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Introduction to Complex Analysis in Several Variables

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Preface

The idea for this book came when I was an assistant at the Department of Mathematics and Computer Science at the Philipps-University Marburg, Germany. Several times I faced the task of supporting lectures and seminars on complex analysis of several variables and found out that there are very few books on the subject, compared to the vast amount of literature on function theory of one variable, let alone on real variables or basic algebra. Even fewer books, to my understanding, were written primarily with the student in mind. So it was quite hard to find supporting examples and exercises that helped the student to become familiar with the fascinating theory of several complex variables.

Of course, there are notable exceptions, like the books of R.M. Range [9] or B. and L. Kaup [6], however, even those excellent books have a drawback: they are quite thick and thus quite expensive for a student's budget. So an additional motivation to write this book was to give a comprehensive introduction to the theory of several complex variables, illustrate it with as many examples as I could find and help the student to get deeper insight by giving lots of exercises, reaching from almost trivial to rather challenging.

There are not many illustrations in this book, in fact, there is exactly one, because in the theory of several complex variables I find most of them either trivial or misleading. The readers are of course free to have a different opinion on these matters.

Exercises are spread throughout the text and their results will often be referred to, so it is highly recommended to work through them.

Above all, I wanted to keep the book short and affordable, recognizing that this results in certain restrictions in the choice of contents. Critics may say that I left out important topics like pseudoconvexity, complex spaces, analytic sheaves or methods of cohomology theory. All of this is true, but inclusion of all that would have resulted in another frighteningly thick book. So I chose topics that assume only a minimum of prerequisites, i.e., holomorphic functions of one complex variable, calculus of several real variables and basic algebra (vector spaces, groups, rings etc.). Everything else is developed from scratch. I also tried to point out some of the relations of complex analysis with other parts of mathematics. For example, the Convergence Theorem of Weierstrass, that a compactly convergent sequence of holomorphic functions has a holomorphic limit is formulated in the language of functional analysis: the algebra of holomorphic functions is a closed subalgebra of the algebra of continuous functions in the compact-open topology.

Also the exercises do not restrict themselves only to topics of complex analysis of several variables in order to show the student that learning the theory of several complex variables is not working in an isolated ivory tower. Putting the knowledge of different fields of mathematics together, I think, is one of the major joys of the subject. Enjoy !

I would like to thank Dr. Thomas Hempfling of Birkhäuser Publishing for his friendly cooperation and his encouragement. Also, my thanks go to my wife Claudia for her love and constant support. This book is for you!

Chapter 1

Elementary theory of several complex variables

In this chapter we study the n-dimensional complex vector space $\mathbb{C}^n$ and introduce some notation used throughout this book. After recalling geometric and topological notions such as connectedness or convexity we will introduce holomorphic functions and mapping of several complex variables and prove the n-dimensional analogues of several theorems well-known from the one-dimensional case. Throughout this book n, m denote natural numbers (including zero). The set of strictly positive naturals will be denoted by $\mathbb{N}_+$, the set of strictly positive reals by $\mathbb{R}_+$.

1.1 Geometry of $\mathbb{C}^n$

The set $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ is the n-dimensional complex vector space consisting of all vectors $z = x+iy$,where $x, y \in \mathbb{R}^n$and i is the imaginary unit satisfying $i^2 = -1$.By $\overline{z} = x - iy$ we denote the complex conjugate. $\mathbb{C}^n$ is endowed with the Euclidian inner product

$$(z|w) := \sum_{j=1}^{n} z_j \overline{w_j} \tag{1.1}$$

and the Euclidian norm

$$\|z\|_2 := \sqrt{(z|z)}. \tag{1.2}$$

$\mathbb{C}^n$ endowed with the inner product (1.1) is a complex Hilbert space and the mapping

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n, (x, y) \mapsto x + iy$$

is an isometry. Due to the isometry between $\mathbb{C}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ all metric and topological notions of these spaces coincide.

Remark 1.1.1. Let $p \in \mathbb{N}$ be a natural number ≥ 1. For $z \in \mathbb{C}^n$ the following settings define norms on $\mathbb{C}^n$:

$$\|z\|_\infty := \max_{j=1}^{n} |z_j|$$

and

$$\|z\|_p := \left(\sum_{j=1}^{n} |z_j|^p \right)^{\frac{1}{p}}.$$

$\|.\|_\infty$ is called the *maximum norm*, $\|.\|_p$ is called the p- *norm*. All norms define the same topology on $\mathbb{C}^n$. This is a consequence of the fact that, as we will show now, in finite dimensional space all norms are equivalent.

Definition 1.1.2. Two norms N_1, N_2 on a vector space V are called *equivalent*, if there are constants $c, c' > 0$ such that

$$cN_1(x) \leq N_2(x) \leq c' N_1(x) \text{ for all } x \in V.$$

Proposition 1.1.3. On a finite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$) all norms are equivalent.

Proof. It suffices to show that an arbitrary norm $\|.\|$ on V is equivalent to the Euclidian norm (1.2), because one shows easily that equivalence of norms is an equivalence relation (Exercise !). Let $\{b_1, \ldots, b_n\}$ be a basis of V and put

$$M := \max \{\|b_1\|, \ldots, \|b_n\|\}.$$

Let $x \in V$, $x = \sum_{j=1}^n \alpha_j b_j$ with coefficients $\alpha_j \in \mathbb{C}$. The triangle inequality and Hölder's inequality yield

$$\begin{aligned} \|x\| &\leq \sum_{j=1}^{n} |\alpha_j| \|b_j\| \\ &\leq \left(\sum_{j=1}^{n} |\alpha_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|b_j\|^2 \right)^{\frac{1}{2}} \\ &\leq \|x\|_2 \sqrt{n} M. \end{aligned}$$

Every norm is a continuous mapping, because $|\|x\| - \|y\|| \leq \|x - y\|$, hence, $\|.\|$ attains a minimum $s \geq 0$ on the compact unit sphere

$$S := \{x \in V \mid \|x\|_2 = 1\}.$$

S is compact by the Heine–Borel Theorem, because $\dim V < \infty$. Since $0 \notin S$ the identity property of a norm, i.e. that $\|x\| = 0$ if and only if $x = 0$, implies that $s > 0$. For every $x \neq 0$ we have

$$\frac{x}{\|x\|_2} \in S,$$

which implies

$$\left\| \frac{x}{\|x\|_2} \right\| \geq s > 0.$$

This is equivalent to $\|x\| \geq s\,\|x\|_2$. Putting both estimates together gives

$$s\,\|x\|_2 \leq \|x\| \leq \sqrt{n}M\,\|x\|_2\,,$$

which shows the equivalence of $\|.\|$ and $\|.\|_2$. □

Exercise 1.1.4. Give an alternative proof of Proposition 1.1.3 using the 1-norm.

Exercise 1.1.5. Show that $\lim_{p\to\infty} \|z\|_p = \|z\|_\infty$ for all $z \in \mathbb{C}^n$.

If we do not refer to a special norm, we will use the notation $\|.\|$ for any norm (not only p-norms).

Example 1.1.6. On infinite-dimensional vector spaces *not* all norms are equivalent. Consider the infinite-dimensional real vector space $\mathcal{C}^1[0,1]$ of all real differentiable functions on the interval $[0,1]$. Then we can define two norms by

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$$

and

$$\|f\|_{\mathcal{C}^1} := \|f\|_\infty + \|f'\|_\infty\,.$$

The function $f(x) := x^n$, $n \in \mathbb{N}$, satisfies

$$\|f\|_\infty = 1,\ \|f\|_{\mathcal{C}^1} = 1 + n.$$

Since n can be arbitrarily large, there is no constant $c > 0$ such that

$$\|f\|_{\mathcal{C}^1} \leq c\,\|f\|_\infty$$

for all $f \in \mathcal{C}^1[0,1]$.

Exercise 1.1.7. Show that $\mathcal{C}^1[0,1]$ is a Banach space with respect to $\|.\|_{\mathcal{C}^1}$, but not with respect to $\|.\|_\infty$.

Let us recall some definitions.

Definition 1.1.8. Let E be a real vector space and $x, y \in E$.

1. The *closed segment* $[x,y]$ is the set

$$[x,y] := \{tx + (1-t)\,y \mid 0 \leq t \leq 1\}\,.$$

2. The *open segment* $]x,y[$ is the set

$$]x,y[:= \{tx + (1-t)\,y \mid 0 < t < 1\}\,.$$

3. A subset $C \subset E$ is called *convex* if $[x, y] \subset C$ for all $x, y \in C$.
4. Let $M \subset V$ be an arbitrary subset. The *convex hull* $\operatorname{conv}(M)$ of M is the intersection of all convex sets containing M.
5. An element x of a compact and convex set C is called an *extremal point* of C if the condition $x \in \,]y, z[$ for some $y, z \in C$ implies that $x = y = z$. The subset of extremal points of C is denoted by $\partial_{ex} C$.

Example 1.1.9. Let $r > 0$ and $a \in \mathbb{C}^n$. The set

$$B_r^n(a) := \{ z \in \mathbb{C}^n \mid \|z - a\| < r \} \tag{1.3}$$

is called the n-dimensional *open ball* with center a and radius r with respect to the norm $\|.\|$. It is a convex set, since for all $z, w \in B_r(a)$ and $t \in [0, 1]$ it follows from the triangle inequality that

$$\|tz + (1 - t) w\| \leq t \|z\| + (1 - t) \|w\| < tr + (1 - t) r = r.$$

The *closed ball* is defined by replacing the $<$ by $\leq$ in (1.3).

Exercise 1.1.10. Show that the closed ball with respect to the p-norm coincides with the topological closure of the open ball. Show that the closed ball is compact and determine all its extremal points.

The open (closed) ball in $\mathbb{C}^n$ is a natural generalization of the open (closed) disc in $\mathbb{C}$. It is, however, not the only one.

Definition 1.1.11. We denote by $\mathbb{R}_+^n$ the set of real vectors of strictly positive components. Let $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ and $a \in \mathbb{C}^n$.

1. The set

$$P_r^n(a) := \{ z \in \mathbb{C}^n | \ |z_j - a_j| < r_j \text{ for all } j = 1, \ldots, n \}$$

is called the open *polycylinder* with center a and polyradius r.

2. The set

$$T_r^n(a) := \{ z \in \mathbb{C}^n | \ |z_j - a_j| = r_j \text{ for all } j = 1, \ldots, n \}$$

is called the *polytorus* with center a and polyradius r. If $r_j = 1$ for all j and $a = 0$ it is called the unit polytorus and denoted $\mathbb{T}^n$.

Remark 1.1.12. The open polycylinder is another generalization of the one- dimensional open disc, since it is the Cartesian product of n open discs in $\mathbb{C}$.Therefore we also use the expression *polydisc*. For $n = 1$, open polycylinder and open ball coincide. $P_r^n(a)$ is also convex.

Lemma 1.1.13. Let C be a convex subset of $\mathbb{C}^n$. Then C is simply connected.

Proof. Let $\gamma : [0,1] \to C$ be a closed curve. Then

$$H : [0,1] \times [0,1] \to \mathbb{C}^n, (s,t) \mapsto s\gamma(0) + (1-s)\gamma(t)$$

defines a homotopy from γ to $\gamma(0)$. Since C is convex we have

$$H(s,t) \in C$$

for all $s, t \in [0,1]$. □

As in the one-dimensional case, the notion of connectedness and of a domain is important in several complex variables. We recall the definition for a general topological space.

Definition 1.1.14. Let X be a topological space.

1. The space X is called *connected*, if X cannot be represented as the disjoint union of two nonempty open subsets of X, i.e., if A, B are open subsets of X, $A \neq \emptyset, A \cap B = \emptyset$ and $X = A \cup B$, then $B = \emptyset$.
2. An open and connected subset $D \subset X$ is called a *domain*.

There are different equivalent characterizations of connected sets stated in the following lemma.

Lemma 1.1.15. Let X be a topological space and $D \subset X$ an open subset. The following statements are equivalent:

1. The set D is a domain.
2. If $A \neq \emptyset$ is a subset of D which is both open and closed, then $A = D$.
3. Every locally constant function $f : D \to \mathbb{C}$ is constant.

Proof. 1. $\Rightarrow$ 2. Let A be a nonempty subset of D which is both open and closed in D. Put $B := D \setminus A$. Then B is open in D, for A is closed, $A \cap B = \emptyset$ and $D = A \cup B$. Since D is connected and $A \neq \emptyset$ we conclude $B = \emptyset$, hence, $A = D$.

2. $\Rightarrow$ 3. Let $c \in D$ and $A := f^{-1}(\{f(c)\})$. In $\mathbb{C}$, sets consisting of a single point are closed (this holds for any Hausdorff space). f is continuous, because f is locally constant, so A is closed in D.Since $c \in A$, the set A is nonempty. Let $p \in A$. Then there is an open neighbourhood U of p, such that $f(x) = f(p) = f(c)$ for all $x \in U$, i.e., $U \subset A$. Thus, A is open. We conclude that $A = D$, so f is constant.

3. $\Rightarrow$ 1. If D can be decomposed into disjoint open nonempty subsets A, B, then

$$f : D \to \mathbb{C}, z \mapsto \begin{cases} 1, & z \in A \\ 0, & z \in B \end{cases}$$

defines a locally constant, yet not constant function □

Remark 1.1.16. In the one-variable case the celebrated Riemann Mapping Theorem states that all connected, simply connected domains in $\mathbb{C}$ are biholomorphically equivalent to either $\mathbb{C}$ or to the unit disc. This theorem is false in the multivariable case. We will later show that even the two natural generalizations of the unit disc, i.e., the unit ball and the unit polycylinder, are not biholomorphically equivalent. This is one example of the far-reaching differences between complex analysis in one and in more than one variable.

Exercise 1.1.17. Let X be a topological space.

1. If $A, B \subset X$, such that $A \subset B \subset \overline{A}$ and A is connected, then B is connected.
2. If X is connected and $f : X \to Y$ is a continuous mapping into some other topological space Y, then $f(X)$ is also connected.
3. The space X is called *pathwise connected*, if to every pair $x, y \in X$ there exists a continuous curve
$$\gamma_{x,y} : [0,1] \to X$$
with $\gamma_{x,y}(0) = x, \gamma_{x,y}(1) = y$. Show that a subset D of $\mathbb{C}^n$ is a domain if and only if D is open and pathwise connected. (Hint: You can use the fact that real intervals are connected.)
4. If $(U_j)_{j \in J}$ is a family of (pathwise) connected sets which satisfies
$$\bigcap_{j \in J} U_j \neq \emptyset,$$
then $\bigcup_{j \in J} U_j$ is (pathwise) connected.
5. Show that for every $R > 0$ and every $n \geq 1$ the set $\mathbb{C}^n \setminus B_R^n(0)$ is pathwise connected.
6. Check the set
$$M := \left\{ z \in \mathbb{C} \;\middle|\; 0 < \operatorname{Re} z \leq 1, \operatorname{Im} z = \sin \frac{1}{\operatorname{Re} z} \right\} \cup [-i, i]$$
for connectedness and pathwise connectedness.

Exercise 1.1.18. We identify the space $M(n, n; \mathbb{C})$ of complex $n \times n$ matrices as a topological space with $\mathbb{C}^{n^2}$ with the usual (metric) topology

1. Show that the set $GL_n(\mathbb{C})$ of invertible matrices is a domain in $M(n, n; \mathbb{C})$.
2. Show that the set $U_n(\mathbb{C})$ of unitary matrices is compact and pathwise connected.
3. Show that the set $P_n(\mathbb{C})$ of self-adjoint positive definite matrices is convex.

Exercise 1.1.19. Let C be a compact convex set.

1. Show that
$$\partial_{ex}C \subset \partial C.$$
2. Let $\overline{P_r^n(a)}$ be a compact polydisc in $\mathbb{C}^n$ and $T_r(a)$ the corresponding polytorus. Show that
$$\partial_{ex}P_r^n(a) = T_r^n(a).$$

Remark 1.1.20. By the celebrated Krein–Milman Theorem (see, e.g.,[11] Theorem VIII.4.4) every compact convex subset C of a locally convex vector space possesses extremal points. Moreover, C can be reconstructed as the closed convex hull of its subset of extremal points:
$$C = \overline{\operatorname{conv}(\partial_{ex}C)}$$

Notation 1.1.21. In the following we will use the expression that some proposition holds *near* a point a or near a set X if there is an open neighbourhood of a resp. X on which it holds.

1.2 Holomorphic functions in several complex variables

1.2.1 Definition of a holomorphic function

Definition 1.2.1. Let $U \subset \mathbb{C}^n$ be an open subset, $f : U \to \mathbb{C}^m, a \in U$ and $\|.\|$ an arbitrary norm in $\mathbb{C}^n$.

1. The function f is called *complex differentiable* at a, if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, a) > 0$ and a $\mathbb{C}$-linear mapping
$$Df(a) : \mathbb{C}^n \to \mathbb{C}^m,$$
such that for all $z \in U$ with $\|z - a\| < \delta$ the inequality
$$\|f(z) - f(a) - Df(a)(z - a)\| \leq \varepsilon \|z - a\|$$
holds. If $Df(a)$ exists, it is called the *complex derivative* of f in a.
2. The function f is called *holomorphic* on U, if f is complex differentiable at all $a \in U$.
3. The set
$$\mathcal{O}(U, \mathbb{C}^n) := \{f : U \to \mathbb{C}^m | \text{ f holomorphic}\}$$
is called the set of holomorphic mappings on U. If $m = 1$ we write
$$\mathcal{O}(U) := \mathcal{O}(U, \mathbb{C})$$
and call this set the set of *holomorphic functions* on U.

This definition is independent of the choice of a norm, since all norms on $\mathbb{C}^n$ are equivalent. The proofs of the following propositions are analogous to the real variable case, so we can leave them out.

Proposition 1.2.2.

1. If f is $\mathbb{C}$-differentiable in a, then f is continuous in a.
2. The derivative $Df(a)$ is unique.
3. The set $\mathcal{O}(U, \mathbb{C}^m)$ is a $\mathbb{C}-$ vector space and
$$D(\lambda f + \mu g)(a) = \lambda Df(a) + \mu Dg(a)$$
for all $f, g \in \mathcal{O}(U, \mathbb{C}^m)$ and all $\lambda, \mu \in \mathbb{C}$.
4. (Chain Rule) Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ be open sets, $a \in U$ and
$$f \in \mathcal{O}(U, V) := \{\varphi : U \to V \mid \varphi \text{ holomorphic}\},$$
$g \in \mathcal{O}\left(V, \mathbb{C}^k\right)$. Then $g \circ f \in \mathcal{O}\left(U, \mathbb{C}^k\right)$ and
$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$
5. Let $U \subset \mathbb{C}^n$ be an open set. A mapping
$$f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$$
is holomorphic if and only if all components $f_1, \ldots, f_m$ are holomorphic functions on U.
6. $\mathcal{O}(U)$ is a $\mathbb{C}-$ algebra. If $f, g \in \mathcal{O}(U)$ and $g(z) \neq 0$ for all $z \in U$, then $\frac{f}{g} \in \mathcal{O}(U)$.

Example 1.2.3. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be a locally constant function. Then f is holomorphic and $Df(a) = 0$ for all $a \in U$.

Proof. Let $a \in U$ and $\varepsilon > 0$. Since f is locally constant there is some $\delta > 0$, such that $f(z) = f(a)$ for all $z \in U$ with $\|z - a\| < \delta$. Therefore
$$\|f(z) - f(a)\| = 0 \leq \varepsilon \|z - a\|$$
for all $z \in U$ with $\|z - a\| < \delta$, i.e., f is holomorphic with $Df(a) = 0$ for all $a \in U$. □

Example 1.2.4. For every $k = 1, \ldots, n$ the projection
$$\mathrm{pr}_k : \mathbb{C}^n \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto z_k$$
is holomorphic and $D\,\mathrm{pr}_k(a) = e_k$ (the k-th canonical basis vector) for all $a \in \mathbb{C}^n$.

Proof. Let $\varepsilon > 0$ and $a \in \mathbb{C}^n$. Then

$$|\mathrm{pr}_k(z) - \mathrm{pr}_k(a) - (z - a|e_k)| = 0 \leq \varepsilon \|z - a\|$$

for all $z \in \mathbb{C}^n$. □

Example 1.2.5. The complex subalgebra $\mathbb{C}[z_1, \ldots, z_n]$ of $\mathcal{O}(\mathbb{C}^n)$ generated by the constants and the projections is called the *algebra of polynomials*. Its elements are sums of the form

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

with $c_\alpha \neq 0$ only for finitely many $c_\alpha \in \mathbb{C}$, where for $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}^n$ we use the notation

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}.$$

The *degree* of a polynomial

$$p(z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ c_\alpha = 0 \text{ for almost all } \alpha}} c_\alpha z^\alpha$$

is defined as

$$\deg p := \max\{\alpha_1 + \cdots + \alpha_n \mid \alpha \in \mathbb{N}^n,\ c_\alpha \neq 0\}.$$

For example, the polynomial $p(z_1, z_2) := z_1^5 + z_1^3 z_2^3$ has degree 6. By convention the zero polynomial has degree $-\infty$.The following formulas for the degree are easily verified:

$$\begin{aligned} \deg(pq) &= \deg p + \deg q, \\ \deg(p+q) &\leq \max\{\deg p, \deg q\}. \end{aligned}$$

Exercise 1.2.6. Show that for all $z, w \in \mathbb{C}^n$ and all $\alpha \in \mathbb{N}^n$ there exists a polynomial $q \in \mathbb{C}[z, w]$ of degree $|\alpha| := \|\alpha\|_1$ such that

$$(z + w)^\alpha = z^\alpha + q(z, w).$$

Exercise 1.2.7. Show that the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ has no zero divisors.

Exercise 1.2.8. Show that the zero set of a complex polynomial in $n \geq 2$ variables is not compact in $\mathbb{C}^n$. (*Hint*: Use the Fundamental Theorem of Algebra). Compare this to the case $n = 1$.

Exercise 1.2.9. Show that every (affine) linear mapping $L : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic. Compute $DL(a)$ for all $a \in \mathbb{C}^n$.

Exercise 1.2.10. Let $U_1, \ldots, U_n$ be open sets in $\mathbb{C}$ and let $f_j : U_j \to \mathbb{C}$ be holomorphic functions, $j = 1, \ldots, n$.

1. Show that $U := U_1 \times \cdots \times U_n$ is open in $\mathbb{C}^n$.
2. Show that the functions

$$f : U \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto \prod_{j=1}^{n} f_j(z_j)$$

and

$$g : U \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto \sum_{j=1}^{n} f_j(z_j)$$

are holomorphic on U.

1.2.2 Basic properties of holomorphic functions

We turn to the multidimensional analogues of some important theorems from the one variable case. The basic tool to this end is the following observation.

Lemma 1.2.11. Let $U \subset \mathbb{C}^n$ be open, $a \in U, f \in \mathcal{O}(U), b \in \mathbb{C}^n$ and $V := V_{a,b;U} := \{t \in \mathbb{C} \mid a + tb \in U\}$. Then V is open in $\mathbb{C}$, $0 \in V$ and the function

$$g_{a,b} : V \to \mathbb{C}, t \mapsto f(a + tb)$$

is holomorphic.

Proof. From $a \in U$ follows that $0 \in V$. If $b = 0$ then $V = \mathbb{C}$. Let $b \neq 0$. If $t_0 \in V$ then $z_0 := a + t_0 b \in U$. Since U is open, there is some $\varepsilon > 0$, such that $B_\varepsilon(z_0) \in U$. Put $z_t := a + tb$. Then

$$\|z_0 - z_t\| = \|b\| \, |t_0 - t| < \varepsilon$$

for all t with $|t_0 - t| < \frac{\varepsilon}{\|b\|}$, i.e., $B_{\frac{\varepsilon}{\|b\|}}(t_0) \subset V$. Since $g_{a,b}$ is the composition of the affine linear mapping $t \mapsto a + tb$ and the holomorphic function f, holomorphy of $g_{a,b}$ follows from the chain rule. ☐

Conclusion 1.2.12. *We have analogues of the following results from the one-dimensional theory.*

1. **Liouville's Theorem***: Every bounded holomorphic function*

$$f : \mathbb{C}^n \to \mathbb{C}$$

is constant.

2. **Identity Theorem***: Let $D \subset \mathbb{C}^n$ be a domain, $a \in D, f \in \mathcal{O}(D)$, such that $f = 0$ near a. Then f is the zero function.*
3. **Open Mapping Theorem***: Let $D \subset \mathbb{C}^n$ be a domain, $U \subset D$ an open subset and $f \in \mathcal{O}(D)$ a non-constant function. Then $f(U)$ is open, i.e., every holomorphic function is an open mapping. In particular, $f(D)$ is a domain in $\mathbb{C}$.*

4. **Maximum Modulus Theorem**: *If $D \subset \mathbb{C}^n$ is a domain, $a \in D$ and $f \in \mathcal{O}(D)$, such that $|f|$ has a local maximum at a, then f is constant.*

Proof. 1. Let $a, b \in \mathbb{C}^n$. The function $g_{a,b-a}$ from Lemma 1.2.11 is holomorphic on $\mathbb{C}$, satisfies

$$g_{a,b-a}(0) = f(a), g_{a,b-a}(1) = f(b)$$

and

$$g_{a,b-a}(\mathbb{C}) \subset f(\mathbb{C}^n).$$

Since f is bounded, $g_{a,b-a}$ is bounded. By the one-dimensional version of Liouville's Theorem $g_{a,b-a}$ is constant, hence, $f(a) = f(b)$ for all $a, b \in \mathbb{C}^n$.

2. Let

$$U := \{z \in D \mid f = 0 \text{ near } z\}.$$

By prerequisite $a \in U$. U is closed in D, because either $U = D$ (if f is the zero function) or, by continuity of f, to every $z \in D \setminus U$ there exists a neighbourhood W, on which f does not vanish, i.e., $W \subset D \setminus U$. Let $c \in U \cap D$. There is a polyradius $r \in \mathbb{R}^n_+$, such that the polycylinder $P_r(c)$ is contained in D and such that $P_r(c) \cap U \neq \emptyset$. Choose some $z \in P_r(c)$ and $w \in P_r(c) \cap U$. From Lemma 1.2.11 we obtain that the set $V_{w,z-w;D}$ is open in $\mathbb{C}$ and because $P_r(c)$ is convex, we have $[0,1] \subset V_{w,z-w;D}$. Since f vanishes near w, there exists an open and connected neighbourhood $W \subset \mathbb{C}$ of $[0,1]$ on which $g_{w,z-w}$ vanishes. This implies that $P_r(c) \subset U$, so U is open in D. However, since D is connected, the only nonempty open and closed subset of D is D itself. Hence, $U = D$, i.e., $f = 0$ on D.

3. $f(D)$ is connected, because D is connected and f is continuous (cf. Exercise 1.1.17). We have to show that $f(U)$ is open. Let $b \in f(U)$. There is some $a \in U$ with $b = f(a)$. Since U is open, there is a polycylinder $P_r(a) \subset U$. By the Identity Theorem f is not constant on $P_r(c)$, since otherwise f would be constant on all of D, contradicting the prerequisites. This implies that there is some $w \in \mathbb{C}^n, w \neq 0$, such that $g_{a,w}$ from Lemma 1.2.11 is not constant on $V = V_{a,w;P_r(a)}$. From the one-dimensional theory we obtain that $g_{a,w}(V)$ is an open neighbourhood of b. Because

$$b \in g_{a,w}(V) \subset f(P_r(a)) \subset f(U),$$

$f(U)$ is a neighbourhood of b. Since b was arbitrary, $f(U)$ is open in $\mathbb{C}$.

4. $f(D)$ is open in $\mathbb{C}$. Since

$$|.| : \mathbb{C} \to [0, +\infty[$$

is an open mapping (Exercise !), the assertion follows. □

Corollary 1.2.13 (Maximal Modulus Principle for bounded domains). *Let $D \subset \mathbb{C}^n$ be a bounded domain and $f : \overline{D} \to \mathbb{C}$ be a continuous function, whose restriction to D is holomorphic. Then $|f|$ attains a maximum on the boundary ∂D.*

Proof. Since D is bounded, the closure $\overline{D}$ is compact by the Heine–Borel Theorem. Thus, the continuous real-valued function $|f|$ attains a maximum in a point $p \in \overline{D}$. If $p \in \partial D$ we are done. If $p \in D$ the Maximum Modulus Theorem says that $f|_D$ is constant. By continuity, f is constant on $\overline{D}$ and thus $|f|$ attains a maximum also on ∂D. $\square$

In the one-dimensional version of the Identity Theorem it is sufficient to know the values of a holomorphic function on a subset of a domain, which has an accumulation point. This is no longer true in more than one dimension.

Example 1.2.14. The holomorphic function

$$f : \mathbb{C}^2 \to \mathbb{C}, (z, w) \mapsto zw$$

is not identically zero, yet it vanishes on the subsets $\mathbb{C}\times \{0\}$ and $\{0\} \times \mathbb{C}$ of $\mathbb{C}^2$, which clearly have accumulation points in $\mathbb{C}^2$.

Exercise 1.2.15. Let $U \subset \mathbb{C}^n$ be an open set. Show that U is a domain if and only if the ring $\mathcal{O}(U)$ is an integral domain, i.e., it has no zero divisors.

Exercise 1.2.16. Let $D \subset \mathbb{C}^n$ be a domain and $\mathcal{F} \subset \mathcal{O}(D)$ be a family of holomorphic functions. We denote by

$$N(\mathcal{F}) := \{z \in D \mid f(z) = 0 \text{ for all } f \in \mathcal{F}\}$$

the common zero set of the family $\mathcal{F}$.

1. Show that either $D \setminus N(\mathcal{F}) = \emptyset$ or $D \setminus N(\mathcal{F})$ is dense in D.
2. Show that $GL_n(\mathbb{C})$ is dense in $M(n, n; \mathbb{C})$.

Exercise 1.2.17. Consider the mapping

$$f : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (z, zw)$$

Show that f is holomorphic, but is not an open mapping. Does this contradict the Open Mapping Theorem?

Exercise 1.2.18. Let

$$f : X \to E$$

be an open mapping from a topological space X to a normed space E. State and prove a Maximum Modulus Theorem for f.

Exercise 1.2.19. Let $D \subset \mathbb{C}^n$ be a domain, $B \subset D$ an open and bounded subset, such that also the closure $\overline{B}$ is contained in D. Let ∂B denote the topological boundary of B and $f \in \mathcal{O}(D)$. Show that

$$\partial(f(B)) \subset f(\partial B).$$

Does this also hold in general, if B is unbounded?

Exercise 1.2.20. Let $B_1^n(0)$ be the n-dimensional unit ball and $f : B_1^n(0) \to \mathbb{C}$ be holomorphic with $f(0) = 0$. Let $M > 0$ be a constant satisfying $|f(z)| \leq M$ for all $z \in B_1^n(0)$. Prove the following n-dimensional generalization of Schwarz' Lemma:

1. The estimate $|f(z)| \leq M \|z\|$ holds for all $z \in B_1^n(0)$.
2. The following estimate holds:

$$\|Df(0)\| := \sup_{\|z\|=1} |Df(0)z| \leq M.$$

1.2.3 Partially holomorphic functions and the Cauchy–Riemann differential equations

As in real calculus one may consider all but one variable of a given holomorphic function

$$(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n)$$

as fixed. This leads to the concept of partial holomorphy.

Definition 1.2.21. Let $U \subset \mathbb{C}^n$ be an open set, $a \in U$ and $f : U \to \mathbb{C}$. For $j = 1, \ldots, n$ define

$$U_j := \{z \in \mathbb{C} \mid (a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n) \in U\}$$

and

$$\widehat{f_j} : U_j \to \mathbb{C}, z \mapsto f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n).$$

f is called *partially holomorphic* on U, if all $\widehat{f_j}$ are holomorphic.

A function f holomorphic on an open set $U \subset \mathbb{C}^n$ can also be considered as a totally differentiable function of $2n$ real variables. Taking this point of view we define

$$\mathcal{C}^k(U) := \{f : U \to \mathbb{C} \mid f \text{ is } k - \text{times } \mathbb{R} - \text{differentiable}\},$$

where, as usual, $k = 0$ denotes the continuous functions. In this case we leave out the superscript. Let $a \in U$ and $f \in \mathcal{C}^1(U)$. Then there is an $\mathbb{R}$-linear function

$$d_a f : \mathbb{R}^{2n} \to \mathbb{C}$$

called the *real differential of f at a*, such that

$$f(z) = f(a) + d_a f(z - a) + O\left(\|z - a\|^2\right).$$

Comparing this to Definition 1.2.1 we can say that f is $\mathbb{C}$-differentiable at a if and only if $d_a f$ is $\mathbb{C}$-linear. This shows that it makes sense at this point to look a little closer at the relationships between $\mathbb{R}$-linear and $\mathbb{C}$-linear functions of complex vector spaces.

Lemma 1.2.22. Let V be a vector space over $\mathbb{C}$ and $V^{\#}$ its algebraic dual, i.e.,

$$V^{\#} := \{\mu : V \to \mathbb{C} \mid \mu \text{ is } \mathbb{C}- \text{ linear}\}.$$

Further, we define

$$\begin{aligned} \overline{V}^{\#} \quad &: = \{\mu : V \to \mathbb{C} \mid \mu \text{ is } \mathbb{C}- \text{ antilinear}\} \\ &= \{\mu : V \to \mathbb{C} \mid \overline{\mu} \text{ is } \mathbb{C}- \text{ linear}\} \end{aligned}$$

and

$$V_{\mathbb{R}}^{\#} := \{\mu : V \to \mathbb{C} \mid \mu \text{ is } \mathbb{R}- \text{ linear}\}.$$

Then $V_{\mathbb{R}}^{\#}$ is a complex vector space, $V^{\#}, \overline{V}^{\#}$ are subspaces of $V_{\mathbb{R}}^{\#}$ and we have the direct decomposition

$$V_{\mathbb{R}}^{\#} = V^{\#} \oplus \overline{V}^{\#}.$$

Proof. The first propositions are clear. We only have to prove the direct decomposition. To this end let $\mu \in V^{\#} \cap \overline{V}^{\#}$ and $z \in V$. Since μ is both complex linear and antilinear we have

$$\mu(iz) = i\mu(z) = -i\mu(z),$$

which holds only if $\mu = 0$. To prove the decomposition property let $\mu \in V_{\mathbb{R}}^{\#}$. We define

$$\begin{aligned} \mu_1(z) &:= \frac{1}{2}(\mu(z) - i\mu(iz)), \\ \mu_2(z) &:= \frac{1}{2}(\mu(z) + i\mu(iz)). \end{aligned}$$

An easy computation shows that μ_1, μ_2 are $\mathbb{R}$-linear and that $\mu_1 + \mu_2 = \mu$. Now

$$\begin{aligned} \mu_1(iz) &= \frac{1}{2}(\mu(iz) - i\mu(-z)) \\ &= i\frac{1}{2}(\mu(z) - i\mu(iz)) = i\mu_1(z) \end{aligned}$$

and

$$\begin{aligned} \mu_2(iz) &= \frac{1}{2}(\mu(iz) + i\mu(-z)) \\ &= -i\frac{1}{2}(\mu(z) + i\mu(iz)) = -i\mu_2(z), \end{aligned}$$

which shows that $\mu_1 \in V^{\#}$ and $\mu_2 \in \overline{V}^{\#}$. □

We use this lemma in the special case

$$V := \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n.$$

Let $w = u + iv \in V$. For $j = 1, \ldots, n$ consider the linear functionals

$$dx_j, dy_j : \mathbb{C}^n \to \mathbb{C},\ dx_j(w) := u_j,\ dy_j(w) := v_j.$$

Clearly, $dx_j, dy_j \in V_{\mathbb{R}}^{\#}$ and

$$dx_j(iw) = -v_j,\ dy_j(iw) = u_j.$$

Now define

$$dz_j(w) := dx_j(w) + idy_j(w),\ d\overline{z_j}(w) := dx_j(w) - idy_j(w).$$

We then have

$$\begin{aligned} dz_j(w) &= u_j + iv_j = dx_j(w) - idx_j(iw), \\ d\overline{z_j}(w) &= u_j - iv_j = dx_j(w) + idx_j(iw). \end{aligned}$$

As in Lemma 1.2.22 we obtain

$$dz_j \in V^{\#},\ \ d\overline{z_j} \in \overline{V}^{\#}.$$

By applying linear combinations of the dz_j resp. $d\overline{z_j}$ to the canonical basis vectors $e_1, \ldots, e_n$ of $\mathbb{C}^n$ we find that the sets $\{dz_1, \ldots, dz_n\}$ resp. $\{d\overline{z_1}, \ldots, d\overline{z_n}\}$ are linearly independent over $\mathbb{C}$, thus forming bases for $V^{\#}$ resp. $\overline{V}^{\#}$. Their union then forms a basis for $V_{\mathbb{R}}^{\#}$ by Lemma 1.2.22. This leads to the following representation of the real differential $d_a f$:

$$d_a f = \sum_{j=1}^{n} \left(\alpha_j(f,a)\, dz_j + \beta_j(f,a)\, d\overline{z_j}\right) \tag{1.4}$$

with unique coefficients $\alpha_j(f,a), \beta_j(f,a) \in \mathbb{C}$.

Notation 1.2.23. Let $\alpha_j(f,a), \beta_j(f,a)$ be the unique coefficients in the representation (1.4). We write

$$\partial_j f(a) := \frac{\partial f}{\partial z_j}(a) := \alpha_j(f,a),\ \overline{\partial_j} f(a) := \frac{\partial f}{\partial \overline{z_j}}(a) := \beta_j(f,a).$$

Definition 1.2.24. The linear functional

$$\partial_a f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)\, dz_j : \mathbb{C}^n \to \mathbb{C}, w \mapsto \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a)\, w_j$$

is called the *complex differential* of f at a. The antilinear functional

$$\overline{\partial_a} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z_j}}(a)\, d\overline{z_j} : \mathbb{C}^n \to \mathbb{C}, w \mapsto \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z_j}}(a)\, \overline{w_j}$$

is called the *complex-conjugate differential* of f at a.

With these definitions we can decompose the real differential

$$d_a f = \partial_a f + \overline{\partial}_a f \in (\mathbb{C}^n)^{\#} \oplus \overline{(\mathbb{C}^n)}^{\#}. \tag{1.5}$$

These results can be summarized in

Theorem 1.2.25 (Cauchy–Riemann). *Let $U \subset \mathbb{C}^n$ be an open set and $f \in \mathcal{C}^1(U)$. Then the following statements are equivalent:*

1. *The function f is holomorphic on U.*
2. *For every $a \in U$ the differential $d_a f$ is $\mathbb{C}$-linear, i.e., $d_a f \in (\mathbb{C}^n)^{\#}$.*
3. *For every $a \in U$ the equation $\overline{\partial_a} f = 0$ holds.*
4. *For every $a \in U$ the function f satisfies the* Cauchy–Riemann differential equations
$$\frac{\partial f}{\partial \overline{z_j}}(a) = 0 \textit{ for all } j = 1, \dots, n$$
on U.

Exercise 1.2.26. (Wirtinger derivatives) Let $U \subset \mathbb{C}^n$ be open, $a \in U$ and $f \in \mathcal{C}^1(U)$.

1. Show that
$$\frac{\partial f}{\partial z_j}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)(a), \quad \frac{\partial f}{\partial \overline{z_j}}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)(a),$$
where $\frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial y_j}$ denote the real partial derivatives.
2. Let $U \subset \mathbb{C}^n$ be open, $a \in U$ and $f = (f_1, \dots, f_m) \in \mathcal{O}(U, \mathbb{C}^m)$. Let $Df(a) = (\alpha_{kl}) \in M(m, n; \mathbb{C})$ be the complex derivative of f in a. Show that
$$\alpha_{kl} = \frac{\partial f_k}{\partial z_l}(a)$$
for all $k = 1, \dots, m$ and $l = 1, \dots, n$.
3. Let $\widehat{f_j}$ be defined as in Definition 1.2.21. Show that if f is holomorphic on U then f is partially holomorphic and satisfies the equations
$$\frac{\partial f}{\partial z_j}(a) = \widehat{f_j}'(a_j) \text{ for all } j = 1, \dots, n.$$

Exercise 1.2.27. Let $U \subset \mathbb{C}^n$ be open and $f = (f_1, \dots, f_m) : U \to \mathbb{C}^m$ differentiable in the real sense. Prove the formulas

$$\frac{\partial \overline{f_k}}{\partial \overline{z_j}} = \overline{\left(\frac{\partial f_k}{\partial z_j}\right)}, \quad \frac{\partial \overline{f_k}}{\partial z_j} = \overline{\left(\frac{\partial f_k}{\partial \overline{z_j}}\right)} \text{ for } j = 1, \dots, n \text{ and } k = 1, \dots, m.$$

Remark 1.2.28. It is a deep theorem of Hartogs [5] that the converse of Exercise 1.2.26.3 also holds: *Every partially holomorphic function is already holomorphic.* The proof of this theorem is beyond the scope of this book, however, we will use the result. Note the fundamental difference from the real case, where a partially differentiable function need not even be continuous, as the well-known example

$$\varphi : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \begin{cases} 0, & \text{if } (x, y) = (0, 0) \\ \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \end{cases}$$

shows. Readers who are not familiar with this example should consider $\lim\limits_{x \to 0} \varphi(x, x)$.

Exercise 1.2.29. Show that the inversion of matrices

$$\operatorname{inv} : GL_n(\mathbb{C}) \to GL_n(\mathbb{C}), Z \mapsto Z^{-1}$$

is a holomorphic mapping (*Hint*: Cramer's rule).

Exercise 1.2.30. Let $m \in \mathbb{N}_+$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be homogenous of degree m, i.e., f satisfies the condition

$$f(tz) = t^m f(z)$$

for all $z \in \mathbb{C}^n$ and all $t \in \mathbb{C}$. Prove *Euler's identity*

$$\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) z_j = mf(z)$$

for all $z \in \mathbb{C}^n$.

1.3 The Cauchy Integral Formula

Probably the most celebrated formula in complex analysis in one variable is Cauchy's Integral Formula, since it implies many fundamental theorems in the one-dimensional theory. Cauchy's Integral Formula allows a generalization to dimension n in a sense of multiple line integrals. We start by considering the polytorus $T_r^n(a)$. For $a \in \mathbb{C}^n$ and $r \in \mathbb{R}_+^n$ let

$$T_r^n(a) := \{z \in \mathbb{C}^n \mid |z_j - a_j| = r_j,\ j = 1, \ldots, n\}.$$

$T_r^n(a)$ is a copy of n circles in the complex plane and is contained in the boundary of the polydisc $P_r^n(a)$. Let

$$f : T_r^n(a) \to \mathbb{C}$$

be continuous and define $h : P_r^n(a) \to \mathbb{C}$ by the iterated line integral

$$\begin{aligned} h(z) &:= \left(\frac{1}{2\pi i}\right)^n \int\limits_{T_r^n(a)} \frac{f(w)}{(w-z)^1} dw \\ &:= \left(\frac{1}{2\pi i}\right)^n \int\limits_{|w_n - a_n| = r_n} \cdots \int\limits_{|w_1 - a_1| = r_1} \frac{f(w)\, dw_1 \cdots dw_n}{(w_n - z_n) \cdots (w_1 - z_1)}, \end{aligned}$$

where the notation $\int_{|w_j-a_j|=r_j}$ stands for the line integral over the circle around a_j of radius r_j. Recall that the integral is independent of a particular parametrization, so we may use this symbolic notation.

Lemma 1.3.1. The function h is partially holomorphic on $P_r^n(a)$.

Proof. Let $b \in P_r^n(a)$. Choose some $\delta > 0$, such that $|z_j - a_j| < r_j$ for all z satisfying $|z_j - b_j| < \delta$, $j = 1, \dots, n$. Then the function

$$\widehat{h_j} : B_\delta^1(b_j) \to \mathbb{C}, z_j \mapsto h(b_1, \dots, b_{j-1}, z_j, b_{j+1}, \dots, b_n)$$

is continuous. Choose a closed triangle $\Delta \subset B_\delta^1(b_j)$. The theorems of Fubini–Tonelli and Goursat yield that

$$\int_{\partial\Delta} \widehat{h_j}(z_j)\, dz_j = 0.$$

By Morera's theorem $\widehat{h_j}$ is holomorphic. $\square$

Applying Hartogs' theorem we see that h is actually holomorphic.

Notation 1.3.2. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We call α a *multiindex* and define

$$\begin{aligned} |\alpha| \ &: = \alpha_1 + \cdots + \alpha_n, \\ \alpha + 1 \ &: = (\alpha_1 + 1, \dots, \alpha_n + 1), \\ \alpha! \ &: = \alpha_1! \cdots \alpha_n!. \end{aligned}$$

For $z \in \mathbb{C}^n$ and a multiindex α we write

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

and we define the partial derivative operators

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

Theorem 1.3.3. *Let $U \subset \mathbb{C}^n$ be open, $a \in U$, $r \in \mathbb{R}_+^n$, such that the closed polycylinder $\overline{P_r^n(a)}$ is contained in U. Let $f : U \to \mathbb{C}$ be partially holomorphic. Then for all $\alpha \in \mathbb{N}^n$ and all $z \in P_r^n(a)$:*

1. **Cauchy's Integral Formula, CIF***:*

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{T_r^n(a)} \frac{f(w)}{(w-z)^{\alpha+1}} dw.$$

2. **Cauchy inequalities**:

$$|D^\alpha f(z)| \leq \frac{\alpha!}{r^\alpha} \left\| f|_{T_r^n(a)} \right\|_\infty .$$

Proof. 1. We prove the formula by induction on n. The case $n = 1$ is known. Chose $r' \in \mathbb{R}_+^n$, such that $r_j < r_j'$ for all j and such that $\overline{P_r^n(a)} \subset P_{r'}^n(a) \subset U$. Let $z = (z_1, \dots, z_n) \in P_{r'}^n(a)$. Then the function of one complex variable

$$\zeta \mapsto f(z_1, \dots, z_{n-1}, \zeta)$$

is holomorphic on the open disc defined by $|\zeta - a_n| < r_n'$. The one-dimensional version of CIF yields

$$f(z_1, \dots, z_n) = \frac{1}{2\pi i} \int\limits_{|w_n - a_n|} \frac{f(z_1, \dots, z_{n-1}, w_n)}{w_n - z_n} dw_n.$$

For fixed w_n the function

$$(z_1, \dots, z_{n-1}) \mapsto f(z_1, \dots, z_{n-1}, w_n)$$

is partially holomorphic on the polydisc defined by $|z_j| < r_j'$, $j = 1, \dots, n-1$. The rest of the proof now follows by the induction hypothesis and differentiation under the integral.

2. Standard estimate for line integrals over circles, as in one dimension. □

Exercise 1.3.4. Let $\mathbb{T}^n \subset \mathbb{C}^n$ be the unit polytorus and $\mu, \nu \in \mathbb{N}^n$. Compute the integral

$$\left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{\zeta^\mu}{\zeta^{\nu+1}} d\zeta.$$

1.4 $\mathcal{O}(U)$ as a topological space

This section studies convergence in the space $\mathcal{O}(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}^n$. To this end we introduce the compact-open topology on $\mathcal{O}(U)$, which turns $\mathcal{O}(U)$ into a Fréchet space. The major results of this section are Weierstrass' Convergence Theorem and Montel's Theorem. Readers with a profound knowledge of functional analysis may skip the part about locally convex spaces.

1.4.1 Locally convex spaces

We collect some basic facts about locally convex spaces, i.e., topological vector spaces whose topologies are defined by a family of seminorms.

Definition 1.4.1. Let k be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and V a k-vector space. A *seminorm* on V is a mapping $p : V \to [0, +\infty[$ with the following properties:

1. The mapping p is *positively homogenous*, i.e., $p(\alpha x) = |\alpha| \, p(x)$ for all $\alpha \in k$ and all $x \in V$.
2. The mapping p is *subadditive*, i.e., $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Example 1.4.2. Every norm $\|.\|$ on a vector space V is a seminorm.

Example 1.4.3. Let $\mathcal{R}[a, b]$ be the space of (Riemann-)integrable functions on the interval $[a, b]$ and let

$$p : \mathcal{R}[a, b] \to \mathbb{R}, f \mapsto \int_a^b |f(t)| \, dt.$$

Then p is a seminorm, but not a norm, because $p(f) = 0$ does not imply $f = 0$. For instance, take the function

$$f : [a, b] \to \mathbb{R}, t \mapsto \begin{cases} 1, & t = a \\ 0, & t \neq a \end{cases} .$$

Seminorms on vector spaces can be used to construct topologies on these spaces. In contrast to the topologies defined by norms, topologies defined by seminorms need not be Hausdorff.

Lemma 1.4.4. Let I be an index set, V a (real or complex) vector space and $(p_i)_{i \in I}$ a family of seminorms on V. For a finite subset $F \subset I$ and $\varepsilon > 0$ put

$$U_{F;\varepsilon} := \bigcap_{i \in F} \{x \in V \mid p_i(x) < \varepsilon\}$$

and define

$$\mathfrak{U} := \{U_{F;\varepsilon} | \ \varepsilon > 0, F \subset I, \# F < \infty\}.$$

Then the set

$$\mathcal{T} := \{O \subset V \mid \text{For all } x \in O \text{ there is some } U \in \mathfrak{U}, \text{ such that } x + O \subset U\}$$

defines a topology on V.

Remark 1.4.5. A vector space with a topology induced by a family of seminorms as above is called a *locally convex space*. The topology $\mathcal{T}$ turns V into a topological vector space, i.e., vector addition and multiplication with scalars are continuous mappings with respect to this topology. Locally convex spaces are studied in depth in functional analysis.

Proof. Trivially, $\mathcal{T}$ contains V and the empty set. Let J be an index set and $(O_j)_{j\in J}$ be an arbitrary family in $\mathcal{T}$. Put

$$O := \bigcup_{j\in J} O_j$$

and let $x \in O$. Then $x \in O_j$ for some index j. Since $O_j \in \mathcal{T}$, there is some $U \in \mathfrak{U}$ with

$$x + U \subset O_j \subset O,$$

hence, $O \in \mathcal{T}$. Now let $O_1, \dots, O_n$ be open sets, i.e., elements of $\mathcal{T}$. Put

$$O := \bigcap_{j=1}^{n} O_j$$

and let $x \in O$. Since every O_j is open, there are sets $U_j \in \mathfrak{U}$, such that

$$x + U_j \in O_j.$$

Every U_j is of the form

$$U_j = U_{F_j;\varepsilon_j}$$

with F_j a finite subset of I and $\varepsilon_j > 0$. Put

$$\varepsilon := \min_{j=1}^{n} \varepsilon_j, \;\; F := \bigcup_{j=1}^{n} F_j.$$

Then $\varepsilon > 0$ and F is also a finite subset of I. Further

$$U_{F;\varepsilon} \subset O,$$

thus,

$$x + U_{F;\varepsilon} \subset \bigcap_{j=1}^{n} (x + U_j) \subset O. \qquad \square$$

Exercise 1.4.6. Show that a locally convex space $(V, \mathcal{T})$ is a Hausdorff space if and only if the family $(p_i)_{i\in I}$ of seminorms separates points, i.e., if for all $x \in V$, $x \neq 0$ there is some index $i \in I$, such that $p_i(x) > 0$.

Exercise 1.4.7. Let V be a locally convex Hausdorff space whose topology $\mathcal{T}$ is induced by a countable family $(p_i)_{i\in\mathbb{N}}$ of seminorms. Show that the definition

$$d(x, y) := \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)} \text{ for all } x, y \in V$$

defines a metric on V and that the topology $\mathcal{T}_d$ induced by this metric coincides with $\mathcal{T}$.

Remark 1.4.8. Topological vector spaces whose topologies can be induced by a metric are called *metrizable*. If the topology can be induced by a norm they are called *normable*. Note that the metric of Exercise 1.4.7 is not induced by a norm, since

$$d(0, x) \leq 2$$

for all $x \in V$. If d were induced by a norm $\|.\|$, then for all $\lambda \in \mathbb{C}$ and all $x \in V$,

$$d(0, \lambda x) = \|\lambda x\| = |\lambda| \, \|x\| \, .$$

If $x \neq 0$ we can choose λ with $|\lambda| > \frac{2}{\|x\|}$.

In the abstract setting of general locally convex spaces the sets $U_{F;\varepsilon}$ play the role of the open ε-balls $B_\varepsilon^n(0)$ in $\mathbb{C}^n$ or $\mathbb{R}^n$, i.e., they form a basis of open neighbourhoods of zero. This analogy is mirrored in the definition of convergence.

Definition 1.4.9. Let $\left(V, (p_i)_{i\in I}\right)$ be a locally convex space, $\mathcal{T}$ the topology induced by the family $(p_i)_{i\in I}$ of seminorms and $\mathfrak{U}$ the basis of neighbourhoods of zero as defined in Lemma 1.4.4.

1. A sequence $(x_j)_{j\in\mathbb{N}}$ in V *converges to the limit* $x \in V$, if for every $U \in \mathfrak{U}$ there is some $N = N_U \in \mathbb{N}$, such that $x - x_j \in U$ for all $j \geq N$.
2. The sequence $(x_j)_{j\in\mathbb{N}}$ is called a *Cauchy sequence*, if for every $U \in \mathfrak{U}$ there is some $N = N_U \in \mathbb{N}$, such that $x_k - x_l \in U$ for all $k, l \geq N$.
3. The space V is called *sequentially complete* with respect to the topology $\mathcal{T}$, if every Cauchy sequence converges in V.

Remark 1.4.10. In functional analysis the general notion of completeness is defined by means of so-called Cauchy nets, which are a generalization of Cauchy sequences. The interested reader may refer to standard literature on functional analysis, e.g., [11]. For our purposes the notion of sequential completeness suffices.

Lemma 1.4.11. A sequence $(x_j)_{j\in\mathbb{N}}$ converges to $x \in V$ if and only if

$$\lim_{j\to\infty} p_i(x_j - x) = 0$$

for all $i \in I$.

Proof. The sequence $(x_j)_{j\in\mathbb{N}}$ converges to $x \in V$ if and only if for all $U \in \mathfrak{U}$ there is some $N_U \in \mathbb{N}$, such that

$$x - x_j \in U$$

for all $j \geq N_U$. By definition of $\mathfrak{U}$ this is equivalent to the condition that for every $\varepsilon > 0$ and every $i \in I$ there is some $N_i \in \mathbb{N}$, such that

$$p_i(x - x_j) < \varepsilon$$

for all $j \geq N_i$. $\square$

Exercise 1.4.12. Let $D \subset \mathbb{C}^n$ be a domain and $K \subset D$ be a compact subset. Define $p_K : \mathcal{C}(D) \to \mathbb{R}$ by

$$p_K(f) := \| f|_K \|_\infty .$$

1. Show that p_K defines a seminorm on $\mathcal{C}(D)$.
2. If K has interior points, then p_K defines a norm on the subspace $\mathcal{O}(D)$ of holomorphic functions.
3. Is $\mathcal{O}(D)$ complete with respect to p_K ?

Exercise 1.4.13. Let $\mathcal{R}[a,b], p$ be as in Example 1.4.3 and let $\mathcal{R}[a,b]$ be equipped with the locally convex topology induced by p. Please show:

1. Restricted to the subspace $\mathcal{C}[a,b] \subset \mathcal{R}[a,b]$ of continuous functions, p defines a norm on $\mathcal{C}[a,b]$.
2. Is $(\mathcal{C}[a,b], p)$ a Banach space?

1.4.2 The compact-open topology on $\mathcal{C}(U,E)$

The results about locally convex spaces and the notion of convergence in these spaces will be applied to the space $\mathcal{C}(U,E)$ of continuous mappings on an open set $U \subset \mathbb{C}^n$ with values in a Banach space $(E, \|.\|_E)$. The reason to consider Banach-space-valued mappings here is mainly that the results apply to scalar-valued and vector-valued mappings at the same time. It is well known that for a compact set K the space $\mathcal{C}(K,E)$ is a Banach space with respect to the norm

$$\|f\|_{E,\infty} := \sup_{x \in K} \|f(x)\|_E .$$

The important fact here is the completeness of $\mathcal{C}(K,E)$. This can be generalized to continuous functions defined on open sets by choosing a compact exhaustion of U.

Lemma 1.4.14. Let $U \subset \mathbb{C}^n$ be open. For every $j \in \mathbb{N}_+$ define

$$K_j := \left\{ z \in U \mid \|z\|_\infty \leq j, \operatorname{dist}_{\|.\|_\infty}(z, \mathbb{C}^n \setminus U) \geq \frac{1}{j} \right\}. \tag{1.6}$$

Then the following holds:

1. Every K_j is a compact set.
2. For all $j \geq 1$ we have $K_j \subset K^\circ_{j+1}$.
3. The set U is the union of the K_j :

$$U = \bigcup_{j=1}^\infty K_j.$$

4. If K is an arbitrary compact subset of U, then there is some $j_K \in \mathbb{N}_+$, such that $K \subset K_{j_K}$.

Proof. 1. Every K_j is the intersection of two closed sets and is contained in the ball of radius j. By the Heine–Borel theorem K_j is compact.

2. The set

$$\left\{ z \in U \mid \|z\|_2 < j+1, \operatorname{dist}_{\|.\|_2}(z, \mathbb{C}^n \setminus U) > \frac{1}{j+1} \right\}$$

is open, contains K_j and is contained in K_{j+1}, thus $K_j \subset K_{j+1}^\circ$.

3. If $U = \emptyset$ there is nothing to show. Let $U \neq \emptyset$. Since U is an open set, every $z \in U$ has a positive distance to the complement of U, i.e., there is some $j_1 \in \mathbb{N}_+$, such that

$$\operatorname{dist}_{\|.\|_\infty}(z, \mathbb{C}^n \setminus U) \geq \frac{1}{j_1}.$$

Also, there is some $j_2 \in \mathbb{N}_+$,such that

$$\|z\|_\infty \leq j_2.$$

Then $z \in K_{j_3}$, where $j_3 := \max\{j_1, j_2\}$. Since every K_j is a subset of U, 3. follows.

4. If K is a compact subset of U the sets $\left\{ K_j^\circ \middle| \, j \in \mathbb{N}_+ \right\}$ form an open cover of K. Compactness of K implies the existence of some $j_K \in \mathbb{N}_+$, such that

$$K \subset \bigcup_{j=1}^{j_K} K_j,$$

but since the $K_j \subset K_{j+1}$ for all j this implies $K \subset K_{j_K}$. □

Remark 1.4.15. A sequence of compact sets having the properties 2. and 3. from Lemma 1.4.14 is called a *compact exhaustion.* A locally compact Hausdorff space X having a compact exhaustion is called *countable at infinity.* Thus, in the language of general topology, Lemma 1.4.14 states that every open set in $\mathbb{C}^n$ is countable at infinity.

Using the normal exhaustion of Lemma 1.4.14 we define a topology $\mathcal{T}_{co}$ on the space $\mathcal{C}(U,E)$ by the family $\left(p_{K_j}\right)_{j\in\mathbb{N}_+}$ of seminorms

$$p_{K_j} : \mathcal{C}(U,E) \to \mathbb{R}, f \mapsto \sup_{x \in K_j} \|f(x)\|_E$$

(cf. Exercise 1.4.12). Note that this turns $\mathcal{C}(U,E)$ into a metric space as was shown in Exercise 1.4.7.

Definition 1.4.16. The topology $\mathcal{T}_{co}$ on $\mathcal{C}(U,E)$ is called the *compact-open topology* or the *topology of compact convergence.*

The name topology of compact convergence stems from the following result.

Proposition 1.4.17. A sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{C}(U,E)$ converges with respect to the topology $\mathcal{T}_{co}$ if and only if $(f_j)_{j\in\mathbb{N}}$ converges compactly on U.

Proof. Let $K \subset U$ be compact. By Lemma 1.4.14 there is an index j_K such that $K \subset K_{j_K}$. If $f_j \to f$ with respect to $\mathcal{T}_{co}$ then

$$\sup_{x\in K} \|f_j(x) - f(x)\|_E = p_K(f_j - f) \le p_{K_{j_K}}(f_j - f) \to 0,$$

hence the sequence $(f_j)_{j\in\mathbb{N}}$ converges uniformly on K. Conversely, if

$$\lim_{j\to\infty} \sup_{x\in K} \|f_j(x) - f(x)\|_E = 0$$

for every compact set $K \subset U$, then this holds in particular for the compact exhaustion (1.6). Lemma 1.4.11 implies that $f_j \to f$ with respect to $\mathcal{T}_{co}$. $\square$

It can be shown that the topology $\mathcal{T}_{co}$ does not depend upon the special choice of the compact exhaustion, i.e., if $\left(K'_j\right)_{j\in\mathbb{N}}$ is any compact exhaustion of U, then the topology induced by the seminorms $p_{K'_j}$ coincides with $\mathcal{T}_{co}$ defined above. We skip the proof of this, because we do not need the result in the following. The interested reader may refer to [3] for details.

Proposition 1.4.18. $(\mathcal{C}(U,E), \mathcal{T}_{co})$ is complete.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}(U,E)$ and $K \subset U$ be a compact set. Then $\left(f_j|_K\right)_{j\in\mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|.\|_{E,\infty}$. Since $\left(\mathcal{C}(K,E), \|.\|_{E,\infty}\right)$ is a Banach space there is an $f^K \in \mathcal{C}(K,E)$, which is the uniform limit of $\left(f_j|_K\right)_{j\in\mathbb{N}}$. Let $z \in U$ be an arbitrary point. There is an open neighbourhood U_z of z such that the closure $\overline{U_z}$ is compact and is contained in U. By the above argument we have $\lim_{j\to\infty} f_j|_{\overline{U_z}} = f^{\overline{U_z}}$. When we define

$$f : U \to \mathbb{C}, z \mapsto \lim_{j\to\infty} f_j(z)$$

we find that $f_j|_{\overline{U_z}}$ converges uniformly towards $f|_{\overline{U_z}} = f^{\overline{U_z}} \in \mathcal{C}\left(\overline{U_z}\right)$, hence, f is continuous at z. Since $z \in U$ is arbitrary we conclude $f \in \mathcal{C}(U,E)$. $\square$

Remark 1.4.19. Looking at the above proof we find that the only characteristic we needed from the open set U was the fact that every point $z \in U$ has a compact neighbourhood contained in U. Therefore Proposition 1.4.18 holds for the space $\mathcal{C}(X,E)$ of continuous mappings on every locally compact Hausdorff space X. A complete metrizable topological vector space is called a *Fréchet space*. Thus, $\mathcal{C}(X,E)$ is a Fréchet space. More precisely, since multiplication in $\mathcal{C}(X,E)$ is continuous, it is a Fréchet algebra.

We return now to the investigation of the space $\mathcal{O}(U,\mathbb{C}^m)$ of holomorphic mappings. As a subspace of $\mathcal{C}(U,\mathbb{C}^m)$ it inherits the topology of compact convergence from $\mathcal{C}(U,\mathbb{C}^m)$.

Theorem 1.4.20 (Weierstrass). *Let $U \subset \mathbb{C}^n$ be an open set. Then $\mathcal{O}(U, \mathbb{C}^m)$ is a closed subspace of $\mathcal{C}(U, \mathbb{C}^m)$ with respect to the topology of compact convergence. For every $\alpha \in \mathbb{N}^n$ the linear operator*

$$D^\alpha : \mathcal{O}(U, \mathbb{C}^m) \to \mathcal{O}(U, \mathbb{C}^m), f \mapsto D^\alpha f$$

is continuous. In case $m > 1$, i.e., $f = (f_1, \ldots, f_m)$ the operator D^α has to be applied to every component.

Proof. Since the assertion holds if and only if it holds in each component separately, we may without loss of generality assume $m = 1$. Since $\mathcal{C}(U)$ is metrizable it suffices to show that for every sequence $(f_j)_{j \in \mathbb{N}^n} \subset \mathcal{O}(U)$ converging compactly towards some $f \in \mathcal{C}(U)$ we have $f \in \mathcal{O}(U)$ and the sequence $(D^\alpha f_j)_{j \in \mathbb{N}}$ converges compactly towards $D^\alpha f$. As in one variable the major tool used here is Cauchy's integral formula. If $a \in U$ there is a polydisc $P_r^n(a)$, which is relatively compact in U. The sequence $\left(f_j|_{\overline{P_r^n(a)}} \right)_{j \in \mathbb{N}^n}$ converges uniformly towards $f|_{\overline{P_r^n(a)}}$. Let $T_r^n(a)$ be the polytorus contained in the boundary of $P_r^n(a)$. For all $w \in T_r^n(a)$ and all $z \in P_r^n(a)$,

$$\lim_{j \to \infty} \frac{f_j(w)}{(w-z)^1} = \frac{f(w)}{(w-z)^1}$$

uniformly in w. Cauchy's integral formula yields

$$\begin{aligned} f(z) &= \lim_{j \to \infty} f_j(z) \\ &= \lim_{j \to \infty} \left(\frac{1}{2\pi i} \right)^n \int\limits_{T_a^n(r)} \frac{f_j(w)}{(w-z)^1} dw \\ &= \left(\frac{1}{2\pi i} \right)^n \int\limits_{T_a^n(r)} \frac{f(w)}{(w-z)^1} dw \end{aligned}$$

and Lemma 1.3.1 says that f is holomorphic. Moreover, by CIF we have

$$D^\alpha f(z) = \alpha! \left(\frac{1}{2\pi i} \right)^n \int\limits_{T_a^n(r)} \frac{f(w)}{(w-z)^{\alpha+1}} dw.$$

Let $K \subset U$ be compact. There is a compact set $K' \subset U$ and a polyradius $r' \in \mathbb{R}_+^n$ such that for all $a \in K$ the polydisc $P_{r'}^n(a)$ is relatively compact in K'. By the Cauchy inequalities,

$$\begin{aligned} \left| \frac{D^\alpha f(a)}{\alpha!} - \frac{D^\alpha f_j(a)}{\alpha!} \right| &\leq \frac{1}{(r')^\alpha} \sup_{T_{r'}^n(a)} |f - f_j| \\ &\leq \frac{1}{(r')^\alpha} \sup_{K'} |f - f_j|. \end{aligned}$$

Since f_j converges compactly towards f and $a \in K$ is arbitrary we find that $D^\alpha f_j$ converges towards $D^\alpha f$ compactly on U, hence, D^α is continuous. $\square$

Exercise 1.4.21. Let $D := \left\{ (z, w) \in \mathbb{C}^2 \middle| \operatorname{Re} z > 0, \operatorname{Re} w > 0 \right\}$ and

$$B : D \to \mathbb{C}, (z, w) \mapsto \int_0^1 t^{w-1} (1-t)^{z-1} \, dt.$$

B is called *Euler's beta function*. Please show:

1. D is a convex domain.
2. B is holomorphic on D. (*Hint*: Construct a sequence $(B_j)_{j \geq 1} \subset \mathcal{O}(D)$ converging compactly to B.)
3. B satisfies the functional equation

$$B(1, w) = \frac{1}{w}, \; B(z+1, w) = \frac{z}{z+w} B(z, w)$$

   and the estimate

$$|B(z, w)| \leq B(\operatorname{Re} z, \operatorname{Re} w)$$

   on D.

Exercise 1.4.22. Let $D \subset \mathbb{C}^n$ be a bounded domain with boundary ∂D and

$$\mathcal{A}(D) := \left\{ f \in \mathcal{C}\left(\overline{D}\right) \mid f|_D \in \mathcal{O}(D) \right\}.$$

1. Show that $\mathcal{A}(D)$, equipped with the norm

$$\|f\|_\infty := \sup_{z \in \overline{D}} |f(z)|$$

   is a complex *Banach algebra*, i.e., a $\mathbb{C}-$ algebra, a Banach space and for all $f, g \in \mathcal{A}(D)$ the inequality

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$$

   holds.
2. The restriction mapping

$$\rho : \mathcal{A}(D) \to \mathcal{C}(\partial D), f \mapsto f|_{\partial D}$$

   is an isometric[1] homomorphism of complex algebras. Is it bijective?
3. The image $\rho(\mathcal{A}(D))$ is a closed subalgebra of $\mathcal{C}(\partial D)$ with respect to the topology induced by $\|.\|_\infty$ (i.e., the topology of uniform convergence) on $\mathcal{C}(\partial D)$. Is $\rho(\mathcal{A}(D))$ an ideal in $\mathcal{C}(\partial D)$?

[1] i.e., $\|f\|_\infty = \|\rho(f)\|_\infty$ for all $f \in \mathcal{A}(D)$.

1.4.3 The Theorems of Arzelà–Ascoli and Montel

We consider the question of compactness in the spaces $\mathcal{C}(U, E)$ and $\mathcal{O}(U, \mathbb{C}^m)$ leading to the multivariable version of Montel's Theorem. Recall that in one variable theory Montel's Theorem is an important tool in the proof of the Riemann Mapping Theorem. Ironically, in Chapter III we will use the multidimensional version of Montel's Theorem to give a proof that the Riemann Mapping Theorem does *not* hold in $\mathbb{C}^n$ if $n > 1$. The proof of Montel's Theorem uses the Arzelà–Ascoli Theorem, which we include in a rather general form, and which has applications outside complex analysis as well.

Definition 1.4.23. Let (X, d_X) be a metric space, $(E, \|.\|_E)$ a (real or complex) Banach space, $U \subset X$ an open subset and $\mathcal{F}$ a family of functions $f : U \to E$.

1. The family $\mathcal{F}$ is called *bounded* if for every compact subset $K \subset U$ there exists a constant $0 \leq M_K < \infty$ such that
$$\sup_{f\in\mathcal{F}} \sup_{x\in K} \|f(x)\|_E \leq M_K.$$
2. The family $\mathcal{F}$ is called *locally bounded* if for all $a \in U$ there exists an open neighbourhood $V = V(a)$ such that
$$\mathcal{F}|_V := \{ f|_A \mid f \in \mathcal{F}\}$$
is bounded.
3. The family $\mathcal{F}$ is called *equicontinuous* if for every $\varepsilon > 0$ there exists some $\delta = \delta_\varepsilon > 0$ such that
$$\|f(x) - f(y)\| < \varepsilon$$
for all $f \in \mathcal{F}$ and all $x, y \in U$ satisfying $d_X(x, y) < \delta$.
4. The family $\mathcal{F}$ is called *locally equicontinuous* if for all $a \in U$ there is an open neighbourhood $V = V(a)$ such that $\mathcal{F}|_V$ is equicontinuous.

Trivially, if $\mathcal{F}$ is equicontinuous then $\mathcal{F} \subset \mathcal{C}(X, E)$. The next proposition shows that for families of holomorphic functions the weaker condition of local boundedness already implies equicontinuity. This will be used in the proof of Montel's Theorem.

Proposition 1.4.24. Let $U \subset \mathbb{C}^n$ be open and $\mathcal{F} \subset \mathcal{O}(U, \mathbb{C}^m)$ be a locally bounded family. Then $\mathcal{F}$ is locally equicontinuous on U.

Proof. Since all norms in $\mathbb{C}^m$ are equivalent it suffices to consider the maximum norm $\|.\|_\infty$ in $\mathbb{C}^m, a \in U, r > 0, M > 0$ such that
$$\overline{P_r(a)} = \{z \in \mathbb{C}^n \mid \|z - a\|_\infty \leq r\} \subset U$$

and

$$\left\| f|_{\overline{P_r(a)}} \right\|_\infty \leq M$$

for all $f \in \mathcal{F}$. This is possible, because U is open, $\overline{P_r(a)}$ is compact and $\mathcal{F}$ is locally bounded. Put

$$K := \overline{P_{\frac{r}{2}}(a)}$$

and let $\varepsilon > 0, f \in \mathcal{F}$ and $x, y \in K$ be arbitrary. Applying the Mean Value Theorem in $\mathbb{R}^{2n}$ to f gives

$$\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty \sup_{z \in K} \|Df(z)\|,$$

where

$$\sup_{z \in K} \|Df(z)\| = \sup_{z \in K} \sup_{\|x\|_\infty = 1} \|Df(z)x\|_\infty.$$

Assume without loss of generality that

$$Df(z)x = \left(\sum_{k=1}^n \frac{\partial f_1(z)}{\partial z_k} x_k, \ldots, \sum_{k=1}^n \frac{\partial f_m(z)}{\partial z_k} x_k \right)$$

attains its maximum absolute value over K and the sphere $\|x\|_\infty = 1$ in the first component, i.e.,

$$\begin{aligned} \sup_{z \in K} \sup_{\|x\|_\infty = 1} \|Df(z)x\|_\infty &= \sup_{z \in K} \sup_{\|x\|_\infty = 1} \left| \sum_{k=1}^n \frac{\partial f_1(z)}{\partial z_k} x_k \right| \\ &\leq \sup_{z \in K} \sum_{k=1}^n \left| \frac{\partial f_1(z)}{\partial z_k} \right| \\ &\leq n \max_{k=1}^n \sup_{z \in K} \left| \frac{\partial f_1(z)}{\partial z_k} \right|. \end{aligned}$$

For $z \in K = \overline{P_{\frac{r}{2}}(a)}$ and $\zeta \in T_r^n(a)$ we have for all k,

$$|\zeta_k - z_k| \geq \frac{r}{2}.$$

This, together with the Cauchy Integral Formula, implies

$$\begin{aligned} \left| \frac{\partial f_1(z)}{\partial z_k} \right| &= \left| \left(\frac{1}{2\pi i} \right)^n \int_{T_r^n(a)} \frac{f_1(\zeta)}{(\zeta_k - z_k)^2} d\zeta \right| \\ &\leq 4r^{n-2} \sup_{\zeta \in T_r^n(a)} |f_1(\zeta)| \leq 4r^{n-2} M. \end{aligned}$$

Choosing
$$\delta := \frac{\varepsilon}{4r^{n-2}M}$$
gives that for all $x, y \in K$ with $\|x - y\|_\infty < \delta$ we have
$$\|f(x) - f(y)\|_\infty < \varepsilon,$$
hence, $\mathcal{F}$ is locally equicontinuous. $\square$

Lemma 1.4.25. Let X be a locally compact and connected metric space and $K \subsetneq X$ a proper compact subset. Then there is a compact set $H \subset X$ such that $K \subset H^\circ$ and $\operatorname{dist}(K, \partial H) > 0$.

Proof. X is locally compact. Therefore, every $x \in K$ has an open neighbourhood $U = U(x) \subset\subset X$. The set
$$\mathcal{U} := \{U(z) \mid x \in K\}$$
then forms an open cover of K containing a finite subcover
$$U_1(x_1), \ldots, U_l(x_l).$$
Since X is connected it contains no proper subsets that are both open and closed. This implies that
$$K \subset \bigcup_{j=1}^{l} U_j(x_j) \subsetneq \bigcup_{j=1}^{l} \overline{U_j(x_j)} =: H.$$
By construction H is compact and contains K in its interior. Since the boundary ∂H is closed we have
$$K \cap \partial H = \emptyset,$$
thus,
$$\operatorname{dist}(K, \partial H) > 0. \qquad \square$$

Lemma 1.4.26. Let X be a locally compact and connected metric space, $K \subsetneq X$ a compact subset, $A \subset K$ a dense subset, E a Banach space, $\mathcal{F} \subset \mathcal{C}(K, E)$ an equicontinuous family of continuous mappings and $(f_j)_{j \geq 1} \subset \mathcal{F}$ a sequence converging pointwise on A. Then $(f_j)_{j \geq 1}$ converges uniformly on K.

Proof. We only have to show that $(f_j)_{j \geq 1}$ is a uniform Cauchy sequence on K. Convergence then follows from completeness of E. Let $\varepsilon > 0$. From Lemma 1.4.25 we know that there is a compact set H containing K in its interior such that K has a positive distance to the boundary of H. Equicontinuity of $\mathcal{F}$ on K implies that we can find some $\delta = \delta_\varepsilon$ such that
$$0 < \delta \leq \frac{1}{2} \operatorname{dist}(K, \partial H)$$

and

$$\|f_j(x) - f_j(y)\| < \frac{\varepsilon}{3}$$

for all $x, y \in K$ satisfying $d(x,y) \leq \delta$ and all $j \geq 0$. The open δ-balls

$$\{B_\delta(x) \mid x \in K\}$$

form an open cover of K and because $\delta \leq \frac{1}{2}\operatorname{dist}(K, \partial H)$ we have.

$$K \subset \bigcup_{x \in K} B_\delta(x) \subset \bigcup_{x \in K} \overline{B_\delta(x)} \subset H.$$

This open cover contains a finite subcover and because A is dense in K we can choose finitely many points

$$x_1, \ldots, x_p \in A$$

such that

$$K \subset \bigcup_{j=1}^{p} \underbrace{\overline{B_\delta(x_j)}}_{=:K_j}.$$

$(f_j)_{j\geq 0}$ is pointwise a Cauchy sequence on A, hence, there is $N = N_\varepsilon \in \mathbb{N}$ such that

$$\|f_k(x_j) - f_l(x_j)\| < \frac{\varepsilon}{3}$$

for all $k, l \geq N$ and all $j = 1, \ldots, p$. If $x \in K$ there is an index $q \in \{1, \ldots, p\}$ such that $x \in K_q$, which implies $d(x, x_q) \leq \delta$. Now we can conclude that for all $k, l \geq N$ and all $x \in K$,

$$\begin{aligned} \|f_k(x) - f_l(x)\| &\leq \|f_k(x) - f_k(x_q)\| + \|f_k(x_q) - f_l(x_q)\| \\ &\quad + \|f_l(x_q) - f_l(x)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Theorem 1.4.27 (Arzelà–Ascoli). *Let X be a locally compact connected metric space, $K \subset X$ a compact subset containing a countable dense subset $A \subset K$, E a finite dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$, $\mathcal{F} \subset \mathcal{C}(K, E)$ a family of equicontinuous functions and $(f_j)_{j\geq 1} \subset \mathcal{F}$ a bounded sequence, i.e., there is a constant $C > 0$ such that*

$$\sup_{j\geq 1} \sup_{x \in K} \|f_j(x)\| \leq C.$$

Then $(f_j)_{j\geq 0}$ contains a subsequence converging uniformly on K.

Proof. Since A is countable, we may write $A = \{x_1, x_2, \dots\}$. Because of

$$\sup_{j\geq 1}\sup_{l\geq 1}\|f_j(x_l)\| \leq C$$

the sequence $(f_j(x_1))_{j\geq 1}$ is bounded in E. Since E is of finite dimension the Theorem of Bolzano–Weierstrass is valid in E. Hence, there is a convergent subsequence $(f_{p_k}(x_1))_{k\geq 1}$ of $(f_j(x_1))_{j\geq 1}$. Because of

$$\left\|\sup_{k\geq 1} f_{p_k}(x_2)\right\| \leq C,$$

by the same argument, we obtain a convergent subsequence

$$(f_{q_k}(x_2))_{k\geq 1}$$

of $(f_{p_k}(x_1))_{k\geq 1}$, where $(q_k)_{k\geq 1}$ is a subsequence of $(p_k)_{k\geq 1}$. An iteration of this process yields a series of subsequences of $(f_j)_{j\geq 1}$,

$$\begin{array}{ccccc}
f_{p_1} & f_{p_2} & f_{p_3} & \cdots & \text{convergent in } x_1 \\
f_{q_1} & f_{q_2} & f_{q_3} & \cdots & \text{convergent in } x_1, x_2 \\
f_{r_1} & f_{r_2} & f_{r_3} & \cdots & \text{convergent in } x_1, x_2, x_3 \\
\vdots & \vdots & \vdots & \ddots &
\end{array}$$

Here, the sequence in the $k-th$ row forms a subsequence of the sequence in the $(k-1)-th$ row. Hence, the diagonal sequence $f_{p_1}, f_{q_2}, f_{r_3}, \ldots$ converges pointwise on all of A. Lemma 1.4.26, however, states that the convergence is uniform on K. □

Corollary 1.4.28. *If $U \subset X$ is open and $(f_j)_{j\geq 1}$ is a locally bounded and locally equicontinuous sequence, then $(f_j)_{j\geq 1}$ contains a subsequence converging locally uniformly on U.*

Proof. Since X is locally compact every $a \in U$ possesses a relatively compact neighbourhood $V \subset\subset U$. Apply the Arzelà–Ascoli Theorem on the compact set $\overline{V}$. □

Remark 1.4.29. In the case $X = \mathbb{C}^n$ a suitable choice for the set A is the set of points in K with rational coordinates, i.e.,

$$A = K \cap (\mathbb{Q} + i\mathbb{Q})$$

We are now ready to consider compact sets in the space $\mathcal{O}(U, \mathbb{C}^m)$ of holomorphic mappings. Remember that in a metric space the notions of compactness and sequential compactness coincide. In the space $\mathcal{O}(U, \mathbb{C}^m)$, provided with the topology of compact convergence, a sequence (f_j) converges if and only if it converges uniformly on every compact subset K of D. The next proposition states that $\mathcal{O}(U, \mathbb{C}^m)$ has the Bolzano–Weierstrass property.

Proposition 1.4.30. Every locally bounded sequence $(f_j) \subset \mathcal{O}(U, \mathbb{C}^m)$ contains a convergent subsequence.

Proof. From Proposition 1.4.24 we know that every locally bounded sequence in $\mathcal{O}(U, \mathbb{C}^m)$ is locally equicontinuous, hence, it contains a subsequence converging locally uniformly on D by Corollary 1.4.28. □

Theorem 1.4.31 (Montel). *Let $U \subset \mathbb{C}^n$ be an open set and $\mathcal{F} \subset \mathcal{O}(U, \mathbb{C}^m)$ be a family of holomorphic mappings. Then the following are equivalent:*

1. *The family $\mathcal{F}$ is locally bounded.*
2. *The family $\mathcal{F}$ is relatively compact in $\mathcal{O}(U, \mathbb{C}^m)$.*

Proof. 1. ⇒ 2. Let $(f_j) \subset \overline{\mathcal{F}}$ be a sequence. Then (f_j) is locally bounded and from Proposition 1.4.30 we know that (f_j) contains a convergent subsequence. Thus, $\overline{\mathcal{F}}$ is sequentially compact, hence, compact.

2. ⇒ 1. Let $K \subset U$ be a compact set. The function

$$\varphi : \overline{\mathcal{F}} \to \mathbb{R}_+, f \mapsto \| f|_K \|_\infty$$

is continuous and since $\overline{\mathcal{F}}$ is compact, $\varphi\left(\overline{\mathcal{F}}\right)$ is bounded. Hence,

$$\sup_{f \in \mathcal{F}} \| f|_K \|_\infty < \infty.$$

□

Corollary 1.4.32. *A subset $\mathcal{F} \subset \mathcal{O}(U, \mathbb{C}^m)$ is compact if and only if $\mathcal{F}$ is closed and bounded.*

Proof. If $\mathcal{F}$ is compact $\mathcal{F}$ is closed and bounded. This holds in every metric space. If $\mathcal{F}$ is closed and bounded, Montel's Theorem states that $\mathcal{F}$ is compact. □

Remark 1.4.33. Locally convex spaces in which the compact sets are exactly those which are closed and bounded are called *Montel spaces*. Note the analogy to the characterization of compact sets in $\mathbb{R}^n$ or $\mathbb{C}^n$ given by the Heine–Borel Theorem.

Example 1.4.34. Montel's Theorem does not hold for real-analytic functions. For instance, the family $\mathcal{F}$ consisting of the functions

$$f_j : \mathbb{R} \to \mathbb{R}, x \mapsto \cos(jx)$$

is bounded in $\mathcal{C}(\mathbb{R})$, but not relatively compact. This can be seen as follows. Since

$$\|f_j\|_\infty \leq 1 \text{ for all } j \in \mathbb{N},$$

the family $\mathcal{F}$ is bounded. If $\overline{\mathcal{F}}$ were compact, every sequence in $\overline{\mathcal{F}}$ would contain a convergent subsequence. However, on the compact set $K := \{\pi\}$ we have

$$\cos(k\pi) - \cos(l\pi) = \begin{cases} -2, & \text{if } k \equiv 1 \bmod 2, l \equiv 0 \bmod 2 \\ 0 & \text{if } k \equiv 0 \bmod 2, l \equiv 0 \bmod 2 \\ 2 & \text{if } k \equiv 0 \bmod 2, l \equiv 1 \bmod 2 \end{cases},$$

thus $(f_j|_K)_{j\in\mathbb{N}}$ contains no Cauchy subsequence.

Definition 1.4.35. Let $D \subset \mathbb{C}^n$ be a domain. A subset $A \subset D$ is called a *set of uniqueness* if for every $f \in \mathcal{O}(D)$ the condition $f|_A = 0$ implies that $f = 0$.

Example 1.4.36. If $D \subset \mathbb{C}^n$ is a domain, then every open subset $A \subset D$ is a set of uniqueness by the Identity Theorem. Also, every dense subset of D is a set of uniqueness, because $\mathcal{O}(D) \subset \mathcal{C}(D)$.

Exercise 1.4.37. Prove *Vitali's Theorem*: Let $D \subset \mathbb{C}^n$ be a domain, $A \subset D$ a set of uniqueness and $(f_k)_{k\in\mathbb{N}}$ a bounded sequence in $\mathcal{O}(D)$, which converges pointwise on A. Then the sequence $(f_k)_{k\in\mathbb{N}}$ converges towards a holomorphic function $f \in \mathcal{O}(D)$. *Hint*: Apply Montel's Theorem to the set

$$\overline{\{\, f_k |\, k \in \mathbb{N}\}}.$$

1.5 Power series and Taylor series

In one dimension Cauchy's integral formula is used to prove that every holomorphic function can be expanded into its Taylor series everywhere it is defined. The same holds in several variables and the proof is also analogous. Before we come to this we introduce general power series of n complex variables. Since $\mathbb{N}^n$ does not carry a natural ordering, we start by recalling some well-known facts about summability in Banach spaces.

1.5.1 Summable families in Banach spaces

Let $(E, \|.\|)$ be a Banach space and I be a countable set.

Definition 1.5.1. Let $(a_\alpha)_{\alpha\in I}$ be a countable family in E. $(a_\alpha)_{\alpha\in I}$ is called *absolutely summable*, if there exists a bijection

$$\tau : \mathbb{N} \to I,$$

such that the series $\sum_{n\geq 0} \left\|a_{\tau(n)}\right\|$ converges.

The following two lemmas are well-known (see, e.g., [1] Ch. 5.5)

Lemma 1.5.2. Let $(a_\alpha)_{\alpha\in I}$ be absolutely summable in the Banach space E. Then the following holds:

1. $\sum_{n\geq 0} a_{\varphi(n)}$ converges for every bijection $\varphi : \mathbb{N} \to I$ to the same limit.
2. If $I = \mathbb{N}^n$, the limit can be computed from the homogenous expansion

$$\sum_{\alpha\in\mathbb{N}^n} a_\alpha = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} a_\alpha \right).$$

Remark 1.5.3. Thanks to this lemma the expression

$$\sum_{\alpha \in I} a_\alpha := \sum_{n \geq 0} a_{\tau(n)}$$

is well defined. We say that $\sum_{\alpha \in I} a_\alpha$ converges absolutely.

Lemma 1.5.4. The following statements are equivalent.

1. The family $(a_\alpha)_{\alpha \in I}$ is absolutely summable.
2. The series $\sum_{\alpha \in I} a_\alpha$ converges absolutely.
3. There exists a constant $C \geq 0$, such that $\sum_{\alpha \in F} \|a_\alpha\| < C$ for all finite subsets $F \subset I$.

1.5.2 Power series

Definition 1.5.5. Let $(c_\alpha)_{\alpha \in \mathbb{N}^n}$ be a family of complex numbers. The expression

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha (z-a)^\alpha = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha_1 \ldots \alpha_n} (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}$$

is called a *power series* in n complex variables $z_1, \ldots, z_n$ centered at $a \in \mathbb{C}^n$.

For simpler notation we will mainly consider $a = 0$.As in one dimension, a power series always converges for $z = a$, so it makes sense to ask for the set of points in $\mathbb{C}^n$, on which a power series converges.

Definition 1.5.6. Let $\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$ be a power series. The interior of the set of points on which the series converges is called the *domain of convergence* of the series.

Example 1.5.7. The *geometric series*

$$\sum_{\alpha \in \mathbb{N}^n} z^\alpha$$

converges on the unit polycylinder. This can be seen as follows. Let $z \in P_1^n(0)$ and F a finite subset of $\mathbb{N}^n$. Then there is some $q \in [0,1[$, such that $|z_j| \leq q$ for all $j = 1, \ldots, n$. Hence,

$$\sum_{\alpha \in F} |z^\alpha| = \sum_{\alpha \in F} |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \leq \sum_{\alpha \in F} q^{\alpha_1 + \ldots \alpha_n} \leq \sum_{j=0}^{\infty} q^j = \frac{1}{1-q}.$$

Moreover, the limit can be computed from the homogenous expansion

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}^n} z^\alpha &= \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k} z^\alpha\right) \\
&= \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} z_1^{\alpha_1}\cdots z_n^{\alpha_n} \\
&= \left(\sum_{k_1=0}^{\infty} z_1^{\alpha_n}\right)\cdots\left(\sum_{k_n=0}^{\infty} z_n^{\alpha_n}\right) = \prod_{j=1}^{n}\frac{1}{1-z_j}.
\end{aligned}$$

As in one variable, this example can be generalized, which finally leads to the Taylor expansion formula for holomorphic functions in n variables.

Lemma 1.5.8 (Abel). Let $(c_\alpha)_{\alpha\in\mathbb{N}^n}$ be a family of complex numbers, $t\in\mathbb{R}_+^n$, such that the family $(c_\alpha t^\alpha)_{\alpha\in\mathbb{N}^n}$ is bounded and $r\in\mathbb{R}_+^n$, such that $r_k < t_k$ for all $k=1,\dots,n$. Then the power series

$$\sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$$

converges absolutely and uniformly on the closed polycylinder $\overline{P_r^n(0)}$.

Proof. Put $s := \sup_{\alpha\in\mathbb{N}^n}|c_\alpha t_1^{\alpha_1}\cdots t_n^{\alpha_n}|$. Since $(c_\alpha t^\alpha)_{\alpha\in\mathbb{N}^n}$ is bounded, we have $0\le s<\infty$. Put

$$q := \max_{k=1}^{n}\frac{r_k}{t_k}.$$

For $z\in\overline{P_r^n(0)}$ we have

$$|c_\alpha z^\alpha| \le |c_\alpha t^\alpha|\left|\frac{r^\alpha}{t^\alpha}\right| \le sq^{|\alpha|},$$

hence,

$$\sum_{\alpha\in\mathbb{N}^n}|c_\alpha z^\alpha| \le \sum_{\alpha\in\mathbb{N}^n} sq^{|\alpha|} = \frac{s}{(1-q)^n} < \infty.$$ □

Corollary 1.5.9 (Taylor expansion). *Let $U\subset\mathbb{C}^n$ be an open set, $a\in U$, $r\in\mathbb{R}_+^n$, such that $\overline{P_r^n(a)}\subset U$ and $f\in\mathcal{O}(U)$. Then the following holds:*

1. *The power series*

$$\sum_{\alpha\in\mathbb{N}^n}\frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha$$

converges compactly on $P_r^n(a)$.

2. *The function*

$$j_f : P_r^n(a) \to \mathbb{C}, z \mapsto \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(a)}{\alpha!} (z-a)^\alpha$$

is holomorphic.

3. *For all $z \in P_r^n(a)$ the equation*

$$f(z) = j_f(z)$$

holds.

Proof. 1. Let $K \subset P_r^n(a)$ be a compact subset and $T_r^n(a)$ be the compact polytorus. By the Cauchy inequalities

$$\left\| \frac{D^\alpha f(a)}{\alpha!} \right\| \leq \frac{\alpha!}{r^\alpha} \left\| f|_{T_r^n(a)} \right\|_\infty .$$

Since K is compact it is bounded. Hence, for $z \in K$ the family

$$\left(\frac{D^\alpha f(a)}{\alpha!} (z-a)^\alpha \right)_{\alpha \in \mathbb{N}^n}$$

is bounded. The proposition now follows from Abel's lemma.

2. It follows from 1. that

$$j_f(z) = \lim_{k \to \infty} \sum_{|\alpha| \leq k} \frac{D^\alpha f(a)}{\alpha!} (z-a)^\alpha$$

compactly on $P_r(a)$. Holomorphy of polynomials and Weierstrass' theorem imply that j_f is holomorphic on $P_r^n(a)$.

3. Let $z \in P_r^n(a)$ be fixed. Choose $\rho \in \mathbb{R}_+^n$ such that $\rho_j < r_j, j = 1, \ldots, n$ and

$$z \in \overline{P_\rho^n(a)} \subset P_r^n(a).$$

If $w \in T_r^n(a)$ then

$$\frac{1}{w-z} = \frac{1}{w-a} \sum_{\alpha \in \mathbb{N}^n} \left(\frac{z-a}{w-a} \right)^\alpha$$

uniformly in $w \in T_r^n(a)$. Cauchy's integral formula yields

$$\begin{aligned} f(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{T_r^n(a)} \frac{f(w)}{w-z} dw \\ &= \left(\frac{1}{2\pi i}\right)^n \sum_{\alpha \in \mathbb{N}^n} (z-a)^\alpha \int_{T_r^n(a)} \frac{f(w)}{(w-a)^{\alpha+1}} dw \\ &= j_f(z). \end{aligned}$$

□

Exercise 1.5.10. Let $f \in \mathcal{O}(\mathbb{C}^n)$ and $\nu \in \mathbb{N}^n$ be a fixed multiindex. Show that if there is a constant $C > 0$ such that

$$|f(z)| \leq C |z^\nu|$$

for all $z \in \mathbb{C}^n$, then f is a polynomial of degree at most $|\nu|$. Which result follows for $|\nu| = 0$?

Exercise 1.5.11. Show that for all $\nu \in \mathbb{N}^n$ the domains of convergence of a power series $\sum_\alpha c_\alpha z^\alpha$ and its derived series $\sum_\alpha c_\alpha (D^\nu z^\alpha)$ coincide.

Exercise 1.5.12. Let $f(z) := \sum_\alpha a_\alpha z^\alpha$ and $g(z) := \sum_\alpha b_\alpha z^\alpha$ be defined by convergent power series and let $f(z) = g(z)$ for all z in some open set $U \subset \mathbb{C}^n$. Then $\alpha_\alpha = b_\alpha$ for all $\alpha \in \mathbb{N}^n$.

Exercise 1.5.13. Let $w \in \mathbb{C}^n$ and let $(c_\alpha)_{\alpha \in \mathbb{N}^n}$ be a family in $\mathbb{C}$ such that $(c_\alpha w^\alpha)_{\alpha \in \mathbb{N}^n}$ is unbounded in $\mathbb{C}$. Show that the power series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

diverges for every $z \in \mathbb{C}^n$, which satisfies $|z_j| > |w_j|$, $j = 1, \ldots, n$.

Exercise 1.5.14. Let D be a domain and $f \in \mathcal{O}(D)$ such that the function

$$\|f\|_2 : D \to \mathbb{R}, z \mapsto \|f(z)\|_2$$

is constant. Show that f must already be constant (*Hint*: Use Exercise 1.2.27).

1.5.3 Reinhardt domains and Laurent expansion

This section studies ways to expand a holomorphic function f into a series other than a power series. It turns out that the geometry of the underlying domain on which f is defined plays a crucial role. The results of this chapter will lead to a phenomenon alien to the theory in one variable, i.e., in more than one variable there are domains $D \subset \mathbb{C}^n$ such that *every* function $f \in \mathcal{O}(D)$ extends holomorphically to a strictly larger domain. Extension phenomena will be examined in depth later in this book. We start by taking a closer look at the geometry of the domain of convergence of a power series. This will lead to the definition of Reinhardt domains.

For the study of subsets of $\mathbb{C}^n$, sometimes a picture of the so-called absolute space is useful. We regard the mapping

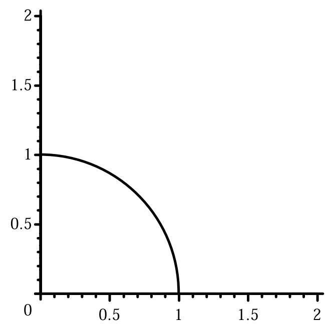
$$\tau : \mathbb{C}^n \to \mathbb{R}^n, (z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|). \tag{1.7}$$

If $U \subset \mathbb{C}^n$ the set $\tau(U)$ is called the *absolute space* of U.

Example 1.5.15. Let B be the unit ball in $\mathbb{C}^2$ with respect to the Euclidian norm $\|.\|_2$. The absolute space of B is the set

$$\tau(B) = \left\{ (|z_1|, |z_2|) \in \mathbb{R}^2 \;\middle|\; |z_1|^2 + |z_2|^2 < 1 \right\}$$

which is the planar domain enclosed by the two axes and the quarter circle in the positive quadrant in $\mathbb{R}^2$.



Absolute space $\tau(B)$

Note that every point in absolute space in this picture represents a 2-torus.

Exercise 1.5.16. Determine the domains D_1, D_2 of convergence of the power series

$$f_1(z,w) := \sum_{k,l \geq 0} z^k w^l$$

and

$$f_2(z,w) := \sum_{k,l \geq 0} \frac{k^2}{l!} z^k w^l$$

and sketch the absolute spaces $\tau(D_1), \tau(D_2)$.

If a power series $\sum_\alpha c_\alpha z^\alpha$ converges absolutely in $z \in \mathbb{C}^n$, then by Abel's Lemma also the power series

$$\sum_\alpha c_\alpha (\zeta_1 z_1)^{\alpha_1} \cdots (\zeta_n z_n)^{\alpha_n}$$

converges for all $\zeta_1, \ldots, \zeta_n \in \mathbb{T}^1$. Let

$$\mathbb{T}^n = \underbrace{\mathbb{T}^1 \times \cdots \times \mathbb{T}^1}_{n \text{ times}} = \left\{ \zeta \in \mathbb{C}^n \mid |\zeta_j| = 1, j = 1, \ldots, n \right\}$$

be the unit polytorus in $\mathbb{C}^n$. $\mathbb{T}^n$ carries an abelian group structure defined by componentwise multiplication

$$\zeta \cdot \xi := (\zeta_1 \xi_1, \ldots, \zeta_n \xi_n).$$

The neutral element is the vector $(1, \ldots, 1) \in \mathbb{T}^n$.

Definition 1.5.17. Let G be a group with neutral element e and X an arbitrary set.

1. G is said to *act on X from the left* if there is a mapping
$$\mu : G \times X \to X$$
with the properties
$$\mu(e, x) = x \text{ for all } x \in X$$
and
$$\mu(ab, x) = \mu(a, \mu(b, x)) \text{ for all } a, b \in G \text{ and all } x \in X.$$
Actions from the right are defined in the obvious way.

2. Let E be a vector space and $\mathcal{L}(E)$ be the vector space of linear endomorphisms of E. A homomorphism
$$\mu : G \to \mathcal{L}(E)$$
is called a *representation* of G on E.

Example 1.5.18. Representations of groups always exist, since the mapping defined by
$$\mu(g) := \mathrm{id}_E \text{ for all } g \in G,$$
where id_E denotes the identity mapping of E, is a representation, called the *trivial representation.*

Remark 1.5.19. The concepts of group action and group representation are equivalent in the following way. If $\mu : G \times E \to E$ determines a left group action on a vector space E, which has the additional property that μ is linear in the second argument, we can define a representation of G on E by
$$G \to \mathcal{L}(E), g \mapsto \mu_g,$$
where $\mu_g(x) := \mu(g, x)$ for all $x \in E$. Vice versa, if $\widetilde{\mu} : G \to \mathcal{L}(E)$ is a representation of G on E, then G acts on E from the left by
$$\mu : G \times E \to E, (g, x) \mapsto \widetilde{\mu}(g)(x).$$

If $D \subset \mathbb{C}^n$ is the domain of convergence of some power series it is easy to see that $\mathbb{T}^n$ acts on D by componentwise multiplication
$$\mu : \mathbb{T}^n \times D \to D, (\zeta, z) \mapsto (\zeta_1 z_1, \ldots, \zeta_n z_n). \tag{1.8}$$
Furthermore, for every $\zeta \in \mathbb{T}^n$ the mapping
$$\mu_\zeta : D \to D, z \mapsto \mu(\zeta, z)$$
is holomorphic. Note that $\mathbb{T}^n$ does not act on arbitrary subsets of $\mathbb{C}^n$. We use the following notation. If $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ we write
$$e^{i\theta} \cdot z := \left(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n\right).$$

Example 1.5.20. Let $D \subset \mathbb{C}$ be a domain. If $\mathbb{T}^1$ acts on D, then D is either the whole complex plane or a disc centered at zero or an annulus centered at zero.

Proof. If $\mathbb{T}^{\ 1}$acts on D and $z \in D$, then D also contains the circle

$$C_{|z|} := \left\{ |z|\, e^{it} \mid 0 \leq t \leq 2\pi \right\}.$$

Because $z \in C_{|z|} \subset D$ for all $z \in D$ we have

$$D = \bigcup_{z \in D} C_{|z|},$$

which is either the whole plane or a disc centered at zero or an annulus centered at zero. □

Definition 1.5.21. A subset $D \subset \mathbb{C}^n$ is called *polycircular* if $\mathbb{T}^n$ acts on D by componentwise multiplication. If D is a domain then D is called a *Reinhardt domain.*

With the above definition we can say that the domains of convergence of power series are Reinhardt domains. Note that the property of being a Reinhardt domain is not invariant under biholomorphic equivalence. For instance, the Riemann Mapping Theorem in one variable says that every simply connected domain D properly contained in $\mathbb{C}$ is biholomorphically equivalent to the open unit disk, which is a Reinhardt domain, even if D isn't.

Example 1.5.22. Let $p \in \mathbb{N} \cup \{\infty\}$ and $r > 0$. Then the ball of radius r with respect to $\|.\|_p$ centered at zero

$$B_r^n(0) = \left\{ z \in \mathbb{C}^n \;\middle|\; \|z\|_p < R \right\}$$

is a Reinhardt domain, because for all $\zeta \in \mathbb{T}^n$ and $p \in \mathbb{N}$,

$$\|\zeta \cdot z\|_p = \left(\sum_{j=1}^{n} \left|\zeta_j z_j\right|^p \right)^{\frac{1}{p}} = \left(\sum_{j=1}^{n} |z_j|^p \right)^{\frac{1}{p}} = \|z\|_p .$$

The case $p = \infty$ follows from Exercise 1.1.5.

Exercise 1.5.23. Let $D \subset \mathbb{C}^n$ be a domain and τ be the mapping (1.7) into absolute space. Show that D is a Reinhardt domain if and only if $\tau^{-1}(\tau(D)) = D$.

Definition 1.5.24. For $j = 1, \ldots, n$ let $0 \leq r_j < R_j \leq +\infty$. Let $a \in \mathbb{C}^n$. The set

$$A_{r,R}^n(a) := \{\, z \in \mathbb{C}^n \mid r_j < |z_j - a_j| < R_j,\ j = 1, \ldots, n\}$$

is called a *polyannulus* with polyradii r and R centered at a.

A polyannulus is the Cartesian product $A_1 \times \cdots \times A_n$ of one-dimensional annuli. If $a = 0$ then a polyannulus is a Reinhardt domain.

Exercise 1.5.25. Let D be a Reinhardt domain. Show that for every $a \in D$ there are polyradii $r < R$ such that

$$a \in A^n_{r,R}(0) \subset D.$$

In particular, every Reinhardt domain is the union of polyannuli.

Theorem 1.5.26 (Laurent expansion). *Let $A := A^n_{r,R}(a)$ be a polyannulus centered at $a \in \mathbb{C}^n$ and $f \in \mathcal{O}(A)$. Let $r_j < \rho_j < R_j, .j = 1, \ldots, n$. Then f has a* Laurent expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha (z-a)^\alpha$$

converging compactly on A. The coefficients c_α are given by

$$c_\alpha = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|w_n - a_n| = \rho_n} \cdots \int\limits_{|w_1 - a_1| = \rho_1} \frac{f(w)}{(w-a)^{\alpha+1}} dw.$$

Proof. We prove the theorem by induction on n. The case $n = 1$ is known. Suppose $n \geq 2$. For $z \in \mathbb{C}^n$ we write

$$z = (z_1, z')$$

where $z' = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$. For fixed w_1 with $|w_1 - a_1| = \rho_1$ the function $f(w_1, .)$ is holomorphic on $A_2 \times \cdots \times A_n$. By induction hypothesis we have

$$\begin{aligned} f(w_1, z') &= \sum_{\alpha' \in \mathbb{Z}^{n-1}} (z' - a')^{\alpha'} \left(\frac{1}{2\pi i}\right)^{n-1} \\ &\quad \times \int\limits_{|w_n - a_n| = \rho_n} \cdots \int\limits_{|w_2 - a_2| = \rho_2} \frac{f(w_1, w')}{(w' - a')^{\alpha'+1}} dw'. \end{aligned}$$

Also, for fixed z' we have the one-dimensional Laurent expansion

$$f(z_1, z') = \sum_{\alpha_1 \in \mathbb{Z}} (z_1 - a_1)^{\alpha_1} \frac{1}{2\pi i} \int\limits_{|w_1 - a_1| = \rho_1} \frac{f(w_1, z')}{(w_1 - a_1)^{\alpha_1 + 1}} dw_1.$$

Combining these two expansions yields that for all z satisfying

$$r_j < |z_j - a_j| < \rho_j,$$

for all j the equation

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} \left(\frac{1}{2\pi i}\right)^n \int\limits_{|w_n - a_n| = \rho_n} \cdots \int\limits_{|w_1 - a_1| = \rho_1} \frac{f(w)}{(w-a)^{\alpha+1}} (z-a)^\alpha dw$$

holds pointwise on A. To prove the compact convergence let $K \subset A$ be a compact set. If we put

$$\rho_j^- := \min_{z \in K} |z_j - a_j|, \ \rho_j^+ := \max_{z \in K} |z_j - a_j|$$

and choose r_j^-, R_j^- such that

$$r_j < r_j^- < \rho_j^- \leq \rho_j^+ < R_j^- < R_j,$$

we conclude from Cauchy's integral theorem (homotopy version) that the coefficients c_α satisfy

$$c_\alpha = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|w_n - a_n| = r_n^{sgn\alpha_n}} \cdots \int\limits_{|w_1 - a_1| = r_1^{sgn\alpha_1}} \frac{f(w)}{(w-a)^{\alpha+1}} dw,$$

where

$$r_j^{sgn\alpha_j} := \begin{cases} r_j^+, & \text{if } \alpha_j \geq 0 \\ r_j^-, & \text{if } \alpha_j < 0 \end{cases}.$$

Then for all $z \in K$ and all $\alpha \in \mathbb{Z}^n$ we obtain from the standard estimate for line integrals over circles

$$|c_\alpha (z-a)^\alpha| \leq \prod_{\substack{j=1 \\ \alpha_j \geq 0}}^n \left(\frac{\rho_j^+}{R_j^-}\right)^{\alpha_j} \prod_{\substack{j=1 \\ \alpha_j < 0}}^n \left(\frac{r_j^-}{\rho_j^-}\right)^{|\alpha_j|} \sup_{w \in \overline{A^n_{r^-, R^-}(a)}} |f(w)|.$$

With

$$q := \max_{j=1}^n \left\{\frac{\rho_j^+}{R_j^-}, \frac{r_j^-}{\rho_j^-}\right\}$$

and

$$M_K := \sup_{w \in \overline{A^n_{r^-, R^-}(a)}} |f(w)|$$

we have

$$\sum_{\alpha \in \mathbb{Z}^n} |c_\alpha (z-a)^\alpha| \leq M_K \sum_{\alpha \in \mathbb{Z}^n} q^{|\alpha|} = \frac{M_K}{(1-q)^n} < \infty$$

on K. □

With Exercise 1.5.25 in mind we can say that Theorem 1.5.26 holds locally for every function holomorphic on a Reinhardt domain. We use the Laurent expansion to give another series expansion for holomorphic functions on Reinhardt domains.

Theorem 1.5.27. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and $f \in \mathcal{O}(D)$. For every $\alpha \in \mathbb{Z}^n$ define*

$$f_\alpha : D \to \mathbb{C}, \ z \mapsto \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{f(\zeta \cdot z)}{\zeta^{\alpha+1}} d\zeta, \tag{1.9}$$

where $\zeta \cdot z = (\zeta_1 z_1, \ldots, \zeta_n z_n)$. *Then* $f_\alpha \in \mathcal{O}(D)$ *and the series*

$$\sum_{\alpha \in \mathbb{Z}^n} f_\alpha(z)$$

converges compactly on D *towards* f.

Proof. Since D is a Reinhardt domain for every $z \in D$, the set

$$X_z := \{\zeta \in \mathbb{C}^n \mid \zeta \cdot z \in D\}$$

is polycircular and contains the unit polytorus $\mathbb{T}^n$. It is also open in $\mathbb{C}^n$, because the mapping

$$\zeta \mapsto \zeta \cdot z$$

is continuous. Thus, there exists an open polyannulus A_z centered at zero such that

$$\mathbb{T}^n \subset A_z \subset X_z.$$

Define

$$g_z : A_z \to \mathbb{C}, \zeta \mapsto f(\zeta \cdot z).$$

Then g_z is holomorphic on A_z. From Theorem 1.5.26 we obtain the Laurent expansion at zero

$$g_z(\zeta) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha(z) \zeta^\alpha,$$

with

$$\begin{aligned} c_\alpha(z) &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{g_z(\zeta)}{\zeta^{\alpha+1}} d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{f(\zeta \cdot z)}{\zeta^{\alpha+1}} d\zeta \\ &= f_\alpha(z). \end{aligned}$$

Choosing $\zeta = (1, \ldots, 1)$ we find that

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha(z)$$

pointwise on D. Let $K \subset D$ be compact. Then also

$$\mathbb{T}^n \cdot K := \{\zeta \cdot z \mid \zeta \in \mathbb{T}^n, z \in K\}$$

is compact by continuity of the action and is contained in D. This implies the existence of some $R > 1$ such that

$$(R-1) \sup_{z \in K} \|z\|_\infty < \delta := \operatorname{dist}_{\|.\|_\infty}(\mathbb{T}^n \cdot K, \partial D).$$

We claim that for all $z \in K$ the compact annulus

$$A := \left\{ \zeta \in \mathbb{C}^n \mid \frac{1}{R} \leq \left|\zeta_j\right| \leq R, j = 1, \ldots, n \right\}$$

is contained in A_z. This can be seen as follows. If $\frac{1}{R} < \left|\zeta_j\right| < R$, then

$$\left|\zeta_j\right| - 1 \leq R - 1,\ 1 - \left|\zeta_j\right| \leq 1 - \frac{1}{R} = \frac{R-1}{R} < R - 1,$$

hence, $\left|\left|\zeta_j\right| - 1\right| \leq R - 1$. If $z \in K$, then for all $j = 1, \ldots, n$,

$$\begin{aligned} \left|\zeta_j z_j - \frac{\zeta_j}{\left|\zeta_j\right|} z_j\right| &= \left|\left(\left|\zeta_j\right| - 1\right) \frac{\zeta_j}{\left|\zeta_j\right|} z_j\right| \\ &= \left|\left|\zeta_j\right| - 1\right| \left|z_j\right| \\ &\leq (R-1) \left\|z\right\|_\infty \\ &< \delta, \end{aligned}$$

thus $\zeta \cdot z \in D$. This implies $\zeta \in A_z$. By the homotopy version of Cauchy's Integral Theorem we have

$$\begin{aligned} f_\alpha (z) &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{f(\zeta \cdot z)}{\zeta^{\alpha+1}} d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\left|\zeta_n\right| = R^{sgn \alpha_n}} \cdots \int\limits_{\left|\zeta_1\right| = R^{sgn \alpha_1}} \frac{f(\zeta \cdot z)}{\zeta^{\alpha+1}}. \end{aligned}$$

Put

$$M_K := \max \left\{ \left|f(\zeta z)\right| \mid z \in K, \frac{1}{R} \leq \left|\zeta_j\right| \leq R, j = 1, \ldots, n \right\}.$$

Then the standard estimate yields

$$\left|f_\alpha (z)\right| \leq \frac{M_K}{R^{|\alpha|}},$$

which implies that $\sum_{\alpha \in \mathbb{Z}^n} f_\alpha (z)$ converges absolutely and uniformly on K. □

Exercise 1.5.28. Let f_α be the function (1.9).

1. Show that f_α is homogenous of order α on $\mathbb{T}^n$, i.e.,

$$f_\alpha (\zeta \cdot z) = \zeta^\alpha f_\alpha (z)$$

for all $\zeta \in \mathbb{T}^n$ and $z \in D$.

2. Prove the formula
$$f_\alpha(z) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{[0,2\pi]^n} f\left(e^{i\theta}\cdot z\right) e^{-i(\alpha|\theta)} d\theta.$$

Exercise 1.5.29. A domain $D \subset \mathbb{C}^n$ is called circular if $\mathbb{T}^1$ acts on D by scalar multiplication, i.e.,
$$\mu : \mathbb{T}^1 \times D \to D, (\zeta, z) \mapsto \zeta z := (\zeta z_1, \dots \zeta z_n)$$

Let D be a circular domain and $f \in \mathcal{O}(D)$.

1. Show that for all $k \in \mathbb{Z}$ the formula
$$f_k(z) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}^1} \frac{f(\zeta z)}{\zeta^{k+1}} d\zeta$$
defines a holomorphic function on D which satisfies
$$f_k(\zeta z) = \zeta^k f_k(z)$$
for all $\zeta \in \mathbb{T}^1, z \in D$.
2. The series
$$\sum_{k\in\mathbb{Z}} f_k$$
converges compactly on D to the function f.
3. If $0 \in D$ then $f_k = 0$ for all $k < 0$.

Chapter 2

Continuation on circular and polycircular domains

In one-dimensional function theory it is known that for every domain $D \in \mathbb{C}$ there exists a function $f \in \mathcal{O}(D)$ which cannot be holomorphically extended to a strictly larger domain (see, e.g., [4], Satz VIII, 5.2. We will prove this result in a general setting in Chapter 7.). For example, the function f defined by $f(z) = \sum_{k=0}^{\infty} z^{k!}$ is holomorphic on the unit disc $B_1^1(0)$ and is unbounded near every boundary point of $B_1^1(0)$. Thus, it cannot be expanded to any domain strictly larger than $B_1^1(0)$.[1] In $n \geq 2$ dimensions the situation changes. Here we encounter domains D having the property that *every* function $f \in \mathcal{O}(D)$ can be holomorphically extended to a larger domain. This has many interesting consequences, which illustrate a big difference between complex analysis in one and in more than one variable. This chapter studies extension phenomena for holomorphic functions based on the geometry of their domain of definition. We will encounter deeper extension theorems (Riemann's Removable Singularity Theorems, Hartogs' Kugelsatz, Bochner's Extension Theorem for tube domains) in the next chapters, finally leading to the notion of holomorphy domains and the theory of Cartan and Thullen.

2.1 Holomorphic continuation

Definition 2.1.1. Let D_1, D be two domains in $\mathbb{C}^n$ with $D_1 \subset D$ and $f \in \mathcal{O}(D_1)$. We say that f can be *holomorphically extended (or continued)* to D if there is a function $F \in \mathcal{O}(D)$ whose restriction to D_1 coincides with f, i.e.,

$$F|_{D_1} = f.$$

[1] The reader who is unfamiliar with this example should carry out a proof by considering the values $f\left(te^{i\theta}\right)$ with $0 \leq t < 1$ and $\theta \in \mathbb{Q}$.

Example 2.1.2. Let $P_1^n(0)$ be the unit polydisc in $\mathbb{C}^n$. The function

$$f : P_1^n(0) \to \mathbb{C}$$

defined by the geometric series

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} z^\alpha$$

can be holomorphically extended to $\mathbb{C}^n \setminus A$ by

$$F(z) := \prod_{j=1}^{n} \frac{1}{1 - z_j},$$

where

$$A := \{z \in \mathbb{C}^n \mid \exists j : 1 \le j \le n : z_j = 1\}.$$

Proposition 2.1.3. Let $\emptyset \neq D_1 \subset D \subset \mathbb{C}^n$ be two domains. If a function $f \in \mathcal{O}(D_1)$ allows a holomorphic extension $F \in \mathcal{O}(D)$, this extension is necessarily unique.

Proof. If $F, G \in \mathcal{O}(D)$ are two holomorphic extensions they coincide on D_1, which is open. By the Identity Theorem $F = G$ on D. □

An equivalent formulation of Proposition 2.1.3 is that the restriction mapping

$$\rho : \mathcal{O}(D) \to \mathcal{O}(D_1), F \mapsto F|_{D_1} \tag{2.1}$$

is injective. ρ is a (continuous) homomorphism of complex algebras, so a natural question is under what circumstances is ρ an isomorphism? ρ is surjective if and only if every $f \in \mathcal{O}(D_1)$ is the restriction of some $F \in \mathcal{O}(D)$, that is, if every $f \in \mathcal{O}(D_1)$ allows a holomorphic continuation to D.

Definition 2.1.4. A domain $D \subset \mathbb{C}^n$ is called *balanced* if $0 \in D, D$ is circular and star-shaped with respect to the origin.

Example 2.1.5. For every $p \in \mathbb{N}_+ \cup \{\infty\}$ and every $r > 0$ the ball

$$\left\{z \in \mathbb{C}^n \mid \|z\|_p < r\right\}$$

is a balanced domain.

Lemma 2.1.6. Let $D \subset \mathbb{C}^n$ be a circular domain and $0 \in D$. Then there exists a unique domain $\widetilde{D}$, called the *balanced hull* of D, with the following properties:

1. The set $\widetilde{D}$ is balanced and contains D.
2. If D' is any balanced domain containing D then $\widetilde{D} \subset D'$.

Proof. 1. We define $\widetilde{D}$ by

$$\widetilde{D} := [0,1] \cdot D := \{tz \mid t \in [0,1], z \in D\}.$$

Then $0 \in D \subset \widetilde{D}$ and $\widetilde{D}$ is star-shaped with respect to 0, hence, connected. Since $0 \in D$ we have

$$\widetilde{D} = \bigcup_{0<t\leq 1} tD.$$

The sets $tD = \{tz \mid z \in D\}$ are open, because for $t > 0$ the mapping $z \mapsto tz$ is a homeomorphism. This shows that $\widetilde{D}$ is also open.

2. Let $D' \supset D$ be a balanced domain. Since D' is star-shaped with respect to the origin we have

$$tz \in D'$$

for all $t \in [0,1]$ and all $z \in D$, hence, $\widetilde{D} \subset D'$. $\square$

Exercise 2.1.7. The domain

$$D_H := \left\{ z \in \mathbb{C}^2 \middle| \ |z_1| < \frac{1}{2}, |z_2| < 1 \right\} \cup \left\{ z \in \mathbb{C}^2 \middle| \ |z_1| < 1, \frac{1}{2} < |z_2| < 1 \right\} \tag{2.2}$$

is called a *Hartogs figure*. Determine the balanced hull $\widetilde{D_H}$ and sketch the sets $\tau(D_H)$ and $\tau\left(\widetilde{D_H}\right)$ in absolute space.

Theorem 2.1.8 (Continuation on circular domains). *Let $D \subset \mathbb{C}^n$ be a circular domain with $0 \in D$ and let $\widetilde{D}$ be the balanced hull of D. Then the restriction*

$$\rho : \mathcal{O}\left(\widetilde{D}\right) \to \mathcal{O}(D), F \mapsto F|_D$$

is an isomorphism of algebras.

Proof. By Lemma 2.1.6 $\widetilde{D}$ is a domain. Together with Proposition 2.1.3 this shows that ρ is injective. Let $f \in \mathcal{O}(D)$. For $k \in \mathbb{N}$ define $f_k : D \to \mathbb{C}$ by

$$f_k(z) := \frac{1}{2\pi i} \int_{\mathbb{T}^1} \frac{f(\zeta z)}{\zeta^{k+1}} d\zeta.$$

As was shown in Exercise 1.5.29 each f_k is holomorphic and the series

$$\sum_{k=0}^{\infty} f_k$$

converges to f compactly on D. We show that this series even converges compactly on $\widetilde{D}$ and thus defines a holomorphic extension F of f. A domain in $\mathbb{C}^n$ is locally compact. Therefore, there is an open cover

$$(U_\lambda)_{\lambda \in \Lambda}$$

of relatively compact subsets of D. Let $K' \subset \widetilde{D}$ be a compact subset. Then

$$K' \subset \bigcup_{0<t\leq 1} \bigcup_{\lambda\in\Lambda} tU_\lambda = \widetilde{D}.$$

Since all sets tU_λ are open and K' is compact, this open cover contains a finite subcover, i.e., there are $M, N \in \mathbb{N}$ such that

$$K' \subset \bigcup_{k=1}^{M} \bigcup_{l=1}^{N} t_k U_{\lambda_l}.$$

The set

$$K := \bigcup_{l=1}^{N} \overline{U_{\lambda_l}}$$

is then a compact subset of D. Hence, the series $\sum_{k\geq 0} f_k$ converges absolutely and uniformly on K. By construction we have

$$K' \subset [0,1] \cdot K.$$

Since for $t \in [0,1]$ and $z \in K$,

$$|f_k(tz)| = \left|t^k f_k(z)\right| \leq |f_k(z)|$$

we have

$$\sup_{z\in K'} |f_k(z)| \leq \sup_{z\in K} |f_k(z)|.$$

This implies that $\sum_{k\geq 0} f_k$ converges absolutely and uniformly on K'. Thus the function

$$F : \widetilde{D} \to \mathbb{C}, z \mapsto \sum_{k\geq 0} f_k(z)$$

defines a holomorphic continuation of f to the balanced hull $\widetilde{D}$, which proves the surjectivity of ρ. □

Note that this theorem reveals its strength only if $n \geq 2$, because if $n = 1$ then every circular domain $D \subset \mathbb{C}$ containing zero coincides with its balanced hull.

Definition 2.1.9. If $D \subset \mathbb{C}^n$ is a Reinhardt domain containing the origin, its *polybalanced hull* $\widetilde{\widetilde{D}}$ is defined as

$$\widetilde{\widetilde{D}} := \{(t_1 z_1, \ldots, t_n z_n) \mid z \in D, t_j \in [0,1], j = 1, \ldots, n\}.$$

Exercise 2.1.10. Show that the polybalanced hull of a Reinhardt domain is again a Reinhardt domain.

Exercise 2.1.11. Determine the polybalanced hull of the Hartogs figure (2.2), sketch $\tau\left(\widetilde{\widetilde{D}}_H\right)$ and compare this to $\tau\left(\widetilde{D}_H\right)$.

Theorem 2.1.12 (Continuation on Reinhardt domains). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain containing the origin and $\widetilde{\widetilde{D}}$ its polybalanced hull. Then the following holds:*

1. *For all $f \in \mathcal{O}(D)$ the series*

$$\sum_{\alpha\in\mathbb{N}^n} \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

converges to f compactly on D.

2. *The restriction*

$$\rho : \mathcal{O}\left(\widetilde{\widetilde{D}}\right) \to \mathcal{O}(D), F \mapsto F|_D$$

is an isomorphism of algebras.

Proof. 1. Since $0 \in D$ there is a polydisc $P \subset D$ containing 0 such that the Taylor expansion

$$f(z) = \sum_{\nu\in\mathbb{N}^n} \frac{D^\nu f(0)}{\nu!} z^\nu$$

converges compactly on P. Thus, for all $\alpha \in \mathbb{Z}^n$ and all $z \in P$ we have

$$\begin{aligned}
f_\alpha(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}^n} \frac{f(\zeta z)}{\zeta^{\alpha+1}} d\zeta \\
&= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}^n} \frac{d\zeta}{\zeta^{\alpha+1}} \underbrace{\left(\sum_{\nu\in\mathbb{N}^n} \frac{D^\nu f(0)}{\nu!}\zeta^\nu z^\nu\right)}_{\text{converges uniformly for } \zeta\in\mathbb{T}^n} \\
&= \sum_{\nu\in\mathbb{N}^n} \frac{D^\nu f(0)}{\nu!} z^\nu \underbrace{\left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}^n} \frac{\zeta^\nu}{\zeta^{\alpha+1}} d\zeta}_{=\delta_{\alpha\nu} \text{ by Exercise 1.3.4}} \\
&= \begin{cases} \frac{D^\alpha f(0) z^\alpha}{\alpha!}, & \text{if } \alpha \in \mathbb{N}^n \\ 0, & \text{if } \alpha \notin \mathbb{N}^n \end{cases}.
\end{aligned}$$

Since D is a domain and $P \subset D$ is open this equality holds on all of D. Theorem 1.5.27 implies

$$f(z) = \sum_{\alpha\in\mathbb{Z}^n} f_\alpha(z) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

compactly on D.

2. Since D is open in $\mathbb{C}^n$ there is an open cover $(U_\lambda)_{\lambda\in\Lambda}$ of D consisting of relatively compact subsets of D. Then the polybalanced hulls $\widetilde{\widetilde{U_\lambda}}$ are relatively compact subsets of $\widetilde{\widetilde{D}}$ and form an open cover of $\widetilde{\widetilde{D}}$. If $K \subset \widetilde{\widetilde{D}}$ is a compact set, then there is a finite subcover $\left(\widetilde{\widetilde{U_{\lambda_j}}}\right)_{j=1,\dots,m}$ of K. Put

$$K' := \bigcup_{j=1}^{m} \overline{U_{\lambda_j}}.$$

Then K' is compact in D and $K \subset \widetilde{\widetilde{K'}}$. Now for all $\alpha \in \mathbb{N}^n$ we have the estimates

$$\sup_{z\in K}\left|\frac{D^\alpha f(0)}{\alpha!}z^\alpha\right| \le \sup_{z\in \widetilde{\widetilde{K'}}}\left|\frac{D^\alpha f(0)}{\alpha!}z^\alpha\right| = \sup_{z\in K'}\left|\frac{D^\alpha f(0)}{\alpha!}z^\alpha\right|.$$

From 1. we know that $\sum_{\alpha\in\mathbb{N}^n}\frac{D^\alpha f(0)}{\alpha!}z^\alpha$ converges to f compactly on D, so by the above estimate this holds on the polybalanced hull $\widetilde{\widetilde{D}}$, too. Hence, the function

$$F : \widetilde{\widetilde{D}} \to \mathbb{C}, z \mapsto \sum_{\alpha\in\mathbb{N}^n}\frac{D^\alpha f(0)}{\alpha!}z^\alpha$$

defines a holomorphic continuation of f from D to $\widetilde{\widetilde{D}}$. □

Exercise 2.1.13. Show that the isomorphism constructed in the above theorem is also a homeomorphism.

In the next continuation result we can weaken the prerequisite that D contains the origin. The key to this is the following theorem about Laurent expansion on Reinhardt domains.

Theorem 2.1.14. *Let D be a polycircular subset of $\mathbb{C}^n$. We define a partition of the set $\{1,\dots,n\}$ by*

$$\begin{aligned} I_D &: = \{k\in\{1,\dots,n\} \mid \exists z\in D : z_k = 0\},\\ J_D &: = \{l\in\{1,\dots,n\} \mid \forall z\in D : z_l \ne 0\} \end{aligned}$$

and put

$$\mathbb{N}^{I_D}\times\mathbb{Z}^{J_D} := \{\alpha\in\mathbb{Z}^n \mid \alpha_j \ge 0 \text{ for all } j\in I_D\}.$$

If D is a Reinhardt domain and $f\in\mathcal{O}(D)$ then

$$f(z) = \sum_{\alpha\in\mathbb{N}^{I_D}\times\mathbb{Z}^{J_D}}\frac{D^\alpha f(0)}{\alpha!}z^\alpha$$

compactly on D.

Proof. From Theorem 1.5.27 we know that $\sum_{\alpha\in\mathbb{Z}^n} f_\alpha$ converges to f compactly on D, where, as usual,

$$f_\alpha(z) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{f(\zeta z)}{\zeta^{\alpha+1}} d\zeta.$$

We show that for every $\alpha \in \mathbb{Z}^n$ the following holds:

1. If there is some $j \in I_D$ with $\alpha_j < 0$ then $f_\alpha = 0$.
2. If $\alpha_j \geq 0$ for all $j \in I_D$ then $f_\alpha(z) = \frac{D^\alpha f(0)}{\alpha!} z^\alpha$.

Let $\alpha \in \mathbb{Z}^n$ be fixed such that $\alpha_j < 0$ for some $j \in I_D$. Without loss of generality we may assume $j = 1$, i.e., there is some $z \in D$ with $z_1 = 0$.Since D is open there is an open disc $B_\varepsilon(0) \subset \mathbb{C}$ and an open polydisc $P' \subset \mathbb{C}^{n-1}$ such that

$$D' := B_\varepsilon(0) \times P' \subset D.$$

Laurent expansion on D' gives

$$f(z) = \sum_{\substack{\alpha\in\mathbb{Z}^n \\ \alpha_1\geq 0}} c_\alpha z^\alpha$$

implying

$$\begin{aligned} f_\alpha(z) &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{f(\zeta z)}{\zeta^{\alpha+1}} d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{\mathbb{T}^n} \frac{1}{\zeta^{\alpha+1}} \sum_{\substack{\mu\in\mathbb{Z}^n \\ \mu_1\geq 0}} c_\mu \zeta^\mu z^\mu d\zeta \\ &= \sum_{\substack{\mu\in\mathbb{Z}^n \\ \mu_1\geq 0}} c_\mu z^\mu \left(\frac{1}{2\pi i}\right)^n \underbrace{\int\limits_{\mathbb{T}^n} \frac{\zeta^\mu}{\zeta^{\alpha+1}} d\zeta}_{=\delta_{\alpha\mu}}. \end{aligned}$$

Since $\alpha_1 < 0$ we conclude that $f_\alpha = 0$ on D'. By the Identity Theorem $f_\alpha = 0$ on D.
Now let $\alpha \in \mathbb{Z}^n$ such that $\alpha_j \geq 0$ for all $j \in I_D$. Then the function

$$z \mapsto \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

is holomorphic on D, because by definition of J_D we have $z_j \neq 0$ for all $j \in J_D$. Choosing a polydisc $P \subset D$ and comparing the Laurent expansion of f on P

with the homogenous expansion $\sum_\alpha f_\alpha$ from Theorem 1.5.27 gives by analogous arguments as above

$$f_\alpha(z) = \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

on P. Apply the Identity Theorem again to see that this equation holds on all of D. $\square$

Corollary 2.1.15. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Let $1 \le p \le n$ such that for all $k = 1, \ldots, p$ there is some $z \in D$ with $z_k = 0$ and for all $l = p+1, \ldots, n$ we have $z_l \neq 0$ for all $z \in D$. Further, let*

$$\widehat{D} := \{(t_1 z_1, \ldots, t_p z_p, z_{p+1}, \ldots, z_n) \mid z \in D, 0 \le t_j \le 1, j = 1, \ldots, p\}.$$

Then

$$\rho : \mathcal{O}\left(\widehat{D}\right) \to \mathcal{O}(D), F \mapsto F|_D$$

is an isomorphism.

Proof. The proof is analogous to the case $0 \in D$, i.e., $p = n$. $\square$

Remark 2.1.16. Note that if $n = 1$, Corollary 2.1.15 is trivial, because in that case we have $0 \in D$, which implies $\widehat{D} = D$.

2.2 Representation-theoretic interpretation of the Laurent series

The results about Laurent expansion can be interpreted in a representation- theoretic framework. If D is a Reinhardt domain the action

$$\mathbb{T}^n \times D \to D, (\zeta, z) \mapsto \zeta \cdot z$$

defines a representation

$$\pi : \mathbb{T}^n \to \mathcal{L}(\mathcal{O}(D))$$

of $\mathbb{T}^n$ on the space $\mathcal{O}(D)$ of holomorphic functions on D by

$$\pi(\zeta)(f)(z) := f(\zeta \cdot z). \tag{2.3}$$

For simplicity, put $G := \mathbb{T}^n$. Generally, in representation theory, one is interested in the simultaneous eigenvectors of the representation, i.e., in this case, in all holomorphic functions $f \in \mathcal{O}(D), f \neq 0$, for which there is a function $\chi : G \to \mathbb{C}$ such that

$$\pi(g)(f) = \chi(g) f$$

for all $g \in G$. This notion, of course, applies to representations of generic groups G on a vector space V.From the fact that π is a representation it follows that χ is a group homomorphism from G to $(\mathbb{C}^\times, \cdot)$. Such group homomorphisms are called *group characters* of G. For every $\alpha \in \mathbb{Z}^n$ we have a continuous group character

$$\chi_\alpha : G \to \mathbb{C}^\times, \zeta \mapsto \zeta^\alpha.$$

Theorem 2.2.1. *Let D be a Reinhardt domain, let π be the representation* (2.3) *and for $\alpha \in \mathbb{Z}^n$ let f_α be the function* (1.9). *Then the following holds:*

1. *The eigenvectors of π are exactly those monomials $z^\alpha, z \in D, \alpha \in \mathbb{Z}^n$, which can be extended to all of D. In other words, if there is a $z \in D$ with $z_j = 0$ for some j, then only those $\alpha \in \mathbb{Z}^n$ with $\alpha_j \geq 0$ appear.*
2. *For every $\alpha \in \mathbb{Z}^n$ the function $p_\alpha : \mathcal{O}(D) \to \mathcal{O}(D)$ defined by $p_\alpha(f) := f_\alpha$ defines a continuous projection onto the eigenspace corresponding to the character χ_α.*
3. *The eigenvectors of π span a dense subset of $\mathcal{O}(D)$.*

Proof. 1. If $f \in \mathcal{O}(D)$ with Laurent expansion $f(z) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha z^\alpha$, then

$$\pi(\zeta)(f)(z) = f(\zeta \cdot z) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \zeta^\alpha z^\alpha.$$

It follows from the uniqueness of the Laurent coefficients that f is an eigenvector if and only if there is an $\alpha \in \mathbb{Z}^n$ such that $f(z) = c_\alpha z^\alpha$. From Theorem 2.1.14 we obtain that $\alpha_j \geq 0$ if $z_j = 0$.

2. From the uniform convergence of the Laurent series on the compact set $\mathbb{T}^n \cdot z$ we obtain

$$\begin{aligned} p_\alpha(f)(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}^n} \frac{f(\zeta \cdot z)}{\zeta^{\alpha+1}} d\zeta \\ &= \sum_{\beta\in\mathbb{Z}^n} \frac{c_\beta}{(2\pi i)^n} z^\beta \int_{\mathbb{T}^n} \zeta^{\beta-\alpha-1} d\zeta \\ &= \sum_{\beta\in\mathbb{Z}^n} \frac{c_\beta}{(2\pi i)^n} z^\beta \delta_{\beta\alpha} = c_\alpha z^\alpha. \end{aligned}$$

In particular, $p_\alpha(c_\alpha \mathrm{id}^\alpha)(z) = c_\alpha z^\alpha$.

3. This is an immediate consequence of Theorem 1.5.27. $\square$

Remark 2.2.2. Theorem 2.2.1 gives a representation-theoretic interpretation of the Laurent expansion of holomorphic functions on Reinhardt domains. Within this framework there are theorems which contain the Laurent expansion theorems on Reinhardt domains as special cases. This theory, however, requires a much deeper background in functional analysis. The reader familiar with this theory might have noticed that one could use the theorems of Peter–Weyl and Harish–Chandra (see [10], for instance) about continuous representations of compact Lie groups on Fréchet spaces and the convergence of the corresponding Fourier series.

2.3 Hartogs' Kugelsatz, Special case

As an application of Corollary 2.1.15 we prove a special form of Hartogs' Kugelsatz. It states that every function holomorphic on a ball shell can be extended to the full ball. As a consequence it follows that holomorphic functions of $n \geq 2$ variables have neither isolated singularities nor isolated zeros. Taking into account that in one variable a good part of the theory is based on the existence of isolated singularities, especially the theory of residues with all its applications, for instance in analytic number theory, this is a rather remarkable fact.

Theorem 2.3.1 (Kugelsatz, special case). *Let $n \geq 2$, $0 < r < R \leq \infty$, $\|.\|$ an arbitrary norm in $\mathbb{C}^n$ and $B_r^n(0), B_R^n(0)$ the balls of radius r resp. R with respect to $\|.\|$ (in case $R = \infty$ this means all of $\mathbb{C}^n$). Let*

$$B^n(r,R) := B_R^n(0) \setminus \overline{B_r^n(0)}.$$

Then the restriction

$$\rho : \mathcal{O}(B_R^n(0)) \to \mathcal{O}(B^n(r,R)), F \mapsto F|_{B^n(r,R)}$$

is an isomorphism of topological algebras.

Proof. Clearly, $B^n(r,R)$ is a Reinhardt domain. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and fix some $\lambda \in]r,R[$. Then $B^n(r,R)$ contains all vectors λe_j having λ in the j-th component and zero otherwise, i.e., we have $p = n$ in Corollary 2.1.15. Hence,

$$\begin{aligned}
\widehat{B^n(r,R)} &= \{(t_1 z_1, \ldots, t_n z_n) \mid z \in B(r,R), 0 \leq t_j \leq 1, j = 1, \ldots, n\} \\
&= \{(t_1 z_1, \ldots, t_n z_n) \mid r < \|z\| < R, 0 \leq t_j \leq 1, j = 1, \ldots, n\} \\
&= \{z \in \mathbb{C}^n \mid 0 \leq \|z\| < R\} \\
&= B_R^n(0).
\end{aligned}$$

Corollary 2.1.15 says that

$$\rho : \mathcal{O}(B_R^n(0)) \to \mathcal{O}(B^n(r,R)), F \mapsto F|_{B(r,R)}$$

is an isomorphism of algebras. It follows from Exercise 2.1.13 that this isomorphism is also topological. □

This theorem has a striking consequence.

Corollary 2.3.2. *A holomorphic function of $n \geq 2$ variables has no isolated non-removable singularities.*

Proof. Let f be holomorphic on an open set $U \subset \mathbb{C}^n$ with $n \geq 2$ and let $a \in U$. By a shift of coordinates we may assume $a = 0$ without loss of generality. Since U is open we can choose $0 < r < R$ such that

$$0 \notin B^n(r,R) \subset U.$$

Since $f \in \mathcal{O}(U)$ the restriction $f|_{B^n(r,R)}$ is holomorphic. By Theorem 2.3.1 there is a unique $F \in \mathcal{O}(B_R^n(0))$, which extends f holomorphically, hence the singularity 0 is removable. $\square$

Example 2.3.3. Let $\Omega := B^2\left(\frac{1}{2}, 1\right) \cap (\mathbb{C} \times \{0\})$ and

$$f : \Omega \to \mathbb{C}, (z, 0) \mapsto \frac{1}{z - \frac{1}{2}}.$$

f allows no holomorphic extension to $B^2\left(\frac{1}{2}, 1\right)$, because if there was an extension

$$F \in \mathcal{O}\left(B^2\left(\frac{1}{2}, 1\right)\right)$$

of f, application of Theorem 2.3.1 would produce another holomorphic extension

$$G \in \mathcal{O}\left(B_1^2(0)\right)$$

of F. This extension, however, cannot exist, because G would have to be bounded near the point $\left(\frac{1}{2}, 0\right) \in B_1^2(0)$, while f isn't. Contradiction!

Exercise 2.3.4. Let D_H be the Hartogs figure (2.2). Show that every $f \in \mathcal{O}(D_H)$ has a unique holomorphic extension to the unit polydisc in $\mathbb{C}^2$.

Exercise 2.3.5. Give an example that Theorem 2.3.1 does not hold for the case $n = 1$.

Exercise 2.3.6. Let $n \geq 2, r, R \in \mathbb{R}_+^n$ such that $0 < r_j < R_j \leq \infty$ for all $j = 1, \ldots, n$ and

$$P^n(r, R) := P_R^n(0) \setminus \overline{P_r^n(0)}.$$

Show that

$$\rho : \mathcal{O}(P_R^n(0)) \to \mathcal{O}(P^n(r, R)), f \mapsto f|_{P^n(r,R)}$$

is an isomorphism.

Exercise 2.3.7. Let $n \geq 2$, P a polydisc in $\mathbb{C}^n$ and D an open and connected neighbourhood of the boundary ∂P. Show that

$$\mathcal{O}(D) \cong \mathcal{O}(D \cup P)$$

Exercise 2.3.8. Show that a holomorphic function in $n \geq 2$ variables has no isolated zeroes.

Chapter 3

Biholomorphic maps

In this chapter we study biholomorphic maps of domains in $\mathbb{C}^n$ and prove the biholomorphic inequivalence of unit ball and unit polydisc if $n \geq 2$. We start with examining the question when a holomorphic map locally has a holomorphic inverse.

3.1 The Inverse Function Theorem and Implicit Functions

Definition 3.1.1. Let $U \subset \mathbb{C}^n$ be an open set and $f \in \mathcal{O}(U, \mathbb{C}^m)$. f is called *biholomorphic* if there is a holomorphic map

$$g : f(U) \to U$$

such that $g \circ f = \mathrm{id}_U$ and $f \circ g = \mathrm{id}_{f(U)}$. If g exists we write

$$f^{-1} := g.$$

Biholomorphic maps can only exist between equidimensional spaces.

Lemma 3.1.2. Let $U \subset \mathbb{C}^n$ be open and $f \in \mathcal{O}(U, \mathbb{C}^m)$ be a biholomorphic map. Then $n = m$ and $\det Df(a) \neq 0$.

Proof. Let $a \in U$ and $b := f(a)$. It follows from the chain rule that for the derivatives $D(f \circ g)$ and $D(g \circ f)$ we have

$$\begin{aligned} \mathrm{id}_{\mathbb{C}^n} &= D(g \circ f)(a) = (Dg(f(a)))\, Df(a), \\ \mathrm{id}_{\mathbb{C}^m} &= D(f \circ g)(b) = (Df(b))\, Dg(b). \end{aligned}$$

If $n \neq m$ then either Df or Dg would not be of maximum rank. By symmetry of the above equations we may without loss of generality assume $\operatorname{rg} Df(a) < n$.

Then there would exist a nonzero vector $z \in \mathbb{C}^n$ with

$$Df(a)\, z = 0,$$

which is a contradiction to $\mathrm{id}_{\mathbb{C}^n} = D(g \circ f)(a)$. Hence, $n = m$ and $\operatorname{rg} Df(a)$ is maximal, which is equivalent to $\det Df(a) \neq 0$. $\square$

The next theorem is the holomorphic version of the Inverse Function Theorem from real calculus.

Theorem 3.1.3 (Inverse Function). *Let $X \subset \mathbb{C}^n$ be open, $a \in X$ and $f \in \mathcal{O}(X, \mathbb{C}^n)$. Then the following statements are equivalent:*

1. *The functional determinant* $\det Df(a) \neq 0$.
2. *There exist open neighbourhoods*

   $$U = U(a) \subset X, V = V(f(a)) \subset \mathbb{C}^n$$

   such that $f(U) \subset V$ and

   $$f|U : U \to V$$

   is biholomorphic.

Proof. The direction 2. $\Rightarrow$ 1. was just proven. Let $\det f(a) \neq 0$. From the real version of the Inverse Function Theorem we obtain the existence of open neighbourhoods

$$U = U(a) \subset X, V = V(f(a)) \subset \mathbb{C}^n$$

and a real differentiable map

$$g : V \to U$$

such that

$$f \circ g = \mathrm{id}_V,\ g \circ f = \mathrm{id}_U\,.$$

We have to show that g is actually holomorphic. To this end we note that $f \circ g = \mathrm{id}_V$ is holomorphic. The Cauchy–Riemann differential equations then say that near a we have

$$0 = \frac{\partial}{\partial \overline{w}} f \circ g = \left(\frac{\partial f}{\partial z} \circ g\right) \circ \frac{\partial g}{\partial \overline{w}}.$$

$\frac{\partial f}{\partial z}$ is invertible near $a = g(f(a))$ by the real Inverse Function Theorem, hence,

$$\frac{\partial g}{\partial \overline{w}} = 0$$

near $f(a)$, i.e., g is holomorphic near $f(a)$. $\square$

As in real calculus we obtain the holomorphic version of the Implicit Function Theorem.

Theorem 3.1.4 (Implicit Functions). *Let $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ be open sets, $f \in \mathcal{O}(X \times Y, \mathbb{C}^m)$,*

$$N(f) := \{(z,w) \in X \times Y \mid f(z,w) = 0\},$$

and $(a,b) \in N(f)$ such that $\operatorname{rg} \frac{\partial f}{\partial w}(a,b) = m$. Then there are open neighbourhoods

$$\begin{aligned} U &= U(a) \subset X, \\ W &= W(b) \subset Y \end{aligned}$$

and a holomorphic function

$$h : U \to W$$

having the property that $(z,w) \in (U \times W) \cap N(f)$ if and only if $w = h(z)$, $(z,w) \in U \times W$.

Proof. We apply the Inverse Function Theorem. From

$$\operatorname{rg} \frac{\partial f}{\partial w}(a,b) = m$$

it follows that there are open neighbourhoods

$$U = U(a) \subset X, W = W(b) \subset Y, V = V((a,0)) \subset X \times Y$$

such that the mapping

$$F : U \times W \to V, (z,w) \mapsto (z, f(z,w))$$

is biholomorphic. By eventually shrinking U we may assume that $U \times \{0\} \subset V$. Let $\operatorname{pr}_{\mathbb{C}^n}$ denote the projection onto the first n coordinates. Then we can write

$$F^{-1} = (\operatorname{pr}_{\mathbb{C}^n}, H) : V \to U \times W$$

and we can define a holomorphic function $h : U \to W$ by

$$h(z) := H(z,0).$$

Now if $(z,w) \in U \times W$ such that $w = h(z) = H(z,0)$, then $(z,w) = F^{-1}(z,0)$. Hence,

$$(z,0) = F\left(F^{-1}(z,0)\right) = F(z,w) = (z, f(z,w)),$$

i.e., $f(z,w) = 0$. Conversely, if $(z,w) \in (U \times W) \cap N(f)$, then there is some $v \in V$ such that

$$(z,w) = F^{-1}(v) = (\operatorname{pr}_{\mathbb{C}^n}(v), H(v)),$$

thus,

$$F(z,w) = (z, f(z,w)) = F\left(F^{-1}(v)\right) = v = (v', v''),$$

where

$$v' = (v_1, \ldots, v_n), \; v'' = (v_{n+1}, \ldots, v_{n+m}).$$

This implies

$$v' = z, \; v'' = f(z, w) = 0,$$

thus,

$$w = H(v) = H(v', 0) = h(v') = h(z).$$

□

Corollary 3.1.5. *Let* $f \in \mathcal{O}\left(\mathbb{C}^2\right)$,

$$Y := N(f) := \left\{z \in \mathbb{C}^2 \mid f(z) = 0\right\},$$

$$X := \left\{z \in Y \mid \frac{\partial f}{\partial z_2}(z) \neq 0\right\}$$

and let pr_k *denote the projection onto the* k*-th coordinate. Then the following holds:*

1. *The set* X *is open in* Y.
2. *Every* $a \in X$ *has an open neighbourhood* $W = W(a) \subset \mathbb{C}^2$ *such that*

$$\mathrm{pr}_1 |_{X \cap W} : X \cap W \to \mathrm{pr}_1(X \cap W)$$

   is a homeomorphism.

Proof. 1. The mapping $\psi : \mathbb{C}^2 \to \mathbb{C} : z \mapsto \frac{\partial f}{\partial z_2}(z)$ is continuous, thus,

$$X = Y \cap \psi^{-1}(\mathbb{C} \setminus \{0\})$$

is open in Y.

2. Let $a = (a_1, a_2) \in X$. By the Implicit Function Theorem there are open neighbourhoods $W_j = W_j(a_j) \subset \mathbb{C}, j = 1, 2$ and a holomorphic function

$$h : W_1 \to W_2$$

such that

$$(b_1, b_2) \in (W_1 \times W_2) \cap Y$$

if and only if

$$b_2 = h(b_1), \; (b_1, b_2) \in W_1 \times W_2 =: W.$$

Let $z, w \in W \cap X$ satisfy $\mathrm{pr}_1(z) = \mathrm{pr}_1(w)$. Then

$$\mathrm{pr}_2(z) = h(\mathrm{pr}_1(z)) = h(\mathrm{pr}_1(w)) = \mathrm{pr}_2(w),$$

i.e., $z = w$. This shows that

$$\mathrm{pr}_1 |_{X \cap W} : X \cap W \to \mathrm{pr}_1(X \cap W)$$

is bijective. Since pr_1 is holomorphic and non-constant, it is a continuous and open mapping. Therefore, the inverse mapping

$$(\mathrm{pr}_1|_{X\cap W})^{-1} : \mathrm{pr}_1(X\cap W) \to X\cap W$$

is continuous. $\square$

Example 3.1.6. Consider the function

$$f : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1 - \exp(z_2).$$

Then $Y = N(f) = \{(\exp\zeta, \zeta) \mid \zeta \in \mathbb{C}\}$ and because of

$$\frac{\partial f}{\partial z_2}(z_1, z_2) = -\exp(z_2) \neq 0 \text{ for all } (z_1, z_2) \in \mathbb{C}^2$$

we have in this case $X = Y$. X carries the structure of an Abelian group by

$$(z_1, z_2)\cdot(w_1, w_2) := (z_1 w_1, z_2 + w_2)$$

with neutral element

$$(1, 0) = (\exp 0, 0).$$

The inverse element to $(\exp z, z)$ is $(\exp(-z), -z)$. If we define the functions

$$Exp : \mathbb{C} \to X, \zeta \mapsto (\exp\zeta, \zeta)$$

and

$$\log : X \to \mathbb{C}, (\exp w, w) \mapsto w,$$

then log is holomorphic[1] and satisfies

$$\log(z\cdot w) = \log z + \log w$$

and

$$\log\circ Exp = \mathrm{id}_{\mathbb{C}},\ Exp\circ\log = \mathrm{id}_X.$$

X is called the *Riemann surface of the logarithm.*

Exercise 3.1.7 (Riemann surface of the square root). Let

$$f : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1 - z_2^2.$$

1. Determine X, Y as in Example 3.1.6.

[1] i.e., to every $a \in X$ exists an open neighbourhood $U \subset \mathbb{C}^2$ and a holomorphic function $f : U \to \mathbb{C}$ such that $f|_{X\cap U} = \log|_{X\cap U}$.

2. Define an Abelian group structure on X and a holomorphic function
$$\sqrt{\cdot} : X \to \mathbb{C} \setminus \{0\}$$
such that
$$\sqrt{z \cdot w} = \sqrt{z}\sqrt{w}$$
for all $z, w \in X$.
3. Determine a holomorphic function $(.)^2 : \mathbb{C} \setminus \{0\} \to X$ which satisfies
$$(.)^2 \circ \sqrt{\cdot} = \mathrm{id}_X, \ \sqrt{\cdot} \circ (.)^2 = \mathrm{id}_{\mathbb{C}\setminus\{0\}} .$$

Exercise 3.1.8. Let $A \subset \mathbb{C}^3$ be the set of solutions of the system of holomorphic equations
$$\begin{aligned} \sin(z_1 + z_2) - z_2^2 z_3 &= 0, \\ \exp(z_1) - \exp(z_2) &= z_3. \end{aligned}$$
Show that there is an open neighbourhood $U \subset \mathbb{C}$ of zero and a holomorphic mapping
$$\varphi : U \to A$$
such that $\varphi(0) = 0$ and $\varphi : U \to \varphi(U)$ is a homeomorphism.

3.2 The Riemann Mapping Problem

A high spot in every introductory course on function theory in one variable is Riemann's Mapping Theorem, which states that every simply connected domain D properly contained in $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. This is especially remarkable, because a purely topological property — simple connectedness — implies a very restrictive analytical property. In more than one variable, however, there are simply connected domains which are not biholomorphically equivalent. In particular, this holds for unit ball and unit polydisc. This fact was first discovered by H. Poincaré in 1907 [8] by proving that the groups of holomorphic automorphisms of $B_1^n(0)$ and $P_1^n(0)$ are not isomorphic if $n > 1$.

Theorem 3.2.1. *Let*
$$B_1^n(0) = \{ z \in \mathbb{C}^n \mid \|z\|_2 < 1\}$$
be the Euclidian unit ball and
$$P_1^n(0) = \{ z \in \mathbb{C}^n \mid \|z\|_\infty < 1\}$$
the unit polydisc in $\mathbb{C}^n$. Then there exists a biholomorphic map
$$F : P_1^n(0) \to B_1^n(0)$$
if and only if $n = 1$.

Proof. Since $B_1^1(0) = P_1^1(0)$ we can choose $F = \mathrm{id}$, if $n = 1$. Let $n > 1$. Assume that F is holomorphic. For fixed $w \in P_1^1(0) \subset \mathbb{C}$ we define the mapping

$$F_w : P_1^{n-1}(0) \to \mathbb{C}^n, z' \mapsto \frac{\partial F}{\partial z_n}(z', w).$$

Let $\left(z_j'\right)_{j\in\mathbb{N}} \subset P_1^{n-1}(0)$ be a sequence converging towards the boundary $\partial P_1^{n-1}(0)$ and define

$$F_j : P_1^1(0) \to B_1^n(0), w \mapsto F\left(z_j', w\right).$$

Then

$$\sup_{j\in\mathbb{N}} \sup_{w\in P_1^1(0)} \|F_j(w)\| \leq 1,$$

i.e., the sequence $(F_j)_{j\in\mathbb{N}}$ is bounded in $\mathcal{O}\left(P_1^1(0), \mathbb{C}^n\right)$. By Montel's Theorem there exists a subsequence $(F_{j_k})_{k\in\mathbb{N}}$ converging compactly on $P_1^1(0)$ to a function

$$\Phi : P_1^1(0) \to \overline{B_1^n(0)}.$$

By Weierstrass' Theorem Φ is holomorphic. Because of

$$\lim_{j\to\infty} z_j' \in \partial P_1^{n-1}(0)$$

for every $w \in P_1^1(0)$ the sequence

$$\left(\left(z_j', w\right)\right)_{j\in\mathbb{N}}$$

converges towards the boundary

$$\partial P_1^n(0) = \{z \in \mathbb{C}^n \mid |z_k| = 1 \text{ for at least one } k\}.$$

Put

$$a_w := \lim_{j\to\infty} F\left(z_j', w\right) \in \overline{B_1^n(0)}.$$

We claim that $a_w \in \partial B_1^n(0)$. This can be seen as follows: if $a_w \notin \partial B_1^n(0)$ then

$$\partial P_1^n(0) \ni \lim_{j\to\infty} \left(z_j', w\right) = F^{-1}(a_w) \in F^{-1}\left(B_1^n(0)\right) \subset P_1^n(0),$$

which is a contradiction, because $P_1^n(0)$ is an open set. Hence,

$$\Phi\left(P_1^1(0)\right) \subset \partial B_1^n(0).$$

This shows that the function

$$w \mapsto \|\Phi(w)\|_2$$

is constant on $P_1^1(0)$, which is equivalent to $\Phi = \mathit{const.} = 1.$, as was shown in Exercise 1.5.14 This implies

$$0 = \Phi'(w) = \lim_{k\to\infty} F'_{j_k}(w) = \lim_{k\to\infty} F_w\left(z'_{j_k}\right),$$

thus

$$\lim_{z'\to\partial P_1^{n-1}(0)} F_w(z') = 0.$$

Hence, F_w is continuously extendible to the boundary $\partial P_1^{n-1}(0)$ by

$$F_w|_{\partial P_1^{n-1}(0)} := 0.$$

Applying the Maximum Modulus Theorem to every coordinate function of F_w we find that $F_w = 0$. Now if $e_n = (0,\ldots,0,1)^T$ denotes the n-th canonical basis vector of $\mathbb{C}^n$, we have

$$0 = F_w(z') = DF(z',w)\,e_n,$$

thus,

$$\det DF(z',w) = 0.$$

This, finally, contradicts the Inverse Function Theorem. □

Exercise 3.2.2. Show that $P_1^n(0)$ and $B_1^n(0)$ are homeomorphic for all $n \geq 1$. Hint: Consider the mapping

$$\Phi : P_1^n(0) \to \mathbb{C}^n, z \mapsto \begin{cases} 0, & \text{if } z = 0 \\ z\frac{\|z\|_\infty}{\|z\|_2}, & \text{if } z \neq 0. \end{cases}$$

Exercise 3.2.3. Show that the mapping

$$\varphi : P_1^2(0) \to B_1^2(0), (z,w) \mapsto \left(z, w\sqrt{1 - z\overline{w}}\right)$$

is bijective and that φ and φ^{-1} are real-analytic.

Exercise 3.2.4. An *automorphism* of a domain $D \subset \mathbb{C}^n$ is a biholomorphic self-map $f : D \to D$.

1. Show that the set $\mathrm{Aut}(D)$ of automorphisms of D forms a group with respect to composition.
2. Show that if D_1, D_2 are biholomorphic domains, then $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$ are isomorphic as groups.
3. Prove that $\mathrm{Aut}(D)$ acts on D by the mapping

$$\mathrm{Aut}(D) \times D \to D, (f,z) \mapsto f(z)$$

Exercise 3.2.5. Show without using Theorem 3.2.1 that if

$$f : P_1^n(0) \to B_1^n(0)$$

is a linear isomorphism then $n = 1$. (*Hint*: Note that f has an extension F to all of $\mathbb{C}^n$. Consider $F(\partial P_1^n(0))$.

3.3 Cartan's Uniqueness Theorem

The following theorem by H.Cartan can be viewed as an n-dimensional analogue of Schwarz' Lemma.

Theorem 3.3.1 (Cartan). *Let $D \subset \mathbb{C}^n$ be a bounded domain, $a \in D$ and*

$$f : D \to D$$

a holomorphic map satisfying $f(a) = a$ and $Df(a) = \mathrm{id}_{\mathbb{C}^n}$. Then f is the identity mapping.

Proof. Without loss of generality we may assume $a = 0$. Let r be a polyradius with $0 < \|r\|_\infty < 1$ such that the polydisc $P := P_r(0) \subset \overline{P} \subset D$. Let $p_j(z)$ be the homogenous polynomial of degree j defined by

$$p_j(z) := \sum_{|\alpha|=j} \frac{D^\alpha f(0)}{\alpha!} z^\alpha.$$[2]

Then we have the expansion on P,

$$f(z) = \sum_{j\geq 0} p_j(z).$$

From $f(0) = 0$ and $Df(0) = \mathrm{id}_{\mathbb{C}^n}$ we deduce that this expansion actually has the form

$$f = \mathrm{id} + \sum_{j\geq k} p_j$$

with $k \geq 2$. Since $f(D) \subset D$ we can consider the iterated mapping

$$f^m := \underbrace{f \circ \cdots \circ f}_{m \text{ times}} : D \to D.$$

Computation gives

$$f^m = \mathrm{id}_D + m p_k + \text{ terms of higher order.} \tag{3.1}$$

Now we can estimate for all $m \in \mathbb{N}$,

$$\begin{aligned} \|m p_k|_{\overline{P}}\|_\infty &= \sup_{z\in\overline{P}} \left| \sum_{|\alpha|=k} \frac{D^\alpha f(0)}{\alpha!} z^\alpha \right| \\ &\leq \|f^m|_{\overline{P}}\|_\infty \\ &\leq \|f\|_\infty . \end{aligned}$$

Since D is a bounded domain and $f(D) \subset D$ we have $\|f\|_\infty < \infty$. The right-hand side of the above estimate is independent of m. From this we conclude $p_k = 0$ for all $k \geq 2$. Hence, $f = \mathrm{id}_D$. ☐

[2] i.e., the vector with polynomial coefficients $\sum_{|\alpha|\leq j} \frac{D^\alpha f_k(0)}{\alpha!} z^\alpha$

Remark 3.3.2. The theorem is false, if D is not bounded. This can already be seen in one dimension. For instance, consider the function

$$f : \mathbb{C} \to \mathbb{C}, z \mapsto \exp z - 1.$$

Corollary 3.3.3. *Let D_1, D_2 be bounded circular domains in $\mathbb{C}^n$ such that $0 \in D_1 \cap D_2$ and $f : D_1 \to D_2$ a biholomorphic mapping with $f(0) = 0$. Then f is linear.*

Proof. For $t \in \mathbb{R}$ define

$$F_t : D_1 \to D_1, z \mapsto f^{-1}\left(e^{-it} f\left(e^{it} z\right)\right).$$

Then $F_t(0) = 0$ and $DF_t(0) = \mathrm{id}_{\mathbb{C}^n}$, hence, by Cartan's Uniqueness Theorem, $F_t = \mathrm{id}_{D_1}$. This implies

$$f\left(e^{it} z\right) = e^{it} f(z). \tag{3.2}$$

Expansion of f into a series of homogenous polynomials gives

$$f = \sum_{j \geq 0} p_j,$$

thus

$$f\left(e^{it} z\right) = \sum_{j \geq 0} p_j\left(e^{it} z\right) = \sum_{j \geq 0} e^{ijt} p_j(z) \overset{(3.2)}{=} \sum_{j \geq 0} e^{it} p_j(z).$$

Hence, for all $t \in \mathbb{R}$,

$$0 = \sum_{j \geq 0} p_j(z) e^{it}\left(e^{i(j-1)t} - 1\right).$$

Comparing coefficients we find that $p_j = 0$ for all $j \neq 1$, i.e., $f = p_1$, which is linear. □

Exercise 3.3.4. In this exercise we give an alternative proof of Theorem 3.2.1. For $a \in B_1^n(0)$ let $[a]$ denote the subspace of $\mathbb{C}^n$ spanned by a. Let

$$P_a : \mathbb{C}^n \to [a]$$

denote the orthogonal projection onto $[a]$ and

$$P_a^{\perp} : \mathbb{C}^n \to [a]^{\perp}$$

the projection onto the orthogonal complement of $[a]$. Let

$$s_a := \left(1 - \|a\|_2^2\right)^2$$

and define

$$\varphi_a : B_1^n(0) \to \mathbb{C}^n, z \mapsto \frac{a - P_a z - s_a P_a^{\perp} z}{1 - (z|a)},$$

where $(z|a)$ denotes the standard inner product in $\mathbb{C}^n$.

1. Show that $\varphi_a\left(B_1^n(0)\right) \subset B_1^n(0)$.
2. Prove the equations
$$D\varphi_a(0) = -s_a^2 P_a - s_a P_a^{\perp}$$
and
$$D\varphi_a(a) = -s_a^2 P_a - s_a^{-1} P_a^{\perp}.$$
(*Hint*: It is useful to determine explicit formulas for $P_a z$ and $P_a^{\perp} z$.)
3. Show that $\varphi_a \circ \varphi_a = \mathrm{id}_{B_1^n(0)}$.
4. The group $\mathrm{Aut}\left(B_1^n(0)\right)$ acts transitively on $B_1^n(0)$, i.e., for every pair
$$(z, w) \in B_1^n(0) \times B_1^n(0)$$
there is some $g \in \mathrm{Aut}\left(B_1^n(0)\right)$ such that $g(z) = w$.
5. If $D \subset \mathbb{C}^n$ is a circular, bounded domain containing the origin and
$$F : D \to B_1^n(0)$$
a biholomorphic map, then there exists some $T \in GL_n(\mathbb{C})$ such that
$$T\left(B_1^n(0)\right) = D.$$
6. Conclude again that $B_1^n(0)$ and $P_1^n(0)$ are not biholomorphically equivalent if $n \geq 2$.

Chapter 4

Analytic Sets

In this chapter we give an introduction to analytic sets. Roughly speaking, analytic sets are sets whose elements are locally solutions of holomorphic systems of equations. We will introduce the notion of codimension of an analytic set and show that in most cases analytic sets are "thin enough" that holomorphic functions defined outside an analytic set can be extended across the analytic set.

4.1 Elementary properties of analytic sets

Definition 4.1.1. Let $D \subset \mathbb{C}^n$ be open. A subset $A \subset D$ is called *analytic in* D, if A is closed and for every $a \in A$ there exists an open neighbourhood $U \subset D$ and finitely many functions $f_1, \dots, f_m \in \mathcal{O}(U)$ such that

$$A \cap U = \{z \in U \mid f_1(z) = \dots = f_m(z) = 0\}.$$

Example 4.1.2. The empty set $\emptyset$ and D itself are analytic in D. D is the zero set of the zero function on D, $\emptyset$ is the zero set of the constant function $f = 1$. If A_1, A_2 are analytic in D then so are $A_1 \cup A_2$ and $A_1 \cap A_2$. If an analytic set A is the union of two proper analytic subsets, then A is said to be *reducible*, otherwise *irreducible*.

Example 4.1.3. If V is a k-dimensional (affine) subspace of $\mathbb{C}^n$, then V is an analytic in $\mathbb{C}^n$.

Proof. By a shift of coordinates we may assume that V is a proper, non-affine subspace of $\mathbb{C}^n$. Let $\{b_1, \dots, b_k\}$ be a basis of V. Then there are vectors $b_{k+1}, \dots, b_n$ such that $\{b_1, \dots, b_n\}$ forms a basis of $\mathbb{C}^n$. For $z = \sum_{l=1}^{n} \alpha_l b_l$ let $\mathrm{pr}_j : \mathbb{C}^n \to \mathbb{C}$ denote the linear projection defined by

$$\mathrm{pr}_j(z) := \alpha_j.$$

Then pr_j is a holomorphic function for all j and

$$V = N\left(\mathrm{pr}_{k+1}, \ldots, \mathrm{pr}_n\right) = \bigcap_{j=k+1}^{n} N\left(\mathrm{pr}_j\right).$$

Since intersections of closed sets are closed, V is closed in $\mathbb{C}^n$. $\square$

A k-dimensional subspace of $\mathbb{C}^n$ has codimension $n-k$. This notion can be generalized to analytic sets. In the case of a k-dimensional (affine) subspace of $\mathbb{C}^n$ both notions coincide.

Definition 4.1.4. Let $A \subset D$ be an analytic set, $a \in A$ and $m \in \mathbb{N}$. We say that

$$\operatorname{codim}_a A \geq m$$

if there is an m-dimensional affine complex subspace E containing a such that a is an isolated point in $E \cap A$. We say that

$$\operatorname{codim}_a A = m$$

if $\operatorname{codim}_a A \geq m$, but $\operatorname{codim}_a A \not> m$. If $A \neq \emptyset$ we define

$$\operatorname{codim} A := \inf_{a \in A} \operatorname{codim}_a A.$$

Example 4.1.5. If $f : D \to \mathbb{C}$ is a holomorphic function, $f \neq 0$, then the zero set

$$N(f) := f^{-1}(\{0\})$$

has codimension 1.

Proof. If $n = 1$ the assertion is trivial. Let $n > 1$. Let $a \in N(f)$. Since $f \neq 0$ there is an open and connected neighbourhood $U = U(a) \subset D$ such that $f(b) \neq 0$ for some $b \in U, b \neq a$. Let

$$E_1 := a + \mathbb{C}b$$

be the complex line through a and $b . E_1 \cap U$ is connected and one-dimensional, thus, a is an isolated zero of $f|_{E_1 \cap U}$, i.e.,

$$\operatorname{codim}_a N(f) \geq 1.$$

If $\operatorname{codim}_a N(f) \geq 2$ there would exist a two-dimensional complex subspace E_2 containing a such that a is an isolated point in $E_2 \cap N(f)$. From Chapter 2, however, we already know that holomorphic functions in more than one variable have no isolated zeroes, thus $\operatorname{codim}_a N(f) \not\geq 2$. $\square$

Proposition 4.1.6. Let $D \subset \mathbb{C}^n$ be open and $A \subset D$ be an analytic set. Then the following holds.

1. If $\operatorname{codim} A \geq 1$, then $D \setminus A$ is dense in D.
2. If D is connected and if the interior A° of A is non-empty, then $A = D$.
3. If D is a domain and $A \neq D$, then $\operatorname{codim} A \geq 1$.

Proof. 1. Let $a \in A$. There is a complex affine subspace E of dimension ≥ 1 such that a is an isolated point in $E \cap A$. This implies that for every $k \geq 1$ there is some

$$z_k \in B^n_{\frac{1}{k}}(a) \cap E,$$

which is not contained in A. Hence, the sequence $(z_k)_{k\geq 1}$ is contained in $D \setminus A$ and satisfies

$$\lim_{k\to\infty} z_k = a.$$

2. Let $a \in \overline{A^\circ}$. There are an open and connected neighbourhood $U = U(a) \subset D$ and holomorphic functions $f_1, \ldots, f_m \in \mathcal{O}(U)$ such that

$$U \cap A = \{z \in U \mid f_j(z) = 0,\ j = 1, \ldots, m\}.$$

Since $a \in \overline{A^\circ}$ we have $U \cap A^\circ \neq \emptyset$ and $U \cap A^\circ$ is open in D. The Identity Theorem implies

$$f_1 = \cdots = f_m = 0,$$

i.e., $U = U \cap A \subset A$, hence, $a \in A^\circ$. Since a was an arbitrary element of $\overline{A^\circ}$ this shows that

$$A^\circ = \overline{A^\circ}.$$

As was shown in Lemma 1.1.15, the only non-empty subset of a domain D which is both open and closed is D itself.

3. Let $a \in A$, U, $f_1, \ldots, f_m \in \mathcal{O}(U)$ as above. 2. implies $A^\circ = \emptyset$, because $A \neq D$ by prerequisite. Then there is some $b \in U$ such that $f_k(b) \neq 0$ for some $k \in \{1, \ldots, n\}$. Let E be the complex line through a and b, i.e.,

$$E := a + \mathbb{C}b.$$

Then $A \cap U \subset N(f_k)$ and

$$(A \cap E) \cap U \subset N(f_k) \cap U \cap E.$$

$N(f_k) \cap U \cap E$ consists only of isolated points, because otherwise the holomorphic function of one variable

$$f : \mathbb{C} \to \mathbb{C}, \lambda \mapsto f_k(a + \lambda b)$$

would have a non-isolated zero, while not being the zero function, which contradicts the Identity Theorem. $\square$

Exercise 4.1.7. Which of the following sets A are analytic in D ?

1. $D = \mathbb{C}^n,\ A = \mathbb{S}^{2n-1} := \{z \in \mathbb{C}^n \mid \|z\|_2 = 1\}$.
2. $D = M(n, n; \mathbb{C}),\ A = GL_n(\mathbb{C})$.
3. $D = \mathbb{C}^n,\ A \subset \mathbb{C}^n$a discrete subset.

Decide in each case, if A is analytic in D, whether A is the zero set of some $f \in \mathcal{O}(D)$.

Exercise 4.1.8. Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ be open sets and $f \in \mathcal{O}(U, V)$. Let $A \subset V$ be analytic in V. Show that $f^{-1}(A)$ is analytic in U.

Exercise 4.1.9. Show that a k-dimensional (affine) subspace V of $\mathbb{C}^n$ has codimension $n - k$ in the sense of analytic sets.

Exercise 4.1.10. Let A_1, A_2 be analytic sets. What can you say about

$$\operatorname{codim}(A_1 \cap A_2)$$

and

$$\operatorname{codim}(A_1 \cup A_2)?$$

Exercise 4.1.11. Let $p, q \in \mathbb{N}$, $p, q > 1$, $\gcd(p, q) = 1$ and consider the analytic set

$$A_{p,q} := \left\{(z, w) \in \mathbb{C}^2 \mid z^p - w^q = 0\right\}.$$

1. Show that the mapping

   $$\phi : \mathbb{C} \to A_{p,q}, \zeta \mapsto (\zeta^q, \zeta^p)$$

   is a holomorphic homeomorphism.

2. Show that there is no function ψ holomorphic in a neighbourhood U of $0 \in \mathbb{C}^2$ such that

   $$\psi \circ \phi|_{\phi^{-1}(U)} = \mathrm{id}_{\phi^{-1}(U)}.$$

   (*Hint*: By Euclid's Algorithm there are $a, b \in \mathbb{Z}$ such that $\gcd(p, q) = ap+bq$.)

3. Show that $\phi(\mathbb{T}) \subset \mathbb{T}^2$.

4. Let $\phi_j := \mathrm{pr}_j \circ \phi, j = 1, 2$ and

   $$\gamma : [0, 1] \to \mathbb{T}, t \mapsto e^{2\pi i t}.$$

   Compute the winding numbers of the curves $\phi_j \circ \gamma$ at zero and interpret this result geometrically. Can you sketch $\phi(\mathbb{T})$ in the case $(p, q) = (2, 3)$? (Imagine the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$.)

4.2 The Riemann Removable Singularity Theorems

Analytic sets give rise to continuation results for holomorphic functions, which are independent of the geometry of the underlying domain of definition. In one variable theory the Riemann Removable Singularity Theorem states that holomorphic functions which are defined outside an isolated point a and which are bounded near a can be extended across a. The next theorem generalizes this result to analytic sets.

Theorem 4.2.1 (1^{st} Riemann Removable Singularity Theorem). *Let $D \subset \mathbb{C}^n$ be open, $A \subset D$ an analytic set with $\operatorname{codim} A \geq 1$ and $f \in \mathcal{O}(D \setminus A)$ a holomorphic function, which is locally bounded at A, i.e., for every $a \in A$ there is an open neighbourhood U such that*

$$f|_{(D \setminus A) \cap U}$$

is bounded. Then f has a unique holomorphic continuation $F : D \to \mathbb{C}$.

Proof. If F exists it is unique, because $D \setminus A$ is dense in D and F is continuous. (Note that D need not be connected, hence, we cannot apply the Identity Theorem on D!). If $n = 1$ the result is known, so we consider the case $n \geq 2$. Since a global extension is also a local extension, uniqueness of F implies that it is enough to prove the theorem in a neighbourhood of each $a \in A$. Hence, without loss of generality, we may assume that

$$A = \{z \in D \mid f_1(z) = \cdots = f_m(z) = 0\}.$$

From $\operatorname{codim} A \geq 1$ we deduce that there is a complex line E through a such that a is an isolated point in $A \cap E$. By change of coordinates and shift of the origin we may assume that $a = 0$ and E is the z_n-axis. Choose some $r > 0$ such that the polydisc

$$P_r^n(0) = \{z \in \mathbb{C}^n \mid \|z\|_\infty < r\}$$

is contained in D. Then there is $0 < \delta < r$ such that

$$\{z \in \mathbb{C}^n \mid z_1 = \cdots = z_{n-1} = 0, |z_n| = \delta\} \subset P_r^n(0) \setminus A.$$

Since $D \setminus A$ is an open set there is some $\varepsilon > 0$ such that

$$R := \{z \in \mathbb{C}^n \mid |z_j| < \varepsilon, j = 1, \ldots, n-1, |z_n| = \delta\} \subset D \setminus A.$$

For every $w' \in \mathbb{C}^{n-1}$ with $\left|w_j'\right| < \varepsilon, j = 1, \ldots, n-1$ the set

$$W_r := \{(w', z_n) \mid |z_n| < r\}$$

intersects A only in isolated points, because $f_j(w', z_n) \neq 0$ for $|z_n| = \delta$. Define

$$U := \{z \in \mathbb{C}^n \mid |z_j| < \varepsilon, j = 1, \ldots, n-1, |z_n| < \delta\}$$

and
$$F : U \to \mathbb{C}, z = (z', z_n) \mapsto \frac{1}{2\pi i} \int\limits_{|\zeta|=\delta} \frac{f(z', \zeta)}{\zeta - z_n} d\zeta.$$

Then F is continuous and partially holomorphic in the variables $z_1, \ldots, z_{n-1}$. We show that F is also partially holomorphic in z_n. Let $z' \in \mathbb{C}^{n-1}$ be fixed such that $\|z'\|_\infty < \varepsilon$. Then the function of one complex variable
$$\varphi(\zeta) := f(z', \zeta)$$
is holomorphic on the disc defined by $|\zeta| < r$ with the possible exception of isolated points. Near these points φ is bounded, because f is bounded. The Riemann Removable Singularity Theorem in one dimension states that φ can be holomorphically extended to the whole disc. Cauchy's Integral Formula yields
$$f(z) = \varphi(z_n) = \frac{1}{2\pi i} \int\limits_{|\zeta|=\delta} \frac{\varphi(\zeta)}{\zeta - z_n} d\zeta = F(z)$$
on $U \setminus A$. □

Corollary 4.2.2. *Let $D \subset \mathbb{C}^n$ be a domain, $A \subset D$ an analytic subset with* $\operatorname{codim} A \geq 1$. *Then $D \setminus A$ is connected.*

Proof. Assume $D \setminus A = U \cup V$, where U, V are disjoint, nonempty open subsets of $D \setminus A$. Then the function
$$f : U \cup V \to \mathbb{C}, z \mapsto \begin{cases} 1, & \text{if } z \in U \\ 0, & \text{if } z \in V \end{cases}$$
is holomorphic and bounded near A. By Theorem 4.2.1, f has a holomorphic extension $F \in \mathcal{O}(D)$. Since D is connected the Identity Theorem then implies that $F = 0$ and $F = 1$, which is absurd. □

Theorem 4.2.3 (2^{nd} Riemann Removable Singularity Theorem). *Let $D \subset \mathbb{C}^n$ be open and $A \subset D$ an analytic set with* $\operatorname{codim} A \geq 2$. *Then the restriction*
$$\rho : \mathcal{O}(D) \to \mathcal{O}(D \setminus A), f \mapsto f|_{D \setminus A}$$
is an isomorphism of complex algebras.

Proof. Let $a \in A$ and $f \in \mathcal{O}(D \setminus A)$. By applying Theorem 4.2.1 we see that it suffices to show that f is bounded near a. Without loss of generality assume $a = 0$ and that a is an isolated point in $A \cap E$, where
$$E := \{z \in \mathbb{C}^n \mid z_1 = \cdots = z_{n-2} = 0\}.$$
Hence, there are $0 < r' < r$ such that
$$\{z \in \mathbb{C}^n \mid z_1 = \cdots = z_{n-2} = 0, \ r' < |z_{n-1}|, |z_n| < r\} \subset D \setminus A.$$

Since $D \setminus A$ is open there is $\varepsilon > 0$ such that the compact set

$$K := \{z \in \mathbb{C}^n \mid |z_j| \le \varepsilon, j = 1, \dots, n-2,\ r' < |z_{n-1}|, |z_n| < r\}$$

is contained in $D \setminus A$. Compactness of K implies that f is bounded on K. f can be holomorphically extended to

$$K' := \{z \in \mathbb{C}^n \mid |z_j| < \varepsilon, j = 1, \dots, n-2,\ |z_{n-1}|, |z_n| < r\}$$

and from the Maximum Modulus Theorem we deduce

$$\left\| f|_{(D \setminus A) \cap K'} \right\|_\infty \le \| f|_K \|_\infty < \infty.$$

Hence, f is bounded near zero. □

Exercise 4.2.4. Give an example of a real-analytic function $f : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\mathbb{R}^{2n} \setminus N(f)$ is not connected.

Exercise 4.2.5. Let $\Delta_n \subset GL_n(\mathbb{C})$ denote the set of regular right upper triangular matrices. For which $n \ge 2$ is the restriction

$$\mathcal{O}(GL_n(\mathbb{C})) \to \mathcal{O}(GL_n(\mathbb{C}) \setminus \Delta_n),\ f \mapsto f|_{GL_n(\mathbb{C}) \setminus \Delta_n}$$

an isomorphism of complex algebras?

Exercise 4.2.6. Let $f, g \in O(\mathbb{C}^n)$ such that $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}^n$. Show that f is a multiple of g, i.e., there is some $\lambda \in \mathbb{C}$ such that $f = \lambda g$.

Chapter 5

Hartogs' Kugelsatz

The goal of this chapter is to state and prove a theorem of Hartogs known as the "Kugelsatz". We encountered a special case of the Kugelsatz already in Chapter 2, where it was shown that every function holomorphic on a ball shell $B(r,R) = \{z \in \mathbb{C}^n \mid r < \|z\|_2 < R\}$ has a unique holomorphic extension to the full ball $B_R(0)$. While this special case is the origin of the name "Kugelsatz" (Kugel is one possible German word for ball) the general case of the Kugelsatz yields the result that every function holomorphic on the complement $D \setminus K \subset \mathbb{C}^n$ ($n \geq 2$) of some compact set K, provided this complement is connected, can be holomorphically extended across K. As a consequence we will see that the zero set of a holomorphic function in more than one variable is never compact unless it is empty.

5.1 Holomorphic Differential Forms

We start by giving a brief introduction to holomorphic differential forms in $\mathbb{C}^n$, but assume that the reader is (at least roughly) familiar with the calculus of differential forms in $\mathbb{R}^n$. Let us first collect the necessary basics from linear algebra.

5.1.1 Multilinear forms

Let V be an n-dimensional real vector space and denote by

$$V^{\#} := \{\mu : V \to \mathbb{R} \mid \mu \ \mathbb{R}\text{-linear}\}$$

its algebraic dual. For $k \in \mathbb{N}$ we then consider the so-called k-th outer product

$$\bigwedge^k V^{\#} := \left\{\mu : V^k \to \mathbb{R} \mid \mu \ \mathbb{R}\text{-multilinear and alternating}\right\}, \tag{5.1}$$

where

$$\bigwedge^0 V^{\#} := \mathbb{R} \text{ and } \bigwedge^1 V^{\#} := V^{\#}.$$

An element $\mu \in \bigwedge^k V^{\#}$ is called an *alternating* k*-form* or simply a k*-form.* k is called the degree $\deg \mu$ of μ. As is known from linear algebra, if $v_1, \ldots, v_k$ are linearly dependent then

$$\mu (v_1, \ldots, v_k) = 0$$

for every $\mu \in \bigwedge^k V^{\#}$, which implies

$$\bigwedge^k V^{\#} = \{0\} \text{ if } k > n.$$

This in mind we define

$$\bigwedge V^{\#} := \bigoplus_{k \geq 0} \bigwedge^k V^{\#}. \tag{5.2}$$

If $T : V \to W$ is a linear mapping and $\omega \in \bigwedge^k W^{\#}$ we define a k-form $T^*\omega \in \bigwedge^k V^{\#}$, called the pullback by T, by

$$T^*\omega (v_1, \ldots, v_k) := \omega (Tv_1, \ldots, Tv_k). \tag{5.3}$$

Denote by S_k the symmetric group of k variables, i.e., the set of permutations of the numbers $1, \ldots, k$. For $\sigma \in S_k$ the signum of σ is defined by

$$\operatorname{sign} \sigma := \prod_{1 \leq i < j \leq k} \frac{i - j}{\sigma (i) - \sigma (j)}. \tag{5.4}$$

$\operatorname{sign} \sigma$ equals $+1$ if it takes an even number of transpositions to reorder the numbers

$$\sigma (1), \ldots, \sigma (k)$$

into their original order $1, \ldots, k$ and -1 if this number of transpositions is odd. It is easy to see that a k-form satisfies

$$\mu (v_1, \ldots, v_k) = \operatorname{sign} \sigma \cdot \mu \left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right).$$

If $\mu \in \bigwedge^p V^{\#}$ and $\omega \in \bigwedge^q V^{\#}$ are two p- resp. q-forms we define their product $\mu \wedge \omega \in \bigwedge^{p+q} V^{\#}$ by

$$\begin{aligned} & \mu \wedge \omega (v_1, \ldots, v_p, v_{p+1}, v_{p+q}) \\ : & = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign} \sigma \cdot \mu (v_1, \ldots, v_p) \, \omega (v_{p+1}, \ldots, v_{p+q}). \end{aligned} \tag{5.5}$$

By bilinear continuation this defines a mapping

$$\wedge : \bigwedge V^{\#} \times \bigwedge V^{\#} \to \bigwedge V^{\#}, (\omega, \omega') \mapsto \omega \wedge \omega',$$

which turns $\bigwedge V^{\#}$ into an associative unital algebra called the exterior algebra of $V^{\#}$.

Lemma 5.1.1. The mapping $(\omega, \omega') \mapsto \omega \wedge \omega'$ has the following properties:

1. The wedge-product $\wedge$ is $\mathbb{R}$-bilinear and associative.
2. The wedge-product $\wedge$ is a natural mapping, i.e., if T is a linear mapping, then
$$T^* (\omega \wedge \omega') = T^*\omega \wedge T^*\omega'.$$
3. The equation $1 \wedge \omega = \omega$ holds.
4. Commutation formula:
$$\omega \wedge \omega' = (-1)^{\deg \omega \deg \omega'} \omega' \wedge \omega.$$
5. If $\deg \omega$ is even, then $\omega \wedge \omega = 0$.
6. If $\mu_1, \ldots, \mu_k \in V^\#$ and $v_1, \ldots, v_k \in V$, then
$$(\mu_1 \wedge \cdots \wedge \mu_k)(v_1, \ldots, v_k) = \det (\mu_i (v_j))_{i,j=1}^k .$$
7. If $\{\varepsilon_1, \ldots, \varepsilon_k\}$ is a basis of $V^\#$, then
$$\{\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k} \mid 1 \leq j_1 < \cdots < j_k \leq n\}$$
is a basis of $\bigwedge^k V^\#$. In particular,
$$\dim_{\mathbb{R}} \bigwedge^k V^\# = \binom{n}{k}.$$

Proof. Left to the reader. □

We now turn to the complexification of a real vector space V. To this end we note that the mapping
$$J : V \times V \to V \times V, (u, v) \mapsto (-v, u)$$
is bilinear and satisfies $J^2 = -\operatorname{id}$. Hence, $V \times V$ becomes a complex vector space $V_{\mathbb{C}}$ by the definition
$$i \cdot (u, v) := J(u, v) = (-v, u)$$
satisfying $i \cdot (V \times \{0\}) = \{0\} \times V$, hence,
$$V_{\mathbb{C}} = V \oplus iV. \tag{5.6}$$
If $V_{\mathbb{C}}^\#$ denotes the real dual space of $V_{\mathbb{C}}$ then
$$\begin{aligned} \bigwedge^k V_{\mathbb{C}}^\# &= \left\{\mu : V_{\mathbb{C}}^k \mid \mu \text{ is } \mathbb{R}\text{-multilinear and alternating}\right\} \\ &= \bigwedge^k V^\# \oplus i \bigwedge^k V^\# \end{aligned}$$
is a complex vector space. We define conjugation in $\bigwedge^k V_{\mathbb{C}}^\#$ by
$$\bar{\mu}(v_1, \ldots, v_k) := \overline{\mu (v_1, \ldots, v_k)}. \tag{5.7}$$

Exercise 5.1.2. Let $\omega_p, \omega_q, \omega_r$ be alternating p-, q- and r-forms. Under what condition does the equation

$$\omega_p \wedge \omega_q \wedge \omega_r = \omega_r \wedge \omega_q \wedge \omega_p$$

hold?

5.1.2 Complex differential forms

We introduce complex differential forms on open sets in $\mathbb{C}^n$. All definitions and results can be transferred to complex manifolds as well.

Definition 5.1.3. Let $X \subset \mathbb{C}^n$ be an open set and $a \in X$.

1. If $\gamma :]-\varepsilon, \varepsilon[\to X$ is a real differentiable curve with $\gamma(0) = a$, then the vector

$$\dot{\gamma}(0) := \left.\frac{d}{dt}\right|_{t=0} \gamma(t)$$

is called a *tangent vector* at a.

2. The set $T_a X$ of all tangent vectors at a is called the *real tangent space* of X at a.

Exercise 5.1.4. Show that the tangent space $T_a X$ of an open set X at $a \in X$ is all of $\mathbb{C}^n$.

Remark 5.1.5. One may ask why we give a rather complicated definition of tangent space of an open set in $\mathbb{C}^n$ if it turns out to coincide with all of $\mathbb{C}^n$. The reason is that this definition of tangent space also generalizes to differentiable manifolds. The real dual space $T_a^{\#} X$ of $T_a X$ is called the *real cotangent space* of X at a.

Definition 5.1.6. A mapping ω from X into the disjoint union

$$\omega : X \to \dot{\bigcup_{a \in X}} \bigwedge \left(T_a^{\#} X\right)_{\mathbb{C}}$$

is called a *complex differential form* of degree k if for all $a \in X$,

$$\omega(a) \in \bigwedge^k \left(T_a^{\#} X\right)_{\mathbb{C}},$$

i.e., if for all $a \in X$,

$$\omega(a) : (T_a X)^k \to \mathbb{C}$$

is $\mathbb{R}$-multilinear and alternating.

Example 5.1.7. If $f \in \mathcal{C}^1(X)$, then the real differential

$$\begin{aligned} df \quad &: \quad X \to \dot{\bigcup_{a\in X}} \bigwedge \left(T_a^{\#}X\right)_{\mathbb{C}}, \\ a \quad &\mapsto \quad d_a f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}(a)\, dz_j + \frac{\partial f}{\partial \overline{z_j}}(a)\, d\overline{z_j} \right) \end{aligned}$$

of f is a complex differential form (cf. (1.4)). Especially the differentials

$$\begin{aligned} dz_j &= d\,\mathrm{pr}_j = dx_j + i\,dy_j, \\ d\overline{z_j} &= d\overline{\mathrm{pr}_j} = dx_j - i\;dy_j \end{aligned}$$

are complex differential forms.

Combining Lemma 1.2.22 and Lemma 5.1.1 we see that a complex differential form ω of degree k can generally be written as a finite sum

$$\omega = \sum_{\text{finite}} a_{I,J} dz_I \wedge d\overline{z_J}, \tag{5.8}$$

where for $k = p + q$,

$$\begin{aligned} dz_I \quad &: \quad = dz_{i_1} \wedge \cdots \wedge dz_{i_p},\ 1 \le i_1 < \cdots < i_p \le n, \\ d\overline{z_J} \quad &: \quad = dz_{j_1} \wedge \cdots \wedge dz_{j_q},\ 1 \le j_1 < \cdots < j_q \le n, \\ a_{I,J} \quad &: \quad X \to \mathbb{C}. \end{aligned}$$

Definition 5.1.8. ω is called a *smooth* differential form of degree k, if all functions $a_{I,J}$ in (5.8) are smooth, i.e., $a_{I,J} \in \mathcal{C}^\infty(X, \mathbb{C})$.

Notation 5.1.9. We write

$$\begin{aligned} \mathcal{E}(X) \quad &: \quad = \mathcal{C}^\infty(X, \mathbb{C}), \\ \mathcal{E}^k(X) \quad &: \quad = \{\omega \mid \omega \text{ smooth complex differential form of degree } k \text{ on } X\}, \\ \mathcal{E}^{p,q}(X) \quad &: \quad = \left\{ \omega = \sum_{finite} a_{I,J} dz_I \wedge d\overline{z_J} \;\middle|\; a_{I,J} \text{ smooth}, \#I = p, \#J = q \right\}. \end{aligned}$$

Remark 5.1.10. It is easy to see that

$$\mathcal{E}^k(X) = \bigoplus_{p+q=k} \mathcal{E}^{p,q}(X).$$

If $(K_j)_{j\in\mathbb{N}}$ is a compact exhaustion of X and $\alpha \in \mathbb{N}^n$ we can define a countable family of seminorms

$$p_{j_\alpha} : \mathcal{E}(X) \to \mathbb{R}, f \mapsto \sup_{z\in K_j} |D^\alpha f(z)|$$

which turns each $\mathcal{E}^{p,q}(X)$ into a Fréchet space. The Fréchet structure of $\mathcal{E}^{p,q}(X)$ is induced by the Fréchet structure of $\mathcal{E}(X)$, since every differential form

$$\omega = \sum_{finite} a_{I,J} dz_I \wedge d\overline{z_J} \in \mathcal{E}^{p,q}(X)$$

can be identified with its set of coefficients $a_{I,J} \in \mathcal{E}(X)$. In particular, this means that a sequence

$$\omega_n = \sum_{finite} a_{I,J;n} dz_I \wedge d\overline{z_J} \in \mathcal{E}^{p,q}(X)$$

converges to some $\omega = \sum_{finite} a_{I,J} dz_I \wedge d\overline{z_J} \in \mathcal{E}^{p,q}(X)$ if and only if

$$\lim_{n\to\infty} a_{I,J;n} = a_{I,J} \text{ for all } I, J.$$

The exterior derivative

If $\omega = \sum_{finite} a_{I,J} dz_I \wedge d\overline{z_J}$ is a smooth complex differential form all $a_{I,J}$ are differentiable functions. Thus we can define a mapping

$$\omega \mapsto d\omega := \sum_{finite} da_{I,J} \wedge dz_I \wedge d\overline{z_J}, \tag{5.9}$$

called the *exterior derivative* of ω. If $\omega = a_{I,J}$ is a differential form of degree zero, i.e., a smooth function, the exterior derivative is simply the real differential of $a_{I,J}$. The following properties are easily verified and the proof is left to the reader.

Lemma 5.1.11. The exterior derivative satisfies:

1. The equation $d \circ d = 0$ holds.
2. The exterior derivative of a wedge-product is given by
$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega'.$$
3. For all $f \in \mathcal{E}(X)$ we have $d\overline{f} = \overline{df}$.
4. The mapping d is $\mathbb{C}$-linear.
5. The mapping d can be decomposed into $d = d' + d''$, where d', d'' are induced by
$$\begin{aligned} d' &: \mathcal{E}(X) \to \mathcal{E}^{1,0}(X), f \mapsto d'f := \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j, \\ d'' &: \mathcal{E}(X) \to \mathcal{E}^{0,1}(X), f \mapsto d''f := \sum_{j=1}^n \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j}. \end{aligned}$$

6. The equations

$$\begin{aligned} d' \circ d' &= d'' \circ d'' = 0, \\ d' \circ d'' + d'' \circ d' &= 0 \end{aligned}$$

hold.

7. We have

$$\begin{aligned} \overline{d' f} &= d'' \overline{f}, \\ \overline{d'' f} &= d' \overline{f}. \end{aligned}$$

Remark 5.1.12. We encountered the mappings d', d'' already in Chapter 1, where we used the notation $\partial f = d' f$ and $\overline{\partial} f = d'' f$ in (1.5). For the rest of the book we will use the more general d', d'', because they apply to both functions and differential forms. We also already know that a function $f \in \mathcal{E}(X)$ is holomorphic if and only if $d'' f = 0$, which is just a short formulation of the Cauchy–Riemann Differential Equations. This leads to the notation

$$\begin{aligned} \Omega^p(X) &:= \ker\left(\mathcal{E}^{p,0}(X) \xrightarrow{d''} \mathcal{E}^{p,1}(X)\right) \\ &= \left\{ \omega = \sum_{finite} a_I dz_I \;\middle|\; a_I \in \mathcal{O}(X) \right\}. \end{aligned} \tag{5.10}$$

$\Omega^p(X)$ is called the space of *holomorphic differential forms of degree* p on X.

Definition 5.1.13. A holomorphic differential form $\omega \in \Omega^p(X)$ is called *exact* or *total* if there exists a $\eta \in \Omega^{p-1}(X)$, called a *primitive* of ω, such that $\omega = d\eta$. ω is called *closed* if $d\omega = 0$.

Lemma 5.1.14. Every exact differential form is closed.

Proof. This follows from $d \circ d = 0$. □

Example 5.1.15. The differential form

$$\omega = dz_1 \wedge dz_2 \in \Omega^2(\mathbb{C}^n), n \geq 2,$$

is exact, because $\omega = d\eta$ with $\eta = z_1 dz_2$. The differential form

$$\omega = z_1 dz_2 \wedge dz_3 \in \Omega^2(\mathbb{C}^n), n \geq 3$$

is not exact, because

$$d\omega = dz_1 \wedge dz_2 \wedge dz_3 \neq 0,$$

thus, ω is not closed.

Remark 5.1.16. It can be shown that on an open subset X of $\mathbb{C}^n$ the equation $d\omega = \eta$ can always be solved locally, i.e., for every $a \in X$ there is an open neighbourhood U, such that the equation $d\omega = \eta$ has a solution that is valid on U. This follows from a result known as *Poincaré's Lemma*, which will be discussed in the exercises. For this and also in the following chapters we need the following notion from general algebra(see, for example, [7]).

Definition 5.1.17. A sequence $\left(E_j, \varphi_j\right)_{j\in\mathbb{N}}$ of vector spaces (groups, rings, modules,...) E_j and corresponding homomorphisms $\varphi_j : E_j \to E_{j+1}$ is called an *exact sequence* of vector spaces (groups, rings, modules,...), if $\operatorname{img} \varphi_j = \ker \varphi_{j+1}$ for all j.

Example 5.1.18. Let G be a group with neutral element e and $N \lhd G$ a normal subgroup. Let *inc* denote the natural inclusion, $\pi : G \to G/N$ the canonical projection

$$\pi(g) := g * N := \{\, g * n \mid n \in N \}$$

for all $g \in G$ and let ϕ be the trivial homomorphism $\phi(g) := e$ for all $g \in G$. Then the sequence

$$\{e\} \overset{inc}{\to} N \overset{inc}{\to} G \overset{\pi}{\to} G/N \overset{\phi}{\to} \{e\} \tag{5.11}$$

is exact. It is customary to write 0 instead of $\{e\}$ and to omit the trivial homomorphisms *inc* and ϕ if the choice is clear from the context, so the sequence (5.11) would usually be written

$$0 \to N \to G \overset{\pi}{\to} G/N \to 0.$$

Definition 5.1.19. An exact sequence of the form

$$0 \to E_1 \overset{\varphi_1}{\to} E_2 \overset{\varphi_2}{\to} E_3 \to 0 \tag{5.12}$$

is called a *short exact sequence.*

Lemma 5.1.20. If $0 \to E_1 \overset{\varphi_1}{\to} E_2 \overset{\varphi_2}{\to} E_3 \to 0$ is a short exact sequence, then φ_1 is injective and φ_2 is surjective.

Proof. By definition $\ker \varphi_1 = \operatorname{img} inc = 0$, hence, φ_1 is injective. φ_2 is surjective, because $\operatorname{img} \varphi_2 = \ker \phi = E_3$. □

Exercise 5.1.21. Let $X \subset \mathbb{C}$ be a simply connected domain, $A \subset X$ a discrete subset and

$$\omega := f dz \in \Omega^1 (X \setminus A).$$

Show that ω is exact if and only if $\operatorname{res}_a f = 0$ for all $a \in A$.

Exercise 5.1.22. Let $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be a holomorphic function and $\omega := \frac{f'}{f} dz$. Determine a function $g \in \mathcal{O}(\mathbb{C})$ such that $dg = \omega$.

Exercise 5.1.23. Let $X \subset \mathbb{C}^n$ be an open set and $F = (f_1, \ldots, f_n) \in \mathcal{O}(X, \mathbb{C}^n)$. Find a relation between $df_1 \wedge \cdots \wedge df_n$ and $dz_1 \wedge \cdots \wedge dz_n$.

Exercise 5.1.24. Let $0 \leq p, q \leq n$, X, Y, Z be open subsets of $\mathbb{C}^n$and

$$f : Z \to Y,\ g : Y \to X$$

holomorphic mappings. For $\omega \in \mathcal{E}^{p,q}(Y)$ we define the *pullback* f^* by $f^*\omega := \omega \circ f$. Please show:

1. f^* is a linear mapping
$$f^* : \mathcal{E}^{p,q}(Y) \to \mathcal{E}^{p,q}(Z),$$
which satisfies
$$(g \circ f)^* = f^* \circ g^*,\ \mathrm{id}_X^* = \mathrm{id}_{\mathcal{E}^{p,q}(X)}.$$

2. The pullback commutes with all outer derivatives, i.e., if $\partial \in \{d, d', d''\}$ then
$$f^*(\partial\omega) = \partial(f^*\omega).$$

Exercise 5.1.25. Let $1 \leq p \leq n$ and $X \subset \mathbb{C}^n$ be a domain, which is star-shaped with respect to the origin. We put

$$\sigma_{i_1 \ldots i_p} := \sum_{k=1}^{p} (-1)^{k-1} z_{i_k} dz_{i_1} \wedge \cdots \wedge \widehat{dz_{i_k}} \wedge dz_{i_k+1} \wedge \cdots \wedge dz_{i_p},$$

where $\widehat{dz_{i_k}}$ means that this factor is omitted. Let

$$\eta = f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \in \Omega^p(X).$$

We define an operator $I : \Omega^p(X) \to \Omega^{p-1}(X)$ by

$$I\eta(z) := \left(\int_0^1 t^{p-1} f(tz)\, dt\right) \sigma_{i_1 \ldots i_p}$$

and extension by linearity. Please show:

1. The equation $d\sigma_{1 \ldots p} = p dz_1 \wedge \cdots \wedge dz_p$ holds.
2. For $j = 1, \ldots, n$ we have $dz_j \wedge \sigma_{1 \ldots p} = -\sigma_{j1 \ldots p} + z_j dz_1 \wedge \cdots \wedge dz_p$.
3. For all $\omega \in \Omega^p(X)$ we have $I(d\omega) + d(I\omega) = \omega$.
4. (Poincaré's Lemma) The sequence
$$0 \to \mathbb{C} \to \mathcal{O}(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \to 0$$
is exact. ($\mathbb{C}$ is identified with the set of constant functions on X.)

5. Interpret Poincaré's Lemma in the case $n = 1$. Which classical theorem in function theory of one complex variable corresponds to it?

Exercise 5.1.26. Let $X \subset \mathbb{C}^n$ be an open set, which is star-shaped with respect to the origin and let $b_1, b_2, b_3 \in \mathcal{O}(X)$ such that

$$\sum_{j=1}^{3} \frac{\partial b_j}{\partial z_j} = 0 \text{ on } X.$$

Show that the system of partial differential equations

$$\begin{aligned} \frac{\partial f_3}{\partial z_2} - \frac{\partial f_2}{\partial z_3} &= b_1, \\ \frac{\partial f_1}{\partial z_3} - \frac{\partial f_3}{\partial z_1} &= b_2, \\ \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} &= b_3 \end{aligned}$$

has a solution $f = (f_1, f_2, f_3)$, $f_j \in \mathcal{O}(X)$. Is this solution unique? (Hint: Apply Poincaré's Lemma.)

Remark 5.1.27. The first statement of Remark 5.1.16 follows from Poincaré's Lemma in the following way: if X is an open subset of $\mathbb{C}^n$, every $a \in X$ has a star-shaped open neighbourhood $U \subset X$, for instance, an open ball, on which Poincaré's Lemma holds. In particular, Poincaré's Lemma says that the mapping

$$\Omega^{n-1}(U) \xrightarrow{d} \Omega^n(U)$$

is surjective. Note that even though the equation $d\omega = \eta$ always has a local solution near every point $a \in X$ this does not mean that it has a global solution. For example, take $n = 1$, $X = \mathbb{C} \setminus \{0\}$ and $\eta = \frac{dz}{z}$. Finding a global solution ω of the equation $d\omega = \eta$ would mean finding a global logarithm function on all of $\mathbb{C} \setminus \{0\}$, which does not exist. When diving deeper into the question of the existence of global solutions of the equation $d\omega = \eta$ it is well known that one encounters merely topological obstructions, which lead to the notion of the so-called *de Rham cohomology groups*. The interested reader may refer to [9]for more details on this matter. General cohomology theory is studied in depth in Algebraic Topology.

5.2 The inhomogenous Cauchy–Riemann Differential Equations

A differentiable function f of one complex variable is holomorphic if f is a solution of the homogenous Cauchy–Riemann Differential Equation

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

It is natural to ask also for solutions of the more general equation

$$\frac{\partial f}{\partial \overline{z}} = \varphi,$$

where φ is a differentiable function. We will answer this question, at least in a special case.

Proposition 5.2.1. Let $U \subset \mathbb{C}^n$ be an open set and let $\varphi : \mathbb{C}\times U \to \mathbb{C}$ be a smooth function such that for all $w \in U$ the function

$$\zeta \mapsto \varphi(\zeta, w)$$

has compact support. Define $f : \mathbb{C} \times U \to \mathbb{C}$ by

$$f(z, w) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(\zeta, w)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Then f is smooth on $\mathbb{C} \times U$ and satisfies

$$\frac{\partial f}{\partial \overline{z}} = \varphi.$$

Proof. Let $z \in \mathbb{C}$ be fixed. Every $\zeta \in \mathbb{C}$ can be expressed in polar coordinates

$$\zeta = z + r \exp(it),$$

so $d\zeta \wedge d\overline{\zeta} = -2ir\ dr \wedge dt$. Hence,

$$f(z, w) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \varphi(z + r\exp(it), w)\, dr \wedge dt,$$

which shows that f exists and has the stated properties with respect to the parameter w. Now let $(a, w) \in \mathbb{C} \times U$ be fixed. Then, if we replace ζ by $\zeta + z$ in the integral,

$$\frac{\partial f}{\partial \overline{z}}(z, w) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \varphi}{\partial \overline{z}}(z + \zeta, w) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta}.$$

Replacing $\frac{\partial}{\partial \overline{z}}$ by $\frac{\partial}{\partial \overline{\zeta}}$ and ζ by $\zeta - a$ we find

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}}(a, w) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta, w) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - a} \\ &= -\frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int\limits_{|\zeta - a| \geq \varepsilon} d\left(\frac{\varphi(\zeta, w)}{\zeta - a} d\zeta\right). \end{aligned}$$

By Stokes' Theorem, applied to $\overline{B_r(a)} \setminus B_\varepsilon(a)$ for suitable $r > \varepsilon$, the latter equals

$$\begin{aligned} \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int\limits_{|\zeta - a| = \varepsilon} \frac{\varphi(\zeta, w)}{\zeta - a} d\zeta &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int_0^{2\pi} \varphi\left(a + \varepsilon e^{it}, w\right) dt \\ &= \varphi(a, w). \end{aligned}$$

$\square$

5.3 Dolbeaut's Lemma

Proposition 5.3.1. Let $P' \subset P \subset \mathbb{C}^n$ be polydiscs such that P' is concentric and $\overline{P'}$ is a compact subset of P. Let $p \geq 0, q \geq 1$. Then to every $\omega \in \mathcal{E}^{p,q}(P)$ satisfying $d''\omega = 0$ there exists some $\eta \in \mathcal{E}^{p,q-1}(P')$ such that

$$d''\eta = \omega|_{P'}.$$

Proof. For $\nu = 0, \ldots, n$ we use the following notation for index sets $I \subset \mathbb{N}$:

$$I \leq \nu :\Longleftrightarrow \max I \leq \nu.$$

Put

$$A_\nu(P) := \left\{ \omega \in \mathcal{E}^{p,q}(P) \,\middle|\, \omega = \sum_{I,J \leq \nu} a_{I,J} dz_I \wedge d\overline{z_J} \right\}.$$

Then $\{0\} = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{E}^{p,q}(P)$. We use induction on ν. If $\nu = 0$ then $\omega = 0$, because $q \geq 1$, so we can choose $\eta := 0$. Now assume the proposition holds for $j = 0, \ldots, \nu - 1$. Choose some open concentric polydisc P'' such that $\overline{P''}$ is compact and

$$\overline{P'} \subset P'' \subset \overline{P''} \subset P$$

and let $\omega \in A_\nu(P)$ satisfy $d''\omega = 0$. ω can be written in the form

$$\omega = \sum_{I,J \leq \nu} a_{I,J} dz_I \wedge d\overline{z_J}.$$

It follows from $J \leq \nu$ that

$$\frac{\partial a_{I,J}}{\partial \overline{z_k}} = 0 \text{ for all } k > \nu,$$

i.e., the coefficients $a_{I,J}$ depend holomorphically on $z_{\nu+1}, \ldots, z_n$. Now we can decompose ω into

$$\omega = d\overline{z_\nu} \wedge \sigma + \beta,$$

where $\beta \in A_{\nu-1}(P)$ and

$$\sigma = \sum_{I,J \leq \nu-1} b_{I,J} dz_I \wedge d\overline{z_J}$$

with coefficient $b_{I,J}$, which coincide with some $a_{I',J'}$ up to the sign. In particular, they depend holomorphically upon $z_{\nu+1}, \ldots, z_n$. There exists a smooth function $\chi : P \to \mathbb{C}$ with compact support such that

$$\chi|_{\overline{P''}} = 1.$$

Proposition 5.2.1 states that for $\chi \cdot b_{I,J}$ there exist smooth functions $c_{I,J} \in \mathcal{E}(P'')$ such that

$$\frac{\partial c_{I,J}}{\partial \overline{z_\nu}} = b_{I,J}|_{P''},$$

and the $c_{I,J}$ depend holomorphically upon $z_{\nu+1}, \ldots, z_n$. Put

$$\gamma := \sum_{I,J \leq \nu-1} c_{I,J} dz_I \wedge d\overline{z_J}.$$

Then

$$d''\gamma = d\overline{z_\nu} \wedge \sigma|_{P''} + \delta$$

with some $\delta \in A_{\nu-1}(P'')$. Thus, $\omega - d''\gamma|_{P''} = \beta|_{X''} - \delta$. In particular,

$$\beta|_{P''} - \delta \in A_{\nu-1}(P''),\ d''(\beta|_{P''} - \delta) = 0.$$

The induction hypothesis yields that there is $\tau \in \mathcal{E}^{p,q-1}(P')$ such that

$$d''\tau = (\beta - \delta)|_{P'}.$$

Choose $\eta := \gamma|_{P'} + \tau$. Then $\eta \in \mathcal{E}^{p,q-1}(P')$ and

$$\begin{aligned} d''\eta &= d''\gamma|_{P'} + d''\tau \\ &= \omega|_{P'} + (\beta - \delta)|_{P'} + (\beta - \delta)|_{P'} \\ &= \omega|_{P'}, \end{aligned}$$

proving the proposition. □

Lemma 5.3.2. Let $(E_k)_{k\geq 0}$ be a family of complete metric spaces and let

$$\gamma_k : E_k \to E_{k-1},\ k \geq 1$$

be continuous mappings such that $\gamma_k(E_k)$ is dense in E_{k-1} for all $k \geq 1$. Then there exists a sequence $(x_k)_{k\geq 0}$ with $x_k \in E_k$ such that

$$\gamma_k(x_k) = x_{k-1}.$$

Proof. Let $x_1 \in E_1$ be an arbitrary element. Put $x_0 := \gamma_1(x_1)$ and define

$$x_{00} := x_{01} := x_0,\ x_{10} := x_{11} := x_1.$$

Let $\varepsilon > 0$ be arbitrary. Then $x_{jk} \in E_j$, $j = 0, 1$ and

$$\gamma_1(x_{j1}) = x_{j-1,1},\ d(x_{j1}, x_{j0}) = 0 < 2^{-1}\varepsilon.$$

By induction on $k \geq 1$ we obtain elements $x_{jk} \in E_j,\ j = 0, \ldots, k$, such that

$$\gamma_j(x_{jk}) = x_{j-1,k},\ d(x_{jk}, x_{j,k-1}) < 2^{-k}\varepsilon.$$

Then, if $k \geq l+1 \geq j$, the triangle inequality yields

$$\begin{aligned} d(x_{jk}, x_{jl}) &\leq \sum_{m=0}^{k-(l+1)} d(x_{j,k-m}, x_{j,k-m-1}) \\ &\leq \sum_{m=0}^{k-(l+1)} 2^{-k+m}\varepsilon = \varepsilon\left(2^{-l} - 2^{-k}\right), \end{aligned}$$

which shows that $(x_{jk})_{k\geq 0}$ is a Cauchy sequence in E_j. Hence, since E_j is complete, $x_j := \lim_{k\to\infty} x_{jk}$ exists in E_j. x_j satisfies

$$\gamma_j(x_j) = \lim_{k\to\infty} \gamma_j(x_{jk}) = \lim_{k\to\infty} x_{j-1,k} = x_{j-1} \in E_{j-1}$$

by continuity of γ_j. □

Theorem 5.3.3 (Dolbeaut's Lemma). *Let $P \subset \mathbb{C}^n$ be a polydisc and $p+q=n$. Then the sequence*

$$0 \to \Omega^p(P) \to \mathcal{E}^{p,0}(P) \xrightarrow{d''} \mathcal{E}^{p,1}(P) \xrightarrow{d''} \cdots \xrightarrow{d''} \mathcal{E}^{p,q}(P) \to 0$$

is exact.

Proof. Let $q \geq 1$ and $\omega \in \mathcal{E}^{p,q}(P)$ such that $d''\omega = 0$. We have to show that there is some $\eta \in \mathcal{E}^{p,q-1}(P)$ such that $d''\eta = \omega$. Let

$$P_0 \subset \overline{P_0} \subset P_1 \subset \overline{P_1} \subset \cdots \subset P$$

be an exhaustion of P by concentric and relatively compact polydiscs. Define for $k \in \mathbb{N}$,

$$M_k := \left\{\eta \in \mathcal{E}^{p,q-1}(P_k) \mid d''\eta = \omega|_{P_k}\right\}.$$

Lemma 5.3.1 says that $M_k \neq \emptyset$, so we have

$$\emptyset \neq M_0 \subset M_1 \subset M_2 \subset \cdots$$

(with the suitable restrictions of the elements). Let $\eta_k \in M_k$. We proceed by induction on q. If $q=1$ and $\eta', \eta'' \in M_k$, then $d''(\eta' - \eta'') = 0$, i.e., $\eta' - \eta'' \in \Omega^p(P_k)$. Consider the mappings

$$\begin{aligned} \psi_k &: \Omega^p(P_k) \to M_k, \eta \mapsto \eta + \eta_k, \\ \tau_k &: \Omega^p(P_k) \to \Omega^p(P_{k-1}), \eta \mapsto \eta + \left(\eta_k - \eta_{k-1}\right). \end{aligned}$$

ψ_k is surjective, τ_k is continuous and has dense image, because the Taylor polynomials of the coefficients of $\eta+\eta_k-\eta_{k-1}$ approximate the coefficients of an arbitrary element of $\Omega^p(P_{k-1})$. Furthermore, if $\rho_k : M_k \to M_{k-1}$ denotes the restriction, the diagram

$$\begin{array}{ccc} \Omega^p(P_k) & \xrightarrow{\tau_k} & \Omega^p(P_{k-1}) \\ \psi_k \downarrow & & \downarrow \psi_{k-1} \\ M_k & \xrightarrow{\rho_k} & M_{k-1} \end{array}$$

commutes. From Lemma 5.3.2 we obtain holomorphic p-forms $\sigma_k \in \Omega^p(P_k)$, such that $\tau_k(\sigma_k) = \sigma_{k-1}$. Put

$$\widetilde{\eta_k} := \psi_k(\sigma_k) = \sigma_k + \eta_k.$$

Then $\widetilde{\eta_k}|_{P_{k-1}} = \widetilde{\eta_{k-1}}$. Hence, we can well-define $\eta \in \mathcal{E}^{p,0}(P)$ by $\eta|_{P_k} := \widetilde{\eta_k}$ for all k. Then

$$d''\eta|_{P_k} = d''\sigma_k + d''\eta_k = 0 + \omega|_{P_k},$$

i.e., $d''\eta = \omega$. The case $q = 1$ is thus proven. Let now $q \geq 2$ and $\eta', \eta'' \in M_k$. From the induction hypothesis we obtain $\sigma \in \mathcal{E}^{p,q-1}(P)$ such that $d''\sigma = \eta' - \eta''$, i.e., the mapping

$$\psi_k : \mathcal{E}^{p,q-2}(P_k) \to M_k, \sigma \mapsto d''\sigma + \eta_k$$

is surjective. Define

$$\tau_k : \mathcal{E}^{p,q-2}(P_k) \to \mathcal{E}^{p,q-2}(P_{k-1}), \sigma \mapsto \sigma + \gamma_{k-1},$$

where $d''\gamma_{k-1} = \sigma_k - \sigma_{k-1}$. τ_k is continuous and since to every compact set $K \subset P_{k-1}$ there exists a smooth function $s \in \mathcal{E}(P_k)$, whose support is contained in P_{k-1} and which satisfies $s|_K = 1$, τ_k has dense image. The diagram

$$\begin{array}{ccc} \mathcal{E}^{p,q-2}(P_k) & \xrightarrow{\tau_k} & \mathcal{E}^{p,q-2}(P_{k-1}) \\ \psi_k \downarrow & & \downarrow \psi_{k-1} \\ M_k & \xrightarrow{\rho_k} & M_{k-1} \end{array}$$

commutes. It follows again from Lemma 5.3.2 that there are $\sigma_k \in \mathcal{E}^{p,q-2}(P_k)$ such that $\tau_k(\sigma_k) = \sigma_{k-1}$. Finally, define $\eta \in \mathcal{E}^{p,q-1}(P)$ by

$$\eta|_{P_k} := \eta_k + d''\sigma_k.$$

Then $d''\eta|_{P_k} = d''\eta_k = \omega|_{P_k}$, i.e., $d''\eta = \omega$. □

Corollary 5.3.4. *Let $n \geq 2$ and let $\omega \in \mathcal{E}^{0,1}(\mathbb{C}^n)$ have compact support and satisfy $d''\omega = 0$. Then there is a smooth function $f : \mathbb{C}^n \to \mathbb{C}$ with compact support such that $d''f = \omega$.*

Proof. By Dolbeaut's Lemma there is a smooth function $h \in \mathcal{E}(\mathbb{C}^n) = \mathcal{E}^{0,0}(\mathbb{C}^n)$ such that $d''h = \omega$. Let K be the support of ω. There is a bounded ball B such that $K \subset B \subset \overline{B}$. On $\mathbb{C}^n \setminus \overline{B}$ we have $d''h = 0$, i.e., h is holomorphic outside $\overline{B}$. It follows from Theorem 2.3.1 that h has a unique holomorphic continuation g to $\mathbb{C}^n$. Put $f := h - g$. Then the support $\operatorname{supp} f$ of f is closed per definition and contained in $\overline{B}$, so it is compact and $d''f = d''h - d''g = d''h = \omega$. □

5.4 The Kugelsatz of Hartogs

Hartogs' Kugelsatz generalizes Theorem 2.3.1. In order to give a proof we need some basic results from general topology.

Lemma 5.4.1. Let X be a locally connected topological space, $W \subset X$ and $a, b \in W$. Then the following defines an equivalence relation $\sim$ on X :

$$a \sim b :\Longleftrightarrow \text{ There is a connected set } U \subset W \text{ such that } a, b \in U.$$

Proof. $a \sim a$, because X is locally connected. Symmetry is trivial. If $a \sim b$ and $b \sim c$ then there are connected sets $U, V \subset W$ such that $a, b \in U$, $b, c \in V$. Because $b \in U \cap V$ the set $U \cup V$ is connected and contains a, b, c, hence, $a \sim c$. $\square$

Definition 5.4.2. The equivalence class of an element $a \in W$ under the above relation is called the *connected component* of a in W. We denote it by $\widetilde{W}^a$. If a is not specified $\widetilde{W}$ denotes any connected component of W.

Lemma 5.4.3. Let X be a locally connected topological space, $W \subset X$ an open subset and $a \in W$. Then the following holds:

1.
$$\widetilde{W}^a = \bigcup_{\substack{V \subset W \\ V \text{ connected} \\ a \in V}} V. \tag{5.13}$$

2. $\widetilde{W}$ is open in X.

3. Let K be a compact subset of $\mathbb{C}^n$. Then $\mathbb{C}^n \setminus K$ contains exactly one unbounded connected component.

Proof. 1. If $b \in \widetilde{W}^a$ there is a connected subset $U \subset X$ that contains a and b, i.e., the inclusion "$\subset$" holds. On the other hand, if b is contained in some connected set V, which also contains a, then $b \sim a$, hence, $b \in \widetilde{W}^a$.

2. The assertion is trivial if $W = \emptyset$. If $W \neq \emptyset$ let $a \in \widetilde{W}$, i.e., $\widetilde{W} = \widetilde{W}^a$ Since W is open and X is locally connected there is a connected open neighbourhood U of a, which is contained in W. Thus, $U \subset \widetilde{W}^a$, which shows that $\widetilde{W}^a$ is open.

3. Put $V := \mathbb{C}^n \setminus K$. Since K is compact, it is closed and bounded. Therefore, V is open and there is some $r > 0$ such that $K \subset B_r^n(0)$. Then $\mathbb{C}^n \setminus B_r^n(0)$ is an unbounded and connected set, which is contained in V.Choose

$$a \in \mathbb{C}^n \setminus B_r^n(0) \subset V$$

and let $\widetilde{V}^a$ denote its connected component in V. Then $\mathbb{C}^n \setminus B_r^n(0) \subset \widetilde{V}^a$, hence, $\widetilde{V}^a$ is an unbounded connected component in V. If $\widetilde{V}$ is an arbitrary unbounded connected component of V then

$$\widetilde{V} \cap (\mathbb{C}^n \setminus B_r^n(0)) \neq \emptyset.$$

Since $\mathbb{C}^n \setminus B_r^n(0) \subset \widetilde{V}^a$, we conclude $\widetilde{V} = \widetilde{V}^a$. $\square$

Theorem 5.4.4 (Kugelsatz). *Let X be an open set in $\mathbb{C}^n, n \geq 2$ and $K \subset X$ a compact subset such that $X \setminus K$ is connected. Then the restriction*

$$\rho : \mathcal{O}(X) \to \mathcal{O}(X \setminus K)$$

is an isomorphism of $\mathbb{C}$*-algebras.*

Proof. As usual, we only prove that ρ is surjective. Choose an open and relatively compact set C such that $K \subset C \subset X$ and a smooth function $\varphi : X \to [0,1]$ with compact support $\operatorname{supp} \varphi$ such that

$$\varphi|_C = 1.$$

Let $f \in \mathcal{O}(X \setminus K)$ and define the smooth function

$$h : X \to \mathbb{C}, z \mapsto \begin{cases} (1 - \varphi(z)) f(z), & \text{if } z \in X \setminus K \\ 0, & \text{if } z \in K \end{cases}.$$

Then

$$h|_{X \setminus \operatorname{supp} \varphi} = f|_{X \setminus \operatorname{supp} \varphi}. \tag{5.14}$$

In particular, we have $d''f = d''h = 0$ outside $\operatorname{supp} \varphi$. $d''h$ can thus be extended by zero outside X, i.e., $d''h \in \mathcal{E}^{0,1}(\mathbb{C}^n)$. The support of $d''h$ is a closed subset of the compact set $\operatorname{supp} \varphi$, hence it is itself compact. We obtain from Corollary 5.3.4 that there is a function $g \in \mathcal{E}(\mathbb{C}^n)$ with compact support, which satisfies $d''g = d''h$. g is holomorphic on $\mathbb{C}^n \setminus \operatorname{supp} \varphi$, because of $d''h = 0$ outside $\operatorname{supp} \varphi$. Since $d''h = d''g$, we can define a holomorphic function $F : X \to \mathbb{C}$ by

$$F := h - g.$$

We claim that F is the desired holomorphic extension of f. This can be seen as follows. If W denotes the unique unbounded connected component of $\mathbb{C}^n \setminus \operatorname{supp} \varphi$ the Identity Theorem implies that g vanishes on W, because $\operatorname{supp} g$ is compact. It follows from the fact that X is an open set and from

$$\partial W \subset \partial (\mathbb{C}^n \setminus \operatorname{supp} \varphi) = \partial \operatorname{supp} \varphi \subset X$$

that $X \cap W \neq \emptyset$. Now note that $X \cap W$ is an open set and that

$$\emptyset \neq X \cap W \subset X \cap (\mathbb{C}^n \setminus \operatorname{supp} \varphi) = X \setminus \operatorname{supp} \varphi.$$

It follows from (5.14) that

$$F|_{X \cap W} = h|_{X \cap W} = f|_{X \cap W}.$$

From $\varphi|_K = 1$ we obtain $K \subset \operatorname{supp} \varphi$, thus, $X \setminus \operatorname{supp} \varphi \subset X \setminus K$. By prerequisite $X \setminus K$ is a domain and $F|_{X \setminus K}$ and f coincide on the open nonempty subset $X \cap W$. Hence, the Identity Theorem implies

$$F|_{X \setminus K} = f,$$

and the proof is complete. □

Corollary 5.4.5. *Let D be a domain in $\mathbb{C}^n, n \geq 2$, and $f \in \mathcal{O}(D)$. Then the zero set $N(f)$ is not compact.*

Proof. If f is the zero function the proof is trivial. Let $f \neq 0$. It was shown in Example 4.1.5 that in this case the analytic set $N(f)$ has codimension 1. Corollary 4.2.2 states that $D \setminus N(f)$ is connected. The function $\frac{1}{f}$ is holomorphic on $D \setminus N(f)$. If $N(f)$ were compact the Kugelsatz would imply that there is a holomorphic function $F \in \mathcal{O}(D)$ such that

$$F|_{D \setminus N(f)} = \frac{1}{f}.$$

Multiplication by f yields $F \cdot f = 1$ on $D \setminus N(f)$. Since f is not the zero function, $D \setminus N(f)$ is a nonempty open subset of the connected set $D \setminus N(f)$, thus, the Identity Theorem says that $F \cdot f = 1$ on all of D, contradicting $f|_{N(f)} = 0$. $\square$

Example 5.4.6. The statement of the corollary does not hold for real analytic functions. For example, the zero set of the function

$$f : \mathbb{C}^n \to \mathbb{C}, z \mapsto \|z\|_2^2 - 1$$

is the compact unit sphere in $\mathbb{C}^n$.

Example 5.4.7. The zero set of the polynomial $p(z_1, \ldots, z_n) := z_1$ is $\{0\} \times \mathbb{C}^{n-1}$, which is not compact if $n > 1$.

Exercise 5.4.8. Give examples that the Kugelsatz does no longer hold if $X \setminus K$ is not connected or if $n = 1$.

Exercise 5.4.9. Let $n \geq 2$ and $K \subset \mathbb{C}^n$ be a compact and convex set. Let $X := \mathbb{C}^n \setminus K$ and $f \in \mathcal{O}(X)$ be a nonconstant function.

1. Show that there is a sequence $(z_j)_{j \in \mathbb{N}}$ in X such that $\lim\limits_{j \to \infty} |f(z_j)| = \infty$.
2. Can one of the prerequisites *compact* resp. *convex* be weakened such that 1. still holds?
3. Does 1. also hold if we only demand f to be smooth?

Exercise 5.4.10. Let D be a bounded domain in $\mathbb{C}^n, n \geq 2$, with connected boundary ∂D. Let

$$\mathcal{A}(D) = \left\{ f \in \mathcal{C}\left(\overline{D}\right) \mid f|_D \in \mathcal{O}(D) \right\}$$

and

$$\rho : \mathcal{A}(D) \to \mathcal{C}(\partial D), f \mapsto f|_{\partial D}.$$

1. Let $f \in \mathcal{A}(D)$. Find the implications in the following two statements:
   (a) $N(f) \cap X \neq \emptyset$.
   (b) $N(\rho(f)) \neq \emptyset$.
2. Determine the set $\{f \in \mathcal{A}(D) \mid |\rho(f)| = \mathit{const.}\}$.

Chapter 6

Continuation on Tubular Domains

An especially important class of domains in $\mathbb{C}^n$ are *tubular domains (*or *tube domain)*, i.e., domains of the form $D := \Omega + i\mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$. Ω is called the *basis* of the tubular domain D. In particular, $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ is a tubular domain with basis $\mathbb{R}^n$. The aim of this chapter is to give a proof of a continuation theorem due to Bochner, which states that every function holomorphic on a tubular domain D can be holomorphically extended to the convex hull of D. Bochner's Theorem, unlike Hartogs' Kugelsatz, holds also in dimension 1. However, if $n = 1$ the theorem is trivial, because every tube domain in $\mathbb{C}$ coincides with its convex hull.

6.1 Convex hulls

Recall from Chapter 1 that for a subset X of a real vector space the convex hull $\operatorname{conv}(X)$ of X is the smallest convex set that contains X, i.e.,

$$\operatorname{conv}(X) = \bigcap_{\substack{C \text{ convex} \\ C \supset X}} C.$$

If $X = \{x_0, \ldots, x_m\}$ is a finite set, then $\operatorname{conv}(X)$ is called the *simplex* spanned by $x_0, \ldots, x_m$.

Example 6.1.1. The convex hull of the set $\{0, 1, i\} \subset \mathbb{C}$ is the closed triangular surface Δ with corners $0, 1, i$.

In $\mathbb{R}^n$ the convex hull of a set X can be computed as the union of all simplices spanned by a maximum of $n+1$ elements of X. This is the content of the following lemma due to Carathéodory.

Lemma 6.1.2 (Carathéodory). Let $X \subset \mathbb{R}^n$. Then the convex hull of X is given by

$$\operatorname{conv}(X) = \left\{ \sum_{j=0}^{n} \lambda_j x_j \;\middle|\; x_j \in X, \lambda_j \in [0,1], \sum_{j=0}^{n} \lambda_j = 1 \right\}.$$

Proof. "$\supset$": Let $x_0, \ldots, x_m \in X$ and let $\lambda_j \in [0,1], \sum_{j=0}^{m} \lambda_j = 1$. We show by induction on $m \geq 1$ that $\sum_{j=0}^{m} \lambda_j x_j \in \operatorname{conv}(X)$. If $m = 1$ this is an immediate consequence of the convexity of $\operatorname{conv}(X)$. Assume that the proposition holds for $m-1$. If $\lambda_m = 1$ then all other λ_j vanish and nothing is to show. If $\lambda_m < 1$ then

$$\sum_{j=0}^{m} \lambda_j x_j = \lambda_m x_m + (1-\lambda_m) \sum_{j=0}^{m-1} \frac{\lambda_j}{1-\lambda_m} x_j.$$

Because of $\sum_{j=0}^{m-1} \frac{\lambda_j}{1-\lambda_m} = 1$ the induction hypothesis yields that

$$y_m := \sum_{j=0}^{m-1} \frac{\lambda_j}{1-\lambda_m} x_j \in \operatorname{conv}(X).$$

Hence,

$$\sum_{j=0}^{m} \lambda_j x_j = \lambda_m x_m + (1-\lambda_m)\, y_m \in \operatorname{conv}(X),$$

because it is a convex combination of the two elements $x_m, y_m \in \operatorname{conv}(X)$.

"$\supset$": Using the first part of the proof it is easy to see that

$$\operatorname{conv}(X) = \bigcup_{m \geq 1} \left\{ \sum_{j=0}^{m} \lambda_j x_j \;\middle|\; x_j \in X, \lambda_j \in [0,1], \sum_{j=0}^{n} \lambda_j = 1 \right\}.$$

To complete the proof we have to show that for $m > n$ every convex combination $\sum_{j=0}^{m} \lambda_j x_j$ can be written as a convex combination $\sum_{j=1}^{m} \widetilde{\lambda_j} x_j$. We may assume that $\lambda_j > 0$, otherwise we simply drop the summand. Since $m > n$ the vectors

$$x_m - x_0, \ldots, x_1 - x_0$$

are linearly dependent in $\mathbb{R}^n$. Thus, there are coefficients $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, m$, not all of them zero, such that

$$0 = \sum_{j=1}^{m} \alpha_j (x_j - x_0).$$

Put $\alpha_0 := -\sum_{j=1}^{m} \alpha_j$. Then $\sum_{j=0}^{m} \alpha_j = 0$ and

$$\sum_{j=0}^{m} \alpha_j x_j = 0.$$

After an eventual renumbering we may assume that $\frac{|\alpha_j|}{\lambda_j} \leq \frac{|\alpha_0|}{\lambda_0}$ for all $j > 0$. In particular, $\alpha_0 \neq 0$. We can now put

$$\widetilde{\lambda_j} := \lambda_j - \frac{\alpha_j \lambda_0}{\alpha_0}, j = 0, \ldots, m.$$

Then $\widetilde{\lambda_0} = 0$, $\widetilde{\lambda_j} \geq 0$ for $j > 0$ and

$$\sum_{j=1}^{m} \widetilde{\lambda_j} = \sum_{j=0}^{m} \widetilde{\lambda_j} = \sum_{j=0}^{m} \lambda_j - \frac{\lambda_0}{\alpha_0} \sum_{j=0}^{m} \alpha_0 = 1 - 0 = 1.$$

Finally,

$$\sum_{j=1}^{m} \widetilde{\lambda_j} x_j = \sum_{j=0}^{m} \lambda_j x_j - \frac{\lambda_0}{\alpha_0} \sum_{j=0}^{m} \alpha_j x_j = \sum_{j=0}^{m} \lambda_j x_j,$$

which completes the proof. □

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ the lemma is valid for subsets of $\mathbb{C}^n$ as well. Another useful, but less geometric characterization of the convex hull of a compact set can be given by means of linear functionals. Let, as usual, $(\mathbb{C}^n)^\#$ denote the algebraic dual of $\mathbb{C}^n$, i.e., the space of all linear functionals $\mu : \mathbb{C}^n \to \mathbb{C}$.

Lemma 6.1.3. If $K \subset \mathbb{C}^n$ is a compact set, then its convex hull is compact and given by

$$\operatorname{conv}(K) = \bigcap_{\mu \in (\mathbb{C}^n)^\#} \left\{ z \in \mathbb{C}^n \,\middle|\, \operatorname{Re} \mu(z) \leq \sup_{\zeta \in K} \operatorname{Re} \mu(\zeta) \right\}. \tag{6.1}$$

Proof. Denote by $S := \left\{ \lambda \in \mathbb{R}^{2n+1} \,\middle|\, \lambda_j \in [0,1], \sum_{j=1}^{n} \lambda_j = 1 \right\}$ the $2n$-dimensional unit simplex. By Carathéodory's Lemma $\operatorname{conv}(K)$ is the image of the continuous mapping

$$K^{2n+1} \times S \to \mathbb{C}^n, (k_0, \ldots, k_{2n}, \lambda_0, \ldots \lambda_{2n}) \mapsto \sum_{j=0}^{2n} \lambda_j k_j.$$

Since K^{2n+1} and S are compact, $\operatorname{conv}(K)$ is compact. Denote the right-hand side of (6.1) by $\widetilde{K}$. For every $\mu \in (\mathbb{C}^n)^\#$ the linear functional $\operatorname{Re} \mu : \mathbb{C}^n \to \mathbb{R}$ is $\mathbb{R}-$linear. Thus, $\widetilde{K}$ is convex. Since $K \subset \widetilde{K}$, we also have $\operatorname{conv}(K) \subset \widetilde{K}$. On the other hand, if $p \notin \operatorname{conv}(K)$ the Hahn–Banach separation theorem (see, e.g., [11], Theorem III.2.4) states that there is an $\mathbb{R}$-linear functional $\widetilde{\mu} : \mathbb{C}^n \to \mathbb{R}$ such that $\widetilde{\mu}(p) > \sup \widetilde{\mu}(K)$. We define $\mu \in (\mathbb{C}^n)^\#$ by the complex-linear extension

$$\mu(z) := \widetilde{\mu}(z) - i\widetilde{\mu}(iz),$$

which satisfies $\operatorname{Re} \mu(p) = \widetilde{\mu}(p) > \sup \widetilde{\mu}(K) = \sup \operatorname{Re} \mu(K)$. □

Exercise 6.1.4. Let $U \subset \mathbb{R}^n$ be a subset. Please show:

1. If U is open then $\operatorname{conv}(U)$ is open in $\mathbb{R}^n$.
2. If U is bounded then $\operatorname{conv}(U)$ is bounded.

Exercise 6.1.5. Determine the convex hull of the sets

$$X_1 := \left\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\right\}$$

and

$$X_2 := X_1 \cup \{(0,0)\}.$$

Sketch the relevant sets.

Exercise 6.1.6. Let $D = \Omega + i\mathbb{R}^n$ be a tubular domain in $\mathbb{C}^n$. Please show:

1. D is convex if and only if Ω is convex.
2. The convex hull satisfies $\operatorname{conv}(\Omega + i\mathbb{R}^n) = \operatorname{conv}(\Omega) + i\mathbb{R}^n$.

Exercise 6.1.7. Let $D = \Omega + i\mathbb{R}^n$ be a tubular domain in $\mathbb{C}^n$ and let $T \in GL_n(\mathbb{R})$ be a regular real matrix.

1. Show that $D' := T(D)$ is a tubular domain, $D' = \Omega' + i\mathbb{R}^n$.
2. If the basis Ω is star-shaped with respect to the origin then so is Ω'.

6.2 Holomorphically convex hulls

Formula (6.1) describes the convex hull of a compact set in terms of linear functionals. Replacing the linear functionals by holomorphic functions leads to the definition of the holomorphically convex hull of a subset $K \subset \mathbb{C}^n$.

Definition 6.2.1. Let $U \subset \mathbb{C}^n$ be open and $K \subset U$. The set

$$\widehat{K}_U := \bigcap_{f \in \mathcal{O}(U)} \left\{ z \in U \;\middle|\; |f(z)| \le \sup_{\zeta \in K} |f(z)| \right\} \tag{6.2}$$

is called the *holomorphically convex hull* of K with respect to U.

From the definition it is immediate that $K \subset \widehat{K}_U$. If K is a compact set we have $\sup |f(K)| < \infty$ in (6.2), otherwise it might happen that this supremum is infinity. In most cases we will consider compact sets K. It is important to notice that the holomorphically convex hull of a set depends on the surrounding open set U.

Remark 6.2.2. Replacing the set $\mathcal{O}(U)$ by an arbitrary family $\mathcal{F}$ of functions on U leads to the general notion of *$\mathcal{F}$-convex hull*. For instance, if we take $\mathcal{F}$ to be the family of monomials

$$\{z^\alpha \mid \alpha \in \mathbb{N}^n\},$$

we obtain the *monomially convex hull.* If $\mathcal{F}$ is the family of polynomials

$$\left\{ \sum_{|\alpha|\leq k} c_\alpha z^\alpha \;\middle|\; k \in \mathbb{N} \right\}$$

we speak of the *polynomially convex hull* and so on.

Example 6.2.3. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Then

$$\widehat{\mathbb{T}}_{\mathbb{C}^\times} = \mathbb{T}$$

and

$$\widehat{\mathbb{T}}_{\mathbb{C}} = \overline{B_1^1(0)}.$$

Proof. The functions $\mathrm{id}_{\mathbb{C}^\times}$ and $\frac{1}{\mathrm{id}_{\mathbb{C}^\times}}$ are holomorphic on $\mathbb{C}^\times$ and both have constant absolute value 1 on $\mathbb{T}$, which shows that $z \in \widehat{\mathbb{T}}_{\mathbb{C}^\times}$ if and only if $|z| = 1$. Since $\mathrm{id}_{\mathbb{C}}$ is holomorphic and has constant absolute value 1 on $\mathbb{T}$, we see that

$$\widehat{\mathbb{T}}_{\mathbb{C}} \subset \overline{B_1^1(0)}.$$

On the other hand, if $f \in \mathcal{O}(\mathbb{C})$ and $|z| = r > 1$ the Maximum Modulus Theorem implies that the function

$$\left| f|_{\overline{B_r^1(0)}} \right| : \overline{B_r^1(0)} \to \mathbb{R}$$

attains its maximum on the circle $T_r^1(0) = \partial \overline{B_r^1(0)}$. Thus, $z \notin \widehat{\mathbb{T}}_{\mathbb{C}}$, which shows that $\overline{B_1^1(0)} \subset \widehat{\mathbb{T}}_{\mathbb{C}}$. □

To get acquainted with the definition the reader should check the following properties of $\widehat{K}_U$.

Lemma 6.2.4. Let $K \subset L \subset U \subset V \subset \mathbb{C}^n$ be subsets such that U, V are open. Then

1. $\widehat{K}_U = \widehat{\left(\widehat{K}_U\right)}_U$.
2. $\widehat{K}_U \subset \widehat{L}_U$.
3. $\widehat{K}_U$ is closed in U.
4. $\widehat{\left(\overline{K} \cap U\right)}_U = \widehat{K}_U$
5. $\widehat{K}_U \subset \widehat{K}_V$.

Lemma 6.2.5. Let $U \subset \mathbb{C}^n$ be open, $K \subset U$ compact. Then the following holds.

1. The holomorphically convex hull is contained in the convex hull: $\widehat{K}_U \subset \operatorname{conv}(K)$.

2. If $M \subset U \setminus K$ is a connected component, which is relatively compact in U, then $M \subset \widehat{K}_U$.

Proof. 1. If $\mu \in (\mathbb{C}^n)^{\#}$ we regard the holomorphic function

$$f := \exp \circ \mu|_U \in \mathcal{O}(U)$$

which satisfies $|f| = \exp \circ \operatorname{Re} \mu$ and thus we obtain from (6.1) that

$$\widehat{K}_U \subset \operatorname{conv}(K).$$

2. Let $f \in \mathcal{O}(D)$. Since $M \subset U \setminus K$ is a relatively compact connected component, we have $\partial M \subset U$ and thus $\partial M \subset K$. The Maximum Modulus Theorem implies that for all $z \in M$,

$$|f(z)| \leq \sup |f(\partial M)| \leq \sup |f(K)|,$$

hence, $z \in \widehat{K}_U$. $\square$

Example 6.2.6. We generalize Example 6.2.3. Let

$$K := \mathbb{S}^{2n-1} := \{z \in \mathbb{C}^n \mid \|z\|_2 = 1\}$$

be the unit sphere. Then

$$\widehat{K}_{\mathbb{C}^n} = \overline{B_1^n(0)}$$

and

$$\widehat{K}_{\mathbb{C}^n \setminus \{0\}} = \begin{cases} \mathbb{T} = \mathbb{S}^1, & \text{if } n = 1 \\ \overline{B_1^n(0)} \setminus \{0\}, & \text{if } n > 1 \end{cases}.$$

Proof. We have $\operatorname{conv}\left(\mathbb{S}^{2n-1}\right) = \overline{B_1^n(0)}$. The only relatively compact connected component of $\mathbb{C}^n \setminus K$ is $B_1^n(0)$. Lemma 6.2.5 gives

$$B_1^n(0) \subset \widehat{K}_{\mathbb{C}^n} \subset \overline{B_1^n(0)}.$$

Since also $K \subset \widehat{K}_{\mathbb{C}^n}$, it follows that

$$\widehat{K}_{\mathbb{C}^n} = \overline{B_1^n(0)}.$$

We know from Example 6.2.3 that for $n = 1$ we have $\widehat{K}_{\mathbb{C} \setminus \{0\}} = \mathbb{T}$. If $n \geq 2$ we can apply the Kugelsatz to extend every function holomorphic on $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{C}^n$ and can thus apply the above argument, which shows that

$$\widehat{K}_{\mathbb{C}^n \setminus \{0\}} = \overline{B_1^n(0)} \setminus \{0\} \text{ for } n \geq 2.$$

Note that $\widehat{K}_{\mathbb{C}^n \setminus \{0\}}$ is compact in $\mathbb{C}^n$ if and only if $n = 1$. $\square$

For a proper subset $U \subset \mathbb{C}^n$ let $U^c := \mathbb{C}^n \setminus U$ denote the complement with respect to $\mathbb{C}^n$. We consider the *boundary distance* function

$$\operatorname{dist}_{\partial U} : \mathbb{C}^n \to \mathbb{R}, z \mapsto \operatorname{dist}_\infty(\partial U, z) := \inf_{w \in \partial U} \{\|z - w\|_\infty\}.$$

For any subset $A \subset \mathbb{C}^n$ we define

$$\operatorname{dist}_\infty(A, \partial U) := \inf_{z \in A} \operatorname{dist}_{\partial U}(z).$$

$\operatorname{dist}_{\partial U}$ is uniformly continuous and its zero set coincides with $\mathbb{C}^n \setminus U$.

Lemma 6.2.7 (Thullen's Lemma). Let $U \subset \mathbb{C}^n$ be a proper open subset and $K \subset U$ compact. If $u \in \mathcal{O}(U)$ satisfies

$$|u(z)| \leq \operatorname{dist}_{\partial U}(z) \text{ for all } z \in K,$$

then for every $f \in \mathcal{O}(U)$ and all $a \in \widehat{K}_U$ the Taylor series

$$\sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(a)}{\alpha!} (z-a)^\alpha$$

converges to f on $\{z \in U \mid \|z - a\|_\infty < |u(a)|\}$.

Proof. For $t \in \left]0, 1\right[$ put

$$M_t := \{z \in U \mid \text{There is } w \in K \text{ such that } \|z - w\|_\infty \leq t\,|u(w)|\}.$$

We claim that M_t is compact. For every $z \in M_t$ there is some $w \in K$ such that

$$\|z\|_\infty \leq \|w\|_\infty + |u(w)|,$$

so M_t is a bounded set, because K is compact and u is continuous. Let $(z_j)_{j \in \mathbb{N}} \subset M_t$ be a sequence converging to some $z \in \overline{M_t}$. Then there is a sequence $(w_j)_{j \in \mathbb{N}} \subset K$ such that

$$\|z_n - w_n\|_\infty \leq t\,|u(w_n)|.$$

Since K is compact there is a subsequence of $(w_j)_{j \in \mathbb{N}}$ converging to some $w \in K$. Without loss of generality we assume that $(w_j)_{j \in \mathbb{N}}$ itself converges to w. If $w = z$ then $z \in M_t$. Otherwise, we obtain from the triangle inequality

$$\|z - w\|_\infty \leq \|z - z_j\|_\infty + \|z_j - w_j\|_\infty + \|w_j - w\|_\infty.$$

Taking the limit $j \to \infty$ yields

$$0 < \|z - w\|_\infty \leq t\,|u(w)| < |u(w)| \leq \operatorname{dist}_{U^c}(w).$$

It follows from the definition of $\operatorname{dist}_{U^c}$ that $z \in U$, hence, $z \in M_t$, i.e., M_t is closed. By the Heine–Borel theorem, M_t is compact. Let $f \in \mathcal{O}(U)$. The compactness of M_t implies that there is some $C_t > 0$ such that

$$\sup |f(M_t)| \leq C_t.$$

Since M_t contains the polytorus $T_{t|u(w)|}(w)$ the Cauchy Inequalities yield for every $\alpha \in \mathbb{N}^n$ that

$$\sup_{w \in K} \left| \frac{D^\alpha f(w)}{\alpha!} \right| t^{|\alpha|} |u(w)|^{|\alpha|} \leq C_t.$$

The function $D^\alpha f \cdot u^{|\alpha|}$ is holomorphic on U, hence, the same estimates apply to $w \in \widehat{K}_U$. Choosing $w = a$ yields the convergence of the Taylor series for $\|z - a\|_\infty < |u(a)|$. □

Corollary 6.2.8. *Let U, K be as in Thullen's Lemma and let $a \in \widehat{K}_U$ .Then every function $f \in \mathcal{O}(U)$ has a holomorphic extension to the polydisc $P^n_{\operatorname{dist}(K,\partial U)}(a)$.*

Proof. Apply Thullen's Lemma to the constant function

$$u = \operatorname{dist}_\infty(K, \partial U). \qquad \square$$

Lemma 6.2.9. Denote by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{C}^n$ and put

$$\Delta := [0,1]\, e_1 \cup [0,1]\, e_2.$$

For $\varepsilon \in \left]0, \frac{1}{2}\right[$ let

$$Q_\varepsilon := \left\{ x_1 e_1 + x_2 e_2 \;\middle|\; \begin{array}{c} 0 \leq x_1, x_2, x_1 + x_2 \leq 1, \\ x_1 + x_2 - \varepsilon\left(x_1^2 + x_2^2\right) \leq 1 - \varepsilon \end{array} \right\}$$

and

$$\Delta_\varepsilon := \left\{ x + iy \in \Delta \;\middle|\; y_1^2 + y_2^2 \leq \frac{1}{\varepsilon}, y_j = 0 \text{ for } j \geq 3 \right\}.$$

Further, let $D = \Omega + i\mathbb{R}^n$ be a tubular domain such that $\operatorname{conv}(\Delta) \subset \Omega$. Then for all $\eta \in \mathbb{R}^n$,

$$Q_\varepsilon + i\eta \subset (\widehat{\Delta_\varepsilon + i\eta})_D.$$

Proof. We can assume that $\eta = 0$, because $D = D + i\eta$, so the general case follows from the case $\eta = 0$. Regard the set

$$M_\varepsilon := \left\{ z_1 e_1 + z_2 e_2 \in \mathbb{C}^n \;\middle|\; \begin{array}{c} 0 \leq \operatorname{Re} z_1, \operatorname{Re} z_2, \operatorname{Re} z_1 + \operatorname{Re} z_2 \leq 1, \\ z_1 + z_2 - \varepsilon\left(z_1^2 + z_2^2\right) = 1 - \varepsilon \end{array} \right\}.$$

For $z = x + iy \in M_\varepsilon$ we have

$$x_1 + x_2 - \varepsilon\left(x_1^2 + x_2^2\right) + \varepsilon\left(y_1^2 + y_2^2\right) = 1 - \varepsilon,\; 0 \leq x_1, x_2,\; x_1 + x_2 \leq 1.$$

If $z \in M_\varepsilon$, then $x_1^2 + x_2^2 \leq x_1 + x_2 \leq 1$ and

$$\varepsilon\left(1 - \left(x_1^2 + x_2^2\right)\right) + \varepsilon\left(y_1^2 + y_2^2\right) = 1 - x_1 - x_2 \leq 1. \tag{6.3}$$

It follows that $y_1^2 + y_2^2 \leq \frac{1}{\varepsilon}$, hence, M_ε is compact. Since $\varepsilon < \frac{1}{2}$ we have for $z \in M_\varepsilon$,

$$\frac{\partial}{\partial z_1}\left(z_1 + z_2 - \varepsilon\left(z_1^2 + z_2^2\right)\right) = 1 - 2\varepsilon z_2 \neq 0.$$

The Implicit Function Theorem shows that z_1 is locally a function of z_2 (and vice versa). It follows that every nonconstant holomorphic function on M_ε attains its maximum absolute value on

$$\widetilde{\partial} M_\varepsilon := \{z \in M_\varepsilon \mid \operatorname{Re} z_1 = \operatorname{Re} z_2 = 0 \text{ or } \operatorname{Re} z_1 + \operatorname{Re} z_2 = 1\}.$$

Every holomorphic function $f \in \mathcal{O}(D)$ thus satisfies for every $z \in M_\varepsilon$ the estimate

$$|f(z)| \leq \sup\left|f\left(\widetilde{\partial} M_\varepsilon\right)\right|.$$

If $\operatorname{Re} z_1 + \operatorname{Re} z_2 = 1$ for some $z \in M_\varepsilon$, equation (6.3) implies $y_1 = y_2 = 0$ and $x_1^2 + x_2^2 = 1$, i.e., $z \in \{e_1, e_2\}$. Hence, $\widetilde{\partial} M_\varepsilon \subset \Delta_\varepsilon$, so

$$|f(z)| \leq \sup|f(\Delta_\varepsilon)|,$$

which implies $M_\varepsilon \subset \left(\widehat{\Delta_\varepsilon}\right)_D$. In particular, $\left(\widehat{\Delta_\varepsilon}\right)_D$ contains the set

$$Q'_\varepsilon := \left\{x_1 e_1 + x_2 e_2 \;\middle|\; \begin{array}{c} 0 \leq x_1, x_2, x_1 + x_2 \leq 1, \\ x_1 + x_2 - \varepsilon\left(x_1^2 + x_2^2\right) = 1 - \varepsilon \end{array}\right\}.$$

From $t\Delta_\varepsilon \subset \Delta_\varepsilon$ for all $t \in [0,1]$ we obtain

$$Q_\varepsilon = \bigcup_{t \in [0,1]} t Q'_\varepsilon \subset \left(\widehat{\Delta_\varepsilon}\right)_D. \qquad \square$$

Exercise 6.2.10. Let

$$D := \{z \in \mathbb{C}^n \mid 1 < \|z\|_2 < 3\}$$

and

$$K := \{z \in \mathbb{C}^n \mid \|z\|_2 = 2\}.$$

Determine $\widehat{K}_D$.

Exercise 6.2.11. Let $U \subset \mathbb{C}^n$ be an open set and let $K := \{p\} \subset U$ consist of a single point. Determine $\widehat{K}_U$.

Exercise 6.2.12. Let $K \subset \mathbb{C}^n$ be a compact set. Show that $\widehat{K}_{\mathbb{C}^n}$ is compact.

Exercise 6.2.13. Let $D \subset \mathbb{C}^n$ be a domain and $K \subset D$ a compact subset. Show that $\widehat{K}_D$ is the union of K with all relatively compact connected components of $D \setminus K$. (Geometrically: $\widehat{K}_D$ arises from K by "filling the holes of K".)

Exercise 6.2.14. Let $D \subset \mathbb{C}^n$ be a domain and $K \subset D$ a compact subset. Show that

$$\operatorname{dist}_\infty (K, \partial D) = \operatorname{dist}_\infty \left(\widehat{K}_D, \partial D \right)$$

in the following cases:

1. $n = 1$.
2. n arbitrary, $D = B_1^n(0)$.

Exercise 6.2.15. Let

$$K := \left\{ (z, w) \in \mathbb{C}^2 \;\middle|\; \left| z - \frac{1}{2} \right|^2 + |w|^2 \geq \frac{1}{4}, \left| w - \frac{1}{2} \right|^2 + |z|^2 \geq \frac{1}{4} \right\} \cap \overline{B_1^n(0)}$$

and

$$f : \mathbb{C}^n \to \mathbb{R}, (z, w) \mapsto \exp(\operatorname{Re} z - \operatorname{Re} w).$$

Determine all maxima of f on K. (*Hint*: Regard $\widehat{K}_{\mathbb{C}^2}$.)

6.3 Bochner's Theorem

Now we have all the technical tools to prove the main result of this chapter.

Theorem 6.3.1 (Bochner). *Let $D = \Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ be a tube domain. Then the restriction*

$$\rho : \mathcal{O}(\operatorname{conv}(D)) \to \mathcal{O}(D), f \mapsto f|_D$$

is an isomorphism of complex algebras.

Proof. As usual, we only have to show that ρ is surjective, i.e., that every function $f \in \mathcal{O}(D)$ can be extended to the convex hull $\operatorname{conv}(D)$.
Step 1: In this first step we assume that the basis Ω of D is star-shaped with respect to some $p \in \Omega$. By a shift of the coordinate system we may assume $p = 0$. If $D = \mathbb{C}^n$ there is nothing to show, so we assume $D \neq \mathbb{C}^n$. If D_1, D_2 are two tubular domains such that all holomorphic functions on D can be extended to D_1, D_2 and if their bases Ω_1, Ω_2 are star-shaped with respect to the origin, then

$$D_1 \cap D_2 = \Omega_1 \cap \Omega_2 + i\mathbb{R}^n$$

is a tubular domain. In particular, $D_1 \cap D_2 \cap D \neq \emptyset$. If $f_j \in \mathcal{O}(D_j)$ are holomorphic extensions of $f \in \mathcal{O}(D)$, then

$$f_1|_{D_1 \cap D_2 \cap D} = f_2|_{D_1 \cap D_2 \cap D},$$

hence, by the Identity Theorem, they coincide on $D_1 \cap D_2$. We obtain thus a holomorphic continuation to the tubular domain $D_1 \cup D_2$, whose basis is also star-shaped with respect to the origin. Let $\widetilde{D}$ be the union of all tubular domains, whose bases are star-shaped with respect to the origin and to which all holomorphic functions can be extended. By the above argument,

$$\widetilde{D} = \widetilde{\Omega} + i\mathbb{R}^n$$

is a tube domain with the same property and $\widetilde{D}$ contains D. We show that the basis $\widetilde{\Omega}$ is convex. If $x, x' \in \widetilde{\Omega}$ are linearly dependent elements then the segment $[x, x']$ is contained in $\widetilde{\Omega}$, because $\widetilde{\Omega}$ is star-shaped with respect to the origin. If x, x' are linearly independent over $\mathbb{R}$, then there is a regular matrix $T \in GL_n(\mathbb{R})$ such that $Tx = e_1, Tx' = e_2$ and $T\left(\widetilde{D}\right)$ is a tube domain whose basis is star-shaped with respect to the origin (see Exercise 6.1.7). Hence, without loss of generality we may assume $x = e_1, x' = e_2$. Let Δ be chosen as in Lemma 6.2.9 and let $0 < \delta < \inf \operatorname{dist}_{\partial D}(\Delta)$. We consider the set

$$E := \left\{ t \in [0,1] \;\middle|\; t \operatorname{conv}(\Delta) \subset \widetilde{\Omega} \right\} \ni 0,$$

which is open in $[0,1]$, because $\widetilde{\Omega}$ is open and $\operatorname{conv}(\Delta)$ is compact. Let $t \in E$ and $0 < \varepsilon < \frac{1}{2}$ and put

$$L_{t,\varepsilon} := (1-\varepsilon)\, t \operatorname{conv}(\Delta).$$

From Lemma 6.2.9 we obtain that there is a compact set $K \subset \mathbb{R}^n$ such that

$$L_{t,\varepsilon} \subset \widehat{(t\Delta + iK)}_{\widetilde{D}}.$$

Choosing u to be the constant function $u = \operatorname{dist}\left(t\Delta + iK, \widetilde{D}^c\right)$ in Thullen's Lemma 6.2.7 we see that the Taylor series of every function $\widetilde{f} \in \mathcal{O}\left(\widetilde{D}\right)$ converges at some point a with $\operatorname{Re} a \in L_{t,\varepsilon}$ for $\|z-a\|_\infty < \delta$, i.e., on the set $a + \delta P_1^n(0)$. If a' is another point with $\operatorname{Re} a' \in L_{a,\varepsilon}$ then the intersection

$$P_\delta^n(a) \cap P_\delta^n(a')$$

is convex. If this intersection is $\neq \emptyset$ there is some $a'' \in [a, a']$. This implies $\operatorname{Re} a'' \in L_{t,\varepsilon}$, thus, $a'' \in \widetilde{D}$. It follows that the two holomorphic continuations of $\widetilde{f}$ to $P_\delta^n(a)$ and to $P_\delta^n(a')$ coincide. We obtain a holomorphic continuation to the tube domain

$$L_{t,\varepsilon} + P_\delta^n(0) + i\mathbb{R}^n.$$

Maximality of $\widetilde{D}$ implies

$$L_{t,\varepsilon} + P_\delta^n(0) + i\mathbb{R}^n \subset \widetilde{D}.$$

Put $\widetilde{t} := (1-\varepsilon)\,t + \delta$. Then

$$\widetilde{t}\,\mathrm{conv}\,(\Delta) \subset L_{t,\varepsilon} + P^n_\delta\,(0) + i\mathbb{R}^n.$$

Letting $\varepsilon \to 0$ we obtain that

$$\widetilde{t}\,\mathrm{conv}\,(\Delta) \subset \widetilde{D}$$

for every $\widetilde{t} < t + \delta$. It follows that $\sup E = 1$, i.e., $E = [0,1]$. Thus, $\mathrm{conv}\,(\Delta) \subset \widetilde{\Omega}$. Since $\mathrm{conv}\,(\Delta)$ contains the segment $[e_1, e_2]$, $\widetilde{\Omega}$ is convex.

Step 2: In the second step we free ourselves from the assumption that the basis Ω of D is star-shaped. Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. By a shift of coordinates we may assume that $0 \in \Omega$. Let

$$\widetilde{D} = \widetilde{\Omega} + i\mathbb{R}^n$$

be the maximal tube domain, which is star-shaped with respect to the origin, such that to every $f \in \mathcal{O}\,(D)$ there is an $\widetilde{f} \in \mathcal{O}\left(\widetilde{D}\right)$, which coincides with f near the origin. From Step 1 we know that $\widetilde{D}$ is convex. We only have to show that $D \subset \widetilde{D}$. This is clear if $\Omega \subset \widetilde{\Omega}$, otherwise let $x \in \Omega \setminus \widetilde{\Omega}$. The pathwise connectedness of Ω implies that there is a continuous curve

$$\gamma : [0,1] \to \Omega,\ \gamma\,(0) = 0,\ \gamma\,(1) = x.$$

Put $s := \sup\left\{t \in [0,1]\ \middle|\ \gamma\,([0,t]) \subset \widetilde{\Omega}\right\}$ and $x_s := \gamma\,(s)$. Then

$$x_s \notin \widetilde{\Omega}, \tag{6.4}$$

because otherwise openness of $\widetilde{\Omega}$ would imply that $s = 1$, i.e., $x = x_s \in \widetilde{\Omega}$, which contradicts the choice of x. Let $\Omega_s \subset \Omega$ be an open convex neighbourhood of x_s, $f \in \mathcal{O}\,(D)$ and $\widetilde{f} \in \mathcal{O}\left(\widetilde{D}\right)$ such that $\widetilde{f}$ and f coincide near the origin. Put

$$I := \left\{t \in [0,s]\ \middle|\ f = \widetilde{f}\ \text{ near } \gamma\,([0,t])\right\}.$$

Then I is an open subinterval of $[0,1]$ and $0 \in I$. Put $\tau := \sup I$. Assume $\tau < s$. Then $\gamma\,(\kappa) \in \widetilde{\Omega}$. If U is an open connected neighbourhood of $\gamma\,(\tau)$ contained in $D \cap \widetilde{D}$ there is a $\tau_0 < \tau$ with $\tau_0 \in I$ and $\gamma\,(\tau_0) \in U$. Hence, f and $\widetilde{f}$ coincide in an open neighbourhood of $\gamma\,(\tau_0)$ and thus, by the Identity Theorem, on all of U, which contradicts the choice of τ. This shows that $\tau = s$ and there is some $\tau_1 < s$ with $\gamma\,(\tau_1) \in C_s$. Using the same argument as above we see that f and $\widetilde{f}$ coincide on an open subset of the tubular domain

$$D_s := \Omega_s + i\mathbb{R}^n,$$

and thus, on all of the convex domain $D_s \cap \widetilde{D}$, defining a holomorphic continuation of $\widetilde{f}$ to $D_s \cup \widetilde{D}$. In Step 1 we saw that the maximal domain $D_{\max}$, which is star-shaped with respect to x_s and to which all $f \in \mathcal{O}\left(D_s \cup \widetilde{D}\right)$ can be extended, is convex. Hence, $D_{\max}$ is star-shaped with respect to the origin. Because of the maximality of $\widetilde{D}$ we conclude $D_{\max} \subset \widetilde{D}$. This, however, implies $D_s \subset \widetilde{D}$, which contradicts (6.4). Finally, we can conclude that

$$D \subset \widetilde{D},$$

hence, every $f \in \mathcal{O}(D)$ has a holomorphic extension to $\operatorname{conv}(D) \subset \widetilde{D}$. □

Corollary 6.3.2. *Let $f \in \mathcal{O}(\mathbb{C}^n)$ be an entire function and let*

$$\Omega_f := \bigcap_{y \in \mathbb{R}^n} \{x \in \mathbb{R}^n \mid f(x+iy) \neq 0\}.$$

Then every connected component of the interior $\overset{\circ}{\Omega}_f$ of Ω_f is convex.

Proof. Let C_f be a connected component of $\overset{\circ}{\Omega}_f$ and put $D_f := C_f + i\mathbb{R}^n$. Then $\frac{1}{f} \in \mathcal{O}(D_f)$. By Bochner's Theorem $\frac{1}{f}$ has a holomorphic extension to $\operatorname{conv}(D_f)$, hence

$$N(f) \cap \operatorname{conv}(D_f) = \emptyset.$$

This shows that if $x, x' \in C_f$ and $t \in [0,1]$,

$$f(tx + (1-t)x' + iy) \neq 0 \text{ for all } y \in \mathbb{R}^n,$$

hence, $tx + (1-t)x' \in C_f$. □

Chapter 7

Cartan–Thullen Theory

In the preceding chapters we have learned that in $\mathbb{C}^n$ with $n \geq 2$ there are many domains with the property that every holomorphic function on the domain can be holomorphically extended to a strictly larger domain. Those domains were not domains of holomorphy in the intuitive understanding that a domain of holomorphy should be the maximal domain of existence of some holomorphic function. In this chapter we will work on the task to characterize those domains $D \subset \mathbb{C}^n$, on which no simultaneous extension phenomena occur.

7.1 Holomorphically convex sets

In Exercise 6.2.12 we saw that for every compact set $K \subset \mathbb{C}^n$ the holomorphically convex hull $\widehat{K}_{\mathbb{C}^n}$ is compact. It is a trivial observation that no function $f \in \mathcal{O}(\mathbb{C}^n)$ can be extended to a larger domain in $\mathbb{C}^n$. However, we also know from the Kugelsatz, that every function holomorphic on the domain $\mathbb{C}^n \setminus \{0\}, n \geq 2$, can be extended to an entire function. We also saw in Example 6.2.6 that the holomorphically convex hull of the unit sphere in $\mathbb{C}^n \setminus \{0\}$ is not compact if $n \geq 2$. This turns out to be a crucial observation.

Definition 7.1.1. Let $U \subset \mathbb{C}^n$ be an open set. U is called *holomorphically convex* if for all compact subsets $K \subset U$ the holomorphically convex hull $\widehat{K}_U$ is compact.

Remark 7.1.2. In general we speak of $\mathcal{F}$-*convexity* if in the above definition the family $\mathcal{O}(U)$ is replaced by the family $\mathcal{F}$ of functions on U, e.g., monomial convexity, polynomial convexity etc.

Example 7.1.3. $\mathbb{C}^n$ is holomorphically convex (Exercise 6.2.12), while $\mathbb{C}^n \setminus \{0\}$ isn't unless $n = 1$. (see Example 6.2.6).

Holomorphic convexity of a domain D can be used to assert the existence of certain unbounded holomorphic functions on D.

Lemma 7.1.4. Let $D \subset \mathbb{C}^n$ be a holomorphically convex set. Then D has a compact exhaustion $(K_j)_{j\in\mathbb{N}}$ by holomorphically convex sets.

Proof. From Lemma 1.4.14 we know that D has a compact exhaustion $(C_j)_{j\in\mathbb{N}}$. Holomorphic convexity of D implies that the sets $\widehat{(C_j)}_D$ are compact. Put

$$K_1 := \widehat{(C_1)}_D.$$

Then K_1 is compact and K_1 coincides with its holomorphically convex hull. We inductively assume that $K_1, \dots, K_m$ are holomorphically convex compact sets such that $K_j \subset K_{j+1}^\circ$ for all $j = 1, \dots, m-1$. Since $(C_j)_{j\in\mathbb{N}}$ is a compact exhaustion there is some index l_m such that $K_m \subset C_{l_m}^\circ$. Put

$$K_{m+1} := \widehat{(C_{l_m})}_D.$$

This construction yields the desired exhaustion . $\square$

Lemma 7.1.5. Let D be a domain and $C \subset D$ an arbitrary subset. Then for every $\varepsilon > 0, M > 0$ and $p \in D \setminus \widehat{C}_D$ there is a function $f \in \mathcal{O}(D)$ such that

$$\sup |f(C)| < \varepsilon$$

and

$$|f(p)| > M.$$

Proof. Since $p \notin \widehat{C}_D$ there is a function $g \in \mathcal{O}(D)$ such that

$$0 < \sup |g(C)| < |g(p)|.$$

If we put

$$\delta := \frac{1}{2}\left(\frac{|g(p)|}{\sup |g(C)|} - 1\right) \text{ and } \lambda := \frac{1+\delta}{|g(p)|},$$

then $\delta, \lambda > 0$ and

$$\begin{aligned} \lambda |g(p)| &= 1 + \delta > 1, \\ \lambda \sup |g(C)| &= \frac{1}{2}\left(\frac{\sup |g(C)|}{|g(p)|} + 1\right) < 1. \end{aligned}$$

Now put $f := (\lambda g)^k$ for a sufficiently large k. $\square$

Lemma 7.1.6. Let $(K_j)_{j\in\mathbb{N}}$ be a compact exhaustion by holomorphically convex sets of the domain $D \subset \mathbb{C}^n$ and let $w_j \in K_{j+1} \setminus K_j$ for all $j \geq 1$. Then there is a function $f \in \mathcal{O}(D)$ such that

$$\lim_{j\to\infty} |f(w_j)| = \infty.$$

Proof. We inductively construct a sequence f_j by the following. Put $f_1 = 0$. By Lemma 7.1.5 we may assume that we have constructed $f_1, \ldots, f_m$ such that for all $j = 1, \ldots, m$ the following holds:

$$\left\| f_j|_{K_j} \right\|_\infty < 2^{-j}, \tag{7.1}$$

$$|f_j(w_j)| > j + 1 + \sum_{k=1}^{j-1} |f_k(w_j)| . \tag{7.2}$$

Since $w_{m+1} \notin \left(\widehat{K_{m+1}}\right)_D$ Lemma 7.1.5 produces a holomorphic function $f_{m+1} \in \mathcal{O}(D)$, which also satisfies the estimates $(7.1), (7.2)$. It follows from (7.1) that the series

$$\sum_{j \geq 0} f_j$$

converges compactly on D. Hence, by Weierstrass' Theorem, $f := \sum_{j \geq 0} f_j$ defines a holomorphic function on D, which by (7.2) satisfies

$$|f(w_j)| \geq |f_j(w_j)| - \sum_{\substack{k \geq 0 \\ k \neq j}} |f_k(w_j)| > j + 1 - \sum_{k > j} |f_k(w_j)|$$

for all $j \geq 2$. □

Lemma 7.1.4 and Lemma 7.1.6 can be combined to give a characterization of holomorphic convexity that gives a hint of what holomorphic convexity has to do with the problem of finding domains without simultaneous extension phenomena.

Proposition 7.1.7. Let $D \subset \mathbb{C}^n$ be a domain. The following are equivalent:

1. D is holomorphically convex
2. For every sequence $(w_j)_{j \in \mathbb{N}} \subset D$ without accumulation point in D there is an $f \in \mathcal{O}(D)$ such that $\sup\limits_{j \in \mathbb{N}} |f(w_j)| = \infty$.

Proof. 1. ⇒ 2. : Using Lemma 7.1.4 we find a compact exhaustion $(K_j)_{j \geq 0}$ of D by holomorphically convex sets. If $(w_j)_{j \geq 0} \subset D$ has no accumulation point in D there are sequences $(k_j), (l_j) \subset \mathbb{N}$ such that

$$w_{k_j} \in K_{l_{j+1}} \setminus K_{l_j} \text{ for all } j \geq 0.$$

The existence of the desired f follows then from Lemma 7.1.6.

2. ⇒ 1. :Let $K \subset D$ be compact and let $(w_j)_{j \geq 0} \subset \widehat{K}_D$ be a sequence. Then for all $f \in \mathcal{O}(D)$

$$\sup_{j \geq 0} |f(w_j)| \leq \|f|_K\|_\infty < \infty.$$

The condition 2. implies that the sequence $(w_j)_{j \geq 0}$ must have an accumulation point $w \in D$, but since the holomorphically convex hull $\widehat{K}_D$ is closed, we even have $w \in \widehat{K}_D$. Hence, $\widehat{K}_D$ is compact. □

Remark 7.1.8. If D is a bounded holomorphically convex domain, Proposition 7.1.7 especially says that to every boundary point $p \in \partial D$ there is a function $f_p \in \mathcal{O}(D)$, which is unbounded near p and thus cannot be extended across any part of the boundary, which contains p.

Example 7.1.9. Every simply connected domain $D \subset \mathbb{C}$ is holomorphically convex.

Proof. From the Riemann Mapping Theorem we obtain a biholomorphic function

$$\varphi : D \to B_1^1(0).$$

If $(z_j)_{j\geq 0} \subset D$ is a sequence without accumulation point on D then

$$(\varphi(z_j))_{j\geq 0}$$

is a sequence in $B_1^1(0)$ without accumulation point in $B_1^1(0)$. Since the unit disc is bounded this sequence must have an accumulation point in D on the boundary $\mathbb{T}$. Hence, the function

$$z \mapsto \sum_{k\geq 0} z^{k!}$$

is unbounded on $(\varphi(z_j))_{j\geq 0}$. □

Remark 7.1.10. In fact, the assumption that D is simply connected is superfluous: *every* domain in $\mathbb{C}$ is holomorphically convex. This will be a consequence of the Theorem of Cartan–Thullen.

Definition 7.1.11. A continuous mapping $f : X \to Y$ between topological spaces is called *proper* if for every compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is compact in X.

Proposition 7.1.12. Let $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ be domains and $f : X \to Y$ a proper holomorphic mapping.

1. If Y is holomorphically convex then so is X.
2. If f is biholomorphic and X is holomorphically convex then so is Y.

Proof. 1. : Let $K \subset X$ be compact and put $C := f(K)$. C is compact, because f is continuous. Since Y is holomorphically convex, the holomorphically convex hull $\widehat{C}_Y$ is compact. Hence, $f^{-1}\left(\widehat{C}_Y\right)$ is compact, for f is a proper mapping. Since closed subsets of compact sets are again compact, we have that

$$f^{-1}\left(\widehat{C}_Y\right) \cap \overline{X} \subset \mathbb{C}^n$$

is compact. Let $z \in \widehat{K}_X$ and $\varphi \in \mathcal{O}(Y)$. Then

$$|(\varphi \circ f)(z)| \leq \max_{w\in K} |(\varphi \circ f)(z)| = \max_{\zeta\in C} |\varphi(\zeta)|.$$

It follows that

$$f(z) \in \widehat{C}_Y = \{\zeta \in Y \mid |\varphi(\zeta)| \leq \|\varphi|_C\|_\infty \; \forall \varphi \in \mathcal{O}(Y)\},$$

thus, $\widehat{K}_X \subset f^{-1}\left(\widehat{C}_Y\right) \cap \overline{X}$. Since $\widehat{K}_X$ is closed and $f^{-1}\left(\widehat{C}_Y\right) \cap \overline{X}$ is compact, we conclude that $\widehat{K}_X$ is compact. Hence, X is holomorphically convex.

2. : If f is biholomorphic, then f^{-1} is a proper holomorphic mapping and part 1. applies. □

Example 7.1.13. The set $S := \left\{(z', z_n) \in \mathbb{C}^n \mid \operatorname{Im} z_n > \|z'\|_2^2\right\}$ is called the *Siegel upper half plane*. It is holomorphically convex, because S is biholomorphically equivalent to the unit ball $B_1^n(0)$ by the Caley map

$$c : B_1^n(0) \to S, (z', z_n) \mapsto \frac{1}{1 + z_n}(z', i(1 - z_n)).$$

The inverse mapping is given by

$$c^{-1} : S \to B_1^n(0), (w', w_n) \mapsto \frac{2i}{i + w_n}\left(w', -\frac{i}{2}(i - w_n)\right).$$

The reader should carry out the details, i.e., he should show that

$$c(B_1^n(0)) \subset S, \; c^{-1}(S) \subset B_1^n(0)$$

and c, c^{-1} are inverse to each other.

Exercise 7.1.14. Let $D := B_1^1(0) \setminus [0, 1[$. Determine a function $f \in \mathcal{O}(D)$, which is unbounded on every sequence $(z_j)_{j \geq 0} \subset D$ without accumulation point in D.

Exercise 7.1.15. Let $U \subset \mathbb{C}^n$ be an unbounded open set and $f : U \to \mathbb{C}^m$ a continuous mapping such that $\lim\limits_{\|z\| \to \infty} \|f(z)\| = \infty$. Show that f is a proper mapping.

Exercise 7.1.16. Is the set $\{z \in \mathbb{C}^n \mid 1 < \|z\|_2 < 3\}$ holomorphically convex?

7.2 Domains of Holomorphy

Definition 7.2.1. A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy* if the following holds: if $\emptyset \neq V \subset U \subset \mathbb{C}^n$ are domains with the properties

H1 $V \subset D$,

H2 For every $f \in \mathcal{O}(D)$ there is a $F \in \mathcal{O}(U)$ such that $f|_V = F|_V$,

then $U \subset D$.

We shall later see that a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex (Theorem of Cartan–Thullen). Before we come to this deep result, we examine the notion of domain of holomorphy in more detail. One can interpret Definition 7.2.1 in such a way that, on a domain of holomorphy, there are functions which cannot be holomorphically extended across any part of the boundary. We have seen such an example in dimension 1 by the function $f(z) = \sum_{k\geq 0} z^{k!}$, which is holomorphic on the open unit disc, but cannot be extended across the unit circle. So we can say that the unit disc in $\mathbb{C}$ is a domain of holomorphy. In fact, this is true for *every* domain in $\mathbb{C}$.

Example 7.2.2. Every domain $D \subset \mathbb{C}$ is a domain of holomorphy.

Proof. Assume there are domains $V \subset U \subset \mathbb{C}$ with the properties $H1, H2$ of Definition 7.2.1, but such that $U \not\subseteq D$. Then there is some $a \in U \setminus D$. Consider the holomorphic function

$$f : \mathbb{C} \setminus \{a\} \to \mathbb{C}, z \mapsto \frac{1}{z-a}.$$

The restriction $f|_D$ is holomorphic, because $a \notin D$. $H2$ implies that there is a holomorphic function $F \in \mathcal{O}(U)$ such that $F|_V = f|_V$. The Identity Theorem yields that

$$F(z) = \frac{1}{z-a}$$

for all $z \in U \setminus \{a\}$. This, however, is absurd, because F is continuous at a, hence, bounded in a neighbourhood of a, while $z \mapsto \frac{1}{z-a}$ is not. □

Proposition 7.2.3. Let $D \subset \mathbb{C}^n$ be a domain such that for all $a \in \partial D$ there is a larger domain $D_a \supset D$ and a holomorphic function $f \in \mathcal{O}(D_a)$ such that

$$a \in D_a,\ f(a) = 0,\ D \cap N(f) = \emptyset. \tag{7.3}$$

Then D is a domain of holomorphy.

Proof. Assume that D is not a domain of holomorphy. Then there are domains $\emptyset \neq V \subset U \subset \mathbb{C}^n$ with the properties $H1, H2$ of Definition 7.2.1, but $U \not\subseteq D$. Since $V \subset U \subset D$, there is a connected component W of $U \cap D$, which contains V. If $g \in \mathcal{O}(D), F \in \mathcal{O}(U)$ such that $g|_V = F|_V$ the functions coincide on W by the Identity Theorem. Let $\gamma : [0,1] \to U$ be a path such that $\gamma(0) \in V$, $\gamma(1) \notin D$ and put

$$\tau := \inf\{t \in [0,1] \mid \gamma(t) \notin D\}.$$

Then $\tau > 0$, $a := \gamma(\tau) \in \partial D$. By prerequisite there is a domain $D_a \supset D$ and a holomorphic function $f \in \mathcal{O}(D_a)$ such that (7.3) holds. Since $W \cap N(f) = \emptyset$,there is a holomorphic function $F \in \mathcal{O}(U)$ such that

$$F|_W = \left.\frac{1}{f}\right|_W.$$

In particular, $f(\gamma(t))\, F(\gamma(t)) = 1$ for all $t < \tau$. Continuity implies

$$f(a)\, F(a) = 1,$$

which is a contradiction to $f(a) = 0$. $\square$

Corollary 7.2.4. *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy and $f \in \mathcal{O}(D)$, $f \neq 0$. Then $D \setminus N(f)$ is a domain of holomorphy.*

Proof. From Corollary 4.2.2 we know that $D \setminus N(f)$ is a domain. The case $N(f) = \emptyset$ is trivial, so we assume $N(f) \neq \emptyset$. Let $\emptyset \neq V \subset U \subset \mathbb{C}^n$ satisfy the conditions of Definition 7.2.1 and assume $U \not\subset D \setminus N(f)$, i.e., we assume that $D \setminus N(f)$ is not a domain of holomorphy. Then there is some

$$a \in U \cap \partial(D \setminus N(f)).$$

Case 1: $a \in N(f)$.

Then $f(a) = 0$, $f(z) \neq 0$ for all $z \notin N(f)$, hence we can put

$$D_a := D \supset D \setminus N(f)$$

in Proposition 7.2.3.

Case 2: $a \in \partial D$.

Let $g \in \mathcal{O}(D)$. Then the restriction $g|_{D \setminus N(f)}$ is holomorphic. Thus, there is $F \in \mathcal{O}(U)$ such that

$$F|_V = g|_V,\ V \subset D \setminus N(f) \subset D.$$

Since D is a domain of holomorphy, we conclude $U \subset D$. This, however, contradicts $a \in V \cap U \cap \partial D$, because D is open and thus, $D \cap \partial D = \emptyset$. $\square$

Example 7.2.5. The set $GL_n(\mathbb{C}) \subset M(n, n; \mathbb{C}) \cong \mathbb{C}^{n^2}$ of regular matrices is a domain of holomorphy, because the determinant function

$$\det : M(n, n; \mathbb{C}) \to \mathbb{C}$$

is holomorphic and $GL_n(\mathbb{C}) = M(n, n; \mathbb{C}) \setminus N(\det)$.

Our next result shows that all convex domains in $\mathbb{C}^n$ are domains of holomorphy. In particular, balls and polydiscs are domains of holomorphy.

Proposition 7.2.6. If $D \subset \mathbb{C}^n$ is a convex domain, then D is a domain of holomorphy.

Proof. This is clear, if $D = \mathbb{C}^n$. If $D \neq \mathbb{C}^n$ there is some $a \notin D$. By the Hahn–Banach separation theorem there is a real linear form μ satisfying

$$\mu(z) < \mu(a)$$

for all $z \in D$, because D is convex. Put

$$\mu_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C} : z \mapsto \mu(z) - i\mu(iz).$$

Then $\mu_{\mathbb{C}}$ is $\mathbb{C}$-linear and $\mu_{\mathbb{C}}(z) \neq \mu_{\mathbb{C}}(a)$ for all $z \in D$. For every $a \in \partial D$ we put

$$D_a := \mathbb{C}^n, \ f := \mu_{\mathbb{C}} - \mu_{\mathbb{C}}(a)$$

and use Proposition 7.2.3 □

Exercise 7.2.7. Show that a tube domain $\Omega + i\mathbb{R}^n$ is a domain of holomorphy if and only if $\Omega \subset \mathbb{R}^n$ is convex.

Exercise 7.2.8. Which of the following sets is a domain of holomorphy?

1. $\left\{(z,w) \in \mathbb{C}^2 \mid |z| < |w| < 1\right\}$ (Hartogs' triangle).
2. $P_n(\mathbb{C})$ (the set of self-adjoint positive definite matrices).
3. $GL_n(\mathbb{C}) \setminus U_n(\mathbb{C})$.

Exercise 7.2.9. Does Corollary 7.2.4 still hold if we replace the zero set $N(f)$ of a single holomorphic function $f \in \mathcal{O}(D)$ by the simultaneous zero set

$$N(f_1, \dots, f_k) := \bigcap_{j=1}^{k} N(f_j)$$

of k functions $f_1, \dots, f_k \in \mathcal{O}(D)$?

7.3 The Theorem of Cartan–Thullen

The Theorem of H.Cartan and P.Thullen says that the notions of holomorphically convex domain and domain of holomorphy coincide. Another characterization is given by means of the boundary distance function.

Lemma 7.3.1. Let $K \subset D \subset \mathbb{C}^n$ and assume

$$\operatorname{dist}_\infty(K, \partial D) = \operatorname{dist}_\infty\left(\widehat{K}_D, \partial D\right).$$

Then $\widehat{K}_D$ is compact.

Proof. By Lemma 6.2.5 we have $\widehat{K}_D \subset \operatorname{conv}(K)$, which is bounded, because K is. We have to show that $\widehat{K}_D$ is closed in $\mathbb{C}^n$. Since $\widehat{K}_D$ is closed in D by Lemma 6.2.4 we have

$$\overline{\widehat{K}_D} \cap D = \widehat{K}_D.$$

The prerequisite and the compactness of K yield

$$0 < \operatorname{dist}_\infty(K, \partial D) = \operatorname{dist}_\infty\left(\widehat{K}_D, \partial D\right) = \operatorname{dist}_\infty\left(\overline{\widehat{K}_D}, \partial D\right),$$

hence, $\overline{\widehat{K}_D} \cap \partial D = \emptyset$. This implies $\widehat{K}_D = \overline{\widehat{K}_D}$. □

Theorem 7.3.2 (Cartan–Thullen). *Let $D \subset \mathbb{C}^n$ be a domain. Then the following are equivalent:*

1. *The domain D is a domain of holomorphy.*
2. *For all compact subsets $K \subset D$ the equation*
$$\operatorname{dist}_\infty (K, \partial D) = \operatorname{dist}_\infty \left(\widehat{K}_D, \partial D\right)$$
holds.
3. *The domain D is holomorphically convex.*

Proof. 1. $\Rightarrow$ 2. Let D be a domain of holomorphy, $K \subset D$ a compact subset and $r := \operatorname{dist}_\infty (K, \partial D)$. Then

$$r \geq \operatorname{dist}_\infty \left(\widehat{K}_D, \partial D\right), \tag{7.4}$$

because $K \subset \widehat{K}_D$. Let $z \in \widehat{K}_D$ and $U := rP_1^1(z)$ be the concentric polydisc of polyradius r centered at z. Choose V to be the connected component of z in $D \cap U$ and let $f \in \mathcal{O}(D)$. It follows from Thullen's Lemma that

$$g(w) := \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(z)}{\alpha!} (w - z)^\alpha$$

defines a holomorphic function on U and that $g = f$ near z. Hence, we obtain from the the Identity Theorem that $g = f$ on V. Since D is a domain of holomorphy, we can conclude $U \subset D$. This implies

$$\operatorname{dist}_\infty (z, \partial D) \geq r \text{ for all } z \in \widehat{K}_D. \tag{7.5}$$

2. $\Rightarrow$ 3. This was shown in Lemma 7.3.1.

3. $\Rightarrow$ 1. Since it is clear that $\mathbb{C}^n$ is a domain of holomorphy we may assume $D \neq \mathbb{C}^n$. We know from Proposition 7.1.7 that to every sequence in D, which has no accumulation point in D there is a holomorphic function, which is unbounded on this sequence. The idea is now to construct a sequence that has every point of the boundary as limit point. The corresponding function cannot be extended at any point outside D, which shows that D is a domain of holomorphy. In order to construct this sequence we consider the set

$$D \cap (\mathbb{Q}^n + i\mathbb{Q}^n)$$

of *rational* points in D. For $z \in D$ let $\widetilde{P_r^n(z)}^D$ denote the connected component of $P_r^n(z) \cap D$, which contains z. Since $\mathbb{Q}^n + i\mathbb{Q}^n$ is countable, also the set

$$\mathfrak{P} := \left\{ \widetilde{P_r^n(z)}^D \;\middle|\; z \in D \cap (\mathbb{Q}^n + i\mathbb{Q}^n), r \in \mathbb{Q}^n, P_r^n(z) \not\subseteq D \right\}$$

is countable. Let

$$P : \mathbb{N} \to \mathfrak{P}, j \mapsto P_j$$

be bijective and let $(K_j)_{j\geq 0}$ be a compact exhaustion in the sense of Lemma 7.1.6. We recursively construct a sequence $(w_j, l_{j+1})_{j\geq 0}$ in $D \times \mathbb{N}$ such that for every j,

$$w_j \in P_j \cap \left(K_{l_{j+1}} \setminus K_{l_j}\right)$$

in the following way: Choose an arbitrary $w_0 \in P_0 \cap (K_1 \setminus K_0)$. Having chosen (w_{j-1}, l_j) fix some $w_j \in P_j \setminus K_{l_j}$ and a number l_{j+1} such that $w_j \in K_{l_{j+1}}$. This is possible, because $(K_j)_{j\geq 0}$ is an exhaustion of D. Lemma 7.1.6, applied to $(K_j)_{j\geq 0}$ yields a function $f \in \mathcal{O}(D)$ satisfying

$$|f(w_j)| \geq j \text{ for all } j.$$

We show that this function is not extendible at any point outside D. Assume, that this is wrong. Then there is a polydisc $P^n_{t_0}(z_0) \nsubseteq D$ with $z_0 \in D$ on which the Taylor series $j_f(z_0)$ of f at z_0 converges. f and $j_f(z_0)$ coincide on the connected component $\widetilde{P^n_{t_0}(z_0)}^D$. Now we recursively construct a sequence $\left(\widetilde{P^n_{t_k}(z_k)}^D\right)_{k\geq 0}$ in $\mathfrak{P}$ with $z_k \in \widetilde{P^n_{t_k}(z_k)}^D$ and $0 < t_k < \frac{1}{k+1}$ for all $k \geq 0$ such that

$$\widetilde{P^n_{t_{k+1}}(z_{k+1})}^D \subset\subset \widetilde{P^n_{t_k}(z_k)}^D.$$

Then the corresponding $w_{j_k} \in \widetilde{P^n_{t_k}(z_k)}^D$ form a subsequence of $(w_j)_{j\geq 0}$ in $\widetilde{P^n_{t_0}(z_0)}^D$, which converges to some $w \in P^n_{t_0}(z_0)$. Now this implies a contradiction, because although j_f converges on $P^n_{t_0}(z_0)$, we have

$$|j_f(w)| = \lim_{k\to\infty} |f(w_{j_k})| = \infty,$$

which is absurd. □

We use the Cartan–Thullen Theorem to identify several domains as domains of holomorphy.

Proposition 7.3.3. Let $D \subset \mathbb{C}^n, D' \subset \mathbb{C}^m$ be domains of holomorphy and $f : D \to \mathbb{C}^m$ a holomorphic mapping. Then every connected component of $\Omega := f^{-1}(D')$ is a domain of holomorphy.

Proof. Let $K \subset f^{-1}(D')$ be a compact set. For every $g \in \mathcal{O}(D')$ we have $g \circ f \in \mathcal{O}(D)$, hence,

$$|g(f(z))| \leq \sup_{w\in f(K)} |g(w)| \text{ for all } z \in \widehat{K}_\Omega.$$

This implies $f(z) \in \widehat{f(K)}_{D'}$ for all $z \in \widehat{K}_\Omega$. Since D is holomorphically convex by the Theorem of Cartan–Thullen, we have

$$\overline{\widehat{K}}_\Omega \subset \overline{\widehat{K}}_D = \widehat{K}_D \subset D.$$

Continuity of f and holomorphic convexity of D' imply

$$f\left(\overline{\widehat{K}}_\Omega\right) \subset \overline{f\left(\widehat{K}_\Omega\right)} \subset \overline{\widehat{f(K)}}_{D'} = \widehat{f(K)}_{D'} \subset D'.$$

Thus, $\overline{\widehat{K}}_\Omega \subset f^{-1}(D') = \Omega$. It follows that $\widehat{K}_\Omega$ is compact, i.e., Ω is holomorphically convex and we can apply the Cartan–Thullen Theorem. □

Example 7.3.4. Let $f_1, \ldots, f_m \in \mathcal{O}(\mathbb{C}^n)$. The set

$$\Omega := \{z \in \mathbb{C}^n \mid |f_j(z)| < 1,\ j = 1, \ldots, m\}$$

is called an *analytic polyhedron*. Every connected component of Ω is a domain of holomorphy, for we can put $D := \mathbb{C}^n$, $D' := P_1^n(0)$ and $f := (f_1, \ldots, f_m)$ in Proposition 7.3.3.

Exercise 7.3.5. Give an example that the union of two domains of holomorphy (with nonempty intersection) need not be a domain of holomorphy any longer. What can you say about intersections of domains of holomorphy?

Exercise 7.3.6. Show that for a domain $D \subset \mathbb{C}^n$ the following two statements are equivalent:

1. The domain D is a domain of holomorphy.
2. A closed subset $K \subset D$ is compact if and only if $\sup |f(K)| < \infty$ for every $f \in \mathcal{O}(D)$.

7.4 Holomorphically convex Reinhardt domains

We conclude this chapter with the characterization of those Reinhardt domains which are domains of holomorphy. It turns out that a complete Reinhardt domain is a domain of holomorphy if and only if it is the domain of convergence of a power series. Another characterization is given by the geometric notion of logarithmic convexity.

Definition 7.4.1. A Reinhardt domain $D \subset \mathbb{C}^n$ is called *logarithmically convex* if the set

$$\log \tau(D) := \left\{(\log|z_1|, \ldots, \log|z_n|) \mid z \in D \cap \left(\mathbb{C}^\times\right)^n\right\} \subset \mathbb{R}^n$$

is convex. It is called *complete* if $P_{\tau(z)}^n(0) \subset D$ for every $z \in D \cap (\mathbb{C}^\times)^n$.

Proposition 7.4.2. Let $D \subset \mathbb{C}^n$ be the domain of convergence of a power series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$. Then D is a complete logarithmically convex Reinhardt domain.

Proof. Abel's Lemma implies that D is a complete Reinhardt domain. Let $x, y \in \log \tau(D)$ and $t \in [0,1]$. We choose $z, w \in D$ and $\lambda > 1$ such that $\lambda z, \lambda w \in D$ and for all $j = 1, \ldots, n$,

$$x_j = \log |z_j|, \; y_j = \log |w_j|.$$

Since the power series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$ converges at λz and λw there is some constant $0 < C < \infty$ such that

$$\max_{\alpha \in \mathbb{N}^n} \left\{ |c_\alpha| \lambda^{|\alpha|} |z^\alpha|, |c_\alpha| \lambda^{|\alpha|} |w^\alpha| \right\} \leq C.$$

It follows that for all $t \in [0,1]$ and all $\alpha \in \mathbb{N}^n$ we have the estimate

$$|c_\alpha| \lambda^{|\alpha|} |z^\alpha|^t |w^\alpha|^{1-t} \leq C.$$

Abel's Lemma implies that $\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$ converges near

$$\xi_t := \left(|z_1|^t |w_1|^{1-t}, \ldots, |z_n|^t |w_n|^{1-t} \right),$$

i.e., $\xi_t \in D$. We conclude that for all $t \in [0,1]$,

$$tx + (1 - ty) \in \log \tau(D),$$

hence, $\log \tau(D)$ is convex. □

Proposition 7.4.3. Let D be a complete logarithmically convex domain. Then D is monomially convex.

Proof. Let $\mathcal{F} := \{z^\alpha \mid \alpha \in \mathbb{N}^n\}$ be the family of monomials and for $K \subset D$ denote by

$$\widehat{K}_{\mathcal{F}} := \bigcap_{f \in \mathcal{F}} \left\{ z \in D \;\middle|\; |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)| \right\}$$

the $\mathcal{F}$ - convex hull of K in D. We have to show that $\widehat{K}_{\mathcal{F}} \cap D$ is relatively compact in D for every compact set $K \subset D$. By the Heine–Borel theorem $\widehat{K}_{\mathcal{F}}$ is compact, so it suffices to show that $\widehat{K}_{\mathcal{F}}$ does not intersect the boundary of D. Compactness of K implies that there is a finite cover of K consisting of polydiscs $P^n_{\tau\left(w^{(l)}\right)} \subset D, l = 1, \ldots, k$, where $w^{(l)} \in D$ and $w_j^{(l)} \neq 0$ for $j = 1, \ldots, n$. Put

$$W := \left\{ w^{(1)}, \ldots, w^{(k)} \right\}.$$

Then $\widehat{K}_{\mathcal{F}} \subset \widehat{W}_{\mathcal{F}}$. The proof is complete if we show that $\widehat{W}_{\mathcal{F}} \cap \partial D = \emptyset$. Let $w \in \partial D$ and put

$$w^* := (\log |w_1|, \ldots, \log |w_n|).$$

Obviously, $w^* \in \partial (\log \tau (D))$. By prerequisite, $\log \tau (D)$ is convex, thus, the Hahn–Banach separation theorem states that there is a linear functional

$$L(\xi) = \sum_{j=1}^{n} \gamma_j \xi_j, \ \gamma_j \in \mathbb{R}$$

such that

$$L(\xi) < L(w^*) \ \text{ for all } \xi \in \log \tau (D).$$

Observing that

$$\xi \in \log \tau (D) \ \text{ implies } \left\{ x \in \mathbb{R}^n \mid x_j \le \xi_j \text{ for all } j = 1, \ldots, n \right\} \subset \log \tau (D) \quad (7.6)$$

we deduce that the coefficients γ_j are nonnegative. Let $W^* \subset \log \tau (D)$ be the finite set of points which corresponds to W. Density of $\mathbb{Q}$ in $\mathbb{R}$ implies that there are rational numbers

$$q_j > \gamma_j \ge 0$$

such that for $\widetilde{L}(\xi) := \sum_{j=1}^{n} q_j \xi_j$ we have

$$\widetilde{L}(\xi) < \widetilde{L}(w^*) \ \text{ for all } \xi \in W^*. \quad (7.7)$$

This equation remains true if we multiply both sides with the common denominator of the numbers $q_1, \ldots, q_n$, so we may assume that $q_j \in \mathbb{N}$. Let m_α be the monomial

$$m_\alpha (z) := z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Then $\sup |m_\alpha (W)| < |m_\alpha (w)|$, i.e., $w \notin \widehat{W}_{\mathcal{F}}$. For the remaining points $p \in \partial D$ we can renumber the coordinates so that there is an index $1 \le k_p \le n$ such that $p_j \neq 0$ for $1 \le j \le k_p$ and $p_j = 0$ for $j > k_p$. Let

$$\mathrm{pr}_{\mathbb{C}^{k_p}} : \mathbb{C}^n \to \mathbb{C}^{k_p}, (z_1, \ldots, z_n) \mapsto \left(z_1, \ldots, z_{k_p}\right)$$

be the projection onto $\mathbb{C}^{k_p}$. Then $\log \tau (\mathrm{pr}_{\mathbb{C}^{k_p}})$ is convex and satisfies equation (7.6). Hence, we can apply the same argument as above to $\mathrm{pr}_{\mathbb{C}^{k_p}} (p)$ and obtain a monomial m in $z_1, \ldots, z_{k_p}$ satisfying

$$\sup |m (W)| < |m (p)|,$$

i.e., $p \notin \widehat{W}_{\mathcal{F}}$. This completes the proof. □

Corollary 7.4.4. *If D is a complete logarithmically convex Reinhardt domain, then D is holomorphically convex.*

Proof. This is trivial, since all monomials are holomorphic. □

We can now give a complete characterization of the domains of convergence of a power series.

Theorem 7.4.5. *Let D be a complete Reinhardt domain with center* 0. *Then the following are equivalent.*

1. *D is the domain of convergence of a power series.*
2. *D is logarithmically convex.*
3. *D is monomially convex.*
4. *D is holomorphically convex.*
5. *D is a domain of holomorphy.*

Proof. 1. $\Rightarrow$ 2. Proposition 7.4.2.
2. $\Rightarrow$ 3. Proposition 7.4.3.
3. $\Rightarrow$ 4. Corollary 7.4.4.
4. $\Rightarrow$ 5. Theorem of Cartan–Thullen.
5. $\Rightarrow$ 1. By Cartan–Thullen and Proposition 7.1.7 there is a function $f \in \mathcal{O}(D)$ which is unbounded near every boundary point of D. The Taylor series of f converges on D and since f is unbounded near every boundary point, it obviously does not converge on any domain strictly larger than D. ☐

Example 7.4.6. The set $\left\{(z,w) \in \mathbb{C}^2 \mid |z| < 1\right\}$ is a complete logarithmically convex Reinhardt domain, for it is the domain of convergence of the power series

$$\sum_{k,l \geq 0} \frac{k}{l!} z^k w^l$$

(see Exercise 1.5.16).

Example 7.4.7. Every convex complete Reinhardt domain is logarithmically convex, because every convex set is a domain of holomorphy by Proposition 7.2.6.

Exercise 7.4.8. Give an example of a complete Reinhardt domain which is not logarithmically convex.

Exercise 7.4.9. Let $r > 0$.

1. Show that the set
$$D_r := \left\{(z,w) \in \mathbb{C}^2 \mid |zw| < r\right\}$$
is a complete Reinhardt domain.
2. Show by direct computation that D_r is logarithmically convex.
3. State a power series which has D_r as domain of convergence.

Chapter 8

Local Properties of holomorphic functions

Every student remembers from his first (or, at least, second) course on calculus that the function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 0, & \text{if } x = 0 \\ e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \end{cases}$$

is of class $\mathcal{C}^\infty$, but its Taylor series represents f only at the origin, since $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. One major difference between real and complex analysis is that a holomorphic function on a domain is determined completely by local information. If two holomorphic functions $f, g \in \mathcal{O}(D)$ coincide on an open subset U of a domain D they coincide on all of D by the Identity Theorem. Locally, a holomorphic function is represented by its Taylor series. In this final chapter we study holomorphic functions from this local point of view, i.e., we do not take into account the domain of definition of a function, but only its local representation. This leads to the concept of germ of a holomorphic function. An equivalent approach is to examine the rings $\mathbb{C}\{z_1, \ldots, z_n\}$ resp. $\mathbb{C}[[z_1, \ldots, z_n]]$ of convergent resp. of formal power series and to study algebraic properties of these rings. The reader not familiar with the algebraic concepts needed in this chapter may refer to [7].

8.1 Local representation of a holomorphic function

8.1.1 Germ of a holomorphic function

Definition 8.1.1. Let $X \subset \mathbb{C}^n$ be an arbitrary subset and $f : X \to \mathbb{C}^m$ a mapping. We say that f is *holomorphic on* X, if there is an open set $U_f \subset \mathbb{C}^n$ with $X \subset U_f$ and a holomorphic mapping $g_f : U_f \to \mathbb{C}^m$ such that $g_f|_X = f$.

Notation 8.1.2. We write $\mathcal{O}^X$ for the set of all holomorphic functions on X in the sense of the above definition, i.e., $\mathcal{O}^X$ consists of all pairs (U, g), where U is an open set containing X and $g \in \mathcal{O}(U)$.

$\mathcal{O}^X$ has no natural algebraic structure, because of the variable domains of definition of the functions in $\mathcal{O}^X$. To get rid of this problem we define an equivalence relation on $\mathcal{O}^X$ by the following:

Definition 8.1.3. We call $(U, f) \in \mathcal{O}^X$ and $(V, g) \in \mathcal{O}^X$ equivalent modulo X, written

$$f \underset{X}{\sim} g, \tag{8.1}$$

if there is an open set $W \subset U \cap V$ with $X \subset W$ such that $f|_W = g|_W$.

The reader easily checks the next lemma.

Lemma 8.1.4. Formula (8.1) defines an equivalence relation on $\mathcal{O}^X$. An equivalence class of (U, f), denoted by

$$\overset{\bullet}{f}_X,$$

is called a *germ of* f *modulo* X. The quotient

$$\mathcal{O}_X := \mathcal{O}^X \Big/ \underset{X}{\sim}$$

carries the structure of a commutative unital complex algebra by the definitions

$$\begin{aligned} \overset{\bullet}{f}_X + \overset{\bullet}{g}_X \quad &: \quad = (f \overset{\bullet}{+} g)_X, \\ \overset{\bullet}{f}_X \cdot \overset{\bullet}{g}_X \quad &: \quad = (f \overset{\bullet}{\cdot} g)_X, \\ \lambda \overset{\bullet}{f}_X \quad &: \quad = (\lambda \overset{\bullet}{f})_X, \ \lambda \in \mathbb{C}. \end{aligned}$$

Every germ in $\mathcal{O}_X$ has a well-defined value at every point of X. If we want to stress the dimension of the surrounding space we also write ${}_n\mathcal{O}_X$. We are especially interested in the case where X consists of a single point a. In this case we write $\mathcal{O}_a$ resp. ${}_n\mathcal{O}_a$. A function germ $\overset{\bullet}{f}_a$carries all the local information about f in a neighbourhood of a and so does the Taylor series of f at a. It is a natural question to ask about the relationship between the algebra of germs at a and the algebra of convergent power series at a. Before dealing with this question in detail we note that we can restrict ourselves to the case where a is the origin: if f is holomorphic near a we consider the translation

$$\tau_a f(z) := f(a - z). \tag{8.2}$$

Then $\tau_a f(0) = f(a)$ and $\tau_a f$ is holomorphic near zero. Hence, τ_a induces an isomorphism of algebras

$$\mathcal{O}_a \to \mathcal{O}_0, \overset{\bullet}{f}_a \mapsto (\tau_a \overset{\bullet}{f})_0. \tag{8.3}$$

Lemma 8.1.5. $\mathcal{O}_0$ is an integral domain.

Proof. Let $\overset{\bullet}{f}_0, \overset{\bullet}{g}_0$ be nonzero function germs. Then no representative f, g of those germs is the zero function near 0, hence, $fg \neq 0$ by the Identity Theorem. This implies $(\overset{\bullet}{fg})_0 \neq 0$. $\square$

8.1.2 The algebras of formal and of convergent power series

Abel's Lemma states that a power series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

converges near zero if and only if there is some $r \in \mathbb{R}^n$ with $r_j > 0$ for all $j = 1, \ldots, n$ such that $(c_\alpha r^\alpha)_{\alpha \in \mathbb{N}^n}$ is bounded. By the Identity Theorem a convergent power series is determined completely by its set of coefficients $(c_\alpha)_{\alpha \in \mathbb{N}^n}$, thus, it can be identified with the mapping

$$\mathbb{N}^n \to \mathbb{C}, \alpha \mapsto c_\alpha,$$

where the numbers c_α satisfy the conditions of Abel's Lemma. Dropping these convergence conditions leads to the concept of formal power series. Formal power series can be considered over arbitrary commutative unital rings R.

Definition 8.1.6. Let R be a commutative ring with unit 1. A *formal power series with coefficients in* R is a mapping

$$\mathbb{N}^n \to R, \alpha \mapsto c_\alpha$$

where $c_\alpha \in R$. This mapping is denoted by the symbol

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha,$$

where $X = (X_1, \ldots, X_n)$ and $X_1, \ldots, X_n$ are indeterminates over R. The set of all formal power series with coefficients in R is denoted by

$$R\,[[X_1, \ldots, X_n]]\,.$$

A formal power series $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha$ has a well-defined "value at zero"

$$f\,(0) := c_{0\ldots 0}.$$

Notation 8.1.7. If R is a ring and $a_1, \ldots, a_l \in R$ we denote by

$$(a_1, \ldots, a_l)$$

the ideal in R generated by $a_1, \dots, a_l$. If we want to emphasize the ring or if there is danger of confusion with the tuple with components $a_1, \dots, a_l$ we will write

$$(a_1, \dots, a_l)_R$$

to make clear the difference.

For the proof of the following we refer to [7], Ch. IV, §9.

Theorem 8.1.8. *Let R be a commutative unital ring. Then the following holds.*

1. $R[[X_1, \dots, X]]$ *is a commutative unital ring by the definitions*

$$\begin{aligned} \sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha + \sum_{\alpha\in\mathbb{N}^n} b_\alpha X^\alpha \quad &: = \sum_{\alpha\in\mathbb{N}^n} (a_\alpha + b_\alpha) X^\alpha, \\ \left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha\right)\cdot\left(\sum_{\beta\in\mathbb{N}^n} b_\beta X^\beta\right) \quad &: = \left(\sum_{\gamma\in\mathbb{N}^n} c_\gamma X^\gamma\right) \end{aligned}$$

where

$$c_\gamma := \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta.$$

Moreover, if R is an algebra over a field $\mathbb{K}$, then $R[[X_1, \dots, X_n]]$ becomes a $\mathbb{K}$-algebra by putting

$$\lambda \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha := \sum_{\alpha\in\mathbb{N}^n} (\lambda c_\alpha) X^\alpha,\ \lambda \in \mathbb{K}.$$

2. $R[[X_1, \dots, X_n]]$ *contains the polynomial ring (algebra)* $R[X_1, \dots, X_n]$ *as a subring (subalgebra).*

3.

$$\begin{aligned} R[[X_1, \dots, X_n]] &= (R[[X_1, \dots, X_{n-1}]])[[X_n]] \\ &= \dots = ((R[[X_1]]) \dots [[X_n]]). \end{aligned}$$

4. *If R is Noetherian, then so is* $R[[X_1, \dots, X_n]]$.

5. *If $R = \mathbb{K}$ is a field, then* $\mathbb{K}[[X_1, \dots, X_n]]$ *is a factorial algebra.*

The statements of the above theorem can be proved in a purely algebraic manner, since no notion of convergence is involved. When considering questions of local complex analysis one works over the field $\mathbb{C}$ of complex numbers and in the subalgebra $\mathbb{C}\{z_1, \dots, z_n\}$ of convergent power series. While the first three statements of the above theorem are proven in exactly the same way, besides the fact that convergence of the relevant series must be checked separately, the fourth and fifth statements require deeper insight into the structure of $\mathbb{C}\{z_1, \dots, z_n\}$, which is defined as follows.

Definition 8.1.9. The set of convergent power series at zero is defined by

$$\mathbb{C}\{z_1,\dots,z_n\} := \left\{ \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha \;\middle|\; \exists r \in \mathbb{R}_+^n,\ \sup_{\alpha\in\mathbb{N}^n} |c_\alpha| r^\alpha < \infty \right\}.$$

We have a chain of strict subalgebras

$$\mathbb{C}[z_1,\dots,z_n] \subsetneq \mathbb{C}\{z_1,\dots,z_n\} \subsetneq \mathbb{C}[[z_1,\dots,z_n]].$$

In an obvious fashion one can define power series at an arbitrary point a and obtain the set $\mathbb{C}\{z_1-a_1,\dots,z_n-a_n\}$, but just as in the case of germs of holomorphic functions it is immediate that the algebras

$$\mathbb{C}\{z_1-a_1,\dots,z_n-a_n\}$$

and

$$\mathbb{C}\{z_1,\dots,z_n\}$$

are isomorphic. The following result is no surprise.

Lemma 8.1.10. The complex algebras ${}_n\mathcal{O}_0$ and $\mathbb{C}\{z_1,\dots,z_n\}$ are isomorphic. An isomorphism is given by

$$j_0 :\ {}_n\mathcal{O}_0 \to \mathbb{C}\{z_1,\dots,z_n\},\ \overset{\bullet}{f}_0 \mapsto j_0 f,$$

where $j_0 f$ is the unique Taylor series expansion of a representative f of $\overset{\bullet}{f}_0$.

Proof. Left to the reader. □

Thanks to this lemma we need not distinguish between germs of holomorphic functions and convergent power series. The next lemma gives us first insight into the analytic structure of the algebra of convergent power series. These results will be needed in the proof of the Weiserstrass Preparation Theorem.

Lemma 8.1.11. For $r \in \mathbb{R}_+^n$ let

$$B(r) := \left\{ \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha \;\middle|\; \sup_{\alpha\in\mathbb{N}^n} |c_\alpha| r^\alpha < \infty \right\}$$

and

$$\|.\|_{(r)} : B(r) \to \mathbb{R},\ \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha \mapsto \sum_{\alpha\in\mathbb{N}^n} |c_\alpha| r^\alpha.$$

Then the following holds:

1. The set $B(r)$ is a complex algebra without zero divisors, which satisfies

$$\mathbb{C}\{z_1,\dots,z_n\} = \bigcup_{r\in\mathbb{R}_+^n} B(r).$$

2. If $s < r$ (componentwise) then $B(r) \subset B(s)$.
3. The inclusion $B(r) \subset \mathcal{O}(P_r^n(0))$ holds.
4. The mapping $\|.\|_{(r)}$ defines a norm on $B(r)$.
5. The norm $\left(B(r), \|.\|_{(r)}\right)$ is a Banach algebra.
6. If $f = \sum_\alpha c_\alpha z^\alpha \in B(r)$ then
$$|c_\alpha| \leq \frac{\|f\|_{(r)}}{r^\alpha}$$
for every $\alpha \in \mathbb{N}^n$.

Proof. 1. and 2. are left to the reader.

3. Since every convergent power series determines a holomorphic function on a polydisc we have
$$B(r) \subset \mathcal{O}(P_r^n(0)).$$

4. It is clear that $\|f\|_{(r)} = 0$ if and only if $f = 0$ and that $\|\lambda f\|_{(r)} = |\lambda| \, \|f\|_{(r)}$ for all $\lambda \in \mathbb{C}$. To see that the triangle inequality holds let $f = \sum_\alpha a_\alpha z^\alpha$ and $g = \sum_\alpha b_\alpha z^\alpha$. Then
$$\begin{aligned}
\|f+g\|_{(r)} &= \sum_{\alpha\in\mathbb{N}^n} |a_\alpha + b_\alpha| \, r^\alpha \\
&\leq \sum_{\alpha\in\mathbb{N}^n} |a_\alpha| \, r^\alpha + \sum_{\alpha\in\mathbb{N}^n} |b_\alpha| \, r^\alpha \\
&= \|f\|_{(r)} + \|g\|_{(r)}.
\end{aligned}$$

5. With the notation as in 4. we have
$$\begin{aligned}
\|fg\|_{(r)} &= \left\| \left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha\right)\left(\sum_{\beta\in\mathbb{N}^n} a_\beta z^\beta\right) \right\|_{(r)} \\
&= \left\| \sum_{\gamma\in\mathbb{N}^n} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) z^\gamma \right\|_{(r)} \\
&= \sum_{\gamma\in\mathbb{N}^n} \left| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right| r^\gamma \\
&\leq \sum_{\gamma\in\mathbb{N}^n} \sum_{\alpha+\beta=\gamma} |a_\alpha b_\beta| \, r^\gamma \quad \text{by the triangle inequality} \\
&= \left(\sum_{\alpha\in\mathbb{N}^n} |a_\alpha| \, r^\alpha\right)\left(\sum_{\beta\in\mathbb{N}^n} |b_\beta| \, r^\beta\right) = \|f\|_{(r)} \, \|g\|_{(r)}.
\end{aligned}$$

Let $(f_j)_{j\geq 0} \subset B(r)$ be a Cauchy sequence with respect to $\|.\|_{(r)}$. We identify f_j with the holomorphic function on $P_r^n(0)$ defined by $z \mapsto f_j(z)$. If $K \subset P_r^n(0)$ is a compact subset then

$$\begin{aligned} \sup_{z\in K} |f_j(z)| &= \sup_{z\in K} \left| \sum_{\alpha\in\mathbb{N}^n} c_\alpha^{(j)} z^\alpha \right| \\ &\leq \sum_{\alpha\in\mathbb{N}^n} \left| c_\alpha^{(j)} \right| r^\alpha \\ &= \|f_j\|_{(r)}. \end{aligned}$$

Hence, since $(f_j)_{j\geq 0}$ is a Cauchy sequence with respect to $\|.\|_{(r)}$, $(f_j)_{j\geq 0}$ converges compactly on $P_r^n(0)$. We apply Weierstrass' Theorem on compact convergence to see that there is some holomorphic function

$$f \in \mathcal{O}(P_r^n(0))$$

such that

$$\lim_{j\to\infty} \sup_{z\in K} |f_j(z) - f(z)| = 0.$$

The Taylor series of f at zero is an element of $B(r)$ by Corollary 1.5.9. Hence, $B(r)$ is complete with respect to $\|.\|_{(r)}$.

6. If $f = \sum_\beta c_\beta z^\beta$ and $\alpha \in \mathbb{N}^n$ then

$$\frac{\|f\|_{(r)}}{r^\alpha} = |c_\alpha| + \sum_{\substack{\beta\in\mathbb{N}^n \\ \beta\neq\alpha}} |c_\alpha| r^\alpha,$$

i.e.,

$$|c_\alpha| = \frac{\|f\|_{(r)}}{r^\alpha} - \sum_{\substack{\beta\in\mathbb{N}^n \\ \beta\neq\alpha}} |c_\alpha| r^\alpha \leq \frac{\|f\|_{(r)}}{r^\alpha}. \qquad \square$$

Definition 8.1.12. Let $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha \in \mathbb{C}[[z_1, \ldots, z_n]]$ be a formal power series. The *order (or subdegree) of* f is defined as

$$\operatorname{ord} f := \begin{cases} +\infty, & \text{if } f = 0 \\ \min\{|\alpha| \mid c_\alpha \neq 0\}, & \text{if } f \neq 0 \end{cases}.$$

Example 8.1.13.

$$\operatorname{ord}\left(z_1^4 + z_2 \sin(z_1 z_2)\right) = 3,$$

because

$$\sin(z_1 z_2) = z_1 z_2 + \sum_{k\geq 1} \frac{(-1)^k}{(2k+1)!} (z_1 z_2)^{2k+1}.$$

Lemma 8.1.14. Let $f, g \in \mathbb{C}[[z_1, \ldots, z_n]]$. Then the following holds:

1. The order satisfies
$$\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g.$$
2. The estimate
$$\operatorname{ord}(f+g) \geq \min\{\operatorname{ord} f, \operatorname{ord} g\}$$
holds.
3. The algebra $\mathbb{C}[[z_1, \ldots, z_n]]$ of formal power series is an integral domain.
4. We have $\operatorname{ord} f = 0$ if and only if f is a unit in $\mathbb{C}[[z_1, \ldots, z_n]]$.

Proof. 1. Every formal power series $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$ has a unique expression as a series of k-homogenous polynomials
$$f = \sum_{k=0}^{\infty} P_k(f),$$
where
$$P_k(f) := \sum_{|\alpha|=k} c_\alpha z^\alpha.$$
Then $\operatorname{ord} f = \min\{k \geq 0 \mid P_k(f) \neq 0\}$. It follows that
$$\begin{aligned} fg &= \left(\sum_{k=0}^{\infty} P_k(f)\right) \cdot \left(\sum_{l=0}^{\infty} P_l(g)\right) \\ &= \sum_{j=0}^{\infty} \sum_{k+l=j} P_k(f) P_l(g) \\ &= \sum_{j=0}^{\infty} P_j(fg), \end{aligned}$$
which implies
$$\begin{aligned} \operatorname{ord} fg &= \min\{k+l \in \mathbb{N} \mid P_k(f) P_l(g) \neq 0\} \\ &= \min\{k \in \mathbb{N} \mid P_k(f) \neq 0\} + \min\{l \in \mathbb{N} \mid P_l(g) \neq 0\} \\ &= \operatorname{ord} f + \operatorname{ord} g. \end{aligned}$$

2. Since
$$f + g = \sum_{\alpha \in \mathbb{N}^n} (a_\alpha + b_\alpha) z^\alpha,$$
we have
$$\begin{aligned} \operatorname{ord}(f+g) &= \min\{|\alpha| \mid a_\alpha + b_\alpha \neq 0\} \\ &\geq \min\{\min\{|\alpha| \mid a_\alpha \neq 0\}, \min\{|\alpha| \mid b_\alpha \neq 0\}\} \\ &= \min\{\operatorname{ord} f, \operatorname{ord} g\}. \end{aligned}$$

3. If $f, g \in \mathbb{C}[[z_1, \ldots, z_n]]$ are $\neq 0$ they are of finite order. From 1. we deduce

$$\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g < \infty,$$

thus, $fg \neq 0$.

4. If $\operatorname{ord} f = 0$ then $f = f(0) \in \mathbb{C}^\times$, thus $\frac{1}{f} \in \mathbb{C}^\times$, i.e., f is a unit. Vice versa, if f is a unit there is a formal power series g such that $fg = 1$. It follows from 1. that

$$0 = \operatorname{ord} 1 = \operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g.$$

Since the order of a power series is always nonnegative it follows that $\operatorname{ord} f = 0$. □

Proposition 8.1.15. Let $\mathfrak{m} := \{f \in \mathbb{C}[[z_1, \ldots, z_n]] \mid f(0) = 0\}$ be the set of nonunits in $\mathbb{C}[[z_1, \ldots, z_n]]$. Then $\mathfrak{m}$ is a maximal ideal in $\mathbb{C}[[z_1, \ldots, z_n]]$. Moreover, $\mathfrak{m}$ is unique.

Proof. The evaluation mapping

$$\varepsilon_0 : \mathbb{C}[[z_1, \ldots, z_n]] \to \mathbb{C}, f \mapsto f(0)$$

is a surjective algebra homomorphism with kernel $\ker \varepsilon_0 = \mathfrak{m}$. The quotient

$$\mathbb{C}[[z_1, \ldots, z_n]] / \mathfrak{m}$$

is isomorphic to the image of ε_0 by the well-known isomorphy theorem from general algebra. Since ε_0 is surjective this image is $\mathbb{C}$, which is a field. Hence, $\mathfrak{m}$ is a maximal ideal. Let $\mathfrak{n}$ be another maximal ideal in $\mathbb{C}[[z_1, \ldots, z_n]]$. Then $\mathfrak{n}$ contains no unit of $\mathbb{C}[[z_1, \ldots, z_n]]$, i.e., $\mathfrak{n} \subset \mathfrak{m}$. However, since $\mathfrak{n}$ is maximal, it follows that $\mathfrak{n} = \mathfrak{m}$. □

Remark 8.1.16. A ring (algebra) with a unique maximal ideal is called a *local ring* (algebra). We have thus proven that $\mathbb{C}[[z_1, \ldots, z_n]]$ is a local algebra. The same is true for the algebra of convergent power series and the proof is identical.

Exercise 8.1.17. Show that a ring (algebra) is local if and only if its nonunits form an ideal. Prove that this ideal of nonunits is necessarily the unique maximal ideal.

Proposition 8.1.18. The maximal ideal $\mathfrak{m}$ is generated by $z_1, \ldots, z_n$.

Proof. Clearly, $z_1, \ldots, z_n \in \mathfrak{m}$. Since $\mathfrak{m}$ is an ideal, $\mathfrak{m}$ also contains the ideal $(z_1, \ldots, z_n)$ generated by $z_1, \ldots, z_n$. On the other hand every $f \in \mathfrak{m}$ has order > 0, hence, f is divided by some z_j, which proves that $\mathfrak{m} \subset (z_1, \ldots, z_n)$. □

A common misperception, especially for students who are just beginning to learn algebra, is that properties of a ring (or some other algebraic structure) etc. do not necessarily hold for a subring. We have just seen that $\mathbb{C}[[z_1, \ldots, z_n]]$ is a local algebra. The set of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ is a subalgebra of $\mathbb{C}[[z_1, \ldots, z_n]]$, however, we have the following:

Example 8.1.19. $\mathbb{C}[z_1, \ldots, z_n]$ is not a local algebra.

Proof. If $\mathbb{C}[z_1, \ldots, z_n]$ were a local algebra its nonunits would form an ideal in $\mathbb{C}[z_1, \ldots, z_n]$. However, the units in $\mathbb{C}[z_1, \ldots, z_n]$ are precisely the nonzero constant polynomials. For example, the polynomial

$$f = 1 + z_1$$

is not a unit. By the same argument as in Proposition 8.1.15 it is shown that a maximal ideal $\mathfrak{m}$ in $\mathbb{C}[z_1, \ldots, z_n]$ is the ideal generated by $z_1, \ldots, z_n$. If $\mathbb{C}[z_1, \ldots, z_n]$ were a local algebra $\mathfrak{m}$ would contain f. However, then $\mathfrak{m}$ would also contain the constant 1, because $\mathfrak{m}$ contains z_1 and since it is an ideal it also contains the difference

$$f - z_1 = 1.$$

This implies $\mathfrak{m} = \mathbb{C}[z_1, \ldots, z_n]$, which is false. $\square$

Exercise 8.1.20. Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$. Show that

$$\lim_{r \to 0} \|f\|_{(r)} = |f(0)|.$$

Exercise 8.1.21. Let R_n be the complex algebra of either formal or of convergent power series and let $\mathfrak{m}$ be the maximal ideal in R_n. For $k \in \mathbb{N}_+$ we define

$$\mathfrak{m}^k := \left\{ \sum_{finite} f_1 \cdots f_k \;\middle|\; f_j \in \mathfrak{m}, j = 1, \ldots, k. \right\}.$$

Prove the following statements:

1. $$\mathfrak{m}^k = \{f \in R_n \mid \operatorname{ord} f \geq k\}.$$
2. Each $\mathfrak{m}^k$ is an ideal in R_n.
3. Determine a minimal set M_k of generators for $\mathfrak{m}^k$ and prove an explicit formula for the number of elements of M_k.
4. Show that $\mathfrak{m}$ is a principal ideal in R_n if and only if $n = 1$.

Exercise 8.1.22. Let $R_n, \mathfrak{m}$ be defined as in the previous exercise and put

$$\mathcal{U} := \left\{\mathfrak{m}^k \mid k \in \mathbb{N}_+\right\}.$$

We define a set $\mathcal{T}$ of subsets of R_n by

$$M \in \mathcal{T} :\Longleftrightarrow \text{For all } f \in M \text{ there is some } U \in \mathcal{U}, \text{ such that } f + U \subset M,$$

where $f + U := \{f + u \mid u \in U\}$. Prove the following statements:

1. The set $\mathcal{T}$ defines a topology on R_n.

2. The set $\mathcal{U}$ is a fundamental system of neighbourhoods of 0, i.e., every neighbourhood of $0 \in R_n$ contains an element of $\mathcal{U}$.
3. The equation
$$\bigcap_{k\geq 0} \mathfrak{m}^k = \{0\}$$
holds.
4. The topology $\mathcal{T}$ is Hausdorff.

($\mathcal{T}$ is called the $\mathfrak{m}$-*adic* or *Krull topology* on R_n.)

5. A sequence $(f_j)_{j\geq 1} \subset R_n$ is called a *Cauchy sequence* if to every $U \in \mathcal{U}$ there is some $n_0 \in \mathbb{N}$ such that
$$f_k - f_l \in U \text{ for all } k, l \geq n_0.$$
It is called *convergent with limit* $f \in R_n$ if to every $U \in U$ there is some $n_0 \in \mathbb{N}$ such that
$$f - f_k \in U \text{ for all } k \geq n_0.$$
Show that every Cauchy sequence in $(R_n, \mathcal{T})$ converges, i.e., R_n is complete with respect to the Krull topology.
6. Let $f \in R_n$, $\operatorname{ord} f > 0$. Show that the series
$$\sum_{k=0}^{\infty} f^k$$
converges with respect to the Krull topology and determine the limit.

8.2 The Weierstrass Theorems

In this and the following sections $R_n = \mathbb{C}\{z_1, \ldots, z_n\}$ denotes the algebra of convergent power series. In case $n = 1$ it is a simple consequence of the Taylor expansion that a holomorphic function which has a zero of order k at the origin can locally be written in the form
$$f(z) = z^k h(z) \tag{8.4}$$
with a holomorphic function h satisfying $h(0) \neq 0$. In particular, h is a unit in $\mathbb{C}\{z\}$. The zero set of f near the origin thus coincides with the zero set of the monomial z^k. The generalization of this rather basic observation to dimension n is the celebrated Weierstrass Preparation Theorem, which, together with the Weierstrass Division Formula, is the subject of this section. These theorems and their consequences are of fundamental importance in the local investigation of the zero sets of holomorphic functions. We start by generalizing formula (8.4).

Definition 8.2.1. An $f \in R_n$ is called z_n*-general of order* m if there is an $h \in \mathbb{C}\{z_n\}$, $h(0) \neq 0$ such that

$$f(0, \ldots, 0, z_n) = z_n^m h(z_n).$$

Example 8.2.2. If $n = 1$, by (8.4), every $f \in R_1$ is z_1 - general. However, if $n > 1$ this need not be the case. For instance

$$f(z_1, z_2) := z_1 z_2 \in R_2$$

is neither z_1- nor z_2-general.

Lemma 8.2.3. Let $f_1, \ldots, f_k \in R_n \setminus \{0\}$. Then $f_1, \ldots, f_k$ are z_n-general if and only if their product $f_1 \cdots f_k$ is z_n-general.

Proof. "$\Rightarrow$": If $f_1, \ldots, f_k$ are z_n-general of order m_j then

$$f_1 \cdots f_k(0, \ldots, 0, z_n) = z_n^{m_1 + \cdots + m_k} \prod_{j=1}^{k} h_j(z_n)$$

with $h_j \in R_n$, $h(0) \neq 0$.

"$\Leftarrow$": Let

$$f_1 \cdots f_k(0, \ldots, 0, z_n) = z_n^m h(z_n)$$

with some $h \in R_n$, $h(0) \neq 0$. Let $1 \leq j \leq k$. Then

$$f_j(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha^{(j)} z^\alpha,$$

hence,

$$f_j(0, \ldots, 0, z_n) = c_{0\ldots0}^{(j)} + \sum_{k \geq 1} \gamma_k^{(j)} z_n^k.$$

If $f_j(0, \ldots, 0, z_n) = 0$ then also $f_1 \cdots f_k(0, \ldots, 0, z_n) = 0$, contradicting the z_n-generality of $f_1 \cdots f_k$. Thus, $f_j(0, \ldots, 0, z_n) \neq 0$. If $c_{0\ldots0}^{(j)} \neq 0$ then f_j is a unit in R_n, thus, z_n-general of order 0. Otherwise, f_j is z_n-general of some order $m_j > 0$. ☐

It turns out that in the Weierstrass Theorems z_n-generality of some power series is a crucial prerequisite. Therefore, the following result is important.

Proposition 8.2.4. Let $h_1, \ldots, h_k \in R_n$ with $h_1 \cdots h_k \neq 0$. Then there are complex numbers $c_1, \ldots, c_{n-1}$ and a *shearing*

$$\sigma : \mathbb{C}^n \to \mathbb{C}^n$$

defined by

$$\sigma_j(z) := \begin{cases} z_j + c_j z_n, & 1 \leq j < n \\ z_n, & j = n \end{cases}$$

such that for all $j = 1, \ldots, n$,

$$\sigma^* h_j := h_j \circ \sigma$$

is z_n-general of some order m_j. In particular, since σ is linearly invertible, thus biholomorphic mapping it induces an algebra automorphism

$$\sigma^* : R_n \to R_n. \tag{8.5}$$

Proof. Let $g \in R_n$, $g \neq 0$. Because of the absolute convergence we can freely reorder terms and can thus write g as a sum of j-homogenous polynomials

$$g(z) = \sum_{j \geq 0} g_j(z).$$

Let $m := \operatorname{ord} g$.Then $g_j = 0$ for $j < m$ and $g_m \neq 0$. Hence, there is a vector

$$(w_1, \ldots, w_n) \in \mathbb{C}^n$$

such that $g_m(w_1, \ldots, w_n) \neq 0$. By continuity we may without loss of generality assume $w_n \neq 0$. Put

$$c_j := \frac{w_j}{w_n},\ j = 1, \ldots, n-1.$$

Then

$$\begin{aligned} \sigma^* g_j(0, \ldots, 0, z_n) &= g_j(c_1 z_n, \ldots, c_{n-1} z_n, z_n) \\ &= z_n^j g_j(c_1, \ldots, c_{n-1}, 1), \end{aligned}$$

hence,

$$\begin{aligned} \sigma^* g(0, \ldots, 0, z_n) &= z_n^m \left(\sum_{j \geq 0} g_{j+m}(c_1, \ldots, c_{n-1}, 1) z_n^j \right) =: z_n^m f(z_n), \\ f(0) &= g_m(c_1, \ldots, c_{n-1}, 1) \neq 0. \end{aligned}$$

In particular, this holds for $g := h_1 \cdots h_k$.

$$\sigma^*(h_1 \cdots h_n) = \sigma^* h_1 \cdots \sigma^* h_k$$

is z_n-general, which is equivalent to all $\sigma^* h_1, \ldots, \sigma^* h_k$ being z_n-general by Lemma 8.2.3. □

Example 8.2.5. The germ defined by

$$f(z_1, z_2) := \sin\left(z_1^2 z_2\right)$$

is neither z_1- nor z_2-general, because

$$f(0, z_2) = f(z_1, 0) = 0.$$

However, with the shearing σ defined by

$$\sigma(z_1, z_2) := (z_1 + z_2, z_2)$$

we have

$$\begin{aligned}
\sigma^* f(z_1, z_2) &= \sin\left((z_1 + z_2)^2 z_2\right) \\
&= \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} (z_1 + z_2)^{4k+2} z_2^{2k+1} \\
&= \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} \sum_{j=0}^{2k+1} \binom{2k+1}{j} z_1^j z_2^{6k+3-j},
\end{aligned}$$

hence,

$$\begin{aligned}
\sigma^* f(0, z_2) &= \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z_2^{6k+3} \\
&= z_2^3 \left(1 + \sum_{k \geq 1} \frac{(-1)^k}{(2k+1)!} z_2^{6k}\right),
\end{aligned}$$

i.e., $\sigma^* f$ is z_2-general of order 3.

Exercise 8.2.6. Let $f(z_1, z_2, z_3) := z_1 z_2 z_3 \cos(z_1 + 2z_3)$. Determine a shearing σ such that $\sigma^* f$ becomes z_3-general of order 3.

Exercise 8.2.7. Let $f \in R_n$. Show that f is z_n-general of order m if and only if there are $f_k \in R_{n-1}, k \geq 0$, such that for $z = (z', z_n)$ we have

$$f(z) = \sum_{k \geq 0} f_k(z') z_n^k$$

and $\min\{k \in \mathbb{N} \mid f_k(0) \neq 0\} = m$.

Exercise 8.2.8. Prove in detail that for a shearing σ the induced map σ^* from (8.5) is an algebra automorphism.

8.2.1 The Weierstrass Division Formula

Weierstrass' Division Formula is an analogue for the division with remainder in $\mathbb{Z}$ or in the polynomial ring $\mathbb{K}[X]$, where $\mathbb{K}$ is a field. The proof of the theorem requires the following auxiliary result.

Lemma 8.2.9. Let $r \in \mathbb{R}_+^n$ and let

$$f \in \mathcal{O}(P_r^n(0)), f \neq 0$$

be z_n-general of order $m \geq 0$. Then there is an $s \in \mathbb{R}_+^n$ and polydisc

$$P_s^n(0) \subset P_r^n(0)$$

such that for all $a \in \mathbb{C}^{n-1}$ satisfying $|a_j| < s_j$, $j = 1, \dots, n-1$, the mapping

$$z_n \mapsto f(a_1, \dots, a_{n-1}, z_n)$$

has m zeros in the disc $B_{s_n}^1(0)$.

Proof. Since f is z_n-general of order m, the function

$$z_n \mapsto f(0, \dots, 0, z_n)$$

has an isolated zero of order m in $z_n = 0$. Therefore, there is some $s_n \in \left]0, r_n\right[$ such that

$$f(0, \dots, 0, z_n) \neq 0$$

for all z_n satisfying $0 < |z_n| \leq s_n$. Compactness of the circle

$$\rho := \{z_n \in \mathbb{C} \mid \ |z_n| = s_n\}$$

implies

$$\varepsilon := \min_{z_n \in \rho} |f(0, \dots, 0, z_n)| > 0.$$

Let $r' := (r_1, \dots, r_{n-1})$. There is a compact set $K \subset P_{r'}^{n-1}(0)$ with $0 \in K$ such that $K \times \rho$ is a compact subset of $P_r^n(0)$, hence, $f|_{K \times \rho}$ is uniformly continuous. This implies that to the chosen ε there is some $\delta > 0$ such that for all $z \in \mathbb{C}^n$ with $\|z\|_\infty < \delta$ we have

$$|f(z_1, \dots, z_n) - f(0, \dots, 0, z_n)| < \frac{\varepsilon}{2}.$$

Let $a \in \mathbb{C}^{n-1}$ such that $\|a\|_\infty < \delta$ and put

$$\begin{aligned}
D &: \ = \{z_n \in \mathbb{C} \mid 0 < |z_n| < s_n\}, \\
F &: \ D \to \mathbb{C}, z_n \mapsto f(a_1, \dots, a_{n-1}, z_n), \\
G &: \ D \to \mathbb{C}, z_n \mapsto f(0, \dots, 0, z_n).
\end{aligned}$$

Then for all $z_n \in \rho$ we have the inequality

$$|F(z_n) - G(z_n)| < |G(z_n)|.$$

By Rouché's Theorem, F and G have the same number of zeros in D. □

For the proof of the Weierstrass Division Formula we need a simple, but useful result from functional analysis, which is an analogue of the classical geometric series.

Lemma 8.2.10. Let $(A, \|.\|)$ be a Banach algebra with neutral element e and let $a \in A$ be an element such that

$$\|e-a\| < 1.$$

Then a is invertible in A and the inverse element is given by the *Neumann series*

$$a^{-1} = \sum_{k \geq 0} (e-a)^k .$$

Proof. Let $s_k := \sum_{j=0}^{k} (e-a)^j$ and let $k \geq l$. Then

$$\|s_k - s_l\| = \left\| \sum_{j=m+1}^{n} (e-a)^j \right\| \leq \sum_{j=m+1}^{n} \|e-a\|^k .$$

Since $\|e-a\| < 1$ we see as in the case of the geometric series that $(s_k)_{k\geq 0}$ is a Cauchy sequence in A. A is complete, thus, $(s_k)_{k\geq 0}$ has a limit $s \in A$. If we regard

$$\begin{aligned} s_k a &= \sum_{j=0}^{k} (e-a)^j a = \sum_{j=0}^{k} (e-a)^j (e-(e-a)) \\ &= \sum_{j=0}^{k} (e-a)^j - \sum_{j=1}^{k+1} (e-a)^j \\ &= e - (e-a)^{k+1} , \end{aligned}$$

since $\lim\limits_{k\to\infty} \|e-a\|^{k+1} = 0$ we conclude that

$$\lim_{k\to\infty} s_k = a^{-1}.$$

□

Theorem 8.2.11 (Weierstrass Division Formula). *Let $f \in R_n$ be z_n-general of order m and let $g \in R_n$. Then there is a unique power series $q \in R_n$ and a unique polynomial $r \in R_{n-1}[z_n]$ of degree $< m$ such that*

$$g = qf + r.$$

Proof. Uniqueness: Suppose we have

$$g = q_1 f + r_1 = q_2 f + r_2.$$

Without loss of generality we may assume that f, q_1, q_2, r_1, r_2 are all holomorphic on some polydisc $P_r^n(0)$. By Lemma 8.2.9 there is some $s \in \mathbb{R}_+^n$, $s < r$ (componentwise) such that for fixed $z' := (z_1, \ldots, z_{n-1})$ satisfying $|z_j| < s_j$, $j = 1, \ldots, n-1$, the function

$$z_n \mapsto f(z', z_n)$$

has m zeroes in the disc $B^1_{s_n}(0)$. Hence, the function

$$z_n \mapsto (q_1 - q_2) f(z', z_n)$$

has m zeroes in $B^1_{s_n}(0)$. However, because

$$(q_1 - q_2) f = r_2 - r_1 \in R_{n-1}[z_n]$$

is a polynomial of degree $< m$, $r_2 - r_1$ must be the zero polynomial. Since $f \neq 0$ and because R_n has no zero divisors, we conclude

$$q_1 = q_2.$$

Existence: If $m = 0$ then f is a unit in R_n and we can put

$$q := gf^{-1}, r := 0.$$

If $m > 0$ we write

$$f(z) = \sum_{k \geq 0} f_k(z') z_n^k$$

with $f_k \in R_{n-1}, \min\{k \in \mathbb{N} \mid f_k(0) \neq 0\} = m$. After multiplication with a suitable unit we may assume that $f_m = 1$. We define $g_k \in R_{n-1}$ by

$$\begin{aligned} g(z) &= \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha \\ &=: \sum_{k \geq 0} g_k(z') z_n^k \\ &= \sum_{k=0}^{m-1} g_k(z') z_n^k + \sum_{k \geq m} g_k(z') z_n^k, \end{aligned}$$

i.e., if we put

$$b_g(z) := \sum_{k=0}^{m-1} g_k(z') z_n^k, \ a_g(z) := \sum_{k=0}^{\infty} g_{k+m}(z') z_n^k$$

we have a unique decomposition

$$g(z) = z_n^m a_g(z) + b_g(z). \tag{8.6}$$

We apply Lemma 8.1.11 to find some $r \in \mathbb{R}^n_+$ such that $g, f \in B(r)$. Consider the linear operator

$$T_f : B(r) \to B(r), g \mapsto a_g f + b_g.$$

Then T_f is injective, because a_g, b_g are uniquely determined by (8.6). With the notation of Lemma 8.1.11 we have

$$\|g\|_{(r)} = \sum_{k \geq 0} \|g_k\|_{(r')} r_n^k = \|a_g\|_{(r)} r_n^m + \|b_g\|_{(r)}.$$

Thus,

$$\begin{aligned}\|T_f g - g\|_{(r)} &= \|a_g (f - z_n^m)\|_{(r)}\\ &\leq \|a_g\|_{(r)} \|f - z_n^m\|_{(r)}.\end{aligned}$$

We can choose $0 < \varepsilon < 1$ and $\|r\|_\infty$ small enough such that

$$\|f - z_n^m\|_{(r)} \leq \varepsilon r_n^m.$$

This implies that

$$\|T_f g - g\|_{(r)} \leq \varepsilon \|g\|_{(r)},$$

thus, the linear operator $\mathrm{id} - T_f$ has operator norm < 1. Applying Lemma 8.2.10 we conclude that the Neumann series

$$\sum_{k \geq 0} (\mathrm{id} - T_f)^k$$

converges to T_f^{-1}. This shows that the operator T_f is bijective, which completes the proof. □

Remark 8.2.12. If we drop the condition that f must be z_n-general, the division formula becomes false. For instance, we regard the function $f(z_1, z_2) := z_1 z_2$ and decompose z_1 in two different ways:

$$\begin{aligned} z_1 &= 1 \cdot z_1 z_2 + (-z_1) z_2 + z_1\\ &= (z_2 + 1) z_1 z_2 + (-z_1) z_2^2 + (-z_1) z_2 + z_2,\end{aligned}$$

thus, uniqueness is no longer given.

8.2.2 The Weierstrass Preparation Theorem

The Weierstrass Preparation Theorem is fundamental in the local study of the zero set of a holomorphic function, because, among other things, it says that near the origin the zero set $N(f)$ of a function f coincides with the zero set of a so-called Weierstrass polynomial ω, which we obtain from a unique decomposition $f = u\omega$, where $u \in R_n$ is a unit. This means that $N(f)$, together with the natural projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ is locally a branched covering of $\mathbb{C}^{n-1}$ with m branches. Furthermore, it can be used to obtain deeper insight into the algebraic structure of the ring of convergent power series or, if you prefer, the ring of germs of holomorphic functions.

Definition 8.2.13. A polynomial

$$\omega = z_n^m + a_{m-1} z_n^{m-1} + \cdots + a_0 \in R_{n-1}[z_n]$$

which satisfies $a_j(0) = 0$ for all $j = 0, \ldots, m-1$ is called a *Weierstrass polynomial* of degree m.

Example 8.2.14.

$$\omega(z_1, z_2) := z_2^3 + z_2\left(1 - \cos^3 z_1\right) + z_1^5 + \exp z_1 - 1$$

is a Weierstrass polynomial of degree 3 with

$$a_2(z_1) = 0,\ a_1(z_1) = 1 - \cos^3 z_1,\ a_0(z_1) = z_1^5 + \exp z_1 - 1.$$

Theorem 8.2.15 (Weierstrass Preparation Theorem). *Let $f \in R_n$ be z_n-general of order $m \in \mathbb{N}$. Then there are a unique Weierstrass polynomial $\omega \in R_{n-1}[z_n]$ and a unique unit $u \in R_n$ such that*

$$f = u \cdot \omega.$$

Proof. We apply the Weierstrass Division Formula to the function germ defined by

$$g(z_1, \ldots, z_n) := z_n^m$$

to obtain unique elements $q \in R_n, r \in R_{n-1}[z_n]$ with $\deg r < m$ such that

$$z_n^m = qf + r.$$

We write

$$r(z) = r(z', z_n) = \sum_{j=1}^{m} b_j(z')\, z_n^{m-j}.$$

From the z_n-generality of f we obtain that there is a unit $h \in \mathbb{C}\{z_n\}$ such that

$$\begin{aligned} q(0, \ldots, 0, z_n)\, f(0, \ldots, 0, z_n) &= q(0, \ldots, 0, z_n)\, z_n^m h(z_n) \\ &= z_n^m - r(0, z_n). \end{aligned}$$

If we put $a_j := -b_j$ for all $j = 1, \ldots, m$ and

$$r_1(z) := \sum_{j=1}^{m} a_j(z')\, z_n^{m-j},$$

we see that

$$q(0, \ldots, 0, z_n)\, z_n^m h(z_n) = z_n^m + \sum_{j=1}^{m} a_j(0)\, z_n^{m-j}.$$

If $z_n = 0$ this implies $a_m(0) = 0$. Iterating this process m times gives that

$$a_j(0) = 0,\ j = 1, \ldots, m,$$

i.e., $z_n^m - r$ is a Weierstrass polynomial. Hence,

$$z_n^m = q(0, \ldots, 0, z_n)\, z_n^m h(z_n),$$

which for $z_n \neq 0$ implies

$$1 = q\left(0, \ldots, 0, z_n\right) h\left(z_n\right).$$

Since $h(0) \neq 0$ we conclude that $q(0) \neq 0$, i.e., q is a unit in R_n. Now we can put

$$u := q^{-1},\ \omega := z_n^m - r,$$

which completes the proof. $\square$

Remark 8.2.16. There are analogues of the Weierstrass Division Theorem and the Weierstrass Preparation Theorem for differentiable real-valued germs. Those analogues are known in the literature as *Malgrange's Theorems*. The interested reader may find details in [2].

Example 8.2.17. Let $f\left(z_1, z_2\right) := e^{3z_1}\left(e^{z_2} - e^{-z_1}\right)^3 \in \mathbb{C}\left\{z_1, z_2\right\}$. Then

$$\begin{aligned} f\left(0, z_2\right) &= \left(e^{z_2} - 1\right)^3 = \left(\sum_{k=1}^{\infty} \frac{z_2^k}{k!}\right)^3 \\ &= z_2^3 \left(1 + \sum_{k \geq 2} \frac{z_2^k}{k!}\right)^3, \end{aligned}$$

i.e., f is z_2-general of order 3 and we obtain the decomposition $f = u\omega$, where

$$\begin{aligned} u\left(z_1, z_2\right) &:= \left(\sum_{k \geq 0} \frac{\left(z_1 + z_2\right)^k}{(k+1)!}\right)^3, \\ \omega\left(z_1, z_2\right) &:= \left(z_1 + z_2\right)^3 = z_1^3 + 3z_2 z_1^2 + 3z_2^2 z_1 + z_2^3. \end{aligned}$$

Exercise 8.2.18. Let f be defined as in Example 8.2.17 and let $g\left(z_1, z_2\right) := z_2^3$. Find the decomposition

$$g = qf + r$$

according to the Weierstrass Division Formula.

Exercise 8.2.19. Let $\omega_1, \ldots, \omega_s \in R_{n-1}\left[z_n\right]$ be normed polynomials. Show that every ω_j is a Weierstrass polynomial if and only if their product

$$\omega := \prod_{j=1}^{s} \omega_j$$

is a Weierstrass polynomial.

Exercise 8.2.20. Let $f \in R_n$ be z_n-general of order 1.

1. Deduce the Weierstrass Preparation Theorem from the Implicit Function Theorem.
2. Vice versa, use the Weierstrass Preparation Theorem to prove the following version of the Implicit Function Theorem: Let $f \in R_n$ satisfy

$$f(0) = 0, \ \frac{\partial f}{\partial z_n}(0) \neq 0.$$

Show that there is a unique $\varphi \in R_{n-1}$, which satisfies

$$\varphi(0) = 0, \ f(z', \varphi(z')) = 0 \text{ near } 0 \in \mathbb{C}^{n-1}.$$

8.3 Algebraic properties of $\mathbb{C}\{z_1, \ldots, z_n\}$

Two well-known important theorems in the theory of polynomial rings in indeterminates $X_1, \ldots, X_n$ are the following results:

Theorem 8.3.1 (Hilbert's Basis Theorem). *Let R be a commutative ring with unit* 1*. Then R is Noetherian if and only if the polynomial ring $R[X_1, \ldots, X_n]$ is Noetherian.*

Theorem 8.3.2 (Gauss). *Let R be a commutative ring with unit* 1*. Then R is a factorial ring if and only if the polynomial ring $R[X_1, \ldots, X_n]$ is factorial.*

The Weierstrass Theorems can be used to obtain equivalent results for the algebra R_n of convergent power series. Before we come to that let us stress the importance of Hilbert's Basis Theorem for the investigation of systems of polynomial equations.

Example 8.3.3. Let $\mathcal{F} \subset R := \mathbb{C}[z_1, \ldots, z_n]$ be an arbitrary family of polynomials and

$$N(\mathcal{F}) := \bigcap_{f \in \mathcal{F}} N(f)$$

their common zero set. Then there is a *finite* family

$$\mathcal{F}_1 \subset R,$$

which has the same zero set, i.e., the eventually infinite (or even uncountable) system of equations has a solution equivalent to the solution of a finite system of equations. This can be seen as follows. If we put

$$\mathfrak{I} := I(N(\mathcal{F})) := \bigcap_{\substack{\mathcal{F} \subset \mathfrak{a} \\ a \text{ Ideal in } R}} \mathfrak{a},$$

then $\mathfrak{I}$ is an ideal in R, the so-called *vanishing ideal* of $N(\mathcal{F})$. By Hilbert's Basis Theorem, $\mathfrak{I}$ is finitely generated, say, by functions $f_1, \ldots, f_m \in R$. Hence,

$$N(\mathcal{F}) = N(f_1, \ldots, f_m).$$

We now come back to R_n.

Theorem 8.3.4. *R_n is a Noetherian algebra.*

Proof. We use induction on n. If $n = 0$ then $R_n \cong \mathbb{C}$, which is a field, thus Noetherian. Now assume that R_{n-1} is Noetherian and let $\mathfrak{a} \lhd R_n$ be an ideal. Let $f \in \mathfrak{a}, f \neq 0$. By Proposition 8.2.4 we may assume without loss of generality that f is z_n-general of order m, because every shearing induces an automorphism of R_n. If $m = 0$ then f is a unit, hence, $\mathfrak{a} = R_n \cdot f = R_n$. If $m > 0$ the Weierstrass Preparation Theorem says that there are a unit $u \in R_n$ and a Weierstrass polynomial $\omega \in R_{n-1}[z_n]$ such that

$$f = u\omega,$$

i.e.,

$$u^{-1} f = \omega \in \mathfrak{a} \cap R_{n-1}[z_n] \lhd R_{n-1}[z_n].$$

R_{n-1} is Noetherian by induction hypothesis, so $R_{n-1}[z_n]$ is Noetherian by Hilbert's Basis Theorem. Hence, there are finitely many polynomials

$$g_1, \ldots, g_l \in R_{n-1}[z_n]$$

which span $\mathfrak{I} := \mathfrak{a} \cap R_{n-1}[z_n]$. If $g \in \mathfrak{a}$ is an arbitrary element we obtain from the Weierstrass Division Formula a unique decomposition

$$g = q \cdot \left(u^{-1} f\right) + r$$

with $r \in \mathfrak{I}$, hence,

$$g \in (f, g_1, \ldots, g_l) = (g_1, \ldots, g_l).$$

This shows that

$$\mathfrak{a} \subset (g_1, \ldots, g_l) = \mathfrak{I} \subset \mathfrak{a},$$

i.e., $\mathfrak{a}$ is finitely generated. $\square$

It is important to notice that this is a *local* result, which becomes false in the global case.

Example 8.3.5. $\mathcal{O}(\mathbb{C})$ is not Noetherian.

Proof. For $k \in \mathbb{N}$ regard

$$\mathfrak{I}_k := \{f \in \mathcal{O}(\mathbb{C}) \mid f(m) = 0 \text{ for all } m \in \mathbb{N}, m > k\}.$$

Then $\mathfrak{I}_k$ is an ideal in $\mathcal{O}(\mathbb{C})$ and we have an ascending chain

$$\mathfrak{I}_0 \subset \mathfrak{I}_1 \subset \mathfrak{I}_2 \subset \cdots. \tag{8.7}$$

By the Weierstrass Factorization Theorem there is an $f_k \in \mathcal{O}(\mathbb{C})$ such that

$$N(f_k) = \{k + l \mid l \in \mathbb{N}\},$$

i.e., $f_k \in \mathfrak{I}_{k+1} \setminus \mathfrak{I}_k$. This means that the chain (8.7) does not terminate. Hence, $\mathcal{O}(\mathbb{C})$ is not Noetherian. $\square$

Lemma 8.3.6. Let $\omega \in R_{n-1}[z_n]$ be a Weierstrass polynomial. Then the following are equivalent:

1. The polynomial ω is prime in $R_{n-1}[z_n]$.
2. The polynomial ω is prime in R_n.

Proof. We first note that an element p of a ring R is a prime element if and only if the quotient $R/_{(p)}$ is an integral domain and that for every unit $u \in R_n$ the element $g := u\omega$ is z_n-general. Furthermore, ω and g generate the same ideal in R_n. Consider the inclusion

$$i : R_{n-1}[z_n] \to R_n$$

and natural projection

$$\pi : R_n \to R_n /_{(g)} .$$

We also have a projection

$$\pi' : R_{n-1}[z_n] \to R_{n-1}[z_n] /_{(\omega)} .$$

Put $\alpha := \pi \circ i$ and let $f \in R_n$. The Weierstrass Division Formula gives a unique decomposition

$$f = qg + r$$

with $q \in R_n$ and $r \in R_{n-1}[z_n]$. Since $qg \in (g)$ we see that α is surjective. We have $g = u\omega$, thus, $\omega \in \ker \alpha$, i.e.,

$$(\omega) \subset \ker \alpha.$$

Also, if $h \in \ker \alpha$ there is some $b \in R_n$ such that

$$h = bg = bu\omega \in (\omega),$$

i.e., $\ker \alpha \subset (\omega)$. Hence, by the Isomorphism Theorem from algebra, we obtain an isomorphism

$$\phi : R_{n-1}[z_n] /_{(\omega)} \to R_n /_{(g)} ,$$

which makes the diagram

$$\begin{array}{ccc} R_{n-1}[z_n] & \xrightarrow{\alpha} & R_n /_{(g)} \\ \pi' \downarrow & \nearrow \phi & \\ R_{n-1}[z_n] /_{(\omega)} & & \end{array}$$

commutative. This shows that ω is a prime element in $R_{n-1}[z]$ if and only if it is prime in R_n. $\square$

Theorem 8.3.7. R_n *is a factorial algebra.*

Proof. We use induction on n. If $n = 0$ then $R_0 \cong \mathbb{C}$. Since $\mathbb{C}$ is a field it contains no elements $\neq 0$, which are not units, so it is trivially a factorial ring. Now suppose that R_{n-1} is factorial and let $f \in R_n$. Once again, by Proposition 8.2.4, we may assume without loss of generality that f is z_n-general of some order m, because f can be factored into prime components

$$f = p_1 \cdots p_l$$

if and only if the sheared germ $\sigma^* f$ can be factored into prime components

$$\sigma^* f = (\sigma^* p_1) \cdots (\sigma^* p_l) .$$

The Weierstrass Preparation Theorem gives a unique decomposition

$$f = u\omega$$

with a unit $u \in R_n$ and a Weierstrass polynomial $\omega \in R_{n-1} [z_n]$ of degree m. By induction hypothesis R_{n-1} is factorial, so we can use Gauss' Theorem to see that $R_{n-1} [z_n]$ is factorial, too. Hence, ω can be factored

$$\omega = \omega_1^{\nu_1} \cdots \omega_s^{\nu_s} .$$

into a product of prime elements $\omega_1, \ldots, \omega_s \in R_{n-1} [z_n]$. We have seen in Exercise 8.2.19 that all ω_j are Weierstrass polynomials, because ω is one. Lemma 8.3.6 finally states that all ω_j are also prime in R_n. $\square$

Remark 8.3.8. This theorem gives even deeper insight into the local structure of the zero set of a holomorphic function near the origin. If $f \in R_n$ then there is a shearing σ such that $\sigma^* f$ is z_n-general and $\sigma^* f = u\omega$ by the Weierstrass Preparation Theorem, where the unit u has no zeroes near the origin. Furthermore, ω factors into prime components $\omega_1, \ldots, \omega_s$. Thus, we have near the origin

$$\begin{aligned} N(f) &= \sigma^{-1} (N(\sigma^* f)) = \sigma^{-1} (N(u\omega)) \\ &= \sigma^{-1} (N(\omega)) = \sigma^{-1} (N(\omega_1 \cdots \omega_s)) \\ &= \sigma^{-1} \left(\bigcup_{j=1}^{s} N\left(\omega_j^{\nu_j}\right) \right) = \sigma^{-1} \left(\bigcup_{j=1}^{s} N(\omega_j) \right) \\ &= \bigcup_{j=1}^{s} \sigma^{-1} (N(\omega_j)) . \end{aligned}$$

Example 8.3.9. Let $f \in R_2$ be defined by

$$f(z_1, z_2) := e^{z_1 - z_2} - 1 - z_1 + z_2 .$$

Because of $f(0,0) = 0$, f is not a unit. Taylor expansion gives

$$f(z_1, z_2) = (z_1 - z_2)^2 \sum_{j \geq 0} \frac{(z_1 - z_2)^j}{(j+2)!} .$$

The series

$$u(z_1, z_2) := \sum_{j \geq 0} \frac{(z_1 - z_2)^j}{(j+2)!}$$

is a unit in R_2, because $u(0,0) = \frac{1}{2} \neq 0$. The factor

$$g(z_1, z_2) := z_1 - z_2$$

is prime in R_2, because it is a Weierstrass polynomial in z_1. Hence, by Lemma 8.3.6 it is prime in R_2 if and only if it is prime in $R_1[z_1]$. Since the degree of g with respect to z_1 is 1, g is irreducible in $R_1[z_1]$ and since $R_1[z_1]$ is a factorial ring, it is also prime. Hence, we have the prime decomposition of f in R_2 given by

$$f = u \cdot g^2.$$

Theorem 8.3.10 (Hensel's Lemma). *Let $\omega \in R_{n-1}[z_n]$ be a normed polynomial of degree $m > 0$, such that for some $s \in \mathbb{N}$,*

$$\omega(0, \ldots, 0, z_n) = \prod_{j=1}^{s} (z_n - c_j)^{m_j} \tag{8.8}$$

with pairwise distinct complex numbers c_j and $m_j \in \mathbb{N}$, $m_1 + \cdots + m_s = m$. Then there are uniquely determined normed polynomials

$$\omega_1, \ldots, \omega_s \in R_{n-1}[z_n]$$

such that the following holds:

1. *The polynomial ω has a factorization $\omega = \prod_{j=1}^{s} \omega_j$.*
2. *For each ω_j we have $\omega_j(0, \ldots, 0, z_n) = (z_n - c_j)^{m_j}$, $j = 1, \ldots, s$.*
3. *The ω_j are pairwise relatively prime in $R_{n-1}[z_n]$.*

Proof. We use induction on s. If $s = 1$ the proposition is clear, for we can choose $\omega_1 := \omega$. If $s > 1$ we consider

$$q(z_1, \ldots, z_n) := \omega(z_1, \ldots, z_{n-1}, z_n + c_1).$$

Then $q \in R_{n-1}[z_n]$ is a normed polynomial and q is z_n-general of order m_1, because

$$q(0, \ldots, 0, z_n) = z_n^{m_1} \prod_{j=2}^{s} (z_n + c_1 - c_j)$$

and $c_j \neq c_1$ for $j > 1$. From the Weierstrass Preparation Theorem we obtain a unit $u \in R_n$ and a Weierstrass polynomial $q_1 \in R_{n-1}[z_n]$ of order m_1 such that $q = uq_1$. Define

$$\begin{aligned} \omega_1(z) \quad &: \quad = q_1(z_1, \ldots, z_{n-1}, z_n - c_1), \\ \omega'(z) \quad &: \quad = u(z_1, \ldots, z_{n-1}, z_n - c_1). \end{aligned}$$

Then $\omega = \omega_1\omega'$, ω_1 is normed and $\omega'(c_1) \neq 0$. By construction, ω' must also be a normed polynomial in $R_{n-1}[z_n]$ and it follows from $\omega = \omega_1\omega'$ and (8.8) that

$$\omega'(0, \ldots, 0, z_n) = \prod_{j=2} (z_n - c_j)^{m_j}.$$

Now we proceed by induction to uniquely factor ω' in the same way, which proves 1. and 2. We also may assume by induction that $\omega_2, \ldots, \omega_s$ are pairwise relatively prime. We only need to show that ω_1 is relatively prime to all ω_j, $j > 1$. This, however, is clear, because otherwise $\omega_1(0, \ldots, 0, z_n)$ and $\omega_j(0, \ldots, 0, z_n)$ would have a common factor, which is not the case, because all c_j are distinct, i.e., 3. also holds. $\square$

Definition 8.3.11. Let $\mathcal{A}$ be a local Noetherian complex algebra with maximal ideal $\mathfrak{m}_\mathcal{A}$, X an indeterminate over $\mathcal{A}$ and let

$$\pi : \mathcal{A}[X] \to (\mathcal{A}[X]/_{\mathfrak{m}_\mathcal{A}})[X]$$

be the canonical projection. $\mathcal{A}$ is called *Henselian* if the following holds: if $f \in \mathcal{A}[X]$ and

$$\pi(f) = p_1 \cdot p_2$$

is a decomposition into relatively prime normed polynomials

$$p_1, p_2 \in (\mathcal{A}[X]/_{\mathfrak{m}_\mathcal{A}})[X],$$

then there are polynomials $f_1, f_2 \in \mathcal{A}[X]$ such that $f = f_1 \cdot f_2$ and $\pi(f_j) = p_j$, $j = 1, 2$.

Remark 8.3.12. Consider the inclusion

$$i : \mathbb{C} \to \mathcal{A}$$

and the projection

$$\pi : \mathcal{A} \to \mathcal{A}\Big/_{\mathfrak{m}_\mathcal{A}}.$$

Since $\mathfrak{m}_\mathcal{A}$ is a maximal ideal, $\mathcal{A}\Big/_{\mathfrak{m}_\mathcal{A}}$ is a field, which contains $\pi(\mathbb{C}) \cong \mathbb{C}$ as a subfield. Because $\mathcal{A}$ is Noetherian

$$\mathcal{A}\Big/_{\mathfrak{m}_\mathcal{A}} \mid \pi(\mathbb{C})$$

is a finite field extension and because $\mathbb{C}$ has characteristic zero, this extension is simple, i.e., there is some $a \in \mathcal{A}\Big/_{\mathfrak{m}_\mathcal{A}}$ such that

$$\mathcal{A}\Big/_{\mathfrak{m}_\mathcal{A}} \cong \mathbb{C}(a).$$

Let $p_a \in \mathbb{C}[X]$ be the minimal polynomial of a. By the Fundamental Theorem of Algebra, p_a splits into linear factors over $\mathbb{C}$, from which we conclude that $a \in \mathbb{C}$, i.e.,

$$\mathcal{A} \Big/ _{\mathfrak{m}_{\mathcal{A}}} \cong \mathbb{C}.$$

Theorem 8.3.13. *The algebra R_n is Henselian.*

Proof. The mapping

$$\pi : R_{n-1}[z_n] \to \mathbb{C}[z_n], \omega(z) \mapsto \omega(0, \ldots, 0, z_n)$$

satisfies the prerequisites of Hensel's Lemma. □

Exercise 8.3.14. Give another proof that $\mathcal{O}(\mathbb{C})$ is not Noetherian by showing that

$$\mathfrak{I} := \{f \in \mathcal{O}(\mathbb{C}) \mid f(k) = 0 \text{ for almost all } k \in \mathbb{N}\}$$

is an ideal in $\mathcal{O}(\mathbb{C})$, which is not finitely generated.

Exercise 8.3.15. Is $\mathcal{O}(\mathbb{C})$ a factorial ring?

Exercise 8.3.16. Let $p, q \in \mathbb{N}$ and $f(z_1, z_2) := z_2^p - z_1^q$.

1. Show that f is irreducible in R_2 if $\gcd(p, q) = 1$.
2. Let $p = q$. Determine the prime factorization of f in R_2.

8.4 Hilbert's Nullstellensatz

Let $D \subset \mathbb{C}$ be a domain, V a nonempty open subset of D and $f, g \in \mathcal{O}(D)$ such that

$$N(f|_V) \subset N(g|_V) \neq V. \tag{8.9}$$

For all $\varphi \in \mathcal{O}(V)$ we define a divisor on V by

$$\operatorname{div}\varphi : V \to \mathbb{N}, z \mapsto \operatorname{div}\varphi(z) := \text{ zero order of } \varphi \text{ at } z.$$

Then it is easily seen that

$$\operatorname{div}\varphi\psi = \operatorname{div}\varphi + \operatorname{div}\psi. \tag{8.10}$$

Let $a \in V$ and $K \subset V$ be a compact neighbourhood of a. Then $\operatorname{div} f$ and $\operatorname{div} g$ attain at most finitely many positive values on K by the Identity Theorem. From this and from (8.9) it follows that there is some $m \in \mathbb{N}$ such that

$$m \operatorname{div} g(z) \geq \operatorname{div} f(z) \text{ for all } z \in V.$$

From (8.10) we obtain that on V the equation

$$0 \leq m \operatorname{div} g - \operatorname{div} f = \operatorname{div}\left(\frac{g^m}{f}\right)$$

holds, which means that the meromorphic function $\frac{g^m}{f}$ has at most removable singularities on V. By the Riemann Removable Singularities Theorem there is an $h \in \mathcal{O}(V)$ such that

$$hf = g^m,$$

i.e., $f|_V$ divides $g^m|_V$ in $\mathcal{O}(V)$. In the words of Algebra we can say that $g^m|_V$ lies in the principal ideal of $\mathcal{O}(V)$ generated by $f|_V$. Hilbert's Nullstellensatz (zero set theorem) generalizes this result to power series in n variables.

8.4.1 Germs of a set

As with germs of functions we define an equivalence relation on subsets of $\mathbb{C}^n$ by the following.

Definition 8.4.1. Let $X, Y \subset \mathbb{C}^n$ and $a \in X \cap Y$.

1. We call X and Y *equivalent at* a if there is an open neighbourhood U of a such that
$$X \cap U = Y \cap U.$$

2. An equivalence class of X under this equivalence relation is called a *germ of* X *at* a and will be denoted by
$$\overset{\bullet}{X}_a.$$
If the point a is not of importance, we simply write $\overset{\bullet}{X}$ for any germ of X.

3. If $\overset{\bullet}{X}, \overset{\bullet}{Y}$ are set germs, we define
$$\begin{aligned} \overset{\bullet}{X} \cap \overset{\bullet}{Y} \quad &: \quad = \overset{\bullet}{(X \cap Y)}, \\ \overset{\bullet}{X} \cup \overset{\bullet}{Y} \quad &: \quad = \overset{\bullet}{(X \cup Y)}, \\ \overset{\bullet}{X} \quad &\subset \quad \overset{\bullet}{Y} :\Longleftrightarrow \overset{\bullet}{X} \cap \overset{\bullet}{Y} = \overset{\bullet}{X}. \end{aligned}$$

4. If $\overset{\bullet}{f} \in {}_n\mathcal{O}_0$ is represented by some $f \in \mathcal{O}(U)$, where $U \subset \mathbb{C}^n$ is a neighbourhood of the origin, we put
$$N\left(\overset{\bullet}{f}\right) := \overset{\bullet}{N(f)}.$$

5. If $\overset{\bullet}{X}_0$ is a germ at zero and $\overset{\bullet}{f} \in {}_n\mathcal{O}_0$, we say that $\overset{\bullet}{f}$ *vanishes on* $\overset{\bullet}{X}$ if $\overset{\bullet}{X} \subset N\left(\overset{\bullet}{f}\right)$.

6. The germ $\overset{\bullet}{X}$ is called an *analytic germ* if there are finitely many germs $\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k \in_n \mathcal{O}_0$ such that

$$\overset{\bullet}{X} = \bigcap_{j=1}^{k} N\left(\overset{\bullet}{f}_j\right) = N\left(f_1, \ldots, f_k\right)^{\bullet}.$$

Lemma 8.4.2. If $\overset{\bullet}{X}$ and $\overset{\bullet}{Y}$ are analytic germs, then so are $\overset{\bullet}{X} \cap \overset{\bullet}{Y}$ and $\overset{\bullet}{X} \cup \overset{\bullet}{Y}$.

Proof. Let U, V be neighbourhoods of zero such that $\overset{\bullet}{X}$ and $\overset{\bullet}{Y}$ can be written as

$$\begin{aligned} \overset{\bullet}{X} &= \{z \in U \mid f_1(z) = \cdots = f_k(z) = 0\}^{\bullet}, \\ \overset{\bullet}{Y} &= \{z \in V \mid g_1(z) = \cdots = g_l(z) = 0\}^{\bullet}. \end{aligned}$$

Then

$$\overset{\bullet}{X} \cap \overset{\bullet}{Y} = \{z \in U \cap V \mid f_1(z) = \cdots = f_k(z) = g_1(z) = \cdots = g_l(z) = 0\}^{\bullet}$$

and

$$\overset{\bullet}{X} \cup \overset{\bullet}{Y} = N\left(\left\{\overset{\bullet}{f}_i \overset{\bullet}{g}_j \mid 1 \le i \le k, 1 \le j \le l\right\}\right). \qquad \square$$

One of the problems when dealing with analytic sets, i.e., solutions of holomorphic equations, is that there is no structure on these sets that allow one to make computations. A very common way to solve this problem is to translate questions of analysis or of geometry into the language of Algebra. This is basically what algebraic geometry is about, where solutions of polynomial equations are studied by examination of associated ideals in the ring of polynomials. We copy this idea for our purposes.

Proposition 8.4.3. Let $\overset{\bullet}{X}$ be an analytic germ at zero and $A \subset \mathcal{O}_0$. We define

$$\mathfrak{I}\left(\overset{\bullet}{X}\right) := \left\{\overset{\bullet}{f} \in \mathcal{O}_0 \,\middle|\, \overset{\bullet}{X} \subset N\left(\overset{\bullet}{f}\right)\right\}$$

and

$$N(A) := \bigcap_{\overset{\bullet}{f} \in A} N\left(\overset{\bullet}{f}\right).$$

Then the following holds:

1. The set $\mathfrak{I}\left(\overset{\bullet}{X}\right)$ is an ideal on $\mathcal{O}_0$, called the *vanishing ideal of* $\overset{\bullet}{X}$.

2. The set $N(A) = N((A))$ is an analytic germ.

Proof. 1. Let $\overset{\bullet}{f}, \overset{\bullet}{g} \in \mathfrak{I}\left(\overset{\bullet}{X}\right)$ and $\overset{\bullet}{h} \in \mathcal{O}_0$. Then

$$\overset{\bullet}{X} \subset N\left(\overset{\bullet}{f}\right) \cap N\left(\overset{\bullet}{g}\right) = N\left(\overset{\bullet}{f}, \overset{\bullet}{g}\right) \subset N\left(\overset{\bullet}{f} + \overset{\bullet}{g}\right)$$

and

$$\overset{\bullet}{X} \subset N\left(\overset{\bullet}{f}\right) \cup N\left(\overset{\bullet}{h}\right) = N\left(\overset{\bullet}{f}\overset{\bullet}{h}\right).$$

This shows that $\overset{\bullet}{f} + \overset{\bullet}{g} \in \mathfrak{I}\left(\overset{\bullet}{X}\right)$ and $\overset{\bullet}{f}\ \overset{\bullet}{h} \in \mathfrak{I}\left(\overset{\bullet}{X}\right)$, i.e., $\mathfrak{I}\left(\overset{\bullet}{X}\right)$ is an ideal in $\mathcal{O}_0$.

2. The ideal (A) generated by the elements of A can be written as

$$(A) = \left\{ \sum_{\text{finite}} \overset{\bullet}{f}_j \overset{\bullet}{h}_j \,\middle|\, \overset{\bullet}{f}_j \in A, \overset{\bullet}{h}_j \in \mathcal{O}_0 \right\},$$

since $\mathcal{O}_0$ is a commutative ring with 1. From Theorem 8.3.4 we know that $\mathcal{O}_0$ is Noetherian, so there are finitely many function germs $\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k \in A$, which generate (A). Hence,

$$N\left((A)\right) = N\left(\left(\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k\right)\right).$$

Let $\overset{\bullet}{f} \in (A)$. Then

$$\overset{\bullet}{f} = \sum_{j=1}^{k} \overset{\bullet}{f}_j \overset{\bullet}{h}_j,$$

which implies

$$N\left(\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k\right) \subset N\left(\overset{\bullet}{f}\right),$$

i.e., $N\left((A)\right) \subset N\left(A\right)$. Now, for all $j = 1, \ldots, k$, we have $\left\{\overset{\bullet}{f}_j\right\} \subset A$. From this we conclude

$$N\left(A\right) \subset N\left(\left\{\overset{\bullet}{f}_j\right\}\right) = N\left(\overset{\bullet}{f}_j\right),$$

so

$$N\left(A\right) \subset \bigcap_{j=1}^{k} N\left(\overset{\bullet}{f}_j\right) = N\left((A)\right).$$

It can be shown easily that $N\left((A)\right)$ does not depend on the choice of generators of (A). The reader may carry out the details. $\square$

Example 8.4.4. The vanishing ideal of the set germ defined by the origin is

$$\begin{aligned} \mathfrak{I}\left(\overset{\bullet}{\{0\}}\right) &= \left\{\overset{\bullet}{f} \in \mathcal{O}_0 \mid \overset{\bullet}{f} \text{ vanishes on } \overset{\bullet}{\{0\}}\right\} \\ &= \{f \text{ holomorphic near } 0 \mid f(0) = 0\} \\ &= \mathfrak{m}, \end{aligned}$$

the unique maximal ideal in $\mathcal{O}_0$.

Example 8.4.5.

$$N(\mathcal{O}_0) = \bigcap_{\overset{\bullet}{f} \in \mathcal{O}_0} N\left(\overset{\bullet}{f}\right) = \emptyset,$$

because $\mathcal{O}_0$ contains the constants.

Exercise 8.4.6. Let $\overset{\bullet}{X}, \overset{\bullet}{Y}$ be analytic germs and let $\mathfrak{a}, \mathfrak{b}$ be ideals in $\mathcal{O}_0$. Show the following:

1. If $\overset{\bullet}{X} \subset \overset{\bullet}{Y}$, then $\mathfrak{I}\left(\overset{\bullet}{Y}\right) \subset \mathfrak{I}\left(\overset{\bullet}{X}\right)$.
2. If $\overset{\bullet}{X} \neq \overset{\bullet}{Y}$, then $\mathfrak{I}\left(\overset{\bullet}{X}\right) \neq \mathfrak{I}\left(\overset{\bullet}{Y}\right)$.
3. If $\mathfrak{a} \subset \mathfrak{b}$, then $N(\mathfrak{b}) \subset N(\mathfrak{a})$.
4. The inclusion $\mathfrak{a} \subset \mathfrak{I}(N(\mathfrak{a}))$ holds.

Exercise 8.4.7. An analytic germ $\overset{\bullet}{X}$ is said to be *reducible* if there is a decomposition

$$\overset{\bullet}{X} = \overset{\bullet}{Y} \cup \overset{\bullet}{Z},$$

where $\overset{\bullet}{Y}, \overset{\bullet}{Z}$ are analytic germs properly contained in $\overset{\bullet}{X}$. If no such decomposition exists, $\overset{\bullet}{X}$ is said to be *irreducible.*

1. Show that an analytic germ $\overset{\bullet}{X}$ is irreducible if and only if its vanishing ideal $\mathfrak{I}\left(\overset{\bullet}{X}\right)$ is a prime ideal in $\mathcal{O}_0$.
2. Is the germ $\overset{\bullet}{X}$ at zero defined by the set

$$X := \left\{(z, w) \in \mathbb{C}^2 \mid zw = 0\right\}$$

irreducible? If X is reducible, determine its irreducible components.

8.4.2 The radical of an ideal

In the introductory example we saw that the condition (8.9) that the zero set of some function f is contained in the zero set of a function g leads to the algebraic statement that f divides a certain power of g, or, equivalently, that a power g^m of g lies in the ideal (f) generated by f. This observation leads us to the following definition.

Definition 8.4.8. Let R be a commutative ring with 1 and let $\mathfrak{I}$ be an ideal in R. The *radical of* $\mathfrak{I}$ *in* R is defined as

$$\operatorname{rad}\mathfrak{I} := \{x \in R \mid \text{There is some } m = m_x \in \mathbb{N}, \text{ such that } x^m \in \mathfrak{I}\}.$$

Example 8.4.9. Consider the ideal $4\mathbb{Z} = (4) \subset \mathbb{Z}$. Then

$$\operatorname{rad}(4) = \{k \in \mathbb{Z} \mid \text{There is some } m \in \mathbb{N}, \text{ such that } 4|k^m\} = (2) = 2\mathbb{Z}.$$

Example 8.4.10. Let $\mathfrak{m} \subset R_n$ be the maximal ideal. Then

$$\begin{aligned} \operatorname{rad}\mathfrak{m} &= \{f \in R_n \mid \text{There is some } m \in \mathbb{N}, \text{ such that } f^m(0) = 0\} \\ &= \{f \in R_n \mid f(0) = 0\} \\ &= \mathfrak{m}. \end{aligned}$$

Lemma 8.4.11. Let R and $\mathfrak{I}$ be as in Definition 8.4.8. Then

1. $\mathfrak{I} \subset \operatorname{rad}\mathfrak{I}$.
2. The radical $\operatorname{rad}\mathfrak{I}$ is an ideal in R.
3. If $\mathfrak{I}$ is a prime ideal, then $\operatorname{rad}\mathfrak{I} = \mathfrak{I}$.

Proof. 1. This is clear, because $x = x^1$ for all $x \in R$.

2. Let $x, y \in \operatorname{rad}\mathfrak{I}$. Then there are $m_x, m_y \in \mathbb{N}$ such that

$$x^{m_x}, y^{m_y} \in \mathfrak{I}.$$

Let $m := m_x + m_y$. Since R is commutative we can apply the Binomial Theorem:

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}.$$

If $k < m_x$ then $m - k > m_y$ and since $\mathfrak{I}$ is an ideal this implies that

$$x^k y^{m-k} \in \mathfrak{I}$$

for all $k = 0, \ldots, m$, hence, since $\mathfrak{I}$ is additively closed, we have

$$x + y \in \operatorname{rad}\mathfrak{I}.$$

Now let $r \in R$ be an arbitrary element. Since $x^{m_x} \in \mathfrak{I}$ and $\mathfrak{I}$ is an ideal we also have

$$r^{m_x} x^{m_x} \in \mathfrak{I},$$

i.e., $rx \in \operatorname{rad} \mathfrak{I}$. Hence, $\operatorname{rad} \mathfrak{I}$ is an ideal in R.

3. Let $\mathfrak{I}$ be a prime ideal and let $x \in \operatorname{rad} \mathfrak{I}$. Then $x^m \in \mathfrak{I}$ for some $m \in \mathbb{N}$. Assume that $m > 1$. Then

$$x^m = x x^{m-1} \in \mathfrak{I}.$$

Since $\mathfrak{I}$ is a prime ideal, this implies that $x \in \mathfrak{I}$ or $x^{m-1} \in \mathfrak{I}$. If $x \in \mathfrak{I}$ we are done, otherwise we proceed by induction to see that $x \in \mathfrak{I}$. Thus,

$$\operatorname{rad} \mathfrak{I} \subset \mathfrak{I}.$$

Together with 1. this proves 3. □

Proposition 8.4.12. Let $\overset{\bullet}{X}$ be an analytic germ. Then

$$N\left(\mathfrak{I}\left(\overset{\bullet}{X}\right)\right) = \overset{\bullet}{X}.$$

Proof. Let $\mathfrak{I}\left(\overset{\bullet}{X}\right)$ be generated by $\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k$. Then

$$\begin{aligned} N\left(\mathfrak{I}\left(\overset{\bullet}{X}\right)\right) &= N\left(\overset{\bullet}{f}_1, \ldots, \overset{\bullet}{f}_k\right) \supset \overset{\bullet}{X}, \\ \overset{\bullet}{X} &= N\left(\overset{\bullet}{g}_1, \ldots, \overset{\bullet}{g}_l\right) \subset N\left(\overset{\bullet}{g}_j\right) \text{ for all } j = 1, \ldots, l. \end{aligned}$$

Therefore, $\overset{\bullet}{g}_j \in \mathfrak{I}\left(\overset{\bullet}{X}\right)$ for all $j = 1, \ldots, l$. If $N\left(\mathfrak{I}\left(\overset{\bullet}{X}\right)\right)$ were not contained in $\overset{\bullet}{X}$ there would be some

$$Y \in N\left(\mathfrak{I}\left(\overset{\bullet}{X}\right)\right) \setminus \overset{\bullet}{X}.$$

This would imply that $Y \notin N\left(\overset{\bullet}{g}_j\right)$ for all j, i.e.,

$$\overset{\bullet}{g}_j|_Y \neq 0,$$

which contradicts the fact that $\overset{\bullet}{g}_j \in \mathfrak{I}\left(\overset{\bullet}{X}\right)$ and $Y \in N\left(\mathfrak{I}\left(\overset{\bullet}{X}\right)\right)$. □

Remark 8.4.13. Proposition 8.4.12 gives rise to considering the following. Let $\mathfrak{X}$ be the set of all analytic set germs at zero and let $\mathfrak{J}$ be the set of all ideals in $\mathcal{O}_0$. Then we have mappings

$$N : \mathfrak{J} \to \mathfrak{X}, \mathfrak{I} \mapsto N(\mathfrak{I})$$

and

$$\mathfrak{I} : \mathfrak{X} \to \mathfrak{J}, \overset{\bullet}{X} \mapsto \mathfrak{I}\left(\overset{\bullet}{X}\right).$$

Proposition 8.4.12 says that the composition $N \circ \mathfrak{I}$ is the identity mapping on $\mathfrak{X}$. In particular, $\mathfrak{I}$ is injective and N is surjective. It is now a natural question to ask what the composition $\mathfrak{I} \circ N$ looks like and whether there is a one-to-one relation between analytic germs and ideals in $\mathcal{O}_0$. Hilbert's Nullstellensatz answers this question.

As we have seen in Example 8.4.9 it is not always true that $\mathfrak{I} = \operatorname{rad} \mathfrak{I}$ for an ideal $\mathfrak{I}$, whereas this equation holds, for example, if $\mathfrak{I}$ is a prime ideal. Our next aim is to find a general way to express the radical of an ideal.

Proposition 8.4.14. Let R be a commutative ring with 1 and let $x \in R$. Then the following are equivalent:

1. x is nilpotent, i.e., there is some $m \in \mathbb{N}$ such that $x^m = 0$.
2. $x \in \mathfrak{p}$ for all prime ideals $\mathfrak{p} \lhd R$.

Proof. 1. $\Rightarrow$ 2. If $x^m = 0$ for some $m \in \mathbb{N}$ then

$$xx^{m-1} \in \mathfrak{p}$$

for all prime ideals $\mathfrak{p}$. Since $\mathfrak{p}$ is prime we have $x \in \mathfrak{p}$ or $x^{m-1} \in \mathfrak{p}$. If $x \in \mathfrak{p}$ we are done. If $x^{m-1} \in \mathfrak{p}$ we proceed by induction to find again that $x \in \mathfrak{p}$ for all prime ideals $\mathfrak{p}$.

2. $\Rightarrow$ 1. Assume that $x^m \neq 0$ for all $m \in \mathbb{N}$. We consider the multiplicatively closed set

$$S := \left\{x^j \mid j \in \mathbb{N}\right\}$$

and the set

$$G := \{\, \mathfrak{b} | \, \mathfrak{b} \text{ ideal in } R, \mathfrak{b} \cap S = \emptyset\}.$$

Then G is inductively ordered by inclusion and $G \neq \emptyset$, because $(0) \in G$. If

$$\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \cdots$$

is an ascending chain of elements of G, their union

$$\mathfrak{b} := \bigcup_{j \geq 0} \mathfrak{b}_j$$

is an ideal, which satisfies

$$\mathfrak{b} \cap S = \bigcup_{j \geq 0} (\mathfrak{b}_j \cap S) = \emptyset,$$

so $\mathfrak{b} \in G$. By Zorn's Lemma, G has a maximal element $\widetilde{\mathfrak{p}}$. We claim that $\widetilde{\mathfrak{p}}$ is a prime ideal. Assume the contrary. Then there are $u, v \in R \setminus \widetilde{\mathfrak{p}}$ such that

$$uv \in \widetilde{\mathfrak{p}}.$$

Denote by $\left(\widetilde{\mathfrak{p}}, u\right)$ the smallest ideal in R which contains $\widetilde{\mathfrak{p}}$ and u. Then

$$\widetilde{\mathfrak{p}} \subsetneq \left(\widetilde{\mathfrak{p}}, u\right) \notin G,$$

because $\widetilde{\mathfrak{p}}$ is maximal in G. This implies that

$$\left(\widetilde{\mathfrak{p}}, u\right) \cap S \neq \emptyset.$$

Analogously,

$$\left(\widetilde{\mathfrak{p}}, v\right) \cap S \neq \emptyset.$$

Then there are $m_1, m_2 \in \mathbb{N}$ such that

$$x^{m_1} \in \left(\widetilde{\mathfrak{p}}, u\right) \cap S,\ x^{m_2} \in \left(\widetilde{\mathfrak{p}}, v\right) \cap S.$$

Hence, there are $\alpha, \beta, \alpha', \beta' \in R$, $p, p' \in \widetilde{\mathfrak{p}}$ such that

$$x^{m_1} = \alpha p + \beta u,\ x^{m_2} = \alpha' p' + \beta' v.$$

Since S is multiplicatively closed we have

$$S \ni x^{m_1} x^{m_2} = \alpha\beta' vu + \beta\alpha' up' + \alpha\alpha' pp' + \beta\beta' uv \in \widetilde{\mathfrak{p}},$$

i.e., $S \cap \widetilde{\mathfrak{p}} \neq \emptyset$, which contradicts $\widetilde{\mathfrak{p}} \in G$. Hence, $\widetilde{\mathfrak{p}}$ is a prime ideal. By prerequisite this implies that $x \in \widetilde{\mathfrak{p}}$, but this contradicts the fact that no power of x meets $\widetilde{\mathfrak{p}}$. Hence, $x^m = 0$ for some $m \in \mathbb{N}$. $\square$

Corollary 8.4.15. *The radical* $\operatorname{rad}\mathfrak{J}$ *of an ideal* $\mathfrak{J}$ *is the intersection of all prime ideals* $\mathfrak{p}$ *which contain* $\mathfrak{J}$:

$$\operatorname{rad}\mathfrak{J} = \bigcap_{\substack{\mathfrak{p} \supset \mathfrak{J} \\ \mathfrak{p} \text{ prime ideal } \subset R}} \mathfrak{p}.$$

Proof. Apply Proposition 8.4.14 to the factor ring $R/_{\mathfrak{J}}$. $\square$

Exercise 8.4.16. Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_0$.

1. Prove the equation $N(\mathfrak{a}) = N(\operatorname{rad}\mathfrak{a})$.
2. Show that $\operatorname{rad}\mathfrak{a} \subset \mathfrak{J}(N(\mathfrak{a}))$.

Exercise 8.4.17. Find an example of a commutative ring R with unit element 1 and an ideal $\mathfrak{J} \lhd R$ with the following two properties:

1. The equation $\operatorname{rad}\mathfrak{J} = \mathfrak{J}$ holds.
2. The ideal $\mathfrak{J}$ is not a prime ideal in R.

Exercise 8.4.18. Let $\overset{\bullet}{f} \in \mathcal{O}_0, \overset{\bullet}{f} \neq \overset{\bullet}{0}$. Show that there is a prime ideal $\mathfrak{p}$ in $\mathcal{O}_0$, which contains no power of $\overset{\bullet}{f}$.

8.4.3 Hilbert's Nullstellensatz for principal ideals

Hilbert's Nullstellensatz in the analytic version states that

$$\mathfrak{I}(N(\mathfrak{a})) = \operatorname{rad}\mathfrak{a}$$

for all ideals $\mathfrak{a}$ in $\mathcal{O}_0$, i.e., there is a bijective relationship between analytic germs and radical ideals, i.e., ideals, which coincide with their radicals. In particular, this establishes a one-to-one correspondence between irreducible germs of analytic sets and their vanishing ideals, as was shown in Exercise 8.4.7. With the knowledge we have gained so far, however, we are only able to prove Hilbert's Nullstellensatz in the case where $\mathfrak{a}$ is a principal ideal. Those readers interested in a proof of the general case are referred to [6], §47. We need another auxiliary result.

Lemma 8.4.19. Let $f, g \in R_n$ have greatest common divisor $\gcd(f,g) = 1$. Then there are a shearing σ, germs $\lambda, \mu \in R_n$ and $p \in R_{n-1}$, $p \neq 0$ such that

$$p = \lambda\sigma^*(f) + \mu\sigma^*(g).$$

Proof. From Proposition 8.2.4 we obtain a shearing σ such that $\sigma^*(f)$ and $\sigma^*(g)$ are z_n-general, so by the Weierstrass Preparation Theorem we find units $u, v \in_n \mathcal{O}_0$ and Weierstrass polynomials $P, Q \in R_{n-1}[z_n]$ such that

$$\sigma^*(f) = uP,\ \sigma^*(g) = vQ. \tag{8.11}$$

Then

$$\begin{aligned} 1 &= \gcd(f,g) \\ &= \gcd(\sigma^*(f), \sigma^*(g)) \\ &= \gcd(uP, vQ) \\ &= \gcd(P,Q), \end{aligned}$$

i.e., P, Q are relatively prime in R_n. Thus, they are also relatively prime in $R_{n-1}[z_n]$. Let $\mathbb{F}$ be the quotient field of R_{n-1}. Then there are some $P', Q', T \in \mathbb{F}[z_n]$ with

$$\deg T < \min\{\deg P, \deg Q\}$$

such that

$$P = P'T,\ Q = Q'T.$$

If we write

$$T(z_1,\ldots,z_n) = \sum_{j=0}^{\deg T} \frac{f_j(z_1,\ldots,z_{n-1})}{g_j(z_1,\ldots,z_{n-1})} z_n^j,$$

we find that

$$\left(\prod_{j=0}^{\deg T} g_j\right) T(z) = \sum_{j=0}^{\deg T} f_j(z') z_n^j \prod_{\substack{k=0 \\ k\neq j}}^{\deg T} g_k(z').$$

This implies that P, Q are also relatively prime in $\mathbb{F}[z_n]$. Since $\mathbb{F}[z_n]$ is a principal ideal ring we conclude that there are $A, B \in \mathbb{F}[z_n]$ such that

$$\gcd(P, Q) = 1 = AP + BQ.$$

We can then find $\alpha, \beta \in R_{n-1}$ and $\lambda', \mu' \in R_{n-1}[z_n]$ such that

$$A = \frac{\lambda'}{\alpha},\ B = \frac{\mu'}{\beta}. \tag{8.12}$$

Putting

$$p := \alpha\beta,\ \lambda := \lambda'\beta u^{-1},\ \mu := \mu'\alpha v^{-1}$$

we obtain from (8.11) and (8.12) that

$$p = \lambda\sigma^*(f) + \mu\sigma^*(g),$$

which proves the lemma. □

Theorem 8.4.20 (Hilbert's Nullstellensatz). *For all $\overset{\bullet}{f} \in \mathcal{O}_0$ we have*

$$\mathfrak{I}\left(N\left(\overset{\bullet}{f}\right)\right) = \operatorname{rad}\left(\overset{\bullet}{f}\right).$$

Proof. We identify the germ $\overset{\bullet}{f} \in \mathcal{O}_0$ with the Taylor series of a representantive $f \in R_n$. Since R_n is a factorial ring we can decompose f into prime powers

$$f = f_1^{\nu_1} \cdots f_r^{\nu_r},$$

so

$$N(f) = \bigcup_{j=1}^{r} N\left(p_j^{\nu_j}\right) = \bigcup_{j=1}^{r} N(p_j).$$

Thus, using the result from Exercise 8.4.16 we have

$$\begin{aligned}
\operatorname{rad}(f) \quad &\subset \quad \mathfrak{I}(N(f)) = \mathfrak{I}\left(\bigcup_{j=1}^{r} N(p_j)\right) \\
&= \quad \bigcap_{j=1}^{r} \mathfrak{I}(N(p_j)).
\end{aligned}$$

Let $\mathfrak{p}_j := (p_j)$ be the principal ideal generated by p_j and let

$$x \in \bigcap_{j=1}^{r} \mathfrak{p}_j.$$

Then for every $j = 1, \ldots, r$ there is some $r_j \in R_n$ such that

$$x^{\nu_j} = r_j^{\nu_j} p_j^{\nu_j},$$

thus,

$$x^{|\nu|} = \prod_{j=1}^{r} x^{\nu_j} = \underbrace{\prod_{j=1}^{r} r_j^{\nu_j}}_{=:r} \prod_{j=1}^{r} p_j^{\nu_j} = rf \in (f).$$

It follows that

$$\bigcap_{j=1}^{r} \mathfrak{p}_j \subset \operatorname{rad}(f),$$

so what is left is a proof that $\mathfrak{I}(N(p)) = (p)$ for an irreducible $p \in R_n$. Furthermore, we know from Proposition 8.2.4 and from the Weierstrass Preparation Theorem that it is no loss of generality if we assume that p is an irreducible Weierstrass polynomial. Let $f \in \mathfrak{I}(N(p))$. We have to distinguish two cases.

Case 1: $\gcd(p, f) = p$.

This means that p divides f in R_n, so $f \in (p)$.

Case 2: $\gcd(p, f) = 1$.

We claim that this cannot happen. Use Lemma 8.4.19 to find a shearing σ and the mentioned $\lambda, \mu \in R_n, q \in R_{n-1}[z_n]$, $q \neq 0$, such that

$$q = \lambda\sigma^*(f) + \mu\sigma^*(p). \tag{8.13}$$

There is some $s > 0$ such that all representatives from equation (8.13) are holomorphic on the symmetric polydisc $P_s^n(0)$. On $P_s^n(0)$ we have

$$\sigma(0) = 0,\ \sigma^* p(0, \ldots, 0, z_n) = z_n^m$$

for some $m \in \mathbb{N}$. Lemma 8.2.9 yields that there is a polydisc

$$P_\rho^n(0) \subset P_s^n(0)$$

such that for all $(z_1, \ldots, z_{n-1}) \in P^{n-1}_{(\rho_1, \ldots, \rho_{n-1})}(0')$ the function

$$z_n \mapsto \sigma^*(p)(z_1, \ldots, z_n)$$

has exactly m zeroes in the one-dimensional disc defined by $|z_n| < \rho_n$. Now we have the result that if $f \in \mathfrak{I}(N(p))$, then

$$\sigma^*(f) \in \mathfrak{I}(N(\sigma^*(p))),$$

so

$$\sigma^*(f)|_{N(\sigma^*(p))} = 0.$$

Applying this to (8.13) we find that

$$q(z') = 0 \text{ for all } z' \in P^{n-1}_{(\rho_1,\ldots,\rho_{n-1})}(0').$$

The Identity Theorem implies $q = 0$, which contradicts the choice of q. This shows that the case $\gcd(p, f) = 1$ is impossible. $\square$

Exercise 8.4.21. Determine generators for the vanishing ideals of the following analytic set germs at zero:

1. $$\overset{\bullet}{X} := \left\{(z, w) \in \mathbb{C}^2 \mid z^2 = w^3\right\}^{\bullet}$$

2. $$\overset{\bullet}{Y} := \left\{(z, w) \in \mathbb{C}^2 \mid z^2 = w^2\right\}^{\bullet}$$

Register of Symbols

$(z\|w)$	standard inner product
$\\|z\\|$	norm
$[x,y]\,,\,]x,y[$	closed resp. open segment or interval
$B_r^n(a)$	ball in $\mathbb{C}^n$ with center a and radius r
$P_r^n(a)$	in $\mathbb{C}^n$ with center a and polyradius r
$T_r^n(a)$	in $\mathbb{C}^n$ with center a and polyradius r
$\mathbb{T}^n$	unit polytorus
$M(m,n;\mathbb{C})$	set of complex $m \times n-$ matrices
$GL_n(\mathbb{C})$	group of regular $n \times n-$ matrices
$U_n(\mathbb{C})$	group of unitary $n \times n-$ matrices
$P_n(\mathbb{C})$	set of positively definite $n \times n-$ matrices
∂X	boundary of the topological space X
$\partial_{ex} X$	set of extremal points of X
$Df(a)\,, f'(a)$	derivative of f at a
$\mathcal{O}(X,Y)$	space of holomorphic mappings $f : X \to Y$
$\mathcal{O}(D)$	algebra of holomorphic functions $f : D \to \mathbb{C}$
pr_k	projection onto the $k-th$ coordinate
$\mathbb{C}[z_1,\ldots,z_n]$	algebra of complex polynomials in n variables
$\mathbb{C}[[z_1,\ldots,z_n]]$	algebra of formal power series
$\mathbb{C}\{z_1,\ldots,z_n\}$ or R_n	algebra of convergent power series
$N(\mathcal{F})$	zero set of the family $\mathcal{F}$
$\mathcal{C}^k(X,Y)$	$k-$ times real differentiable mappings $f : X \to Y$
$d_a f$	real differential of f at a
$V^\#$	algebraic dual space of the vector space V
$\partial f, \overline{\partial} f$	complex rsp. conjugate-complex differential of f
$D^\alpha f, \frac{\partial^{\|\alpha\|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$	partial derivatives of f
$\tau(D)$	absolute space of D
ρ	restriction mapping
$B^n(r,R)$	ball shell in $\mathbb{C}^n$ with radii r and R

$P^n(r,R)$	polydisc shell in $\mathbb{C}^n$ with polyradii r and R
$\operatorname{Aut}(D)$	group of automorphisms of D
$\operatorname{codim}_a A$	codimension of the analytic set A at the point a
$\operatorname{codim} A$	codimension of the analytic set A
$\mathbb{S}^{2n-1}$	Euclidian unit sphere in $\mathbb{C}^n$
$\bigwedge^k V$	$k-th$ outer product of the vector space V
$d\omega$	outer derivative of the differential form ω
$\Omega^p(X)$	set of holomorphic differential forms of degree p
$\widetilde{W}^a$	connected component of the point a in W
$\operatorname{con}(X)$	convex hull of X
$\Omega + i\mathbb{R}^n$	tubular domain with basis Ω
$\widehat{K}_U$	holomorphically convex hull of K in U
$\operatorname{dist}_\infty(X,Y)$	distance of X and Y with respect to $\Vert.\Vert_\infty$
$\mathcal{O}^X$	set of holomorphic functions on a subset $X \subset \mathbb{C}^n$
$\overset{\bullet}{f}_a, \overset{\bullet}{f}$	germ of the function f at a resp. at an arbitrary point
$\mathcal{O}_X, \mathcal{O}_a$	algebra of function germs on the set X resp. at a
$\tau_a f$	translate of f by a
$\operatorname{ord} f$	order of a power series f
$\mathfrak{m}$	maximal ideal in R_n
σ	shearing or permutation
$\overset{\bullet}{X}$	germ of the set X
$\mathfrak{I}\left(\overset{\bullet}{X}\right)$	vanishing ideal of the germ $\overset{\bullet}{X}$
$\operatorname{rad}\mathfrak{I}$	radical of the ideal $\mathfrak{I}$
$A \subset\subset B$	A is a relatively compact subset of B

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