices $A \in \mathrm{SL}(2, R)$, row vectors $\alpha \in R^2$ and $n \in R$, equipped with the multiplication law

$$(A, \alpha, n) \cdot (B, \beta, n') = (AB, \alpha B + \beta, n + n' + \det \begin{pmatrix} \alpha B \\ \beta \end{pmatrix}).$$

The Jacobi group $\mathbb{J}(\mathbb{R})$ acts on functions $\psi(z, x)$ defined on $\mathbb{H} \times \mathbb{C}$ by

$$\begin{aligned}&(\psi|(A, \alpha, n))(z, x)\\ &= e_2\left(-\frac{cx^2}{(cz+d)} + a_1^2 z + 2a_1 x + a_1 a_2 + n\right) \psi\left(Az, \frac{x + a_1 z + a_2}{cz + d}\right)\end{aligned}$$

(with A and α as in the above Lemma, and with $e_2(\dots) = e(\frac{1}{2}[\dots])$). That this is indeed an action can be verified by a direct (though subtle) computation [E-Z; Theorem 1.4]. Using this group action the formulas of Lemma can now be reinterpreted as

$$\phi|g = \rho(g)\phi,$$

for all $g \in \mathbb{J}(\mathbb{Z})$, where

$$\rho\left((A, \alpha, n)\right) = (-1)^{a_1 + a_2 + a_1 a_2 + n} \epsilon(A).$$

From this transformation law for ϕ it is clear that ρ defines a character of $\mathbb{J}(\mathbb{Z})$, as can, of course, also be checked directly.

Proof of Theorem 6.1. For $\alpha = (a_1, a_2) \in \mathbb{Q}^2$ and $\beta = (b_1, b_2) \in \mathbb{Z}^2$ we have

$$\begin{aligned}\phi|(1, \beta + \alpha, 0) &= \phi|[(1, \beta, 0) \cdot (1, \alpha, 0) \cdot (1, 0, \det \begin{pmatrix} \alpha \\ \beta \end{pmatrix})]\\ &= \rho((1, \beta, 0))\, e_2\left(\det \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)\ \phi|(1, \alpha, 0).\end{aligned}$$

Call the factor in front of $\phi|(1, \alpha, 0)$ on the right hand side $C(\alpha, \beta)$. It can easily be checked that

$$C(\alpha, \beta) = \delta(\alpha + \beta)/\delta(\alpha),$$

where

$$\delta(\alpha) = -\rho((1, \lfloor\alpha\rfloor, 0))\, e_2\left(\det \begin{pmatrix} \alpha \\ \lfloor\alpha\rfloor \end{pmatrix}\right)\ e_2\left(-(a_2 - \lfloor a_2\rfloor)(a_1 - \lfloor a_1\rfloor - 1)\right),$$

with $\lfloor \alpha \rfloor = (\lfloor a_1 \rfloor, \lfloor a_2 \rfloor)$. Thus, if we set, for $\alpha \in \mathbb{Q}^2$, $\alpha \notin \mathbb{Z}^2$,

$$S_\alpha = \delta(\alpha)\phi^{-1}|(1, \alpha, 0),$$

then $S_\alpha = S_{\alpha+\beta}$ for $\beta \in \mathbb{Z}^2$. From the transformation law for ϕ under $\mathbb{J}(\mathbb{Z})$ we obtain

$$S_\alpha|(A, 0, 0) = \epsilon(A)^{-1}\frac{\delta(\alpha)}{\delta(\alpha A)}S_{\alpha A}.$$

A simple calculation shows that $s_\alpha(z) = S_\alpha(z, 0)$. From this Theorem 6.1 follows. □

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AUTOMORPHIC FORMS AND ZETA FUNCTIONS

This volume contains a valuable collection of articles presented at a conference on Automorphic Forms and Zeta Functions in memory of Tsuneo Arakawa, an eminent researcher in modular forms in several variables and zeta functions. The book begins with a review of his works, followed by 16 articles by experts in the fields including H. Aoki, R. Berndt, K. Hashimoto, S. Hayashida, Y. Hironaka, H. Katsurada, W. Kohnen, A. Krieg, A. Murase, H. Narita, T. Oda, B. Roberts, R. Schmidt, R. Schulze-Pillot, N. Skoruppa, T. Sugano, and D. Zagier. A variety of topics in the theory of modular forms and zeta functions are covered: Theta series and the basis problems, Jacobi forms, automorphic forms on $Sp(1, q)$, double zeta functions, special values of zeta and L-functions, many of which are closely related to Arakawa's works.

This collection of papers illustrates Arakawa's contributions and the current trends in modular forms in several variables and related zeta functions.

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